Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 58-5 (2024) 152–157

Removing Chaos while preserving equilibria by means of 2-contraction ⋆

David Angeli [*,**] **Davide Martini** [**] **Giacomo Innocenti** [**] **Alberto Tesi** [**]

[*] *Department of Electrical and Electronic Engineering, Imperial College London, London, SW7 2AZ, UK (e-mail: d.angeli@imperial.ac.uk).*

[**] *Dept of Information Engineering of University of Florence (e-mail: david.angeli@unifi.it, davide.martini@unifi.it, giacomo.innocenti@unifi.it, alberto.tesi@unifi.it).*

Abstract: Conditions for designing a feedback control law to remove the dense set of Unstable Periodic Orbits and chaotic attractors while preserving equilibria are formulated in terms of 2-contractivity of the closed loop system, by means of the second additive compound of the system's Jacobian. Matrix inequalities to allow computing the control gain matrix are discussed. An example of application of the method to the Thomas' system is presented.

Copyright © 2024 The Authors. This is an open access article under the CC BY-NC-ND license (https://creativecommons.org/licenses/by-nc-nd/4.0/)

Keywords: controlling chaos; chaotic attractors; compound matrices; chaotic behaviour; Lyapunov exponents; Thomas system.

1. INTRODUCTION

For decades now, it is well known that chaos and chaotic systems are present in the world around us. The peculiarity of some systems to exhibit high sensitivity with respect to initial conditions, thus making any long-term prediction inaccurate, was known from the beginning of the last century. Despite this, chaos caught the attention of the scientific community only once it was experimentally discovered by Edward N. Lorenz, in his seminal work on deterministic non-periodic flows (Lorenz (1963)). Since then, chaotic systems were discovered in several fields of biological, ecological, chemical, physical sciences and engineering (Holden (1986)). Moreover, other peculiar aspects of chaotic systems started to be clear, such as the coexistence of unstable periodic orbits (UPOs) and the fact that, due to ergodicity, the neighborhood of each UPO is continuosly visited during the system time evolution (Li and Yorke (1975)).

Subsequently, the idea of taking advantage of chaos gained momentum. In particular, it was observed that many different oscillatory behaviours can be derived from a single chaotic system, without significant input injections in steady state, by stabilizing individual UPOs via the application of the right perturbation to the system. The first approach to stabilize a given UPO of a chaotic system was introduced by Ott-Grebogi-Yorke (OGY) in Ott et al. (1990). The method suggests to tackle the problem of stabilizing UPOs as a local stabilization of a fixed point of the system's Poincaré map. Therefore, knowledge of the right perturbation to apply as well as of the considered Poincaré section is needed. Since then, the field of chaos control has gained ground and different alternative methods were introduced. For example, in Pyragas (1992) the author suggests the well known delayed feedback control (DFC), which has undergone many changes and modifications over the years (Pyragas (2006)). The Pyragas' approach needs a priori knowledge of the UPO period and it exploits as a control input a signal proportional to the difference of the actual system's output and a delayed one, with some constant gain that has to be properly tuned numerically or experimentally. Unfortunately, the stability analysis is complex since the resulting controller and closed-loop system is not finite dimensional (Tian et al. (2005)). To overcome this issue, in the case of harmonically forced nonlinear systems, finite-dimensional controllers have been introduced, which are obtained by solving linear matrix inequalities (LMIs) in Basso et al. (2002) and Bagni et al. (2005). It is worth noting that both the OGY and DFC methods are non-invasive methods, since the stabilized periodic orbit coincides with the uncontrolled UPO.

A related problem to chaos control and predictability of system's trajectories, is the ability of characterizing and analysing the long-term behaviour of solutions of a dynamical system. Many approaches were introduced among the years, such as Lyapunov based methods (Sastry (2013)), dual Lyapunov analysis (Rantzer (2001)), contraction analysis (Lohmiller and Slotine (1998)) and so on. The last one suggests to exploit the *variational equation* to postulate LMI-based sufficient stability conditions within some "contraction regions" where the convergence of solutions towards each other is ensured. A related approach, introduced to deal with the problem of ruling out the presence of oscillatory behaviour of a nonlinear system, was introduced by James Muldowney (Muldowney (1990)). The method suggests to make use of a contraction condition on the *second additive compound* of the Jacobian to forbid oscillatory behaviours of the system by preventing the existence of invariant bidimensional surfaces. Following this line, the so called k-Contraction Theory was recently introduced (Wu et al. (2022)). This provides conditions for which arbitrary k-dimensional submanifolds of state space

⋆ The work of D. Martini was partially supported by the grant Faber 3 - Fondazione CR Firenze.

2405-8963 Copyright © 2024 The Authors. This is an open access article under the CC BY-NC-ND license.
Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2024.07.078

contract in volume along solutions of the system (for a more in depth explanation on k-Contraction Theory and compound matrices see Bar-Shalom et al. (2023)). This approach, for $k = 1$, recovers the classical Contraction Theory, while for $k = 2$ we obtain Muldowney's conditions. In Martini et al. (2022), authors show how the machinery of compound matrices and the variational equation can be exploited to find a priori bounds on the maximal Lyapunov exponents, and, as a consequence, to rule out unstable limit cycles and more complex behaviours. The same authors in Martini et al. (2023) gave some remarks on the method as well as some applications on the study of the system dynamics, also in the case of systems with a first integral of motion. Furthermore, since the dimension of the second additive compound grows as $\binom{n}{2}$, in Angeli et al. (2023) it is shown how the second additive compound of the Jacobian can be exploited to find small-gain like conditions considering the system as the interconnection of two different subsystems and, as a consequence, it allows tackling the problem via LMIs of lower dimension.
In this context, our contribution provides a different perspective on chaos control by exploiting the variational equation associated to the second additive compound of the Jacobian. In particular, the method allows designing nonlinear time-invariant state feedback controllers capable of removing the dense set of UPOs and chaotic attractors, while preserving equilibria, by making the closed loop system 2-contractive. Moreover, it will be shown how the feedback gain matrix can be obtained by solving some matrix inequalities which involve the second additive compound of the Jacobian of the closed loop system. The rest of the paper is organized as follows. In Section 2 some preliminary results are introduced, while in Section 3 the main results are presented. Section 4 concerns an example of application to the Thomas' system. Finally, some conclusions end the paper in Section 5.

2. BACKGROUND AND PRELIMINARY RESULTS

The notation adopted in the paper is as follows:

- $A^{(k)}$: multiplicative compound matrix of A of k order;
- $A^{[k]}$: additive compound matrix of A of k order;
- $J(x)$: Jacobian $\partial f(x)/\partial x$ of $f(x)$;
- $\text{conv}(A_1, \dots, A_l)$ (resp. $A > 0$): convex hull of matrices $A_1, A_2, \dots, A_l$;
- $\text{Ker}(A)$: kernel of matrix A;
- I_n: $n \times n$ identity matrix;
- $[v_1, v_2, \dots, v_m]$: matrix with column vectors $v_1, v_2, \dots, v_m$;
- $v_1 \wedge v_2$: wedge product of vectors v_1 and v_2.

Definition 1. Consider a rectangular matrix $A \in \mathbb{R}^{n\times m}$ and an integer k such that $k \leq \min\{n, m\}$. The k-multiplicative compound matrix of A, denoted as $A^{(k)} \in \mathbb{R}^{\binom{n}{k}\times\binom{m}{k}}$, is the matrix whose entries are all k-dimensional minors of the matrix A, when listing columns and rows subsets of cardinality k in lexicographical order.

From Definition 1, it follows that $A^{(1)} = A$ and, for a square matrix, $A^{(n)} = \det(A)$. Multiplicative compound matrices allow denoting in a synthetic way the *wedge* product between vectors, as it is shown in the next statement.

Proposition 2. Let v_i, $v_j \in \mathbb{C}^n$ be any complex vectors. The wedge product between vectors can be computed through multiplicative compound matrices as:

$$v_i \wedge v_j = [v_i, v_j]^{(2)}.$$

In the case of square matrices, multiplicative compound matrices have an important spectral property. Let $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be the eigenvalues of A. Then, the spectrum of the k-multiplicative compound matrix of A is equal to:

$$\{\lambda_1\lambda_2\cdots\lambda_k,\ 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}.$$

Definition 3. Let A be a square matrix, $A \in \mathbb{R}^{n\times n}$, and consider any positive integer k such that $k \leq n$. The k-additive compound matrix, denoted with $A^{[k]}$, is the $\binom{n}{k} \times \binom{n}{k}$ matrix defined as:

$$A^{[k]} = \frac{d}{d\varepsilon}(I + \varepsilon A)^{(k)}\bigg|_{\varepsilon=0}.$$

Similar to multiplicative compound, also additive compound matrices have an important spectral property. Let $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be the eigenvalues of A, then:

$$\{\lambda_1 + \lambda_2 + \cdots + \lambda_k,\ 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

is the spectrum of $A^{[k]}$.
The main reason for the interest about compound matrices is due to their connection with linear differential equations. Indeed, let $X(t) \in \mathbb{R}^{n\times m}$ be any solution of the matrix differential equation

$$\dot{X}(t) = A(t)X(t), \tag{1}$$

where $A(t)$ is a square matrix of compatible dimension. For any positive integer $k \leq \min\{n, m\}$ the following holds:

$$\dot{X}^{(k)}(t) = A^{[k]}(t)X^{(k)}(t). \tag{2}$$

Equation (2) highlights how the k order minors of the matrix $X(t)$ evolve according to linear dynamics, which are determined by the k-additive compound of the matrix $A(t)$. It is worth noting that this can be applied to linear time-varying systems and, as a consequence, can be also exploited for nonlinear systems through their variational equation. Indeed, let us consider the nonlinear dynamical system described by the following set of first-order differential equations:

$$\dot{x}(t) = f(x(t)) \tag{3}$$

where $x \in X \subset \mathbb{R}^n$ is the state vector, $f : X \to \mathbb{R}^n$ is at least of class $\mathcal{C}^1$, $x(t)$ is the state time evolution with initial condition x_0 and $\omega(x_0)$ is the system's ω-limit set. One may associate to equation (3) its variational equation, obtaining the following system of differential equations:

$$\begin{aligned} \dot{x}(t) &= f(x(t)), \\ \dot{\delta}(t) &= \tfrac{\partial f}{\partial x}(x(t))\delta(t), \end{aligned} \tag{4}$$

where $\delta(t)$ is a vector, $\delta(t) \in \mathbb{R}^n$, whose components propagate along solutions the effect of an infinitesimal perturbation of initial conditions in the direction $\delta(0)$ and $\frac{\partial f}{\partial x}(x(t)) \in \mathbb{R}^{n\times n}$ is the Jacobian of system (3) evaluated along such solutions. The spectral analysis of the matrix version of (4) can be exploited to find a-priori bounds on the Lyapunov exponents of system (3) (see Li and Xia (2004)), which are a powerful tool to get insight about the attractor generated by its solution $x(t)$, and whose existence is nowadays well known (see Oseledec (1968); Ruelle (1979)). In Martini et al. (2022), authors

propose to exploit the algebra of compound matrices to deal with the problem of finding a-priori bounds on the Lyapunov exponents of a nonlinear system. In particular, the approach suggests to use the following linearized dynamics, possibly higher-order, involving the 2-additive compound matrix of the Jacobian of system (3):

$$\begin{aligned}\dot{x}(t) &= f(x(t)),\\ \dot{\eta}(t) &= \frac{\partial f}{\partial x}^{[2]}(x(t))\eta(t),\end{aligned} \tag{5}$$

where $\eta(t) \in \mathbb{R}^{\binom{n}{2}}$, while $\frac{\partial f}{\partial x}^{[2]}(x(t)) \in \mathbb{R}^{\binom{n}{2}\times\binom{n}{2}}$ is the 2-additive compound of the Jacobian. To understand the meaning of $\eta(t)$, let us suppose that $\delta_1(t)$ and $\delta_2(t)$ are two different solutions of (4) relative to initial perturbations $\delta_1(0)$ and $\delta_2(0)$, respectively. Then, the infinitesimal initial area perturbation in the direction represented by $\delta_1(0) \wedge \delta_2(0)$, i.e $\eta(0) = [\delta_1(0), \delta_2(0)]^{(2)}$, is propagated along solution by the components of the vector $\eta(t)$, which can be obtained as $\eta(t) = \delta_1(t) \wedge \delta_2(t) = [\delta_1(t), \delta_2(t)]^{(2)}$. The spectral properties of the matrix version of (5) are strictly related to the Lyapunov exponents of (3) and can be exploited to rule out positive Lyapunov exponents inside a certain known invariant set of system (3). The next are the main results of the method.

Theorem 4. Martini et al. (2023)
Consider system (3) with $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ of class $\mathcal{C}^1$. Let $\mathcal{D} \subseteq \mathbb{R}^n$ be a positively invariant set of (3), i.e., $x(t) \in \mathcal{D}\ \forall t \geq 0$ if $x(0) = x_0 \in \mathcal{D}$. If there exists a state-dependent symmetric matrix $P(x) \in \mathbb{R}^{\binom{n}{2}\times\binom{n}{2}}$ which is positive definite for all $x \in \mathcal{D}$ and such that:

$$\left(\frac{\partial f}{\partial x}^{[2]}(x)\right)^T P(x) + P(x)\left(\frac{\partial f}{\partial x}^{[2]}(x)\right) + \frac{\partial P(x)}{\partial x} f(x) \leq 0 \quad \forall x \in \mathcal{D}, \tag{6}$$

then all the Lyapunov exponents relative to each solution $x(t)$ of (3), with initial condition $x_0 \in \mathcal{D}$ such that $f(x) \neq 0$ for all $x \in \omega(x_0)$, are less or equal than 0.

Proposition 5. Martini et al. (2023)

(1) If $P(x) = P$, $P \in \mathbb{R}^{\binom{n}{2}\times\binom{n}{2}}$ symmetric and positive definite, then condition (6) boils down to

$$\left(\frac{\partial f}{\partial x}^{[2]}(x)\right)^T P + P\left(\frac{\partial f}{\partial x}^{[2]}(x)\right) \leq 0 \quad \forall x \in \mathcal{D}. \tag{7}$$

(2) If $P(x) = P \cdot \exp[V(x)]$, $P \in \mathbb{R}^{\binom{n}{2}\times\binom{n}{2}}$ symmetric and positive definite, and $V(x) : \mathbb{R}^n \to \mathbb{R}$ is of class $\mathcal{C}^1$. Then, condition (6) reduces to

$$\left(\frac{\partial f}{\partial x}^{[2]}(x)\right)^T P + P\left(\frac{\partial f}{\partial x}^{[2]}(x)\right) + \dot{V}(x)P \leq 0 \quad \forall x \in \mathcal{D}. \tag{8}$$

It is worth noting that from the results reported above, after choosing a suitable Lyapunov function, it is possible to find conditions to rule out the presence of positive Lyapunov exponents inside the ω-limit set of system (3), i.e. the presence of unstable limit cycles and more complex behaviours. Therefore, the results can be interpreted as a relaxed form of those presented in Angeli et al. (2021), in which equation (5) is exploited to exclude the presence of an invariant torus of arbitrary dimension.

3. PROBLEM FORMULATION AND MAIN RESULTS

Based on the previous results, our aim is to design a feedback control law for system (3) that allows removing chaotic behaviours while altering the system dynamics as little as possible. Specifically, we will follow the previous approach to study the behaviour of a second order perturbation η along solutions of a system confined within a certain forward invariant subset $\mathcal{D}$ of state-space. In particular, we are looking for conditions that guarantee 2-contraction by means of a feedback control law that preserves equilibria, thus becoming inactive in steady-state. To this end, consider a nonlinear control system of the following form:

$$\dot{x} = f(x) + Bu. \tag{9}$$

where $x \in X \subset \mathbb{R}^n$ is the state vector, $f : X \to \mathbb{R}^n$ is at least of class $\mathcal{C}^1$, $B \in \mathbb{R}^{n\times m}$ is the input matrix and $u \in \mathbb{R}^m$ is the input vector. We assume, for simplicity, that the state of the system is measured and instantaneously available, and that the following feedback law is implemented

$$u = Kf(x), \tag{10}$$

where $K \in \mathbb{R}^{m\times n}$ is the matrix gain. Therefore, the closed loop control system assumes the form

$$\dot{x} = f(x) + BKf(x) = (I + BK)f(x), \tag{11}$$

while, the Jacobian of the closed loop system, named $J_{cl}(x)$, becomes:

$$J_{cl}(x) = (I + BK)J(x), \tag{12}$$

where $J(x) = \frac{\partial f}{\partial x}(x)$ is the Jacobian of system (3).
Consider the case of non-state dependent Jacobian, i.e. $J(x) \equiv A,\ \forall x \in X$. For the matrix pair (A, B) we introduce the following definition:

Definition 6. We say that the pair (A, B) is 2-contraction stabilizable if there exists a matrix K, of suitable dimension, such that $(A + BK)^{[2]}$ is Hurwitz stable.

Similarly, we may consider a related notion of stabilizability while adopting a feedback of the form

$$u = KAx, \tag{13}$$

which preserves the Kernel of A (similar in spirit to the notion of equilibrium preservation considered throughout the paper).

Definition 7. We say that the pair (A, B) is 2-contraction stabilizable with derivative feedback if there exists a matrix K of suitable dimension, such that $[(I + BK)A]^{[2]}$ is Hurwitz stable.

The following result relates the concepts just introduced.

Proposition 8. The following facts are equivalent for a pair (A, B) with A invertible:

(1) (A, B) is 2-contraction stabilizable.
(2) $A_u^{[2]}$ is Hurwitz, where A_u denotes the submatrix describing the unreachable dynamics in any Kalman's canonical reachability decomposition.
(3) (A, B) is 2-contraction stabilizable with derivative feedback.

Proof. See the appendix.

The next results hold true.

Proposition 9. Consider a closed loop control system as in (11). Then, the feedback control law does not modify the location of open loop equilibria.

Proof. The proof directly follows from observing that equilibria of system (11) trivially fulfill:

$$\{x : f(x) \in \mathrm{Ker}(I + BK)\} \supset \{x : f(x) = 0\}.$$

The next propositions provide sufficient conditions to rule out the existence of solutions with positive Lyapunov exponents in a closed loop system of the form (11), that can also be exploited to compute the matrix gains. In particular, while in Proposition 10 the condition (14) has to be verified for all x inside the state space, Proposition 11 states that when the Jacobian of the system belongs to a convex hull of given vertices, the number of linear matrix inequality to be verified significantly decreases.

Proposition 10. The following matrix inequality, fulfilled for all $x \in X \subset \mathbb{R}^n$, is a sufficient condition to ensure non positive Lyapunov exponents for the system (11):

$$\left([(I + BK)J(x)]^{[2]}\right)^T P + P\left([(I + BK)J(x)]^{[2]}\right) \leq 0,$$
$$P > 0\ , \tag{14}$$

where $P = P^T$ and K is the gain matrix of the derivative feedback (10) for system (9).

Proof. The proof directly follows from Theorem 4.

Proposition 11. Consider the control system in equation (11) and let $\mathcal{D}$ be a globally attractive and forward invariant set for system (3). Assume that:

$$J(x) \in \mathrm{conv}(J_1, \ldots, J_l), \qquad \forall\, x \in \mathcal{D}$$

for fixed constant matrices $J_i, \ldots, J_l$ of suitable dimension. Then, the following matrix inequality for $i = 1 \ldots l$

$$\left([(I + BK)J_i]^{[2]}\right)^T P + P\left([(I + BK)J_i]^{[2]}\right) \leq 0 \tag{15}$$

is a sufficient condition to ensure non-positive Lyapunov exponents system (11).

4. CASE STUDY: THOMAS' SYSTEM

As an example of application, we consider the Thomas' system, also known as the Thomas' cyclically symmetrical attractor. In the third-order case, the system dynamics is described by the following system of differential equations:

$$\begin{aligned} \dot{x}_1 &= \sin(x_2) - b\,x_1, \\ \dot{x}_2 &= \sin(x_3) - b\,x_2, \\ \dot{x}_3 &= \sin(x_1) - b\,x_3, \end{aligned} \tag{16}$$

where $x = (x_1, x_2, x_3)^T$ is the space vector and b is a positive parameter. It is well known that for all initial conditions the solutions of (16) are eventually confined inside the invariant set

$$\mathcal{D} := \{x \in \mathbb{R}^3 : b\,||x||_\infty \leq 1\}. \tag{17}$$

The Thomas' system exhibits a unique equilibrium point at $x = 0$ for $b > 1$, which undergoes to a supercritical pitchfork bifurcation for $b = 1$, giving rise to two additional stable equilibrium points, while the equilibrium point at $x = 0$ becomes unstable. The stable equilibrium points have coordinates equal to $x_\pm = \pm(\gamma, \gamma, \gamma)$, where γ is such that $\gamma = \sin\gamma/b$. The two stable branches display a supercritical Hopf bifurcation when γ fulfills the equation $\tan\gamma = -0.5\gamma$, which yields $\gamma = \gamma_{\mathcal{H}} = 2.2889$ and corresponds to $b = b_{\mathcal{H}} = 0.3290$. As b decreases within the range between $0.11 < b < b_{\mathcal{H}}$, the system displays quite a rich dynamical scenario characterized by a succession of period-doubling bifurcations and the presence of strange attractors (for a detailed analysis of the route to chaos see Sprott and Chlouverakis (2007)).

For removing chaos from the system, we set the parameter $b = 0.11$ and we choose an input vector B and a gain row vector K of the form

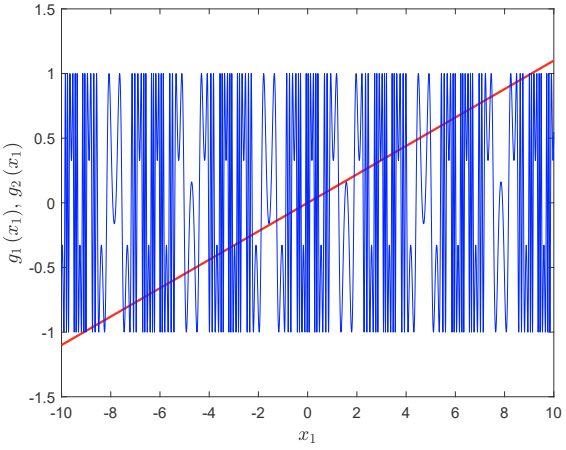


Fig. 1. Functions $g_1(x_1)$ (red) and $g_2(x_2)$ (blue): their intersections provide the coordinate x_1 of the equilibrium points of (19).

$$\begin{aligned} B &= (0, 0, 1)^T \\ K &= (0, 0, k) \end{aligned} \tag{18}$$

where $k \in \mathbb{R}$. Therefore, the closed-loop system assumes the form:

$$\dot{x} = f_{cl}(x) = \begin{pmatrix} \sin(x_2) - 0.11\,x_1 \\ \sin(x_3) - 0.11\,x_2 \\ (k+1)(\sin(x_1) - 0.11\,x_3) \end{pmatrix}. \tag{19}$$

The equilibrium points of system (19) are all the solutions of the system of equations provided by $f_{cl} = 0$. The closed-loop system has the same equilibria of the open loop-system for the same value of b, as it can be easily verified from equation (19). The coordinates of the equilibrium points are computed by first solving for x_1 the following equation

$$g_1(x_1) := bx_1 = \sin\left(\frac{\sin\left(\frac{\sin(x_1)}{b}\right)}{b}\right) =: g_2(x_1), \tag{20}$$

which amounts to find all the intersections between the functions $g_1(x_1)$ and $g_2(x_1)$, as it is shown in Fig. 1. The other two coordinates are automatically calculated as

$$\begin{aligned} x_3 &= \tfrac{\sin(x_1)}{b}, \\ x_2 &= \tfrac{\sin(x_3)}{b} = \tfrac{\sin\left(\frac{\sin(x_1)}{b}\right)}{b}. \end{aligned} \tag{21}$$

It turns out that the system displays 175 equilibria, 87 of which are unstable (including the point $x = 0$) while 88 are stable. It is worth noting that all equilibria are unstable for the open-loop system and that only 7 equilibrium points follow the system's symmetry, i.e. the points $x_{e_i} =$

$(\bar{x}_i, \bar{x}_i, \bar{x}_i)^T$ where $\bar{x}_i$ are all the solutions of the equation $b\bar{x} = \sin(\bar{x})$, meaning that most of them break the system's symmetry.

The Jacobian of the open-loop system (16) reads:

$$J(x) = \begin{pmatrix} -b & c_2 & 0 \\ 0 & -b & c_3 \\ c_1 & 0 & -b \end{pmatrix}, \tag{22}$$

where $c_i = \cos(x_i)$. Since the functions $\cos(x_i) \in [-1, 1]$, it follows that the matrix $J(x)$ fulfills:

$$J(x) \in \text{conv}(J_1, J_2, \ldots, J_8)$$

where

$$J_h = \begin{pmatrix} -b & v_2^{(h)} & 0 \\ 0 & -b & v_3^{(h)} \\ v_1^{(h)} & 0 & -b \end{pmatrix}$$

is the Jacobian calculated on the vertices $v^{(h)}$ of the cube $[-1, 1]^3$. Hence, the 2-additive compound of the closed loop system satisfies:

$$J_{cl}^{[2]}(x) \in \text{conv}([(I + BK)J_1]^{[2]}, [(I + BK)J_2]^{[2]}, \ldots, [(I + BK)J_8]^{[2]}). \tag{23}$$

In order to find the value of the control gain k, we wish to apply Proposition 11 inside the invariant set (17). Therefore, we need to find a matrix $P = P^T > 0$ and a control gain k such that the following minimization problem turns out to be feasible:

$$\begin{aligned} &\min_{P,K} \quad q \\ &\text{subject to} \\ &Q_h \leq qI, \qquad h = 1, \ldots, 8 \\ &P = P^T \geq \varepsilon I \\ &q \geq -10 \\ &k \leq k_{max} \\ &k_{max} = 400 \end{aligned} \tag{24}$$

where

$$Q_h = \left([(I + BK)\, J_h]^{[2]}\right)^T P + P\left([(I + BK)\, J_h]^{[2]}\right).$$

The constraint $q \geq -10$ has been imposed to ensure that the minimization problem is bounded, while the constraint $k \leq k_{max}$, where $k_{max} = 400$, is introduced to prevent an excessively high gain. As it can be noted from (24), since we have to solve the problem with respect to both the control gain k and the matrix P, the problem turns out to be a Bilinear Matrix Inequality (BMI). To solve it, the software YALMIP (Löfberg (2004)) can be used. The optimiser provides the control gain $k = 395.393$, while the matrix P has the following form

$$P = \begin{pmatrix} 49050.771 & 0 & 0 \\ 0 & 4357.775 & 0 \\ 0 & 0 & 3.163 \end{pmatrix}. \tag{25}$$

Figure 2 reports some MATLAB simulations of the controlled system with $b = 0.11$ and $k = 395.393$, showing convergence of the trajectories towards the equilibrium points.

5. CONCLUSION

The paper introduces a general notion of 2-contraction stabilization both for state and derivative feedback. Additionally, it presents a feedback control law based on 2-contraction concepts that allows removing chaos while preserving equilibria. Furthermore, sufficient condition to synthesize matrix gains in the form of matrix inequality is provided. Finally, the case in which the matrix inequality reduces to Bilinear Matrix Inequality (BMI) is also discussed.

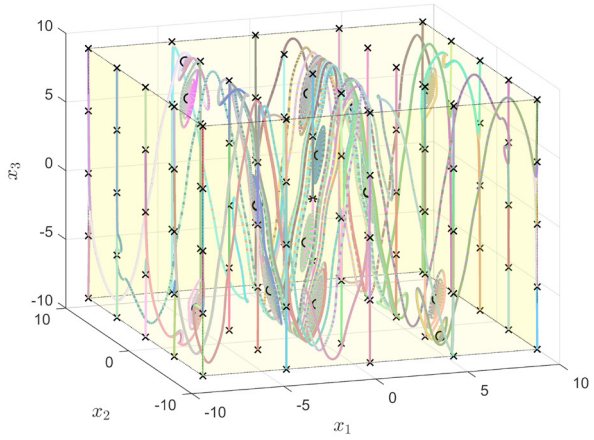


Fig. 2. Trajectories of the controlled Thomas system for $b = 0.11$. The initial conditions are marked with $\times$. Stable equilibrium points are marked by $\circ$, while the unstable equilibrium point at $x = 0$ is marked by $*$.

REFERENCES

Angeli, D., Al-Radhawi, M.A., and Sontag, E.D. (2021). A robust Lyapunov criterion for nonoscillatory behaviors in biological interaction networks. *IEEE Transactions on Automatic Control*, 67(7), 3305–3320.

Angeli, D., Martini, D., Innocenti, G., and Tesi, A. (2023). A small-gain theorem for 2-contraction of nonlinear interconnected systems. *arXiv preprint arXiv:2305.03211*.

Bagni, G., Basso, M., Genesio, R., and Tesi, A. (2005). Synthesis of MIMO controllers for extending the stability range of periodic solutions in forced nonlinear systems. *Automatica*, 41(4), 645–654.

Bar-Shalom, E., Dalin, O., and Margaliot, M. (2023). Compound matrices in systems and control theory: a tutorial. *Mathematics of Control, Signals, and Systems*, 1–55.

Basso, M., Genesio, R., and Tesi, A. (2002). An LMI-based controller synthesis for periodic trajectories in a class of nonlinear systems. *IEEE transactions on automatic control*, 47(10), 1740–1744.

Holden, A. (1986). *Chaos*. History and Related Disciplines Select Bibliographies. Manchester University Press.

Li, C. and Xia, X. (2004). On the bound of the Lyapunov exponents for continuous systems. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 14(3), 557–561.

Li, T.Y. and Yorke, J.A. (1975). Period Three Implies Chaos. *The American Mathematical Monthly*, 82(10), 985–992.

Löfberg, J. (2004). YALMIP : A Toolbox for Modeling and Optimization in MATLAB. In *In Proceedings of the CACSD Conference*. Taipei, Taiwan.

Lohmiller, W. and Slotine, J.J.E. (1998). On contraction analysis for non-linear systems. *Automatica*, 34(6), 683–696.

Lorenz, E.N. (1963). Deterministic nonperiodic flow. *Journal of atmospheric sciences*, 20(2), 130–141.
Martini, D., Angeli, D., Innocenti, G., and Tesi, A. (2022). Ruling Out Positive Lyapunov Exponents by Using the Jacobian's Second Additive Compound Matrix. *IEEE Control Systems Letters*, 6, 2924–2928.
Martini, D., Angeli, D., Innocenti, G., and Tesi, A. (2023). Bounding Lyapunov Exponents Through Second Additive Compound Matrices: Case Studies and Application to Systems with First Integral. *International Journal of Bifurcation and Chaos*, 33(10), 2350114.
Muldowney, J.S. (1990). Compound matrices and ordinary differential equations. *The Rocky Mountain Journal of Mathematics*, 857–872.
Oseledec, V.I. (1968). A multiplicative ergodic theorem, Lyapunov characteristic numbers for dynamical systems. *Transactions of the Moscow Mathematical Society*, 19, 197–231.
Ott, E., Grebogi, C., and Yorke, J.A. (1990). Controlling chaos. *Physical review letters*, 64(11), 1196.
Pyragas, K. (1992). Continuous control of chaos by self-controlling feedback. *Physics letters A*, 170(6), 421–428.
Pyragas, K. (2006). Delayed feedback control of chaos. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 364(1846), 2309–2334.
Rantzer, A. (2001). A dual to Lyapunov's stability theorem. *Systems & Control Letters*, 42(3), 161–168.
Ruelle, D. (1979). Ergodic theory of differentiable dynamical systems. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 50(1), 27–58.
Sastry, S. (2013). *Nonlinear systems: analysis, stability, and control*, volume 10. Springer Science & Business Media.
Sprott, J.C. and Chlouverakis, K.E. (2007). Labyrinth chaos. *International Journal of Bifurcation and Chaos*, 17(06), 2097–2108.
Tian, Y., Zhu, J., and Chen, G. (2005). A survey on delayed feedback control of chaos. *Journal of Control Theory and Applications*, 3, 311–319.
Wu, C., Kanevskiy, I., and Margaliot, M. (2022). k-contraction: Theory and applications. *Automatica*, 136, 110048.

Appendix A. PROOF OF PROPOSITION 8

For the proof of Proposition 8, we consider, without loss of generality, a matrix A in Kalman's canonical reachability decomposition, viz.

$$A = \begin{bmatrix} A_r & * \\ 0 & A_u \end{bmatrix}, \qquad B = \begin{bmatrix} B_r \\ 0 \end{bmatrix},$$

where (A_r, B_r) is a completely reachable pair, and A_r, A_u represent the reachable and unreachable dynamics, while the symbol $*$ denotes an arbitrary matrix whose value is irrelevant to the subsequent discussion.

We show first the implications (1) $\iff$ (2). Considering the matrix of the closed loop system $A_{cl} = (A + BK)$. After the application of a suitable permutation matrix, it can be decomposed in the following reachability canonical form:

$$\tilde{A}_{cl} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_r + B_r K & * \\ 0 & A_u \end{bmatrix}.$$

Following the same calculations as in Angeli et al. (2023), after the application of a suitable permutation matrix, the 2-additive compound of $(A + BK)$ can be written as

$$\bar{A}_{cl}^{[2]} = \begin{bmatrix} A_{11}^{[2]} & \tilde{B}_1 & 0 \\ \tilde{G}_1 & A_{11} \oplus A_{22} & \tilde{G}_2 \\ 0 & \tilde{B}_2 & A_{22}^{[2]} \end{bmatrix} = \begin{bmatrix} (A_r + B_r K)^{[2]} & * & 0 \\ 0 & (A_r + B_r K) \oplus A_u & * \\ 0 & 0 & A_u^{[2]} \end{bmatrix},$$

where (Angeli et al. (2023)):

$$\tilde{G}_1 = (I \otimes A_{21}) M_{n_1}$$
$$\tilde{B}_2 = L_{n_2}(I_{n_2} \otimes A_{21}) - L_{n_2}(A_{21} \otimes I_{n_2}) H_{n_2,n_1}$$

are identically zero since $A_{21} \equiv 0$. Therefore, it can be readily shown that the pair (A, B) is 2-contraction stabilizable if and only if the matrix $A_u^{[2]}$ is Hurwitz. This is indeed necessary, since $A_u^{[2]}$ is a block along the diagonal of the upper triangular matrix $\bar{A}_{cl}^{[2]}$. It is also sufficient, since by complete reachability of the pair (A_r, B_r), eigenvalues of $A_r + B_r K_r$ can be allocated arbitrarily to the left of any real abscissas, and in particular of $\min\{0, -\gamma\}$, where γ is the maximum real part of the eigenvalues of A_u, so that matrices $(A_r + B_r K_r)^{[2]}$ and $(A_r + B_r K_r) \oplus A_u$ are simultaneously stabilized.

We show now the implications (2) $\iff$ (3). The proof follows the same lines as the previous one by replacing the matrix $(A_r + B_r K)$ with $(I + B_r K) A_r$.

We wish to show now that (3) $\implies$ (1). Carrying out the following calculations

$$[(I + BK)\, A]^{[2]} = (A + BKA)^{[2]}$$

and defining the control matrix $\tilde{K} = KA$ we obtain the equality:

$$[(I + BK)A]^{[2]} = (A + B\tilde{K})^{[2]},$$

thus proving that the gain $\tilde{K}$ is suitable to achieve 2-contraction stability for the pair (A, B), which yields (1). Finally, for the implication (1) $\implies$ (3), if the matrix A is invertible, we have:

$$(A + BK)^{[2]} = \left(A + BKA^{-1}A\right)^{[2]} = \left[\left(I + B\bar{K}\right) A\right]^{[2]},$$

where $\bar{K}$ is a suitable gain to achieve 2-contraction stabilization by means of derivative feedback for the pair (A, B), thus proving (3).