

Imperial College of Science, Technology and Medicine
Department of Physics

Non-Gaussian quantum states of optical and mechanical modes: from generating and verifying entanglement to wavefunction collapse

Lydia A. Kanari-Naish

Supervised by
Dr. Michael R. Vanner & Dr. Sofia Qvarfort

Submitted in part fulfilment of the requirements for the degree of
Doctor of Philosophy in Physics at Imperial College London, April 2023

Abstract

Non-Gaussianity plays an important role in quantum information theory and testing outstanding foundational problems. Cavity quantum optomechanics provides a promising experimental platform for quantum science and technology. An outstanding challenge is the generation and verification of a two-mode Schrödinger-cat state in the displacement of two mechanical oscillators. We introduce a protocol to create this state and verify its non-Gaussian entanglement. These entangled states may find exciting applications including fault-tolerant quantum computation, and quantum metrology.

To verify the presence of entanglement, inseparability criteria are explored. The second portion of research builds on this theme: we examine the broader question of choosing appropriate inseparability criteria for arbitrary non-Gaussian states — a typically challenging problem as higher-order moments are required. Focusing on inseparability criteria derived from the property that separable states have a positive partial transpose, we present a statistical framework to quantify the error and consequently the success of such tests. To demonstrate the scope of this method, we identify experimentally-amenable criteria which are tailored towards verifying entanglement in two specific non-Gaussian states; one mechanical and one optical. We also study an optical scheme to measure the relevant bipartite statistical moments needed to assess these criteria.

Thirdly, we explore whether the displacemon device (an electromechanical device consisting of a vibrating nanobeam coupled to a superconducting qubit) can be used to test objective collapse models which have been offered as possible resolutions to the measurement problem and the quantum-to-classical transition. We study two prominent examples of collapse models: Continuous Spontaneous Localization (CSL), and Diosi-Penrose (DP). To investigate these models, a protocol involving qubit manipulations is developed to both generate mechanical non-Gaussian states and probe the subsequent decoherence. We study the experimental requirements necessary to isolate CSL and DP collapse over conventional decoherence.

Acknowledgements

I would like to express immense gratitude to my supervisor Michael Vanner for being ever-supportive and providing clear and consistent guidance. From the start of my PhD he helped to instil in me a comprehensive vision of the research landscape, while also encouraging me to explore topics that piqued my interest. My second supervisor Sofia Qvarfort has been incredibly generous with her time and advice and was an excellent host in Stockholm. In addition to her supervisory role, Sofia has been an incredible role-model and mentor, being ever-patient and always taking my silly questions seriously. I am also indebted to my friend and fellow student Jack Clarke who introduced me to so many of the mathematical tools I used throughout my PhD and continues to be a source of inspiration and knowledge.

I am thankful to have had a network of friends and colleagues that have motivated me during uncertain times. I was fortunate to have the chance to collaborate with the more experimentally-inclined members of my research group: Andreas Svela, John Price, Lars Freisem, and Niall Moroney. I am thankful to more recent members who continue to make the group an intellectually-nurturing environment: Evan Cryer-Jenkins, Harsh Rathee, and Kyle Major. Thanks for all the memories (of India, 18 Gloucester Road, SW7 4RB)! Beyond the research group, the collaboration with Edward Laird has been one of the most fast-paced and exciting projects of my academic career.

I would like to also thank the Controlled Quantum Dynamics CDT for easing the transition from undergraduate to postgraduate life and the EPSRC for the generous funding I received. I would also like to thank Nordita for hosting my research visit to Stockholm and giving me several opportunities to present my work. Through a happy coincidence this coincided with International Cinnamon Bun Day (who knew this was a thing?), ensuring that my visit involved many sugar-fuelled seminars (fikas) and interesting discussions.

On a personal note, I am forever grateful to Alistair Smith for his support these past four years. And of course, I probably would not be submitting my thesis without the enduring love of my mother, Soteroula, who will be very glad I mentioned her last!

Statement of Originality

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except where specified. Parts of this work have appeared in the following publications and conferences:

Publications

- *Towards phase-sensitive measurement-based mechanical time-bin state preparation using Franson-type interferometry*
J. J. Price, L. A. Kanari-Naish, A. Ø. Svela, L. Freisem, M. R. Vanner, (Manuscript in preparation)
- *Negative partial transpose entanglement criteria: a framework for selection*
L. A. Kanari-Naish, J. Clarke, S. Qvarfort, M. R. Vanner, (Manuscript in preparation)
- *Two-mode Schrödinger-cat states with nonlinear optomechanics: generation and verification of non-Gaussian mechanical entanglement*
L. A. Kanari-Naish, J. Clarke, S. Qvarfort, M. R. Vanner, Quantum Sci. Technol. **7** (3), 035012, (2022)
- *Can the displacemon device test objective collapse models?*
L. A. Kanari-Naish, J. Clarke, M. R. Vanner, E. A. Laird, AVS Quantum Sci. **3** (4), 045603, (2021)

Conferences

Two-mode Schrodinger-cat states with nonlinear optomechanics: generation and verification of non-Gaussian mechanical entanglement:

- Presented poster at UniKORN virtual poster session (2021)
- Presented prize-winning poster at WOMBAT (Workshop on Optomechanics and Brillouin Scattering at the Max-Planck Institute for the Science of Light in Erlangen) (2022)
- Presented poster at Quantum characterization and control of quantum complex systems workshop, organised by Lake Como School of Advanced Studies (2022)

Copyright Declaration

The copyright of this thesis rests with the author. Unless otherwise indicated, its content are licensed under a Creative Commons Attribution-NonCommercial 4.0 International Licence(CC BY-NC). Under this licence, you may copy and redistribute the material in any medium or format. You may also create and distribute modified versions of the work. This is on the condition that: you credit the author and do not use it, or any derivative works, for a commercial purpose. When reusing or sharing this work, ensure you make the licence terms clear to others by naming the licence and linking to the licence text. Where a work has been adapted, you should indicate that the work has been changed and describe those changes. Please seek permission from the copyright holder for uses of this work that are not included in this licence or permitted under UK Copyright Law.

Thesis summary

In classical physics, a particle can exist in a well-defined state. We can precisely specify all the attributes needed to deterministically recreate it; for example, its position, momentum, energy, spin, and so on. In contrast, a quantum mechanical particle, or more generally a quantum system can exist in a superposition of observable outcomes. This superposition principle gives rise to distinctly quantum phenomena. For example, a massive object existing in a superposition exhibits wave-like properties with testable signatures such as a diffraction pattern when passed through a double-slit. When a measurement is performed on the system, the detector registers only one of many permitted outcomes. This outcome is randomly selected from a probability distribution which depends on both the state of the system and the measurement device or setting used.

A natural question arises when considering particles in a superposition: how many particles can exist in a coherent superposition? This question has resided at the heart of quantum mechanics since its inception in the early 20th century. Schrödinger’s famous thought experiment envisions an extreme version of the superposition principle in which a cat, composed of around 200 septillion atoms¹, is prepared in a superposition of being ‘alive’ and ‘dead’ simultaneously in an unopened box. In continuous variable quantum mechanics, the cat’s fate is closely akin to an optical, or mechanical, harmonic oscillator in a superposition of two coherent states, giving the canonical even (+) and odd (−) Schrödinger-cat states

$$|\psi\rangle^{\pm} \propto |\alpha\rangle \pm |-\alpha\rangle . \quad (1)$$

We will explore the mathematics of this in subsequent chapters however here it is sufficient to associate the component $|\alpha\rangle$ with the ‘alive’ state while $|-\alpha\rangle$ is the ‘dead’ state. Such a state can be realised in optical systems comprised of many photons [1, 2] and can be used to encode quantum information in a manner which is resilient to errors induced by photon losses [3].

In mechanical systems the generation of macroscopic cat states, consisting of billions of atoms in a spatial superposition, is an exciting challenge. The motivation to study these non-Gaussian states is fueled in part by their promising quantum information and sensing applications. The

¹Assuming our (un)lucky cat weighs 4 kg and is composed entirely of carbon atoms, a quick calculation yields 2×10^{26} atoms.

desire to explore the classical-to-quantum boundary also motivates the study of such states. Although there can be many particles in a cat state, we describe the system in terms of a single degree of freedom, for example the centre of mass position of the bulk object. Such states can be used to test our conventional understanding of decoherence [4] and, if any deviation from standard quantum theory is detected, determine whether or not it can be ascribed to other phenomena such as gravity. If the system in superposition is of sufficiently large mass, the interplay between gravity and quantum mechanics cannot be neglected [5]. Consequently, the field of cavity quantum optomechanics is well-placed to investigate these fundamental questions with numerous exciting proposals striving to realise a macroscopic cat-state in a mechanical system [6, 7].

So far we have only considered a superposition of a single degree of freedom in a (potentially multi-atom) system. Things become more complicated when our system possesses multiple degrees of freedom, which could be associated to the system as a whole or individual subsystems. When applied to systems with multiple degrees of freedom, the superposition principle leads to the phenomenon of entanglement. An entangled system cannot be described in terms of its individual subsystems ² and can only be described as a whole. Hence, entangled states are often dubbed inseparable. Bell showed that entangled systems can exhibit stronger correlations than are permitted with classical (local hidden variable) theories of physics [8]. One can show that if a Bell-CHSH inequality is violated we can conclude that a state is entangled [9, 10]. However, proving that any entangled state violates a Bell-type inequality is an open research question with important implications for our interpretation of quantum mechanics [11, 12, 13]. The violation of Bell-type inequalities is not the only way to test for entanglement which can also be detected via operational criteria. Such entanglement (or inseparability) criteria are a recurring theme and focus of this thesis.

Entangled states show up in nature, usually with a small number of particles. However, entangling degrees of freedom of microscopic states, comprised of thousands or millions of particles, is much more challenging. Decoherence caused by coupling to a noisy environment makes these states short-lived; nevertheless, the decoherence itself is of great interest. Thus, the generation of macroscopic entangled states is hugely interesting due to their potential to probe the quantum-to-classical boundary. Beyond resolving such foundational questions, quantum entanglement is

²While we use the term subsystem here, which usually refers to a constituent object/particle within a multipartite system, the same description can be used for a single object with many degrees of freedom.

a key resource for surpassing some of the limitations set by classical physics. It has widespread applications, including sensing [14, 15], and quantum communications; for example, quantum key distribution [16], computing [17, 18], networking [19, 20].

This thesis is structured in two parts; in Part I we introduce all the relevant literature and background theory required to follow the results presented in Part II. These results constitute the majority of work completed as part of my PhD research. The focus of this research falls into three principle strands which share common themes.

The first strand of research examines a proposal to generate and verify entanglement in the degrees of freedom of two mechanical oscillators. Specifically, we study a scheme to generate the two-mode version of the Schrödinger-cat state given in Eq. (1); for an introduction to these quantum states see Chapter 1. Our scheme uses cavity quantum optomechanics, specifically exploiting nonlinear optomechanics, to achieve this non-Gaussian entangled state. The physics of optomechanics is outlined in Chapter 2; here we emphasise that cavity quantum optomechanics is an exciting platform in quantum science and technology with applications ranging from quantum-information processing to tests of the foundations of physics. In Chapter 5 we outline and examine our protocol which focuses on the outstanding challenge in optomechanics of observing a canonical two-mode Schrödinger-cat state in the displacement of two mechanical oscillators. Briefly, our scheme uses pulsed light in order to exploit the nonlinearity of the radiation-pressure interaction combined with photon-counting measurements in order to generate this entangled non-Gaussian mechanical state. Importantly, we also describe a protocol using subsequent pulsed interactions to verify the non-Gaussian entanglement generated. Our pulsed verification protocol allows quadrature moments of the two mechanical oscillators to be measured up to any finite order. These statistical moments can be used to detect bipartite entanglement using entanglement criteria. We specifically examine two entanglement criteria and assess the feasibility of experimentally evaluating them. More broadly, our verification protocol provides a tool-set for the experimental characterisation of bipartite mechanical quantum states and furthermore allows for a broad range of inseparability criteria to be evaluated.

Choosing a suitable entanglement criterion for a non-Gaussian entangled state can be a formidable task. The body of work outlined in Chapter 5 uses two entanglement criteria, one of which is frequently-cited in the context of two-mode cat states. Using this criterion is sufficient for the aforementioned protocol since we wished to demonstrate that our verification scheme of

measuring moments could feasibly detect entanglement. Indeed, one of the strengths of our verification protocol is that arbitrary moments can be measured. Thus, using the same experimental set-up, we have access to a plethora of entanglement criteria which are also constructed using moments. This naturally leads to the second strand of my research which was motivated by the desire to thoroughly examine if there exist suitable entanglement criteria (based on moments) which are more closely-tailored to our specific two-mode Schrödinger-cat state, or, more broadly, other non-Gaussian entangled states.

Central to the entanglement criteria which we examine in this thesis is the requirement for a separable quantum state to possess a positive partial transpose. The violation of this condition indicates entanglement and can be translated to a number of experimentally amenable tests. The infinite hierarchy of negative partial transpose (NPT) inequalities introduced by Vogel and Shchukin gives us a necessary and sufficient criterion to test this positivity [21]. The theoretical methods of this approach are outlined in Chapter 3. However, the interplay between experimental factors and feasibility must also be considered when choosing such a condition. In Chapter 6, we provide a method of error propagation which can be used to evaluate the feasibility of an NPT criterion for an arbitrary bipartite state. This feasibility is quantified through hypothesis testing and uses techniques from classical statistics in order to model the sampling error; these methods are discussed in Chapter 4. We carefully consider coupling to the environment through a beam-splitter model, and also estimate the number of measurements required to unambiguously verify entanglement. We use this statistical framework to identify simple NPT criteria which are experimentally-viable for identifying entanglement in both two-mode Schrödinger-cat states, and photon-subtracted two-mode squeezed states. Moreover, the scope of the statistical framework is not limited to these two states alone. Therefore, this work builds on Chapter 5, specifically, how to choose a suitable NPT entanglement criterion for a non-Gaussian entangled state.

As has been emphasised, the moments of a bipartite quantum state are important in characterising its entanglement; therefore, it is crucial that schemes to measure such moments are feasible. While in Chapter 5 we outline a method for measuring arbitrary mechanical moments using a sequence of pulsed verification pulses, more generally it is also important to be able to measure arbitrary moments of a two-mode light state (which perhaps has information pertaining to a mechanical state encoded in its modes). In the latter half of Chapter 6 we examine

a proposal to measure such two-mode moments of light composed of two spatially separated modes. This borrows some key mathematical steps from the verification scheme of Chapter 5, and hinges on homodyne detection to iteratively obtain any moment.

Finally, the third portion of this thesis is dedicated to an electromechanical protocol which probes for wavefunction collapse by generating a single-mode non-Gaussian state encoded in the displacement of a mechanical resonator. Proposals such as this aim to test the limits of the applicability of quantum mechanics and deepen our understanding of the universe. Successful experiments may shed light on the interplay between quantum mechanics and gravity.

The question of how or even whether wavefunctions collapse is often termed the measurement problem. This, together with the precise nature of the quantum-to-classical interface, are fascinating and highly-debated foundational topics in quantum mechanics. Objective collapse models have been offered as a possible solution to these unsolved questions and have gained wide interest in the community as, crucially, they can be tested with experiments. Two prominent examples of collapse models are continuous spontaneous localization (CSL) and the Diósi-Penrose (DP) collapse theory. In CSL theory the Schrödinger equation is supplemented with a nonlinear and stochastic diffusion process which localizes the wavefunction even in the absence of a measurement. The larger the system, the stronger the collapse, thus explaining the emergence of classical phenomena when the system’s mass increases. In the DP collapse model, wavefunction collapse is specifically attributed to gravity. Penrose has argued that this collapse manifests as a result of the mismatch in the spacetimes of particles in spatial superpositions and the incompatibility of general relativity and quantum mechanics.

Tests of objective collapse models aim to detect deviations from standard quantum theory, often through sensing additional sources of heating, which are predicted by collapse models. Numerous exciting proposals and experiments have been produced exploiting a large range of physical platforms including but not limited to optomechanics, superconducting devices, and ultracold atoms. Additionally, many exciting matter-wave interferometric experiments have been designed and proposed to detect unconventional decoherence mechanisms in quantum superposition states.

The “displacemon” is a proposed electromechanical device consisting of a mechanical resonator flux-coupled to a superconducting qubit enabling generation and readout of mechanical quantum states. The design features and Hamiltonian of the displacemon device are introduced

in Chapter 2. In the original displacemon proposal [22] the mechanical resonator was a carbon nanotube, containing 10^6 nucleons. In Chapter 7 we directly build on this work with the specific intention of probing quantum mechanics at a more macroscopic scale. We thus propose using an aluminium mechanical resonator on two larger mass scales, one inspired by the Marshall-Simon-Penrose-Bouwmeester moving-mirror proposal, and one set by the Planck mass. For such a device, we examine the experimental requirements needed to perform a more macroscopic quantum test and thus feasibly detect the decoherence effects predicted by both CSL and DP collapse models. Our protocol for testing these two theories takes advantage of the displacemon architecture to create non-Gaussian mechanical states out of equilibrium with their environment and then analyze the measurement statistics of a superconducting qubit to determine the timescale over which the mechanical state completely decoheres.

Contents

Abstract	i
Acknowledgements	iii
Thesis summary	vii
I Background	1
1 Concepts and tools	2
1.1 Mathematical formalism	2
1.2 Continuous variable systems	8
1.3 Non-Gaussian states	16
1.3.1 Non-Gaussianity quantifier	17
1.3.2 Two-mode Schrödinger-cat state	19
1.3.3 Phonon added/subtracted two-mode squeezed vacuum states	21
2 Cavity optomechanics	22
2.1 Cavity quantum electrodynamics	24

2.2	Quantum optomechanics	25
2.3	Quantum Langevin equations	30
2.3.1	Quantum harmonic oscillator	31
2.3.2	Rotating wave approximation	37
2.4	Pulsed optomechanics	38
2.4.1	Nonlinear pulsed interaction	40
2.4.2	Linear pulsed interaction	45
2.5	Electromechanics	48
2.5.1	Transmon physics	49
2.5.2	The displacemon device	51
2.5.3	The resonator	53
3	Inseparability criteria for non-Gaussian entangled states	56
3.1	Higher-order inseparability criteria	57
3.2	Vogel and Shchukin inseparability criteria	58
4	Classical statistics of quantum measurements	62
4.1	Sample statistics	63
4.1.1	Estimators and sampling distributions	64
4.1.2	Hypothesis testing	66
4.2	Error propagation	68

II	Results	72
5	Generating and verifying two-mode mechanical Schrödinger-cat states	73
5.1	Introduction	73
5.2	Entanglement protocol	75
5.2.1	Finite mechanical evolution during entanglement preparation	84
5.2.2	Optical losses, detector inefficiencies and resolution	88
5.3	Entanglement Verification	96
5.3.1	Inseparability Criteria	97
5.3.2	Verification protocol	99
5.3.3	Open-system dynamics	103
5.4	Results	106
5.5	Discussion	111
5.5.1	Verification of non-Gaussian entanglement	111
5.5.2	Optical effects	113
5.6	Conclusion	114
6	Constructing entanglement criteria for bipartite non-Gaussian states	116
6.1	Introduction	116
6.2	Entanglement criteria	119
6.2.1	Characterizing determinants	119
6.2.2	Invariance of an NPT criterion under rotation	123

6.3	Propagation of uncertainty in NPT criteria	125
6.3.1	Error on the determinant	126
6.3.2	Minimizing the error of an NPT criterion	130
6.4	Loss and open-system dynamics	131
6.5	Simulations	133
6.5.1	Two-mode Schrödinger-cat state	137
6.5.2	Photon-subtracted two-mode squeezed vacuum state	142
6.5.3	Discussion	144
6.6	Towards the measurement of arbitrary two-mode moments of an optical field . .	147
6.6.1	Calculating single-mode moments	150
6.6.2	Alternative measurement scheme	152
6.6.3	Discussion	157
6.7	Conclusion	159
7	Testing wavefunction collapse with the displacemon device	162
7.1	Introduction	162
7.2	Grating operations with the displacemon	165
7.2.1	Relationship between the displacemon and a conventional matter-wave interferometer	165
7.2.2	The displacemon protocol	166
7.2.3	Physical parameters	168
7.3	Non-Gaussian mechanical state preparation via qubit measurement	169

7.4	Testing collapse models	171
7.4.1	Conventional decoherence	171
7.4.2	Diósi-Penrose objective collapse	172
7.4.3	Continuous Spontaneous Localisation (CSL)	177
7.5	Testing protocol	179
7.6	Experimental considerations	184
7.7	Conclusion	186
8	Conclusion	188
	Bibliography	191
A	Generating and verifying two-mode mechanical entanglement	213
A.1	Verification set-up	213
A.1.1	Parallel set-up	214
A.1.2	Series set-up	219
A.2	Example experimental parameter set	220
B	Error propagation in matrices	222
B.1	Derivation of error on the determinant	222
B.2	Optimal way of allocating measurements	225
B.3	Open-system model	227
B.3.1	Correlations between input operators	227
B.3.2	Beam splitter model	227

Part I

Background

Chapter 1

Concepts and tools

In this chapter we outline some physical principles and mathematical tools that will be utilized in later chapters of this thesis. Following a brief review of some basic concepts and tools of quantum mechanics, we describe the field of Gaussianity in quantum mechanics in order to formally define what is meant by a non-Gaussian state, and discuss the motivation behind studying non-Gaussian entanglement. Finally, we present two specific non-Gaussian entangled states which will be used extensively in Part II of the thesis.

1.1 Mathematical formalism

To describe a pure quantum state we introduce the quantum state vector $|\psi\rangle$ which contains all knowledge about the system, for the degrees of freedom of interest. From the superposition principle, this vector can be represented as a linear combination of orthogonal basis vectors $|\psi_i\rangle$ in a Hilbert space

$$|\psi\rangle = \sum_i c_i |\psi_i\rangle \quad , \quad (1.1)$$

where c_i are complex coefficients. We should not infer from this vector that a our state is in one of states $|\psi_i\rangle$ and that knowledge of the precise state is obscured. Instead, the state is in a coherent superposition which is distinct from the notion of a classical distribution of component

states. This coherence manifests in distinctly quantum phenomena such as interference effects or the notion of a projective measurement.

It is usually convenient to use the density matrix which, for the pure state $|\psi\rangle$, is defined as

$$\rho \equiv |\psi\rangle\langle\psi| . \quad (1.2)$$

Using Eq. (1.1), the density matrix in this particular basis $\{|\psi_i\rangle\}$ can be written as

$$\rho = \sum_{ij} c_i c_j^* |\psi_i\rangle\langle\psi_j| , \quad (1.3)$$

where the terms $i \neq j$ are usually referred to as interference, or off-diagonal, terms. A further crucial tool is the trace operation which acts in the following way on some quantum operator $\hat{A}$

$$\text{Tr}[\hat{A}] \equiv \sum_i \langle\phi_i| \hat{A} |\phi_i\rangle , \quad (1.4)$$

where $\{|\phi_i\rangle\}$ form an orthonormal basis of the Hilbert space of the system. While density matrices are themselves operators, we will denote them without the hat for simplicity. The trace operation is basis independent and thus we are not constrained to a particular choice of basis vectors. The trace operation has a physical interpretation when we consider $\hat{A} = \rho \hat{O}$, where $\hat{O}$ is a Hermitian operator representing an observable that is measured on the system. Decomposing $\hat{O}$ into its orthonormal eigenstates $\hat{O} = \sum_i o_i |\phi_i\rangle\langle\phi_i|$, where o_i are its eigenvalues, we can evaluate the trace in this eigenbasis

$$\begin{aligned} \text{Tr}[\rho \hat{O}] &= \sum_{ij} o_i \langle\phi_j| \rho |\phi_i\rangle \langle\phi_i| \phi_j\rangle \\ &= \sum_i o_i \langle\phi_i| \rho |\phi_i\rangle \\ &= \sum_i o_i P(o_i) = \langle\hat{O}\rangle . \end{aligned} \quad (1.5)$$

From Born's rule ¹ we can identify $\langle\phi_i| \rho |\phi_i\rangle$ as the probability of measuring the outcome o_i .

¹This key axiom of quantum mechanics states that for a pure state $|\psi\rangle$ the Born probability of the outcome o_i is simply $|\langle\phi_i|\psi\rangle|^2$ assuming the states $|\phi_i\rangle$ with eigenvalues o_i are the only possible outcomes of an experiment.

Therefore, we see that $\text{Tr}[\rho\hat{O}]$ gives the expectation value of the observable $\hat{O}$. This expectation is the average over all possible outcomes o_i of this measurement weighted by their probabilities $P(o_i)$.

Density matrices are particularly powerful tools for describing mixed states. Mixed states can be understood as classical ensembles of different pure quantum states. This could be as the result of probabilistic state preparation, either by design or as a result of experimental imperfections, or through interactions with a system's environment of which we have insufficient knowledge for a joint pure state description. This is distinct from a coherent superposition of pure states in that while for any pure state there exists a measurement with a deterministic outcome; this is not true for a mixed state whose measurement outcomes are fundamentally probabilistic. A typical example of a mixed state is a thermal state, the quantum description of a system in thermal equilibrium, as described by the Bose-Einstein distribution.

Suppose we have an experimental set-up which prepares the state $|\psi_i\rangle$ with probability p_i (each $|\psi_i\rangle$ may be a superposition state in a given basis). The density matrix describing this scenario is written as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| , \quad (1.6)$$

where $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$. We see that the mixed-state density matrix is a convex combination of pure-state density matrices. If $p_j = 1$ (so that all other $p_i = 0$) we recover the pure state density matrix $|\psi_j\rangle\langle\psi_j|$, showing that the pure-state description is fully encompassed by the density matrix formalism. The trace operation defined in Eq. (1.5) is also valid for mixed-state density matrices. Density matrices describing both pure and mixed states obey the following normalization condition

$$\text{Tr}[\rho] = 1 . \quad (1.7)$$

From the definition in Eq. (1.2), we see that a pure-state density matrix is simply the projection operator of the pure state. From this it follows that $\rho^2 = \rho$. This property is not shared with mixed states. We can define measures of mixedness which quantify the observer's uncertainty

about which pure state a system is in. One example is the purity of a quantum state, defined as $\text{Tr}[\rho^2]$. Unsurprisingly, a pure state has purity equal to 1, while the purity of a mixed state can range between $1/N \leq \text{Tr}[\rho^2] \leq 1$, where N is dimension of the system in question.

We introduce an additional measure of purity, namely, the von Neumann entropy. This measure is an extension of the notion of entropy in classical statistical mechanics and is defined as

$$S(\rho) \equiv -\text{Tr}[\rho \log_2 \rho] = \sum_i \lambda_i \log_2 \lambda_i , \quad (1.8)$$

where λ_i are the eigenvalues of ρ . Note that the logarithm is taken to base 2 in accordance with the convention that entropy is measured in bits [23]. It follows that a pure state has $S(\rho) = 0$, while for a maximally mixed state we have $S(\rho) = \log_2(N)$.

Bipartite states

So far we have treated the state vector $|\psi\rangle$ completely generally. However, it is often important to divide our total system into two (or more) subsystems. For example, in order to describe the phenomenon of quantum entanglement we must consider correlations between two subsystems. We now we consider a bipartite state describing two composite subsystems $\mathcal{A}$ and $\mathcal{B}$, each with its own degree(s) of freedom. Such a state is described by vectors in the Hilbert vector space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. For now, we will consider only discrete-variable systems but in Section 1.2 we will extend these notions to systems with continuous variables. A state vector of the composite system is now written as,

$$|\psi\rangle = \sum_{ij} c_{ij} (|a_i\rangle \otimes |b_j\rangle) , \quad (1.9)$$

where $\{|a_i\rangle\} \in \mathcal{H}_A$ and $\{|b_j\rangle\} \in \mathcal{H}_B$. Often the tensor product is dropped to keep notation more succinct. We can express the density matrix of the pure state $|\psi\rangle$ as

$$\rho = \sum_{ijkl} c_{ij} c_{kl}^* |a_i\rangle\langle a_k| \otimes |b_j\rangle\langle b_l| . \quad (1.10)$$

We now arrive at another powerful tool which is integral in understanding quantum entanglement: the partial trace. The physical motivation which underpins this mathematical operation is to describe everything that can be learnt about the composite system if we only take measurements on, say, system $\mathcal{A}$ and make no observations of $\mathcal{B}$. This scenario is described by the reduced density matrix of subsystem $\mathcal{A}$

$$\rho_{\mathcal{A}} = \text{Tr}_{\mathcal{B}}[\rho] , \quad (1.11)$$

where the trace operation is performed using only an orthonormal basis in the Hilbert space $\mathcal{H}_{\mathcal{B}}$. This is often referred to as “tracing out $\mathcal{B}$ ”.

Entangled bipartite states

Having introduced the concept of a bipartite pure state, it is now natural to consider entanglement. If it is possible to express the state vector of a bipartite state as a tensor product $|\psi\rangle = |a\rangle \otimes |b\rangle$, then this state is said to be separable. If it is not possible, then it is said to be entangled. The partial trace of a separable state can then be calculated as

$$\rho_{\mathcal{A}} = \text{Tr}_{\mathcal{B}} [|a\rangle\langle a| \otimes |b\rangle\langle b|] = |a\rangle\langle a| , \quad (1.12)$$

which is itself a pure state. Using the definition of the von Neumann entropy in Eq. (1.8), it follows that $S(\rho_{\mathcal{A}}) = 0$. Therefore, the von Neumann entropy of the reduced state is non-zero $S(\rho_{\mathcal{A}}) \neq 0$ only if the state is entangled. The von Neumann entropy of entanglement thus provides a proper measure of entanglement and is defined as

$$S(\rho_{\mathcal{A}}) = \text{Tr}[\rho_{\mathcal{A}} \log_2 \rho_{\mathcal{A}}] = \text{Tr}[\rho_{\mathcal{B}} \log_2 \rho_{\mathcal{B}}] = S(\rho_{\mathcal{B}}) . \quad (1.13)$$

For mixed states, a state is said to be separable if it can be written, in some basis, as

$$\rho = \sum_k p_k \rho_{\mathcal{A}}^k \otimes \rho_{\mathcal{B}}^k , \quad (1.14)$$

where $\sum_k \rho_k = 1$, and $\{\rho_A^k\}$ and $\{\rho_B^k\}$ are pure ensembles of each subsystem. This formulation is valid also for continuous variable systems and Eq.(1.14) will be used extensively in this Thesis. Here, we note the important caveat that the von Neumann entropy of entanglement is no longer a valid measure of entanglement for mixed states. This will be discussed further in Section 1.2.

Quantum measurements

We now introduce a description for the act of taking a quantum measurement. Any quantum measurement is characterised by a ‘positive operator valued measurement’ (POVM). A POVM is a set of positive semi-definite matrices $\{M_i\}$ which describe the outcomes of the measurement that together obey the following constraint

$$\sum_m \hat{M}_m = \hat{1} . \quad (1.15)$$

The values $\{m\}$ corresponds to the different outcomes of the measurement. If the state is initially prepared in state ρ , then the probability of obtaining the outcome m is

$$P(m) = \text{Tr} \left[\hat{M}_m \rho \right] , \quad (1.16)$$

where $P(m) \geq 0$ and $\sum_m P(m) = 1$ are satisfied. Since $\hat{M}_m$ is a positive semi-definite matrix it can be decomposed in the form

$$\hat{M}_m = \hat{K}_m^\dagger \hat{K}_m , \quad (1.17)$$

where $\{\hat{K}_m\}$ are sometimes referred to as Kraus operators. This decomposition is not unique since $\hat{K}'_m = \hat{V} \hat{K}_m$, where $\hat{V}^\dagger \hat{V} = \hat{1}$, also gives the same $\hat{M}_m$. However, the choice of $\hat{K}_m$ corresponds to the physical implementation of the measurement and so determines the post-measurement state.

Following a measurement where the outcome m is obtained, the state conditioned on this outcome is

$$\rho_m = \frac{\hat{K}_m \rho \hat{K}_m^\dagger}{\text{Tr} \left[\hat{K}_m^\dagger \hat{K}_m \rho \right]} . \quad (1.18)$$

We can identify the denominator as $P(m)$ which renormalizes the state following the non-unitary action of a measurement.. This denominator is referred to as the heralding probability by which ρ_m is obtained.

We can condition on such a state by performing the POVM described and keeping only outcomes where m is obtained. This post-selection can be generalized to some filtering map where we keep a set of measurement outcomes $\{m\}$ and discard the rest. Since, in general, this state is only obtained probabilistically, such a post-selection process is referred to as ‘non-deterministic’. This filtering map is nonlinear in ρ and its nonlinearity will be exploited in Chapter 5 to create a non-Gaussian bipartite quantum state.

In contrast, deterministic processes are described by quantum maps which are completely positive (CP). CP-maps which act on the density matrix $\Phi(\rho)$ can be described using the Kraus operator formalism

$$\Phi(\rho) = \sum_m \hat{K}_m \rho \hat{K}_m^\dagger, \quad (1.19)$$

where, like for a POVM, $\sum_m \hat{K}_m^\dagger \hat{K}_m = \mathbb{1}$ and $\hat{K}_m^\dagger \hat{K}_m \geq 0$. One of the most well-known CP-maps is obtained by considering a POVM with a single Kraus operator given by a unitary operator $\hat{U}$. The condition $\sum_\mu \hat{K}_\mu^\dagger \hat{K}_\mu = \mathbb{1}$ is automatically satisfied since $\hat{U}^\dagger \hat{U} = \mathbb{1}$. Therefore, all unitary evolution described by evolving a Hamiltonian in a closed-system $\hat{U} = e^{-i\hat{H}t}$ is in fact a CP-map and is thus deterministic. Another example would be a POVM where the measurement outcomes are not recorded or are inaccessible.

1.2 Continuous variable systems

In the previous sections we restricted ourselves to considering subsystems which are described by discrete variables. This is a suitable description for discrete systems such as spins or systems that are restricted to finite-dimensional subspaces, for example the logical degree of freedom of a qubit. However, many states are encoded in continuous variables, for example, the quantized electromagnetic field in a cavity which we will explore in greater detail in subsequent chapters.

Here we adopt some of the notation from Ref. [24].

A continuous variable system of N canonical bosonic modes is described by the Hilbert space $\mathcal{H} = \otimes_{k=1}^N \mathcal{H}_k$ where $\mathcal{H}_k$ an infinite dimensional Fock space associated with mode k . For a quantum harmonic oscillator, the energy levels are discrete (or quantized). These discrete excitations are known as phonons for a mechanical system and photons for an optical system. The Hamiltonian for a system of N harmonic oscillators each with a different frequency ω_k is

$$\hat{H} = \sum_k^N \frac{\hat{p}_k^2}{2m} + \frac{m\omega_k^2 \hat{x}_k^2}{2} . \quad (1.20)$$

Each frequency constitutes a mode of the field which has an associated momentum $\hat{p}_k$ and position $\hat{x}_k$ operator.

The quantized harmonic oscillator forms a ladder of permitted eigen-energies, and so the space $\mathcal{H}_k$ is spanned by the Fock basis $\{|n\rangle_k\}$. Therefore, the system is naturally characterised by ladder operators $\hat{a}_k$ and $\hat{a}_k^\dagger$ which describe the effect of annihilation and creation of a quantum of energy in mode k with frequency ω_k . These operators obey the bosonic commutation relations

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'} , \quad (1.21a)$$

$$[\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0 . \quad (1.21b)$$

If the k^{th} mode is initially prepared in the Fock state $|n\rangle_k$, the ladder operators act on the Fock states in the following manner

$$\hat{a}_k^\dagger |n\rangle_k = \sqrt{n+1} |n+1\rangle , \quad (1.22a)$$

$$\hat{a}_k |n\rangle_k = \sqrt{n} |n-1\rangle . \quad (1.22b)$$

Thus, these operators describe transitions between the rungs of the energy-level ladder. Each mode has its own vacuum state $|0\rangle_k$ such that $\hat{a}_k |0\rangle_k = 0$, since we cannot remove a phonon (or photon) from an oscillator already in its ground state. From these equations it follows that $\hat{a}_k^\dagger \hat{a}_k |n\rangle_k = n |n\rangle_k$ and so it is convenient to introduce the number operator $\hat{n}_k = \hat{a}_k^\dagger \hat{a}_k$ whose

eigenvalue when acting upon $|n\rangle_k$ is simply n .

The ladder operators are linked to the position and momentum operators via the following relations

$$\hat{x}_k = x_{\text{zp},k}(\hat{a}_k^\dagger + \hat{a}_k) , \quad (1.23a)$$

$$\hat{p}_k = ip_{\text{zp},k}(\hat{a}_k^\dagger - \hat{a}_k) , \quad (1.23b)$$

where the prefactors are the zero-point motion and momentum of the k^{th} mode

$$x_{\text{zp},k} = \sqrt{\frac{\hbar}{2m\omega_k}} , \quad (1.24a)$$

$$p_{\text{zp},k} = \sqrt{\frac{\hbar m\omega_k}{2}} . \quad (1.24b)$$

These prefactors are linked to the zero point energy of the oscillator. It is straightforward to demonstrate, using $[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'}$ that the position and momentum operators obey the following commutation rule

$$[\hat{x}_k, \hat{p}_{k'}] = i\hbar\delta_{kk'} . \quad (1.25)$$

While discussing the properties of the ladder operators, it is important to also introduce the coherent state $|\alpha\rangle$. This state is the eigenvector of the annihilation operator $\hat{a}_k$ with eigenvalue α such that $\hat{a}_k |\alpha\rangle_k = \alpha |\alpha\rangle_k$, where, in general, $\alpha \in \mathbb{C}$. Coherent states can be generated by applying the single-mode Weyl displacement operator to the vacuum state

$$\hat{D}_k(\alpha) |0\rangle_k = |\alpha\rangle_k , \quad (1.26)$$

where the displacement operator is defined as

$$\hat{D}_k(\alpha) = e^{\alpha\hat{a}_k^\dagger - \alpha^*\hat{a}_k} . \quad (1.27)$$

Coherent states are an important set of states which, while not orthogonal, form a basis in $\mathcal{H}_k$. In fact, coherent states form an overcomplete set and must be treated accordingly. They can

be expressed in the Fock basis as follows

$$|\alpha\rangle_k = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle_k . \quad (1.28)$$

Optical coherent states have an important physical interpretation: they are the quantum-mechanical equivalent a classical monochromatic electromagnetic wave.

Phase space methods

It is often convenient to define a pair of dimensionless, self-adjoint position and momentum operators for each mode

$$\hat{X}_k = \frac{\hat{a}_k + \hat{a}_k^\dagger}{\sqrt{2}} , \quad (1.29a)$$

$$\hat{P}_k = i \frac{\hat{a}_k^\dagger - \hat{a}_k}{\sqrt{2}} . \quad (1.29b)$$

These quadrature phase operators are often referred to as a pair of canonical operators which arises from their classical analogue in Hamiltonian dynamics. Grouping together the canonical operators into the following vector

$$\mathbf{R} = (\hat{X}_1, \hat{P}_1, \dots, \hat{X}_N, \hat{P}_N)^T , \quad (1.30)$$

these non-commuting operators obey commutation relations which can be expressed in the compact form

$$[R_k, R_l] = i\Omega_{kl} \quad (1.31)$$

where

$$\Omega = \bigoplus_{k=1}^N \omega , \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \quad (1.32)$$

The phase space representation of a continuous variable state corresponds to quantities measured in homodyne experiments. Therefore, it is often amenable to study the dynamics of the state in phase space rather than the density matrix representation. The vector $\mathbf{R}$ permits us to

represent the state as vector on a phasor diagram with dimensionless position and momentum on each axis. However, due to the uncertainty principle, the direction and length of the vector cannot be precisely known leading to the ‘ball-and-stick’ representation on a phasor diagram, for a single mode. The stick indicates the vector while the ball illustrates the quantum mechanical uncertainty in knowing both X and P simultaneously. When we consider higher-dimensional spaces, phasor diagrams become increasingly difficult to visualise. It is thus often appropriate to use a phase space distribution to entirely characterise the state. One such class of phase space distributions is derived from the s -ordered characteristic functions. For a single mode this is defined as

$$\chi_s(\alpha) = \text{Tr} \left[\rho \hat{D}(\alpha) \right] e^{s|\alpha|^2/2} , \quad (1.33)$$

where s is real valued and α is complex². Taking the Fourier transform of this family of characteristic functions gives us the s -ordered distributions

$$W_s(\alpha) = \frac{1}{\pi^2} \int_{\mathbb{C}} d^2\beta e^{\alpha\beta^* - \alpha^*\beta} \chi_s(\beta) . \quad (1.34)$$

Setting $s = -1$ gives the Husimi ‘Q-function’ [25], while $s = 1$ gives the Glauber-Sudarshan P-representation [26, 27]. Here we emphasise the case where $s = 0$, which yields the Wigner function [28]. For a single mode, the Wigner function can be defined more concisely by setting $\alpha = (X + iP)/\sqrt{2}$ and $\beta = (X' + iP')/\sqrt{2}$ to give

$$W_0(X, P) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dX' e^{iPX'} \left\langle X + \frac{X'}{2} \right| \rho \left| X - \frac{X'}{2} \right\rangle . \quad (1.35)$$

The Wigner function has features similar to a classical probability distribution and is normalised such that $\int dX \int dP W_0(X, P) = 1$. The probability distributions of X and P can be recovered from the marginals of $W(X, P)$

$$P(X) = \int dP W_0(X, P) , \quad (1.36a)$$

$$P(P) = \int dX W_0(X, P) . \quad (1.36b)$$

²The s -ordered characteristic functions can be readily extended to N mode systems by replacing $D(\alpha) = \otimes_k^N D_k(\alpha_k)$ where $\alpha = (\alpha_1, \dots, \alpha_N)$, $\alpha_k \in \mathbb{C}$.

These are the same positive-definite quantum distributions predicted by Born's rule. Therefore, the Wigner function is often interpreted as a probability distribution in phase space and the marginals of the Wigner distribution are related to the results of homodyne detection. However, intriguingly, the Wigner distribution can be negative in some regions, in contrast to what is permitted in classical probability theory. Hence, this function is often dubbed a 'quasi-probability distribution'.

Gaussian states

We will now apply these mathematical tools directly to define and study an important class of quantum states: Gaussian states. These states are integral to the field of quantum optics and quantum information theory. The class of Gaussian states encompasses states such as coherent states, thermal states, and squeezed states; but also combinations of these such as squeezed thermal states.

By definition, a state is considered Gaussian if it has a Gaussian characteristic function or Wigner function. In direct analogy with classical statistical theory, such a distribution is completely characterized by its first and second statistical moments in phase space. Using the notation we have introduced, this means an N -mode Gaussian state can be fully characterized with just knowledge of the vector of first moments

$$\overline{\mathbf{R}} = (\langle \hat{X}_1 \rangle, \langle \hat{P}_1 \rangle, \dots, \langle \hat{X}_N \rangle, \langle \hat{P}_N \rangle)^T, \quad (1.37)$$

in conjunction with the covariance matrix $\boldsymbol{\sigma}$

$$\sigma_{ij} = \frac{1}{2} \langle \mathbf{R}_i \mathbf{R}_j + \mathbf{R}_j \mathbf{R}_i \rangle - \overline{\mathbf{R}}_i \overline{\mathbf{R}}_j, \quad (1.38)$$

where $\mathbf{R}$ is defined in Eq. (1.30). Therefore, despite the density matrix of a Gaussian N -mode system residing in the infinite dimensional Hilbert space $\mathcal{H} = \otimes_{k=1}^N \mathcal{H}_k$, we can form a complete description using a $2N \times 2N$ covariance matrix $\boldsymbol{\sigma}$. In this notation, the Wigner function of a

Gaussian state can be written in the following manner

$$W_0(\mathbf{R}) = \frac{2^N}{\pi^N \sqrt{\text{Det}(\boldsymbol{\sigma})}} \exp \left[- (\mathbf{R} - \bar{\mathbf{R}})^T \boldsymbol{\sigma}^{-1} (\mathbf{R} - \bar{\mathbf{R}}) \right], \quad (1.39)$$

which is clearly a Gaussian function in $\mathbf{R}$. Gaussian states must also have Gaussian characteristic functions. These states can be mathematically explored in a very clean framework offered by symplectic formalism; however, the richness of this approach is beyond the scope of this Thesis.

Gaussian entanglement

Here, we state some key results in the theory of Gaussian entanglement; an overview of these results can be found in Ref. [29]. In Section 1.1 we introduced one way of quantifying bipartite entanglement: the von Neumann entropy of the reduced density matrix, see Eq. (1.13). Crucially, the von Neumann entropy of entanglement cannot be increased by performing only local operations on a state. This arises from the fact that entanglement cannot be generated with local operations and classical communication (LOCC) operations alone [30]. However, this entanglement monotone fails for mixed states because there is no unique way of decomposing the density matrix defined in Eq. (1.6). In demonstrating that a bipartite mixed state is entangled, we must show that the correlations between the two subsystems could not be induced by LOCC operations. Due to the ambiguity in the state preparation, it is often more amenable to use alternative operational criteria for mixed quantum states.

One approach is offered by entanglement witnesses, where methods have been explored for finding optimal continuous-variable entanglement witnesses [31, 32, 33]. However, one of the most powerful of these operational criteria – valid for both pure and mixed states – is the requirement that a separable state has a positive partial transpose (PPT). This is the continuous variable extension of the Peres-Horodecki condition [34, 35]. To demonstrate this let us first consider a separable mixed state defined in Eq. (1.14). We now apply the partial transpose

map which has the effect of transposing one of the subsystems

$$\rho_{AB}^{\text{PT}} = \sum_k p_k \rho_A^k \otimes (\rho_B^k)^{\text{T}} , \quad (1.40)$$

where we have chosen to apply the transpose operation to subsystem $\mathcal{B}$. The same results can be demonstrated by applying the operation to either subsystem. Since the transposition map preserves eigenvalues, the spectrum of $(\rho_B^k)^{\text{T}}$ must be the same as that of ρ_B^k and thus $(\rho_B^k)^{\text{T}}$ is still a positive semi-definite matrix. Therefore ρ_{AB}^{PT} must also still be positive semi-definite. And so PPT is a necessary condition for separability. The PPT criterion was extended to continuous variable systems by Simon [36] who also showed that the PPT criterion is a necessary and sufficient criterion for bipartite Gaussian states where each subsystem is composed of one mode (i.e. a (1×1) -mode entangled Gaussian state will not be positive under the partial transpose operation). For these states, it is possible to reformulate the PPT criterion in terms of first and second-order moments of canonical operators to form a necessary and sufficient entanglement criterion, known as Simon's criterion³. This criterion will be formally introduced in Section 6.2.

We note that the partial transpose is an example of a positive but not completely positive map (PNCP) and so cannot be realised by a physical channel such as those described by a POVM (see Section 1.1). In fact, the partial transpose physically corresponds to time inversion in one subsystem from the mirror reflection of the momentum operator.

In addition to the PPT criterion, a number of other operational criteria exist such as entanglement witnesses and several experimentally-amenable criteria. The latter are based on the violation of inequalities composed of variances of canonical operators. A well-known example of this is the Duan inequality [38] which is calculated via the total variance of a pair of Einstein-Podolsky-Rosen type operators. Like Simon's criterion, this criterion is in fact necessary and sufficient for (1×1) -mode entangled Gaussian states.

Gaussian entanglement can be routinely generated in a range of different systems. For example,

³In fact, the PPT criterion is necessary and sufficient for the separability of all $(1 \times N)$ -mode Gaussian states [36, 37]. The PPT criterion is no longer necessary and sufficient for (2×2) -mode Gaussian states however this is beyond the scope of this Thesis. We will limit our study to (1×1) -mode Gaussian states, where each subsystem consists of one mode.

entangled electromagnetic fields can be made using parametric amplifiers or oscillators which have the following linear Hamiltonian $\hat{a}^\dagger \hat{b}^\dagger + \hat{a} \hat{b}$ [39]. Assuming the initial state is a Gaussian separable state, then after this linear interaction, the final state will be an entangled Gaussian state whose entanglement properties are completely encoded in the covariance matrix σ . Furthermore, two-mode squeezed states, generated by Hamiltonians of the form $i(\hat{a}^\dagger \hat{b}^\dagger - \hat{a} \hat{b})$, are important entangled resources for quantum information protocols [40]. Squeezing is a process that decreases the variance of one continuous variable (e.g. position) at the cost of an increased variance in its conjugate variable (e.g. momentum). This state can be generated in the output of an idealised degenerate parametric down conversion process. In the limit of infinite squeezing the state approaches the EPR state. For example, squeezed light can be used for enhanced metrology and sensing, such as in the detection of gravitational waves. Another elegant example illustrating the power of Gaussian states is in providing secure quantum key distribution [41]. Many quantum information protocols which were first envisioned for finite-dimensional Hilbert spaces have been re-imagined for CV platforms which exploit Gaussian states [42].

1.3 Non-Gaussian states

Gaussian states and Gaussian operations are the foundational building blocks of many quantum-optical tasks. However for many applications, non-Gaussianity can be desirable or, in fact, a key resource. In some instances, tasks for which Gaussian states are sufficient can be improved with non-Gaussian states or non-Gaussian operations. For example, the entanglement of Gaussian states cannot be increased, or distilled, by local Gaussian operations alone [43]; a non-Gaussian operation can therefore improve a quantum protocol which relies on enhanced entanglement. A prime example of a nonlinear, local operation on a Gaussian state is photon subtraction [44, 45], which helps achieve the optimal fidelity when cloning coherent states [46], and also improves the fidelity in the teleportation of coherent [47] and other non-classical input states [48]. Non-Gaussian states also offer advantages in parameter estimation [49, 50].

On the other hand, non-Gaussianity is required for certain tasks where Gaussianity alone does

not suffice. Gaussian states can be reproduced by stochastic classical systems and thus it is argued they cannot offer a genuine quantum advantage. For example, non-Gaussian entangled states are a key resource in the actualisation of universal quantum computing on a continuous variable platform [51]. It has been proposed that Gaussian cluster states could provide a route for universal one-way quantum computing; however, a single-mode non-Gaussian measurement is needed to achieve this [18, 52]. Non-Gaussianity is also crucial in quantum error correction; with no-go theorems proving that Gaussian operations do not protect Gaussian states against Gaussian errors in quantum communication protocols [53]. Furthermore, non-Gaussian entangled states are necessary for testing foundational physics; for example, in detecting violations of loophole-free Bell tests using a continuous variable platform [54, 55, 56]. Non-Gaussian states can be achieved either through a nonlinear interaction whose Hamiltonian has order greater than two (e.g. the Kerr effect), or through conditional measurements such as heralded photon subtraction(addition). The mathematics of conditional measurements were discussed in Section 1.1.

1.3.1 Non-Gaussianity quantifier

There exist two closely related measures of non-Gaussianity both of which aim to quantify the difference between a state ρ and a reference Gaussian state ρ_G . The first quantifies the difference using the Hilbert-Schmitt distance [57], while the second uses the quantum relative entropy [58]. Both measures share the same qualitative properties however are not strictly monotone to each other. We will focus on the quantum relative entropy measure which has been shown to have several useful applications in quantum information theory. The subtle differences between the two measures are discussed in Ref. [59]. This entropic measure requires knowledge of the full density matrix. Therefore, in this Thesis it should be viewed a theoretical tool rather than an operational measure.

The quantum relative entropy of the state ρ with respect to another state ρ_1 is defined as

$$S(\rho||\rho_1) = \text{Tr}[\rho(\ln(\rho) - \ln(\rho_1))] , \quad (1.41)$$

such that $S(\rho||\rho_1) \geq 0$. If we set $\rho_1 = \rho_G$, where ρ_G is a reference Gaussian state with the first and second moments of ρ , we can use this quantity as an exact measure of non-Gaussianity. Letting $\delta(\rho) = S(\rho||\rho_G)$ this reduces to [59, 60]

$$S(\rho||\rho_G) = S(\rho_G) - S(\rho) , \quad (1.42)$$

where S is the standard von Neumann entropy defined in Eq. (1.8). This can be viewed as a quantum generalisation of negentropy in classical statistics which quantifies the distance from a distribution to normality. This simple expression as the difference of two entropies stems from the fact that of all distributions with a fixed variance (i.e. first and second moments), the one with the largest entropy is Gaussian. This further reinforces the property $\delta(\rho) \geq 0$.

As noted in Section 1.1, the von Neumann entropy of a pure state is 0. Supposing our state ρ is pure and Gaussian, it follows that $S(\rho) = S(\rho_G) = 0$ and consequently $\delta = 0$ implying the state is exactly Gaussian. If ρ is pure and non-Gaussian, then $S(\rho) = 0$ but $S(\rho_G) > 0$ and so $\delta > 0$. Finally, let us consider the case where ρ is mixed. Firstly, if ρ is mixed and Gaussian then $S(\rho) = S(\rho_G)$ and so $\delta = 0$ as expected. Secondly, if ρ is mixed and non-Gaussian then $S(\rho) \neq 0$ but $S(\rho_G) > S(\rho)$ and so $\delta > 0$. Since δ is an exact measure of non-Gaussianity [60], a greater δ suggests a greater degree of non-Gaussianity, while $\delta = 0$ implies the state is exactly Gaussian. However, the measure has no upper bound (there is no maximally non-Gaussian state) so we can only comment on a state being more non-Gaussian than another.

Using the covariance matrix formalism introduced in Section 1.2, we can calculate the non-Gaussian quantifier without needing to find the full spectrum of ρ . Firstly, we calculate σ for the state ρ , and then we compute the symplectic eigenvalues

$$\nu = \text{Eig}[i\Omega\sigma] , \quad (1.43)$$

where Ω is defined in Eq. (1.32). For a bipartite state, this yields 4 symplectic eigenvalues. The symplectic eigenvalues come as repeated pairs of numbers which are greater than 1; and so for

a bipartite state there are 2 unique eigenvalues, ν_1 and ν_2 . Using the following function [61]

$$s(x) = \frac{(x+1)}{2} \log\left(\frac{x+1}{2}\right) + \frac{(x-1)}{2} \log\left(\frac{x-1}{2}\right), \quad (1.44)$$

we can compute the non-Gaussianity measure in a straightforward manner,

$$\delta = s(\nu_1) - s(\nu_2). \quad (1.45)$$

1.3.2 Two-mode Schrödinger-cat state

We now introduce a specific non-Gaussian entangled state that is one of the central themes of this Thesis. We will define the two-mode even (+) and odd (−) Schrödinger-cat states as

$$|\Psi\rangle_{\text{cat}}^{\pm} = \mathcal{N} \left(|\alpha\rangle_1 |0\rangle_2 \pm |0\rangle_1 |\alpha\rangle_2 \right), \quad (1.46)$$

with normalisation

$$\mathcal{N} = \frac{1}{\sqrt{2 \pm 2e^{-|\alpha|^2}}}. \quad (1.47)$$

This state has various alternative forms in the literature; the most prominent being $|\Psi\rangle \propto |\alpha\rangle |\beta\rangle \pm |-\alpha\rangle |-\beta\rangle$, or $|\Psi\rangle \propto |\alpha\rangle |\alpha\rangle - |-\alpha\rangle |-\alpha\rangle$, up to some normalisation. However, with local displacement and rotation operations all these states can be brought into the same form. We will use the definition in Eq. (1.46) since in Chapter 5 we will introduce a protocol to generate such a non-Gaussian state of two mechanical oscillators. Nevertheless, since the various two-mode Schrödinger-cat states are connected by LOCC operations we expect them to possess similar entanglement structure and therefore applications.

Two-mode Schrödinger-cat states have a wide range of applications with prominent examples including quantum metrology [15], quantum teleportation [62], quantum networks [63], and fault-tolerant quantum computation [64]. These entangled states, when encoded in mechanical degrees of freedom, can also be applied for empirical studies of quantum macroscopicity [65, 66], sensing, and the quantum-to-classical transition [6, 7, 67, 68]. This has motivated proposals to

generate two-mode Schrödinger cat states in mechanical platforms [69, 70].

It can be shown that in the limit of $|\alpha| \rightarrow 0$, the odd cat state $|\Psi\rangle_{\text{cat}}^-$ tends to the Bell state $|\Psi\rangle_{\text{Bell}}^- \propto |1\rangle|0\rangle - |0\rangle|1\rangle$, expressed in the Fock basis. We can demonstrate this by inserting Eq. (1.28) into Eq. (1.46) to give

$$\begin{aligned} |\Psi\rangle_{\text{cat}}^- &= \mathcal{N} e^{-\frac{|\alpha|^2}{2}} \sum_{n=0} \left[\frac{\alpha^n}{\sqrt{n!}} |n\rangle|0\rangle - \frac{\alpha^n}{\sqrt{n!}} |0\rangle|n\rangle \right] \\ &= \mathcal{N} e^{-\frac{|\alpha|^2}{2}} \left[\alpha \left(|1\rangle|0\rangle - |0\rangle|1\rangle \right) + \frac{\alpha^2}{2} \left(|2\rangle|0\rangle - |0\rangle|2\rangle \right) + \dots \right] \\ &\approx \frac{|1\rangle|0\rangle - |0\rangle|1\rangle}{\sqrt{2}}, \end{aligned} \quad (1.48)$$

for small α . We note that in the limit of small α the even cat state $|\Psi\rangle_{\text{cat}}^+$ approaches the vacuum state $|0\rangle|0\rangle$. Although we note that when $\alpha = 0$ we are left with the vacuum state $|0\rangle|0\rangle$.

In the asymptotic limit where $|\alpha| \rightarrow \infty$, both cat states $|\Psi\rangle_{\text{cat}}^\pm$ tend towards a maximally-entangled Bell-like state encoded in the coherent basis. To demonstrate this, we can analytically compute the von Neumann entropy of entanglement using Eq. (1.8). Firstly, the reduced density matrix is found by tracing out one of the subsystems

$$\begin{aligned} \rho_A &= \frac{1}{\pi} \int d\beta^2 \langle \beta|_B \rho_{AB} |\beta\rangle_B \\ &= \frac{1}{2(1 \pm e^{-|\alpha|^2})} \left[|\alpha\rangle\langle\alpha| + |0\rangle\langle 0| \pm e^{-|\alpha|^2/2} \left(|0\rangle\langle\alpha| + |\alpha\rangle\langle 0| \right) \right], \end{aligned} \quad (1.49)$$

where $\rho_{AB} = |\Psi\rangle\langle\Psi|_{\text{cat}}^\pm$. In the limit of $\alpha \rightarrow \infty$ this tends to $\rho_A = \frac{1}{2} |\alpha\rangle\langle\alpha| + \frac{1}{2} |0\rangle\langle 0|$. The overlap $\langle\alpha|0\rangle = e^{-|\alpha|^2/2} \rightarrow 0$ as $\alpha \rightarrow \infty$ and so the states $|\alpha\rangle$ and $|0\rangle$ become increasingly orthogonal. The von Neumann entropy can be trivially calculated in this limit as $S(\rho_A) = \log_2 2$ which is equivalent to the von Neumann entropy of the Bell states. Such maximally entangled states have applications in quantum teleportation schemes, for example in optical platforms [71, 62], and hybrid teleportation in circuit quantum electrodynamics [72].

1.3.3 Phonon added/subtracted two-mode squeezed vacuum states

As previously discussed, non-Gaussian entangled quantum states are hugely important in quantum information theory, for example in quantum error correction [53] and achieving universal quantum computing. Deterministically producing a non-Gaussian quantum optical state by using the Kerr effect is not always feasible as it requires high optical nonlinearities. However, photon subtraction (and addition) operations can be exploited to create highly non-Gaussian states and give rise to curious statistical properties in bosonic systems [73]. These non-unitary interactions can be achieved probabilistically by conditioning on a particular photon detection outcome. This heralded addition or subtraction has been experimentally demonstrated on a range of physical platforms. For example, single-photon detection has been exploited to create optical ‘kitten’ states by subtracting a single photon from a squeezed vacuum state [74, 75]. The technique has been explored in other bosonic systems including cavity quantum optomechanics [76], with recent experiments confirming the non-Gaussianity and statistics which arise from phonon subtraction operations [77, 78, 79]. In a multipartite setting, photon addition/subtraction can have further interesting effects. For example, it can be used to create photon-(added)subtracted two-mode squeezed vacuum (TMSV) states. Up to some normalisation, this subtracted state is defined as

$$|\Psi\rangle_{\text{sub}} \propto \hat{a}^n \hat{b}^m e^{\zeta^* \hat{a} \hat{b} - \zeta \hat{a}^\dagger \hat{b}^\dagger} |0\rangle_1 |0\rangle_2, \quad (1.50)$$

which describes a TMSV state with squeezing parameter ζ which undergoes n - and m -photon subtractions on modes $\mathcal{A}$ and $\mathcal{B}$, respectively. The normalisation factor is cumbersome to write out fully but can be found in Ref. [80]. The non-Gaussian operation of photon-subtraction(addition) has been found to enhance entanglement as n and m are increased [80]. Therefore we will also consider this class of non-Gaussian entangled states in Chapter 6.

Chapter 2

Cavity optomechanics

In this Chapter we will introduce some of the key concepts in cavity optomechanics. This term encapsulates the phenomenon of a mechanical element coupling to the optical field inside a cavity via the radiation pressure of the light. Interest in this field has been fueled partly by precision sensing applications within and beyond gravitational wave detection [81, 82], and also by the radiation-pressure interaction itself which enables us to generate nonclassical states in optical and mechanical modes [83]. Optomechanical devices offer exciting applications in the ever-burgeoning field of quantum technologies; for example as light-matter interfaces [84], or quantum memories [85]. Furthermore, the nonclassical mechanical states could be exploited in tests for fundamental physics such as probing for decoherence [6, 7, 86] and testing collapse models [67], enabling us to explore parameter sets which are tantalisingly close to the quantum-to-classical interface.

Transitioning from Maxwell’s description of electromagnetic waves to the quantum picture was a necessary step taken at the start of the 20th century to explain outstanding problems such as black body radiation and the photoelectric effect. In quantum theory, light is composed of discrete quanta, or photons, which have energy given by the Planck-Einstein relation $E = hf$. A photon has momentum $p = h/\lambda$ and so a photon bouncing elastically off a surface will impart a momentum kick of $2h/\lambda$. This radiation pressure had been remarked upon before the advent of quantum mechanics. The often-cited example being Kepler’s observation in 1619 of the tail

of a comet being directed away from the sun. Other spectacular manifestations of radiation pressure in astronomy include the Pillars of Creation in the Eagle Nebula. The radiation pressure phenomenon was first observed on Earth in 1899 by Lebedev and independently in 1901 by Nichols and Hull. The radiation pressure interaction from a laser can be exploited for purposes such as optical tweezers [87], which have important applications in biology and medicine.

Another application includes solar sails which have even been used to weakly propel spacecraft by exploiting the radiation of the sun hitting mirrors on the sails. The radiation pressure we experience on Earth from the sun is on the order of a micropascal [88]. Unfortunately, the radiation pressure force is so weak that even with a laser, launching rockets with light is currently constrained to the annals of science fiction.

Nevertheless, let us consider a suspended mirror which reflects light elastically. The radiation pressure effect can be observed in the displacement of the mirror. Particularly interesting for quantum mechanics would be the case where a single photon has reflected off a mirror which is initially prepared in its ground state of motion. That way, a displacement in the mirror would directly correspond to the number of photons in a field. However, the mirror itself has an intrinsic position uncertainty given by its zero-point motion, see Eq. 1.24a. For the light-induced displacement to exceed x_{zp} we require the mass of the mirror to be very small or the intensity of light to be very bright [89]. One way to mitigate these challenges is to trap the light and resonantly enhance the interaction by recirculating the photons. Optical cavities offer a platform to achieve this where light reverberates between two mirrors. One mirror is fixed, while the other is free to oscillate and thus its displacement responds to the momentum transferred by the photons.

In this Chapter we will first explore cavity quantum electrodynamics, which concerns itself with the quantized optical fields inside a cavity. We will then examine quantum optomechanics, where one mirror is free to oscillate. We also consider the quantum treatment of optical fields entering and leaving the cavity which will lead us to the quantum Langevin equations. Using this formalism, we will introduce a particular regime known as pulsed optomechanics

whereby the cavity decay rate is much faster than the mechanical frequency. This has important implications on the light-matter interaction and the information imprinted on the outgoing optical field. Finally, we will examine the field of electromechanics where microwave photons are used to couple to the displacement of a mechanical element instead of visible light. Such an electromechanical system can be coupled to a superconducting qubit via the mechanical component enabling one to create and measure the mechanical state via qubit manipulations.

We will use the physical principles outlined in this chapter to describe two experimental proposals that form key parts of this thesis. Firstly, in Chapter 5 we will outline a cavity optomechanics protocol to generate a non-Gaussian entangled state encoded in the displacement of a mechanical resonator. Specifically, we will study experimental scheme to generate two-mode Schrödinger-cat states and verify their entanglement (we introduced non-Gaussian entanglement in Chapter 1.3). Secondly, in Chapter 7 we will examine an electromechanical protocol which can be used to generate a single-mode Schrödinger-cat state in the centre-of-mass displacement of a mechanical nanotube.

2.1 Cavity quantum electrodynamics

Here we will highlight the key steps in the quantization of light in a cavity. This is not a rigorous proof, and more detailed derivations are covered in many quantum optics text books. We will summarize the steps taken in Ref. [90].

The behaviour of classical electromagnetic waves in a cavity can be fully described by Maxwell's equations for the electric $\mathbf{E}$ and the magnetic $\mathbf{B}$ fields. By considering a single mode of light, i.e. one wavelength, which is confined between two conducting mirrors separated by a distance L_{cav} along the z axis, the resonant modes of the electric and magnetic fields are described by the following wave solutions

$$E_x(z, t) \propto q(t) \sin(kz) , \quad (2.1a)$$

$$B_y(z, t) \propto \dot{q}(t) \cos(kz) , \quad (2.1b)$$

with both fields propagating along the z -axis. We assume the electric and magnetic fields are polarized in the x and y directions, respectively. We identify the time-dependent variable $q(t)$ with position as it has dimensions of length, and $\dot{q}(t)$ with momentum, since $p(t) = \dot{q}(t)$. The classical energy is defined as

$$H = \frac{1}{2} \int dV \left(\epsilon_0 E_x^2(z, t) + \frac{1}{\mu_0} B_y^2(z, t) \right), \quad (2.2)$$

where we have used the symbol H for the classical Hamiltonian of the system. Integrating the wave solutions yields the following result,

$$H = \frac{1}{2} (p^2 + \omega^2 q^2). \quad (2.3)$$

This is immediately identifiable as the Hamiltonian of a harmonic oscillator with frequency ω and unit mass. Therefore, q (the electric field) and p (the magnetic field) are the canonical position and momentum variables.

We will now assume that when going from the classical to the quantum picture, we can replace the canonical variables with their operator equivalents $q \rightarrow \hat{x}$ and $p \rightarrow \hat{p}$. This yields the Hamiltonian for a quantum harmonic oscillator defined in Eq. 1.20 with $m = 1$.

With these new variables, it is apparent that the quantum Hamiltonian for light in a cavity can be rewritten as

$$\hat{H}_L = \hbar\omega_L \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \quad (2.4)$$

where $\hat{a}$ and $\hat{a}^\dagger$ are discussed in Section 1.2. The subscript L will be used throughout to denote variables pertaining to the optical system.

2.2 Quantum optomechanics

In this section, we will examine quantum optomechanics through the pedagogical model provided by the Fabry-Pérot cavity. We emphasise that optical cavities can be actualised in a

plethora of physical systems which are underpinned by the same quantum mechanical description. Examples include whispering-gallery-mode resonators, optically levitated nanoparticles, and photonic crystals.

Let us consider the simple Fabry-Pérot cavity which consists of two flat mirrors. Each mirror has an associated transmittivity and reflectivity and they are separated by a distance L_{cav} . Thus, the mirrors act as beam-splitters admitting light into and out of the cavity. The mirrors are parallel such that inside the cavity light bounces backwards and forward in a medium with refractive index n . These Fabry-Pérot cavities have a rich history in the field of classical optics, particularly as interferometers. Here we will consider the electromagnetic fields which exist in cavities and their coupling to matter. This is captured by the subject of cavity quantum electrodynamics. We will begin by reminding the reader of some basic properties of cavities which are derived from a classical optics perspective and then building on these ideas present a quantum mechanical description of an optomechanical cavity.

Optical cavities: a classical perspective

In classical optics, we envision light of wavelength λ bouncing around inside a cavity, accumulating a phase each round trip $\phi = 4\pi n L_{\text{cav}}/\lambda$, where n is the refractive index between the mirrors. Assuming there are no absorption or scattering losses in the cavity, using the plane wave solutions to Maxwell's equations, we can derive a transmission through the cavity

$$T = \frac{1}{1 + (4\mathcal{F}^3/\pi^2) \sin^2(\phi/2)} . \quad (2.5)$$

The cavity finesse is defined as

$$\mathcal{F} = \frac{\pi(R_1 R_2)^{1/4}}{1 - \sqrt{R_1 R_2}} , \quad (2.6)$$

and R_1 and R_2 are the reflectivity coefficients of each mirror. Here, we will introduce the notion of a ‘single-sided’ cavity where the back mirror has a reflectivity of unity. The second mirror is referred to as the input mirror and has some reflectivity bounded between 0 and 1. A cavity is said to be on resonance when the cavity length is equal to an integer multiple m of intracavity

half wavelengths

$$L_{\text{cav}} = m\lambda/(2n) . \quad (2.7)$$

When the cavity is on-resonance the light bouncing around is in phase each round trip and can be intuitively understood as the case where standing waves are supported.

As we scan the frequency of monochromatic light inside the cavity, we find that the transmitted light is sharply peaked at the so-called resonant modes. This can be seen from Eq. (2.5) which has maxima when ϕ equals integer multiples of 2π . This naturally leads to the concept of resonant modes which satisfy the following condition

$$\omega_{\text{L},j} = j \frac{\pi c}{n L_{\text{cav}}} , \quad (2.8)$$

where j is an integer and c is the speed of light in a vacuum. These modes are preferentially selected by the cavity since these light waves will constructively interfere. It is evident that the resonant modes depend on the refractive index of the cavity medium n , and the length of the cavity L_{cav} . The spacing between frequency modes is known as the free spectral range. The spectral width $\Delta\omega$ is defined as the full-width at half maximum of the transmission response centred at the resonant modes. This is commonly referred to as the linewidth and using the above equations one can show that $\Delta\omega = \pi c/(n\mathcal{F}L_{\text{cav}})$. Therefore, a greater finesse leads to narrower peaks in the cavity spectrum. Since finesse is determined by the reflectivity of each mirror, it follows that if photons are lost from the cavity each round trip, this will lead to a broadening of the linewidth. The process of photons leaking out of a cavity is an exponential decay process where κ is the photon decay rate. This leads to the concept of a photon lifetime defined as $\kappa \equiv \tau_{\text{cav}}^{-1}$. Combining these equations we can show that $\Delta\omega = \tau_{\text{cav}}^{-1}$.

It is important to consider the factors which affect the linewidth: intrinsic losses and extrinsic coupling. In the case of the idealised one-sided Fabry-Pérot cavity, the back mirror has a reflectivity of unity. However, in practice this will not be the case as we cannot have a perfectly one-sided cavity. Furthermore, there will be scattering and absorption events at the boundary of both mirrors. These factors are collectively referred to as intrinsic losses. On the other hand,

extrinsic coupling is dictated by the reflectivity of the input mirror and describes the rate at which cavity modes couple to the desired input and output modes. These ideas also feed into our quantum description of an optomechanical cavity.

Derivation of the quantum optomechanical Hamiltonian

We have already seen how a photon reflecting off a mirror will impart a total momentum of $2h/\lambda$. In the usual description of a cavity we assume that these mirrors have infinite mass and so are not displaced by the reflected photons. Now we consider the case where one of the mirrors is a harmonic oscillator whose harmonic motion will be affected by the imparted momentum. Of course, this will now change the cavity length L_{cav} which will in turn affect the resonant cavity modes (see Eq. (2.8)). Since the total energy, or Hamiltonian, of the system depends on the frequency of the cavity resonances, we say the cavity light is coupled to the displacement of the mechanical oscillator. Here we will derive the light-mechanics Hamiltonian for such an optomechanical cavity, first by considering the classical Hamiltonian and then by promoting the conjugate variables to quantum operators.

Now we examine how replacing one end mirror with part of a mechanical oscillator will affect the resonant modes of the cavity. In Eq. (2.8) we defined the resonant frequency for the j^{th} cavity mode. However, the length of the cavity will now change as the oscillator moves $L(x) = L_{\text{cav}} - x$, where L_{cav} is the cavity length in equilibrium. Therefore the resonant frequency is now a function of the oscillator displacement x

$$\omega_{\text{L},j}(x) = \frac{j\pi c}{nL_{\text{cav}}} \frac{1}{1 - x/L_{\text{cav}}} . \quad (2.9)$$

For small displacements $x \ll L_{\text{cav}}$ we consider the expansion to first order around the equilibrium position

$$\omega_{\text{L},j}(x) \approx \omega_{\text{L},j}(0) \left(1 + \frac{x}{L_{\text{cav}}} \right) . \quad (2.10)$$

Therefore the frequency has a linear shift dependent on the displacement of the moving mirror. Eq. (2.9) gives the resonant frequencies in a planar cavity with two parallel mirrors. Therefore,

Eq. (2.10) is valid for the geometry of a Fabry-Pérot cavity. However, more generally, we may instead Taylor expand $\omega_{L,j}$ to obtain

$$\omega_{L,j}(x) \approx \omega_{L,j}(0) + Gx , \quad (2.11)$$

where

$$G \equiv \left. \frac{\partial \omega_{L,j}(x)}{\partial x} \right|_{x=0} , \quad (2.12)$$

quantifies the frequency shift per meter. As noted already, $G = \omega_{L,j}(0)/L_{\text{cav}}$ for a Fabry-Pérot cavity.

Although a mechanical oscillator could range between a single particle up to millions of constituent atoms, we will only consider the macroscopic degrees of freedom and how they couple with light. An oscillator may be vibrating in a linear sum of many harmonic modes, also called phonons. We will focus on a single mode parameterised by frequency ω_M with effective mass m . In classical mechanics, a harmonic oscillator has the following Hamiltonian

$$H_M = \frac{p^2}{2m} + \frac{m\omega_M^2 x^2}{2} , \quad (2.13)$$

where x is displacement, and p is momentum. As with the derivation of the cavity Hamiltonian, we can promote these variables to quantum operators $x \rightarrow \hat{x}$ and $p \rightarrow \hat{p}$. The total quantum Hamiltonian is simply $\hat{H} = \hat{H}_M + \hat{H}_L$, thus inserting Eq. (2.4) and ignoring the zero-point energy of the light oscillator, this is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega_M^2 \hat{x}^2}{2} + \hbar\omega_L(x)\hat{a}^\dagger\hat{a} . \quad (2.14)$$

Inserting and quantising the linear shift in frequency Eq. (2.10) gives the light-matter Hamiltonian for an optomechanical cavity

$$\hat{H} = \hbar\omega_L\hat{a}^\dagger\hat{a} + \omega_M\hat{b}^\dagger\hat{b} + \hbar g_0\hat{a}^\dagger\hat{a}(\hat{b}^\dagger + \hat{b}) , \quad (2.15)$$

where we have used the shorthand $\omega_L = \omega_{L,j}(0)$. Unless otherwise stated, we will generally use

the ladder operators $\hat{a}(\hat{a}^\dagger)$ for light and reserve $\hat{b}(\hat{b}^\dagger)$ for mechanics. We have also introduced the optomechanical coupling rate

$$g_0 \equiv Gx_{\text{zp}} , \quad (2.16)$$

which is a ubiquitous parameter in the field of optomechanics and has units of angular frequency.

Hamiltonian in a rotating frame

One final consideration is the laser that we use to drive the cavity which may excite the photons in the cavity. The laser frequency ω_D is typically close to the resonant cavity frequency ω_L ; as such, both frequencies are much larger than the other system rates such as the mechanical frequency ω_M , mechanical and optical decay rates. Therefore, it is often more convenient to consider a frame rotating with the laser frequency ω_D .

To switch to a rotating frame, we perform the following transformation on the Hamiltonian

$$\tilde{H} = \hat{U}^\dagger \hat{H} \hat{U} - \hat{A} , \quad (2.17)$$

where $\hat{A} = \hbar\omega_D \hat{a}^\dagger \hat{a}$ and $\hat{U} = e^{-i\hat{A}t/\hbar}$. Applying this transformation gives the following Hamiltonian

$$\tilde{H} = \Delta \hbar \hat{a}^\dagger \hat{a} + \hbar\omega_M \hat{b}^\dagger \hat{b} + \hbar g_0 \hat{a}^\dagger \hat{a} (\hat{b}^\dagger + \hat{b}) , \quad (2.18)$$

where $\Delta = \omega_L - \omega_D$ is the detuning between the cavity and drive frequencies. This Hamiltonian is used extensively so we will drop the tilde notation.

2.3 Quantum Langevin equations

From classical physics, we are familiar with the mechanics of a damped harmonic oscillator subjected to external driving forces

$$m\ddot{x} + m\gamma\dot{x} + m\omega^2 x = F(t) , \quad (2.19)$$

where γ is the friction term. The driving force $F(t)$ is the sum of all external forces, including forces from the thermal bath in which the oscillator is in contact. In the absence of driving forces and at non-zero temperature, the oscillator experiences a stochastic force from the environment causing it to undergo Brownian motion. In our classical model, this coupling between the system and the environment leads to dissipation (loss of energy to the environment) and thermalization. In our quantum model, the open-system dynamics lead to a further entirely quantum phenomenon: decoherence. Briefly, if a state is prepared in a quantum superposition, decoherence causes it to tend towards a classical ensemble of states. If we consider a density matrix, decoherence causes the off diagonal elements to decay until only a diagonal matrix remains. Decoherence is discussed further in Chapter 7 where we introduce a protocol for testing decoherence in the context of objective collapse models.

2.3.1 Quantum harmonic oscillator

In order to model a harmonic oscillator more realistically, we couple the system to a bath consisting of a large number of oscillators. We shall explore the quantum Langevin methods to model this open system and derive equations of motion for optical and mechanical operators. As we shall see, there are two quantum Langevin equations we will use: (i) equations derived under the assumptions of the rotating wave approximation which are valid for optical frequencies which oscillate much more quickly than the other system parameters, (ii) the ‘non-rotating wave approximation’ equations which are often used for mechanical operators. We will first consider these two regimes for a generic harmonic oscillator system without making explicit reference whether the system is mechanical or optical.

The first step in both regimes (rotating and non-rotating wave approximations) is to consider the Hamiltonian of the system, in addition to the environment and the coupling with the environment. The environment is typically modelled as a system of independent oscillators where the quantum system is coupled to the j^{th} mode of the bath via springs with spring constant k_j . Each bath oscillator has an associated position $\hat{x}_j$, momentum $\hat{p}_j$, mass m_j , and angular frequency ω_j . The usual definition of the spring constant applies $k_j = m_j\omega_j^2$. We are

primarily concerned with the aggregate effect of the bath rather than the exact dynamics of each environmental oscillator. The total system Hamiltonian is thus

$$\hat{H} = \hat{H}_{\text{sys}} + \hat{H}_{\text{bath}} + \hat{H}_{\text{sys-bath}} . \quad (2.20)$$

We wish to solve this for an arbitrary system, so we will express the Hamiltonian of the quantum system in question as

$$\hat{H}_{\text{sys}} = \frac{\hat{p}^2}{2m} + \hat{V}(\hat{x}) \quad (2.21)$$

where the first term is the kinetic energy and the second term is the potential energy as a function of the position of the harmonic oscillator (we have not explicitly defined the platform yet). The independent-oscillator model dictates the following Hamiltonian for the bath

$$\hat{H}_{\text{bath}} = \sum_j \frac{\hat{p}_j^2}{2m_j} , \quad (2.22)$$

and system-bath interaction

$$H_{\text{sys-bath}} = \sum_j \frac{k_j}{2} (\hat{x}_j - \hat{x})^2 . \quad (2.23)$$

Physically this coupling means that the potential energy depends on the deviation of $\hat{x}$ from all positions $\hat{x}_j$, and so the position of the oscillator is harmonically bound to each bath mode. This important step assumes linear coupling between system and bath position operators $k_j \hat{x}_j \hat{x}$, although more exotic couplings exist which we will not examine in this Thesis. This implies that energy is transferred between the two systems by absorption or emission of bath quanta, which is valid for the optomechanical systems under consideration here. Inserting the total Hamiltonian in Eq. (2.21) into the Heisenberg equation for an arbitrary operator $\hat{\mathcal{O}}$ gives

$$\dot{\hat{\mathcal{O}}} = -\frac{i}{\hbar} [\hat{\mathcal{O}}, \hat{H}] . \quad (2.24)$$

We can derive expressions for $\dot{\hat{x}}$, $\dot{\hat{p}}$, and for the j^{th} bath oscillator $\dot{\hat{x}}_j$, $\dot{\hat{p}}_j$. Eliminating momentum

by using the fact that $\hat{p} = m\dot{\hat{x}}$, we obtain two coupled, second-order differential equations

$$m\ddot{\hat{x}} = \frac{\partial \hat{V}(\hat{x})}{\partial \hat{x}} + \sum_j k_j(\hat{x}_j - \hat{x}) , \quad (2.25a)$$

$$m_j\ddot{\hat{x}}_j = -k_j(\hat{x}_j - \hat{x}) . \quad (2.25b)$$

We begin by solving Eq. (2.25b) for $\hat{x}_j$. We remind the reader that when solving second-order ordinary differential equations, we must first solve the homogeneous equation $m\ddot{\hat{x}}_j + k_j\hat{x}_j = 0$ to obtain the complementary function

$$\hat{x}_j^{\text{CF}}(t) = \hat{x}_j(0) \cos(\omega_j t) + \frac{\hat{p}_j(0)}{\omega_j m_j} \sin(\omega_j t) . \quad (2.26)$$

The particular integral must then be obtained by finding a suitable ansatz for the non-homogeneous equation. The general solution is the sum of the complementary function and particular integral. Although it may not seem entirely obvious, the following solution can be shown to be valid by substituting it back into the differential equation

$$\hat{x}_j(t) = \hat{x}_j^{\text{CF}}(t) + \hat{x} - \int_{-\infty}^t dt' \cos[\omega_j(t - t')] \dot{\hat{x}}(t') , \quad (2.27)$$

where one must use Leibniz rule to differentiate under the integral. Now we insert this solution into Eq. (2.25a) to give the quantum Langevin equation

$$m\ddot{\hat{x}} + \int_{-\infty}^t dt' \mu(t - t') \dot{\hat{x}}(t') + \frac{\partial \hat{V}(\hat{x})}{\partial \hat{x}} = \hat{F}(t) , \quad (2.28)$$

where $\mu(t)$ is the memory kernel, and $\hat{F}(t) = \sum_j k_j \hat{x}_j^{\text{CF}}(t)$ is a stochastic force from the environment with zero expectation value. The memory kernel dictates the timescale over which the bath remembers its interaction with the oscillator. One can see from Eq. (2.28) that the equations of motion at time t depend on the values of $\hat{x}$ at earlier times. Therefore, if $\mu(t)$ is fast-decaying then the equations have a short memory. Explicitly it is given by

$$\mu(t) = \sum_j k_j \cos(\omega_j t) . \quad (2.29)$$

We now approximate the spectrum of phonon (or photon) modes ω_j to a continuous spectrum

$$\mu(t) = \int_0^\infty d\omega \rho(\omega) k(\omega) \cos(\omega t) , \quad (2.30)$$

where $\rho(\omega)$ is the density of oscillators and the spring constant for a given mode is now a continuous function of the frequency of the oscillators $k(\omega)$. It is possible that a number of modes may have the same frequency and so the density function $\rho(\omega)$ is needed. The continuum limit is valid provided the bath is very large implying the normal modes of the environment are very close together in frequency.

Fourier analysis of the integral in Eq. (2.28) shows that it can be written as

$$\int_{-\infty}^t dt' \mu(t-t') \dot{x}(t') = k_{\text{bath}} \hat{x}(t) + m \int_{-\infty}^{\infty} dt' \gamma(t-t') \dot{x}(t'), \quad (2.31)$$

where

$$k_{\text{bath}} = \int_0^\infty d\omega \rho(\omega) k(\omega) , \quad (2.32)$$

is an additional effective spring constant from the bath, and $\gamma(t)$ is the Fourier transform of $\gamma(\omega) = k(\omega)\rho(\omega)/4m$.

In order for the bath to be memoryless we make the so-called first Markov approximation. This is achieved by setting $\gamma(t) = \gamma\delta(t)$ where γ is a constant capturing the dissipation rate. Inserting this into Eq. (2.28) gives the following Markovian Langevin equation for the position of the oscillator

$$m\ddot{\hat{x}} + \gamma\dot{\hat{x}}(t) + \frac{\partial \hat{V}(\hat{x})}{\partial \hat{x}} = \hat{F}(t) , \quad (2.33)$$

which is the quantum analogue of Langevin's original description of a particle moving in a viscous fluid, cf. Eq. (2.19). This implies that the coupling between our system and the normal modes of the bath are independent of mode frequency. We note that the potential which the particle experiences is now replaced $\hat{V}(\hat{x}) \rightarrow \hat{V}(\hat{x}) + k_{\text{bath}}\hat{x}^2$. This describes a potential in addition to the the initial potential felt by the particle. This shifts the energy levels of the harmonic oscillator and is similar to the Lamb shift in atomic physics. This effect is present

even when vacuum noise is the sole input from the environment.

Instead of inserting $\hat{x}$ into the Heisenberg equation (see Eq. (2.24), we can proceed with an arbitrary operator $\hat{\mathcal{O}}$. We can follow the same steps, using the solution $\hat{x}_j(t)$ and making the first Markov approximation, to derive the quantum Langevin equation for $\hat{\mathcal{O}}$

$$\dot{\hat{\mathcal{O}}} = -\frac{i}{\hbar}[\hat{\mathcal{O}}, H_{\text{sys}}] + \frac{i}{\hbar}[\hat{\mathcal{O}}, \hat{x}]\hat{F}(t) - \frac{im}{2\hbar}\{[\hat{\mathcal{O}}, \hat{x}], \gamma\dot{\hat{x}}(t)\}_+ . \quad (2.34)$$

The final term contains the anticommutator notation whose operation is defined as $\{\hat{A}, \hat{B}\}_+ = \hat{A}\hat{B} + \hat{B}\hat{A}$. We note, without going into the details, that the first Markov approximation does not result in a completely memoryless bath. As the system becomes more classical, i.e. in the limit $\hbar \rightarrow 0$, the force from the bath on the oscillator does indeed become memoryless. However, in the quantum model the bath has a memory of the oscillator which is sharply peaked at $t - t'$ with a characteristic time scale $\hbar/k_{\text{B}}T$. Provided $\hbar\gamma \ll k_{\text{B}}T$ we can safely assume that the bath coupling is in fact Markovian.

Finally, we note that it is often easier to work in phase space with dimensionless position and momentum quadratures $\hat{X}$ and $\hat{P}$ defined previously for a harmonic oscillator in Eqs. (1.29a)-(1.29b). Anticipating this, we define an input operator for momentum

$$\hat{P}_{\text{in}} = \frac{i}{\sqrt{2}}(\hat{a}_{\text{in}}^\dagger(t) - \hat{a}_{\text{in}}(t)) = \frac{x_{\text{zp}}\hat{F}(t)}{\hbar\sqrt{\gamma}} , \quad (2.35)$$

which dimensional analysis shows to have units of $\text{s}^{-1/2}$, as do the input annihilation/creation operators. Substituting this into Eq. (2.34), we obtain an alternative form of the quantum Langevin equation

$$\dot{\hat{\mathcal{O}}} = -\frac{i}{\hbar}[\hat{\mathcal{O}}, \hat{H}_{\text{sys}}] + i\sqrt{2\gamma}[\hat{\mathcal{O}}, \hat{X}]\hat{P}_{\text{in}}(t) - \frac{i}{2Q}\{[\hat{\mathcal{O}}, \hat{X}], \dot{\hat{\mathcal{O}}}\}_+ . \quad (2.36)$$

The quality factor of the oscillator $Q = \omega/\gamma$ is analogous with the classical oscillator and describes the amount of energy dissipated per cycle. We will use this equation to examine the open-system dynamics of a mechanical oscillator.

Correlation relations

Here we will briefly examine the properties of the momentum noise operator $\hat{P}_{\text{in}}$, and the inputs and outputs of the system. The incoming modes induce random noise effects and provide energy to the particle; whereas the outgoing modes dissipate energy and give rise to radiation damping. The outgoing channel provides us with a means to perform measurements related to the light in the cavity or the state of a mechanical oscillator.

The input operator can consist of electromagnetic radiation from a number of sources. In the case of a driven cavity, the input includes the laser field. The laser light will have some noisy statistics from random fluctuations in the laser. The input modes also include vacuum noise which is present even in the absence of any light. The vacuum contribution preserves the canonical commutation relations and prevents them decaying to zero and thus violating quantum mechanical foundations.

The input noise operator $\hat{a}_{\text{in}}(t)$ and its adjoint have the the following commutation relations

$$[\hat{a}_{\text{in}}, \hat{a}_{\text{in}}^\dagger] = \delta(t - t') , \quad (2.37a)$$

$$[\hat{a}_{\text{in}}, \hat{a}_{\text{in}}] = [\hat{a}_{\text{in}}^\dagger, \hat{a}_{\text{in}}^\dagger] = 0 . \quad (2.37b)$$

Assuming the input is a thermal state it follows that

$$\langle \hat{a}_{\text{in}}^\dagger(t) \hat{a}_{\text{in}}(t') \rangle = \bar{n} \delta(t - t') , \quad (2.38a)$$

$$\langle \hat{a}_{\text{in}}(t) \hat{a}_{\text{in}}^\dagger(t') \rangle = (\bar{n} + 1) \delta(t - t') , \quad (2.38b)$$

$$\langle \hat{a}_{\text{in}}(t) \hat{a}_{\text{in}}(t') \rangle = \langle \hat{a}_{\text{in}}^\dagger(t) \hat{a}_{\text{in}}^\dagger(t') \rangle = 0 , \quad (2.38c)$$

where $\bar{n}$ is the thermal occupation of the thermal state given by the Bose-Einstein distribution

$$\bar{n} = \frac{1}{e^{\hbar\omega/k_B T} - 1} , \quad (2.39)$$

where ω is the angular frequency of the oscillator, and T is the temperature of the bath. In the

limit where $k_B T \gg \hbar \omega$ we can use the following approximation

$$\bar{n} \approx k_B T / \hbar \omega . \quad (2.40)$$

From these relations we can derive the following correlation relations for the dimensionless bath momentum operator

$$\langle \hat{P}_{\text{in}}(t) \hat{P}_{\text{in}}(t') \rangle = \left(\bar{n} + \frac{1}{2} \right) \delta(t - t') . \quad (2.41)$$

2.3.2 Rotating wave approximation

The general Markovian Langevin equation for an arbitrary operator in Eq. (2.34) holds for any system described by a harmonic oscillator potential which is assumed to be linearly coupled to a bath, i.e. both quantised light in a cavity and also a quantum mechanical resonator. However, it is more convenient when considering optical fields to invoke a rotating wave approximation. In quantum optics, the bath coupling rate is often much smaller than other relevant rates in the system. Transforming to a frame rotating at the frequency ω using Eq. (2.17) and neglecting the fast rotating terms gives the following Hamiltonian for the system and bath

$$\hat{H}_{\text{RWA}} = \hat{H}_{\text{sys}} + \sum_j \left[\hbar \omega_j \hat{a}_j^\dagger \hat{a}_j + i \hbar \gamma_j (\hat{a}_j \hat{a}^\dagger - \hat{a}_j^\dagger \hat{a}) \right] , \quad (2.42)$$

where γ_j is the coupling to each bath oscillator. This is valid provided we are only interested in the behaviour of the oscillator within a relatively narrow bandwidth around the resonant frequency ω . Following the same steps as in Section 2.3.1, using the Heisenberg equation and applying the Markovian approximation, we can derive the following equation of motion for an arbitrary optical operator $\hat{O}$

$$\dot{\hat{O}} = -\frac{i}{\hbar} [\hat{O}, \hat{H}_{\text{sys}}] - [\hat{O}, \hat{a}^\dagger] \left(\frac{\gamma}{2} \hat{a} - \sqrt{\gamma} \hat{a}_{\text{in}}(t) \right) + \left(\frac{\gamma}{2} \hat{a}^\dagger - \sqrt{\gamma} \hat{a}_{\text{in}}^\dagger \right) [\hat{O}, \hat{a}] . \quad (2.43)$$

For the field operator $\hat{a}$, this yields the equation [91]

$$\dot{\hat{a}} = -\frac{i}{\hbar}[\hat{a}, \hat{H}_{\text{sys}}] - \frac{\gamma}{2}\hat{a} + \sqrt{\gamma}\hat{a}_{\text{in}}(t) , \quad (2.44)$$

which will be used extensively in the subsequent section of this chapter. It is customary to reserve the symbol γ for mechanical damping, and κ for optical decay rates. As such, we will make the following substitution

$$\gamma \rightarrow 2\kappa . \quad (2.45)$$

Finally, the Langevin equation for an optical system coupled to the environment is

$$\dot{\hat{a}} = -\frac{i}{\hbar}[\hat{a}, \hat{H}_{\text{sys}}] - \kappa\hat{a} + \sqrt{2\kappa}\hat{a}_{\text{in}}(t) . \quad (2.46)$$

By following the same steps but deriving a time-reversed Langevin equation for the output operators we can define the following input-output relation

$$\hat{a}_{\text{out}}(t) + \hat{a}_{\text{in}}(t) = \sqrt{2\kappa}\hat{a}(t) , \quad (2.47)$$

where we have used the convention that the input field is left-going and negative while the output field is right-going and positive [92]. We will use these equations to solve for the cavity optical field when we examine pulsed optomechanics protocols.

2.4 Pulsed optomechanics

In this chapter we explore the optomechanical unitary interactions which underpin in the area of pulsed optomechanics [93]. Pulsed optomechanics is a class of cavity optomechanical protocols where the pulse length is much shorter than the mechanical period such that the optomechanical interaction occurs in a window of time in which the mechanical oscillator remains effectively stationary. Protocols that exploit the pulsed regime could be used to test commutation relations [94], and to achieve a range of exciting nonlinear states such as single-mode

Schrödinger-cat states [95]. It has been demonstrated that this back-action-evading interaction can be exploited for precision measurements of the position of a mechanical oscillator which can be used to perform full state tomography of the mechanical motional state [81, 82].

Driving an optomechanical cavity with a laser ω_D leads to sidebands in the optical field at $\omega_D \pm \omega_M$, where ω_M is the resonant mechanical frequency. In the resolved-sideband regime, where $\kappa < \omega_M$, the optical sidebands can be resolved as they lie outside of the optical cavity bandwidth. This enables efficient laser cooling of the mechanics when the laser is red-detuned from the cavity resonant frequency $\Delta \equiv \omega_D - \omega_L = -\omega_M$, and the squeezing interaction when the laser is blue detuned $\Delta = \omega_M$.

In contrast to the resolved-sideband regime, one consequence of the pulsed regime is that it operates in the non-resolved sideband regime such that $\kappa \gg \omega_M$, where κ is the optical decay rate of the cavity. The sidebands created lie within the bandwidth of the optical cavity and so are not well-resolved. In this regime, and when $\Delta = 0$, high-precision measurements on the position of the mechanical motion can be made. The radiation-pressure interaction itself can also be exploited to create non-linearity as we shall explore.

In Eq. (2.18) we introduced the light mechanics Hamiltonian in a frame rotating at the optical frequency. When the drive laser is on resonance with the cavity frequency we obtain the following Hamiltonian

$$\hat{H}/\hbar = \omega_M b^\dagger b - g_0 a^\dagger a (b^\dagger + b) . \quad (2.48)$$

We remind the reader that ω_M is the mechanical angular frequency, and g_0 is the optomechanical coupling rate. We now proceed using the Langevin equations derived in Section 2.3 to describe the open system dynamics of a cavity optomechanical system in the pulsed setting. We treat both optical and mechanical baths as Markovian however for the optical field we use Eq. (2.46) which assumed the rotating wave approximation, whereas for the mechanical operators we use Eq. (2.34).

Operating in the unresolved sideband regime means the interaction happens on a scale where $\omega_M \ll \kappa$. Physically, the mechanical position remains constant during the pulse interaction.

We assume that our oscillator has a high quality factor $Q = \omega_M/\gamma \gg 1$, and thus we can neglect mechanical harmonic motion, damping and open system dynamics during the light-matter interaction. Therefore, under these assumptions, the equation of motion of the field operator during this short interaction is found from Eq. (2.34) to be

$$\dot{\hat{b}} \approx ig_0 \hat{a}^\dagger \hat{a} . \quad (2.49)$$

Separating this into real and imaginary parts yields the Langevin equations for the dimensionless $\hat{X}_M$ and $\hat{P}_M$ quadratures

$$\dot{\hat{X}}_M = 0 , \quad (2.50a)$$

$$\dot{\hat{P}}_M = \sqrt{2}g_0 \hat{a}^\dagger \hat{a} , \quad (2.50b)$$

where we can instantly see that the position of the mechanical oscillator remains stationary but the momentum will change. The magnitude of the momentum which is imparted to the resonator clearly depends on the cavity field $\hat{a}$. Therefore, we now examine the dynamics of the cavity field for a pulsed interaction. As already discussed, the cavity field is governed by the optical Langevin equation Eq. (2.46) where we insert the Hamiltonian in Eq. (2.48) to obtain

$$\dot{\hat{a}} = (ig_0\sqrt{2}\hat{X}_M - \kappa)\hat{a} + \sqrt{2\kappa}\hat{a}_{\text{in}} , \quad (2.51)$$

$\hat{a}_{\text{in}}$ is the field operator for the input field and obeys the input-output boundary condition defined in Eq. (2.47). By considering the transformation of $\hat{a}_{\text{in}}$ to the output field operator $\hat{a}_{\text{out}}$ we can derive a unitary transformation for this interaction and thus model the effect on the mechanical mode.

2.4.1 Nonlinear pulsed interaction

Supposing we address a mechanical mode such that g_0 is weak, we can keep the full nonlinear interaction given in Eq. (2.48) without approximating this to a linear interaction. As previously

mentioned, our goal is to derive a unitary to model the effect which the light-matter interaction has on the mechanical system. We now follow the same arguments outlined in Ref. [94].

If we work in a regime where the characteristic pulse length τ is much longer than the characteristic time it takes for the light to leave the cavity then $\kappa \gg \tau^{-1}$. This is equivalent to saying the cavity linewidth is large compared to the pulse bandwidth. This is in addition to the non-resolved sideband constraint and so we require $\kappa \gg \tau^{-1} \gg \omega_M$ to ensure the mechanical position remains constant during the pulse interaction. Under these assumptions, the cavity field can be adiabatically eliminated $\dot{\hat{a}} = 0$, see for example Refs. [96] and [97], and so from Eq. (2.51) it follows that

$$\hat{a}(t) = \frac{\sqrt{2\kappa}}{\kappa - i\sqrt{2}g_0\hat{X}_M} \hat{a}_{\text{in}}(t) . \quad (2.52)$$

Using the input-output relations in Eq. (2.47) gives the following transformation for light entering and leaving the cavity

$$\begin{aligned} \hat{a}_{\text{out}}(t) &= \frac{\kappa + i\sqrt{2}g_0\hat{X}_M}{\kappa - i\sqrt{2}g_0\hat{X}_M} \hat{a}_{\text{in}}(t) \\ \hat{a}_{\text{out}}(t) &= \frac{1 + i\mu\hat{X}_M/2}{1 - i\mu\hat{X}_M/2} \hat{a}_{\text{in}}(t) , \end{aligned} \quad (2.53)$$

where we have introduced the dimensionless parameter $\mu = \sqrt{8}g_0/\kappa$. The physical interpretation of μ will shortly be discussed. Since we assume that the position of the oscillator remains essentially unchanged for the duration of the interaction, we can Taylor expand the prefactor of $\hat{a}_{\text{in}}(t)$. This agrees with the series expansion of the exponential function up to second order in $\mu\hat{X}_M$, and so

$$\hat{a}_{\text{out}}(t) \approx e^{i\mu\hat{X}_M} \hat{a}_{\text{in}}(t) . \quad (2.54)$$

Now let us consider the effect that such a pulsed nonlinear interaction has on the mechanical momentum quadrature. Let us begin by integrating Eqs. (2.50a) and (2.50b)

$$\hat{X}'_M - \hat{X}_M = 0 , \quad (2.55a)$$

$$\hat{P}'_M - \hat{P}_M = \int_0^\tau dt \sqrt{2}g_0 \hat{a}^\dagger(t) \hat{a}(t) , \quad (2.55b)$$

where we use primed notation to indicate the final quadratures following the light-matter interaction. Note that we have set the limits of integration to cover only the duration of the pulse rather than $[-\infty, \infty]$. This choice of integration limits is valid provided we are operating in the adiabatic regime, $\kappa \gg \tau^{-1}$, such that the cavity fills quickly, saturates such that $\dot{\hat{a}} = 0$ for the pulse's duration, and quickly empties after the pulse has passed. From these preliminary solutions we can see that the mechanical position does not move on the time length of the interaction, whereas a non-zero momentum kick has been imparted on the mechanics. We now define a time-integrated input photon number operator n_L to be

$$\begin{aligned}\hat{n}'_L &= \int_0^\tau dt \hat{a}_{\text{in}}^\dagger(t) \hat{a}_{\text{in}}(t) , \\ &= \int_0^\tau dt \hat{a}_{\text{out}}^\dagger(t) \hat{a}_{\text{out}}(t) ,\end{aligned}\tag{2.56}$$

where the second equality holds assuming a lossless pulsed interaction. Therefore the change in momentum is

$$\hat{P}'_M - \hat{P}_M = \frac{2\sqrt{2}g_0\kappa}{\kappa^2 + 2g_0^2\hat{X}_M^2} \hat{n}'_L ,\tag{2.57}$$

where we substituted $\hat{a}_{\text{in}}(t)$ using Eq. (2.52). Using the variable $\mu = 2\sqrt{2}g_0/\kappa$ it follows that

$$\hat{P}'_M - \hat{P}_M = \frac{\mu}{1 + (\mu\hat{X}_M/2)^2} \hat{n}'_L .\tag{2.58}$$

Expanding this to second order in $\mu\hat{X}_M$ (as was done in the derivation of Eq. (2.54)) gives

$$\hat{P}'_M - \hat{P}_M = \mu\hat{n}'_L .\tag{2.59}$$

So we achieve a momentum kick of size $\mu\hat{n}'_L$ meaning that the effective unitary for such an interaction can be approximated to

$$\hat{U} = e^{i\mu\hat{n}'_L\hat{X}_M} .\tag{2.60}$$

We have implicitly assumed that the input optical pulse shape is Gaussian with temporal width $\tau \gg 1/\kappa$, as this allows one to adiabatically eliminate the cavity field even at early times in the cavity's dynamics. This value of $\mu = 2\sqrt{2}g_0/\kappa$ is the result of the coherent sum over all

possible trajectories the photons may take as they enter and then leave the cavity. Certain trajectories where the photons reflect only a small number of times from the mechanical mode, and trajectories where the photons remain in the cavity for a very long time compared to $1/\kappa$, lead to smaller or larger optomechanical couplings, respectively. However, in the adiabatic regime these few-reflection and high-reflection-number photon trajectories occur with a negligible probability amplitude and so the optomechanical coupling strength is well approximated by $\mu = 2\sqrt{2}g_0/\kappa$. This nonlinear pulsed interaction is explored in more mathematical rigour in Ref. [98].

We can satisfy ourselves that this captures the correct dynamics by examining the transformations of the light and mechanical operators before and after the interaction. The unitary maps the incident light $\hat{a}_{\text{in}}$ onto the output light $\hat{a}_{\text{out}}$ in the following way

$$\hat{a}_{\text{out}} = \hat{U}^\dagger \hat{a}_{\text{in}} \hat{U} , \quad (2.61)$$

which gives Eq. (2.54) as expected. Similarly, the mechanics are transformed as follows

$$\begin{aligned} \hat{P}'_{\text{M}} &= \hat{U}^\dagger \hat{P}_{\text{M}} \hat{U} \\ \hat{X}'_{\text{M}} &= \hat{U}^\dagger \hat{X}_{\text{M}} \hat{U} \end{aligned} \quad (2.62)$$

which agrees with Eq. (2.59) and the fact that the mechanical position remains unchanged $\hat{X}'_{\text{M}} = \hat{X}_{\text{M}}$. Therefore we can see that μ plays the role of a dimensionless optomechanical coupling strength.

A note on μ

In our derivation of the pulsed nonlinear interaction we made the assumption that the pulse length is much longer than the cavity decay rate $\tau \gg \kappa^{-1}$. This is a key step in the adiabatic approximation which allows us assume that the intracavity field is effectively constant during the interaction to set $\dot{\hat{a}} = 0$.

More generally, we must solve Eqs. (2.49) and (2.51) for a given input pulse $a_{\text{in}}(t)$. Assuming

we are still operating in the unresolved sideband regime, the unitary interaction is still given by Eq. (2.60) however more generally $\mu \propto g_0/\kappa$. The expected change in momentum can be found by integrating $\dot{\hat{P}}_M = \sqrt{2}g_0\hat{a}^\dagger\hat{a}$ from which it follows that

$$\langle \hat{P}'_M - \hat{P}_M \rangle = \int_{-\infty}^{\infty} dt \sqrt{2}g_0 \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle = \mu \langle \hat{n}'_L \rangle , \quad (2.63)$$

where $a(t)$ are the time-dependent cavity field operators and $\langle \hat{n}'_L \rangle$ is the expectation of the time-integrated input number operator of the pulse defined in Eq. (2.56). For a single photon input $\langle \hat{n}'_L \rangle = 1$ and so,

$$\mu = \sqrt{2}g_0 \int_{-\infty}^{\infty} dt \langle \hat{a}^\dagger(t)\hat{a}(t) \rangle . \quad (2.64)$$

Before we can solve for μ we first directly integrate Eq. (2.51) giving,

$$\hat{a}(t) = \sqrt{2\kappa} \int_{-\infty}^t dt' \hat{a}_{\text{in}}(t') e^{-(\kappa - i\sqrt{2}g_0\hat{X}_M)|t-t'|}, \quad (2.65)$$

where we have used the technique of an integrating factor. (We note that the adiabatic approximation can be intuited from this equation since under the assumption $\tau \gg \kappa^{-1}$ this integral can be approximately evaluated to give Eq. (2.52).) Inserting Eq. (2.65) into Eq. (2.64) gives,

$$\mu = \sqrt{8}g_0\kappa \int_{-\infty}^{\infty} dt e^{-2\kappa t} \int_{-\infty}^t dt' \int_{-\infty}^t dt'' e^{\kappa(t'+t'')} \langle \hat{a}_{\text{in}}^\dagger(t')\hat{a}_{\text{in}}(t'') \rangle , \quad (2.66)$$

where we have used the fact that $\langle \hat{X}_M \rangle = 0$ because the mechanical displacement remains constant during the optomechanical interactions. Therefore the coupling strength μ is dependant on the input pulse shape. As demonstrated in the Appendix of Ref. [95], it can be shown that for a single photon input with a temporal envelope $f(t)$ which matches the cavity spectrum

$$f(t) = \sqrt{\kappa} e^{-\kappa|t|}, \quad (2.67)$$

then $\langle \hat{a}_{\text{in}}^\dagger(t)\hat{a}_{\text{in}}(t) \rangle = f^*(t)f(t)$ and the integral can be evaluated to give $\mu = 3g_0/\sqrt{2}\kappa$. In the subsequent work outlined in this Thesis we will use the greater value of $\mu = 2\sqrt{2}g_0/\kappa$ (obtained by using a Gaussian pulse with $\kappa \gg \tau^{-1}$ as previously discussed) as this gives rise to

a greater optomechanical coupling strength. Furthermore, it is more experimentally challenging to optimise the input pulse to match the cavity spectrum.

2.4.2 Linear pulsed interaction

Now we examine the pulsed interaction for the case of a linearized optomechanical Hamiltonian [93]. We will follow the derivation from Refs. [97] and [96]. Starting with the same optomechanical interaction Hamiltonian defined in Eq. (2.48), we can linearize this Hamiltonian. To physically switch from the nonlinear interaction outlined in Section 2.4.1 to a linearised interaction, we can use light of different wavelength to address a different cavity resonance with a weaker coupling strength g_0 . The linearized Hamiltonian is derived by writing the optical field as a small quantum noise operator $\delta\hat{a}$ around a coherent amplitude $\alpha(t) = \langle\hat{a}(t)\rangle$. We therefore make the following substitution

$$\hat{a} = \delta\hat{a}(t) + \alpha(t) . \quad (2.68)$$

Inserting this into Eq. (2.48) gives the linearized Hamiltonian

$$\hat{H}/\hbar = -\sqrt{2}g_0\hat{X}_M|\alpha(t)|^2 - 2g_0|\alpha(t)|\hat{X}_M\bar{X}_L(t) , \quad (2.69)$$

where we have defined an optical fluctuation operator

$$\bar{X}_L(t) = \frac{\alpha^*\delta\hat{a}(t) + \alpha\delta\hat{a}^\dagger(t)}{\sqrt{2}|\alpha(t)|} , \quad (2.70)$$

with an similar equation for $\bar{P}_L$. We assume that we are operating in the weak coupling regime $g_0 \ll \kappa$ such that the coupling between the mechanical element and optical fluctuations $\delta\hat{a}^\dagger\delta\hat{a}X_M$ is negligible and so the term quadratic in $\delta\hat{a}$ is dropped.

Substituting (2.68) into the Langevin equation, Eq. (2.46), for the optical field we can identify

equations of motion for α and $\delta\hat{a}$

$$\dot{\alpha} = -\kappa\alpha(t) + \sqrt{2\kappa}\alpha_{\text{in}}(t) , \quad (2.71a)$$

$$\delta\dot{\hat{a}} = ig_0\sqrt{2}[\alpha(t) + \delta\hat{a}]\hat{X}_{\text{M}} - \kappa\delta\hat{a}(t) + \sqrt{2\kappa}\delta\hat{a}_{\text{in}}(t) , \quad (2.71b)$$

Note that $\hat{X}_{\text{M}}$ has no explicit time-dependency as in the non-resolved sideband regime the operator does not have time to evolve over the course of the interaction.

From the input-output relation Eq. (2.47), we obtain equivalent relations for the optical fluctuation quadratures

$$\hat{X}_{\text{L,out}} + \hat{X}_{\text{L,in}} = \sqrt{2\kappa}\bar{X}_{\text{L}} , \quad (2.72a)$$

$$\hat{P}_{\text{L,out}} + \hat{P}_{\text{L,in}} = \sqrt{2\kappa}\bar{P}_{\text{L}} . \quad (2.72b)$$

Using the same methods as in Section 2.4.1, we proceed by making the adiabatic approximation $\dot{\alpha} = 0$ which is valid for a pulse whose width is much larger than the cavity decay rate $\tau \gg \kappa^{-1}$. First, by setting $\dot{\alpha} = 0$ in Eq. (2.71a) we obtain $\alpha(t) = \alpha_{\text{in}}(t)\sqrt{2/\kappa}$. Assuming a real α and considering the real and imaginary parts of the equation $\delta\dot{\hat{a}} = 0$ separately leads to the following transformations for the optical fluctuations

$$\hat{X}_{\text{L,out}} = \hat{X}_{\text{L,in}} , \quad (2.73a)$$

$$\hat{P}_{\text{L,out}} = \hat{P}_{\text{L,in}} + \frac{4g_0\alpha_{\text{in}}(t)}{\kappa}\hat{X}_{\text{M}} . \quad (2.73b)$$

Using $\dot{\hat{b}} = ig_0\hat{a}^\dagger\hat{a}$ (cf. Eq. (2.49)) we can obtain equations of motion for the mechanical phase quadratures

$$\dot{\hat{X}}_{\text{M}} = 0 , \quad (2.74a)$$

$$\dot{\hat{P}}_{\text{M}} = \frac{4g_0\alpha_{\text{in}}(t)}{\kappa}\left(\hat{X}_{\text{L,in}} + \frac{\alpha_{\text{in}}(t)}{\sqrt{2}}\right) , \quad (2.74b)$$

where we have taken α to be real and positive. Following the steps outlined in Ref. [97], we

can integrate these equations to obtain the post-interaction mechanical fluctuations (indicated by primed variables)

$$\hat{X}'_{\text{M}} = \hat{X}_{\text{M}} , \quad (2.75\text{a})$$

$$\hat{P}'_{\text{M}} = \hat{P}_{\text{M}} + \chi X_{\text{L}} + (\chi\sqrt{N/2}) , \quad (2.75\text{b})$$

where the dimensionless coupling strength χ is

$$\chi = \frac{4g_0\sqrt{N}}{\kappa} , \quad (2.76)$$

the collective pulse quadrature X_{L} is defined as $X_{\text{L}} = \int_{-\infty}^{\infty} dt \alpha_{\text{in}}(t) \hat{X}_{\text{L},\text{in}}(t)/\sqrt{N}$, and $N = \int_{-\infty}^{\infty} dt \alpha_{\text{in}}(t)^2$ is the average number of photons in the input pulse. As noted in Ref. [97], the transformed $\hat{P}'_{\text{M}}$ contains a classical momentum kick $\chi\sqrt{N/2}$ which can be removed via a post-interaction mechanical displacement. We henceforth ignore this classical kick.

Performing similar analysis shows that one can arrive at the post-interaction light fluctuations (again indicated by primed variables) by integrating Eqs. (2.73a)-(2.73b) multiplied by $\alpha_{\text{in}}(t)$ to give

$$\hat{X}'_{\text{L}} = \hat{X}_{\text{L}} , \quad (2.77\text{a})$$

$$\hat{P}'_{\text{L}} = \hat{P}_{\text{L}} + \chi X_{\text{M}} , \quad (2.77\text{b})$$

These transformations are captured by the following light-matter unitary¹

$$\hat{U}_{\text{Lin}} = e^{i\chi\hat{X}_{\text{M}}\hat{X}_{\text{L}}} . \quad (2.78)$$

This is an example of a quantum non-demolition interaction between the light and mechanics and can be used to obtain precise information about the displacement of the mechanical

¹If one does not ignore the classical momentum kick of $\chi\sqrt{N/2}$ the full unitary to describe the interaction is $\hat{U} = e^{i\lambda\hat{X}_{\text{M}}} e^{i\chi\hat{X}_{\text{M}}\hat{X}_{\text{L}}}$, where the first exponential describes a deterministic momentum transfer to the mechanics [93]. This is not important in the transformation of the light quadratures and so has been omitted in this discussion. If one wishes to mitigate this effect on the mechanics it is possible to subtract the classical kick by performing a displacement on the mechanical oscillator immediately after the interaction.

resonator. We can see that each oscillator acquires information about one quadrature of the other during this interaction. In particular, the phase of the output light $\hat{P}'_L$ has received a shift proportional to the initial displacement of the mechanical oscillator $\hat{X}_M$. Therefore, χ quantifies the position measurement strength; in other words how much mechanical position information is imparted onto the optical phase quadrature.

As argued in Section 2.4.1, the coupling strength χ is dependant on the input pulse shape. If we were to similarly consider a single-photon input whose temporal shape matches the cavity spectrum then we would obtain $\chi = 2\sqrt{5}\sqrt{N}g_0/\kappa$ as demonstrated in Ref. [93].

2.5 Electromechanics

So far we have implicitly assumed that optomechanics is the study of visible light interacting with a mechanical system. The high frequency of visible light has the advantage of not interacting with the surrounding thermal bath; furthermore, the small wavelength also means that smaller cavities can be fabricated with enhanced coupling strengths. Nevertheless, the microwave domain can also be used to achieve electromechanical coupling. While microwaves are not impervious to the thermal environment and exert weaker radiation pressure, capacitive coupling means electromechanical systems can be created comprising of a mechanical resonator coupled with a superconducting microwave circuit. The coupling strength between the resonator and superconducting qubit can exceed the qubit decoherence rate, enabling us to control and read out the state of the mechanical oscillator via qubit operations [99].

The authors in Ref. [22], propose a device called the ‘displacemon’ which enables strong coupling between a nanomechanical resonator and a superconducting qubit. The design is based on a split transmon qubit that can achieve strong and coherent coupling to the mechanical motion of a carbon nanotube. Specifically, a concentric transmon architecture is proposed where one of the Josephson junctions is replaced with a vibrating nanotube. This design favours strong coupling to nanomechanical displacement, hence the name: displacemon. In this Section we will briefly introduce transmon devices in general before discussing the displacemon device

specifically. The advantages of this geometry are outlined in greater detail in Ref. [22]. We will also derive the displacemon Hamiltonian which consists of a qubit, a resonator and a qubit-resonator interaction term. This Hamiltonian will then be used extensively in Chapter 7 where we will show how qubit rotations, initialization, and projective measurement can be exploited to generate and measure quantum states in the centre-of-mass displacement of the nanotube. The purpose of the protocol outlined in Chapter 7 is to generate a non-Gaussian mechanical state which can be used to test for wavefunction collapse.

2.5.1 Transmon physics

The first iteration of a superconducting qubit was the Cooper Pair Box (or charge qubit) [100], whose qubit is a tiny superconducting island which is coupled via a Josephson junction to a superconducting reservoir. The state of the qubit is determined by the number of Cooper pairs which have tunneled across the junction. If the superconducting island has an extra Cooper pair with charge $2e$ it is in the state $|0\rangle$, otherwise it is $|1\rangle$. The mathematics is similar to the classical LC circuit where energy oscillates between electrical energy stored in the capacitor and magnetic energy in the linear inductor. The quantum analogy of the classical LC circuit gives rise to the harmonic oscillator Hamiltonian introduced in Eq. (1.20) with regularly spaced energy levels $|0\rangle, |1\rangle, |2\rangle, \dots$ in the energy eigenbasis. Instead of phonons or photons, here, the basis states are charge states. However, crucially, in a superconducting qubit the Josephson junction introduces a nonlinearity in the Hamiltonian which in turn introduces an anharmonicity in the transition frequencies between the energy levels. This anharmonicity enables us to individually address two energy levels and achieve a subspace consisting of only the two lowest-energy states (i.e. a qubit system) which we henceforth denote as $|+\rangle$ and $|-\rangle$. Therefore, the superconducting qubit Hamiltonian is

$$\hat{H} = 4E_C(\hat{n} - n_g)^2 - E_J \cos \hat{\phi} , \quad (2.79)$$

where $E_C = e^2/2C$ is the charging energy required to add each electron of the Cooper-pair to the island, E_J is the Josephson energy, $\hat{n}$ is the excess number of Cooper pairs on the island

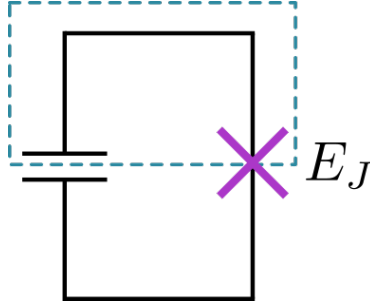


Figure 2.1: Circuit diagram of a charge qubit circuit consisting of a superconducting island (dotted line) coupled via superconducting electrodes to a capacitor and a Josephson junction (purple cross). The charge qubit is a LC circuit with the Josephson junction acting as a non-linear inductor. The anharmonicity in the energy states allows us to exploit the lowest energy states to achieve a qubit system. A transmon is a type of superconducting charge qubit in the regime $E_J \gg E_C$ which is less sensitive to charge noise.

and $\hat{\phi}$ is the phase across the junctions. The gate offset charge n_g is unimportant in this work.

The original Cooper Pair Box was designed to operate in the regime $E_J \leq E_C$ meaning the charge states were particularly sensitive to charge noise; and so the system is governed by the important ratio E_J/E_C . To mitigate the charge noise, these days the regime $E_J \gg E_C$ is favoured, which can be achieved by shunting the junction with a large capacitor as is accomplished in the transmon device [101].

One widely-used technique is to replace the single Josephson junction of the transmon with a superconducting quantum interference device (SQUID) comprised of a loop interrupted by two identical Josephson junctions. This so-called ‘split transmon’ design is shown in Figure 2.2. In the conventional design, we assume each junction has energy $E_J^0/2$, where $E_J^0 = I_c \Phi_0/\pi$. I_c is the critical current in each junction which is assumed to be equal, and $\Phi_0 = h/2e$ is the superconducting flux quantum. The combined Josephson energy of the SQUID is then

$$E_J = E_J^0 \left| \cos \frac{\pi \Phi}{2 \Phi_0} \right|, \quad (2.80)$$

where Φ is the magnetic flux through the SQUID loop. Mathematically, the SQUID-loop can still be treated as a single junction, with the important modification that E_J is tunable via an external flux [102]. Mechanical dependence can be introduced to E_J by replacing one of the Josephson junctions with a superconducting beam. The qubit can be controlled by a

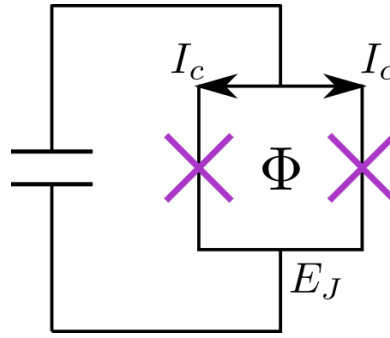


Figure 2.2: Circuit diagram of the split transmon where the single Josephson junction is replaced with a SQUID. The Josephson energy E_J depends on the flux through the SQUID. We can introduce mechanical dependence by replacing one of the Josephson junctions with a vibrating nanotube. The position X of the resonator centre-of-mass is now flux-coupled to the qubit energy.

static magnetic field while the displacement of the beam X modulates Φ and therefore E_J and consequently the qubit state. Such electromechanical coupling between an oscillating beam and a transmon was recently demonstrated [103].

In the transmon limit $E_J \gg E_C$ the qubit Hamiltonian can be approximated to

$$\hat{H} \approx \frac{1}{2} \omega_q \hat{\sigma}_z, \quad (2.81)$$

where $\omega_q = \sqrt{8E_C E_J}/\hbar$ is the qubit frequency, and $\hat{\sigma}_z$ is the Pauli matrix acting in the basis $\{|+\rangle, |-\rangle\}$. The external magnetic field can be controlled to implement qubit algorithms [104].

2.5.2 The displacemon device

A drawback of the split transmon shown in Figure 2.2 is that the qubit is sensitive to flux noise which enters through the term Φ . In the displacemon proposal in Ref. [22] the design is adapted so that it contains two flux loops. This can be achieved with two inner and outer superconductors which operate as the two electrodes of the capacitor. A superconducting beam is introduced into one arm of the loop and the qubit is operated in a static magnetic field, so that the displacement of the beam modulates Φ and therefore E_J . This set-up is shown in Figure 2.3 and is similar to the concentric transmon design [105]. The Josephson energy now depends on the flux difference $\Delta\Phi_0 \equiv \Phi_1 - \Phi_2$ between the two junctions

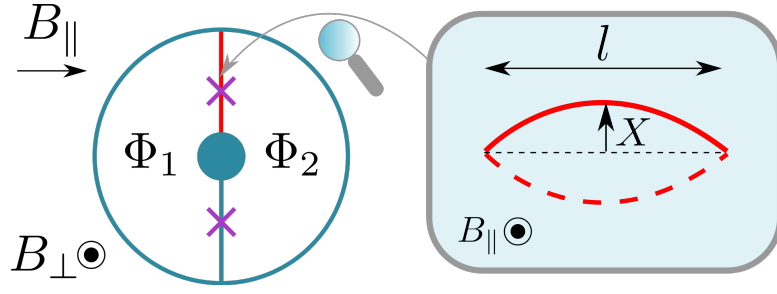


Figure 2.3: The displacemon differs from the split transmon design (see Figure 2.2) by incorporating two enclosed flux regions so that the qubit energy depends on $\Delta\Phi \equiv \Phi_1 - \Phi_2$. One of the Josephson junctions (purple) is a vibrating superconducting nanotube (red). The blue indicates a superconductor with the inner and outer circles effectively being the electrodes of the capacitor. Compared with the split transmon design, this has the additional advantage of decoupling the qubit from homogenous magnetic field fluctuations and thus decrease the sensitivity of E_J to flux noise. The perpendicular magnetic field $B_\perp$ tunes the flux difference $\Delta\Phi$ while the parallel static magnetic field $B_\parallel$ couples to X and in turn also tunes $\Delta\Phi$. The qubit and mechanical nanotube are thus flux-coupled.

$$E_J = E_J^0 \left| \cos \frac{\pi \Delta\Phi}{2\Phi_0} \right|. \quad (2.82)$$

The qubit frequency can therefore be tuned via an externally applied perpendicular magnetic field $B_\perp$, such as in a traditional split transmon. In the displacemon device, an additional in-plane magnetic field $B_\parallel$ which is perpendicular to the plane of the oscillating conducting nanotube also tunes the flux difference. Therefore

$$\Delta\Phi = A\Delta B_\perp + 2\beta_0 l B_\parallel X, \quad (2.83)$$

where A is the area of the SQUID loop, X is the position of the nanotube's centre-of-mass and l is the length of the nanotube. The coupling coefficient β_0 relates the displacement of the beam to the area it sweeps out. We will consider cuboid beams which have $\beta_0 \approx 0.831$ [106]. We note that the symbol X is usually dimensionless while the symbol x usually has units of $[m]$. However, to maintain consistent notation with Refs. [22] and [106], X has units of displacement in chapters pertaining to the displacemon device.

As the position of the nanotube is swept, the qubit frequency will be modulated accordingly

$$\frac{d\omega_q}{dX} = -\omega_q^0 \frac{\pi\beta_0 l B_\parallel}{2\Phi_0} \frac{\sin(\pi\Delta\Phi/2\Phi_0)}{\sqrt{|\cos(\pi\Delta\Phi/2\Phi_0)|}}, \quad (2.84)$$

where we have inserted Eqs. (2.82) and (2.83) into $\omega_q = \sqrt{8E_C E_J}/\hbar$ and differentiated with respect to X . The maximal qubit frequency is $\omega_q^0 = \sqrt{8E_C E_J^0}/\hbar$. The resonator-dependent qubit frequency shift can be inserted into the qubit Hamiltonian, which together with the usual Hamiltonian of a mechanical harmonic oscillator of angular frequency Ω , gives the following qubit-resonator Hamiltonian

$$\hat{H} = \frac{1}{2}\omega_q\hat{\sigma}_z + \hbar\Omega\hat{b}^\dagger\hat{b} + \hbar\lambda(t)(\hat{b} + \hat{b}^\dagger)\hat{\sigma}_z, \quad (2.85)$$

where

$$\begin{aligned} \lambda(t) &= \frac{X_{\text{zp}}}{2} \frac{d\omega_q}{dX} \\ &= -\omega_q \frac{\pi\beta_0 X_{\text{zp}} l B_{\parallel}}{4\Phi_0} \tan \frac{\pi\Delta\Phi(t)}{2\Phi_0}, \end{aligned} \quad (2.86)$$

and the time dependence arises through the tunable flux offset $\Delta\Phi(t)$. Therefore, flux coupling between the qubit and the mechanical resonator is achieved. Qubit operations and read-out are performed using established techniques for concentric transmons [105]. Importantly, by initialising, controlling and measuring the qubit using standard techniques of circuit quantum electrodynamics we can generate mechanical states of interest in the resonator and furthermore perform measurements on the mechanical state.

In principle, Eq. (2.86) allows arbitrarily large $\lambda(t)$. In practice, the inevitable asymmetry between the two Josephson junctions has the effect of limiting the useful range of $\Delta\Phi$, meaning that the $\tan \frac{\pi\Delta\Phi(t)}{2\Phi_0}$ factor is likely to be of order unity.

2.5.3 The resonator

In the Hamiltonian for the combined qubit-resonator system (see Eq. (2.85)), we assumed that the resonator can be modelled as a mechanical oscillator in the quantum regime $\hbar\Omega\hat{b}^\dagger\hat{b}$. Here, we will explore how the resonator frequency Ω depends on its geometry.

The resonator is modelled as a superconducting cuboid beam with length ℓ , and a square

cross-section with area $D \times D$ (see Fig. 2.3 for the orientation of beam). For simplicity, we assume that the beam has been fabricated without tension. Its motion is described by the Euler-Bernoulli beam equation [107]

$$\left(\rho_0 D^2 \frac{\partial^2}{\partial t^2} + \frac{ED^4}{12} \frac{\partial^4}{\partial Z^4} \right) u(Z, t) = 0 , \quad (2.87)$$

where $u(Z, t)$ is the displacement profile of the beam as a function of time t and coordinate along the beam Z , with $Z = 0$ being the centre. We note again that contrary to the usual convention, Z has units of $[m]$ in keeping with published works on this topic. Here ρ_0 is the density of the beam and E is its Young's modulus, and Poisson's ratio has been neglected.

To solve for the frequency of the fundamental displacement mode, we make the following ansatz

$$u(Z, t) = u_0(Z) e^{i\Omega t} , \quad (2.88)$$

which means that $u_0(Z)$ must satisfy

$$\frac{d^4}{dZ^4} u_0(Z) = k^4 u_0(Z) , \quad (2.89)$$

where

$$k \equiv \left(12 \Omega^2 \frac{\rho_0}{ED^2} \right)^{1/4} . \quad (2.90)$$

The solution, respecting the boundary conditions $u_0(\pm\ell/2) = 0$ and normalising [107] so that $\int u_0^2(Z) dZ = 1$, is

$$u_0(Z) = \sqrt{2} \frac{\cos \frac{k\ell}{2} \cosh kZ - \cosh \frac{k\ell}{2} \cos kZ}{\sqrt{\cos^2 \frac{k\ell}{2} + \cosh^2 \frac{k\ell}{2}}} . \quad (2.91)$$

The additional boundary conditions $u'_0(\pm\ell/2) = 0$ constrain k to obey

$$\cos \frac{k\ell}{2} \sinh \frac{k\ell}{2} + \cosh \frac{k\ell}{2} \sin \frac{k\ell}{2} = 0 , \quad (2.92)$$

whose lowest solution is $k\ell \approx 4.730$. Therefore, for a cuboid beam the frequency of the funda-

mental mode is approximately

$$\Omega \approx 6.47 \frac{D}{\ell^2} \sqrt{\frac{E}{\rho_0}}. \quad (2.93)$$

An ideal square beam has two degenerate fundamental modes corresponding to orthogonal directions of motion. However, only the out-of-plane mode couples strongly to the qubit. Furthermore, fabrication imperfections will generally break the degeneracy of the two modes. (If necessary, this can be enforced by designing the beam width to be different from its thickness.) We therefore consider only the out-of-plane mode in the rest of this work.

In Chapter 7 we will examine two devices with two different parameter sets for D , l , E , and ρ_0 . Therefore, the two devices are characterised by different fundamental angular frequencies Ω and, importantly, different mass density distributions. These parameters are proposed with the intention of probing the quantum-to-classical boundary and potentially observing the effects of gravity.

Chapter 3

Inseparability criteria for non-Gaussian entangled states

The phenomenon of quantum entanglement, describing correlations which go beyond our classical understanding of the universe, is a key concept in quantum information theory, quantum communication, and fundamental studies. In Section 1.2 we introduced the definition of a Gaussian state and furthermore the important class of Gaussian entangled states. Nevertheless, there are a plethora of non-Gaussian entangled states which permit exciting avenues in fields such as quantum information and foundations of physics. We have already introduced two such states; the two-mode Schrödinger-cat state in Section 1.3.2, and the photon-removed two-mode squeezed vacuum state in Section 1.3.3. While in Section 1.2 we introduced some methods for testing entanglement in the context Gaussian states, in general, characterising non-Gaussian entanglement is challenging and remains a lively research question.

Firstly, we note that Gaussian entanglement is very well-studied. By Gaussian entanglement we mean entangled states which have Gaussian characteristic functions and Wigner distributions (see Section 1.2). In Section 1.2 we introduced the positive partial transpose requirement for a separable state (see Eq. (1.40)). For Gaussian states, this condition can be elegantly recast in both the covariance matrix and symplectic representations. Furthermore, one can even define a monotonic measure of entanglement by quantifying the degree to which the PPT criterion is

violated, known as the negativity. The closely-related logarithmic negativity is often employed in the study of Gaussian entanglement [108].

We now turn our attention to the problem of observing entanglement in two-mode non-Gaussian states. It is possible to show that any two-mode state with the same covariance matrix as an entangled Gaussian state will be entangled [29]. Therefore the criteria developed for Gaussian entanglement will also be sufficient conditions for non-Gaussian entanglement. However, all the Gaussian tests rely on first and second order moments, but non-Gaussian states cannot be characterized in this manner. It will not always be possible to detect entanglement in non-Gaussian states using these same entanglement tests; for example, if the covariance matrix of a non-Gaussian entangled state does not correspond to an entangled Gaussian state. Therefore, in order to capture the entanglement in non-Gaussian states we must consider higher-order moments of the state.

3.1 Higher-order inseparability criteria

There are a number of inseparability criteria that utilise higher-order statistical moments to verify entanglement in non-Gaussian states. Examples of this methodology include the entanglement criteria developed by Argawal and Biswas [109], and the hierarchy of inequalities put forward by Hillery and Zubairy [110]. We will discuss the overarching and unifying criteria presented by Vogel and Shchukin (VS) [21] which exploit the PPT requirement. The criteria take the form of an infinite set of inequalities. In its entirety, this hierarchy gives us a necessary and sufficient criterion for PPT in any bipartite state, Gaussian or non-Gaussian¹. An individual

¹A negative partial transpose is a sufficient criterion for entanglement but not all entangled states necessarily have a negative partial transpose. Entanglement detected via a negative partial transpose is often called ‘free’ entanglement since this entanglement can be distilled [111]. Entanglement distillation refers to protocols that allow us to transform noisy entangled states to maximally entangled Bell states. However, if a partially transposed density matrix ρ has only positive eigenvalues then either it is separable, or it describes a ‘bound’ entangled state. If the dimension of a bipartite system is larger than six [112], there exist bound-entangled (or PPT-entangled) states that cannot be distilled. It is an open question whether or not there exist bound entangled states with negative eigenvalues after partial transpose [113, 114]. Bound entanglement is beyond the scope of this thesis and we emphasise that the VS hierarchy in its entirety is still equivalent to assessing the negativity of a density matrix.

inequality, when violated, indicates the state has a negative partial transpose (NPT) thus provides a sufficient condition for entanglement. Therefore, the hierarchy of inequalities provides us with an operational way of deducing the positivity of ρ_{AB}^{PT} defined in Eq.(1.40) for a bipartite mixed state. By operational, we mean that these NPT inequalities are constructed from measurable moments (expectation values of field operators) and so by scanning the hierarchy of criteria we can identify experimentally suitable entanglement tests.

Finally, we note that a number of well-known entanglement criteria, including the aforementioned tests for Gaussian and non-Gaussian entanglement [115, 109, 116, 110, 38, 36], are shown to be individual NPT conditions within the VS hierarchy.

3.2 Vogel and Shchukin inseparability criteria

Here we will briefly outline the key steps in deriving the VS criteria following the methods of the original paper [21]. Let us begin by reiterating the PPT condition for a bipartite state ρ_{AB} composed of two single-mode subsystems $\mathcal{A}$ and $\mathcal{B}$. The condition states that the partially transposed density matrix ρ_{AB}^{PT} , defined in Eq. (1.40), is a positive semi-definite matrix for a separable state. By definition, positive semi-definite matrices have eigenvalues which are all greater than or equal to zero. In general, a Hermitian matrix $\mathbf{A}$ (not necessarily the density matrix) is positive-semidefinite if and only if all the principal minors of $\mathbf{A}$ are nonnegative. A principal minor is the determinant of the matrix which remains when we delete rows and columns from $\mathbf{A}$ in a pairwise manner. This is known as Sylvester's criterion which states that the existence of at least one negative principal minor is a sufficient criterion for indicating the matrix is not positive semi-definite. We will use these matrix properties to demonstrate an equivalence between PPT and the negativity of a Hermitian matrix $\mathbf{A}$ composed of moments of the state.

We recall that a Hermitian operator $\hat{A}$ is positive semi-definite provided it satisfies

$$\langle \psi | \hat{A} | \psi \rangle = \text{Tr} [\hat{A} | \psi \rangle \langle \psi |] \geq 0 , \quad (3.1)$$

for all states $|\psi\rangle$. This positivity condition can be rewritten as

$$\langle \hat{f}^\dagger \hat{f} \rangle_{\hat{A}} = \text{Tr} \left[\hat{A} \hat{f}^\dagger \hat{f} \right] \geq 0 , \quad (3.2)$$

where $\hat{f}$ is any operator whose normally ordered form exists. Applying the PPT condition to the partially transposed density matrix of a separable state gives

$$\langle \hat{f}^\dagger \hat{f} \rangle^{\text{PT}} \equiv \text{Tr} \left[\rho_{\mathcal{AB}}^{\text{PT}} \hat{f}^\dagger \hat{f} \right] \geq 0 . \quad (3.3)$$

Since $\hat{f}$ is an arbitrary normally ordered operator, we can use the following definition

$$\hat{f} = \sum_{n,m,k,l=0}^{\infty} c_{nmkl} \hat{a}^{\dagger n} \hat{a}^m \hat{b}^{\dagger k} \hat{b}^l , \quad (3.4)$$

where $\hat{a}$ and $\hat{b}$ are the annihilation operators associated with subsystem $\mathcal{A}$ and $\mathcal{B}$, respectively, and c_{nmkl} are complex coefficients. Inserting this into the PPT criterion in Eq. (3.3) gives

$$\langle \hat{f}^\dagger \hat{f} \rangle^{\text{PT}} = \sum_{n,m,k,l,p,k,r,s=0}^{\infty} c_{pqrs}^* c_{nmkl} M_{pqrs,nmkl} \geq 0 , \quad (3.5)$$

where M contains the field operator moments

$$M_{pqrs,nmkl} = \langle \hat{a}^{\dagger q} \hat{a}^p \hat{a}^{\dagger n} \hat{a}^m \hat{b}^{\dagger s} \hat{b}^r \hat{b}^{\dagger k} \hat{b}^l \rangle . \quad (3.6)$$

It is convenient to fold these indices into multi-indices to give an equivalent form of the PPT criterion

$$\sum_{ij} c_i^* c_j \mathbf{M}_{ij} \geq 0 , \quad (3.7)$$

where $\mathbf{M}$ is an infinite-dimensional Hermitian matrix consisting of expectation values of operators of a bipartite state $\rho_{\mathcal{AB}}$ such that the i^{th} and j^{th} element is

$$\mathbf{M}_{ij} = \langle \hat{a}^{\dagger q_i} \hat{a}^{p_i} \hat{a}^{\dagger n_j} \hat{a}^{m_j} \hat{b}^{\dagger l_j} \hat{b}^{k_j} \hat{b}^{\dagger r_i} \hat{b}^{s_i} \rangle . \quad (3.8)$$

$$\mathbf{A} = \begin{pmatrix} \boxed{M_{11}} & \boxed{M_{12}} & \cancel{M_{13}} & \boxed{M_{14}} & \cancel{M_{15}} & \dots \\ \boxed{M_{21}} & \boxed{M_{22}} & \cancel{M_{23}} & \boxed{M_{24}} & \cancel{M_{25}} & \dots \\ \cancel{M_{31}} & \cancel{M_{32}} & \cancel{M_{33}} & \cancel{M_{34}} & \cancel{M_{35}} & \dots \\ \boxed{M_{41}} & \boxed{M_{42}} & \cancel{M_{43}} & \boxed{M_{44}} & \cancel{M_{45}} & \dots \\ \cancel{M_{51}} & \cancel{M_{52}} & \cancel{M_{53}} & \cancel{M_{54}} & \cancel{M_{55}} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Figure 3.1: An example of how the matrix $\mathbf{A}$ can be constructed. Here, we have deleted all the rows and columns in matrix $\mathbf{M}$ except the first, second and fourth rows and columns to constitute $\mathbf{A}$. Since $\mathbf{M}$ is infinite there are an infinite number of ways of constructing $\mathbf{A}$.

The i^{th} index is equal to the ordinal number ² of the ordered, but infinite, list of multi-indices (p_i, q_i, r_i, s_i) ; likewise, the j^{th} index corresponds to the ordinal number of the multi-indices (n_j, m_j, k_j, l_j) . The ordering of the multi-indices is arbitrary but once a convention is picked it is consistent between the two sets of multi-indices. For example, one might choose an ordering convention where $i = 1$ indicates $(p_1, q_1, r_1, s_1) = (0, 0, 0, 0)$, $i = 2$ indicates $(p_2, q_2, r_2, s_2) = (0, 0, 0, 1)$, and so on. Since the ordering must be consistent between the two sets of multi-indices then $(p_N, q_N, r_N, s_N) = (n_N, m_N, k_N, l_N)$ for all positive integer values of N .

Therefore we have demonstrated that the PPT criterion is equivalent to the statement that the Hermitian matrix $\mathbf{M}$ is a positive semi-definite matrix. We will subsequently discuss how to determine the positivity of $\mathbf{M}$.

Let us consider the main minors (or leading principal minors) $\mathbf{M}_N$ which are constructed by keeping the first N rows and N columns of $\mathbf{M}$,

$$\mathbf{M}_N = \begin{pmatrix} M_{11} & \dots & M_{1N} \\ \vdots & & \vdots \\ M_{N1} & \dots & M_{NN} \end{pmatrix}. \quad (3.9)$$

This is motivated by the practical desire to deal with a finite-dimensional matrix. Following from the definition of a positive semi-definite matrix and the application of Sylvester's criterion, $\mathbf{M}$ is positive semi-definite if and only if all the principal minors are positive. In the context of entanglement, ρ_{AB} is separable if and only if $\det[\mathbf{M}_N] > 0$ for all N . Conversely, if $\det[\mathbf{M}_N] < 0$

²Ordinal numbers indicate the position or rank of an element in some list, or set of objects; e.g. first, second, and so on, are ordinal numbers. Whereas cardinal numbers convey a quantity; e.g. one, two, and so on.

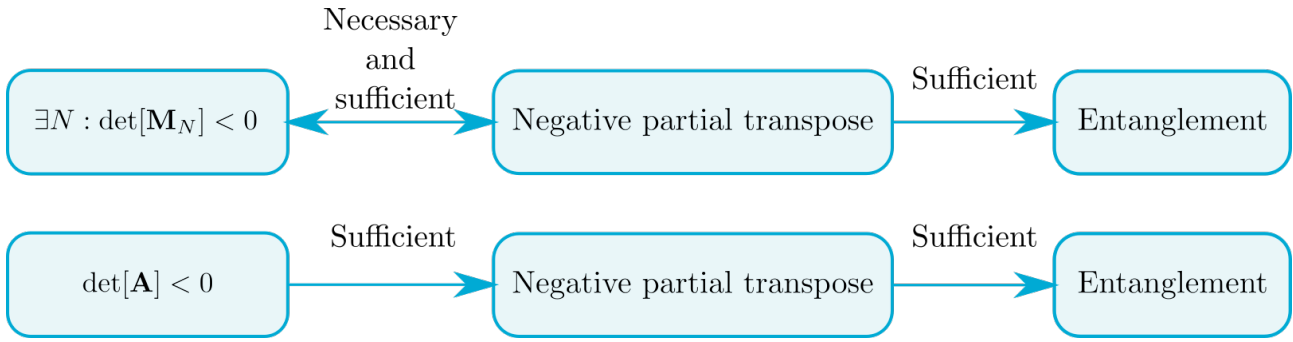


Figure 3.2: Schematic outlining the hierarchy of entanglement inequalities as derived in Ref. [21]. The upper panel captures the infinite hierarchy of inequalities which together form a necessary and sufficient condition for NPT. If there exists a principal minor which is negative $\det[\mathbf{M}_N] < 0$ then the partial transpose is not positive and thus entanglement is demonstrated. As shown in the upper panel, NPT is only a sufficient criterion as bound entangled states constitute a small class of entangled states which have a positive partial transpose; see footnote 1. In the lower panel we demonstrate how an individual inequality can be used as a sufficient criterion for entanglement. The matrix $\mathbf{A}$ is constructed by deleting rows and columns of $\mathbf{M}$ in a pairwise manner.

for any N , then the state has a negative partial transpose and is thus inseparable. These notions are schematically demonstrated in the upper pane of Fig. 3.2.

More generally, we can consider the submatrix $\mathbf{A}$ which is constructed by deleting rows and columns of $\mathbf{M}$ in a pairwise fashion. The process of constructing such a matrix is shown in Fig. 3.1. From Sylvester’s criterion it follows that $\det[\mathbf{A}] < 0$ also indicates NPT and so is a sufficient criterion to demonstrate entanglement. This condition will be used extensively in our analysis and has been outlined schematically in the lower pane of Fig. 3.2.

It should be emphasised that the magnitude of the negativity is not significant and should not be interpreted as a monotonic measure of entanglement. The only meaningful information is whether or not $\det[\mathbf{A}]$ is negative. Nevertheless, it is crucial to consider a confidence interval on the value of the determinant; a negative determinant with an uncertainty that is greater than its absolute value is clearly not a conclusive entanglement test. Therefore in this thesis, the interplay between the error and the magnitude of the negativity of a determinant is carefully examined when searching through a potentially infinite number of entanglement tests. The question of how to choose a suitably negative determinant is discussed in Chapter 6 and is a key theme of this thesis.

Chapter 4

Classical statistics of quantum measurements

In order to verify the inseparability of a quantum state using the criteria outlined in Chapter 3 we must perform measurements to ascertain moments of the field operators given in Eq. (3.8). At first this might sound rather abstract; however, ultimately we are finding the expectation value of a quantum operator $\langle \hat{O} \rangle$. For an observable $\hat{O}$ with a continuous spectrum, we can write its spectral decomposition as $\hat{O} = \int do \, o |o\rangle\langle o|$ where o is some continuous variable measurement outcome, and $\langle o|o' \rangle = \delta(o - o')$. If we were to assemble identical quantum states and measure $\hat{O}$ on each we would find that the outcome of an individual measurement o is random but the probability of an outcome $P(o)$ is dictated by Born's rule

$$P(o)do = \langle o | \rho | o \rangle do . \quad (4.1)$$

Following the same steps as in Section 1.1, the expectation value of a continuous variable observable can be found through the trace operation

$$\begin{aligned} \text{Tr}[\rho \hat{O}] &= \int do' \int do \, o \langle o'|o \rangle \langle o | \rho | o' \rangle \\ &= \int do \, o P(o) = \langle \hat{O} \rangle , \end{aligned} \quad (4.2)$$

where we note that in going from a discrete to continuous spectrum we replace the sum in the trace operation with an integral (c.f. Eq. (1.4) for discrete trace operation). Therefore, to find statistical quantities such as $\langle \hat{O} \rangle$, or higher-order moments, we may experimentally infer these from $P(o)$.

Indeed, one of the key goals in science more generally is to uncover parameters of an unknown probability distribution, such as $P(o)$, which describes some physical phenomenon. If the probability distribution were known then we would be able to deduce all system parameters and would not need to conduct further measurements! If we work in a regime where a single measurement of $\hat{O}$ is obtained per experimental run, then the only way to uncover parameters relating to the probability distribution is by repeated measurements. In each repeated measurement, the quantum state must be recreated since the very act of performing a strong measurement of $\hat{O}$ will project the quantum state into one of the measurement eigenstates. Subsequent measurements with $\hat{O}$ on the conditioned state (see Eq. (1.18)) will not yield any useful information since the measurement outcomes will not be governed by $P(o)$.

Once the desired measurements have been performed on a quantum state and a string of numbers is recorded, measurement noise and classical statistics will also be present in the data. In this chapter we will explore some of these classical statistics which feed into estimating quantities derived from $P(o)$.

4.1 Sample statistics

Suppose we have a physical process and we wish to measure some quantity x from a population which is characterised by a probability density function $P(x)$. If we perform N measurements to obtain the dataset $\{x_1, \dots, x_N\}$, we expect each outcome to be slightly different every time an experiment is performed. If the experiment is repeated from scratch every time then $\{x_i\}$ form a set of N random variables. Assuming that the accuracy of the measuring procedure remains constant then each measurement will be drawn independently from the same population $P(x)$. The collection of random variables $\{x_i\}$ are thus referred to as being independent and identically

distributed (i.i.d).

Any function of $\{x_1, \dots, x_N\}$ is referred to as a sample statistic, or simply a statistic. One of the best known statistics is the mean

$$\bar{x} = \frac{1}{N} \sum_{i=1}^N x_i . \quad (4.3)$$

The next natural statistic to construct from one's data is the variance

$$s^2 = \frac{1}{N} \sum_{i=1}^N (x_i - \bar{x})^2 , \quad (4.4)$$

which describes the spread of data around the mean, $\bar{x}$. Closely related is the standard deviation which can be found by taking the square root of the variance.

4.1.1 Estimators and sampling distributions

In general, the distribution $P(x)$ of a population from which the sample is drawn is unknown. The central aim of experimental physics is to use sample values to infer properties about the unknown population. Populations are characterised by certain parameters; for example, populations which follow Gaussian distributions are characterised by the mean μ , and standard deviation σ . Let us denote some population parameter as a ; then if we wish to estimate a we must estimate this using our sample $\{x_1, \dots, x_N\}$. Therefore, sample statistics which aim to estimate a are known as estimators and are usually denoted as $\hat{a}$. As this is a self-contained chapter on classical statistics, we will use a hat to indicate an estimator in this chapter alone in keeping with mathematical literature; elsewhere in the thesis a hat denotes a quantum operator. However, $\hat{a}$ is a function of the random variables $\{x_1, \dots, x_N\}$ and therefore estimators are themselves random variables. The estimator $\hat{a}$ will vary from one sample to the next and so will not necessarily equal a . The variation in the estimator is described by its sampling distribution ¹.

¹An estimator is usually judged by three criteria: consistency, bias, and efficiency. A consistent estimator $\hat{a}$ should tend towards the population parameter a as $N \rightarrow \infty$. Whereas an unbiased estimator should have an expectation value equivalent to the population parameter we are trying to estimate, $E[\hat{a}] = a$. Efficiency

Since an estimator might vary from one data set to the next, it is often necessary to quantify the uncertainty associated with the estimator $\hat{a}$. One way of characterising this uncertainty is via the standard deviation of the sampling distribution

$$\sigma_{\hat{a}} = (\text{V}[\hat{a}])^{1/2} . \quad (4.5)$$

And so when we try to estimate the true parameter a using an estimator we must include the uncertainty which is associated with it; i.e. $a = \hat{a} \pm \sigma_{\hat{a}}$.

Suppose the distribution $P(x)$ has a known mean μ and variance σ^2 . If we use the sample mean $\bar{x}$ defined in Eq. (4.3) as our estimator of the mean $\hat{\mu}$, then we may apply the central limit theorem to show that

$$\text{E}[\bar{x}] = \mu , \quad (4.6a)$$

$$\text{V}[\bar{x}] = \frac{\sigma^2}{N} . \quad (4.6b)$$

In this limit of large N , we can obtain an approximate form for the full sampling distribution of $\bar{x}$

$$P(\bar{x}) \approx \frac{1}{\sqrt{2\pi\sigma^2/N}} \exp\left[-\frac{(\bar{x} - \mu)^2}{2\sigma^2/N}\right] , \quad (4.7)$$

which follows from the central limit theorem.

However, finding suitable estimators is complicated by the fact that one does not usually know the functional form of the distribution from which the sample is drawn. If we already knew μ and σ there would be no need to conduct the experiment and obtain a sample. Therefore, if σ is unknown we must replace σ^2 in Eq. (4.7) with an estimated value of the variance $\hat{\sigma}^2$. This comes at the cost of substituting the equality sign with an approximate sign

$$\text{V}[\bar{x}] \approx \frac{\hat{\sigma}^2}{N} , \quad (4.8)$$

where the hat symbol spans σ^2 to symbolise that we desire an estimator for the variance $\hat{\sigma}^2$; not

quantifies the variance of an estimator with population size N and is closely related to the Fisher information.

the square of the estimator for the standard deviation $\hat{\sigma}^2$. A natural first guess at the form of $\hat{\sigma}^2$ is that it should be the sample variance s^2 given in Eq. (4.4). However, this yields a biased estimator since $E[s^2] = \sigma^2(N-1)/N$. This can be easily adjusted with the following correction to give an unbiased estimator

$$\hat{\sigma}^2 = \frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{x})^2, \quad (4.9)$$

such that $E[\hat{\sigma}^2] = \sigma^2$. Then it is common practice to use the following estimator for standard deviation

$$\hat{\sigma} = (\hat{\sigma}^2)^{1/2}, \quad (4.10)$$

which we note is biased. Indeed, no known unbiased estimator exists for standard deviation. However, this is of little concern as for large N these biases become negligible.

4.1.2 Hypothesis testing

Samples can be used to estimate various population parameters. We can also use the information contained in the sample to test whether or not our estimated parameter equals a particular value of interest, or lies in a particular range of values of interest.

This method is formally known as hypothesis testing. A hypothesis is a statement made about the value of a population parameter. It can be tested by carrying out an experiment or taking a sample from the population. The statistic calculated from the sample is called the test statistic. The null hypothesis (H_0) is the hypothesis assumed to be correct. This is rejected if the test statistics is lower than a given threshold, called the significance level. The alternative hypothesis (H_1) tells us about the parameter if your assumption is shown to be wrong.

As a simple example, let us assume that $X_1, \dots, X_n$ are i.i.d random variables drawn from a population which is normally distributed $\mathcal{N}(\mu, \sigma^2)$. A particular value has been suggested for the population mean $\mu = \mu_0$. We wish to see if the data supports this or if the mean has a lower value $\mu < \mu_0$, according to some other model proposed by another scientist. And so, the

null and alternative hypotheses are

$$H_0 : \mu = \mu_0 , \quad (4.11a)$$

$$H_1 : \mu < \mu_0 . \quad (4.11b)$$

To test this we must use an estimator of μ . A natural estimator is $\bar{x}$ defined in Eq. (4.3) which will act as our test statistic. From the central limit theorem, this follows a Normal distribution $\mathcal{N}(\mu, \sigma^2/N)$ (cf. Eq. (4.7)). From this we can calculate the probability of observing this test statistic under the assumptions of the null hypothesis. This quantity is known as the p -value where $p = P(\bar{x} < \mu_0 | H_0)$. This is usually calculated via the univariate Normal distribution where the test statistic is

$$Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} , \quad (4.12)$$

and it follows that $p = \Phi(Z)$, where $\Phi(Z)$ is the cumulative probability function for the standard Gaussian $\mathcal{N}(0, 1)$. If the p -value exceeds some threshold known as the significance level then we reject the null hypothesis (e.g. if $p > 95\%$ we might choose to reject H_0). Since the null and alternative hypothesis form complementary sets of parameter space, if we reject H_0 , then we must accept H_1 as our new hypothesis.

However, if the samples $x_1, \dots, x_N$ are drawn from a Gaussian distribution with unknown mean or standard deviation we must instead use the following test statistic

$$t = \frac{\bar{x} - \mu_0}{\hat{\sigma}/\sqrt{N}} , \quad (4.13)$$

where $\hat{\sigma}$ is the estimate of the standard deviation, given in Eq. (4.10). Under the null hypothesis, the test statistic now follows a Student's t -test distribution with $N - 1$ degrees of freedom². The cumulative probability function can be found in statistics tables (or statistics packages for the more technologically-oriented). The methods of hypothesis testing are the same as the ones described for the Normal distribution. The t -distribution tends towards the Gaussian

²This distribution was first derived by William Gossett who, while working as a mathematician at Guinness Breweries, wrote using the pseudonym 'Student' and is thus unrelated to the countless students who have since studied this method!

distribution for large values of N .

4.2 Error propagation

Summarizing the results from the previous section, we conclude that in an experiment to estimate some population parameter a , the result is given by

$$a = \hat{a} \pm \sigma_{\hat{a}} , \quad (4.14)$$

where $\hat{a}$ is the estimator we have chosen for a , and $\sigma_{\hat{a}}$ is the standard error on the estimator. If we do not know all the parameters of our population, it may be necessary to use an estimated standard error, in which case $a = \hat{a} \pm \hat{\sigma}_{\hat{a}}$. This estimated standard error is found empirically.

Suppose instead of just estimating a single parameter a , we instead wish to estimate some other parameter which is computed by combining other population parameters, each of which it is possible to observe. We introduce a scalar-valued function $f(\mathbf{a}) = f(a_1, a_2, \dots, a_d)$, where $\mathbf{a}$ is a vector containing d population parameters. Using the arguments of the previous section we expect the final result of an experiment to find this quantity to be

$$f(\mathbf{a}) = \hat{f}(\hat{\mathbf{a}}) \pm \Delta , \quad (4.15)$$

where Δ is the standard error on the estimator $\hat{f}(\hat{\mathbf{a}})$ and is found via $\Delta^2 = V[\hat{f}(\hat{\mathbf{a}})]$. We have included a hat to demonstrate that in some cases we might use a different function $\hat{f}$ to estimate f , and the hats on $\hat{\mathbf{a}}$ indicate that the arguments of the function $\hat{f}$ are also estimates of the population parameters $\mathbf{a}$. We now assume the estimator to be the function in question, with arguments given by estimators of the input parameters such that $\hat{f}(\hat{\mathbf{a}}) = f(\hat{\mathbf{a}})$. The arguments of the function are the estimated values of each population parameter where $\hat{\mathbf{a}} = (\hat{a}_1, \dots, \hat{a}_d)$.

We now outline how to calculate Δ for an arbitrary function.

In order to find $V[f(\hat{\mathbf{a}})]$, we start by Taylor expanding this function to first order

$$f(\hat{\mathbf{a}}) \approx f(\boldsymbol{\mu}) + \sum_i^d \left. \frac{\partial f}{\partial \hat{a}_i} \right|_{\hat{a}_i = \mu_i} (\hat{\mathbf{a}} - \boldsymbol{\mu})_i, \quad (4.16)$$

where the d -length vector $\boldsymbol{\mu}$ contains the expectation of each estimator such that $\mu_i = E[\hat{a}_i]$. Following from Eq. (4.7), if each estimator is constructed by finding the mean of lots of repeated independent measurements, it follows that an estimator is a normally distributed random variable $\hat{a}_i \sim \mathcal{N}(E[\hat{a}_i], V[\hat{a}_i])$. Now, in order to find $V[f(\hat{\mathbf{a}})]$ we use the definition of variance

$$V[f(\hat{\mathbf{a}})] = E \left[(f(\hat{\mathbf{a}}) - E[f(\hat{\mathbf{a}})])^2 \right]. \quad (4.17)$$

Inserting Eq. (4.16) into the expression for variance, we obtain the following standard result

$$V[f(\hat{\mathbf{a}})] = \sum_i^d \sum_j^d \frac{\partial f}{\partial \hat{a}_i} \text{Cov}[\hat{a}_i, \hat{a}_j] \frac{\partial f}{\partial \hat{a}_j}, \quad (4.18)$$

where the first partial derivative is evaluated at $\hat{a}_i = \mu_i$ and the second at $\hat{a}_j = \mu_j$. The covariance matrix is

$$\text{Cov}[\hat{a}_i, \hat{a}_j] = E[(\hat{a}_i - \mu_i)(\hat{a}_j - \mu_j)]. \quad (4.19)$$

Contrary to a quantum physicist's intuition, it is again stressed that in this chapter hats denote sample estimators constructed from repeated measurements, and *not* quantum operators. If the estimators $\hat{a}_i$ and $\hat{a}_j$ are found through independent data sets where no joint measurements are made, then $\text{Cov}[\hat{a}_i, \hat{a}_j] = 0$. Under these assumptions, this statement is valid even the population parameters a_i and a_j have a non-zero covariance.

To demonstrate this we take a trivial example of tossing a fair coin. Suppose a Heads (H) detector records 1 every time there is a Head or 0 if there is not a Head; similarly for the Tails (T) detector. In an experiment where joint measurements of H and T are taken we might have the following results $\{H, T\}$: $\{1, 0\}, \{1, 0\}, \{0, 1\}$, and so on. Clearly, when considering the population parameters H and T , it follows that $\text{Cov}[H, T] = -1/4$ as they are perfectly anticorrelated and can be described by a joint probability distribution $P(H, T) \neq P(H)P(T)$.

However, if we adapt our experiment and instead only measure H then we might obtain the following results $\{0\}, \{1\}, \{0\}$, etc. With sufficiently large samples, we would conclude that $\bar{H} = 1/2$ and similarly $\bar{T} = 1/2$, recall that the bar indicates these are estimators for the mean. But if we now consider $\text{Cov}[\bar{H}, \bar{T}] = \text{E}[(\bar{H} - \text{E}[\bar{H}])(\bar{T} - \text{E}[\bar{T}])]$, then under the assumption that $\bar{H}$ is found through repeated measurements where only H is measured, then $\bar{H}$ is itself a random variable which follows a normal distribution $\bar{H} \sim \mathcal{N}(\text{E}[\bar{H}], \text{V}[\bar{H}])$, which is independent of $\bar{T}$. The joint sampling probability density function is thus entirely separable $P(\bar{H}, \bar{T}) = P(\bar{H})P(\bar{T})$, and $\text{Cov}[\bar{H}, \bar{T}] = 0$.

Extending this simple example, it follows that $\text{Cov}[\hat{a}_i, \hat{a}_j] = \delta_{ij} \text{V}[\hat{a}_i]$, under the assumption that independent data sets are used to estimate $\hat{a}_i$ and $\hat{a}_j$. If $\hat{a}_i$ is the estimator for the mean then $\text{Cov}[\hat{a}_i, \hat{a}_j] = \delta_{ij} \sigma^2 / N$, from Eq. (4.7), provided the population variance σ^2 is known.

When all the population parameters $a_1, \dots, a_d$ are estimated independently in this manner, then the variance of the function we are estimating $f(\hat{\mathbf{a}})$ reduces to the simple form

$$\text{V}[f(\hat{\mathbf{a}})] = \sum_i^d \left(\frac{\partial f(\hat{\mathbf{a}})}{\partial \hat{a}_i} \right)^2 s_i^2, \quad (4.20)$$

where $s_i = \sqrt{\text{V}[\hat{a}_i]}$ is the standard deviation of each estimator. This is in fact the well-known ‘adding-in-quadrature’ formula used for uncorrelated variables. If we take a straightforward linear function, $f(\hat{\mathbf{a}}) = \sum_i^d \hat{a}_i$ then the standard error of the estimator of this function is

$$\Delta = \sqrt{\sum_i^d s_i^2}. \quad (4.21)$$

However, if the function estimator is not simply a linear combination of the estimated quantities $\hat{a}_i$ then we must take care to differentiate this function appropriately. A particularly useful non-linear function is the determinant. Suppose we wish to estimate the error on a determinant calculated from some matrix $\mathbf{A}$. We start with the simple assumption that each element $\mathbf{A}_{ij}$ of matrix $\mathbf{A}$ is estimated independently. The determinant of $\mathbf{A}$ is a function of all $\mathbf{A}_{ij}$ and we denote this as $\det[\mathbf{A}]$. Technically, $\det[\mathbf{A}]$ and $\mathbf{A}_{ij}$ are estimators and so should have hats. However, this notation is unnecessarily cumbersome. Having introduced the key concepts of

error propagation we henceforth drop the hats and in doing so avoid confusion with quantum mechanical operators in subsequent sections. Then the standard deviation of the estimator of the determinant can be found using Eq. (4.21)

$$V[\det[\mathbf{A}]] = \sum_{ij} \left(\frac{\partial \det[\mathbf{A}]}{\partial \mathbf{A}_{ij}} \right)^2 s_{ij}^2, \quad (4.22)$$

where s_{ij} is the standard deviation of each estimator $\mathbf{A}_{ij}$. As a special case

$$\frac{\partial \det[\mathbf{A}]}{\partial \mathbf{A}_{ij}} = \text{adj}[\mathbf{A}]_{ji}, \quad (4.23)$$

where $\text{adj}[\mathbf{A}]$ is the matrix adjugate. The adjugate (or classical adjunct) is defined as

$$\text{adj}[\mathbf{A}] = \mathbf{C}^T, \quad (4.24)$$

$\mathbf{C}$ is the cofactor matrix of $\mathbf{A}$. This should not be confused with the Hermitian adjoint commonly used in quantum mechanics. From this we obtain the following formula for the standard error on the determinant,

$$\Delta = \sqrt{\sum_{ij} \left(\text{adj}[\mathbf{A}]_{ji} s_{ij} \right)^2}, \quad (4.25)$$

which is sometimes referred to as Jacobi's rule. We will use these ideas in Chapter 6 where we consider the specific case where $\mathbf{A}$ is a Hermitian matrix. Importantly, we will examine the standard error when some of the estimators are found from the same sample set which precludes us from using Jacobi's rule. This, in conjunction with the hypothesis testing, will allow us to compute confidence levels on a determinant which has been found via repeated measurements of each matrix element. This purpose of this is to establish the suitability of an entanglement criterion (which is related to the aforementioned determinant) at detecting entanglement in a given non-Gaussian state.

Part II

Results

Chapter 5

Generating and verifying two-mode mechanical Schrödinger-cat states

The work described in this chapter was conducted in collaboration with Jack Clarke, Sofia Qvarfort and Michael R. Vanner and has been published in *Quantum Sci. Technol.* **7** (3), 035012, 2022 [117]. This chapter is partly based on this publication.

5.1 Introduction

Quantum mechanics allows for two or more objects to exhibit correlations that are stronger than what is permitted by classical physics. The existence of such correlations—or quantum entangled states—are one of the most counter-intuitive and powerful aspects of quantum mechanics, and offer exciting prospects to develop quantum technologies and to explore fundamental physics.

As we have explored in Chapter 2, cavity quantum optomechanics now provides a rapidly progressing new platform for quantum science [83], with quantum entanglement in macroscopic mechanical oscillators currently emerging as an active avenue of study. In particular, the radiation-pressure optomechanical interaction defined in Eq. 2.18 can be utilized to generate

a rich variety of different entangled states. Notably, this interaction, when linearized such as in Eq. 2.69, gives rise to Gaussian entanglement. In Section 1.2 we defined bipartite Gaussian entanglement as a property of two arbitrary bosonic subsystems whose composite Wigner function is Gaussian and inseparable. More specifically, such Gaussian entanglement has been studied both theoretically and experimentally between optical and mechanical modes. Theoretically, entanglement between both field and mechanics, and also between two mechanical elements has been explored in this regime [115, 118, 119, 120, 121, 122, 123, 124, 125, 126]. While experimentally, two-mode squeezed states have been prepared between a microwave field and a mechanical oscillator [127], and excitingly, continuous-variable entanglement between two mechanical oscillators has also been generated [128, 129, 130].

In addition to such Gaussian states, the linearized regime combined with photon-counting enables the preparation of single- and two-mode non-Gaussian mechanical states [131, 76, 132, 133]. Photon-counting generates a conditioned state defined in Eq. (1.18) and as such introduces a nonlinearity in the state preparation. Recent experimental progress using this combination has enabled single-phonon addition and subtraction to single-mode mechanical systems [134, 84, 135, 136, 78, 77, 79], and, similarly, to generate entangled mechanical states that share a single quanta [137, 138]. Furthermore, photon-counting protocols have been proposed to create macroscopic superposition states which can help to overcome the challenge of single-photon weak coupling [86, 139]. Beyond the linearized regime, continued experimental progress has enabled early signatures of the intrinsically cubic radiation-pressure interaction to be observed [140, 141], and there has also been increasing theoretical interest in this nonlinear regime [142, 143, 144, 145, 146]. Theoretical progress in this direction has examined entangling operations and the properties of the mechanical states generated [147, 69, 70], and the nonlinear regime remains as a rich avenue of study for mechanical quantum states. In particular, an experimental recipe for mechanical continuous-variable non-Gaussian entanglement verification remains outstanding. For a discussion on the far-reaching implications of non-Gaussian entanglement see Chapter 1.3.

In this Chapter, we propose a protocol for generating two-mode mechanical Schrödinger-cat states encoded in the displacement of two mechanical oscillators. This class of entangled states

has a wide range of applications which are discussed in Section 1.3.2. Importantly, we also introduce an operational technique to verify the non-Gaussian entanglement with a scheme that enables any bipartite mechanical moment to be measured up to any finite order. Our protocol comprises a state preparation stage followed by a verification stage, which both utilize pulsed optomechanical interactions [93] in the unresolved sideband regime. For entanglement generation, we take advantage of the nonlinear radiation-pressure interaction together with photon-counting measurements with an interferometric set-up to herald a two-mode mechanical Schrödinger-cat state. For the entanglement verification stage, we propose a method to determine arbitrary mechanical moments utilizing subsequent pulsed measurements with an iterative verification protocol whereby lower-order moments are used to unlock higher-order moments. We then use the moments obtained to evaluate inseparability criteria discussed in Section 3.2 which identify the presence of non-Gaussian mechanical-mechanical entanglement.

Our scheme is applicable to current experimental approaches and non-Gaussian entanglement can be created and verified with only minor improvements to current experiments operating outside the resolved sideband regime. To establish the feasibility of this scheme, we carefully model the key experimental factors including those arising from the system's interaction with the environment which occur during the verification stage; and optical losses, detection inefficiency, and dark counts in the heralding stage.

5.2 Entanglement protocol

We first outline our scheme to generate non-Gaussian entanglement between two mechanical oscillators before exploring each stage in more mathematical detail. This entanglement scheme can be applied to optomechanical systems that operate outside the resolved sideband regime $\kappa \gg \omega_M$ (where κ is the cavity amplitude decay rate and ω_M is the mechanical angular frequency). We consider two distinct configurations involving a Mach-Zehnder interferometer set-up: (i) the ‘parallel’ case where light in the two paths interact with separate mechanical oscillators (see Fig. 5.1a), and (ii) the ‘series’ case where only light in the first path interacts

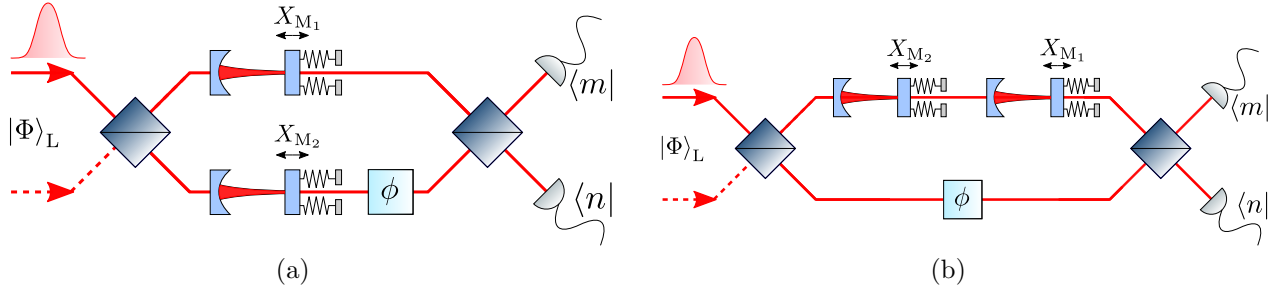


Figure 5.1: Proposed experimental schematics to prepare mechanical oscillators in a two-mode Schrödinger-cat state. Both schemes involve injecting a weak coherent state $|\Phi\rangle_L = |\alpha\rangle_{L_1} |0\rangle_{L_2}$, or a single photon $|\Phi\rangle_L = |1\rangle_{L_1} |0\rangle_{L_2}$ into a Mach-Zehnder interferometer containing two optomechanical cavities. Following a nonlinear interaction, photon-counting is then performed at the two outputs of the interferometer, with counts $\{m, n\}$, that heralds the creation of the mechanical state $\rho_{\text{out}}^{\{m,n\}}$. The phase ϕ influences the properties of the state. (a) The ‘parallel’ set-up described by the measurement operator in Eq. (5.4a). (b) The ‘series’ set-up described by the measurement operator in Eq. (5.4b). The phase ϕ in both configurations is used to control the properties of the resulting non-Gaussian entangled mechanical state.

with two mechanical oscillators sequentially (see Fig. 5.1b). In both cases, a pulse of light is first injected into one input of the interferometer. The light in the lower interferometer-path is subjected to a phase shift ϕ , which is an experimental handle that allows us to directly influence the the properties of the resulting mechanical quantum states. Following a pulsed nonlinear optomechanical interaction [94, 148, 95], the two optical modes recombine at another 50:50 beam splitter and are subsequently measured by photon-counting detectors. We denote the event where m and n photons are detected in the two outputs as $\{m, n\}$ thus heralding the creation of the entangled mechanical state $\rho_{\text{out}}^{\{m,n\}}$.

We consider separately two initial optical states to implement our scheme. Firstly, a weak coherent state, i.e. $|\Phi\rangle_L = |\alpha\rangle_{L_1} |0\rangle_{L_2}$, where subscripts 1 and 2 denote upper and lower light modes in the interferometer. And, secondly, a single-photon input, i.e. $|\Phi\rangle_L = |1\rangle_{L_1} |0\rangle_{L_2}$ which offers advantages for the quantum measurement process, however with the increased experimental complexity of requiring a single-photon source. At the first beam splitter the light interacts with the vacuum mode and undergoes the following transformations: $\hat{U}_{12}^\dagger \hat{a}_1 \hat{U}_{12} \rightarrow (\hat{a}_1 + \hat{a}_2)/\sqrt{2}$ and $\hat{U}_{12}^\dagger \hat{a}_2 \hat{U}_{12} \rightarrow (\hat{a}_1 - \hat{a}_2)/\sqrt{2}$, where $\hat{a}_1$ and $\hat{a}_2$ are the annihilation operators associated with optical modes 1 and 2. A 50:50 beam splitter unitary is given by

$$\hat{U}_{12} = e^{i(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_1 \hat{a}_2^\dagger)/\sqrt{2}}. \quad (5.1)$$

The subsequent interaction between the optical pulse and the mechanical oscillator is captured by the following Hamiltonian:

$$\hat{H}_{\text{int}} = -\hbar g_0 \hat{a}_i^\dagger \hat{a}_i (\hat{b}_j + \hat{b}_j^\dagger) , \quad (5.2)$$

where g_0 is the optomechanical coupling rate, and $\hat{b}_j$ is the mechanical annihilation operator for the j^{th} oscillator. The justification for this Hamiltonian can be found in the text preceeding Eq. (2.15). We do not require strong coupling and furthermore assume the same g_0 for the two optomechanical cavities for brevity and clarity of presentation, but that can be readily generalized if needed. We consider this interaction happening in the pulsed regime where the pulse duration is much shorter than the mechanical period. The framework underpinning the pulsed regime is outlined in Section 2.4. This assumption allows us to model the optomechanical interaction by the unitary $e^{i\mu \hat{a}_i^\dagger \hat{a}_i \hat{X}_{M_j}}$; where $\mu \propto g_0/\kappa$ is the dimensionless coupling strength (and depends on the shape of the pulse), and $\hat{X}_{M_j} = (\hat{b}_j + \hat{b}_j^\dagger)/\sqrt{2}$ is the position-quadrature operator of the j^{th} mechanical system. We remind the reader that the pulsed regime assumes $\kappa \gg \omega_M$, however, the interplay between the unitary $e^{i\mu \hat{a}_i^\dagger \hat{a}_i \hat{X}_{M_j}}$ and a finite value of the ratio ω_M/κ is analysed in Section 5.2.1, where we demonstrate that the entanglement generation protocol is robust with respect to this ratio.

Physically, this unitary has the effect of imparting a momentum kick onto the mechanical oscillator but leaves the position unchanged. This can be demonstrated by examining how the dimensionless position and momentum phase quadratures transform as

$$e^{-i\mu \hat{a}^\dagger \hat{a} \hat{X}_M} \hat{X}_M e^{i\mu \hat{a}^\dagger \hat{a} \hat{X}_M} = \hat{X}_M , \quad (5.3a)$$

$$e^{-i\mu \hat{a}^\dagger \hat{a} \hat{X}_M} \hat{P}_M e^{i\mu \hat{a}^\dagger \hat{a} \hat{X}_M} = \hat{P}_M + \mu , \quad (5.3b)$$

for the case of a single photon. Therefore, in phase space, the quantum state has been kicked along the momentum axis with a displacement of magnitude μ . For a visual representation of the phase space see Fig. 5.3.

For a pulse duration τ which satisfies $\omega_M \ll \tau^{-1} \ll \kappa$, it can be shown that $\mu = 2\sqrt{2}g_0/\kappa$ as

the cavity mode can be adiabatically eliminated [94]. We have assumed that the input pulse shape is Gaussian with temporal width τ . The details of this approximation are outlined in Section 2.4. Furthermore, the input-output theory [149] used to derive $e^{i\mu\hat{a}_i^\dagger\hat{a}_i\hat{X}_{M_j}}$ in Ref. [94] demonstrates that in this adiabatic regime the output pulse from the optomechanical cavity experiences minimal distortion. This has the advantage of more easily ensuring spatio-temporal mode matching at the second beam splitter if the two optomechanical cavities are not totally identical, e.g. slightly different values of κ between the two cavities.

After the pulse has imparted a momentum kick, an $\{m, n\}$ photon click heralds the mechanical state $\rho_{\text{out}}^{\{m, n\}} = \hat{\Upsilon}_{mn}\rho_{\text{in}}\hat{\Upsilon}_{mn}^\dagger/P_{mn}$, where ρ_{in} is the initial mechanical density operator, and P_{mn} is the heralding probability of an $\{m, n\}$ click event. The measurement operator $\hat{\Upsilon}_{mn}$ captures the propagation of the light through the interferometer, its interaction with the mechanical systems, and the final measurement of $\{m, n\}$ photons. For the parallel configuration, this is given by $\hat{\Upsilon}_{mn} = \langle m | \langle n | \hat{U}_{12}^\dagger e^{i\mu\hat{a}_1^\dagger\hat{a}_1\hat{X}_{M_1}} e^{i\mu\hat{a}_2^\dagger\hat{a}_2\hat{X}_{M_2}} e^{i\phi\hat{a}_2^\dagger\hat{a}_2} \hat{U}_{12} | \Phi \rangle_L$, where $|m\rangle$ and $|n\rangle$ are Fock states. The measurement operator for the series configuration can be deduced in a similar manner. For $|\Phi\rangle_L = |\alpha\rangle_{L_1} |0\rangle_{L_2}$, the measurement operators are:

$$\hat{\Upsilon}_{mn} = \mathcal{N}_{mn} [e^{i\mu\hat{X}_{M_1}} + e^{i\mu\hat{X}_{M_2}+i\phi}]^m [e^{i\mu\hat{X}_{M_1}} - e^{i\mu\hat{X}_{M_2}+i\phi}]^n, \quad (5.4a)$$

$$\hat{\Upsilon}_{mn} = \mathcal{N}_{mn} [e^{i\mu(\hat{X}_{M_1}+\hat{X}_{M_2})} + e^{i\phi}]^m [e^{i\mu(\hat{X}_{M_1}+\hat{X}_{M_2})} - e^{i\phi}]^n, \quad (5.4b)$$

for the parallel and series set-ups, respectively. For a single-photon input state $|\Phi\rangle_L = |1\rangle_{L_1} |0\rangle_{L_2}$, the measurement operators are:

$$\hat{\Upsilon}_{mn} = \mathcal{N}_{mn} [(e^{i\mu\hat{X}_{M_1}} + e^{i\mu\hat{X}_{M_2}+i\phi})\delta_{m,1}\delta_{n,0} + (e^{i\mu\hat{X}_{M_1}} - e^{i\mu\hat{X}_{M_2}+i\phi})\delta_{m,0}\delta_{n,1}], \quad (5.5a)$$

$$\hat{\Upsilon}_{mn} = \mathcal{N}_{mn} [(e^{i\mu(\hat{X}_{M_1}+\hat{X}_{M_2})} + e^{i\phi})\delta_{m,1}\delta_{n,0} + (e^{i\mu(\hat{X}_{M_1}+\hat{X}_{M_2})} - e^{i\phi})\delta_{m,0}\delta_{n,1}], \quad (5.5b)$$

for the parallel and series cases, respectively. The prefactor $\mathcal{N}_{mn}$ depends on the injected light

state $|\Phi\rangle_L$:

$$\mathcal{N}_{mn} = \begin{cases} e^{-\frac{|\alpha|^2}{2}} \left(\frac{\alpha}{2}\right)^{m+n} \frac{1}{\sqrt{n!m!}} & \text{for } |\Phi\rangle_L = |\alpha\rangle_{L_1} |0\rangle_{L_2}, \\ 1/2 & \text{for } |\Phi\rangle_L = |1\rangle_{L_1} |0\rangle_{L_2}. \end{cases} \quad (5.6)$$

While the assumptions of the non-resolved sideband regime are implicit through our choice of optomechanical unitary, this analysis is exact for α and consequently these results are valid for α of any magnitude.

Two-mode mechanical Schrödinger-cat states of interest are heralded by the click events $\{1, 0\}$ or $\{0, 1\}$, although we note that with high efficiency and photon-resolving detectors, $\{m, n\}$ photon click events can be exploited to herald a range of other mechanical states. In fact, the operators $\hat{\Upsilon}_{10}$ and $\hat{\Upsilon}_{01}$ can be made identical with a π phase shift in the interferometer. As ϕ is an experimental control, we only need to consider one of these operators to probe the entanglement structure of the state. Therefore, without loss of generality, subsequent analysis will focus on the click event $\{1, 0\}$, which heralds the mechanical state $\rho_{\text{out}}^{\{1,0\}}$.

As a motivating example, let us consider the case where the mechanical oscillators are initially in their ground states $|0\rangle_{M_1} |0\rangle_{M_2}$, and an optical coherent state is used $|\Phi\rangle_L = |\alpha\rangle_{L_1} |0\rangle_{L_2}$. After the input pulse travels through the first beam splitter the resulting product state is $|\alpha/\sqrt{2}\rangle_{L_1} |e^{i\phi}\alpha/\sqrt{2}\rangle_{L_2}$, where the lower path has also experienced a phase shift. In the parallel set-up shown in Fig. 5.1a, the upper path L1 leads to an interaction with oscillator M1 while the lower path L2 interacts with oscillator M2. The total light-mechanics state is now proportional to $\hat{D}_{L_1}(e^{i\mu\hat{X}_{M_1}}\alpha/\sqrt{2})\hat{D}_{L_2}(e^{i\mu\hat{X}_{M_2}+i\phi}\alpha/\sqrt{2})|0\rangle_{L_1}|0\rangle_{L_2}|0\rangle_{M_1}|0\rangle_{M_2}$, where D_{L_j} is the Weyl displacement operator acting on the j^{th} optical mode (this operator is first introduced in Eq. (1.27)). At the second beam splitter the light beams are recombined yielding the state $\hat{D}_{L_1}((e^{i\mu\hat{X}_{M_1}} + e^{i\mu\hat{X}_{M_2}+i\phi})\alpha/2)\hat{D}_{L_2}((e^{i\mu\hat{X}_{M_1}} - e^{i\mu\hat{X}_{M_2}+i\phi})\alpha/2)|0\rangle_{L_1}|0\rangle_{L_2}|0\rangle_{M_1}|0\rangle_{M_2}$. At this point, we have a light-mechanics correlated state and the mechanical state of interest is not yet produced until the photon measurement is performed. The detection of a single photon in one detector but no photon in the other detector within the coincidence window dictated by the pulse duration gives the desired $\{1, 0\}$ click, while we discard runs which have different click events. A

$\{1, 0\}$ click projects the mechanical state into the form $e^{i\mu\hat{X}_{M_1}} |0\rangle_{M_1} |0\rangle_{M_2} + e^{i\mu\hat{X}_{M_2} + i\phi} |0\rangle_{M_1} |0\rangle_{M_2}$. Since μ is a real number, the operators $e^{i\mu\hat{X}_{M_j}}$ can be expressed as $\exp\left[(i\mu/\sqrt{2})\hat{b}_j^\dagger - (i\mu/\sqrt{2})^*\hat{b}_j\right]$ which is the usual displacement operator $\hat{D}_{M_j}(i\mu/\sqrt{2})$. These displacement operators act on the mechanical ground state of the j^{th} oscillator in the usual manner: $\hat{D}_{M_j}(i\mu/\sqrt{2}) |0\rangle_{M_j} = |i\mu/\sqrt{2}\rangle_{M_j}$. As the amplitude of this displacement is purely imaginary, this physically corresponds to a momentum kick of magnitude μ . Therefore, the heralded mechanical states in the parallel and series set-ups, respectively, are given by the two-mode Schrödinger-cat states:

$$|\Psi\rangle_{10} \propto \left(\left| \frac{i\mu}{\sqrt{2}} \right\rangle |0\rangle + e^{i\phi} |0\rangle \left| \frac{i\mu}{\sqrt{2}} \right\rangle \right), \quad (5.7a)$$

$$|\Psi\rangle_{10} \propto \left(\left| \frac{i\mu}{\sqrt{2}} \right\rangle \left| \frac{i\mu}{\sqrt{2}} \right\rangle + e^{i\phi} |0\rangle |0\rangle \right), \quad (5.7b)$$

where, for now, we have omitted the proportionality constants. We note that these states are independent of whether we use a coherent state or a single photon for the input optical pulse $|\Phi\rangle_L$. For convenience we have dropped the labels M1 and M2 when describing the bipartite mechanical state. In the parallel example we have been considering, one oscillator receives a single-photon momentum kick while the other remains in its ground state but whichway information has been erased.

Magnitude of momentum displacement

As a brief aside, we note that in order for the momentum transfer from a single photon to exceed the momentum zero-point fluctuations we would require $\mu > 1$. As discussed in the Introduction to Chapter 2, reaching this regime is a highly sought-after goal within the quantum optomechanics community and remains outstanding outside of cold-atom and trapped-ion implementations. It must be stressed, however, that our proposal does not require $\mu > 1$. As we will demonstrate in this Chapter, one can still create and verify entanglement with $\mu < 1$. In fact, this is one of the strengths of our scheme. The central idea of our protocol is that we can still create a quantum interference effect with ϕ -dependency through the superposition of a ‘kick’ and ‘no-kick’ for each oscillator. The subsequent state is inseparable because the which-way information of the photon is erased by the interferometer, and we will demonstrate

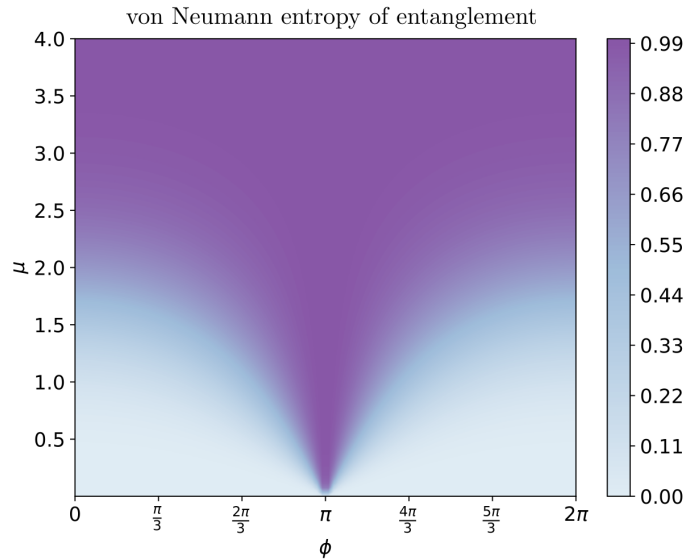


Figure 5.2: The von Neumann entropy of entanglement calculated for the pure states defined in Eqs. (5.7) which are generated following a $\{1,0\}$ photon click. This proper measure of entanglement is plotted as a function of the interferometer phase ϕ and optomechanical coupling strength μ . For large μ the entanglement tends towards its maximal value of 1 regardless of phase ϕ . For small μ the entanglement is maximal when $\phi = \pi$.

that even with very small μ non-Gaussian entanglement can be verified.

Pure state: a closer look

Two-mode Schrödinger-cat states are usually defined as being proportional to $|\alpha\rangle|\beta\rangle \pm |-\alpha\rangle|-\beta\rangle$ where $\alpha, \beta \in \mathbb{C}$. Acting upon such a state with the local displacement operations $\hat{D}_{M_1}(\alpha)\hat{D}_{M_2}(-\beta)$ and setting $2\alpha = -2\beta = i\mu/\sqrt{2}$ gives rise to Eq. 5.7a, while $\hat{D}_{M_1}(\alpha)\hat{D}_{M_2}(\beta)$ with $2\alpha = 2\beta = i\mu/\sqrt{2}$ gives Eq. 5.7b. Again, we denote the displacement operator acting on the j^{th} oscillator as $\hat{D}_{M_j} = e^{\alpha\hat{b}_j^\dagger - \alpha^*\hat{b}_j}$. As these are entirely local operations, we expect the von Neumann entanglement entropy to be equivalent to that of a traditional Schrödinger-cat state; this is also discussed in Section 1.3.2. Therefore, we refer to the generated mechanical states as Schrödinger-cat states; although one could also call them *displaced* Schrödinger-cat states.

We now examine the two states in Eq. (5.7) and discern some properties related to their entanglement. Firstly, we calculate the von Neumann entropy of entanglement which is a proper measure of entanglement and is defined in Eq. (1.13). This can be done using Python packages

like QuTip but we will outline a way to do this analytically too. Acting on the bipartite mechanical state produced in the parallel set-up Eq. (5.7a) with the unitary $\hat{D}_{M_2}(i\mu/\sqrt{2})\hat{R}_{M_2}(\pi)$ will map the state onto the density matrix produced via the series set-up in Fig. 5.7b, where $\hat{R}_{M_2}(\pi) = e^{-i\pi\hat{b}_2^\dagger\hat{b}_2}$. Therefore, the von Neumann entropies of entanglement of the states produced via the parallel and series configurations are identical [150]. We complete the von Neumann entropy calculation for the parallel case however the result will be the same for the series state too. The mechanical state after an optical measurement of $\{1,0\}$ is given in Eq. (5.7a). Including the normalization, this is

$$|\Psi\rangle_{10} = \frac{1}{1 + e^{-\mu^2/2} \cos \phi} \left(\left| \frac{i\mu}{\sqrt{2}} \right\rangle |0\rangle + e^{i\phi} |0\rangle \left| \frac{i\mu}{\sqrt{2}} \right\rangle \right). \quad (5.8)$$

The state is not in an orthogonal basis ($\langle \frac{i\mu}{\sqrt{2}} | 0 \rangle \neq 0$) and furthermore the coherent state $|\frac{i\mu}{\sqrt{2}}\rangle$ has infinite dimensions in the Fock basis. Instead of truncating the density matrix we construct an orthogonal basis:

$$|\chi\rangle = \frac{|\frac{i\mu}{\sqrt{2}}\rangle + |0\rangle}{\mathcal{N}}, \quad (5.9a)$$

$$|\chi_\perp\rangle = \frac{|\frac{i\mu}{\sqrt{2}}\rangle - |0\rangle}{\mathcal{M}}, \quad (5.9b)$$

where

$$\mathcal{N}^2 = 2 + 2e^{-\frac{\mu^2}{4}}, \quad (5.10a)$$

$$\mathcal{M}^2 = 2 - 2e^{-\frac{\mu^2}{4}}. \quad (5.10b)$$

After recasting the state in this orthogonal basis and performing a partial trace over the second oscillator, the density matrix of the first oscillator in the new basis is simply

$$\rho_{M_1} = \frac{1}{4 \left(1 + e^{-\frac{\mu^2}{2}} \cos \phi \right)} \begin{pmatrix} \mathcal{N}^2(1 + e^{-\frac{\mu^2}{4}} \cos \phi) & -i\mathcal{N}\mathcal{M}e^{-\frac{\mu^2}{4}} \sin \phi \\ i\mathcal{N}\mathcal{M}e^{-\frac{\mu^2}{4}} \sin \phi & \mathcal{M}^2(1 - e^{-\frac{\mu^2}{4}} \cos \phi) \end{pmatrix}. \quad (5.11)$$

By finding the two eigenvalues of this matrix λ_i we can compute the von Neumann entropy of

entanglement using

$$S(\rho_{M1}) = \sum_i -\lambda_i \log_2 \lambda_i, \quad (5.12)$$

where

$$\lambda_{1,2} = \frac{1}{2} \pm \frac{e^{-\mu^2/4}}{2} \frac{\sqrt{(1 + \cos \phi)(2 - e^{-\mu^2/2} + e^{-\mu^2/2} \cos \phi)}}{1 + e^{-\mu^2/2} \cos \phi}. \quad (5.13)$$

This function is plotted in Fig. 5.2 and we will now discuss some of the features.

- **Magnitude of μ**

We can see from Figure. 5.2 that for large μ the von Neumann tends towards its maximal value of 1. We know that the entanglement entropy for a two-mode state cannot exceed $S(\rho_{M1}) \leq \log_2 N$, where N is the dimension of the Hilbert space; in this case $N = 2$. This agrees with our calculation in Section 1.3.2 where we showed that the two-mode Schrödinger-cat states have an upper bound of $\log_2(2)$. Recalling that since $\langle 0 | i\mu / \sqrt{2} \rangle = e^{-\mu^2/4}$, for large μ the two components become increasingly orthogonal $\langle 0 | i\mu / \sqrt{2} \rangle \rightarrow 0$, and thus both Eqs. (5.7a) and (5.7b) approach maximally-entangled Bell states in the coherent basis. While the phase ϕ will dictate which Bell-like state we approach, it does not affect the magnitude of the entanglement in the limit of large μ .

- **Phase dependency** For the regime where $\mu \ll 1$, the entanglement entropy is highly dependant on the phase ϕ . Setting $\phi = \pi$, we can expand Eqs. (5.7a) and (5.7b) in the Fock basis. In the limit of $\mu \rightarrow 0$ this also results in the Bell states $|\Psi^\pm\rangle = (|0\rangle |1\rangle \pm |1\rangle |0\rangle) / \sqrt{2}$ in the Fock basis, where $|\Psi^-\rangle$ corresponds to the parallel state in Eq. (5.7a) and $|\Psi^+\rangle$ the series state in (5.7b). This calculation is demonstrated in Eq. (1.48). However, setting $\phi = 0$, and performing a similar expansion results in a state that tends to $|0\rangle |0\rangle$ as $\mu \rightarrow 0$. Indeed, we can differentiate the analytical expression for the von Neumann entropy of entanglement and show that for a fixed μ , the entropy is always maximised when $\phi = \pi$.

Thermal state

We now turn our attention to an initial thermal state $\rho_{\text{in}} = \rho_{\bar{n}_1} \otimes \rho_{\bar{n}_2}$, where $\rho_{\bar{n}_j}$ denotes the j^{th} oscillator being in a thermal state with thermal occupation number $\bar{n}_j$. A thermal state is defined in the Fock basis as

$$\rho_{\bar{n}} = \frac{1}{1 + \bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1 + \bar{n}} \right)^n |n\rangle\langle n| , \quad (5.14)$$

where $\bar{n}$ is the expected phonon occupation number $\langle \hat{n} \rangle = \bar{n} = (e^{\hbar\omega/k_B T} - 1)^{-1}$. The resulting state $\rho_{\text{out}}^{\{1,0\}}$ approaches the desired two-mode Schrödinger-cat state as $\bar{n}_1$ and $\bar{n}_2$ are reduced and the probability of generating $\rho_{\text{out}}^{\{1,0\}}$ is:

$$P_{10} = \mathcal{N}_{10} [1 + e^{-\mu^2(1+\bar{n}_1+\bar{n}_2)/2} \cos(\phi)] , \quad (5.15)$$

where the prefactor $\mathcal{N}_{10}$ is defined in Eq. 5.6 and depends on whether the input state $|\Phi\rangle_{\text{L}}$ is a coherent state or single photon. With a coherent state input pulse, P_{10} is maximised when $|\alpha| = 1$. As we have already noted, the mechanical states produced by the parallel (Fig. 5.1a) and series schemes (Fig. 5.1b) are related by a local unitary operation. Nevertheless, implementing the parallel set-up may be more experimentally convenient as it more easily ensures temporal-mode matching at the beam splitter. Therefore, subsequent analysis will consider the entangled mechanical state produced by the parallel configuration.

5.2.1 Finite mechanical evolution during entanglement preparation

The assumption that we operate in the unresolved sideband $\omega_{\text{M}}/\kappa \ll 1$ is a crucial step in the derivation of the optomechanical interaction unitaries both in the generation and verification stages of our protocol. Here we discuss the interplay between the sideband resolution ratio ω_{M}/κ and state generation and verification and find that our protocol is robust with respect to this ratio.

The requirement that $\omega_{\text{M}}/\kappa \ll 1$ implies that the cavity fills and empties on a much faster

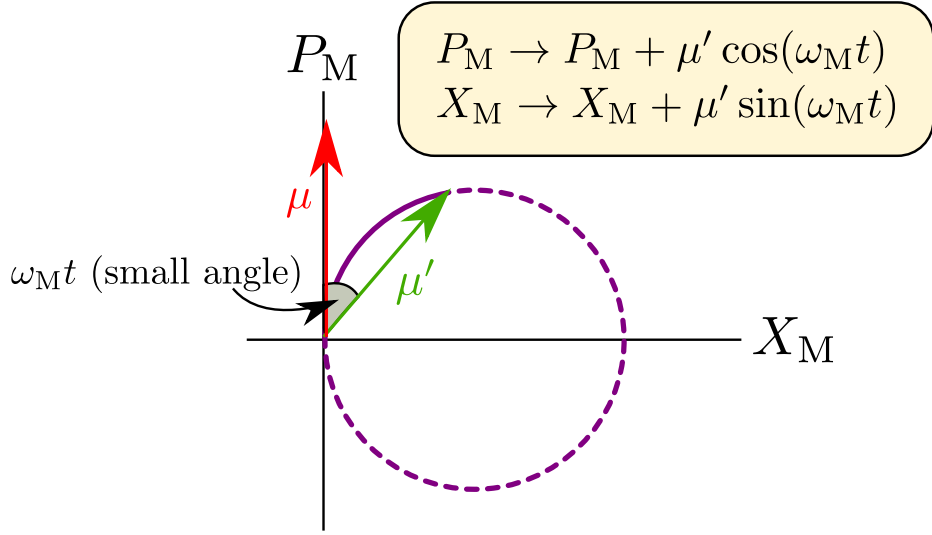


Figure 5.3: In the manuscript we assume $\kappa \gg \omega_M$ allowing us to model the optomechanical interaction unitary as $U = e^{i\mu a^\dagger a X_M}$. For a single-photon click event, this results in a displacement in P_M of size μ shown by the vertical red arrow. If the sideband ratio is not sufficiently large, we cannot assume $\dot{X}_M = 0$ during the interaction and we have to consider the evolution of both X_M and P_M during the pulse window. The purple dotted line shows the trajectory of the mechanical state in phase space as a function of time; this trajectory is a circle centred at $(\sqrt{2}g_0/\omega_M, 0)$ with radius $\sqrt{2}g_0/\omega_M$. The resulting displacement is shown by the green line and has magnitude μ' ; the P_M quadrature is shifted by $\mu' \cos(\omega_M t)$ while X_M by $-\mu' \sin(\omega_M t)$.

length of time than the mechanical period, and so in the derivation of the state-generation unitary $\hat{U} = e^{i\mu \hat{a}^\dagger \hat{a} \hat{X}_M}$ it is assumed that $\dot{\hat{X}}_M = 0$. However, if the sideband ratio is relaxed then we have to consider the mechanical evolution during the timescale of the pulse. For the purposes of exploring $\hat{U}$ we focus on a single light mode interacting with a single mechanical mode, but these results can be generalised to the multimode unitary $\hat{U} = e^{i\mu \hat{a}_i^\dagger \hat{a}_i \hat{X}_{M_j}}$ which is used in Section 5.2. Starting with the light-mechanics Hamiltonian in a frame rotating at optical input frequency which is on resonance with the cavity:

$$\hat{H}/\hbar = \omega_M \hat{b}^\dagger \hat{b} - g_0 \hat{a}^\dagger \hat{a} (\hat{b}^\dagger + \hat{b}) , \quad (5.16)$$

we can formulate the Heisenberg equations of motion for $\hat{X}_M$ and $\hat{P}_M$:

$$\dot{\hat{X}}_M = \frac{i}{\hbar} [\hat{H}, \hat{X}_M] = \omega_M \hat{P}_M \quad (5.17a)$$

$$\dot{\hat{P}}_M = \frac{i}{\hbar} [\hat{H}, \hat{P}_M] = -\omega_M \hat{X}_M + \sqrt{2}g_0 \hat{a}^\dagger \hat{a} . \quad (5.17b)$$

These coupled differential equations can be solved for $\hat{X}_M(t)$ and $\hat{P}_M(t)$. Strictly, $\hat{a}^\dagger \hat{a}(t)$ should be solved using the Heisenberg-Langevin equation that follows from Eq. (5.16) however for purposes of this order-of-magnitude calculation we can set $\hat{a}^\dagger \hat{a} = 1$ which follows from a single photon input. This gives:

$$\hat{X}_M(t) = -\frac{\sqrt{2}g_0}{\omega_M} \cos(\omega_M t) + \frac{\sqrt{2}g_0}{\omega_M} \quad (5.18a)$$

$$\hat{P}_M(t) = \frac{\sqrt{2}g_0}{\omega_M} \sin(\omega_M t) . \quad (5.18b)$$

Therefore at time t , the optomechanical unitary interaction U should displace the mechanical quadratures according to the following transformations: $\hat{X}_M \rightarrow \hat{X}_M + \mu' \sin(\omega_M t)$ and $\hat{P}_M \rightarrow \hat{P}_M + \mu' \cos(\omega_M t)$, where $\mu' = 2g_0\omega_M^{-1}\sqrt{1 - \cos(\omega_M t)}$ is the effective dimensionless coupling strength. Thus, the displacement predicted by the Hamiltonian in Eq. (5.16) is captured by the following unitary:

$$\hat{U} = \exp \left[i\mu' (\cos(\omega_M t)\hat{X}_M + \sin(\omega_M t)\hat{P}_M) \right] . \quad (5.19)$$

We can see from this unapproximated unitary that there are two effects which are dependant on the ratio ω_M/κ : (i) the effective size of the displacement μ' , such that as $\omega_M/\kappa \ll 1$ is relaxed μ' is reduced, (ii) the direction of the displacement in phase space. Therefore, without making any assumptions on the size of ω_M/κ we can still generate an entangled bipartite mechanical state (albeit with a different displacement magnitude and direction).

The displacement direction is freely rotating in phase space during the pulsed interaction. This effect can be accommodated by adjusting the verification times in order perform homodyne measurements relative to the rotated displacement direction. We can keep track of this rotation by introducing the time-dependent variable $\hat{X}_M^\theta = \cos(\omega_M t)\hat{X}_M + \sin(\omega_M t)\hat{P}_M$.

We now focus on the effect which the sideband parameter has on the magnitude of the displacement μ . Relaxing $\omega_M/\kappa \ll 1$ leads to a reduction in the effective size of μ which is described by μ' . This is not necessarily a disadvantage as the inseparability criteria outlined in the verification stage are better-suited to smaller values of μ . Therefore, to analyse the effect

Table 5.1: Mechanical parameters taken from recent experiments (cf. Table 5.2). Parameters considered are optomechanical coupling rate g_0 , cavity amplitude decay rate κ , mechanical frequency ω_M , and sideband resolution ratio ω_M/κ . We can see that strong coupling is not required for our protocol. Using the formula $\mu = 2\sqrt{2}g_0/\kappa$ we have derived values for μ in Table 5.2. This formula comes from assuming the pulse length τ satisfies $\omega_M \ll \tau^{-1} \ll \kappa$. In the final column, the percentage reduction in μ is calculated using the expression Eq. (5.21) which results from the ratio ω_M/κ .

Mechanical parameters					
Refs.	$g_0/2\pi$	$\kappa/2\pi$	$\omega_M/2\pi$	ω_M/κ	% reduction in μ
[151]	127 Hz	15.9 MHz	1.139 MHz	7.16×10^{-2}	8.6×10^{-2}
[152]	20 kHz	0.44 GHz	4.3 MHz	9.77×10^{-3}	1.6×10^{-3}
[141]	35 MHz	8.8 GHz	3.74 MHz	4.25×10^{-4}	3.0×10^{-6}

of the sideband ratio on the size of μ' we now consider the unitary which acts on the mechanics $\hat{U} = \exp\left[i\mu' \hat{X}_M^\theta\right]$ given a single-photon has been detected. Let us take $t = C/\kappa$, where C is of order unity and depends on the pulse shape and duration (the details here are unimportant). Inserting the expression for μ' and Taylor expanding gives:

$$\hat{U} = \exp\left[\frac{i\sqrt{2}g_0C}{\kappa}\left(1 - \frac{\omega_M^2 C^2}{24\kappa^2} + \dots\right)\hat{X}_M^\theta\right]. \quad (5.20)$$

We can see that for $\omega_M/\kappa \ll 1$ we recover the mechanical displacement after a single photon measurement $\hat{U} = e^{i\mu\hat{X}_M}$ where $\mu = \sqrt{2}Cg_0/\kappa$ (this is used in Section 5.2). For a pulse which satisfies $\omega_M \ll \tau^{-1} \ll \kappa$ then $C = 2$, and the percentage reduction in μ to second order is:

$$\% \text{ reduction in } \mu = \frac{\omega_M^2}{6\kappa^2} \times 100. \quad (5.21)$$

This reduction is presented in Table 5.1 for the experimental parameters we consider in Table 5.2.

The sideband resolution ratio is also important for the verification stage. The verification stage uses a different wavelength and cavity mode, so will have a different sideband resolution ratio compared with the generation stage. The effect that significant ω_M/κ has on the linear interaction $\hat{U}_V = e^{i\chi\hat{X}_L\hat{X}_M}$ is analyzed in Ref. [93], where it is demonstrated that the correction to χ is likewise second order in ω_M/κ .

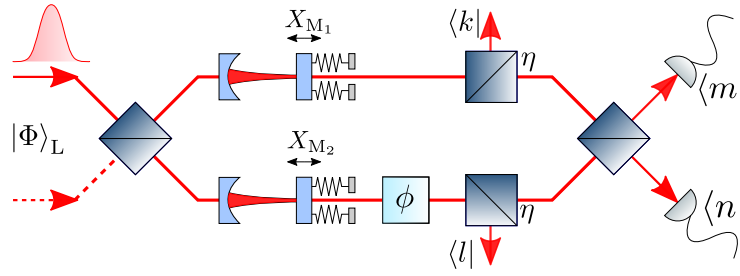


Figure 5.4: Two beam splitters are introduced into the state generation optical set-up in order to model optical losses and detector inefficiencies (cf. Fig. 5.1a). The beam splitters have intensity transmission η , and couple the entangling pulse with the environment (which is described by vacuum states). In our formalism, k and l photons are lost to the environment in the upper and lower arms, respectively; m and n photons are measured by the photodetectors. When $|\Phi\rangle_L$ is a coherent state it is possible that more than one photon interacts with the two oscillators but the photodetectors measure $\{1,0\}$.

5.2.2 Optical losses, detector inefficiencies and resolution

With a single photon input state, assuming no optical losses in the apparatus, the use of a $\{1,0\}$ click event to herald the mechanical state $\rho_{\text{out}}^{\{1,0\}}$ is robust against optical losses: if a photon is lost in the interferometer, no click is detected and this run of the experiment is discarded. In contrast, a coherent state input risks creating a mechanical state which is different to that indicated by the photodetector clicks. Reverting to the single photon input case, if there are optical losses then it is possible to lose the input photon in the apparatus but accidentally register a dark count, thus incorrectly concluding that the desired mechanical state has been generated. Therefore, in addition to the notion of heralding probability defined in Eq. (5.15), we must also consider the possibility that a different mechanical state $\rho_{\text{out}}^{\{m,n\}}$ (where $\{m,n\} \neq \{1,0\}$) is created even when the detectors have recorded a $\{1,0\}$ click due to optical losses, detector inefficiencies, or dark counts. This would lead us to incorrectly identify this state as $\rho_{\text{out}}^{\{1,0\}}$, resulting in a ‘false positive’ event. Conversely, if a $\{1,0\}$ click does successfully herald the creation of $\rho_{\text{out}}^{\{1,0\}}$ this is a ‘true positive’ event.

Here we examine how experimental imperfections can affect the success with which we herald our desired mechanical state. We can model optical losses and detector inefficiencies which are present during the state generation stage by introducing loss-model beam splitters with intensity transmission η in the lower and upper arms of the interferometer in Fig. 5.1a, such

that when $\eta = 1$ there are no optical losses and the detectors are perfectly efficient. The optical loss model is shown schematically in Fig. 5.4 where we have introduced a_3 and a_4 as the annihilation operators of modes of the environment to which entangling pulse couples to at these additional beam splitters. At optical frequencies we can assume that the initial environment mode is well described by the vacuum state. Considering that we performed photon counting in the output ports of modes 3 and 4, the full measurement operator for the parallel configuration with an injected coherent state is given by:

$$\hat{\Upsilon}_{mnkl} = \langle m|_1 \langle n|_2 \langle k|_3 \langle l|_4 U_{12}U_{24}U_{13} e^{i\mu\hat{a}_1^\dagger\hat{a}_1\hat{X}_{M1}} e^{i\mu\hat{a}_2^\dagger\hat{a}_2\hat{X}_{M2} + i\hat{a}_2^\dagger\hat{a}_2\phi} \hat{U}_{12} |\alpha\rangle_1 |0\rangle_2 |0\rangle_3 |0\rangle_4, \quad (5.22)$$

where $\hat{U}_{13}$ and $\hat{U}_{24}$ describe the unitary beam splitter interactions which couple modes 1 and 2 to the environment, with the subscript indicating which modes they act upon:

$$\hat{U}_{13}^\dagger \hat{a}_1 \hat{U}_{13} = \sqrt{\eta} \hat{a}_1 + \sqrt{1-\eta} \hat{a}_3, \quad (5.23a)$$

$$\hat{U}_{24}^\dagger \hat{a}_2 \hat{U}_{24} = \sqrt{\eta} \hat{a}_2 + \sqrt{1-\eta} \hat{a}_4. \quad (5.23b)$$

We have assumed the same beam splitter parameter η between the environment and both the upper and lower arms of the interferometer. This could be generalised however if we can distinguish between the upper and lower path due to differing η then this asymmetry is a greater problem for entanglement generation than false positives. The measurement operator is then:

$$\begin{aligned} \hat{\Upsilon}_{mnkl} = & e^{-\frac{|\alpha|^2}{2}} \left(\frac{\sqrt{\eta}\alpha}{2} \right)^{m+n} \left(\frac{\sqrt{1-\eta}\alpha}{\sqrt{2}} \right)^{k+l} \frac{1}{\sqrt{m!n!k!l!}} \\ & \times (e^{i\mu\hat{X}_{M1}} + e^{i\mu\hat{X}_{M2}+i\phi})^m (e^{i\mu\hat{X}_{M1}} - e^{i\mu\hat{X}_{M2}+i\phi})^n \\ & \times e^{i\mu k\hat{X}_{M1}} e^{i\mu l\hat{X}_{M2}+il\phi}, \end{aligned} \quad (5.24)$$

and can be interpreted as losing k and l photons to the environment via each mode (either due to optical losses or detector inefficiencies) but measuring m and n photons at the detectors.

In addition to the case where $\{k, l\}$ photons are lost to the environment, we must consider dark counts which occur independently to the photons in the experiment. In the time window of the

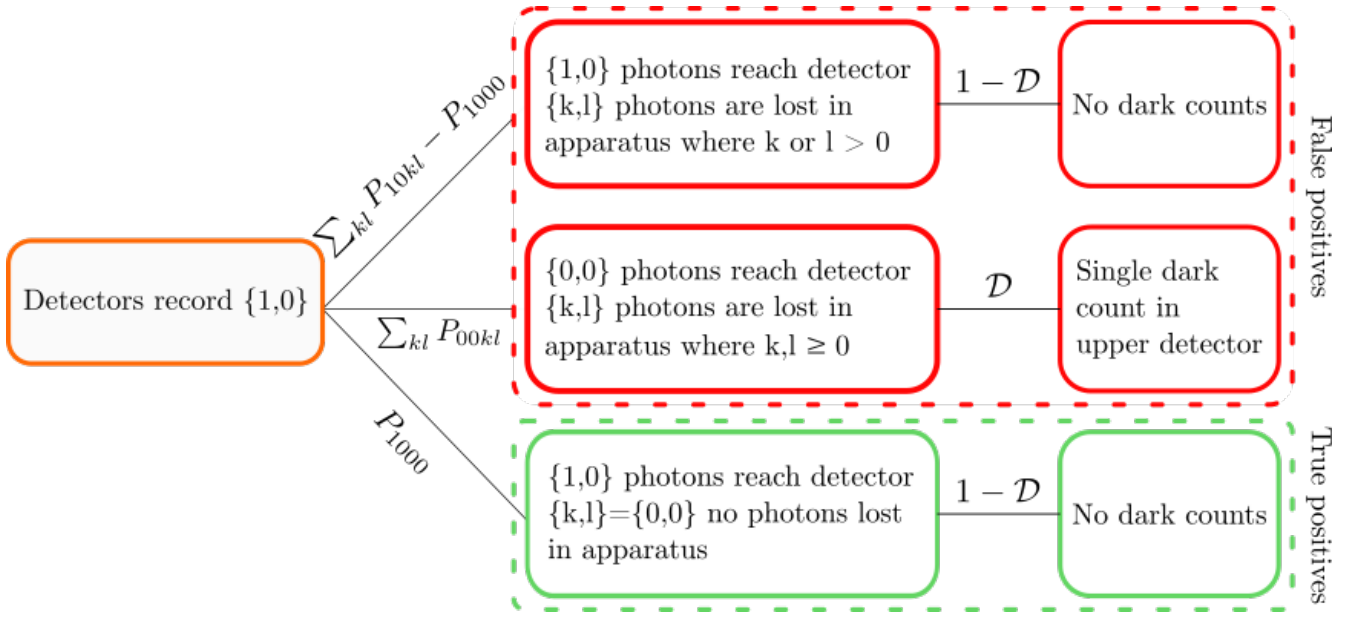


Figure 5.5: Probability tree outlining the photon pathways which could lead to photo-detectors recording $\{1,0\}$. We assume detectors have the ability to resolve single photons and so a single photon exists in the output of the upper arm of the interferometer and there are no photons in the lower. Events outlined in red are pathways which lead to heralding the incorrect mechanical state while the green pathway is the only route to truly heralding the desired mechanical state. The probabilities of each pathway are labelled on the branches. We emphasise that this tree is valid given a $\{1,0\}$ photon measurement. Therefore, one must use Bayes' theorem to normalise the probabilities of false or true positive outcomes. For the notation of the pathways see Fig. 5.4.

pulsed interaction there are 3 potential dark count outcomes: 0 dark counts, 1 dark count, 2 or more dark counts. Assuming the dark count rate is 1 s^{-1} and the detectors are on for 10 ns we expect 10^{-8} dark counts. The probability $P(k)$ of k dark counts being detected during the experiment is captured by a Poisson distribution $P(k) = \lambda^k e^{-\lambda} / k!$ with mean $\lambda = 10^{-7}$. It then follows that $P(0) = 0.9999$, $P(1) = 9.99 \times 10^{-9}$, and $P(k > 1) \approx 10^{-14}$. We henceforth assume the probability of more than 1 dark count is negligible on this timescale. Therefore, we introduce the independent, binomial probability $\mathcal{D}$ that a single dark count is detected during the detection window, where $\mathcal{D} \approx 10^{-8}$.

To calculate the probability of successfully heralding the desired mechanical state given a $\{1,0\}$ click, let $P_{mnkl} = \text{Tr}[\hat{\Upsilon}_{mnkl} \rho_{\text{in}} \hat{\Upsilon}_{mnkl}^\dagger]$, such that P_{mnkl} is the probability of losing $\{m, n, k, l\}$ photons at each of the outputs labelled accordingly in Fig. 5.4. The probability for resolving

detectors to herald a mechanical state other than $\rho_{\text{out}}^{\{1,0\}}$, given a $\{1,0\}$ click, is:

$$P(\text{False}|\{1,0\}\text{click}) = \frac{(1-D)\sum_{kl}(P_{10kl} - P_{1000}) + \mathcal{D}\sum_{kl}P_{00kl}}{(1-D)\sum_{kl}P_{10kl} + \mathcal{D}\sum_{kl}P_{00kl}}, \quad (5.25)$$

where we have made use of Bayes' Theorem to normalise the probability. The probability tree in Fig. 5.5 outlines the possible routes that could lead to heralding an undesired mechanical state. The probability of heralding the correct $\rho_{\text{out}}^{\{1,0\}}$ state, given a $\{1,0\}$ detector click, is:

$$P(\text{True}|\{1,0\}\text{click}) = \frac{(1-\mathcal{D})P_{1000}}{(1-D)\sum_{kl}P_{10kl} + \mathcal{D}\sum_{kl}P_{00kl}}. \quad (5.26)$$

Let us denote the ratio of false positives (Eq. 5.25) to true positives (Eq. 5.26) as $\mathcal{R}$. Then the fraction of true positives we expect for a given number of $\{1,0\}$ clicks is $\mathcal{F} = 1/(1 + \mathcal{R})$. Therefore, when using number-resolving photodetectors, the fraction $\mathcal{F}$ of true positive events per total number of $\{1,0\}$ clicks is expected to be:

$$\mathcal{F} = \left[e^{(1-\eta)|\alpha|^2} + \frac{e^{-\eta|\alpha|^2}\mathcal{D}}{\eta P_{10}(1-\mathcal{D})} \right]^{-1}, \quad (5.27)$$

where P_{10} is defined in Eq. (5.15) (for $|\Phi\rangle_{\text{L}} = |\alpha\rangle_1 |0\rangle_2$). In order to neglect false positives and confidently ensure that a $\{1,0\}$ click heralds the $\rho_{\text{out}}^{\{1,0\}}$ state, $\mathcal{F}$ must be as close to unity as possible. Therefore, we require that $(1-\eta)|\alpha|^2 \ll 1$, and that $\mathcal{D}$ is sufficiently low compared to P_{10} . Thus, Eqs (5.15) and (5.27) can be used to establish the feasibility for the heralding process, in particular, for the dependence on μ .

We now consider the case where the photodetectors in the heralding stage are non-resolving. The combination of a click in the upper detector and no-click in the lower detector would indicate at least 1 photon has been detected in the upper detector, but there could be more photons present. We still label this as a $\{1,0\}$ click event. Similarly we can construct a probability tree for the possible photon events that lead to such a detection outcome, shown in

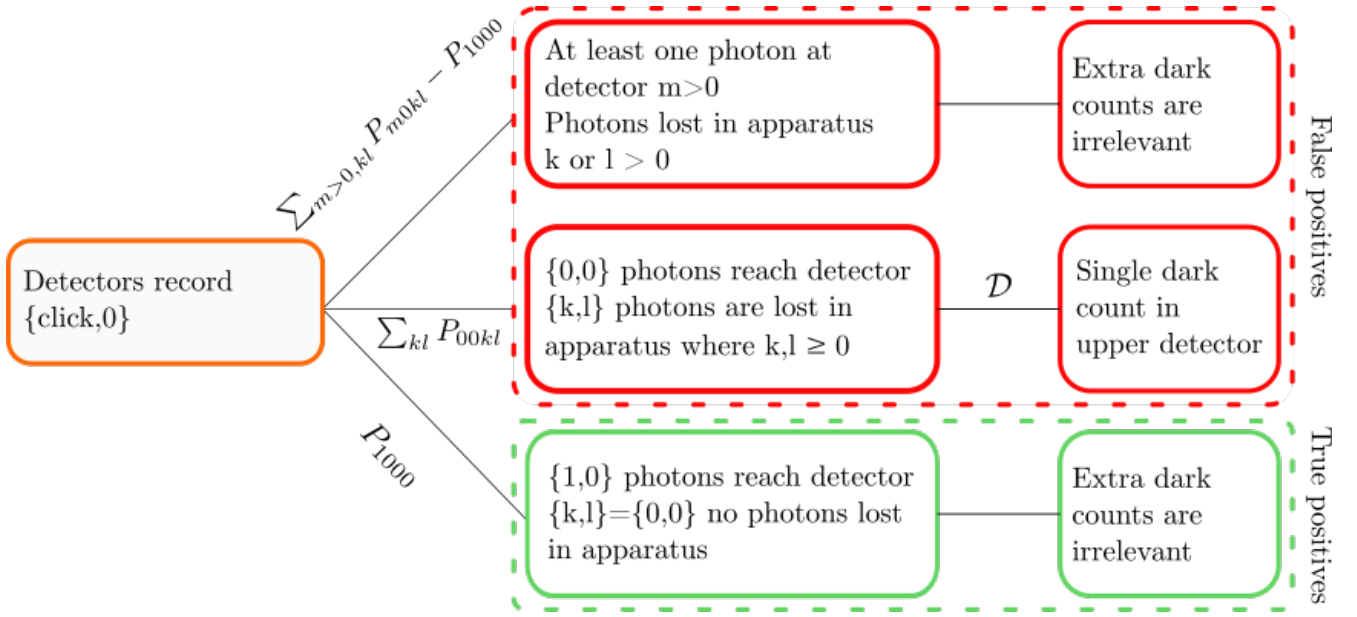


Figure 5.6: Probability tree outlining photon pathways which could lead to photo-detectors recording $\{\text{click}, 0\}$. We now model non-resolving detectors which cannot distinguish between one or more photons. Events outlined in red are pathways which can lead to heralding the incorrect state while the green pathway is the only route through which it is possible to herald the correct state. The diagram is very similar to Fig. 5.5 but with subtle differences due to non-resolving nature of detectors.

Fig. 5.6. Following the same analysis as above, the analogous equations are:

$$P(\text{False}|\{1, 0\}\text{click}) = \frac{\sum_{m>0,kl} P_{m0kl} - P_{1000} + \mathcal{D} \sum_{kl} P_{00kl}}{\sum_{m>0,kl} P_{m0kl} + \mathcal{D} \sum_{kl} P_{00kl}}, \quad (5.28)$$

$$P(\text{True}|\{1, 0\}\text{click}) = \frac{P_{1000}}{\sum_{m>0,kl} P_{m0kl} + \mathcal{D} \sum_{kl} P_{00kl}}. \quad (5.29)$$

The ratio of false positive (Eq. (5.28)) to true positive events (Eq. (5.29)) is again denoted as $\mathcal{R}$. Using $\mathcal{F} = 1/(1 + \mathcal{R})$ we can find $\mathcal{F}$ for non-resolving detectors:

$$\mathcal{F} = \left[\frac{e^{-\eta|\alpha|^2} \mathcal{L}}{\eta P_{10}} + \frac{e^{-\eta|\alpha|^2} \mathcal{D}}{\eta P_{10}} \right]^{-1}, \quad (5.30)$$

where P_{10} is defined in Eq. (5.15) (for $|\Phi\rangle_L = |\alpha\rangle_1 |0\rangle_2$), and

$$\mathcal{L} = \sum_{m=1}^{\infty} \sum_{k=0}^{2m} \binom{2m}{k} \left(\frac{\eta|\alpha|^2}{4} \right)^m \frac{1}{m!} e^{-i(m-k)\phi} \lambda^{(m-k)^2}, \quad (5.31)$$

with $\lambda = e^{-\frac{\mu^2}{2}(1+\bar{n}_1+\bar{n}_2)}$. Similar to the model for resolving photodetectors, the non-resolving model similarly requires $\mathcal{D}$ and optical losses to be sufficiently low such that we can neglect false positives.

By examining the fraction of true positives $\mathcal{F}$ for resolving (Eq. 5.27) and non-resolving (Eq. 5.30) detectors we can see that it is parameterised by the optomechanical coupling strength μ , the initial occupation number of the mechanics $\bar{n}$, the amplitude of the coherent state input α , the optical transmission probability through each arm of the interferometer η , and the dark count probability $\mathcal{D}$. Assuming steps have been taken to mitigate $\mathcal{D}$ and η and the state has been prepared as close to the the ground state as possible, then the only parameter we have freedom over is α , through attenuation methods. Therefore, one should also optimise α in order to maximise the probability of generating the correct state. We noted that the heralding probability of a $\{1,0\}$ photon click is optimised when $\alpha = 1$ and so tuning α can come at the cost of a diminished overall heralding probability which means more experimental runs are required before the desired $\{1,0\}$ photon detection event is recorded.

To investigate this behaviour we have plotted Eq. 5.27 as a function of μ in Fig 5.7a, while we have plotted Eq. 5.30 in Fig. 5.7b. In both plots we fix $\bar{n} = 0.1$ and $\mathcal{D} = 10^{-8}$. For both resolving and non-resolving models we consider separately two cases: (i) we fix $\alpha = 1$, (ii) α is chosen in order to maximise $\mathcal{F}$. In general, $\mathcal{F}$ increases with μ before saturating. The exception is for the non-resolving detector case when $\alpha = 1$. Here, we see that when $\mu > 10^{-1}$ the probability of generating the correct state given the desired detection outcome starts to deteriorate. This is because as the optomechanical coupling μ grows, multiphoton interactions between the light and mechanics become more likely than single photon interactions. These multiphoton kicks would still be recorded as a correct photon detection by the non-resolving detectors.

Single-photon input state

So far we have only considered the case where the injected light used to generate entanglement is $|\Phi\rangle_L = |\alpha\rangle_1 |0\rangle_2$. We stated that in the idealised case, a single photon input is robust against

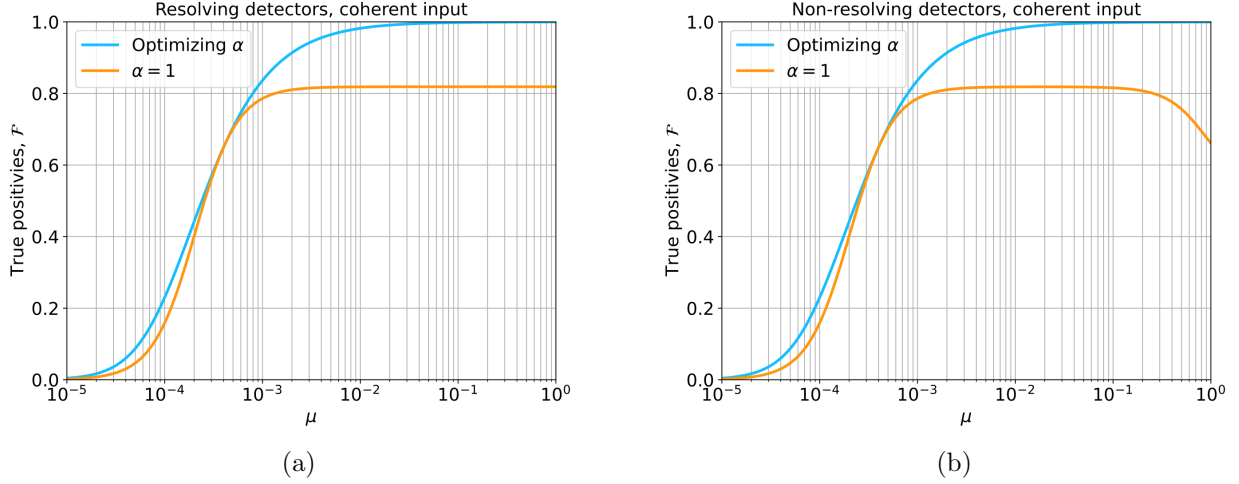


Figure 5.7: True positive fraction $\mathcal{F}$ as function of coupling strength μ . The following parameters have been assumed throughout: $\phi = \pi$, initial thermal occupation number of oscillators $\bar{n} = 0.1$, number of dark counts during detection window of 10 ns is $\mathcal{D} = 10^{-8}$. (a) $\mathcal{F}$ with coherent input state $|\Phi\rangle_L = |\alpha\rangle|0\rangle$ for the case of number-resolving photodetectors. Orange line is for a fixed coherent state amplitude $\alpha = 1$; blue line is when α is allowed to vary in order to maximise $\mathcal{F}$. (b) $\mathcal{F}$ for coherent input state $|\Phi\rangle_L = |\alpha\rangle|0\rangle$ with non-resolving photodetectors. Similarly, orange line is for a fixed coherent state amplitude $\alpha = 1$; blue line is when α is allowed to vary in order to maximise $\mathcal{F}$. As μ increases, $\mathcal{F}$ increases exponentially before saturating. Note: in (b) for $\alpha = 1$, as μ increases above $\approx 10^{-1}$ $\mathcal{F}$ begins to deteriorate; this is due to the increased likelihood of multiphoton interactions with the oscillator which cannot be distinguished by non-resolving detectors.

false positives since a $\{1, 0\}$ detection means one photon has passed through the interferometer. However we can now also consider dark counts in conjunction with optical losses both in the case of number resolving detectors and non-resolving detectors given a single photon input state $|\Phi\rangle_L = |1\rangle_1|0\rangle_2$.

Following the same method as we used for the case where the injected light is prepared in a coherent state, we introduce two beam splitters into each optical arm in order to model optical losses and detector inefficiencies. We can derive the following measurement operator:

$$\begin{aligned} \hat{\Upsilon}_{mnkl} = & \frac{\sqrt{\eta}}{2} (e^{i\mu\hat{X}_{M1}} + e^{i\mu\hat{X}_{M2}+i\phi}) \delta_{m1} \delta_{n0} \delta_{k0} \delta_{l0} + \frac{\sqrt{\eta}}{2} (e^{i\mu\hat{X}_{M1}} - e^{i\mu\hat{X}_{M2}+i\phi}) \delta_{m0} \delta_{n1} \delta_{k0} \delta_{l0} \\ & + \frac{\sqrt{1-\eta}}{\sqrt{2}} e^{i\mu\hat{X}_{M1}} \delta_{m0} \delta_{n0} \delta_{k1} \delta_{l0} + \frac{\sqrt{1-\eta}}{\sqrt{2}} e^{i\mu\hat{X}_{M2}+i\phi} \delta_{m0} \delta_{n0} \delta_{k0} \delta_{l1} . \end{aligned} \quad (5.32)$$

For the case where detectors can resolve single photons, the probability of a false positive

detection event is:

$$\begin{aligned} P(\text{False}|\{1,0\}\text{click}) &= \frac{\mathcal{D} \sum_{kl} P_{00kl}}{\mathcal{D} \sum_{kl} P_{00kl} + (1 - \mathcal{D})P_{1000}} , \\ &= \frac{(1 - \eta)\mathcal{D}}{(1 - \eta)\mathcal{D} + \eta(1 - \mathcal{D})P_{10}} , \end{aligned} \quad (5.33)$$

and for correctly detecting the desired state given a $\{1,0\}$ click

$$\begin{aligned} P(\text{True}|\{1,0\}\text{click}) &= \frac{(1 - \mathcal{D})P_{1000}}{\mathcal{D} \sum_{kl} P_{00kl} + (1 - \mathcal{D})P_{1000}} , \\ &= \frac{\eta(1 - \mathcal{D})P_{10}}{(1 - \eta)\mathcal{D} + \eta(1 - \mathcal{D})P_{10}} . \end{aligned} \quad (5.34)$$

We can similarly define the ratio $\mathcal{R}$ between false and true positive probabilities

$$\mathcal{R} = \frac{(1 - \eta)\mathcal{D}}{\eta(1 - \mathcal{D})P_{10}} , \quad (5.35)$$

where P_{10} is the heralding probability for $\{1,0\}$ defined in Eq. (5.15) (for $|\Psi\rangle_L = |1\rangle_1 |0\rangle_2$). Therefore, it follows from $\mathcal{F} = 1/(1 + \mathcal{R})$ that the probability of obtaining a true positive event is

$$\mathcal{F} = \frac{\eta P_{10}(1 - \mathcal{D})}{\eta P_{10}(1 - \mathcal{D}) + (1 - \eta)\mathcal{D}} . \quad (5.36)$$

In the non-resolving case where the detector just clicks when a photon is detected but we can't distinguish between 1 or more photons, then the false positive ratio is as follows:

$$\mathcal{R} = \frac{\mathcal{D} \sum_{kl} P_{00kl}}{P_{1000}} = \frac{(1 - \eta)\mathcal{D}}{\eta P_{10}} . \quad (5.37)$$

Again, $\mathcal{F}$ can be calculated

$$\mathcal{F} = \frac{\eta P_{10}}{\eta P_{10} + (1 - \eta)\mathcal{D}} . \quad (5.38)$$

The fraction of true positives for photon resolving (Eq. (5.36)) and non-resolving (Eq. (5.38)) are both plotted in Fig. 5.8 assuming a single photon input. For the parameters considered in the plot ($\bar{n} = 0.1$ and $\mathcal{D} = 10^{-8}$) the two functions are approximately equal, therefore it is unimportant whether one uses resolving or non-resolving detectors. This reinforces the

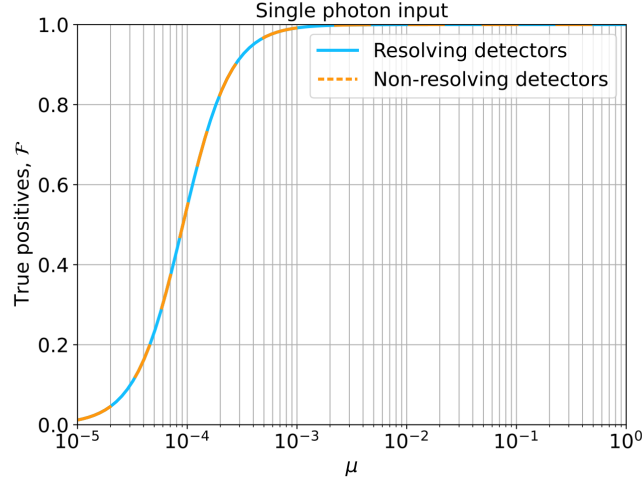


Figure 5.8: Plot of fraction of true positives we can expect for a given set of $\{1,0\}$ photon heralding events for a single photon input. The functions show the probability that the desired photon measurement event actually generates the expected mechanical state as a function of μ . The solid blue line is for the case of resolving photodetectors (Eq. (5.36)) while the dashed orange line is for non-resolving photodetectors (Eq. (5.38)). These lines are further parameterised by assuming the interferometer phase of $\phi = \pi$, mechanical thermal occupation number of $\bar{n} = 0.1$, and a dark count probability of $\mathcal{D} = 10^{-8}$. The two functions are almost identical so appear to be superimposed.

desirability of a single photon pulsed input source in this protocol.

5.3 Entanglement Verification

In order to verify the mechanical-mechanical entanglement generated, we propose an experimental procedure in order to measure the components needed to test inseparability criteria for the state produced via the parallel-configuration (Fig. 5.1a). We include the analogous verification set-up for the series configuration in A.1.2. The verification protocol is tailored towards assessing inseparability criteria belonging to the class of criteria in Ref. [21]. These inseparability tests take the form of inequalities (constructed from physical observables) which when violated verify the existence of entanglement. The experimental procedure we propose here can provide access to all the inequalities belonging to this class of inseparability criteria. To demonstrate the strength of our scheme we have selected two such inequalities from this class; one of which is guaranteed to detect entanglement in Gaussian states, while the other is

able to detect entanglement in states with a greater degree of non-Gaussianity.

5.3.1 Inseparability Criteria

In Chapter 3 we discussed the challenges of finding a suitable inseparability criterion for non-Gaussian entangled states. While many continuous-variable entanglement tests are necessary and sufficient for detecting entanglement in Gaussian states, not all are capable of capturing entanglement in highly non-Gaussian states (to our knowledge there is no single test that is guaranteed to detect all forms of non-Gaussian entanglement in a state). Since higher-order moments are needed to characterise non-Gaussian states, criteria formed from higher-order moments are a natural choice for confirming non-Gaussian entanglement. Shchukin and Vogel introduced a class of inseparability criteria derived from the negative partial transpose (NPT) of the state which take the form of inequalities constructed from arbitrary moments of a continuous-variable quantum state [21]. Therefore, this class of operational criteria lends itself well to the identification of entanglement in non-Gaussian states. The mathematical formalism of these criteria is discussed in detail in Section 3.2.

We have previously discussed that a bipartite state ρ_{AB} is separable if and only if $\det[\mathbf{M}_N] > 0$ for all N ; the matrix $\mathbf{M}_N$ is defined in Eq. 3.9. We will henceforth denote this determinant more succinctly as D_N . Therefore, if $D_N < 0$ for any N then NPT has been demonstrated and the state is entangled. Using the ordering convention in Ref. [21], the first few rows and columns of the matrix from which D_N is calculated are shown here:

$$D_N = \begin{vmatrix} 1 & \langle \hat{b}_1 \rangle & \langle \hat{b}_1^\dagger \rangle & \langle \hat{b}_2^\dagger \rangle & \langle \hat{b}_2 \rangle & \dots \\ \langle \hat{b}_1^\dagger \rangle & \langle \hat{b}_1^\dagger \hat{b}_1 \rangle & \langle \hat{b}_1^{\dagger 2} \rangle & \langle \hat{b}_1^\dagger \hat{b}_2^\dagger \rangle & \langle \hat{b}_1^\dagger \hat{b}_2 \rangle & \dots \\ \langle \hat{b}_1 \rangle & \langle \hat{b}_1^2 \rangle & \langle \hat{b}_1 \hat{b}_1^\dagger \rangle & \langle \hat{b}_1 \hat{b}_2^\dagger \rangle & \langle \hat{b}_1 \hat{b}_2 \rangle & \dots \\ \langle \hat{b}_2 \rangle & \langle \hat{b}_1 \hat{b}_2 \rangle & \langle \hat{b}_1^\dagger \hat{b}_2 \rangle & \langle \hat{b}_2^\dagger \hat{b}_2 \rangle & \langle \hat{b}_2^2 \rangle & \dots \\ \langle \hat{b}_2^\dagger \rangle & \langle \hat{b}_1 \hat{b}_2^\dagger \rangle & \langle \hat{b}_1^\dagger \hat{b}_2^\dagger \rangle & \langle \hat{b}_2^{\dagger 2} \rangle & \langle \hat{b}_2 \hat{b}_2^\dagger \rangle & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}. \quad (5.39)$$

We have previously discussed how many other entanglement criteria are special cases of this

overarching hierarchy of inequalities. In particular, D_5 (the determinant of the first 5 rows and columns of the matrix in Eq. (5.39)) is a reformulation of Simon's criterion [36] which is a necessary and sufficient entanglement test for single-mode bipartite Gaussian states [37]. Therefore, applying D_5 to a Gaussian state will always verify entanglement (although some non-Gaussian entangled states might also satisfy $D_5 < 0$ since a negative D_5 is only a sufficient condition for non-Gaussian entangled states).

We can construct other inseparability criteria by deleting rows and columns of the matrix in Eq. (5.39) in a pairwise fashion. This leaves behind a submatrix $\mathbf{A}$, see Fig. 3.1 for a visual representation of this. If the determinant of the resulting matrix is negative then this also fulfils the NPT criterion and the state is entangled. In this way, we can delete entries in D_N to arrive at the following specific subdeterminant which we will label S_3

$$S_3 = \begin{vmatrix} 1 & \langle \hat{b}_2^\dagger \rangle & \langle \hat{b}_1 \hat{b}_2^\dagger \rangle \\ \langle \hat{b}_2 \rangle & \langle \hat{b}_2^\dagger \hat{b}_2 \rangle & \langle \hat{b}_1 \hat{b}_2^\dagger \hat{b}_2 \rangle \\ \langle \hat{b}_1^\dagger \hat{b}_2 \rangle & \langle \hat{b}_1^\dagger \hat{b}_2^\dagger \hat{b}_2 \rangle & \langle \hat{b}_1^\dagger \hat{b}_1 \hat{b}_2^\dagger \hat{b}_2 \rangle \end{vmatrix}, \quad (5.40)$$

such that if $S_3 < 0$ we have a sufficient criterion for entanglement. S_3 is one of the simplest subdeterminants that goes beyond combinations of quadratic moments, thus allowing non-Gaussian entanglement to be detected. As a result of its simplicity, it has been previously highlighted in the context of two-mode Schrödinger-cat states [21, 153]. We note that lower-dimensional matrices, from which the determinant is computed, offer a more practical route to experimental verification.

Since D_5 captures all entanglement for Gaussian states, if S_3 indicates entanglement in a region of state space for which D_5 does not, we can conclude that these entangled states are non-Gaussian. Therefore, applying these two inequalities together to a state, we can identify parameter regions where the state is non-Gaussian and entangled. We note that there may be entangled non-Gaussian states that are not detected by D_5 or S_3 , since these are only sufficient entanglement criteria. Furthermore, for states where both $S_3 < 0$ and $D_5 < 0$, or $D_5 < 0$ but $S_3 > 0$, we cannot infer if the entangled state is non-Gaussian. It should also

be highlighted that the magnitude of a determinant has no relevance to the NPT criterion; only the sign of a determinant matters for entanglement verification. Nevertheless, a more negative determinant could be easier to experimentally confirm since each expectation value has an associated experimental uncertainty. We will explore the effect of errors arising from environment on the expectation values in Section 5.3.3.

5.3.2 Verification protocol

We now detail our experimental scheme to obtain the moments of the mechanical state which are needed for the inseparability criteria in Section 5.3.1. Our verification protocol can be used to extract arbitrary mechanical moments. These can then be used to calculate D_5 and S_3 or more complicated subdeterminants composed of higher-order moments.

The experimental schematic for the verification protocol is illustrated in Fig. 5.9. After the entangling pulse has interacted with the two oscillators and photon-detection has been performed, we are left with a mechanical-mechanical entangled state. After this photon measurement, up to four verification pulses are sent into each arm of the apparatus. For the verification pulses to provide an independent readout, and to operate in the linearized regime, these fields are resonant with a different optical mode. The pulses also address each oscillator individually further improving the independence of the readout. Assuming the verification pulses are strong and the optical phase shift imparted to the light is small, we get the following linearized optomechanical interaction for each pulse: $\hat{U}_v = e^{i\chi\hat{X}_{L_i}\hat{X}_{M_j}}$, where $\hat{X}_{L_i}$ and $\hat{X}_{M_j}$ are the position operators of the i^{th} light mode and j^{th} oscillator respectively [122, 123, 126]. The assumptions which underpin the derivation of this unitary are discussed in Section 2.4.2. We emphasise that the verification pulses are independent from the generation pulse, but to avoid overloading the notation we simply use L_i in this section to refer to the verification light modes. The dimensionless interaction strength is given by $\chi \propto g_0\sqrt{N_p}/\kappa$ where N_p is the mean photon number in the input field. The coupling rate here g_0 differs to the coupling rate for the state-preparation stage as a different cavity mode is used, and the proportionality constant is of order unity, which depends on the properties of the input pulse. If we use a long verification pulse with duration τ that

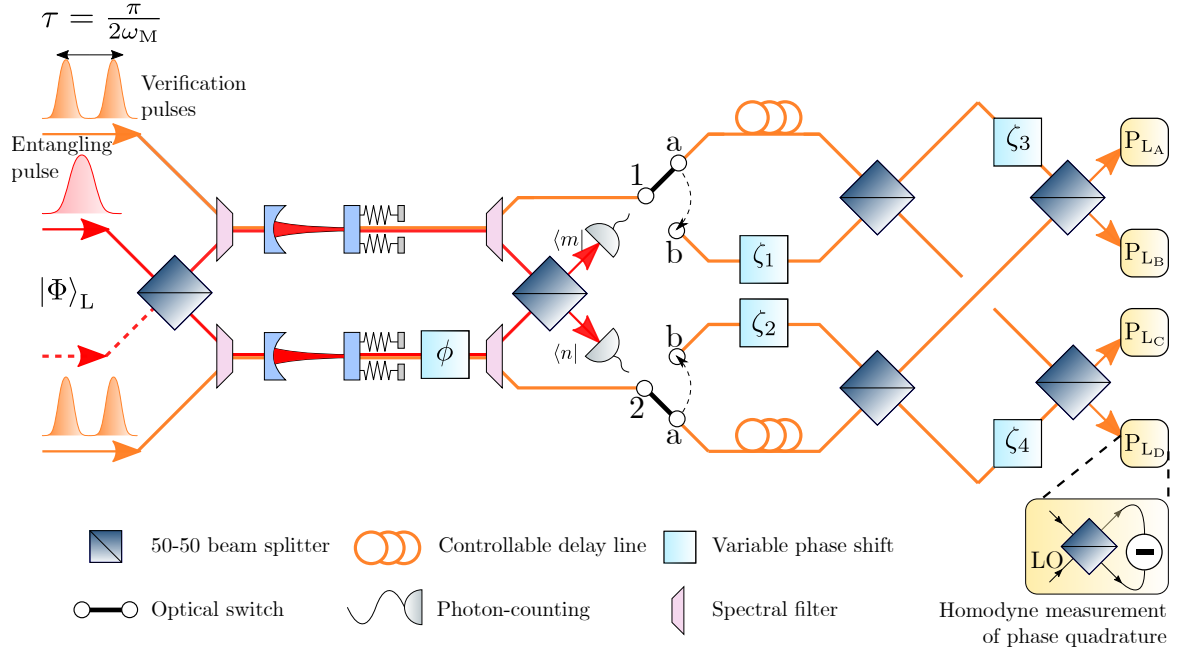


Figure 5.9: Proposed verification scheme to measure arbitrary quadrature moments of the mechanical state produced in the parallel configuration. These moments allow us to compute the inseparability criteria D_5 (Eq. (5.39)) and S_3 (Eq. (5.40)). The entangling pulse (red) generates a mechanical state. Following a $\{1,0\}$ click event, the verification stage is subsequently conducted. Up to four pulses of light are sent into apparatus at times $\tau = 0$ and/or $\tau = \pi/(2\omega_M)$. These verification pulses (orange) have a different wavelength to the entangling pulse and so spectral filters ensure that the verification and entangling pulses follow different optical paths. Both the entangling and verification pulses are on resonance with a cavity mode that interacts in the unresolved sideband regime. Following an optomechanical interaction, the light quadratures of the verification pulse transform according to Eqs. (5.41a)-(5.41b); mechanical quadrature information is transferred to the momentum quadrature of light. Switches 1 and 2 provide the option to hold earlier pulses in a delay line, ensuring all pulses are incident at the beam splitters concurrently. Pulses which do not require a delay follow path 1b or 2b. Variable phase shifts ζ_l (where $l = 1, 2, 3, 4$) can be changed between pulses and runs, providing us with the degrees of freedom necessary to solve for the desired moments. Homodyne measurements of $\hat{P}_{L_k}$ (where $k = A, B, C, D$) are performed on the momentum quadrature of light as it emerges from the final set of beam splitters. For each configuration of pulses, switches, and phase shifts, we require many runs (each requiring state creation followed by a measurement of $\hat{P}_{L_k}$) in order to construct a data set of measurements $\{P_{L_k}^{(i)}\}_{i=1}^N$. From this data set one can calculate moments such as $\langle \hat{P}_{L_k}^d \rangle$ using Eq. (5.42) which can in turn be expanded in terms of mechanical quadrature moments, thus allowing us to find all moments of the form $\langle \hat{X}_{M_1}^p \hat{P}_{M_1}^q \hat{X}_{M_2}^r \hat{P}_{M_2}^s \rangle$.

satisfies $\omega_M \ll \tau^{-1} \ll \kappa$, then $\chi = 4g_0\sqrt{N_p}/\kappa$ [96, 97]. Similar to the generation stage, using such a pulse has the advantage of more easily ensuring spatio-temporal mode matching at the second beam splitter if the two cavities are not identical. Note also that an additional deterministic momentum kick is also imparted on the system per verification pulse that depends on the optical intensity, which may be accounted for in post-processing or cancelled with an appropriate feedback force.

Following the interaction $\hat{U}_v$, the i^{th} -mode light quadratures $\hat{X}_{L_i}$ and $\hat{P}_{L_i}$ are given by:

$$\hat{X}_{L_i}(\theta_j) = \hat{X}_{L_{\text{in}}} , \quad (5.41a)$$

$$\hat{P}_{L_i}(\theta_j) = \hat{P}_{L_{\text{in}}} + \chi \hat{X}_{M_j}(\theta_j) , \quad (5.41b)$$

where $\hat{X}_{M_j}(\theta_j)$ is the mechanical position operator before the interaction $\hat{U}_v$. Here, we have assumed the input light quadratures $\hat{X}_{L_{\text{in}}}$ and $\hat{P}_{L_{\text{in}}}$ are quantum noise limited with variances of $1/2$, and the mechanical quadratures freely evolve as a function of $\theta_j = \omega_{M_j}\tau$, where ω_{M_j} is the mechanical frequency of the j^{th} oscillator, and τ is the time elapsed between state generation and the interaction $\hat{U}_v$. We henceforth assume that the two oscillators have the same parameters for brevity and drop the j label on ω_M . For a system isolated from its environment, $\hat{X}_{M_j}(\pi/2) = \hat{P}_{M_j}(0)$, so by switching on the verification pulse at different times we can imprint different mechanical quadratures on $\hat{P}_{L_i}$.

As $\hat{X}_M$ and $\hat{P}_M$ are more experimentally accessible than $\hat{b}$ and $\hat{b}^\dagger$ we will recast the determinants in Eqs. (5.39) and (5.40) to be in terms of these operators using the relation $\hat{b}_i = (\hat{X}_{M_i} + i\hat{P}_{M_i})/\sqrt{2}$. From this, each element of D_N (Eq. (5.39)) can be expressed as a linear combination of moments in $\hat{X}_M$ and $\hat{P}_M$: $\sum_{pqrs} c_{pqrs} \langle \hat{X}_{M_1}^p \hat{P}_{M_1}^q \hat{X}_{M_2}^r \hat{P}_{M_2}^s \rangle$ where $c_{pqrs} \in \mathbb{C}$. This is convenient as the optomechanical interaction in Eq. (5.41b) is dependent on the mechanical quadratures.

To obtain all the terms in the inseparability criteria we must measure moments of the form $\langle \hat{X}_{M_1}^p \hat{P}_{M_1}^q \hat{X}_{M_2}^r \hat{P}_{M_2}^s \rangle$. Since we cannot simultaneously measure $\hat{X}_{M_i}$ and $\hat{P}_{M_i}$, we exploit a network of switches, time delays, and phase shifts $\{\zeta_l\}$ in the optical set-up (Fig. 5.9) to probe these quantities. This network should be integrated with the optical apparatus used for entanglement

generation as each verification run involves recreating the entangled state. Our proposed scheme allows us to control which combination of mechanical quadratures are obtained via homodyne measurement of the output light quadratures P_{L_k} , where $k = \{A, B, C, D\}$ (see Fig. 5.9).

We can build a data set of N homodyne measurement results $\{P_{L_k}^{(i)}\}_{i=1}^N$ for a particular optical pathway (a specific combination of switches, time-delays, and phase shifts characterises a pathway) by repeatedly generating the mechanical state, and then performing a verification pulse with homodyne detection. From the data we can directly compute moments using the formula:

$$\langle \hat{P}_{L_k}^d \rangle = \frac{1}{N} \sum_i^N (P_{L_k}^{(i)})^d. \quad (5.42)$$

Eq. (5.41b) allows us to expand the measured $\langle \hat{P}_{L_k}^d \rangle$ as a linear combination of mechanical quadrature moments with coefficients determined by the phase shifts $\{\zeta_l\}$. Repeating these steps with sufficiently many appropriate combinations of $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ will provide us with enough linearly independent equations to solve for any of the mechanical quadrature moments that appear in the expansion of $\langle \hat{P}_{L_k}^d \rangle$ [154]. From these moments D_5 , S_3 , or any other subdeterminant can then be calculated.

As a simple example we can consider finding $\langle \hat{X}_{M_1} \rangle$. After a $\{1,0\}$ click heralds the state $\rho_{\text{out}}^{\{1,0\}}$, the verification pulses are allowed to interact with the mechanical oscillators. Spectral filters ensure the verification pulses follow a different pathway to that of the entangling pulse, since the entangling and verification pulses have a different wavelength (see Fig. 5.9). For this mechanical quadrature moment, a single verification pulse is injected at time $\tau = 0$ into the upper arm and interacts with oscillator 1. The pulse then follows path 1b, passes through a set of beam splitters (for this moment $\{\zeta_l\}$ are unimportant). Finally, we perform a homodyne measurement of $\hat{P}_{L_k}$ (Eq. (5.41b)). Over numerous runs (each time recreating the entangled state) we build a data set of N homodyne measurements $\{P_{L_k}^{(i)}\}_{i=1}^N$. According to Eq. (5.42), by summing over the data set we can deduce the first moment $\langle \hat{P}_{L_k} \rangle = \chi \langle \hat{X}_{M_1} \rangle / 2$ (where we have assumed vacuum noise statistics $\langle \hat{P}_{L_{\text{in}}} \rangle = 0$, and the factor $1/2$ arises from the beam splitters). The value of χ is assumed to be known accurately via an initial calibration stage (see A.1.1), allowing us to extract the value of $\langle \hat{X}_{M_1} \rangle$. Repeating this entire sequence but sending in a single

pulse at time $\tau = \pi/(2\omega_M)$ will yield $\langle \hat{P}_{M_1} \rangle$. Using Eq. (5.42) we can calculate higher-order moments of $\hat{P}_{L_k}$ from the same data set and thus find higher order mechanical moments. For example, the third moment can be expressed as $\langle \hat{P}_{L_k}^3 \rangle = \chi^3 \langle \hat{X}_{M_1}^3 \rangle / 8 + \chi \langle \hat{X}_{M_1} \rangle / 4$ (where we have assumed $\langle \hat{P}_{L_{in}}^3 \rangle = 0$ and $\langle \hat{P}_{L_{in}}^2 \rangle = 1/2$). This contains the already-calculated term $\langle \hat{X}_{M_1} \rangle$, and $\langle \hat{P}_{L_k}^3 \rangle$ is found from the data set of homodyne measurements; and so we can now determine $\langle \hat{X}_{M_1}^3 \rangle$. Continuing this iterative procedure, we can use lower-order moments to iteratively find those of higher order.

The full method for finding an arbitrary mechanical moment $\langle \hat{X}_{M_1}^p \hat{P}_{M_1}^q \hat{X}_{M_2}^r \hat{P}_{M_2}^s \rangle$ is outlined in A.1.1, while a simpler experimental approach for finding some of the lower-order moments in D_5 and S_3 is presented in A.1.1. The iterative nature of our verification scheme (using lower-order moments to find higher ones) means that, by determining all the moments for S_3 , one also unlocks all the lower-order moments for D_5 , making it experimentally convenient to calculate both determinants together. We note that, in practice, to confidently measure higher-order moments using Eq. (5.42) and thus unambiguously determine the sign of the determinants D_5 and S_3 , we must ensure a sufficiently large data set of homodyne measurement results $\{P_{L_k}^{(i)}\}_{i=1}^N$ for each optical pathway considered. The size of the data set will affect the standard error of $\langle \hat{P}_{L_k}^d \rangle$ and in turn will affect the overall error on D_5 and S_3 and subsequently our confidence in entanglement verification. These factors are discussed in great detail in Chapter 6. In general, increasing N will reduce this error and consequently mitigate the propagation of errors as we iteratively solve for higher-order moments. We note that unlike pure optical schemes [155], our protocol allows increasingly high-order moments of the bipartite state to be found without requiring additional optical apparatus (only those in Fig. 5.9).

5.3.3 Open-system dynamics

In the verification protocol described in the previous section, we assumed that the mechanical state is isolated from its environment while the verification stage is performed. This idealised version captures the essence of the scheme; however, we now consider the effects that occur from the system interacting with a thermal environment during the time elapsed between state

preparation and the verification stage. We henceforth assume that the two oscillators are in identical thermal environments, and share the same mechanical properties and initial thermal state ($\bar{n}_1 = \bar{n}_2 = \bar{n}$), prior to the entangling pulse. The mechanical quadratures evolve between $\hat{X}_{M_i}$ and $\hat{P}_{M_i}$ and undergo damping and rethermalisation due to their coupling with an external heat bath.

We can find equations of motion for each oscillator by considering the evolution of the dimensionless operator $\hat{X}_{M_i}$ and $\hat{P}_{M_i}$. Inserting these into Eq. (2.34) yields the following quantum Langevin equations

$$\dot{\hat{X}}_{M_i} = \omega_M \hat{P}_{M_i} , \quad (5.43a)$$

$$\dot{\hat{P}}_{M_i} = -\omega_M \hat{X}_{M_i} - \gamma \hat{P}_{M_i} + \sqrt{2\gamma} \hat{\xi}_i , \quad (5.43b)$$

where γ is the damping rate, and $\hat{\xi}_i$ is Brownian force term acting on the i^{th} oscillator which models random excitations from the bath [156]. Note that we have made the substitution $\hat{P}_{\text{in}} = \sqrt{2\gamma} \hat{\xi}$ where $\hat{P}_{\text{in}}$ is defined in Eq. (2.35). We observe that the damping and fluctuations only appear in the equation of motion of the momentum. The Brownian force has the following properties:

$$\langle \hat{\xi}_i(t) \rangle = 0 , \quad (5.44a)$$

$$\langle \hat{\xi}_i(t) \hat{\xi}_j(t') \rangle = (2\bar{n}_B + 1) \delta(t - t') \delta_{ij} , \quad (5.44b)$$

where $\bar{n}_B$ is the thermal occupation number of the bath. In general, $\bar{n} \neq \bar{n}_B$ if cooling strategies are implemented. The time-dependent solutions to these coupled differential equations are:

$$\hat{X}_M(t) = e^{-\gamma t/2} [\cos(\omega_M t) + \epsilon \sin(\omega_M t)] \hat{X}_M(0) + e^{-\gamma t/2} [\sin(\omega_M t) \hat{P}_M(0) + \Delta X_M(t)] , \quad (5.45a)$$

$$\hat{P}_M(t) = e^{-\gamma t/2} [\cos(\omega_M t) - \epsilon \sin(\omega_M t)] \hat{P}_M(0) + e^{-\gamma t/2} [-\sin(\omega_M t) \hat{X}_M(0) + \Delta P_M(t)] , \quad (5.45b)$$

where γ and ω_M are the mechanical damping rate and angular frequency, respectively; and $\epsilon = \gamma/2\omega_M$. We have assumed the optomechanical device has a high quality factor such that

$Q = \omega_M/\gamma \gg 1$. The operators $\Delta X_M(t)$ and $\Delta P_M(t)$ contain random excitations entering from the thermal bath:

$$\Delta X_{M_i}(t) = \sqrt{2\gamma} \int_0^t dt' e^{\gamma t'/2} \sin(\omega_M(t-t')) \hat{\xi}_i(t') , \quad (5.46a)$$

$$\Delta P_{M_i}(t) = \sqrt{2\gamma} \int_0^t dt' e^{\gamma t'/2} \hat{\xi}_i(t') [\cos(\omega_M(t-t')) - \epsilon \sin(\omega_M(t-t'))] \quad (5.46b)$$

where $\hat{\xi}_i$ is the Brownian force on the i^{th} oscillators whose properties are captured in Eqs. (5.44a) and (5.44b).

The influence of the environment will manifest during the time delays between verification pulses and we characterise this effect through $\bar{n}_B$, and the oscillators' quality factors $Q = \omega_M/\gamma$. Since the optomechanical interaction is governed by Eq. (5.41b), the $\hat{X}_M$ quadrature of a particular oscillator is imprinted on the verification pulse, and so we only make use of Eq. (5.45a). For example, the moment $\langle \hat{P}_{M_i} \rangle$ is measured at a time that has allowed the $\hat{X}_{M_i}$ quadrature to evolve by a quarter of a mechanical oscillation (via the use of a delay line). However, in the presence of damping, the time which is equivalent to a quarter of an mechanical cycle is no longer $\tau = \pi/(2\omega_M)$. Instead, it takes time τ' for the $\hat{X}_{M_i}$ quadrature to evolve into the $\hat{P}_{M_i}$ quadrature, where $\tau' = \arctan(-\epsilon^{-1}) + \pi/\omega_M$. In the limit that $\gamma \rightarrow 0$ then $\tau' \rightarrow \pi/(2\omega_M)$ as is expected in the absence of damping. Based on our proposed verification scheme, the value which is measured is in fact $\langle \hat{P}_{M_i} \rangle = \langle \hat{P}_{M_i}(0) \rangle e^{-\gamma\tau'/2} \sin(\omega_M\tau')$, where $\langle \hat{P}_{M_i}(0) \rangle$ is the expectation of the initial momentum quadrature at the time of state generation. The excitations from the bath via the Brownian force appear in moments which contain second order terms of the same mode. For example

$$\langle P_{M_i}^2 \rangle = \langle P_{M_i}^2(0) \rangle e^{-\gamma\tau'} \sin^2(\omega_M\tau') + 2\gamma(2\bar{n}_B + 1)e^{-\gamma\tau'} I(\tau') , \quad (5.47)$$

where

$$I(\tau') = \frac{-\gamma^2 + 4(e^{\gamma\tau'} - 1)\omega_{M_i}^2 + \gamma^2 \cos(2\omega_{M_i}\tau') - 2\gamma\omega_{M_i} \sin(2\omega_{M_i}\tau')}{2(\gamma^3 + 4\gamma\omega_{M_i}^2)} . \quad (5.48)$$

Similar expressions can be found for terms such as $\langle \hat{X}_{M_i} \hat{P}_{M_i} \rangle$ and $\langle \hat{P}_{M_i} \hat{X}_{M_i} \rangle$. These open-system effects have been taken into account for all the matrix elements in D_5 and S_3 throughout the

Table 5.2: Parameter sets of proposed and experimentally-realised values; considered for the parallel entanglement configuration with $\phi = \pi$, where ϕ is the phase in Fig. 5.1a. The two oscillators are characterised by the dimensionless optomechanical coupling strength μ , their quality factor Q , their initial thermal occupation number $\bar{n}$, and the occupation of the thermal bath $\bar{n}_B$. The inseparability criteria D_5 and S_3 have been calculated to incorporate the open-system dynamics which the system undergoes in the time elapsed between state generation and verification (see Section 5.3.3). We assume the following throughout: $\eta = 0.8$ (overall optical intensity efficiency), $\mathcal{D} = 10^{-8}$ (the probability of a single dark count during the detection window; assuming a dark count rate of 1 s^{-1} and a detection window of 10 ns). The fraction $\mathcal{F}$ of true positive $\{1,0\}$ events is shown both for the case of number-resolving photodetectors (Eq. (5.27)) and non-resolving (Eq. (5.30)). The main values of $\mathcal{F}$ are calculated with $\alpha = 1$ (the amplitude of the injected coherent state in the entanglement stage) and the values in brackets show $\mathcal{F}$ maximised by tuning α . (i)-(iv) are theoretically proposed parameters. Experiment-inspired parameter sets are used based on Refs [151], a micro-mechanical membrane; [141], a nanobeam oscillator; and [152], a sliced photonic-crystal structure. The rows corresponding to each of these three references are divided into two sub-rows: the first sub-row contains published protocol parameters, in the second-row we use values for $\bar{n}$ and $\bar{n}_B$ based on near-future improvements (involving additional cooling techniques) of existing systems. For a pulse which satisfies $\omega_M \ll \tau^{-1} \ll \kappa$, it follows that $\mu = 2\sqrt{2}g_0/\kappa$ (for example experimental parameters see Table 5.1). *These $\bar{n}_B$ values have been calculated assuming the thermal bath has been cooled to 100 mK.

Refs.	Protocol parameters				Determinants		True positives $\mathcal{F}$ [%]	
	μ	Q	$\bar{n}$	$\bar{n}_B$	D_5	S_3	Resolving	Non-resolving
(i)	10^{-3}	10^5	0.1	1000	0.56	-0.080	79% (84%)	79% (84%)
(ii)	10^{-2}	10^5	0.1	1000	0.56	-0.080	82% (98%)	82% (98%)
(iii)	10^{-1}	10^5	0.1	1000	0.56	-0.080	82% (> 99%)	82% (> 99%)
(iv)	10^0	10^5	0.1	1000	0.54	-0.029	82% (> 99%)	66% (> 99%)
[151]	2.26×10^{-5}	1.03×10^9	0.29	10^5	1.4	0.084	1.3% (2.8%)	1.3% (2.8%)
	2.26×10^{-5}	1.03×10^9	0.1	10^{3*}	0.51	-0.089	0.99% (2.1%)	0.99% (2.1%)
[152]	1.29×10^{-4}	7.54×10^5	5.3	2.1×10^4	3500	420	65% (65%)	65% (65%)
	1.29×10^{-4}	7.54×10^5	0.1	484*	0.51	-0.089	23% (30%)	23% (30%)
[141]	1.12×10^{-2}	3.74×10^4	1.7×10^4	1.7×10^4	1.3×10^{17}	6.3×10^{12}	82% (> 99%)	60% (> 99%)
	1.12×10^{-2}	3.74×10^4	0.1	559*	0.58	-0.074	82% (98%)	82% (98%)

results presented in Figs. 5.10a, 5.10b, 5.12, and Table 5.2.

5.4 Results

In this section we present our theoretical model results for the verification of entanglement between two optomechanical oscillators which have been prepared in the state $\rho_{\text{out}}^{\{1,0\}}$ using the parallel set-up shown in Fig. 5.1a. The oscillators are initialized in a separable thermal state

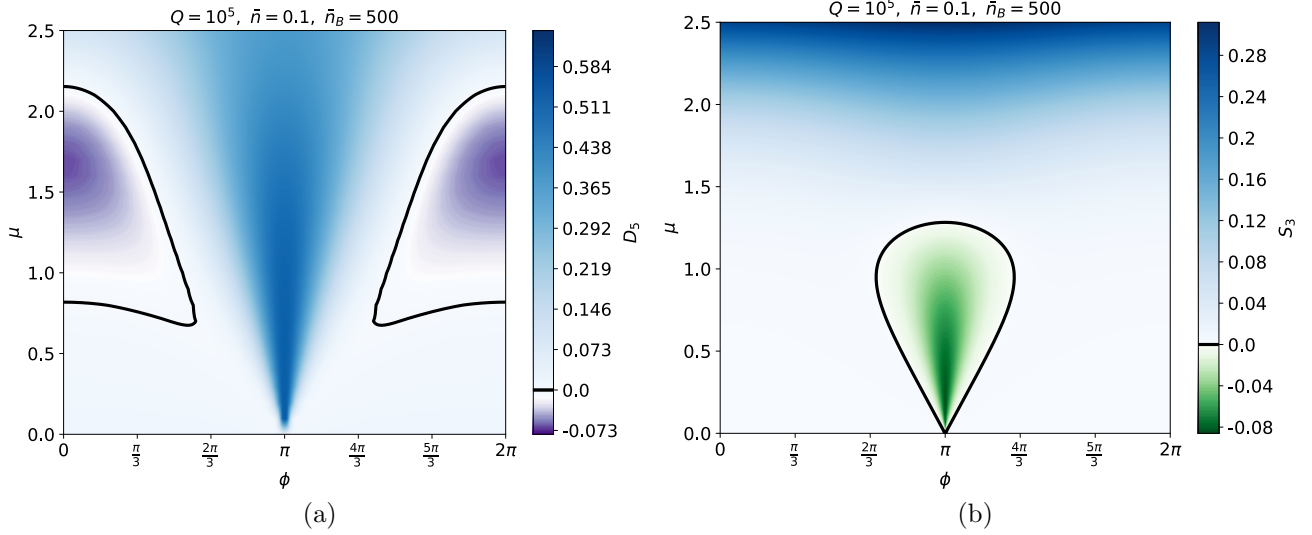


Figure 5.10: Inseparability criteria outcomes as obtained via the verification method for the state $\rho_{\text{out}}^{\{1,0\}}$ produced via the parallel configuration (Fig. 5.1a). Plots of D_5 (a) and S_3 (b) as functions of the interferometer phase ϕ , and the optomechanical coupling strength μ . The open-system dynamics predicted to occur while measuring the determinants (see Fig. 5.9) is parameterised by $Q = 10^5$, $\bar{n} = 0.1$, and $\bar{n}_B = 500$; for a summary of the experimental parameters see Table 5.2. Both determinants are periodic with ϕ and repeat every 2π . The black line demarcates regions where determinants are negative therefore indicating regions where the state is found to be entangled. All Gaussian entangled states fall within the purple region of (a). Non-Gaussian entangled states are captured by the Green region of (b) since it does not overlap with the purple region of (a).

ρ_{in} with equal $\bar{n}$. For a summary of the key experimental parameters used to calculate the results see Table 5.2. For an example experimental parameter set that goes into Table 5.2 see Table A.1 in A.2.

Firstly we present our results for the inseparability criteria D_5 (Eq. (5.39)) and S_3 (Eq. (5.40)). The moments for these criteria were calculated using the open-system dynamics outlined in Section 5.3.3. This model allows us to simulate the loss of energy to and random excitations from the environment that would arise in an experimental implementation of the verification scheme outlined in Fig. 5.9. As previously mentioned, we assume that both oscillators are initially identical with the same quality factor Q , thermal mechanical occupation number $\bar{n}$, and bath occupation number $\bar{n}_B$.

Figs. 5.10a and 5.10b show D_5 and S_3 , respectively, as functions of the experimentally controlled phase ϕ , and the optomechanical coupling μ . We have chosen $Q = 10^5$, $\bar{n}_B = 500$, and $\bar{n} = 0.1$. The state $\rho_{\text{out}}^{\{1,0\}}$ has a phase periodicity of 2π in ϕ , which is reflected in the determinants. The

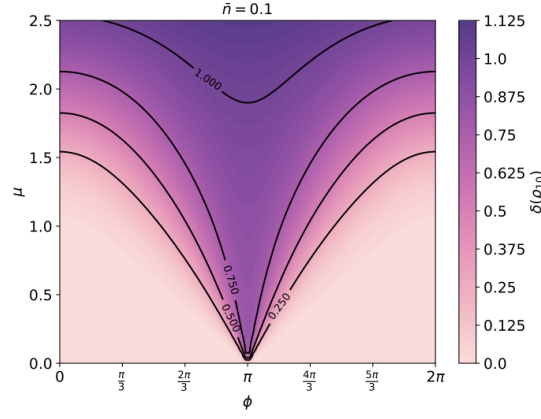


Figure 5.11: Measure of non-Gaussianity computed on the mechanical state $\rho_{\text{out}}^{\{1,0\}}$ created via the entanglement scheme in Fig. 5.1a. The mechanical oscillators have initial thermal occupation number $\bar{n} = 0.1$. The state becomes more non-Gaussian as $\phi \rightarrow \pi$ or as μ is increased. We note that parameter regions which are more Gaussian qualitatively correlate with the negative contours in Fig. 5.10a. Regions which are more non-Gaussian are centred around the same phase ($\phi = \pi$) as the negative contours in Fig 5.10b.

parameter regions for which entanglement can be verified using D_5 and S_3 are indicated by the negative values and demarcated by solid black lines. For D_5 , the negative region is centred at $\phi = \{0, 2\pi\}$ while for S_3 , it is concentrated around $\phi = \pi$. We reiterate that it is the sign of the determinants, not their magnitude which indicates entanglement. Nevertheless, a more negative determinant could be more resistant to experimental errors. That is, for successful verification, the magnitude of a negative determinant should be greater than the uncertainty of the value.

To better understand the features of Figs. 5.10a and 5.10b we examine the non-Gaussianity of the state $\rho_{\text{out}}^{\{1,0\}}$ in terms of a non-Gaussian quantifier $\delta(\rho_{\text{out}}^{\{1,0\}})$ defined in Eq. (??). Since δ is an exact measure of non-Gaussianity [60], a greater δ suggests a greater degree of non-Gaussianity, while $\delta = 0$ implies our state is exactly Gaussian. However, the measure has no upper bound (there is no maximally non-Gaussian state) so we can only comment on a state being more non-Gaussian than another.

While we do not propose an experimental way to obtain δ , the measure $\delta(\rho_{\text{out}}^{\{1,0\}})$ is shown in Fig. 5.11 as a function of μ and ϕ for $\bar{n} = 0.1$. For simplicity, we have computed the non-Gaussianity of the heralded state before verification has been conducted (i.e. for a closed system). As $\mu \rightarrow 0$,

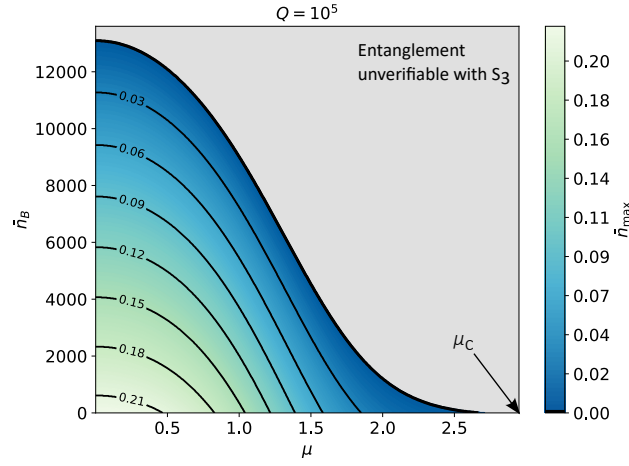


Figure 5.12: Contour plot of the maximum oscillator thermal occupation $\bar{n}_{\max}$ for which S_3 can verify entanglement. This maximum is plotted with μ and $\bar{n}_B$ for an oscillator of $Q = 10^5$. The phase in the entanglement stage is set to $\phi = \pi$. The gray region corresponds to the parameter space for which S_3 is unable to verify entanglement even at $\bar{n} = 0$. Note, for coupling strengths greater than μ_c , the scheme is unable to verify entanglement using S_3 even at $\bar{n} = \bar{n}_B = 0$ due to the interactions with the environment present with finite γ .

the state becomes more Gaussian; physically at $\mu = 0$ there is no optomechanical interaction, therefore the state $\rho_{\text{out}}^{\{1,0\}}$ is comprised of two separable, fully Gaussian thermal states. For a given μ , the state becomes more Gaussian as $\phi \rightarrow 0, 2\pi$; likewise, the negative regions of D_5 where entanglement can be verified are centred at $\phi = 0, 2\pi$. This is in line with the established notion that D_5 can detect all entanglement in bipartite Gaussian states. In contrast, for a given μ the state becomes increasingly non-Gaussian as $\phi \rightarrow \pi$. Comparing Figs. 5.10b and 5.11, we see that S_3 performs best around the values of ϕ that correspond to higher non-Gaussianity. However, in high μ regimes, we are unable to verify entanglement with S_3 despite the state becoming more non-Gaussian as μ increases (we discuss this feature in Section 5.5.1).

The ability to verify entanglement is strongly dependent on the mechanical initial thermal occupation $\bar{n}$. This can be reduced by lowering the temperature of the oscillator using laser cooling techniques and/or cryogenics. Here we determine the levels of cooling (in terms of $\bar{n}$) required for entanglement to be verified. We are specifically interested in non-Gaussian entanglement and therefore investigate how high we can allow $\bar{n}$ to be while still achieving a negative S_3 . For this analysis, the phase in the entanglement stage is fixed to $\phi = \pi$ (at which point the mechanical state is most non-Gaussian). For a given μ , $\bar{n}_B$, and Q , the oscillators can take a range of thermal occupation numbers between $0 < \bar{n} < \bar{n}_{\max}$ over which S_3 is still

negative. When $\bar{n} = \bar{n}_{\max}$, $S_3 = 0$ and we can no longer verify entanglement. The maximum mechanical occupation number $\bar{n}_{\max}$ is shown as a function of μ and $\bar{n}_B$ in Fig. 5.12 for $Q = 10^5$. As expected, in order to compensate for an increasing bath temperature one must pre-cool the oscillator more. The gray region in Fig. 5.12 bounded by the black line designates the values of μ and $\bar{n}_B$ for which S_3 is not capable of verifying entanglement, even as the oscillator approaches the ground state ($\bar{n} \rightarrow 0$). We note that beyond a cut-off value of $\mu_c \approx 3.6$, S_3 cannot verify entanglement regardless of bath temperature and oscillator cooling. This does not preclude anything about the entanglement structure of the state at higher μ but would suggest S_3 is more suitable for lower μ regimes (this behaviour is further discussed in Section 5.5.1). In Table 5.2, we give the values of D_5 and S_3 calculated for seven chosen parameter sets (four theoretically proposed and three taken from state-of-the-art experiments). The experiments that we consider operate in the unresolved sideband regime and so are suitable for our protocol. For the parameter sets taken from these experiments, we include both published parameters and suggested near-future improvements to the oscillator occupation number $\bar{n}$, and the thermal bath occupation number $\bar{n}_B$. Comparing the entries in the S_3 column of Table 5.2, we see that a negative S_3 is achievable with modest amounts of additional cooling.

The final two columns of Table 5.2 give the fraction $\mathcal{F}$ of true positive $\{1,0\}$ clicks for the corresponding experimental parameters. The ‘resolving’ photodetectors column is calculated using Eq. (5.27) while the ‘non-resolving’ column uses Eq. (5.30). In the latter equation, we must calculate a sum to infinity over the index m . When numerically computed, the sum over the m^{th} index is truncated at a sufficiently high m such that higher order terms are negligible. Physically, m corresponds to the number of photons at the detector, and since the entangling pulse is a weak coherent state with low mean photon number it is reasonable to neglect very high m .

We have mentioned already that setting the amplitude of the entangling pulse to be $|\alpha| = 1$ maximises the probability P_{10} of heralding the state $\rho_{\text{out}}^{\{1,0\}}$ (Eq. (5.15)), therefore the main set of results in these two columns assumes $\alpha = 1$. However, in order to be confident that we have successfully heralded state $\rho_{\text{out}}^{\{1,0\}}$, we require $\mathcal{F}$ to be as high as possible. We can experimentally tune α by attenuation, and so the values of $\mathcal{F}$ parentheses have been optimised, having been

calculated with the value of α that maximises Eqs. (5.27) and (5.28), with all other variables in the equations kept constant.

5.5 Discussion

5.5.1 Verification of non-Gaussian entanglement

Here we discuss the interplay of the non-Gaussian nature of the state and our ability to detect entanglement. As established in Section 5.3.1, the equivalence between $D_5 < 0$ and Simon's criterion provides a necessary and sufficient condition for the existence of entanglement in Gaussian states. It must also be noted though that $D_5 < 0$ is only a sufficient criterion for entanglement in non-Gaussian states. As different moments require different time delays between verification pulses they are affected by open-system dynamics to differing degrees. If there is a high degree of environmental interactions across the time-scales we consider this could impede our ability to identify all Gaussian entanglement through use of D_5 . A sufficiently high quality factor, and low bath occupation $\bar{n}_B$, ensure that the system is well-isolated from the environment, meaning that a region of state space where S_3 is negative but D_5 is positive corresponds to states which are guaranteed to be non-Gaussian. On the other hand, if the consequences of an open system do prevent us from identifying Gaussian entanglement with D_5 , then we can no longer assume merely from studying D_5 and S_3 that the state is entangled and non-Gaussian. This ambiguity could potentially be resolved using post-processing mitigation schemes that correct for the effects of noise and open-system dynamics on measured moments [157, 158]. Within the parameter set we have explored in Figs. 5.10a and 5.10b, the region for which $S_3 < 0$ does not overlap with that where $D_5 < 0$. As the quality factor considered is large ($Q = 10^5$) we can be confident that all the states which fall in the negative (Green) region of Fig. 5.10b are non-Gaussian, and this is confirmed by the non-Gaussianity measure shown in Fig. 5.11.

As discussed in Section 5.4, both the regions of high non-Gaussianity and negative S_3 are

centred at $\phi = \pi$. However, we are unable to verify entanglement at higher values of μ using S_3 (Fig. 5.10b), despite the measure $\delta(\rho_{\text{out}}^{\{1,0\}})$ suggesting that non-Gaussianity increases with μ (Fig. 5.11). This is reinforced by Fig. 5.12 which suggests that there is a cut-off μ_c beyond which S_3 fails regardless of cooling (even when $\bar{n} = \bar{n}_B = 0$).

To aid our understanding of the behaviour of S_3 in the higher μ regime, we can examine the analytic formula for S_3 calculated for the idealised state $\rho_{\text{out}}^{\{1,0\}}$ with $\bar{n} = 0$ (defined in Eq. (5.7a)). Assuming a closed system, S_3 is given by:

$$S_3 = -\frac{\mu^6 e^{-\mu^2}}{64(1 + e^{-\frac{\mu^2}{2}} \cos \phi)^3}. \quad (5.49)$$

Notably, this expression is negative for all choices of μ and ϕ , approaching 0 from below as $\mu \rightarrow \infty$. Also note that, Eq. (5.49) counter-intuitively suggests that at $\phi = \pi$, S_3 tends to $-1/8$ as $\mu \rightarrow 0$ which indicates that there is entanglement even when there is no optomechanical coupling. However, this must be considered in conjunction with the heralding probability in Eq. (5.15) which tends to 0 as $\mu \rightarrow 0$, meaning that such a state cannot be created.

From studying this ground state behaviour (for a closed system), we see that the magnitude of S_3 decays exponentially with increasing μ . Therefore, in the regime where we consider the effects of $\bar{n}$ and open-system dynamics, at greater values of μ the possible negative contribution from entanglement detection is too small to compete with these effects causing S_3 to become positive. Even at $\bar{n} = \bar{n}_B = 0$ there are interactions with the environment present due to finite γ , thus for coupling strengths greater than μ_c , the scheme is unable to verify entanglement using S_3 . Indeed, as Q is improved to values greater than 10^5 , the gray region of Fig. 5.12 shrinks: μ_c increases beyond 3.6, and entangled states in environments of even higher $\bar{n}_B$ can be verified using S_3 . This analysis suggests our protocol is better suited to parameter regimes where μ is up to approximately unity, which also corresponds to what is experimentally accessible at present. The experimentally viable systems we have considered [151, 152, 141] (see Table 5.2) are well within the range of μ where S_3 captures entanglement (Fig. 5.12).

In addition to open-system dynamics, additional mechanical modes that couple to the optical

field within the bandwidth of the pulsed interaction will lead to unwanted correlations between the mechanical mode of interest and other mechanical modes, and affect the non-Gaussian entanglement criteria used here. In the pulsed optomechanics literature, this unwanted effect was qualitatively noted in Ref. [93] and then such contributions were experimentally investigated for pulsed position measurements [81, 82]. This type of multimode contribution was also an experimental factor and carefully studied in Ref. [125]. These unwanted contributions may be minimized by engineering the optical and mechanical mode profiles to provide negligible coupling to unwanted mechanical modes, which has, for instance, been used successfully for photonic crystal devices [159]. Additionally, optical trapping based approaches to optomechanics provide excellent coupling to the centre-of-mass motion with few other modes contributing. Moreover, co-levitation [160, 124] provides a promising means to implement the ‘series’ optical circuit we have in Fig. 5.1b. Lastly, we would like to note that, rather than optimizing the interaction to couple to a single mechanical mode per device, one can utilize this type of pulsed interaction to entangle multiple mechanical modes of a single device. In this direction, bipartite Gaussian entanglement has been explored [126], and this proposal can also provide tools and techniques for studies of non-Gaussian entanglement in such a single mechanical device regime.

5.5.2 Optical effects

The following discussion addresses the effect of optical imperfections for the coherent state input case $|\Phi\rangle_L = |\alpha\rangle_1 |0\rangle_2$ (as opposed to the single-photon input case which provides more resilience to the effects considered). Optical losses, dark counts, and the entangling pulse amplitude do not directly affect the values of the moments used to calculate D_5 and S_3 , however they do have experimental implications. For example, the entangling pulse amplitude α dictates the probability of heralding the desired mechanical state $\rho_{\text{out}}^{\{1,0\}}$, see Eq. (5.15). In the presence of dark counts, the fraction of false positives also depends on α , as well as on η , and $\mathcal{D}$. These false positive events result in mixing of the desired state with the initial state and higher photon-number contributions. This could introduce errors in the measurement of the moments for D_5 and S_3 , which can be readily mitigated by operating in a parameter regime where $\mathcal{F}$ is close to

unity ($>\approx 95\%$).

The rows (i) to (iv) of Table 5.2 show that $\mathcal{F}$ is highly dependent on μ . For photon-number-resolving detectors, $\mathcal{F}$ improves with increasing μ (keeping other experimental parameters constant). From Eq. (5.27) we see that as μ decreases, so does the heralding probability P_{10} (with $\phi = \pi$ in Eq. (5.15)), meaning that the desired state $\rho_{\text{out}}^{\{1,0\}}$ is less likely to be created. Once P_{10} is small enough that it is comparable to the dark count rate $\mathcal{D}$, dark counts become significant, thus reducing $\mathcal{F}$. By maximising Eq. (5.27) with respect to α , we can compensate for this reduction in $\mathcal{F}$, as demonstrated by the values in parentheses. For the same reasons as the photon-number-resolving case, the reduction in $\mathcal{F}$ for small μ is also present for non-resolving photodetectors. However, non-resolving photodetectors suffer an additional drawback and thus never outperform resolving photodetectors: as μ is increased the probability of a multi-photon interaction grows more quickly than one involving a single photon. As a non-resolving detector cannot distinguish between these and a true $\{1,0\}$ click, this reduces $\mathcal{F}$ compared with a resolving detector (this is demonstrated in row (iv)). One can find the optimal $\mathcal{F}$ (shown in brackets in Table 5.2) by adjusting α in Eq. (5.30) to balance the competing effects of dark counts and multi-photon events. In a similar manner to μ , increasing $\bar{n}$ both reduces P_{10} (for $\phi = \pi$) and increases the probability of multi-photon interactions. This means that the qualitative effect of $\bar{n}$ on $\mathcal{F}$ can be explained by the same arguments as for μ .

5.6 Conclusion

We have proposed a scheme to generate two-mode mechanical Schrödinger-cat states using pulsed nonlinear optomechanical interactions in conjunction with a photon-counting heralding scheme. The heralding scheme is based on an optical interferometer set-up, with a variable phase that is used to control the form of the entanglement generated. To verify the presence of entanglement we have introduced an experimental protocol which exploits subsequent pulsed interactions and measurements in order to obtain moments of the bipartite mechanical state. Inseparability criteria can be computed from these moments and we have considered the D_5

and S_3 criteria. When used together, the two criteria allow us to identify non-Gaussian entangled states. To assess the feasibility of the protocol we have included the key experimental factors including optical losses, detection inefficiency, and dark counts, as well as open-system dynamics.

Our findings indicate that the protocol presented here provides a realistic means of generating non-Gaussian entanglement between two mechanical oscillators. The inseparability criterion $S_3 < 0$ can verify entanglement in parameter regimes accessible in state-of-the-art experiments with only modest additional cooling required. While the influence of the environment degrades the entanglement, we have demonstrated that for realistic experimental parameters, including a low optomechanical coupling strength, non-Gaussian mechanical entanglement may still be generated and verified.

Furthermore, the experimental verification scheme proposed here enables the measurement of bipartite mechanical moments of arbitrarily high order. While we have focused on extracting quadrature moments used for the two suggested inseparability criteria, any entanglement or non-classicality test that relies on measuring higher-order moments [161, 155, 162] can be applied without making any changes to our set-up.

In this chapter we specifically used S_3 as a proof-of-principle to demonstrate the feasibility of our entanglement verification scheme in detecting non-Gaussian entanglement. Nevertheless, we have emphasised throughout that this verification protocol can be extended to a plethora of other NPT criteria constructed from bipartite moments of the state. In the following chapter we will investigate whether there are other matrices that can outperform S_3 for this specific entangled state. More broadly, we will identify factors which influence the choice of a suitable NPT criterion.

Chapter 6

Constructing entanglement criteria for bipartite non-Gaussian states

The work described in this chapter was conducted in collaboration with Jack Clarke, Sofia Qvarfort and Michael R. Vanner. Some of the work in this chapter is currently preparation for peer-review.

6.1 Introduction

In continuous variable systems, entangled non-Gaussian states are of crucial importance in wide-ranging applications such as enhancing entanglement in teleportation schemes [74], and Bell tests of nonlocality. Nevertheless, choosing a suitable entanglement criterion for a non-Gaussian state can be a formidable task. Entanglement can be detected by the application of all positive but not completely positive (PNCP) maps. To this end, methods of constructing optimized entanglement witnesses have been proposed [33, 163]. Though an entanglement criterion may be guaranteed to detect entanglement in a given state, if experimental verification is too resource-intensive then this must be factored into the utility of this criterion. Moreover, the experimental uncertainty on a criterion may lead to an inconclusive entanglement test. One such PNCP map is the partial transposition discussed at length in Chapter 3. We remind the reader

that bipartite separable quantum states possess a positive partial transpose (PPT) [34, 35]. Entanglement criteria which aim to detect any violation of the PPT requirement of a separable state are especially useful as they are valid for both pure and mixed states. Such criteria are interchangeably referred to as negative partial transpose (NPT) tests.

In contrast to Gaussian states, which only require second-order moments to detect entanglement, identifying a suitable NPT criterion for non-Gaussian states can be challenging as we must consider higher-order correlations. The entire class of NPT tests are encompassed by the Vogel and Shchukin (VS) hierarchy discussed in detail in Section 3.2. Subsets of the VS hierarchy have been applied to specific non-Gaussian states. For example, the authors in Ref. [164] evaluate higher-order Hillery-Zubairy criteria in order to detect entanglement in two-mode squeezed number states. The authors benchmark the success of a criterion against the von Neumann entropy of entanglement, made possible by fact that these are pure states. Using another subset of the VS criteria which are equivalent to high-order covariance matrices, the authors in Ref. [162] have proposed systematic frameworks for assessing PPT in states generated from nonlinear Hamiltonians, such as those that generate partially degenerate N-mode spontaneous parametric down conversion. These works demonstrate that higher-order correlations play a key role in entanglement tests for highly non-Gaussian states. Nevertheless, even if a suitable criterion is identified for a specific non-Gaussian state, we must also consider the interplay with experimental feasibility since the criterion may require a prohibitively large number of measurements to evaluate it with a high degree of certainty. Given that there are an infinite number of potential NPT inequalities to explore it is likely that more than one inequality will register entanglement in a given state. Therefore, it is crucial to also assess the ability of an NPT condition to unambiguously evaluate entanglement even in the face of experimental factors such as losses to the environment.

In the first half of this chapter, we will outline a statistical framework for quantifying the uncertainty on any NPT criterion which exists in the infinite VS hierarchy. This will build on the entanglement theory introduced in Section 3.2 and will use the statistical theory concerning samples which is covered in Chapter 4. We will shortly explore each assumption in mathematical detail; however, here we will very broadly outline the methodology used to quantify the

uncertainty of an NPT criterion. The methodology consists of five steps:

1. The NPT criteria are constructed by finding the determinant of a Hermitian matrix of bipartite moments such as in Eq. (3.8). We assume each moment is estimated by repeatedly measuring the quantum state (which is recreated between runs) and taking the appropriate statistical average over the measurements; we adopted a similar approach for calculating moments from measurement data in Section 5.3.2 cf. Eq.(5.42).
2. The mean of these measurements is thus subject to sampling errors which propagate into the inequality itself. Sample errors arise when we estimate a population parameter via the use of a sample.
3. These statistical considerations together with losses to the environment allow us to calculate an uncertainty associated with an NPT inequality.
4. With this uncertainty, our method provides a confidence level in rejecting the no-entanglement hypothesis for a specific NPT criterion.
5. By searching through all possible criteria (matrices) we can identify state-dependent NPT criteria which offer the most experimentally-feasible route to entanglement detection. Assuming experimental resources are finite, the most experimentally feasible NPT criterion ideally has the smallest number of total measurements for a given confidence level.

We apply our statistical framework to two states: (i) the two-mode Schrödinger-cat state, and (ii) the phonon subtracted two-mode squeezed vacuum state. This enables us to determine the most suitable NPT test for a given set of state parameters.

In the second half of the chapter, we examine how one might experimentally measure an arbitrary two-mode moment of optical fields. Measuring these moments is crucial in practical applications of the NPT entanglement tests which we identify in this chapter. We examine previously proposed schemes and also identify an alternative measurement scheme based on homodyne detection which is potentially experimentally straightforward especially for lower-order moments.

6.2 Entanglement criteria

Here we will reiterate the key results laid out in Section 3.2 and establish our notation. First, consider an infinite-dimensional square matrix $\mathbf{M}$ whose entries consist of expectation values of operators of a bipartite state $\rho_{\mathcal{AB}}$, also called moments. The matrix elements given by indices i and j are

$$\mathbf{M}_{ij} = \langle \hat{a}^{\dagger q_i} \hat{a}^{p_i} \hat{a}^{\dagger n_j} \hat{a}^{m_j} \hat{b}^{\dagger l_j} \hat{b}^{k_j} \hat{b}^{\dagger r_i} \hat{b}^{s_i} \rangle, \quad (6.1)$$

where $\hat{a}$ and $\hat{b}$ are the annihilation operators associated with subsystem $\mathcal{A}$ and $\mathcal{B}$, respectively. The multi-indices i and j are discussed in Section 3.2 and we will define the ordering of the multi-indices shortly.

The main minors $\mathbf{M}_N$ are constructed by keeping the first N rows and N columns of $\mathbf{M}$. The state $\rho_{\mathcal{AB}}$ has a positive partial transpose if and only if $\det[\mathbf{M}_N] \geq 0$ for all N . Conversely, if $\det[\mathbf{M}_N] < 0$ for any N , then the state has a negative partial transpose and is thus inseparable. Since swapping rows and columns in a matrix in a pairwise way leaves the determinant invariant, we can more generally consider the submatrix $\mathbf{A}$ which is constructed by deleting rows and columns of $\mathbf{M}$ in a pairwise fashion (see Figure 3.1). If $\det[\mathbf{A}] < 0$ this also indicates NPT and so is a sufficient criterion to demonstrate entanglement. Therefore, we consider the determinant of matrix $\mathbf{A}$ to be a more general way of evaluating individual NPT criteria within the VS hierarchy as these also encompass the leading principle minors $\det[\mathbf{M}_N]$.

6.2.1 Characterizing determinants

Finding suitable NPT tests, and indeed a systematic method, for successfully identifying entanglement in an arbitrary non-Gaussian state can be challenging since it is not always obvious which moments one should consider. Nevertheless, by considering the entire hierarchy of criteria generated by all possible $\mathbf{A}$ and testing the negativity of $\det[\mathbf{A}]$ we have access to a necessary and sufficient NPT test for any bipartite state (Gaussian or not)¹. All NPT tests, which

¹A negative partial transpose is a sufficient entanglement test for any state however not all entangled states possess negative eigenvalues. These bound entangled states are briefly discussed in Section 3 but we do not concern ourselves with this class of states here.

are only sufficient, are contained within this hierarchy. We could theoretically calculate every possible $\mathbf{A}$ for a given state. However this must be done in a systematic way that incorporates errors and experimental factors which impact the overall success of entanglement detection.

The matrix $\mathbf{A}$ is generated by deleting rows and columns from the matrix $\mathbf{M}$ in a pairwise way; therefore $\mathbf{A}$ can be fully described by the indices of the rows (and columns) that remain after this deletion. See Fig. 3.1 for a visual representation of this process. An important parameter in our characterization of $\mathbf{A}$ is therefore its matrix dimensions d . It is desirable for d to be as small as possible as it means fewer moments must be measured in total. Consequently, we anticipate (and will later prove) that the overall error on $\det[\mathbf{A}]$ reduces with d . Furthermore, it is favourable to keep the order of the moments which appear in $\mathbf{A}$ as low as possible. For example, examining the moment in Eq. (6.1) we define its order as $q_i + p_i + n_j + m_j + l_j + k_j + r_i + s_i$. From an experimental perspective, measuring higher-order moments is more challenging as it may require a more elaborate measurement procedure. Therefore, we introduce the parameter n which is defined as the order of the highest-order moment in $\mathbf{A}$

$$n = \max[q_i + p_i + n_j + m_j + l_j + k_j + r_i + s_i] . \quad (6.2)$$

In Section 6.5 we will search through all suitable $\mathbf{A}$ which are able to detect entanglement for specific states, while keeping d and n constrained.

Multi-indices

The matrix $\mathbf{M}$ has entries dictated by Eq. (6.1). The indices i and j are the ordinal numbers of two ordered sets of multi-indices. The precise ordering is arbitrary but once it is picked it is consistent across the two sets of multi-indices. The first few ordered multi-indices according to the ordering we have adopted are listed in Table 6.2.1.

We can generate entanglement criteria by deleting rows and columns of $\mathbf{M}$ in a pairwise fashion to leave behind a submatrix $\mathbf{A}$ with dimensions $d \times d$. Therefore, we can denote matrix $\mathbf{A}$ as a vector of the rows (or, equivalently, columns) of $\mathbf{M}$ which we have chosen to keep. We have

Table 6.1: The ordering convention we have adopted to generate the indices of the matrix $\mathbf{M}$ defined in Eq. (6.1). We have only listed $i, j \leq 15$ as these indices are sufficient to specify all moments which have a maximum order less than or equal to 4. Any submatrix $\mathbf{A}$ generated by keeping rows and columns of $\mathbf{M}$ up to $i, j \leq 15$ will have $n \leq 4$. Including $i, j > 15$ will generate higher order moments.

i	(p_i, q_i, r_i, s_i)	j	(n_j, m_j, k_j, l_j)
1	(0, 0, 0, 0)	1	(0, 0, 0, 0)
2	(1, 0, 0, 0)	2	(1, 0, 0, 0)
3	(0, 1, 0, 0)	3	(0, 1, 0, 0)
4	(0, 0, 1, 0)	4	(0, 0, 1, 0)
5	(0, 0, 0, 1)	5	(0, 0, 0, 1)
6	(2, 0, 0, 0)	6	(2, 0, 0, 0)
7	(1, 1, 0, 0)	7	(1, 1, 0, 0)
8	(0, 2, 0, 0)	8	(0, 2, 0, 0)
9	(1, 0, 1, 0)	9	(1, 0, 1, 0)
10	(0, 1, 1, 0)	10	(0, 1, 1, 0)
11	(0, 0, 2, 0)	11	(0, 0, 2, 0)
12	(1, 0, 0, 1)	12	(1, 0, 0, 1)
13	(0, 1, 0, 1)	13	(0, 1, 0, 1)
14	(0, 0, 1, 1)	14	(0, 0, 1, 1)
15	(0, 0, 0, 2)	15	(0, 0, 0, 2)

chosen this precise ordering for i and j because it enables us to systematically increase the order of the moments in $\mathbf{M}$ as i and j are increased. For example, restricting ourselves to considering only $i, j \leq 5$ gives us matrices $\mathbf{A}$ made of only first and second moments. In our notation, this gives us matrices characterised by $n \leq 2$. If $i, j \leq 15$ we can generate all matrices $\mathbf{A}$ which contain moments up to and including fourth order $n \leq 4$, but no higher moments will feature. Concretely, the full 15×15 matrix will contain every possible first, second, third, and fourth order moment.

Using the ordering in Table 6.2.1, we can generate the first few row and columns of the matrix

M

$$\mathbf{M} = \begin{pmatrix} 1 & \langle \hat{a}^\dagger \rangle & \langle \hat{a} \rangle & \langle \hat{b} \rangle & \langle \hat{b}^\dagger \rangle & \langle \hat{a}^{\dagger 2} \rangle & \langle \hat{a}^\dagger \hat{a} \rangle & \dots \\ \langle \hat{a} \rangle & \langle \hat{a} \hat{a}^\dagger \rangle & \langle \hat{a}^2 \rangle & \langle \hat{a} \hat{b} \rangle & \langle \hat{a} \hat{b}^\dagger \rangle & \langle \hat{a} \hat{a}^{\dagger 2} \rangle & \langle \hat{a} \hat{a}^\dagger \hat{a} \rangle & \dots \\ \langle \hat{a}^\dagger \rangle & \langle \hat{a}^{\dagger 2} \rangle & \langle \hat{a}^\dagger \hat{a} \rangle & \langle \hat{a}^\dagger \hat{b} \rangle & \langle \hat{a}^\dagger \hat{b}^\dagger \rangle & \langle \hat{a}^{\dagger 3} \rangle & \langle \hat{a}^{\dagger 2} \hat{a} \rangle & \dots \\ \langle \hat{b}^\dagger \rangle & \langle \hat{a}^\dagger \hat{b}^\dagger \rangle & \langle \hat{a} \hat{b}^\dagger \rangle & \langle \hat{b} \hat{b}^\dagger \rangle & \langle \hat{b}^2 \rangle & \langle \hat{a}^{\dagger 2} \hat{b}^\dagger \rangle & \langle \hat{a}^\dagger \hat{b} \hat{b}^\dagger \rangle & \dots \\ \langle \hat{b} \rangle & \langle \hat{a}^\dagger \hat{b} \rangle & \langle \hat{a} \hat{b} \rangle & \langle \hat{b}^2 \rangle & \langle \hat{b}^\dagger \hat{b} \rangle & \langle \hat{a}^{\dagger 2} \hat{b} \rangle & \langle \hat{a}^\dagger \hat{a} \hat{b} \rangle & \dots \\ \langle \hat{a}^2 \rangle & \langle \hat{a}^2 \hat{a}^\dagger \rangle & \langle \hat{a}^3 \rangle & \langle \hat{a}^2 \hat{b} \rangle & \langle \hat{a}^2 \hat{b}^\dagger \rangle & \langle \hat{a}^2 \hat{a}^{\dagger 2} \rangle & \langle \hat{a}^2 \hat{a}^\dagger \hat{a} \rangle & \dots \\ \langle \hat{a}^\dagger \hat{a} \rangle & \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \rangle & \langle \hat{a}^\dagger \hat{a}^2 \rangle & \langle \hat{a}^\dagger \hat{a} \hat{b} \rangle & \langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \rangle & \langle \hat{a}^\dagger \hat{a} \hat{a}^2 \rangle & \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (6.3)$$

We can assure ourselves that taking just the first 5 rows and columns leaves the minor matrix $\mathbf{M}_5$ which contains all possible first and second order moments. Therefore, this submatrix gives us a necessary and sufficient criterion for Gaussian states: an entangled Gaussian state is guaranteed to have $\det[\mathbf{M}_5] < 0$ due to its equivalence with Simon's criterion [36].

In Table 6.2.1, we have only listed $i, j \leq 15$ as these indices are sufficient to specify any submatrix $\mathbf{A}$ with $n \leq 4$. Including $i, j > 15$ will generate higher-order moments which are more challenging to experimentally measure. We will also demonstrate that higher-order moments can suffer more greatly from losses to the environment. Nevertheless, we emphasise that the error propagation method we outline in this chapter can be extended to arbitrarily high i and j but at the cost of increased experimental complexity and thus sampling error. And so, for the purposes of demonstrating our statistical framework we have limited the search to $n \leq 4$.

Example

We illustrate these notation conventions with the following example. It has been noted [21, 153] that the following determinant

$$D_V = \begin{vmatrix} 1 & \langle \hat{b}^\dagger \rangle & \langle \hat{a} \hat{b}^\dagger \rangle \\ \langle \hat{b} \rangle & \langle \hat{b}^\dagger \hat{b} \rangle & \langle \hat{a} \hat{b}^\dagger \hat{b} \rangle \\ \langle \hat{a}^\dagger \hat{b} \rangle & \langle \hat{a}^\dagger \hat{b}^\dagger \hat{b} \rangle & \langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle \end{vmatrix}, \quad (6.4)$$

is negative for any canonical two-mode Schrödinger-cat state $|\psi\rangle = \mathcal{N}(|\alpha\rangle|\beta\rangle - |-\alpha\rangle|-\beta\rangle)$, where $\mathcal{N}$ is the normalisation factor. This is the same determinant as that defined in Chapter 5 as S_3 , see Eq.(5.40). Here, the highest-order moment in the matrix from which the determinant is then computed is $\langle\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}\rangle$ and so $n = 4$; the dimension of the matrix from which the determinant is calculated is $d = 3$. If we generate $\mathbf{M}$ using the multi-index ordering summarized in Table 6.2.1, then this criterion is generated by keeping the first, fifth and thirteenth rows/columns. In other words, the submatrix $\mathbf{A}$ is generated by keeping the vector of rows/columns (1, 5, 13). Then the NPT criterion is established by calculating $\det[\mathbf{A}]$ and evaluating whether or not it is negative. We have specifically labelled the determinant in Eq. (6.4) as D_V as it will feature in later sections.

6.2.2 Invariance of an NPT criterion under rotation

In experiments and for many states, it may be challenging to control the local phase on an individual subsystem to a high degree of precision. Here we note the theoretical result that an NPT criterion $\det[\mathbf{A}] < 0$, which has been constructed from a particular matrix $\mathbf{A}$ for a bipartite state ρ_{AB} , is invariant as the state is mapped onto the state $\tilde{\rho}_{AB}$ via local rotations. More specifically, we define $\tilde{\rho}_{AB} = \hat{U}(\theta_A, \theta_B)\rho_{AB}\hat{U}^\dagger(\theta_A, \theta_B)$ where $\hat{U}(\theta_A, \theta_B) = \exp(-i\theta_A\hat{a}^\dagger\hat{a})\exp(-i\theta_B\hat{b}^\dagger\hat{b})$ describe local rotations of θ_A and θ_B on subsystems $\mathcal{A}$ and $\mathcal{B}$, respectively. Considering a specific matrix of moments $\mathbf{A}$ from which we construct the entanglement test $\det[\mathbf{A}] < 0$ for the state ρ_{AB} , the i^{th} and j^{th} entry is $\mathbf{A}_{ij} = \langle\hat{a}^{\dagger q_i}\hat{a}^{p_i}\hat{a}^{\dagger n_j}\hat{a}^{m_j}\hat{b}^{\dagger l_j}\hat{b}^{k_j}\hat{b}^{\dagger r_i}\hat{b}^{s_i}\rangle$. Calculating the same moments for state $\tilde{\rho}_{AB}$ will yield a different matrix of moments denoted by $\tilde{\mathbf{A}}$. The entries of the matrix $\tilde{\mathbf{A}}$ for the rotated state are

$$\tilde{\mathbf{A}}_{ij} = \langle\hat{U}^\dagger(\theta_A, \theta_B)\hat{a}^{\dagger q_i}\hat{a}^{p_i}\hat{a}^{\dagger n_j}\hat{a}^{m_j}\hat{b}^{\dagger l_j}\hat{b}^{k_j}\hat{b}^{\dagger r_i}\hat{b}^{s_i}\hat{U}(\theta_A, \theta_B)\rangle . \quad (6.5)$$

Using the identity $\exp(i\hat{a}^\dagger\hat{a})\hat{a}\exp(-i\hat{a}^\dagger\hat{a}) = \hat{a}e^{-i\theta}$, and similarly for $\hat{b}$, it follows that

$$\tilde{\mathbf{A}}_{ij} = \mathbf{A}_{ij}e^{-i\theta_A(p_i - q_i + m_j - n_j) - i\theta_B(k_j - l_j + s_i - r_i)} . \quad (6.6)$$

As discussed in Section 6.2.1, the two sets of multi-indices have the same ordering such that the v^{th} multi-index is the same for both $(p_v, q_v, r_v, s_v) = (n_v, m_v, k_v, l_v)$. From this property we can group the indices in the following manner

$$\tilde{\mathbf{A}}_{ij} = \mathbf{A}_{ij} e^{-i\theta_A(p_i - q_i) - i\theta_B(s_i - r_i)} e^{i\theta_A(p_j - q_j) + i\theta_B(s_j - r_j)} , \quad (6.7)$$

and using the substitution $\phi_u = \theta_A(p_u - q_u) + \theta_B(s_u - r_u)$ where $u = \{i, j\}$, then

$$\tilde{\mathbf{A}}_{ij} = \mathbf{A}_{ij} e^{-i(\phi_i - \phi_j)} . \quad (6.8)$$

Hence, matrix $\mathbf{A}$ is mapped onto $\tilde{\mathbf{A}}$ by the following transformation $\tilde{\mathbf{A}} = \mathbf{U}^\dagger \mathbf{A} \mathbf{U}$ where $\mathbf{U}$ is a diagonal, unitary matrix whose i^{th} and j^{th} elements are $\mathbf{U}_{ij} = e^{i\phi_i} \delta_{ij}$. Since $\det[\mathbf{U}^\dagger] = \det[\mathbf{U}]^{-1}$, it is evident that $\det[\tilde{\mathbf{A}}] = \det[\mathbf{A}]$. Therefore, the NPT criterion $\det[\mathbf{A}] < 0$ is invariant as a state undergoes local rotations. In general, and notably, this result does not hold for local operations such as displacements. The statement that these NPT criteria are not invariant under local displacement operators may initially seem surprising since entanglement cannot be generated by local operations, nor can the entanglement entropy be enhanced by local unitaries [30]; however, a single NPT criterion in isolation is only a sufficient test and is not a proper measure of entanglement.

In Section 6.5 we will apply these NPT criteria to specific states and examine all such criteria which are capable of detecting entanglement. One state we will consider is the two-mode odd Schrödinger-cat state

$$|\Psi\rangle_{\text{cat}}^- = \mathcal{N} (|\alpha\rangle |0\rangle - |0\rangle |\alpha\rangle) , \quad (6.9)$$

where $\mathcal{N}$ is the normalisation constant. This state was first introduced in Section 1.3.2 and in Chapter 5 we introduced a protocol to generate this state in the displacement of two mechanical oscillators. In general $\alpha \in \mathbb{C}$; therefore, the NPT criteria which are the most suitable candidates for entanglement verification in this state must be evaluated as α is varied in complex parameter space. However, using the rotational invariance simplifies our search: a particular NPT criterion $\det[\mathbf{A}]$ will be the same for two states which are separated by local rotations. For example, with

$\alpha = |\alpha|e^{i\theta}$ where $\theta = \arg(\alpha)$, then rotating this state by $\hat{U}(-\theta, -\theta)$ to give $|\Psi\rangle = \mathcal{N}(|\alpha\rangle|0\rangle - |0\rangle|\alpha\rangle)$ will yield the same value for $\det[\mathbf{A}]$. We henceforth restrict our search to $\alpha \in \mathbb{R}^+$ for the state in Eq. (6.9). The values we find for $\det[\mathbf{A}]$ will be the same for all rotated states $\hat{U}(\theta_A, \theta_B)|\Psi\rangle_{\text{cat}}^-$, even when open-system dynamics are incorporated.

We will also analyse the photon-subtracted TMSV state which we introduced in Section 1.3.3. The squeezing parameter is usually a complex number $\zeta = |\zeta|e^{i\phi}$ where $\phi = \arg(\zeta)$. Applying the rotation $\hat{U}(-i\phi, 0)$ gives the following state up to a global phase

$$U(-i\phi, 0)|\Psi\rangle_{\text{sub}} \propto \hat{a}^n \hat{b}^m e^{|\zeta|\hat{a}\hat{b} - |\zeta|\hat{a}^\dagger \hat{b}^\dagger} |0, 0\rangle . \quad (6.10)$$

As outlined previously, we will likewise only consider $\zeta \in \mathbb{R}^+$ in order to simplify our parameter search.

6.3 Propagation of uncertainty in NPT criteria

In Chapter 4 we discussed sampling errors relevant to the quantum measurements considered in this thesis. To summarize, a collection of random variables is independent and identically distributed (i.i.d) if each random variable has the same probability distribution as the others and all are mutually independent. We consider such a variable X whose mean is to be estimated via many repeated measurements. The sample mean $\bar{X}$ is a type of statistic which follows its own probability distribution. The error on this statistic is quantified by the standard deviation of its distribution which is often dubbed the standard error. The distribution of the sample mean tends towards a normal distribution when a large number of measurements are taken.

For instance, suppose the variable X has population mean μ and standard deviation $\sigma[X]$. The expectation of the sample mean is given in Eq. (4.6a) and is the same as the population mean, $E[\bar{X}] = \mu$. However, the standard deviation of the sample mean $\sigma[\bar{X}]$ is generally not equal to the standard deviation of the population $\sigma[X]$, see Eq. (4.6b). Assuming independent observations of X (an assumption underpinning i.i.d variables), it follows that $\sigma[\bar{X}] = \sigma[X]/\sqrt{M}$

where M is the number of measurements made.

The same principles are applicable for a quantum observable $\hat{X}$. We will use hats to denote quantum operators, and a bar to denote the expectation value of an observable calculated from repeated measurements. It follows that the quantum expectation value $\langle \hat{X} \rangle$ is equivalent to the sample mean $\bar{X}$ found from repeated observations of $\hat{X}$. Crucially, we assume only one quantum observable is measured per experimental run, and that the quantum state is perfectly recreated between runs. We use these concepts in the following section to formulate a method for finding the standard error on $\det[\mathbf{A}]$.

6.3.1 Error on the determinant

We henceforth denote the error on a determinant as $\Delta \det[\mathbf{A}]$. The i^{th} and j^{th} entry of the d -dimensional square matrix $\mathbf{A}$ is a moment given by $\mathbf{A}_{ij} = \langle \hat{O}_{ij} \rangle$, where $\hat{O}_{ij} = \hat{a}^{\dagger q_i} \hat{a}^{p_i} \hat{a}^{\dagger n_j} \hat{a}^{m_j} \hat{b}^{\dagger l_j} \hat{b}^{k_j} \hat{b}^{\dagger r_i} \hat{b}^{s_i}$. For a discussion on the multi-indexing see the paragraph below Eq. (3.8).

There are d^2 such moments which appear in $\mathbf{A}$. One might naively assume that a moment $\langle \hat{O}_{ij} \rangle$ can be found independently from another moment $\langle \hat{O}_{i'j'} \rangle$ (where $i \neq i'$ and $j \neq j'$) by taking repeated measurements of $\hat{O}_{ij}$ and computing the sample mean $\bar{O}_{ij}$ which is equivalent to $\langle \hat{O}_{ij} \rangle$. Then the standard error on each of these means can be predicted and we may use Jacobi's rule given in Eq. (4.25) to quantify the overall error $\Delta \det[\mathbf{A}]$. Furthermore, $\mathbf{A}$ is a Hermitian matrix and so a second naive assumption is that it is only necessary to experimentally calculate the upper triangle of this matrix plus diagonal elements which comes to a total of $d + d(d-1)/2$ such operators. We are now presented with two challenges. Firstly, Jacobi's rule no longer holds since we can no longer assume that all the entries of $\mathbf{A}$ are calculated independently. Secondly, $\hat{O}_{ij}$ is not necessarily a Hermitian operator and so cannot always be directly measured to obtain $\bar{O}_{ij}$ in the laboratory. In general, an off-diagonal element of $\mathbf{A}$ is complex $\mathbf{A}_{ij} = \text{Re}[\mathbf{A}_{ij}] + i \text{Im}[\mathbf{A}_{ij}]$ (the diagonal entries of $\mathbf{A}$ are all real from the Hermiticity that arises by construction). In order to analyse how measurement errors propagate through

the determinant, we decompose the operator $\hat{O}_{ij}$ into two Hermitian operators

$$\begin{aligned}\hat{O}_{ij} &= \frac{\hat{O}_{ij} + \hat{O}_{ij}^\dagger}{2} - \frac{(\hat{O}_{ij}^\dagger - \hat{O}_{ij})}{2} , \\ &= \hat{B}_{ij,0} + i\hat{B}_{ij,1} .\end{aligned}\tag{6.11}$$

We use the notation $\hat{B}_{ij,p}$ to collectively refer to either of these two Hermitian operators. The label $p = \{0, 1\}$ specifies each operator such that

$$\hat{B}_{ij,p} = \frac{(i)^p [\hat{O}_{ij}^\dagger + (-1)^p \hat{O}_{ij}]}{2} .\tag{6.12}$$

From this it follows that $\text{Re}[\mathbf{A}_{ij}] = \langle \hat{B}_{ij,0} \rangle$ and $\text{Im}[\mathbf{A}_{ij}] = \langle \hat{B}_{ij,1} \rangle$. Since the operators $\hat{B}_{ij,p}$ are Hermitian they are all observable in theory and so we can repeatedly measure an operator to obtain $\bar{B}_{ij,p}$. Depending on the particular quantum platform, post-processing is potentially required to obtain $\bar{B}_{ij,p}$. For example, measurements schemes have been proposed for general moments of the form $\langle \hat{a}^{\dagger k} \hat{a}^l \hat{b}^{\dagger m} \hat{b}^n \rangle$ [155]. We will examine these measurement schemes in Section 6.6.

Therefore, assuming only one Hermitian operator operator $\hat{B}_{ij,p}$ is measured in a given run, there are d^2 such operators which must be independently measured in order to compute $\mathbf{A}$. If $M_{ij,p}$ measurements are allocated to the computation of each expectation value [165], then the expected value of the operator $\hat{B}_{ij,p}$ is

$$\langle \hat{B}_{ij,p} \rangle = \bar{B}_{ij,p} = \frac{1}{M_{ij,p}} \sum_m b_{ij,p}^{(m)} ,\tag{6.13}$$

where $b_{ij,p}^{(m)}$ is the result of a single measurement (by definition $b_{ij,p}^{(m)} \in \mathbb{R}$). The sample mean follows its own sampling distribution. As previously discussed, with sufficient measurements the sample mean is approximately normally distributed $\bar{B}_{ij,p} \sim \mathcal{N}(\text{E}[\bar{B}_{ij,p}], \text{V}[\bar{B}_{ij,p}])$. The variance of this distribution is

$$\text{V}[\bar{B}_{ij,p}] = \frac{\text{V}[\hat{B}_{ij,p}]}{M_{ij,p}} ,\tag{6.14}$$

which follows from Eq. (4.6b). We emphasise that $\bar{B}_{ij,p}$ is a classical statistic but $\hat{B}_{ij,p}$ is a

quantum operator and so its variance is given by $V[\hat{B}_{ij,p}] = \langle \hat{B}_{ij,p}^2 \rangle - \langle \hat{B}_{ij,p} \rangle^2$, where the quantum expectation values are evaluated using Born's rule. In general, $V[\hat{B}_{ij,p}]$ is not known but can be approximated using the unbiased estimator of the sample variance defined in Eq. (4.9). Therefore, we can empirically approximate the variance of the sample mean to

$$V[\bar{B}_{ij,p}] \approx \frac{1}{M_{ij,p}} \sum_{m=1}^{M_{ij,p}} \frac{(\bar{B}_{ij,p} - b_{ij,p}^{(m)})^2}{M_{ij,p} - 1}. \quad (6.15)$$

As previously mentioned, obtaining a single measurement $b_{ij,p}^{(m)}$ may require post-processing depending on the platform. Therefore, m might not strictly correspond to a single experimental run; however $M_{ij,p}$ quantifies the size of the data set from which the mean is then calculated thus providing an indication of the resources required. Nevertheless, we will assume that each data set — from which each statistic $\bar{B}_{ij,p}$ is calculated — is obtained in isolation from data sets for other operators.

In general, quantum operators have a non-zero covariance $\text{Cov}[\hat{B}_{ij,p}, \hat{B}_{kl,q}] \neq 0$. In order to capture this quantum covariance, joint measurements of $\hat{B}_{ij,p}$ and $\hat{B}_{kl,q}$ must be made. However, the covariance between the sampled means of two operators is necessarily zero if they are calculated independently $\text{Cov}[\bar{B}_{ij,p}, \bar{B}_{kl,q}] = \delta_{ik}\delta_{jl}\delta_{pq}V[\bar{B}_{ij,p}]$, where $V[\bar{B}_{ij,p}]$ is defined in Eq. (6.14). This is because $\bar{B}_{ij,p}$ and $\bar{B}_{kl,q}$ are not quantum operators but instead are two random variables which follow two independent normal distributions; as already emphasised $\bar{B}_{ij,p} \sim \mathcal{N}(E[\bar{B}_{ij,p}], V[\bar{B}_{ij,p}])$ while $\bar{B}_{kl,q} \sim \mathcal{N}(E[\bar{B}_{kl,q}], V[\bar{B}_{kl,q}])$.

Using the aforementioned definitions, we can calculate $\Delta\text{det}[\mathbf{A}]$ using the first order Taylor expansion which has been previously introduced in Eq. (4.18). We will reiterate it here without the hat notation to avoid confusion between estimators and quantum operators. First, we consider some multivariable function $f(\mathbf{x})$ expanded around $\mathbf{x} = \mathbf{x}_0$ where $\mathbf{x}$ is a vector of random real variables, and $\mathbf{x}_0$ is the vector of expectation values of each variable $\mathbf{x}_0 = E[\mathbf{x}]$. The standard error of this function is denoted as $\Delta f(\mathbf{x})$ where the error squared is given by

the following expression

$$(\Delta f[\mathbf{x}])^2 = \sum_{ij} \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}_i} \text{Cov}[\mathbf{x}_i, \mathbf{x}_j] \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}_j} . \quad (6.16)$$

Now we replace $f(\mathbf{x})$ with $\det[\mathbf{A}]$ which is a function of d^2 variables. The arguments of this function are the expectation values $\bar{B}_{ij,p}$ which appear the upper triangle and diagonal of the matrix $\mathbf{A}$, thus

$$(\Delta \det[\mathbf{A}])^2 = \sum_i^d \sum_{j \geq i}^d \sum_k^d \sum_{l \geq k}^d \sum_{p=0}^1 \sum_{q=0}^1 \frac{\partial \det[\mathbf{A}]}{\partial \bar{B}_{ij,p}} \text{Cov}[\bar{B}_{ij,p}, \bar{B}_{kl,q}] \frac{\partial \det[\mathbf{A}]}{\partial \bar{B}_{kl,q}} . \quad (6.17)$$

We proceed using the identity $\partial \det[\mathbf{A}]/\partial \mathbf{A}_{ij} = \text{adj}[\mathbf{A}]_{ji}$ where $\text{adj}[\mathbf{A}]$ is the adjugate of matrix $\mathbf{A}$, defined in Eq.(4.24). The adjugate of a matrix equivalent to the transpose of its cofactor matrix and so for non-singular matrices this can be expressed as $\text{adj}[\mathbf{A}]_{ij} = \det[\mathbf{A}]\mathbf{A}^{-1}$. Following the steps outlined in Appendix B.1, we derive the following result for the error on the determinant squared

$$\begin{aligned} (\Delta \det[\mathbf{A}])^2 = & \sum_i^d (\text{adj}[\mathbf{A}]_{ii})^2 \text{V}[\bar{B}_{ii,0}] \\ & + 4 \sum_i^d \sum_{j > i}^d \sum_{p=0}^1 \left[\delta_{p0} (\text{Re}[\text{adj}[\mathbf{A}]_{ij}])^2 + \delta_{p1} (\text{Im}[\text{adj}[\mathbf{A}]_{ij}])^2 \right] \text{V}[\bar{B}_{ij,p}] . \end{aligned} \quad (6.18)$$

We remind the reader that $\text{Re}[\mathbf{A}_{ij}] = \langle \hat{B}_{ij,0} \rangle$ and $\text{Im}[\mathbf{A}_{ij}] = \langle \hat{B}_{ij,1} \rangle$. For the purposes of investigating this formula and identifying feasible inseparability criteria for a specific state, we assume knowledge of the quantum state itself. Therefore, in our subsequent simulations, terms such as $\langle \hat{B}_{ij,p} \rangle$ and $\text{V}[\bar{B}_{ij,p}] = \text{V}[\hat{B}_{ij,p}]/M_{ij,p}$ will be evaluated using Born's rule. Nevertheless, the quantity $\Delta \det[\mathbf{A}]$ can also be empirically determined using the measurement data set $\{b_{ij,p}^{(m)}\}$ with no assumptions made concerning the state itself. The equivalent expression for the error on the determinant calculated from a data set can be found by inserting Eqs. (6.13) and (6.15) into $\langle \hat{B}_{ij,p} \rangle$ and $\text{V}[\bar{B}_{ij,p}]$, respectively. When no knowledge of the state is assumed and we only use the data set, the quantity $\Delta \det[\mathbf{A}]$ is approximate due to the approximate relation in Eq. (6.15). However, in the limit of a large number of measurements $M_{ij,p}$, the discrepancy

between the $V[\bar{B}_{ij,p}]$ and its estimator is negligible. We emphasise that, since $\hat{B}_{ij,p}$ are Hermitian observables, it follows from Eq. (6.18) that $\Delta\det[\mathbf{A}]$ is real and positive. This is in keeping with the statistical interpretation of the standard error on a real quantity such as $\det[\mathbf{A}]$.

While we noted in Section 6.2.2 that $\det[\mathbf{A}]$ is invariant when the quantum state undergoes local rotations; in general, the quantity $\Delta\det[\mathbf{A}]$ is not invariant under such rotations. This feature will be discussed in Section 6.7.

6.3.2 Minimizing the error of an NPT criterion

In the previous section we introduced $M_{ij,p}$ as the number of measurements allocated to calculating the expectation of each Hermitian operator $\hat{B}_{ij,p}$. However, some operators may have larger $V[\hat{B}_{ij,p}]$ and thus may require a larger data set (and consequently additional measurements) in order to unambiguously calculate $\bar{B}_{ij,p}$. Here, we consider the optimal choice of $M_{ij,p}$ in order to minimise the overall error $\Delta\det[\mathbf{A}]$, while keeping the total number of measurements fixed. The total number of measurements used to compute $\det[\mathbf{A}]$ is M_{tot} and is defined as

$$M_{\text{tot}} = \sum_i^d \sum_{j>i}^d \sum_{p=0}^1 M_{ij,p} , \quad (6.19)$$

where we have summed over all the measurements used for each of the observable operators which appear in the upper triangle and diagonal of $\mathbf{A}$ (according to our decomposition in Eq. (6.11)). Using the method of Lagrange multipliers outlined in Appendix B.2 we can minimise $\Delta\det[\mathbf{A}]$ under the constraint of a fixed M_{tot} . This allows us to compare the performance of entanglement criteria assuming the same total number of resources are allocated to each criterion. The minimum error on the determinant calculated in this way is

$$\Delta\det[\mathbf{A}] = \frac{\Gamma}{\sqrt{M_{\text{tot}}}} , \quad (6.20)$$

where

$$\Gamma = \sum_i^d |\text{adj}[\mathbf{A}]_{ii}| \sigma[\hat{B}_{ii,p=0}] + 2 \sum_i^d \sum_{j>i}^d |\text{Re}\{\text{adj}[\mathbf{A}]_{ij}\}| \sigma[\hat{B}_{ij,0}] + |\text{Im}\{\text{adj}[\mathbf{A}]_{ij}\}| \sigma[\hat{B}_{ij,1}] , \quad (6.21)$$

where $\sigma[\hat{B}_{ij,p}] = V[\hat{B}_{ij,p}]^{1/2}$. The quantity Γ depends on the quantum state, the choice of subdeterminant, but is independent of the number of measurements per operator $M_{ij,p}$ and total number of measurements M_{tot} . If knowledge of the quantum state is not assumed *a priori*, a similar optimisation method can be adopted where $M_{ij,p}$ are assigned in proportion to the estimator of the sample variance defined in Eq. (6.15).

By examining Eq. (6.21), we note qualitatively that as the dimension d of the matrix $\mathbf{A}$ increases (while keeping M_{tot} fixed), the number of terms in Γ also increases. Consequently, the error $\Delta\det[\mathbf{A}]$ will be larger if more rows and columns are added to $\mathbf{A}$. Therefore, while matrices with higher d might be able to detect entanglement in a greater number of non-Gaussian states, this comes at the cost of a greater uncertainty in the conclusion of the inseparability test. This uncertainty can be reduced but at the experimental cost of more measurements. Therefore, it is desirable from the perspective of experimental feasibility to have matrices with smaller d if possible.

6.4 Loss and open-system dynamics

In the laboratory, all quantum systems interact with their environment, which leads to loss of quantum information in the form of dissipation, thermalisation and decoherence. Such losses affect the entanglement between two systems, and thus have a detrimental effect on the success with which the inseparability criteria can detect entanglement. Here we explore the effect which the environment has on the moments of a bipartite state. Interactions with the environment in the time that elapses between state generation and verification will change the value of $\det[\mathbf{A}]$ and $\Delta\det[\mathbf{A}]$.

We use a beam splitter model to capture the decoherence of a state $\rho_{\mathcal{AB}}$ [166], which is applicable

to the optical and mechanical systems we consider. Considering subsystem $\mathcal{A}$, the field operator $\hat{a}$ at the point of measurement will be

$$\hat{a} = \hat{a}_0\sqrt{\eta} + \hat{a}_E\sqrt{1-\eta} , \quad (6.22)$$

where $\hat{a}_0$ is the initial mode in the absence of loss, η is the beam splitter parameter, and $\hat{a}_E$ is the annihilation operator for the environment mode. The environment is uncorrelated with $\hat{a}_0$, and obeys the following correlation-relations

$$\langle \hat{a}_E^\dagger \hat{a}_E \rangle = \bar{n}_B , \quad (6.23a)$$

$$\langle \hat{a}_E \hat{a}_E^\dagger \rangle = \bar{n}_B + 1 , \quad (6.23b)$$

where $\bar{n}_B$ is the thermal occupation number of the bath, and for optical systems we set $\bar{n}_B = 0$. There are analogous equations for subsystem $\mathcal{B}$ where $\hat{a}$ is substituted for $\hat{b}$. The environmental modes, $\hat{a}_E$ and $\hat{b}_E$, are independent of each other. In this work, we assume that both subsystems couple to the environment with the same η but this can be generalized if necessary.

In order to experimentally calculate the entries in $\mathbf{A}$, moments such as $\langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle$ must be measured. To predict the value of this measurement in the presence of loss, we can use Eq. (6.22) to write these moments in terms of the beam splitter parameter η , the initial moment $\langle \hat{a}_0^\dagger \hat{a}_0 \hat{b}_0^\dagger \hat{b}_0 \rangle$, and correlations between field operators of the same environment mode. In calculating the latter correlations, we use the identities such as

$$\langle \hat{a}_E^{\dagger p} \hat{a}_E^q \rangle = \delta_{pq} p! \bar{n}_B^p , \quad (6.24)$$

which are derived in Appendix (B.3.1). By studying Eq. (6.21) we can see that the dependence of the quantum state and the measured moment on η will propagate into $\det[\mathbf{A}]$ and indeed $\Delta \det[\mathbf{A}]$.

A Langevin-equation-based approach can be adopted instead of a beam splitter model for platforms such as optical cavities and optomechanics where there are time-continuous dynamics.

We can switch between the two frameworks by substituting $\sqrt{\eta} = e^{-\kappa t}$, where κ is the amplitude decay rate and t is the time of measurement. We note that this assumes that moments are all measured at the same time t . In this formalism, the determinant and its error are parameterized by κt . The equivalence between the two approaches is outlined in Appendix B.3. This holds for an initially entangled state (which has been created through some interaction Hamiltonian and/or performing some conditional measurement), but in time t that elapses between state generation and verification, this interaction is switched off.

6.5 Simulations

Using the error propagation outlined in Section 6.3 and incorporating the beam splitter decoherence model from Section 6.4, we assess the success with which a specific NPT criterion within this class of tests can detect entanglement in: (i) a two-mode Schrödinger-cat state, (ii) a phonon-subtracted TMSV state. Through our analysis we identify specific entanglement criteria which are the most successful at unambiguously verifying entanglement in these states.

We limit our analysis to $n \leq 4$ and $d = 2$; as discussed in Section 6.2.1 keeping these as small as possible and thus reducing experimental complexity. Nevertheless, the method outlined can be readily extended to search all possible matrices $\mathbf{A}$ characterized by $n > 4$ or $d > 2$; especially if higher-order moments are required to unambiguously verify entanglement in a given state. Numerics to analyse the confidence of a particular NPT criterion and thus select the criterion that is best suited to a particular quantum state is written in Python and uses the following steps.

1. There are 105 possible matrices $\mathbf{A}$ which are characterized by $d = 2$ and $n \leq 4$, some of which will potentially be capable of detecting entanglement. We perform a preliminary step to narrow the number of candidate matrices $\mathbf{A}$ by first assuming there is no coupling to the environment and no sampling error. Only determinants that are negative in absence of decoherence and errors will possibly be negative when decoherence and sampling errors are later included in our calculations.

2. All 105 matrices are numerically calculated across the desired parameter space of the state under consideration. For example, with $|\Psi\rangle_{\text{cat}}^-$ we consider α between some range of real, positive numbers. As shown in Section 6.2.2, the NPT criteria are invariant under local rotations, which means we can constrain our parameter space to only consider real α . Using QuTip, we compute the lossless moments (for example $\langle \hat{a}_0^{\dagger m} \hat{a}_0^n \hat{b}_0^{\dagger k} \hat{b}_0^l \rangle$) in order to calculate 105 possible $\det[\mathbf{A}]$ for the state when isolated from the environment. We identify any determinants that exhibit negativity for any value of α , consequently discarding the rest.
3. We now investigate the effects of the environment and sampling errors on this subset of determinants. With this subset of matrices $\mathbf{A}$, we calculate $\det[\mathbf{A}]$ and also $\Delta\det[\mathbf{A}]$ as functions of the beam splitter parameter η and the total number of measurements M_{tot} . The effect of decoherence is calculated both using the analytical expressions in Eqs. (B.18) and (B.19), and using the lossless moments which are found numerically with QuTip.
4. We then examine the relationship between a given determinant, and the loss and uncertainty. Consequently, we quantify the success of that determinant at detecting entanglement by considering the confidence of rejecting the no-entanglement null hypothesis, in the face of decoherence and sampling errors. A confidence which is below some critical threshold implies that there is insufficient evidence to reject the no-entanglement hypothesis; however does not necessarily preclude a state from being entangled. The hypothesis testing is discussed in greater detail immediately below.

Confidence in NPT entanglement tests

To compute the confidence with which we can conclude that a negative determinant indicates entanglement we can conduct a hypothesis test. The null hypothesis is that the state is not entangled $\det[\mathbf{A}] \geq 0$. The alternative hypothesis is that the state is entangled $\det[\mathbf{A}] < 0$. For now, let us assume that, since our test statistic X is constructed from sample means, it follows a normal distribution $X \sim N(\mu, \sigma^2)$ with mean $\mu = \det[\mathbf{A}]$ and standard deviation $\sigma = \Gamma/\sqrt{M_{\text{tot}}}$ (this is equivalent to the standard error of the determinant, see Eq. (6.20)). We

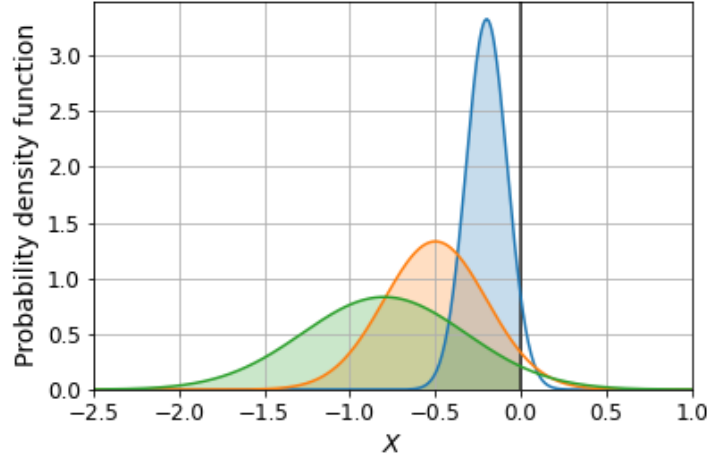


Figure 6.1: Cartoon plots to demonstrate the probability density functions of the test statistic X of three different NPT criteria shown in green, orange, and blue. The area of the shaded regions corresponds to their respective p -values defined in Eq. (6.25). We assume that each test statistic follows a normal distribution $X \sim \mathcal{N}(\det[\mathbf{A}], \Gamma^2/M_{\text{tot}})$. All three NPT criteria have the same p -value of 95% despite having different expectation values.

can conduct a one-tailed hypothesis test by finding the probability that X is negative under the null hypothesis. Therefore the probability of rejecting no-entanglement hypothesis is

$$p = P(X < 0) = \Phi\left(\frac{-\mu}{\sigma}\right), \quad (6.25)$$

where Φ is the CDF of the standard normal distribution. We will refer to p as our confidence in rejecting the no-entanglement null hypothesis. If p exceeds some threshold confidence level, say 95%, then we can assuredly conclude the state is entangled. We note that even if an entanglement criterion is not sufficient to reject the null-hypothesis at a specified confidence level, it does not imply the state is necessarily separable. It simply means that we have insufficient evidence to reject the hypothesis that the state is separable.

This is visually demonstrated in Figure 6.1 where we consider the probability density functions of the test statistic of three different NPT criteria in green, orange, and blue. The area of the shaded regions corresponds to their respective p -values. We have made the assumption that each test statistic follows a normal distribution given in the previous paragraph. In this example, we have chosen the standard deviation of each distribution in such a way that their p -values are all 95%. This shows that despite a determinant potentially being more negative,

it is crucial to also take its error into account. It is also important to note that if, in our simulations, the test statistic is distributed around a positive mean, one should not perform the hypothesis test. This is partly because a positive mean value is not insightful with regards to entanglement so should not be used as an entanglement test. More importantly, if we admit positive expected values of $\det[\mathbf{A}]$ into our hypothesis test we obtain intriguing results in our simulations which can lead to misconstruing the underlying physics. For example, let us consider an NPT criterion which happens to yield a determinant whose mean value is positive for a given entangled quantum state, with some degree of loss to the environment $0 < \eta < 1$. Repeated measurements of the determinant are assumed to be distributed normally $\mathcal{N}(\mu_1, \sigma_1^2)$, where $\mu_1 > 0$. Our hypothesis test will tell us that the p -value, say p_1 , is some number less than a half, i.e. $p_1 < 50\%$. Now let us take the extreme case where $\eta = 0$ and each mode of the quantum state is replaced with the environment modes (see Eq. (6.22)). The state is decidedly separable and so repeated measurements of the determinant are normally distributed around a positive mean $\mathcal{N}(\mu_2, \sigma_2^2)$, where $\mu_2 > 0$. We can find the p -value which will again be constrained $p_2 < 50\%$. However, in some cases it is feasible that $p_2 > p_1$, due to the complicated interplay between $\Gamma/\sqrt{M_{\text{tot}}}$ and the quantum state. One might incorrectly think that since p_2 is larger, the NPT criterion is better-suited at detecting entanglement for this quantum state, despite it being separable for these parameters! Of course, $p_1, p_2 < 50\%$ so it is unlikely one would ever be satisfied with such a low confidence level for either of the two tests. However, when we present our simulations we will omit any NPT criteria whose expected determinant is distributed around a positive mean value to avoid any confusion concerning these potential ‘revivals’ in confidence. This is merely a feature of the hypothesis test and does not impact the key results. Considering only $\mu < 0$ also means by construction that the confidence levels will range between 50% to 100%.

In our simulations we assume that we know the quantum state and use Born’s rule to accurately calculate the mean and variance of the distribution of the test statistic. Therefore, we conduct a hypothesis test under a normal distribution. Assuming the test statistic is normally distributed but the mean and variance are estimated using the data set, then it is more appropriate to assume that the test statistic follows a Student’s t -distribution if one has to estimate the mean

and variance of the distribution from the data. This is discussed in Section 4.1.2. In the limit of a large M_{tot} this converges to a normal distribution.

6.5.1 Two-mode Schrödinger-cat state

We now present the results for the two-mode odd Schrödinger-cat state $|\Psi\rangle_{\text{cat}}^- \propto |\alpha\rangle|0\rangle - |0\rangle|\alpha\rangle$. In Fig. 6.2 we display the four entanglement criteria $\det[\mathbf{A}]$ (under the constraint $d = 2$ and $n \leq 4$) which exhibit negativity as α is varied in our region of interest. We use the labels $D_{\text{I-IV}}$ for these determinants which are summarized in Table 6.2. We remind the reader that a negative determinant is a sufficient criterion for entanglement. We note that D_{I} is in fact the simplest Hillery-Zubairy condition [110]. The beam splitter parameter η , and the thermal occupation number of the bath $\bar{n}_{\text{B}}$ affect the values of these determinants; these are kept fixed with $\eta = 0.9$, and $\bar{n}_{\text{B}} = 0$. As previously discussed, the value of $\det[\mathbf{A}]$ is invariant for any state generated by local rotations $\hat{U}(\theta_A, \theta_B)|\Psi\rangle_{\text{cat}}^-$. We have also included D_{V} defined in Eq. (6.4). This determinant is potentially more experimentally demanding to measure because it has $d = 3$; nevertheless, we have included this criterion in our analysis as it has been previously highlighted in the context of this state (see Section 6.2.1). Also in Fig. 6.2, superimposed on top of the thick lines of $D_{\text{I-V}}$, are semi-transparent error bars representing $\Delta\det[\mathbf{A}]$ for each determinant, defined in Eq. (6.20). This uncertainty is parameterised by the total number of measurements M_{tot} in addition to η and $\bar{n}_{\text{B}}$. We note that this uncertainty in the determinant is not invariant for states generated by local rotations; rotating the state changes the value of Γ in Eq. (6.21).

In theory, $D_{\text{I-V}} < 0$ should be sufficient criteria for entanglement. Indeed, we initially observe that D_{I} , D_{III} , and D_{IV} are the most negative as we scan α in this range. However these tests alone are not entirely conclusive in determining entanglement due to $\Delta\det[\mathbf{A}]$. From this plot, this problem is especially apparent for larger values of α where $\Delta\det[\mathbf{A}] > |\det[\mathbf{A}]|$.

Therefore, the success of a criterion in detecting entanglement is displayed in Fig. 6.3 where we have plotted confidence p (see Eq. (6.25)) as a function of α . As previously discussed, this is the confidence in rejecting the no-entanglement null-hypothesis. A low confidence level

Table 6.2: Suitable NPT criteria for the two-mode Schrödinger-cat state $|\Psi\rangle_{\text{cat}} \propto |\alpha\rangle|0\rangle - |0\rangle|\alpha\rangle$. Determinant negativity is a sufficient criterion for entanglement. These determinants are plotted in Fig. 6.2-6.5 where sampling errors and open-system dynamics are considered. The matrix $\mathbf{A}$ from which the determinant is calculated is provided; furthermore we consider the dimension d of the matrices and the order n of the highest-order moment which appears. Valid matrices can be constructed by keeping pairwise rows and columns from the super matrix $\mathbf{M}$ in Eq. 3.8 to leave a submatrix $\mathbf{A}$. However not all matrices $\mathbf{A}$ are guaranteed to be negative even if a state is entangled. Determinants $D_{\text{I-IV}}$ are the only determinants with $d = 2$ and $n \leq 4$ which exhibit negativity in the region of parameter space have plotted. The indices of the rows and columns which are kept in order to generate $\mathbf{A}$ are shown; $\mathbf{M}$ is generated using the ordering in Table 6.2.1. The determinants for the pure state $|\Psi\rangle_{\text{cat}}^-$ isolated from the environment are also included ($\eta = 0$, $\bar{n}_{\text{B}} = 0$). We note that these determinants are negative for all values of α however when experimental factors are considered negativity alone is sufficient to detect entanglement; we should instead evaluate the confidence of rejecting the no-entanglement hypothesis. Note that D_{I} is the first-order Hillery-Zubairy criterion. We have also included D_{V} (see Eq. (6.4), for comparison, despite having dimensions $d = 3$.

Determinant	Matrix form $\mathbf{A}$	Dimension d	Order n	Rows/ Columns	Pure state
D_{I}	$\begin{vmatrix} 1 & \langle \hat{a}\hat{b}^\dagger \rangle \\ \langle \hat{a}^\dagger \hat{b} \rangle & \langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle \end{vmatrix}$	2	4	(1,13)	$-\frac{ \alpha ^4 e^{-2 \alpha ^2}}{4(1 + \cos \phi e^{- \alpha ^2})^2}$
D_{II}	$\begin{vmatrix} \langle \hat{a}\hat{a}^\dagger \rangle & \langle \hat{a}\hat{a}\hat{b}^\dagger \rangle \\ \langle \hat{a}^\dagger \hat{a}^\dagger \hat{b} \rangle & \langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle \end{vmatrix}$	2	4	(2,13)	$-\frac{ \alpha ^6 e^{-2 \alpha ^2}}{4(1 - e^{- \alpha ^2})^2}$
D_{III}	$\begin{vmatrix} \langle \hat{a}\hat{a}\hat{a}^\dagger \hat{a}^\dagger \rangle & \langle \hat{a}\hat{a}\hat{a}\hat{b}^\dagger \rangle \\ \langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{b} \rangle & \langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle \end{vmatrix}$	2	4	(6,13)	$-\frac{ \alpha ^8 e^{-2 \alpha ^2}}{4(1 - e^{- \alpha ^2})^2}$
D_{IV}	$\begin{vmatrix} \langle \hat{a}\hat{a}^\dagger \hat{b} \hat{b}^\dagger \rangle & \langle \hat{a}\hat{a} \hat{b}^\dagger \hat{b}^\dagger \rangle \\ \langle \hat{a}^\dagger \hat{a}^\dagger \hat{b} \hat{b} \rangle & \langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle \end{vmatrix}$	2	4	(9, 13)	$-\frac{ \alpha ^8 e^{-2 \alpha ^2}}{4(1 - e^{- \alpha ^2})^2}$
D_{V}	$\begin{vmatrix} 1 & \langle \hat{b}^\dagger \rangle & \langle \hat{a}\hat{b}^\dagger \rangle \\ \langle \hat{b} \rangle & \langle \hat{b}^\dagger \hat{b} \rangle & \langle \hat{a}\hat{b}^\dagger \hat{b} \rangle \\ \langle \hat{a}^\dagger \hat{b} \rangle & \langle \hat{a}^\dagger \hat{b}^\dagger \hat{b} \rangle & \langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle \end{vmatrix}$	3	4	(1, 5, 14)	$-\frac{ \alpha ^6 e^{-2 \alpha ^2}}{8(1 - e^{- \alpha ^2})^3}$

does not indicate an increased certainty in separability; it only suggests there is insufficient evidence to conclude the state is inseparable when that particular determinant is used. We have included the 95% confidence level as a dotted black line. For these specific values of η , $\bar{n}_{\text{B}}$, M_{tot} , and considering only matrices which satisfy $d = 2$ and $n \leq 4$, we can conclude that D_{I} is best-suited for states with smaller α , while D_{IV} is capable of detecting entanglement with the highest degree of certainty for the highest α . We also note that D_{V} is able to detect entanglement for small α with very high certainty. However, all tests degrade in confidence as α is further increased. This can be qualitatively understood by referring to Figure 5.11 which depicts how the non-Gaussianity of the state increases as α is increased. To confidently

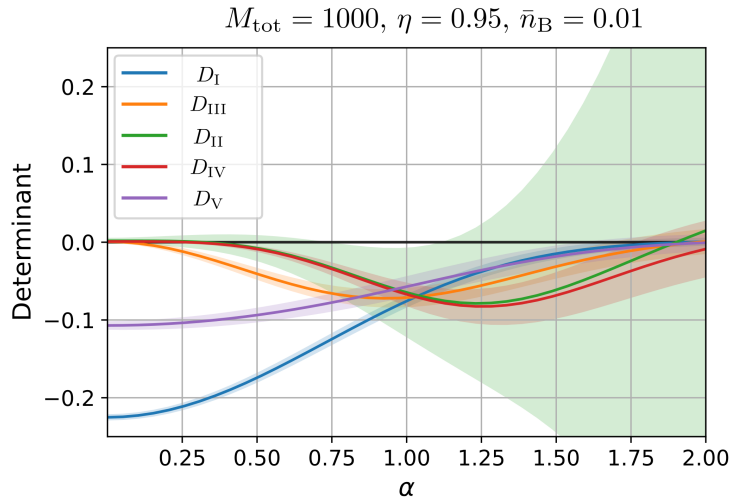


Figure 6.2: Determinants $\det[\mathbf{A}]$ for the two-mode Schrödinger-cat state $|\Psi\rangle_{\text{cat}}$ are plotted as functions of α . By first considering all matrices $\mathbf{A}$ which have $d = 2$ and $n \leq 4$, we have selected the determinants $D_{\text{I-IV}}$ which exhibit negativity in this range of α . We have also plotted D_{V} defined in Eq. (6.4) which has $d = 3$ and $n = 4$. The criterion $D_{\text{I-V}} < 0$ is not entirely conclusive in detecting entanglement without also considering the finite errors on each determinant $\Delta\det[\mathbf{A}]$ which are shown as semi-transparent error bars. In this plot, the experimental factors are kept fixed $M_{\text{tot}} = 1000$, $\eta = 0.9$, and $\bar{n}_{\text{B}} = 0$.

detect entanglement in states with higher α we must therefore consider more complicated $\mathbf{A}$ with higher d and n . This can be done with our statistical framework and leads to many more feasible determinants. When considering more complicated $\mathbf{A}$ this comes at the cost of a higher M_{tot} to attain the same confidence (assuming the same η and $\bar{n}_{\text{B}}$), since more moments must be measured and each moment could be of higher order.

In Fig. 6.4 we plot the confidence in rejecting the no-entanglement hypothesis as a function of the total number of measurements M_{tot} allocated to calculating each determinant. We set $\alpha = 1, \eta = 0.95, \bar{n}_{\text{B}} = 0.01$. Provided $\det[\mathbf{A}] < 0$, it is evident that confidence increases with increasing M_{tot} . This follows directly from Eqs. (6.20) and (6.21).

Finally, in Fig. 6.5 we have calculated confidence as a function of η ; we recall that $\eta = 1$ is a lossless system while $\eta = 0$ is entirely lossy. As the system is made more lossy it becomes more entangled with the environment and consequently the bipartite state which was initially entangled becomes more separable. Therefore, one might intuitively expect the confidence of each determinant to decrease as the system becomes more susceptible to losses. This behaviour is visible in $D_{\text{I-V}}$. We also qualitatively note that D_{III} and D_{IV} degrade the most quickly as

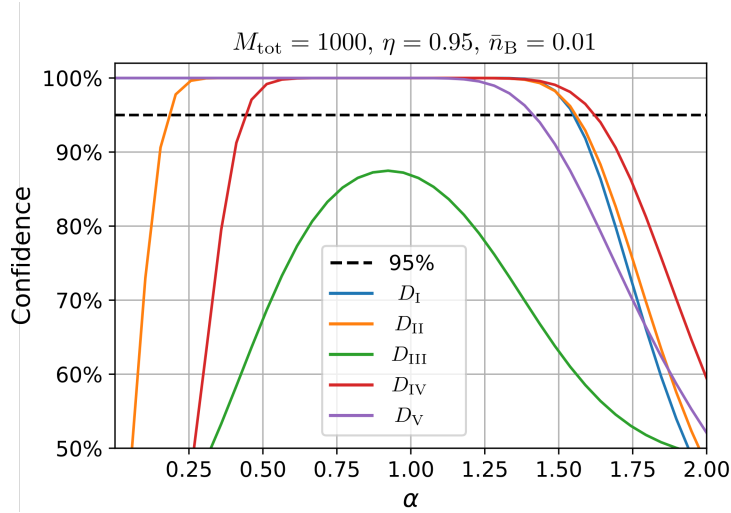


Figure 6.3: Confidence in rejecting the no-entanglement null hypothesis for each determinant D_{I-V} . We find that for the given parameter set, $M_{\text{tot}} = 1000$, $\eta = 0.9$, $\bar{n}_B = 0$, D_I and D_V are best-suited for small α (the two lines overlap completely for small α). While for higher α the determinant D_{IV} can most confidently detect entanglement.

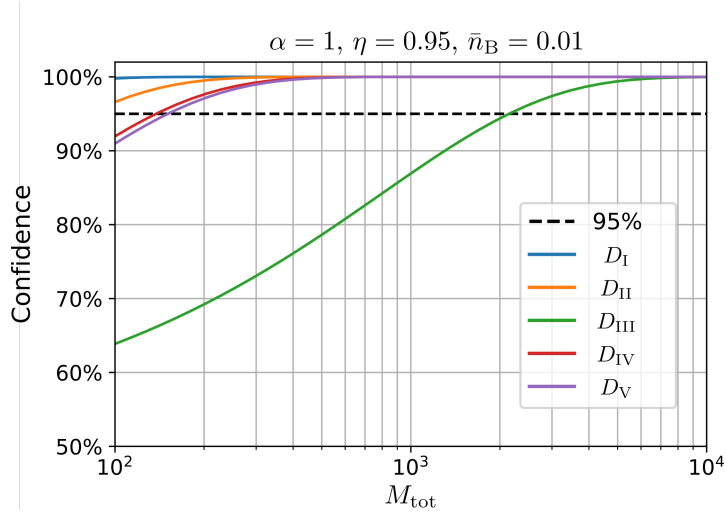


Figure 6.4: Confidence in rejecting the no-entanglement null hypothesis as a function of M_{tot} , the total number of measurements allocated to determining a single determinant. Experimental factors are kept constant $\eta = 0.9$, $\bar{n}_B = 0$. $\alpha = 1$ has been chosen since at this value all D_{I-V} are negative. At this value of α , D_I requires the fewest number of measurements to approach 100% confidence, while D_{III} requires the most. For different values of α different determinants would perform better. If we are in a region where D is greater than 0, increasing M_{tot} does not lead to a confidence which approaches 100%.

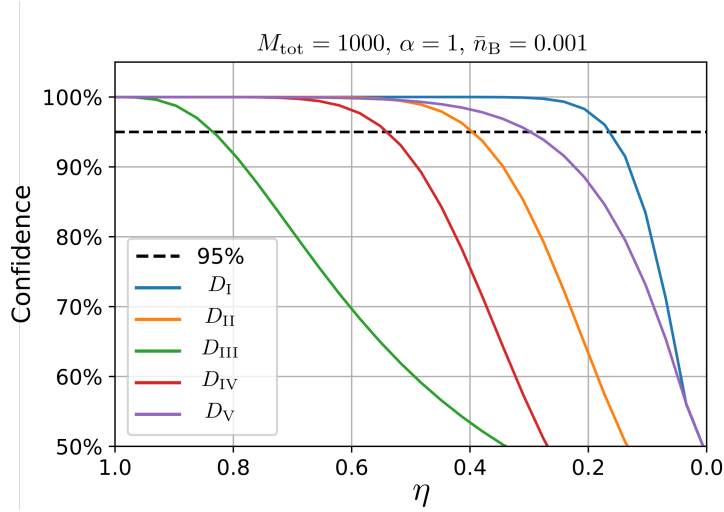


Figure 6.5: Confidence in rejecting the no-entanglement null hypothesis as a function of loss parameter η . For a lossless system with $\eta = 1$ all determinants have 100% confidence. As decoherence is introduced, the confidence of D_{III} in detecting entanglement tends to just below 50% because the expected value of this determinant (with $M_{\text{tot}} = 1000$, $\alpha = 1$, $\bar{n}_{\text{B}} = 0$) is a positive number with a bigger uncertainty than its value; whereas the other determinants are positive numbers with errors larger in value than the magnitude of the determinant, and so tend to 0%.

η is increased since these are the only determinants where all four entries are comprised of fourth-order moments. Therefore, determinants composed of increasingly high n will be more susceptible to losses. This dependence can be seen by studying Eq. (6.22).

We have also computed analytical expressions for $D_{\text{I-V}}$ which are summarized in Table 6.2 for the idealized state $|\Psi\rangle_{\text{cat}}^-$ in isolation from the environment and with no sampling errors. We find that these determinants are negative for all values of α . However, as demonstrated by the results in Fig. 6.3, when experimental factors are considered, negativity in isolation is not sufficient to confidently detect entanglement.

We also note that because of the symmetry between the two subsystems of the state $|\Psi\rangle_{\text{cat}}^-$, we can swap the operators $\hat{a}$ ($\hat{a}^\dagger$) with $\hat{b}$ ($\hat{b}^\dagger$) which appear in the moments in $\mathbf{A}$ and we expect the same results. This means we can generate three further determinants by swapping the field operators corresponding to each subsystem in D_{II} , D_{III} , and D_{V} with no effect on the results; however the same statement could not be made for a state which is not invariant when the subsystem labels are swapped.

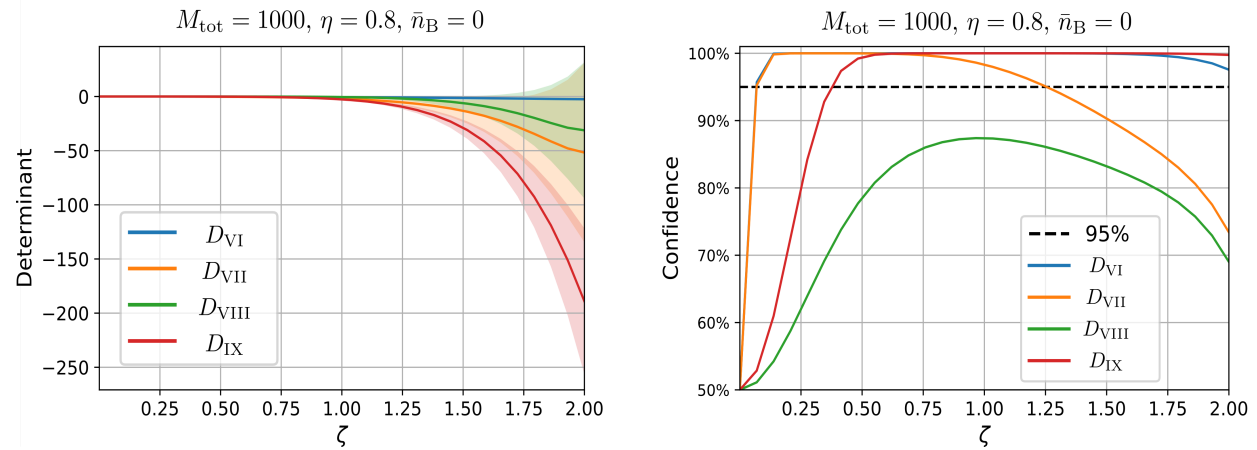
6.5.2 Photon-subtracted two-mode squeezed vacuum state

Here we conduct the same analysis as outlined in Section 6.5.1 but for the state $|\Psi\rangle_{\text{sub}}$ which is generated by subtracting 1 photon from each of the two subsystems of a TMSV state. This state can be mathematically expressed by setting $n = 1$ and $m = 1$ in Eq. (1.50). We note very similar characteristics between these results and the plots in Section 6.5.1 and so have summarized all the results in Fig. 6.6. We use the labels $D_{\text{VI-IX}}$ to denote the relevant determinants, these are summarized in Table 6.3. We will now examine each subfigure in turn.

In Fig. 6.6a we have plotted the subset of determinants which exhibit negativity as we vary the squeezing parameter ζ but keep all other parameters fixed. These determinants are labelled as $D_{\text{VI-IX}}$. We reiterate that this subset of four determinants is found by first considering all possible matrices $\mathbf{A}$ which satisfy $d = 2$ and $n \leq 4$ and identifying which are determinants are negative as we sweep ζ in the absence of losses and sampling errors. Using this narrower set, we then evaluate the determinants when losses and sampling errors are introduced. We thus set $M_{\text{tot}} = 1000$, $\eta = 0.8$, and $\bar{n}_{\text{B}} = 0$ and identify the subset for which $\det[\mathbf{A}] < 0$ for any value of ζ which leaves us with $D_{\text{VI-IX}}$. We set $\bar{n}_{\text{B}} = 0$ as we consider an optical state in this thesis. The error bars are quantified by $\Delta\det[\mathbf{A}]$. We can qualitatively see that as ζ increases the determinants $D_{\text{VII-IX}}$ become increasingly negative, with the exception of D_{VI} which does not appear to decrease as dramatically. However, the error bars on all four determinants also increase therefore this plot in isolation is not conclusive in establishing which of the four NPT criteria are best-suited to this particular state.

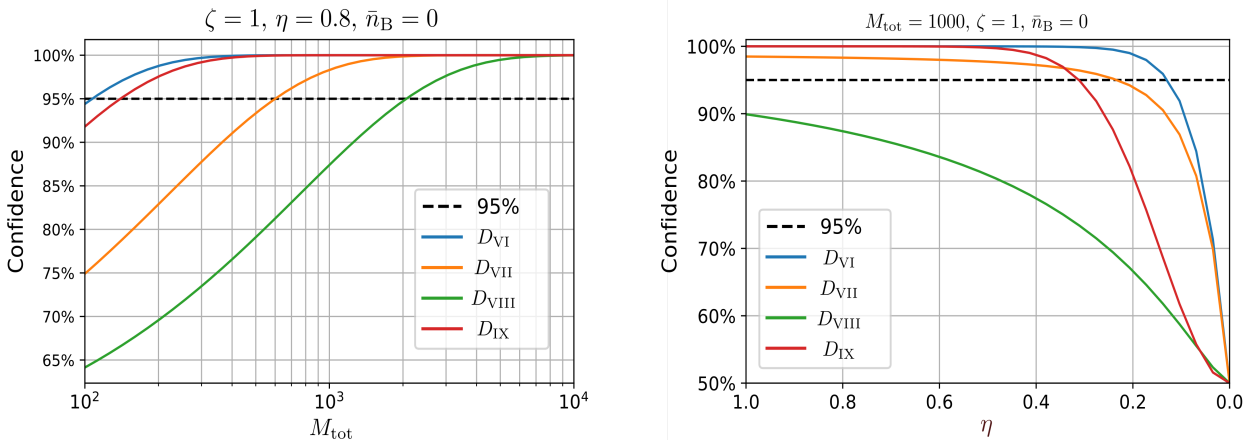
Similar to Figure. 6.3, we therefore plot confidence as a function of ζ in Fig. 6.6b. For very small ζ , confidently detecting entanglement is challenging. This is understood to be due to the fact that the degree of entanglement is very small with small ζ . However, for greater ζ we find that D_{VI} and D_{IX} can detect entanglement with the greatest degree of confidence.

We can see in Fig. 6.6c that as the total number of measurements M_{tot} is increased, the confidence also increases. This is plotted for a fixed $\zeta = 1$, $\eta = 0.8$, and $\bar{n}_{\text{B}} = 0$. This same behaviour is exhibited in Fig. 6.4, which attributed to the fact that as M_{tot} is increased the



(a) Determinants $\det[\mathbf{A}]$ for the TMSV state are plotted as functions of ζ . By first considering all matrices $\mathbf{A}$ which have $d = 2$ and $n \leq 4$, we have selected the determinants $D_{\text{VI-IX}}$ which exhibit negativity in this range of ζ . The criterion $D_{\text{VI-IX}} < 0$ is not entirely conclusive in detecting entanglement without also considering the finite errors on each determinant $\Delta \det[\mathbf{A}]$ which are shown as semi-transparent error bars. In this plot, the experimental factors are kept fixed $M_{\text{tot}} = 1000$, $\eta = 0.8$, $\bar{n}_B = 0$.

(b) Confidence in rejecting the no-entanglement null hypothesis using one of the negative determinants. Confidence is calculated using hypothesis test as ζ is varied but M_{tot} , η , and N are kept fixed. We note that for small ζ , the lines for D_{VI} and D_{VII} are almost overlapping.



(c) Confidence in rejecting the no-entanglement null hypothesis as a function of M_{tot} , the total number of measurements allocated to determining a single determinant. Experimental factors are kept constant $\eta = 0.9$, $\bar{n}_B = 0$. $\zeta = 1$ has been chosen since at this value all $D_{\text{VI-IX}}$ are negative. At this value of ζ , D_{VI} requires the fewest number of measurements to approach 100% confidence, while D_{VIII} requires the most. For different values of ζ different determinants would perform better.

(d) Confidence in rejecting the no-entanglement null hypothesis as a function of beam splitter parameter η . As η tends towards 0, the system becomes more lossy. The confidence of $D_{\text{VI-IX}}$ tend towards 50% because the expected value of these determinant (with $M_{\text{tot}} = 1000$, $\zeta = 1$, $\bar{n}_B = 0$) tend towards 0 but with a finite error.

Figure 6.6: Results for the photon subtracted TMSV state $|\Psi\rangle_{\text{sub}}$. A negative determinant implies an inseparable state however it should be emphasised that the magnitude of negativity of the determinant has no physical interpretation. The error on the determinant is an important consideration in concluding whether or not a state is entangled therefore also plotted is the interplay between the confidence in an entanglement criterion and the state-dependent ζ , and experimental factors such as the beam splitter parameter η and the total number of measurements M_{tot} . Confidence should be interpreted as the confidence in rejecting the no-entanglement null hypothesis.

Table 6.3: Suitable NPT criteria for the photon subtracted TMSV state, $|\Psi\rangle_{\text{sub}}$. Determinant negativity is a sufficient criterion for entanglement. These determinants are plotted in Fig. 6.6 where sampling errors and open-system dynamics are considered. The matrix $\mathbf{A}$ from which the determinant is calculated is provided; furthermore we consider the dimension d of the matrices and the order n of the highest-order moment which appears. Valid matrices can be constructed by keeping pairwise rows and columns from the super matrix $\mathbf{M}$ in Eq. 3.8 to leave a submatrix $\mathbf{A}$. However not all matrices $\mathbf{A}$ are guaranteed to be negative even if a state is entangled. Determinants $D_{\text{VI-IX}}$ are the only determinants with $d = 2$ and $n \leq 4$ which exhibit negativity in the region of parameter space have plotted. The indices of the rows and columns which are kept in order to generate $\mathbf{A}$ are shown; $\mathbf{M}$ is generated using the ordering in Table 6.2.1.

Determinant	Matrix form $\mathbf{A}$	Dimension d	Order n	Rows/ Columns
D_{VI}	$\begin{vmatrix} \langle \hat{a}^\dagger \hat{a} \rangle & \langle \hat{a}^\dagger \hat{b} \rangle \\ \langle \hat{a} \hat{b} \rangle & \langle \hat{b}^\dagger \hat{b} \rangle \end{vmatrix}$	2	4	(3,5)
D_{VII}	$\begin{vmatrix} \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle & \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{b}^\dagger \rangle \\ \langle \hat{a} \hat{a}^\dagger \hat{a} \hat{b} \rangle & \langle \hat{a} \hat{a}^\dagger \hat{b}^\dagger \hat{b} \rangle \end{vmatrix}$	2	4	(7,12)
D_{IIX}	$\begin{vmatrix} \langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle & \langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{b}^\dagger \rangle \\ \langle \hat{a}^\dagger \hat{a} \hat{a} \hat{b} \rangle & \langle \hat{a}^\dagger \hat{a} \hat{b}^\dagger \hat{b} \rangle \end{vmatrix}$	2	4	(8,13)
D_{IX}	$\begin{vmatrix} \langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle & \langle \hat{a}^\dagger \hat{a}^\dagger \hat{b}^\dagger \hat{b}^\dagger \rangle \\ \langle \hat{a} \hat{a} \hat{b} \hat{b} \rangle & \langle \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} \rangle \end{vmatrix}$	2	4	(8,15)

overall error on a given determinant decreases. Finally, in Figure 6.6d, we plot the confidence of detecting entanglement using these four NPT criteria as η is reduced from 1 to 0. The confidence of $D_{\text{VI-IX}}$ reduces from 100% down to 50% as the system is made more lossy.

As with $|\Psi\rangle_{\text{cat}}^-$, the state $|\Psi\rangle_{\text{sub}}$ generated when $m = 1$, $n = 1$, is invariant when labels $\mathcal{A}$ and $\mathcal{B}$ of each subsystem are swapped. Therefore, we could construct two further determinants by swapping $\hat{a}$ ($\hat{a}^\dagger$) with $\hat{b}$ ($\hat{b}^\dagger$) in D_{VII} and D_{IIX} and produce identical results. Unlike in Table 6.2, we have not analytically computed expressions for each determinant for the pure state when it is isolated from the environment and there are no sampling errors. This would be mathematically cumbersome to succinctly express.

6.5.3 Discussion

In finding the standard error of an NPT criterion, $\Delta \det[\mathbf{A}]$, we assume that we first find the expected value of each relevant Hermitian operator $\bar{B}_{ij,p}$ which appears in the determinant

by taking repeated measurements of $\hat{B}_{ij,p}$ and directly averaging. This enabled us to find an analytical formula for the first-order error correction $\Delta \det[\mathbf{A}] = \Gamma / \sqrt{M_{\text{tot}}}$. The sample mean of each Hermitian operator follows a normal distribution $\bar{B}_{ij,p} \sim \mathcal{N}(\text{E}[\bar{B}_{ij,p}], \text{V}[\bar{B}_{ij,p}])$ which follows from the central limit theorem. In our hypothesis test we made the further assumption that repeated measurements of $\det[\mathbf{A}]$ will also follow a normal distribution. Thus, our confidence of rejecting the no-entanglement hypothesis was given by $P(X < 0)$ where X is a test statistic which follows a normal distribution $X \sim \mathcal{N}(\det[\mathbf{A}], \Gamma^2 / M_{\text{tot}})$.

However, are we justified in assuming that the test statistic follows a normal distribution under the null hypothesis? Even though the sample means $\bar{B}_{ij,p}$ are normally distributed, the determinant $\det[\mathbf{A}]$ is a nonlinear function of these variables.

We can predict the probability density function of the test statistic X by numerically simulating the process of experimentally measuring each $\bar{B}_{ij,p}$. A single sample of X is made by randomly sampling values of $\bar{B}_{ij,p}$. We randomly sample $\bar{B}_{ij,p}$ under the assumption that it is normally distributed as discussed above; we assume that $\text{V}[\bar{B}_{ij,p}] = \text{V}[\hat{B}_{ij,p}] / M_{ij,p}$ and $M_{ij,p}$ have been allocated in the optimum way to minimize the overall error on the determinant (see Appendix B.2). From a set of sample values $\{\bar{B}_{ij,p}\}$ we calculate a matrix determinant which constitutes a single sample of X . Repeating this sampling process many times we can construct a histogram which corresponds to the probability density function of X .

In Figure 6.7, we have simulated the probability density function of a test statistic X for two separate NPT criteria calculated for the pure cat-state $|\Psi\rangle_{\text{cat}}^-$ with $\alpha = 1$, in the absence of losses. The blue histogram is the simulated probability density function of X assuming the values $\bar{B}_{ij,p}$ are randomly drawn from normal distributions as discussed above. We have fitted a normal distribution from the mean and standard deviation of the histogram which is shown by the solid blue line. We have also superimposed a normal distribution under the assumption that the test statistic follows the normal distribution $X \sim \mathcal{N}(\det[\mathbf{A}], \Gamma^2 / M_{\text{tot}})$, which was a key assumption made in the results presented in Section 5.4. This is shown by the dotted orange line for each NPT criterion. For some NPT criteria the histogram and orange line coincide remarkably well. For example, in Fig. 6.7a we have plotted D_{I} (the determinants are defined in

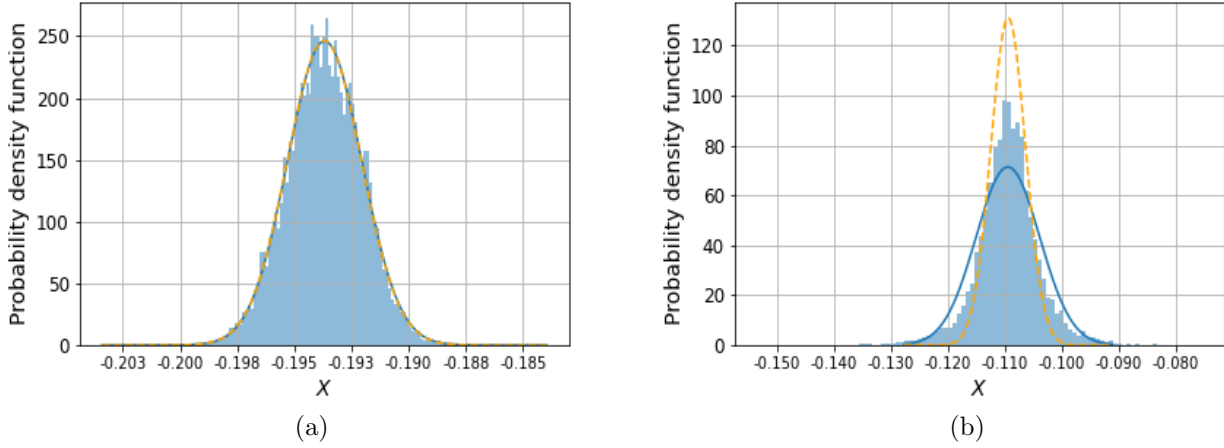


Figure 6.7: Empirical estimates of the distributions of the test statistic X for D_I and D_V obtained through numerical simulations of the sampling process used in their calculation. This involved simulating a measurement of each $\bar{B}_{ij,p}$ which are needed to compute a specific NPT criterion. A single sample of X is obtained by sampling each value of $\bar{B}_{ij,p}$ under the assumption that it is normally distributed according to its sampling distribution $\mathcal{N}(\mathbb{E}[\bar{B}_{ij,p}], \mathbb{V}[\bar{B}_{ij,p}])$. A sampled set of $\{\bar{B}_{ij,p}\}$ values are used to compute a matrix determinant, providing a single sample of X . For each histogram 10^4 samples are grouped into 100 bins. When drawing a value of $\bar{B}_{ij,p}$ from its sample distribution we assume that $\mathbb{V}[\bar{B}_{ij,p}] = \mathbb{V}[\hat{B}_{ij,p}]/M_{ij,p}$ and $M_{ij,p}$ have been allocated in the optimum way to minimize the overall error on the determinant. The solid blue line indicates a reference Gaussian distribution corresponding to the mean and variance of the empirical data. The orange dotted line shows the distribution of X under the assumption that it follows $X \sim \mathcal{N}(\det[\mathbf{A}], \Gamma^2/M_{\text{tot}})$ as assumed in the results presented in Section 6.5. (a) Empirical distribution of X for D_I . (b) Empirical distribution of X for D_V . The histogram deviates from the solid blue line indicating that X is not well-described by a Gaussian distribution. While the yellow dotted line does not fully match the histogram it provides a decent estimate of the spread of X .

Table 6.2). The agreement between the orange dotted line and the histogram means that for this criterion we are justified assuming that X is distributed as $X \sim \mathcal{N}(\det[\mathbf{A}], \Gamma/M_{\text{tot}})$. However, for other NPT criteria, such as in Fig. 6.7b where we have plotted D_V , the histogram of the simulated probability density function is not Gaussian. This non-Gaussianity is qualitatively evident from the fact that the fitted normal distribution (solid blue line) does not match with to the histogram. Interestingly, the assumption that X follows the normal distribution $X \sim \mathcal{N}(\det[\mathbf{A}], \Gamma^2/M_{\text{tot}})$ shown by the dotted orange line is a better fit to the histogram than the solid blue line. The widths of the orange dotted Gaussian and the histogram are fairly close suggesting that we have accurately predicted the first-order error correction on a given NPT criterion.

Therefore, a more thorough hypothesis test for a given NPT criterion would assume that the test statistic is distributed according to the blue histogram under the null hypothesis, and one would compute $p = P(X < 0)$ by finding the area of the histogram that corresponds to this probability. One would have to simulate a histogram for each NPT criterion and perform the hypothesis test against this histogram. Nevertheless, the formula we have derived for the standard deviation of the expected value of the determinant $\Delta(\det[\mathbf{A}]) = \Gamma/\sqrt{M_{\text{tot}}}$ is still a valid first-order approximation of the standard error; however, we cannot always assume that the null hypothesis follows a normal distribution. As a general method of comparing the expected value of a determinant to its standard error, a normal distribution hypothesis test is still beneficial in identifying a smaller subset of suitable NPT criteria for a given quantum state from the infinite class of inseparability criteria.

6.6 Towards the measurement of arbitrary two-mode moments of an optical field

As we have seen throughout this chapter, knowledge of bipartite moments of the form $\langle \hat{a}^{\dagger p} \hat{a}^q \hat{b}^{\dagger r} \hat{b}^s \rangle$ is hugely important in detecting entanglement. Specifically, higher-order moments are crucial in the verification of non-Gaussian entanglement where states cannot be described by first and second moments alone. These moments can also be used to evaluate nonclassicality, similarly, through the violation of inequalities constructed from moments [167]. We will now examine schemes which have been proposed for measuring single- and two-mode moments. We will also present an alternative measurement scheme which may be advantageous in certain experimental settings.

Firstly, let us examine the method of optical homodyne tomography which in principle gives us access to all statistical quantities pertaining to the state, including $\langle \hat{a}^{\dagger p} \hat{a}^q \hat{b}^{\dagger r} \hat{b}^s \rangle$. For a single-mode state, homodyne tomography requires a balanced homodyne detector (BHD) whereby the field of interest $\hat{a}$ is combined with a local oscillator $\alpha_{\text{LO}} e^{i\phi_1}$ at a 50:50 beam splitter. The difference in signal between two photodetectors at the outputs of the beam splitter provide the

quadrature measurement $\hat{q}_1(\phi_1) = (\hat{a}^\dagger e^{i\phi_1} + \hat{a} e^{-i\phi_1})/\sqrt{2}$. An ensemble of repeated measurements of the quadratures for various phases ϕ_1 is obtained [168]. This is repeated as the phase is varied discretely and for each phase a probability density function $P_{q_1}(\phi_1)$ (or histogram) is found. The Wigner function is related to these sets of probabilities through the inverse Radon transform which can be achieved by using a filtered backprojection and also via maximum likelihood methods. This reconstruction algorithm acts as a smoothing filter which dictates the resolution with which the Wigner function is determined and introduces statistical errors. The density matrix of the field can then be obtained by Fourier transforming the Wigner distribution. This can be straightforwardly extended to two-modes by introducing a second BHD which measures $\hat{q}_2(\phi_2) = (\hat{b}^\dagger e^{i\phi_2} + \hat{b} e^{-i\phi_2})/\sqrt{2}$. Measuring $\hat{q}_1(\phi_1)$ and $\hat{q}_2(\phi_2)$ many times and then discretely and independently varying the phases allows us to build a joint probability distribution $P_{q_1 q_2}(\phi_1, \phi_2)$. Again the inverse Radon transform can be applied. However, for two-modes this is much more computationally expensive and many more inversion operations are required.

It is also possible to obtain the density matrix by directly averaging experimental data [169], and this method has been examined for reconstructing a two-mode density matrix in the Fock basis [170]. Using a direct sampling method to obtain information pertaining to a two-mode state is therefore a powerful tool as it avoids going through the Wigner function and does not require the storage of large amounts of data as the sample averages (and therefore the state reconstruction) can be found in real time. Therefore, we henceforth consider quantum state measurement schemes which do not rely on tomographically recreating the Wigner function or density matrix. Since two BHDs are sufficient to obtain a tomographically complete measurement for determining the density matrix of a bipartite state [171], this set-up should also be sufficient in the measurement of other bipartite statistical quantities.

Assuming the modes of light are spatio-temporal it is important to distinguish two cases: the case where the two modes cannot be spatially separated, and the case where the two modes can be spatially separated (e.g. via filtering, polarizers, propagation direction). In the former case, the internal correlation statistics of optical pulses can be measured using a single BHD [170, 154]. In the latter case where the two modes can be separated in space, one must send the two separated modes into two separate homodyne detectors and average over the experimentally

determined joint quadrature measurements. To this end, there exist a number of schemes for measuring the quantum state of light based on homodyne detection [172, 173, 171], which exploit correlations between the two homodyne measurements in order to be used to reconstruct the quantum state. For example, the authors in Ref. [171] examine the reconstruction of an N -mode optical state. Using N -BHDs and retaining joint count distributions and joint difference-count distributions once can reconstruct the density matrix and any moment of the annihilation and creation operators. Crucially, in this scheme two Fourier integrals per mode are required. In all the schemes we have referenced, Fourier transforms are a recurring theme and one must vary the phases of the local oscillators of each BHD between 0 and 2π . Building on these homodyne protocols it has been suggested that in some applications it advantageous to use a weak local oscillator impinging on an unbalanced beam splitter [174]. It can be shown that this set-up can still extract information about the quantum state even when detection efficiencies are small.

Let us narrow the discussion and focus specifically on the measurement of moments. Intuitively, we anticipate fewer measurements are necessary for calculating a subset of specific moments compared with reconstructing a two-mode density matrix and subsequently finding the moments from that. The aforementioned weak local oscillator scheme has been examined in detail with regards to measuring single-moments of the form $\langle \hat{a}^{\dagger k} \hat{a}^l \rangle$ [155]. It has even been suggested [110, 21] that this single-moment protocol can be extended to two-modes in order to solve for any bipartite moment of the form $\langle \hat{a}^{\dagger p} \hat{a}^q \hat{b}^{\dagger r} \hat{b}^s \rangle$. This is not entirely straightforward to outline here however we can be confident that two independent local oscillators will be required (in order to be tomographically complete). Repeated measurements must subsequently be taken to construct some joint-count probability density function with fixed ϕ_1 and ϕ_2 . Furthermore, by studying the single-mode scheme we can deduce that one must perform two Fourier transforms by varying ϕ_1 and ϕ_2 continuously and independently between 0 and 2π . Below, we will briefly outline the scheme proposed in Ref. [155] for measuring $\langle \hat{a}^{\dagger p} \hat{a}^q \rangle$. We will then outline an alternative method for measuring $\langle \hat{a}^{\dagger p} \hat{a}^q \hat{b}^{\dagger r} \hat{b}^s \rangle$.

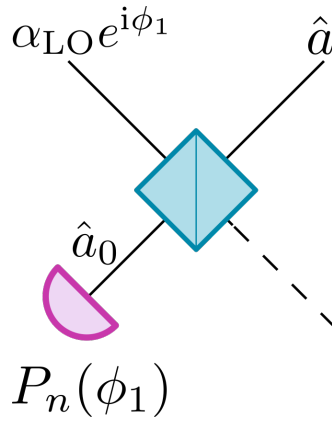


Figure 6.8: Homodyne detection scheme to determine single-mode moments $\langle a^{\dagger p} a^q \rangle$. A weak local oscillator is combined with the field of interest $\hat{a}$ at a beam splitter with transmissivity T_0 and reflectivity R_0 where $T_0 \gg R_0$. Through repeated photon number-resolving measurements we can build a function P_n for the probability of n clicks. Scanning ϕ_1 between 0 and 2π allows us to subsequently perform a Fourier transform which yields the function $f_n(n)$. Assuming we have calculated all the moments contained in $f_n(m)$ we can iteratively find increasingly high order single-mode moments by then calculating $f_{n+1}(m)$.

6.6.1 Calculating single-mode moments

The single-mode scheme outlined in Fig.1 of Ref. [155] is able to calculate moments of the form $\langle \hat{a}^{\dagger k} \hat{a}^l \rangle$ where $n = k + l$ is the order of the moment. In the protocol a local oscillator $\hat{a}_{LO}$ combines with the mode of interest $\hat{a}$ at a beam splitter of transmissivity T_0 reflectivity R_0 . The intensity of the local oscillator is typically of the same order of magnitude as the signal field, which can be realized with $T_0 \gg R_0$. The signal then passes through a sequence of 50:50 beam splitters where the number of beam splitters scales with the order of the moment. Coincidence clicks between n photodetectors are registered to obtain a correlation function $\Gamma^{(n)}$. The optical set-up is constructed under the assumption that one does not have access to photon-resolving detectors. We can achieve the same effect and bypass some of the mathematics by replacing the latter part of the apparatus with a photon-resolving detector. Instead, one can deduce the probability of an n photon detection event P_n via repeated measurements. The correlation function is defined as

$$\Gamma^{(n)} \propto \langle \hat{a}_0^{\dagger n} \hat{a}_0^n \rangle, \quad (6.26)$$

where

$$\hat{a}_0 = (T_0 \hat{a} + R_0 \hat{a}_{LO}) / \sqrt{2}. \quad (6.27)$$

However, from Glauber's Theorem, an idealized photon counting device absorbs n quanta when n photons are detected. The field transitions from the initial state $|i\rangle$ to the final state $|f\rangle$. The probability amplitude of this transition is given by $\langle f | \hat{a}_0^{\dagger n} | i \rangle$. Therefore, the probability of an n photon detection is found by taking the modulus squared, summing over all possible final states, and inserting the resolution of identity to obtain

$$P_n = \sum_f |\langle f | \hat{a}_0^{\dagger n} | i \rangle|^2 = \langle \hat{a}_0^{\dagger n} \hat{a}_0^n \rangle, \quad (6.28)$$

which is equivalent to $\Gamma^{(n)}$ up to some normalization. Therefore we will circumvent some of the mathematical details of the linear optics by just assuming we have access to photon-resolving detectors and consider the function P_n .

With access to the probability P_n one can iteratively obtain all the moments $\langle \hat{a}^{\dagger k} \hat{a}^l \rangle$. This can be seen by inserting Eq. (6.27) into Eq. (6.28). Assuming the local oscillator is simply $\hat{a}_{\text{LO}} = \alpha e^{i\phi_{\text{LO}}}$ we obtain

$$P_n = \sum_{m=-n}^n f_n(m) e^{im\phi}, \quad (6.29)$$

where $\phi = \phi_{\text{LO}} - \pi/2$. The term

$$f_n(m) = \frac{1}{2^n} \sum_{k+l=m} \binom{n}{k} \binom{n}{l} |T_0|^{k+l} |R_0 \alpha|^{2n-k-l} \langle \hat{a}^{\dagger k} \hat{a}^l \rangle, \quad (6.30)$$

contains the moments of interest. Therefore, by Fourier transforming the correlation function we obtain

$$f_n(m) = \frac{1}{2\pi} \int_0^{2\pi} d\phi P_n e^{-im\phi}. \quad (6.31)$$

An individual moment, say $\langle \hat{a}^{\dagger k'} \hat{a}^{l'} \rangle$ can be found iteratively. First one must calculate all the first order terms which are found via $f_1(-1), f_1(0), f_1(1)$ before progressing onto the second order terms $f_2(-2), \dots, f_2(2)$.

We will take inspiration from these schemes and construct a similar measurement BHD protocol that does not involve scanning the local oscillator phases continuously. The protocol is iterative, similar to Ref. [155] however we will also exploit some of the techniques outlined in Section. 5.3.2.

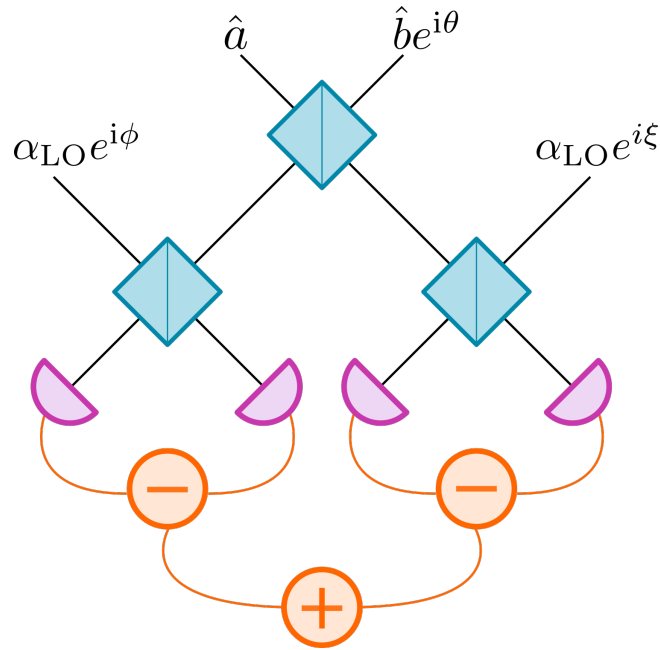


Figure 6.9: Optical schematic outlining an alternative method for obtaining arbitrary two-mode moments $\langle \hat{a}^{dagp} \hat{a}^q \hat{b}^{\dagger r} \hat{b}^s \rangle$. Assuming the modes are spatially separated they are initially combined at a 50:50 beam splitter whose phase is θ . The outputs of the beam splitter are then mixed with two independent local oscillator fields whose phases, ϕ and ξ , are independently controlled. The difference between the two photocurrents of each detector (shown in purple) are then added to obtain a final joint ‘homodyne’ measurement given in Eq. (6.33).

6.6.2 Alternative measurement scheme

Here we outline another method inspired by homodyne measurements. This will allow us to find any moment $\langle \hat{a}^{\dagger p} \hat{a}^q \hat{b}^{\dagger r} \hat{b}^s \rangle$ of order $n = p + q + r + s$ quantum states comprised of two spatially separable modes. The scheme works in an iterative manner whereby we must first find all the moments of order less than or equal to n first before progressing to $n + 1$.

Our set-up is outlined in Fig. 6.9, whereby each mode of interest $\hat{a}$ and $\hat{b}$ is combined at a 50:50 beam splitter with two independent local oscillators $\alpha_{LO}e^{i\phi_1}$ and $\alpha_{LO}e^{i\phi_2}$, respectively. We have assumed both local oscillators have the same real amplitude but this can be generalised and is unimportant here. Two photodetectors are connected together so that their photocurrents are subtracted giving an output which can be renormalized to be equivalent to the following

quadratures:

$$\hat{q}_1(\phi, \theta) = (\hat{a}^\dagger + \hat{b}^\dagger e^{-i\phi})e^{i\theta} + (\hat{a} + \hat{b}e^{i\phi})e^{-i\theta} , \quad (6.32a)$$

$$\hat{q}_2(\phi, \xi) = (\hat{a}^\dagger - \hat{b}^\dagger e^{-i\phi})e^{i\xi} + (\hat{a} - \hat{b}e^{i\phi})e^{-i\xi} . \quad (6.32b)$$

We can combine these quadratures by adding the photocurrents again. This is essentially a four-detector correlation where we add two of the outputs and subtract the other two outputs. Therefore the final ‘homodyne’ measurement made each run is

$$\hat{q}_{12}(\phi, \theta, \xi) = \hat{a}^\dagger(e^{i\theta} + e^{i\xi}) + \hat{a}(e^{-i\theta} + e^{-i\xi}) + \hat{b}^\dagger(e^{i(\theta-\phi)} - e^{i(\xi-\phi)}) + \hat{b}(e^{-i(\theta-\phi)} - e^{-i(\xi-\phi)}) . \quad (6.33)$$

Keeping the phases fixed we can take repeated measurements (recreating the state between runs) in order to build up a probability distribution for the joint quadrature measurement $q_{12}(\phi, \theta, \xi)$. From the data set of repeated measurements we can deduce the n^{th} -order moment. This can be done by directly averaging over each data point raised to the power of n which is discussed in detail in Section 5.3.2 (see Eq. (5.42)). The n^{th} -order moment of the joint quadrature distribution is

$$\begin{aligned} \langle \hat{q}_{12}^n(\phi, \theta, \xi) \rangle &= \sum_{p'+q'+r'+s'=n} (e^{i\theta} + e^{i\xi})^{p'} (e^{-i\theta} + e^{-i\xi})^{q'} (e^{i(\theta-\phi)} - e^{i(\xi-\phi)})^{r'} (e^{-i(\theta-\phi)} - e^{-i(\xi-\phi)})^{s'} \\ &\quad \times \mathcal{S}[\langle \hat{a}^{\dagger p'} \hat{a}^{q'} \hat{b}^{\dagger r'} \hat{b}^{s'} \rangle] , \end{aligned} \quad (6.34)$$

where $\mathcal{S}[\langle \hat{a}^{\dagger p'} \hat{a}^{q'} \hat{b}^{\dagger r'} \hat{b}^{s'} \rangle]$ is the symmetrized sum of the possible moments which can be made from these annihilation and creation operators. For example, $\mathcal{S}[\langle \hat{a}^{\dagger 2} \hat{a} \rangle] = \langle \hat{a}^{\dagger 2} \hat{a} \rangle + \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \rangle + \langle \hat{a} \hat{a}^{\dagger 2} \rangle$. Assuming the measuring procedure is conducted iteratively and we have access to all lower order moments, we can consequently compute every moment that appears in $\mathcal{S}[\langle \hat{a}^{\dagger 2} \hat{a} \rangle]$. For example, suppose we wish to calculate $\langle \hat{a}^{\dagger 2} \hat{a} \rangle$ then we note that $\langle \hat{a}^{\dagger 2} \hat{a} \rangle = 1/3(\mathcal{S}[\langle \hat{a}^{\dagger 2} \hat{a} \rangle] - \langle \hat{a}^\dagger \rangle)$. Provided we know $\mathcal{S}[\langle \hat{a}^{\dagger 2} \hat{a} \rangle]$ and $\langle \hat{a}^\dagger \rangle$ then we can compute $\langle \hat{a}^{\dagger 2} \hat{a} \rangle$. Therefore, the problem of finding an arbitrary two-mode moment $\langle \hat{a}^{\dagger p} \hat{a}^q \hat{b}^{\dagger r} \hat{b}^s \rangle$ is reduced to calculating the relevant

quantity $\mathcal{S}[\langle \hat{a}^{\dagger p} \hat{a}^q \hat{b}^{\dagger r} \hat{b}^s \rangle]$ which contains the desired moment. However, before we progress to higher-order moments $n + 1$ we must calculate all terms $\mathcal{S}[\langle \hat{a}^{\dagger p'} \hat{a}^{q'} \hat{b}^{\dagger r'} \hat{b}^{s'} \rangle]$ which appear in the expansion of $\langle \hat{q}_{12}^n(\phi, \theta, \xi) \rangle$.

Studying Eq. (6.34), there are N unknown terms of the form $\mathcal{S}[\langle \hat{a}^{\dagger p'} \hat{a}^{q'} \hat{b}^{\dagger r'} \hat{b}^{s'} \rangle]$ where $N = \sum_{p+q+r+s} 1$. Therefore, in order to solve each of these unknown symmeterized terms, we must repeat the whole process with N sets of phases $\{\phi_k, \theta_k, \xi_k\}$ where the $k = 1, \dots, N$. For each set of phases, we must calculate $\langle \hat{q}_{12}^n(\phi_k, \theta_k, \xi_k) \rangle$.

These phases must be picked appropriately such that we obtain a set of N linearly independent equations which can be solved for each $\mathcal{S}[\langle \hat{a}^{\dagger p'} \hat{a}^{q'} \hat{b}^{\dagger r'} \hat{b}^{s'} \rangle]$. If we consider the example of $n = 2$, then we have a vector of 10 unknown symmeterized terms

$$\mathbf{x} = \begin{pmatrix} \mathcal{S}[\langle \hat{b}^2 \rangle] \\ \mathcal{S}[\langle \hat{b}^\dagger \hat{b} \rangle] \\ \mathcal{S}[\langle \hat{b}^{\dagger 2} \rangle] \\ \mathcal{S}[\langle \hat{a} \hat{b} \rangle] \\ \mathcal{S}[\langle \hat{a} \hat{b}^\dagger \rangle] \\ \mathcal{S}[\langle \hat{a}^2 \rangle] \\ \mathcal{S}[\langle \hat{a}^\dagger \hat{b} \rangle] \\ \mathcal{S}[\langle \hat{a}^\dagger \hat{b}^\dagger \rangle] \\ \mathcal{S}[\langle \hat{a}^\dagger \hat{a} \rangle] \\ \mathcal{S}[\langle \hat{a}^{\dagger 2} \rangle] \end{pmatrix}. \quad (6.35)$$

With a set of 10 phase sets $\{\phi_k, \theta_k, \xi_k\}$ we can construct a 10×10 matrix $\mathbf{P}$ whose entries can be found from the coefficients in Eq. (6.34). By inverting $\mathbf{P}$ we can solve the equation

$$\mathbf{P}\mathbf{x} = \mathbf{y}, \quad (6.36)$$

where $\mathbf{y}$ is a vector of length 10 containing the second order moments such that the i^{th} element is $\mathbf{y}_i = \langle \hat{q}_{12}^2(\phi_i, \theta_i, \xi_i) \rangle$. In order to solve for each symmetric term $\mathcal{S}[\langle \hat{a}^{\dagger p'} \hat{a}^{q'} \hat{b}^{\dagger r'} \hat{b}^{s'} \rangle]$ in $\mathbf{x}$ where $p' + q' + r' + s' = 2$ we must ensure that $\mathbf{P}$ has rank 10.

Let us examine $\mathbf{P}$ in more detail. Making the substitution $A_k = e^{i\theta_k} + e^{i\xi_k}$ and $B_k = e^{i(\theta_k - \phi_k)} - e^{i(\xi_k - \phi_k)}$, where $A_k, B_k \in \mathbb{C}$, we can fill out the first few rows of $\mathbf{P}$ noting that as we go down the rows the index k increases, such that

$$\mathbf{P} = \begin{pmatrix} (B_1^*)^2 & |B_1|^2 & (B_1)^2 & A_1^* B_1^* & A_1^* B_1 & (A_1^*)^2 & A_1 B_1^* & A_1 B_1 & |A_1|^2 & (A_1)^2 \\ (B_2^*)^2 & |B_2|^2 & (B_2)^2 & A_2^* B_2^* & A_2^* B_2 & (A_2^*)^2 & A_2 B_2^* & A_2 B_2 & |A_2|^2 & (A_2)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (B_{10}^*)^2 & |B_{10}|^2 & (B_{10})^2 & A_{10}^* B_{10}^* & A_{10}^* B_{10} & (A_{10}^*)^2 & A_{10} B_{10}^* & A_{10} B_{10} & |A_{10}|^2 & (A_{10})^2 \end{pmatrix}. \quad (6.37)$$

Choosing the 10 phase sets $\{\phi_k, \theta_k, \xi_k\}$ in a suitable manner is therefore crucially important in order to solve for each symmetric term. Although we have now parameterised the matrix $\mathbf{P}$ in terms of 20 complex numbers $\{A_k, B_k\}$, do we have complete freedom over our choice of A_k and B_k within a pair corresponding to k^{th} index? By rewriting A_k and B_k in terms of their magnitudes and complex phase

$$A_k = |e^{i\theta_k} + e^{i\xi_k}| e^{i \arg(A_k)}, \quad (6.38a)$$

$$B_k = |e^{i\theta_k} - e^{i\xi_k}| e^{i \arg(B_k)}, \quad (6.38b)$$

it is evident that the magnitudes are not independent. Furthermore, the argument of the complex number A_k is some function of θ_k and ξ_k . However, the argument of B_k can be separated into part that depends on θ_k and ξ_k , and an independent part which depends on ϕ_k alone. Assuming one picks A_k first then it follows that

$$A_k = |A_k| e^{i \arg(A_k)}, \quad (6.39a)$$

$$B_k = \sqrt{1 - |A_k|^2} e^{i f(\theta_k, \xi_k)} e^{i \phi_k}, \quad (6.39b)$$

where $f(\theta_k, \xi_k) = \arg(e^{i\theta_k} - e^{i\xi_k})$. Since θ_k and ξ_k are dictated by the choice of A_k , but ϕ_k is defined only for B_k , we can conclude that for a given A_k , B_k is a complex number whose magnitude depends on $|A_k|$ but whose direction in the complex plane can be arbitrarily chosen by altering ϕ_k . This is visually demonstrated in Figure 6.10. Therefore, choosing the entries of

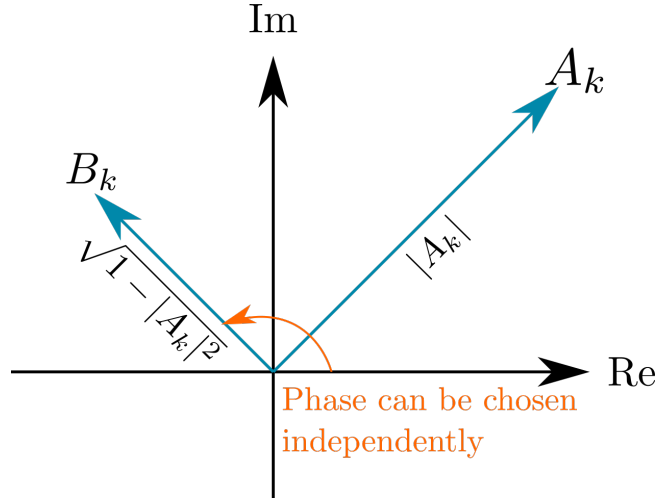


Figure 6.10: Argand diagram representing two complex parameters A_k and B_k as vectors. We can parameterise the k^{th} row of the matrix $\mathbf{P}$ using A_k and B_k . These complex numbers depend on the choice of phases $\{\phi_k, \theta_k, \xi_k\}$. As demonstrated in the diagram, for a given set of phases the magnitude of B_k depends on the magnitude of A_k . However, the phase B_k can be independently picked by varying ϕ_k .

$\mathbf{P}$ is now reduced to selecting appropriate pairs $\{|A_k|, \phi_k\}$ to ensure that $\mathbf{P}$ is a rank 10 matrix, for the case where we consider moments of order $n = 2$. This method can be extended for an arbitrary n in which case the matrix $\mathbf{P}$ must be of rank N where we remind the reader that $N = \sum_{p+q+r+s=n} 1$.

Generating a matrix $\mathbf{P}$ of rank N is a formidable task due to the nonlinear interplay between A_k and B_k . Checking the linear independence of each row of $\mathbf{P}$ analytically is challenging especially for large n as one has to perform Gaussian row elimination on a $N \times N$ matrix. To demonstrate how N scales with n we have listed the first few terms in Table 6.4.

Intriguingly, choosing $|A_k|$ and ϕ_k randomly between the ranges $[-2, 2]$ and $[0, 2\pi]$, respectively, is particularly effective. Through simulations, one can show that it is possible to generate a rank N matrix in this manner and thus solve for all N unknown $\mathcal{S}[\langle \hat{a}^{\dagger p} \hat{a}^q \hat{b}^{\dagger r} \hat{b}^s \rangle]$. One potential drawback of this approach arises from the precision of ϕ_k (and the precision with which we can control θ_k and ξ_k which in turn affect the error on $|A_k|$). When randomly picking phases, it is possible that two phases overlap when their precision is taken into account.

We can attempt to introduce a greater degree of structure in our choice of $\{|A_k|, \phi_k\}$. For a matrix to have linearly independent rows it is not sufficient for each entry in the matrix to

Table 6.4: We present how N scales with n up to $n \leq 9$. In order to calculate all the moments which have order n one must calculate $\langle q_{12}^n(\phi_k, \theta_k, \xi_k) \rangle$ via repeated measurements for a set of N phases $\{\phi_k, \theta_k, \xi_k\}$, where $N = \sum_{p+q+r+s=n} 1$. Assuming all the moments of order $n-1$ have previously been calculated, one can solve for all moments $\langle a^{\dagger p} a^q b^{\dagger r} b^s \rangle$ for which $p+q+r+s=n$ by choosing the phase sets in an appropriate manner such that we obtain N linearly independent equations which can be stacked into the matrix $\mathbf{P}$ such that the rank of this matrix is N .

n	1	2	3	4	5	6	7	8	9
N	4	10	20	35	56	84	120	165	220

simply be different. We can begin by picking a different A_k as k is varied. To introduce a nonlinearity between the rows we pick ϕ_k to be some nonlinear function of k . We can construct A_k and B_k in the following manner, for $k = 1, \dots, N$,

$$A_k = \frac{k}{N} e^{ik2\pi/N}, \quad (6.40a)$$

$$B_k = \sqrt{1 - |A_k|^2} e^{ik^2 2\pi/N}. \quad (6.40b)$$

This choice of $\{A_k, B_k\}$ is sufficient for solving all the symmetrized terms in $n = 8$ where one is able to generate a rank 165 matrix. Consequently all moments of order 8 can also be deduced from the symmetrized terms. However, for $n = 9$, this method only generates a matrix $\mathbf{P}$ of rank 217 instead of the desired rank of 220. This rank deficiency could be due to numerical error since the eigenvalues of $\mathbf{P}$ become extremely small. Interestingly, randomly choosing A_k and B_k works for calculating moments higher than $n = 8$ and the size of the eigenvalues does not reduce as quickly.

6.6.3 Discussion

Measuring arbitrary two-mode optical moments of the form $\langle \hat{a}^{\dagger p} \hat{a}^q \hat{b}^{\dagger r} \hat{b}^s \rangle$ is crucial for the verification of entanglement using the inseparability criteria we first introduced in Chapter 3. We have examined the methods by which we can measure such moments which fall into two categories: (i) first reconstructing the quantum state using the Wigner distribution or density matrix and consequently deducing moments. State reconstruction can be achieved either through state tomography or through the less-resource intensive method of direct sampling. Or

(ii), by performing homodyne measurements and measuring statistical averages, one can infer the moments through post-processing. The latter methods are less resource-intensive since we do not have to reconstruct the entire quantum state. Proposals which fall into the second category rely on homodyne detection (either with photon counting or difference-counting at the output) to build up joint distributions. By Fourier transforming the average of this joint distribution we can find the desired moment, usually in an iterative manner. Crucially, the Fourier transform is performed with respect to the phase of the local oscillator used in the homodyne set-up. Therefore, one must scan the phase making sure that one takes a sufficient number of repeated measurements for each phase.

We have introduced an alternative protocol for calculating an arbitrary moment $\langle \hat{a}^{\dagger p} \hat{a}^q \hat{b}^{\dagger r} \hat{b}^s \rangle$ assuming the two optical modes can be spatially separated. This approach also relies on a homodyne set-up where the two modes are first combined at 50:50 beam splitter with a controllable phase shift. The two outputs of the beam splitter are then combined with two local oscillators at 50:50 beam splitters. The phases of the local oscillators are independently controlled. The four outputs are then measured by photodetectors and by combining the photocurrents we can obtain a joint quadrature measurement given in Eq. (6.33). Taking repeated measurements allows us to find the n^{th} order moment of this distribution $\langle \hat{q}_{12}^n(\phi_k, \theta_k, \xi_k) \rangle$ for a given set of phases $\{\phi_k, \theta_k, \xi_k\}$. We must repeat the process with N distinct phase sets such that $k = 1, \dots, N$, where $N = \sum_{p+q+r+s=n} 1$. This enables to solve an arbitrary moment $\langle \hat{a}^{\dagger p} \hat{a}^q \hat{b}^{\dagger r} \hat{b}^s \rangle$ where $p+q+r+s = n$, assuming we have already found all moments with order $n-1$. In order to have sufficient information to solve for any moment of order n we must choose the phases carefully.

Therefore, one advantage of our approach is the fact that a Fourier transform is not required. We can identify a smaller set of relevant phases without needing to scan each phase independently between 0 and 2π . However, for higher n we need an increasingly large number of phase sets. For example, with $n = 9$ we require 220 phase sets. There will come a point where it is no longer advantageous to consider distinct sets of phases due to the precision error of each phase. For such high order it may be more straightforward to scan each phase and perform a Fourier transform.

For small n it is relatively easy to identify a suitable sets of phases $\{\phi_k, \theta_k, \xi_k\}$ such that we have a sufficient number of linearly independent equations to solve for each moment. However, for $n \geq 9$ using a systematic method of phase allocation becomes increasingly challenging. While we have identified a structure which works for $n \leq 8$, due to numerical errors it is inconclusive whether this method can work for higher order moments. However, choosing the phases randomly is surprisingly effective even for $n \geq 9$. Nevertheless, for such high-order moments even when we choose phases randomly, we again encounter the problem of ensuring phase sets do not unintentionally overlap due to their precision error.

Finally, a strength of our approach is its relative experimental simplicity. Our optical set-up (see Figure. 6.9) relies on established homodyne detection techniques which do not require photon-number-resolving detectors. Furthermore, as the order of the moments n increases, we require more measurements to be taken, however there is no increased complexity in the set-up.

6.7 Conclusion

In the first half of this chapter we simulated the experimental process of verifying entanglement using an NPT criterion to test the entanglement of a two-mode state. We construct an entanglement criterion by assessing the negativity of the determinant of a valid matrix. The matrix is composed of moments of annihilation and creation operators of the modes. However, as we have demonstrated, the negativity in isolation is not necessarily a conclusive entanglement test as it is important to also consider the magnitude of its uncertainty. Therefore, we have derived an analytical expression for the first-order error on an NPT criterion, taking into account sampling errors and interactions with the open system. This is done on the assumption that the expectation value of each Hermitian operator which appears in the criterion is independently obtained by averaging a data set of repeated measurements (with the caveat that we reuse the same data set to also calculate the expectation of its Hermitian conjugate). Therefore, the interplay between the magnitude of the negativity and its error is an important factor which we have explored through hypothesis testing. This has enabled us to identify state-specific NPT

criteria which are best suited at detecting entanglement with a high degree of confidence for a given number of experimental runs and degree of coupling to environment.

Through our simulations, we have identified specific NPT criteria which are optimal at detecting entanglement in two states for the parameters we have considered. The two-mode Schrodinger-cat state criteria are listed in Table 6.2; while criteria for the photon-subtracted TMSV state are presented Table 6.3. We have investigated the interplay between the total number of measurements allocated to the evaluation of an NPT criterion; as the number of measurements increases the error associated with an entanglement criterion decreases. We have also examined the success of these criteria when we introduce coupling between the bipartite state and a noisy environment. We have used the confidence in rejecting the no-entanglement hypothesis as our metric for success. This determines how sure we are that a criterion is negative under the assumption that the repeated measurements of the criterion are normally distributed with a standard deviation given by its first-order standard error. We have therefore also discussed the validity of assuming our test statistic follows a normal distribution.

More generally, our method of error propagation can be applied to other non-classicality tests [167] and separability tests [153] which are based on the positivity of a determinant found from a matrix of moments.

The latter half of this chapter has been dedicated to experimental protocols for measuring arbitrary two-mode moments of annihilation and creation operators. We have examined the literature concerning the reconstruction of a two-mode quantum state and also proposals for homodyne detection schemes which aim to measure these moments without knowledge of the density matrix. A common feature of these schemes is their reliance on scanning the phase of the local oscillator in order to perform a Fourier transform. We have therefore proposed an alternative homodyne scheme which relies on repeated measurements but with a sets of fixed phases. This is potentially more experimentally amenable in some cases as fewer phases are required for lower-order moments compared with other measurement schemes. However, for extremely high-order moments, analytically choosing suitable phases sets can be challenging. Surprisingly, randomly picking phases sets can be more successful. However, high-order mo-

ments become increasingly difficult to measure as many more phase sets, and therefore more total measurements, are required. Nevertheless, for the states we have considered in the first section of this chapter, one does not require such high-order moments to confidently detect entanglement.

A final caveat which should be addressed is that the measurement schemes for measuring two-mode moments (both in the literature and the scheme we have proposed) are iterative. One must possess knowledge of all moments of order n before progressing to moments of order $n + 1$. This means that a data set of repeated measurements which is used to calculate the moment of a Hermitian operator of order n is not necessarily independent from the data set of another Hermitian operator of order $n + 1$. Analytically computing the covariance of these data sets is challenging and also depends on the precise measurement scheme employed. Mathematically, the assumption that $\text{Cov}[\bar{B}_{ij,p}\bar{B}_{k,q}] = \delta_{ik}\delta_{jl}\delta_{pq}V[\bar{B}_{ij,p}]$ used in Eq. (6.18) is no longer valid; one instead must use $\text{Cov}[\bar{B}_{ij,p}\bar{B}_{kl,q}] = E[(\bar{B}_{ij,p} - E[\bar{B}_{ij,p}])(\bar{B}_{kl,q} - E[\bar{B}_{kl,q}])]$. The error propagation would have to be simulated in accordance with the way the data sets are combined. In fact, for an iterative homodyne detection scheme, one might use the same data set for the calculation of many moments. However, one could still assume that the data sets for each moment are independent if they are partitioned in an unbiased manner and each data subset is only used in the calculation of one moment. However more measurements would be required so that each smaller data set has sufficient measurements. Our analytical formula may not be entirely accurate for these iterative homodyne schemes; however, it still provides a method of quantifying the anticipated error in order to compare suitable NPT criteria. Our formula for the error of a criterion $\Delta\det[\mathbf{A}] = \Gamma/\sqrt{M_{\text{tot}}}$ is parameterised by the total number of measurements allocated to determining the criterion. In these iterative schemes we are likely to require a fewer number of total measurements since data can be reused for more than one moment. However, the value of Γ may be greater due to the covariances between the data sets.

Chapter 7

Testing wavefunction collapse with the displacemon device

The work described in this chapter was conducted in collaboration with Jack Clarke, Michael R. Vanner, and Edward A. Laird and has been published in *AVS Quantum Sci.* **3** (4), 045603, 2021 [106]. This publication is part of a Special Collection of topics involving quantum theory that have been inspired by Roger Penrose’s work. This chapter is partly based on this publication and examines whether the device first introduced in Ref. [22] could be exploited to test objective-collapse theories.

7.1 Introduction

The quantum measurement problem and the quantum-to-classical interface have been much debated since the inception of quantum mechanics. This debate has sparked numerous questions such as: “Is there a limit to the extent that the Schrödinger equation may be applied?” and: “What are the mechanisms that cause a coherent quantum superposition to lose its interference features and result in classical behaviour?” This debate and these particular questions explore the two key dynamical processes in quantum mechanics, namely the linear deterministic Schrödinger evolution and the nonlinear stochastic wavefunction collapse owing to a measure-

ment. Several different physical interpretations of these two processes have become popular among various groups within the physics and philosophy communities. However it is often very difficult or even impossible to experimentally distinguish between these differing views of the world.

One resolution to the measurement problem and the quantum-to-classical interface is provided by objective collapse theories. These propose physical mechanisms, not encompassed by standard Schrödinger evolution, which cause wavefunctions to collapse without the need for collapse by a measurement; hence the terminology ‘objective’ collapse.

Following early progress in modifying the Schrödinger equation [175, 176, 177, 178], spontaneous localization models were studied by Ghirardi, Rimini, and Weber (GRW) [179]. GRW theory satisfies the requirements of a physical collapse model with the key ingredient being that an N -particle wavefunction $\psi_N(x_1, \dots, x_N)$ experiences a spontaneous Gaussian localization with the effect of premultiplying it by an operator of the form $\exp\left(-(\hat{X} - x_k)^2/(2r^2)\right)$, where r is the localisation length and $\hat{X}$ is the position operator. These Gaussian jumps happen at randomly distributed times with a mean rate λ which is chosen to preserve unitary time evolution for microscopic systems, but importantly accounts for classical features emerging as the number of particles in an object is increased [180].

A further refinement of this type of collapse theory combines the ideas of stochastic dynamical reduction together with GRW to obtain continuous spontaneous localization (CSL) [181], which is now a widely studied model. In contrast to GRW theory, CSL assumes the localization operations happen continuously in time. Like GRW, CSL behaviour is characterised by two phenomenological parameters, the collapse rate λ_{CSL} and the localization length r_{CSL} . The goal of many experimental tests of CSL is to impose bounds on these constants. To this end, $r_{\text{CSL}} = 100 \text{ nm}$ and $\lambda_{\text{CSL}} = 10^{-8 \pm 2} \text{ Hz}$ have been identified as an interesting part of parameter space to explore [182, 183, 184, 185].

Another contender to reconcile the issues presented by the measurement problem, offered by Diósi [186, 187] and later rederived through a different line of reasoning by Penrose [188], is now referred to as Diósi-Penrose (DP) collapse. This gravity-induced collapse model is argued

by Penrose to arise from the incompatibility of the theories of general relativity and quantum mechanics. The essence of the theory is that for a massive particle in a superposition of two well-localised states, each state has an associated space-time geometry leading to a conflict with Einstein's theory of general covariance [189]. This superposition of differing gravitational fields leads to an instability which induces wave-function collapse after a characteristic lifetime t_{DP} . To quantify the difference between two space-times, Penrose has suggested using the gravitational self-energy from classical physics E_{G} , and relating this to t_{DP} via $t_{\text{DP}} = \hbar/E_{\text{G}}$.

Tests of objective collapse models aim to detect deviations from standard open-quantum system dynamics. Non-interferometric tests focus on sensing the additional sources of decoherence which are predicted by collapse models, usually in the form of heating. Searching for this excess noise has led to experimental realisations including test masses on ultracold cantilevers [190], underground tests [191], and entangling macroscopic diamonds [137, 192]. Meanwhile, a host of non-interferometric experiments have been proposed using a range of platforms including optomechanics [193], superconducting devices [194], ultracold atoms [195], and levitated nanospheres [196]. Notably in optomechanics, building on the Bose-Jacobs-Knight proposal [6], the Marshall-Simon-Penrose-Bouwmeester mirror-superposition scheme proposes how to study spatial superposition states involving 10^{14} atoms [7]. On the other hand, interferometric tests strive to create large superpositions with increasingly massive objects in order to probe the quantum-to-classical transition [68]; these experiments include atom interferometry [197], superpositions of large molecules [198, 199], interferometry of free-falling nano-particles [200], and also proposals with ultracold atomic systems [201]. Meanwhile, electromechanical devices have been used in several experiments that, while not testing collapse models directly, nevertheless probe fundamental aspects of quantum behaviour, such as coherent exchange between a mechanical resonator and a qubit [202], and entanglement between a resonator and a microwave field [127] or another resonator [128]. Moreover, there is growing evidence that electromechanical devices are a viable route for tests of collapse, particularly with mechanical oscillators coupled to superconducting circuits [203]. We have already introduced the field of electromechanics in Section 2.5.

In this Chapter we propose an experimental scheme to probe collapse models using the displace-

mon platform [22]. This device consists of a vibrating nanobeam coupled to a superconducting qubit such that operations performed on the qubit allow us to generate non-Gaussian states of the centre-of-mass of the mechanical system. The displacemon device and Hamiltonian are discussed in detail in Section 2.5. Building on this theory, here we will outline the qubit operations required to produce mechanical superpositions. A scheme is then presented that allows us to monitor the decoherence of these large-scale mechanical superpositions via qubit measurement and thereby test for sources of decoherence that could be attributed to DP or CSL collapse theories. We carefully examine the feasibility of our protocol given current and near-term experimental parameters and find that with technical improvements in isolating the device from vibrational noise, the displacemon device provides an experimentally viable route for testing DP and CSL models of objective collapse beyond existing bounds.

7.2 Grating operations with the displacemon

7.2.1 Relationship between the displacemon and a conventional matter-wave interferometer

A matter-wave interferometer works by passing the test particle through a series of diffraction gratings and then measuring its position. A well-known example is the electron double-slit interferometer. Quantum superposition manifests itself as a series of spatial fringes, which arise from interference between different paths a particle could take through the gratings. This kind of matter-wave interferometry, now realised using increasingly massive test particles in sophisticated interferometers, allows the most stringent interferometry-based exclusions of spontaneous collapse theories [204]. The largest mass tested so far is 27,000 amu, surviving in a superposition for 7 ms [199]. However, this mass is still too small to test gravitational decoherence and some theories of spontaneous collapse.

Progress in this area continues to be rapid, but to explore the gravitational-decoherence scale using conventional matter-wave interferometers, two major challenges need to be addressed.

First, as test masses become larger, it becomes increasingly difficult to create suitable diffraction gratings. Existing state-of-the-art interferometers use finely patterned membranes in the first and final step. However, the test particles interact with these gratings by van der Waals interaction, which becomes stronger as the particle diameter increases and introduces unwanted smearing of the wavefunction [205]. Future proposed interferometers will use all-optical gratings [200], but here the large mismatch between the optical wavelength and the de Broglie wavelength of a massive test particle imposes other stringent requirements on the interferometer. The second challenge is the requirement for the particle to evolve in free-fall, so that the wavefunction has time to spread out before the final measurement.

The displacemon [22] provides a very different approach to test objective collapse models by using as its test particle a doubly-clamped cantilever coupled to a superconducting qubit. The key idea utilizes a “synthetic diffraction grating”, realised through a series of operations on the qubit.

7.2.2 The displacemon protocol

In Section 2.5.2 we derived the following Hamiltonian for the flux coupled qubit-resonator system

$$\hat{H} = \hbar \frac{\omega_q}{2} \hat{\sigma}_z + \hbar \Omega \hat{b}^\dagger \hat{b} + \hbar \lambda(t) (\hat{b} + \hat{b}^\dagger) \hat{\sigma}_z, \quad (7.1)$$

where ω_q is the bare qubit frequency, σ_z is the Pauli operator operating in the qubit basis with eigenvectors $|+\rangle$ and $|-\rangle$, and λ is the qubit-mechanical coupling parameter, which may depend on time t . The mechanical resonator has creation (annihilation) operator $\hat{b}$ ($\hat{b}^\dagger$) and mechanical frequency $\Omega/2\pi$. The displacement is $\hat{X} \equiv X_{\text{ZP}}(\hat{b} + \hat{b}^\dagger)$, where $X_{\text{ZP}} \equiv \sqrt{\hbar/2m\Omega}$. We emphasise that elsewhere in this Thesis $\hat{X}$ and $\hat{P}$ are the dimensionless phase quadratures; however, in this chapter $\hat{X}$ and $\hat{P}$ have dimensions of length and momentum, respectively. This is to maintain consistency with the work in Ref. [22] and [106] which we repeatedly refer to. The interaction between the mechanical oscillator and the qubit shifts the energy splitting of the qubit by an amount proportional to the mechanical displacement $\hat{X}$ and the coupling rate

λ .

In the simplest implementation of a diffraction grating, the required projective measurement is realised as follows:

1. The qubit is prepared in an eigenstate $(|+\rangle + |-\rangle)/\sqrt{2}$. Such a qubit can be initialised, controlled, and measured using standard techniques of circuit quantum electrodynamics (cQED) [101].
2. The system evolves under the Hamiltonian (7.1) for time τ_R . In the qubit's rotating frame, it precesses at a rate proportional to the mechanical displacement $\hat{X}$. (In the parameter regime considered here, the mechanical frequency is small enough that the particle can be assumed stationary during this step. Ref. [22] explains how the protocol should be modified if this assumption is violated.) At the end of this step, the accumulated qubit precession is

$$\Theta = \lambda \frac{\hat{X}}{X_{\text{ZP}}} \tau_R . \quad (7.2)$$

3. A microwave burst generates a $\pi/2$ rotation and the qubit is then projectively measured in the $\{|+\rangle, |-\rangle\}$ basis. The probability to obtain the result $|+\rangle$ is

$$p_+ = \cos^2 \Theta . \quad (7.3)$$

Conditioning on this outcome (i.e. analysing subsequent data only in those experimental runs for which this outcome was obtained) realises an effective grating. Conditioning on $|-\rangle$ results in a π phase-shifted interference fringe. One does not necessarily need to discard all $|-\rangle$ outcomes however the subsequent analysis in this Chapter must be suitably adapted to reflect it is contingent on the $|-\rangle$ outcome. For simplicity, we will henceforth condition our diffraction grating on the $|+\rangle$ outcome.

In a conventional interferometer, the particle encounters gratings that are located at different positions along its path. In the displacemon, the effective grating is turned on and off in time.

Table 7.1: Parameters of the devices considered here, both suspended aluminium beams. Device A is designed to have the same mass as the vibrating mirror proposed in Ref. [7]. Device B is designed to have the Planck mass, of $22 \mu\text{g}$. For both devices, the magnetic field is assumed to be $B_{\parallel} = 4 \text{ mT}$, since that field is found not to affect transmon operation in Ref. [103], and the qubit dephasing time is taken as $T_2^* = 1 \mu\text{s}$, which is typical of optimised transmons [206]. Material parameters for aluminium are density $\rho_0 = 2700 \text{ kg m}^{-3}$ and Young's modulus $E = 68 \text{ GPa}$. ($\dagger$ To properly distinguish between CSL and purely thermal decoherence, device B requires a smaller $|\alpha|$ which can be achieved by reducing $B_{\parallel}$ by a factor of ~ 400 to $\sim 0.01 \text{ mT}$.)

Parameter		Device A	Device B
Beam width	$D \text{ (m)}$	1×10^{-6}	1×10^{-5}
Beam length	$\ell \text{ (m)}$	1.9×10^{-3}	8×10^{-2}
Mass	$m \text{ (kg)}$	5.1×10^{-12}	2.2×10^{-8}
Frequency	$\Omega/2\pi \text{ (Hz)}$	1400	8.1
Zero-point amplitude	$X_{\text{ZP}} \text{ (m)}$	3.4×10^{-14}	6.9×10^{-15}
Electromechanical coupling	$ \lambda_{\text{on}} /2\pi \text{ (Hz)}$	7.2×10^5	6.2×10^6
Grating parameter	$ \alpha $	4.5	39 (0.1 †)
Maximum separation after single grating	$\Delta X_{\text{max}} \text{ (m)}$	6.1×10^{-13}	1.1×10^{-12}
Energy of gravitational incompatibility (Eq. (7.26))	$E_G \text{ (eV)}$	3.8×10^{-14}	1.6×10^{-10}
Corresponding gravitational lifetime	$\tau_{\text{DP}} \text{ (s)}$	1.8×10^{-2}	4.2×10^{-6}
Energy of gravitational incompatibility (Eq. (7.27))	$E_G \text{ (eV)}$	3.4×10^{-15}	7.0×10^{-11}
Corresponding gravitational lifetime	$\tau_{\text{DP}} \text{ (s)}$	1.9×10^{-2}	9.4×10^{-6}
Energy of gravitational incompatibility (Eq. (7.31))	$E_G \text{ (eV)}$	1.4×10^{-19}	1.9×10^{-15}
Corresponding gravitational lifetime	$\tau_{\text{DP}} \text{ (s)}$	4.7×10^3	3.4×10^{-1}

In both cases, state preparation of the matter degree of freedom manifests itself as a diffraction pattern in the transverse coordinate X .

7.2.3 Physical parameters

We consider two realisations of the device in Fig. 2.3, with different levels of ambition. The parameters of these devices are shown in Table 7.2.3. Device A is designed to have the same mass as the vibrating mirror in the Marshall-Simon-Penrose-Bouwmeester proposal to test for gravitational collapse [7]. Device B is designed to have the Planck mass. The dimensions of Devices A and B are chosen to be within the capabilities of microfabrication; obviously, other combinations of length and width could be chosen to give the same mass. For both devices, the mechanical quality factor is taken to be $Q = 1.1 \times 10^6$, the same as in Ref. [103] and typical for aluminium nanobeams.

The qubit frequency is set by the geometry (which determines E_C) and the Josephson junction parameters (which determine E_J). We again take parameters close to those experimentally measured in Ref. [103] by assuming $\omega_q^0/2\pi = 8$ GHz and $\omega_q/2\pi = 6$ GHz, which requires $\Delta\Phi \approx 0.620\Phi_0$. We reiterate the definitions from Section 2.5: $E_C = e^2/2C$ is the charging energy, E_J is the Josephson energy defined for the displacemon in Eq. (2.82), and $\Phi_0 = h/2e$ is the superconducting flux quantum.

7.3 Non-Gaussian mechanical state preparation via qubit measurement

We now consider the evolution of the resonator following the conditioning step described in Section 7.2.2. The spatial wavefunction of the conditioned state is proportional to the initial wavefunction multiplied by a sinusoid, meaning that the effective grating performs the transformation [22]:

$$|\psi\rangle_M \rightarrow \hat{\Upsilon}_+ |\psi\rangle_M = \hat{R}(\Omega t) \cos\left(|\alpha| \frac{\hat{X}}{X_{\text{ZP}}}\right) |\psi\rangle_M. \quad (7.4)$$

Here $|\psi\rangle_M$ is the mechanical wavefunction, $\hat{\Upsilon}_+$ is a measurement operator corresponding to the sequence of operations described in Section 7.2.2, $\hat{R}(\theta) \equiv e^{-i\theta b^\dagger b}$ is the resonator rotation operator, and

$$\alpha \equiv i \int dt' e^{i\Omega t'} \lambda(t'). \quad (7.5)$$

For a derivation of Eq. (7.4), see Ref. [22]. Multiplying the wavefunction $|\psi\rangle_M$ by a sinusoid is equivalent to dividing it into two branches, one of which has received an impulse $+\hbar|\alpha|/X_{\text{ZP}}$ and the other which has received $-\hbar|\alpha|/X_{\text{ZP}}$. This is apparent if we rewrite the measurement operator in the following form

$$\begin{aligned} \hat{\Upsilon}_+ &= \hat{R}(\Omega t) \cos\left(|\alpha| \frac{\hat{X}}{X_{\text{ZP}}}\right) \\ &= \frac{\hat{R}(\Omega t)}{2} \left[\hat{D}(|\alpha|) + \hat{D}^\dagger(|\alpha|) \right], \end{aligned} \quad (7.6)$$

where $\hat{D}(\beta) = e^{\beta\hat{b}^\dagger - \beta^*\hat{b}}$ is the single-mode displacement operator. This kind of measurement operator is familiar from the work in Section 5.2 where we demonstrated in Eqs (5.3a)-(5.3b) that the unitary $e^{\pm i|\alpha|\hat{X}/X_{\text{ZP}}}$ imparts a momentum kick but has no effect on the position of the resonator. In Section 5.2 we used two displacement operators each acting on two separate modes to create a two-mode Schrödinger-cat state. In contrast, the operators in Eq. (7.6) act on a single mechanical mode and therefore create a superposition of two branches one of which has received a momentum kick in the positive direction, while the other in the negative direction. If the mechanical resonator is initially prepared in its ground state $|\psi\rangle_{\text{M}} = |0\rangle$, then

$$\hat{\Upsilon}_+ |\psi\rangle_{\text{M}} = \frac{\hat{R}(\Omega)}{2} (|\alpha\rangle + |-\alpha\rangle) . \quad (7.7)$$

Therefore, in this work, a single-mode mechanical Schrödinger-cat state is realised via measurement on the superconducting qubit. We note that the state in Eq. (7.7) is the even cat state; the odd cat state is conditioned on the $\hat{\Upsilon}_-$ outcome.

After the qubit measurement is performed the resonator is decoupled from the qubit and freely evolves. The coupled equations of motion are thus $P(t) = P(0)\cos(\Omega t)$ and $X(t) = P(0)\sin(\Omega t)/(m\Omega)$. The initial momenta of each branch are $P(0) = \pm\hbar|\alpha|/X_{\text{ZP}}$. Therefore, after a quarter of a mechanical period, the centroids of these two branches are spatially separated by a distance

$$\Delta X = \frac{2\hbar|\alpha|}{m\Omega X_{\text{ZP}}} \quad (7.8)$$

$$= 4|\alpha|X_{\text{ZP}} . \quad (7.9)$$

The largest separation will occur if $|\alpha|$ is maximised. In Eq. (7.5), the time integral extends at most over the qubit decoherence time T_2^* , since otherwise the qubit fringes required to create the effective grating are washed out. Thus, during the qubit precession step, if $\lambda(t)$ is set to its largest accessible value λ_{max} , then the largest accessible separation is

$$\Delta X_{\text{max}} = 4T_2^* \lambda_{\text{on}} X_{\text{ZP}} . \quad (7.10)$$

Taking $\Delta\Phi = 0.620\Phi_0$ during this step, as explained in Section 7.2.3, gives

$$\lambda_{\text{on}} = -1.16 \times \omega_q \frac{\beta_0 X_{\text{ZP}} \ell B_{\parallel}}{\Phi_0}, \quad (7.11)$$

and therefore

$$\alpha = -1.16i \times \omega_q T_2^* \frac{\beta_0 X_{\text{ZP}} \ell B_{\parallel}}{\Phi_0}, \quad (7.12)$$

and

$$\Delta X_{\text{max}} = 4.64 \times \omega_q T_2^* \frac{\beta_0 X_{\text{ZP}}^2 \ell B_{\parallel}}{\Phi_0} \quad (7.13)$$

$$= 2.32 \times \omega_q T_2^* \frac{\hbar}{m\Omega} \frac{\beta_0 \ell B_{\parallel}}{\Phi_0}. \quad (7.14)$$

The largest achievable separation between these two branches is shown for devices A and B in Table 7.2.3.

7.4 Testing collapse models

7.4.1 Conventional decoherence

In the standard theory of quantum mechanics, the thermal decoherence of the mechanical state ρ may be modelled by the Caldeira–Leggett master equation

$$\dot{\rho} = -\frac{i}{\hbar} [\hat{H}_0, \rho] - \frac{i\gamma}{\hbar} [\hat{X}, \{\hat{P}, \rho\}] - \mathcal{D}_{\text{th}}[\hat{X}, [\hat{X}, \rho]], \quad (7.15)$$

which describes quantum Brownian motion [207, 208]. The first term in Eq. (7.15) describes unitary evolution of the mechanical oscillator with Hamiltonian $\hat{H}_0 = \hbar\Omega\hat{b}^\dagger\hat{b}$, while the term $-i\gamma[\hat{X}, \{\hat{P}, \rho\}]/\hbar$ describes dissipation, which occurs over a timescale $1/\gamma = Q/\Omega$. Again, we emphasise that $\hat{X}$ and $\hat{P}$ have dimensions of distance and momentum.

The final term $-\mathcal{D}_{\text{th}}[\hat{X}, [\hat{X}, \rho]]$ in Eq. (7.15) describes momentum diffusion and leads to the decoherence of quantum superpositions. Namely, the effect of this term is to cause spatial

superpositions of size $2\Delta X$ to decohere over a timescale [4]

$$t_{\text{th}} = \frac{1}{4\mathcal{D}_{\text{th}}(\Delta X)^2}. \quad (7.16)$$

Here, the thermal decoherence rate is $\mathcal{D}_{\text{th}} = 2m\gamma k_{\text{B}}T_{\text{eff}}/\hbar^2$ and the thermal occupation of the oscillator is $\bar{N} = k_{\text{B}}T_{\text{eff}}/\hbar\Omega$, which is valid in the limit [180] $k_{\text{B}}T_{\text{eff}} \gg \hbar\Omega$. Here, T_{eff} is the effective temperature of the environment, which incorporates the cryogenic temperature and other effects such as ambient mechanical noise, which we discuss further in Section 7.6.

A term $-\mathcal{D}_{\text{P}}[\hat{P}, [\hat{P}, \rho]]$ may be added to Eq. (7.15) to bring the dynamics into Lindblad form [209], where $\hat{P}$ is the mechanical momentum operator. This term describes spatial diffusion at a rate $\mathcal{D}_{\text{P}} = \gamma/(8mk_{\text{B}}T_{\text{eff}})$. However, in the temperature regime considered in this work, $k_{\text{B}}T_{\text{eff}} \gg \hbar\Omega$, this term is negligible. We use the corresponding Langevin equations, which describe this quantum Brownian motion in the Heisenberg picture, in order to solve for the open-system dynamics of the mechanical state in Section 7.5. This enables us to properly model the effects of dissipation and quantum decoherence while avoiding any violation of positivity when solving Eq. (7.15) at short times with a non-negligible spatial diffusion term [210].

7.4.2 Diósi-Penrose objective collapse

In Diósi-Penrose theory [211, 188], the collapse of a superposition arises from the difference in the mass distributions $\rho(\mathbf{r})$ and $\rho'(\mathbf{r})$ in the two branches of the superposition. The lifetime of a superposition is

$$\tau_{\text{DP}} = \frac{\hbar}{E_G}, \quad (7.17)$$

where [189]

$$E_G \equiv 4\pi G\gamma_{\text{DP}} \iint \frac{[\rho(\mathbf{r}) - \rho'(\mathbf{r})][\rho(\mathbf{r}') - \rho'(\mathbf{r}')]d^3\mathbf{r}d^3\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \quad (7.18)$$

parameterizes the incompatibility of the space-time generated by the two branches of the superposition, and $\mathbf{r}$ and $\mathbf{r}'$ are spatial coordinates. Equation (7.18) is proportional to the gravitational self-energy of the difference between the two mass distributions, and γ_{DP} is a con-

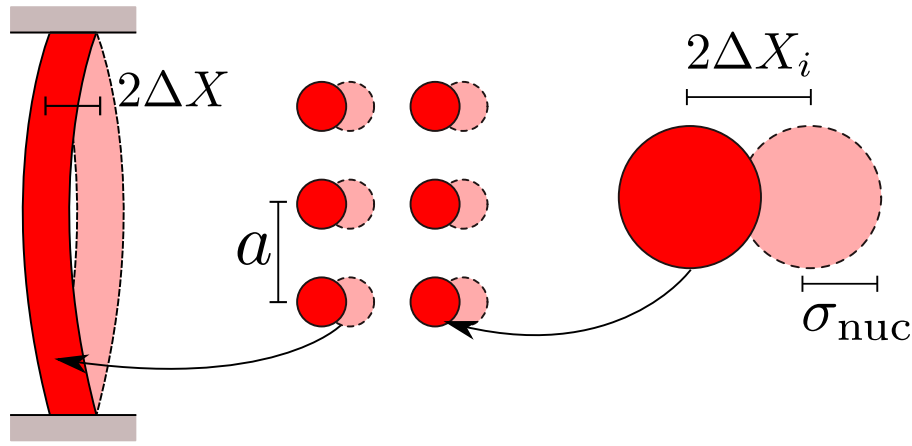


Figure 7.1: Cartoon interpretation of the displacemon resonator in a spatial superposition of size $2\Delta X$. Here, ΔX is the root-mean-square displacement of the beam from its equilibrium position. The resonator is a cuboid of volume $V = \ell D^2$ with inter-atomic distance given by $a = 2.9 \times 10^{-10} \text{m}$ for aluminium. Since the majority of an atom's mass resides in the nucleus, the gravitational radius σ_{nuc} labels the nuclear radius. The role that the parameter σ_{nuc} plays in objective collapse is subject to debate; see Section 7.4.2.

stant [189] which takes the value $\gamma_{\text{DP}} = 1/8\pi$ if $|E_G|$ is equal to the gravitational self-energy (rather than only proportional to it).

For the devices considered here, the spatial separation $2\Delta X$ is much smaller than the distance between adjacent atoms a . (Compare Table 7.2.3 with the nearest-neighbour distance in aluminium, which is $a = 2.9 \times 10^{-10} \text{ m}$.) This means that Eq. (7.18) can be broken down into a sum of self-energies created by the individual nuclei, each in a superposition:

$$E_G \approx \sum_i E_G^{(1)}(2\Delta X_i). \quad (7.19)$$

Here $E_G^{(1)}(2\Delta X_i)$ is the self-energy of the superposition of a single nucleus separated by $2\Delta X_i$, which is the separation of the entire beam at the location of the i^{th} nucleus (see Fig. 7.1). Since

$$X_i(Z, t) = X(t)u_0(Z_i), \quad (7.20)$$

with $u_0(Z)$ defined by Eq. (2.91), Eq. (7.19) becomes

$$E_G(t) = 8\pi\gamma_{\text{DP}} \frac{D^2 \rho_0}{m_a} \int E_G^{(1)}(2X(t)u_0(Z)) \, dZ \quad (7.21)$$

where ρ_0 and m_a are the density and atomic mass of the material from which the beam is made. The factor 2 takes account of the fact that the two branches of the wavefunction correspond to beam displacements in different directions.

A full expression for $E_G^{(1)}$ is given in Ref. [67]; modelling the nuclei as hard spheres with radius σ , it is

$$E_G^{(1)}(2\Delta X_i) = 8\pi\gamma_{\text{DP}} Gm_a^2 \left(\frac{6}{5\sigma} - \frac{1}{2\Delta X_i} \right). \quad (7.22)$$

The value of σ is the size of the nuclear mass distribution that acts as the gravitational source. The simplest choice is that σ is the nuclear radius, which in aluminium [212] is $\sigma_{\text{nuc}} = 3.06 \times 10^{-15}$ m. Since Table 7.2.3 shows that $\Delta X \gg \sigma_{\text{nuc}}$, the first term in Eq. (7.22) dominates, meaning that

$$E_G^{(1)} \approx 8\pi\gamma_{\text{DP}} \frac{6}{5} \frac{Gm_a^2}{\sigma_{\text{nuc}}}, \quad (7.23)$$

and therefore

$$E_G = 8\pi\gamma_{\text{DP}} \frac{6}{5} \frac{GD^2\ell\rho_0m_a}{\sigma} \quad (7.24)$$

$$= 8\pi\gamma_{\text{DP}} \frac{6}{5} \frac{Gmm_a}{\sigma_{\text{nuc}}} \quad (7.25)$$

$$= \frac{6}{5} \frac{Gmm_a}{\sigma_{\text{nuc}}}, \quad (7.26)$$

where m is the mass of the beam, and the last line makes the assumption [189] that $\gamma_{\text{DP}} = 1/8\pi$. This leads to the predicted collapse lifetime τ_{DP} shown in Table 7.2.3. Any superposition in device A would collapse due to gravity in approximately 18 ms, or 25 mechanical periods; for device B, the collapse time is much less than the mechanical period, so that a superposition that survives even for half a period would be enough to exclude the set of assumptions with this collapse model.

For gravitational collapse to be observable, the conventional decoherence rate needs to be reduced below the gravitational rate $1/\tau_{\text{DP}}$. In Section 7.5, we outline a scheme to detect this decoherence rate using the displacemon device.

Alternative choices for the gravitational radius σ

Above, we assumed that σ is the same as the nuclear radius σ_{nuc} . It has also been suggested [67] that σ should be the radius of the ground-state wavepacket if the mechanical state is initialized in the ground state prior to superposition state preparation. There are two ways of defining this. One is to say that the wavepacket size is equal to the beam's zero-point amplitude, i.e. $\sigma = X_{\text{ZP}}$. If we make the crude approximation that the wavepacket gravitates in the same way as a uniform sphere with this radius, then we still have $\Delta X_{\text{max}} \gg \sigma$ and so Eq. (7.26) is modified to

$$E_G = \frac{6}{5} \frac{Gmm_a}{X_{\text{ZP}}}. \quad (7.27)$$

Alternatively, the wavepacket size may be taken as the zero-point amplitude of a single atom in the crystal, which is

$$\sigma = X_{\text{ZP}}^{(1)} = \sqrt{\frac{\hbar}{2m_a\Omega_p}} \quad (7.28)$$

$$\approx 8 \times 10^{-12} \text{ m in aluminium} \quad (7.29)$$

where Ω_p , the maximum of the lower phonon dispersion branch, which in aluminium [213], is $\Omega_p \approx 2\pi \times 3 \text{ THz}$. We now have $\Delta X_{\text{max}} < \sigma$, and so (in the approximation that the wavepacket still gravitates as a uniform sphere), Eq. (7.22) is modified to

$$E_G^{(1)}(2\Delta X_i) \approx 8\pi\gamma_{\text{DP}} \frac{2Gm_a^2(\Delta X_i)^2}{\left(X_{\text{ZP}}^{(1)}\right)^3} \quad (7.30)$$

which follows an expression from Ref [189], keeping only the leading term in $\Delta X_i/X_{\text{ZP}}^{(1)}$. A full analysis should take into account that ΔX_i depends on both Z and t , but in the gross approximation that $\Delta X_i \sim \Delta X_{\text{max}}$ then we find

$$E_G \sim 2Gmm_a \frac{(\Delta X_{\text{max}})^2}{\left(X_{\text{ZP}}^{(1)}\right)^3} \quad (7.31)$$

again assuming $\gamma_{\text{DP}} = 1/8\pi$.

For completeness the gravitational energies and lifetimes that follow from Eqs. (7.27) and (7.31) are listed in Table 7.2.3.

Comparison to thermal decoherence rate

To observe the collapse predicted by the DP model we require that $t_{\text{th}} > \tau_{\text{DP}}$. Using Eqs. (7.16) and (7.17) this condition may be rewritten as

$$\bar{N} < \frac{E_G Q}{8m\Omega^2(\Delta X)^2} , \quad (7.32)$$

where $\bar{N} = k_B T_{\text{eff}}/\hbar\Omega$ is the thermal occupation of the oscillator, and $Q = \Omega/\gamma$ is the quality factor of the oscillator. The right-hand side of Eq. (7.32) is a function of the separation ΔX . Note that ΔX is a dynamical variable that varies between an initial value of zero and ΔX_{max} over one quarter of a mechanical cycle. Over this timescale, the dissipation rate is much smaller than the mechanical frequency $\Omega/2\pi$ and the decoherence rates, so may be ignored. This assumption is valid provided the oscillator has a high quality factor $Q \gg 1$. Furthermore, the separation ΔX_{max} may be controlled with the magnetic field $B_{\parallel}$. Therefore, we may maximize Eq. (7.32) with respect to ΔX to estimate the maximum thermal occupation of the environment $\bar{N}^{\text{max}}$ that still allows DP collapse to be observed, and the corresponding effective environmental temperature $T_{\text{eff}}^{\text{max}}$, which includes both the temperature and environmental vibrations at the relevant frequency band..

To maximize the right-hand side of Eq. (7.32) we account for the dependence of E_G on ΔX through Eq. (7.22) in the approximation $\Delta X_i \approx \Delta X$, which is valid for $\ell \gg D$. The term $E_G/(\Delta X)^2$ is then a decreasing function of ΔX for [189] $\Delta X \geq \sigma$. Hence, in the region $\Delta X \geq \sigma$ we find that Eq. (7.32) is maximized at $\Delta X = \sigma$ to give $\bar{N} < \bar{N}^{\text{max}}$ with

$$\bar{N}^{\text{max}} = 7Gm_a Q/(80\sigma^3\Omega^2) . \quad (7.33)$$

Firstly, we take the gravitational radius to be the nuclear radius $\sigma = \sigma_{\text{nuc}}$, in which case Device A requires $\bar{N}^{\text{max}} = 1.3 \times 10^5$ ($T_{\text{eff}}^{\text{max}} = 8.7 \text{ mK}$) while Device B requires $\bar{N}^{\text{max}} = 3.9 \times 10^9$

($T_{\text{eff}}^{\text{max}} = 1.5 \text{ K}$). Secondly, we take σ to be the zero-point amplitude of the beam $\sigma = X_{\text{ZP}}$. In this case we find a more stringent requirement for Device A of $\bar{N}^{\text{max}} = 95$ ($T_{\text{eff}}^{\text{max}} = 6.3 \mu\text{K}$), while Device B may operate at $\bar{N}^{\text{max}} = 3.4 \times 10^8$ ($T_{\text{eff}}^{\text{max}} = 0.13 \text{ K}$). Device B can operate at a higher effective temperature for both choices of σ , which is attributed to the $1/\Omega^2$ dependence of $\bar{N}^{\text{max}}$. These relatively high effective temperatures required by Device B are encouraging given recent experimental progress in cooling Planck mass objects [214].

Finally, we consider the case when the gravitational radius is given by the zero-point amplitude of a single atom in the crystal $\sigma = X_{\text{ZP}}^{(1)}$. As this corresponds to the region $\Delta X < \sigma$ we take $\Delta X = \Delta X_{\text{max}}$ to provide an estimate for $\bar{N}^{\text{max}}$. Combining Eqs. (7.31) and (7.32) then gives $\bar{N} < \bar{N}^{\text{max}} = Gm_a Q / \left(4\Omega^2 (X_{\text{ZP}}^{(1)})^3 \right)$, which gives $\bar{N}^{\text{max}} = 2.1 \times 10^{-5}$ ($T_{\text{eff}}^{\text{max}} = 1.4 \times 10^{-12} \text{ K}$) for Device A and $\bar{N}^{\text{max}} = 0.62$ ($T_{\text{eff}}^{\text{max}} = 2.4 \times 10^{-10} \text{ K}$) for Device B. These results break our initial assumption that $k_B T_{\text{eff}} \gg \hbar\Omega$ and do not correspond to experimentally practical temperatures. Hence the displacemon is not capable of sensing DP collapse with $\sigma = X_{\text{ZP}}^{(1)}$.

Treating the constituents as uniform spheres [189], the term $E_G/(\Delta X)^2$ is maximized in the limit $\Delta X \rightarrow 0$, which seems to increase the $\bar{N}$ our devices can tolerate in (7.32). However, we note that in this limit, both decoherence timescales τ_{DP} and t_{th} tend to infinity and we do not expect to see any collapse as there is no spatial superposition. Furthermore, in our testing protocol—see Section 7.5—in the limit $\Delta X \rightarrow 0$ the mechanical state is initialized in a thermal state. In this limit, there is no quantum decoherence and the thermal state simply equilibrates with its environment.

7.4.3 Continuous Spontaneous Localisation (CSL)

In order to model the dynamical evolution of the displacemon resonator under the assumptions of CSL, we can add the term $-\mathcal{D}_{\text{CSL}}[\hat{X}, [\hat{X}, \rho]]$ to the right-hand side of the master equation in Eq. (7.15), corresponding to a momentum diffusion process with rate $\mathcal{D}_{\text{CSL}}$. This additional term is valid for regimes where the resonator has an oscillation amplitude [215] $X_{\text{ZP}} \ll r_{\text{CSL}}$, and would only be detectable over conventional thermal noise if $\mathcal{D}_{\text{CSL}} \gtrsim \mathcal{D}_{\text{th}}$. The geometry of the superposed macroscopic object dictates $\mathcal{D}_{\text{CSL}}$. Assuming that the superposed displacemon

beam is a cuboid of length ℓ and width D such that the volume is given by $V = \ell D^2$, it follows that [215, 216]:

$$\mathcal{D}_{\text{CSL}} = \lambda_{\text{CSL}} \frac{N_{\text{n}}^2}{D^2} \Gamma_1\left(\frac{D}{\sqrt{2}r_{\text{CSL}}}\right) \Gamma_1\left(\frac{\ell}{\sqrt{2}r_{\text{CSL}}}\right) \times [1 - e^{-D^2/4r_{\text{CSL}}^2}] , \quad (7.34)$$

where

$$\Gamma_1(x) = \frac{2}{x^2} \left[e^{-x^2/2} - 1 + \sqrt{\frac{\pi}{2}} x \text{erf}\left(\frac{x}{\sqrt{2}}\right) \right] , \quad (7.35)$$

and N_{n} (not to be confused with $\bar{N}$) is the total number of nucleons that make up the resonator. Using $N_{\text{n}} = m/\text{amu}$, we have that $N_{\text{n}} = 3.1 \times 10^{15}$ for Device A, and 1.3×10^{19} for Device B. The evolution of the resonator is governed by Eq. (7.15) (adapted to include CSL decoherence, as discussed) and is captured by the following quantum Langevin equations:

$$\dot{\hat{X}} = \hat{P}/m , \quad (7.36a)$$

$$\dot{\hat{P}} = -m\Omega^2 \hat{X} - \gamma \hat{P} + \hat{f}_{\text{th}}(t) + \hat{f}_{\text{CSL}}(t) , \quad (7.36b)$$

where $\hat{f}_{\text{th}}$ and $\hat{f}_{\text{CSL}}$ are the stochastic force terms arising from the standard theory of thermal decoherence, and CSL, respectively. These operators obey the following relations:

$$\langle \hat{f}_{\text{th}}(t) \hat{f}_{\text{th}}(t') \rangle = \hbar^2 \mathcal{D}_{\text{th}} \delta(t - t') , \quad (7.37a)$$

$$\langle \hat{f}_{\text{CSL}}(t) \hat{f}_{\text{CSL}}(t') \rangle = \hbar^2 \mathcal{D}_{\text{CSL}} \delta(t - t') , \quad (7.37b)$$

and $\langle \hat{f}_{\text{th}}(t) \hat{f}_{\text{CSL}}(t') \rangle = 0$ at all times since the two noise sources are uncorrelated. By exploiting qubit measurements on the displacemon device, we can test the thermal decoherence of a mechanical resonator and deduce whether the rate agrees with conventional decoherence theory or whether there is a deviation which can be attributed to a non-zero $\mathcal{D}_{\text{CSL}}$. The proposed procedure for this is described in detail in the next section.

7.5 Testing protocol

The displacemon device has several features which can be exploited to dynamically quantify the degree of decoherence of the resonator. From the statistics of qubit measurements taken at regular time intervals one can infer both the extent of decoherence in the mechanical system and the rate at which this occurs. The following method should be read with CSL in mind as it exploits the dynamical description provided by the quantum Langevin equations (see Section 2.3.1). While Penrose does not offer a dynamical description for gravity-induced collapse, we shall demonstrate that the following protocol can still be used to determine whether or not total decoherence of the mechanical state occurs on the timescale of t_{DP} or t_{th} .

Our method to test the decoherence rate of a mechanical superposition using the displacemon device is as follows:

1. The resonator is cooled towards its ground state using the qubit-measurement pre-cooling scheme outlined in Ref. [22]. By using a sequence of pulses and selecting specific qubit outcomes, we can discard higher energy states and select the lowest-energy thermal states. Thus, the mechanical state $\rho_{\bar{n}}$ is prepared in a thermal state with $\bar{n}$ being the initial thermal occupation number.
2. The qubit-mechanical interaction is switched on (Eq. (7.1)) and a measurement of the qubit in the state $|+\rangle$ heralds an effective diffraction grating (see Eq. (7.4)) in the mechanical system. This then leads to a spatial mechanical superposition (as outlined in Section 7.3). The post-measurement state of the mechanical system is given by:

$$\rho_{\text{in}} = \frac{\hat{\Upsilon}_{+\rho_{\bar{n}}} \hat{\Upsilon}_{+}^{\dagger}}{\text{Tr}[\hat{\Upsilon}_{+}^{\dagger} \hat{\Upsilon}_{+\rho_{\bar{n}}}]}, \quad (7.38)$$

where the denominator is the heralding probability of creating such a state: $\text{Tr}[\hat{\Upsilon}_{+}^{\dagger} \hat{\Upsilon}_{+\rho_{\bar{n}}}] = (1 + e^{-2(1+2\bar{n}|\alpha|^2)})/2$. We neglect any mechanical decoherence which may occur during the implementation of the measurement operator since $\hat{\Upsilon}_{+}$ is enacted on the timescale of the qubit lifetime T_2^* which is much shorter than the mechanical period.

3. Following the qubit measurement outcome $|+\rangle$, the mechanical state $\rho(t)$ is decoupled from the qubit and the position and momentum operators evolve according to Eqs. (C.2a)-(C.2b), in the Heisenberg picture.
4. Finally, after a time t_k , we apply a second qubit measurement $\hat{\Upsilon}_+$ by again switching on the qubit-mechanical interaction and measuring the qubit in the state $|+\rangle$. The time elapsed between applying the first and second gratings is $t_k = k\pi/\Omega$ where k is the integer number of mechanical half-periods. By waiting for mechanical half-periods we can neglect the $\hat{R}(\Omega t)$ operator in Eq. (7.4). As the qubit is only coupled to the mechanical system during the fast measurement operations, we are not constrained by the qubit coherence time; a key strength of our protocol. Over many experimental runs, we calculate the probability of measuring the qubit in the $|+\rangle$ state at time intervals t_k after it was first measured in the $|+\rangle$ state. This conditional probability is denoted by $P(k, \mathcal{D}) = \text{Tr}[\hat{\Upsilon}_+^\dagger \hat{\Upsilon}_+ \rho(t_k)]$, where $\mathcal{D} = \mathcal{D}_{\text{th}}$ according to conventional decoherence theory, or $\mathcal{D} = \mathcal{D}_{\text{th}} + \mathcal{D}_{\text{CSL}}$ if we incorporate the predictions from CSL theory. In the absence of any decoherence effects ($\mathcal{D} = 0$) this expression can be readily evaluated to give:

$$P(k, 0) = \frac{3 + 4e^{-2(1+2\bar{n})|\alpha|^2} + e^{-8(1+2\bar{n})|\alpha|^2}}{4(1 + e^{-2(1+2\bar{n})|\alpha|^2})}, \quad (7.39)$$

and is bounded between $[0.75, 1]$ for all $\bar{n}$ and $|\alpha|$. Therefore, observing $P(k, \mathcal{D}) < 0.75$ necessarily implies the presence of decoherence either by conventional means, or via an additional collapse-theory mechanism. If $P(k, \mathcal{D}) = 0.5$ then the qubit is equally likely to be in the $|+\rangle$ or $|-\rangle$ state and so the mechanical state has completely decohered.

5. Alternatively, to test DP-style collapse we can measure the timescale required for $P(k, \mathcal{D})$ to reduce to an appropriate value (depending on the initial state) indicating that the mechanical state has undergone decoherence, and compare this with the timescales t_{DP} and t_{th} .
6. Note: We are interested in the probability that a second qubit measurement yields $|+\rangle$. As this is a binary-outcome measurement, instances where the second measurement is $|-\rangle$ also contribute to our empirical estimate of $P(k, \mathcal{D})$ and so these experimental runs

need not be discarded.

Using this experimental recipe we can investigate how the probability $P(k, \mathcal{D})$ scales with various parameters as we tune $\mathcal{D}_{\text{CSL}}$ and $\mathcal{D}_{\text{th}}$. From this we can identify the regions of parameter space where the predicted effects of CSL are greater than those of conventional decoherence and thus can be feasibly tested.

We can derive an analytical expression for $P(k, \mathcal{D})$ by recasting it in the Heisenberg picture $P(k, \mathcal{D}) = \langle \hat{\Upsilon}_+^\dagger(t_k) \hat{\Upsilon}_+(t_k) \rangle$, where $\hat{\Upsilon}_+(t_k) = \cos(|\alpha| \hat{X}(t_k)/X_{\text{ZP}})$. By inserting the solutions to the Langevin equations into the definition of $\hat{\Upsilon}_+(t_k)$ (see Eq. (7.4)) we expect the probability to decay with k according to:

$$P(k, \mathcal{D}) = \frac{1}{2} + \frac{1}{2} P_0(t_k) \exp \left[-8\mathcal{D}|\alpha|^2 \frac{(1 - e^{-k\pi\gamma/\Omega})}{\gamma(4 + \gamma/\Omega)} \right]. \quad (7.40)$$

For a derivation of this formula see Appendix C. In regimes, such as ours, with high quality factors $Q = \Omega/\gamma \gg 1$ we can make the following approximation

$$P(k, \mathcal{D}) \approx \frac{1}{2} + \frac{1}{2} P_0(k) \exp \left(-2\mathcal{D}|\alpha|^2 \frac{k\pi}{\Omega} \right), \quad (7.41)$$

demonstrating that the probability decays exponentially with time towards a value of 0.5. The slowly-decaying envelope function $P_0(k)$ (defined in Eq. (C.6) of Appendix C) determines the initial probability $P(0, \mathcal{D})$ and depends on k , the grating parameter $|\alpha|$, and the properties of the mechanical oscillator γ , Ω , and $\bar{n}$.

The theoretical decay of the probabilities $P(k, \mathcal{D}_{\text{th}})$ and $P(k, \mathcal{D}_{\text{th}} + \mathcal{D}_{\text{CSL}})$ are plotted for Devices A and B in Figs. 7.2a and 7.2b. For these plots, the CSL decay rate has been calculated assuming $r_{\text{CSL}} = 10^{-7}\text{m}$ and $\lambda_{\text{CSL}} = 10^{-11}\text{ Hz}$ (a factor of 10^3 times smaller than Adler's predicted value [183], thus probing a more challenging region of parameter space). The mechanical parameters and $|\alpha|$ have been set to the values given in Table 7.2.3. The probability in Eq. (7.41) decays exponentially with $|\alpha|^2$ and measurements are only performed after at least half a mechanical period has elapsed. Accordingly, we have used $|\alpha| = 0.1$ for Device B so that CSL is properly distinguishable from purely thermal decoherence; this can be achieved by setting

$B_{||} = 0.01$ mT (see Eq. (7.12)). The initial thermal occupation numbers of the resonators are set to $\bar{n} = 100$, assuming both devices have been pre-cooled. The bath environment is assumed to have a thermal occupation number $\bar{N} = k_B T_{\text{eff}}/(\hbar\Omega)$ with values $\bar{N} = 10^3$ (equivalent to an effective temperature of 0.07 mK) for Device A and $\bar{N} = 5 \times 10^6$ (or 1.9 mK) for Device B. We have chosen these values of $\bar{N}$ for these plots in order to most clearly visually demonstrate the exponential decay of $P(k, \mathcal{D})$. However, we discuss below the experimental challenge of achieving such low temperatures, and the interplay between the bath temperature and testable CSL parameters.

To maximally distinguish between decoherence effects in standard quantum theory versus CSL, we can investigate the percentage difference between the expected probabilities of a second $|+\rangle$ qubit measurement as predicted by the two theories:

$$\Delta = 2 \frac{P(k, \mathcal{D}_{\text{th}}) - P(k, \mathcal{D}_{\text{th}} + \mathcal{D}_{\text{CSL}})}{P(k, \mathcal{D}_{\text{th}}) + P(k, \mathcal{D}_{\text{th}} + \mathcal{D}_{\text{CSL}})} . \quad (7.42)$$

For this we use the full analytic result for $P(k, \mathcal{D})$ given in Eq. (7.40). As we vary the number of mechanical half periods k we can find the maximum percentage difference between the two probabilities Δ_{Max} for different values of $\mathcal{D}_{\text{th}}$ and $\mathcal{D}_{\text{CSL}}$. This quantity is plotted as a function of λ_{CSL} and $\bar{N}$ for Devices A and B in Figs. 7.3a and 7.3b, using the parameters in Table 7.2.3. As before, we have used initial resonator occupation numbers of $\bar{n} = 100$. We only consider $\bar{N} > \bar{n} = 100$ as we expect the bath to always be warmer than the device itself.

For both devices, Fig. 7.3 shows that one should expect appreciable differences between $P(k, \mathcal{D})$ for the two theories, provided sufficiently low environmental temperatures can be achieved. Device B is more suitable for probing lower λ_{CSL} values at more feasible bath temperatures. One can see that even at $\bar{N} = 2 \times 10^7$ (a feasible effective temperature of 7.7 mK), we expect Device B to exhibit at least an $\approx 5\%$ difference between CSL and standard quantum theory for $\lambda_{\text{CSL}} > 10^{-10}$ Hz. In contrast, much lower bath temperatures are required for Device A to probe similar CSL rates. To confidently distinguish between CSL and standard thermal decoherence we require $\mathcal{D}_{\text{CSL}} > \mathcal{D}_{\text{th}} \propto 2\bar{N}\Omega/Q$ and so have only included the region in which this is true. Since $\mathcal{D}_{\text{CSL}}$ is directly proportional to λ_{CSL} , the smallest value of λ_{CSL} that we can conclusively

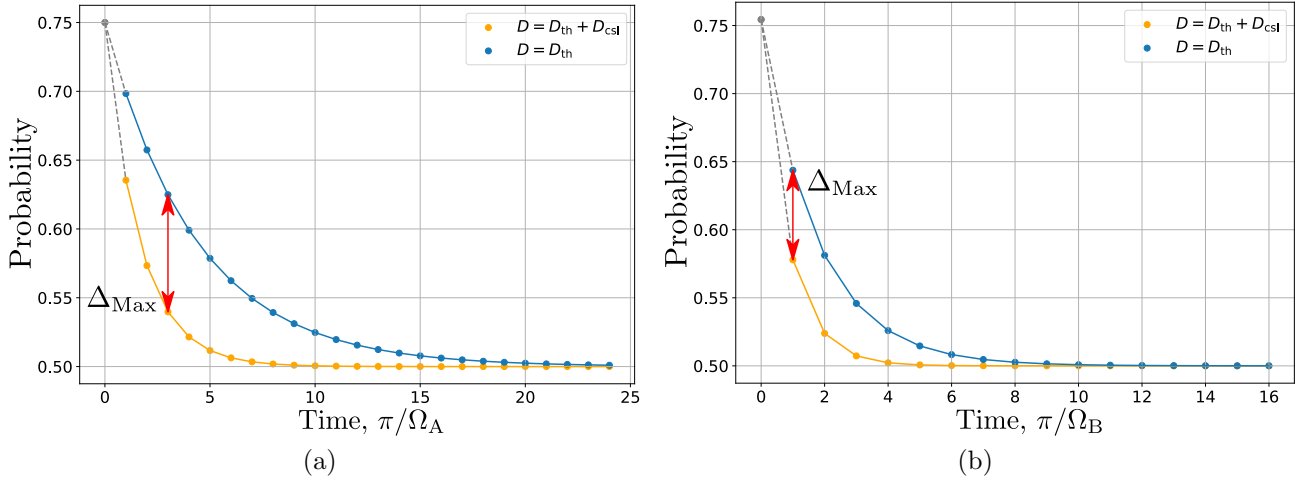


Figure 7.2: Probability of measuring the displacemon qubit in the $|+\rangle$ state, given it was initially measured in $|+\rangle$, as a function of the the number of mechanical half-cycles k elapsed between the two qubit measurements (see Eq. (7.41)). The blue line shows the probability decaying as predicted by standard quantum mechanics with momentum diffusion rate given by $D = D_{th}$. The orange line has incorporated the CSL decoherence rate giving a total momentum diffusion rate of $D = D_{th} + D_{CSL}$. The red arrow indicates the position of maximum percentage difference Δ_{Max} we can expect between the two theories for this parameter set; these points correspond to the orange dots in Figs. 7.3a - 7.3b. (a) Plot for Device A using the values in Table 7.2.3, $\lambda_{CSL} = 10^{-11}$ Hz and $r_{CSL} = 10^{-7}$ m, initial mechanical occupation $\bar{n} = 100$, and a bath occupation number $\bar{N} = 10^3$ (or 0.07 mK). (b) Plot for Device B, again using values from Table 7.2.3, $\lambda_{CSL} = 10^{-11}$ Hz, $r_{CSL} = 10^{-7}$ m, $\bar{n} = 100$, and $\bar{N} = 5 \times 10^6$ (or 1.9 mK).

probe is proportional to the mechanical frequency Ω , which is much higher for Device A.

While we have focused on the displacemon device, we note that in principle our testing protocol is applicable to other optomechanical platforms that achieve an equivalent measurement operator to Eq. (7.4). For example, pulsed optomechanics [93] provides a path to achieve such operations [139] and grow large spatial superpositions [95]. In this case, the superconducting qubit would be replaced by an optical mode and the qubit measurement statistics by photon counting statistics. Furthermore, as well as being a potential platform to test the collapse of quantum superpositions, the displacemon device may also offer a promising route towards tests of unconventional decoherence mechanisms in entangled systems. In Ref. [217], it was proposed that Furry's hypothesis [218]—regarding spatially dependent decoherence mechanisms—could be tested in entangled optomechanical systems. Meanwhile, in this Thesis, we have also examined a method for generating two-mode entangled Schrödinger-cat states via a measurement operator equivalent to a two-mode version of Eq. (7.4), which suggests a two-mode displacemon

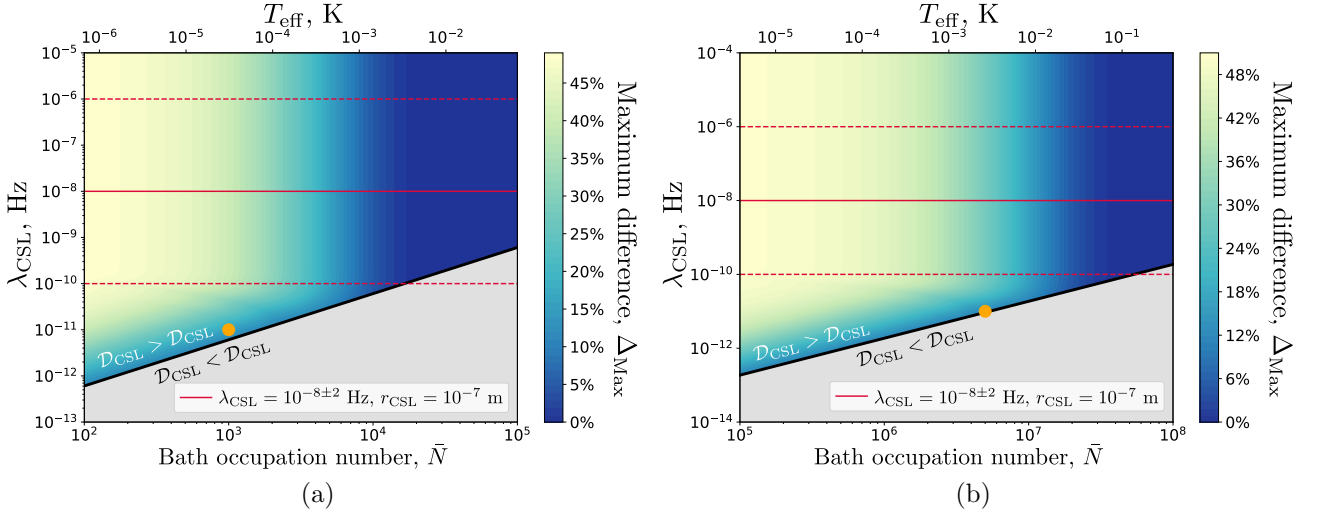


Figure 7.3: Maximum percentage difference Δ_{Max} between the probabilities predicted by standard quantum mechanics versus CSL theory, found by maximising Eq. (7.42) with respect to k . Results are plotted as a function of λ_{CSL} and bath thermal occupation number $\bar{N}$, with the equivalent effective temperature T_{eff} shown along the top axis. (a) Plot for Device A using parameters in the corresponding column of Table 7.2.3. (b) Plot for Device B, again using parameters in the corresponding column of Table 7.2.3. For both plots we use $r_{\text{CSL}} = 10^{-7}$ m and assume the mechanical resonator has been pre-cooled to a thermal state with initial occupation number $\bar{n} = 100$. The light yellow indicates regions of parameter space which we expect to be most sensitive to CSL effects as they exhibit the greatest deviation from standard quantum theory. The grey exclusion region, demarcated by the black line, indicates that $\mathcal{D}_{\text{CSL}} < \mathcal{D}_{\text{th}}$, meaning that the effects of CSL cannot be detected above thermal decoherence. The red horizontal lines show Adler’s suggested theoretical parameters for λ_{CSL} (and r_{CSL}). The orange dots correspond to the parameters used in Figs. 7.2a-7.2b which explicitly show the decay of the probabilities $P(k, \mathcal{D}_{\text{th}})$ and $P(k, \mathcal{D}_{\text{th}} + \mathcal{D}_{\text{CSL}})$ with time.

system could be used to test Furry’s hypothesis.

7.6 Experimental considerations

Here we discuss briefly the main experimental challenges. First there is the challenge of integrating comparatively large moving elements into a transmon qubit. In three-dimensional cQED [219], the transmon is typically ~ 1 mm long, so it would need to be made 1 or 2 orders of magnitude larger in order to accommodate resonators A and B. There is no fundamental reason why this should not work, but it would require modifications to the geometry in order to keep the various capacitances involved within the desirable range for cQED. Another geometrical challenge is that the thickness D envisaged in Table 7.2.3 is quite large compared to

the normal thickness of the superconducting layer in a transmon, which is typically less than 100 nm. Having fabricated the qubit, it will also be necessary to operate it coherently away from its flux sweet spot in order to achieve good electromechanical coupling.

The requirement to suppress thermal decoherence to the point that other mechanisms become observable means that the experiment must be performed at a very low temperature. In a dilution refrigerator, if vibrational noise is sufficiently suppressed, a mechanical resonator typically [220] thermalises to $T_{\text{eff}} \sim 50$ mK. With the aid of nuclear magnetic demagnetization, a 15 MHz resonator was recently [221] cooled to $T_{\text{eff}} = 0.5$ mK.

Equally important with the cryostat temperature is the mechanical noise at the resonator frequency, which heats the resonator in the same way as an elevated environmental temperature. The environmental noise can be quantified by a single-sided spectral density $S_{XX}(f)$, which describes the motion of the substrate to which the resonator is clamped. Converting this position spectral density to a force spectral density $S_{FF}(f) = m\Omega^2 S_{XX}(f)$, and using equipartition to relate the resulting motion to an effective temperature leads to the relation [107]

$$T_{\text{eff}}^{\text{mech}} = \frac{1}{k_B} \frac{Q\Omega^3}{4} S_{XX}(2\pi\Omega) \quad (7.43)$$

where $T_{\text{eff}}^{\text{mech}}$ is the mechanical contribution to the effective environmental temperature T_{eff} . In general the resonator will experience an environmental temperature that is determined by the larger of $T_{\text{eff}}^{\text{mech}}$ and the cryostat temperature. For example, the 8.7 mK effective temperature necessary to test gravitational collapse in Device A with $\sigma = \sigma_{\text{nuc}}$ requires

$$S_{XX}(2\pi\Omega) \lesssim \left(3.5 \times 10^{-16} \text{ m}/\sqrt{\text{Hz}}\right)^2, \quad (7.44)$$

while for Device B, with $T_{\text{eff}} = 1.5$ K, the requirement is

$$S_{XX}(2\pi\Omega) \lesssim \left(1.6 \times 10^{-13} \text{ m}/\sqrt{\text{Hz}}\right)^2. \quad (7.45)$$

For comparison, the isolation platform built for Advanced LIGO has a vibration level below $10^{-12} \text{ m}/\sqrt{\text{Hz}}$ at 8 Hz and below $10^{-13} \text{ m}/\sqrt{\text{Hz}}$ at 100 Hz, the highest frequency measured [222].

Thus, minimizing acoustic noise by mechanically isolating the sample and even the cryostat, and operating the resonator at an optimal frequency, will be important to testing collapse mechanisms in this frequency band.

7.7 Conclusion

We have shown the potential of the displacemon device for tests of unconventional decoherence mechanisms as predicted by objective collapse models, while carefully considering the effects of conventional decoherence and dissipation predicted by standard quantum theory. The displacemon-based collapse testing protocol we introduce consists of a cooling stage followed by a grating operation on the resonator to prepare a non-Gaussian mechanical state of motion. The measurement on the mechanically-coupled superconducting qubit induces this grating operation and occurs over a timescale much less than the mechanical period. Therefore, our protocol is not constrained by the decoherence time of the qubit. The application of a second grating after an integer number of half-mechanical periods allows obtaining qubit measurement statistics that are sensitive to the underlying decoherence mechanisms. In principle, an experiment with the displacemon could be used to sense deviations from standard quantum mechanics predicted by any collapse model, regardless of whether a dynamical theory is proposed (as in GRW, CSL, and Diósi's theory) or not (as in Penrose's theory). In this paper, we have explored the sensitivity of the displacemon device to the DP and CSL collapse models specifically.

For the DP model, we have considered three possibilities for the gravitational radius, namely the nuclear radius, the zero-point amplitude of the beam resonator, and the zero-point amplitude of a single constituent aluminium atom. We find that by minimising their effective temperature, Devices A and B may be able to test for collapse in the first case (gravitational radius equals nuclear radius). Due to its lower frequency, Device B may also be suitable for testing the second case (gravitational radius equals zero-point amplitude of the beam). However, owing to the much larger length scale, our proposal is not capable of testing collapse when the gravitational

radius equals the zero-point amplitude of a single aluminium atom within the lattice.

For the CSL model, we have taken into account the effect of the beam geometry on the momentum diffusion rate induced by CSL collapse. Moreover, we have derived an analytic result for the qubit measurement statistics in our testing protocol. We do so using solutions to the quantum Langevin equations for the system. This allows us to study the sensitivity of our testing protocol to the displacemon and CSL parameters, and identify regions of parameter space where CSL can be investigated. While the parameters chosen for Device A only permit CSL to be tested at temperatures which are beyond current capabilities, the smaller mechanical frequency of Device B allows a larger range of λ_{CSL} to be explored in temperature regimes which could be reached with near-future devices. In tests of both DP and CSL collapse, minimising the sensitivity of the devices to mechanical noise is likely to be the biggest challenge a future experiment will face.

We have shown that with technical advances in the fabrication of hybrid nanomechanical devices and minimisation of vibration sensitivities, the displacemon device provides an experimentally feasible route for testing collapse mechanisms in certain regions of parameter space. Finally, our testing protocol is applicable to other mechanical resonator devices, which can implement an equivalent grating operation and demonstrate sensitivity to decoherence in their measurement statistics. Thus, our proposal opens new avenues for tests of objective collapse models, complementary to matter-wave interferometry experiments.

Chapter 8

Conclusion

The results of this Thesis fall into three principle sections linked by the common thread of non-Gaussian quantum states. Firstly, we introduce an optomechanics protocol to generate a non-Gaussian entangled mechanical state. Of critical importance for optomechanics is the generation and verification of non-Gaussian states of motion. Indeed, an outstanding challenge in optomechanics is the generation and verification of a canonical two-mode Schrodinger-cat state of motion of two mechanical oscillators. In this thesis, we introduced a protocol to create this state in the displacement of two mechanical oscillators and verify the non-Gaussian entanglement generated (non-Gaussianity is a key resource in quantum information). Here, we introduced a protocol to create this state utilizing the nonlinearity of the radiation-pressure interaction combined with photon counting measurements. The entangled mechanical states generated by our scheme may find exciting applications including fault-tolerant quantum computation, and quantum metrology.

Furthermore, we introduced a powerful new scheme for entanglement verification using moment-based entanglement criteria. The verification scheme uses pulsed linearized optomechanical interactions to iteratively deduce mechanical moments by taking homodyne measurements of the output light. Our proposed experimental toolbox for state verification opens a new avenue within optomechanics (and quantum optics more generally) for a wide range of further studies relating to single- and multi-mode mechanical state characterisation.

Key experimental factors, such as optical loss and open-system dynamics, are carefully analyzed and we show that the scheme is feasible with only minor improvements to current experiments that operate outside the resolved-sideband regime. Our scheme provides a new avenue for quantum experiments with entangled mechanical oscillators and offers significant potential for further research and development that utilizes such non-Gaussian states for quantum-information and sensing applications, and for studying the quantum-to-classical transition.

In the second section of the results, we studied the problem of choosing the optimal entanglement criterion to detect entanglement in a non-Gaussian state. In the aforementioned optomechanical scheme, we used a specific non-Gaussian entanglement criterion based on detecting a negative partial transpose (NPT). We examined the feasibility of this particular criterion as a proof-of-concept that our verification scheme could detect non-Gaussian entanglement. However, we did not consider further inseparability criteria which are also attainable with our verification scheme. We addressed this issue in the subsequent chapter of this Thesis in which we assessed the suitability of a given NPT criterion (from a hierarchy of possible criteria) by formulating a statistical toolkit with which to quantify its associated error. The error incorporates both the impact of entanglement-degrading interactions with a system's environment, and of taking only a finite number of measurements when experimentally determining an NPT criterion. This allows us to assess the confidence level in an NPT criterion in detecting entanglement. With this statistical framework we assessed many NPT criteria to determine which are most appropriate for two specific non-Gaussian states: the two-mode Schrödinger cat-state from the previous chapter, and the photon-subtracted two-mode squeezed vacuum state. We verify that the choice of an NPT criterion depends on the state itself, the number of measurements, and the coupling to the environment. Beyond inseparability criteria, our toolkit can be used to find the error on a determinant which has been calculated from a Hermitian matrix of observable moments and thus can be applied to other nonclassicality criteria constructed in this way.

In this chapter we also studied schemes for measuring arbitrary two-mode moments of optical states. We began by examining methods of obtaining moments by initially reconstructing the quantum states, and also more direct methods of finding moments. Additionally, we presented

an alternative method of measuring two-mode moments of spatially separated modes of light. Our method relies on joint homodyne measurements to iteratively find increasingly higher-order moments. Crucially, the scheme uses sets of optical phases without the need to discretely scan a phase in order to perform a Fourier transform, as is the case in many measurement schemes. We have provided a method to choose the phase sets for lower order moments and surprisingly find that randomly choosing the phase sets can also be successful. However, further work is potentially needed here in order to provide a deterministic method of choosing the phase sets for moments of arbitrarily high order.

Finally, in the third chapter of our results, we examined whether the displacemon device can be used to test for objective collapse models. The displacemon is an electromechanical device consisting of a vibrating nanobeam coupled to a superconducting qubit. Through qubit operations and measurements, we can implement synthetic diffraction gratings and thus create a mechanical superposition in the centre-of-mass trajectories of the coupled resonator. Thus we can prepare the nanobeam in a Schrödinger-cat state, which is sensitive to unconventional decoherence mechanisms. Specifically, we propose a scheme that exploits qubit measurement statistics in order to monitor the decoherence of a displacemon resonator which has been prepared in a Schrödinger cat state. These measurement statistics allow us to quantify the degree of decoherence at regular time intervals and thereby test for sources of unconventional decoherence. Moreover, as the mechanical period is much larger than all relevant timescales of the auxiliary qubit, our testing protocol is not limited by the decoherence time of the qubit.

In particular, we examine the feasibility of using a displacemon with an aluminium nanobeam on two mass scales: one inspired by the Marshall-Simon-Penrose-Bouwmeester moving-mirror proposal, and the other set by the Planck mass. We calculate the lifetimes of the mechanical state predicted by DP theory and calculate the temperatures necessary to observe this collapse over conventional decoherence. Moreover, we consider different possibilities for the key parameter in the DP theory—the gravitational radius—which determines the rate of the DP collapse. From the literature there are three different suggested parameters one might assume for the gravitational radius: (i) the nuclear radius, (ii) the zero-point amplitude of the beam, and (iii) the zero-point amplitude of a single atom in the crystal. One might experimentally

probe the first parameter using a resonator of either mass scale; the cooling requirements on the Planck mass object are less stringent. Only the Planck mass object is capable of potentially investigating the second gravitational radius parameter, with further cooling required. It is not feasible to investigate the third parameter with either device.

For CSL, we solve the dynamics of the initial mechanical Schrödinger cat state in the presence of the CSL collapse mechanism to arrive at an analytic formula for the qubit measurement statistics, which are sensitive to the underlying collapse mechanics. By varying the displacemon and CSL parameters in our dynamical model, we examine the feasibility of testing CSL with near-future experimental capabilities. We find that lower temperatures are required for the Marshall-Simon-Penrose-Bouwmeester mass device to probe the same set of CSL parameters compared with the Planck mass device.

Furthermore, we discuss the importance of minimising the vibration sensitivities of the displacemon in order to achieve the effective temperatures required to test objective collapse. We find that the lower frequency device based on the Planck mass can probe DP and CSL theories with near-term improvements in these electromechanical devices, and we quantify the conditions to observe collapse with the higher frequency device. Although we have focused on the displacemon, we note that our testing protocol is device independent and we identify another optomechanical platform that can achieve equivalent grating operations. We find that with improvements to the fabrication and vibration sensitivities of these electromechanical devices, the displacemon device provides a new route to feasibly test decoherence mechanisms beyond standard quantum theory.

Bibliography

- [1] A. Ourjoumtsev, H. Jeong, R. Tualle-Broui, and P. Grangier. Generation of optical ‘Schrödinger cats’ from photon number states. *Nature*, 448(7155):784, 2007.
- [2] B. Vlastakis, G. Kirchmair, Z. Leghtas, S. E. Nigg, L. Frunzio, S. M. Girvin, M. Mirrahimi, M. H. Devoret, and R. J. Schoelkopf. Deterministically encoding quantum information using 100-photon schrödinger cat states. *Science*, 342(6158):607–610, 2013.
- [3] M. Mirrahimi, Z. Leghtas, V. V. Albert, S. Touzard, R. J. Schoelkopf, L. Jiang, and M. H. Devoret. Dynamically protected cat-qubits: a new paradigm for universal quantum computation. *New J. Phys.*, 16(4):045014, 2014.
- [4] W. H. Zurek. From quantum to classical. *Phys. Today*, 44(20):37, 1991.
- [5] G. Amelino-Camelia. Gravity in quantum mechanics. *Nat. Phys.*, 10(4):254–255, 2014.
- [6] S. Bose, K. Jacobs, and P. L. Knight. Scheme to probe the decoherence of a macroscopic object. *Phys. Rev. A*, 59(5):3204, 1999.
- [7] W. Marshall, C. Simon, R. Penrose, and D. Bouwmeester. Towards quantum superpositions of a mirror. *Phys. Rev. Lett.*, 91(13):130401, 2003.
- [8] J. S. Bell. On the Einstein Podolsky Rosen paradox. *Phys. Phys. Fiz.*, 1(3):195, 1964.
- [9] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt. Proposed experiment to test local hidden-variable theories. *Phys. Rev. Lett.*, 23(15):880, 1969.
- [10] N. Gisin. Bell’s inequality holds for all non-product states. *Phys. Lett. A*, 154(5-6):201–202, 1991.

- [11] R. F. Werner. Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model. *Phys. Rev. A*, 40(8):4277, 1989.
- [12] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters. Quantum nonlocality without entanglement. *Phys. Rev. A*, 59(2):1070, 1999.
- [13] T. Vértesi and N. Brunner. Disproving the Peres conjecture by showing bell nonlocality from bound entanglement. *Nat. Commun.*, 5(1):5297, 2014.
- [14] V. Giovannetti, S. Lloyd, and L. Maccone. Advances in quantum metrology. *Nat. Photonics*, 5(4):222, 2011.
- [15] J. Joo, W. J. Munro, and T. P. Spiller. Quantum metrology with entangled coherent states. *Phys. Rev. Lett.*, 107(8):083601, 2011.
- [16] A. K. Ekert. Quantum cryptography based on Bell’s theorem. *Phys. Rev. Lett.*, 67(6):661, 1991.
- [17] D. Gottesman and I. L. Chuang. Demonstrating the viability of universal quantum computation using teleportation and single-qubit operations. *Nature*, 402(6760):390–393, 1999.
- [18] N. C. Menicucci, P. Van Loock, M. Gu, C. Weedbrook, T. C. Ralph, and M. A. Nielsen. Universal quantum computation with continuous-variable cluster states. *Phys. Rev. Lett.*, 97(11):110501, 2006.
- [19] H.-J. Briegel, W. Dür, J. I. Cirac, and P. Zoller. Quantum repeaters: The role of imperfect local operations in quantum communication. *Phys. Rev. Lett.*, 81(26):5932, 1998.
- [20] S. Wehner, D. Elkouss, and R. Hanson. Quantum internet: A vision for the road ahead. *Science*, 362(6412), 2018.
- [21] E. Shchukin and W. Vogel. Inseparability criteria for continuous bipartite quantum states. *Phys. Rev. Lett.*, 95(23):230502, 2005.

- [22] K. E. Khosla, M R Vanner, N Ares, and E A Laird. Displacemon electromechanics: how to detect quantum interference in a nanomechanical resonator. *Phys. Rev. X*, 8(2):21052, 2018.
- [23] M. A. Nielsen and I. L. Chuang. *Quantum computation and quantum information*. Cambridge University Press, 2010.
- [24] A. Serafini. *Quantum Continuous Variables: A Primer of Theoretical Methods (1st ed.)*. CRC Press, 2017.
- [25] K. Husimi. Some formal properties of the density matrix. *Proceedings of the Physico-Mathematical Society of Japan. 3rd Series*, 22(4):264, 1940.
- [26] R. J. Glauber. Coherent and incoherent states of the radiation field. *Physical Review*, 131(6):2766, 1963.
- [27] E.C.G. Sudarshan. Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams. *Phys. Rev. Lett.*, 10(7):277, 1963.
- [28] E. P. Wigner. On the quantum correction for thermodynamic equilibrium. In *Part I: Physical Chemistry. Part II: Solid State Physics*, pages 110–120. Springer, 1997.
- [29] G Adesso and F. Illuminati. Entanglement in continuous-variable systems: recent advances and current perspectives. *J. Phys. A: Math. Theor.*, 40(28):7821, 2007.
- [30] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight. Quantifying entanglement. *Phys. Rev. Lett.*, 78(12):2275, 1997.
- [31] M. Lewenstein, B. Kraus, J. I. Cirac, and P. Horodecki. Optimization of entanglement witnesses. *Phys. Rev. A*, 62(5):052310, 2000.
- [32] O. Gühne and N. Lütkenhaus. Nonlinear entanglement witnesses. *Phys. Rev. Lett.*, 96(17):170502, 2006.
- [33] P. Hyllus and J. Eisert. Optimal entanglement witnesses for continuous-variable systems. *New J. Phys.*, 8(4):51, 2006.

- [34] A. Peres. Separability criterion for density matrices. *Phys. Rev. Lett.*, 77(8):1413, 1996.
- [35] M. Horodecki, P. Horodecki, and R. Horodecki. Separability of mixed states: necessary and sufficient conditions. *Phys. Lett. A*, 223(1):1, 1996.
- [36] R. Simon. Peres-Horodecki separability criterion for continuous variable systems. *Phys. Rev. Lett.*, 84(12):2726, 2000.
- [37] R. F. Werner and M. M. Wolf. Bound entangled Gaussian states. *Phys. Rev. Lett.*, 86(16):3658–3661, 2001.
- [38] L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller. Inseparability criterion for continuous variable systems. *Phys. Rev. Lett.*, 84(12):2722, 2000.
- [39] A. Heidmann, R. J. Horowicz, S. Reynaud, E. Giacobino, C. Fabre, and G. Camy. Observation of quantum noise reduction on twin laser beams. *Phys. Rev. Lett.*, 59(22):2555, 1987.
- [40] G. J. Milburn. Multimode minimum uncertainty squeezed states. *J. Phys. A: Math. Gen.*, 17(4):737, 1984.
- [41] F. Grosshans. Collective Attacks and unconditional security in continuous variable quantum keyDistribution. *Phys. Rev. Lett.*, 94(2):020504, 2005.
- [42] S. L. Braunstein and P. van Loock. Quantum information with continuous variables. *Rev. Mod. Phys.*, 77(2):51, 2005.
- [43] J. Eisert, S. Scheel, and M. B. Plenio. Distilling Gaussian states with Gaussian operations is impossible. *Phys. Rev. Lett.*, 89(13):137903, 2002.
- [44] A. Kitagawa, M. Takeoka, M. Sasaki, and A. Chefles. Entanglement evaluation of non-Gaussian states generated by photon subtraction from squeezed states. *Phys. Rev. A*, 73(4):042310, 2006.
- [45] A. Ourjoumtsev, A. Dantan, R. Tualle-Broui, and P. Grangier. Increasing entanglement between Gaussian states by coherent photon subtraction. *Phys. Rev. Lett.*, 98(3):030502, 2007.

- [46] N. J. Cerf, O. Krüger, P. Navez, R. F. Werner, and M. M. Wolf. Non-Gaussian cloning of quantum coherent states is optimal. *Phys. Rev. Lett.*, 95(7):070501, 2005.
- [47] S. Olivares, M. G. A. Paris, and R. Bonifacio. Teleportation improvement by inconclusive photon subtraction. *Phys. Rev. A*, 67(3):032314, 2003.
- [48] F. Dell’Anno, S. De Siena, L. Albano, and F. Illuminati. Continuous-variable quantum teleportation with non-Gaussian resources. *Phys. Rev. A*, 76(2):022301, 2007.
- [49] M. G. Genoni, C. Invernizzi, and M. G. A. Paris. Enhancement of parameter estimation by kerr interaction. *Phys. Rev. A*, 80(3):033842, 2009.
- [50] G. Adesso, F. Dell’Anno, S. De Siena, F. Illuminati, and L. A. M. Souza. Optimal estimation of losses at the ultimate quantum limit with non-Gaussian states. *Phys. Rev. A*, 79(4):040305, 2009.
- [51] S. Lloyd and S. L. Braunstein. Quantum computation over continuous variables. *Phys. Rev. Lett.*, 82(8):1784, 1999.
- [52] M. Ohliger, K. Kieling, and J. Eisert. Limitations of quantum computing with Gaussian cluster states. *Phys. Rev. A*, 82(4):042336, 2010.
- [53] J. Niset, J. Fiurášek, and N. J. Cerf. No-go theorem for Gaussian quantum error correction. *Phys. Rev. Lett.*, 102(12):120501, 2009.
- [54] B. Hensen et al. Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres. *Nature*, 526(7575):682, 2015.
- [55] L. K. Shalm et al. Strong loophole-free test of local realism. *Phys. Rev. Lett.*, 115(25):250402, 2015.
- [56] M. Giustina et al. Significant-loophole-free test of Bell’s theorem with entangled photons. *Phys. Rev. Lett.*, 115(25):250401, 2015.
- [57] M. G. Genoni, M. G. A. Paris, and K. Banaszek. Measure of the non-Gaussian character of a quantum state. *Phys. Rev. A*, 76(4):042327, 2007.

- [58] M. G. Genoni, M. G. A. Paris, and K. Banaszek. Quantifying the non-Gaussian character of a quantum state by quantum relative entropy. *Phys. Rev. A*, 78(6):060303, 2008.
- [59] M. G. Genoni and M. G. A. Paris. Quantifying non-Gaussianity for quantum information. *Phys. Rev. A*, 82(5):052341, 2010.
- [60] P. Marian and T. A. Marian. Relative entropy is an exact measure of non-Gaussianity. *Phys. Rev. A*, 88(1):012322, 2013.
- [61] S. Qvarfort, A. Serafini, A. Xuereb, D. Rätzel, and D. E. Bruschi. Enhanced continuous generation of non-Gaussianity through optomechanical modulation. *New J. Phys.*, 21(5):055004, 2019.
- [62] S. J. van Enk and O. Hirota. Entangled coherent states: Teleportation and decoherence. *Phys. Rev. A*, 64(2):022313, 2001.
- [63] N. Fiaschi, B. Hensen, A. Wallucks, R. Benevides, J. Li, T. P. M. Alegre, and S. Gröblacher. Optomechanical quantum teleportation. *Nat. Photonics*, 15(11):817, 2021.
- [64] C. Wang et al. A Schrödinger cat living in two boxes. *Science*, 352(6289):1087, 2016.
- [65] C.-W. Lee and H. Jeong. Quantification of macroscopic quantum superpositions within phase space. *Phys. Rev. Lett.*, 106(22):220401, 2011.
- [66] B. Yadin and V. Vedral. General framework for quantum macroscopicity in terms of coherence. *Phys. Rev. A*, 93(2):022122, 2016.
- [67] D. Kleckner, I. Pikovski, E. Jeffrey, L. Ament, E. Eliel, J. Van Den Brink, and D. Bouwmeester. Creating and verifying a quantum superposition in a micro-optomechanical system. *New J. Phys.*, 10(9):095020, 2008.
- [68] A. Bassi, K. Lochan, S. Satin, T. P. Singh, and H. Ulbricht. Models of wave-function collapse, underlying theories, and experimental tests. *Rev. Mod. Phys.*, 85(2):471S, 2013.
- [69] U. Akram, W. P. Bowen, and G. J. Milburn. Entangled mechanical cat states via conditional single photon optomechanics. *New J. Phys.*, 15(9):093007, 2013.

- [70] B. Xiong, X. Li, S.-L. Chao, Z. Yang, W.-Z Zhang, and L. Zhou. Generation of entangled Schrödinger cat state of two macroscopic mirrors. *Opt. Express*, 27(9):13547, 2019.
- [71] X Wang. Quantum teleportation of entangled coherent states. *Phys. Rev. A*, 64(2):022302, 2001.
- [72] J. Joo and E. Ginossar. Efficient scheme for hybrid teleportation via entangled coherent states in circuit quantum electrodynamics. *Sci. Rep.*, 6(1):1, 2016.
- [73] S. M. Barnett, G. Ferenczi, C. R. Gilson, and F. C. Speirits. Statistics of photon-subtracted and photon-added states. *Phys. Rev. A*, 98(1):013809, 2018.
- [74] A. Ourjoumtsev, R. Tualle-Brouri, J. Laurat, and P. Grangier. Generating optical Schrödinger kittens for quantum information processing. *Science*, 312(5770):83, 2006.
- [75] B. M. Neergaard-Nielsen, J. S .and Nielsen, C. Hettich, K. Mølmer, and E. S. Polzik. Generation of a superposition of odd photon number states for quantum information networks. *Phys. Rev. Lett.*, 97(8):083604, 2006.
- [76] M. R. Vanner, M. Aspelmeyer, and M. S. Kim. Quantum state orthogonalization and a toolset for quantum optomechanical phonon control. *Phys. Rev. Lett.*, 110(1):010504, 2013.
- [77] R. N. Patel, T. P. McKenna, Z. Wang, J. D. Witmer, W. Jiang, R. Van Laer, C. J. Sarabalis, and A. H. Safavi-Naeini. Room-temperature mechanical resonator with a single added or subtracted phonon. *Phys. Rev. Lett.*, 127(13):133602, 2021.
- [78] G.ENZIAN, J. J. Price, L. Freisem, J. Nunn, J. Janousek, B. C. Buchler, P. K. Lam, and M. R. Vanner. Single-phonon addition and subtraction to a mechanical thermal state. *Phys. Rev. Lett.*, 126(3):033601, 2021.
- [79] G.ENZIAN, L. Freisem, J. J. Price, A. Ø. Svela, J. Clarke, B. Shajilal, J. Janousek, B. C. Buchler, P. K. Lam, and M. R. Vanner. Non-Gaussian mechanical motion via single and multiphonon subtraction from a thermal state. *Phys. Rev. Lett.*, 127(24):243601, 2021.

- [80] C. Navarrete-Benlloch, R. García-Patrón, J. H. Shapiro, and N. J. Cerf. Enhancing quantum entanglement by photon addition and subtraction. *Phys. Rev. A*, 86(1):012328, 2012.
- [81] M. R. Vanner, J. Hofer, G. D. Cole, and M. Aspelmeyer. Cooling-by-measurement and mechanical state tomography via pulsed optomechanics. *Nat. Commun.*, 4(1):2295, 2013.
- [82] J. T. Muhonen, G. R. La Gala, R. Leijssen, and E. Verhagen. State preparation and tomography of a nanomechanical resonator with fast light pulses. *Phys. Rev. Lett.*, 123(11):113601, 2019.
- [83] M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt. Cavity optomechanics. *Rev. Mod. Phys.*, 86(4):1391, 2014.
- [84] R. Riedinger, S. Hong, R. A. Norte, J. A. Slater, J. Shang, A. G. Krause, V. Anant, M. Aspelmeyer, and S. Gröblacher. Non-classical correlations between single photons and phonons from a mechanical oscillator. *Nature*, 530(7590):313, 2016.
- [85] K. Heshami, D. G. England, P. C. Humphreys, P. J. Bustard, V. M. Acosta, J. Nunn, and B. J. Sussman. Quantum memories: emerging applications and recent advances. *Journal of modern optics*, 63(20):2005, 2016.
- [86] B. Pepper, R. Ghobadi, Evan Jeffrey, Christoph S., and D. Bouwmeester. Optomechanical superpositions via nested interferometry. *Phys. Rev. Lett.*, 109(2):023601, 2012.
- [87] A. Ashkin. Acceleration and trapping of particles by radiation pressure. *Phys. Rev. Lett.*, 24(4):156, 1970.
- [88] G. Kopp and J. L. Lean. A new, lower value of total solar irradiance: Evidence and climate significance. *Geophys. Res. Lett.*, 38(1), 2011.
- [89] W. P. Bowen and G. J. Milburn. *Quantum optomechanics*. CRC press, 2015.
- [90] C. Gerry, P. Knight, and P. L. Knight. *Introductory quantum optics*. Cambridge University Press, 2005.

- [91] M. J. Collett and C. W. Gardiner. Squeezing of intracavity and traveling-wave light fields produced in parametric amplification. *Phys. Rev. A*, 30(3):1386, 1984.
- [92] D. F. Walls and G. J. Milburn. *Quantum Optics*. Springer, 2011.
- [93] M. R. Vanner, I. Pikovski, G. D. Cole, M. S. Kim, C. Brukner, K. Hammerer, G. J. Milburn, and M. Aspelmeyer. Pulsed quantum optomechanics. *Proc. Natl. Acad. Sci.*, 108(39):16182, 2011.
- [94] I. Pikovski, M. R. Vanner, M. Aspelmeyer, M. S. Kim, and Č. Brukner. Probing Planck-scale physics with quantum optics. *Nat. Phys.*, 8(5):393, 2012.
- [95] J. Clarke and M. R. Vanner. Growing macroscopic superposition states via cavity quantum optomechanics. *Quantum Sci. Technol.*, 4(1):014003, 2019.
- [96] K. E. Khosla, M. R. Vanner, W. P. Bowen, and G. J. Milburn. Quantum state preparation of a mechanical resonator using an optomechanical geometric phase. *New J. Phys.*, 15(4):043025, 2013.
- [97] J. S. Bennett, K. Khosla, L. S. Madsen, M. R. Vanner, H. Rubinsztein-Dunlop, and W. P. Bowen. A quantum optomechanical interface beyond the resolved sideband limit. *New J. Phys.*, 18(5):053030, 2016.
- [98] J. Clarke, P. Neveu, K. E. Khosla, E. Verhagen, and M. R. Vanner. Position measurement and the nonlinear regime of cavity quantum optomechanics. *arXiv preprint arXiv:2207.11153*, 2022.
- [99] A. D. Armour and M. P. Blencowe. Probing the quantum coherence of a nanomechanical resonator using a superconducting qubit: I. echo scheme. *New J. Phys.*, 10(9):095004, 2008.
- [100] Y. Nakamura, Y. A Pashkin, and J. S. Tsai. Coherent control of macroscopic quantum states in a single-Cooper-pair box. *Nature*, 398(6730):786, 1999.

- [101] P. Krantz, M. Kjaergaard, F. Yan, T. P. Orlando, S. Gustavsson, and W. D. Oliver. A quantum engineer's guide to superconducting qubits. *Appl. Phys. Rev.*, 6(2):021318, 2019.
- [102] Jens Koch, Terri Yu, J M Gambetta, A A Houck, D I Schuster, J Majer, A Blais, M H Devoret, S M Girvin, and R J Schoelkopf. Charge-insensitive qubit design derived from the Cooper pair box. *Phys. Rev. A*, 76(4):042319, 2007.
- [103] T. Bera, S. Majumder, S. K. Sahu, and V. Singh. Large flux-mediated coupling in hybrid electromechanical system with a transmon qubit. *Commun. Phys.*, 4(1):1, 2021.
- [104] L. DiCarlo et al. Demonstration of two-qubit algorithms with a superconducting quantum processor. *Nature*, 460(7252):240–244, 2009.
- [105] J. Braumüller et al. Concentric transmon qubit featuring fast tunability and an anisotropic magnetic dipole moment. *Appl. Phys. Lett.*, 108(3):032601, 2016.
- [106] L. A. Kanari-Naish, J. Clarke, M. R. Vanner, and E. A. Laird. Can the displacemon device test objective collapse models? *AVS Quantum Sci.*, 3(4):045603, 2021.
- [107] M. Poot and H. S. J. van der Zant. Mechanical systems in the quantum regime. *Phys. Rep.*, 511(5):273, 2012.
- [108] G. Vidal and R. F. Werner. Computable measure of entanglement. *Phys. Rev. A*, 65(3):032314, 2002.
- [109] G. S. Agarwal and A. Biswas. Inseparability inequalities for higher order moments for bipartite systems. *New J. Phys.*, 7(1):211, 2005.
- [110] M. Hillery and M. S. Zubairy. Entanglement conditions for two-mode states: Applications. *Phys. Rev. A*, 74(3):032333, 2006.
- [111] M. Horodecki, P. Horodecki, and R. Horodecki. Mixed-state entanglement and distillation: Is there a “bound” entanglement in nature? *Phys. Rev. Lett.*, 80(24):5239, 1998.

- [112] P. Horodecki. Separability criterion and inseparable mixed states with positive partial transposition. *Phys. Lett. A*, 232(5):333, 1997.
- [113] P. Horodecki, Ł. Rudnicki, and K. Życzkowski. Five open problems in quantum information. *arXiv preprint arXiv:2002.03233*, 2020.
- [114] B. C. Hiesmayr. Free versus bound entanglement, a np-hard problem tackled by machine learning. *Sci. Rep.*, 11(1):1, 2021.
- [115] S. Mancini, V. Giovannetti, D. Vitali, and P. Tombesi. Entangling macroscopic oscillators exploiting radiation pressure. *Phys. Rev. Lett.*, 88(12):120401, 2002.
- [116] M. G. Raymer, A. C. Funk, B. C. Sanders, and H. de Guise. Separability criterion for separate quantum systems. *Phys. Rev. A*, 67(5):052104, 2003.
- [117] L. A. Kanari-Naish, J. Clarke, S. Qvarfort, and M. R. Vanner. Two-mode Schrödinger-cat states with nonlinear optomechanics: generation and verification of non-Gaussian mechanical entanglement. *Quantum Sci. Technol.*, 7(3):035012, 2022.
- [118] S. Pirandola, D. Vitali, P. Tombesi, and S. Lloyd. Macroscopic entanglement by entanglement swapping. *Phys. Rev. Lett.*, 97(15):150403, 2006.
- [119] Y.-D. Wang and A. A. Clerk. Reservoir-engineered entanglement in optomechanical systems. *Phys. Rev. Lett.*, 110(25):253601, 2013.
- [120] M. J. Woolley and A. A. Clerk. Two-mode squeezed states in cavity optomechanics via engineering of a single reservoir. *Phys. Rev. A*, 89(6):063805, 2014.
- [121] N. Vostrosablin, A. A. Rakhubovsky, and R. Filip. Pulsed quantum interaction between two distant mechanical oscillators. *Phys. Rev. A*, 94(6):063801, 2016.
- [122] J. Clarke, P. Sahium, K. E. Khosla, I. Pikovski, M. S. Kim, and M. R. Vanner. Generating mechanical and optomechanical entanglement via pulsed interaction and measurement. *New J. Phys.*, 22(6):063001, 2020.

- [123] Matteo Brunelli, Daniel Malz, Albert Schliesser, and Andreas Nunnenkamp. Stroboscopic quantum optomechanics. *Phys. Rev. Research*, 2(2):023241, 2020.
- [124] I. Brandão, B. Suassuna, B. Melo, and T. Guerreiro. Entanglement dynamics in dispersive optomechanics: Nonclassicality and revival. *Phys. Rev. Research*, 2(4):043421, 2020.
- [125] C. Gut et al. Stationary optomechanical entanglement between a mechanical oscillator and its measurement apparatus. *Phys. Rev. Research*, 2(3):033244, 2020.
- [126] P. Neveu, J. Clarke, M. R. Vanner, and E. Verhagen. Preparation and verification of two-mode mechanical entanglement through pulsed optomechanical measurements. *New J. Phys.*, 23(2):023026, 2021.
- [127] T. A. Palomaki, J. D. Teufel, R. W. Simmonds, and K. W. Lehnert. Entangling mechanical motion with microwave fields. *Science*, 342(6159):710, 2013.
- [128] C. F. Ockeloen-Korppi, E. Damskägg, J.-M. Pirkkalainen, M. Asjad, A. A. Clerk, F. Massel, M. J. Woolley, and M. A. Sillanpää. Stabilized entanglement of massive mechanical oscillators. *Nature*, 556(7702):478, 2018.
- [129] L. Mercier de Lépinay, C. F. Ockeloen-Korppi, M. J. Woolley, and M. A. Sillanpää. Quantum mechanics-free subsystem with mechanical oscillators. *Science*, 372(6542):625, 2021.
- [130] S. Kotler et al. Direct observation of deterministic macroscopic entanglement. *Science*, 372(6542):622, 2021.
- [131] K. Børkje, A. Nunnenkamp, and S. M. Girvin. Proposal for entangling remote micromechanical oscillators via optical measurements. *Phys. Rev. Lett.*, 107(12):123601, 2011.
- [132] H. Flayac and V. Savona. Heralded preparation and readout of entangled phonons in a photonic crystal cavity. *Phys. Rev. Lett.*, 113(14):143603, 2014.
- [133] T. J. Milburn, M. S. Kim, and M. R. Vanner. Nonclassical-state generation in macroscopic systems via hybrid discrete-continuous quantum measurements. *Phys. Rev. A*, 93(5):053818, 2016.

- [134] K. C. Lee, B. J. Sussman, M. R. Sprague, P. Michelberger, K. F. Reim, J. Nunn, N. K. Langford, P. J. Bustard, D. Jaksch, and I. A. Walmsley. Macroscopic non-classical states and terahertz quantum processing in room-temperature diamond. *Nat. Photonics*, 6(1):41, 2012.
- [135] K. A. G. Fisher, D. G. England, J.-P. W. MacLean, P. J. Bustard, K. Heshami, K. J. Resch, and B. J. Sussman. Storage of polarization-entangled THz-bandwidth photons in a diamond quantum memory. *Phys. Rev. A*, 96(1):012324, 2017.
- [136] S. T. Velez, K. Seibold, N. Kipfer, M. D. Anderson, V. Sudhir, and C. Galland. Preparation and decay of a single quantum of vibration at ambient conditions. *Phys. Rev. X*, 9(4):041007, 2019.
- [137] K. C. Lee et al. Entangling macroscopic diamonds at room temperature. *Science*, 334(6060):1253, 2011.
- [138] R. Riedinger, A. Wallucks, I. Marinković, C. Löschner, M. Aspelmeyer, S. Hong, and S. Gröblacher. Remote quantum entanglement between two micromechanical oscillators. *Nature*, 556(7702):473, 2018.
- [139] M. Ringbauer, T. J. Weinhold, L. A. Howard, A. G. White, and M. R. Vanner. Generation of mechanical interference fringes by multi-photon counting. *New J. Phys.*, 20(5):053042, 2018.
- [140] G. A. Brawley, M. R. Vanner, P. E. Larsen, S. Schmid, A. Boisen, and W. P. Bowen. Non-linear optomechanical measurement of mechanical motion. *Nat. Commun.*, 7(1):10988, 2016.
- [141] R. Leijssen, G. R. La Gala, L. Freisem, J. T. Muhonen, and E. Verhagen. Nonlinear cavity optomechanics with nanomechanical thermal fluctuations. *Nat. Commun.*, 8(1):16024, 2017.
- [142] S. Mancini, V. I. Man’ko, and P. Tombesi. Ponderomotive control of quantum macroscopic coherence. *Phys. Rev. A*, 55(4):3042–3050, 1997.

- [143] S. Bose, K. Jacobs, and P. L. Knight. Preparation of nonclassical states in cavities with a moving mirror. *Phys. Rev. A*, 56(5):4175–4186, 1997.
- [144] P. Rabl. Photon blockade effect in optomechanical systems. *Phys. Rev. Lett.*, 107(6):063601, 2011.
- [145] A. Nunnenkamp, K. Børkje, and S. M. Girvin. Single-photon optomechanics. *Phys. Rev. Lett.*, 107(6):063602, 2011.
- [146] S. Qvarfort, M. R. Vanner, P. F. Barker, and D. E. Bruschi. Master-equation treatment of nonlinear optomechanical systems with optical loss. *Phys. Rev. A*, 104(1):013501, 2021.
- [147] G. Vacanti, M. Paternostro, G. M. Palma, and V. Vedral. Optomechanical to mechanical entanglement transformation. *New J. Phys.*, 10(9):095014, 2008.
- [148] Z. Wang and A. H. Safavi-Naeini. Enhancing a slow and weak optomechanical nonlinearity with delayed quantum feedback. *Nat. Commun.*, 8(1):15886, 2017.
- [149] C. W Gardiner and M. J Collett. Input and output in damped quantum systems: Quantum stochastic differential equations and the master equation. *Phys. Rev. A*, 31(6):3761, 1985.
- [150] V. Guifré. Entanglement monotones. *Journal of Modern Optics*, 47(2-3):355, 2000.
- [151] M. Rossi, D. Mason, J. Chen, Y. Tsaturyan, and A. Schliesser. Measurement-based quantum control of mechanical motion. *Nature*, 563(7565):53, 2018.
- [152] D. J. Wilson, V. Sudhir, N. Piro, R. Schilling, A. Ghadimi, and T. J. Kippenberg. Measurement-based control of a mechanical oscillator at its thermal decoherence rate. *Nature*, 524(7729):32, 2015.
- [153] A. Miranowicz, M. Piani, P. Horodecki, and R. Horodecki. Inseparability criteria based on matrices of moments. *Phys. Rev. A*, 80(5):052303, 2009.
- [154] T. Opatrný, D.-G. Welsch, and W. Vogel. Homodyne detection for measuring internal quantum correlations of optical pulses. *Phys. Rev. A*, 55(2):1416, 1997.

- [155] E. V. Shchukin and W. Vogel. Nonclassical moments and their measurement. *Phys. Rev. A*, 72(4):043808, 2005.
- [156] G.J. Milburn and M.J. Woolley. An introduction to quantum optomechanics. *Acta Physica Slovaca*, 61:483, 2011.
- [157] Y. Li and S. C. Benjamin. Efficient variational quantum simulator incorporating active error minimization. *Phys. Rev. X*, 7(2):021050, 2017.
- [158] K. Temme, S. Bravyi, and J. M. Gambetta. Error mitigation for short-depth quantum circuits. *Phys. Rev. Lett.*, 119(18):180509, 2017.
- [159] G. La Gala, J. P. Mathew, P. Neveu, and E. Verhagen. Nanomechanical design strategy for single-mode optomechanical measurement. *JJ. Phys. D: Appl. Phys.*, 55(22):225101, 2022.
- [160] A. A. Rakhubovsky, D. W. Moore, U. Delić, N. Kiesel, M. Aspelmeyer, and R. Filip. Detecting nonclassical correlations in levitated cavity optomechanics. *Phys. Rev. Applied*, 14(5):054052, 2020.
- [161] S. P. Walborn, B. G. Taketani, A. Salles, F. Toscano, and R. L. de Matos Filho. Entropic entanglement criteria for continuous variables. *Phys. Rev. Lett.*, 103(16):160505, 2009.
- [162] D. Zhang, D. Barral, Y. Cai, Y. Zhang, M. Xiao, and K. Bencheikh. Hierarchy of nonlinear entanglement dynamics for continuous variables. *Phys. Rev. Lett.*, 127(15):150502, 2021.
- [163] J. Sperling and W. Vogel. Necessary and sufficient conditions for bipartite entanglement. *Phys. Rev. A*, 79(2):022318, 2009.
- [164] F. Dell’Anno, S. De Siena, and F. Illuminati. Test of inseparability criteria for squeezed number states of the radiation field. *Open Systems & Information Dynamics*, 13(4):383, 2006.
- [165] J. Tilly, R. J Marshman, A. Mazumdar, and S. Bose. Qudits for witnessing quantum-gravity-induced entanglement of masses under decoherence. *Phys. Rev. A*, 104(5):052416, 2021.

- [166] Ulf Leonhardt. *Essential Quantum Optics: From quantum measurements to black holes*. Cambridge University Press, 2010.
- [167] E. Shchukin, T. Richter, and W. Vogel. Nonclassicality criteria in terms of moments. *Phys. Rev. A*, 71(1):011802, 2005.
- [168] D. T. Smithey, M. Beck, M. G. Raymer, and A. Faridani. Measurement of the Wigner distribution and the density matrix of a light mode using optical homodyne tomography: Application to squeezed states and the vacuum. *Phys. Rev. Lett.*, 70(9):1244, 1993.
- [169] G. M. D’Ariano, C. Macchiavello, and M. G. A. Paris. Detection of the density matrix through optical homodyne tomography without filtered back projection. *Phys. Rev. A*, 50(5):4298, 1994.
- [170] M. G. Raymer, D. F. McAlister, and U. Leonhardt. Two-mode quantum-optical state measurement: Sampling the joint density matrix. *Phys. Rev. A*, 54(3):2397, 1996.
- [171] H. Kühn, D.-G. Welsch, and W. Vogel. Reconstruction of the quantum state of multimode light. *Phys. Rev. A*, 51(5):4240, 1995.
- [172] U. Leonhardt and H. Paul. Realistic optical homodyne measurements and quasiprobability distributions. *Phys. Rev. A*, 48(6):4598, 1993.
- [173] G. S. Agarwal and S. Chaturvedi. Scheme to measure the positive P distribution. *Phys. Rev. A*, 49(2):R665, 1994.
- [174] W. Vogel. Homodyne correlation measurements with weak local oscillators. *Phys. Rev. A*, 51(5):4160–4171, 1995.
- [175] P. Pearle. Reduction of the state vector by a nonlinear Schrödinger equation. *Phys. Rev. D*, 13(4):857, 1976.
- [176] P. Pearle. Toward explaining why events occur. *International Journal of Theoretical Physics*, 18(7):489, 1979.

- [177] P. Pearle. Experimental tests of dynamical state-vector reduction. *Phys. Rev. D*, 29(2):235, 1984.
- [178] N. Gisin. Quantum measurements and stochastic processes. *Phys. Rev. Lett.*, 52(19):1657–1660, 1984.
- [179] G. C. Ghirardi, A. Rimini, and T. Weber. Unified dynamics for microscopic and macroscopic systems. *Phys. Rev. D*, 34(2):470, 1986.
- [180] M. A Schlosshauer. *Decoherence: and the quantum-to-classical transition*. Springer Science & Business Media, 2007.
- [181] G. C. Ghirardi, P. Pearle, and A. Rimini. Markov processes in Hilbert space and continuous spontaneous localization of systems of identical particles. *Phys. Rev. A*, 42(1):78, 1990.
- [182] Q. Fu. Spontaneous radiation of free electrons in a nonrelativistic collapse model. *Phys. Rev. A*, 56(3):1806, 1997.
- [183] S. L. Adler. Lower and upper bounds on CSL parameters from latent image formation and IGM heating. *J. Phys. A: Math. Theor.*, 40(12):2935, 2007.
- [184] S. L. Adler and F. M. Ramazanoğlu. Photon-emission rate from atomic systems in the CSL model. *J. Phys. A: Math. Theor.*, 40(44):13395, 2007.
- [185] M. Carlesso and A. Bassi. Current tests of collapse models: How far can we push the limits of quantum mechanics? In *Quantum Information and Measurement (QIM) V: Quantum Technologies*, page S1C.3. Optical Society of America, 2019.
- [186] L. Diósi. Gravitation and quantum mechanical localization of Macro-Objects. *Phys. Lett. A*, 105(4):199, 1984.
- [187] L. Diósi. A universal master equation for the gravitational violation of quantum mechanics. *Phys. Lett. A*, 120(8):377–381, 1987.

- [188] R. Penrose. On Gravity's role in Quantum State Reduction. *Gen. Relativ. Gravit.*, 28(5):581, 1996.
- [189] R. Howl, R. Penrose, and I. Fuentes. Exploring the unification of quantum theory and general relativity with a Bose–Einstein condensate. *New J. Phys.*, 21(4):043047, 2019.
- [190] A. Vinante, M. Carlesso, A. Bassi, A. Chiasera, S. Varas, P. Falferi, B. Margesin, R. Mezzena, and H. Ulbricht. Narrowing the parameter space of collapse models with ultracold layered force sensors. *Phys. Rev. Lett.*, 125(10):100404, 2020.
- [191] S. Donadi, K. Piscicchia, Catalina Curceanu, L. Diósi, M. Laubenstein, and A. Bassi. Underground test of gravity-related wave function collapse. *Nat. Phys.*, 17(1):74, 2021.
- [192] S. Belli, R. Bonsignori, G. D'Auria, L. Fant, M. Martini, S. Peirone, S. Donadi, and A. Bassi. Entangling macroscopic diamonds at room temperature: Bounds on the continuous-spontaneous-localization parameters. *Phys. Rev. A*, 94(1):012108, 2016.
- [193] M. Bahrami, M. Paternostro, A. Bassi, and H. Ulbricht. Proposal for a noninterferometric test of collapse models in optomechanical systems. *Phys. Rev. Lett.*, 112(21):210404, 2014.
- [194] C. H. Van Der Wal, A. C. J. Ter Haar, F. K. Wilhelm, R. N. Schouten, C. J. P. M. Harman, T. P. Orlando, S. Lloyd, and J. E. Mooij. Quantum superposition of macroscopic persistent-current states. *Science*, 290(5492):773, 2000.
- [195] F. Laloë, W. J. Mullin, and P. Pearle. Heating of trapped ultracold atoms by collapse dynamics. *Phys. Rev. A*, 90(5):052119, 2014.
- [196] J. Li, S. Zippilli, J. Zhang, and D. Vitali. Discriminating the effects of collapse models from environmental diffusion with levitated nanospheres. *Phys. Rev. A*, 93(5):050102, 2016.
- [197] T. Kovachy, P. Asenbaum, C. Overstreet, C. A. Donnelly, S. M. Dickerson, A. Sugarbaker, J. M. Hogan, and M. A. Kasevich. Quantum superposition at the half-metre scale. *Nature*, 528(7583):530–533, 2015.

- [198] S. Eibenberger, S. Gerlich, M. Arndt, M. Mayor, and J. Tüxen. Matter-wave interference of particles selected from a molecular library with masses exceeding 10,000 amu. *Phys. Chem. Chem. Phys.*, 15(35):14696, 2013.
- [199] Y. Y. Fein, P. Geyer, P. Zwick, F. Kiałka, S. Pedalino, M. Mayor, S. Gerlich, and M. Arndt. Quantum superposition of molecules beyond 25 kDa. *Nat. Phys.*, 15(12):1242, 2019.
- [200] J. Bateman, S. Nimmrichter, K. Hornberger, and H. Ulbricht. Near-field interferometry of a free-falling nanoparticle from a point-like source. *Nat. Commun.*, 5(1):4788, 2014.
- [201] M. Bilardello, A. Trombettoni, and A. Bassi. Collapse in ultracold Bose Josephson junctions. *Phys. Rev. A*, 95(3):032134, 2017.
- [202] A. O’Connell et al. Quantum ground state and single-phonon control of a mechanical resonator. *Nature*, 464(7289):697, 2010.
- [203] M. F. Gely and G. A. Steele. Superconducting electro-mechanics to test Diósi–Penrose effects of general relativity in massive superpositions. *AVS Quantum Sci.*, 3(3):035601, 2021.
- [204] M. Arndt and K. Hornberger. Testing the limits of quantum mechanical superpositions. *Nat. Phys.*, 10(4):271, 2014.
- [205] B. Brezger, L. Hackermüller, S. Uttenthaler, J. Petschinka, M. Arndt, and A. Zeilinger. Matter-wave interferometer for large molecules. *Phys. Rev. Lett.*, 88(10):100404, 2002.
- [206] A. A. Houck et al. Controlling the Spontaneous Emission of a Superconducting Transmon Qubit. *Phys. Rev. Lett.*, 101(8):080502, 2008.
- [207] A. O. Caldeira and A. J. Leggett. Influence of dissipation on quantum tunneling in macroscopic systems. *Phys. Rev. Lett.*, 46(4):211, 1981.
- [208] A. O. Caldeira and A. J. Leggett. Quantum tunnelling in a dissipative system. *Ann. Phys.*, 149(2):374, 1983.

- [209] H. P. Breuer and F. Petruccione. *The theory of open quantum systems*. Oxford University Press on Demand, 2002.
- [210] L. Diósi. On high-temperature Markovian equation for quantum Brownian motion. *EPL*, 22(1):1, 1993.
- [211] L. Diósi. Models for universal reduction of macroscopic quantum fluctuations. *Phys. Rev. A*, 40(3):1165, 1989.
- [212] I. Angeli and K. P. Marinova. Table of experimental nuclear ground state charge radii: An update. *At. Data Nucl. Data Tables*, 99(1):69, 2013.
- [213] R. Stedman and G. Nilsson. Dispersion relations for phonons in aluminum at 80 and 300 K. *Phys. Rev.*, 145(2):492, 1966.
- [214] Leonhard Neuhaus et al. Laser cooling of a Planck mass object close to the quantum ground state. *e-print arXiv:2104.11648*, 2021.
- [215] S. Nimmrichter, K. Hornberger, and K. Hammerer. Optomechanical sensing of spontaneous wave-function collapse. *Phys. Rev. Lett.*, 113(2):020405, 2014.
- [216] L. Ferialdi and A. Bassi. Continuous spontaneous localization reduction rate for rigid bodies. *Phys. Rev. A*, 102(4):042213, 2020.
- [217] S. Kieseewetter, R. Y. Teh, P. D. Drummond, and M. D. Reid. Pulsed entanglement of two optomechanical oscillators and Furry’s hypothesis. *Phys. Rev. Lett.*, 119(2):023601, 2017.
- [218] W. H. Furry. Note on the quantum-mechanical theory of measurement. *Phys. Rev.*, 49(5):393, 1936.
- [219] H. Paik et al. Observation of high coherence in Josephson junction qubits measured in a three-dimensional circuit QED architecture. *Phys. Rev. Lett.*, 107(24):240501, 2011.
- [220] C. A. Regal, J. D. Teufel, and K. W. Lehnert. Measuring nanomechanical motion with a microwave cavity interferometer. *Nat. Phys.*, 4(7):555, 2008.

- [221] D Cattiaux et al. A macroscopic object passively cooled into its quantum ground state of motion beyond single-mode cooling. *Nat. Commun.*, 12(1):6182, 2021.
- [222] F. Matichard et al. Advanced LIGO two-stage twelve-axis vibration isolation and positioning platform. Part 2: Experimental investigation and tests results. *Precis. Eng.*, 40:287, 2015.
- [223] M. R. Vanner, I. Pikovski, and M.S. Kim. Towards optomechanical quantum state reconstruction of mechanical motion. *Annalen der Physik*, 527(1-2):15, 2015.

Appendix A

Generating and verifying two-mode mechanical entanglement

A.1 Verification set-up

In this Appendix we detail the key experimental steps required to calculate arbitrary mechanical moments of an entangled state, which are needed to evaluate the inseparability criteria outlined in Section 5.3.1. We begin by introducing the notation used to describe the moments, and then we focus on each of the two verification set-ups shown in Figs. 5.9 and A.1. For a given mechanical moment we define the order as the sum of the exponents on the operators, e.g. the order of $\langle \hat{X}_{M_1}^p \hat{P}_{M_1}^q \hat{X}_{M_2}^r \hat{P}_{M_2}^s \rangle$ is $d = p + q + r + s$. We use the notation $\mathcal{S}[\langle \hat{X}_{M_1}^p \hat{P}_{M_1}^q \hat{X}_{M_2}^r \hat{P}_{M_2}^s \rangle]$ to denote the sum of all possible permutations of p lots of $\hat{X}_{M_1}$ -operators, q lots of $\hat{P}_{M_1}$ -operators etc., taking into account the fact that operators from different oscillators commute. For example, $\mathcal{S}[\langle \hat{X}_{M_1}^2 \hat{P}_{M_1} \hat{X}_{M_2} \rangle] = \langle \hat{X}_{M_1}^2 \hat{P}_{M_1} \hat{X}_{M_2} \rangle + \langle \hat{X}_{M_1} \hat{P}_{M_1} \hat{X}_{M_1} \hat{X}_{M_2} \rangle + \langle \hat{P}_{M_1} \hat{X}_{M_1}^2 \hat{X}_{M_2} \rangle$. Knowing the value of $\mathcal{S}[\langle \hat{X}_{M_1}^2 \hat{P}_{M_1} \hat{X}_{M_2} \rangle]$ and the lower-order moment $\langle \hat{X}_{M_1} \hat{X}_{M_2} \rangle$ is sufficient to gain access to $\langle \hat{X}_{M_1}^2 \hat{P}_{M_1} \hat{X}_{M_2} \rangle$, $\langle \hat{X}_{M_1} \hat{P}_{M_1} \hat{X}_{M_1} \hat{X}_{M_2} \rangle$, and $\langle \hat{P}_{M_1} \hat{X}_{M_1}^2 \hat{X}_{M_2} \rangle$ individually as these can be found via the commutation relation $[\hat{X}_{M_i}, \hat{P}_{M_j}] = i\delta_{ij}$. For example, if our desired moment is $\langle \hat{X}_{M_1}^2 \hat{P}_{M_1} \hat{X}_{M_2} \rangle$, then $\langle \hat{X}_{M_1}^2 \hat{P}_{M_1} \hat{X}_{M_2} \rangle = \mathcal{S}[\langle \hat{X}_{M_1}^2 \hat{P}_{M_1} \hat{X}_{M_2} \rangle]/3 + i \langle \hat{X}_{M_1} \hat{X}_{M_2} \rangle$. Hence any desired moment – composed of a sequence of mechanical quadrature operators $\langle \hat{X}_{M_1}^p \hat{X}_{M_2}^q \hat{X}_{M_2}^r \hat{X}_{M_2}^s \rangle -$

can be written as the sum over the moments of all the distinct permutations of those operators $\mathcal{S}[\langle \hat{X}_{M_1}^p \hat{X}_{M_2}^q \hat{X}_{M_2}^r \hat{X}_{M_2}^s \rangle]$, plus lower-order moments (which as we shall see are already known due to the iterative nature of the procedure). This iterative procedure involves finding the moments of all the permutations of sequences of operators belonging to order d (i.e. finding values for all possible sums $\mathcal{S}$ of order d , and consequently individual moments) before being able to unlock moments of order $d + 1$. The method prescribed can be used to find a moment with arbitrarily large d .

A.1.1 Parallel set-up

Here we will outline the key experimental steps necessary for the entanglement verification (using the set-up in Fig. 5.9) of a mechanical state created by the parallel set-up in Fig. 5.1a. We will first demonstrate that any arbitrary moment of the form $\langle \hat{X}_{M_1}^p \hat{P}_{M_1}^q \hat{X}_{M_2}^r \hat{P}_{M_2}^s \rangle$ can be calculated using our scheme. Then we will show that for some lower-order moments, which appear in the subdeterminants of D_5 and S_3 , a simpler sequence of verification pulses can be used.

Generalised scheme

First, let us consider the case where our desired moment is contained within the expression $\mathcal{S}[\langle \hat{X}_{M_1}^p \hat{P}_{M_1}^q \hat{X}_{M_2}^r \hat{P}_{M_2}^s \rangle]$. The entangled mechanical state $\rho_{\text{out}}^{\{1,0\}}$ is generated following a $\{1,0\}$ click (see Fig. 5.1a). As discussed in the main text, this mechanical state depends on the phase ϕ in the interferometer and so this phase is kept fixed throughout the verification of the state. For generality, we have included ϕ in our expressions but it has no relevance in the verification scheme. All other phases ζ_l (where $l = 1, 2, 3, 4$) are controllable phases. After the state has been created, a verification pulse is then sent into oscillator 1 at time $\tau = 0$ followed by a second at $\tau = \pi/(2\omega_M)$. Each pulse interacts sequentially with the mechanical oscillator, and the phase quadratures of the verification pulses transform according to Eq. (5.41b). These verification pulses have a different wavelength to the entangling pulse to operate in the linearized regime and

so spectral filters ensure that after the optomechanical interaction all the verification pulses are diverted away from the photodetectors in Fig. 5.9. The first pulse follows path 1a and is held in a delay line for $\tau = \pi/(2\omega_M)$ while a switch ensures the second, later pulse is directed along 1b and thereby receives an additional phase shift ζ_1 . The two pulses are both incident at the first beam splitter simultaneously, which allows us to measure moments comprised of both position and momentum quadratures. Similarly, two verification pulses are also sent into oscillator 2 at time $\tau = 0$ and $\tau = \pi/(2\omega_M)$. After the optomechanical interaction with oscillator 2, with the aid of a spectral filter, the first pulse follows path 2a and is held in a delay line while the second, later pulse travels along 2b and experiences a phase shift of ζ_2 at the beam splitter. The light from each oscillator is then combined at two further beam splitters as depicted in Fig. A.1. There is an additional phase at each of these two beam splitters, ζ_3 and ζ_4 . After the second set of beam splitters, a homodyne measurement is performed on each of the four output modes, and the phase quadratures are measured. The phase quadratures at the 4 beam splitter ports are given by:

$$\hat{P}_{L_A} = [e^{i\zeta_3}(\hat{X}_{M_1} + e^{i\zeta_1}\hat{P}_{M_1}) + e^{i\phi}(\hat{X}_{M_2} + e^{i\zeta_2}\hat{P}_{M_2})]\chi/2 + \hat{P}'_{L_A} , \quad (\text{A.1a})$$

$$\hat{P}_{L_B} = [e^{i\zeta_3}(\hat{X}_{M_1} + e^{i\zeta_1}\hat{P}_{M_1}) - e^{i\phi}(\hat{X}_{M_2} + e^{i\zeta_2}\hat{P}_{M_2})]\chi/2 + \hat{P}'_{L_B} , \quad (\text{A.1b})$$

$$\hat{P}_{L_C} = [\hat{X}_{M_1} - e^{i\zeta_1}\hat{P}_{M_1} - e^{i(\phi+\zeta_4)}(\hat{X}_{M_2} + e^{i\zeta_2}\hat{P}_{M_2})]\chi/2 + \hat{P}'_{L_C} , \quad (\text{A.1c})$$

$$\hat{P}_{L_D} = [\hat{X}_{M_1} - e^{i\zeta_1}\hat{P}_{M_1} + e^{i(\phi+\zeta_4)}(\hat{X}_{M_2} - e^{i\zeta_2}\hat{P}_{M_2})]\chi/2 + \hat{P}'_{L_D} . \quad (\text{A.1d})$$

Here, $\hat{P}'_{L_k}$ (where $k = \{A, B, C, D\}$) can be understood as the momentum quadrature of the pulse in the absence of the oscillators, and commutes with all mechanical quadrature operators. The momentum quadratures $\hat{P}'_{L_k} \propto \hat{P}_{L_{\text{in}}}$; for example, the topmost output in Fig. 5.9 has $\hat{P}'_{L_A} = (e^{i\zeta_3}(1+e^{i\zeta_1})+e^{i\phi}(1+e^{i\zeta_2}))\hat{P}_{L_{\text{in}}}/2$. In the main text we assumed $\hat{P}_{L_{\text{in}}}$ has vacuum noise statistics, however here we generalise the expressions and assume only that all 4 verification pulses have identical initial statistics, and that their momentum quadratures after each optomechanical interaction are described by Eq. (5.41b). The process of creating the entangled states, followed by sending in these four pulses of light and performing a homodyne measurement of the four outputs to measure the momentum quadratures (A.1a)-(A.1d) is repeated over many runs with

a fixed set of phases $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$. This allows us to build up a data set for each of the homodyne measurements in Eqs. (A.1a)-(A.1d). We can sum over this data set $\{P_{L_k}\}_{i=1}^N$ using Eq. (5.42) to extract increasing orders of moments. For example, considering Eq. (A.1a) which describes the topmost output in Fig. 5.9, the d^{th} order moment of the optical momentum quadrature can be expanded in terms of mechanical quadratures and $\hat{P}'_{L_A}$:

$$\langle \hat{P}_{L_A}^d \rangle = (1/2)^d \sum_{j=0}^d \binom{d}{j} \langle [e^{i\zeta_3}(\hat{X}_{M_1} + e^{i\zeta_1}\hat{P}_{M_1}) + e^{i\phi}(\hat{X}_{M_2} + e^{i\zeta_2}\hat{P}_{M_2})]^j \rangle (\chi/2)^j \langle (\hat{P}'_{L_A})^{j-d} \rangle, \quad (\text{A.2})$$

where we have used the fact that $\hat{P}'_{L_A}$ commutes with the mechanical quadratures. The value of $\langle \hat{P}_{L_A}^d \rangle$ is calculated summing over the data set of $\hat{P}_{L_A}$ measurements according to Eq. (5.42). The process also assumes we know χ accurately and have already obtained full statistics on $\hat{P}_{L_{\text{in}}}$ and therefore $\hat{P}'_{L_A}$ in an initial calibration stage (we discuss this in Section A.1.1). The remaining terms in the sum of Eq. (A.2) are quadrature moments of order $j = 1, \dots, d$. However, the iterative nature of the process means we have already calculated $\langle \hat{P}_{L_A}^{d-1} \rangle$ and so all mechanical moments up to order $j = d - 1$ are known. Therefore, the only unknown moments in the expansion (A.2) are contained in the d^{th} order term:

$$\begin{aligned} \langle [e^{i\zeta_3}(\hat{X}_{M_1} + e^{i\zeta_1}\hat{P}_{M_1}) + e^{i\phi}(\hat{X}_{M_2} + e^{i\zeta_2}\hat{P}_{M_2})]^d \rangle = \dots \\ + e^{ip\zeta_3+iq(\zeta_1+\zeta_3)+ir\phi+is(\zeta_2+\phi)} \mathcal{S}[\langle \hat{X}_{M_1}^p \hat{P}_{M_1}^q \hat{X}_{M_2}^r \hat{P}_{M_2}^s \rangle] \\ + \dots, \end{aligned} \quad (\text{A.3})$$

where $\mathcal{S}[\langle \hat{X}_{M_1}^p \hat{P}_{M_1}^q \hat{X}_{M_2}^r \hat{P}_{M_2}^s \rangle]$ is the term we are trying to calculate. However, the other terms in the sum Eq. (A.3) are of the form $\mathcal{S}[\langle \hat{X}_{M_1}^{\tilde{p}} \hat{P}_{M_1}^{\tilde{q}} \hat{X}_{M_2}^{\tilde{r}} \hat{P}_{M_2}^{\tilde{s}} \rangle]$, with $\tilde{p} + \tilde{q} + \tilde{r} + \tilde{s} = d$ and $\{\tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}\} \neq \{p, q, r, s\}$. The coefficients of the terms $\mathcal{S}[\langle \hat{X}_{M_1}^{\tilde{p}} \hat{P}_{M_1}^{\tilde{q}} \hat{X}_{M_2}^{\tilde{r}} \hat{P}_{M_2}^{\tilde{s}} \rangle]$ are dictated by the combination of phases chosen $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$. Therefore, we repeat the entire process numerous times with a different set of phases until we obtain the sufficient number of linearly independent equations required to solve for the term $\mathcal{S}[\langle \hat{X}_{M_1}^p \hat{P}_{M_1}^q \hat{X}_{M_2}^r \hat{P}_{M_2}^s \rangle]$. We reiterate that knowledge of the value of $\mathcal{S}[\langle \hat{X}_{M_1}^p \hat{P}_{M_1}^q \hat{X}_{M_2}^r \hat{P}_{M_2}^s \rangle]$ is sufficient to calculate the moment of interest via canonical

commutation relations. Due to the 4 distinct beam splitter outputs, for a given set of phases $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$, we in fact have access to four linearly independent equations by applying the same analysis to equations (A.1b)-(A.1d). This reduces the number of distinct sets of phases required by a factor of 4.

Special cases

We will now discuss some special cases of moments that require fewer than 4 verification pulses per run. In fact, all of the moments in the subdeterminants of D_5 and S_3 fall under this category except $\langle \hat{b}_1^\dagger \hat{b}_1 \hat{b}_2^\dagger \hat{b}_2 \rangle$ for which we must use the generalised scheme. For single-mode moments $\langle \hat{X}_{M_i}^d(\theta) \rangle$, we require a single verification pulse sent into the i^{th} oscillator at time $\tau = \theta/\omega_M$, where $\theta = \{0, \pi/2\}$ (no verification pulses are sent to the j^{th} oscillator, where $j \neq i$). The entangled state $\rho_{\text{out}}^{\{1,0\}}$ is first created in the usual way (see Fig. 5.1a). For the verification stage, the switches are position 1b and 2b since no delay lines are required. All optical phases ζ_l are set to 0. Sending in the verification pulse at time $\tau = 0$ will allow us to find $\langle \hat{X}_{M_i}^d \rangle$ while sending in the pulse at time $\tau = \pi/(2\omega_M)$ will give us $\langle \hat{P}_{M_i}^d \rangle$. The optical momentum quadratures are measured at each of the four beam splitter outputs, for example $\hat{P}_{L_A}(\theta) = \eta_V(\hat{P}_{L_{\text{in}}} + \chi \hat{X}_{M_i}(\theta)) + \sqrt{1 - \eta_V}(1 + \sqrt{\eta_V})\hat{P}_V$, where the vacuum noise $\hat{P}_V$ has been included from each beam splitter interaction (each beam splitter transmits $\sqrt{\eta_V}$ of the pulse, and each mode passes through two beam splitters). We build up data set of homodyne measurements $\{\hat{P}_{L_k}(\theta)\}_{i=1}^N$ and find all moments up to order d using Eq. (5.42). Rearranging the equation for $\langle \hat{P}_{L_A}^d(\theta) \rangle$ gives:

$$\langle \hat{X}_{M_i}^d(\theta) \rangle = (\eta_V \chi)^{-d} [\langle \hat{P}_{L_A}^d(\theta) \rangle - \sum_{j=0}^{d-1} \binom{d}{j} (\eta_V \chi)^j \langle \hat{X}_{M_i}^j(\theta) \rangle \langle (\hat{P}_{L_A}'')^{d-j} \rangle], \quad (\text{A.4})$$

where $\hat{P}_{L_A}'' = \eta_V \hat{P}_{L_{\text{in}}} + \sqrt{1 - \eta_V}(1 + \sqrt{\eta_V})\hat{P}_V$. Again, this calculation must be done in an iterative way starting with $d = 1$ (hence all the moments $\langle \hat{X}_{M_i}^j(\theta) \rangle$, where $j < d$, are presumed to be already known).

For moments of the same oscillator $\langle \hat{X}_{M_i}^p \hat{P}_{M_i}^q \rangle$, after creating the entangled state $\rho_{\text{out}}^{\{1,0\}}$ in the

usual manner, the switches are moved to position 1a for $i = 1$ (or position 2a for $i = 2$). Two pulses are sent into the i^{th} oscillator, one at time $\tau = 0$ followed by one at $\tau = \pi/(2\omega_M)$ (no pulses are sent to the j^{th} oscillator). The first pulse is held in a delay line with the switch in position 1a (or 2a). The second, later pulse follows path 1b (or 2b), experiences a ζ_i phase shift, and coincides with the first pulse at the beam splitter. All other phases can be neglected: $\zeta_j = \zeta_3 = \zeta_4 = 0$. For moments like $\langle \hat{X}_{M_1}^p(\theta_1) \hat{X}_{M_2}^q(\theta_2) \rangle$ one pulse is sent to oscillator 1 at time $\tau = \theta_1/\omega_M$ and a second pulse is sent to oscillator 2 at $\tau = \theta_2/\omega_M$. The phases $\zeta_1 = \zeta_2 = 0$, but the set $\{\zeta_3, \zeta_4\}$ must be varied. Finally, for a moment such as $\langle \hat{X}_{M_i}^p(\theta_i) \hat{X}_{M_j}^q \hat{P}_{M_j}^r \rangle$, one pulse is sent into the i^{th} oscillator at $\tau = \theta_i/\omega_M$ and two pulses are sent into the j^{th} oscillator at time $\tau = 0$ and $\tau = \pi/(2\omega_M)$. All four phases in the set $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ must be varied. With all these special cases, the phase quadratures $\hat{P}_{L_k}$ are measured over many runs in order to construct a data set of homodyne measurement results $\{P_{L_k}^{(i)}\}_{i=0}^N$ for given set of phases $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$. From the data set all the moments up to $\langle \hat{P}_{L_k}^d \rangle$ are calculated using Eq. (5.42). The phases within the set are then varied (and we repeat previous step of homodyning over many runs to construct further data sets of homodyne measurements) until we have a sufficient number of linearly independent equations to solve for the term of interest. The mathematics of solving the equations is the same as the general case of $\mathcal{S}[\langle \hat{X}_{M_1}^p \hat{P}_{M_1}^q \hat{X}_{M_2}^r \hat{P}_{M_2}^s \rangle]$ and has been examined in detail in A.1.1.

Calibration

We have assumed throughout A.1.1 and A.1.1 that we already know the full statistics of $\hat{P}'_{L_k}$ where $k = \{A, B, C, D\}$ via a calibration step. This is an important first stage in order to accurately eliminate initial pulse moments from equations such as (A.2). In the absence of any oscillators, we have that $\hat{P}_{L_k} = \hat{P}'_{L_k}$. Therefore, still using the optical set-up of Fig. 5.9, we send in the verification pulses which we would normally use to obtain a specific moment but in the absence of any optomechanical oscillators (the experimental set-up would need to be adapted further from Fig. 5.9 in order to bypass the oscillators). For example, if we are using the generalised regime we would send in 4 verification pulses, 2 in each arm and time $\tau = 0$ and $\tau = \pi/(2\omega_M)$. The phases ζ_l are set to whichever set of phases $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ we are intending

on using for that particular run. Measuring the quadratures $\hat{P}_{L_k}$ many times to build up a data set of homodyne measurements from which we can extract $\langle(\hat{P}'_{L_k})^d\rangle$ for a given set of ζ_l phases.

Another assumption in the verification protocol is precise knowledge of χ , which appears in Eq. (5.41b) and is the interaction strength between the verification pulses and the mechanical system. One can measure χ precisely in a calibration stage which is separate from the entangling and verification steps. This involves sending pulses of light towards a mechanical state that is in thermal equilibrium and then phase-homodyning the output light. Knowledge of the statistics of the input pulses and the bath temperature allows a value for χ to be extracted from the value of $\langle\hat{P}_L^2\rangle$ which is in turn computed by summing over a data set of many phase-homodyne measurements. Furthermore, phase fluctuations will lead to additional phase noise on top of the intrinsic noise of the verification pulses. However, this can be seen as reduction in χ and, provided the variance of this additional phase noise is narrow, this can be accounted in the calibration stage [223].

A.1.2 Series set-up

Our proposed set-up to verify the type of entanglement produced when the two oscillators are in series with each other is displayed in Fig. A.1. In this scheme, we integrate the verification apparatus into the entanglement set-up of Fig. 5.1b. The mathematical method is identical to the prescribed method for the parallel set-up in A.1.1. Sending in up to two verification pulses per run and repeating numerous times allows us to build data sets of homodyne measurements of $\hat{P}_{L_A}$ and $\hat{P}_{L_B}$. From the data, the moments $\langle\hat{P}_{L_A}^j\rangle$ and $\langle\hat{P}_{L_B}^j\rangle$ for $j = 1, \dots, d$ are directly calculated using Eq. (5.42). We can expand $\langle\hat{P}_{L_A}^d\rangle$ and $\langle\hat{P}_{L_B}^d\rangle$ in terms of mechanical moments of order d plus lower-order moments. The process is iterative so lower-order moments are already known, leaving only moments of order d to calculate. Repeating with different sets of phases $\{\zeta_1, \zeta_2, \zeta_3\}$ gives us access to a sufficient number of linearly independent equations which can be solved to obtain values such as $\mathcal{S}[\langle\hat{X}_{M_1}^p \hat{P}_{M_1}^q \hat{X}_{M_1}^r \hat{P}_{M_2}^s\rangle]$, and consequently any moment that appears in the expansion of $\mathcal{S}[\langle\hat{X}_{M_1}^p \hat{P}_{M_1}^q \hat{X}_{M_1}^r P_{M_2}^s\rangle]$ can be deduced using canonical commutation relations.

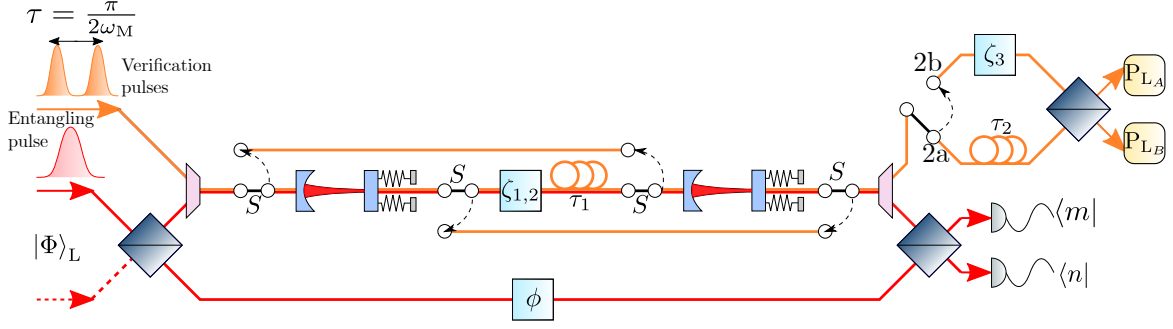


Figure A.1: Proposed verification scheme to measure mechanical moments of a state produced via the series set-up. The first stage is to create the entangled state $\rho_{\text{out}}^{\{1,0\}}$ shown by the red path on the diagram; all the switches labelled S are positioned accordingly. After a click event of $\{1,0\}$, verification pulses shown in orange interact with the mechanical state. Spectral filters ensure that the verification pulses (which have a different wavelength) can follow different paths to that of the entangling pulse. In order to calculate a moment which appears in the sum $\mathcal{S}[\langle \hat{X}_{M_1}^p \hat{P}_{M_1}^q \hat{X}_{M_2}^r \hat{P}_{M_2}^s \rangle]$ we send up to two verification pulses per run into the upper arm of the apparatus and then perform homodyne measurements on the verification pulses. This process is repeated many times in order to obtain a data set of homodyne measurements from which moments can be calculated. The delay line τ_1 is useful if we wish to allow the second oscillator to freely evolve relative to the first oscillator. The switches S allow us to bypass an oscillator in order to get a single-mode mechanical moment. τ_2 allows us to delay the first verification pulse so that the two verification pulses are incident on the final beam splitter coincidentally. The other features of the set-up serve the same purpose as they do in Fig. 5.9.

A.2 Example experimental parameter set

Here we summarize an example experimental parameter set drawn from [141] together with additional parameters to give the values in the last row of Table 5.2. To illustrate the feasibility of our protocol, in Section 5.4 we focused on the dimensionless, system-independent parameters μ , Q , $\bar{n}$, and $\bar{n}_B$. This demonstrated that our protocol of generating and verifying non-Gaussian entanglement is valid for very small μ of order 10^{-5} up to μ of order 1. Even with present-day experiments and only improved cooling we can generate and verify non-Gaussian entangled states.

Table A.1: Example experimental parameter set with key parameters taken from Ref [141]. These parameters demonstrate in detail how μ and Q are calculated in the last row of Table 5.2. Additional experimental parameters are also proposed and listed in order calculate $\bar{n}$ and $\bar{n}_B$, enabling us to calculate D_5 , S_3 and $\mathcal{F}$.

Example experimental parameters		
m	Effective mass	1.5 pg
$\omega_M/2\pi$	Mechanical frequency	3.74 MHz
x_{zpf}	Zero-point fluctuation $x_{\text{zpf}} = \sqrt{\hbar/2m\omega_M}$	43 fm
$G/2\pi$	Parametric coupling $G = \partial\omega_c/\partial x$; cavity resonance frequency ω_c	0.8 THz nm ⁻¹
$g_0/2\pi$	Optomechanical coupling rate $g_0 = Gx_{\text{zpf}}$	35 MHz
$\kappa/2\pi$	Cavity amplitude decay rate	8.8 GHz
$\gamma/2\pi$	Mechanical decay rate	100 Hz
Q	Mechanical quality factor $Q = \omega_M/\gamma$	3.74×10^4
μ	Dimensionless optomechanical coupling strength $\mu = 2\sqrt{2}g_0/\kappa$	1.12×10^{-2}
Additional experimental parameters		
$\bar{n}$	Initial mechanical phonon occupation number	0.1
T_B	Bath temperature	100 mK
$\bar{n}_B$	Bath occupation number calculated from T_B	559
Δt	Detection window	10 ns
$\lambda_{\mathcal{D}}$	Dark count rate	1 s ⁻¹
$\mathcal{D}$	Probability of single dark count in Δt	10 ⁻⁸

Appendix B

Error propagation in matrices

B.1 Derivation of error on the determinant

As discussed in the Section 6.3.1, the matrix $\mathbf{A}_{ij}$ whose negative determinant would indicate has entries $\mathbf{A}_{ij} = \langle \hat{O}_{ij} \rangle$ where $\hat{O}_{ij} = \hat{a}^{\dagger q_i} \hat{a}^{p_i} \hat{a}^{\dagger n_j} \hat{a}^{m_j} \hat{b}^{\dagger l_j} \hat{b}^{k_j} \hat{b}^{\dagger r_i} \hat{b}^{s_i}$. However, $\hat{O}_{ij}$ is not necessarily Hermitian. We can split the matrix into real and imaginary parts which can be measured via two Hermitian operators $\hat{B}_{ij,0}$ and $\hat{B}_{ij,1}$, such that $\mathbf{A}_{ij} = \langle \hat{B}_{ij,0} \rangle + i \langle \hat{B}_{ij,1} \rangle$. If these moments are calculated by taking repeated measurements then $\mathbf{A}_{ij} = \bar{B}_{ij,0} + i\bar{B}_{ij,1}$. We can exploit the fact that the matrix is Hermitian, and in an experiment we only need to calculate the moments in the upper triangle and diagonal of $\mathbf{A}$, and then use these numbers to complete the rest of the matrix and compute its determinant. It follows that lower triangle elements are given by $\mathbf{A}_{ji} = \bar{B}_{ij,0} - i\bar{B}_{ij,1}$ and the diagonal elements are real $\mathbf{A}_{ii} = \bar{B}_{ii,0}$ such that all $\bar{B}_{ii,1} = 0$. Therefore the determinant of this Hermitian matrix is a function of d^2 independent random variables $\bar{B}_{ij,p}$ which appear in the upper triangle and diagonal $\det[\mathbf{A}] = f(\bar{B}_{ij,p})$ where $j \geq 1$. We assume the means of conjugate operators $\bar{B}_{ij,0}$ and $\bar{B}_{ij,1}$ are independent random variables since they are experimentally determined separately.

Reiterating the method laid out in Section 6.3.1, we proceed by using the first order the Taylor

expansion, and taking care to only sum over the upper triangle of the matrix

$$(\Delta \det[\mathbf{A}])^2 = \sum_i^d \sum_{j \geq i}^d \sum_k^d \sum_{l \geq k}^d \sum_{p=0}^1 \sum_{q=0}^1 \frac{\partial \det[\mathbf{A}]}{\partial \bar{B}_{ij,p}} \text{Cov}[\bar{B}_{ij,p}, \bar{B}_{kl,q}] \frac{\partial \det[\mathbf{A}]}{\partial \bar{B}_{kl,q}} \quad (\text{B.1})$$

Let us first calculate these derivatives. From the multivariable chain rule,

$$\begin{aligned} \frac{\partial \det[\mathbf{A}]}{\partial \bar{B}_{ij,p}} &= \sum_{rs} \frac{\partial \det[\mathbf{A}]}{\partial \mathbf{A}_{rs}} \frac{\partial \mathbf{A}_{rs}}{\partial \bar{B}_{ij,p}} \\ &= \sum_{rs} \frac{\partial \det[\mathbf{A}]}{\partial \mathbf{A}_{rs}} \left(\delta_{p0} (\delta_{ir} \delta_{js} + \delta_{is} \delta_{jr}) + i \delta_{p1} (\delta_{ir} \delta_{js} - \delta_{is} \delta_{jr}) \right) \\ &= \delta_{p0} \left(\frac{\partial \det(\mathbf{A})}{\partial \mathbf{A}_{ij}} + \frac{\partial \det(\mathbf{A})}{\partial \mathbf{A}_{ji}} \right) + i \delta_{p1} \left(\frac{\partial \det(\mathbf{A})}{\partial \mathbf{A}_{ij}} - \frac{\partial \det(\mathbf{A})}{\partial \mathbf{A}_{ji}} \right) \\ &= \delta_{p0} \left(\text{adj}[\mathbf{A}]_{ji} + \text{adj}[\mathbf{A}]_{ij} \right) + i \delta_{p1} \left(\text{adj}[\mathbf{A}]_{ji} - \text{adj}[\mathbf{A}]_{ij} \right) \end{aligned} \quad (\text{B.2})$$

Where we have used the identity that $\partial \det[\mathbf{A}]/\partial \mathbf{A}_{ij} = \text{adj}[\mathbf{A}]_{ji}$. Now we can insert this into the expression for $\det[\mathbf{A}]$. We will split the sum into diagonal (and real) elements of the matrix $\mathbf{A}$ and off-diagonal elements which exist in the upper triangle of $\mathbf{A}$:

$$\begin{aligned} \Delta \det[\mathbf{A}] &= \sum_i (\text{adj}[\mathbf{A}]_{ii})^2 \text{Cov}[\bar{B}_{ii,0}, \bar{B}_{ii,0}] \\ &+ \sum_i^d \sum_{j > i}^d \sum_k^d \sum_{l > k}^d \sum_{p=0}^1 \sum_{q=0}^1 \left[\delta_{p0} (\text{adj}[\mathbf{A}]_{ji} + \text{adj}[\mathbf{A}]_{ij}) + i \delta_{p1} (\text{adj}[\mathbf{A}]_{ji} - \text{adj}[\mathbf{A}]_{ij}) \right] \\ &\quad \text{Cov}[\bar{B}_{ij,p}, \bar{B}_{kl,q}] \left[\delta_{p0} (\text{adj}[\mathbf{A}]_{lk} + \text{adj}[\mathbf{A}]_{kl}) + i \delta_{p1} (\text{adj}[\mathbf{A}]_{lk} - \text{adj}[\mathbf{A}]_{kl}) \right] \end{aligned} \quad (\text{B.3})$$

However, since we measure every operator $\hat{B}_{ij,p}$ independently, then $\text{Cov}[\bar{B}_{ij,p}, \bar{B}_{kl,q}] =$

$$\text{Var}[\bar{B}_{ij,p}]\delta_{ik}\delta_{jl}\delta_{pq}.$$

$$\begin{aligned}\Delta\det[\mathbf{A}] &= \sum_i \left(\text{adj}[\mathbf{A}]_{ii} \right)^2 \text{Var}[\bar{B}_{ii,0}] \\ &\quad + \sum_i^d \sum_{j>i}^d \sum_{p=0}^1 \left[\delta_{p0} (\text{adj}[\mathbf{A}]_{ji} + \text{adj}[\mathbf{A}]_{ij}) + i\delta_{p1} (\text{adj}[\mathbf{A}]_{ji} - \text{adj}[\mathbf{A}]_{ij}) \right]^2 \text{Var}[\bar{B}_{ij,p}] \\ &= \sum_i \left(\text{adj}[\mathbf{A}]_{ii} \right)^2 \text{Var}[\bar{B}_{ii,0}] \\ &\quad + \sum_i^d \sum_{j>i}^d \sum_{p=0}^1 \left[\delta_{p0} (\text{adj}[\mathbf{A}]_{ji} + \text{adj}[\mathbf{A}]_{ij})^2 - \delta_{p1} (\text{adj}[\mathbf{A}]_{ji} - \text{adj}[\mathbf{A}]_{ij})^2 \right] \text{Var}[\bar{B}_{ij,p}]\end{aligned}\tag{B.4}$$

Using the properties of the adjugate and the Hermiticity of $\mathbf{A}$ we can demonstrate that $\mathbf{A}$ is itself a Hermitian matrix. This follows from the fact that $\text{adj}(\mathbf{M}^\dagger) = \text{adj}(\mathbf{M})^\dagger$ where $\mathbf{M}$ is any $n \times n$ matrix and $\dagger$ denotes the complex transpose. However if $\mathbf{M}$ is Hermitian then $\mathbf{M}^\dagger = \mathbf{M}$ and so $\text{adj}(\mathbf{M})^\dagger = \text{adj}(\mathbf{M})$. Therefore $\text{adj}[\mathbf{A}]^T = \text{adj}[\mathbf{A}]^*$ where T is the standard matrix transpose and $*$ is the complex conjugate. Using this

$$\begin{aligned}(\Delta\det[\mathbf{A}])^2 &= \sum_i^d \left(\text{adj}[\mathbf{A}]_{ii} \right)^2 \text{Var}[\bar{B}_{ii,0}] \\ &\quad + \sum_i^d \sum_{j>i}^d \sum_{p=0}^1 \left[\delta_{p0} (\text{adj}[\mathbf{A}]_{ij}^* + \text{adj}[\mathbf{A}]_{ij})^2 - \delta_{p1} (\text{adj}[\mathbf{A}]_{ij}^* - \text{adj}[\mathbf{A}]_{ij})^2 \right] \text{Var}[\bar{B}_{ij,p}] \\ &= \sum_i^d \left(\text{adj}[\mathbf{A}]_{ii} \right)^2 \text{Var}[\bar{B}_{ii,0}] \\ &\quad + 4 \sum_i^d \sum_{j>i}^d \sum_{p=0}^1 \left[\delta_{p0} (\text{Re}[\text{adj}[\mathbf{A}]_{ij}])^2 + \delta_{p1} (\text{Im}[\text{adj}[\mathbf{A}]_{ij}])^2 \right] \text{Var}[\bar{B}_{ij,p}]\end{aligned}\tag{B.5}$$

This guaranteed to give you a real error in the determinant. Which is good since the determinant is a real number too!

B.2 Optimal way of allocating measurements

The variance in the sample mean is found by the usual standard error formula $\text{Var}[\bar{B}_{ij,p}] = \text{Var}[\hat{B}_{ij,p}]/M_{ij,p}$. Inserting this gives

$$(\Delta \det[\mathbf{A}])^2 = \sum_i^d (\text{adj}[\mathbf{A}]_{ii})^2 \frac{\text{Var}[\hat{B}_{ii,p=0}]}{M_{ii,p=0}} + 4 \sum_i^d \sum_{j>i}^d \sum_{p=0}^1 \left[\delta_{p0} (\text{Re}\{\text{adj}[\mathbf{A}]_{ij}\})^2 + \delta_{p1} (\text{Im}\{\text{adj}[\mathbf{A}]_{ij}\})^2 \right] \frac{\text{Var}[\hat{B}_{ij,p}]}{M_{ij,p}} \quad (\text{B.6})$$

We wish to minimise $\Delta \det[\mathbf{A}]$ under the constraint of the total number of measurements $M_{\text{tot}} = \sum_i M_{ii,p=0} + \sum_i^n \sum_{j>i}^n \sum_{p=0}^1 M_{ij,p}$. Use method of Lagrange multipliers; define new function

$$f(M_{11,0}, \dots, M_{ij,p}, \dots, \mu) = (\Delta \det[\mathbf{A}])^2 + \mu \left(\sum_i M_{ii,p=0} + \sum_i^d \sum_{j>i}^d \sum_{p=0}^1 M_{ij,p} - M_{\text{tot}} \right) \quad (\text{B.7})$$

Differentiating with respect to the number of measurements allocated to measuring the moments along the diagonal of $\mathbf{A}$

$$\frac{\partial f}{\partial M_{ii,0}} = - \frac{(\text{adj}[\mathbf{A}]_{ii})^2 \text{Var}[\hat{B}_{ii,p=0}]}{M_{ii,p=0}^2} + \mu \quad (\text{B.8})$$

Now differentiating with respect to the number of measurements required to measure the moments in the upper triangle and non-diagonal elements of $\mathbf{A}$

$$\frac{\partial f}{\partial M_{ij,p}} = -4 \sum_i^d \sum_{j>i}^d \sum_{p=0}^1 \left[\delta_{p0} (\text{Re}\{\text{adj}[\mathbf{A}]_{ij}\})^2 + \delta_{p1} (\text{Im}\{\text{adj}[\mathbf{A}]_{ij}\})^2 \right] \frac{\text{Var}[\hat{B}_{ij,p}]}{M_{ij,p}^2} + \mu \quad (\text{B.9})$$

Therefore the function f is minimised by picking the following values for the diagonal measurements $M_{ii,0}$

$$M_{ii,0} = \frac{\text{adj}[\mathbf{A}]_{ii} \sigma[\hat{B}_{ii,p=0}]}{\sqrt{\mu}} \quad (\text{B.10})$$

and for the off diagonals where $j > i$

$$M_{ij,p} = \frac{2 \left[\delta_{p0} (\text{Re}\{\text{adj}[\mathbf{A}]_{ij}\})^2 + \delta_{p1} (\text{Im}\{\text{adj}[\mathbf{A}]_{ij}\})^2 \right]^{1/2} \sigma[\hat{B}_{ij,p}]}{\sqrt{\mu}} \quad (\text{B.11})$$

However, from the original constraint we can show that

$$\sqrt{\mu} = \frac{\Gamma}{M_{\text{tot}}} \quad (\text{B.12})$$

where

$$\Gamma = \sum_i^d |\text{adj}[\mathbf{A}]_{ii}| \sigma[\hat{B}_{ii,p=0}] + 2 \sum_i^d \sum_{j>i}^d |\text{Re}\{\text{adj}[\mathbf{A}]_{ij}\}| \sigma[\hat{B}_{ij,0}] + |\text{Im}\{\text{adj}[\mathbf{A}]_{ij}\}| \sigma[\hat{B}_{ij,1}] \quad (\text{B.13})$$

Substituting this into the expression for $(\Delta \det[\mathbf{A}])^2$ gives

$$(\Delta \det[\mathbf{A}])^2 = \frac{\Gamma^2}{M_{\text{tot}}} , \quad (\text{B.14})$$

which is equivalent to Eq. 6.20 thus concluding the derivation of the error on a given determinant.

B.3 Open-system model

B.3.1 Correlations between input operators

Here we derive the result in Eq. (B.19). Making use of the Isserlis-Wick theorem.

$$\begin{aligned}
& \left\langle \left[\sqrt{2\kappa} \int_0^t dt' \hat{a}_{\text{in}}^\dagger(t') e^{-\kappa(t-t')} \right]^p \left[\sqrt{2\kappa} \int_0^t dt' \hat{a}_{\text{in}}(t') e^{-\kappa(t-t')} \right]^q \right\rangle \\
&= (2\kappa)^{\frac{p+q}{2}} \int_0^t \dots \int_0^t dt_1^{(1)} \dots dt_p^{(1)} dt_1^{(2)} \dots dt_q^{(2)} e^{-\kappa(t-t_1^{(1)})} \dots e^{-\kappa(t-t_p^{(1)})} e^{-\kappa(t-t_1^{(2)})} \dots e^{-\kappa(t-t_q^{(2)})} \\
&\quad \langle a_{\text{in}}^\dagger(t_1^{(1)}) \dots a_{\text{in}}^\dagger(t_p^{(2)}) a_{\text{in}}(t_1^{(2)}) \dots a_{\text{in}}(t_q^{(2)}) \rangle \\
&= \delta_{p,q} (2\kappa)^{\frac{p+q}{2}} p! \left[\int_0^t \int_0^t dt' dt'' e^{-\kappa(t-t')} e^{-\kappa(t-t'')} \langle \hat{a}_{\text{in}}^\dagger(t') \hat{a}_{\text{in}}(t'') \rangle \right]^p \\
&= \delta_{p,q} (2\kappa)^p p! \left[\int_0^t \int_0^t dt' dt'' e^{-\kappa(t-t')} e^{-\kappa(t-t'')} n_{\text{B}} \delta(t-t') \right]^p \\
&= \delta_{p,q} (2\kappa)^p p! \left[\int_0^t dt' e^{-2\kappa(t-t')} n_{\text{B}} \right]^p \\
&= \delta_{p,q} (2\kappa)^p p! n_{\text{B}}^p \left[\frac{1 - e^{-2\kappa t}}{2\kappa} \right]^p \\
&= \delta_{p,q} p! [n_{\text{B}} (1 - e^{-2\kappa t})]^p
\end{aligned} \tag{B.15}$$

B.3.2 Beam splitter model

We use the following Heisenberg-Langevin equation

$$\dot{\mathbf{a}}(t) = \frac{i}{\hbar} [\hat{H}, \mathbf{a}(t)] - \kappa \mathbf{a}(t) + \sqrt{2\kappa} \mathbf{a}_{\text{in}}(t) , \tag{B.16}$$

where $\mathbf{a} = (\hat{a}(t), \hat{a}^\dagger(t))^T$, the input field operators are $\mathbf{a}_{\text{in}} = (\hat{a}_{\text{in}}(t), \hat{a}_{\text{in}}^\dagger(t))^T$, and κ is amplitude decay rate [91]. The input operators obey the following relations:

$$\langle \hat{a}_{\text{in}}^\dagger(t) \hat{a}_{\text{in}}(t') \rangle = n_{\text{B}} \delta(t - t') \tag{B.17a}$$

$$\langle \hat{a}_{\text{in}}(t) \hat{a}_{\text{in}}^\dagger(t') \rangle = (n_{\text{B}} + 1) \delta(t - t') , \tag{B.17b}$$

where n_B is the thermal occupation number of the bath. Also $\langle \mathbf{a}_{\text{in}}(t) \rangle = 0$. The solution to Eq. (B.16) assuming a free-evolution Hamiltonian is:

$$\mathbf{a}(t) = \mathbf{a}(0)e^{-\kappa t} + \sqrt{2\kappa} \int_0^t dt' \mathbf{a}_{\text{in}}(t')e^{-\kappa(t-t')}, \quad (\text{B.18})$$

The equations have been listed for one mode of the system but can be applied to the second mode by substituting $\mathbf{a}$ for $\mathbf{b}$. We have assumed both systems have the same κ but this can be generalised so they have different amplitude decay rates. Throughout we assume that the noise operators acting on each subsystem are independent.

An arbitrary element of the matrix from which the determinant is calculated, as a function of time is $\mathbf{A}_{ij}(t) = \langle \hat{B}_{ij,0}(t) \rangle + i \langle \hat{B}_{ij,1}(t) \rangle$. Therefore, to calculate $\det[\mathbf{A}(t)]$ we can directly calculate all the moments $\langle \hat{B}_{ij,p}(t) \rangle$ by directly inserting the Heisenberg operators in Eq. (B.18) and computing the quantum expectation values directly.

Similarly, to find $\Delta \det[\mathbf{A}]$ we can insert these time-dependent solutions into Eq. (6.18), replacing $\mathbf{A}$ and $\text{Var}[\hat{B}_{ij,p}]$ with $\mathbf{A}(t)$ and $\text{Var}[\hat{B}_{ij,p}(t)]$, respectively.

All the moments required to calculate $\det[\mathbf{A}]$ and $\Delta \det[\mathbf{A}]$ are all in the form $\langle \hat{a}^{\dagger m}(t) \hat{a}^n(t) \hat{b}^{\dagger k}(t) \hat{b}^l(t) \rangle$. It can be analytically demonstrated that the expectation of these time-dependent operators depends only on the expectation with respect to the initial quantum state $\langle \hat{a}^{\dagger m}(0) \hat{a}^n(0) \hat{b}^{\dagger k}(0) \hat{b}^l(0) \rangle$ and the decoherence model parameters. The more correlations which arise from input operators can be analytically computed using the fact that input operators of different modes are independent. It can be shown that Appendix that

$$\begin{aligned} & \left\langle \left[\int_0^t dt' \hat{a}_{\text{in}}^{\dagger}(t') e^{-\kappa(t-t')} \right]^p \left[\int_0^t dt'' \hat{a}_{\text{in}}(t'') e^{-\kappa(t-t'')} \right]^q \right\rangle \\ &= \delta_{pq} \frac{p!}{(2\kappa)^p} n_B^p (1 - e^{-2\kappa t})^p \end{aligned} \quad (\text{B.19})$$

Appendix C

CSL: probability of conditional measurement

Here we outline the steps taken to obtain the probability given in Eq. (7.41). Using Eq. (7.4), the probability $P(k, \mathcal{D}) = \langle \hat{Y}_+^\dagger(t_k) \hat{Y}_+(t_k) \rangle$ can be written as:

$$\begin{aligned} P(k, \mathcal{D}) &= \frac{1}{4} \langle \cos^2 (|\alpha| \hat{x}(t_k)) \rangle \\ &= \frac{1}{2} + \frac{1}{2} \langle \cos (2|\alpha| \hat{x}(t_k)) \rangle \end{aligned} \tag{C.1}$$

where the operator $\hat{x}(t) = \hat{X}(t)/X_{\text{ZP}}$ is dimensionless. Therefore we can solve the quantum Langevin equations (written in dimensionless units) to obtain the following solutions evaluated at $t_k = k\pi/\Omega$:

$$\hat{x}(t_k) = e^{-\gamma t/2} \hat{x}(0) + e^{-\gamma t/2} \Delta x(t_k) , \tag{C.2a}$$

$$\hat{p}(t_k) = e^{-\gamma t/2} \hat{p}(0) + e^{-\gamma t/2} \Delta p(t_k) , \tag{C.2b}$$

where

$$\Delta x(t_k) = \int_0^{t_k} dt' e^{\gamma t'/2} \sin(\Omega(t_k - t')) \hat{\xi}(t') , \quad (\text{C.3a})$$

$$\Delta p(t_k) = \int_0^{t_k} dt' e^{\gamma t'/2} \hat{\xi}(t') [\cos(\Omega(t_k - t')) - \frac{\gamma}{2\Omega} \sin(\Omega(t_k - t'))] \quad (\text{C.3b})$$

and $\hat{\xi}(t) = \sqrt{2}X_{\text{ZP}}f(t)/\hbar$ such that $\hat{f}(t) = \hat{f}_{\text{th}}(t) + \hat{f}_{\text{CSL}}(t)$. Substituting in these solutions into Eq. (C.1) and making use of the Isserlis-Wick theorem:

$$\begin{aligned} \langle \hat{\xi}(t_1) \hat{\xi}(t_2) \dots \hat{\xi}(t_n) \rangle &= \frac{(2s)!}{2^s s!} \langle \hat{\xi}(t) \hat{\xi}(t') \rangle \\ &= \frac{(2s)!}{2^s s!} (\sqrt{2}X_{\text{ZP}})^2 (D_{\text{th}} + D_{\text{CSL}}) , \end{aligned} \quad (\text{C.4})$$

(for $n = 2s$, otherwise the expression evaluates to 0) we arrive at the solution:

$$P(k, \mathcal{D}) = \frac{1}{2} + \frac{1}{2} P_0(t_k) \exp \left[-8\mathcal{D}|\alpha|^2 \frac{(1 - e^{-k\pi\gamma/\Omega})}{\gamma(4 + \gamma/\Omega)} \right] \quad (\text{C.5})$$

with

$$\begin{aligned} P_0(0) &= \frac{1}{2(1 + e^{-2(1+2\bar{n})|\alpha|^2})} \times \\ &\left[e^{-2(1+2\bar{n})|\alpha|^2(\kappa^2+2\kappa+1)} + e^{-2(1+2\bar{n})|\alpha|^2(\kappa^2-2\kappa+1)} + e^{-2(1+2\bar{n})|\alpha|^2\kappa^2} \right] \end{aligned} \quad (\text{C.6})$$

and $\kappa = e^{k\pi\gamma/(2\Omega)}$. In the limit of a high quality factor $Q = \Omega/\gamma \gg 1$ we can expand the term in the exponential in Eq. (7.40):

$$-8\mathcal{D}|\alpha|^2 \frac{(1 - e^{-k\pi/Q})}{\gamma(4 + 1/Q)} \approx -2\mathcal{D} \frac{k\pi}{\Omega} \quad (\text{C.7})$$

and thus obtain Eq. (7.41).