Arithmetic topological models for the attractors of infinitely satellite renormalisable maps

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I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

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Abstract

In this thesis we propose an arithmetic topological model for the post-critical set of infinitely quadratic-like renormalisable quadratic polynomials of satellite type. This is inspired by the near-parabolic renormalisation scheme of Inou and Shishikura. This model only depends on the combinatorics of the renormalisations, which is a sequence of rational numbers in $(-1/2, 1/2]\backslash\{0\}$. We also introduce an optimal arithmetic condition on such sequences that determines when the model is a Cantor set. We show that these topological models enjoy a universal property. That is, either they are a Cantor set of points or they are determined by some topological axioms similar to the topological characterisation of the Cantor set in the plane. We introduce a new topological object, *hairy Cantor sets*, and investigate their topological properties. We also define a model for the dynamics on the topological model, designed to reflect the behaviour of the quadratic map on its post- critical set. We identify all closed invariant subsets and describe their topology. We also show that the model is uniquely ergodic, and identify the unique invariant probability.

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1. Introduction

The systematic study of the global dynamics of holomorphic maps of the Riemann sphere started by the seminal works of Pierre Fatou and Gaston Julia in the 1920's [Fat19, Jul18]. For a rational map f of the sphere, with degree at least two, the Fatou set of f is defined as

$$F_f := \{z \in \hat{\mathbb{C}} \mid (f^n)_{n=1}^{\infty} \text{ is a normal family on a neighbourhood of } z\},$$

and the Julia set is defined as $J_f := \hat{\mathbb{C}} \setminus F_f$. On every periodic connected component of the Fatou set, the iterates of f form a normal family, and the dynamics is well understood. In the 1970's, Sullivan showed that for any rational function, all connected components of the Fatou set are pre-periodic [Sul87]. On the other hand, the dynamics on the Julia set is often non-trivial. The family of quadratic polynomials is specially interesting. Although maps in this family only possess a single critical point of order two, and are parametrised with a single complex parameter, they display fairly rich and complicated behaviour that are not well-understood.

Any quadratic polynomial is affinely conjugate to a unique polynomial of the form

$$P_c(z) = z^2 + c,$$

for some $c \in \mathbb{C}$. If the critical point tends to infinity under the iterates of P_c, then the Julia set of P_c is a Cantor set of points, and the dynamics on the Julia set is topologically conjugate to the shift map on two symbols. On the other hand, if the critical orbit remains bounded in the complex plane, then the Julia set is connected. The Mandelbrot set is defined as

$$\mathcal{M} := \{c \in \mathbb{C} \mid J_{P_c} \text{ is connected}\}.$$

If $p \in \mathbb{C}$ is a periodic point of period k for some quadratic polynomial P_c, then $\lambda := (P_c^k)'(p)$ is called the *multiplier* of P_c at p. A periodic orbit is called *attracting*, *indifferent*, or *repelling* if we have $|\lambda| < 1$, $|\lambda| = 1$ or $|\lambda| > 1$, respectively. If $|\lambda| = 1$ and λ is a root of unity, then p is called a *parabolic* periodic orbit. If $\lambda = e^{2\pi i\alpha}$, for some irrational number α, then the orbit is called *irrationally indifferent*. For quadratic polynomials, there is at most one periodic orbit that is either attracting or indifferent. If P_c has an attracting periodic point, it is known that the critical point tends to that cycle. If P_c has an attracting orbit, or J_{P_c} is a Cantor set, then it is called a *hyperbolic* map and the dynamics is fairly easy to describe.

A central conjecture in the filed of holomorphic dynamics by Fatou suggests that the set of complex numbers c such that P_c is hyperbolic is dense in the complex plane. Douady and Hubbard [DH84] conjectured that the Mandelbrot set is locally connected and showed that this conjecture implies the conjecture of Fatou. If P_c has an attracting periodic orbit, then c belongs to the interior of the Mandelbrot set and the Mandelbrot set is locally connected at c. Quadratic maps with a parabolic periodic orbit are also well-understood. If P_c has an indifferent periodic orbit with multiplier $\lambda = e^{2\pi i\alpha}$, then the arithmetic of α plays a key role in the dynamical behaviour of P_c near the fixed point. For example, Brjuno condition determines whether P_c^k is locally analytically conjugate to a rigid rotation on a neighbourhood of the periodic orbit [Sie42, Brj71, Yoc95] Research in this direction has revealed many fascinating results on the nature of these maps, see for instance [McM98, PZ04, BC12]. A key arithmetic condition which has recently appeared in the study of such maps is the Herman condition, [Che17]. The set of irrational numbers of Herman type were discovered in the study of the linearisation problem for analytic circle maps, [Her79, Yoc84]. There is a rather complicated characterisation of such irrational numbers identified by Yoccoz. It appears that the Brjuno and Herman condition determine the topology of the measure theoretic attractor of the quadratic polynomial with indifferent fixed points [Che17].

A central concept in the study of quadratic maps is the idea of *quadratic-like renormalisation* structures. As we will explain precisely in Section 2, for a *quadratic-like renormalisable* map

P_c, there are simply connected domains U and V as well as an integer $q \geq 2$ such that U is compactly contained in V and $P_c^q|_U : U \to V$ is a proper branched covering map of degree two with a non-escaping critical point. By a fundamental theorem of Douady and Hubbard [DH85], there is a quadratic map $P_{c'}$, for some $c' \in \mathbb{C}$, which is quasi-conformally conjugate to $P_c^q|_U$ on a neighbourhood of its Julia set. The map $P_{c'}$ is called the renormalisation of P_c. When a quadratic polynomial has no quadratic-like renormalisation structures, there is a natural way to divide a neighbourhood of the Julia set into dynamically well-behaved pieces called the *Yoccoz puzzles*. If a quadratic map with no attracting or indifferent periodic points is not renormalisable, then the dynamics on its puzzle pieces behaves in an almost Markov fashion. Yoccoz introduced this tool to explain the dynamics of such map, when there are no attracting or indifferent periodic points. In particular, he showed that J_{P_c} is locally connected and the Mandelbrot set is locally connected at c, see [Hub93].

According to Douady and Hubbard, quadratic-like renormalisation occurs in two fashions. One may define the Julia set of $P_c^q|_U$ using the conjugacy to $P_{c'}$. Then the renormalisation is called *satellite* type if the Julia set of $P_c^q|_U$ contains a fixed point of P_c, and otherwise it is called *primitive* type. If the renormalisation is of satellite type, then there is a natural way to assign a rational number $p/q \in (-1/2, 1/2] \setminus \{0\}$ that describes the combinatorial nature of the behaviour of P_c near the featured fixed point. Therefore, to any infinitely satellite renormalisable quadratic polynomial, we may assign a sequence of rational numbers $\tau = (p_i/q_i)_{i=1}^{\infty}$ in $(-1/2, 1/2] \setminus \{0\}$. This is often referred as the combinatorics of the renormalisations.

The first rigorous breakthrough in regards to the infinitely renormalisable quadratics came in the pioneering work of Sullivan [Sul87, Sul92]. He introduced the key notion of *a priori bound* for the renormalisation levels. This idea was thoroughly investigated by the ground-breaking works of McMullen, Lyubich, Graczyk-Świątek and Levin-van Strien, see for instance [McM94, McM96], [Lyu97, Lyu99, Lyu02], [GS98] and [LvS98], among others. Briefly speaking, a priori bounds is employed to prove the local connectivity of the Mandelbrot set, local connectivity of the Julia set [Jia00], and the ergodicity of conformal measures [Pra98].

By the work of Douady and Hubbard in [DH84], for each sequence of rational numbers $\tau = (p_i/q_i)_{i=1}^{\infty}$ in $(-1/2, 1/2] \setminus \{0\}$, there is $c \in \mathbb{C}$ such that P_c is infinitely satellite renormalisable of type τ. The uniqueness of this parameter is a non-trivial problem and is equivalent to the local connectivity of the Mandelbrot set at the corresponding point. It is known that when the denominators q_i tend to infinity fast enough, the Julia set fails to be locally connected, [Sr00]. However, their argument works in the limit and does not provide an explicit condition. In [Mil06b], Milnor proposed an optimal arithmetic condition for the local connectivity of the Julia sets. However, later Levin showed that this condition is necessary but not sufficient [Lev11]. He also described a condition that implies non-local connectivity of the Julia set. These results suggested that new tools distinct from *a priori bounds* are required to investigate the dynamics of infinitely satellite renormalisable quadratics.

In 2006, Inou and Shihikura [IS06] introduced the *near-parabolic* renormalisation. This is a powerful tool that has been used by Buff and Cheritat [BC12] to find the first known examples of positive area Julia sets in the quadratic family. Cheraghi and Shishikura [CS15] showed that for a wide class of sequences of rational numbers $(p_i/q_i)_{i=1}^{\infty}$ in $(-1/2, 1/2] \setminus \{0\}$, there are infinitely satellite renormalisable quadratic maps which are also infinitely near-parabolic renormalisable. Moreover, they establish the local connectivity of the Mandelbrot set under a certain growth condition on the sequence p_i/q_i.

For $c \in \mathbb{C}$, let us denote the post-critical set of P_c by $\mathcal{PC}(P_c)$, that is,

$$\mathcal{PC}(P_c) = \overline{\cup_{i=0}^{\infty} P_c^i(0)},$$

where 0 is the critical point of P_c. It is well-known [Lyu83] that $\mathcal{PC}(P_c)$ is the measure theoretic attractor of the dynamics of P_c on its Julia set. That is, the orbit of Lebesgue almost every point in J_{P_c} eventually stays in any given neighbourhood of $\mathcal{PC}(P_c)$. To understand the long term behaviour of typical orbits in the Julia set, one needs to understand the topological and geometric structure of $\mathcal{PC}(P_c)$, as well as the iterates of P_c near $\mathcal{PC}(P_c)$. If the Julia set of an infinitely renormalisable quadratic map is locally connected, then the post-critical set is a Cantor set, (See Subsection 2.3).

A key problem on the topological aspect of the set $\mathcal{PC}(P_c)$ is to identify an arithmetic condition that implies $\mathcal{PC}(P_c)$ is a Cantor set. In this work we construct an arithmetic model M_τ, for the post-critical set of an infinitely satellite renormalisable quadratic with combinatorics τ. We also describe the topological properties of that model. Our work is inspired by the near-parabolic renormalisation scheme of Inou and Shishikura [IS06], and draws on tools from [CS15]. In particular, we introduce an arithmetic condition on τ, namely *Condition* $\mathcal{C}$, which determines whether M_τ is a Cantor set. This condition is rather complicated and is in the spirit of the Herman condition identified by Yoccoz. See Section 3 for the precise definition of condition $\mathcal{C}$.

Theorem A (Dichotomy)**.** *Let $\tau = (p_i/q_i)_{i=0}^{\infty}$ be a sequence of rational numbers in $(-1/2, 1/2] \setminus \{0\}$. Then, one of the following holds:*

(i) the sequence τ satisfies condition $\mathcal{C}$, and M_τ is a Cantor set of points,
(ii) the sequence τ does not satisfy condition $\mathcal{C}$, and M_τ is a hairy Cantor set.

A hairy Cantor set, which we introduce here for the first time, is a topological object in the plane which enjoys a similar level of universal features as objects like Jordan curve, Cantor set, Cantor bouquet, and hairy circles, [Veb05, MK19, Lel61, Cha89, BO90, Nad92, HO16]. As we prove here, hairy Cantor sets are topological objects uniquely determined, up to ambient homeomorphisms, by some simple axioms.

We say a compact set $X \subseteq \mathbb{R}^2$ is a *hairy Cantor set* if it satisfies the following axioms.

(A_1) Any connected component of X is either a single point or a simple arc[1];
(A_2) The closure of the set of point components of X is a Cantor set of points, say B;
(A_3) Any arc component of X meets B at one of its end points;
(A_4) Whenever $x_i \to x$ within X, the unique arc in X connecting x_i to B converges to the unique arc in X connecting x to B, with the convergence in the Hausdorff topology;[2]
(A_5) The set $X \setminus B$ is dense in X;
(A_6) For every arc component γ of X, points in the interior of γ are not accessible from $\mathbb{R}^2 \setminus X$;

Theorem B. *All hairy Cantor sets are ambiently homeomorphic as subsets of the plane.*

Note that only axiom (A_6) requires X to be a subset of the plane. The other axioms may be stated for a compact metric space. We say a compact metric space X is an *abstract hairy Cantor set* if it satisfies axioms (A_1), (A_2), (A_3), (A_4) and the set of end points of the arc components of X is dense in X.

Theorem C. *Any hairy Cantor set is an abstract hairy Cantor set. Moreover, all abstract hairy Cantor sets are homeomorphic.*

[1]We say a compact metric space is a *simple arc* if it is the image of a homeomorphism γ from the interval $[0, 1]$. The *end points* of this simple arc are $\gamma(0)$ and $\gamma(1)$ and its *interior* is $\gamma((0, 1))$.

[2]In a metric space (M, d), given two sets $X, Y \subseteq M$, the *Hausdorff distance* of X and Y is defined as

$$d_H(X, Y) := \inf\{\varepsilon > 0 \mid D(X, \varepsilon) \subseteq Y, D(Y, \varepsilon) \subseteq X\}.$$

The Hausdorff distance is a metric on the set of non-empty compact subsets of M. The induced topology is called the *Hausdorff topology*.

We also propose a topological model for the dynamics of an infinitely renormalisable map of satellite type on its post-critical set. That is, a homeomorphism $f_\tau : M_\tau \to M_\tau$. We completely describe the dynamics of f_τ on M_τ.

Theorem D. *For any sequence of rational numbers $\tau = (p_i/q_i)_{i=0}^{\infty}$ in $(-1/2, 1/2] \setminus \{0\}$, every point in M_τ is forward and backward recurrent under the action of $f_\tau : M_\tau \to M_\tau$.*

We also classify the closed invariant subsets of $f_\tau : M_\tau \to M_\tau$.

Theorem E. *There is a unique closed invariant set Cantor set $B_\tau \subseteq M_\tau$ under f_τ. If M_τ is a Cantor set, then f_τ is minimal and $B_\tau = M_\tau$. Otherwise, there is a homeomorphism*

$$I_\tau : [0, 1] \to \{\text{closed invariant sets for } f_\tau : M_\tau \to M_\tau\},$$

such that the following hold:

(i) $I_\tau(0) = B_\tau$ is the base Cantor set of the hairy Cantor set M_τ, and $I_\tau(1) = M_\tau$;
(ii) for $0 < x \leq 1$, $I_\tau(x)$ is a hairy Cantor set;
(iii) for $0 \leq x < y \leq 1$, we have $I_\tau(x) \subsetneq I_\tau(y)$.

Theorem F. *For any sequence of rational numbers $\tau = (p_i/q_i)_{i=0}^{\infty}$ in $(-1/2, 1/2] \setminus \{0\}$, the map $f_\tau : M_\tau \to M_\tau$ is uniquely ergodic. Moreover, the unique invariant probability measure is supported on the Cantor set B_τ.*

In [DR14], DeZotti and Roesch propose a model for the post-critical set of a class of infinitely satellite renormalisable quadratics. They also describe a conjectural condition for the local connectivity of the Julia set for the case when $\tau = (1/q_i)_{i=1}^{\infty}$. In a forthcoming paper [CP18], we prove that for a wide class of τ, the corresponding quadratic map P_c on $\mathcal{PC}(P_c)$ is conjugate to $f_\tau : M_\tau \to M_\tau$.

In Section 2, we review some of the background material. In particular, we review some of the properties of quadratic-like and near-parabolic renormalisation and we describe some aspects of the relation between these renormalisation schemes. We will also define odometers and describe how one may assign an "address" to every connected component of the post-critical set of an infinitely renormalisable quadratic map.

In Section 3, we describe the combinatorial data needed to define the topological model. We also use this data to define condition $\mathcal{C}$.

In Section 4, we define the topological model. The definition of the model is given in Subsection 4.2. We need the combinatorial data defined in Section 3 and the model maps defined in Subsection 4.1 to construct the model. In the remainder of this section, we investigate some of the properties of the model. These results culminate in Subsection 4.8 where we prove Theorem A.

In Subection 5.1, we define the dynamics on the model and prove Theorem D. We analyse invariant subsets and prove Theorem E in Subsection 5.2. We finish this section by proving Theorem F in Subsection 5.3.

Section 6 consists of a purely topological classification of the model. We define *straight hairy Cantor sets*, a special class of hairy Cantor sets that are easier to analyse, in Subsection 6.1. We prove Theorem B in Subsection 6.3 and we finish Subsection 6.4 with the proof of Theorem C.

2. Renormalisation structures of quadratic maps

In this section, we study quadratic polynomials and describe some of their basic properties. One may refer to [Mil06a] or [Bea91] for the basic notions in complex dynamics. We also define quadratic-like renormalisation and some combinatorial aspects of this operator. Our description of this renormalisation operator follows the seminal work of Douady and Hubbard in [DH84] and [DH85]. For more details on the combinatorics of quadratic maps and their quadratic-like renormalisation, one may consult [McM94], [Mil00] or [Sch04].

2.1. **Quadratic maps.** Consider the class of quadratic polynomials in the complex plane. Every quadratic polynomial is affinely conjugate to a unique map of the form

$$P_c(z) := z^2 + c,$$

where $c \in \mathbb{C}$. For a quadratic polynomial P_c, we may define the *filled in Julia set* and the *Julia set* by

$$K_{P_c} := \{z \in \mathbb{C} \mid (P_c^n(z))_{n=0}^{\infty} \text{ is bounded}\} \quad \text{and} \quad J_{P_c} := \partial K_{P_c},$$

respectively. At any point outside of the Julia set, the iterates of the map form a normal family and the dynamics is easier to describe. The Julia set, on the other hand, is the locus of chaos. For a quadratic map, the Julia set is either connected, or a Cantor set. The Julia set is connected if and only if the the orbit of the critical point is bounded. When the Julia set is a Cantor set, the dynamics on the Julia set is topologically conjugate to a shift map.

The dynamics is more complicated when the Julia set is connected. In this case, K_{P_c} is a compact, full subset of the complex plane. Hence $\hat{\mathbb{C}} \setminus K_{P_c}$ is a topological disk. [3] We may define the *Böttcher coordinate* for P_c as the biholomorphic isomorphism

$$\phi_{P_c} : \mathbb{C} \setminus K_{P_c} \to \mathbb{C} \setminus \overline{\mathbb{D}}.$$

that satisfies

$$\lim_{z \to \infty} \phi'_{P_c}(z) = 1.$$

The map ϕ_{P_c} holomorphically conjugates $P_c|_{\mathbb{C}\setminus K_{P_c}}$ to $z^2|_{\mathbb{C}\setminus\overline{\mathbb{D}}}$. For any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, the set

$$R_{P_c}(\theta) := \{\phi_{P_c}^{-1}(re^{i\theta}) \mid r \in (1, \infty)\}$$

is called the *external ray* of P_c with angle θ. We say an external ray $R_{P_c}(\theta)$ lands at $z \in J_{P_c}$ if

$$\lim_{r \to 1} \phi_{P_c}^{-1}(re^{i\theta}) = z.$$

In other words, if $\overline{R_{P_c}(\theta)} = R_{P_c}(\theta) \cup \{z\}$. For any $r > 1$, the set

$$E_{P_c}(r) := \{\phi_{P_c}^{-1}(re^{i\theta}) \mid \theta \in \mathbb{R}/(2\pi\mathbb{Z})\}$$

is called the *equipotential* of radius r.

Any quadratic polynomial P_c has two fixed points which agree only when P_c has a parabolic fixed point. Assume J_{P_c} is connected. As shown by Douady and Hubbard in [DH84], for all $\theta \in 2\pi\mathbb{Q}$, the external ray $R_{P_c}(\theta)$ lands. We know that $R_{P_c}(0)$ is mapped into itself under P_c. Therefore the landing point of $R_{P_c}(0)$ is a fixed point of P_c. This fixed point is either repelling or parabolic. We say this point is the *β-fixed point of P_c* and we will refer to the other fixed point of P_c as its α-fixed point.

In [DH84], Douady and Hubbard proved that any parabolic or repelling periodic point of a quadratic map is the landing point of at least one, but at most finitely many of its external rays. Let P_c be a quadratic map with connected Julia set such that the α-fixed point of P_c

[3] A *topological disk* is a topological space homeomorphic to the open unit disk in the plane.

is repelling. Then there are $q > 1$ rays landing at the fixed point α. Let us assume that $R_{P_c}(\theta_0), R_{P_c}(\theta_1), \cdots, R_{P_c}(\theta_{q-1})$ are the external rays landing at α. We may rearrange the indices so that we have $\theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{q-1} < \theta_0$ with respect to the cyclic order of $\mathbb{R}/2\pi\mathbb{Z}$. There is a non-zero integer $-q/2 < p \leq q/2$ such that

$$P_c(R_{P_c}(\theta_i)) = R_{P_c}(\theta_{i+p \pmod q}),$$

for all $0 \leq i < q$. We will say that $p/q \in (-1/2, 1/2] \setminus \{0\}$ is the *combinatorial rotation number* of P_c.

2.2. **Quadratic-like renormalisation.** To define quadratic-like renormalisation, first we define quadratic-like maps.

Definition 2.1. *A holomorphic map $f : U \to f(U) = V$ is called a quadratic-like map if $U, V \subseteq \mathbb{C}$ are topological disks, U is compactly contained in V and f is a proper map of degree two.*

Note that for any quadratic polynomial P_c with a connected Julia set and any large disk $V \supseteq J_{P_c}$ centred at the origin, the map $P_c|_{P_c^{-1}(V)} : P_c^{-1}(V) \to V$ is quadratic-like. For a quadratic-like map f, we may define the filled in Julia set and the Julia set similar to a quadratic map.

$$K_f := \{z \in U \mid \forall n \geq 1, f^n(z) \in U\}, \quad J_f := \partial K_f.$$

The Julia set is connected if and only if it contains the critical point of f. A *hybrid conjugacy* between two quadratic-like maps $f : U \to V$ and $g : U' \to V'$ is a quasi-conformal map h from a neighbourhood of K_f to a neighbourhood of K_g that conjugates f and g and $\partial h/\partial \overline{z} = 0$ for Lebesgue almost every point $z \in K_f$. If such a conjugacy exists, we say f and g are hybrid equivalent.

Theorem 2.2. [DH85, Straightening theorem] *Any quadratic-like map f is hybrid conjugate to a quadratic polynomial. Moreover, if J_f is connected, then any two such quadratics are affinely conjugate.*

Using the straightening theorem, for any quadratic-like map f with repelling fixed points and a connected Julia set, we may uniquely define the α-fixed point, the β-fixed point and the combinatorial rotation number.

Recall that $\mathcal{M}$ denotes the Mandelbrot set, the set of points $c \in \mathbb{C}$ such that J_{P_c} is connected. Let $c \in \mathcal{M}$ be a complex number such that that the α-fixed point of the map P_c is repelling. The closure of the q rays landing at α cut the complex plane into q open connected components which we denote by Y_j, for $0 \leq j \leq q-1$. By a simple topological argument, P_c on these pieces has a simple covering property. One of these components, which we denote by Y_{q-1}, contains both critical point 0 and the pre-fixed point $-\alpha$. Moreover, $P_c(Y_{q-1})$ covers $\cup_{j=0}^{q-1} Y_j$. We may relabel the other components so that $P_c(Y_j) = Y_{j+1}$, for $0 \leq j \leq q-2$. Thus, the critical value is contained in Y_1 and the critical point is mapped back into Y_{q-1} under q iterates of P_c.

Fix $r > 1$. The equipotential $E_{P_c}(r)$ divides each piece Y_j, for $0 \leq j \leq q-1$, into two connected components. We denote by Y_j^1, for $0 \leq j \leq q-1$, the closure of the bounded connected component of $Y_j \setminus E_{P_c}(r)$. These are called *puzzle pieces of level* 1. The closure of the connected components of $P_c^{-i}(\operatorname{int}(Y_j^1))$, for $i \in \mathbb{N}$ and $0 \leq j \leq q-1$, are called *puzzle pieces of level i*. They form nests of pieces breaking the Julia set into components. As $P_c(Y_{q-1})$ covers $\cup_{j=0}^{q-1} \operatorname{int}(Y_j^1)$, the connected components of $P_c^{-1}(\operatorname{int}(Y_j^1))$ contained in Y_{q-1} divide Y_{q-1} into q pieces. We denote the one containing 0 by Z_{q-1}^2 and the remaining ones by Z_j^2, for $0 \leq j \leq q-2$. For c in the Mandelbrot set with α-fixed point repelling and q rays landing at α, the map P_c is called *quadratic-like renormalisable of satellite type*, if $P_c^{nq}(0) \in Z_{q-1}^2$, for all $n \in \mathbb{N}$. Using $|P_c'(\alpha)| > 1$,

we may build a simply connected domain $\tilde{Z}^2_{q-1}$ containing the closure of Z^2_{q-1} such that $P^q_c(\tilde{Z}^2_{q-1})$ contains the closure of $\tilde{Z}^2_{q-1}$ and $P^q_c : \tilde{Z}^2_{q-1} \to P^q_c(\tilde{Z}^2_{q-1})$ is a quadratic-like map. Since the orbit of 0 does not escape the domain of this quadratic-like map under iteration, its Julia set is connected. We use $\mathcal{R}_{QL}(P_c)$ to denote the unique quadratic map $P_{c'}$ that is hybrid conjugate to $P^q_c : \tilde{Z}^2_{q-1} \to P^q_c(\tilde{Z}^2_{q-1})$. In this case, we say q is the *renormalisation period*. If $f := P^q_c|_{\tilde{Z}^2_{q-1}}$, we say that $J_f, P_c(J_f), \cdots, P^{q-1}_c(J_f)$ are the *little Julia sets* of P_c. Note that α is contained in all of the little Julia sets and the sets $J_f \setminus \{\alpha\}, P_c(J_f) \setminus \{\alpha\}, \cdots, P^{q-1}_c(J_f) \setminus \{\alpha\}$ are pair-wise disjoint. Moreover, the point α is the β-fixed point of all of the little Julia sets. Clearly the β-fixed point of P_c is not contained in any of the little Julia sets.

For c as in the above paragraph, if P_c is not quadratic-like renormalisable of satellite type, there is the smallest $n \in \mathbb{N}$ such that $P^{nq}_c(0) \in Z^2_j$, for some $0 \le j \le q-2$. Then, let V_1 denote the closure of the connected component of $P^{-nq}_c(\mathrm{int}\,(Z^2_j))$ containing 0 that is obtained by pulling back along the orbit $0, P_c(0), \cdots, P^{nq}_c(0)$. If the orbit of 0 under P_c never enters $\mathrm{int}\,(V_1)$, then P_c is not quadratic-like renormalisable. Otherwise, let $j_1 \in \mathbb{N}$ be the smallest integer with $P^{j_1}_c(0) \in \mathrm{int}\,(V_1)$, and denote by V_2 the closure of the component of $P^{-j_1}_c(\mathrm{int}\,(V_1))$ that is obtained by pulling back along $0, P_c(0), \cdots, P^{j_1}_c(0)$. Here, $P^{j_1}_c : \mathrm{int}\,(V_2) \to \mathrm{int}\,(V_1)$ is a quadratic-like map. Inductively, we define domains $V_1 \supset V_2 \supset V_3 \supset \cdots$, and positive integers $j_1, j_2, j_3, \cdots$ such that each j_m is the smallest positive integer with $P^{j_m}_c(0) \in \mathrm{int}\,(V_m)$ and V_{m+1} is the closure of the pull-back of $\mathrm{int}\,(V_m)$ along the orbit $0, P_c(0), \cdots, P^{j_m}_c(0)$. For all $m \ge 2$, the puzzle piece V_m is compactly contained in the interior of V_{m-1} and $P^{j_m}_c : V_m \to V_{m-1}$ is a quadratic-like map. The quadratic map P_c is *quadratic-like renormalisable of primitive type*, if there is $m \ge 2$ such that the orbit of 0 under $P^{j_m}_c : V_m \to V_{m-1}$ remains in V_m. Note that if this occurs, the sequence j_m is eventually constant. For the smallest positive integer m satisfying this property, we denote the unique quadratic map $P_{c'}$ that is hybrid conjugate to $P^{j_m}_c : V_m \to V_{m-1}$ by $\mathcal{R}_{QL}(P_c)$. In this case, we say j_m is the *renormalisation period*. If $f := P^{j_m}_c|_{V_m}$, we say that $J_f, P_c(J_f), \cdots, P^{j_m-1}_c(J_f)$ are the *little Julia sets* of P_c. Note that, in this case, little Julia sets are pair-wise disjoint and none of the little Julia sets contain any fixed point of P_c.

A quadratic map is *quadratic-like renormalisable*, if it is quadratic-like renormalisable of either satellite or primitive type.

2.3. **Infinitely renormalisable maps and odometers.** Let c_1 be a complex number. If P_{c_1} is renormalisable, then there is a complex number c_2 such that $P_{c_2} = \mathcal{R}_{QL}(P_{c_1})$. We say P_{c_1} is twice renormalisable if there is $c_3 \in \mathcal{M}$ such that $P_{c_3} = \mathcal{R}_{QL}(P_{c_2})$. Similarly we may define k-times renormalisable quadratics, for all $k \in \mathbb{N} \cup \{\infty\}$. Assume P_c is infinitely renormalisable where every renormalisation is of satellite type. Then there is a sequence $(c_i)_{i=1}^{\infty}$ in $\mathcal{M}$ such that

$$c_1 = c, \quad \forall i \ge 1, \quad P_{c_{i+1}} = \mathcal{R}_{QL}(P_{c_i}).$$

Moreover, every periodic orbit of P_c is repelling and there is a sequence of non-zero rational numbers $(p_i/q_i)_{i=1}^{\infty}$ in $(-1/2, 1/2] \setminus \{0\}$ where p_i/q_i is the combinatorial rotation number of the α-fixed point of P_{c_i}. We say P_c is infinitely satellite renormalisable with combinatorics $(p_i/q_i)_{i=1}^{\infty}$. As mentioned before, Douady and Hubbard [DH84] showed that for each sequence of rational numbers $\tau = (p_i/q_i)_{i=1}^{\infty}$ in $(-1/2, 1/2] \setminus \{0\}$, there are some parameters $c \in \mathcal{M}$ such that P_c is infinitely satellite renormalisable of type τ.

Recall that for all $c \in \mathbb{C}$, the post-critical set of P_c is denoted by $\mathcal{PC}(P_c)$, that is,

$$\mathcal{PC}(P_c) = \overline{\bigcup_{t=0}^{\infty} P^t_c(0)},$$

where 0 is the critical point of P_c. As we will describe, when P_c is infinitely renormalisable, there is a natural way to assign an "address" to every connected component of $\mathcal{PC}(P_c)$. Note that in general, it is not proven that every such address uniquely identifies a connected component. The addresses are built using only the renormalisation periods of P_c. The space of addresses is called an *odometer*.

Definition 2.3. *Let $\boldsymbol{q} := (q_i)_{i=1}^{\infty}$ be a sequence of natural numbers greater than one. Consider the sequence of canonical maps between discrete topological groups*

$$\mathbb{Z}/q_1 \leftarrow \mathbb{Z}/(q_1q_2) \leftarrow \cdots \leftarrow \mathbb{Z}/(q_1q_2\cdots q_i) \leftarrow \cdots,$$

where the map from $\mathbb{Z}/(q_1q_2\cdots q_i) \to \mathbb{Z}/(q_1q_2\cdots q_{i-1})$ sends any number $t \pmod{q_1q_2\cdots q_i}$ to $t \pmod{q_1q_2\cdots q_{i-1}}$. The "odometer", or "adding machine", with combinatorics $\boldsymbol{q}$ is defined as

$$\mathcal{O}_{\boldsymbol{q}} := \operatorname*{proj\,lim}_{i\to\infty} \mathbb{Z}/(q_1q_2\cdots q_i) = \{(t_i)_{i=1}^{\infty} \mid \forall j \geq i \geq 1,\ t_i = t_j \pmod{q_1q_2\cdots q_i}\}.$$

For all $i \geq 1$, we use the notation $t \mapsto t \pmod{q_1q_2\cdots q_i}$ to denote the projection from $\mathcal{O}_{\mathbf{q}}$ onto $\mathbb{Z}/(q_1q_2\cdots q_i)$. There is a canonical way to define topology and group structure on $\mathcal{O}_{\mathbf{q}}$ that is compatible with the projective limit. Hence $\mathcal{O}_{\mathbf{q}}$ is a topological group.

There is a unique element $\mathcal{O}_{\mathbf{q}}$ that maps into $1 \pmod{q_1q_2\cdots q_i}$, for all $i \geq 1$. We use the notation $1 \in \mathcal{O}_{\mathbf{q}}$ to denote this element. It follows that the map

$$t \mapsto t + 1 : \mathcal{O}_{\mathbf{q}} \to \mathcal{O}_{\mathbf{q}},$$

is the projective limit of the homeomorphisms $t \mapsto t + 1 : \mathbb{Z}/(q_1q_2\cdots q_i) \to \mathbb{Z}/(q_1q_2\cdots q_i)$, for $i \geq 1$. Hence it is a homeomorphism. We include a proof of the following classical statement for completeness.

Lemma 2.4. *For any combinatorics $\boldsymbol{q} := (q_i)_{i=1}^{\infty}$, the odometer $\mathcal{O}_{\boldsymbol{q}}$ is a Cantor set.*

Proof. According to a classical theorem of Brower, any non-empty compact Hausdorff space without isolated points that has a countable base consisting of clopen sets is a Cantor set. The odometer $\mathcal{O}_{\mathbf{q}}$ is a closed subspace of the product space $\prod_{i=1}^{\infty} \mathbb{Z}/(q_1q_2\cdots q_i)$ and the product space is compact. Hence $\mathcal{O}_{\mathbf{q}}$ is compact. If $\pi_i : \mathcal{O}_{\mathbf{q}} \to \mathbb{Z}/(q_1q_2\cdots q_i)$, for $i \geq 1$, denotes the projection map, then the family

$$\{\pi_i^{-1}(t) \mid i \geq 1,\ t \in \mathbb{Z}/(q_1q_2\cdots q_i)\}$$

is a countable base for the topology of $\mathcal{O}_{\mathbf{q}}$, consisting of clopen sets. If $t \neq t'$ are distinct points in $\mathcal{O}_{\mathbf{q}}$, then there is an integer $i \geq 1$ such that $\pi_i(t) \neq \pi_i(t')$. Hence the the open sets $\pi_i^{-1}(\pi_i(t))$ and $\pi_i^{-1}(\pi_i(t'))$ are disjoint. Therefore $\mathcal{O}_{\mathbf{q}}$ is Hausdorff.

To complete the proof, we need to show that $\mathcal{O}_{\mathbf{q}}$ has no isolated points. Let $t \in \mathcal{O}_{\mathbf{q}}$ be a point. We will show that the sequence $(t + q_1, t + q_1q_2, t + q_1q_2q_3, \cdots)$ tends to t. For $i \geq 1$, let $U_i := \pi_i^{-1}(\pi_i(t))$. The family of open sets $(U_i)_{i=1}^{\infty}$ forms a basis around the point t. Let $i \geq 1$ be an integer. For all $j > i$, we have

$$t = t + q_1q_2\cdots q_j \pmod{q_1q_2\cdots q_i}.$$

Hence $t + q_1q_2\cdots q_j \in U_i$, for all $j > i$. Therefore we have

$$\lim_{j\to\infty} t + q_1q_2\cdots q_j = t. \qquad \square$$

When $c \in \mathbb{R}$ such that P_c is infinitely renormalisable, it is a well-known result that the post-critical set is a Cantor set and the map $P_c : \mathcal{PC}(P_c) \to \mathcal{PC}(P_c)$ is homeomorphic to an odometer. (See [dMvS93]) Theorem 9.4 in [McM94] states a similar for some class of infinitely renormalisable quadratics that includes cases when $c \notin \mathbb{R}$. The following results follow from simple applications of the standard techniques in the field. We include the proof for completeness.

Proposition 2.5. *Let $c \in \mathcal{M}$ be a complex number such that P_c is infinitely quadratic-like renormalisable with $\boldsymbol{q} := (q_i)_{i=1}^{\infty}$ as its sequence of renormalisation periods. Then there is a surjective continuous map*

$$\boldsymbol{t}_{P_c} : \mathcal{PC}(P_c) \to \mathcal{O}_{\boldsymbol{q}},$$

that is constant on each connected component of $\mathcal{PC}(P_c)$ and we have

$$\boldsymbol{t}_{P_c}(P_c(z)) = \boldsymbol{t}_{P_c}(z) + 1,$$

for all $z \in \mathcal{PC}(P_c)$.

Proof. For any $i \geq 1$, there are q_i little Julia sets from the renormalisation of P_{c_i} within $J_{P_{c_i}}$ that form a periodic orbit. We may use the hybrid conjugacy in the definition of the renormalisation to obtain $q_1 q_2 \cdots q_i$ little Julia sets from the i-th renormalisation of P_c within J_{P_c} that form a periodic orbit. Let $J^i_{P_c}$ be the little Julia set in the i-th level that contains the critical value. Then, the set of little Julia sets in the i-th level is

$$\{P_c^t(J^i_{P_c}) \mid t \in \mathbb{Z}/(q_1 q_2 \cdots q_i)\}.$$

If $t \in \mathbb{Z}/(q_1 q_2 \cdots q_i)$, let

$$J_{P_c}(t) := P_c^t(J^i_{P_c}), \quad \mathcal{PC}(P_c)(t) := J_{P_c}(t) \cap \mathcal{PC}(P_c).$$

Since $J_{P_{c_i}}$ is connected, for all $i \geq 1$, we see that the critical orbit is contained in the union of the little Julia sets. Therefore we have

$$\mathcal{PC}(P_c) = \bigcap_{i=1}^{\infty} \bigcup_{t \in \mathbb{Z}/(q_1 q_2 \cdots q_i)} \mathcal{PC}(P_c)(t) \subseteq \bigcap_{i=1}^{\infty} \bigcup_{t \in \mathbb{Z}/(q_1 q_2 \cdots q_i)} J_{P_c}(t).$$

It follows from the definition of the renormalisation that for any $t \neq t'$ in $t \in \mathbb{Z}/q_1$, the sets $J_{P_c}(t) \cap J_{P_c}(t')$ is either empty or it consists of a single point σ. If this intersection is non-empty, then σ corresponds to the α-fixed point of P_{c_1} and the β-fixed point of the map P_{c_2}. Therefore σ is not contained in the set $\bigcup_{t \in \mathbb{Z}/(q_1 q_2)} J_{P_c}(t)$. Hence we see that for all $t \neq t'$ in $t \in \mathbb{Z}/q_1$, we have $\mathcal{PC}(P_c)(t) \cap \mathcal{PC}(t') = \emptyset$. Using a similar argument, we see that for all $i \geq 1$ and $t \neq t'$ in $\mathbb{Z}/(q_1 q_2 \cdots q_i)$, we have

$$\mathcal{PC}(P_c)(t) \cap \mathcal{PC}(t') = \emptyset.$$

Therefore, for any point $z \in \mathcal{PC}(P_c)$ and any $i \geq 1$, there is a unique $t \in \mathbb{Z}/(q_1 q_2 \cdots q_i)$ such that $z \in \mathcal{PC}(P_c)(t)$. The map $\mathbf{t}^i_{P_c} : \mathcal{PC}(P_c) \to \mathbb{Z}/(q_1 q_2 \cdots q_i)$ is continuous and surjective and we have

$$\mathbf{t}^i_{P_c}(P_c(z)) = \mathbf{t}^i_{P_c}(z) + 1,$$

for all $z \in \mathcal{PC}(P_c)$. Hence we may define $\mathbf{t}_{P_c}$ as the projective limit of $\mathbf{t}^i_{P_c}$ and we obtain a surjective continuous map $\mathbf{t}_{P_c} : \mathcal{PC}(P_c) \to \operatorname{proj\,lim}_{i \to \infty} \mathbb{Z}/(q_1 q_2 \cdots q_i) = \mathcal{O}_{\mathbf{q}}$ such that for all $z \in \mathcal{PC}(P_c)$, we have

$$\mathbf{t}_{P_c}(P_c(z)) = \mathbf{t}_{P_c}(z) + 1. \qquad \square$$

Lemma 2.6. *Under the assumption of Proposition 2.5, if J_{P_c} is locally connected, then any nest of little Julia sets, as defined in the proof of Proposition 2.5, shrinks to a single point. In other words, for all $t \in \mathcal{O}_{\boldsymbol{q}}$, the set*

$$J_{P_c}(t) := \bigcap_{i \geq 1} J_{P_c}(t \pmod{q_1 q_2 \cdots q_i})$$

consists of a single point.

Proof. We say an open domain S in the plane is a *sector* based on a point $p \in \mathbb{C}$ for the map P_c if ∂S consist of the point p together with two external rays which land at p. For any domain $S' \subseteq \mathbb{C}$ which is bounded by finitely many external rays together with their landing points, we define the *angular width* of S' as the Lebesgue measure of the open set $I(S') \subseteq \mathbb{R}/(2\pi\mathbb{Z})$ consisting of all the angles θ such that $R_{P_c}(\theta) \subseteq S'$. We use $|I(S')|$ to denote the angular width. Note that $I(S')$ is a union of finitely many open intervals. Moreover, if S' is a sector, then $I(S')$ is an open interval.

Let $i \geq 1$ and $q := q_1 q_2 \cdots q_i$. For all $t \in \mathbb{Z}/q$, let $\beta(t)$ and $\alpha(t)$ denote the β-fixed point and the α-fixed point of the little Julia set $J_{P_c}(t)$, respectively. If the i-th renormalisation of P_c is primitive, then $(\beta(t))_{t\in\mathbb{Z}/q}$ forms a periodic orbit of period q. Otherwise, for any $t \in \mathbb{Z}/q$, we have $\beta(t) = \alpha(t \pmod{q/q_i})$ and $(\beta(t))_{t\in\mathbb{Z}/q}$ forms a periodic orbit of period q/q_i. [4] According to Theorem 1.1 in [Mil00], among all the sectors based on $(\beta(t))_{t\in\mathbb{Z}/q}$, there is a unique sector which contains the critical value of P_c and whose closure contains no other point in the periodic orbit $(\beta(t))_{t\in\mathbb{Z}/q}$ other than its base. This sector is called the *critical value sector* of the orbit $(\beta(t))_{t\in\mathbb{Z}/q}$ and is denoted by S_i. Let $k_i \in \mathbb{Z}/q$ be a number such that $\beta(k_i)$ is the base of this sector. As we will see later in this proof, we may choose $k_i = 0 \pmod{q}$.

Let t_i^- and t_i^+ be angles in $\mathbb{R}/(2\pi\mathbb{Z})$ such that $I(S_i) = (t_i^-, t_i^+)$ and let $d_i = t_i^+ - t_i^-$ be the angular width of the critical value sector. According to the discussions in Section 8 of [Mil00], in particular Lemma 8.1 in [Mil00], the external rays of angle $t_i^- + d_i/2^q$ and $t_i^+ - d_i/2^q$ land at the same point in $S_i \cap P_c^{-q}(\beta(k_i))$. Moreover, if we define $S'(0 \pmod{q}) \subseteq S_i$ to be the open domain bounded by ∂S_i together with these two rays and their common landing point, then the map P_c^q carries $S'(0 \pmod{q})$ onto S_i by a proper map of degree two, with the critical value equal to the critical value of P_c. Therefore the map P_c^{q-1} is univalent on $S'(0 \pmod{q})$ and $P_c^{q-1}(S'(0 \pmod{q})) = P_c^{-1}(S_i)$. As described in [Mil00], we may slightly thicken the sets S_i and $S'(0 \pmod{q})$ and cut them along appropriate equipotentials to obtain $\tilde{S}_i$ and $\tilde{S}'(0 \pmod{q})$ such that $P_c^q : \tilde{S}'(0 \pmod{q}) \to \tilde{S}_i$ is a quadratic-like map. This procedure is similar to the definition of satellite renormalisation described in Subsection 2.2. The Julia set of this quadratic-like map contains the critical value of P_c. Therefore, according to Theorem 7.1 in [McM94], its Julia set is equal to $J_{P_c}(0 \pmod{q})$ and we have $\beta(k_i) = \beta(0 \pmod{q})$. In other words, we may choose $k_i = 0 \pmod{q}$. Hence we have

$$J_{P_c}(0 \pmod{q}) \subseteq \overline{S'(0 \pmod{q})}.$$

Define

$$S'(n \pmod{q}) := P_c^n(S'(0 \pmod{q})),$$

for all $0 \leq n < q$ and

$$S''(-1 \pmod{q}) := P_c^{-1}(S'(0 \pmod{q})) \subseteq P_c^{q-1}(S'(0 \pmod{q})) = S'(-1 \pmod{q}).$$

Since $S'(0 \pmod{q})$ is a domain bounded by 4 external rays, we see that $S''(-1 \pmod{q})$ is a domain bounded by 8 external rays. We also see that $I(S'(0 \pmod{q}))$ is a union of 2 open intervals, while $I(S''(-1 \pmod{q}))$ is a union of 4 open intervals. For $2 \leq n \leq q$, let

$$S''(-n \pmod{q}) := P_c^{-n+1}(S''(-1 \pmod{q})) \cap P_c^{q-n}(S'(0 \pmod{q})).$$

Since P_c^{q-1} is univalent on $S'(0 \pmod{q})$, we see that P_c^{q-1} is also univalent on $S''(0 \pmod{q})$ and

$$P_c^q : S''(0 \pmod{q})) \to S'(0 \pmod{q})$$

[4] When $i = 1$, the group $\mathbb{Z}/(q_1 q_2 \cdots q_{i-1}) = \mathbb{Z}/(1)$ is the trivial group with one element. Moreover, $J_{P_c}(0 \pmod{1}) = J_{P_c}$ and $\alpha(0 \pmod{1})$ is the α-fixed point of P_c.

is a proper branched double cover. Since $J_{P_c}(0 \pmod q) \subseteq \overline{S'(0 \pmod q)}$, we conclude that $J_{P_c}(0 \pmod q) \subseteq \overline{S''(0 \pmod q)}$. Therefore, we have

$$J_{P_c}(t) \subseteq \overline{S''(t)},$$

for all $t \in \mathbb{Z}/(q_1 q_2 \cdots q_i)$.

The map P_c sends any external ray of angle θ to the external ray of angle 2θ. Therefore, the univalence of P_c^{q-1} on $S''(0 \pmod q) \subseteq S'(0 \pmod q)$ implies that

$$\begin{aligned}|I(S''(0 \pmod q))| = |I(S''(1 \pmod q))|/2 &= |I(S''(2 \pmod q))|/2^2 = \cdots \\ &= |I(S''(q-1 \pmod q))|/2^{q-1},\end{aligned}$$

and

$$\begin{aligned}|I(S'(0 \pmod q))| = |I(S'(1 \pmod q))|/2 &= |I(S'(2 \pmod q))|/2^2 = \cdots \\ &= |I(S'(q-1 \pmod q))|/2^{q-1}.\end{aligned}$$

On the other hand, since $P_c : S''(-1 \pmod q)) \to S'(0 \pmod q))$ and $P_c : S'(-1 \pmod q)) \to S_i$ are both double covers, we see that

$$|I(S''(-1 \pmod q))| = |I(S'(0 \pmod q))|, \quad |I(S'(-1 \pmod q))| = |I(S_i)|.$$

Therefore, for all $t \in \mathbb{Z}/(q_1 q_2 \cdots q_i)$, the set $I(S''(t))$ is the union of 4 open intervals and

$$|I(S''(t))| \leq 2^{-q_1 q_2 \cdots q_i + 1} |I(S_i)| \leq 2^{-q_1 q_2 \cdots q_i + 2} \pi.$$

Since the Julia set is locally connected, every external ray lands and every point in the Julia set is the landing point of at least one external ray. Let $t \in \mathcal{O}_{\mathbf{q}}$ and let $I(t) \subseteq \mathbb{R}/(2\pi\mathbb{Z})$ denote the set of angles θ such that the external ray $R_{P_c}(\theta)$ lands on the set $J_{P_c}(t)$. We have

$$J_{P_c}(t) = \bigcap_{i \geq 1} J_{P_c}(t \pmod{q_1 q_2 \cdots q_i}) \subseteq \bigcap_{i \geq 1} \overline{S''(t \pmod{q_1 q_2 \cdots q_i})}.$$

Therefore

$$I(t) \subseteq \bigcap_{i \geq 1} \overline{I(S''(t \pmod{q_1 q_2 \cdots q_i}))}.$$

Assume $I(t)$ contains at least 5 distinct points and let ε denote the minimum distance between these points in $\mathbb{R}/(2\pi\mathbb{Z})$. If $i \geq 1$ is large enough so that $2^{-q_1 q_2 \cdots q_i + 2}\pi < \varepsilon/3$, then we see that the set $\overline{I(t \pmod{q_1 q_2 \cdots q_i})} \supseteq I(t)$ which is the union of at most 4 closed intervals can not contain all of these points. Hence $I(t)$ contains at most 4 points. Since every point in $J_{P_c}(t)$ is the landing point of an external ray with an angle in $I(t)$, we see that $J_{P_c}(t)$ contains at most 4 points. The set $J_{P_c}(t)$ is the intersection of a nest of non-empty compact connected sets, so it is also non-empty and connected. Therefore $J_{P_c}(t)$ consists of a single point. □

Corollary 2.7. *If P_c is an infinitely renormalisable quadratic map such that J_{P_c} is locally connected, then the map $\mathbf{t}_{P_c}$ defined in Proposition 2.5 is a homeomorphism and $\mathcal{PC}(P_c)$ is a Cantor set.*

Proof. According to Lemma 2.6, any nest of little Julia sets shrinks to a point. In other words, for all $t \in \mathcal{O}_{\mathbf{q}}$, the set

$$\emptyset \neq \bigcap_{i \geq 1} \mathcal{PC}(P_c)(t \pmod{q_1 q_2 \cdots q_i}) \subseteq \bigcap_{i \geq 1} J_{P_c}(t \pmod{q_1 q_2 \cdots q_i}) = J_{P_c}(t)$$

consists of a single point. Therefore every connected component of $\mathcal{PC}(P_c)$ is a single point and $\mathbf{t}_{P_c}$ is bijective. Since $\mathcal{PC}(P_c)$ is compact, we see that $\mathbf{t}_{P_c}$ is a homeomorphism and $\mathcal{PC}(P_c)$ is a Cantor set. □

2.4. **Near-parabolic renormalisation.** In 2006, Inou and Shishikura introduced the *near-parabolic renormalisation*. One may consult [IS06] or [CS15] for a precise definition of this renormalisation scheme and its relation to quadratic-like satellite renormalisation. Here we will describe the general picture and explain how it motivates our topological model.

There are two class of holomorphic maps $\Gamma \subseteq \Gamma'$ such that every map $f \in \Gamma'$ is defined on an open subset of the complex plane and $f(0) = 0$. Moreover, if $f \in \Gamma$, then there is a complex number α_f such that

$$0 < |\alpha_f| < 1/2, \quad |\operatorname{Im}(\alpha_f)| \leq |\operatorname{Re}(\alpha_f)|$$

and $f'(0) = e^{2\pi i \alpha_f}$. The class of maps Γ contains a subset of all quadratic maps. Inou and Shishikura define two operators

$$\mathcal{R}^{\mathrm{t}}_{NP} : \Gamma \to \Gamma' \quad \mathcal{R}^{\mathrm{b}}_{NP} : \Gamma \to \Gamma',$$

the *top renormalisation* and the *bottom renormalisation*, respectively. If for some map $f \in \Gamma$, either $\mathcal{R}^{\mathrm{t}}_{NP}$ or $\mathcal{R}^{\mathrm{b}}_{NP}$ belong to Γ, we may consider the second renormalisations of f. We say a map $f \in \Gamma$ is *infinitely near-parabolic renormalisable* if there is a sequence of symbols $(\kappa_n)_{n=1}^{\infty}$, where $\kappa_n \in \{\mathrm{t}, \mathrm{b}\}$ such that the sequence of maps

$$f_1 := f, \quad \forall n \geq 1,\ f_{n+1} := \mathcal{R}^{\kappa_n}_{NP}(f_n)$$

is well-defined. For any such f, there is a unique sequence $(\kappa_n)_{n=1}^{\infty}$ satisfying this property.

For all $f \in \Gamma$ with $\mathcal{R}^{\mathrm{t}}_{NP}(f) \in \Gamma$, we have

$$\alpha_{\mathcal{R}^{\mathrm{t}}_{NP}(f)} = -1/\alpha \ (\mathrm{mod}\ \mathbb{Z}).$$

Therefore, if $f \in \Gamma$ with $f'(0) = e^{2\pi i \alpha}$ is an infinitely near-parabolic renormalisable map with only top renormalisations, then the sequence

$$\alpha_1 := \alpha, \quad \forall n \geq 1,\ \alpha_{n+1} = -1/\alpha_n \ (\mathrm{mod}\ \mathbb{Z})$$

is well-defined and $f_n'(0) = e^{2\pi i \alpha_n}$, for all $n \geq 1$. Therefore 0 is an indifferent fixed point. One may consult [Che17] for a detailed analysis of such maps. If the sequence of renormalisation of f contains at most finitely many bottom renormalisation, then f has an indifferent periodic point. Here we only focus on the maps $f \in \Gamma$ where the sequence of renormalisation contains infinitely many bottom renormalisation.

As described in [CS15], one may assign a sequence of rational numbers $(p_n/q_n)_{n=1}^{\infty}$ to any infinitely near-parabolic renormalisable map $f \in \Gamma$. In the same paper, Cheraghi and Shishikura showed that for a wide class of sequences of rational numbers $(\hat{p}_n/\hat{q}_n)_{n=1}^{\infty}$, there is a quadratic map that is both infinitely quadratic-like renormalisable of satellite type with combinatorics $(\hat{p}_n/\hat{q}_n)_{n=1}^{\infty}$ and infinitely near-parabolic renormalisable. Moreover, near-parabolic combinatorics and the quadratic-like combinatorics are closely related. The notion of *basic* and *compound* sequences of rational numbers defined in Section 3 correspond to near-parabolic combinatorics and the quadratic-like combinatorics, respectively.

For $z \in \mathbb{R}$, let $[z]$ denote the closest integer to z. If there is an integer n such that $z = n + 1/2$, we define

$$[z] := \begin{cases} n+1 & \text{if } n \geq 0, \\ n & \text{if } n < 0. \end{cases}$$

Hence, for all $z \in \mathbb{C}$, we have $[-z] = -[z]$. We define the *modified Gauss map* as

$$G : \mathbb{R} \setminus \{0\} \to [-1/2, 1/2], \quad G(z) := -1/z + [1/z].$$

Therefore for all $f \in \Gamma$ with $\mathcal{R}^{\mathrm{t}}_{NP}(f) \in \Gamma$, we have

$$\alpha_{\mathcal{R}^{\mathrm{t}}_{NP}(f)} = G(\alpha).$$

Assume f is a quadratic map that is both infinitely quadratic-like renormalisable of satellite type and infinitely near-parabolic renormalisable. Let $(\hat{p}_n/\hat{q}_n)_{n=1}^{\infty}$ denote its quadratic-like combinatorics and let $(\kappa_n)_{n=1}^{\infty}$ be the sequences associated to f as defined above. The map $\mathcal{R}_{QL}(f)$, is an infinitely quadratic-like renormalisable map of satellite type with combinatorics $(\hat{p}_n/\hat{q}_n)_{n=2}^{\infty}$. If $|\hat{p}_1| \neq 1$, then we have $\kappa_1 = \mathrm{t}$ and the map $\mathcal{R}_{NP}^{\mathrm{t}}(f)$ behaves similar to an infinitely quadratic-like renormalisable map of satellite type with combinatorics

$$(G(\hat{p}_1/\hat{q}_1), \hat{p}_2/\hat{q}_2, \hat{p}_3/\hat{q}_3, \cdots).$$

On the other hand, if $|\hat{p}_1| = 1$, then $\kappa_1 = \mathrm{b}$ and $\mathcal{R}_{NP}^{\mathrm{b}}(f)$ behaves similar to an infinitely quadratic-like renormalisable map of satellite type with combinatorics

$$(\hat{p}_2/\hat{q}_2, \hat{p}_3/\hat{q}_3, \hat{p}_4/\hat{q}_4, \cdots).$$

For any rational number p/q, the sequence $p/q, G(p/q), G^2(p/q), \cdots$ terminates in zero in finite number of steps. Assume $j \geq 1$ is the integer such that $G^j(\hat{p}_1/\hat{q}_1) = 0$. It follows that the first satellite renormalisation corresponds to $j-1$ top renormalisations and one bottom renormalisation. [5] In other words, near-parabolic renormalisation provides a way to split one satellite renormalisation into a finite sequence of near-parabolic renormalisations which leads to a finer analysis of such maps. In the following sections, we describe a topological model for the post-critical set of infinitely satellite renormalisable quadratics inspired by this powerful method.

[5]For any such map f, we may always apply a simple transformation so that we have $\kappa_1 = \mathrm{b}$. For simplicity, in the following sections, we always assume $\kappa_1 = \mathrm{b}$.

3. Condition $\mathcal{C}$

In this section, we will describe the combinatorial data that will be used in the construction of the topological model in Subsection 4.2. We will also define *condition* $\mathcal{C}$. This condition will determine the topology of the topological model for the post-critical set. We start by defining *basic* and *compound* sequences of rational numbers.

3.1. **Basic and compound sequences.** Let $\hat{\tau} = (\hat{p}_i/\hat{q}_i)_{i=1}^{\infty}$ be an arbitrary sequence of rational numbers in $(-1/2, 1/2]\backslash\{0\}$. Evidently, if $p/q \in \mathbb{Q}$, the iterates $p/q, G(p/q), G^2(p/q), ...$ terminate in 0, in a finite number of steps, where G is the modified Gauss map defined in Subsection 2.4. Using $\hat{\tau}$, we may generate a new sequence of non-zero rational numbers $(p_n/q_n)_{n=0}^{\infty}$ in the interval $[-1/2, 1/2]$. Let

$$\begin{aligned}\tau = (p_n/q_n)_{n=1}^{\infty} = \big(&\hat{p}_1/\hat{q}_1, G(\hat{p}_1/\hat{q}_1), G^2(\hat{p}_1/\hat{q}_1), \dots,\\ &\hat{p}_2/\hat{q}_2, G(\hat{p}_2/\hat{q}_2), G^2(\hat{p}_2/\hat{q}_2), \dots,\\ &\hat{p}_3/\hat{q}_3, G(\hat{p}_3/\hat{q}_3), G^2(\hat{p}_3/\hat{q}_3), \dots,\\ &, \dots)\end{aligned} \tag{1}$$

where in the above equation, the k-th line is a finite sequence of rational numbers starting with $\hat{p}_k/\hat{q}_k$ and stopping just before the iterate of G terminates in 0. To be more precise, for each $i \geq 1$, let j_i be the smallest positive integer with $G^{j_i}(\hat{p}_i/\hat{q}_i) = 0$. Define the integers

$$l_1 := 1, \quad l_i := 1 + \sum_{n=1}^{i-1} j_n, \text{ for } i > 1. \tag{2}$$

For every $i \geq 1$, we have

$$\frac{p_{l_i}}{q_{l_i}} = \frac{\hat{p}_i}{\hat{q}_i}$$

and for every $n \geq 1$, we have

$$G(p_n/q_n) = \begin{cases} 0 & \text{if } n = l_i - 1 \text{ for some } i > 1,\\ p_{n+1}/q_{n+1} & \text{otherwise.}\end{cases}$$

For example, if $\hat{p}_1/\hat{q}_1 = 8/27$ and $\hat{p}_2/\hat{q}_2 = 3/5$, then the sequence $(p_n/q_n)_{n=1}^{\infty}$ starts with $(8/27, -3/8, -1/3, 3/5, 1/3)$ and we have $l_1 = 1$ and $l_2 = 4$. Here we say that $\tau = \tau(\hat{\tau})$ is the *basic sequence* associated to $\hat{\tau}$ and $\hat{\tau} = \hat{\tau}(\tau)$ is the *compound sequence* associated to τ. For all $i \geq 1$, we will always assume that $\hat{q}_i > 0$ and $\gcd(\hat{p}_i, \hat{q}_i) = 1$. Similarly, for all $n \geq 1$, we assume $q_n > 0$ and $\gcd(p_n, q_n) = 1$.

Given a basic sequence $\tau = (p_n/q_n)_{n=1}^{\infty}$ and the associated sequence $(l_i)_{i\geq 1}$, for all $n \geq 1$, we define the sequence of symbols

$$\kappa_n := \begin{cases} \text{b} & \text{if } n = 1 \text{ or } G(\frac{p_{n-1}}{q_{n-1}}) = 0\\ \text{t} & \text{otherwise}\end{cases} = \begin{cases} \text{b} & \text{if } n = l_i \text{ for some } i \geq 1\\ \text{t} & \text{otherwise}\end{cases}.$$

We also define $\kappa_0 := \text{t}$. For any $n \geq 1$, we define the positive integer b_n as well as the signs $e_n, \varepsilon_n' \in \{-1, 1\}$ according to

$$b_n := \left|\left[\frac{q_n}{p_n}\right]\right| \geq 2, \quad e_n := \begin{cases} -1 & \text{if } \kappa_n = \text{b}\\ 1 & \text{if } \kappa_n = \text{t}\end{cases}, \quad \varepsilon_n' := e_n \operatorname{sgn}(p_n).$$

Since $q_n > 0$, for all $n \geq 1$, it follows that

$$\forall n \geq 1, \quad \left[\frac{q_n}{p_n}\right] = \operatorname{sgn}(p_n) b_n = e_n \varepsilon'_n b_n.$$

We also define

$$e_0 := 1, \quad \varepsilon'_0 := 1, \quad \forall n \geq 1, \quad \varepsilon_n := \varepsilon'_n \varepsilon'_{n-1}.$$

Lemma 3.1. *If $\kappa_{n+1} = \mathrm{t}$ for some $n \geq 1$, then*

- $q_{n+1} = |p_n| \leq q_n/2$,
- $q_n = b_n q_{n+1} - e_n \varepsilon'_n p_{n+1} = b_n q_{n+1} - e_n \varepsilon_{n+1} |p_{n+1}|$.

Proof. The condition $\kappa_{n+1} = \mathrm{t}$ implies $G(p_n/q_n) = p_{n+1}/q_{n+1}$ and $e_{n+1} = 1$. We also have $[q_n/p_n] = b_n \operatorname{sgn}(p_n)$ and $\operatorname{sgn}(p_n) = e_n \varepsilon'_n$. Therefore

$$\frac{p_{n+1}}{q_{n+1}} = G(\frac{p_n}{q_n}) = -\frac{q_n}{p_n} + \left[\frac{q_n}{p_n}\right] = \frac{-q_n \operatorname{sgn}(p_n) + b_n |p_n| \operatorname{sgn}(p_n)}{|p_n|}.$$

Since $\gcd(p_n, q_n) = \gcd(p_{n+1}, q_{n+1}) = 1$ and $|p_n/q_n| \leq 1/2$, we have

$$q_{n+1} = |p_n| \leq q_n/2, \quad p_{n+1} = -q_n \operatorname{sgn}(p_n) + b_n |p_n| \operatorname{sgn}(p_n).$$

Hence

$$q_n = b_n q_{n+1} - p_{n+1} \operatorname{sgn}(p_n) = b_n q_{n+1} - e_n \varepsilon'_n p_{n+1} = b_n q_{n+1} - e_n \varepsilon_{n+1} |p_{n+1}|. \qquad \square$$

3.2. **Condition $\mathcal{C}$.** To define condition $\mathcal{C}$, we first describe a one parameter family of functions on $\mathbb{R}$. For $r \in (0, 1/2]$, we define the function $g^r : \mathbb{R} \to [0, +\infty)$ as

$$g^r(x) := \begin{cases} \frac{1}{2\pi} \log(1 + 1/r) + rx - 1 & \text{if } x \geq 1/r, \\ \frac{1}{2\pi} \log(1 + |x|) & \text{if } |x| < 1/r, \\ \frac{1}{2\pi} \log(1 + 1/r) & \text{if } x \leq -1/r. \end{cases}$$

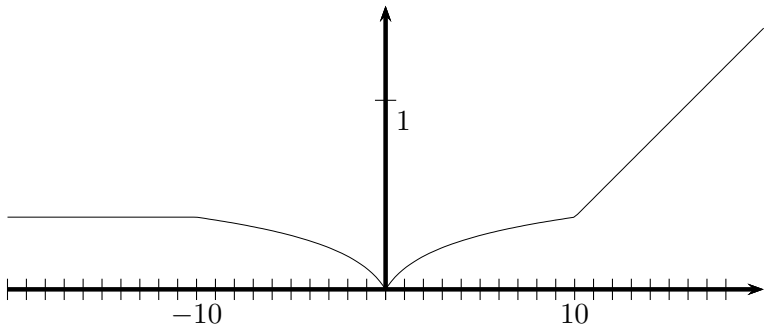


FIGURE 1. The graph of the map $g^r : \mathbb{R} \to [0, +\infty)$ for $r = 1/10$.

Lemma 3.2. *For every $r \in (0, 1/2]$, the function $g^r : \mathbb{R} \to [0, +\infty)$ is a continuous functions which is non-increasing on $(-\infty, 0]$ and increasing on $[0, \infty)$. Moreover, for all $x, y \in \mathbb{R}$ we have*

$$|g^r(x) - g^r(y)| \leq \frac{|x - y|}{2}.$$

Proof. Continuity follows from the definition. We have

$$(g^r)'(z) = \begin{cases} r & z > 1/r, \\ \frac{\operatorname{sgn}(z)}{2\pi(1+|z|)} & 0 < |z| < 1/r, \\ 0 & z < -1/r. \end{cases}$$

It follows that g^r is non-increasing on $(-\infty, 0]$ and increasing on $[0, \infty)$. It also follows that $|g_r'(z)| \leq 1/2$, for all $z \in \mathbb{R} \setminus \{-1/r, 0, 1/r\}$ which completes the proof. □

Definition 3.3. *Define $g_n : \mathbb{R} \to \mathbb{R}$, for $n \geq 1$, as*

$$g_n(x) := e_{n-1} g^{|p_n/q_n|}(x).$$

We say that $(\hat{p}_i/\hat{q}_i)_{i=1}^{\infty}$ satisfies condition $\mathcal{C}$, if

$$\lim_{i\to\infty} g_1 \circ g_2 \circ \cdots \circ g_{l_i}(|q_{l_i}/p_{l_i}|) = 0.$$

For $i \geq 1$, let

$$h_{l_i}^i := \left| \frac{q_{l_i}}{p_{l_i}} \right|,$$

and for all $n \geq 0$, $i \geq 1$ with $n < l_i$, let

$$h_n^i := g_{n+1} \circ g_{n+2} \circ \cdots \circ g_{l_i}(h_{l_i}^i) = g_{n+1} \circ g_{n+2} \circ \cdots \circ g_{l_i}\left(|q_{l_i}/p_{l_i}|\right).$$

It follows that

$$h_n^i = g_{n+1}(h_{n+1}^i)$$

whenever both sides are defined. Using Lemma 3.2, we see that $|h_n^i| \leq |h_{n+1}^i|/2$. Therefore we have

$$\limsup_{i\to\infty} |h_0^i| \leq \limsup_{i\to\infty} \frac{|h_{l_i}^i|}{2^{l_i}} = \limsup_{i\to\infty} \frac{|\hat{q}_i/\hat{p}_i|}{2^{l_i}}.$$

Hence, any sequence $(\hat{p}_i/\hat{q}_i)_{i=1}^{\infty}$ that does not satisfy the condition must have a fast convergence towards zero. In particular, any constant sequence satisfies condition $\mathcal{C}$. For an example of a sequence not satisfying this condition, we may consider the sequence

$$\frac{\hat{p}_1}{\hat{q}_1} := \frac{1}{2}, \quad \forall i \geq 1, \ \frac{\hat{p}_{i+1}}{\hat{q}_{i+1}} := \frac{1}{\lceil e^{2\pi \hat{q}_i/\hat{p}_i} \rceil}.$$

Also note that, for all $r \in (0, 1/2]$, the map g^r is continuous and we have $x = 0$ if and only if $g^r(x) = 0$. Therefore,

$$\lim_{i\to\infty} g_1 \circ g_2 \circ \cdots \circ g_{l_i}(|q_{l_i}/p_{l_i}|) = g_1 \circ g_2 \circ \cdots \circ g_{l_2-1}\left(\lim_{i\to\infty} g_{l_2} \circ g_{l_2+1} \circ \cdots \circ g_{l_i}(|q_{l_i}/p_{l_i}|)\right).$$

Hence, the sequence $(\hat{p}_i/\hat{q}_i)_{i=1}^{\infty}$ satisfies condition $\mathcal{C}$ if and only if $(\hat{p}_i/\hat{q}_i)_{i=2}^{\infty}$ satisfies this condition. In other words, condition $\mathcal{C}$ only depends on the tail of the sequence $(\hat{p}_i/\hat{q}_i)_{i=1}^{\infty}$.

Remark 1. The topological model is constructed as the intersection of a shrinking sequence of open sets. The $(i+1)$-th open set in this sequence corresponds to the i-th satellite renormalisation and contains $\hat{q}_1\hat{q}_2 \cdots \hat{q}_i$ connected components. These connected components correspond to the sets $\mathcal{PC}(P_c)(t)$, for $t \in \mathbb{Z}/(\hat{q}_1\hat{q}_2 \cdots \hat{q}_i)$ as described in Subsection 2.3. In Corollary 4.26, we show that there is a connected component that is somewhat the "largest". This connected component corresponds to the set $\mathcal{PC}(P_c)(0 \pmod{\hat{q}_1\hat{q}_2 \cdots \hat{q}_i})$, which contains the critical value. As we will see in Proposition 4.21, the number $|h_0^i|$ estimates the diameter of this connected component.(See Figure 4.2) For an infinitely satellite renormalisable quadratic map, $|h_0^i|$ corresponds to $\log(1+d_i)$, where d_i is the distance between the critical value and the β-fixed point of the little Julia set of level i that contains the critical value.

Remark 2. Condition $\mathcal{C}$ is closely related to the Herman condition for irrational numbers. The set of irrational numbers of Herman type were discovered in the study of the linearisation problem for analytic circle maps, [Her79, Yoc84]. As we described in Subsection 2.4, infinitely top near-parabolic renormalisable maps have an indifferent fixed point. By analysing these maps, Cheraghi showed that the Brjuno and Herman condition determine the topology of the post-critical set of a wide class of quadratic polynomial with indifferent fixed points,[Che17]. The maps $g^r|_{[0,\infty)}$ and $g^r|_{(-\infty,0]}$ correspond to top and bottom renormalisations respectively. The sign $e_{n-1} \in \{1,-1\}$ in the definition of g_n ensures that we use $g^r|_{(-\infty,0]}$ for $n \geq 1$ that corresponds to a bottom renormalisation. In fact, the Herman condition may be defined using the family of maps $(g^r|_{[0,\infty)})_{r\in(0,1/2]}$. One may consult [Che17] for more details.

The following lemma describes an equivalent formulation for condition $\mathcal{C}$.

Lemma 3.4. *The sequence $(\hat{p}_i/\hat{q}_i)_{i=1}^{\infty}$ satisfies condition $\mathcal{C}$, if and only if there is a constant $C > 0$ such that for all $N > 0$, there are integers $m, k > 0$ such that $N < m < l_k$ and*

$$|h_m^k| < C.$$

Proof. First we show that $|h_n^{i+1}| \leq |h_n^i|$ for all $n \geq 0, i \geq 1$ such that $n \leq l_i$. Note that if x and y have the same sign and $|x| \leq |y|$, then $g_n(x)$ and $g_n(y)$ have the same sign and $|g_n(x)| \leq |g_n(y)|$. Fix $n \geq 0$, $i \geq 1$ and let

$$x := h_{l_i}^{i+1} = g_{l_i+1} \circ g_{l_i+2} \circ \cdots g_{l_{i+1}} \left(\left|q_{l_{i+1}}/p_{l_{i+1}}\right|\right).$$

Since $e_{l_i} = -1$, we have $x \leq 0$. Choose a real number $y < \min(x, -|q_{l_i}/p_{l_i}|) < 0$. Then

$$h_n^i = g_{n+1} \circ g_{n+2} \circ ... \circ g_{l_i}(|q_{l_i}/p_{l_i}|) = g_{n+1} \circ g_{n+2} \circ ... \circ g_{l_i}(y)$$

and

$$h_n^{i+1} = g_{n+1} \circ g_{n+2} \circ ... \circ g_{l_i}(x)$$

have the same sign and $|h_n^{i+1}| \leq |h_n^i|$.

Assume $C > 1$ and $k \geq 1, m \geq 0$ such that $m < l_k$ and $|h_m^k| < C$. We know that $h_n^i = g_{n+1}(h_{n+1}^i)$, whenever both sides are defined. Therefore Lemma 3.2 implies that $|h_0^k| < \frac{C}{2^m}$. Hence

$$\limsup_{i\to\infty} |h_0^i| \leq |h_0^k| < \frac{C}{2^m},$$

which shows that the condition in the lemma implies condition $\mathcal{C}$. On the other hand assume condition $\mathcal{C}$ holds and choose $C = 1$. Then for all $m \geq 1$, we have

$$\lim_{i\to\infty} g_1 \circ \cdots \circ g_m(h_m^i) = 0.$$

Therefore $\lim_{i\to\infty} h_m^i = 0$ for all $m \geq 1$. So for all $m \geq 1$, there is $k \geq 1$ such that $m < l_k$ and

$$|h_m^k| < 1. \qquad \square$$

The following lemma states a useful property of the maps g^r that we will need later.

Lemma 3.5. *For all $r \in (0, 1/2]$, $0 \leq h \leq x \leq y$ and $e \in \{1,-1\}$, we have*

$$0 \leq g^r(ey) - g^r(ey - eh) < g^r(ex) - g^r(ex - eh) + 1.$$

Proof. Define the function $f : [h, \infty) \to \mathbb{R}$ by

$$\forall z \in [h, \infty), \quad f(z) := g^r(ez) - g^r(ez - eh).$$

We need to show that $0 \le f(y) < f(x) + 1$. Since g^r is increasing on $[0, \infty)$ and non-increasing on $(-\infty, 0]$, we have $f(y) \ge 0$. The function f is piecewise smooth, so we have

$$f(y) - f(x) = \int_x^y f'(t)dt.$$

If $e = -1$, direct calculation shows that $f'(z) \le 0$ for all $z \in (h, \infty) \setminus \{1/r, h + 1/r\}$, which implies $f(y) \le f(x) < f(x) + 1$. Let us assume that $e = 1$. Then for all $z \in (h, \infty) \setminus \{1/r, h + 1/r\}$, we have

$$f'(z) = \begin{cases} 0 & z - h > 1/r, \\ r - \frac{1}{2\pi(1+z-h)} & z > 1/r > z - h, \\ \frac{1}{2\pi(1+z)} - \frac{1}{2\pi(1+z-h)} & z < 1/r. \end{cases}$$

If $f'(z) > 0$, then $z > 1/r > z - h > 0$ and $r > \frac{1}{2\pi(1+z-h)}$. Hence we have

$$\max(1/r, h, h + 1/(2\pi r) - 1) \le z \le h + 1/r.$$

Let $a := \max(1/r, h, h + 1/(2\pi r) - 1)$ and $b := h + 1/r$. We have $f(b) = rh$ and

$$f(y) - f(x) = \int_x^y f'(t)dt \le \int_a^b f'(t)dt = f(b) - f(a)$$

If $a = h \ge 1/r$, then $f(a) = g^r(a)$ and

$$f(b) - f(a) = rh - \left(\frac{1}{2\pi} \log(1 + 1/r) + rh - 1 \right) < 1.$$

If $a = 1/r > h$, then we have

$$f(b) - f(a) \le f(b) = rh < 1.$$

Lastly, if $a = h + 1/(2\pi r) - 1 > h$, then we have $2\pi r < 1$. Therefore

$$\begin{aligned} f(a) &= g^r(a) - g^r(a - h) = \frac{1}{2\pi} \log(1 + 1/r) + ra - 1 + \frac{1}{2\pi} \log(2\pi r) \\ &= \frac{1}{2\pi} \log(2\pi(1 + r)) + rh + \frac{1}{2\pi} - r - 1 > rh + \frac{1}{2\pi} - r - 1 > rh - 1 = f(b) - 1. \end{aligned}$$

□

4. Arithmetic topological model for the post-critical set

In this section we build a topological model for the post-critical set of infinitely satellite-renormalisable quadratic polynomials and study the topological features of this model. Let $\hat{\tau} = (\hat{p}_i/\hat{q}_i)_{i=1}^{\infty}$ be a sequence of rational numbers in $(-1/2, 1/2] \setminus \{0\}$. As described in Subsection 3.1, we may define

$$(p_i/q_i)_{i=1}^{\infty} := \tau(\hat{\tau}),$$

and the associated sequences $(\kappa_n)_{n=0}^{\infty}$, $(e_n)_{n=0}^{\infty}$, $(\varepsilon_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ and $(l_n)_{n=1}^{\infty}$. In Subsection 4.2, we use these sequences, together with the maps defined in Subsection 4.1, to defined the topological model.

4.1. The maps $T^{\mathrm{t},r}$ and $T^{\mathrm{b},r}$.

Here we define and analyse the maps that we will later use to construct our topological model. These maps model the action of the maps between consecutive levels in the renormalisation tower of an infinitely near-parabolic renormalisable map. These maps are chosen so that they satisfy appropriate covering properties and quantitative behaviours. These properties will be needed to construct a model that reflects the complexity of the post-critical set while remaining as simple as possible. Define the sets

$$\mathbb{H}^{\mathrm{t}} = \{w \in \mathbb{C} \mid \operatorname{Im}(w) > -2\} \quad \text{and} \quad \mathbb{H}^{\mathrm{b}} = \{w \in \mathbb{C} \mid \operatorname{Im}(w) < 2\}.$$

For $r \in (0, 1/2]$, we consider the continuous maps

$$T^{\mathrm{t},r} : \mathbb{H}^{\mathrm{t}} \to \mathbb{C} \quad \text{and} \quad T^{\mathrm{b},r} : \mathbb{H}^{\mathrm{b}} \to \mathbb{C}$$

given by the formulae

$$T^{\mathrm{t},r}(w) = r \operatorname{Re}(w) + \frac{1}{2\pi i} \log \left| \frac{r}{1 - e^{-2\pi i r(w+4i)}} \right| + C_1^{\mathrm{t}}(r)$$

and

$$T^{\mathrm{b},r}(w) = \frac{1}{2\pi i} \log \left(\frac{r}{1 - e^{-2\pi i r(w-4i)}} \right) + C_1^{\mathrm{b}}(r).$$

Here log denotes the principal branch of logarithm and the numbers $C_1^{\mathrm{t}}(r)$ and $C_1^{\mathrm{b}}(r)$ are chosen such that we have

$$T^{\mathrm{t},r}(0) = 0 \quad \text{and} \quad T^{\mathrm{b},r}(0) = 1/2.$$

In Lemma 4.1 and Lemma 4.4, we will show that these maps are well-defined. See Figures 4.1 and 4.1 for a visualisation of these maps.

Lemma 4.1. *The following statements hold for every $r \in (0, 1/2]$.*

(i) The map $T^{\mathrm{t},r} : \mathbb{H}^{\mathrm{t}} \to T^{\mathrm{t},r}(\mathbb{H}^{\mathrm{t}})$ is a well-defined C^{∞}-diffeomorphism. Moreover, for all $w \in \mathbb{H}^{\mathrm{t}}$, we have

$$T^{\mathrm{t},r}(w + 1/r) = T^{\mathrm{t},r}(w) + 1.$$

(ii) The map $T^{\mathrm{t},r}$ sends vertical lines into vertical lines. Moreover, for any $x \in \mathbb{R}$, the map $y \mapsto \operatorname{Im}(T^{\mathrm{t},r}(x + iy)) : (-2, \infty) \to \mathbb{R}$ is increasing.

(iii) We have $|C_1^{\mathrm{t}}(r)| < 4$.

(iv) The image, $T^{\mathrm{t},r}(\mathbb{H}^{\mathrm{t}})$, is compactly contained in $\mathbb{H}^{\mathrm{t}}$.

(v) For all $w \in \mathbb{H}^{\mathrm{t}}$, we have $\operatorname{Im}(T^{\mathrm{t},r}(i \operatorname{Im}(w))) \leq \operatorname{Im}(T^{\mathrm{t},r}(w))$.

Proof. **(i, ii):** To show that $T^{\mathrm{t},r}$ is well-defined, we only need to show that $e^{-2\pi i r(w+4i)} \neq 1$, which is true since $-4i \notin \mathbb{H}^{\mathrm{t}}$. In fact, $T^{\mathrm{t},r}$ may be defined on the set $\overline{\mathbb{H}^{\mathrm{t}}}$. From the definition we see that $T^{\mathrm{t},r}$ is C^{∞} smooth and

$$T^{\mathrm{t},r}(w + 1/r) = T^{\mathrm{t},r}(w) + 1, \quad \operatorname{Re}(T^{\mathrm{t},r}(w)) = r \operatorname{Re}(w),$$

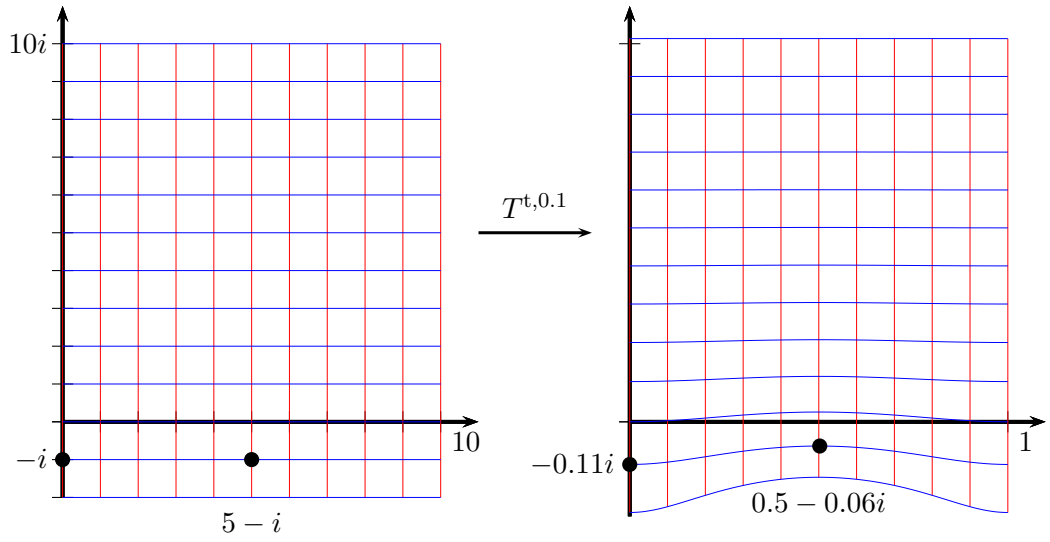


FIGURE 2. A visualisation of the map $T^{\mathrm{t},r}$ for $r = 0.1$. By applying Lemma 4.1(v) to the point $w = 5 - i$, we obtain

$$-0.11 \approx \operatorname{Im}(T^{\mathrm{t},r}(-i)) \leq \operatorname{Im}(T^{\mathrm{t},r}(5-i)) \approx -0.06.$$

for all $w \in \overline{\mathbb{H}^{\mathrm{t}}}$. Therefore $T^{\mathrm{t},r}$ maps vertical lines into vertical lines. Next, we prove injectivity on $\overline{\mathbb{H}^{\mathrm{t}}}$. Assume $T^{\mathrm{t},r}(w) = T^{\mathrm{t},r}(w')$. Then we have

$$r \operatorname{Re}(w) = \operatorname{Re}(T^{\mathrm{t},r}(w)) = \operatorname{Re}(T^{\mathrm{t},r}(w')) = r \operatorname{Re}(w')$$

and

$$\left|1 - e^{-2\pi i r(w+4i)}\right| = \left|1 - e^{-2\pi i r(w'+4i)}\right|.$$

If $w = x + iy$ and $w' = x + iy'$, then

$$\left|1 - e^{-2\pi i r(w+4i)}\right| = \left|1 - e^{-2\pi i r(x+iy+4i)}\right| = \left|1 - e^{-2\pi i r x}.e^{2\pi r(y+4)}\right|.$$

Hence we may define

$$d := \left|e^{2\pi i r x} - e^{2\pi r(y+4)}\right| = \left|e^{2\pi i r x} - e^{2\pi r(y'+4)}\right|.$$

The circle of radius d centred at the point $e^{2\pi i r x}$ in the unit circle intersects $[1, \infty)$ in at most one point. Therefore we have $y = y'$. Now consider $T^{\mathrm{t},r}$ as a map from $\overline{\mathbb{H}^{\mathrm{t}}} \cup \{\infty\} \subseteq \hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$ that sends ∞ into ∞. This is a continuous injective map defined on a compact set. Therefore it is a homeomorphism onto its image. Hence $T^{\mathrm{t},r} : \mathbb{H}^{\mathrm{t}} \to T^{\mathrm{t},r}(\mathbb{H}^{\mathrm{t}})$ is a C^∞-diffeomorphism.

Since $\operatorname{Re}(T^{\mathrm{t},r}(w)) = r \operatorname{Re}(w)$ and $T^{\mathrm{t},r}$ is injective, the map $y \mapsto \operatorname{Im}(T^{\mathrm{t},r}(x+iy))$ is also injective, for all $x \in \mathbb{R}$. Therefore it is monotone. We have $\lim_{\operatorname{Im}(w)\to\infty} \operatorname{Im}(T^{\mathrm{t},r}(w)) = \infty$, so it is in fact increasing.

(iii): Since $T^{\mathrm{t},r}(0) = 0$, we have

$$C_1^{\mathrm{t}}(r) = \frac{1}{2\pi i} \log\left(\frac{e^{8\pi r} - 1}{r}\right).$$

The expression $(e^{8\pi r}-1)/r$ is increasing as a function of r for $r \in (0,\infty)$. Therefore

$$|C_1^{\mathrm{t}}(r)| \le \frac{1}{2\pi}\log\left(\frac{e^{4\pi}-1}{1/2}\right) \le \frac{1}{2\pi}\log(e^{8\pi}-1) < 4.$$

(iv): In order to prove $\overline{T^{\mathrm{t},r}(\mathbb{H}^{\mathrm{t}})} \subseteq \mathbb{H}^{\mathrm{t}}$, it is sufficient to show that $\operatorname{Im}(T^{\mathrm{t},r}(w)) > -3/2$. Let $w = x+iy \in \mathbb{H}^{\mathrm{t}}$. Since $C_1^{\mathrm{t}}(r)$ is chosen so that $T^{\mathrm{t},r}(0)=0$, we have

$$\operatorname{Im}(T^{\mathrm{t},r}(w)) = \operatorname{Im}(T^{\mathrm{t},r}(w)-T^{\mathrm{t},r}(0)) = \frac{1}{2\pi}\left(\log\left|1-e^{-2\pi i r(w+4i)}\right| - \log\left|1-e^{8\pi r}\right|\right).$$

Since $\operatorname{Im}(w) \ge -2$, we have

$$\left|1-e^{-2\pi i r(w+4i)}\right| \ge \left|e^{-2\pi i r(w+4i)}\right| - 1 \ge e^{8\pi r + 2\pi r\operatorname{Im}(w)} - 1 \ge e^{4\pi r}-1.$$

Therefore

$$\begin{aligned}\operatorname{Im}(T^{\mathrm{t},r}(w)) &\ge \frac{1}{2\pi}\left(\log\left(e^{4\pi r}-1\right) - \log\left(e^{8\pi r}-1\right)\right) = \frac{-1}{2\pi}\log\left(e^{4\pi r}+1\right)\\ &\ge \frac{-1}{2\pi}\log\left(e^{2\pi}+1\right) > -\frac{3}{2}.\end{aligned}$$

(v): Let $w' := i\operatorname{Im}(w)$. We have

$$\left|1-e^{-2\pi i r(w+4i)}\right| \ge \left|e^{-2\pi i r(w+4i)}\right| - 1 = e^{-2\pi i r(w'+4i)} - 1.$$

Hence

$$\begin{aligned}\operatorname{Im}(T^{\mathrm{t},r}(w)) &= \frac{1}{2\pi}\log\left|1-e^{-2\pi i r(w+4i)}\right| - \frac{1}{2\pi}\log\left|1-e^{8\pi r}\right|\\ &\ge \frac{1}{2\pi}\log\left|1-e^{-2\pi i r(w'+4i)}\right| - \frac{1}{2\pi}\log\left|1-e^{8\pi r}\right| = \operatorname{Im}(T^{\mathrm{t},r}(w')).\end{aligned}$$ □

Next we show that $T^{\mathrm{t},r}$ is uniformly contracting and the least vertical contraction of $T^{\mathrm{t},r}(w)$ occurs along the line $\operatorname{Re}(w)=0$.

Lemma 4.2. *The following statements hold for every $r \in (0,1/2]$.*

(i) For all $w, w' \in \mathbb{H}^{\mathrm{t}}$, we have $|T^{\mathrm{t},r}(w) - T^{\mathrm{t},r}(w')| \le \frac{3}{4}|w-w'|$.

(ii) For all $w, w' \in \mathbb{H}^{\mathrm{t}}$ such that $\operatorname{Re}(w) = \operatorname{Re}(w')$, we have

$$|T^{\mathrm{t},r}(i\operatorname{Im}(w)) - T^{\mathrm{t},r}(i\operatorname{Im}(w'))| \ge |T^{\mathrm{t},r}(w) - T^{\mathrm{t},r}(w')|.$$

Proof. **(i):** To prove uniform contraction, let $w = x+iy \in \mathbb{H}^{\mathrm{t}}$. Clearly

$$\left|\frac{\partial}{\partial x}(T^{\mathrm{t},r})(w)\right| = |r| \le \frac{1}{2}.$$

On the other hand,

$$\begin{aligned}\frac{\partial}{\partial y}(T^{\mathrm{t},r})(w) = \frac{\partial}{\partial y}\left(\frac{1}{2\pi i}\log\left|\frac{r}{1-e^{-2\pi i r(w+4i)}}\right|\right) &= \frac{\partial}{\partial y}\left(\frac{1}{2\pi i}\operatorname{Re}\left(\log\left(\frac{r}{1-e^{-2\pi i r(w+4i)}}\right)\right)\right)\\ &= \operatorname{Im}\left(\frac{\partial}{\partial y}\left(\frac{1}{2\pi i}\log\left(\frac{r}{1-e^{-2\pi i r(w+4i)}}\right)\right)\right).\end{aligned}$$

Note that the value inside log is never zero. Therefore a branch of log is locally well-defined up to an additive constant. For holomorphic maps, we have $\frac{\partial}{\partial y} = i\frac{\partial}{\partial w}$. Therefore

$$\frac{\partial}{\partial y}(T^{\mathrm{t},r})(w) = \operatorname{Im}\left(\frac{\partial}{\partial w}\left(\frac{1}{2\pi}\log\left(\frac{r}{1-e^{-2\pi ir(w+4i)}}\right)\right)\right) = \frac{-1}{2\pi}\operatorname{Im}\left(\frac{\partial}{\partial w}\left(\log\left(1-e^{-2\pi ir(w+4i)}\right)\right)\right)$$
$$= \operatorname{Im}\left(\frac{-ire^{-2\pi ir(w+4i)}}{1-e^{-2\pi ir(w+4i)}}\right) = \operatorname{Im}\left(ir - \frac{ir}{1-e^{-2\pi ir(w+4i)}}\right) = r + \operatorname{Re}\left(\frac{r}{e^{-2\pi ir(w+4i)}-1}\right).$$

Since $|e^{-2\pi ir(w+4i)}| > e^{4\pi r}$, we have $|e^{-2\pi ir(w+4i)} - 1| \geq e^{4\pi r} - 1$. Therefore

$$\left|\frac{\partial}{\partial y}(T^{\mathrm{t},r})(w)\right| \leq r + \frac{r}{|e^{-2\pi ir(w+4i)}-1|} \leq r + \frac{r}{e^{4\pi r}-1} \leq r + \frac{1}{4\pi} < \frac{3}{4}.$$

These bounds on partial derivatives imply

$$\forall w, w' \in \mathbb{H}^{\mathrm{t}}, \quad |T^{\mathrm{t},r}(w) - T^{\mathrm{t},r}(w')| \leq \frac{3}{4}|w - w'|.$$

(ii): Let $w \in \mathbb{H}^{\mathrm{t}}$ and $w'' := i\operatorname{Im}(w)$. The point $e^{-2\pi ir(w+4i)} - 1$ belongs to the circle of radius $e^{-2\pi ir(w''+4i)} > 1$ centred at -1. Therefore $1/(e^{-2\pi ir(w+4i)}-1)$ belongs to a circle that is bounded between the vertical lines $\operatorname{Re}(w) = 1/(e^{-2\pi ir(w''+4i)}-1) > 0$ and $\operatorname{Re}(w) = 1/(-e^{-2\pi ir(w''+4i)}-1) < 0$. Hence we have

$$\operatorname{Re}\left(\frac{r}{e^{-2\pi ir(w+4i)}-1}\right) \leq \operatorname{Re}\left(\frac{r}{e^{-2\pi ir(w''+4i)}-1}\right).$$

Therefore, according to the formula for $\frac{\partial}{\partial y}(T^{\mathrm{t},r})(w)$, we have

$$\frac{\partial}{\partial y}(T^{\mathrm{t},r})(w) \leq \frac{\partial}{\partial y}(T^{\mathrm{t},r})(i\operatorname{Im}(w)),$$

for all $w \in \mathbb{H}^{\mathrm{t}}$. Lemma 4.1(ii) implies that these derivatives are non-negative. Hence, we have

$$0 \leq T^{\mathrm{t},r}(w) - T^{\mathrm{t},r}(w') \leq T^{\mathrm{t},r}(i\operatorname{Im}(w)) - T^{\mathrm{t},r}(i\operatorname{Im}(w')),$$

for all $w, w' \in \mathbb{H}^{\mathrm{t}}$ where $\operatorname{Re}(w) = \operatorname{Re}(w')$ and $\operatorname{Im}(w) \geq \operatorname{Im}(w')$. □

We define the map $\tilde{T}^{\mathrm{t},r} : \mathbb{H}^{\mathrm{t}} \to \mathbb{C}$ for all $r \in (0, 1/2]$ by

$$\forall w \in \mathbb{H}^{\mathrm{t}}, \quad \tilde{T}^{\mathrm{t},r}(w) := \begin{cases} \frac{i}{2\pi}\log(d(w+4i, \mathbb{Z}/r)) + r\operatorname{Re}(w) & \text{if } -2 < \operatorname{Im}(w) \leq 1/r, \\ \frac{i}{2\pi}\log(1/r) + rw & \text{if } 1/r < \operatorname{Im}(w). \end{cases}$$

The map $\tilde{T}^{\mathrm{t},r}$ is not continuous and can not be used to construct the model. However, as the following lemma shows, it provides a uniform estimate for $T^{\mathrm{t},r}$ that will simplify our calculations.

Lemma 4.3. *The map $\tilde{T}^{\mathrm{t},r}$ is uniformly close to $T^{\mathrm{t},r}$. In other words, there is a constant C_2^{t}, independent of w and r, such that*

$$\forall w \in \mathbb{H}^{\mathrm{t}}, \quad |\tilde{T}^{\mathrm{t},r}(w) - T^{\mathrm{t},r}(w)| < C_2^{\mathrm{t}}.$$

Proof. We have $|C_1^{\mathrm{t}}(r)| < 4$. Therefore,

$$\left|T^{\mathrm{t},r}(w) - \left(\frac{i}{2\pi}\log(d(w+4i, \mathbb{Z}/r)) + r\operatorname{Re}(w)\right)\right| \leq \frac{1}{2\pi}\left|\log\left|\frac{r.d(w+4i, \mathbb{Z}/r)}{1-e^{-2\pi ir(w+4i)}}\right|\right| + 4.$$

The right hand side is periodic in w with period $1/r$, so we restrict ourselves to the set

$$\{w \in \mathbb{C} \mid |\operatorname{Re}(w)| \leq 1/2r, \operatorname{Im}(w) \geq -2\}$$

and therefore replace $d(w+4i,\mathbb{Z}/r)$ with $|w+4i|$. We have

$$\frac{1}{2\pi}\left|\log\left|\frac{r(w+4i)}{1-e^{-2\pi ir(w+4i)}}\right|\right| \le \frac{1}{2\pi}\left|\log\left|\frac{-2\pi ir(w+4i)}{1-e^{-2\pi ir(w+4i)}}\right|\right| + \frac{1}{2\pi}\log(2\pi).$$

If we define $z=-2\pi ir(w+4i)$, for all $w\in[-1/2r,1/2r]\times i(-2,1/r]$, we have

$$0<4\pi r\le \operatorname{Re}(z)=2\pi r(\operatorname{Im}(w)+4)\le 2\pi+8\pi r\le 6\pi, \quad |\operatorname{Im}(z)|\le\pi.$$

Therefore, for $w\in[-1/2r,1/2r]\times i(-2,1/r]$, we have

$$\left|T^{\mathrm{t},r}(w)-\left(\frac{i}{2\pi}\log(d(w+4i,\mathbb{Z}/r))+r\operatorname{Re}(w)\right)\right| \le \frac{1}{2\pi}\sup\left\{\left|\log\left|\frac{z}{1-e^z}\right|\right|\right\}+\frac{1}{2\pi}\log(2\pi)+4,$$

where supremum is taken over z in $[0,6\pi]\times i[-\pi,\pi]$. It is straightforward to see that $|\log|\frac{z}{1-e^z}||$ is uniformly bounded from above over this set.

To work with the case $\operatorname{Im}(w)\ge 1/r$, first note that

$$\frac{1}{2\pi i}\log\left|e^{-2\pi ir(w+4i)}\right| = \frac{1}{2\pi i}\log\left(e^{-2\pi ir(i\operatorname{Im}(w)+4i)}\right) = -r(i\operatorname{Im}(w)+4i)\in i\mathbb{R}.$$

On the other hand

$$\begin{aligned}
\left|T^{\mathrm{t},r}(w)-\left(\frac{i}{2\pi}\log(1/r)+rw\right)\right| &\le \left|\frac{1}{2\pi i}\log\left|\frac{1}{1-e^{-2\pi ir(w+4i)}}\right| - ri\operatorname{Im}(w)\right| + 4\\
&\le \left|\frac{1}{2\pi i}\log\left|\frac{1}{1-e^{-2\pi ir(w+4i)}}\right| + \frac{1}{2\pi i}\log\left|e^{-2\pi ir(w+4i)}\right|\right| + 4r+4\\
&\le \left|\frac{1}{2\pi i}\log\left|\frac{e^{-2\pi ir(w+4i)}}{1-e^{-2\pi ir(w+4i)}}\right|\right| + 6\\
&= \frac{1}{2\pi}\left|\log\left|1-e^{2\pi ir(w+4i)}\right|\right| + 6.
\end{aligned}$$

For $\operatorname{Im}(w)\ge 1/r$,

$$\left|e^{2\pi ir(w+4i)}\right| = e^{-2\pi r(\operatorname{Im}(w)+4)} \le e^{-2\pi-8\pi r} < e^{-2\pi}.$$

Therefore the value inside the log is within the interval $[1-e^{-2\pi},1+e^{-2\pi}]$. Hence the expression is uniformly bounded from above, independent of w and r. □

Lemma 4.4. *The following statements hold for every $r\in(0,1/2]$.*

(i) The map $T^{\mathrm{b},r}:\mathbb{H}^{\mathrm{b}}\to T^{\mathrm{b},r}(\mathbb{H}^{\mathrm{b}})$ is a well-defined holomorphic covering map, with the deck transformation group generated by the translation $w\mapsto w+1/r$. Moreover, the set $T^{\mathrm{b},r}(\mathbb{H}^{\mathrm{b}})$ is a topological disk with a single point p removed, where

$$p=\lim_{\operatorname{Im}(w)\to-\infty}T^{\mathrm{b},r}(w)=\frac{1}{2\pi i}\log(r)+C_1^{\mathrm{b}}(r).$$

(ii) The image, $T^{\mathrm{b},r}(\mathbb{H}^{\mathrm{b}})$, is compactly contained in

$$\{w\in\mathbb{H}^{\mathrm{t}} \mid 0<\operatorname{Re}(w)<1\}.$$

(iii) We have $|C_1^{\mathrm{b}}(r)|<2$.

(iv) For all $w\in\mathbb{H}^{\mathrm{b}}$, we have $\operatorname{Im}(T^{\mathrm{b},r}(i\operatorname{Im}(w)))\le\operatorname{Im}(T^{\mathrm{b},r}(w))$.

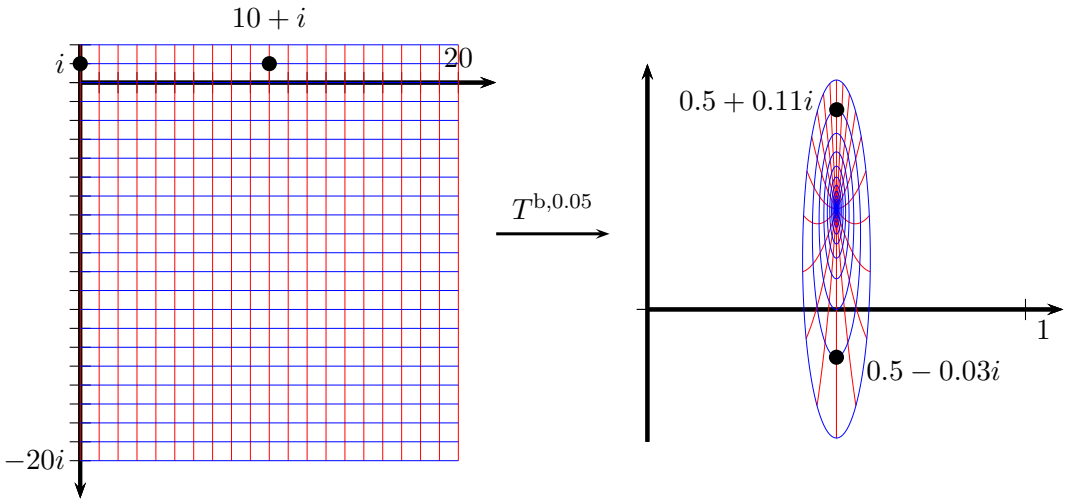


FIGURE 3. A visualisation of the map $T^{\mathrm{b},r}$ for $r = 0.05$. By applying Lemma 4.4(iv) to the point $w = 10 + i$, we obtain

$$-0.03 \approx \mathrm{Im}(T^{\mathrm{b},r}(i)) \leq \mathrm{Im}(T^{\mathrm{b},r}(10+i)) \approx 0.11.$$

Proof. **(i):** Define the maps

$$f_1 : \mathbb{C} \to \mathbb{C}, \quad f_1(w) = e^{-2\pi i r(w-4i)}, \quad f_2 : \mathbb{C} \setminus \{1\} \to \mathbb{C} \setminus \{0\}, \quad f_2(w) := \frac{r}{1-w}$$

and

$$f_3 : \{w \in \mathbb{C} \mid \mathrm{Re}(w) > 0\} \to (-1/4, 1/4) \times i\mathbb{R}, \quad f_3(w) := \frac{1}{2\pi i}\log(w),$$

where log denotes the principal branch of the logarithm. We have $f_1(\overline{\mathbb{H}^{\mathrm{b}}}) = \{w \in \mathbb{C} \mid 0 < |w| \leq e^{-4\pi r}\} \subseteq \mathbb{D} \subseteq \mathbb{C} \setminus \{1\}$. Therefore $f_1(\overline{\mathbb{H}^{\mathrm{b}}}) \subseteq \mathrm{Dom}(f_2)$. For all $w \in f_1(\overline{\mathbb{H}^{\mathrm{b}}})$, we have

$$\mathrm{Re}(1-w) = 1 - \mathrm{Re}(w) \geq 1 - |w| > 0,$$

which implies that $\mathrm{Re}(f_2(w)) > 0$. Hence $f_2(f_1(\overline{\mathbb{H}^{\mathrm{b}}})) \subseteq \mathrm{Dom}(f_3)$. In other words, the map $f := f_3 \circ f_2 \circ f_1$ is well-defined on $\overline{\mathbb{H}^{\mathrm{b}}}$. Clearly, for all $w \in \mathbb{H}^{\mathrm{b}}$,

$$C_1^{\mathrm{b}}(r) = -f(0) + 1/2, \quad T^{\mathrm{b},r}(w) = f(w) + C_1^{\mathrm{b}}(r).$$

The map $f_1|_{\mathbb{H}^{\mathrm{b}}} : \mathbb{H}^{\mathrm{b}} \to \{w \in \mathbb{C} \mid 0 < |w| < e^{-4\pi r}\}$ is a covering map, with deck transformation generated by the translation $w \mapsto w + 1/r$. The maps f_2 and f_3 are univalent on their respective domains. Hence $T^{\mathrm{b},r}$ is a covering map with the same deck transformation group.

The set $f_1(\mathbb{H}^{\mathrm{b}}) = \{w \in \mathbb{C} \mid 0 < |w| < e^{-4\pi r}\}$ is a topological disk with the point zero removed from its interior. Since f_2 and f_3 are univalent, the set $T^{\mathrm{b},r}(\mathbb{H}^{\mathrm{b}})$ is also a topological disk with a point p removed from its interior where

$$p = f_3(f_2(0)) + C_1^{\mathrm{b}}(r) = \lim_{\mathrm{Im}(w)\to-\infty} T^{\mathrm{b},r}(w) = \frac{1}{2\pi i}\log(r) + C_1^{\mathrm{b}}(r).$$

(ii, iii): We have $\{1 - f_1(w) \mid w \in \mathbb{H}^{\mathrm{b}}\} = D(1, e^{-4\pi r})$. [6] Hence

$$\forall w \in \mathbb{H}^{\mathrm{b}}, \quad |f_2(f_1(w))| < \frac{r}{1-e^{-4\pi r}} \leq 1.$$

[6] If $p \in \mathbb{C}$ and $r > 0$, we use $D(p, r)$ to denote the open ball of radius r centred at p.

The last inequality easily follows from the fact that $1 - r - e^{-4\pi r} \geq 0$ at $r = 0, 1/2$ and also at $r = \log(4\pi)/(4\pi)$ which is the only point where this expression has zero derivative. Therefore

$$\forall w \in \mathbb{H}^{\mathrm{b}}, \quad \mathrm{Im}(f(w)) = \frac{-1}{2\pi} \log |f_2(f_1(w))| > 0.$$

On the other hand, we have

$$f(0) = \frac{1}{2\pi i} \log \left(\frac{r}{1 - e^{-8\pi r}} \right) \in i\mathbb{R}.$$

The inequality $e^{-8\pi r} \geq 1 - 8\pi r$ implies that

$$\frac{r}{1 - e^{-8\pi r}} \geq \frac{1}{8\pi} > e^{-2\pi},$$

so $\mathrm{Im}(f(0)) \in (0, 1)$. Hence, for all $w \in \mathbb{H}^{\mathrm{b}}$, we have

$$\mathrm{Im}(T^{\mathrm{b},r}(w)) = \mathrm{Im}(f(w)) - \mathrm{Im}(f(0)) > -1$$

and

$$\mathrm{Re}(T^{\mathrm{b},r}(w)) = \mathrm{Re}(f(w)) + 1/2 \in (1/4, 3/4).$$

Therefore the set $T^{\mathrm{b},r}(\mathbb{H}^{\mathrm{b}})$ is compactly contained in $\{w \in \mathbb{H}^{\mathrm{t}} \mid 0 < \mathrm{Re}(w) < 1\}$. We also have

$$|C_1^{\mathrm{b}}(r)| = | - f(0) + 1/2| < 2.$$

(iv): For all $w \in \mathbb{H}^{\mathrm{b}}$, we have

$$\mathrm{Im}(T^{\mathrm{b},r}(w)) = \frac{1}{2\pi i} \log \left| \frac{r}{1 - e^{-2\pi i r(w-4i)}} \right| + \mathrm{Im}(C_1^{\mathrm{b}}(r)).$$

The rest of the proof follows from an argument similar to the proof of Lemma 4.1(v). □

Next we show that $T^{\mathrm{b},r}$ is uniformly contracting and the least vertical contraction of $T^{\mathrm{b},r}(w)$ occurs along the line $\mathrm{Re}(w) = 0$.

Lemma 4.5. *The following statements hold for every $r \in (0, 1/2]$.*

(i) For all $w, w' \in \mathbb{H}^{\mathrm{b}}$, we have $|T^{\mathrm{b},r}(w) - T^{\mathrm{b},r}(w')| \leq \frac{3}{4}|w - w'|$.

(ii) For all $w, w' \in \mathbb{H}^{\mathrm{b}}$ such that $\mathrm{Re}(w) = \mathrm{Re}(w')$, we have

$$|\,\mathrm{Im}(T^{\mathrm{b},r}(i\,\mathrm{Im}(w))) - \mathrm{Im}(T^{\mathrm{b},r}(i\,\mathrm{Im}(w')))| \geq |\,\mathrm{Im}(T^{\mathrm{b},r}(w)) - \mathrm{Im}(T^{\mathrm{b},r}(w'))|.$$

Proof. **(i):** First we compute the derivative.

$$\frac{\partial}{\partial w}(T^{\mathrm{b},r})(w) = \frac{-1}{2\pi i} \frac{\partial}{\partial w} \left(\log \left(1 - e^{-2\pi i r(w-4i)} \right) \right) = \frac{-r e^{-2\pi i r(w-4i)}}{1 - e^{-2\pi i r(w-4i)}} = \frac{r}{1 - e^{2\pi i r(w-4i)}}.$$

Therefore we have

$$\left| \frac{\partial}{\partial w}(T^{\mathrm{b},r})(w) \right| \leq \left| \frac{r}{1 - e^{4\pi r}} \right| \leq \frac{1}{4\pi} < \frac{3}{4},$$

which proves the uniform contraction.

(ii): Let $w = x + iy \in \mathbb{H}^{\mathrm{t}}$ and $w'' := iy$. For holomorphic maps, we have $\frac{\partial}{\partial y} = i\frac{\partial}{\partial w}$. Therefore

$$\mathrm{Im} \left(\frac{\partial}{\partial y}(T^{\mathrm{b},r})(w) \right) = \mathrm{Re} \left(\frac{\partial}{\partial w}(T^{\mathrm{b},r})(w) \right) = \mathrm{Re} \left(\frac{r}{1 - e^{2\pi i r(w-4i)}} \right).$$

The point $1 - e^{2\pi i r(w-4i)}$ belongs to a circle with radius $e^{2\pi i r(w''-4i)} > 1$ centred at 1. Therefore $1/(1 - e^{2\pi i r(w-4i)})$ belongs to a circle that is bounded between the vertical lines $\operatorname{Re}(w) = 1/(1 - e^{2\pi i r(w''-4i)})$ and $\operatorname{Re}(w) = 1/(1 + e^{2\pi i r(w''-4i)})$. We have

$$\frac{1}{1 - e^{2\pi i r(w''-4i)}} < 0 < \frac{1}{1 + e^{2\pi i r(w''-4i)}} < \frac{-1}{1 - e^{2\pi i r(w''-4i)}}.$$

Therefore

$$\left|\operatorname{Re}\left(\frac{r}{1 - e^{2\pi i r(w-4i)}}\right)\right| \leq -\operatorname{Re}\left(\frac{r}{1 - e^{2\pi i r(w''-4i)}}\right).$$

In other words,

$$\left|\operatorname{Im}\left(\frac{\partial}{\partial y}(T^{\mathrm{b},r})(w)\right)\right| \leq -\operatorname{Im}\left(\frac{\partial}{\partial y}(T^{\mathrm{b},r})(i\operatorname{Im}(w))\right),$$

for all $w \in \mathbb{H}^{\mathrm{b}}$. Now the inequality in the lemma follows from integration along vertical lines. □

To simplify calculations later on, we define the map $\tilde{T}^{\mathrm{b},r} : \mathbb{H}^{\mathrm{b}} \to \mathbb{C}$, for $r \in (0, 1/2]$, by

$$\forall w \in \mathbb{H}^{\mathrm{b}}, \quad \tilde{T}^{\mathrm{b},r}(w) = \begin{cases} \frac{i}{2\pi}\log(d(w - 4i, \mathbb{Z}/r)) & \text{if } -1/r \leq \operatorname{Im}(w) < 2, \\ \frac{i}{2\pi}\log(1/r) & \text{if } \operatorname{Im}(w) < -1/r. \end{cases}$$

Note that $\tilde{T}^{\mathrm{b},r}$ is not continuous.

Lemma 4.6. *The map $\tilde{T}^{\mathrm{b},r}$ is uniformly close to $T^{\mathrm{b},r}$. In other words, there is a constant C_2^{b}, independent of w and r, such that*

$$\forall w \in \mathbb{H}^{\mathrm{b}}, \quad |\tilde{T}^{\mathrm{b},r}(w) - T^{\mathrm{b},r}(w)| < C_2^{\mathrm{b}}.$$

Proof. First note that both $T^{\mathrm{b},r}$ and $\tilde{T}^{\mathrm{b},r}$ are periodic with period $1/r$. Hence we may restrict our attention to the set $\{w \in \mathbb{H}^{\mathrm{b}} \mid |\operatorname{Re}(w)| \leq 1/2r\}$. On this set we have $d(w-4i, \mathbb{Z}/r) = |w-4i|$. In Lemma 4.4(ii) and 4.4(iii), we saw that $\operatorname{Re}(T^{\mathrm{b},r}(w)) \in (0, 1)$ and $|C_1^{\mathrm{b}}(r)| < 2$. Therefore

$$\left|T^{\mathrm{b},r}(w) - i\operatorname{Im}\left(\frac{1}{2\pi i}\log\left(\frac{r}{1 - e^{-2\pi i r(w-4i)}}\right)\right)\right| = |\operatorname{Re}(T^{\mathrm{b},r}(w)) + \operatorname{Im}(C_1^{\mathrm{b}}(r))| < 3.$$

Hence

$$\begin{aligned}\left|T^{\mathrm{b},r}(w) - \frac{i}{2\pi}\log(d(w - 4i, \mathbb{Z}/r))\right| &\leq \left|\frac{1}{2\pi i}\log\left|\frac{r}{1 - e^{-2\pi i r(w-4i)}}\right| - \frac{i}{2\pi}\log(d(w - 4i, \mathbb{Z}/r))\right| + 3 \\ &= \frac{1}{2\pi}\left|\log\left|\frac{r|w - 4i|}{1 - e^{-2\pi i r(w-4i)}}\right|\right| + 3\end{aligned}$$

For $-1/r \leq \operatorname{Im}(w) < 2$, one may find a uniform upper bound for this expression as in the proof of Lemma 4.3.

On the other hand, we have

$$\begin{aligned}\left|T^{\mathrm{b},r}(w) - \frac{i}{2\pi}\log(1/r)\right| &\leq \left|i\operatorname{Im}\left(\frac{1}{2\pi i}\log\left(\frac{r}{1 - e^{-2\pi i r(w-4i)}}\right)\right) - \frac{i}{2\pi}\log(1/r)\right| + 3 \\ &= \left|\frac{1}{2\pi i}\log\left|1 - e^{-2\pi i r(w-4i)}\right|\right| + 3.\end{aligned}$$

Again, for $\operatorname{Im}(w) < -1/r$, this expression is uniformly bounded from above by an argument similar to the one for Lemma 4.3. □

Using the maps $\tilde{T}^{\mathrm{b},r}$, we see that $\operatorname{Im}(T^{\mathrm{b},r}(w))$ is "almost decreasing" on vertical lines.

Lemma 4.7. *Assume $r \in (0, 1/2]$ and $x \in \mathbb{R}$. If $y_1 \le y_2 < -2$, then*

$$\mathrm{Im}(T^{\mathrm{b},r}(x+iy_1)) > \mathrm{Im}(T^{\mathrm{b},r}(x+iy_2)) - (2C_2^{\mathrm{b}}+1), \quad \mathrm{Im}(\tilde{T}^{\mathrm{b},r}(x+iy_1)) > \mathrm{Im}(\tilde{T}^{\mathrm{b},r}(x+iy_2)) - 1.$$

Moreover, if $x = 0$, then $y \mapsto \mathrm{Im}(T^{\mathrm{b},r}(x+iy))$ is decreasing.

Proof. For a fixed $x \in \mathbb{R}$, the map $l_x : (-\infty, 2) \to \mathbb{R}$, defined by $l_x(y) := \mathrm{Im}(\tilde{T}^{\mathrm{b},r}(x+iy))$ is, by definition, decreasing in $[-1/r, 2)$. Moreover, for $y \in (-\infty, -1/r)$, we know that $l_x(y) = \frac{1}{2\pi}\log(1/r)$ is a constant. Now assume $y_1 < -1/r \le y_2 < 2$. The maximum value of l_x in $[-1/r, 2)$ is attained when $y = -1/r$. Hence we have

$$\begin{aligned} l_x(y_2) \le l_x(-1/r) &= \frac{1}{2\pi}\log(d(x+i(-1/r-4), \mathbb{Z}/r)) \\ &= \frac{1}{2\pi}\log(1/r) + \frac{1}{2\pi}\log(d(xr+i(-1-4r), \mathbb{Z})) < \frac{1}{2\pi}\log(1/r) + 1 = l_x(y_1) + 1. \end{aligned}$$

Using Lemma 4.6, we may approximate $T^{\mathrm{b},r}$ with $\tilde{T}^{\mathrm{b},r}$ with a uniform error smaller than C_2^{b} and obtain the result.

For the case $x = 0$, note that

$$\mathrm{Im}(T^{\mathrm{b},r}(iy)) = \frac{1}{2\pi}\log\left(\frac{1-e^{2\pi r(y-4)}}{r}\right) + \mathrm{Im}(C_1^{\mathrm{b}}(r)),$$

which is clearly decreasing. □

4.2. **The topological model.** Here we use the sequences described in Subsection 3.1 and the maps defined in Subsection 4.1 to define the topological model. For $n \ge 0$, define

$$\mathbb{H}_n := \mathbb{H}^{\kappa_n},$$

and for $n \ge 1$, define the function $s_n : \mathbb{C} \to \mathbb{C}$ by

$$s_n(z) := \begin{cases} z & \text{if } \varepsilon_n = +1, \\ \overline{z} & \text{if } \varepsilon_n = -1. \end{cases}$$

For $n \ge 1$, we also define the map $T_n : \mathbb{H}_n \to \mathbb{H}_{n-1}$ by

$$T_n(w) := \begin{cases} s_n(T^{\kappa_n, |p_n/q_n|}(w)) & \text{if } \varepsilon_n e_{n-1} = 1, \\ -s_n(T^{\kappa_n, |p_n/q_n|}(w)) + 1 & \text{if } \varepsilon_n e_{n-1} = -1. \end{cases}$$

Note that we use $T^{\mathrm{b},r}$ to define T_n exactly when $n = l_i$, for some $i \ge 1$. Also note that we have

$$\mathrm{Im}(T_n(w)) = \begin{cases} \varepsilon_n \,\mathrm{Im}(T^{\kappa_n, |p_n/q_n|}(w)) & \text{if } \varepsilon_n e_{n-1} = 1 \\ -\varepsilon_n \,\mathrm{Im}(T^{\kappa_n, |p_n/q_n|}(w)) & \text{if } \varepsilon_n e_{n-1} = -1 \end{cases} = e_{n-1}\,\mathrm{Im}(T^{\kappa_n, |p_n/q_n|}(w)),$$

and $T_n(0) \in \{0, 1/2, 1\}$.

Remark 3. The map T_n is orientation-preserving when $\varepsilon_n = 1$ and it is orientation-reversing otherwise. The corresponding map between consecutive levels in the renormalisation tower of an infinitely near-parabolic renormalisable map is either holomorphic or anti-holomorphic. Whether it is holomorphic or not is determined by ε_n.

To simplify the calculations, in analogy with the maps T_n, we also define

$$\tilde{T}_n(w) = \begin{cases} s_n(\tilde{T}^{\kappa_n, |p_n/q_n|}(w)) & \text{if } \varepsilon_n e_{n-1} = 1, \\ -s_n(\tilde{T}^{\kappa_n, |p_n/q_n|}(w)) + 1 & \text{if } \varepsilon_n e_{n-1} = -1, \end{cases}$$

for all $n \ge 1$.

We gather some of the most useful statements about these maps in the following lemma.

Lemma 4.8. *There is a constant $C_2 > 0$ such that for any $n \geq 1$, the following statements hold.*

(i) The map $T_n : \mathbb{H}_n \to \mathbb{H}_{n-1}$ is a C^∞-smooth map. Moreover, the restriction of the map T_n on the set $\{w \in \mathbb{H}_n \mid 0 < \mathrm{Re}(w) < |q_n/p_n|\}$ is a homeomorphism.

(ii) The image, $T_n(\mathbb{H}_n)$, is compactly contained in $\mathbb{H}_{n-1}$.

(iii) For all $w, w' \in \mathbb{H}_n$, we have $|T_n(w) - T_n(w')| \leq \frac{3}{4}|w - w'|$.

(iv) For all $w \in \mathbb{H}_n$, we have

$$e_{n-1}\,\mathrm{Im}(T_n(w)) \geq e_{n-1}\,\mathrm{Im}(T_n(i\,\mathrm{Im}(w))).$$

(v) For all $w, w' \in \mathbb{H}_n$ such that $\mathrm{Re}(w) = \mathrm{Re}(w')$, we have

$$|\,\mathrm{Im}(T_n(i\,\mathrm{Im}(w))) - \mathrm{Im}(T_n(i\,\mathrm{Im}(w')))| \geq |\,\mathrm{Im}(T_n(w)) - \mathrm{Im}(T_n(w'))|.$$

(vi) For all $w \in \mathbb{H}_n$, we have $|\tilde{T}_n(w) - T_n(w)| < C_2$.

(vii) For all $x \in \mathbb{R}$ and $-2 < y_1 < y_2$, we have

$$e_{n-1}\,\mathrm{Im}(T_n(x + ie_n y_1)) < e_{n-1}\,\mathrm{Im}(T_n(x + ie_n y_2)) + 2C_2 + 1.$$

Moreover, if $x = 0$, then this map is increasing.

(viii) For all $y \in \mathbb{R}$ with $iy \in \mathbb{H}_n$, we have

$$|\,\mathrm{Im}(T_n(iy)) - g_n(y)| < C_2 + 2.$$

Proof. Define $C_2 := \max(C_2^{\mathrm{t}}, C_2^{\mathrm{b}})$. Statements (i)-(vii) follow directly from the definition and Lemmas 4.1, 4.2, 4.3, 4.4, 4.5, 4.6 and 4.7. To prove (viii), first note that

$$\mathrm{Im}(\tilde{T}_n(iy)) = e_{n-1}\,\mathrm{Im}(\tilde{T}^{\kappa_n,|p_n/q_n|}(iy)), \quad g_n(y) = e_{n-1}g^{|p_n/q_n|}(y).$$

If $\kappa_n = \mathrm{t}$ and $iy \in \mathbb{H}_n \cap i\mathbb{R} = i(-2,\infty)$, then

$$\begin{aligned}
|\,\mathrm{Im}(T_n(iy)) - g_n(y)| &< C_2 + \left|\mathrm{Im}(\tilde{T}^{\mathrm{t},|p_n/q_n|}(iy)) - g^{|p_n/q_n|}(y)\right| \\
&= C_2 + \begin{cases} \left|\frac{1}{2\pi}\log(y+4) - \frac{1}{2\pi}\log(|y|+1)\right| & \text{if } -2 < y \leq |\frac{q_n}{p_n}| \\ \left|\frac{1}{2\pi}\log(|\frac{q_n}{p_n}|) - \frac{1}{2\pi}\log(|\frac{q_n}{p_n}|+1) + 1\right| & \text{if } y > |\frac{q_n}{p_n}| \end{cases} \\
&< C_2 + 2,
\end{aligned}$$

where the last inequality is elementary. The case $\kappa_n = \mathrm{b}$ is similar. □

Let $i \geq 1$ be an integer. Define

$$M_{l_i}^i := \left\{ w \in \mathbb{H}^{\mathrm{b}} \;\middle|\; 0 < \mathrm{Re}(w) < |q_{l_i}/p_{l_i}| \right\} \subseteq \mathbb{H}_{l_i}.$$

Recall that $b_n = \left|\left[\frac{q_n}{p_n}\right]\right| \geq 2$, for all $n \geq 1$. If $n = l_i - 1 \geq 1$, we also define

$$M_n^i := \bigcup_{t=0}^{b_n-1} \left(\left(T_{l_i}(\mathbb{H}^{\mathrm{b}}) \cup \left\{ \lim_{\mathrm{Im}(w)\to-\infty} T_{l_i}(w) \right\} \right) + t \right),$$

and

$$J_n^i := \left\{ w \in M_n^i \middle|\; b_n - 1 < \mathrm{Re}(w) < b_n \right\}, \quad K_n^i := \left\{ w \in M_n^i \middle|\; 0 < \mathrm{Re}(w) < b_n - 1 \right\}.$$

Thus, M_n^i is the union of b_n topological disks. Then, for $1 \leq n < l_i - 1$, we shall recursively define the domains M_n^i, J_n^i and K_n^i from M_{n+1}^i, J_{n+1}^i and K_{n+1}^i. We consider three cases. If $\kappa_{n+1} = \mathrm{b}$, then

$$M_n^i := \bigcup_{t=0}^{b_n-1} \left(T_{n+1}(M_{n+1}^i) + t \right).$$

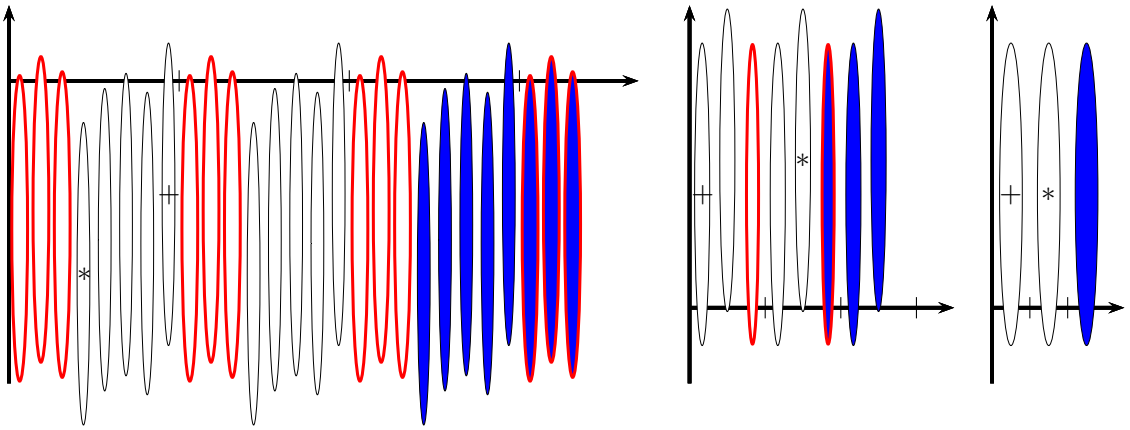


FIGURE 4. If $\hat{p}_1/\hat{q}_1 = 8/27$, then we have

$$(p_1/q_1, p_2/q_2, p_3/q_3) = (8/27, -3/8, -1/3).$$

Moreover, we have $b_1 = b_2 = b_3 = 3$, $(e_1, e_2, e_3, e_4) = (-1, 1, 1, -1)$ and $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (-1, 1, 1)$. Since $\varepsilon_2 = \varepsilon_3 = 1$, the maps T_3 and T_2 are both orientation preserving. From left to right, this figure shows the schematic picture of the domains M_1^4, M_2^4 and M_3^4. The sets J_n^4 are blue and the translations of their lift under T_n have a red boundary. The address of the connected component with the "+" mark starts with $(t_0, t_1, t_2, t_3) = (0, 0, 0, 0)$ while the address of the connected component with "*" mark starts with $(t_0, t_1, t_2, t_3) = (0, 0, 1, 1)$. Note that there is no admissible address starting with $(t_0, t_1, t_2) = (0, 3, 0)$. As we will see in Proposition 4.21, h_n^4 estimates the height of the connected component of M_n^4 that is marked with "+".

If $\kappa_{n+1} = \mathrm{t}$ and $e_n\varepsilon_{n+1} = 1$, then

$$M_n^i := \bigcup_{t=0}^{b_n-2} \left(T_{n+1}(M_{n+1}^i) + t\right) \bigcup \left(T_{n+1}(K_{n+1}^i) + b_n - 1\right).$$

Otherwise, if $\kappa_{n+1} = \mathrm{t}$ and $e_n\varepsilon_{n+1} = -1$, then

$$M_n^i := \bigcup_{t=0}^{b_n-1} \left(T_{n+1}(M_{n+1}^i) + t\right) \bigcup \left(T_{n+1}(J_{n+1}^i) + b_n\right).$$

We also define

$$J_n^i = \left\{w \in M_n^i \middle| \, |q_n/p_n| - 1 < \mathrm{Re}(w) < |q_n/p_n|\right\}, \quad K_n^i = \left\{w \in M_n^i \middle| \, 0 < \mathrm{Re}(w) < |q_n/p_n| - 1\right\}.$$

For all $i \geq 1$, we also define

$$M_0^i := T_1(M_1^i).$$

This completes the definition of the sets M_n^i for all $n \geq 0$, $i \geq 1$ with $n \leq l_i$.

Lemma 4.9. *Let $i \geq j > 1$ and $l_{j-1} \leq n < l_j \leq l_i$.*

(i) The set M_n^j is compactly contained in $\mathbb{H}_n$. Therefore the map $T_n|_{\overline{M_n^j}}$ is a homeomorphism.

(ii) The set M_n^{i+1} is compactly contained in M_n^i.

(iii) The set M_n^j has q_n connected components. Moreover, if A is a connected component of M_n^j, then there is an integer $0 \leq k < q_n$ such that

$$\overline{A} = \left\{ w \in \overline{M_n^j} \;\middle|\; k/|p_n| < \mathrm{Re}(w) < (k+1)/|p_n| \right\}.$$

(iv) If $n < l_i - 1$ and A is a connected component of M_n^i, then there is a connected component of M_{n+1}^i, namely A' and an integer $0 \leq t \leq b_n$ such that

$$A = t + T_{n+1}(A').$$

(v) If A is a connected component of M_{n+1}^j, then there is a unique integer $b_n - 2 \leq t \leq b_n$ such that

$$t + T_{n+1}(A) \subseteq J_n^j.$$

(vi) The set M_n^i has $q_n q_{l_j} q_{l_{j+1}} \cdots q_{l_{i-1}}$ connected components.

Proof. **(i):** This follows directly from Lemma 4.8(ii), the definition of M_n^j and Lemma 4.8(i).

(ii): From (i), we know that the set $M_{l_i}^{i+1}$ is compactly contained in $\mathbb{H}^{\mathrm{b}}$. It follows from Lemma 4.8(ii) that $M_{l_i-1}^{i+1}$ is compactly contained in $M_{l_i-1}^i$. Therefore M_n^{i+1} is compactly contained in M_n^i.

(iii): We use backward induction on n to prove the statement for $l_{j-1} \leq n < l_j$. If $n = l_j - 1$, the statement follows directly from Lemma 4.4(ii) and the fact that $q_{l_j-1} = b_{l_j-1}$. Assume $l_{j-1} \leq n < l_j - 1$ and the statement is true for $n+1$. Therefore we have $\kappa_{n+1} = \mathrm{t}$. We will write the proof for the case $e_n \varepsilon_{n+1} = 1$. The other case is similar. We have $\mathrm{Re}(T_{n+1}(w)) = |p_{n+1}/q_{n+1}| \,\mathrm{Re}(w)$, for all $w \in \mathbb{H}_{n+1}$. Therefore

$$\begin{aligned}\overline{T_{n+1}(K_{n+1}^j)} \subseteq \overline{T_{n+1}(M_{n+1}^j)} &= T_{n+1}\left(\overline{M_{n+1}^j}\right) \\ &\subseteq T_{n+1}(\{w \in \mathbb{H}_{n+1} \mid 0 < \mathrm{Re}(w) < |q_{n+1}/p_{n+1}|\}) \subseteq \{w \in \mathbb{H}_{n+1} \mid 0 < \mathrm{Re}(w) < 1\}.\end{aligned}$$

Hence the sets $T_{n+1}(M_{n+1}^i) + t$, for $0 \leq t \leq b_n - 2$, and $T_{n+1}(K_{n+1}^i) + b_n - 1$ are pairwise disjoint. The induction hypothesis implies that M_{n+1}^j has q_{n+1} and K_{n+1}^j has $q_{n+1} - |p_{n+1}|$ connected components. Therefore the number of connected components of M_n^j is equal to

$$(b_n - 1)q_{n+1} + (q_{n+1} - |p_{n+1}|) = q_n,$$

where the equality follows from Lemma 3.1.

Assume A is a connected component of M_n^j. Based on the discussions above, either $A \subseteq T_{n+1}(M_{n+1}^i) + t$, for some $0 \leq t \leq b_n - 2$ or $A \subseteq T_{n+1}(K_{n+1}^i) + b_n - 1$. Assume the first case is correct. The map T_{n+1} is a homeomorphism from $\mathbb{H}^{\mathrm{t}}$ into its image. Therefore the set $T_{n+1}^{-1}(A - t)$ is a connected component of M_{n+1}^j. So there is an integer $0 \leq k' < q_{n+1}$ such that

$$T_{n+1}^{-1}\left(\overline{A} - t\right) = \overline{T_{n+1}^{-1}(A - t)} = \left\{ w \in \overline{M_{n+1}^j} \;\middle|\; k'/|p_{n+1}| < \mathrm{Re}(w) < (k'+1)/|p_{n+1}| \right\}.$$

According to Lemma 3.1, we have $q_{n+1} = |p_n|$. Therefore, horizontal contraction of T_{n+1} is equal to $|p_{n+1}/p_n|$. Hence

$$\begin{aligned}\overline{A} &= \left\{ w \in T_{n+1}\left(\overline{M_{n+1}^j}\right) + t \;\middle|\; k'/|p_n| + t < \mathrm{Re}(w) < (k'+1)/|p_n| + t \right\} \\ &\subseteq \left\{ w \in \overline{M_n^j} \;\middle|\; k'/|p_n| + t < \mathrm{Re}(w) < (k'+1)/|p_n| + t \right\}.\end{aligned}$$

Let $k := k' + t|p_n|$. Since $0 \leq k' < q_{n+1} = |p_n|$, we have

$$0 \leq k = k' + t|p_n| < (t+1)|p_n| \leq (b_n - 1)|p_n| = \left([|q_n/p_n|] - 1\right)|p_n| < q_n.$$

Let $w \in \overline{M_n^j}$ be a point such that $k/|p_n| < \mathrm{Re}(w) < (k+1)/|p_n|$. Since $0 \le k' < |p_n|$, this implies that $t < \mathrm{Re}(w) < t+1$. Therefore we have $w \in T_{n+1}\left(\overline{M_{n+1}^j}\right) + t$. Hence

$$\overline{A} = \left\{ w \in \overline{M_n^j} \;\middle|\; k/|p_n| < \mathrm{Re}(w) < (k+1)/|p_n| \right\}.$$

The remaining case is when $A \subseteq T_{n+1}(K_{n+1}^i) + b_n - 1$. Let $t := b_n - 1$. In this case $T_{n+1}^{-1}(A-t)$ is a connected component of $K_{n+1}^j = \{w \in M_{n+1}^j \mid 0 < \mathrm{Re}(w) < |q_{n+1}/p_{n+1}| - 1\}$. Therefore, there is $0 \le k' < q_{n+1} - |p_{n+1}|$ such that

$$T_{n+1}^{-1}\left(\overline{A} - t\right) = \overline{T_{n+1}^{-1}(A-t)} = \left\{ w \in \overline{M_{n+1}^j} \;\middle|\; k'/|p_{n+1}| < \mathrm{Re}(w) < (k'+1)/|p_{n+1}| \right\}.$$

Hence

$$\begin{aligned} \overline{A} &= \left\{ w \in T_{n+1}\left(\overline{M_{n+1}^j}\right) + t \;\middle|\; k'/|p_n| + t < \mathrm{Re}(w) < (k'+1)/|p_n| + t \right\} \\ &\subseteq \left\{ w \in \overline{M_n^j} \;\middle|\; k'/|p_n| + t < \mathrm{Re}(w) < (k'+1)/|p_n| + t \right\}. \end{aligned}$$

Let $k := k' + t|p_n|$. By an argument similar to above, we see that

$$\overline{A} = \left\{ w \in \overline{M_n^j} \;\middle|\; k/|p_n| < \mathrm{Re}(w) < (k+1)/|p_n| \right\}.$$

We also have

$$0 \le k = k' + tq_{n+1} < (q_{n+1} - |p_{n+1}|) + (b_n - 1)q_{n+1} = q_n.$$

(iv): According to (ii) and (iii), there is an integer $0 \le k < q_n$ such that

$$k/|p_n| < \mathrm{Re}(w) < (k+1)/|p_n|,$$

for all $w \in A$. Let $t := \lfloor k/|p_n| \rfloor \le \lfloor (q_n - 1)/|p_n| \rfloor \le [q_n/|p_n|] = b_n$. According to the definition, the set $A - t$ is also a connected component of M_n^i. Therefore $A - t$ is a connected component of the set

$$\{w \in M_n^i \mid 0 < \mathrm{Re}(w) < 1\} = T_{n+1}(M_{n+1}^i).$$

Since T_{n+1} is a homeomorphism on M_{n+1}^i, the set $A' := T_{n+1}^{-1}(A - t)$ is a connected component of M_{n+1}^i and we have

$$A = t + T_{n+1}(A').$$

(v): If $n = l_j - 1$, then we have $|p_n| = 1$ and $\mathrm{Re}(T_{n+1}(M_n^j)) \subseteq (0,1)$. Therefore $t = q_n - 1 = b_n - 1$ is the unique integer with the given property. Now assume $n < l_j - 1$, which implies that $\kappa_{n+1} = \mathrm{t}$. If $e_n \varepsilon_{n+1} = 1$, we have $\mathrm{Re}(T_{n+1}(w)) = |p_{n+1}/q_{n+1}| \,\mathrm{Re}(w)$, for all $w \in \mathbb{H}^{\mathrm{t}}$. Hence we have

$$\mathrm{Re}(T_{n+1}(J_{n+1}^j)) \subseteq (1 - |p_{n+1}/q_{n+1}|, 1), \quad \mathrm{Re}(T_{n+1}(K_{n+1}^j)) \subseteq (0, 1 - |p_{n+1}/q_{n+1}|).$$

Therefore

$$J_n^j = (T_{n+1}(J_{n+1}^j) + (b_n - 2)) \cup (T_{n+1}(K_{n+1}^j) + (b_n - 1)).$$

It follows that $t = b_n - 2$ if $A \subseteq J_{n+1}^j$ and $t = b_n - 1$ otherwise. On the other hand, if $e_n \varepsilon_{n+1} = -1$, then we have

$$J_n^j = (T_{n+1}(K_{n+1}^j) + (b_n - 1)) \cup (T_{n+1}(J_{n+1}^j) + b_n).$$

Therefore $t = b_n$ if $A \subseteq J_{n+1}^j$ and $t = b_n - 1$ otherwise.

(vi): We use induction on $i - j$. If $i = j$, then the statement follows from part (iii). Assume the statement is true for some $i \ge j$. According to (iii) and (i), the set $M_{l_i}^{i+1}$ has q_{l_i} connected components and $T_{l_i}|_{M_{l_i}^{i+1}}$ is a homeomorphism. Hence the set $A' := T_{l_i}(M_{l_i}^{i+1})$ has q_{l_i} connected

components. Let A be a connected component of M_n^i. According to (iv), there are integers $t_n, t_{n+1}, \cdots, t_{l_i-1}$ such that

$$A = t_n + T_{n+1}(t_{n+1} + T_{n+2}(\cdots t_{l_i-2} + T_{l_i-1}(t_{l_i-1} + A')\cdots)).$$

Therefore $A \cap M_n^{i+1}$ has q_{l_i} connected components. Hence the number of connected components of M_n^{i+1} is q_{l_i} times the number of connected components of M_n^i and the induction step is complete. □

For all $n \geq 0$, define

$$M_n := \bigcap_{i \geq 1, l_i > n} M_n^i.$$

Since M_n^{i+1} is compactly contained in M_n^i for all $i \geq 1$ with $l_i > n$, the set M_n is compact. The set M_0 projects under exponential map $\zeta \mapsto e^{2\pi i\zeta}$ to a compact set M_* in the plane. This will be our topological model for the post-critical set, that is,

$$M_* = M_*((\hat{p}_i/\hat{q}_i)_{i=1}^\infty) := \{e^{2\pi i w} \mid w \in M_0\}.$$

It should be noted that this set only depends on the sequence $(\hat{p}_i/\hat{q}_i)_{i=1}^\infty$.

Recall that two sets A and B in the plane are called *ambiently homeomorphic* if there is a homeomorphism Ψ of the plane that such that $\Psi(A) = B$. Since M_0 is compactly contained in the strip $\mathrm{Re}(w) \in (0,1)$, there is a Jordan domain D that is also compactly contained in the same vertical strip and $M_0 \subseteq D$. Clearly the set $D_* := \{e^{2\pi i w} \mid w \in D\} \supseteq M_*$ is also a Jordan domain. Therefore, there is a homeomorphism $\Psi : \mathbb{C} \to \mathbb{C}$ such that $\Psi(w) = e^{2\pi i w}$, for all $w \in D$. Hence M_0 and M_* are ambiently homeomorphic.

4.3. **Admissible Addresses.** In Subsection 2.3, we assigned an address to every connected component of the post-critical set of an infinitely renormalisable quadratic maps. Here we will assign a sequence of integers to every connected component of M_0. We say a sequence is *admissible* if it is assigned to a connected component of M_0. We will investigate some of the properties of the space of admissible addresses. In particular, we define a metric on this space and show that it is homeomorphic to an odometer.

Evidently, M_0 has uncountably many connected components. By our construction, we may assign a unique address to each connected component. First note that for all $n \geq 1$, we have

$$M_n \subseteq \bigcup_{t=0}^{b_n} \left(T_{n+1}(M_{n+1}) + t\right).$$

Since we have $\mathrm{Re}(T_{n+1}(M_{n+1})) \subseteq (0,1)$, we see that for all $z_n \in M_n$, we have

$$z_n - \lfloor \mathrm{Re}(z_n) \rfloor \in T_{n+1}(M_{n+1}).$$

Therefore, for any $z_0 \in M_0$, there is a unique sequence $(z_n)_{n=0}^\infty$, with $z_n \in M_n$ and

$$z_n - \lfloor \mathrm{Re}(z_n) \rfloor = T_{n+1}(z_{n+1}).$$

We call $(z_n)_{n=0}^\infty$ *the sequence associated to* z_0. The *address of the point* z_0 is then defined as

$$\mathbf{t}(z_0) = (t_n)_{n=0}^\infty, \quad \forall n \geq 0, \quad t_n := \lfloor \mathrm{Re}(z_n) \rfloor.$$

Note that we always have $t_0 = 0$. For $n \geq 1$, we have $0 \leq t_n \leq b_n$. As we see in Figure 4.2, with this address we may identify a unique connected component of M_n^i for every $n \geq 0, i \geq 1$ with

$n < l_i$. We will call this connected component $M_n^i(\mathbf{t})$. To be precise, for all $n \geq 0$ and $i \geq 1$ with $n < l_i$, we define $M_n^i(\mathbf{t})$ as the set

$$t_n + T_{n+1}\left(t_{n+1} + T_{n+2}\left(\cdots t_{l_i-2} + T_{l_i-1}\left(t_{l_i-1} + \left(T_{l_i}(\mathbb{H}^{\mathrm{b}}) \cup \left\{\lim_{\mathrm{Im}(w)\to-\infty} T_{l_i}(w)\right\}\right)\right)\cdots\right)\right).$$

It follows that for all integers m with $n \leq m < l_i$, we have

$$M_n^i(\mathbf{t}) = t_n + T_{n+1}(t_{n+1} + T_{n+2}(\cdots t_{m-2} + T_{m-1}(t_{m-1} + T_m(M_m^i(\mathbf{t})))\cdots)).$$

Therefore we may also assign an address to every connected component of M_n, for all $n \geq 0$.

$$M_n(\mathbf{t}) := \bigcap_{i\geq 1, l_i > n} M_n^i(\mathbf{t}) \subseteq \mathbb{H}_n.$$

It follows that for all $m \geq n \geq 0$, we have

$$M_n(\mathbf{t}) = t_n + T_{n+1}(t_{n+1} + T_{n+2}(\cdots t_{m-2} + T_{m-1}(t_{m-1} + T_m(M_m(\mathbf{t})))\cdots)).$$

A given sequence $\mathbf{t} = (t_n)_{n=0}^{\infty}$ is an *admissible address* if it can be realised as the address of a connected component of M_0. We will use the notation $\mathbf{0}$ to denote the zero address,

$$\mathbf{0} := (0, 0, 0, \cdots).$$

Let $\mathbf{t}$ be an admissible address. If $\mathrm{Re}(M_n(\mathbf{t})) \subseteq (q_n/|p_n| - 1, q_n/|p_n|)$, for all $n \geq 1$, then we define $\mathbf{t} + 1 := \mathbf{0}$. Otherwise, there is a smallest integer $n \geq 1$ such that $\mathrm{Re}(M_n(\mathbf{t})) \nsubseteq (q_n/|p_n| - 1, q_n/|p_n|)$. In this case, we define

$$\mathbf{t} + 1 := (0, 0, \cdots, 0, t_n + 1, t_{n+1}, t_{n+2}, \cdots).$$

Clearly $\mathbf{t} + 1$ is always admissible.

Lemma 4.10. *The map $\boldsymbol{t} \mapsto \boldsymbol{t}+1$ is a bijection on the set of admissible addresses. In particular, there is a unique admissible address, namely **-1**, such that **-1** $+ 1 =$ **0**.*

Proof. Let $\mathbf{t}'$ be an admissible address. We will find an admissible address $\mathbf{t}$ such that $\mathbf{t} + 1 = \mathbf{t}'$ and show that it is unique. First consider the case $\mathbf{t}' = \mathbf{0}$. According to the definition, we have $\mathrm{Re}(M_n(\mathbf{t})) \subseteq (q_n/|p_n| - 1, q_n/|p_n|)$, for all $n \geq 0$. If $n = l_i - 1$, for some $i > 1$, then we have $t_n = b_n - 1$. Now we may use Lemma 4.9(v) to recursively find the unique value of t_{n-1} using t_n, for the remaining values of n.

On the other hand, if $\mathbf{t}' \neq \mathbf{0}$, then $\mathrm{Re}(M_m(\mathbf{t})) \nsubseteq (q_m/|p_m| - 1, q_m/|p_m|)$, for some $m \geq 0$. Let m be the smallest integer with this property. We have

$$t_m = t'_m - 1, \quad \forall n > m,\ t_m = t'_m.$$

Now we may use Lemma 4.9(v) again to recursively find the unique value of t_{n-1} using t_n, for the remaining values of n. □

In the remainder of this section and the next section, whenever we have a character such as w, where w_0 is used to denote a point in M_0, we will use $(w_n)_{n=0}^{\infty}$ to denote its associated sequence.

Lemma 4.11. *For any admissible address $\boldsymbol{t} = (t_n)_{n=0}^{\infty}$ and any $i \geq 1$, the following hold.*

(i) $\left\{M_0^i(\boldsymbol{t}') \mid \boldsymbol{t}' \text{ admissible } \right\} = \left\{M_0^i(\boldsymbol{t}+k) \mid 0 \leq k < q_{l_1} q_{l_2} \cdots q_{l_{i-1}}\right\}$.

(ii) The address $\boldsymbol{t}' := (t_0, t_1, t_2, \cdots, t_{l_i-2}, t_{l_i-1}, 0, 0, 0, \cdots)$ is admissible and we have

$$M_0^i(\boldsymbol{t}) = M_0^i(\boldsymbol{t}').$$

(iii) We have $M_0^i(\boldsymbol{t}+k) = M_0^i(\boldsymbol{t}+k')$ if and only if

$$k = k' \pmod{q_{l_1} q_{l_2} \cdots q_{l_{i-1}}}.$$

Proof. Clearly (iii) follows from (i). Therefore we only need to prove (i) and (ii). First we prove the lemma for $\mathbf{t} = \mathbf{0}$. Let

$$A := \{\mathbf{t} = (t_n)_{n=0}^{\infty} \mid \mathbf{t} \text{ is an admissible address, } \forall k \geq l_i, t_k = 0\}.$$

All addresses in A have the same tail as $\mathbf{0}$. Therefore, for every $\mathbf{t}' \in A$, there is an integer $k \geq 0$ such that $\mathbf{t}' = \mathbf{0} + k$. On the other hand, if $\mathbf{0} \neq \mathbf{t}' \in A$, then we have $\mathbf{t}' - 1 \in A$. Therefore, there exists $q > 0$ such that $A = \{\mathbf{0} + k \mid 0 \leq k < q\}$. For any admissible address $\mathbf{t}' = (t'_n)_{n=0}^{\infty}$, the definition of $M_0^j(\mathbf{t}')$ only depends on the values of $t'_0, t'_1, \cdots, t'_{l_i-1}$. Therefore we have a clear bijection between the set A and the connected components of M_0^i given by $\mathbf{t}' \mapsto M_0^i(\mathbf{t}')$. Hence, according to Lemma 4.9(vi), we have $|A| = q_{l_1} q_{l_2} \cdots q_{l_{i-1}}$. Therefore we have

$$A = \{\mathbf{0} + k \mid 0 \leq k < q_{l_1} q_{l_2} \cdots q_{l_{i-1}}\},$$

and

$$\left\{M_0^i(\mathbf{0} + k) \mid 0 \leq k < q_{l_1} q_{l_2} \cdots q_{l_{i-1}}\right\} = \left\{M_0^i(\mathbf{t}') \mid \mathbf{t}' \in A\right\} = \left\{M_0^i(\mathbf{t}') \mid \mathbf{t}' \text{ admissible }\right\}.$$

The address $\mathbf{0} + |A| - 1$ belongs to A, while $\mathbf{0} + |A| \notin A$. Therefore we have $\mathbf{0} + |A| = (0, 0, \cdots, 0, 1, 0, \cdots)$ where the only 1 in the sequence is at the point l_i. Therefore we have $M_0^i(\mathbf{0} + |A|) = M_0^i(\mathbf{0})$.

Define the function Γ on the space of admissible addresses by

$$\Gamma((t_n)_{n=0}^{\infty}) := (t_0, t_1, \cdots, t_{l_i-1}, 0, 0, \cdots).$$

Let $\mathbf{t}' = (t'_n)_{n=0}^{\infty}$ be an admissible address. We have

$$M_0(\Gamma(\mathbf{t}')) = t'_0 + T_1(t'_1 + T_2(\cdots T_{l_i}(M_{l_i}(\mathbf{0}))\cdots)) \subseteq t'_0 + T_1(t'_1 + T_2(\cdots T_{l_i}(M_{l_i}^i(\mathbf{0}))\cdots)) = M_0^i(\mathbf{t}').$$

Therefore $\Gamma(\mathbf{t}')$ is admissible. Clearly we also have $\Gamma(\mathbf{t}') \in A$ and

$$M_0^i(\mathbf{t}') = M_0^i(\Gamma(\mathbf{t}')),$$

which completes the proof of (ii). In fact $\Gamma(\mathbf{t}')$ is the only address in A for which this equation holds. Therefore the equation $M_0^i(\mathbf{0} + |A|) = M_0^i(\mathbf{0})$ implies that $\Gamma(\mathbf{0} + |A|) = \mathbf{0}$. It now follows from the definition of Γ and the addition operation that

$$\Gamma(\mathbf{t}' + 1) = \Gamma(\Gamma(\mathbf{t}') + 1).$$

Let $\mathbf{t} \neq \mathbf{0}$ be an admissible address. We have $\Gamma(\mathbf{t}) = \mathbf{0} + k$, for some $0 \leq k < |A|$. Clearly we have $\Gamma(\mathbf{t} + l) = \mathbf{0} + k + l$, for all $l < |A| - k$. We also have

$$\Gamma(\mathbf{t} + |A| - k) = \Gamma(\Gamma(\mathbf{t}) + |A| - k) = \Gamma(\mathbf{0} + |A|) = \mathbf{0}.$$

Therefore $\Gamma(\mathbf{t} + l) = \mathbf{0} + l - |A| + k$, for all $|A| - k \leq l < |A|$. Hence we have

$$A = \{\Gamma(\mathbf{t} + k) \mid 0 \leq k < |A|\}.$$

Therefore

$$\left\{M_0^i(\mathbf{t} + k) \mid 0 \leq k < |A|\right\} = \left\{M_0^i(\Gamma(\mathbf{t} + k)) \mid 0 \leq k < |A|\right\} = \left\{M_0^i(\mathbf{0} + k) \mid 0 \leq k < |A|\right\}. \quad \square$$

Definition 4.12. *Let $\boldsymbol{t} = (t_n)_{n=0}^{\infty}$ and $\boldsymbol{t}' = (t'_n)_{n=0}^{\infty}$ be two admissible addresses. We define the distance between them to be $d(\boldsymbol{t}, \boldsymbol{t}') := 0$, if $\boldsymbol{t} = \boldsymbol{t}'$ or*

$$d(\boldsymbol{t}, \boldsymbol{t}') := \frac{1}{2^n}$$

where $n \geq 1$ is the smallest integer for which we have $t_n \neq t'_n$.

If $(z_0^i)_{i=1}^\infty$ is a sequence of points in M_0 that converges to a point $z_0 \in M_0$, then we clearly have $\mathbf{t}(z_0^i) \to \mathbf{t}(z_0)$. In other words, the map $\mathbf{t} : M_0 \to \{\mathbf{t} \mid \mathbf{t} \text{ is admissible}\}$ is continuous.

In Subsection 2.3, we defined odometers and showed that they are Cantor sets. We also showed that they may be used as "addresses" for the connected components of the post-critical set of infinitely renormalisable quadratics. Recall that $\hat{q}_i = q_{l_i}$, for all $i \geq 1$. Here we will show that the odometer with the combinatorics $(\hat{q}_i)_{i=1}^\infty$ is homeomorphic to the space of admissible addresses.

Proposition 4.13. *Let $\hat{\boldsymbol{q}} := (\hat{q}_i)_{i=1}^\infty$. There is a homeomorphism $\pi : \{\boldsymbol{t} \mid \boldsymbol{t} \text{ is admissible}\} \to \mathcal{O}_{\hat{\boldsymbol{q}}}$ such that*

$$\pi(\boldsymbol{t}+1) = \pi(\boldsymbol{t}) + 1,$$

for all admissible addresses $\boldsymbol{t}$.

Proof. First we define continuous maps $\pi_i : \{\mathbf{t} \mid \mathbf{t} \text{ is admissible}\} \to \mathbb{Z}/(\hat{q}_1\hat{q}_2\cdots\hat{q}_i)$, for all $i \geq 1$, such that

$$\pi_i(\mathbf{t}+1) = \pi_i(\mathbf{t}) + 1,$$

for all admissible addresses $\mathbf{t}$. Let $i \geq 1$ and let $\mathbf{t}$ be an admissible address. The set M_0^{i+1} has $\hat{q}_1\hat{q}_2\cdots\hat{q}_i$ connected components. Therefore, using Lemma 4.11, we see that there is a unique number $0 \leq \hat{t}_i < \hat{q}_1\hat{q}_2\cdots\hat{q}_i$ such that $M_0^{i+1}(\mathbf{t}) = M_0^{i+1}(\mathbf{0}+k)$, if and only if $k = \hat{t}_i \pmod{\hat{q}_1\hat{q}_2\cdots\hat{q}_i}$. In other words, we may consider $\hat{t}_i$ as an element of $\mathbb{Z}/(\hat{q}_1\hat{q}_2\cdots\hat{q}_i)$. We define $\pi_i(\mathbf{t}) := \hat{t}_i$. Since $M_0^{i+1}(\mathbf{t}) = M_0^{i+1}(\mathbf{0}+\pi_i(\mathbf{t}))$, we have $M_0^{i+1}(\mathbf{t}+1) = M_0^{i+1}(\mathbf{0}+\pi_i(\mathbf{t})+1)$. Hence

$$M_0^{i+1}(\mathbf{0}+\pi_i(\mathbf{t}+1)) = M_0^{i+1}(\mathbf{t}+1) = M_0^{i+1}(\mathbf{0}+\pi_i(\mathbf{t})+1).$$

Therefore we have $\pi_i(\mathbf{t}+1) = \pi_i(\mathbf{t}) + 1$.

If $i > j \geq 1$ and $\mathbf{t}$ is an admissible address, then we have

$$M_0(\mathbf{t}) \subseteq M_0^{i+1}(\mathbf{t}) = M_0^{i+1}(\pi_i(\mathbf{t})) \subseteq M_0^{j+1}(\mathbf{t}) = M_0^{j+1}(\pi_j(\mathbf{t})).$$

Therefore, using Lemma 4.11, we see that

$$\pi_j(\mathbf{t}) = \pi_i(\mathbf{t}) \pmod{\hat{q}_1\hat{q}_2\cdots\hat{q}_j}.$$

Hence, we may take the projective limit of the sequence $(\pi_i)_{i=1}^\infty$ to obtain a map

$$\pi : \{\mathbf{t} \mid \mathbf{t} \text{ is admissible}\} \to \operatorname*{proj\,lim}_{i\to\infty} \mathbb{Z}/(\hat{q}_1\hat{q}_2\cdots\hat{q}_i) = \mathcal{O}_{\hat{\mathbf{q}}}.$$

Clearly we have $\pi(\mathbf{t}+1) = \pi(\mathbf{t}) + 1$, for all admissible addresses $\mathbf{t}$. Since π_i is surjective for all $i \geq 1$, we conclude that π is also surjective. If $\mathbf{t}$ and $\mathbf{t}'$ are two admissible addresses such that $\pi(\mathbf{t}) = \pi(\mathbf{t}')$, then we have $\pi_i(\mathbf{t}) = \pi_i(\mathbf{t}')$, for all $i \geq 1$. Therefore

$$M_0(\mathbf{t}) = \bigcap_{i=1}^\infty M_0^{i+1}(\mathbf{t}) = \bigcap_{i=1}^\infty M_0^{i+1}(\pi_i(\mathbf{t})) = \bigcap_{i=1}^\infty M_0^{i+1}(\pi_i(\mathbf{t}')) = \bigcap_{i=1}^\infty M_0^{i+1}(\mathbf{t}') = M_0(\mathbf{t}').$$

Hence $\mathbf{t} = \mathbf{t}'$ and π is injective.

To complete the proof, we need to show that π is a homeomorphism. Since π is bijective and $\mathcal{O}_{\hat{\mathbf{q}}}$ is compact, it is sufficient to show that π^{-1} is continuous. Let $(\mathbf{t}^k)_{k=1}^\infty = ((t_n^k)_{n=0}^\infty)_{k=1}^\infty$ be a sequence of admissible addresses such that $(\pi(\mathbf{t}^k))_{k=1}^\infty$ converges to a point $\hat{t} \in \mathcal{O}_{\hat{\mathbf{q}}}$. Let $\mathbf{t} := \pi^{-1}(\hat{t}) = (t_n)_{n=0}^\infty$ and let $i \geq 1$ be an integer. We have

$$\lim_{k\to\infty} \pi_i(\mathbf{t}^k) = \lim_{k\to\infty} \Big(\pi(\mathbf{t}^k) \pmod{\hat{q}_1\hat{q}_2\cdots\hat{q}_i}\Big) = \pi(\mathbf{t}) \pmod{\hat{q}_1\hat{q}_2\cdots\hat{q}_i} = \pi_i(\mathbf{t}) \in \mathbb{Z}/(\hat{q}_1\hat{q}_2\cdots\hat{q}_i).$$

Hence, for large k, we have $\pi_i(\mathbf{t}^k) = \pi_i(\mathbf{t})$ and $M_0^{i+1}(\mathbf{t}) = M_0^{i+1}(\mathbf{t}^k)$. Therefore, according to Lemma 4.11, we see that $t_n = t_n^k$, for all $0 \leq n < \hat{q}_1\hat{q}_2\cdots\hat{q}_i$. In other words, for large k, we have

$$d(\mathbf{t}, \mathbf{t}^k) \leq 2^{-\hat{q}_1\hat{q}_2\cdots\hat{q}_i}.$$

Hence $\mathbf{t}^k \to \mathbf{t}$ and the proof is complete. □

4.4. **Topology of the connected components.** In this section we aim to show that every connected component of M_0 is either a single point or a Jordan arc. A continuous injective map $\gamma : [0,1] \to \mathbb{C}$ is called a *Jordan arc* and a continuous map $\gamma' : [0,1] \to \mathbb{C}$ that is injective on $[0,1)$ and $\gamma'(0) = \gamma'(1)$ is called a *Jordan curve*. If $\gamma : [0,1] \to \mathbb{C}$ is a Jordan arc or a Jordan curve, we may use γ to denote both the map and its image, if there is no ambiguity. The *interior* of a Jordan arc γ is defined to be the set $\gamma((0,1))$ and the points $\gamma(0)$ and $\gamma(1)$ are called the *end points* of γ. We shall employ a classical characterisation of the Jordan arcs in plane.

Let $X \subseteq \mathbb{C}$ be a compact connected set. A point $z \in X$ is called a *cut-point*, if $X \setminus \{z\}$ is disconnected. It is a classical result that every compact connected subset of the plane with at most two non cut-points is either a single point or a Jordan arc. (see [Nad92, Theorem 6.6 and Theorem 6.17])

We start with a definition.

Definition 4.14. *Let $\boldsymbol{t}$ be an admissible address and assume $z_0, w_0 \in M_0(\boldsymbol{t})$. Also assume $(z_n)_{n=0}^{\infty}$ and $(w_n)_{n=0}^{\infty}$ are the sequences associated to z_0 and w_0. We say $z_0 \succcurlyeq w_0$ if either $z_0 = w_0$ or*

$$\lim_{n\to\infty} e_n \operatorname{Im}(z_n - w_n) = +\infty.$$

Recall that a binary relation R on a set X is called a *partial order* if for all $x, y, z \in X$, we have

- xRx,
- If xRy and yRz, then xRy,
- If xRy and yRx, then $x = y$.

A partial order is called a *total order* if for any pair x and y in X either xRy or yRx.

Lemma 4.15. *For every admissible address $\boldsymbol{t}$, the relation "$\succcurlyeq$" is a total order on $M_0(\boldsymbol{t})$.*

Proof. Let $\mathbf{t} = (t_n)_{n=0}^{\infty}$ be an admissible address. First note that "$\succcurlyeq$" is clearly a partial order on $M_0(\mathbf{t})$. The only non-trivial statement left to prove is that for every $z_0, w_0 \in M_0(\mathbf{t})$, we have either $z_0 \succcurlyeq w_0$ or $w_0 \succcurlyeq z_0$. If $z_0 = w_0$, there is nothing to prove. Assuming $z_0 \neq w_0$, uniform contraction of T_n implies that

$$|z_n - w_n| \geq (4/3)^n |z_0 - w_0|.$$

Therefore

$$\lim_{n\to\infty} |z_n - w_n| = +\infty.$$

Since $|\operatorname{Re}(z_n) - \operatorname{Re}(w_n)| < 1$, this implies that

$$\lim_{n\to\infty} |\operatorname{Im}(z_n - w_n)| = +\infty.$$

So there is $N > 0$ such that for $n \geq N$ we have $|\operatorname{Im}(z_n) - \operatorname{Im}(w_n)| > 2C_2 + 3$. Assume $e_n \operatorname{Im}(z_n) > e_n \operatorname{Im}(w_n)$, for some $n > N$. Then, using uniform contraction of T_n and Lemma 4.8(vii), we have

$$\begin{aligned} e_{n-1} \operatorname{Im}(z_{n-1} - w_{n-1}) &= e_{n-1} \operatorname{Im}(T_n(z_n) - T_n(w_n)) \\ &> e_{n-1} \operatorname{Im}(T_n(t_n + i \operatorname{Im}(z_n)) - T_n(t_n + i \operatorname{Im}(w_n))) - 2 \\ &> -2C_2 - 3. \end{aligned}$$

Therefore $e_{n-1} \operatorname{Im}(z_{n-1} - w_{n-1}) > 0$. By a similar argument for the case $e_n \operatorname{Im}(z_n) < e_n \operatorname{Im}(w_n)$, we see that

$$\forall n > N, \quad \operatorname{sgn}(e_n \operatorname{Im}(z_n - w_n)) = \operatorname{sgn}(e_{n-1} \operatorname{Im}(z_{n-1} - w_{n-1})).$$

Hence the limit

$$\lim_{n\to\infty} e_n \operatorname{Im}(z_n - w_n)$$

exists and is either $+\infty$ or $-\infty$. □

Proposition 4.16. *For any admissible address* $\boldsymbol{t}$*, the set* $M_0(\boldsymbol{t})$ *is either a single point or a Jordan arc.*

Proof. Let us assume $M_0(\mathbf{t})$ is not a point and let $z_0 \in M_0(\mathbf{t})$ be a point that is neither the smallest nor the largest point in $M_0(\mathbf{t})$ with respect to "$\succcurlyeq$". Note that the smallest and largest points are unique, if they exist. If any such z_0 is a cut-point for $M_0(\mathbf{t})$, then $M_0(\mathbf{t})$ has at most two non cut-points and therefore it is a Jordan arc.

Let $C := 8C_2 + 7$ and, for $n \geq 0$, define the sets

$$A^n := \{x_0 \in M_0(\mathbf{t}) \mid e_n \operatorname{Im}(x_n) \geq e_n \operatorname{Im}(z_n) - C\}$$

and

$$A := \bigcap_{N\geq 0} \bigcup_{n\geq N} A^n = \{x_0 \in M_0(\mathbf{t}) \mid \forall N \geq 0, \exists n \geq N, e_n \operatorname{Im}(x_n - z_n) \geq -C\}.$$

Note that each A^n is compact. By definition, we have $A \supseteq \{x_0 \in M_0(\mathbf{t}) \mid x_0 \succcurlyeq z_0\}$. On the other hand, if there is $z_0 \neq x_0 \in A$ such that $x_0 \preccurlyeq z_0$, then we have

$$\lim_{n\to\infty} e_n \operatorname{Im}(x_n - z_n) = -\infty,$$

which is not possible. So we have

$$A = \{x_0 \in M_0(\mathbf{t}) \mid x_0 \succcurlyeq z_0\}.$$

We will show that $A^n \subseteq A^{n-1}$ for all $n \geq 1$. Let $n \geq 1$ and $x_0 \in A^n \setminus \{z_0\}$. Choose $z_n' \in \mathbb{C}$ such that $\operatorname{Im}(z_n') = \operatorname{Im}(z_n)$ and $\operatorname{Re}(z_n') = \operatorname{Re}(x_n)$. Since x_0 and z_0 have the same address, we have $|z_n - z_n'| < 1$. Hence

$$e_n \operatorname{Im}(x_n) \geq e_n \operatorname{Im}(z_n) - C = e_n \operatorname{Im}(z_n' - ie_nC).$$

Using Lemma 4.8(vii) and Lemma 4.8(iii), we have

$$\begin{aligned} e_{n-1} \operatorname{Im}(T_n(x_n)) &> e_{n-1} \operatorname{Im}(T_n(z_n' - ie_nC)) - 2C_2 - 1 \geq e_{n-1} \operatorname{Im}(T_n(z_n')) - 3C/4 - 2C_2 - 1 \\ &\geq e_{n-1} \operatorname{Im}(T_n(z_n)) - 3(C+1)/4 - 2C_2 - 1 = e_{n-1} \operatorname{Im}(T_n(z_n)) - C. \end{aligned}$$

Therefore $x_0 \in A^{n-1}$.

We have

$$\{x_0 \in M_0(\mathbf{t}) \mid x_0 \succcurlyeq z_0\} = A = \bigcap_{N\geq 0} \bigcup_{n\geq N} A^n = \bigcap_{n\geq 0} A^n.$$

Therefore $\{x_0 \in M_0(\mathbf{t}) \mid x_0 \succcurlyeq z_0\}$ is compact. A similar argument shows that $\{x_0 \in M_0(\mathbf{t}) \mid x_0 \preccurlyeq z_0\}$ is also compact. Hence the two sets $\{x_0 \in M_0(\mathbf{t}) \setminus \{z_0\} \mid x_0 \preccurlyeq z_0\}$ and $\{x_0 \in M_0(\mathbf{t}) \setminus \{z_0\} \mid x_0 \succcurlyeq z_0\}$ are non-empty, disjoint subsets of $M_0(\mathbf{t}) \setminus \{z_0\}$ that are closed in $M_0(\mathbf{t}) \setminus \{z_0\}$. Therefore z_0 is a cut-point of $M_0(\mathbf{t})$. □

Corollary 4.17. *For any admissible address* $\boldsymbol{t}$ *and any* $w_0, w_0' \in M_0(\boldsymbol{t})$*, we have* $w_0 \succcurlyeq w_0'$ *if and only if*

$$e_n \operatorname{Im}(w_n) \geq e_n \operatorname{Im}(w_n') - (8C_2 + 7),$$

for all $n \geq 0$*.*

Proof. As we saw in the proof of Lemma 4.16, we have

$$\{x_0 \in M_0(\mathbf{t}) \mid x_0 \succcurlyeq w_0'\} = \{x_0 \in M_0(\mathbf{t}) \mid \forall n \geq 0, e_n \operatorname{Im}(x_n) \geq e_n \operatorname{Im}(w_n') - (8C_2 + 7)\}. \qquad \square$$

4.5. **Sizes of the connected components.** Here we will analyse the maps T_n to control the sizes of the connected components of the model. We will find many admissible addresses $\mathbf{t}$ such that $M_0(\mathbf{t})$ is a single point. We will show that the connected component $M_0(\mathbf{0})$ is a single point if and only if condition $\mathcal{C}$ is satisfied. As we will see, the reason is that the compositions of T_n near the imaginary axis is closely approximated with compositions of g_n. Finally, we will define a map $P : M_0 \to M_0(\mathbf{0})$ that is injective on every connected component and respects the order "$\succcurlyeq$". Using this map, we show that $M_0(\mathbf{0})$ is the "longest" connected component of M_0. To be more precise, if $M_0(\mathbf{0})$ is a single point, then every connected component of M_0 is a single point.

For integers $n \geq 0$ and $i \geq 1$ with $n \leq l_i$ and for any admissible address $\mathbf{t}$, we define the following quantities associated to the connected set $M_n^i(\mathbf{t})$.

$$u_n^i(\mathbf{t}) := \sup\{e_n \operatorname{Im}(z) \mid z \in M_n^i(\mathbf{t})\},$$

$$d_n^i(\mathbf{t}) := \inf\{e_n \operatorname{Im}(z) \mid z \in M_n^i(\mathbf{t})\}.$$

Clearly we always have $-2 \leq d_n^i(\mathbf{t}) \leq u_n^i(\mathbf{t}) \leq \infty$. The following lemma gives us a criteria for determining when a connected component of M_0 is a point.

Lemma 4.18. *For a given admissible address $\boldsymbol{t}$, the set $M_0(\boldsymbol{t})$ is a single point if and only if there exists a constant $C > 0$ such that for every $N \geq 1$ there are integers n, i with $N \leq n < l_i$ for which we have*

$$|u_n^i(\boldsymbol{t}) - d_n^i(\boldsymbol{t})| < C.$$

Proof. First note that for any $n \geq 0$, $i \geq 1$ with $n < l_i$, we have $\{\operatorname{Re}(z) \mid z \in M_n^i(\mathbf{t})\} \subseteq (t_n, t_n+1)$, where $\mathbf{t} = (t_n)_{n\geq 0}$. Hence

$$|u_n^i(\mathbf{t}) - d_n^i(\mathbf{t})| \leq \operatorname{diam}\left(M_n^i(\mathbf{t})\right) \leq |u_n^i(\mathbf{t}) - d_n^i(\mathbf{t})| + 1.$$

Assume the condition in the lemma is satisfied with a constant $C > 0$. Then we have

$$\forall N \geq 1, \quad \exists n, i, \quad N \leq n < l_i, \quad \operatorname{diam}\left(M_n^i(\mathbf{t})\right) < C + 1.$$

We know that $M_0^i(\mathbf{t}) = T_1(t_1 + T_2(t_2 + \cdots T_n(M_n^i(\mathbf{t}))\cdots))$. Therefore Lemma 4.8(iii) implies that

$$\forall N \geq 1, \quad \exists n, i, \quad N \leq n < l_i, \quad \operatorname{diam}\left(M_0^i(\mathbf{t})\right) \leq (\frac{3}{4})^n(C+1) \leq (\frac{3}{4})^N(C+1).$$

But we have

$$\operatorname{diam}\left(M_0(\mathbf{t})\right) = \operatorname{diam}\left(\bigcap_{j\geq 1} M_0^j(\mathbf{t})\right) \leq \inf\left\{\operatorname{diam}\left(M_0^j(\mathbf{t})\right) \;\middle|\; j \geq 1\right\}.$$

Therefore $M_0(\mathbf{t})$ is a point.

To prove the other direction, assume $M_0(\mathbf{t})$ is a point and let $C = 1$. In this case $M_n(\mathbf{t})$ is also a point for all $n \geq 0$. For any sequence of nested compact sets $(K_n)_{n=1}^{\infty}$ in the plane that shrinks to a point, we have $\operatorname{diam}(K_n) \to 0$. Therefore

$$\forall n \geq 0, \quad \lim_{i\to\infty} \operatorname{diam}\left(\overline{M_n^i(\mathbf{t})}\right) = 0.$$

As a consequence, for any $n \geq 0$, there is $i \geq 1$ such that $\operatorname{diam}\left(M_n^i(\mathbf{t})\right) < C$. □

We know that for any set $M_n^i(\mathbf{t})$, where $n \geq 0$ and $\mathbf{t} = (t_n)_{n\geq 0}$ is an admissible address, we have

$$M_n^i(\mathbf{t}) \subseteq \left\{z \in \mathbb{C} \mid t_n < \operatorname{Re}(z) < t_n + 1, d_n^i(\mathbf{t}) \leq e_n \operatorname{Im}(z) \leq u_n^i(\mathbf{t})\right\}.$$

The next lemma shows roughly how the corners of this rectangle are mapped by T_n.

Lemma 4.19. *For any admissible address $\boldsymbol{t} = (t_n)_{n\geq 0}$ and any $n, i \geq 1$ with $n \leq l_i$, we have*

$$|u^i_{n-1}(\boldsymbol{t}) - e_{n-1}\operatorname{Im}(T_n(t_n + ie_n u^i_n(\boldsymbol{t})))| < 2C_2 + 2,$$

$$|d^i_{n-1}(\boldsymbol{t}) - e_{n-1}\operatorname{Im}(T_n(t_n + ie_n d^i_n(\boldsymbol{t})))| < 2C_2 + 2.$$

Here we interpret $T_n(t_n + ie_n\infty)$ as $\lim_{y\to\infty} T_n(t_n + ie_n y)$.

Proof. First note that whenever $u^i_n(\mathbf{t}) = \infty$, we have $n = l_i$. Therefore

$$\lim_{e_n \operatorname{Im}(z)\to\infty} e_{n-1}T_n(z) = \frac{1}{2\pi i}\log\left|\frac{p_{l_i}}{q_{l_i}}\right| + C_1^{\mathrm{b}}\left(\left|\frac{p_{l_i}}{q_{l_i}}\right|\right),$$

according to Lemma 4.4(i). Hence the expression $e_{n-1}\operatorname{Im}(T_n(t_n+ie_n u^i_n(\mathbf{t})))$ is always well-defined. Let

$$A := \{t_n + ie_n y \mid y \in (d^i_n(\mathbf{t}), u^i_n(\mathbf{t}))\}.$$

If d_H denotes the Hausdorff distance associated to the Euclidian distance on the plane, then we have

$$d_H(A, M^i_n(\mathbf{t})) \leq 1.$$

Since T_n is uniformly contracting with a factor of $3/4$, the bound on the Hausdorff distance implies that

$$\left|\sup\left\{e_{n-1}\operatorname{Im}(T_n(z)) \mid z \in M^i_n(\mathbf{t})\right\} - \sup\left\{e_{n-1}\operatorname{Im}(T_n(z)) \mid z \in A\right\}\right| \leq \frac{3}{4}.$$

Hence, according to Lemma 4.8(vii), we have

$$\begin{aligned}
&\left|u^i_{n-1}(\mathbf{t}) - e_{n-1}\operatorname{Im}(T_n(t_n + ie_n u^i_n(\mathbf{t})))\right| \\
&\quad= \left|\sup\left\{e_{n-1}\operatorname{Im}(T_n(z)) \mid z \in M^i_n(\mathbf{t})\right\} - e_{n-1}\operatorname{Im}(T_n(t_n + ie_n u^i_n(\mathbf{t})))\right| \\
&\quad< \left|\sup\left\{e_{n-1}\operatorname{Im}(T_n(z)) \mid z \in A\right\} - e_{n-1}\operatorname{Im}(T_n(t_n + ie_n u^i_n(\mathbf{t})))\right| + 1 \\
&\quad= \left|\sup\left\{e_{n-1}\operatorname{Im}(T_n(t_n + ie_n y)) \mid y \in (d^i_n(\mathbf{t}), u^i_n(\mathbf{t}))\right\} - e_{n-1}\operatorname{Im}(T_n(t_n + ie_n u^i_n(\mathbf{t})))\right| + 1 \\
&\quad< 2C_2 + 2.
\end{aligned}$$

A similar argument proves the estimate for $d^i_{n-1}(\mathbf{t})$. □

Next we show that there are many connected components of M_0 that are single points.

Lemma 4.20. *Let $\boldsymbol{t} = (t_n)_{n=0}^{\infty}$ be an admissible address such that*

$$\limsup_{k\to\infty} d\left(\frac{t_{l_k}}{q_{l_k}/|p_{l_k}|}, \mathbb{Z}\right) > 0.$$

Then $M_0(\boldsymbol{t})$ is a single point.

Proof. For all $i \geq 1$, define

$$\lambda_i := d\left(\frac{t_{l_i}}{q_{l_i}/|p_{l_i}|}, \mathbb{Z}\right) \in [0, 1).$$

Let $k \geq 1$ be an integer such that $\lambda_k > 0$ and let $n = l_k$. According to Lemma 4.7, the map $y \mapsto e_{n-1}\operatorname{Im}(\tilde{T}_n(t_n + iy)) : (-\infty, 2) \to \mathbb{R}$ is "almost increasing". Considering this and the fact

that $u_n^{k+1}(\mathbf{t}) \geq d_n^{k+1}(\mathbf{t}) \geq -2$, for $x \in \left\{u_n^{k+1}(\mathbf{t}), d_n^{k+1}(\mathbf{t})\right\}$, we have

$$\begin{aligned}\frac{1}{2\pi}\log\left(\frac{q_n}{|p_n|}\right) &> e_{n-1}\operatorname{Im}(\tilde{T}_n(t_n - ix)) - 1 > e_{n-1}\operatorname{Im}(\tilde{T}_n(t_n + 2i)) - 2\\ &= \frac{1}{2\pi}\log\left(d\left(t_n - 2i, \frac{q_n}{p_n}\mathbb{Z}\right)\right) - 2 > \frac{1}{2\pi}\log\left(d\left(t_n, \frac{q_n}{p_n}\mathbb{Z}\right)\right) - 2\\ &= \frac{1}{2\pi}\log\left(\frac{q_n}{|p_n|}\right) + \frac{1}{2\pi}\log\left(d\left(\frac{t_n}{q_n/|p_n|}, \mathbb{Z}\right)\right) - 2\\ &= \frac{1}{2\pi}\log\left(\frac{q_n}{|p_n|}\right) + \frac{1}{2\pi}\log(\lambda_k) - 2.\end{aligned}$$

Since $\kappa_n = \mathrm{b}$, we have $e_n = -1$. Hence

$$\begin{aligned}&|u_{n-1}^{k+1}(\mathbf{t}) - d_{n-1}^{k+1}(\mathbf{t})|\\ &\quad< |e_{n-1}\operatorname{Im}(T_n(t_n - iu_n^{k+1}(\mathbf{t}))) - e_{n-1}\operatorname{Im}(T_n(t_n - id_n^{k+1}(\mathbf{t})))| + 4C_2 + 4 \quad \text{(Lemma 4.19)}\\ &\quad= |e_{n-1}\operatorname{Im}(\tilde{T}_n(t_n - iu_n^{k+1}(\mathbf{t}))) - e_{n-1}\operatorname{Im}(\tilde{T}_n(t_n - id_n^{k+1}(\mathbf{t})))| + 6C_2 + 4 \quad \text{(Lemma 4.8(vi))}\\ &\quad< \left(\frac{1}{2\pi}\log(1/\lambda_k) + 2\right) + 6C_2 + 4.\end{aligned}$$

Let $C := \limsup_{i\to\infty} \lambda_i/2 > 0$. Then there are infinitely many $k \geq 1$ with $\lambda_k > C$. For any such k, we have

$$|u_{l_k-1}^{k+1}(\mathbf{t}) - d_{l_k-1}^{k+1}(\mathbf{t})| < \frac{1}{2\pi}\log(1/\lambda_k) + 6C_2 + 6 < \frac{1}{2\pi}\log(1/C) + 6C_2 + 6.$$

Now the statement follows from Lemma 4.18. □

Recall that

$$h_{l_i}^i = \left|\frac{q_{l_i}}{p_{l_i}}\right| \quad \text{and} \quad h_n^i = g_{n+1}(h_{n+1}^i) = g_{n+1} \circ g_{n+2} \circ \ldots \circ g_{l_i}(|q_{l_i}/p_{l_i}|),$$

for all $i \geq 1$, $n \geq 0$ with $n < l_i$. In the following proposition, we will show that the compositions of T_n near the imaginary axis closely approximates the compositions of g_n. It follows that $|h_n^i|$ uniformly approximates the diameter of the set $M_n^i(\mathbf{0})$, for $n \geq 0$, $i \geq 1$ with $n < l_i$. We use this to conclude that $M_0(\mathbf{0})$ is a single point if and only if condition $\mathcal{C}$ is satisfied.

Proposition 4.21. *Let $\boldsymbol{t} = (t_n)_{n\geq 0}$ be an admissible address such that $\sup_{n\geq 0} t_n < \infty$. Then we have*

(i) $\sup\{|d_n^i(\boldsymbol{t})| \mid n \geq 0, i \geq 1, n < l_i\} < \infty$,
(ii) $\sup\{|u_n^i(\boldsymbol{t}) - e_n h_n^i| \mid n \geq 0, i \geq 1, n < l_i\} < \infty$.

As a consequence, $M_0(\boldsymbol{t})$ is a point if and only if condition $\mathcal{C}$ is satisfied.

Proof. We first show that assuming (i) and (ii), condition $\mathcal{C}$ determines whether $M_0(\mathbf{t})$ is a point or not. Let

$$D := \sup_{n\geq 0, i\geq 1, n<l_i} |d_n^i(\mathbf{t})| < \infty, \quad D' := \sup_{n\geq 0, i\geq 1, n<l_i} |u_n^i(\mathbf{t}) - e_n h_n^i| < \infty.$$

Then

$$|u_n^i(\mathbf{t}) - d_n^i(\mathbf{t})| - D - D' \leq |h_n^i| \leq |u_n^i(\mathbf{t}) - d_n^i(\mathbf{t})| + D + D'.$$

Now the claim follows from Lemma 3.4 and Lemma 4.18.

To prove (i) and (ii), let

$$C := \sup_{n\geq 0} t_n < \infty.$$

(i): Let $D := 3C+8C_2+26$. We use induction on l_i-n to show that $|d_n^i(\mathbf{t})| < D$. For $n = l_i-1$, we have $d_n^i(\mathbf{t}) \in [-2, 0]$ and the statement is true. According to the induction hypothesis, there is $N \geq 0$ such that $|d_m^i(\mathbf{t})| < D$ for all $m \geq 0$, $i \geq 1$ with $l_i - m \leq N$. Let $n := l_i - N \geq 1$.

Choose a point $z_n^i \in M_n^i(\mathbf{t})$ such that $|d_n^i(\mathbf{t}) - e_n \operatorname{Im}(z_n^i)| < 1$. We have

$$|\operatorname{Re}(z_n^i)| < C + 1, \quad |\operatorname{Im}(z_n^i)| < D + 1, \quad |z_n^i - (t_n + ie_n d_n^i(\mathbf{t}))| < 2.$$

Based on Lemma 4.19 and Lemma 4.8(iii),

$$|d_{n-1}^i(\mathbf{t}) - e_{n-1} \operatorname{Im}(T_n(z_n^i))| < |d_{n-1}^i(\mathbf{t}) - e_{n-1} \operatorname{Im}(T_n(t_n + ie_n d_n^i(\mathbf{t})))| + 3/2 < 2C_2 + 4.$$

Hence, using Lemma 4.8(iii) and the fact that $T_n(0) \in \{0, 1/2, 1\}$, we have

$$\begin{aligned} |d_{n-1}^i(\mathbf{t})| &\leq |\operatorname{Im} T_n(z_n^i)| + |d_{n-1}^i(\mathbf{t}) - e_{n-1} \operatorname{Im}(T_n(z_n^i))| \\ &< (3|z_n^i|/4 + 1) + 2C_2 + 4 \\ &\leq 3(|\operatorname{Re}(z_n^i)| + |\operatorname{Im}(z_n^i)|)/4 + 2C_2 + 5 \\ &< 3(C + D + 2)/4 + 2C_2 + 5 = D. \end{aligned}$$

(ii): Let $D' := 3C + 12C_2 + 16$. We use induction on $l_i - n$ to prove $|u_n^i(\mathbf{t}) - e_n h_n^i| < D'$. The base of induction is when $n = l_i - 1$. According to Lemma 4.4(i), we have

$$\lim_{e_{l_i} \operatorname{Im}(z) \to \infty} T_{l_i}(z) = \frac{e_{l_i-1}}{2\pi i} \log \left| \frac{p_{l_i}}{q_{l_i}} \right| + e_{l_i-1} C_1^{\mathrm{b}} \left(\left| \frac{p_{l_i}}{q_{l_i}} \right| \right).$$

Therefore using the definition of $h_{l_i-1}^i$ and Lemmas 4.19 and 4.4(iii), we have

$$\begin{aligned} |u_{l_i-1}^i(\mathbf{t}) - h_{l_i-1}^i| &= \left| u_{l_i-1}^i(\mathbf{t}) - \frac{e_{l_i-1}}{2\pi} \log \left| \frac{q_n}{p_n} \right| \right| \\ &< \left| \lim_{e_{l_i} \operatorname{Im}(z) \to \infty} T_{l_i}(z) - \frac{e_{l_i-1}}{2\pi} \log \left| \frac{q_n}{p_n} \right| \right| + 2C_2 + 2 < 2C_2 + 4 < D'. \end{aligned}$$

By induction hypothesis, there is $N \geq 0$ such that $|u_m^i(\mathbf{t}) - e_m h_m^i| < D'$ for all $m, i \geq 0$ with $l_i - m \leq N$. Let $n := l_i - N \geq 1$. We have

$$\begin{aligned} |u_{n-1}^i(\mathbf{t}) - e_{n-1} h_{n-1}^i| &< |e_{n-1} \operatorname{Im}(T_n(t_n + ie_n u_n^i(\mathbf{t}))) - e_{n-1} h_{n-1}^i| + 2C_2 + 2 && \text{(Lemma 4.19)} \\ &\leq |\operatorname{Im}(T_n(ie_n u_n^i(\mathbf{t}))) - g_n(h_n^i)| + 3C/4 + 2C_2 + 2 && \text{(Lemma 4.8(iii))} \\ &\leq |\operatorname{Im}(T_n(ih_n^i))) - g_n(h_n^i)| + 3D'/4 + 3C/4 + 2C_2 + 2 && \text{(Lemma 4.8(iii))} \\ &< (C_2 + 2) + 3D'/4 + 3C/4 + 2C_2 + 2 = D'. && \text{(Lemma 4.8(viii))} \end{aligned}$$

□

Before we prove that every connected component of M_0 is a point if $M_0(\mathbf{0})$ is a point, we state two lemmas that will help us compare the effect of compositions of T_n between the zero address and an arbitrary address. The next lemma may be considered as a generalisation of Lemma 4.8(iv) to compositions of T_n with a uniform additive error.

Lemma 4.22. *Let $w_0 \in M_0$, $m \geq 0$ and define*

$$\forall 0 \leq n \leq m, \quad w_n^m := T_{n+1} \circ T_{n+2} \circ \cdots \circ T_m(w_m).$$

Then, for all $0 \leq n \leq m$, we have

$$e_n \operatorname{Im}(w_n^m - w_n) < 8C_2 + 12.$$

Proof. We use induction on $m-n$ to prove the lemma. If $n=m$, then $w_n^m = w_n$ and there is nothing to prove. If $m = n+1$, then $w_n^m = T_m(w_m^m) = T_m(w_m) = w_n - t_n$, and the statement is true. Assume the statement is true for some $0 < n < m$. Then we have

$$e_n \operatorname{Im}(w_n^m) < e_n \operatorname{Im}(w_n) + C = e_n \operatorname{Im}(w_n + ie_nC),$$

where $C := 8C_2 + 12$. Hence

$$\begin{aligned}
e_{n-1}\operatorname{Im}(w_{n-1}^m) &= e_{n-1}\operatorname{Im}(T_n(w_n^m)) \\
&< e_{n-1}\operatorname{Im}(T_n(i\operatorname{Im}(w_n^m))) + 1 && \text{(Lemma 4.8(iii))} \\
&\le e_{n-1}\operatorname{Im}(T_n(t_n + i\operatorname{Im}(w_n^m))) + 1 && \text{(Lemma 4.8(iv))} \\
&< e_{n-1}\operatorname{Im}(T_n(t_n + i\operatorname{Im}(w_n + ie_nC))) + 2C_2 + 2 && \text{(Lemma 4.8(vii))} \\
&< e_{n-1}\operatorname{Im}(T_n(t_n + i\operatorname{Im}(w_n))) + 3C/4 + 2C_2 + 2 && \text{(Lemma 4.8(iii))} \\
&< e_{n-1}\operatorname{Im}(T_n(w_n)) + 3C/4 + 2C_2 + 3 && \text{(Lemma 4.8(iii))} \\
&= e_{n-1}\operatorname{Im}(w_{n-1}) + C. && \square
\end{aligned}$$

The following lemma compares the vertical contraction of compositions of T_n's between an arbitrary address and the zero address.(cf. Lemma 4.8(v))

Lemma 4.23. *Let $m \ge 0$ be an integer and let $\boldsymbol{t} = (t_n)_{n=0}^{\infty}$ and $\hat{\boldsymbol{t}} = (\hat{t}_n)_{n=0}^{\infty}$ be admissible addresses such that $t_n = \hat{t}_n$ for all $0 \le n < m$. Let $w_0 \in M_0(\boldsymbol{t})$ and $\hat{w}_0 \in M_0(\hat{\boldsymbol{t}})$ such that*

$$e_n \operatorname{Im}(\hat{w}_n) \ge e_n \operatorname{Im}(w_n) - C,$$

for all $0 \le n \le m$ and for some constant $C > 0$. Define

$$\forall 0 \le n \le m, \quad w_n^m := T_{n+1} \circ T_{n+2} \circ \cdots \circ T_m(w_m), \quad \hat{w}_n^m := T_{n+1} \circ T_{n+2} \circ \cdots \circ T_m(\hat{w}_m).$$

Then we have

$$e_n \operatorname{Im}(\hat{w}_n) - e_n \operatorname{Im}(w_n) < e_n \operatorname{Im}(\hat{w}_n^m) - e_n \operatorname{Im}(w_n^m) + 6C + 64C_2 + 148,$$

for all $0 \le n \le m$.

Proof. We use backward induction on n to prove the lemma. If $n = m$, then $w_n^m = w_n$ and $\hat{w}_n^m = \hat{w}_n$ and there is nothing to prove. If $n = m-1$, then $\operatorname{Im}(w_n) = \operatorname{Im}(w_n^m)$ and $\operatorname{Im}(\hat{w}_n) = \operatorname{Im}(\hat{w}_n^m)$. Hence the statement is true. Assume the statement is true for some $0 < n < m$ and define

$$h := e_n \operatorname{Im}(\hat{w}_n) - e_n \operatorname{Im}(w_n) + C \ge 0, \quad y := e_n \operatorname{Im}(\hat{w}_n) + C' + C + 2,$$

$$x := h + e_n \operatorname{Im}(w_n^m) + 2 \ge h, \quad x' := e_n \operatorname{Im}(\hat{w}_n^m) + C'' + C + 2,$$

where $C' := 8C_2 + 12$ and $C'' := 6C + 64C_2 + 148$. Using Lemma 4.22, we have

$$\begin{aligned}
y - x &= (e_n \operatorname{Im}(\hat{w}_n) + C' + C + 2) - (e_n \operatorname{Im}(\hat{w}_n) - e_n \operatorname{Im}(w_n) + C) - (e_n \operatorname{Im}(w_n^m) + 2) \\
&= C' + e_n \operatorname{Im}(w_n) - e_n \operatorname{Im}(w_n^m) > 0.
\end{aligned}$$

Induction hypothesis implies that

$$x' - x = e_n \operatorname{Im}(\hat{w}_n^m) - e_n \operatorname{Im}(w_n^m) + C'' - (e_n \operatorname{Im}(\hat{w}_n) - e_n \operatorname{Im}(w_n)) > 0.$$

Therefore, according to Lemma 3.5 and Lemma 3.2, we have

$$\begin{aligned}
e_{n-1}g_n(e_ny) - e_{n-1}g_n(e_ny - e_nh) &< e_{n-1}g_n(e_nx) - e_{n-1}g_n(e_nx - e_nh) + 1 \\
&\le e_{n-1}g_n(e_nx') - e_{n-1}g_n(e_nx - e_nh) + 1
\end{aligned}$$

Now we may apply Lemma 4.8(viii) and Lemma 4.8(iii) to see that

$$\begin{aligned}
&e_{n-1}\operatorname{Im}(T_n(i\operatorname{Im}(\hat{w}_n))) - e_{n-1}\operatorname{Im}(T_n(i\operatorname{Im}(w_n)))\\
&\quad< e_{n-1}\operatorname{Im}(T_n(ie_ny)) - e_{n-1}\operatorname{Im}(T_n(ie_ny - ie_nh)) + 3C'/2 + 3C/4 + 3\\
&\quad< e_{n-1}g_n(e_ny) - e_{n-1}g_n(e_ny - e_nh) + 3C'/2 + 3C/4 + 2C_2 + 7\\
&\quad< e_{n-1}g_n(e_nx') - e_{n-1}g_n(e_nx - e_nh) + 3C'/2 + 3C/4 + 2C_2 + 8\\
&\quad< e_{n-1}\operatorname{Im}(T_n(ie_nx')) - e_{n-1}\operatorname{Im}(T_n(ie_nx - ie_nh)) + 3C'/2 + 3C/4 + 4C_2 + 12\\
&\quad< e_{n-1}\operatorname{Im}(T_n(i\operatorname{Im}(\hat{w}_n^m))) - e_{n-1}\operatorname{Im}(T_n(i\operatorname{Im}(w_n^m))) + 3C''/4 + 3C'/2 + 3C/2 + 4C_2 + 15\\
&\quad= e_{n-1}\operatorname{Im}(T_n(i\operatorname{Im}(\hat{w}_n^m))) - e_{n-1}\operatorname{Im}(T_n(i\operatorname{Im}(w_n^m))) + C'' - 4.
\end{aligned}$$

Hence we may use Lemma 4.8(iii) and Lemma 4.8(v) to obtain

$$\begin{aligned}
&e_{n-1}\operatorname{Im}(\hat{w}_{n-1}) - e_{n-1}\operatorname{Im}(w_{n-1})\\
&\qquad= e_{n-1}\operatorname{Im}(T_n(\hat{w}_n)) - e_{n-1}\operatorname{Im}(T_n(w_n))\\
&\qquad< e_{n-1}\operatorname{Im}(T_n(t_n + i\operatorname{Im}(\hat{w}_n))) - e_{n-1}\operatorname{Im}(T_n(t_n + i\operatorname{Im}(w_n))) + 2\\
&\qquad\le e_{n-1}\operatorname{Im}(T_n(i\operatorname{Im}(\hat{w}_n))) - e_{n-1}\operatorname{Im}(T_n(i\operatorname{Im}(w_n))) + 2\\
&\qquad< e_{n-1}\operatorname{Im}(T_n(i\operatorname{Im}(\hat{w}_n^m))) - e_{n-1}\operatorname{Im}(T_n(i\operatorname{Im}(w_n^m))) + C'' - 2\\
&\qquad< e_{n-1}\operatorname{Im}(\hat{w}_{n-1}^m) - e_{n-1}\operatorname{Im}(w_{n-1}^m) + C''.
\end{aligned}$$

□

Definition 4.24. *Let $P : M_0 \to M_0(\boldsymbol{0})$ be the map defined as*

$$\forall z_0 \in M_0, \quad P(z_0) = \min\{M_0(\boldsymbol{0}) \cap \overline{\{T_1 \circ T_2 \circ \cdots \circ T_n(z_n) \mid n \ge 0\}}\},$$

where the minimum is taken with respect to the total order "$\succcurlyeq$" on $M_0(\boldsymbol{0})$.

For any admissible address $\mathbf{t}$, the relation "$\succcurlyeq$" is a total order on $M_0(\mathbf{t})$. This relation allows us to compare different points on $M_0(\mathbf{0})$. As we will see in the following lemma, the map P provides a way to compare arbitrary points of M_0 by mapping them into the set $M_0(\mathbf{0})$. We will also use this map to define the dynamics on the model in Section 5.

Lemma 4.25. *For any admissible address $\boldsymbol{t}$, the map $P|_{M_0(\boldsymbol{t})} : M_0(\boldsymbol{t}) \to M_0(\boldsymbol{0})$ is injective and monotone with respect to the order "$\preccurlyeq$".*

Proof. Let $\mathbf{t} = (t_n)_{n=0}^{\infty}$ be an admissible address and w_0 and $\hat{w}_0$ be two distinct points in $M_0(\mathbf{t})$ such that $w_0 \preccurlyeq \hat{w}_0$. According to Corollary 4.17, we have $e_n\operatorname{Im}(\hat{w}_n) \ge e_n\operatorname{Im}(w_n) - C$, for all $n \ge 0$, where $C := 8C_2 + 7$. For all $m \ge n \ge 0$, define

$$\begin{aligned}
w_n^m &:= T_{n+1} \circ T_{n+2} \circ \cdots \circ T_m(w_m),\\
\hat{w}_n^m &:= T_{n+1} \circ T_{n+2} \circ \cdots \circ T_m(\hat{w}_m).
\end{aligned}$$

According to Lemma 4.23, we have

$$e_n\operatorname{Im}(\hat{w}_n^m) - e_n\operatorname{Im}(w_n^m) > e_n\operatorname{Im}(\hat{w}_n) - e_n\operatorname{Im}(w_n) - C',$$

for all $m \ge n \ge 0$, where $C' := 6C + 64C_2 + 148$. Hence

$$\liminf_{m\to\infty} e_n\operatorname{Im}(\hat{w}_n^m) \ge e_n\operatorname{Im}(\hat{w}_n) - e_n\operatorname{Im}(w_n) - C' + \liminf_{m\to\infty} e_n\operatorname{Im}(w_n^m),$$

for all $n \ge 0$.

Let $z_0 := P(w_0)$ and $\hat{z}_0 := P(\hat{w}_0)$. For any $n \ge 0$, there is a subsequence of $(\hat{w}_n^m)_{m=n}^{\infty}$ that tends to $\hat{z}_n$. Hence we have

$$e_n\operatorname{Im}(\hat{z}_n) \ge \liminf_{m\to\infty} e_n\operatorname{Im}(\hat{w}_n^m)$$

On the other hand, for any $n \geq 0$, any limit point of $(w_n^m)_{m=n}^{\infty}$ belongs to $\{z_n' \in M_n(\mathbf{0}) \mid z_n' \succcurlyeq z_n\}$. Hence, according to Corollary 4.17, we have

$$\liminf_{m\to\infty} e_n \operatorname{Im}(\hat{w}_n^m) \geq e_n \operatorname{Im}(z_n) - C.$$

Therefore we have

$$e_n \operatorname{Im}(\hat{z}_n) \geq e_n \operatorname{Im}(\hat{w}_n) - e_n \operatorname{Im}(w_n) - C' + e_n \operatorname{Im}(z_n) - C,$$

for all $n \geq 0$. Since $\hat{w}_0 \succcurlyeq w_0$ and $\hat{w}_0 \neq w_0$, we have

$$\lim_{n\to\infty} e_n \operatorname{Im}(\hat{w}_n) - e_n \operatorname{Im}(w_n) = \infty.$$

Hence we have

$$\lim_{n\to\infty} e_n \operatorname{Im}(\hat{z}_n) - e_n \operatorname{Im}(z_n) = \infty.$$

Therefore $\hat{z}_0 \succcurlyeq z_0$ and $\hat{z}_0 \neq z_0$. □

Corollary 4.26. *If $M_0(\boldsymbol{0})$ consists of a point, then every connected component of M_0 also consists of a point.*

Proof. Assume $M_0(\mathbf{0})$ is a single point and let $\mathbf{t}$ be an admissible address. Since $P|_{M_0(\mathbf{t})} : M_0(\mathbf{t}) \to M_0(\mathbf{0})$ is injective, $M_0(\mathbf{t})$ is also a single point. □

4.6. **Saturated subsets.** Here we will define *saturated subsets* and describe some of their properties. These sets are crucial in our analysis of the model and its dynamics. In Subsection 5.2, we will show that the set of saturated subsets is equal to the set of closed invariant subsets of the model under the dynamics.

Definition 4.27. *Let z_0 and w_0 be two points in M_0 and let $(z_n)_{n=0}^{\infty}$ and $(w_n)_{n=0}^{\infty}$ be their associated sequences, respectively. We say w_0 and z_0 have the same tail if there is a $N \geq 0$ such that for all $n \geq N$, we have $w_n = z_n$. Similarly, we say two admissible addresses $\boldsymbol{t} = (t_n)_{n=0}^{\infty}$ and $\boldsymbol{t'} = (t_n')_{n=0}^{\infty}$ have the same tail if there is some $N \geq 0$ such that $t_n = t_n'$ for all $n \geq N$.*

Definition 4.28. *We say a set $I \subseteq M_0$ is a saturated set if it is closed and for all two points $z_0, w_0 \in M_0$ that have the same tail, we have $z_0 \in I$ if and only if $w_0 \in I$. For any $z_0 \in M_0$, we define $\mathcal{I}_0(z_0)$ to be the smallest saturated set containing z_0. In other words*

$$\mathcal{I}_0(z_0) := \bigcap_{z_0 \in I \text{ and } I \text{ is saturated}} I.$$

We use the map $z \mapsto e^{2\pi i z}$ from M_0 to M_* to define saturated subsets of M_* and $\mathcal{I}_*$.

Lemma 4.29. *For any $z_0 \in M_0$, we have*

$$\mathcal{I}_0(z_0) = \overline{\{w_0 \in M_0 \mid w_0 \text{ has the same tail as } z_0\}}.$$

Moreover, for any admissible address $\boldsymbol{t}$, we have

$$\mathcal{I}_0(z_0) \cap M_0(\boldsymbol{t}) \neq \emptyset.$$

Proof. Let $\mathbf{t} = (t_n)_{n=0}^{\infty}$ and $\mathbf{t}' = (t_n')_{n=0}^{\infty}$ be two admissible addresses that have the same tail and let $k \geq 1$ be an integer such that $t_n = t_n'$ for all $n \geq l_k$. Then we have $M_{l_k}^{k+1}(\mathbf{t}) = M_{l_k}^{k+1}(\mathbf{t}')$ by definition. We also have

$$M_0^{k+1}(\mathbf{t}) = T_1(t_1 + T_2(t_2 + \cdots T_{l_k}(M_{l_k}^{k+1}(\mathbf{t}))\cdots)),$$

$$M_0^{k+1}(\mathbf{t}') = T_1(t_1 + T_2(t_2 + \cdots T_{l_k}(M_{l_k}^{k+1}(\mathbf{t}'))\cdots)).$$

Hence we obtain a homeomorphism $T_{\mathbf{t},\mathbf{t}'} : M_0^{k+1}(\mathbf{t}) \to M_0^{k+1}(\mathbf{t}')$. Note that for all $w_0 \in M_0^{k+1}(\mathbf{t})$, the points w_0 and $T_{\mathbf{t},\mathbf{t}'}(w_0)$ have the same tail.

Let $K := \{w_0 \in M_0 \mid w_0 \text{ has the same tail as } z_0\}$ and assume $x_0 \in \overline{K}$. Also assume $y_0 \in M_0$ has the same tail as x_0. Therefore there is $k \geq 1$ such that $x_n = y_n$ for all $n \geq l_k$. Let $\mathbf{t}$ and $\mathbf{t}'$ denote the addresses of x_0 and y_0 respectively and define $T_{\mathbf{t},\mathbf{t}'}$ as above. Clearly we have $y_0 = T_{\mathbf{t},\mathbf{t}'}(x_0)$. Since $x_0 \in \overline{K}$, there is a sequence $(z_0^i)_{i=1}^{\infty}$ of points in K that converge to x_0. Possibly after removing finitely many elements, we may assume that $z_0^i \in M_0^{k+1}(\mathbf{t})$ for all $i \geq 1$. Therefore the sequence $(T_{\mathbf{t},\mathbf{t}'}(z_0^i))_{i=1}^{\infty}$ is a sequence in K that converge to $T_{\mathbf{t},\mathbf{t}'}(x_0) = y_0$. Hence the set $\overline{K}$ is saturated. On the other hand, any saturated set that contains z_0 also contains K. Hence we have $\mathcal{I}_0(z_0) = \overline{K}$.

Now assume $\mathbf{t}$ is the address of z_0 and let $\mathbf{t}'$ denote an arbitrary admissible address. For all $k \geq 1$, define

$$\mathbf{t}^k := (t_0', t_1', t_2', \cdots, t_{l_k-1}', t_{l_k}, t_{l_k+1}, \cdots).$$

Therefore we have $M_0^k(\mathbf{t}') = M_0^k(\mathbf{t}^k)$. We may define $T_{\mathbf{t},\mathbf{t}^k} : M_0^{k+1}(\mathbf{t}) \to M_0^{k+1}(\mathbf{t}^k)$ as above. Therefore the sequence $(T_{\mathbf{t},\mathbf{t}^k}(z_0))_{k=1}^{\infty}$ is a sequence in $\mathcal{I}_0(z_0)$ and for all $k \geq 1$, we have

$$T_{\mathbf{t},\mathbf{t}^k}(z_0) \in M_0^{k+1}(\mathbf{t}^k) = M_0^{k+1}(\mathbf{t}') \subseteq M_0^1(\mathbf{t}').$$

Since the set $M_0^1(\mathbf{t}')$ is bounded, this sequence has a convergent subsequence. Clearly any limit of such subsequence belongs to the set

$$\mathcal{I}_0(z_0) \cap M_0(\mathbf{t}') = \bigcap_{k\geq 1} (\mathcal{I}_0(z_0) \cap M_0^{k+1}(\mathbf{t}')). \qquad \square$$

Lemma 4.30. *If $z_0 \in M_0(\boldsymbol{0})$, then*

$$\mathcal{I}_0(z_0) \supseteq \{w_0 \in M_0 \mid \inf_{n\geq 0} e_n \operatorname{Im}(w_n - z_n) > -\infty\}.$$

Proof. Let $\mathbf{t} = (t_n)_{n=0}^{\infty}$ be an admissible address and let $w_0 \in M_0(\mathbf{t})$ be a point such that $e_n \operatorname{Im}(w_n) \geq e_n \operatorname{Im}(z_n) - C$ for all $n \geq 0$ and for some $C > 0$. For each $i \geq 1$, we will find an integer $0 \leq k_{l_i} < b_{l_i}$, such that

$$|T_{l_i}(w_{l_i}) - T_{l_i}(z_{l_i} + k_{l_i})| < 3C/4 + 2C_2 + 5.$$

Fix $i \geq 1$, let $m := l_i$ and $\delta := q_m/|p_m|$. If $\delta \geq 6$ and $e_m \operatorname{Im}(z_m) \leq \delta$, define $c := \lfloor \delta/2 \rfloor \geq 3$. In this case, we have

$$\begin{aligned} e_{m-1} \operatorname{Im}(T_m(z_m + c)) &> e_{m-1} \operatorname{Im}(\tilde{T}_m(z_m + c)) - C_2 = \frac{1}{2\pi} \log\left(d\left(z_m + c, \delta\mathbb{Z}\right)\right) - C_2 \\ &\geq \frac{1}{2\pi} \log(c - 1) - C_2 \geq \frac{1}{2\pi} \log(\delta/2 - 1) - C_2 \\ &\geq \frac{1}{2\pi} \log(\delta/3) - C_2 > \frac{1}{2\pi} \log(\delta) - C_2 - 1. \end{aligned}$$

In other cases, define $c := 0$. If $\delta < 6$, then

$$e_{m-1} \operatorname{Im}(T_m(z_m + c)) > -2 > \frac{1}{2\pi} \log(\delta) - C_2 - 3.$$

On the other hand, if $e_m \operatorname{Im}(z_m) > \delta$, then

$$e_{m-1} \operatorname{Im}(T_m(z_m + c)) > e_{m-1} \operatorname{Im}(\tilde{T}_m(z_m + c)) - C_2 = \frac{1}{2\pi} \log(\delta) - C_2.$$

In any case, we have found an integer $0 \le c < b_m$ such that

$$e_{m-1} \operatorname{Im}(T_m(z_m + c)) > \frac{1}{2\pi} \log(\delta) - C_2 - 3.$$

Consider the set

$$A := [0, \delta] \times i e_m [e_m \operatorname{Im}(z_m), \infty).$$

If $x \in A$, then according to Lemmas 4.8(vi) and 4.7 and the discussion above, we have

$$\begin{aligned} e_{m-1} \operatorname{Im}(T_m(x)) &< C_2 + e_{m-1} \operatorname{Im}(\tilde{T}_m(x)) < C_2 + 1 + \lim_{e_m \operatorname{Im}(y) \to \infty} e_{m-1} \operatorname{Im}(\tilde{T}_m(x)) \\ &= \frac{1}{2\pi} \log(\delta) + C_2 + 1 < e_m \operatorname{Im}(T_m(z_m + c)) + 2C_2 + 4. \end{aligned}$$

On the other hand, using Lemmas 4.8(iv) and 4.8(vii), we see that

$$\begin{aligned} e_{m-1} \operatorname{Im}(T_m(x)) \ge e_{m-1} \operatorname{Im}(T_m(i \operatorname{Im}(x))) &> e_{m-1} \operatorname{Im}(T_m(i \operatorname{Im}(z_m))) - 2C_2 - 1 \\ &> e_{m-1} \operatorname{Im}(T_m(z_m)) - 2C_2 - 2. \end{aligned}$$

Since T_m is uniformly contracting with a factor smaller than one, we see that for any $x \in A$, there is $0 \le k \le c$ such that

$$|e_m \operatorname{Im}(T_m(x)) - e_m \operatorname{Im}(T_m(z_m + k))| < 2C_2 + 4.$$

In particular, since $w_m + i e_m C \in A$, we may choose k_{l_i} such that

$$\begin{aligned} |T_{l_i}(w_{l_i}) - T_{l_i}(z_{l_i} + k_{l_i})| &< |\operatorname{Im}(T_m(w_m)) - \operatorname{Im}(T_m(z_m + k_m))| + 1 \\ &< |\operatorname{Im}(T_m(w_m + i e_m C)) - \operatorname{Im}(T_m(z_m + k_m))| + 3C/4 + 1 \\ &< 3C/4 + 2C_2 + 5. \end{aligned}$$

For all $i \ge 1$,we use k_{l_i} to define

$$z_0^i := T_1(t_1 + T_2(t_2 + \cdots t_{l_i - 1} + T_{l_i}(k_{l_i} + z_{l_i}) \cdots))$$

Since T_n is uniformly contracting, we have

$$|w_0 - z_0^i| < (3/4)^{l_i - 1} |T_{l_i}(w_{l_i}) - T_{l_i}(z_{l_i} + k_{l_i})| < (3/4)^{l_i - 1}(3C/4 + 2C_2 + 5).$$

Hence we have $z_0^i \to w_0$. Since every z_0^i has the same tail as z_0, we have $w_0 \in \mathcal{I}_0(z_0)$. □

Lemma 4.31. *For every real number $C \ge 8C_2 + 7$ and every $z_0 \in M_0(\boldsymbol{0})$, we have*

$$\begin{aligned} \mathcal{I}_0(z_0) &= \{x_0 \in M_0 \mid \forall n \ge 0, e_n \operatorname{Im}(x_n - z_n) \ge -C\} \\ &= \{x_0 \in M_0 \mid \limsup_{n \to \infty} e_n \operatorname{Im}(x_n - z_n) > -\infty\}. \end{aligned}$$

Proof. Let $z_0 \in M_0(\mathbf{0})$ and define

$$A_n^C := \{x_0 \in M_0 \mid e_n \operatorname{Im}(x_n) \ge e_n \operatorname{Im}(z_n) - C\},$$

for $n \ge 0$ and $C \ge 8C_2 + 7$. Also define

$$A^C := \bigcap_{N \ge 0} \bigcup_{n \ge N} A_n^C = \{x_0 \in M_0 \mid \forall N \ge 0, \exists n \ge N, e_n \operatorname{Im}(x_n - z_n) \ge -C\}.$$

The statement of the lemma is equivalent to

$$\mathcal{I}_0(z_0) = \bigcap_{n \ge 0} A_n^C = \bigcup_{C' \ge 8C_2 + 7} A^{C'}.$$

for all $C \geq 8C_2 + 7$. Clearly, it is sufficient to prove that

$$\mathcal{I}_0(z_0) = \bigcap_{n\geq 0} A_n^C = A^C,$$

for all $C \geq 8C_2 + 7$.

Fix such C and consider the set A^C. Any point $w_0 \in M_0$ that has the same tail as z_0 belongs to A^C. Therefore we have $\mathcal{I}_0(z_0) \subseteq \overline{A^C}$. On the other hand, according to Lemma 4.30, we have $\mathcal{I}_0(z_0) \supseteq \cap_{n\geq 0} A_n^C$. Therefore to complete the proof, it is sufficient to show that $A^C = \cap_{n\geq 0} A_n^C$ and A^C is closed.

We will use an argument similar to proposition 4.16 to show that $A_n^C \subseteq A_{n-1}^C$ for all $n \geq 1$. Let $n \geq 1$ and $x_0 \in A_n^C$. If $x_n' := i\operatorname{Im}(x_n)$ and $z_n' := i\operatorname{Im}(z_n)$, then

$$e_n \operatorname{Im}(x_n') = e_n \operatorname{Im}(x_n) \geq e_n \operatorname{Im}(z_n) - C = e_n \operatorname{Im}(z_n' - ie_nC).$$

Since $z_0 \in M_0(\mathbf{0})$, we have $|z_n - z_n'| < 1$. Hence

$$\begin{aligned}
e_{n-1}\operatorname{Im}(T_n(x_n)) &\geq e_{n-1}\operatorname{Im}(T_n(x_n')) && \text{(Lemma 4.8(iv))}\\
&> e_{n-1}\operatorname{Im}(T_n(i\operatorname{Im}(z_n - ie_nC))) - 2C_2 - 1 && \text{(Lemma 4.8(vii))}\\
&> e_{n-1}\operatorname{Im}(T_n(i\operatorname{Im}(z_n))) - 3C/4 - 2C_2 - 1 && \text{(Lemma 4.8(iii))}\\
&> e_{n-1}\operatorname{Im}(T_n(z_n)) - 3/4 - 3C/4 - 2C_2 - 1 && \text{(Lemma 4.8(iii))}\\
&\geq e_{n-1}\operatorname{Im}(T_n(z_n)) - C.
\end{aligned}$$

Therefore we have $x_0 \in A_{n-1}^C$.

It follows that

$$A^C = \bigcap_{N\geq 0}\bigcup_{n\geq N} A_n^C = \bigcap_{n\geq 0} A_n^C$$

On the other hand, since each A_n^C is closed, the set A^C is also closed. □

Corollary 4.32. *For all $z_0 \in M_0(\boldsymbol{0})$ and any admissible address $\boldsymbol{t}$, there is $w_0 \in M_0(\boldsymbol{t})$ such that*

$$\mathcal{I}_0(z_0) \cap M_0(\boldsymbol{t}) = \{x_0 \in M_0(\boldsymbol{t}) \mid x_0 \succcurlyeq w_0\} \neq \emptyset.$$

Moreover, we have

$$\mathcal{I}_0(z_0) \cap M_0(\boldsymbol{0}) = \{x_0 \in M_0(\boldsymbol{t}) \mid x_0 \succcurlyeq z_0\}.$$

Proof. This follows directly from Lemma 4.29 and Lemma 4.31. □

4.7. **Base-points and end-points.**

Definition 4.33. *For any admissible address $\boldsymbol{t}$ and any integer $k \geq 0$, the set $M_k(\boldsymbol{t})$ is homeomorphic to $M_0(\boldsymbol{t})$. Therefore we may define the total order $\succcurlyeq$ on $M_k(\boldsymbol{t})$. Define $\operatorname{End}_k(\boldsymbol{t})$ as the smallest point in $M_k(\boldsymbol{t})$ and $\operatorname{Base}_k(\boldsymbol{t})$ as the largest point in $M_k(\boldsymbol{t})$. We call them end point and base point of $M_k(\boldsymbol{t})$ respectively. If $M_k(\boldsymbol{t})$ is not a point, we say $\operatorname{End}_k(\boldsymbol{t})$ is a proper end-point. For $z_0 \in M_0$, we also use the notation*

$$\operatorname{Base}_k(z_0) := \operatorname{Base}_k(\boldsymbol{t}(z_0)) \text{ and } \operatorname{End}_k(z_0) := \operatorname{End}_k(\boldsymbol{t}(z_0)).$$

Proposition 4.34. *The base of M_0, the set $\operatorname{Base}_0(M_0)$, is a Cantor set. Moreover, we have*

$$\operatorname{Base}_0(M_0) = \overline{\{M_0(\boldsymbol{t}) \mid M_0(\boldsymbol{t}) \text{ is a point }\}} = \mathcal{I}_0(\operatorname{Base}_0(\boldsymbol{0})).$$

Proof. According to Corollary 4.32, we have $\mathrm{Base}_0(M_0) \subseteq \mathcal{I}_0(\mathrm{Base}_0(\mathbf{0}))$ and

$$\mathcal{I}_0(\mathrm{Base}_0(\mathbf{0})) \cap M_0(\mathbf{0}) = \{\mathrm{Base}_0(\mathbf{0})\}.$$

It follows from the definition that for any saturated set I and any point $w_0 \in I$, we have $P(w_0) \in I$. Therefore, for every admissible address $\mathbf{t}$ and every $w_0 \in \mathcal{I}_0(\mathrm{Base}_0(\mathbf{0})) \cap M_0(\mathbf{t})$, we have $P(w_0) = \mathrm{Base}_0(\mathbf{0})$. Hence, it follows from Lemma 4.25 that $\mathcal{I}_0(\mathrm{Base}_0(\mathbf{0})) \cap M_0(\mathbf{t})$ is a single point. Now it follows from Corollary 4.32 that

$$\mathcal{I}_0(\mathrm{Base}_0(\mathbf{0})) \cap M_0(\mathbf{t}) = \{\mathrm{Base}_0(\mathbf{t})\}.$$

Hence we have $\mathrm{Base}_0(M_0) = \mathcal{I}_0(\mathrm{Base}_0(\mathbf{0}))$ and $\mathrm{Base}_0(M_0)$ is closed.

Clearly the set $\mathrm{Base}_0(M_0)$ is totally disconnected and contains every connected component of M_0 that is a point. Assume $z_0 \in \mathrm{Base}_0(M_0)$ with address $\mathbf{t} = (t_n)_{n=0}^{\infty}$. For any $i \geq 1$, let $z_0^i \in M_0$ be a point with address $\mathbf{t}^i := (t_n^i)_{n=0}^{\infty}$ such that

$$\forall n \leq l_i, \quad t_n^i := t_n,$$

and

$$\limsup_{k\to\infty} d\left(\frac{t_{l_k}^i}{q_{l_k}/p_{l_k}}, \mathbb{Z}\right) > 0.$$

According to Lemma 4.20, we have

$$M_0(\mathbf{t}^i) = \{z_0^i\} = \{\mathrm{Base}(\mathbf{t}^i)\}.$$

The set $\mathrm{Base}_0(M_0)$ is compact. Therefore, possibly after passing to a subsequence, we may assume that $(z_0^i)_{i=1}^{\infty}$ is convergent to a point in $\mathrm{Base}_0(M_0)$. The addresses of z^i's tends to $\mathbf{t}$, therefore this limit point is equal to $\mathrm{Base}_0(\mathbf{t}) = z_0$. □

Proposition 4.35. *We have*

$$M_0 = \mathcal{I}_0(\mathrm{End}_0(\boldsymbol{0})) = \overline{\{\mathrm{End}_0(\boldsymbol{t}) \mid \boldsymbol{t} \text{ has the same tail as } \boldsymbol{0}\}}.$$

In particular, if $M_0(\boldsymbol{0})$ is not a point, then the set of proper end-points of M_0 is dense in M_0.

Proof. The second equality follows from Lemma 4.29. According to Proposition 4.21, for any admissible address $\mathbf{t}$ that has the same tail as $\mathbf{0}$, the point $\mathrm{End}_0(\mathbf{t})$ is a proper end-point. Therefore it is sufficient to prove the first equality to complete the proof.

Let $w_0 \in M_0$ be a point and fix $n \geq 0$. Since $M_n(\mathbf{0}) = \bigcap_{i\geq 0, n<l_i} M_n^i(\mathbf{0})$, we have

$$\min\{e_n \operatorname{Im}(z) \mid z \in M_n(\mathbf{0})\} = \lim_{i\to\infty} d_n^i(\mathbf{0}).$$

Therefore, according to the definition of $\mathrm{End}_n(\mathbf{0})$ and Corollary 4.17, we have

$$e_n \operatorname{Im}(\mathrm{End}_n(\mathbf{0})) < \lim_{i\to\infty} d_n^i(\mathbf{0}) + 8C_2 + 7.$$

Hence, according to Proposition 4.21, there is a constant $D > 0$, independent of n, such that $e_n \operatorname{Im}(\mathrm{End}_n(\mathbf{0})) < D$, for all $n \geq 0$. Since $e_n \operatorname{Im}(w_n) > -2$, we have

$$\inf_{n\geq 0} e_n \operatorname{Im}(w_n - \mathrm{End}_n(\mathbf{0})) \geq -2 - \sup_{n\geq 0} e_n \operatorname{Im}(\mathrm{End}_n(\mathbf{0})) \geq -2 - D.$$

Now the statement follows from Lemma 4.30. □

4.8. **Topology of the model.** In this subsection, we characterise the topology of the model uniquely as a subset of the plane. For every admissible address $\mathbf{t}$ and every $n \geq 0$, the set $M_k(\mathbf{t})$ is a point or a Jordan arc. For $x, y \in M_k(\mathbf{t})$, we define $[x, y]$ to be the unique (possibly degenerate) Jordan arc in $M_k(\mathbf{t})$ joining x and y.

Definition 4.36. *In a metric space (M, d), given two sets $X, Y \subseteq M$, the Hausdorff distance of X and Y is defined as*

$$d_H(X, Y) := \inf\{\varepsilon > 0 \mid D(X, \varepsilon) \subseteq Y, D(Y, \varepsilon) \subseteq X\},$$

where $D(A, r) := \{x \in M \mid \exists z \in A, d(x, z) < r\}$, for all $A \subseteq M$ and $r > 0$. We also define

$$\mathrm{H}[M] := \{A \subseteq M \mid A \text{ is compact and non-empty }\}.$$

On $\mathrm{H}[M]$, the Hausdorff distance is a metric. The induced topology is called the Hausdorff topology.

Proposition 4.37. *If $(z_0^j)_{j=1}^\infty$ is a sequence in M_0 and that converges to a point $z_0 \in M_0$, then*

$$\lim_{j\to\infty} d_H([z_0, \mathrm{Base}_0(\boldsymbol{t})], [z_0^j, \mathrm{Base}_0(\boldsymbol{t}^j)]) = 0,$$

where d_H is the Hausdorff distance on subsets of the plane.

Proof. Let $\mathbf{t} = \mathbf{t}(z_0) = (t_n)_{n=0}^\infty$, $\mathbf{t} = \mathbf{t}^j(z_0) = (t_n^j)_{n=0}^\infty$ and $m \geq 0$. Assuming $C := 8C_2 + 7$, for all $j \geq 1$, define

$$K^j := \{t_m^j + iy \mid e_m \operatorname{Im}(z_m^j) - C \leq e_m y \leq e_m \operatorname{Im}(\mathrm{Base}_m(z_0^j)) + C\},$$

$$K := \{t_m + iy \mid e_m \operatorname{Im}(z_m) - C \leq e_m y \leq e_m \operatorname{Im}(\mathrm{Base}_m(z_0)) + C\}.$$

We have $\mathrm{Re}(M_m(\mathbf{t})) \subseteq (t_m, t_{m+1})$. Using this and Corollary 4.17, we see that

$$[z_m, \mathrm{Base}_m(\mathbf{t})] \subseteq D(K, 1) \quad \text{and} \quad K \subseteq D([z_m, \mathrm{Base}_m(\mathbf{t})], C).$$

In other words,

$$d_H([z_m, \mathrm{Base}_m(\mathbf{t})], K) \leq C.$$

Similarly, we have

$$d_H([z_m^j, \mathrm{Base}_m(\mathbf{t}^j)], K^j) \leq C,$$

for all $j \geq 1$.

Since $z_0^j \to z_0$, we have $\mathbf{t}^j \to \mathbf{t}$. Consider the sequence $(\mathrm{Base}_m(\mathbf{t}^j))_{j=1}^\infty$. Since $\mathrm{Base}_m(M_0)$ is compact, any convergent subsequence converges to a point with address $\mathbf{t}$ in $\mathrm{Base}_m(M_0)$. In other words, we have $\mathrm{Base}_m(\mathbf{t}^j) \to \mathrm{Base}_m(\mathbf{t})$. Using these facts, we see that there is $j_m > 0$ such that for all $j \geq j_m$, we have

$$|z_m^j - z_m| < 1, \quad |\mathrm{Base}_m(\mathbf{t}^j) - \mathrm{Base}_m(\mathbf{t})| < 1, \quad \forall 0 \leq n \leq m,\ t_n^j = t_n.$$

Therefore, we have $d_H(K, K^j) \leq 1$, for all $j \geq j_m$. Hence, for all $j \geq j_m$, we have

$$\begin{aligned} d_H([z_m, \mathrm{Base}_m(\mathbf{t})], [z_m^j, \mathrm{Base}_m(\mathbf{t}^j)]) &\leq d_H([z_m, \mathrm{Base}_m(\mathbf{t})], K) + d_H(K, K^j) \\ &\quad + d_H(K^j, [z_m^j, \mathrm{Base}_m(\mathbf{t}^j)]) \leq 2C + 1. \end{aligned}$$

If we define $T^m : \mathbb{H}_m \to \mathbb{H}_0$ as

$$\forall z \in \mathbb{H}_m, \quad T^m(z) := T_1(t_1 + T_2(t_2 + \cdots T_m(z) \cdots)),$$

then, for all $j \geq j_m$, we have

$$[z_0, \mathrm{Base}_0(\mathbf{t})] = T^m([z_m, \mathrm{Base}_m(\mathbf{t})]), \quad [z_0^j, \mathrm{Base}_0(\mathbf{t}^j)] = T^m([z_m^j, \mathrm{Base}_m(\mathbf{t}^j)]).$$

Hence we may use uniform contraction of T_n's to conclude that

$$d_H([z_0, \mathrm{Base}_0(\mathbf{t})], [z_0^j, \mathrm{Base}_0(\mathbf{t}^j)]) < (\frac{3}{4})^m (2C+1),$$

for all $j \geq j_m$. Since this is true for all $m \geq 0$, we see that

$$\lim_{j\to\infty} d_H([z_0, \mathrm{Base}_0(\mathbf{t})], [z_0^j, \mathrm{Base}_0(\mathbf{t}^j)]) = 0. \qquad \square$$

Definition 4.38. *Let $U \subseteq \mathbb{C}$ be an open set. A point $p \in \partial U$ is accessible from U if there is a continuous map $\lambda : [0,1) \to U$ such that*

$$\lim_{x\to 1} \lambda(x) = p.$$

Proposition 4.39. *Let $I \subseteq M_0$ be a saturated set and let $\gamma \subseteq I$ be a Jordan arc. Then any interior point of γ is not accessible from $\mathbb{C} \setminus I$.*

Proof. Let w_0 be an interior point of γ and let $\mathbf{t} := \mathbf{t}(w_0) = (t_n)_{n=0}^{\infty}$. There are points $x_0 \lessdot y_0$ in $I \cap M_0(\mathbf{t})$ such that $\gamma = [x_0, y_0]$. Let $i \geq 1$ and define the map $T^i : \mathbb{H}^{\mathrm{b}} \to M_0^i$ as

$$\forall z \in \mathbb{H}^{\mathrm{b}}, \quad T^i(z) := T_1(t_1 + T_2(t_2 + \cdots T_{l_i}(z) \cdots)).$$

Also define $\gamma^i := (T^i)^{-1}(\gamma) \subseteq M_{l_i}$.

If $t_{l_i} \geq 1$, then let $k_1 := -1$. In this case $\gamma^i + k_1$ is contained in M_{l_i} and therefore $T^i(\gamma^i + k_1) \subseteq M_0$. Since any point in $T^i(\gamma^i + k_1)$ has the same tail as some point in γ, we have $T^i(\gamma^i + k_1) \subseteq I$. Otherwise, if $t_{l_i} = 0$, let $k_1 := \lfloor |q_{l_i}/p_{l_i}| \rfloor - |q_{l_i}/p_{l_i}| - 1 \in [-1, 0)$. Here, we have $\gamma^i + k_1 \subseteq M_{l_i} - |q_{l_i}/p_{l_i}|$. Therefore, since T_{l_i} is periodic with period $|q_{l_i}/p_{l_i}|$, we have

$$T^i(\gamma^i + k_1) = T^i(\gamma^i + k_1 + |q_{l_i}/p_{l_i}|) \subseteq M_0.$$

Similar to before, we have $T^i(\gamma^i + k_1) \subseteq I$. We define k_2 similarly. If $t_{l_i} < \lfloor |q_{l_i}/p_{l_i}| \rfloor$, then let $k_2 := 1$. Otherwise, let $k_2 := -\lfloor |q_{l_i}/p_{l_i}| \rfloor + |q_{l_i}/p_{l_i}| + 1 \in (0, 1]$. Similar to before, we see that $T^i(\gamma^i + k_2) \subseteq I$.

Assume $l \leq l'$ are two real numbers such that $\mathrm{Im}(\gamma^i) = [l, l']$ and consider the set

$$\{x + iy \in \mathbb{C} \mid y \in [l, l']\} \setminus ((\gamma^i + k_1) \cup (\gamma^i + k_2)).$$

There is a unique connected component of this set that contains γ^i in its closure. Let $D_{l_i}^i$ denote the closure of this connected component and define $D_0^i := T^i(D_{l_i}^i)$. Note that we have $D_{l_i}^i \subseteq D(\gamma^i, 2)$. Therefore, using the uniform contraction of T_n's, we have

$$D_0^i \subseteq D(\gamma, 2.(3/4)^{l_i}).$$

We have $e_{l_i} = -1$ and $\gamma = [x_0, y_0]$ where $x_0 \lessdot y_0$. Using Corollary 4.17, we see that

$$\mathrm{Im}(x_{l_i}) \geq l - C, \quad \mathrm{Im}(y_{l_i}) \leq l' + C,$$

where $C := 8C_2 + 7$.

Assume $z_0 \in \partial D_0^i \setminus I$. By construction we have

$$\partial D_{l_i}^i \setminus ((\gamma^i + k_1) \cup (\gamma^i + k_2)) \subseteq (\mathbb{R} + il) \cup (\mathbb{R} + il').$$

Therefore $z_{l_i} \in \partial D_{l_i}^i$ and $\mathrm{Im}(z_{l_i}) \in \{l, l'\}$. We also know that $\gamma^i \subseteq M_{l_i}(\mathbf{t})$ and therefore $\mathrm{Re}(\gamma^i)$ is an interval with length less than one. Hence

$$d(z_{l_i}, \{x_{l_i}, y_{l_i}\}) < C + 3.$$

Therefore, using the uniform contraction of T_n's, we see that

$$d(z_0, \{x_0, y_0\}) < (3/4)^{l_i}(C + 3).$$

We now have the ingredients of the proof. Assume w_0 is accessible from $\mathbb{C} \setminus I$. Therefore there is a Jordan arc λ in the plane such that $\lambda \cap I = \{w_0\}$. We know that $\gamma \subseteq D_0^i \subseteq D(\gamma, 2.(3/4)^{l_i})$, for all $i \geq 1$. Therefore $\gamma = \cap_{i \geq 1} D_0^i$. The set $\lambda \setminus \{w_0\}$ does not intersect γ. Therefore there is $N \geq 1$ such that $\partial D_0^i \cap \lambda \neq \emptyset$, for all $i \geq N$. For every such i, choose $z_0^i \in \partial D_0^i \cap \lambda \subseteq \partial D_0^i \setminus I$. Based on the arguments above, we have

$$d(z_0^i, \{x_0, y_0\}) < (3/4)^{l_i}(C + 3),$$

for all $i \geq N$. Hence either x_0 or y_0 is a limit point of the sequence $(z_0^i)_{i=N}^{\infty}$. Since λ is compact, it follows that either x_0 or y_0 belong to λ. However this is impossible since $\{x_0, y_0\} \subseteq \gamma \subseteq I$. □

In Section 6 we will describe a certain class of topological objects in the plane, called *hairy Cantor Sets*. These sets are characterised by universal topological properties similar to Cantor sets. Moreover, all hairy Cantor sets are ambiently homeomorphic.

Theorem 4.40 (Theorem A)**.** *One the following statements is true.*

(i) The sequence $(\hat{p}_i/\hat{q}_i)_{i=1}^{\infty}$ satisfies condition $\mathcal{C}$ and M_* is a Cantor set.

(ii) The sequence $(\hat{p}_i/\hat{q}_i)_{i=1}^{\infty}$ does not satisfy the condition $\mathcal{C}$ and M_* is a hairy Cantor set.

Proof. Since M_* and M_0 are ambiently homeomorphic, it is sufficient to prove the theorem for M_0. If condition $\mathcal{C}$ is satisfied, then Proposition 4.21 and Corollary 4.26 imply that $M_0 = \text{Base}_0(M_0)$ and Proposition 4.34 implies that M_0 is a Cantor set. Now assume that condition $\mathcal{C}$ is not satisfied. Then $M_0 \subseteq \mathbb{C}$ is a compact set and the following statements are true.

(i) Every connected component of M_0 is either a Jordan arc or a single point. (Proposition 4.16)

(ii) The closure of the union of the point components of M_0 is a Cantor set, namely $\text{Base}_0(M_0)$. (Proposition 4.34)

(iii) Any arc component of M_0 meets $\text{Base}_0(M_0)$ at one of its end-points. (Definition of base-points)

(iv) For any sequence $(z^i)_{i \in \mathbb{N}}$ in M_0 with $z^i \to z$, we have $[\text{Base}_0(z^i), z^i] \to [\text{Base}_0(z), z]$ in the Hausdorff topology.(Proposition 4.37).

(v) The set $M_0 \setminus \text{Base}_0(M_0)$ is dense in $\text{Base}_0(M_0)$. (Proposition 4.35)

(vi) The points in the interior of the arc components of M_0 are not accessible from $\mathbb{C} \setminus M_0$. (Proposition 4.39)

Therefore, according to the definition, M_0 is a hairy Cantor set. □

5. Dynamics on the model

In this section, we define a function f_0 on the model. This function is designed to model the behaviour of infinitely satellite quadratic polynomials on their post-critical set. We will identify the closed invariant subsets of the model under f_0 and describe their topology. We will also show that f_0 is uniquely ergodic and describe the ergodic measure.

5.1. The function f_*.

Definition 5.1. *Let z_0 be a point in M_0. If $\boldsymbol{t}(z_0) = \textbf{-1}$, then we define*

$$f_0(z_0) := P(z_0) \in M_0(\boldsymbol{0}),$$

where $P : M_0 \to M_0(\boldsymbol{0})$ is the map described in Definition 4.24. Otherwise, the addresses $\boldsymbol{t}(z_0)$ and $\boldsymbol{t}(z_0)+1$ have the same tail. In this case, we define $f_0(z_0)$ to be the unique point in $M_0(\boldsymbol{t}(z_0)+1)$ that has the same tail as z_0. Since M_0 and M_* are homeomorphic under the map $z \mapsto e^{2\pi i z}$, we obtain a map $f_* : M_* \to M_*$ conjugate to $f_0 : M_0 \to M_0$.

Lemma 5.2. *If $z_0 \in M_0$ and $w_0 := f_0(z_0)$, then we have*

$$|\operatorname{Im}(z_n - w_n)| \le 12,$$

for all $n \ge 0$.

Proof. Let $\mathbf{t} := \mathbf{t}(z_0) = (t_n)_{n=0}^{\infty}$ and $\mathbf{t}' := \mathbf{t}(w_0) = (t'_n)_{n=0}^{\infty}$. First we consider the case $\mathbf{t} \ne \textbf{-1}$. In this case, the points z_0 and w_0 have the same tail. Let $k \ge 0$ be the smallest integer such that $z_k = w_k$. We have $\mathbf{t}' = \mathbf{t} + 1$. Therefore $t'_{k-1} = t_{k-1} - 1$ and

$$t'_m = 0, \quad q_m/|p_m| - 1 < \operatorname{Re}(z_m) < q_m/|p_m|,$$

for $1 \le m < k-1$. Hence $\operatorname{Re}(z_m - w_m) \le \operatorname{Re}(z_m) < q_m/|p_m|$, and

$$\operatorname{Re}(z_m - w_m) = \operatorname{Re}(z_m) - \operatorname{Re}(w_m) > (q_m/|p_m| - 1) - (t'_m + 1) = q_m/|p_m| - 2.$$

Now we use backward induction to prove the lemma. For $n \ge k$, we have $z_n = w_n$ and we also have $w_{k-1} = z_{k-1} + 1$. Hence, for all $n \ge k-1$, we have $\operatorname{Im}(z_n - w_n) = 0$. If $k \ge 2$, we may use Lemma 4.8(iii) to see that

$$|\operatorname{Im}(z_{k-2} - w_{k-2})| = |\operatorname{Im}(T_{k-1}(z_{k-1}) - T_{k-1}(w_{k-1}))| < \frac{3}{4}|z_{k-1} - w_{k-1}| = \frac{3}{4} < 12.$$

Assume $1 \le m \le k-2$ and the lemma is true for all $n \ge m$. According to Lemmas 4.1(i) and 4.4(i), we have $|T_m(w_m + q_m/|p_m|)) - T_m(w_m))| \le 1$. Therefore

$$\begin{aligned}
|\operatorname{Im}(z_{m-1} - w_{m-1})| &= |\operatorname{Im}(T_m(z_m) - T_m(w_m))| \le |T_m(z_m) - T_m(w_m)| \\
&\le |T_m(z_m) - T_m(w_m + q_m/|p_m|)| + 1 < \frac{3}{4}|z_m - w_m - q_m/|p_m|| + 1 \\
&\le \frac{3}{4}\left(|\operatorname{Im}(z_m - w_m)| + |\operatorname{Re}(z_m - w_m - q_m/|p_m|)|\right) + 1 \\
&< \frac{3}{4}(12+2) + 1 < 12.
\end{aligned}$$

The remaining case is when $\mathbf{t} = \textbf{-1}$. For $m \ge n \ge 0$, define

$$z_n^m := T_{n+1}(T_{n+2}(\cdots T_m(z_m)\cdots)).$$

Note that we have $w_0 = f_0(z_0) = P(z_0) \in \overline{\{z_0^m \mid m \ge 0\}}$ and therefore $w_n \in \overline{\{z_n^m \mid m \ge n\}}$. Hence it is sufficient to prove $|\operatorname{Im}(z_n^m - z_n)| \le 12$, for all $m \ge n \ge 0$. We use induction on $m-n$. If $m-n = 0$, then we have $z_n^m = z_n$ and if $m-n = 1$, then $z_n^m - z_n \in \mathbb{Z}$. Assume $|\operatorname{Im}(z_n^m - z_n)| \le 12$,

for some value of $m > n > 0$. We have $|z_n^m - i\,\mathrm{Im}(z_n^m)| < 1$ and $|z_n - (i\,\mathrm{Im}(z_n) + q_n/|p_n|)| < 1$. Hence

$$\begin{aligned}
|\,\mathrm{Im}(z_{n-1}^m - z_{n-1})| &= |\,\mathrm{Im}(T_n(z_n^m) - T_n(z_n))| \le |T_n(z_n^m) - T_n(z_n)| \\
&< |T_n(i\,\mathrm{Im}(z_n^m)) - T_n(i\,\mathrm{Im}(z_n) + q_n/|p_n|)| + 2 \quad \text{(Lemma 4.8(iii))} \\
&\le |T_n(i\,\mathrm{Im}(z_n^m)) - T_n(i\,\mathrm{Im}(z_n))| + 3 \quad \text{(Lemmas 4.1(i) and 4.4(i))} \\
&< \frac{3}{4}|\,\mathrm{Im}(z_n^m - z_n)| + 3 \le 12. \quad \text{(Lemma 4.8(iii))}
\end{aligned}$$

□

Theorem 5.3. *The function $f_* : M_* \to M_*$ is a homeomorphism.*

Proof. It is sufficient to prove that $f_0 : M_0 \to M_0$ is a homeomorphism. Since M_0 is a compact set, it is sufficient to show that f_0 is continuous and bijective. We start with continuity. Let $(z_0^i)_{i=1}^\infty$ be a sequence in M_0 converging to a point $z_0 \in M_0$. Also let $w_0 := f_0(z_0)$ and $w_0^i := f_0(z_0^i)$, for all $i \ge 1$. If $\mathbf{t}(z_0) = (t_n)_{n=0}^\infty \ne \mathbf{\text{-}1}$, then w_0 and z_0 have the same tail. Let $k \ge 0$ be the smallest integer such that $w_n = z_n$, for all $n \ge k$. We have

$$\begin{aligned}
w_0 &= T_1(T_2(\cdots T_{k-1}(z_{k-1} + 1)\cdots)) \\
&= T_1(T_2(\cdots T_{k-1}(T_{k-1}^{-1}(\cdots T_2^{-1}(T_1^{-1}(z_0 - t_0) - t_1) - t_2 \cdots) + 1)\cdots)).
\end{aligned}$$

Let T denote the function above that sends z_0 to w_0. For large i, we have $d(\mathbf{t}(z_0^i), \mathbf{t}(z_0)) < 2^{-k}$. For such value of i, the first k terms of the sequence $\mathbf{t}(z_0^i)$ is equal to the first k terms of $\mathbf{t}(z_0)$. Therefore the first k terms of the sequence $\mathbf{t}(w_0^i)$ is equal to the first k terms of $\mathbf{t}(w_0)$. Hence we have $T(z_0^i) = f_0(z_0^i) = w_0^i$, for large i. Therefore f_0 is continuous at z_0.

Next we assume $\mathbf{t}(z_0) = \mathbf{\text{-}1}$. Let $n \ge 1$ be an integer. We have

$$|w_n - w_n^i| \le |w_n + q_n/|p_n| - z_n| + |w_n^i + q_n/|p_n| - z_n^i| + |z_n - z_n^i|.$$

According to Lemma 5.2, we have

$$|\,\mathrm{Im}(w_n + q_n/|p_n| - z_n)| \le 12, \quad \forall i \ge 1,\ |\,\mathrm{Im}(w_n^i + q_n/|p_n| - z_n^i)| \le 12.$$

On the other hand, we have $\mathrm{Re}(w_n) \in (0,1)$ and $\mathrm{Re}(q_n/|p_n| - z_n) \in (0,1)$. Similarly, for large i, we also have $\mathrm{Re}(w_n^i) \in (0,1)$ and $\mathrm{Re}(q_n/|p_n| - z_n^i) \in (0,1)$. Therefore $|w_n + q_n/|p_n| - z_n| \le 14$ and

$$\limsup_{i\to\infty} |w_n^i + q_n/|p_n| - z_n^i| \le 14.$$

Therefore, according to Lemma 4.8(iii), we have

$$\begin{aligned}
\limsup_{i\to\infty} |w_0 - w_0^i| &\le (3/4)^n \limsup_{i\to\infty} |w_n - w_n^i| \\
&\le (3/4)^n |w_n + q_n/|p_n| - z_n| + (3/4)^n \limsup_{i\to\infty} |w_n^i + q_n/|p_n| - z_n^i| + (3/4)^n \lim_{i\to\infty} |z_n - z_n^i| \\
&\le 28(3/4)^n,
\end{aligned}$$

for all $n \ge 0$. Hence $\lim_{i\to\infty} w_0^i = w_0$.

To complete the proof, we need to show that f_0 is a bijection. Let w_0 be a point in M_0. If $\mathbf{t}(w_0) \ne \mathbf{0}$, then $\mathbf{t}(w_0) - 1$ has the same tail as $\mathbf{t}(w_0)$. Hence there is a unique point $z_0 \in M_0(\mathbf{t}(w_0) - 1)$ that has the same tail as w_0 and we have $f_0(z_0) = w_0$. Hence the map

$$f_0|_{M_0 \setminus M_0(\mathbf{\text{-}1})} : M_0 \setminus M_0(\mathbf{\text{-}1}) \to M_0 \setminus M_0(\mathbf{0})$$

is a bijection. According to Lemma 4.25, the function $f_0|_{M_0(\mathbf{\text{-}1})} = P|_{M_0(\mathbf{\text{-}1})} : M_0(\mathbf{\text{-}1}) \to M_0(\mathbf{0})$ is injective. Therefore f_0 is injective on M_0. The function f_0 is continuous and M_0 is compact. Hence the set $f_0(M_0)$ is a compact subset of M_0 that contains $M_0 \setminus M_0(\mathbf{0})$. According to Proposition 4.35, the set $\mathrm{End}_0(M_0)$ is dense in M_0. The set M_0 is either a Cantor set or a hairy Cantor set. Therefore

it does not have any isolated points. Hence $\mathrm{End}_0(M_0) \setminus \{\mathrm{End}_0(\mathbf{0})\}$ is also dense in M_0. Therefore we have $f_0(M_0) = M_0$. □

5.2. **Closed invariant subsets.** Here we will describe the topology of the closed invariant subsets and show that these sets form a simple curve in the space of non-empty compact subsets of the plane with the Hausdorff topology. In fact, we will show that closed invariant subsets are exactly saturated subsets, as defined in Subsection 4.6. Recall that for all $z_0 \in M_0$, the set $\mathcal{I}_0(z_0)$ is the smallest saturated subset containing z_0. We start by showing that these saturated subsets are invariant.

Lemma 5.4. *For all $z_0 \in M_0$, we have $\mathcal{I}_0(z_0) = \mathcal{I}_0(P(z_0)) = \mathcal{I}_0(f_0(z_0))$.*

Proof. Let $\mathbf{t} := \mathbf{t}(z_0) = (t_n)_{n=0}^{\infty}$ and $w_0 := P(z_0)$. If $\mathbf{t} \neq \mathbf{\text{-}1}$, then z_0 and $f_0(z_0)$ have the same tail and we have $\mathcal{I}_0(z_0) = \mathcal{I}_0(f_0(z_0))$. On the other hand, if $\mathbf{t} = \mathbf{\text{-}1}$, then we have $\mathcal{I}_0(f_0(z_0)) = \mathcal{I}_0(P(z_0)) = \mathcal{I}_0(w_0)$. Therefore we only need to show that $\mathcal{I}_0(z_0) = \mathcal{I}_0(w_0)$.

We have $w_0 = P(z_0) \in \mathcal{I}_0(z_0)$. Therefore $\mathcal{I}_0(P(z_0)) \subseteq \mathcal{I}_0(z_0)$. To complete the proof, it is sufficient to show that $z_0 \in \mathcal{I}_0(w_0)$. To prove by contradiction, assume $z_0 \notin \mathcal{I}_0(w_0)$. According to Lemma 4.31, we have

$$\limsup_{n\to\infty} e_n \operatorname{Im}(z_n - w_n) = -\infty.$$

Therefore, there is $C > 0$ such that

$$e_n \operatorname{Im}(z_n - w_n) < C,$$

for all $n \geq 0$.

For all $0 \leq n \leq m$, define

$$z_n^m := T_1 \circ T_2 \circ \cdots \circ T_m(z_m), \quad w_n^m := T_1(t_1 + T_2(t_2 + \cdots T_m(w_m) \cdots)).$$

According to Lemma 4.22, we have

$$e_n \operatorname{Im}(w_n) < e_n \operatorname{Im}(w_n^m) + C',$$

for all $0 \leq n \leq m$, where $C' := 8C_2 + 12$. Hence we have

$$e_n \operatorname{Im}(z_n) < e_n \operatorname{Im}(w_n^m) + C + C',$$

for all $0 \leq n \leq m$. For any $m \geq 0$, we have $\mathbf{t}(w_0^m) = (t_n, t_1, t_2, \cdots, t_{m-1}, 0, 0, \cdots)$. Therefore, according to Lemma 4.23, we have

$$-C - C' < e_n \operatorname{Im}(w_n^m - z_n) < e_n \operatorname{Im}(w_n - z_n^m) + C'',$$

for all $0 \leq n < m$, where $C'' := 6(C + C') + 64C_2 + 148$. We have $|\operatorname{Re}(w_n^m - z_n)| < 1$ and $|\operatorname{Re}(w_n - z_n^m)| < 1$. Therefore

$$\begin{aligned} |w_n^m - z_n| &< 1 + |\operatorname{Im}(w_n^m - z_n)| < 1 + (C + C') + |\operatorname{Im}(w_n - z_n^m)| + C'' \\ &< |w_n - z_n^m| + C + C' + C'' + 2. \end{aligned}$$

We have $w_0 = P(z_0)$. Therefore,

$$\liminf_{m\to\infty} |w_0 - z_0^m| = 0.$$

It follows that

$$\liminf_{m\to\infty} |w_n - z_n^m| = 0,$$

for all $n \geq 0$. Hence, using the uniform contraction of T_n's, we have

$$\begin{aligned} \liminf_{m\to\infty} |w_0^m - z_0| \leq (3/4)^n \liminf_{m\to\infty} |w_n^m - z_n| &\leq (3/4)^n (C + C' + C'' + 2 + \liminf_{m\to\infty} |w_n - z_n^m|) \\ &= (3/4)^n (C + C' + C'' + 2), \end{aligned}$$

for all $n \geq 0$. Therefore

$$\liminf_{m\to\infty} |w_0^m - z_0| = 0.$$

Since each w_0^m has the same tail as w_0, this implies that $z_0 \in \mathcal{I}_0(w_0)$. □

Theorem 5.5 (Theorem D)**.** *Every point in M_* is forward and backward recurrent for the function f_*.*

Proof. It is sufficient to prove the result for $f_0 : M_0 \to M_0$. Let $z_0 \in M_0$ and $\mathbf{t} := \mathbf{t}(z_0) = (t_n)_{n=0}^{\infty}$. First we show forward recurrence. Assume $\mathbf{t}$ and **-1** have the same tail. Therefore, there is $k \geq 0$ such that $\mathbf{t} + k = \mathbf{-1}$. Let $w_0 := f_0^{k+1}(z_0) \in M_0(\mathbf{0})$. If $x_0 \in M_0$ has the same tail as $w_0 \in M_0(\mathbf{0})$, then there is an integer $m \geq 0$ such that $\mathbf{t}(x_0) = \mathbf{0} + m$ and $x_0 = f_0^m(w_0)$. Hence, using Lemmas 5.4 and 4.29, we have

$$\begin{aligned} z_0 \in \mathcal{I}_0(z_0) &= \mathcal{I}_0(f_0^{k+1}(z_0)) = \mathcal{I}_0(w_0) \\ &= \overline{\{x_0 \mid x_0 \text{ has the same tail as } w_0\}} = \overline{\{f_0^m(z_0) \mid m \geq k+1\}}. \end{aligned}$$

Therefore z_0 is forward recurrent.

Now assume $\mathbf{t}$ and **-1** do not have the same tail. Therefore, there are infinitely many $k \geq 1$, such that $\mathrm{Re}(M_k(\mathbf{t})) \not\subseteq (q_k/|p_k| - 1, q_k/|p_k|)$. Choose a k with this property and let $i = i_k \geq 1$ be the integer such that $l_i \leq k < l_{i+1}$. Define

$$\begin{aligned} \mathbf{t}^k := (&t_0, t_1, \cdots, t_{l_i-1}, \\ &\max(t_{l_i}, b_{l_i} - 2), \max(t_{l_i+1}, b_{l_i+1} - 2), \cdots, \max(t_{k-1}, b_{k-1} - 2), \\ &t_k + 1, t_{k+1}, t_{k+2}, \cdots). \end{aligned}$$

We want to show that $\mathbf{t}^k$ is admissible. First note that, for all $n \geq 0$, $M_n(\mathbf{t}^k)$ does not depend on the first $n-1$ terms in the sequence $\mathbf{t}^k$. Hence $M_k(\mathbf{t}^k)$ is well-defined and it is equal to $M_k(\mathbf{t})+1$. According to the definition, for all $n \geq 0$, we have

$$\bigcup_{l=0}^{b_n-2} (T_n(M_n) + l) \subseteq M_{n-1}.$$

Therefore the set $M_{l_i}(\mathbf{t}^k) \subseteq M_{l_i}^i$ is well-defined. We may lift the set $M_{l_i}^i$ with $(t_0, t_1, \cdots, t_{l_i-1})$ to obtain $M_0^i(\mathbf{t})$. Therefore we may lift $M_{l_i}(\mathbf{t}^k)$ similarly to obtain $M_0(\mathbf{t}^k)$. In other words, $\mathbf{t}^k$ is admissible. On the other hand, it is clear from the construction that there is an integer $m_k > 0$ such that $\mathbf{t}^k = \mathbf{t} + m_k$. Let $z_0^k := f_0^{m_k}(z_0) \in M_0(\mathbf{t}^k)$. For all $l_i \leq n < k$, we have

$$\begin{aligned} |z_n^k - z_n| &= |T_{n+1}(z_{n+1}^k) + \max(t_n, b_n - 2) - (T_{n+1}(z_{n+1}) + t_n)| \\ &\leq |T_{n+1}(z_{n+1}^k) - T_{n+1}(z_{n+1})| + 2 \leq \frac{3}{4}|z_{n+1}^k - z_{n+1}| + 2. \end{aligned}$$

We also have $z_k^k = z_k + 1$. Therefore, we may use induction to see that $|z_{l_i}^k - z_{l_i}| < 8$. Hence we have $|z_0 - f_0^{m_k}(z_0)| < 8(3/4)^{l_{i_k}}$. As k tends to infinity, i_k also tend to infinity. Therefore z_0 is forward recurrent.

Proving backward recurrence is similar. Assume $\mathbf{t}$ and $\mathbf{0}$ have the same tail. Therefore, there is $k \geq 0$ such that $\mathbf{0} + k = \mathbf{t}$. Let $w_0 := f_0^{-k-1}(z_0) \in M_0(\mathbf{-1})$. If $x_0 \in M_0$ has the same tail as $w_0 \in M_0(\mathbf{-1})$, then there is an integer $m \geq 0$ such that $\mathbf{t}(x_0) + m = \mathbf{-1}$ and therefore $x_0 = f_0^{-m}(w_0)$. Hence we have

$$\begin{aligned} z_0 \in \mathcal{I}_0(z_0) &= \mathcal{I}_0(f_0^{-k-1}(z_0)) = \mathcal{I}_0(w_0) \\ &= \overline{\{x_0 \mid x_0 \text{ has the same tail as } w_0\}} = \overline{\{f_0^{-m}(z_0) \mid m \geq k+1\}}. \end{aligned}$$

Therefore z_0 is backward recurrent.

Now assume $\mathbf{t}$ and $\mathbf{0}$ do not have the same tail. Therefore, there are infinitely many $k \geq 1$, such that $\mathrm{Re}(M_k(\mathbf{t})) \not\subseteq (0,1)$. Choose a k with this property and let $j = j_k \geq 1$ be the integer such that $l_j \leq k < l_{j+1}$. Define

$$\begin{aligned}\hat{\mathbf{t}}^k := (&t_0, t_1, \cdots, t_{l_j-1},\\ &\max(t_{l_j}, b_{l_j} - 2), \max(t_{l_j+1}, b_{l_j+1} - 2), \cdots, \max(t_{k-1}, b_{k-1} - 2),\\ &t_k - 1, t_{k+1}, t_{k+2}, \cdots).\end{aligned}$$

Similar to above, $\hat{\mathbf{t}}^k$ is admissible and there is an integer $n_k > 0$ such that $\hat{\mathbf{t}}^k + n_k = \mathbf{t}$. Hence we may use a similar argument to see that $|z_0 - f_0^{-n_k}(z_0)| < 8(3/4)^{l_{j_k}}$. Therefore z_0 is backward recurrent. □

Corollary 5.6. *Every closed forward or backward invariant subset of M_* is totally invariant. Moreover, any closed invariant subset of M_* such set is a saturated subset.*

Proof. The first statement follows immediately from the previous proposition. Let I be a closed invariant subset of M_* and let $z_* \in I$. If $w_* \in M_*$ has the same tail as z_*, then there is an integer k such that $\mathbf{t}(w_*) = \mathbf{t}(z_*) + k$. Therefore $w_* = f_*^k(z_*) \in I$. □

According to Lemma 5.4, for any $z_* \in M_*$, the set $\mathcal{I}_*(z_*)$ is a closed invariant subset of M_*. The next theorem shows that these are the only closed invariant subsets of M_*.

Theorem 5.7. *The map $\mathcal{I}_*|_{M_*(\boldsymbol{0})} : M_*(\boldsymbol{0}) \to \{$closed invariant subsets of $M_*\}$ is a homeomorphism with respect to the Hausdorff metric on the compact subsets of the plane. Moreover, for all $z_*, z'_* \in M_*(\boldsymbol{0})$, we have $z_* \succcurlyeq z'_*$ if and only if $\mathcal{I}_*(z_*) \subseteq \mathcal{I}_*(z'_*)$.*

Proof. It is sufficient to prove the statement for $f_0 : M_0 \to M_0$. Let $z_0, z'_0 \in M_0(\mathbf{0})$ such that $z_0 \succcurlyeq z'_0$. According to Corollary 4.32, we have

$$z_0 \in \{x_0 \in M_0(\mathbf{0}) \mid x_0 \succcurlyeq z'_0\} \subseteq \mathcal{I}_0(z'_0).$$

Therefore $\mathcal{I}_0(z_0) \subseteq \mathcal{I}_0(z'_0)$ and the proof of the second statement is complete.

The set $M_0(\mathbf{0})$ is compact. Therefore, to prove that $\mathcal{I}_0|_{M_0(\mathbf{0})}$ is a homeomorphism, it is sufficient to show that it is bijective and continuous. Choose a non-empty closed invariant set $I \subseteq M_0$. According to Corollary 5.6, I is a saturated set. We have $P(I) = I \cap M_0(\mathbf{0})$ and $P|_{I \cap M_0(\mathbf{0})} = \mathrm{Id}\,|_{I \cap M_0(\mathbf{0})}$. Therefore, using Lemma 5.4, we have

$$I = \bigcup_{z_0 \in I} \mathcal{I}_0(z_0) = \bigcup_{z_0 \in I} \mathcal{I}_0(P(z_0)) = \bigcup_{z_0 \in I \cap M_0(\mathbf{0})} \mathcal{I}_0(z_0).$$

According to Corollary 4.32, for every $z_0 \in I \cap M_0(\mathbf{0})$, we have $\mathcal{I}_0(z_0) \cap M_0(\mathbf{0}) = [z_0, \mathrm{Base}_0(\mathbf{0})]$. Since $I \cap M_0(\mathbf{0})$ is closed, there is a unique $w_0 \in I \cap M_0(\mathbf{0})$ such that

$$I \cap M_0(\mathbf{0}) = \bigcup_{z_0 \in I \cap M_0(\mathbf{0})} [z_0, \mathrm{Base}_0(\mathbf{0})] = [w_0, \mathrm{Base}_0(\mathbf{0})] = \{z_0 \in M_0(\mathbf{0}) \mid z_0 \succcurlyeq w_0\}.$$

Hence w_0 is the unique point in $M_0(\mathbf{0})$ for which we have $I = \mathcal{I}_0(w_0)$.

To complete the proof, we need to show that the map $\mathcal{I}_0|_{M_0(\mathbf{0})}$ is continuous. We start by finding an upper bound for $d_H(\mathcal{I}_0(x_0), \mathcal{I}_0(y_0))$ whenever x_0 and y_0 are two points in $M_0(\mathbf{0})$. Let $x_0 \succcurlyeq y_0$ be two such points and let y'_0 be a point in M_0 that has the same tail as y_0. Then $\mathbf{t} := \mathbf{t}(y'_0) = (t_n)_{n=0}^{\infty}$ is an address that has the same tail as $\mathbf{0}$. Let x'_0 denote the unique points

in $M_0(\mathbf{t})$ that has the same tail as x_0. and let $m \geq 0$ be the smallest integer such that $t_m = 0$. Then we have

$$\forall 0 \leq n \leq m, \quad y_n = T_{n+1} \circ T_{n+2} \circ \cdots \circ T_m(y'_m), \quad x_n := T_{n+1} \circ T_{n+2} \circ \cdots \circ T_m(x'_m).$$

Consider the homeomorphism

$$z_0 \mapsto T_1(t_1 + T_2(t_2 + \cdots T_m(T_m^{-1}(T_{m-1}^{-1}(\cdots T_1^{-1}(z_0)\cdots)))\cdots)) : M_0(\mathbf{0}) \to M_0(\mathbf{t}).$$

This homeomorphism sends $\mathrm{Base}_0(\mathbf{0})$, x_0 and y_0 to $\mathrm{Base}_0(\mathbf{t})$, x'_0 and y'_0 respectively. The Jordan arc $[y_0, \mathrm{Base}_0(\mathbf{0})]$ is mapped into $[y'_0, \mathrm{Base}_0(\mathbf{t})]$ Hence we have $x'_0 \in [y'_0, \mathrm{Base}_0(\mathbf{t})]$ which means $x'_0 \succcurlyeq y'_0$. Therefore, according to Corollary 4.17, we have

$$\forall n \geq 0, \quad e_n \operatorname{Im}(x'_n) \geq e_n \operatorname{Im}(y'_n) - C,$$

where $C := 8C_2 + 7$. Now we may use Lemma 4.23 to see that

$$\forall n \geq 0, \quad e_n \operatorname{Im}(x'_n - y'_n) \leq e_n \operatorname{Im}(x_n - y_n) + C' - 1,$$

where $C' := 6C + 64C_2 + 149$. We have $\mathbf{t}(x'_0) = \mathbf{t}(y'_0)$. Therefore

$$\forall n \geq 0, \quad |x'_n - y'_n| \leq |\operatorname{Im}(x'_n - y'_n)| + |\operatorname{Re}(x'_n - y'_n)| \leq |x_n - y_n| + C'.$$

Since $x'_0 \in \mathcal{I}_0(x_0)$, this implies that

$$\forall n \geq 0, \quad d(y'_n, \{z_n \in M_n \mid z_0 \in \mathcal{I}_0(x_0)\}) \leq |x_n - y_n| + C'.$$

Recall that we chose y'_0 to be an arbitrary point that has the same tail as y_0. According to Lemma 4.29, we have $\mathcal{I}_0(y_0) = \overline{\{y'_0 \in M_0 \mid y'_0 \text{ has the same tail as } y_0\}}$. Therefore, the above inequality holds whenever $y'_0 \in \mathcal{I}_0(y_0)$. So we have

$$\{z_n \in M_n \mid z_0 \in \mathcal{I}_0(x_0)\} \subseteq \{z_n \in M_n \mid z_0 \in \mathcal{I}_0(y_0)\} \subseteq D(\{z_n \in M_n \mid z_0 \in \mathcal{I}_0(x_0)\}, |x_n - y_n| + C').$$

Hence

$$\begin{aligned} d_H(\mathcal{I}_0(x_0), \mathcal{I}_0(y_0)) &\leq (3/4)^n d_H(\{z_n \in M_n \mid z_0 \in \mathcal{I}_0(x_0)\}, \{z_n \in M_n \mid z_0 \in \mathcal{I}_0(y_0)\}) \\ &\leq (3/4)^n (|x_n - y_n| + C'), \end{aligned}$$

for all $n \geq 0$.

Assume $(z_0^i)_{i=1}^\infty$ is a sequence in $M_0(\mathbf{0})$ that converges to a point $z_0 \in M_0(\mathbf{0})$. Then we have $z_n^i \to z_n$, for all $n \geq 0$. Hence we have

$$\forall n \geq 0, \quad \limsup_{i\to\infty} d_H(\mathcal{I}_0(z_0^i), \mathcal{I}_0(z_0)) \leq \limsup_{i\to\infty} (3/4)^n (|z_n^i - z_n| + C') = (3/4)^n C'.$$

Therefore

$$\lim_{i\to\infty} d_H(\mathcal{I}_0(z_0^i), \mathcal{I}_0(z_0)) = 0. \qquad \square$$

Theorem 5.8 (Theorem E)**.** *The base Cantor set is the only closed invariant subset of M_* which is a Cantor set. If M_* is a Cantor set, then f_* is minimal. Otherwise, there is a homeomorphism*

$$\tilde{\mathcal{I}}_* : [0, 1] \to \{\textit{closed invariant sets for } f_* : M_* \to M_*\},$$

such that the following hold.

(i) The set $\tilde{\mathcal{I}}_*(0)$ is the base Cantor set of the hairy Cantor set M_*, and $\tilde{\mathcal{I}}_*(1) = M_*$;
(ii) For $0 < x \leq 1$, $\tilde{\mathcal{I}}_*(x)$ is a hairy Cantor set.
(iii) For $0 \leq x < y \leq 1$, we have $\tilde{\mathcal{I}}_*(x) \subsetneq \tilde{\mathcal{I}}_*(y)$.

Proof. We will prove the statement for M_0. If M_0 is a Cantor set, then it is equal to its base Cantor set. According to Theorem 5.7, M_0 is the only non-empty closed invariant subsets of itself. Hence f_0 is minimal.

Assume M_0 is not a Cantor set. Therefore, according to Proposition 4.16 and Corollary 4.26, the set $M_0(\mathbf{0})$ is a Jordan arc. Hence, using Theorem 5.7, we may define $\tilde{\mathcal{I}}_0$ such that (i) and (iii) hold. To complete the proof, we need to show that every non-empty closed invariant subset of M_0 that is not the base Cantor set is a hairy Cantor set. Let I be a non-empty closed invariant subset of M_0. According to Corollary 5.6 and Theorem 5.7, there is $z_0 \in M_0(\mathbf{0})$ such that $I = \mathcal{I}_0(z_0)$. If $z_0 = \text{Base}_0(\mathbf{0})$, then $I = \text{Base}_0(M_0)$ is a Cantor set according to Proposition 4.34. Let us assume $z_0 \neq \text{Base}_0(\mathbf{0})$. To prove that I is a hairy Cantor set, we will check the conditions in the definition of the hairy Cantor sets.

(i) Every connected component of I is either a Jordan arc or a single point. (Corollary 4.32)
(ii) Closure of the union of the point components of I is a Cantor set, namely $\text{Base}_0(I) = \text{Base}_0(M_0)$. (Proposition 4.34 and Corollary 4.32)
(iii) Any arc component of I meets $\text{Base}_0(I)$ at one of its end-points.(Corollary 4.32)
(iv) For any sequence $(z^i)_{i\in\mathbb{N}}$ in I with $z^i \to z$, we have $[\text{Base}_0(z^i), z^i] \to [\text{Base}_0(z), z]$ in the Hausdorff topology.(Proposition 4.37).
(v) The set $I \setminus \text{Base}_0(I)$ is dense in $\text{Base}_0(I)$. (Lemma 4.29 and the fact that any point in M_0 that has the same tail as z_0 is not a base point.)
(vi) The points in the interior of the arc components of I are not accessible from $\mathbb{C} \setminus I$. (Proposition 4.39) □

5.3. **Unique Ergodicity.** Here we will show that f_* is uniquely ergodic and describe the ergodic measure. We start with an auxiliary lemma.

Lemma 5.9. *Let $\phi : M_0 \to \mathbb{R}$ be a continuous function. If $q'_j := q_{l_0} q_{l_1} q_{l_2} \cdots q_{l_{j-1}}$, for all $j \geq 1$, then we have*

$$\lim_{j\to\infty} \frac{1}{q'_j} \sum_{k=0}^{q'_j-1} \text{diam}\left(\phi(M_0 \cap M_0^j(\boldsymbol{0}+k))\right) = 0.$$

Proof. Since M_0 is compact, the map ϕ is uniformly continuous on this set and it has a uniform bound. In other words, there is a function $\omega : [0,\infty) \to [0,\infty)$ such that

$$\lim_{\varepsilon\to 0} \omega(\varepsilon) = 0, \quad \forall x, y \in M_0,\ |\phi(x) - \phi(y)| \leq \omega(|x-y|),$$

and there is a constant $C' > 0$ such that

$$\forall x \in M_0, \quad |\phi(x)| < C'.$$

For $j \geq 1$, define

$$A^j := \{\mathbf{t} = (t_n)_{n=0}^{\infty} \mid \mathbf{t} \text{ is an admissible address, } \forall k \geq l_j, t_k = 0\}.$$

According to Lemma 4.11, we have $A^j = \{\mathbf{0} + k \mid 0 \leq k < q'_j\}$. We divide the proof into two cases.

Case (i) - $\limsup_{j\to\infty} |p_{l_j}/q_{l_j}| > 0$ **:**

In this case, there is a constant $C > 0$ and a sequence $(j_i)_{i=1}^{\infty}$ such that $|q_{l_{j_i}}/p_{l_{j_i}}| < C$. Fix $i \geq 0$ and let $m := l_{j_i}$. For any address $\mathbf{t} \in A^{j_i}$, we have

$$\begin{aligned}
\operatorname{diam}\left(M_{m-1}^{j_i}(\mathbf{t})\right) &\leq \left(u_{m-1}^{j_i}(\mathbf{t}) - d_{m-1}^{j_i}(\mathbf{t})\right) + 1 \leq u_{m-1}^{j_i}(\mathbf{t}) + 3 \\
&\leq 2C_2 + 5 + \lim_{e_m \operatorname{Im}(z) \to \infty} e_{m-1} \operatorname{Im}(T_m(z)) && \text{(Lemma 4.19)} \\
&= 2C_2 + 5 + \frac{1}{2\pi} \log |q_{l_i}/p_{l_i}| + \operatorname{Im}(C_1^{\mathrm{b}}(|p_{l_i}/q_{l_i}|)) && \text{(Lemma 4.4(i))} \\
&\leq 2C_2 + \frac{1}{2\pi} \log(C) + 7. && \text{(Lemma 4.4(iii))}
\end{aligned}$$

Therefore

$$\operatorname{diam}\left(M_0 \cap M_0^{j_i}(\mathbf{t})\right) \leq \operatorname{diam}\left(M_0^{j_i}(\mathbf{t})\right) \leq (3/4)^{m-1} \left(2C_2 + \frac{1}{2\pi} \log(C) + 7\right).$$

Hence we have

$$\operatorname{diam}\left(\phi(M_0 \cap M_0^{j_i}(\mathbf{t}))\right) \leq \omega\left((3/4)^{l_{j_i}-1} \left(2C_2 + \frac{1}{2\pi} \log(C) + 7\right)\right),$$

for all $i \geq 1$ and any admissible address $\mathbf{t}$.

As we saw in the proof of Lemma 4.9(vi), every connected component of M_0^j contains q_{l_j} connected component of M_0^{j+1}. Therefore, we have

$$\begin{aligned}
\frac{1}{q'_j} \sum_{\mathbf{t} \in A^j} \operatorname{diam}\left(\phi(M_0 \cap M_0^j(\mathbf{t}))\right) &\geq \frac{1}{q_{l_j} q'_j} \sum_{\mathbf{t} \in A^{j+1}} \operatorname{diam}\left(\phi(M_0 \cap M_0^{j+1}(\mathbf{t}))\right) \\
&= \frac{1}{q'_{j+1}} \sum_{\mathbf{t} \in A^{j+1}} \operatorname{diam}\left(\phi(M_0 \cap M_0^{j+1}(\mathbf{t}))\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\limsup_{j \to \infty} \frac{1}{q'_j} \sum_{\mathbf{t} \in A^j} \operatorname{diam}\left(\phi(M_0 \cap M_0^j(\mathbf{t}))\right) &\leq \limsup_{i \to \infty} \frac{1}{q'_{j_i}} \sum_{\mathbf{t} \in A^{j_i}} \operatorname{diam}\left(\phi(M_0 \cap M_0^{j_i}(\mathbf{t}))\right) \\
&\leq \limsup_{i \to \infty} \omega\left((3/4)^{l_{j_i}-1} \left(2C_2 + \frac{1}{2\pi} \log(C) + 7\right)\right) = 0.
\end{aligned}$$

Case (ii) - $\lim_{j \to \infty} |p_{l_j}/q_{l_j}| = 0$ **:**

Define

$$B^j(r) = \left\{ \mathbf{t} = (t_n)_{n=0}^{\infty} \in A^j \;\middle|\; r < \frac{t_{l_{j-1}}}{q_{l_{j-1}}/|p_{l_{j-1}}|} < 1 - r \right\},$$

for $0 < r < 1/2$ and $j \geq 2$. Let $j \geq 2$ be an integer. In the proof of Lemma 4.20, we showed that

$$\left| u_{l_i-1}^{i+1}(\mathbf{t}) - d_{l_i-1}^{i+1}(\mathbf{t}) \right| \leq \frac{1}{2\pi} \log\left(d\left(\frac{t_{l_i}}{q_{l_i}/|p_{l_i}|}, \mathbb{Z} \right)^{-1} \right) + 6C_2 + 6,$$

for all $i \geq 1$. Therefore for all $\mathbf{t} \in B^j(r)$, we have

$$\begin{aligned}
\operatorname{diam}\left(M_{l_{j-1}-1}^j(\mathbf{t})\right) &\leq \left| u_{l_{j-1}-1}^j(\mathbf{t}) - d_{l_{j-1}-1}^j(\mathbf{t}) \right| + 1 \\
&\leq \frac{1}{2\pi} \log\left(d\left(\frac{t_{l_{j-1}}}{q_{l_{j-1}}/|p_{l_{j-1}}|}, \mathbb{Z} \right)^{-1} \right) + 6C_2 + 7 \\
&< \frac{1}{2\pi} \log(1/r) + 6C_2 + 7.
\end{aligned}$$

Therefore

$$\operatorname{diam}\left(M_0^j(\mathbf{t})\right)\right) \leq (3/4)^{l_{j-1}-1}\left(\frac{1}{2\pi}\log(1/r)+6C_2+7\right),$$

which implies that

$$\operatorname{diam}\left(\phi(M_0\cap M_0^j(\mathbf{t}))\right)\right) \leq \omega\left((3/4)^{l_{j-1}-1}\left(\frac{1}{2\pi}\log(1/r)+6C_2+7\right)\right),$$

for all addresses $\mathbf{t}\in B^j(r)$.

On the other hand, for any admissible address $\mathbf{t}$, we have

$$\operatorname{diam}\left(\phi(M_0\cap M_0^j(\mathbf{t}))\right) \leq 2\sup_{x\in M_0^j}|\phi(x)| \leq 2C'.$$

Putting these together, we obtain

$$\begin{aligned}
&\frac{1}{q'_j}\sum_{\mathbf{t}\in A^j}\operatorname{diam}\left(\phi(M_0\cap M_0^j(\mathbf{t}))\right)\\
&\qquad=\frac{1}{q'_j}\sum_{\mathbf{t}\in A^j\setminus B^j(r)}\operatorname{diam}\left(\phi(M_0\cap M_0^j(\mathbf{t}))\right)+\frac{1}{q'_j}\sum_{\mathbf{t}\in B^j(r)}\operatorname{diam}\left(\phi(M_0\cap M_0^j(\mathbf{t}))\right)\\
&\qquad\leq\left(1-\frac{|B^j(r)|}{q'_j}\right)(2C')+\frac{|B^j(r)|}{q'_j}\omega\left((3/4)^{l_{j-1}-1}\left(\frac{1}{2\pi}\log(1/r)+6C_2+7\right)\right).
\end{aligned}$$

If we take the limit $j\to\infty$, we see that

$$\limsup_{j\to\infty}\frac{1}{q'_j}\sum_{\mathbf{t}\in A^j}\operatorname{diam}\left(\phi(M_0\cap M_0^j(\mathbf{t}))\right)\leq 2C'.\limsup_{j\to\infty}\left(1-\frac{|B^j(r)|}{q'_j}\right).$$

Now, looking at both sides at the limit $r\to 0$, we obtain

$$\limsup_{j\to\infty}\frac{1}{q'_j}\sum_{\mathbf{t}\in A^j}\operatorname{diam}\left(\phi(M_0\cap M_0^j(\mathbf{t}))\right)\leq 2C'.\liminf_{r\to 0}\limsup_{j\to\infty}\left(1-\frac{|B^j(r)|}{q'_j}\right).$$

Therefore, to complete the proof, it is sufficient to show that

$$\lim_{r\to 0}\liminf_{j\to\infty}\frac{|B^j(r)|}{q'_j}=1.$$

Define

$$B_k^j:=\left\{\mathbf{t}=(t_n)_{n=0}^{\infty}\in A^j \;\middle|\; t_{l_{j-1}}=k\right\},$$

for all $0\leq k\leq b_{l_{j-1}}$. Note that $A^j=\cup_{0\leq k\leq b_{l_{j-1}}}B_k^j$. Let

$$k_j:=\left\lceil r\frac{q_{l_{j-1}}}{|p_{l_{j-1}}|}\right\rceil+1\leq r\frac{q_{l_{j-1}}}{|p_{l_{j-1}}|}+2\quad\text{ and }\quad k'_j:=\left\lfloor(1-r)\frac{q_{l_{j-1}}}{|p_{l_{j-1}}|}\right\rfloor-2\geq(1-r)\frac{q_{l_{j-1}}}{|p_{l_{j-1}}|}-3.$$

Clearly we have $B_k^j\subseteq B^j(r)$, for all $k_j\leq k\leq k'_j$.

Let $\mathbf{t}\in B_k^j$ for some $0\leq k\leq b_{l_{j-1}}-3$. The set $M_{l_{j-1}}^j(\mathbf{t})$ is a connected component of $M_{l_{j-1}}^j$ and $\operatorname{Re}(M_{l_{j-1}}^j(\mathbf{t}))\subseteq(0,b_{l_{j-1}}-2)$. Therefore the set $M_{l_{j-1}}^j(\mathbf{t})+1$ is also a connected component of $M_{l_{j-1}}^j\subseteq\mathbb{H}^{\mathrm{b}}$. Hence we may lift this set with the map $z\mapsto T_1(t_1+T_2(t_2+\cdots t_{l_{j-1}-1}+T_{l_{j-1}}(z)\cdots))$

to obtain a connected component of M_0^j. Clearly this connected component has the address $(t_0, t_1, t_2, \cdots, t_{l_{j-1}} + 1, \cdots)$. Therefore the map

$$(t_0, t_1, t_2, \cdots, t_{l_{j-1}}, \cdots) \mapsto (t_0, t_1, t_2, \cdots, t_{l_{j-1}} + 1, \cdots) : B_k^j \to B_{k+1}^j$$

is well defined and injective. Hence we have $|B_k^j| \geq |B_{k+1}^j|$, for all $0 \leq k \leq b_{l_{j-1}} - 3$. On the other hand, for $1 \leq k \leq b_{l_j}$, we may use a similar argument to see that $|B_k^j| \geq |B_{k-1}^j|$. Hence

$$\left|B_0^j\right| = \left|B_1^j\right| = \left|B_2^j\right| = \cdots = \left|B_{b_{l_j}-2}^j\right| \geq \left|B_{b_{l_j}-1}^j\right|, \quad \left|B_0^j\right| \geq \left|B_{b_{l_j}}^j\right|.$$

The equality $A^j = \cup_{0 \leq k \leq b_{l_{j-1}}} B_k^j$ implies that

$$q_j' = |A^j| \leq (b_{l_{j-1}} + 1)|B_0^j| \leq (|q_{l_{j-1}}/p_{l_{j-1}}| + 2)|B_0^j|.$$

Therefore we have

$$|B^j(r)| \geq \sum_{k=k_j}^{k_j'} |B_k^j| = (k_j' - k_j + 1)|B_0^j| \geq \frac{k_j' - k_j + 1}{|q_{l_{j-1}}/p_{l_{j-1}}| + 2} q_j' \geq \frac{(1-2r)|q_{l_{j-1}}/p_{l_{j-1}}| - 4}{|q_{l_{j-1}}/p_{l_{j-1}}| + 2} q_j'.$$

Hence

$$\frac{|B^j(r)|}{q_j'} \geq \frac{(1-2r)|q_{l_{j-1}}/p_{l_{j-1}}| - 4}{|q_{l_{j-1}}/p_{l_{j-1}}| + 2} = \frac{1 - 2r - 4|p_{l_{j-1}}/q_{l_{j-1}}|}{1 + 2|p_{l_{j-1}}/q_{l_{j-1}}|}.$$

If we take the limit $j \to \infty$, we see that

$$\liminf_{j \to \infty} \frac{|B^j(r)|}{q_j'} \geq 1 - 2r.$$

Therefore we may take the limit $r \to \infty$ to obtain

$$\lim_{r \to 0} \liminf_{j \to \infty} \frac{|B^j(r)|}{q_j'} = 1,$$

which completes the proof. □

Proposition 5.10. *The function $f_* : M_* \to M_*$ is uniquely ergodic.*

Proof. It is sufficient to prove that $f_0 : M_0 \to M_0$ is uniquely ergodic. According to [Mañ87, Theorem 9.2], the function f_0 is uniquely ergodic if for any continuous function $\phi : M_0 \to \mathbb{R}$ and every $z \in M_0$, the limit

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(f_0^k(z))$$

exists and is independent of z. Let $\phi : M_0 \to \mathbb{R}$ be a continuous function. Since M_0 is compact, there is a constant $C > 0$ such that $|\phi(z)| < C$, for all $z \in M_0$. For $z \in M_0$ and $n \geq 1$, define

$$S_n(z) := \frac{1}{n} \sum_{k=0}^{n-1} \phi(f_0^k(z)).$$

Clearly we have $|S_n(z)| < C$, for all $z \in M_0$ and $n \geq 1$. Also let $q_j' := q_{l_0} q_{l_1} q_{l_2} \cdots q_{l_{j-1}}$, for all $j \geq 1$ and

$$\delta_j := \frac{1}{q_j'} \sum_{k=0}^{q_j'-1} \operatorname{diam}\left(\phi(M_0 \cap M_0^j(\mathbf{0} + k))\right).$$

Let $w, z \in M_0$, $j \geq 1$ and $q := q'_j$. According to Lemma 4.11, the set $\{f_0^k(z) \mid 0 \leq k < q\}$ has exactly one point in every connected component of M_0^j and the set of connected components of M_0^j is equal to $\{M_0^j(\mathbf{0}+k) \mid 0 \leq k < q-1\}$. The same is true for $\{f_0^k(w) \mid 0 \leq k < q\}$. Therefore there is a permutation σ of $\{0, 1, \cdots q-1\}$ such that $f_0^k(z)$ and $f_0^{\sigma(k)}(w)$ belong to the same connected component of M_0, for all $0 \leq k < q$. Therefore we have

$$|S_q(z) - S_q(w)| \leq \frac{1}{q}\sum_{l=0}^{q-1} |\phi(f_0^k(z)) - \phi(f_0^{\sigma(k)}(z))| \leq \frac{1}{q}\sum_{l=0}^{q-1} \operatorname{diam}\left(\phi(M_0 \cap M_0^j(\mathbf{0}+k))\right) = \delta_j.$$

Let $n \geq 1$ be an integer. There are unique integers $k \geq 0$ and $0 \leq r < q$ such that $n = kq + r$. Therefore we have

$$\begin{aligned}
|S_n(z) - S_{q'_j}(w)| &= \frac{1}{n}\left| q\sum_{i=0}^{k-1}(S_q(f_0^{iq}(z)) - S_q(w)) + \sum_{i=kq}^{n-1}(\phi(f_0^i(z))) - S_q(w))\right| \\
&\leq \frac{q}{n}\sum_{i=0}^{k-1}|S_q(f_0^{iq}(z)) - S_q(w)| + \frac{1}{n}\sum_{i=kq}^{n-1}(|\phi(f_0^i(z)))| + |S_q(w)|) \\
&\leq \frac{qk}{n}\delta_j + \frac{2rC}{n} \leq \delta_j + \frac{2q'_jC}{n}.
\end{aligned}$$

Therefore, for all $m \geq 1$, we have

$$|S_n(z) - S_m(w)| \leq |S_n(z) - S_{q'_j}(w)| + |S_m(w) - S_{q'_j}(w)| \leq 2\delta_j + \frac{4q'_jC}{\min(n,m)}.$$

According to Lemma 5.9, we have $\lim_{j\to\infty}\delta_j = 0$. Hence the sequence

$$S_1(z), S_1(w), S_2(z), S_2(w), \cdots$$

is Cauchy. Therefore the limit $\lim_{n\to\infty} S_n(z)$ exists and it is equal to $\lim_{n\to\infty} S_n(w)$. □

Theorem 5.11 (Theorem F)**.** *There is a unique Borel probability measure μ on $\mathrm{Base}_*(M_*)$ such that for any $j \geq 1$ and any admissible address $\boldsymbol{t}$, we have*

$$\mu\left(\mathrm{Base}_*(M_*) \cap M_*^j(\boldsymbol{t})\right) = \frac{1}{q_1 q_2 \cdots q_{l_{j-1}}}.$$

The measure μ is the unique ergodic probability measure for $f_* : M_* \to M_*$.

Proof. We will prove the statement for M_0. If ν and μ are Borel probability measures on $\mathrm{Base}_0(M_0)$ such that

$$\mu\left(\mathrm{Base}_0(M_0) \cap M_0^j(\mathbf{t})\right) = \nu\left(\mathrm{Base}_0(M_0) \cap M_0^j(\mathbf{t})\right) = \frac{1}{q_1 q_2 \cdots q_{l_{j-1}}},$$

for any $j \geq 1$ and any admissible address $\mathbf{t}$, then we may use Carathèodory extension theorem to see that $\nu = \mu$.

Let $z := \mathrm{Base}_0(\mathbf{0})$. If μ is the unique ergodic probability measure, then we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\phi(f_0^k(z)) = \int_{M_0}\phi d\mu,$$

for every continuous function $\phi : M_0 \to \mathbb{R}$. According to Lemma 5.4, Theorem 5.5 and Proposition 4.34, we have $\overline{\{f_0^k(z) \mid k \geq 0\}} = \mathcal{I}_0(z) = \mathrm{Base}_0(M_0)$. Therefore the value of the limit above only depends on $\phi|_{\mathrm{Base}_0(M_0)}$. Since μ is a Borel measure, the set $M_0 \setminus \mathrm{Base}_0(M_0)$ is measurable

and we have $\mu(M_0 \setminus \mathrm{Base}_0(M_0)) = 0$. In other words, the support of μ is contained in the Cantor set $\mathrm{Base}_0(M_0)$.

For any $j \geq 1$ and any admissible address $\mathbf{t}$, there is a continuous function $\phi : \mathbb{C} \to \mathbb{R}$ such that $\phi(w) = 1$ for $w \in \overline{M_0^j(\mathbf{t})}$ and $\phi(w) = 0$ for $w \in \overline{M_0^j} \setminus \overline{M_0^j(\mathbf{t})}$. Therefore we have

$$\mu\left(\mathrm{Base}_0(M_0) \cap M_0^j(\mathbf{t})\right) = \int_{M_0} \phi d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(f_0^k(z)).$$

According to Lemma 4.11, from every consecutive $q_1 q_2 \cdots q_{l_{j-1}}$ iterates of z under f_0, exactly one belongs to the set $M_0^j(\mathbf{t})$. Therefore

$$\left\lfloor \frac{n}{q_1 q_2 \cdots q_{l_{j-1}}} \right\rfloor \leq \sum_{k=0}^{n-1} \phi(f_0^k(z)) \leq \left\lceil \frac{n}{q_1 q_2 \cdots q_{l_{j-1}}} \right\rceil.$$

Hence we have

$$\mu\left(\mathrm{Base}_0(M_0) \cap M_0^j(\mathbf{t})\right) = \frac{1}{q_1 q_2 \cdots q_{l_{j-1}}}. \qquad \square$$

6. Hairs on a Cantor Set

Here we introduce *"hairy Cantor sets"* and describe some of their properties. Hairy Cantor sets are compact subsets of the plane that satisfy certain axioms similar to Cantor bouquet and hairy circles [Lel61, Cha89, BO90]. However, unlike these topological objects that are connected, hairy Cantor sets have uncountably many connected components.

We start by defining straight hairy Cantor sets. These objects are similar to straight one-sided hairy arcs, as defined in [AO93] and our analysis follows a similar approach. The most technical part of our analysis of hairy Cantor sets is in Subsection 6.2 where we find a Jordan curve that passes through all the connected components of a given hairy Cantor set and satisfies some extra properties. The main tools needed to find such curves are *Whitney maps* and *Borsuk's theorem*. After finding such Jordan curves, we use Caratheodory's theory of prime ends and classical results in complex analysis, such as Gehring-Hayman theorem to show that all hairy Cantor sets are ambiently homeomorphic. Finally, in Subsection 6.4, we characterise all metric spaces that are homeomorphic to a hairy Cantor set.

6.1. Straight hairy Cantor sets.

Here we define *straight hairy Cantor sets* and investigate some of their properties. In particular, we show that all straight hairy Cantor sets are ambiently homeomorphic. The proof is fairly elementary and is based on a careful analysis of bump functions associated to straight hairy sets. A similar analysis has been done in [AO93] for straight one-sided hairy arcs.

Definition 6.1. *A set $X \subseteq \mathbb{R}^2$ is called a straight hairy Cantor set if there is a Cantor set $C \subseteq \mathbb{R}$ and an upper semi-continuous function $l : C \to [0, +\infty)$ such that*

$$X = \{(x, y) \in \mathbb{R}^2 \mid x \in C, 0 \leq y \leq l(x)\},$$

and the following hold.

(i) The set $\{x \in C \mid l(x) \neq 0\}$ is dense in C.

(ii) For any $x \in C$ with $l(x) \neq 0$, there are strictly monotone sequences $(p_n)_{n\in\mathbb{N}}$, $(q_n)_{n\in\mathbb{N}}$ in C such that [7]

$$p_n \uparrow x, \quad q_n \downarrow x, \quad \lim_{n\to\infty} l(p_n) = \lim_{n\to\infty} l(q_n) = l(x).$$

We will say C is the base of X and l is the length function of X.

For simplicity, we will assume that l is defined on $\mathbb{R}$ and we have

$$\forall x \in \mathbb{R} \setminus C, \quad l(x) := 0.$$

Note that since l is upper semi-continuous, it is also bounded. We start our analysis by showing that straight hairy Cantor sets exist.

Proposition 6.2. *There exists a straight hairy Cantor set in the plane.*

Proof. We will build a Cantor set C, a function $l : C \to \mathbb{R}$ and a set X and show that the conditions in the definition of straight hairy Cantor sets hold.

Construction of the Cantor set C : We will use a recursive process similar to the construction of the ternary Cantor set. Let $P_1 = [0, 1]$ and $t_1 : [0, 1] \to \mathbb{Z}$ such that $t_1(x) = 1$ for all $x \in [0, 1]$.

[7] For a sequence $(z_n)_{n\in\mathbb{N}}$ and point z, the term $z_n \uparrow z$ means that $(z_n)_{n\in\mathbb{N}}$ is an increasing sequence that tends to z. Similarly, the term $z_n \downarrow z$ means $(z_n)_{n\in\mathbb{N}}$ is decreasing and tends to z.

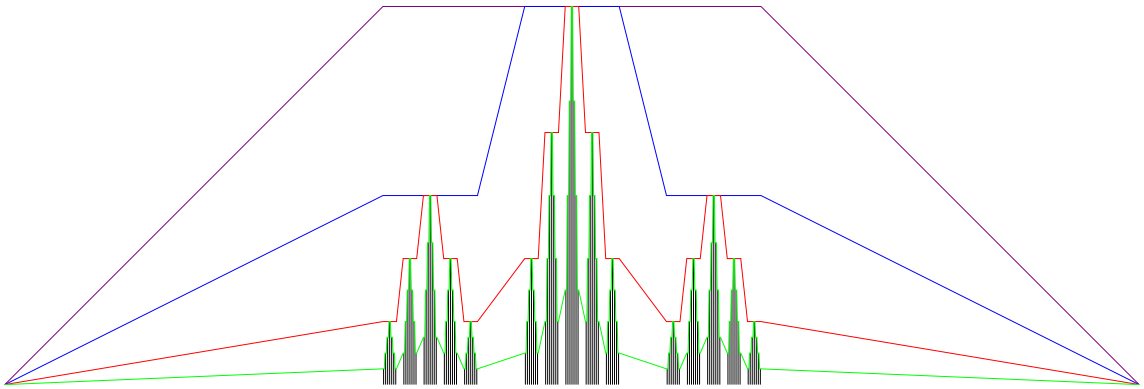


FIGURE 5. The set X, in black, and the functions l_1, l_2, l_3 and l_4 described in Proposition 6.2

Assume $n > 1$ and $P_{n-1} \subseteq [0,1]$ is a disjoint union of finitely many closed proper intervals, namely $I_j = [a_j, b_j]$ for $j \in J$, where J is a finite index set. We define

$$t_n : [0,1] \to \mathbb{Z}, \quad t_n(x) := \begin{cases} k & \text{if } x \in I_j \text{ and } \frac{2k-2}{4n-3} \leq \frac{x-a_j}{b_j-a_j} \leq \frac{2k-1}{4n-3}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$P_n := \{x \in [0,1] \mid t_n(x) > 0\} \subseteq P_{n-1}.$$

In other words, for any connected component of P_{n-1}, we divide it into $4n-3$ equal subintervals and keep $2n-1$ of the them that have disjoint boundaries. The set P_n is the union of the closure of these subintervals.

Define

$$C := \bigcap_{n \geq 1} P_n.$$

For every point $x \in C$, we may assign an address $(t_1(x), t_2(x), \cdots)$ and for all $m \geq 1$, we have $1 \leq t_m(x) \leq 2m-1$. Note that two points in C have the same address if and only if they are identical.

Construction of l and X : We will recursively construct a sequence of real valued continuous functions $(l_n)_{n \geq 0}$ on $\mathbb{R}$ and use them to define l. We start by defining $l_0(x) := 1$, for all $x \in \mathbb{R}$. Given l_{n-1}, we define

$$l_n(x) := \begin{cases} (1 - |t_n(x) - n|/n) l_{n-1}(x) & \text{if } x \in P_n, \\ 0 & \text{if } x \in \mathbb{R} \setminus (-1, 2), \end{cases}$$

and extend l_n to a continuous map on the real line by linear interpolation. For any $x \in P_n$, we have $|t_n(x) - n| \leq n-1$, which implies $l_{n-1}(x)/n \leq l_n(x) \leq l_{n-1}(x)$.

For any $n \geq 0$, define

$$X_n := \{(x,y) \in \mathbb{R}^2 \mid x \in \mathbb{R}, 0 \leq y \leq l_n(x)\}.$$

Since l_n is continuous, X_n is closed. Therefore the set $X := \cap_{n \geq 0} X_n \cap (C \times [0,1])$ is also closed. Moreover, if we define

$$l : C \to [0,1], \quad l(x) := \lim_{n \to \infty} l_n(x),$$

then $X = \{(x,y) \in \mathbb{R}^2 \mid x \in C, 0 \leq y \leq l(x)\}$. Since X is closed, the function l is upper semi-continuous.

Conditions (i) and (ii) : For any $x \in C$, we have

$$l(x) = \prod_{n\geq 1}(1 - |t_n(x) - n|/n).$$

Therefore for any $x \in C$ such that for large n, $t_n(x) = n$, we have $l(x) > 0$. Clearly the set of such points is dense in C and condition (i) is satisfied.

To prove condition (ii), let $x \in C$ such that $l(x) > 0$. Using the above formula, it is easy to see that there is an integer $N > 0$ such that for all $n > N$, we have $1/2 < t_n(x)/n < 3/2$. For any $n > N$, we define p_n to be the unique point in C with the address

$$(t_1(x), \cdots, t_{n-1}(x), t_n(x) - 1, t_{n+1}(x), \cdots).$$

The sequence $(p_n)_{n>N}$ is a strictly increasing sequence that tends to x. Moreover, we have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{l(p_n)}{l(x)} - 1\right| &= \lim_{n\to\infty}\left|\frac{1 - |t_n(x) - 1 - n|/n}{1 - |t_n(x) - n|/n} - 1\right| \\ &= \lim_{n\to\infty}\left|\frac{|t_n(x)/n - 1| - |t_n(x)/n - 1/n - 1|}{1 - |t_n(x)/n - 1|}\right| \\ &\leq \lim_{n\to\infty}\frac{1/n}{1 - |t_n(x)/n - 1|} = 0.\end{aligned}$$

Similarly, we may define a strictly decreasing sequence $(q_n)_{n>N}$ in C such that

$$\lim_{n\to\infty} l(q_n) = l(x). \qquad \square$$

Proposition 6.3. *All straight hairy Cantor sets in the plane are ambiently homeomorphic.*

Let $\mathcal{C}$ be the middle-third Cantor set. For any Cantor set $C \subseteq \mathbb{R}$, there is a homeomorphism of the real line, $\phi : \mathbb{R} \to \mathbb{R}$, such that $\phi(C) = \mathcal{C}$. We may extend ϕ to a homeomorphism of the plane by

$$\phi : \mathbb{R}^2 \to \mathbb{R}^2, \quad \phi(x, y) := (\phi(x), y).$$

Given any straight hairy Cantor set X with base C, we see that $\phi(X)$ is ambiently homeomorphic to X and is based on $\mathcal{C}$. Now we only need to show that every two straight hairy Cantor sets with base $\mathcal{C}$ are ambiently homeomorphic.

From this point onward, unless otherwise stated, we will assume that all straight hairy Cantor sets have base $\mathcal{C}$.

Definition 6.4. *Let $\mathcal{C}$ be the middle-third Cantor set. We call a closed set $P \subseteq [0, 1]$ a Cantor partition, if P is the union of finitely many closed intervals, $\partial P \subseteq \mathcal{C}$ and $\mathcal{C} \subseteq P$. In other words, there is an integer $k \geq 1$ and points $0 = x_1 < y_1 < x_2 < y_2 < \cdots < x_k < y_k = 1$ in $\mathcal{C}$ such that*

$$\mathcal{C} \subseteq P = \bigcup_{i=1}^{k}[x_i, y_i].$$

We say a sequence of partitions $(P_n)_{n=0}^{\infty}$ is a shrinking sequence of Cantor partitions if

$$P_0 \supset P_1 \supset P_2 \supset \cdots$$

and

$$\mathcal{C} = \bigcap_{n\geq 0} P_n.$$

Note that we have only defined Cantor partitions for the middle-third Cantor set $\mathcal{C}$. Given a straight hairy Cantor set X based on $\mathcal{C}$ with length function l^X, if $P = \cup_{i=1}^{k}[x_i, y_i]$ is a Cantor partition, then condition (ii) in the definition of straight hairy Cantor sets implies that

$$l^X(x_1) = l^X(y_1) = \cdots = l^X(x_k) = l^X(y_k) = 0.$$

Definition 6.5. *Let $\mathcal{C}$ be the middle-third Cantor set and let $l : \mathcal{C} \to [0, \infty)$ be an upper semi-continuous function. We use the notation*

$$\mathrm{Max}(l, I) := \max\{l(x) \mid x \in I \cap \mathcal{C}\},$$

for any interval $I := [\alpha, \beta] \subseteq [0, 1]$. Note that such a maximum exists since l is upper semi-continuous. Choose a point $c \in I$ where $l(c) = \mathrm{Max}(l, I)$. Such a point exists, but the choice is not necessarily unique. Using this, we may define the bump function on interval I associated to c.

$$B_I^l : I \to [0, \mathrm{Max}(l, I)], \quad B_I^l(x) := \begin{cases} \max\{l(z) \mid z \in [\alpha, x]\} & \alpha \le x \le c, \\ \max\{l(z) \mid z \in [x, \beta]\} & c < x \le \beta. \end{cases}$$

If $P = \cup_{i=1}^{k}[x_i, y_i] \subseteq [0, 1]$ is a Cantor partition, then a bump function on P is defined as

$$B_P^l(x) := \begin{cases} B_{[x_i, y_i]}^l(x) & \text{if } x \in [x_i, y_i] \subseteq P, \\ 0 & \text{if } x \notin P, \end{cases}$$

where $B_{[x_i, y_i]}^l$, for $1 \le i \le k_n$, are bump functions as defined before.

Lemma 6.6. *Let $\mathcal{C}$ be the middle-third Cantor set and let $l : \mathcal{C} \to [0, \infty)$ be an upper semi-continuous function. Also let $(I_n)_{n \ge 0}$ be a sequence of intervals in $[0, 1]$ that shrink to a point $z \in \mathcal{C}$. Then*

$$l(z) = \lim_{n \to \infty} \mathrm{Max}(l, I_n).$$

Proof. We have $l(z) \le \mathrm{Max}(l, I_n)$, for all $n \ge 0$, which means

$$l(z) \le \liminf_{n \to \infty} \mathrm{Max}(l, I_n).$$

On the other hand, since l is upper semi-continuous, we have

$$l(z) \ge \limsup_{n \to \infty} \mathrm{Max}(l, I_n). \qquad \square$$

Lemma 6.7. *Let $\mathcal{C}$ be the middle-third Cantor set and let X be a straight hairy Cantor set X based on $\mathcal{C}$ with a length function l. Then any bump function, as defined above, is piecewise monotone and continuous.*

Proof. Let $I = [\alpha, \beta], c$ and $B = B_I^l$ be chosen as above. It follows from the definition that the bump function is piecewise monotone.

We will show that B is right continuous at every $x_* \in [\alpha, \beta)$. Left continuity is similar. First consider the case $x_* \in [\alpha, c)$. Let $(x_i)_{i \in \mathbb{N}}$ be a strictly decreasing sequence in $[\alpha, c)$ such that $x_i \downarrow x_*$ and let

$$b := \lim_{i \to \infty} B(x_i) \ge B(x_*).$$

We will show $b = B(x_*)$ by contradiction. Let us assume $b \ne B$. Then we have

$$\varepsilon := (b - B(x_*))/2 > 0.$$

For all $i \in \mathbb{N}$, there exists $x_i' \in (x_*, x_i]$ such that $l(x_i) \ge B(x_*) + \varepsilon > B(x_*) \ge l(x_*)$. However, this contradicts the assumption that l is upper semi-continuous. Hence B is right continuous on $[\alpha, c)$.

Now consider the case $x_* \in [c, \beta)$ and let $(x_i)_{i\in\mathbb{N}}$ be a strictly decreasing sequence in $[c, \beta)$ such that $x_i \downarrow x_*$. Let

$$b := \lim_{i\to\infty} B(x_i) \le B(x_*).$$

Therefore we have

$$b = \sup\{l(x) \mid x \in (x_*, \beta)\}.$$

If $b < B(x_*)$, then $l(x_*) = B(x_*) > 0$. On the other hand, from condition (ii) in the definition of straight hairy Cantor sets, we know that there is a strictly decreasing sequence $(p_i)_{i\in\mathbb{N}}$ such that $l(p_i) \to l(x)$. The contradiction shows that $b = B(x_*)$ and completes the proof. □

First we will show that straight hairy Cantor sets X_1 and X_2 are ambiently homeomorphic if there is a shrinking sequence of Cantor partitions such that the two length functions have the same asymptotics with respect to this shrinking sequence. The precise statement is the following lemma.

Lemma 6.8. *Let X_1 and X_2 be straight hairy Cantor sets based on $\mathcal{C}$ with l^1 and l^2 as their length functions and let $(P_n)_{n=0}^{\infty}$ be a shrinking sequence of Cantor partitions where $P_0 = [0, 1]$ and*

$$P_n = \bigcup_{i=1}^{k_n} I_{n,i}, \quad I_{n,i} = [x_{n,i}, y_{n,i}], \quad 0 = x_{n,1} < y_{n,1} < x_{n,2} < y_{n,2} < \cdots < x_{n,k_n} < y_{n,k_n} = 1,$$

for all $n \ge 0$ and $1 \le i \le k_n$. Let $M^s_{n,i}$ denote $\mathrm{Max}(l^s, I_{n,i})$. If for all integers i, j, n with $I_{n+1,i} \subseteq I_{n,j}$, we have

$$\frac{1}{1 + 2^{-(n+2)}} < \frac{M^2_{n+1,i}/M^2_{n,j}}{M^1_{n+1,i}/M^1_{n,j}} < 1 + 2^{-(n+2)},$$

then X_1 and X_2 are ambiently homeomorphic.

Proof. Without loss of generality, by simply scaling in the vertical direction, we may assume

$$M^1_{0,1} = M^2_{0,1} = 1/2.$$

For $n \ge 1$, let

$$\lambda_n(t) := \begin{cases} \frac{M^2_{n,i}/M^1_{n,i}}{M^2_{n-1,j}/M^1_{n-1,j}} & \text{if } t \in I_{n,i} \subseteq I_{n-1,j}, \\ 1 & \text{if } t \notin P_n. \end{cases}$$

Note that for $n \ge 1$ and $t \in P_n$, there are unique integers i, j such that $t \in I_{n,i} \subseteq I_{n-1,j}$. Hence λ_n is well-defined. It follows from the assumptions that

$$\forall t \in [0, 1], \quad |\lambda_n(t) - 1| < 2^{-(n+1)}.$$

We recursively define homeomorphisms $\phi_n : \mathbb{R}^2 \to \mathbb{R}^2$ in a way that their limit is the desired map. For $n \ge 0$, we will construct the map ϕ_n such that $\phi_n(X_1)$ is a straight hairy Cantor set with the length function $l^{\phi_n(X_1)}$ and for all $t \in I_{n,i}$, we have

$$l^{\phi_n(X_1)}(t) = \frac{M^2_{n,i}}{M^1_{n,i}} l^1(t).$$

For $n = 0$, we use the identity map

$$\forall z \in \mathbb{R}^2, \quad \phi_0(z) := z.$$

Now let us assume that for some $n \geq 1$, we have constructed such a map ϕ_{n-1} with the desired properties. Let B_n be a bump function for $\phi_{n-1}(X_1)$ on the Cantor partition P_n. Note that for all $x \in [0,1]$, we have

$$B_n(x) \leq \max\{l^{\phi_{n-1}(X_1)}(x) \mid x \in [0,1]\} = \max\left\{(M^2_{n-1,j}/M^1_{n-1,j})l^1(x) \mid 1 \leq j \leq k_{n-1}, x \in I_{n-1,i}\right\}$$
$$= \max\{M^2_{n,j} \mid 1 \leq j \leq k_{n-1}\} = 1/2.$$

Therefore

$$\lambda_n(x)B_n(x) \leq \lambda_n(x)/2 < 5/8.$$

Using B_n and λ_n, we define a map H_n as follows. The map H_n is the identity outside of $[0,1]^2$. For $(x,y) \in [0,1]^2$, we let

$$H_n(x,y) := \begin{cases} (x, \lambda_n(x)y) & \text{if } y \leq B_n(x), \\ \left(x, 1 - \frac{1-\lambda_n(x)B_n(x)}{1-B_n(x)}(1-y)\right) & \text{if } y > B_n(x). \end{cases}$$

The map H_n sends $\{(x,y) \mid x \in [0,1], y = B_n(x)\}$ to $\{(x,y) \mid x \in [0,1], y = \lambda_n(x)B_n(x)\}$. On every vertical line with $x = x_0$, this map is piecewise linear and

$$H_n(\{x_0\} \times [0, B_n(x_0)]) = \{x_0\} \times [0, \lambda_n(x_0)B_n(x_0)],$$

$$H_n(\{x_0\} \times [B_n(x_0), 1]) = \{x_0\} \times [\lambda_n(x_0)B_n(x_0), 1].$$

Note that H_n is in fact identity on $\mathbb{R}^2 \setminus (P_n \times [0,1])$. We may now define ϕ_n recursively by

$$\phi_n := H_n \circ \phi_{n-1}.$$

We need to show that $\phi_n(X_1)$ is a straight hairy Cantor set. However, this is clear since the effect of H_n on $\phi_{n-1}(X_1)$ is dividing $\phi_{n-1}(X_n)$ into k_n parts and scaling each part in the vertical direction by a constant factor. For $t \in I_{n,i} \subseteq I_{n-1,j}$, we have

$$l^{\phi_n(X_1)}(t) = \lambda_n(t)l^{\phi_{n-1}(X_1)}(t) = \left(\frac{M^2_{n,i}/M^1_{n,i}}{M^2_{n-1,j}/M^1_{n-1,j}}\right)\left(\frac{M^2_{n-1,j}}{M^1_{n-1,j}}l^1(t)\right) = \frac{M^2_{n,i}}{M^1_{n,i}}l^1(t).$$

Hence ϕ_n has the required properties.

Now that we have defined ϕ_n, we will show that the map

$$\phi := \lim_{n\to\infty} \phi_n$$

exists, it is a homeomorphism and $\phi(X_1) = X_2$. For $(x,y) \in [0,1]^2$, we have

$$|H_n(x,y) - (x,y)| = \begin{cases} |(\lambda_n(x) - 1)y| & \text{if } y \leq B_n(x), \\ \left|(1-y)(\lambda_n(x)-1)\frac{B_n(x)}{1-B_n(x)}\right| & \text{if } y > B_n(x). \end{cases}$$

We also have $B_n(x) \leq 1/2$, for all $x \in [0,1]$, which implies $\frac{B_n(x)}{1-B_n(x)} \leq 1$. Therefore

$$\|\phi_n - \phi_{n-1}\|_\infty = \|H_n - \mathrm{Id}\,\|_\infty \leq 2^{-(n+1)}.$$

Hence the sequence $(\phi_n)_{n=0}^\infty$ is Cauchy with respect to the supremum norm and the limit ϕ is a continuous function.

To prove that ϕ is a homeomorphism, we will show that it is injective and surjective. Let us fix $x \in [0,1]$ and $n \geq 1$. If $y, y' \in [0, B_n(x)]$, then

$$|H_n(x,y) - H_n(x,y')| = \lambda_n(x)|y - y'| \geq (1 - 2^{-(n+1)})|(x,y) - (x,y')|.$$

On the other hand, if $y, y' \in [B_n(x), 1]$, then

$$|H_n(x,y) - H_n(x,y')| = \frac{1-\lambda_n(x)B_n(x)}{1-B_n(x)}|y-y'| = \left(1 - \frac{B_n(x)}{1-B_n(x)}(1-\lambda_n(x))\right)|y-y'|$$
$$\geq \left(1 - \frac{B_n(x)}{1-B_n(x)}2^{-(n+1)}\right)|y-y'| \geq (1-2^{-(n+1)})|(x,y)-(x,y')|.$$

Finally, since $H_n|_{\{x\}\times[0,1]}$ is a homeomorphism from its domain onto itself, if $0 \leq y' < B_n(x) < y \leq 1$, then

$$|H_n(x,y) - H_n(x,y')| = |H_n(x,y) - H_n(x,B_n(x))| + |H_n(x,B_n(x)) - H_n(x,y')|$$
$$\geq (1-2^{-(n+1)})|(x,y)-(x,y')|.$$

Therefore, for $x, y, y' \in [0,1]$ and $n \geq 0$, we have

$$|\phi_n(x,y) - \phi_n(x,y')| = |H_n \circ H_{n-1} \circ \cdots \circ H_1(x,y) - H_n \circ H_{n-1} \circ \cdots \circ H_1(x,y')|$$
$$\geq (1-2^{-(n+1)})|H_{n-2} \circ \cdots \circ H_1(x,y) - H_{n-2} \circ \cdots \circ H_1(x,y')|$$
$$\geq \cdots \geq \left(\prod_{k=2}^{n+1}(1-2^{-k})\right)|(x,y)-(x,y')| \geq \left(\prod_{k=2}^{\infty}(1-2^{-k})\right)|y-y'|.$$

It is easy to see that $\prod_{k=2}^{\infty}(1-2^{-k}) > 0$. Every H_n (and therefore every ϕ_n) sends vertical lines to vertical lines and is identity on the boundary of $[0,1]^2$. Therefore ϕ also has this property. Hence the inequality

$$\forall x, y, y' \in [0,1], \quad |\phi(x,y) - \phi(x,y')| \geq \left(\prod_{k=2}^{\infty}(1-2^{-k})\right)|y-y'|$$

implies that ϕ is injective. To prove that ϕ is surjective, note that $\phi(\mathcal{C}) \subseteq \mathcal{C}$ is compact. Therefore, if $z \in \mathcal{C} \setminus \phi(\mathcal{C})$, then there is $\varepsilon > 0$ such that $(D_\varepsilon(z) \cap \mathcal{C}) \subseteq \mathcal{C} \setminus \phi(\mathcal{C})$. Choose $N \geq 0$ large enough so that $\|\phi_N - \phi\|_\infty < \varepsilon$. Then we have $\phi(\phi_N^{-1}(z)) \in D_\varepsilon(z)$ which is a contradiction.

If we define

$$\forall t \in \mathcal{C}, \quad l^{\phi(X_1)}(t) := \lim_{n\to\infty} l^{\phi_n(X_1)}(t),$$

we immediately see that $\phi(X_1) = \{(x,y) \in \mathbb{R}^2 \mid x \in \mathcal{C}, 0 \leq y \leq l^{\phi(X_1)}\}$. To complete the proof, we need to show that for all $t \in \mathcal{C}$, we have $l^2(t) = l^{\phi(X_1)}(t)$. Fix $t \in \mathcal{C}$ and for all $m \geq 0$ choose i_m such that $t \in I_{m,i_m}$. We have

$$l^{\phi(X_1)}(t) = \lim_{n\to\infty} l^{\phi_n(X_1)}(t) = \lim_{n\to\infty} \frac{M^2_{n,i_n}}{M^1_{n,i_n}} l^1(t).$$

Lemma 6.6 implies that

$$l^1(t) = \lim_{n\to\infty} M^1_{n,i_n}, \quad l^2(t) = \lim_{n\to\infty} M^2_{n,i_n}.$$

Hence, if $l^1(t) \neq 0$, we have

$$l^{\phi(X_1)}(t) = \frac{l^2(t)}{l^1(t)} l^1(t) = l^2(t).$$

If $l^1(t) = 0$, then $l^{\phi(X_1)}(t) = 0$. We will show that $l^2(t) = 0$ as well. Let us assume $l^2(t) > 0$. Then we have

$$\lim_{n\to\infty} \frac{M^1_{n,i_n}}{M^2_{n,i_n}} = \frac{l^1(t)}{l^2(t)} = 0.$$

On the other hand, we have

$$\begin{aligned}\frac{M^1_{n,i_n}}{M^2_{n,i_n}} &= \frac{M^1_{n,i_n}/M^2_{n,i_n}}{M^1_{n-1,i_{n-1}}/M^2_{n-1,i_{n-1}}}\frac{M^1_{n-1,i_{n-1}}/M^2_{n-1,i_{n-1}}}{M^1_{n-2,i_{n-2}}/M^2_{n-2,i_{n-2}}}\cdots\frac{M^1_{1,i_1}/M^2_{1,i_1}}{M^1_{0,1}/M^2_{0,1}}\\ &> \frac{1}{(1+2^{-(n+1)})(1+2^{-n})\cdots(1+2^{-2})} > \frac{1}{e^{2^{-(n+1)}}e^{2^{-n}}\cdots e^{2^{-2}}}\\ &= e^{-(2^{-2}+2^{-3}+\cdots+2^{-(n+1)})} > e^{-1}\end{aligned}$$

which implies

$$\liminf_{n\to\infty}\frac{M^1_{n,i_n}}{M^2_{n,i_n}} \geq e^{-1} > 0.$$

The contradiction shows that $l^2(t) = l^{\phi(X_1)}(t)$, for all $t \in \mathcal{C}$, and completes the proof. □

Now if we have two straight hairy Cantor sets, the next lemma shows that we may find a sequence of shrinking Cantor sets for each one in a way that a condition similar to previous lemma is satisfied.

Lemma 6.9. *Let X_1 and X_2 be straight hairy Cantor sets based on $\mathcal{C}$ with length functions l^1 and l^2. Then, for $s \in \{1,2\}$, there are shrinking sequence of Cantor partitions $(P^s_n)_{n\geq 0}$ with*

$$P^s_n = \bigcup_{i=1}^{k_n} I^s_{n,i}, \quad I^s_{n,i} = [x^s_{n,i}, y^s_{n,i}], \quad 0 = x^s_{n,1} < y^s_{n,1} < \cdots < x^s_{n,k_n} < y^s_{n,k_n} = 1,$$

for all $n \geq 0$ and $1 \leq i \leq k_n$ such that $P^1_0 = P^2_0 = [0,1]$ and the following properties hold.

(i) We have $\mathrm{mesh}(P^s_n) < 2^{-n}$ *for all $n \geq 2$ and $s = 1, 2$.*[8]

(ii) For each $n \geq 0$, the partitions P^1_n and P^2_n have the same number of elements, namely k_n, and for each $i \in \{1, ..., k_{n+1}\}$ and $j \in \{1, ..., k_n\}$, we have

$$I^1_{n+1,i} \subseteq I^1_{n,j} \text{ if and only if } I^2_{n+1,i} \subseteq I^2_{n,j}.$$

(iii) For each $n \geq 0$, if $I^s_{n+1,i} \subseteq I^s_{n,j}$ for some i and j, then

$$\frac{1}{1+2^{-(n+2)}} < \frac{M^2_{n+1,i}/M^2_{n,j}}{M^1_{n+1,i}/M^1_{n,j}} < 1 + 2^{-(n+2)}.$$

Here we used $M^s_{n,i}$ to denote $\mathrm{Max}(l^s, I^s_{n,i})$.

Proof. We have $P^1_0 = P^2_0 = [0,1]$. Given $P^s_0, ..., P^s_n$, for $s = 1, 2$ and some $n \geq 0$, we inductively construct P^1_{n+1} and P^2_{n+1}. First we consider the case where n is even. Choose the partition P^2_{n+1} to be a subset of the partition P^2_n such that

$$\mathrm{mesh}(P^2_{n+1}) < 2^{-(n+2)}.$$

We shall use this to construct P^1_{n+1}. The partition P^2_{n+1} is a subset of P^2_n, so for any $1 \leq t \leq k_n$, there are $0 \leq i \leq j \leq k_{n+1}$ such that

$$x^2_{n+1,i} = x^2_{n,t}, \quad y^2_{n+1,j} = y^2_{n,t}.$$

Therefore we need to set

$$x^1_{n+1,i} = x^1_{n,t}, \quad y^1_{n+1,j} = y^1_{n,t}.$$

[8]The mesh of a Cantor partition $P = \bigcup_{i=1}^k [x_i, y_i]$ is defined as

$$\mathrm{mesh}(P) := \max\{|y_i - x_i| \mid 1 \leq i \leq k\}.$$

Now we need to specify $x^1_{n+1,r}$ and $y^1_{n+1,r-1}$ for $i < r \leq j$. If $i = j$, there is nothing to do. Let us assume $i < j$. In order to do that, we choose $c^s_{n,t} \in I^s_{n,t}$ such that $l^s(c^s_{n,t}) = M^s_{n,t}$. Therefore there is a unique $i \leq w \leq j$ such that

$$x^2_{n+1,w} < c^2_{n,t} < y^2_{n+1,w}.$$

If, $i < w$, then we will choose $y^1_{n+1,r}$ and $x^1_{n+1,r+1}$ recursively by going from $r = i$ to $w - 1$. If $w < j$, we will do a similar construction by going from $r = j - 1$ to w. We will go through this construction for $i \leq r < w$. The other part is done similarly.

Let us assume that for some $i \leq r < w - 1$, we have already defined

$$x^1_{n+1,i} < y^1_{n+1,i} < \cdots < x^1_{n+1,r} < c^1_{n,t}.$$

Using this, we will define $y^1_{n+1,r}$ and $x^1_{n+1,r+1}$. Consider the bump function $B = B^{l^1}_{[x^1_{n+1,r}, y^1_{n+1,j}]}$ associated to the point $c^1_{n,t}$. As we move y from $x^1_{n+1,r}$ to $c^1_{n,t}$, the value of $B(y)$ continuously increases from 0 to $M^1_{n,t}$. Therefore there is $x^1_{n+1,r} < y < c^1_{n,t}$ such that

$$\frac{1}{1 + 2^{-(n+2)}} < \frac{M^2_{n+1,r}/M^2_{n,t}}{B(y)/M^1_{n,t}} < 1 + 2^{-(n+2)}.$$

By slightly moving y to the right, we may find two points $y^1_{n+1,r}, x^1_{n+1,r+1}$ in $\mathcal{C}$ such that

$$x^1_{n+1,r} < y < y^1_{n+1,r} < x^1_{n+1,r+1} < c^1_{n,t}, \quad (y^1_{n+1,r}, x^1_{n+1,r+1}) \subseteq [0,1] \setminus \mathcal{C},$$

$$\frac{1}{1 + 2^{-(n+2)}} < \frac{M^2_{n+1,r}/M^2_{n,t}}{B(y^1_{n+1,r})/M^1_{n,t}} = \frac{M^2_{n+1,r}/M^2_{n,t}}{M^1_{n+1,r}/M^1_{n,t}} < 1 + 2^{-(n+2)}.$$

Here we did the construction for even n. For odd n, we interchange the role of 1 and 2. □

Proof of Proposition 6.3. Based on the discussion after the statement of the proposition, we only need to consider straight hairy Cantor sets that are based on $\mathcal{C}$. Consider two straight hairy Cantor sets X_1 and X_2 with base $\mathcal{C}$ and l^1 and l^2 as their length functions. From Lemma 6.9 we get shrinking sequences of Cantor partitions $(P^1_n)_{n\geq 0}$ and $(P^2_n)_{n\geq 0}$. We first find a homeomorphism of $\mathcal{C}$ that sends the corresponding partitions to each other. We define

$$\phi : \{t \mid t \in \partial P^1_{n,i}, n \geq 0, 1 \leq i \leq k_n\} \to \{t \mid t \in \partial P^2_{n,i}, n \geq 0, 1 \leq i \leq k_n\},$$

$$\forall n \geq 0, 1 \leq i \leq k_n, \quad \phi(x^1_{n,i}) := x^2_{n,i}, \quad \phi(y^1_{n,i}) := y^2_{n,i}.$$

This map is increasing and its domain and range are both dense in $\mathcal{C}$. Therefore ϕ has a unique continuous extension to $\mathcal{C}$. Moreover, ϕ^{-1} also extends continuously to $\mathcal{C}$ and its extension is the inverse of the extension of ϕ. Hence the extended function $\phi : \mathcal{C} \to \mathcal{C}$ is an increasing homeomorphism. We may extend this map to the complement of $\mathcal{C}$ by linear interpolation. This will give us a homeomorphism of the interval $[0, 1]$. Now we extend ϕ to be the identity on $\mathbb{R} \setminus [0, 1]$, and define

$$\forall (x, y) \in \mathbb{R}^2, \quad \phi(x, y) := (\phi(x), y).$$

This is clearly a homeomorphism of the plane. Now consider the two straight hairy Cantor sets X_1 and $\phi^{-1}(X_2)$. Evidently, the shrinking sequence of Cantor partitions $(P^1_n)_{n\geq 0}$ satisfies the condition needed in Lemma 6.8. Hence X_1 and $\phi^{-1}(X_2)$ are ambiently homeomorphic. □

Proposition 6.10. *If X is a straight hairy Cantor set with base Cantor set C and length function $l : C \to [0, \infty)$, then the set*

$$\{(x, l(x)) \mid x \in C, l(x) \neq 0\}$$

is dense in X.

Proof. Let $A = \{(x, l(x)) \mid x \in C, l(x) \neq 0\}$ and choose a point $(x, y) \in X$. We will show that $(x, y) \in \overline{A}$. Without loss of generality, we may assume that $C = \mathcal{C}$ is the middle-third Cantor set. Let $(P_n)_{n=0}^{\infty}$ be a shrinking sequence of Cantor partitions.

Fix $n \geq 0$ and let $I = [a, b]$ be the connected component of P_n that contains x. Choose a bump function $B := B_I^l$ on the interval I. According to the definition of the straight hairy Cantor sets, we know that $B : I \to [0, \infty)$ is not constant and $B(a) = B(b) = 0$. Lemma 6.7 shows that B is continuous. If $y = 0$, we may choose a value $x'_n \in I$ such that $0 < B(x'_n) < 1/n$. Otherwise, if $0 < y \leq l(x)$, then we have $B(a) = 0$ and $B(x) \geq y > 0$, so we may choose $x'_n \in I$ such that $B(x'_n) = y$. In any case, according to the definition of B, we find $x_n \in I$ such that

$$l(x_n) = B(x'_n) \in [y, y + 1/n) \setminus \{0\}.$$

If we repeat this construction for all $n \geq 0$, we obtain a sequence $(x_n)_{n\geq 0}$ such that

$$x_n \to x, \quad l(x_n) \to y,$$

and $l(x_n) \neq 0$ for all $i \geq 0$. In other words, $(x, y) \in \overline{A}$. □

The conditions in the definition of the straight hairy Cantor sets are necessary to prove Proposition 6.3. However, if we are only interested in the homeomorphic images of a straight hairy Cantor set and we ignore its embedding in the plane, we may weaken the conditions as described in the following lemma. We will need this result in our analysis of the *Abstract hairy Cantor sets* in Subsection 6.4.

Proposition 6.11. *Let $C \subseteq \mathbb{R}$ be a Cantor set and let $l : C \to [0, \infty]$ be a function such that*

$$X := \{(x, y) \in \mathbb{R}^2 \mid x \in C, 0 \leq y \leq l(x)\}$$

is compact and the set $\{(x, l(x)) \mid x \in C, l(x) \neq 0\}$ is dense in X. Then X is homeomorphic to a straight hairy Cantor set.

Proof. We will construct a sequence of homeomorphisms $\phi_n : C \to C$, for $n \geq 0$, and use them to define a homeomorphism of $C \times \mathbb{R}$ onto itself that sends X onto a set X_*. Then we will show that X_* is a straight hairy Cantor set.

Without loss of generality, we may assume that C is the middle-third Cantor set. Let $P_0 = [0, 1]$ and recursively define[9]

$$\forall n \geq 1, \quad P_n := \frac{P_{n-1}}{3} \cup (\frac{2}{3} + \frac{P_{n-1}}{3}).$$

Note that $(P_n)_{n=0}^{\infty}$ is a shrinking sequence of Cantor partitions. For any $n \geq 0$, assume that $P_n = \bigcup_{i=1}^{2^n} I_{n,i}$, where $I_{n,i}$'s are the connected components of P_n and $\sup I_{n,i} < \inf I_{n,i+1}$ for all $1 \leq i < 2^n$.

Let ϕ_0 be the identity function on C and let $m_0 = 0$. We will recursively choose integers m_n, define homeomorphisms $\phi_n : C \to C$ and upper semi-continuous functions

$$l_n : C \to \mathbb{R}, \quad l_n(x) := l(\phi_n^{-1}(x)),$$

for all $n \geq 0$. Note that X is closed, therefore l is upper semi-continuous. We want to guarantee that for $n > 0$ and $1 \leq j \leq 2^{m_{n-1}}$,

$$\operatorname{Max}(l_n, I_{m_n, 2^{m_n - m_{n-1}}(j-1)+1}) < 1/n, \quad \operatorname{Max}(l_n, I_{m_n, 2^{m_n - m_{n-1}} j}) < 1/n,$$

and for all $2^{m_n - m_{n-1}}(j-1) + 1 \leq i < 2^{m_n - m_{n-1}} j$, we have

$$|\operatorname{Max}(l_n, I_{m_n,i}) - \operatorname{Max}(l_n, I_{m_n,i+1})| < 1/n.$$

[9]For a set $A \in \mathbb{R}$ and a real number $k \in \mathbb{R}$, we define $A + k := \{a + k \mid a \in A\}$ and $kA := \{ka \mid a \in A\}$.

Note that

$$\{i \in \mathbb{N} \mid 2^{m_n - m_{n-1}}(j-1) + 1 \le i < 2^{m_n - m_{n-1}} j\} = \{i \in \mathbb{N} \mid I_{m_n,i} \subseteq I_{m_{n-1},j}.\}$$

Let us assume that $n \ge 1$ and ϕ_{n-1} and m_{n-1} are defined. We will define a homeomorphism $H_n : C \to C$ such that for every $1 \le j \le 2^{m_{n-1}}$, we have

$$H_n(I_{m_{n-1},j} \cap C) = I_{m_{n-1},j} \cap C,$$

and use H_n to define

$$\phi_n := H_n \circ \phi_{n-1}.$$

If H_n is defined, then the above condition implies that for all $x \in C$,

$$|H_n(x) - x| \le \max\{\operatorname{diam}(I_{m_{n-1},j}) \mid 1 \le j \le 2^{m_{n-1}}\} = 3^{-m_{n-1}}.$$

It also implies that for all $0 \le k < n$ and $1 \le t \le 2^{m_k}$,

$$\begin{aligned} H_n(I_{m_k,t} \cap C) &= H_n\left(\bigcup_{1 \le j \le 2^{m_{n-1}}, I_{m_{n-1},j} \subseteq I_{m_k,t}} (I_{m_{n-1},j} \cap C)\right) \\ &= \bigcup_{1 \le j \le 2^{m_{n-1}}, I_{m_{n-1},j} \subseteq I_{m_k,t}} (I_{m_{n-1},j} \cap C) = I_{m_k,t} \cap C. \end{aligned}$$

In other words, for such k and t, the map $H_n|_{I_{m_k,t} \cap C}$ is a homeomorphism from its domain onto itself.

The map $(x, y) \mapsto (\phi_{n-1}(x), y) : X \to \mathbb{R}^2$ is a homeomorphism from X onto the set

$$X_{n-1} := \{(x, y) \in \mathbb{R}^2 \mid x \in C, 0 \le y \le l_{n-1}(x)\}.$$

Hence the set $\{(x, l_{n-1}(x)) \mid x \in C, l_{n-1}(x) \ne 0\}$ is dense in X_{n-1}. Therefore, for any $1 \le j \le 2^{m_{n-1}}$, we may find a finite set $A_j \subseteq I_{m_{n-1},j} \cap C$, such that

$$D_{1/5n}(l_{n-1}(A_j)) \supseteq [0, \operatorname{Max}(l_{n-1}, I_{m_{n-1},j})].$$

According to Lemma 6.6, $l_{n-1}(x) = \lim_{m \to \infty, x \in I_{m,i}} \operatorname{Max}(l_{n-1}, I_{m,i})$. Therefore we may find an integer $m_n := m > m_{n-1}$ large enough so that

(*) $\quad D_{1/4n}(\{\operatorname{Max}(l_{n-1}, I_{m_n,i}) \mid I_{m_n,i} \subseteq I_{m_{n-1},j}\}) \supseteq D_{1/5n}(l_{n-1}(A_j)) \supseteq [0, \operatorname{Max}(l_{n-1}, I_{m_{n-1},j})]$,

for all $1 \le j \le 2^{m_{n-1}}$.

We will first define $H_n|_{I_{m_{n-1},j} \cap C}$ for $j = 1$. The construction is similar for other values of j. Let

$$K := \{1 \le i \le 2^{m_n} \mid I_{m_n,i} \subseteq I_{m_{n-1},1}\} = \{i \in \mathbb{N} \mid 1 \le i \le 2^{m_n - m_{n-1}}\},$$

and let $\tau : K \to K$ be a permutation such that

$$\operatorname{Max}(l_{n-1}, I_{m_n,\tau(1)}) \le \operatorname{Max}(l_{n-1}, I_{m_n,\tau(2)}) \le \cdots \le \operatorname{Max}(l_{n-1}, I_{m_n,\tau(2^{m_n - m_{n-1}})}).$$

Define a permutation $\sigma : K \to K$ by

$$\begin{aligned} \sigma(1), \sigma(2), \cdots, \sigma(2^{m_n - m_{n-1}}) =& \tau(1), \tau(3), \tau(5), \cdots, \tau(2^{m_n - m_{n-1}} - 1), \\ & \tau(2^{m_n - m_{n-1}}), \tau(2^{m_n - m_{n-1}} - 2), \tau(2^{m_n - m_{n-1}} - 4), \cdots, \tau(2). \end{aligned}$$

According to (*), we see that

$$\operatorname{Max}(l_{n-1}, I_{m_n,\sigma(1)}) < 1/n, \quad \operatorname{Max}(l_{n-1}, I_{m_n,\sigma(2^{m_n - m_{n-1}})}) < 1/n,$$

and for all $1 \le i < 2^{m_n - m_{n-1}}$, we have

$$|\operatorname{Max}(l_{n-1}, I_{m_n,\sigma(i)}) - \operatorname{Max}(l_{n-1}, I_{m_n,\sigma(i+1)})| < 1/n.$$

For any $i \in K$, we want to have

$$H_n(I_{m_n,i} \cap C) = I_{m_n,\sigma^{-1}(i)} \cap C.$$

Moreover, we want $H_n|_{I_{m_{n-1},1} \cap C}$ to act by translation on $I_{m_n,i} \cap C$ for any $i \in K$. These two conditions uniquely define $H_n|_{I_{m_{n-1},1} \cap C}$. As a consequence, we may also define $l_n|_{I_{m_{n-1},1} \cap C}$. Note that for all $i \in K$, we have

$$\mathrm{Max}(l_n, I_{m_n,i}) = \mathrm{Max}(l_{n-1} \circ H_n^{-1}, I_{m_n,i}) = \mathrm{Max}(l_{n-1}, I_{m_n,\sigma(i)}).$$

We may do a similar construction for all $1 < j \leq 2^{m_{n-1}}$ to define H_n and l_n on C.

We will show that the limit

$$\forall x \in C, \quad \phi_*(x) := \lim_{n \to \infty} \phi_n(x),$$

exists and the function ϕ_* is a homeomorphism of C onto itself. For any $n \geq 0$, we have

$$\|\phi_n - \phi_{n-1}\|_\infty = \|H_n - \mathrm{Id}\,\|_\infty \leq 3^{-m_{n-1}}.$$

Therefore the sequence $(\phi_n)_{n \geq 0}$ in uniformly Cauchy, hence the limit ϕ_* exists and it is a continuous map on C. To prove that ϕ_* is surjective, note that $\phi_*(C) \subseteq C$ is compact. Therefore, if $z \in C \setminus \phi_*(C)$, then there is $\varepsilon > 0$ such that

$$D_\varepsilon(z) \cap C \subseteq C \setminus \phi_*(C).$$

Choose $N \geq 0$ large enough so that $\|\phi_N - \phi_*\|_\infty < \varepsilon$. Then we have $\phi_*(\phi_N^{-1}(z)) \in D_\varepsilon(z)$ which is a contradiction. To prove that ϕ_* is injective, let x and y be points in C. If $x \neq y$, then there is $n > 0$ and $1 \leq i < i' \leq 2^{m_n}$ such that $x \in I_{m_n,i}$ and $y \in I_{m_n,i'}$. According to the construction, there are $1 \leq j, j' \leq 2^{m_n}$ with $j \neq j'$ such that $\phi_n(x) \in I_{m_n,j}$ and $\phi_n(y) \in I_{m_n,j'}$. On the other hand, for all $n' > n$, we have

$$H_{n'}(I_{m_n,j} \cap C) = I_{m_n,j} \cap C, \quad H_{n'}(I_{m_n,j'} \cap C) = I_{m_n,j'} \cap C.$$

Therefore $\phi_*(x) = \cdots \circ H_{n+2} \circ H_{n+1} \circ \phi_n(x) \in I_{m_n,j}$ and similarly $\phi_*(y) \in I_{m_n,j'}$. Hence we have $\phi_*(x) \neq \phi_*(y)$ and ϕ_* is injective.

Now we define $l_* := l \circ \phi_*^{-1}$ and

$$X_* := \{(x, y) \in \mathbb{R}^2 \mid x \in C, 0 \leq y \leq l_*(x)\}.$$

Since ϕ_* is a homeomorphism, $(x, y) \mapsto (\phi_*(x), y) : X \to X_*$ is also a homeomorphism.

To complete the proof, we need to prove that X_* is a straight hairy Cantor set. First we will show that for all $n \geq 0$ and $1 \leq i \leq 2^{m_n}$, we have

$$\mathrm{Max}(l_*, I_{m_n,i}) = \mathrm{Max}(l_n, I_{m_n,i}).$$

According to the definition of l_* and l_n, it is sufficient to show that $\phi_*^{-1}(I_{m_n,i} \cap C) = \phi_n^{-1}(I_{m_n,i} \cap C)$, or equivalently, $I_{m_n,i} \cap C = \phi_* \circ \phi_n^{-1}(I_{m_n,i} \cap C)$. However, we know that for all $n' > n$, $H_{n'}(I_{m_n,i} \cap C) = I_{m_n,i} \cap C$ and $\phi_* \circ \phi_n^{-1} = \cdots \circ H_{n+2} \circ H_{n+1}$. Hence $\phi_* \circ \phi_n^{-1}$ maps $I_{m_n,i} \cap C$ onto itself homeomorphically.

The first condition in the definition of the straight hairy Cantor sets states that the set $\{x \in C \mid l_*(x) = 0\}$ is dense in C. Choose $n \geq 0$ and $1 \leq i \leq 2^{m_n}$ and let $C' := I_{m_n,i} \cap C$. Since the choice of n and i is arbitrary, it is sufficient to find a point $z \in C'$ such that $l_*(z) = 0$. According to the construction, we may choose $(i_k)_{k=n}^\infty$ with $i_n = i$ and $1 \leq i_k \leq 2^{m_k}$, for all $k \geq n+1$, such that

$$I_{m_n,i_n} \supseteq I_{m_{n+1},i_{n+1}} \supseteq \cdots \supseteq I_{m_k,i_k} \supseteq \cdots$$

and

$$\forall k \geq n+1, \quad \mathrm{Max}(l_*, I_{m_k,i_k}) < 1/k.$$

The intervals $(I_{m_k,i_k})_{k \geq n}$ shrink to a point $z \in C'$ and by Lemma 6.6 we have $l_*(z) = 0$.

Now let $z \in C$ be a point such that $l_*(z) > 0$ and let $\varepsilon > 0$. Also let $(i_k)_{k\geq 0}$ be the sequence of integers such that $z \in I_{m_k,i_k}$ for all $k \geq 0$. According to Lemma 6.6, we may choose $N > \max(2/\varepsilon, 1/l_*(z))$ large enough so that $3^{-m_{N-1}} < \varepsilon$ and

$$|\operatorname{Max}(l_*, I_{m_N,i_N}) - l_*(z)| < \varepsilon/2.$$

Since $\operatorname{Max}(l_*, I_{m_N,i_N}) \geq l_*(z) > 1/N$, we have

$$|\operatorname{Max}(l_*, I_{m_N,i_N-1}) - \operatorname{Max}(l_*, I_{m_N,i_N})| < 1/N, \quad |\operatorname{Max}(l_*, I_{m_N,i_N}) - \operatorname{Max}(l_*, I_{m_N,i_N+1})| < 1/N,$$

and

$$I_{m_N,i_N-1} \cup I_{m_N,i_N+1} \subseteq I_{m_{N-1},i_{N-1}}.$$

Let α be a point in $I_{m_N,i_N-1} \cap C$ such that $l_*(\alpha) = \operatorname{Max}(l_*, I_{m_N,i_N-1})$. Then we have

$$\alpha < z, \quad |\alpha - z| < \operatorname{diam}(I_{m_{N-1},i_{N-1}}) = 3^{-m_{N-1}} < \varepsilon, \quad |l_*(\alpha) - l_*(z)| < \varepsilon.$$

Similarly we may find $\beta \in C$ such that

$$\beta > z, \quad |\beta - z| < \varepsilon, \quad |l_*(\beta) - l_*(z)| < \varepsilon.$$

Since the choice of $\varepsilon > 0$ was arbitrary, this shows that the second condition in the definition of the straight hairy Cantor sets holds. □

6.2. **Hairy Cantor sets.** Here we will define hairy Cantor sets and analyse some of their properties. We define *base curves* and in Proposition 6.26, we show that every hairy Cantor set in the plane admits a *base curve*. The main ingredients of the proof are *Whitney maps* and *Borsuk's theorem*. Using base curves, we obtain an alternative characterisation of hairy Cantor sets. As a corollary, we will see that all straight hairy Cantor sets, as defined previously, are indeed hairy Cantor sets. We will also find an alternative characterisation of the straight hairy Cantor sets.

Unless otherwise stated, we will be considering the metric space $\mathbb{R}^2$ with the Euclidian distance. We will also use $\mathbb{D}$ to denote the unit disk $D(0,1)$ in the plane and use $\mathbb{S}^1$ to denote its boundary $\partial\mathbb{D}$. For a metric space (M,d), the Hausdorff distance induces a metric on

$$\mathrm{H}[M] := \{A \subseteq M \mid A \text{ is compact and non-empty }\}.$$

It is known that for any compact metric space M, the set $\mathrm{H}[M]$ with the Hausdorff topology is compact.

Now we have the ingredients to define hairy Cantor sets.

Definition 6.12. *A compact set $X \subseteq \mathbb{R}^2$ is called a hairy Cantor set if the following properties are satisfied.*

- *(i) Every connected component of X is either a single point or a Jordan arc.*
- *(ii) Closure of the union of the point components of X is a Cantor set, namely $B \subseteq X$.*
- *(iii) Any arc component of X meets B at exactly one of its end-points.*
- *(iv) Whenever $x_i \to x$ within X, the unique arc in X connecting x_i to B converges to the unique arc in X connecting x to B, with the convergence in the Hausdorff topology.*
- *(v) The set $X \setminus B$ is dense in X.*
- *(vi) The points in the interior of the arc components of X are not accessible from $\mathbb{R}^2 \setminus X$.*

If x, y are points in X that belong to the same connected component of X, we use $[x,y]$ to denote the unique (possibly degenerate) Jordan arc in X joining x and y. We define the base map of X as the map $b : X \to B$ that sends any point $x \in X$ to the unique point in the connected component of X containing x that belongs to B. We will say that $b(x)$ is the base point of x.

Note that the fourth condition implies the continuity of the base map.

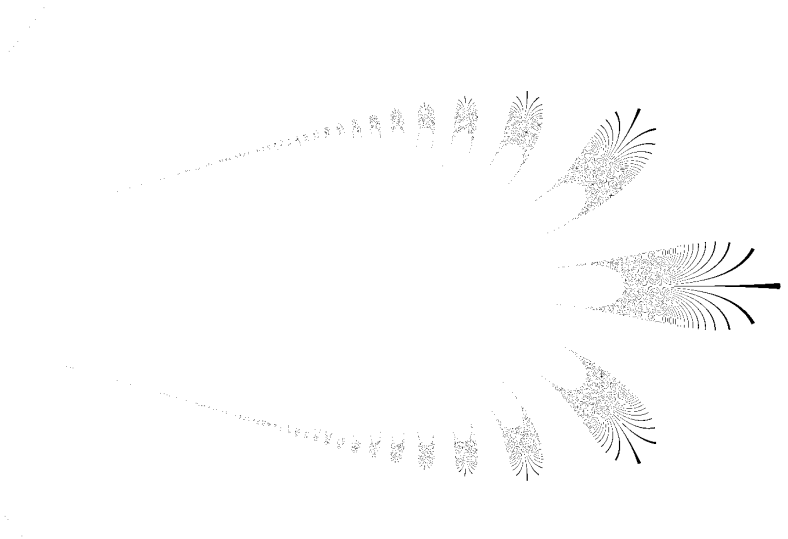


FIGURE 6. An illustration of a hairy Cantor set in the plane.

Definition 6.13. *For any hairy Cantor set X with base Cantor set B, we define a map $p : X \to X$ such that*

- *If $\{x\} \subseteq X$ is a connected component, then $p(x) := x$.*
- *If x is in an arc component in X, then $p(x)$ is the end point of that Jordan arc that does not belong to B.*

We will say that p is the peak map of X and $p(x)$ is the peak of x.

Definition 6.14. *Let X be a hairy Cantor set with the base Cantor set B. A height function on X is a continuous map $h : X \to [0, \infty)$ that is injective on every connected component of X and*

$$h^{-1}(0) = B.$$

As we will see at the end of this subsection, all straight hairy Cantor sets are hairy Cantor sets. For a straight hairy Cantor set X with the base Cantor set C and the length function l, the function

$$h : X \to [0, \infty), \quad h(x, y) := y,$$

is a height function on X. The choice of height function is clearly not unique. To analyse height functions for arbitrary hairy Cantor sets, we need the following definition.

Definition 6.15. *Let X be a non-empty compact metric space. A Whitney map for X is a function $\mu : \mathrm{H}[X] \to [0, \infty)$ that satisfies the following properties.*

- *The map μ is continuous with respect to the Hausdorff topology.*
- *For any point $x \in X$, we have $\mu(\{x\}) = 0$.*
- *For all $A_1, A_2 \in \mathrm{H}[X]$ that $A_1 \subsetneq A_2$, we have $\mu(A_1) < \mu(A_2)$.*

In [Whi33], Whitney proved the existence of such maps for a large family of sets which includes non-empty compact metric spaces. On may consult [Nad92] for a modern treatment of the subject.(See [Nad92, Theorem 13.4])

Lemma 6.16. *Any hairy Cantor set admits a height function.*

Proof. Let X be a hairy Cantor set with base Cantor set B and let μ be a Whitney map for X. Define

$$\forall x \in X, \quad h(x) := \mu([b(x), x]).$$

Clearly h is injective on every connected component of X and

$$h^{-1}(0) = \{x \in X \mid x = b(x)\} = B.$$

In order to prove the continuity of h, let $(x_i)_{i\in\mathbb{N}}$ be a sequence in X where $x_i \to x_*$ for some $x_* \in X$. By the definition of hairy Cantor sets, $[b(x_i), x_i]$ tends to $[b(x_*), x_*]$ in the Hausdorff topology. Therefore

$$\lim_{i\to\infty} h(x_i) = h(x_*). \qquad \square$$

Definition 6.17. *Let X be a hairy Cantor set with base Cantor set B. We call a Jordan curve ω a base curve for X if*

$$B = X \cap \omega$$

and the bounded connected component of $\mathbb{R}^2 \setminus \omega$ does not intersect X.

We will show that any hairy Cantor set admits a base curve. However, as we will describe in Remark 5, the choice of such a base curve is far from canonic.

Lemma 6.18. *Let $X \subseteq \mathbb{R}^2$ be a hairy Cantor set with base Cantor set B and let $\omega \subseteq \mathbb{R}^2$ be a Jordan curve such that $\omega \cap X = B$. Then the set $\mathbb{R}^2 \setminus (X \cup \omega)$ has two connected components.*

Before we start the proof of this lemma, we discuss one of the main tools that we need. *Borsuk's theorem* in two dimensions asserts that for a closed set $Y \subseteq \hat{\mathbb{C}}$, the set $\hat{\mathbb{C}}\setminus Y$ is connected if and only if every continuous map $g : Y \to \mathbb{S}^1$ is null-homotopic.[10] (See [ES52], Chapter XI, Theorem 3.7)

As a simple application of Borsuk's theorem, we will prove that for any Cantor set $C \subseteq \mathbb{R}^2$, $\mathbb{R}^2 \setminus C$ is connected. To see this, let $g : C \to \mathbb{S}^1$ be a continuous map. Since C is compact, g is uniformly continuous. Therefore, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|g(x) - g(y)| < 1/2$. Hence, we may partition C into disjoint compact subsets $C = \cup_{i\in I} C_i$, where I is a finite index set, and for any $i \in I$, we have $\operatorname{diam}(C_i) < \delta$. Therefore $g(C_i) \neq \mathbb{S}^1$, for all $i \in I$. Note that for any $\theta \in \mathbb{S}^1$, the set $\mathbb{S}^1 \setminus \{\theta\}$ is homeomorphic to $\mathbb{R}$. Therefore, any continuous map from a topological space into $\mathbb{S}^1$ that is not surjective is null-homotopic. Hence, for any $i \in I$, there is a homotopy H_i such that

$$H_i : C_i \times [0, 1] \to \mathbb{S}^1, \quad \forall z \in C_i, \quad H_i(z, 0) = g(z), \quad H_i(z, 1) = \theta,$$

where θ is a point in $\mathbb{S}^1$. Since the sets C_i, for $i \in I$, are pairwise disjoint compact sets, we may combine these homotopies to obtain a homotopy between g and a constant map.

Proof of Lemma 6.18. Let U' and U denote the bounded and unbounded connected component of $\hat{\mathbb{C}} \setminus \omega$, respectively. We need to prove that the sets

$$U \setminus X = \hat{\mathbb{C}} \setminus (X \cup \omega \cup U'), \quad U' \setminus X = \hat{\mathbb{C}} \setminus (X \cup \omega \cup U),$$

are both connected. We will prove $U \setminus X$ is connected. The other argument is similar. Let

$$K := X \cup \omega \cup U'.$$

According to Borsuk's theorem, it is sufficient to show that any map $g : K \to \mathbb{S}^1$ is null-homotopic. Let $g : K \to \mathbb{S}^1$ be a continuous map. We will construct a homotopy between g and a constant map. First we need some auxiliary definitions.

[10] For topological spaces C and D, a continuous map $g : C \to D$ is called null-homotopic if g is homotopic to a constant map.

Let $\hat{X} = \overline{X \cap U}$. If $\hat{X} = \emptyset$, then there is nothing to prove. Otherwise, we see that $\hat{X}$ satisfies the conditions in the definition of the hairy Cantor sets. Let $\hat{B} \subseteq \hat{X}$ be the base Cantor set of $\hat{X}$ and let $\hat{b} : \hat{X} \to \hat{B}$ be the corresponding base map. Note that we have $K = \hat{X} \cup \omega \cup U'$ and $\hat{X} \cap \omega = \hat{X} \cap B = \hat{B}$. Let $\psi : \hat{B} \to \psi(\hat{B}) \subseteq \mathbb{R}$ be a homeomorphism from $\hat{B}$ into a Cantor set in $\mathbb{R}$. Choose a height function for $\hat{X}$, namely h, such that $h(\hat{X}) \subseteq [0,1]$ and define

$$\Psi : \hat{X} \to \psi(\hat{B}) \times [0,1], \quad \Psi(z) := (\psi(\hat{b}(z)), h(z)).$$

The map Ψ is a continuous injective map defined on a compact set. Hence it is a homeomorphism onto its image.

Let

$$H_1 : (\psi(\hat{B}) \times [0,1]) \times [0,1] \to \psi(\hat{B}) \times [0,1], \quad H_1((x,y),t) := (x, \min(y, 1-t)),$$

and

$$G_1 : K \times [0,1] \to \mathbb{S}^1, \quad G_1(z,t) := \begin{cases} g(\Psi^{-1}(H_1(\Psi(z),t))) & \text{if } z \in \hat{X}, \\ g(z) & \text{if } z \notin \hat{X}. \end{cases}$$

The map $G_1|_{\hat{X}\times[0,1]}$ is continuous since it is a composition of continuous maps. For $z \in \omega \cup U' = \overline{U'}$ and $t \in [0,1]$, we have $G_1(z,t) = g(z)$. Therefore the maps $G_1|_{\hat{X}\times[0,1]}$ and $G_1|_{\overline{U'}\times[0,1]}$ are continuous maps with closed domains that take the same value in the intersection of their domains. Hence G_1 is continuous.

We have $G_1(z,0) = g(z)$, for all $z \in K$. Therefore G_1 is a homotopy between g and the map

$$g' : K \to \mathbb{S}^1, \quad g'(z) := G_1(z,1).$$

Note that $g'(z) = g(z)$, for all $z \in \overline{U'}$ and $g'(z) = g(\hat{b}(z))$ for all $z \in \hat{X}$.

Now we construct a homotopy between g' and a constant map. The set $\overline{U'} = U' \cup \omega$ is homeomorphic to the closed unit disk. Hence the map $g'|_{\overline{U'}} = g|_{\overline{U'}}$ is null-homotopic. Let H_2 be such a homotopy. In other words, let H_2 be a continuous map such that

$$H_2 : \overline{U'} \times [0,1] \to \mathbb{S}^1, \quad \forall z \in \overline{U'}, \quad H_2(z,0) = g(z), \quad H_2(z,1) = \theta,$$

for some $\theta \in \mathbb{S}^1$. We extend the map $\hat{b}$ to K by setting $\hat{b}(z) = z$, for all $z \in U'$. The maps $\hat{b}|_{\overline{U'}} = \mathrm{Id}_{\overline{U'}}$ and $\hat{b}|_{\hat{X}}$ are continuous maps with closed domains that take the same values on the intersection of their domains. Hence the extended map $\hat{b} : K \to \overline{U'}$ is continuous. Now we define

$$G_2 : K \times [0,1] \to \mathbb{S}^1, \quad G_2(z,t) := H_2(\hat{b}(z)), t).$$

The function G_2 is continuous. Moreover, we have

$$\forall z \in K, \quad G(z,0) = g(\hat{b}(z)) = g'(z), \quad G(z,1) = \theta.$$

Hence g is null-homotopic and the proof is complete. □

Remark 4. One can easily see that the above proof does not use the condition (vi) in the definition of the hairy Cantor sets. In other words, if $X \subseteq \mathbb{R}^2$ is a compact set and $B \subseteq X$ is a Cantor set such that the conditions (i)-(v) in the definition of the hairy Cantor sets are satisfies. Then for any Jordan curve ω with $\omega \cap X = B$, the set $\mathbb{R}^2 \setminus (X \cup \omega)$ has two connected components.

We will show that every hairy Cantor set admits a base curve. We will need a few definitions and lemmas first.

Definition 6.19. *Let X be a hairy Cantor set with base Cantor set B and base map $b : X \to B$. A bounded open set $U \subseteq \mathbb{R}^2$ is admissible for X if the following conditions are satisfied.*

(i) If V_1 and V_2 are distinct connected component of U, then $\overline{V_1} \cap \overline{V_2} = \emptyset$.

(ii) The set U has finitely many connected components, all of which are simply connected.[11]
(iii) We have $B \subseteq U$ and if $x \in X \cap U$, then we have $[b(x), x] \subseteq U$.
(iv) Any point $x \in \partial U \setminus X$ is accessible from both U and $\mathbb{R}^2 \setminus \overline{U}$.

Before we construct appropriate admissible sets that would help us build base curves, we prove a useful property about the admissible sets.

Lemma 6.20. *Let X be a hairy Cantor set, let U be an admissible set for X and let V be a connected component of U. Then $V \setminus X$ is connected.*

Proof. We consider two subsets of X. Let X_1 be the union of the connected components of X that intersect both V and ∂V and let X_2 be the union of the connected components of X that are contained in V. Note that every connected component of X_1 is a Jordan arc.

Step 1 : The set X_1 is closed in $\mathbb{R}^2$.

Let $z \in \overline{X_1}$ and let $(z_i)_{i\in\mathbb{N}}$ be a sequence in X_1 that tends to z. We may choose $z_i' \in b^{-1}(b(z_i)) \cap \partial V$, for all $i \in \mathbb{N}$. Since ∂V is compact, possibly by passing to a subsequence, we may assume that $(z_i')_{i\in\mathbb{N}}$ is convergent. Let z' be the limit of this sequence. We know that

$$b(z) = \lim_{i\to\infty} b(z_i) = \lim_{i\to\infty} b(z_i') = b(z') \in \overline{V}.$$

Hence $z' \in b^{-1}(b(z)) \cap \partial V$. To prove that $z \in X_1$, it is sufficient to show that $b(z) \in V$. However, note that $b(z) \in \overline{V}$. On the other hand, we have $B \subseteq U$, where B is the base Cantor set of X. Hence $B \cap \partial V \subseteq B \cap \partial U = \emptyset$, which implies that $b(z) \in V$.

Step 2 : Every connected component of $V \setminus X_1$ is a topological disk.

Let

$$K := (\hat{\mathbb{C}} \setminus V) \cup X_1 = \hat{\mathbb{C}} \setminus (V \setminus X_1).$$

The set $V \setminus X_1 = \hat{\mathbb{C}} \setminus K$ is open. Let V' be a connected component of $V \setminus X$ that is not simply connected. Then there exists a closed curve $\gamma \subseteq V'$ that is not null-homotopic in V'. We have $K = \hat{\mathbb{C}} \setminus (V \setminus X_1) \subseteq \hat{\mathbb{C}} \setminus \gamma$. Since γ is not null-homotpic, at least one of the bounded connected components of $\hat{\mathbb{C}} \setminus \gamma$ must intersect K. However, this is not possible since K is connected and it intersects the unbounded connected component of $\hat{\mathbb{C}} \setminus \gamma$.

Step 3 : The set $V \setminus X_1$ is connected.

Now we construct an equivalence relation $\sim_1$ on $\hat{\mathbb{C}}$ and use it to show that it is sufficient to prove a simpler statement. Let $\sim_1$ be the equivalence relation on $\hat{\mathbb{C}}$ where $x \sim_1 y$ if and only if $x = y$ or both x and y belong to $\hat{\mathbb{C}} \setminus V$. The quotient space $\hat{\mathbb{C}}/\sim_1$ is the one-point compactification of the topological disk V. Hence it is a topological sphere and we may identify it with $\hat{\mathbb{C}}$. Let $\pi_1 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the quotient map and define

$$v_1 := \pi_1(\hat{\mathbb{C}} \setminus V), \quad K_1 := \pi_1(K) = \pi_1((\hat{\mathbb{C}} \setminus V) \cup X_1) = \{v_1\} \cup \pi_1(X_1 \cap V).$$

The restricted map $\pi_1|_V : V \to \hat{\mathbb{C}} \setminus \{v_1\}$ is an injective map and therefore a homeomorphism. Hence $V \setminus X_1$ is connected if and only if the set

$$\pi_1(V \setminus X_1) = \pi_1(\hat{\mathbb{C}} \setminus K) = \hat{\mathbb{C}} \setminus K_1$$

is connected. Therefore, according to Borsuk's theorem, it is sufficient to prove that any continuous map $g : K_1 \to \mathbb{S}^1$ is null-homotopic.

[11] Recall that a path-connected topological space U is simply connected if every continuous map $g : \mathbb{S}^1 \to U$ is homotopic to a constant map.

Let $g : K_1 \to \mathbb{S}^1$ be a continuous map. We will construct a homotopy between g and a constant map. First we need some auxiliary definitions. We will assign a coordinate to each point in X. Let B be the base Cantor set of X and let $\psi : B \to \psi(B) \subseteq \mathbb{R}$ be a homeomorphism from B into a Cantor set in $\mathbb{R}$. Choose a height function for X, namely h, such that $h(X) \subseteq [0,1]$. Define

$$\Psi : X \to \psi(B) \times [0,1], \quad \Psi(z) := (\psi(b(z)), h(z)).$$

The map Ψ is a continuous injective map defined on a compact set. Hence it is a homeomorphism onto its image.

Let $C := \psi(b(X_1)) \subseteq \mathbb{R}$. For every connected component of X_1, namely γ, there is a point $z \in \gamma$ such that $V \cap \gamma = [b(z), z] \setminus \{z\}$. Therefore, there is a function $u_1 : C \to [0,1]$ such that

$$\Psi(V \cap X_1) = \{(x,y) \in \mathbb{R}^2 \mid x \in C, 0 \leq y < u_1(x)\}.$$

Note that $(C \times \{1\}) \cap \Psi(V \cap X_1) = \emptyset$.

Define

$$\Phi_1 : C \times [0,1] \to K_1, \quad \Phi_1(x,y) := \begin{cases} \pi_1(\Psi^{-1}(x,y)) & \text{if } y < u_1(x), \\ v_1 & \text{if } y \geq u_1(x). \end{cases}$$

We will show that Φ_1 is continuous. Clearly $\Phi_1|_{\Psi(V \cap X_1)} : \Psi(V \cap X_1) \to K_1 \setminus \{v_1\}$ is a homeomorphism onto its image. Hence the map $(\Phi_1|_{K_1 \setminus \{v_1\}})^{-1} : K_1 \setminus \{v_1\} \to C \times [0,1]$ is well-defined and $\Phi_1^{-1}(K_1 \setminus \{v_1\}) \subseteq \Psi(X_1)$. The set $X_1 \setminus V$ is closed in X_1 which is in turn a compact set. Hence the set $\Phi_1^{-1}(v_1) \cap \Psi(X_1) = \Psi(X_1 \setminus V)$ is also compact. Therefore

$$\begin{aligned} \overline{\Phi_1^{-1}(v_1)} \cap \Phi_1^{-1}(K_1 \setminus \{v_1\}) &= (\overline{\Phi_1^{-1}(v_1)} \cap \Psi(X_1)) \cap \Phi_1^{-1}(K_1 \setminus \{v_1\}) \\ &= (\Phi_1^{-1}(v_1) \cap \Psi(X_1)) \cap \Phi_1^{-1}(K_1 \setminus \{v_1\}) = \emptyset. \end{aligned}$$

Hence $\Phi_1^{-1}(v_1) = \overline{\Phi_1^{-1}(v_1)}$ is a closed subset of the plane. On the other hand, the map $\pi_1 \circ (\Psi^{-1})$ is continuous on the closed set $\Psi(X_1)$ and for every $z \in \Psi(X_1 \setminus V)$, we have $\pi_1(\Psi^{-1}(z)) = v_1$. Therefore the maps $\Phi_1|_{\Phi_1^{-1}(v)}$ and $\Phi_1|_{\Psi(X_1)}$ are continuous maps with closed domains that take the same values in the intersection of their domains. Hence Φ_1 is continuous.

Let

$$H_1 : (C \times [0,1]) \times [0,1] \to C \times [0,1], \quad H_1((x,y),t) := (x, \max(y,t)),$$

and

$$G_1 : K_1 \times [0,1] \to \mathbb{S}^1, \quad G_1(w,t) := \begin{cases} g(\Phi_1(H_1(\Phi_1^{-1}(w),t))) & \text{if } w \neq v_1, \\ g(v_1) & \text{if } w = v_1. \end{cases}$$

We have

$$\forall w \in K_1, \quad G_1(w,0) = g(w), \quad G_1(w,1) = g(v_1).$$

Clearly H_1 is well-defined and continuous. If we show that G_1 is continuous, then we have a homotopy between g and a constant map.

By construction, the restriction of G_1 to the set $(K_1 \setminus \{v_1\}) \times [0,1]$, which is open in $K \times [0,1]$, is continuous and the restriction of G_1 to the set $\{v_1\} \times [0,1]$ is the constant map equal to $g(v_1)$. Therefore to prove that G_1 is continuous, it is sufficient to show that for all convergent sequences $((w_i,t_i))_{i \in \mathbb{N}}$ in the set $(K_1 \setminus \{v_1\}) \times [0,1]$ such that $w_i \to v_1$, we have $G_1(w_i,t_i) \to g(v_1)$.

Let $(x_i, y_i) := \Phi_1^{-1}(w_i)$, for all $i \in \mathbb{N}$. Since $w_i \to v_1$, the sequence $((x_i, y_i))_{i\in\mathbb{N}}$ can not have a limit point in $\Phi_1^{-1}(K_1 \setminus \{v_1\})$. Hence we have [12]

$$\lim_{i\to\infty} d((x_i, y_i), \Phi_1^{-1}(v_1)) = 0.$$

Fix $i \in \mathbb{N}$ and let (x_i', y_i') be a point in the compact set $\Phi_1^{-1}(v_1)$ such that $d((x_i, y_i), \Phi_1^{-1}(v_1)) = d((x_i, y_i), (x_i', y_i'))$. We have

$$d(H_1((x_i, y_i), t_i), H_1((x_i', y_i'), t_i)) \le d((x_i, y_i), (x_i', y_i')) = d((x_i, y_i), \Phi_1^{-1}(v_1)).$$

We know that

$$\Phi_1^{-1}(v_1) = \{(x, y) \in \mathbb{R}^2 \mid x \in C, u_1(x) \le y \le 1\}.$$

As a consequence, $(x_i', y_i') \in \Phi_1^{-1}(v_1)$ implies that $H_1((x_i', y_i'), t_i) \in \Phi_1^{-1}(v_1)$. Therefore

$$d(H_1((x_i, y_i), t_i), \Phi_1^{-1}(v_1)) \le d(H_1((x_i, y_i), t_i), H_1((x_i', y_i'), t_i)) \le d((x_i, y_i), \Phi_1^{-1}(v_1)).$$

Hence we have

$$\lim_{i\to\infty} d(H_1((x_i, y_i), t_i), \Phi_1^{-1}(v_1)) \le \lim_{i\to\infty} d((x_i, y_i), \Phi_1^{-1}(v_1)) = 0,$$

which implies $G_1(w_i, t_i) \to g(v_1)$.

Step 4 : The set $V \setminus X$ is connected.

Let $\sim_2$ be the equivalence relation on $\hat{\mathbb{C}}$ where $x \sim_2 y$ if and only if $x = y$ or both x and y belong to K. We have shown that $V \setminus X_1$ is connected and simply connected. The quotient space $\hat{\mathbb{C}}/\sim_2$ is the one-point compactification of the topological disk $V \setminus X_1 = \hat{\mathbb{C}} \setminus K$. Hence it is a topological sphere and we may identify it with $\hat{\mathbb{C}}$. Let $\pi_2 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the quotient map and define

$$v_2 := \pi_2(K), \quad K_2 := \pi_2((\hat{\mathbb{C}} \setminus V) \cup X) = \pi_2(K \cup X_2) = \{v_2\} \cup \pi_2(X_2).$$

The restricted map $\pi_2|_{V\setminus X_1} : V \setminus X_1 \to \hat{\mathbb{C}} \setminus \{v_2\}$ is a homeomorphism. Hence $V \setminus X$ is connected if and only if the set

$$\pi_2(V \setminus X) = \pi_2(\hat{\mathbb{C}} \setminus (K \cup X_2)) = \hat{\mathbb{C}} \setminus K_2$$

is connected. Therefore, according to Borsuk's theorem, it is sufficient to prove that any continuous map $g : K_2 \to \mathbb{S}^1$ is null-homotopic.

Let $g : K_2 \to \mathbb{S}^1$ be a continuous map. We construct a map G_2 and show that if it is continuous, then it is a homotopy between g and the map g' described below. First we need some auxiliary definitions.

Consider the map Ψ defined earlier and note that

$$\Psi(X) = \{(x, y) \in \mathbb{R}^2 \mid x \in \psi(B), 0 \le y \le h(p(\psi^{-1}(x)))\},$$

where p is the peak map of X. Define

$$\Phi_2 : \Psi(X) \to K_2, \quad \Phi_2(z) := \pi_2(\Psi^{-1}(z)).$$

Clearly $\Phi_2|_{\Psi(X_2)}$ is a homeomorphism onto $K_2 \setminus \{v_2\}$. Let

$$H_2 : \Psi(X) \times [0, 1] \to \Psi(X), \quad H_2((x, y), t) := (x, \min(y, 1 - t)),$$

and

$$G_2 : K_2 \times [0, 1] \to \mathbb{S}^1, \quad G_2(w, t) := \begin{cases} g(\Phi_2(H_2(\Phi_2^{-1}(w), t))) & \text{if } w \ne v_2, \\ g(v_2) & \text{if } w = v_2. \end{cases}$$

[12] Given a point $x \in \mathbb{R}^2$ and a non-empty closed set $C \subseteq \mathbb{R}^2$, we define

$$d(x, C) := \min\{d(x, y) \mid y \in C\}.$$

The map H_2 is clearly well-defined and continuous. If G_2 is continuous, then it is a homotopy between g and the map

$$g' : K_2 \to \mathbb{S}^1, \quad g'(w) := G_2(w, 1).$$

Let $((w_i, t_i))_{i\in\mathbb{N}}$ be a convergent sequence in $K_2 \times [0, 1]$. Clearly G_2 is continuous on $(K_2 \setminus \{v_2\}) \times [0, 1]$. Therefore to prove that G_2 is continuous, it is sufficient to consider sequences where $w_i \in K_2 \setminus \{v_2\}$, for all $i \in \mathbb{N}$, and $w_i \to v_2$. We know that $\pi_2^{-1}(w_i) \in X_2$, for all $i \in \mathbb{N}$, and $\pi_2^{-1}(v_2) = K = (\hat{\mathbb{C}} \setminus V) \cup X_1$. Since $w_i \to v_2$, the sequence $(\pi_2^{-1}(w_i))_{i\in\mathbb{N}}$ can not have a limit point in $\pi_2^{-1}(\hat{\mathbb{C}} \setminus \{v_2\}) = \hat{\mathbb{C}} \setminus K$. On the other hand $\pi_2^{-1}(w_i) \in X_2$, for all $i \in \mathbb{N}$. Hence any limit point of this sequence belongs to $\overline{X_2} \cap K$. Note that we have

$$b(\overline{X_2}) = \overline{b(X_2)} \subseteq B \cap \overline{V} = B \cap V = b(X_1 \cup X_2),$$

which implies $\overline{X_2} \subseteq X_1 \cup X_2$. Hence $\overline{X_2} \cap K \subseteq (X_1 \cup X_2) \cap K = X_1$. Therefore we have

$$\lim_{i\to\infty} d(\pi_2^{-1}(w_i), X_1) = 0,$$

which implies

$$\lim_{i\to\infty} d(\Psi(\pi_2^{-1}(w_i)), \Psi(X_1)) = 0.$$

For $i \in \mathbb{N}$, let $(x_i, y_i) := \Phi_2^{-1}(w_i) = \Psi(\pi_2^{-1}(w_i))$ and let (x_i', y_i') be a point in the compact set $\Psi(X_1)$ such that $d((x_i, y_i), \Psi(X_1)) = d((x_i, y_i), (x_i', y_i'))$. We have

$$d(H_2((x_i, y_i), t_i), H_2((x_i', y_i'), t_i)) \le d((x_i, y_i), (x_i', y_i')) = d((x_i, y_i), \Psi(X_1)).$$

Since $H_2((x_i', y_i'), t_i) \in \Psi(X_1)$, for all $i \in \mathbb{N}$, this implies

$$\lim_{i\to\infty} d(H_2((x_i, y_i), t_i), \Psi(X_1)) \le \lim_{i\to\infty} d((x_i, y_i), \Psi(X_1)) = 0.$$

Hence we have $G_2(w_i, t_i) \to g(v_2)$. This completes the proof for the continuity of G_3 and shows that g is homotopic to g'.

Now we construct a map G_3 and show that if it is continuous, then it is a homotopy between g' and a constant map. Let

$$D := \pi_2(K \cup B) = \{v_2\} \cup \pi_2(B \cap X_2) = \{v_2\} \cup \pi_2(b(X_2)).$$

The set D is closed and totally disconnected. The only point in D that might be isolated is v_2. Hence D is either a Cantor set or the union of an isolated point and a Cantor set. Therefore, according to the discussion before this lemma, we know that $g|_D$ is null-homotopic. Let F be such a homotopy. In other words, let F be a continuous map such that

$$F : D \times [0, 1] \to \mathbb{S}^1, \quad \forall w \in D, \quad F(w, 0) = g(w), \quad F(w, 1) = \theta,$$

for some $\theta \in \mathbb{S}^1$. Let

$$G_3 : K_2 \times [0, 1] \to \mathbb{S}^1, \quad G_3(w, t) := \begin{cases} F(\pi_2(b(\pi_2^{-1}(w))), t) & \text{if } w \neq v_2, \\ F(v_2, t) & \text{if } w = v_2. \end{cases}$$

Note that

$$\pi_2(b(\pi_2^{-1}(K_2 \setminus \{v_2\}))) = \pi_2(b(X_2)) \subseteq D.$$

Therefore G_3 is well-defined. For any $w \in K_2 \setminus \{v_2\}$, we have $\Phi_2^{-1}(w) \in \mathbb{R} \times \{0\}$ is and only if $w \in D$. Hence we have

$$\forall w \in K_2 \setminus \{v_2\}, \quad g'(w) = G_2(w, 1) = g(\Phi_2(H_2(\Phi_2^{-1}(w), 1))) = g(\pi_2(b(\pi_2^{-1}(w)))).$$

Therefore

$$\forall w \in K_2, \quad G_3(w, 0) = g'(w), \quad G_3(w, 1) \in F(D, 1) = \{\theta\}.$$

If G_3 is continuous, then it is a homotopy between g' and a constant map and the proof is complete.

Let $((w_i, t_i))_{i\in\mathbb{N}}$ be a convergent sequence in $K_2 \times [0,1]$. To prove that G_3 is continuous, it is sufficient to only consider such sequences where $w_i \in K_2 \setminus \{v_2\}$, for all $i \in \mathbb{N}$, and $w_i \to v_2$. The sequence $(w_i')_{i\in\mathbb{N}}$, where $w_i' := \pi_2(b(\pi_2^{-1}(w_i)))$ for all $i \in \mathbb{N}$, is contained in D. Let us assume that $w' \in D \setminus \{v_2\}$ is a limit point of the sequence $(w_i')_{i\in\mathbb{N}}$. Then $\pi_2^{-1}(w') \in \pi_2^{-1}(D \setminus \{v_2\}) \subseteq X_2$ is a limit point of the sequence $(b(\pi_2^{-1}(w_i)))_{i\in\mathbb{N}}$. Therefore $(\pi_2^{-1}(w_i))_{i\in\mathbb{N}}$ has a limit point in the set $b^{-1}(\pi_2^{-1}(w')) \subseteq X_2$. Hence $(w_i)_{i\in\mathbb{N}}$ has a limit point in $\pi_2(X_2) = K_2 \setminus \{v_2\}$, which is a contradiction. Since D is closed, we conclude that $w_i' \to v_2$. Hence $F(w_i', t_i)$ tends to $F(v_2, \lim_{i\to\infty} t_i)$ as i tends to infinity. Therefore G_3 is continuous. □

Lemma 6.21. *Let X be a hairy Cantor set with base Cantor set B. Also let U be a bounded open set that satisfies the properties (ii)-(iv) of the admissible sets. Then there exists an admissible set $U' \subseteq U$ such that every connected component of U contains exactly one connected component of U'.*

Proof. Let $U = \cup_{j\in J} U_j$, where J is a finite index set and U_j's are the connected components of U. For any $j \in J$, let

$$X_j := \{x \in X \mid b(x) \in U_j\},$$

where b is the base map of X. First note that each X_j is closed. To see this, let $j \in J$ and $x \in \overline{X_j}$. There is a sequence $(x_i)_{i\in\mathbb{N}}$ in X_j such that $x_i \to x$. Therefore, we have $b(x_i) \in U_j$, for all $i \in \mathbb{N}$, and $b(x_i) \to b(x)$. Hence $b(x) \in \overline{U_j}$. The set B does not intersect the boundary of U, therefore $b(x) \in U_j$ and $x \in X_j$.

Fix $j \in J$ and define

$$K_j := (X \setminus X_j) \cap \overline{U_j}.$$

Since U satisfies the condition (iii) in the definition of the admissible sets, we see that

$$K_j = (K_j \cap \partial U_j) \cup \{x \in X \cap U_j \mid b(x) \notin U_j\} = K_j \cap \partial U_j \subseteq \partial U_j.$$

We may cover the compact set K_j with a set V_j such that V_j is a union of finitely many Jordan domains whose closures are disjoint. We may choose V_j small enough so that $\overline{V_j} \cap X_j = \emptyset$. The set U satisfies the condition (iv) in the definition of the admissible sets. Therefore every point $x \in \partial U_j \setminus X$ is accessible from U_j and $\mathbb{R}^2 \setminus \overline{U_j}$. Hence we may assume that every connected component of ∂V_j intersects ∂U_j at exactly two points. It follows that the set

$$U_j^1 := U_j \setminus \overline{V_j} \subseteq U_j$$

is a topological disk and every point in $\partial U_j^1 \setminus X$ is accessible from both U_j^1 and $\mathbb{R}^2 \setminus \overline{U_j^1}$. Moreover, we have $U_j^1 \cap X = U_j \cap X$, which implies that for any $x \in U_j^1 \cap X$, we have $[x, b(x)] \subseteq U_j^1$. We repeat this construction for every $j \in J$ and see that the set $\cup_{j\in J} U_j^1$ satisfies the properties (ii)-(iv) of the admissible sets.

Note that for all $j, j' \in J$ with $j \neq j'$, we have

$$\partial U_j^1 \cap X_{j'} = \emptyset.$$

This is true because

$$\partial U_j^1 \cap X_{j'} = (\overline{U_j} \cap X_{j'}) \setminus U_j \subseteq K_j \setminus U_j = K_j \subseteq V_j,$$

while

$$\partial U_j^1 \cap X_{j'} \subseteq \overline{U_j^1} \subseteq \overline{U_j} \setminus V_j.$$

It follows that for every $j \in J$, we have

$$\left(\bigcup_{j' \in J \setminus \{j\}} \partial U^1_{j'} \cap \partial U^1_j\right) \bigcap X = \left(\bigcup_{j' \in J \setminus \{j\}} \partial U^1_{j'} \cap \partial U^1_j\right) \bigcap \bigcup_{j'' \in J} X_{j''}$$
$$= \bigcup_{j' \in J \setminus \{j\}, j'' \in J} \partial U^1_{j'} \cap \partial U^1_j \cap X_{j''} = \emptyset.$$

Fix $j \in J$ and define

$$K'_j := \bigcup_{j' \in J \setminus \{j\}} \partial U^1_{j'} \cap \partial U^1_j \subseteq \partial U^1_j.$$

We may cover the compact set K'_j with an open set V'_j such that V'_j is a union of finitely many Jordan domains whose closures are disjoint. We may choose V'_j small enough so that $\overline{V'_j} \cap X = \emptyset$. Every point in $\partial U^1_j \setminus X$ is accessible from U^1_j and $\mathbb{R}^2 \setminus \overline{U^1_j}$. Therefore we may choose V'_j such that each connected component of $\partial V'_j$ intersects the ∂U^1_j at exactly two points. It follows that the set

$$U^2_j := U^1_j \setminus \overline{V'_j} \subseteq U^1_j \subseteq U_j$$

is a topological disk and every point in $\partial U^2_j \setminus X$ is accessible from both U^2_j and $\mathbb{R}^2 \setminus \overline{U^2_j}$. Moreover, we have $U^2_j \cap X = U^1_j \cap X$, which implies that for any $x \in U^2_j \cap X$, we have $[x, b(x)] \subseteq U^2_j$. We repeat this construction for every $j \in J$ and define

$$U' := \bigcup_{j \in J} U^2_j.$$

In order to prove that U' is admissible for X, we now only need to show that connected components of U' have disjoint closures. For every $j, j' \in J$ with $j \neq j'$, we have

$$\overline{U^2_j} \cap \overline{U^2_{j'}} \subseteq \overline{U^1_j} \cap \overline{U^1_{j'}} = \partial U^1_j \cap \partial U^1_{j'} \subseteq K'_j \subseteq V'_j,$$

and

$$\overline{U^2_j} \cap \overline{U^2_{j'}} \subseteq \overline{U^2_j} \subseteq \overline{U^1_j} \setminus V'_j,$$

which implies

$$\overline{U^2_j} \cap \overline{U^2_{j'}} = \emptyset.$$

In other words, connected components of U' have disjoint closures. Therefore the set $U' \subseteq U$ is admissible for X and every connected component of U contains exactly one connected component of U'. □

Lemma 6.22. *Let X be a hairy Cantor set with base Cantor set B. For any $\varepsilon > 0$, there is an admissible set for X that is contained in $D(B, \varepsilon)$.*

Proof. Let $\varepsilon > 0$ and consider the open set $D(B, \varepsilon)$. We may find an open set U_0 that is a finite union of Jordan domains with disjoint closures such that

$$B \subseteq U_0 \subseteq D(B, \varepsilon).$$

We will find an admissible set for X that is a subset of U_0. Let h be a height function for X and define

$$h_0 := \min(\{h(x)/2 \mid x \in \partial U_0 \cap X\} \cup \{1\}), \quad X_0 := \{x \in X \mid h(x) < h_0\} \subseteq U_0,$$
$$U_1 := U_0 \setminus (X \setminus X_0) \supseteq B, \quad U_2 := \text{fill}(U_1) \subseteq U_0.$$

The set $X \setminus X_0$ is closed, therefore U_1 is open. For any $x \in U_1 \cap X = X_0$, we have

$$[x, b(x)] \subseteq X_0 \subseteq U_1,$$

where $b : X \to B$ is the base map of X. Therefore U_1 satisfies the third property of the admissible sets. We will show that this property still holds after adding the bounded connected components of $\mathbb{R}^2 \setminus U_1$ to obtain U_2. For $x \in U_1 \cap X$, we have $[x, b(x)] \subseteq U_1 \subseteq U_2$. Let us assume $x \in (U_2 \setminus U_1) \cap X$. We have $b(x) \in B \subseteq U_1$ and for any $y \in [x, b(x)] \cap U_1$, we have $[y, b(x)] \subseteq U_1$. Therefore there exists $z \in [x, b(x)]$ such that

$$[x, b(x)] \cap U_1 = [z, b(x)] \setminus \{z\}.$$

Hence the connected set $[x, z]$ is contained in the bounded connected component of $\mathbb{R}^2 \setminus U_1$ that contains the point x. Therefore

$$[x, b(x)] = [x, z] \cup ([z, b(x)] \setminus \{z\}) \subseteq (U_2 \setminus U_1) \cup U_1 = U_2.$$

Note that every connected component of U_2 is simply connected.

Let U_3 be the union of the connected components of U_2 that intersect B. The collection of the connected components of U_3 forms an open cover for the compact set B. Hence U_3 has finitely many connected components, all of which are simply connected. The discussion in the previous paragraph shows that the third property of the admissible sets also holds for U_3.

We will show that the fourth property of the admissible sets also holds for U_3. First note that X has no interior points and we have $U_0 \setminus X \subseteq U_1 \subseteq U_2 \subseteq U_0$. Hence we have $\overline{U_0} = \overline{U_0 \setminus X} = \overline{U_2}$. Therefore

$$\begin{aligned}\partial U_2 \setminus X = (\overline{U_2} \cap (\overline{\mathbb{R}^2 \setminus U_2})) \setminus X = (\overline{U_2} \cap (\mathbb{R}^2 \setminus U_2)) \setminus X = \overline{U_2} \cap (\mathbb{R}^2 \setminus (U_2 \cup X)) \\ = \overline{U_0} \cap (\mathbb{R}^2 \setminus (U_0 \cup X)) = (\overline{U_0} \cap (\mathbb{R}^2 \setminus U_0)) \setminus X = \partial U_0 \setminus X.\end{aligned}$$

The set ∂U_0 is a disjoint union of Jordan arcs. Hence U_2 satisfies the fourth property of admissible sets. Every connected component of ∂U_3 is a connected component of ∂U_2. Therefore U_3 also satisfies this property.

We have shown that the bounded open set U_3 satisfies the properties (ii)-(iv) of the admissible sets. Therefore, according to Lemma 6.21, there is an admissible set

$$U' \subseteq U_3 \subseteq D(B, \varepsilon). \qquad \square$$

Definition 6.23. *Let X be a hairy Cantor set, U be an admissible set for X and $\boldsymbol{u}$ be a subset of ∂U. Assume that*

$$U = \bigcup_{i \in I} U_i, \quad \boldsymbol{u} = \{u_i \mid i \in I\},$$

where I is an index set, U_i's are the connected components of U and $u_i \in \partial U_i \setminus X$, for all $i \in I$. Then we say $(U, \boldsymbol{u})$ is a marked admissible set for X, and $\boldsymbol{u}$ is the set of its marked points.

Let $(V, \boldsymbol{v})$ be a marked admissible set for X such that $\overline{V} \subseteq U$. Also let V_* be a connected component of V and let $v_* \in \boldsymbol{v}$ be the marked point of V on ∂V_*. We say V_* is a free component of $(V, \boldsymbol{v})$ within $(U, \boldsymbol{u})$ if there is a Jordan arc $\gamma : [0, 1] \to \overline{U}$ such that

$$\gamma(0) \in \boldsymbol{u}, \quad \gamma(1) = v_*, \quad \gamma((0, 1)) \subseteq U \setminus (\overline{V} \cup X).$$

We say $(V, \boldsymbol{v})$ is a free admissible set within $(U, \boldsymbol{u})$ if every connected component of V is free within $(U, \boldsymbol{u})$.

Lemma 6.24. *Let X be a hairy Cantor set with base Cantor set B and let $(U, \boldsymbol{u})$ be a marked admissible set for X. For any $\varepsilon > 0$, there exists a marked admissible set $(U', \boldsymbol{u}')$ for X such that $U' \subseteq D(B, \varepsilon)$ and $(U', \boldsymbol{u}')$ is a free admissible set within $(U, \boldsymbol{u})$.*

Proof. Let ε be a small positive number. Without loss of generality, we may assume $\varepsilon < d(\partial U, B)$. According to Lemma 6.22, there is an admissible set $U_1 \subseteq D(B, \varepsilon/2) \subseteq U$.

Let $\tilde{U}$ be an admissible set such that $\tilde{U} \subseteq U_1$ and let $\tilde{U}_*$ be a connected component of $\tilde{U}$. We say $\tilde{U}_*$ is *visible from* $(U, \boldsymbol{u})$ if there is a Jordan arc $\gamma : [0,1] \to \overline{U}$ such that

$$\gamma(0) \in \mathbf{u}, \quad \gamma(1) \in \partial \tilde{U}_* \setminus X, \quad \gamma((0,1)) \subseteq U \setminus (\overline{\tilde{U}} \cup X).$$

If every connected component of $\tilde{U}$ is visible from $(U, \mathbf{u})$, then it follows from the definition of visibility that there is $\tilde{\mathbf{u}}$ such that $(\tilde{U}, \tilde{\mathbf{u}})$ is a marked admissible set with the desired properties.

Hence we only need to construct an admissible set $\tilde{U} \subseteq U_1$ such that every connected component of $\tilde{U}$ is visible from $(U, \mathbf{u})$. For any $m \geq 1$, we will recursively build a new admissible set $U_{m+1} \subseteq U_m$ in a way that the number of connected components that are invisible from $(U, \mathbf{u})$ decreases. Let $m \geq 1$ and assume that the admissible set U_m has an invisible component, namely V. Let U_* be the connected component of U that contains V and let $u_* \in \mathbf{u} \cap \partial U_*$ be the marked point of $(U, \mathbf{u})$ that is on the boundary of U_*.

The set $U_* \setminus X$ is path-connected and u_* is accessible from it. Therefore we may find a Jordan arc $\gamma : [0,1] \to \overline{U_*}$ such that

$$\gamma(0) = u_*, \quad \gamma(1) \in \partial V \setminus X, \quad \gamma((0,1)) \subseteq U_* \setminus (\overline{V} \cup X).$$

Since V is invisible, $\gamma((0,1))$ intersects $U_m \setminus V$.

Let W be the union of the connected components of U_m that intersect γ and let W' be the union of the connected components of $W \setminus \gamma$ that intersect X. If $x \in W' \cap X$, then we have $b(x) \in [x, b(x)] \subseteq W'$, where b is the base map of X. Hence B, the base Cantor set of X, intersects every connected component of W'. We have $B \subseteq U_m$, which implies $B \cap \partial W = \emptyset$. Therefore the collection of the connected components of W' forms an open cover for the compact set $W' \cap B = W \cap B = \overline{W} \cap B$. Therefore W' has finitely many connected components. Hence the set $(U_m \setminus W) \cup W'$ satisfies all of the properties of the admissible sets except possibly the first condition. Connected components of this set may have intersecting closures. Now we may use Lemma 6.21 to shrink every connected component of the set $(U_m \setminus W) \cup W'$ and obtain an admissible set U_{m+1}.

The connected set $V \cap U_{m+1}$ is a visible component of U_{m+1}. Every visible component of U_m that is not contained in W shrinks into a visible component of U_{m+1} through the process carried out in Lemma 6.21. Now we will show that every connected component of the set $U_{m+1} \cap W$ is also a visible component of U_{m+1}. Let $V' \subseteq W$ be a connected component of U_{m+1}. The set V' is obtained by shrinking a connected component of $W \setminus \gamma$. Therefore we may find a Jordan arc γ' that starts at u_*, traces the curve γ until it gets close to the boundary of V' and then ends in $\partial V' \setminus X$. Existence of γ' shows that V' is visible from $(U, \mathbf{u})$.

We see that the number of invisible components of U_{m+1} is strictly less than the number of invisible components of U_m. Hence there is some $M > 0$ such that every connected component of

$$\tilde{U} := U_M \subseteq U_1$$

is visible within $(U, \mathbf{u})$. □

Lemma 6.25. *Let X be a hairy Cantor set with base Cantor set B. Then there are marked admissible sets $(U_n, \boldsymbol{u}_n)$, for $n \in \mathbb{N}$, such that*

$$U_n = \bigcup_{1 \leq i \leq k_n} U_{n,i}, \quad \boldsymbol{u}_n = \{u_{n,i}\}_{1 \leq i \leq k_n}, \quad \forall 1 \leq i \leq k_n, \quad u_{n,i} \in \partial U_{n,i} \setminus X,$$

where $k_1 = 1 \leq k_2 \leq k_3 \cdots$ and for any $n \geq 1$, the $U_{n,i}$'s are connected components of U_n and the following properties are satisfied.

- *We have* $\bigcap_{n\geq 1} U_n = B$.
- *For any* $n > 1$, $(U_n, \boldsymbol{u}_n)$ *is a free admissible set within* $(U_{n-1}, \boldsymbol{u}_{n-1})$.

Proof. We will recursively construct U_n's such that

$$\forall n > 1, \quad U_n \subseteq D(B, 2^{-n}).$$

For $n = 1$, let $U_1 = U_{1,1}$ be a Jordan domain that contains X and choose $u_{1,1}$ to be any point in $\partial U_{1,1}$. Clearly $(U_1, \{u_{1,1}\})$ is a marked admissible set. Let $n > 1$ and assume that we have built the marked admissible set $(U_m, \mathbf{u}_m)$, for all $m < n$. Let $\varepsilon = \min(d(\partial U_{n-1}, B), 2^{-n})$. According to Lemma 6.24, there is a marked admissible set $(U_n, \mathbf{u}_n)$, such that

$$U_n \subseteq D(B, \varepsilon) \subseteq U_{n-1} \cap D(B, 2^{-n}),$$

and $(U_n, \mathbf{u}_n)$ is a marked admissible set within $(U_{n-1}, \mathbf{u}_{n-1})$. □

Proposition 6.26. *Any hairy Cantor set admits a base curve.*

Proof. Let B be the base Cantor set of X. Our starting point is the construction in Lemma 6.25 and we will use the same notation. According to the definition of the free admissible sets, for any $n > 1$ and $1 \leq i \leq k_n$, there is a Jordan arc $\lambda_{n,i} : [0,1] \to \overline{U_{n-1}}$ such that

$$\lambda_{n,i}(0) \in \mathbf{u}_{n-1}, \quad \lambda_{n,i}(1) = u_{n,i}, \quad \lambda_{n,i}((0,1)) \subseteq U_{n-1} \setminus (\overline{U_n} \cup X).$$

Since $U_{n-1} \setminus (\overline{U_n} \cup X)$ is an open set, without loss of generality, we may assume that the interiors of these Jordan arcs are pairwise disjoint. Let $T_1 := \mathbf{u}_1$ and for $n > 1$, define T_n to be the union of the Jordan arcs $\lambda_{m,i}$, for all $1 < m \leq n$ and $1 \leq i \leq k_m$. Note that T_n is a tree with $k_1 + k_2 + \cdots + k_n$ vertices in the plane.

Now we choose open intervals $C_{n,i} \subseteq [0,1]$, for $n \geq 1$ and $1 \leq i \leq k_n$, such that the following hold.

- Any two intervals $C_{n,i}$ and $C_{n',i'}$ either have disjoint closures or one of them is compactly contained in the other one.
- For all n, i, n', i', we have $\overline{C_{n,i}} \subseteq C_{n',i'}$ if and only if $\overline{U_{n,i}} \subseteq U_{n',i'}$.
- For all n, i, we have $|C_{n,i}| < 2^{-n}$.

Note that these conditions imply that the set $C := \bigcap_n \bigcup_i C_{n,i}$ is a Cantor set.

Using the constructions so far, we will inductively define Jordan curves $\omega_n : [0,1] \to \mathbb{R}^2$ in a way that the following conditions are satisfied.

- For all $n \geq 1$ and $x \notin \bigcup_i C_{n,i}$, we have $\omega_n(x) = \omega_{n+1}(x) = \omega_{n+2}(x) = \cdots$.
- For all $m \geq n$ and $1 \leq i \leq k_n$, we have $\omega_m^{-1}(U_{n,i}) = C_{n,i}$ and $|\omega_m \cap \partial U_{n,i}| = 2$.
- For all $n \geq 1$, the bounded connected component of $\mathbb{R}^2 \setminus \omega_n$ contains T_n and its unbounded connected component contains X.
- For all $n \geq 1$, the bounded connected component of $\mathbb{R}^2 \setminus \omega_{n+1}$ contains the bounded connected component of $\mathbb{R}^2 \setminus \omega_n$.

We know that every point in $\partial U_{1,1} \setminus X$ is accessible from $U_{1,1}$ and $\mathbb{R}^2 \setminus \overline{U_{1,1}}$. We pick two points in $\partial U_{1,1} \setminus X$ and connect them with two Jordan arcs that do not intersect X, one contained in $\overline{U_{1,1}}$ and the other one outside of $U_{1,1}$. These Jordan arcs may be chosen so that their interiors are contained in $\mathbb{R}^2 \setminus \partial U_{1,1}$. Let the Jordan curve ω_1 be a parametrization of the union of these Jordan arcs such that $\omega_1^{-1}(U_{1,1}) = C_{1,1}$.

Now let us assume that we have the Jordan curve ω_{n-1} and we want to build ω_n. For $x \notin \cup_i C_{n-1,i}$, let $\omega_n(x) := \omega_{n-1}(x)$. We will describe the construction for $\omega_n|_{C_{n-1,1}}$ but the same

argument works for $\omega_n|_{C_{n-1,i}}$, for all $1 \leq i \leq k_n$. Let us assume that

$$U_{n-1,1} \cap U_n = \bigcup_{j=1}^{s} U_{n,i_j},$$

for some $1 \leq s \leq k_n$ and $1 \leq i_1 < i_2 < \cdots < i_s \leq k_n$.

Choose $\varepsilon > 0$ small enough so that $\overline{D(T_n, \varepsilon)} \cap X = \emptyset$. The set $U_{n-1,1} \setminus T_n$ is connected and open. Therefore we may connect $\omega(\inf C_{n-1,1})$ to $\omega(\sup C_{n-1,1})$ with a Jordan arc γ inside $\overline{U_{n-1,1}}$ that does not intersect $T_n \cap U_{n-1,1}$ or X, but remains very close to $T_n \cap U_{n-1,1}$. Moreover, we may assume that for all $1 \leq j \leq s$,

$$\gamma \cap \overline{U_{n,i_j}} \subseteq D(T_n, \varepsilon), \quad |\gamma \cap \partial U_{n,i_j}| = 2.$$

We want the function $\omega_n|_{\overline{C_{n-1,1}}}$ to trace the image of γ. We may choose an appropriate parametrisation of γ so that, possibly after rearranging the second indices in C_{n,i_j} for $1 \leq j \leq s$, we have

$$\forall 1 \leq j \leq s, \quad \omega_n^{-1}(U_{n,i_j}) = C_{n,i_j}.$$

Clearly the bounded connected component of $\mathbb{R}^2 \setminus \omega_n$ contains T_n and its unbounded connected component contains X. It is also clear that the bounded connected component of $\mathbb{R}^2 \setminus \omega_n$ contains the bounded connected component of $\mathbb{R}^2 \setminus \omega_{n-1}$.

Now, having constructed the Jordan curves ω_n, we will show that $\omega_n : [0,1] \to \mathbb{R}^2$ tends to a Jordan curve $\omega : [0,1] \to \mathbb{R}^2$ which is a base curve for X. For $n \geq 1$, let

$$r_n := \max\{\operatorname{diam}(U_{n,i}) \mid 1 \leq i \leq k_n\}.$$

The inclusion $U_{n+1} \subseteq U_n$, for all $n \geq 1$, implies that $(r_n)_{n \in \mathbb{N}}$ is a decreasing sequence. The fact that $\bigcap_{n \geq 1} U_n = B$ implies that r_n tends to zero as n tends to infinity.

Let $N \geq 1$ and $m, n \geq N$. For $x \notin \cup_{i=1}^{k_N} C_{N,i}$, we have $\omega_n(x) = \omega_m(x)$. If $x \in C_{N,i}$ for some $1 \leq i \leq k_N$, then $\omega_n(x), \omega_m(x) \in U_{N,i}$. In any case, we have

$$\forall x \in [0,1], \quad |\omega_n(x) - \omega_m(x)| \leq \max\{\operatorname{diam}(U_{N,i}) \mid 1 \leq i \leq k_N\} = r_N.$$

Hence the sequence of functions $(\omega_n)_{n \geq 1}$ is uniformly convergent to a function $\omega : [0,1] \to \mathbb{R}^2$.

Since $\omega_0(0) = \omega_0(1) = \omega_1(0) = \omega_1(1) = \cdots$, we have $\omega(0) = \omega(1)$. Assume $0 \leq x, y < 1$ such that $\omega(x) = \omega(y)$. We will show that $x = y$. If there are integers n and i such that $x \in C_{n,i}$ and $y \notin C_{n,i}$, then

$$\omega(x) \in U_{n,i}, \quad \omega(y) \notin U_{n,i}$$

which is a contradiction. Similarly, we also rule out the possibility of having n and i such that $x \notin C_{n,i}$ and $y \in C_{n,i}$. Hence we may assume that for all n and i, we have $x \in C_{n,i}$ if and only if $y \in C_{n,i}$. If there is some n such that $x \notin \cup_{i=1}^{k_n} C_{n,i}$, then we will also have $y \notin \cup_{i=1}^{k_n} C_{n,i}$. This implies that

$$\omega_n(x) = \omega(x) = \omega(y) = \omega_n(y).$$

Since ω_n is a Jordan curve, we have $x = y$. If $x, y \in C$ are points such that for all n and i, we have $x \in C_{n,i}$ if and only if $y \in C_{n,i}$, then x and y belong to the same connected component of C, which means $x = y$.

We have seen that $\omega : [0,1] \to \mathbb{R}^2$ is a continuous map such that $\omega(0) = \omega(1)$ and $\omega|_{[0,1)}$ is injective. Hence it is a Jordan curve. It is clear from the constructions and the arguments above that

$$\omega \cap X = \omega(C) = \bigcap_{n \geq 1} U_n = B.$$

For all $n \geq 1$, let W_n be the bounded connected component of $\mathbb{R}^2 \setminus \omega_n$. Then the bounded connected component of $\mathbb{R}^2 \setminus \omega$ is

$$W := \bigcup_{n \geq 1} W_n.$$

Since $W_n \cap X = \emptyset$, for all $n \geq 1$, we see that $W \cap X = \emptyset$. In other words, the bounded connected component of the complement of ω does not intersect X. Hence ω is a base curve for X. □

The existence of a base curve for any hairy Cantor set has an interesting implication that leads to an alternative definition of the hairy Cantor sets.

Lemma 6.27. *Let X be a hairy Cantor set with base Cantor set B and let $\omega \subseteq \mathbb{R}^2$ be a Jordan curve such that $\omega \cap X = B$. Also let b denote the base map of X. Then for any $x \in X \setminus B$, there are sequences $(\alpha_i)_{i \in \mathbb{N}}$ and $(\beta_i)_{i \in \mathbb{N}}$ in X such that $\alpha_i \to x$, $\beta_i \to x$ and $b(\alpha_i) < b(x) < b(\beta_i)$, with respect to a fixed cyclic order on ω.*

Proof. Let p denote the peak map of X. Choose a point $x \in X \setminus B$ and continue the Jordan arc $[b(x), p(x)]$ in the direction of $b(x)$ to obtain a Jordan arc γ such that $\gamma \cap (X \cup \omega) = [b(x), p(x)]$. Choose a sequence $(x_n)_{n \in \mathbb{N}}$ in $[b(x), x] \setminus \{b(x), x\}$ that tends to x. For $n \geq 1$, we may find a Jordan domain V_n such that

$$[b(x), x_n] \subseteq V_n \subseteq D_{1/n}([b(x), x_n]),$$

and $p(x) \notin \overline{V_n}$. We may also assume that ∂V_n intersects ω at exactly two points and $V_n \setminus \gamma$ has two connected components, namely V_n^1 and V_n^2. Fix a cyclical order on ω. For each $n \geq 1$, choose the superscripts 1 and 2, so that for all $z \in V_n^1 \cap \omega$ and $w \in V_n^2 \cap \omega$, we have

$$z < b(x) < w.$$

For $n, m \geq 1$, let $U_n^m \subseteq V_n \cap D_{1/m}(x_n)$ be a Jordan domain that contains x_n and does not intersect ω. Since x_n is not accessible from $\mathbb{R}^2 \setminus X$, there exists points

$$\alpha_n^m \in X \cap U_n^m \cap V_n^1, \quad \beta_n^m \in X \cap U_n^m \cap V_n^2.$$

We have $\lim_{m \to \infty} \alpha_n^m = x_n$. Therefore $\lim_{m \to \infty} [\alpha_n^m, b(\alpha_n^m)] = [x_n, b(x)]$ in the Hausdorff topology. Hence for large m, we have

$$[\alpha_n^m, b(\alpha_n^m)] \subseteq V_n \setminus \gamma = V_n^1 \cup V_n^2.$$

Since $\alpha_n^m \in V_n^1$, we conclude that $b(\alpha_n^m) \in V_n^1$. After using a similar argument for β_n^m's, we see that for all $n \geq 1$, there is $M_n \geq n$, such that

$$\alpha_n := \alpha_n^{M_n} \in V_n^1 \cap D_{1/n}(x_n), \quad b(\alpha_n) \in V_n^1,$$
$$\beta_n := \beta_n^{M_n} \in V_n^2 \cap D_{1/n}(x_n), \quad b(\beta_n) \in V_n^2.$$

It follows that

$$\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = \lim_{n \to \infty} x_n = x,$$

and

$$\forall n \in \mathbb{N}, \quad b(\alpha_n) < b(x) < b(\beta_n).$$ □

Theorem 6.28. *Let $X \subseteq \mathbb{R}^2$ be a compact set. Then X is a hairy Cantor set if and only if the following conditions hold:*

- *(i) Every connected component of X is either a single point or a Jordan arc.*
- *(ii) There is a Jordan curve $\omega \subset \mathbb{R}^2$ such that the set $B = \omega \cap X$ is a Cantor set of points.*
- *(iii) Every point component of X is contained in B, and every arc component of X meets B only at one of its end points.*

(iv) *If $x_i \to x_*$ in X, $i \geq 1$, then the unique Jordan arc in X connecting x_i to B converges to the unique Jordan arc in X connecting x_* to B, with respect to the Hausdorff topology.*
(v) *The set $X \setminus B$ is dense in X.*
(vi) *For any $x \in X \setminus B$, there are sequences $(\alpha_i)_{i\in\mathbb{N}}$ and $(\beta_i)_{i\in\mathbb{N}}$ in X such that $\alpha_i \to x$, $\beta_i \to x$, and $b(\alpha_i) < b(x) < b(\beta_i)$, with respect to a fixed cyclic order on ω, where $b(z) \in B$ denotes the unique end point of the component of X containing z.*

Proof. ($\Rightarrow$) First let us assume that X is a hairy Cantor set and let B be the base Cantor set of X. We only need to show that the conditions (ii) and (vi) hold. According to Proposition 6.26, there exists a Jordan curve ω such that $\omega \cap X = B$. Now we may use Lemma 6.27 to complete the proof.

($\Leftarrow$) Let us assume that X is a compact subset of the plane that satisfies the conditions described in the theorem. Since B is a Cantor set, $\omega \setminus B$ is a union of open arcs[13] in ω. Let $x \in B$ be an end-point of a connected component of $\omega \setminus B$. Then condition (vi) implies that $b^{-1}(b(x)) = \{x\}$. Since B is a Cantor set, the set consisting of such x is dense in B. Therefore, the set of point components of X is dense in B. To prove that X is a hairy Cantor set, we now only need to show that the points in the interior of arc components of X are not accessible from $\mathbb{R}^2 \setminus X$. Let $x \in X$ be an interior point of an arc component of X and assume that it is accessible from $\mathbb{R}^2 \setminus X$. According to Remark 4, the proof of Lemma 6.18 does not rely on condition (vi) in the definition of the hairy Cantor sets. Therefore, using the same argument, we see that $\mathbb{R}^2 \setminus (X \cup \omega)$ has two connected components. We consider the case where x belongs to the unbounded connected component of $\mathbb{R}^2 \setminus (X \cup \omega)$. The other case is similar. Since x is accessible from the unbounded connected component of $\mathbb{R}^2 \setminus (X \cup \omega)$, we may find a simple curve $\lambda \subseteq \{x\} \cup (\mathbb{R}^2 \setminus (X \cup \omega))$ that connects x to infinity.

For any two points $y, z \in X$ that belong to the same connected component of X, we define $[x, z]$ to be the unique, possibly degenerate, Jordan arc in X that connects y and z. Let p_x denote the end point of the arc component of X containing x that does not belong to B. We may find a Jordan domain W such that

$$[b(x), x] \subseteq W, \quad p_x \notin \overline{W}.$$

We may also assume that ∂W intersects $[b(x), p_x]$ at exactly one point, namely y, and intersects ω at exactly two points.

According to condition (vi), there are sequences $(\alpha_i)_{i\in\mathbb{N}}$ and $(\beta_i)_{i\in\mathbb{N}}$ in X that converge to p_x such that $b(\alpha_i) < b(x) < b(\beta_i)$ for all $i \in \mathbb{N}$. Possibly after removing finitely many elements from these sequences, we may assume that

$$\forall i \in \mathbb{N}, \quad b(\alpha_i) \in W, \quad \alpha_i \notin W, \quad b(\beta_i) \in W, \quad \beta_i \notin W.$$

For all $i \in \mathbb{N}$, we define $\hat{\alpha}_i$ to be the unique point in $[b(\alpha_i), \alpha_i]$ such that

$$\hat{\alpha}_i \notin W, \quad [b(\alpha_i), \hat{\alpha}_i] \setminus \{\hat{\alpha}_i\} \subseteq W.$$

We may similarly define $\hat{\beta}_i$. Let $\delta_i \subseteq \partial W$ be the Jordan arc that connects $\hat{\alpha}_i$ to $\hat{\beta}_i$ and intersects $[b(x), p_x]$. The sequences $(\hat{\alpha}_i)_{i\in\mathbb{N}}$ and $(\hat{\beta}_i)_{i\in\mathbb{N}}$ both belong to $X \cap \partial W$. Therefore, possibly by passing to a subsequence, we may assume that they are both convergent. We have $b(\hat{\alpha}_i) \to b(x)$, so the limit point of $(\hat{\alpha}_i)_{i\in\mathbb{N}}$ is contained in $[b(x), p_x] \cap \partial W = \{y\}$. Similarly, we also see that $\hat{\beta}_i$ tends to y.

Now note that for any $i \in \mathbb{N}$, the Jordan arc

$$\eta_i := [b(\alpha_i), \hat{\alpha}_i] \cup [b(\beta_i), \hat{\beta}_i] \cup \{w \in \omega \mid b(\alpha_i) \leq w \leq b(\beta_i)\} \subseteq X \cup \omega$$

[13] An *open arc* is a homeomorphic image of an open interval in $\mathbb{R}$.

does not intersect λ. The Jordan domain bounded by $\eta_i \cup \delta_i$ contains the point x. Therefore λ intersects δ_i. For any $i \in \mathbb{N}$, choose $y_i \in \lambda \cap \delta_i$. We know that $\delta_i \subseteq \partial W$ is the Jordan arc connecting $\hat{\alpha}_i$ to $\hat{\beta}_i$ and we have $\hat{\alpha}_i \to y$ and $\hat{\beta}_i \to y$. Therefore we have $y_i \to y \in \partial W \cap X$. Since λ is a closed subset of the plane, we conclude that $y \in \lambda \cap X \neq \emptyset$. This contradiction rules out the possibility of x being accessible from $\mathbb{R}^2 \setminus X$ and completes the proof. □

Corollary 6.29. *Every straight hairy Cantor set is a hairy Cantor set.*

Proof. Every straight hairy Cantor set X satisfies the conditions (i)-(v) in the definition of the hairy Cantor sets. Let $I \subseteq \mathbb{R}$ be a closed interval such that $X \cap \mathbb{R} \subseteq I$ and let ω to be the boundary of the rectangle $I \times [-1, 0] \subseteq \mathbb{R}^2$. Now we may use Theorem 6.28 to see that X is a hairy Cantor set. □

Remark 5. For hairy circles, "the set of base points" is a Jordan curve and it has a canonic cyclic order. In the proof of Corollary 6.29, we choose the base curve as the boundary of the rectangle $I \times [-1, 0]$. In this case, the cyclic order on the base Cantor set B induced by the base curve is similar to the order of the points in B as a subset of the real axis. However, it is straightforward to see that we may find infinitely many base curves such that the cyclic orders induced on B are pair-wise distinct. In other words, the order induced on the base Cantor set by the base curve is not canonic. However, as seen in Lemma 6.27, not every cyclic order is permissible.

We will now describe an alternative definition for straight hairy Cantor sets as well.

Corollary 6.30. *A compact set $X \subseteq \mathbb{R}^2$ is a straight hairy Cantor set if and only if there is a Cantor set $C \subseteq \mathbb{R}$ and a function $l : C \to [0, \infty)$ such that*

$$X = \{(x, y) \in \mathbb{R}^2 \mid x \in C, 0 \leq y \leq l(x)\}$$

and the following conditions hold.

(i) The set $\{x \in C \mid l(x) \neq 0\}$ is dense in C.
(ii) The points in the set $\{(x, y) \in \mathbb{R}^2 \mid x \in C, 0 < y < l(x)\}$ are not accessible from $\mathbb{R}^2 \setminus X$.

Proof. Corollary 6.29 implies that every straight hairy Cantor set is compact and satisfies the conditions (i) and (ii). Now consider a compact set $X \subseteq \mathbb{R}^2$, a Cantor set C and a function $l : C \to [0, \infty)$ satisfying the given properties. Note that X satisfies the properties in the definition of the hairy Cantor set. The function l is bounded and upper semi-continuous since its under-graph is compact. Let $I \subseteq \mathbb{R}$ be a closed interval such that $C \subseteq I$ and let ω to be the boundary of the rectangle $I \times [-1, 0] \subseteq \mathbb{R}^2$. Now we may directly use Lemma 6.27 to see that X satisfies the second condition in the definition of the straight hairy Cantor sets. □

6.3. **Ambient homeomorphism.** Here we will show that all hairy Cantor sets in the plane are ambiently homeomorphic. For this, we will use some classical results from the theory of prime ends. For more details, one may consult [Car13] or [Koe15]. We will use the notations $\mathrm{d}_{\hat{\mathbb{C}}}$ and $\mathrm{diam}_{\hat{\mathbb{C}}}$ for the distance and the diameter with respect to the spherical metric on $\hat{\mathbb{C}}$.

Definition 6.31. *Let $U \subseteq \hat{\mathbb{C}}$ be a simply connected domain such that $|\partial U| > 1$. A fundamental chain in U is a sequence of non-empty, simply connected domains $\Omega_i \subseteq U$, for $i \in \mathbb{N}$, such that the following hold.*

- *For all $i \in \mathbb{N}$, the set $\overline{\partial\Omega_i \cap U}$ is a Jordan arc with its interior contained in U, connecting two points on the boundary of U.*
- *For all $i \in \mathbb{N}$, we have $\overline{\partial\Omega_i \cap U} \cap \overline{\partial\Omega_{i+1} \cap U} = \emptyset$.*
- *For all $i \in \mathbb{N}$, we have $\overline{\Omega}_{i+1} \cap U \subseteq \Omega_i$.*
- *We have $\mathrm{diam}_{\hat{\mathbb{C}}}(\partial\Omega_i \cap U) \to 0$ as i tends to infinity.*

As a consequence, we see that $\left(\overline{\partial\Omega_i \cap U}\right)_{i\in\mathbb{N}}$ *is a sequence of disjoint Jordan arcs, each of which connect two distinct points of* ∂U*. If there is a point* $x \in \overline{U}$ *such that* $x \notin \overline{\partial\Omega_i \cap U}$*, for all* $i \in \mathbb{N}$*, and* $\operatorname{diam}_{\hat{\mathbb{C}}}(\{x\} \cup (\partial\Omega_i \cap U)) \to 0$*, as* i *tends to infinity, then we say that* x *is a principal point of the fundamental chain* $(\Omega_i)_{i\in\mathbb{N}}$*. The impression of a fundamental chain* $(\Omega_i)_{i\in\mathbb{N}}$ *is defined as the set* $\bigcap_{i\in\mathbb{N}} \overline{\Omega_i}$*.*

Note that, in this definition, a fundamental chain $(\Omega_i)_{i\in\mathbb{N}}$ does not necessarily have a principal point. However, if it has a principal point, then it is unique.

Definition 6.32. *Let* $U \subseteq \hat{\mathbb{C}}$ *be a simply connected domain such that* $|\partial U| > 1$*. Two fundamental chains* $(\Omega_i)_{i\in\mathbb{N}}$ *and* $(\Omega'_i)_{i\in\mathbb{N}}$ *in* U *are called equivalent if for every integer* $i > 0$ *there is an integer* $j > 0$ *such that* $\Omega_j \subseteq \Omega'_i$ *and* $\Omega'_j \subseteq \Omega_i$*. A prime end for* U *is an equivalence class of fundamental chains. The set of principal points of a prime end is defined as the set consisting of all the principal points of the fundamental chains in its equivalence class. The impression of a prime end is defined as the impression of any fundamental chain in its equivalence class. This is well-defined by the definition of the equivalence stated above.*

Note that, in this definition, the impression of a prime end for U is a subset of ∂U that contains its principal points.

Throughout this subsection, we will fix a hairy Cantor set X and a base curve ω.

Definition 6.33. *Define*

$$X_\omega := X \cup \omega, \quad \tilde{X}_\omega := \operatorname{fill}(X_\omega).$$

Lemma 6.18 shows that $\hat{\mathbb{C}} \setminus X_\omega$ *has two connected components. Let* U' *and* U *be the bounded and unbounded connected component of* $\hat{\mathbb{C}} \setminus X_\omega$ *respectively. Note that* U' *is the Jordan domain bounded by* ω *and we have*

$$\tilde{X}_\omega = X_\omega \cup U', \quad U = \hat{\mathbb{C}} \setminus \tilde{X}_\omega.$$

We will now extend the domain of the maps p *and* b*, the peak map and the base map respectively, to* X_ω *by setting*

$$\forall x \in X_\omega \setminus X, \quad p(x) := x, \quad b(x) := x.$$

Also define

$$\forall x \in X_\omega \setminus X, \quad [b(x), p(x)] := \{x\}.$$

Note that, in this definition, $b|_\omega = \mathrm{Id}_\omega$ and $b|_X$ are continuous maps with closed domains that take the same values on the intersection of their domains. Hence $b : X_\omega \to \omega$ is continuous.

Lemma 6.34. *The set* U *is a topological disk and we have* $X_\omega = \partial U$*.*

Proof. The set $U = \hat{\mathbb{C}} \setminus \operatorname{fill}(X_\omega)$ is open and simply connected. Hence it is a topological disk. The set $\hat{\mathbb{C}} \setminus \overline{U}$ is an open set that is contained in $\hat{\mathbb{C}} \setminus U = \tilde{X}_\omega = X_\omega \cup U'$. Since X_ω has no interior points and U' is a Jordan domain, we have $\hat{\mathbb{C}} \setminus \overline{U} \subseteq U'$. Hence $X_\omega \subseteq \hat{\mathbb{C}} \setminus U' \subseteq \overline{U}$. Therefore

$$\partial U = \overline{U} \cap \overline{\hat{\mathbb{C}} \setminus U} = \overline{U} \cap (\hat{\mathbb{C}} \setminus U) = \overline{U} \cap (X_\omega \cup U') = \overline{U} \cap X_\omega = X_\omega. \qquad \square$$

Before we state the next lemma, let us recall the following classical results from the theory of prime ends. If $V \subseteq \hat{\mathbb{C}}$ is a simply connected domain such that $|\partial V| > 1$, then the following statements hold.

- If two fundamental chains $(\Omega^1_i)_{i\in\mathbb{N}}$ and $(\Omega^2_i)_{i\in\mathbb{N}}$ in V are not equivalent, then there exists $N \in \mathbb{N}$ such that $\Omega^1_N \cap \Omega^2_N = \emptyset$.
- Every prime end of V has at least one principal point.

- A point $p \in \partial V$ is accessible from V if and only if it is the unique principal point of at least one prime end of V. In this case we say the corresponding prime ends are *accessible*.
- Every point $p \in \partial V$ is contained in the impression of a prime end of V.

Lemma 6.35. *The following statements hold.*

(i) The set of impressions of the prime ends of U is equal to $\{[b(z), p(z)] \mid z \in X_\omega\}$.
(ii) Every prime end of U is accessible and the set of points in $\tilde{X}_\omega$ that are accessible from U is equal to $\{p(z) \mid z \in X_\omega\}$.
(iii) Distinct prime ends of U have disjoint impressions.

Proof. Let P be a prime end for U and let $(\Gamma_i)_{i\in\mathbb{N}}$ be a fundamental chain of P. For any $i \in \mathbb{N}$, the set $\overline{\partial\Gamma_i \cap U}$ is a Jordan arc connecting two points of X_ω. Let us call these points α_i and β_i. Considering the cyclic order on ω and the fact that

$$\overline{\Gamma_{i+1}} \cap U \subseteq \Gamma_i \subseteq \overline{\Gamma_i} \cap U,$$

we may assume that for all $i \in \mathbb{N}$, we have

$$b(\alpha_i) \leq b(\alpha_{i+1}) \leq b(\beta_{i+1}) \leq b(\beta_i).$$

Moreover, for any $i \in \mathbb{N}$ and $z \in \partial\Gamma_i \cap \partial U$, we have $b(\alpha_i) \leq b(z) \leq b(\beta_i)$. On the other hand, for any $z \in X_\omega$, the inequality $b(\alpha_i) < b(z) < b(\beta_i)$ implies that $[b(z), p(z)] \subseteq \partial\Gamma_i$.

We start by finding the impression of P.

(i) : Let x be a point in the impression of P. First we show that the impression of P is contained in $[b(x), p(x)]$. Since $\operatorname{diam}_{\hat{\mathbb{C}}}(\overline{\partial\Gamma_i \cap U}) \to 0$, we have $\mathrm{d}_{\hat{\mathbb{C}}}(\alpha_i, \beta_i) \to 0$. The map $b : X_\omega \to \omega$ is uniformly continuous. Therefore we have $\mathrm{d}_{\hat{\mathbb{C}}}(b(\alpha_i), b(\beta_i)) \to 0$. On the other hand, for any $i \in \mathbb{N}$, the point x belongs to $\partial\Gamma_i$, which implies $b(\alpha_i) \leq b(x) \leq b(\beta_i)$. Therefore we have $b(x) = \lim_{i\to\infty} b(\alpha_i)$, which shows that $b(x)$ does not depend on the choice of x. In other words, for any point y in the impression of P, we have $y \in b^{-1}(b(x)) = [b(x), p(x)]$.

Fix $i \in \mathbb{N}$. We have

$$b(\alpha_i) \leq b(\alpha_{i+1}) \leq b(\alpha_{i+2}) \leq b(x) \leq b(\beta_{i+2}) \leq b(\beta_{i+1}) \leq b(\beta_i).$$

The α_j's and β_j's are distinct points in X_ω that are accessible from U. Therefore they can not be interior points of an arc component of X. Hence the points $b(\alpha_i)$, $b(\alpha_{i+1})$ and $b(\alpha_{i+2})$ can not all be equal. After using a similar argument for β_j's, we see

$$b(\alpha_i) < b(\alpha_{i+2}) \leq b(x) \leq b(\beta_{i+2}) < b(\beta_i),$$

which implies that $[b(x), p(x)]$ is contained in $\partial\Gamma_i$. Since the argument is independent of the choice of i, we conclude that $[b(x), p(x)]$ is the impression of P.

Any point $z \in X_\omega = \partial U$ is contained in the impression of a prime end of U. Therefore we have proved that the set of impressions of the prime ends of U is equal to $\{[b(z), p(z)] \mid z \in X_\omega\}$.

(ii) : Assume $(\Gamma_i)_{i\in\mathbb{N}}$ has a principal point, namely y. We will show that $y = p(y)$. If $p(y) = b(y)$, there is nothing to prove. Let us assume that $[b(y), p(y)]$ is a Jordan arc and $y \neq p(y)$. We continue the Jordan arc $[b(y), p(y)]$ in the direction of $b(y)$ to obtain a slightly larger Jordan arc γ such that

$$\gamma \cap X_\omega = [b(y), p(y)].$$

Since

$$y \in [b(y), p(y)] \setminus \{p(y)\},$$

there is a small Jordan domain D containing $[y, b(y)]$ such that $D \setminus \gamma$ has two connected components, namely D_1 and D_2, and $p(y) \notin D$. We may choose D in a way that ∂D intersects ω at exactly two points. We name the connected components of $D \setminus \gamma$ so that for any $z_1 \in D_1 \cap \omega$ and

$z_2 \in D_2 \cap \omega$, we have $z_1 < b(y) < z_2$. As a consequence, we have $b(\alpha_i) \in D_1$ and $b(\beta_i) \in D_2$ for large i.

According to the definition of hairy Cantor sets and principal points, we have $[\alpha_i, b(\alpha_i)] \to [y, b(y)]$, $[\beta_i, b(\beta_i)] \to [y, b(y)]$ and $\operatorname{diam}_{\hat{\mathbb{C}}}((\partial\Gamma_i \cap U) \cup \{y\}) \to 0$. Therefore we have

$$[\alpha_i, b(\alpha_i)] \cup \overline{\partial\Gamma_i \cap U} \cup [\beta_i, b(\beta_i)] \subseteq D,$$

for large i. Fix a large i consider the set $[\alpha_i, b(\alpha_i)] \cup \overline{\partial\Gamma_i \cap U} \cup [\beta_i, b(\beta_i)]$. This set is a Jordan arc contained in $D \setminus \gamma$ that does not intersect γ. Therefore its end points, $b(\alpha_i)$ and $b(\beta_i)$, belong to the same connected component of $D \setminus \gamma$, which is a contradiction. The contradiction shows that $y = p(y)$.

The impression of P is equal to $[b(x), p(x)]$, for some $x \in X_\omega$. We have proved that for any fundamental chain of P that has a principal point, $p(x)$ is the principal point. Therefore $p(x)$ is the unique principal point of the prime end P. Hence P is accessible. On the other hand, any set of the form $[b(x), p(x)]$, for $x \in X_\omega$, is the impression of a prime end of U. Therefore any prime end of U is accessible and $x \in X_\omega$ is accessible from U if and only if $x = p(x)$.

(iii): Let P^1 and P^2 be two different prime ends for U, such that their impressions intersect. According to (i), this implies that their impressions are equal and equal to $[b(y_*), p(y_*)]$, for some $y_* \in X_\omega$. Let $(\Omega_i^1)_{i\in\mathbb{N}}$ and $(\Omega_i^2)_{i\in\mathbb{N}}$ be fundamental chains in P^1 and P^2 respectively. There is $j \in \mathbb{N}$ such that $\Omega_j^1 \cap \Omega_j^2 = \emptyset$. For all $i \in \mathbb{N}$, choose $y_i \in \Omega_i^1$. Then we have $d(y_i, [b(y_*), p(y_*)]) \to 0$.

Let $(\Gamma_i)_{i\in\mathbb{N}} = (\Omega_i^2)_{i\in\mathbb{N}}$ and define the sequences $(\alpha_i)_{i\in\mathbb{N}}$ and $(\beta_i)_{i\in\mathbb{N}}$ as above. The open set [14]

$$W_j := \operatorname{int}(\overline{\Omega_j^2} \cup \overline{U'})$$

is a Jordan domain bounded by

$$\overline{\partial\Omega_j^2 \cap U} \cup [\alpha_j, b(\alpha_j)] \cup [\beta_j, b(\beta_j)] \cup \{w \in \omega \mid b(\beta_j) \le b(w) \le b(\alpha_j)\}).$$

In particular, we have $[b(y_*), p(y_*)] \subseteq W_j$. Since $d(y_i, [b(y_*), p(y_*)]) \to 0$, there is $N > j$ such that $y_N \in W_j$, which implies $y_N \in W_j \cap U = \Omega_j^2$. Therefore we have

$$y_N \in \Omega_N^1 \cap \Omega_j^2 \subseteq \Omega_j^1 \cap \Omega_j^2 = \emptyset,$$

which is a contradiction. □

In the remainder of this subsection, we will need to use the following classical results. If $V \subseteq \hat{\mathbb{C}}$ is a simply connected domain such that $|\partial V| > 1$ and $\phi : \mathbb{D} \to V$ is a Riemann map, then the following statements hold.

- For any fundamental chain $(\Omega_i)_{i\in\mathbb{N}}$ in V, the sequence $(\phi^{-1}(\Omega_i))_{i\in\mathbb{N}}$ is a fundamental chain in $\mathbb{D}$. Moreover, equivalent fundamental chains of V are mapped into equivalent fundamental chains of $\mathbb{D}$. This correspondence induces a bijection between the set of prime ends of V and the set of prime ends of $\mathbb{D}$.
- Any prime end of $\mathbb{D}$ is accessible. The map that sends a prime end of $\mathbb{D}$ to its unique principal point induces a bijection between prime ends of $\mathbb{D}$ and angles in $\mathbb{R}/2\pi\mathbb{Z}$. As a consequence, there is a one-to-one correspondence between the angles in $\mathbb{R}/2\pi\mathbb{Z}$ and the prime ends of V.
- If a prime end of V has a unique principal point and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is the angle corresponding to P, then the limit

  $$\lim_{r\to 1} \phi(re^{i\theta})$$

  exists and it is equal to the principal point of P. (See Corollary 2.17 in [Pom92]).

[14] We use the notation $\operatorname{int}(A)$ to denote the interior of the set A.

Definition 6.36. *Consider a Riemann map $\phi : \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to U$ with $\phi(\infty) = \infty$ and define*

$$A := \phi\left(\left\{z \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \mid |z| \leq 2\right\}\right) \cup X_\omega.$$

The set A is a compact annulus. In fact, its complement in $\hat{\mathbb{C}}$ is the union of two topological disks, one being the bounded region in the complement of ω and the other one being

$$\phi(\{z \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \mid |z| > 2\}).$$

Consider the continuous function $\theta : U \setminus \{\infty\} \to \mathbb{R}/2\pi\mathbb{Z}$ defined by $\theta(y) = \arg(\phi^{-1}(y))$. We will use θ to extend the domain of b to include the set A. The set of prime ends of U is in one-to-one correspondence with angles in $\mathbb{R}/2\pi\mathbb{Z}$. Every prime end in U is accessible by Lemma 6.35. Therefore, for every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, the limit $\lim_{r\to 1} \phi(re^{i\theta})$ exists and belongs to the set $\{p(z) \mid z \in \omega\}$. Note that we have $x = \phi(|\phi^{-1}(x)|e^{i\theta(x)})$, for any $x \in A \setminus X_\omega$. We define

$$\forall x \in A \setminus X_\omega, \quad b(x) := b(\lim_{r\to 1} \phi(re^{i\theta(x)})).$$

For any $x \in X_\omega$, there is a unique value of $\theta(x) \in \mathbb{R}/2\pi\mathbb{Z}$ for which the above equation holds. This is true since $[b(x), p(b(x))]$ is the impression of a unique prime end in U, which in turn corresponds to a unique angle in $\mathbb{R}/2\pi\mathbb{Z}$. As a consequence, we see that for every $w \in \omega$, the set

$$b^{-1}(w) = [w, p(w)] \cup \{x \in A \setminus X_\omega \mid w = b(\lim_{r\to 1} \phi(re^{i\theta(x)}))\}$$

is a Jordan arc in A that connects the two boundary Jordan curves of A.

Lemma 6.37. *The map $b : A \to \omega$ is continuous.*

Proof. We want to show that $b^{-1}(K)$ is closed for every closed set $K \subseteq \omega$. Since ω is a Jordan curve, it is sufficient to prove that $b^{-1}(K)$ is closed whenever $K \subseteq \omega$ is a either a point or a Jordan arc. If K is a single point, then $b^{-1}(K)$ is a Jordan arc and therefore closed. Let $K \subseteq \omega$ be a Jordan arc and let α and β be its end-points. We know that $b^{-1}(\alpha)$ and $b^{-1}(\beta)$ are disjoint Jordan arcs that connect the two boundary components of A. Therefore the set

$$\operatorname{int}(A) \setminus (b^{-1}(\alpha) \cup b^{-1}(\beta))$$

is a disjoint union of two Jordan domains, namely V_1 and V_2. The sets $\overline{V_1} \cap \omega$ and $\overline{V_2} \cap \omega$ are Jordan arcs in ω that have α and β as their end-points. Hence one of these sets, say $\overline{V_1} \cap \omega$, is equal to K. We know that the Jordan arcs $b^{-1}(w)$, for $w \in \omega$, are disjoint and their union is equal to A. Therefore $\overline{V_1}$ is a union of some of the Jordan arcs in $\{b^{-1}(w) \mid w \in \omega\}$. Hence we have

$$\overline{V_1} = b^{-1}(b(\overline{V_1})) = b^{-1}(\overline{V_1} \cap \omega) = b^{-1}(K). \qquad \square$$

For every $w \in \omega$ and $x, y \in b^{-1}(w)$, we define $[x, y]$ to be the unique Jordan arc in $b^{-1}(w)$ that ends at these points. Note that if $x, y \in X_\omega \cap b^{-1}(w)$, then this notation is consistent with the previous definition of $[x, y]$. Since $b^{-1}(w)$ is a Jordan arc, we may define a natural linear order on it. We will consider the linear order where $w = b(w)$ is the smallest point.

Lemma 6.38. *For any sequence $(x_i)_{i\in\mathbb{N}}$ in A such that $x_i \to x_* \in A$, we have*

$$\lim_{i\to\infty} [x_i, b(x_i)] = [x_*, b(x_*)]$$

in the Hausdorff topology.

Before we start the proof of this lemma, we state a classical result from complex analysis. Let $V \subseteq \hat{\mathbb{C}}$ be a simply connected domain such that $|\partial V| > 1$. Also let $x, y \in V$ and let $\gamma, \lambda \subseteq V$ be Jordan arcs with end points x and y such that γ is a geodesic with respect to the hyperbolic metric on V. Then the *Gehring-Hayman theorem* asserts that

$$\operatorname{diam}_{\hat{\mathbb{C}}}(\gamma) \leq C_{GH} \operatorname{diam}_{\hat{\mathbb{C}}}(\lambda),$$

where C_{GH} is a universal constant. (see Theorem 4.20 in [Pom92])

Proof. The set of non-empty compact subsets of A with the Hausdorff metric is compact. Therefore it is sufficient to prove that for any sequence $x_i \to x_*$ where $([x_i, b(x_i)])_{i\in\mathbb{N}}$ is a convergent sequence in the Hausdorff metric, we have $[x_i, b(x_i)] \to [x_*, b(x_*)]$.

Let $(x_i)_{i\in\mathbb{N}}$ be a convergent sequence in A with the limit $x_* \in A$ such that the sequence $([x_i, b(x_i)])_{i\in\mathbb{N}}$ converges to a set $K \subseteq A$ in the Hausdorff topology. Since b is continuous, K is contained in $b^{-1}(b(x_*))$. The set K is the Hausdorff limit of $[x_i, b(x_i)]$, so we have

$$K = \{y \mid y = \lim_{i\to\infty} y_i, y_i \in [x_i, b(x_i)]\} \supset \{x_*, b(x_*)\}.$$

Moreover, each set $[x_i, b(x_i)]$ is connected so their limit K is also connected. Hence we have $K = [y_*, b(y_*)]$ for some $y_* \in b^{-1}(b(x_*))$. We have $x_* \in K = [y_*, b(y_*)]$, so $x_* \leq y_*$ with respect to the linear order on $b^{-1}(b(x_*))$. We will assume $y_* > x_*$ and prove the lemma by deriving a contradiction.

If $y_* > x_*$ and $y_* \in U$, we define

$$r_0 := \begin{cases} (|\phi^{-1}(y_*)| + |\phi^{-1}(x_*)|)/2 & \text{if } x_* \in U, \\ (|\phi^{-1}(y_*)| + 1)/2 & \text{if } x_* \notin U. \end{cases}$$

Consider the compact set

$$A' := X_\omega \cup \phi(\{z \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}} \mid |z| \leq r_0\}) \subseteq A.$$

Since $x_i \to x_*$, there is an integer N such that for all $i > N$, we have

$$x_i \in X_\omega \text{ or } |\phi^{-1}(x_i)| \leq r_0.$$

In other words, $x_i \in A'$ for $i > N$. Hence we have $[x_i, b(x_i)] \subseteq A'$ for $i > N$. Therefore the Hausdorff limit of the sets $[x_i, b(x_i)]$ also belongs to A' which is a contradiction since we have $|\phi^{-1}(y_*)| > r_0$.

Now consider the case $y_* > x_*$ and $y_* \in X_\omega$. Since $y_* > x_* \geq b(y_*)$, we have $y_* \in X$. Fix a cyclic order on ω. By passing to a subsequence we may assume that the convergent sequence $(b(x_i))_{i\in\mathbb{N}}$ is monotone. Therefore, without any loss of generality, we may assume $b(x_i) \uparrow b(x_*)$. For any $i \in \mathbb{N} \cup \{*\}$, let

$$\gamma_i := b^{-1}(b(x_i)) \quad t_i := \max\{\gamma_i\} \in \gamma_i \cap (\partial A \setminus \omega).$$

and for some b_0 slightly smaller than $b(x_1)$, let $\gamma_0 := b^{-1}(b_0)$. The Jordan arcs γ_0 and γ_* bound a subset of A, namely the set

$$R = \{z \in A \mid b_0 < b(z) < b(x_*)\}.$$

Choose two points $w_*, z_* \in \gamma_*$ such that $x_* < z_* < w_* < y_*$. We may find Jordan arcs $\lambda^w, \lambda^z : [0, 1] \to \overline{R}$ such that

$$\lambda^w(0) = w_*, \quad w_0 := \lambda^w(1) \in \gamma_0, \quad \lambda^w((0, 1)) \subseteq \operatorname{int}(R),$$
$$\lambda^z(0) = z_*, \quad z_0 := \lambda^z(1) \in \gamma_0, \quad \lambda^z((0, 1)) \subseteq \operatorname{int}(R),$$

and $\lambda^w \cap \lambda^z = \emptyset$. The set $R \setminus (\lambda^z \cup \lambda^w)$ has three connected components. We name the connected components R^t, R^m and R^b such that

$$x_* \in \partial R^b, \quad y_* \in \partial R^t.$$

According to the definition of K, there is a sequence $(y_i)_{i\in\mathbb{N}}$ such that $y_i \in [x_i, b(x_i)]$ for all $i \in \mathbb{N}$ and $y_i \to y_*$. After possibly removing finitely many i's, we may assume that $x_i \in R^b$ and $y_i, t_i \in R^t$, for all $i \in \mathbb{N}$. As a consequence, we see that both $[x_i, y_i]$ and $[t_i, x_i]$ intersect λ^w. For any $i \in \mathbb{N}$, we may choose $\alpha_i^w, \beta_i^w \in \lambda^w \cap \gamma_i$ such that

$$\alpha_i^w < x_i < \beta_i^w,$$

and the Jordan arc $\lambda_i^w \subseteq \lambda^w$ that connects α_i^w to β_i^w intersects γ_i at only α_i^w and β_i^w. We define W_i to be the Jordan domain bounded by $[\alpha_i^w, \beta_i^w] \cup \lambda_i^w$.

We know that $x_i \in R^b$ and $\lambda_i^w \cap \overline{R^b} = \emptyset$. Therefore x_i and λ_i^w are contained in different connected components of $\overline{W_i} \setminus \lambda^z$. In other words, λ^z separates x_i from λ_i^w within $\overline{W_i}$. We want to find two points $\alpha_i^z, \beta_i^z \in \gamma_i$ and a Jordan arc $\lambda_i^z \subseteq \overline{W_i} \cap \lambda_z$ that connects them such that

$$\lambda_i^z \cap \gamma_i = \{\alpha_i^z, \beta_i^z\}, \quad \alpha_i^w < \alpha_i^z < x_i < \beta_i^z < \beta_i^w.$$

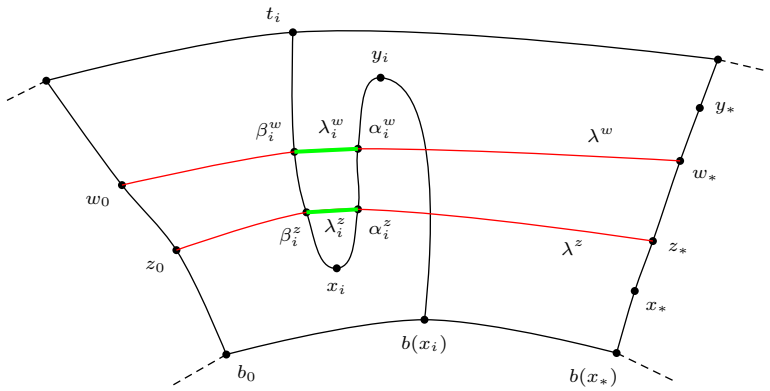
Define

$$\Gamma_i = \{\lambda \subseteq \lambda^z \mid \lambda \text{ is a Jordan arc}, \quad \lambda \cap [x_i, \alpha_i^w] \neq \emptyset, \quad \lambda \cap [x_i, \beta_i^w] \neq \emptyset, \quad \operatorname{int}(\lambda) \cap [\alpha_i^w, \beta_i^w] = \emptyset\},$$

and note that Γ_i is a non-empty set containing finitely many elements. Let E_i denote the set of end-points of Jordan arcs in Γ_i. By cutting λ^z at the points in E_i, we may obtain a finite collection of Jordan arcs, namely Γ'. Since

$$\lambda^z = \bigcup_{\lambda\in\Gamma'} \lambda, \quad |\Gamma'| < \infty,$$

we may find an element $\lambda \in \Gamma'$ that separates x_i from λ_i^w within $\overline{W_i}$. Any Jordan arc in $\Gamma' \setminus \Gamma$ can only intersect one of $[x_i, \alpha_i^w]$ and $[x_i, \beta_i^w]$. Hence we have $\lambda \in \Gamma$. Since λ intersects ∂W_i only at its end-points and $\lambda \cap W_i \neq \emptyset$, it follows that $\lambda_i^z := \lambda \subseteq \overline{W_i} \cap \lambda^z$ has the desired properties.



Now we consider two cases and derive a contradiction in each case. Either there are infinitely many $i \in \mathbb{N}$, where $X \cap \lambda_i^z \neq \emptyset$ or we have $X \cap \lambda_i^z = \emptyset$, for all large $i \in \mathbb{N}$. First we consider the former case. After possibly passing to a subsequence, we may assume that $X \cap \lambda_i^z \neq \emptyset$, for all $i \in \mathbb{N}$. Let z_i be an arbitrary point in $X \cap \lambda_z^i$. If $z_i \in \gamma_i$, then we have $z_i \in \gamma_i \cap \lambda_z^i = \{\alpha_i^z, \beta_i^z\} \subseteq \overline{R^b} \cap \gamma_i$ and $\alpha_i^w < \alpha_i^z < \beta_i^z$. Hence

$$[z_i, b(z_i)] \cap \lambda^w \supseteq \{\alpha_i^w\} \neq \emptyset.$$

If $z_i \notin \gamma_i$, then $z_i \in W_i$ and $b(z_i) \notin W_i$. Hence we have $[z_i, b(z_i)] \cap \partial W_i \neq \emptyset$. However, using the fact that $\gamma_i \cap b^{-1}(b(z_i)) = \emptyset$, we see that

$$\emptyset \neq [z_i, b(z_i)] \cap \partial W_i = [z_i, b(z_i)] \cap ([\alpha_i^w, \beta_i^w] \cup \lambda_i^w) = [z_i, b(z_i)] \cap \lambda_i^w \subseteq [z_i, b(z_i)] \cap \lambda^w.$$

Therefore we may find $w_i \in [z_i, b(z_i)] \cap \lambda^w$ for all $i \in \mathbb{N}$. According to the definition of hairy Cantor sets, we have $[z_i, b(z_i)] \to [z_*, b(z_*)]$. On the other hand $b(w_i) \to b(w_*)$ and $w_i \in \lambda^w$ for all $i \in \mathbb{N}$, which implies $w_i \to w_*$. However $w_i \in [z_i, b(z_i)]$ for all $i \in \mathbb{N}$ and $w_* \notin [z_i, b(z_i)]$, which is a contradiction.

Now we may assume that $X \cap \lambda_i^z = \emptyset$, for all large $i \in \mathbb{N}$. We have $b(\alpha_i^z) \to b(z_*)$ and $\alpha_i^z \in \lambda^z$ for all $i \in \mathbb{N}$, which implies $\alpha_i^z \to z_*$. Similarly, we have $\beta_i^z \to z_*$. Hence

$$\lim_{i\to\infty} \operatorname{diam}_{\hat{\mathbb{C}}}(\lambda_i^z) = 0.$$

The Jordan arc $[\alpha_i^z, \beta_i^z]$ is a geodesic with respect to the hyperbolic metric on U and the Jordan arc λ_i^z is contained in U for large i. Therefore, according to Gehring-Hayman theorem, for large $i \in \mathbb{N}$, we have

$$\operatorname{diam}_{\hat{\mathbb{C}}}([\alpha_i^z, \beta_i^z]) \leq C_{GH} \operatorname{diam}_{\hat{\mathbb{C}}}(\lambda_i^z).$$

On the other hand, $x_i \in [\alpha_i^z, \beta_i^z]$ and $\alpha_i^z \to z_*$, so we have

$$\liminf_{i\to\infty} \operatorname{diam}_{\hat{\mathbb{C}}}([\alpha_i^z, \beta_i^z]) \geq \operatorname{diam}_{\hat{\mathbb{C}}}(\{z_*, x_*\}) > 0.$$

Therefore

$$\liminf_{i\to\infty} \operatorname{diam}_{\hat{\mathbb{C}}}(\lambda_i^z) \geq \liminf_{i\to\infty} \frac{\operatorname{diam}_{\hat{\mathbb{C}}}([\alpha_i^z, \beta_i^z])}{C_{GH}} \geq \frac{\operatorname{diam}_{\hat{\mathbb{C}}}(\{z_*, x_*\})}{C_{GH}} > 0,$$

which is a contradiction. □

Theorem 6.39 (Theorem B)**.** *All hairy Cantor sets in the plane are ambiently homeomorphic.*

In particular, using Corollary 6.29 we see that all hairy Cantor sets are ambiently homeomorphic to the straight hairy Cantor set described in Proposition 6.2.

Proof. According to Proposition 6.3, it is sufficient to prove that every hairy Cantor set in the plane is ambiently homeomorphic to a straight hairy Cantor set. Let X be a hairy Cantor set in the plane, let ω be a base curve for X and recall Definition 6.33 and 6.36.

Consider a Whitney map μ for the set A and define

$$h : A \to [0, \infty), \quad h(x) := \mu([x, b(x)]).$$

Lemma 6.37 and 6.38 show that b and h are continuous. Note that if $b(x) = b(y)$ and $h(x) = h(y)$, then we have $[x, b(x)] = [y, b(x)]$, which implies $x = y$. Consider a Jordan arc $I \subseteq \omega$ such that $B \subseteq I$ and consider a parametrisation of the Jordan curve $\omega : [0, 1] \to \omega([0, 1])$ such that $\omega^{-1}(I) = [0, s]$, for some $0 < s < 1$. Now define

$$\Psi : \{z \in A \mid b(x) \in I\} \to [0, s] \times [0, \infty), \quad \Psi(x) := (\omega^{-1}(b(x)), h(x)).$$

The map Ψ is injective and continuous with a compact domain so it is a homeomorphism. The domain and the image of Ψ are both closures of Jordan domains. Hence Ψ can be extended to a homeomorphism of the plane. According to Corollary 6.30, the set $\Psi(X)$ is a straight hairy Cantor set. □

Corollary 6.40. *For any hairy Cantor set X, the set of the end points of the arc components of X is dense in X.*

Proof. This immediately follows from Proposition 6.10 and Theorem 6.39. □

6.4. **Abstract hairy Cantor sets.** Hairy Cantor sets are topological objects in the plane. In this subsection we generalise the notion of hairy Cantor sets to metric spaces and define *abstract hairy Cantor sets*. In other words, we characterise all metric spaces that are homeomorphic to a hairy Cantor set.

Definition 6.41. *A compact metric space X is called an abstract hairy Cantor set if the following conditions are satisfied.*

(i) Every connected component of X is either a single point or a simple arc.[15]
(ii) Closure of the union of the point components of X is a Cantor set, namely $B \subseteq X$.
(iii) Any arc component of X meets B at exactly one of its end-points.
(iv) Whenever $x_i \to x$ within X, the unique arc in X connecting x_i to B converges to the unique arc in X connecting x to B, with the convergence in the Hausdorff topology.
(v) The set of the end points of the arc components of X is dense in X.

If x, y are points in X that belong to the same connected component of X, we use $[x, y]$ to denote the unique (possibly degenerate) simple arc in X joining x and y. We define the base map of X as the map $b : X \to B$ that sends any point $x \in X$ to the unique point in the connected component of X containing x that belongs to B. We will say that $b(x)$ is the base point of x.

First we show that this notion is indeed a generalisation.

Proposition 6.42. *Any hairy Cantor set is an abstract hairy Cantor set.*

Proof. We only need to check condition (v) in the definition of abstract hairy Cantor sets. However, this is the statement of Corollary 6.40. □

Note that an abstract hairy Cantor set in the plane is not necessarily a hairy Cantor set. In fact, if X is a straight hairy Cantor set with base Cantor set $C \subseteq \mathbb{R}$ and length function $l : C \to [0, \infty)$, then for any $z \in C$ with $l(z) \neq 0$, the set

$$X' := \{(x, y) \in X \mid x \leq z\}$$

is an abstract hairy Cantor set that is not a hairy Cantor set.

Remark 6. Assume X is an abstract hairy Cantor set in the plane. Using arguments similar to the proof of Proposition 6.26 and Theorem 6.39, one can show that X is ambiently homeomorphic to a compact set of the form

$$Y := \{(x, y) \in \mathbb{R}^2 \mid x \in C, 0 \leq y \leq l(x)\},$$

where $l : C \to [0, \infty)$ is a function such that the set $\{x \in C \mid l(x) \neq 0\}$ is dense in C. If all the points in the set $\{(x, y) \in \mathbb{R}^2 \mid x \in C, 0 < y < l(x)\}$ are inaccessible from $\mathbb{R}^2 \setminus Y$, then we may use Corollary 6.30 to see that X is a hairy Cantor set. However, as shown in Figure 6.4, this condition is not always satisfied.

Theorem 6.43 (Theorem C)**.** *All abstract hairy Cantor sets are homeomorphic.*

Proof. Let X be an abstract hairy Cantor set with base Cantor set $B \subseteq X$ and base map $b : X \to B$. According to Theorem 6.39, it is sufficient to show that X is homeomorphic to a straight hairy Cantor set. Consider a Whitney map $\mu : \mathrm{H}[X] \to [0, \infty)$ and let

$$h : X \to [0, \infty), \quad h(x) := \mu([x, b(x)]).$$

[15] We say a compact metric space is a *simple arc* if it is homeomorphic to a closed interval in $\mathbb{R}$.

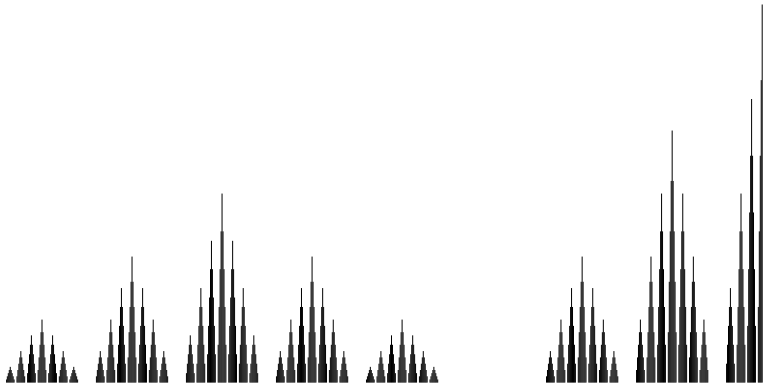


FIGURE 7. This figure illustrates how an abstract hairy Cantor set in the plane may fail to be a hairy Cantor set. The last vertical arc on the right hand side of the picture is not approximated by vertical arcs from both sides.

According to the definition of the abstract hairy Cantor sets and the Whitney maps, h is continuous on X and injective on every connected component of X. Let $C \subseteq \mathbb{R}$ be a Cantor set and let $g : B \to C$ be a homeomorphism. Define

$$\Phi : X \to C \times [0, \infty), \quad \Phi(x) := (g(b(x)), h(x)).$$

If for some $x_1, x_2 \in X$, we have $g(b(x_1)) = g(b(x_2))$ and $h(x_1) = h(x_2)$, then x_1 and x_2 belong to the same connected component of X and we have $[x_1, b(x_1)] = [x_2, b(x_2)]$, which implies that $x_1 = x_2$. In other words, Φ is injective. The function Φ is continuous and injective with a compact domain. Therefore it is a homeomorphism. According to the definition of the Whitney maps, we have

$$h^{-1}(0) = \{x \in X \mid x = b(x)\} = B.$$

Therefore there exists a function $l : C \to [0, \infty)$ such that

$$\Phi(X) = \{(x, y) \in \mathbb{R}^2 \mid x \in C, 0 \leq y \leq l(x)\}.$$

Let

$$E_b := \{(x, 0) \in \mathbb{R}^2 \mid x \in C, l(x) \neq 0\}, \quad E_p := \{(x, l(x)) \in \mathbb{R}^2 \mid x \in C, l(x) \neq 0\}.$$

Condition (v) in the definition of the abstract hairy Cantor sets shows that $E_b \cup E_p$ is dense in X. We have

$$\overline{E_b} \cap \{(x, y) \in \mathbb{R}^2 \mid x \in C, \varepsilon \leq y \leq l(x)\} = \emptyset,$$

for any $\varepsilon > 0$, which implies that

$$\overline{E_p} \supseteq \{(x, y) \in \mathbb{R}^2 \mid x \in C, \varepsilon \leq y \leq l(x)\}.$$

Hence we have

$$\overline{E_p} \supseteq \bigcup_{\varepsilon > 0} \{(x, y) \in \mathbb{R}^2 \mid x \in C, \varepsilon \leq y \leq l(x)\} = \{(x, y) \in \mathbb{R}^2 \mid x \in C, 0 < y \leq l(x)\},$$

which implies that

$$\overline{E_p} \supseteq \{(x, y) \in \mathbb{R}^2 \mid x \in C, 0 \leq y \leq l(x), l(x) \neq 0\} \supseteq E_b.$$

Therefore E_p is dense in X. Now we see that $\Phi(X)$ satisfies the requirements of Proposition 6.11. Hence $\Phi(X)$, and therefore X, is homeomorphic to a straight hairy Cantor set. □

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