

METRIC KINKS IN GENERAL RELATIVITY

by

Martin Francis Lynch Wall

A thesis submitted for the degree of Doctor of Philosophy of
the University of London and for the Diploma of Membership of
Imperial College

Theoretical Physics Group

Imperial College of Science & Technology

LONDON SW7

September 1983

CONTENTS

	<u>Page</u>
Abstract	1
Preface	3
Introduction	4
CHAPTER 1. THE MATHEMATICAL THEORY OF KINK METRICS	7
1.1 Kinks and their classification	7
1.2 Kinks on R^4	12
1.3 A family of kink metrics on R^4	21
CHAPTER 2. THE PHYSICAL PROPERTIES OF KINK METRICS	27
2.1 General results	27
2.2 A class of spherically-symmetric static kink metrics	32
2.3 Further results for the restricted family of kinks	38
CHAPTER 3. EXAMPLES OF KINK METRICS ON R^4	47
3.1 The energy conditions	47
3.2 Examples of one kink metrics	50
3.3 A one-parameter family of 2-kinks	53
3.4 A two-parameter family of 2-kinks	55
CHAPTER 4. KINKS ON $R \times S' \times S' \times S'$	57
4.1 The classification of kinks on $M = R \times S' \times S' \times S'$	57
4.2 The light-cone structure	60
4.3 Geometrical quantities associated to a family of kink metrics on $R \times S' \times S' \times S'$	61
4.4 Examples of kink metrics on $M = R \times S' \times S' \times S'$	63

ABSTRACT

In this thesis we investigate the homotopy properties of the space of Lorentz metrics on a manifold, and in particular its splitting into sectors, the metrics in one sector not being continuously deformable into those of another. These sectors are known as kinks and the metrics in them as kink metrics.

The first chapter deals with the reasons for and the classification of these sectors. We then choose as our base manifold R^4 ; by placing certain boundary conditions on the metrics, this manifold is effectively compactified to $R \times S^3$, and hence the space of Lorentz metrics splits up into sectors. We give a method of determining to which sector any given metric belongs.

The next two chapters deal with the physical properties of kink metrics. After proving some general results, we study in detail the restricted class of spherically-symmetric static kinks. Explicit examples are given in Chapter 3.

The final chapter is concerned with kink metrics on the manifold $R \times S^1 \times S^1 \times S^1$. We restrict our attention to a Z_2 family of kinks, for which we calculate the energy-momentum and Weyl tensors, these are found to have a particularly simple form. We also study the local properties of these metrics, emphasising the curious features of the underlying light-cone structure.

This thesis is dedicated to my mother

PREFACE

The work described in this thesis was carried out in the Theoretical Physics Group, Imperial College, between October 1979 and October 1982 under the supervision of Professor C.J. Isham. Except where otherwise stated, the work is original and has not been submitted before for a degree of this or any other university.

I should especially like to thank my supervisor, Chris Isham, for all the help he has given me over the past years, and for being a true friend. Thanks also go to my friends in the theoretical physics group for providing a pleasant environment in which to work.

A special thanks goes to my mother for providing me with much help and inspiration.

Finally, I wish to thank the Science Research Council for financial support.

INTRODUCTION

This thesis involves the study of certain classes of space-times which have come to be known as kinks.

The formal definition of kinks will be left to the next chapter: for now we shall give a brief discussion to give the term some meaning. Given a fixed four dimensional manifold M^4 we consider the space of all metrics with a lorentzian signature. It is found (how will be explained later) that this space splits up into a number of sectors, such that metrics in one sector cannot be deformed continuously into a metric in another. These classes of metrics, and by an abuse of language any representative of these classes, are known as kinks.

Before indicating the areas we will be investigating, let us first look at what we will not be studying. We shall not take the view that these metrics in some way represent fundamental particles. This view can be found in Misner et al, ref. (5). The second line of argument we will not follow is associated with the above; it is that certain kink sectors exhibit something akin to spin, and hence might represent spin half particles, ref. (3, 4, 10, 11). These lines of approach have never really been taken up after the initial papers. The main reasons are that they both involve considerable interpretational problems, and that other theories to explain the particle spectrum were being developed at that time. These included Yang-Mills, grand unified and super gravity theories. The other reason for not taking the above point of view is that they develop in a direction tangent to our approach, which is to consider any representative of a kink sector as representing a space-time in total, and the theory as representing all possible space-times on a fixed manifold.

The investigation which follows is purely classical. We

will not make any attempt to incorporate our discussion into a quantum picture. This step would be very desirable, but it is beset with problems. As well as the normal problems of quantizing gravity, things are made considerably worse by the intrinsic way space and time are intertwined, in all sectors other than that which contains the flat metric on the manifold. These points can be found in Misner et al, ref. (5), and have been elucidated in a private communication from Dr. Gibbons, for which I am most grateful.

The line of thought we develop is however motivated by the possibility of forming a quantum picture. This comes about in two ways. The first is that kinks only occur in the lorentzian sector, and hence would be lost in any attempt at Riemmanization. We believe that a full picture of quantum gravity will involve lorentzian and not Riemmanian metrics, hence the kink sectors will be a part of the quantum theory of gravity. That we are motivated by a quantum picture can also be seen in the way we set up the problem. We consider the space of lorentzian metrics on a fixed manifold. These metrics will be well defined over the whole of the manifold. We will not try to extend the manifold, so the metrics will not necessarily be geodesically complete.

We now proceed with an outline of the contents of the thesis. It begins with an investigation of the classification scheme for kink sectors. This is not new and can be found in Williams et al, ref. (3). We then fix our venue to be the manifold R^4 , with boundary conditions on the metrics so that R^4 is effectively (for topological purposes) compactified to $R \times S^3$. On this manifold we give a method of determining to which sector any given metric belongs; this step is new and cannot be found anywhere in the literature. We then look at a family

of spherically symmetric static kinks on R^4 .

The second and third chapters constitute a detailed study of the physical properties of kink metrics on R^4 . The general results at the beginning of Chapter 2 are completely new, while for the rest some of the results have been recorded in certain papers, refs. (3, 4, 5, 7, 8, 9), while others have not, and in the main these extend the known results.

Kinks on R^4 are classified by a single integer, this does not hold for other parallelizable manifolds, where the classification is usually $Z \oplus Z_m^m$ where m depends on the manifold. Nowhere has there been a real study of the Z_2 factor. In the fourth chapter we make some remedy for this by choosing $M^4 = R \times S^1 \times S^1 \times S^1$. Here it is found that the kinks are classified by $Z \oplus Z_2 \oplus Z_2 \oplus Z_2$. The approach we take is to write down a family of metrics, and then prove that it is a kink family labelled by Z_2 , i.e. they split up into two sectors, metrics in one not being continuously deformable into metrics in the other. Using this family of metrics, we proceed along the same lines as the case where $M = R^4$, to form a physical picture of what the effect of the Z_2 factor in the classification is. The most interesting phenomenon is that the metrics in one sector are time orientable, while the metrics in the other are not.

To finish, we again state the main aims of this investigation. These are to build a classical picture of kink metrics which clarifies much of the past confusion. That extends most of the old results and adds some new ones, in the hope of obtaining a model that will make the horrendous problem of their quantization more tractable.

CHAPTER 1

THE MATHEMATICAL THEORY OF KINK METRICS

1.1 Kinks and their Classification

When studying the space of Riemannian metrics on a manifold, one finds that there are no obstructions to the deformation of one metric into another. However, in the case of lorentz metrics the situation is radically different: this space decomposes into sectors, the metrics in one sector not being continuously deformable into those in another.

We will consider only parallelizable four-dimensional manifolds. The frame bundle $L(M^4)$ of these manifolds admits cross-sections, which are maps of the form $C: M^4 \rightarrow L(M^4)$ such that $\pi \circ C = I_{M^4}$ where π is the projection map in the fibre bundle.

$$GL(4, R) \rightarrow L(M^4)$$

$$\downarrow \pi$$

$$M^4$$

(1.1.1)

For such bundles both the tangent and cotangent bundles are trivial, as are any bundles associated with them.

The bundle of lorentz metric tensors $T_{2L}(M^4)$ is of particular interest. It is defined as follows. Let $S_{4,3}$ be the space of all real 4x4 non-singular symmetric matrices of signature -2. The group $GL(4, R)$ acts on this space by the congruence $S \rightarrow A^T S A$ where $A \in GL(4, R)$ and $S \in S_{4,3}$. $T_{2L}(M^4)$ is associated to the cotangent bundle $T^*(M^4)$ with the given action of $GL(4, R)$ on the fibres.

Any $g \in T_{2L}(M^4)$ can be represented in the following

way, given that M^4 is parallelizable, we can choose a global basis $\{W^a\}$ of the cotangent bundle, $T^*(M^4)$. With this basis g can be written as $g = \sum_{a,b} g_{ab} W^a \otimes W^b$ where g_{ab} is an element of $S_{4,3}$. As $T_{2L}(M^4)$ is associated with $T^*(M^4)$ from above the bundle is trivial. The cross-sections are the lorentz metric tensor fields on M^4 and it is these we want to classify up to a homotopy.

Two cross-sections $C, C' : M^4 \rightarrow T_{2L}(M^4)$ are said to be homotopic if there exists a one-parameter family of continuous cross-sections $C_s : M^4 \rightarrow T_{2L}(M^4)$ such that $C_0 = C$ and $C_1 = C'$. We shall denote the homotopy classes of these cross-sections by $[M^4, T_{2L}(M^4)]$ and if there are more than one of these M^4 is said to admit kinks.

The homotopy classes we are using are pointed, as only base point preserving maps are being considered. These are maps for which a fixed point of the manifold is always taken into the same element of $S_{4,3}$. For the cases in which we are interested, base-point preserving maps can arise in two ways. The first is where the manifold has a compact spatial section, (for example $R \times S^1 \times S^1 \times S^1$). An arbitrary point on the manifold is taken to be the base point, and this point is always mapped into $G \in S_{4,3}$ where $G = d \cdot a(1, -1, -1, -1)$. For the second case the basic manifold is R^4 , but the boundary conditions on the metrics essentially compactify this to $R \times S^3$; the sphere at spatial infinity is identified to a point which we will select as our base point again mapping it into G .

$$\begin{array}{ccc} \text{As the bundle} & S_{4,3} & \longrightarrow T_{2L}(M^4) \\ & & \downarrow \pi \\ & & M^4 \end{array} \quad (1.1.2)$$

is trivial, it is equivalent to the product bundle

$$\begin{array}{ccc} S_{4,3} & \xrightarrow{i} & M^4 \times S_{4,3} \\ & & \downarrow p \\ & & M^4 \end{array} \quad (1.1.3)$$

where i is the inclusion map and p is the projection map for this product bundle. It is a property of trivial bundles that the cross-sections are in one-one correspondence with continuous maps of M^4 into $S_{4,3}$. We will use the notation

$[M^4, S_{4,3}]$ for the homotopy classes of these maps.

So the classification scheme can be reduced to

$$[M^4, T_{2L}(M^4)] = [M^4, S_{4,3}]. \quad (1.1.4)$$

This result can be simplified further by noting that the space $S_{4,3}$ has the same homotopy type as RP^3 . The proof follows from the fact that $S_{4,3}$ can be viewed as the total space of the fibre-bundle

$$\begin{array}{ccc} C_7 & \longrightarrow & S_{4,3} \\ & & \downarrow p \\ & & RP^3 \end{array} \quad (1.1.5)$$

where C_7 is a seven-dimensional cell^{and} as such is contractible, hence maps of M^4 into RP^3 can be lifted to maps into $S_{4,3}$ and this lifting preserves homotopy; so we have

$$[M^4, T_{2L}(M^4)] = [M^4, S_{4,3}] = [M^4, RP^3]. \quad (1.1.6)$$

We shall restrict our attention to the case where there is a fibration of M^4 by a system of time-lines. This means that it is a vector bundle whose fibre is R and where the base space is a 3-dimensional manifold M . For such a fibration, M is a strong deformation retract of M^4 , giving us $[M^4, RP^3] = [M, RP^3]$

A simple case of this fibration is where $M^4 = M \times R$. M will always be a closed, connected and orientable manifold.

Thus, to classify kinks, the space $[M, R P^3]$ needs to be calculated. This calculation is discussed in detail in ref. (3), where it is shown that $[M, R P^3] = Z \oplus Z_2^L$ where Z_2^L is $Z_2 \oplus Z_2 \oplus \dots Z_2$ (L times). We shall now give a brief discussion of this result. L is determined by $H^1(M, Z_2)$ which varies from manifold to manifold, but will always be of the form $H^1(M, Z_2) = Z_2^k$ where either $L = k$ or $L = k-1$, depending on the nature of the manifold. To distinguish the two possible values of L , we first define the degree of a map $\varphi: M \rightarrow R P^3$. Intuitively it is the number of times the function wraps M around $R P^3$. To be more precise we note that for a compact, connected and orientable three manifold M , the third integral singular homology group

$$H_3(M) = Z \text{ so we have that } H_3(M) = H_3(R P^3) = Z$$

A map $\varphi: M \rightarrow R P^3$ induces a homomorphism

$\varphi_*: H_3(M) \rightarrow H_3(R P^3)$ which must correspond to multiplication by an integer n , known as the degree of the map. If M admits a map into $R P^3$ of degree one then it is known as a manifold of type 1; otherwise it is said to be of type 2. With these definitions the values of L are

$$L = \begin{cases} k-1 & \text{if the manifold is of type 1} \\ k & \text{if the manifold is of type 2} \end{cases} \quad (1.1.7)$$

Finally, let us consider a few examples of manifolds which would be of interest in general relativity:

Table 1

M	TYPE	$H^1(M, \mathbb{Z}_2)$	$[M, \mathbb{RP}^3]$
S^3	2	0	$\mathbb{Z}$
$\mathbb{RP}^3$	1	$\mathbb{Z}_2$	$\mathbb{Z}$
$S^2 \times S^1$	2	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$
$S^1 \times S^1 \times S^1$	2	$\mathbb{Z}_2^3$	$\mathbb{Z} \oplus \mathbb{Z}_2^3$

1.2 Kinks on R^4

When studying the topological properties of the space of lorentz metrics on R^4 one finds that there are no obstructions to the deformation of any metric into another. However, if we restrict our attention to asymptotically flat metrics the situation is very different. The effect of this constraint is to identify the sphere at spatial infinity to a point which leaves us considering the manifold $R \times S^3$. Using the results from the last section, we find that the space of asymptotically flat lorentz metrics decomposes into sectors which are labelled by

$$[R \times S^3, RP^3] = [S^3, RP^3] = \pi_3(RP^3) = \mathbb{Z}$$

and hence giving rise to a countable infinity of kinks. For the rest of the chapter the above shall be referred to as metrics.

We will now develop a method to determine the kink sector in which a given metric lies. This procedure splits up into a number of steps. The first is to associate with the metric a map of $R \times S^3$ into $S_{4,3}$ in the following way. As any metric we consider lies on R^4 , the coordinate system (t, x, y, z) is well defined over the whole manifold. In this coordinate system we can employ the globally defined frame

$$\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

which at each point P spans the tangent space of the manifold. This leads to a dual basis of the cotangent space (dt, dx, dy, dz) in terms of which the metric can be written at each point as $g(P) = \sum_{\mu, \nu} g_{\mu\nu}(P) dx^\mu \otimes dx^\nu$ where the matrix $\|g_{\mu\nu}\|_P \in S_{4,3}$. This gives the desired map

$$\|g_{\mu\nu}\| : R \times S^3 \longrightarrow S_{4,3}$$

$$P \rightsquigarrow \|g_{\mu\nu}\|$$

(1.2.1)

and as we are only considering asymptotically flat metrics we have the constraint

$$\|g_{\mu\nu}\| \rightarrow \|g_{\mu\nu}\| = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{as } r^2 = x^2 + y^2 + z^2 \rightarrow \infty \quad (1.2.2)$$

The map $\|g_{\mu\nu}\| : R \times S^3 \rightarrow S_{4,3}$ can be viewed as a one-parameter family of maps $\|g_{\mu\nu}\|_t : S^3 \rightarrow S_{4,3}$ and as the development of these maps in t is continuous they will all lie in the same homotopy class. Thus it is how S^3 maps into $S_{4,3}$ which determines to which sector the map

$\|g_{\mu\nu}\| : R \times S^3 \rightarrow S_{4,3}$ belongs, so the topological information about this map is carried by the map

$$\|g_{\mu\nu}\|_t : S^3 \rightarrow S_{4,3} \quad \text{evaluated at any } t.$$

From now on we will work with the map corresponding to $t = 0$ which we will denote by $\|g_{\mu\nu}\|^0$.

The next stage involves the study of certain properties of the space $S_{4,3}$. The classic works which cover this are Steenrod and Chevalley, refs. (1, 2). We have already mentioned the projection

$$\begin{array}{ccc} R^7 & \longrightarrow & S_{4,3} \\ & & \downarrow P \\ & & S_{4,3} \cap S_{0,4} = \mathbb{R}P^3 \end{array} \quad (1.2.3)$$

We now want to know how any given element of $S_{4,3}$ is projected onto $S_{4,3} \cap S_{0,4}$. Any $\tau \in S_{4,3}$ can be written uniquely in the form $\tau = G\alpha$ where

$G \in S_{4,3} \wedge SO_4$ and $\alpha \in S_{4,0}$, $S_{4,0}$ being the space of symmetric, positive definite matrices. The desired projection is then $P(\tau) = P(G\alpha) = G$ and the matrix is derived from τ in a number of steps.

- 1) First form the symmetric, positive definite matrix,

$$\alpha_1 = \tau^T \tau$$

- 2) Then diagonalize this matrix to give $\delta_1 = q \alpha_1 q^{-1}$

- 3) Then define δ by $\delta = \sqrt{\delta_1}$.

- 4) Finally we define $\alpha = q^{-1} \delta q$ and note that this satisfies the requirement $\alpha \in S_{4,0}$

Once we have calculated α , G follows easily as

$G = \tau \alpha^{-1}$. We will also need to use the result that any

$G \in S_{4,3} \wedge SO_4$ can be written in the form

$G = R \|g_{\mu\nu}\| R^T$ where $R \in SO_4$. We shall now

return to consider the map

$$\|g_{\mu\nu}\|^0: S^3 \rightarrow S_{4,3}$$

$$p \mapsto \tau p = \|g_{\mu\nu}\|_p^0 \quad (1.2.4)$$

From the first section we know that to work out the homotopy sector in which this map lies we have to investigate the associated map

$$G: S^3 \rightarrow S_{4,3} \wedge SO_4 = RP^3 \quad (1.2.5)$$

$$p \mapsto P(\tau p) = G_p$$

We shall now analyse the topological properties of this map.

The space RP^3 can be viewed as a quotient space

$$RP^3 = SO_4 / O_3 \quad \text{so that} \quad G: S^3 \rightarrow SO_4 / O_3$$

We factorize this map to the form

$$\begin{array}{ccccc} S^3 & \xrightarrow{R} & SO_4 & \xleftarrow{\quad} & O_3 \\ & \searrow G & \downarrow q & & \\ & & SO_4 / O_3 & & \end{array} \quad (1.2.6)$$

using the commutivity of the above diagram, we have that

$G = R\eta$. We then apply a short homotopy exact sequence to

$$\begin{array}{ccc}
 S^3 & \xrightarrow{\quad} & SO_4 \\
 & \searrow G & \downarrow \eta \\
 & & SO_4/O_3
 \end{array} \quad (1.2.7)$$

to give

$$\begin{array}{ccccc}
 G_* : & \mathbb{Z} & \xrightarrow{R_*} & \pi_3(SO_4) & \xrightarrow{\eta_*} & \pi_3(SO_4/O_3) \\
 & \cap & & & & \\
 & \pi_3(S^3) & & (n_1, n_2) & & (n_1 - n_2)
 \end{array} \quad (1.2.8)$$

where η is the generator of $\pi_3(S^3)$ and $G_* = R_* \eta_*$

So the degree of G is $(n_1 - n_2)$. To work out n_1, n_2 we proceed as follows from the map

$$\begin{array}{ccc}
 G : S^3 & \rightarrow & S_{4,3} \wedge SO_4 \\
 p & \leadsto & G_p
 \end{array} \quad (1.2.9)$$

and using the fact that at each point $G_p = R_p \parallel \eta_p \parallel R_p^T$

where $R_p \in SO_4$. We obtain the map

$$\begin{array}{ccc}
 R : S^3 & \rightarrow & SO_4 \\
 p & \leadsto & R_p
 \end{array} \quad (1.2.10)$$

To ascertain n_1, n_2 we will make use of the isomorphism

$$SO_4 \cong SU_L(2) \times SU_R(2) / \mathbb{Z}_2 \quad (1.2.11)$$

n_1 is the degree of the map of S^3 into $SU_L(2) \cong S^3$ and

n_2 is the degree of the map of S^3 into $SU_R(2) \cong S^3$

As we have started from a given metric we know explicitly the map $R : S^3 \rightarrow SO_4$. We need to calculate from this the maps $\varphi : S^3 \rightarrow SU_L(2)$ and $\psi : S^3 \rightarrow SU_R(2)$. We will

deal with this in the following way. We choose two arbitrary maps $\varphi: S^3 \rightarrow SU_2(2)$ and $\psi: S^3 \rightarrow SU_2(2)$ and calculate the associated map $\hat{R}: S^3 \rightarrow SO_4$ and then compare with the given map $R: S^3 \rightarrow SO_4$ to work out the maps φ and ψ .

The effect of $\hat{R}_p \in SO_4$ at any point is to rotate the four vector

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \xrightarrow{\hat{R}_p} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{into another four vector} \quad (1.2.12)$$

We first represent this four vector with a complex 2×2 matrix $(x_0 \mathbb{I} + i \tau_i x_i)$ where the τ_i are the Pauli matrices ($i = 1, 2, 3$). Then associated to $\hat{R}_p$ is a matrix

$$M_R = A (x_0 \mathbb{I} + i \tau_i x_i) B^{-1} \quad \text{where } A, B \in SU(2)$$

Noting that (A, B) and $(-A, -B)$ are associated with the same rotation matrix, we have the desired isomorphism

$SO_4 = SU_2(2) \times SU_2(2) / \mathbb{Z}_2$ Now any $SU(2)$ matrix can be written in the following form

$$\begin{bmatrix} \alpha_0 - i \alpha_3 & \alpha_2 - i \alpha_1 \\ -\alpha_2 - i \alpha_1 & \alpha_0 + i \alpha_3 \end{bmatrix} \quad (1.2.13)$$

where

$$\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$$

$$M_P = \begin{bmatrix} M & N \\ 0 & P \end{bmatrix} = \begin{bmatrix} \varphi_0 - i\varphi_3 & \varphi_2 - i\varphi_1 \\ -\varphi_2 - i\varphi_1 & \varphi_0 + i\varphi_3 \end{bmatrix} \begin{bmatrix} x_0 + ix_3 & -x_2 + ix_1 \\ x_2 + ix_1 & x_0 - ix_3 \end{bmatrix} \begin{bmatrix} \psi_0 + i\psi_3 & -\psi_2 + i\psi_1 \\ -\psi_2 + i\psi_1 & \psi_0 - i\psi_3 \end{bmatrix} \quad (1.2.14)$$

where $(\varphi_0, \varphi_1, \varphi_2, \varphi_3)$ and $(\psi_0, \psi_1, \psi_2, \psi_3)$ are the values of the map at the point P . From this we find that at each point P

$$\begin{aligned} x_0 &\rightarrow \operatorname{Re} M = \sum a_{0i} x_i \\ x_1 &\rightarrow \operatorname{Re} 0 = \sum a_{1i} x_i \\ x_2 &\rightarrow \operatorname{Im} 0 = \sum a_{2i} x_i \\ x_3 &\rightarrow \operatorname{Im} M = \sum a_{3i} x_i \end{aligned} \quad (1.2.15)$$

Reading off the coefficients a_{ij} gives us the rotation matrix

$$\begin{bmatrix} a_{00} & a_{10} & a_{20} & a_{30} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \quad (1.2.16)$$

where

$$a_{00} = [\varphi_0 \psi_0 + \varphi_1 \psi_1 + \varphi_2 \psi_2 + \varphi_3 \psi_3]$$

$$a_{01} = [-\varphi_0 \psi_1 + \varphi_3 \psi_2 - \varphi_2 \psi_3 + \varphi_1 \psi_0]$$

$$a_{02} = [-\varphi_0 \psi_2 - \varphi_2 \psi_1 + \varphi_2 \psi_0 + \varphi_1 \psi_3]$$

$$a_{03} = [-\varphi_0 \psi_3 - \varphi_3 \psi_0 + \varphi_2 \psi_1 - \varphi_1 \psi_2]$$

$$a_{10} = [-\varphi_2 \psi_3 - \varphi_1 \psi_0 + \varphi_0 \psi_1 + \varphi_3 \psi_2]$$

$$a_{11} = [\varphi_0 \psi_0 + \varphi_1 \psi_1 - \varphi_2 \psi_2 - \varphi_3 \psi_3]$$

$$a_{12} = [\varphi_2 \psi_1 + \varphi_1 \psi_2 + \varphi_0 \psi_3 + \varphi_3 \psi_0]$$

$$a_{13} = [-\varphi_2 \psi_0 + \varphi_1 \psi_3 - \varphi_0 \psi_2 + \varphi_3 \psi_1]$$

$$a_{20} = [-\varphi_2 \psi_0 + \varphi_1 \psi_3 + \varphi_0 \psi_2 - \varphi_3 \psi_1]$$

$$a_{21} = [\varphi_2 \psi_1 + \varphi_1 \psi_2 - \varphi_0 \psi_3 - \varphi_3 \psi_0]$$

$$a_{22} = [\varphi_0 \psi_0 - \varphi_1 \psi_1 + \varphi_2 \psi_2 - \varphi_3 \psi_3]$$

$$a_{23} = [\varphi_2 \psi_0 + \varphi_1 \psi_3 + \varphi_0 \psi_2 + \varphi_3 \psi_1]$$

$$a_{30} = [-\varphi_3 \psi_0 + \varphi_0 \psi_3 + \varphi_1 \psi_2 - \varphi_2 \psi_1]$$

$$a_{31} = [\varphi_0 \psi_1 + \varphi_3 \psi_2 + \varphi_2 \psi_0 + \varphi_1 \psi_3]$$

$$a_{32} = [\varphi_3 \psi_2 - \varphi_0 \psi_1 + \varphi_2 \psi_3 - \varphi_1 \psi_0]$$

$$a_{33} = [\varphi_0 \psi_0 - \varphi_1 \psi_1 - \varphi_2 \psi_2 + \varphi_3 \psi_3]$$

By letting $(\varphi_0, \varphi_1, \varphi_2, \varphi_3)$ and $(\psi_0, \psi_1, \psi_2, \psi_3)$ be the components of the maps $\varphi: S^3 \rightarrow SU_2(2)$ and

$\psi: S^3 \rightarrow SU_2(2)$ rather than the values of the map at a point p the matrix above can be viewed as the map

$\hat{R}: S^3 \rightarrow SO_4$ By comparing with ^{the} given map $R: S^3 \rightarrow SO_4$

and using $\sum \varphi_i \varphi_i = \sum \psi_i \psi_i = 1$ we can work out

the explicit maps $\varphi: S^3 \rightarrow SU_2(2)$ and $\psi: S^3 \rightarrow SU_2(2)$

The degrees of these maps are n_1 and n_2 in turn. The

difference $n_1 - n_2$ will label the homotopy sectors of the maps

$G: S^3 \rightarrow SO_4 \cap S_{4,3}$ and hence the kink sector.

The matrix corresponding to the maps φ and ψ is

$$\begin{bmatrix} -1 + 2\alpha_0^2 & 2\alpha_0\alpha_1 & 2\alpha_0\alpha_2 & 2\alpha_0\alpha_3 \\ 2\alpha_0\alpha_1 & -1 + 2\alpha_1^2 & 2\alpha_1\alpha_2 & 2\alpha_1\alpha_3 \\ 2\alpha_0\alpha_2 & 2\alpha_1\alpha_2 & -1 + 2\alpha_2^2 & 2\alpha_2\alpha_3 \\ 2\alpha_0\alpha_3 & 2\alpha_1\alpha_3 & 2\alpha_2\alpha_3 & -1 + 2\alpha_3^2 \end{bmatrix} \quad (1.2.18)$$

where

$$\alpha_1 = (\varphi_0\psi_0 + \varphi_1\psi_1 + \varphi_2\psi_2 + \varphi_3\psi_3)$$

$$\alpha_2 = (-\varphi_2\psi_3 - \varphi_1\psi_0 + \varphi_0\psi_1 + \varphi_3\psi_2)$$

$$\alpha_3 = (-\varphi_2\psi_0 + \varphi_1\psi_2 + \varphi_0\psi_2 - \varphi_3\psi_1)$$

$$\alpha_4 = (-\varphi_3\psi_0 + \varphi_0\psi_3 + \varphi_2\psi_1 - \varphi_1\psi_2) \quad (1.2.19)$$

We note that $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 = 1$ so α is itself a map of S^3 into S^3 .

We now consider what might seem a special case of the

above, where the matrix B in $M_R = A(X_0, 1 + i\tau \cdot X)B^{-1}$ is the identity map at every point p , i.e. $\psi_0 = 1$ and $\psi_1 = \psi_2 = \psi_3 = 0$. Then we obtain the map

$$R: S^3 \longrightarrow SO(4) \\ p \longmapsto R_p$$

where

$$R_p = \begin{bmatrix} \varphi_0 & \varphi_1 & \varphi_2 & \varphi_3 \\ -\varphi_1 & \varphi_0 & \varphi_3 & -\varphi_2 \\ -\varphi_2 & -\varphi_3 & \varphi_0 & \varphi_1 \\ -\varphi_3 & \varphi_2 & \varphi_1 & \varphi_0 \end{bmatrix} \quad (1.2.20)$$

In this case $n_2 = 0$ and the kink number is the degree of the map $\varphi: S^3 \rightarrow SU_L(2) \simeq S^3$. The map $G: S^3 \rightarrow SO(4) \wedge S_{4,3}$ corresponding to the above map is

$$G: S^3 \longrightarrow SO(4) \wedge S_{4,3} \\ p \longmapsto G_p = R_p \cap R_p^T$$

Explicit matrix multiplication gives

$$G_p = \begin{bmatrix} -1 + 2\varphi_0^2 & 2\varphi_0\varphi_1 & 2\varphi_0\varphi_2 & 2\varphi_0\varphi_3 \\ 2\varphi_0\varphi_1 & -1 + 2\varphi_1^2 & 2\varphi_1\varphi_2 & 2\varphi_1\varphi_3 \\ 2\varphi_0\varphi_2 & 2\varphi_1\varphi_2 & -1 + 2\varphi_2^2 & 2\varphi_2\varphi_3 \\ 2\varphi_0\varphi_3 & 2\varphi_1\varphi_3 & 2\varphi_2\varphi_3 & -1 + 2\varphi_3^2 \end{bmatrix} \quad (1.2.21)$$

This is not actually a special case, as the most general G as in eq. (1.2.21) can be written in the above form using α instead of φ . So G can be written as

$$\begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ -\alpha_1 & \alpha_0 & \alpha_3 & -\alpha_2 \\ -\alpha_2 & -\alpha_3 & \alpha_0 & \alpha_1 \\ -\alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_0 & \alpha_1 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix} \quad (1.2.22)$$

leaving the kink number as the degree of the map $\alpha: S^3 \rightarrow S^3$
the degree of α being the difference of degrees of

$$\varphi: S^3 \rightarrow S^3 \quad \text{and} \quad \psi: S^3 \rightarrow S^3$$

1.3 A Family of Kink Metrics on R^4

We now investigate a family of kinks on R^4 where

$$g_{\mu\nu} = -\delta_{\mu\nu} + 2\varphi_\mu\varphi_\nu \quad \mu, \nu = 0, 1, 2, 3$$

$$\varphi_0 = \cos \alpha(r) \quad \text{and} \quad \varphi_i = \frac{x_i}{r} \sin \alpha(r)$$

(1.3.1)

This family of metrics was discussed from a mathematical point of view in ref. (3), and used as a model for space time in ref. (8).

The choice of $\alpha(r)$ gives the precise form of the kink metric. We have the following constraints on $\alpha(r)$, that

$$\alpha(0) = \pi \quad \text{and} \quad \alpha(r) \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty$$

The map φ is initially a map $\varphi: R^3 \rightarrow S^3$, the condition that $\alpha(r) \rightarrow 0$ as $r \rightarrow \infty$ effectively compactifies R^3 to S^3 giving us the map $\varphi: S^3 \rightarrow S^3$. The constraint

that $\mathcal{L}(0) = \pi$ ensures that the map $\varphi: S^3 \rightarrow S^3$ is a degree one map of S^3 onto itself.

We shall analyse this family of kinks. Setting

$$\tau = ||g_{\mu\nu}|| \quad \text{we first have to calculate } G. \quad \text{The given } \tau \text{ can be written in the following way, } \tau = P \eta P^T$$

where $\eta = ||\eta_{\mu\nu}||$ and $P = \begin{bmatrix} \varphi_0 & \varphi_1 & \varphi_2 & \varphi_3 \\ -\varphi_1 & \varphi_0 & \varphi_3 & -\varphi_2 \\ -\varphi_2 & -\varphi_3 & \varphi_0 & \varphi_1 \\ -\varphi_3 & \varphi_2 & -\varphi_1 & \varphi_0 \end{bmatrix} \quad (1.3.2)$

It is possible to check that $P P^T = 1$ so $P \in SO_4$. To find G we need to compute α . We first form α .

$$\alpha = \tau^k \tau = P \eta P^T P \eta P^T = P \eta \eta P^T = P P^T = 1$$

So $\alpha = 1$ and it follows that $\alpha = 1$, giving us $G = \tau \alpha^{-1} = \tau$. This result could have been seen directly as τ is an element of $SO_4 \wedge S_{4,3}$. As G is already in the form

$$G = P \eta P^T \quad \text{we can immediately read off the map}$$

$P: S^3 \rightarrow SO_4$. Using the results of the last section this corresponds to the map $\varphi: S^3 \rightarrow SU_L(2)$ and the trivial map of S^3 into $SU_R(2)$. So the kink number is simply the degree of the map $\varphi: S^3 \rightarrow S^3$, which, from the definition of the map, is one. So the above metrics are a family of one kink metrics.

We now show that $\tau^N = ||g_{\mu\nu}||^N = P^N \eta P^{N^T}$ corresponds to a family of N-kink metrics. The first step is to prove that

$$\rho^{\sim} = \begin{bmatrix} \tilde{\varphi}_0 & \tilde{\varphi}_1 & \tilde{\varphi}_2 & \tilde{\varphi}_3 \\ \tilde{\varphi}_1 & \tilde{\varphi}_0 & \tilde{\varphi}_3 & -\tilde{\varphi}_2 \\ -\tilde{\varphi}_2 & -\tilde{\varphi}_3 & \tilde{\varphi}_0 & \tilde{\varphi}_1 \\ -\tilde{\varphi}_3 & \tilde{\varphi}_2 & -\tilde{\varphi}_1 & \tilde{\varphi}_0 \end{bmatrix}$$

(1.3.3)

where $\tilde{\varphi}_0 = \cos N\alpha$ and $\varphi_i = \frac{x_i}{r} \sin N\alpha$. The proof is as follows. Given

$$P = \begin{bmatrix} \cos \alpha & \frac{x}{r} \sin \alpha & \frac{y}{r} \sin \alpha & \frac{z}{r} \sin \alpha \\ -\frac{x}{r} \sin \alpha & \cos \alpha & \frac{z}{r} \sin \alpha & -\frac{y}{r} \sin \alpha \\ -\frac{y}{r} \sin \alpha & -\frac{z}{r} \sin \alpha & \cos \alpha & \frac{x}{r} \sin \alpha \\ -\frac{z}{r} \sin \alpha & \frac{y}{r} \sin \alpha & -\frac{x}{r} \sin \alpha & \cos \alpha \end{bmatrix} \quad (1.3.4)$$

we have by matrix multiplication

$$\begin{bmatrix} \cos 2\alpha & \frac{x}{r} \sin 2\alpha & \frac{y}{r} \sin 2\alpha & \frac{z}{r} \sin 2\alpha \\ -\frac{x}{r} \sin 2\alpha & \cos 2\alpha & \frac{z}{r} \sin 2\alpha & -\frac{y}{r} \sin 2\alpha \\ -\frac{y}{r} \sin 2\alpha & -\frac{z}{r} \sin 2\alpha & \cos 2\alpha & \frac{x}{r} \sin 2\alpha \\ -\frac{z}{r} \sin 2\alpha & \frac{y}{r} \sin 2\alpha & -\frac{x}{r} \sin 2\alpha & \cos 2\alpha \end{bmatrix} \quad (1.3.5)$$

noting that if

$$P^m = \begin{bmatrix} \cos m\alpha & \frac{x}{r} \sin m\alpha & \frac{y}{r} \sin m\alpha & \frac{z}{r} \sin m\alpha \\ -\frac{x}{r} \sin m\alpha & \cos m\alpha & \frac{z}{r} \sin m\alpha & -\frac{y}{r} \sin m\alpha \\ -\frac{y}{r} \sin m\alpha & -\frac{z}{r} \sin m\alpha & \cos m\alpha & \frac{x}{r} \sin m\alpha \\ -\frac{z}{r} \sin m\alpha & \frac{y}{r} \sin m\alpha & -\frac{x}{r} \sin m\alpha & \cos m\alpha \end{bmatrix} \quad (1.3.6)$$

Then it follows from matrix multiplication

$$P^{m+1} = \begin{bmatrix} \cos (m+1)\alpha & \frac{x}{r} \sin (m+1)\alpha & \frac{y}{r} \sin (m+1)\alpha & \frac{z}{r} \sin (m+1)\alpha \\ -\frac{x}{r} \sin (m+1)\alpha & \cos (m+1)\alpha & \frac{z}{r} \sin (m+1)\alpha & -\frac{y}{r} \sin (m+1)\alpha \\ -\frac{y}{r} \sin (m+1)\alpha & -\frac{z}{r} \sin (m+1)\alpha & \cos (m+1)\alpha & \frac{x}{r} \sin (m+1)\alpha \\ -\frac{z}{r} \sin (m+1)\alpha & \frac{y}{r} \sin (m+1)\alpha & -\frac{x}{r} \sin (m+1)\alpha & \cos (m+1)\alpha \end{bmatrix} \quad (1.3.7)$$

So it follows by induction that

$$P^N = \begin{bmatrix} \cos N\alpha & \frac{x}{r} \sin N\alpha & \frac{y}{r} \sin N\alpha & \frac{z}{r} \sin N\alpha \\ -\frac{x}{r} \sin N\alpha & \cos N\alpha & \frac{z}{r} \sin N\alpha & -\frac{y}{r} \sin N\alpha \\ -\frac{y}{r} \sin N\alpha & -\frac{z}{r} \sin N\alpha & \cos N\alpha & \frac{x}{r} \sin N\alpha \\ -\frac{z}{r} \sin N\alpha & \frac{y}{r} \sin N\alpha & -\frac{x}{r} \sin N\alpha & \cos N\alpha \end{bmatrix} \quad (1.3.8)$$

Following the same line of argument used for the $N = 1$ case, we find that the kink number is the degree of the map

$$\tilde{\varphi} : S^3 \rightarrow S^3 \quad \text{From the definition of } \tilde{\varphi} \text{ we find that the}$$

degree of this map is N . So we have a family of N -kink metrics.

We shall end this section by deriving a result which will be used in a following chapter. Given the two maps $\varphi : S^3 \rightarrow S^3$

$$\text{where } \varphi_0 = \cos \alpha, \varphi_i = \frac{x_i}{r} \sin \alpha \text{ and } \psi : S^3 \rightarrow S^3$$

$$\text{where } \psi_0 = \cos \beta, \psi_i = \frac{x_i}{r} \sin \beta \text{ then the metric associated}$$

$$\text{with these maps is } \tau = p_\alpha p_\beta \eta p^\alpha p^\beta \text{ where}$$

$$p_\alpha = \begin{bmatrix} \varphi_0 & \varphi_1 & \varphi_2 & \varphi_3 \\ -\varphi_1 & \varphi_0 & \varphi_3 & -\varphi_2 \\ -\varphi_2 & -\varphi_3 & \varphi_0 & \varphi_1 \\ -\varphi_3 & \varphi_2 & -\varphi_1 & \varphi_0 \end{bmatrix} \quad p_\beta = \begin{bmatrix} \psi_0 & \psi_1 & \psi_2 & \psi_3 \\ -\psi_1 & \psi_0 & \psi_3 & -\psi_2 \\ -\psi_2 & -\psi_3 & \psi_0 & \psi_1 \\ -\psi_3 & \psi_2 & -\psi_1 & \psi_0 \end{bmatrix}$$

(1.3.9)

is a two-kink metric. The proof is as follows by matrix multiplication.

$$p_\alpha p_\beta = \begin{bmatrix} \cos(\alpha+\beta) & \frac{x}{r} \sin(\alpha+\beta) & \frac{y}{r} \sin(\alpha+\beta) & \frac{z}{r} \sin(\alpha+\beta) \\ -\frac{x}{r} \sin(\alpha+\beta) & \cos(\alpha+\beta) & \frac{z}{r} \sin(\alpha+\beta) & -\frac{y}{r} \sin(\alpha+\beta) \\ -\frac{y}{r} \sin(\alpha+\beta) & -\frac{z}{r} \sin(\alpha+\beta) & \cos(\alpha+\beta) & \frac{x}{r} \sin(\alpha+\beta) \\ -\frac{z}{r} \sin(\alpha+\beta) & \frac{y}{r} \sin(\alpha+\beta) & -\frac{x}{r} \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

(1.3.10)

We again follow the same line of argument previously used to obtain the result that the kink number is the degree of the

map $\Theta: S^3 \rightarrow S^3$ where $\Theta_0 = \cos(\alpha + \beta)$ $\Theta_i = \frac{x_i}{r} \sin(\alpha + \beta)$

The separate conditions on α and β are $\alpha(0) = \pi$ and

$\alpha(r) \rightarrow 0$ as $r \rightarrow \infty$ and $\beta(0) = \pi$ and $\beta(r) \rightarrow 0$
as $r \rightarrow \infty$ so the degree of the map is two, giving us a
two kink metric.

CHAPTER 2

THE PHYSICAL PROPERTIES OF KINK METRICS

2.1 General Results

The aim of this chapter is to build up a physical picture of kink metrics. We start by proving some general results before proceeding to the study of the restricted class of spherically-symmetric, ^{stationary} kink metrics which take the form

$$ds^2 = (-1 + 2 \cos^2 N \alpha) dt^2 + \sum_i 2 \frac{x^i}{r} \sin 2 N \alpha dx^i dt + \sum_{i,j} (-1 + 2 \frac{x^i x^j}{r^2} \sin^2 N \alpha) dx^i dx^j \quad (2.1.1)$$

Firstly, we shall prove that there are no ^{stationary} kink metrics on R^4 . From the above metric it is clear that ^{stationary} kinks exist, so the proof will consist of showing that the second requirement for a ^{static} metric, which is the vanishing of the cross-terms g_{0i} ($i = 1, 2, 3$) can not hold if the metric is in a non-trivial kink sector.

We will approach this by considering metrics where the cross-terms do vanish, in this case $\|g_{\mu\nu}\|$ can be written in the following way:

$$\|g_{\mu\nu}\| = \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & & & \\ 0 & & -G & \\ 0 & & & \end{bmatrix} \quad (2.1.2)$$

where G is a 3×3 matrix. We again emphasize that the indices μ, ν are matrix indices. The relationship between these

indices and coordinate indices is as follows. Given any metric g on a manifold, it can always be written in a neighbourhood of P in the form $g_P = \sum_b g_{ab}(P) W^a W^b$ where W^a_P and W^b_P are basis frames and $\|g_{ab}\|_P \in S_{4,3}$. For a parallelizable manifold, there exists a basis frame that is globally defined, so then it is possible to write g in the form $\sum_{a,b} g_{ab} W^a W^b$, where now $\|g_{ab}\|: M \rightarrow S_{4,3}$. In the case of pinned R^4 , the coordinate frame is such a globally defined frame, hence it is possible to write g as $g = \sum_{\mu,\nu} g_{\mu\nu} dx^\mu dx^\nu$ where $\|g_{\mu\nu}\|: M \rightarrow S_{4,3}$. This result is meaningful globally.

The first result we have to prove is that $g_{00} > 0$ everywhere. From 2.1.2 we find that $\det \|g_{\mu\nu}\| = -g_{00} \det G$. As g_{00} is a continuous function, if g_{00} changes sign it must go through zero somewhere, and at this point $\det \|g_{\mu\nu}\| = 0$ which is not admissible as we have demanded that $\|g_{\mu\nu}\|$ be non-singular. With this condition that $g_{00} > 0$ it follows that G is a 3×3 symmetric positive definite matrix, so 2.1.2 can be written as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & & & \\ 0 & & G & \\ 0 & & & \end{bmatrix} \quad (2.1.3)$$

where

$$\begin{bmatrix} g_{00} & 0 & 0 & 0 \\ 0 & & & \\ 0 & & G & \\ 0 & & & \end{bmatrix} \text{ is positive definite}$$

So the projection of $\|g_{\mu\nu}\| \in C^0(M, S_{4,3})$ onto $C^0(M, S_{4,3} \cap S_{0,4})$ will be $\|\gamma_{\mu\nu}\|$. Hence the metric $\sum_{\mu,\nu} g_{\mu\nu} dx^\mu dx^\nu$ is in the trivial kink sector. From this we have that no static metric can be a kink metric.

The existence of these cross-terms leads us to explore a certain feature of kink metrics, the tumbling of their light-cone. We take the general N-kink in the form $ds^2 = \sum_{\mu,\nu} g_{\mu\nu}(p) dx^\mu dx^\nu$. We note that though $\|g_{\mu\nu}\|_p \in S_{4,3}$, $\|g_{\mu\nu}\|$ is not restricted to lie in $S_{4,3} \cap S_{0,4}$. Following the standard method the light-cone structure is the solution to the equation $\sum_{\mu,\nu} g_{\mu\nu}(p) dx^\mu dx^\nu = 0$. To simplify the problem we restrict ourselves to work in the $t-x$ plane ^{$x=x'$} , so the above equation reduces to

$$g_{00}(p) dt^2 + 2g_{01}(p) dx dt + g_{11}(p) dx^2 = 0$$

The solution follows easily and is

$$\frac{dt}{dx} = \frac{-g_{01}(p) \pm \sqrt{g_{01}^2(p) - g_{00}(p)g_{11}(p)}}{g_{00}(p)}$$

(2.1.4)

giving us the two branches of the light-cone

$$\left(\frac{dt}{dx}\right)_p^+ = \frac{-g_{01}(p) + \sqrt{g_{01}^2(p) - g_{00}(p)g_{11}(p)}}{g_{00}(p)}$$

$$\left(\frac{dt}{dx}\right)_p^- = \frac{-g_{01}(p) - \sqrt{g_{01}^2(p) - g_{00}(p)g_{11}(p)}}{g_{00}(p)}$$

(2.1.5)

From this we see that for a general point p $\left(\frac{dt}{dx}\right)_p^+ \neq -\left(\frac{dt}{dx}\right)_p^-$. So the light-cone has tilted, that is, the central time-like direction does not point along the t -axis, but at some angle to it.

This result can be extended considerably, for this we need to return to a study of the matrix $||g_{\mu\nu}||_p$ for a general N-kink metric. Using the results of the last chapter this matrix can be written as

$$||g_{\mu\nu}||_p = \begin{bmatrix} -1+2\varphi_0^2 & 2\varphi_0\varphi_1 & 2\varphi_0\varphi_2 & 2\varphi_0\varphi_3 \\ 2\varphi_0\varphi_1 & -1+2\varphi_1^2 & 2\varphi_1\varphi_2 & 2\varphi_1\varphi_3 \\ 2\varphi_0\varphi_2 & 2\varphi_1\varphi_2 & -1+2\varphi_2^2 & 2\varphi_2\varphi_3 \\ 2\varphi_0\varphi_3 & 2\varphi_1\varphi_3 & 2\varphi_2\varphi_3 & -1+2\varphi_3^2 \end{bmatrix}_p \begin{bmatrix} f & g & h & k \\ g & L & M & N \\ h & M & P & Q \\ k & N & Q & R \end{bmatrix}_p \quad (2.1.6)$$

where φ is a mapping of degree N of S^3 into S^3 , and as such satisfies the constraint $\varphi_0^2 + \varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 1$. For a degree N-map φ_0 must pass through ± 1 N-times, at these points P_i $i = 1 \dots 2N$. We have

$$||g_{\mu\nu}||_{P_i} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} f & g & h & k \\ g & L & M & N \\ h & M & P & Q \\ k & N & Q & R \end{bmatrix}_{P_i} \quad (2.1.7)$$

The constraint that $||g_{\mu\nu}||_{P_i}$ is symmetric forces upon us the condition $g(P_i) = h(P_i) = k(P_i) = 0$ leaving us with

$$||g_{\mu\nu}||_{p_i} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & -L & -M & -N \\ 0 & -M & -P & -Q \\ 0 & -N & -Q & -R \end{bmatrix} \quad (2.1.8)$$

So we are led to the conclusion that at these $2N$ points the cross-terms $g_{01}(p_i)$, $g_{02}(p_i)$ and $g_{03}(p_i)$ vanish. At these points the solution of the light-cone structure equation gives the two branches to be

$$\left. \frac{dt}{dx} \right|_{p_i}^{\pm} = \pm \frac{\sqrt{-g_{00}(p_i) g_{11}(p_i)}}{g_{00}(p_i)}$$

and hence we have $\left. \frac{dt}{dx} \right|_{p_i}^{+} = \left. \frac{dt}{dx} \right|_{p_i}^{-}$. This implies at each p_i the central time-like direction of the light-cone points along the t -axis.

Taken together, these results give us a complete description of the tumbling of the light-cone structure in the x - t plane. At $r = \infty$ the central time-like vector of the light cone points along the $+t$ direction, and as r decreases the light-cone rotates. When the value of r is such that $\varphi_0(r) = -1$, the angle of rotation of the light-cone will be

π radians, with the central time like vector pointing along the $-t$ direction. For a one kink metric this point will be the origin. If N is greater than one, then this point

p_i will be at some distance r_i away from the origin, and the light-cone will continue to rotate with decreasing r until it points along the $-t$ direction at the origin. For the general N -kink the total angle of rotation of the light-cone is $N\pi$ radians.

It follows immediately from the above proof that the same behaviour will take place in the t-y and t-z planes.

2.2 A Class of Spherically-Symmetric ^{stationary} Kinks

In order to proceed with our investigations we shall now make two restrictions on the space of metrics we consider. The first is to only work with kinks where

$\|g_{\mu\nu}\|_p \in S_{4,3} \cap SO_4$ These metrics have the form

$$ds^2 = \sum_{\mu\nu} (-S_{\mu\nu} + 2\varphi_\mu \varphi_\nu) dx^\mu dx^\nu \quad (2.2.1)$$

The second restriction is to consider only spherically-symmetric ^{stationary} kinks. This is achieved by imposing the following constraints on φ_μ , $\varphi_0 = \cos N \alpha(r)$ and $\varphi_i = \frac{x_i}{r} \sin N \alpha(r)$ where $\alpha(r)$ is an arbitrary function of r.

The normal procedure would now be to transform to a spherical polar coordinate system. This we shall do. However, when dealing with kink metrics this step deserves careful consideration. In this coordinate system $\det \|g_{\mu\nu}\| = -r^4 \sin^2 \theta$ which has zeroes at $\theta = 0$ and $\theta = \pi$ that is $\|g_{\mu\nu}\|$ is singular on these two surfaces. But the method we use to determine to which kink sector a given metric lies in demanded that the matrix $\|g_{\mu\nu}\|$ was non-singular everywhere. This might seem a small point that could make no real difference, however ^{using} the techniques outlined in the first chapter, we find that in this coordinate system the kink "vanishes". This would affect global results but not the local results that we will be considering for the rest of the chapter.

In a spherical polar coordinate system (t, r, θ, φ) the above family of metrics takes the form

$$ds^2 = (\cos 2N\alpha) (dt^2 - dr^2) + 2\sin 2N\alpha dr dt - r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \quad (2.2.2)$$

The first point of interest for our discussion is the zeroes of g_{00} , these are the solutions to the equation $g_{00} = \cos 2N\alpha = 0$ for $N = 1$. This becomes $g_{00} = \cos 2\alpha = 0$ which has two solutions, one at $\alpha = \frac{\pi}{4}$, the other at $\alpha = \frac{3\pi}{4}$. This means that the manifold splits up into three regions:

$$\begin{aligned} \text{a) } & 0 \leq \alpha(r) < \frac{\pi}{4} & \text{where } g_{00} > 0 \\ \text{b) } & \frac{\pi}{4} < \alpha(r) < \frac{3\pi}{4} & \text{where } g_{00} < 0 \text{ and} \\ \text{c) } & \frac{3\pi}{4} < \alpha(r) \leq \pi & \text{where } g_{00} > 0 \end{aligned} \quad (2.2.3)$$

We see that the surfaces of $g_{00} = 0$ bound a region where $g_{00} < 0$. In this region t becomes a spatial coordinate and r a temporal one, this region is very similar to the egosphere for the Kerr solution.

For these metrics $\frac{\partial}{\partial t}$ is a killing vector as the components of the metric are not functions of t . However it is not time-like everywhere; in the region $g_{00} < 0$ it is a space-like killing vector.

The extension of these results to the general case where $N \neq 1$ follows naturally. We now have to solve the equation $g_{00} = \cos 2N\alpha = 0$. There are $2N$ solutions at $\alpha_m = \frac{m\pi}{4N}$ where $m = 1 \dots 4N - 1$, giving us $2N - 1$ regions of alternating sign of g_{00} .

We have already seen that the light-cone structure of kink metrics is very interesting, so we go on to give the complete solution to the light-cone equation in the r - t plane for the above family of metrics. The equation of interest is $ds^2 = 0$, which gives us

$$\begin{aligned} & (\cos 2N\alpha (dt^2 - dr^2) + 2\sin 2N\alpha dr dt - r^2 (d\theta^2 + \sin^2\theta d\phi^2)) \\ & = 0 \end{aligned} \quad (2.2.4)$$

The restriction to the r-t plane leads to the equation

$$(\cos 2N\alpha (dt^2 - dr^2) + 2\sin 2N\alpha dr dt = 0 \quad (2.2.5)$$

The solution is obtained by the following method. We rewrite (2.2.5) in the form

$$\begin{aligned} & [(\cos N\alpha - \sin N\alpha) dt + (\cos N\alpha + \sin N\alpha) dr] \times \\ & [(\cos N\alpha + \sin N\alpha) dt - (\cos N\alpha - \sin N\alpha) dr] \end{aligned} \quad (2.2.6)$$

This gives us two equations, the solution of each leads to a branch of the light-cone. These equations are

$$-(\cos N\alpha - \sin N\alpha) dt = (\cos N\alpha + \sin N\alpha) dr \quad (2.2.7)$$

$$(\cos N\alpha + \sin N\alpha) dt = (\cos N\alpha - \sin N\alpha) dr \quad (2.2.8)$$

The solutions to these in turn are

$$\begin{aligned} \left. \frac{dt}{dr} \right)^+ &= \frac{\cos N\alpha + \sin N\alpha}{\sin N\alpha - \cos N\alpha} = \frac{\sin(N\alpha + \pi/4)}{\cos(N\alpha + \pi/4)} \\ &= \tan(N\alpha + \frac{\pi}{4}) \end{aligned} \quad (2.2.9)$$

$$\begin{aligned} \left. \frac{dt}{dr} \right)^- &= \frac{\cos N\alpha - \sin N\alpha}{\cos N\alpha + \sin N\alpha} = \frac{\sin(N\alpha - \pi/4)}{\cos(N\alpha - \pi/4)} \\ &= \tan(N\alpha - \frac{\pi}{4}) \end{aligned} \quad (2.2.10)$$

From this we can calculate the angle between the branches of

the light-cone and the positive t-axis, which are

$$\alpha_1 = \tan^{-1} \left(\frac{dt}{dr} \right)^+ = \tan^{-1} \tan \left(N\alpha + \frac{\pi}{4} \right) = N\alpha + \frac{\pi}{4} \quad (2.2.11)$$

$$\alpha_2 = \tan^{-1} \left(\frac{dt}{dr} \right)^- = \tan^{-1} \tan \left(N\alpha - \frac{\pi}{4} \right) = N\alpha - \frac{\pi}{4} \quad (2.2.12)$$

the angle between the branches being $\alpha_1 - \alpha_2 = \pi/2$. We now have the basic features of the light-cone structure in the r-t plane. These are that the angle between the branches of the light-cone is always $\frac{\pi}{2}$ radians. While the central time-like direction rotates the angle of rotation measured from infinity being $N\alpha$.

For the case where $N = 1$, we find that the central time-like direction points along the positive t-axis. As r decreases α increases. When $r = r_1$, the position of the first zero of g_{00} we have that $\alpha(r_1) = \frac{\pi}{4}$ radians and one branch of the light-cone points along the r-axis. This surface of $g_{00} = 0$ has the curious property that all the time-like directions point along the negative r-axis, so this surface is a trapped surface, particles and light can cross this surface in one direction only. This is similar to the famous $g_{00} = 0$ surface for the Schwarzschild solution.

We now consider what happens for a radius $r < r_1$. We find that α continues to increase and when we reach $\alpha = \frac{3\pi}{4}$ at a radius of r_2 , g_{00} is again zero with now the second branch of the light-cone pointing along the r-axis. For this radius and all smaller radii the positive half of the light-cone points along the negative t-axis; this strange phenomenon is in direct

variance with our normal ideas of causality.

To develop our understanding of these results, we shall now investigate the equations for the null geodesics associated with these metrics. These equations are obtained by varying the action for particles in the space time. This will give us

$$\delta \int \{ \cos 2\alpha (\dot{t}^2 - \dot{r}^2) + 2 \sin 2\alpha \dot{r} \dot{t} - r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \} = 0$$

where $\dot{A} = \frac{dA}{ds}$. This leads to Euler-Lagrange equations

$$\frac{d}{ds} [\cos 2\alpha \dot{t} + \sin 2\alpha \dot{r}] = 0$$

$$\frac{d}{ds} (r^2 \dot{\theta}) = r \sin \theta \cos \theta \dot{\phi}$$

$$\frac{d}{ds} (r^2 \sin \theta \dot{\phi}) = 0 \quad \text{and}$$

$$\begin{aligned} \frac{d}{ds} [2 \sin 2\alpha \dot{t} - 2 \cos 2\alpha \dot{r}] = \\ - 2\alpha' \sin 2\alpha \dot{t} + 2\alpha' \sin 2\alpha \dot{r}^2 - 4\alpha' \cos 2\alpha \dot{r} \dot{t} \\ - 2r (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \end{aligned}$$

(2.2.12)

where $\alpha' = \frac{d\alpha}{dr}$. We shall not use this last equation, but instead we will use

$$\begin{aligned} \cos 2\alpha (\dot{t}^2 - \dot{r}^2) + 2 \sin 2\alpha \dot{r} \dot{t} \\ - r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) = 0 \end{aligned} \quad (2.2.13)$$

This set of equations completely determines the motion of a light-like particle in the above space time. However, we are mainly interested in the motion in the r - t plane, so we make

the restriction $\dot{\Theta} = \dot{\Phi} = 0$ With this the set of equations reduces to

$$\frac{d}{ds} [\cos 2\alpha \dot{t} + \sin 2\alpha \dot{r}] = 0 \quad (2.2.14)$$

$$\cos 2\alpha (\dot{t}^2 - \dot{r}^2) + 2 \sin 2\alpha \dot{r} \dot{t} = 0 \quad (2.2.15)$$

The solution to equation (2.2.14) is $\cos 2\alpha \dot{t} + \sin 2\alpha \dot{r} = L$ where L is a constant. This result can be rearranged to give

$$\dot{t} = \frac{L - \sin 2\alpha \dot{r}}{\cos 2\alpha} \quad (2.2.16)$$

Substituting this into equation (2.2.15) leads to the result $L^2 - r^2 = 0$, which, upon integration, yields

$$r = \pm Ls + d \quad (2.2.17)$$

Finally, when (2.2.17) is substituted into equation (2.2.16), we obtain

$$\begin{aligned} \dot{t} &= \frac{L \pm L \sin 2\alpha}{\cos 2\alpha} = \frac{L (\cos \alpha \mp \sin \alpha)}{(\cos \alpha \pm \sin \alpha)} \\ &= L \tan \left(\alpha \mp \frac{\pi}{4} \right) \end{aligned} \quad (2.2.18)$$

Consider a particle of light moving along the negative r -direction from some point r_0 where $\alpha(r_0) < \frac{\pi}{4}$. Then $r = r_0 - Ls$ and $\dot{t} = L \tan \left(\alpha + \frac{\pi}{4} \right)$. An examination of these results shows that with increasing s , r decreases and t increases, both of these results seem physically reasonable. However, we also have the following result, that as

$\alpha(r) \rightarrow \frac{\pi}{4}$, $t \rightarrow \infty$. This means it takes an infinite time to approach this surface of $g_{00}=0$, by looking at the other approaches to surfaces of $g_{00}=0$. We always find that it takes an infinite time for a light-like particle to

cross from a region where the future is measured by $+t$ to a region where it is measured by $-t$. We can also conclude that these space-times are incomplete, and that singularities exist.

2.3 Further Results for the Restricted Family of Kinks

We will now examine some of the geometrical properties of the following family of kink metrics:

$$ds^2 = \sum_{\mu, \nu} (-\delta_{\mu\nu} + 2\varphi_\mu \varphi_\nu) dx^\mu dx^\nu \quad (2.3.1)$$

where $\varphi_0 = -\cos N\alpha$ and $\varphi_i = \frac{x_i}{r} \sin N\alpha$

This involves the calculation of their curvature, energy-momentum and Weyl tensors. From these results we can prove that the energy-momentum tensor can not vanish everywhere. We also derive an even stronger result that the curvature scalar R does not vanish. The consequence of this is that the above family of metrics can not be solutions to Einstein-equations of the form $G_\mu^\nu = -T_\mu^\nu$ where T_μ^ν the energy-momentum tensor is generated by a conformal field, such as the electromagnetic or Yang-Mills fields.

We will work in a spherical polar coordinate system which is permissible as we are only studying local properties. In this system the family of metrics takes the form

$$ds^2 = \cos 2N\alpha (dt^2 - dr^2) - 2 \sin 2N\alpha dr dt - r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \quad (2.3.2)$$

The most effective way to approach these calculations is to use the tetrad notation. In this notation we have

$$ds^2 = (w^0)^2 - (w^1)^2 - (w^2)^2 - (w^3)^2$$

where

$$W^0 = \cos N\alpha \, dt - \sin N\alpha \, dr$$

$$W^1 = \sin N\alpha \, dt + \cos N\alpha \, dr$$

$$W^2 = r \, d\theta$$

$$W^3 = r \sin \theta \, d\varphi$$

(2.3.3)

We proceed in the following manner. By using Cartan's first structure equation $dW^i = -W^i_p \wedge W^p$ we can obtain the W^i_p . These are

$$W^0_1 = -N\alpha' \sin N\alpha \, W^0 - N\alpha' \cos N\alpha \, W^1$$

$$W^0_2 = -\frac{1}{r} \sin N\alpha \, W^3$$

$$W^0_3 = -\frac{1}{r} \sin N\alpha \, W^3$$

$$W^1_2 = -\frac{1}{r} \cos N\alpha \, W^2$$

$$W^1_3 = -\frac{1}{r} \cos N\alpha \, W^3$$

$$W^2_3 = -\frac{1}{r} \cot \theta \, W^3$$

2.3.4

The rest follow from the symmetry relations,

$$W^0_0 = W^1_1 = W^2_2 = W^3_3 = 0, \quad W^0_1 = W^1_0, \quad W^0_2 = W^2_0,$$

$$W^0_3 = W^3_0, \quad W^1_2 = -W^2_1, \quad W^1_3 = -W^3_1, \quad W^2_3 = -W^3_2$$

Using the W^i_p and the second cartan structure equation we can calculate the components of the curvature tensor. The equation of interest is

$$dW_L^i + W_L^i \wedge W_L^P = \frac{1}{2} R_{LJK}^i W^J \wedge W^K \quad (2.3.5)$$

So the components of the curvature tensor are

$$R_{110}^0 = \frac{1}{2} [\cos 2N\alpha]''$$

$$R_{202}^0 = R_{212}^1 = R_{313}^1 = -\frac{1}{2r} [\cos 2N\alpha]'$$

$$R_{002}^2 = R_{003}^3 = -\frac{1}{2r} [\cos 2N\alpha]'$$

$$R_{010}^1 = \frac{1}{2} [\cos 2N\alpha]''$$

$$R_{112}^2 = R_{113}^3 = \frac{1}{2r} [\cos 2N\alpha]'$$

$$R_{223}^3 = \frac{1}{r^2} [1 - \cos 2N\alpha]$$

(2.3.6)

Where we have $A' = \frac{d}{dr} A$. Using these results we can obtain the Ricci tensor by the following contraction,

$$R_{ij} = R_{ikj}^k \quad \text{giving}$$

$$R_{00} = -\frac{1}{2} [\cos 2N\alpha]'' - \frac{1}{r} [\cos 2N\alpha]'$$

$$R_{11} = \frac{1}{2} [\cos 2N\alpha]'' + \frac{1}{r} [\cos 2N\alpha]'$$

$$R_{22} = \frac{1}{r} [\cos 2N\alpha]' - \frac{1}{r^2} [1 - \cos 2N\alpha]$$

$$R_{33} = \frac{1}{r} [\cos 2N\alpha]' - \frac{1}{r^2} [1 - \cos 2N\alpha] \quad (2.3.7)$$

all terms of the form R_{ij} where $i \neq j$ vanish. By using a second contraction we obtain the curvature scalar $R = R^i_i$

$$R = - [\cos 2N\alpha]'' - \frac{4}{r} [\cos 2N\alpha]' + \frac{2}{r^2} [1 - \cos 2N\alpha] \quad (2.3.8)$$

To calculate the energy-momentum tensor we first find the Einstein tensor $G^\sim_\nu$. We do this via

$$G^\sim_\nu = R^\sim_\nu - \frac{1}{2} \delta^\sim_\nu R$$

leading to

$$\begin{aligned} G^0_0 &= \frac{1}{r} [\cos 2N\alpha]' - \frac{1}{r^2} [1 - \cos 2N\alpha] \\ G^1_1 &= \frac{1}{r} [\cos 2N\alpha]' - \frac{1}{r^2} [1 - \cos 2N\alpha] \\ G^2_2 &= \frac{1}{2} [\cos 2N\alpha]'' + \frac{1}{r} [\cos 2N\alpha]' \\ G^3_3 &= \frac{1}{2} [\cos 2N\alpha]'' + \frac{1}{r} [\cos 2N\alpha]' \end{aligned}$$

(2.3.9)

We now introduce the function $\nu^\sim(r) = r \sin^2 N\alpha$.

At this stage of the discussion this will be for purely notational convenience. However, as we proceed it will become clear that it plays a much more important role. In terms of the function $\nu^\sim(r)$ we rewrite $G^\sim_\nu$ in the following way:

$$G^0_0 = G^1_1 = -\frac{2}{r^2} \frac{d\nu}{dr}$$

$$G^2_2 = G^3_3 = -\frac{1}{r} \frac{d^2\nu}{dr^2}$$

$$\text{and } R = \frac{2}{r^2} \frac{d^2}{dr^2} [r\nu] \quad (2.3.10)$$

Knowing G^μ_ν the energy-momentum tensor follows from Einstein's equation $G^\mu_\nu = -T^\mu_\nu$ giving

$$T^0_0 = T^1_1 = \frac{2}{r^2} \frac{d\nu}{dr}$$

$$T^2_2 = T^3_3 = \frac{1}{r} \frac{d^2\nu}{dr^2}$$

and also

$$T = \sum_i T^i_i = R = \frac{2}{r^2} \frac{d^2}{dr^2} [r\nu] \quad (2.3.11)$$

The energy-momentum tensor describes the non-propagation part of the curvature tensor. To investigate the part of the tensor which propagates, we would need to construct the Weyl tensor. This is done via

$$\begin{aligned} C_{abcd} = & R_{abcd} + \frac{1}{2} (g_{ac} R_{bd} + \\ & g_{bd} R_{ac} - g_{ad} R_{bc} - g_{bc} R_{ad}) - \\ & \frac{R}{6} (g_{ab} g_{cd} - g_{ad} g_{bc}) \end{aligned} \quad (2.3.12)$$

For the special family of kink metrics we are working with, we find that the components of the Weyl tensor are

$$C_{0101} = -2A$$

$$C_{0202} = A$$

$$C_{0303} = A$$

$$C_{1212} = -A$$

$$C_{1313} = -A$$

$$C_{2323} = 2A$$

(2.3.13)

where $A = \frac{1}{12} [\cos 2N\alpha]'' - \frac{1}{6r} [\cos 2N\alpha]' - \frac{1}{6r^2} [1 - \cos 2N\alpha]$. These components, with those that are derived from the symmetry of the Weyl tensor, are the only non-zero components. As we pointed out earlier, these calculations were carried out in the tetrad notation. We now return to the spherical-polar coordinate system to study the energy-momentum tensor. The formula that relates these two notations is the following: $T_{\nu}^{\mu} = g^{\alpha\beta} \eta_{\alpha c} W_{\beta}^c W_{\nu}^d T_d^c$ where the Greek indices refer to the spherical-polars and the Latin indices to the tetrad basis. W_{μ}^c is defined by $W^c = W_{\mu}^c dx^{\mu}$. With this result the components of the energy-momentum tensor in a spherical polar coordinate system are

$$T_0^0 = T_1^1 = \frac{2}{r^2} \frac{dN}{dr}$$

$$T_2^2 = T_3^3 = \frac{1}{r} \frac{d^2 N}{dr^2}$$

(2.3.14)

There is one very striking consequence of these results, as

$N(r) = r \sin^2 \alpha$ is invariant under the transformation $N \rightarrow -N$, it follows that T_{ν}^{μ} is also invariant.

Hence both the kink and anti-kink are solutions to the same Einstein's equation. Here we have an example where an Einstein equation has two solutions that cannot be deformed into each other.

A less immediate but equally important consequence of the above results is that $T^{\sim}_{\nu}$ cannot vanish. The proof is as follows. If it did vanish, then $T^{\sim}_{\nu} = \frac{2}{r^2} \frac{d}{dr} \mu^{\sim} = 0$ implying that $\frac{d}{dr} \mu^{\sim} = 0$, which, on integration, gives $\mu^{\sim}(r) = c$. Hence we obtain the result $\sin \mu^{\sim} = \sqrt{\frac{E}{r}}$. The right hand side of this equation tends to infinity as $r \rightarrow 0$, while the conditions on μ lead to the left hand side being equal to zero at $r = 0$. So the two sides cannot be matched; the implication of this is that there are no kink solutions that satisfy $T^{\sim}_{\nu} = 0$. The only exception is when $N = 0$, and here the metric reduces to the Minkowski metric, which has an energy-momentum tensor that is zero everywhere.

The proof of the stronger result that R cannot vanish is basically the same. If $R = 0$, then it follows that

$$R = \frac{2}{r^2} \frac{d^2}{dr^2} (r \mu^{\sim}(r)) = 0$$

which would imply $\frac{d^2}{dr^2} (r \mu^{\sim}) = 0$, leading to $\mu^{\sim} = c + \frac{d}{r}$ where c and d are constants. Using $\mu^{\sim} = r \sin^2 \mu$ We obtain the result that $\sin \mu = \sqrt{\frac{E}{r} + \frac{d}{r^2}}$. We again have the right hand side tending to infinity as $r \rightarrow 0$, while the conditions on μ lead to the left hand side becoming zero when $r = 0$. So the two sides cannot be matched (except when $N = 0$). This means that there are no kink solutions, which satisfies $R = 0$. The consequence of this result can be obtained by using Einstein's equation,

$$G^\sim_\nu = R^\sim_\nu - \frac{1}{2} \delta^\sim_\nu R = -T^\sim_\nu$$

Taking the trace gives $-T^\sim_\nu = R^\sim_\nu - \frac{1}{2} \delta^\sim_\nu R = R - 2R = -R$. So $R = T^\sim_\nu$, hence the fact that R cannot equal zero everywhere implies the same is true for the trace of the energy-momentum tensor. This leads us to the conclusion that the energy-momentum tensor for kink metrics cannot be generated by conformal fields, because if it was its trace would vanish.

The function $\mu^\sim(r) = r \sin^2 N \alpha$ has so far played an important notational role. We shall now show it plays an important physical role as well. From $T^\circ_0 = \frac{2}{r} \frac{d\mu^\sim}{dr}$ it follows that $\mu^\sim(R) = \int_0^R \frac{r^2}{2} T^\circ_0 dr$. The right hand-side is proportional to the mass inside a radius R ; the constant of proportionality we shall set to be one. So $\mu^\sim(R) = R \sin^2 N \alpha(R)$ will be the mass of the N -kink inside the radius R . This gives us a useful relationship between the mass inside a given radius and the angle of rotation of the light-cone at this radius. For $N = 1$ we have $\mu^1(R) = R \sin^2 \alpha(R)$ and it is possible to choose $\alpha(r)$ such that $\mu^1(R)$ is monotonically increasing. This means that the mass of the kink will always increase with increasing radius. It also would imply that $T^\circ_0 > 0$ everywhere. Examples of this point will be given in the next chapter. This situation cannot be maintained for $n > 1$ where $\mu^\sim(R) = R \sin^2 N \alpha(R)$. Here it is impossible to choose $\alpha(r)$ so that $\mu^\sim(R)$ is monotonically increasing. This leaves us with the curious phenomenon that as the radius of the kink increases, the mass of the kink inside the radius at first increases, then decreases. The implication of this is that $T^\circ_0 < 0$ for some region of the manifold.

We will now use the above to derive a relationship between the mass of an N-kink and the mass of a one-kink.

We define the mass of the kink to be $\mu^{\sim}(\infty) = \lim_{R \rightarrow \infty} \mu^{\sim}(R)$

We also assume that the mass of the one-kink is non-zero

and finite $\mu'(\infty) = \lim_{R \rightarrow \infty} \mu'(R) = \lim_{R \rightarrow \infty} R \sin^2 \alpha = c \neq 0$

The claim is that the mass of the N-kink will be $N^2 c$. The proof is by induction. Firstly,

$$\begin{aligned} \mu^2(\infty) &= \lim_{R \rightarrow \infty} [R \sin^2 2\alpha] = \lim_{R \rightarrow \infty} 4R \sin^2 \alpha \cos^2 \alpha \\ &= 4 \lim_{R \rightarrow \infty} R \sin^2 \alpha \cos^2 \alpha = 4c \end{aligned} \quad (2.3.15)$$

The next step is to show, ^{that} if the mass of an M-kink is $m^2 c$, then the mass of an $(M + 1)$ kink is $(M + 1)^2 c$. This is done as follows:

$$\begin{aligned} \mu^{m+1}(\infty) &= \lim_{R \rightarrow \infty} [R \sin^2 (m+1)\alpha] = \\ &= \lim_{R \rightarrow \infty} [R (c \sin^2 m\alpha + 2 \sin m\alpha \sin \alpha + \sin^2 \alpha)] \\ &= \mu^m(\infty) + 2 \sqrt{\mu^m(\infty)} \sqrt{\mu'(\infty)} + \mu'(\infty) \\ &= m^2 c + 2mc + c = (m+1)^2 c \end{aligned}$$

(2.3.16)

CHAPTER 3

EXAMPLES OF KINK METRICS ON R^4

3.1 The Energy Conditions

Reviewing the structure of the last chapter, we find that we started with a study of the general kink metric on R^4 (with the appropriate boundary conditions), this was followed by a more detailed investigation of the restricted class of stationary spherically-symmetric kinks. These take the form

$$ds^2 = \cos 2N\alpha (dt^2 - dr^2) - 2\sin 2N\alpha dr dt$$

$- r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$ The corresponding energy-momentum tensor is

$$T_0^0 = T_1^1 = \frac{2}{r^2} \frac{dN}{dr}$$

$$T_2^2 = T_3^3 = \frac{1}{r} \frac{d^2 N}{dr^2}$$

where

$$N(r) = r \sin^2 \alpha$$

We shall now continue this process of increasing specialization by going on to explore certain examples where a definite form for $N(r)$ is chosen. After we have formulated the example our main concern will be the structure of its energy-momentum tensor.

To carry out the above program we first have to study certain conditions that T_ν^ν has to satisfy to be considered physical. The minimal requirement is that T_ν^ν satisfies the weak-energy condition that is $T_\nu^\nu \eta_\nu \eta^\nu \geq 0$ where η^ν is either time like or null. The indices refer to a spherical polar coordinate system. We shall choose η_ν to be a time-

like or null linear combination of the W^μ_ν where as before

$$W^0 = \cos N\alpha \, dt - \sin N\alpha \, dr$$

$$W^1 = \sin N\alpha \, dt + \cos N\alpha \, dr$$

$$W^2 = r \, d\theta$$

$$W^3 = r \sin\theta \, d\phi$$

(3.1.1)

From the W^α we can find the W^α_ν by using

$$W^\alpha = W^\alpha_\nu dx^\nu \text{ We obtain}$$

$$W^0_0 = \cos N\alpha \quad W^0_1 = -\sin N\alpha \quad W^0_2 = 0 \quad W^0_3 = 0$$

$$W^1_0 = \sin N\alpha \quad W^1_1 = \cos N\alpha \quad W^1_2 = 0 \quad W^1_3 = 0$$

$$W^2_0 = 0 \quad W^2_1 = 0 \quad W^2_2 = r \quad W^2_3 = 0$$

$$W^3_0 = 0 \quad W^3_1 = 0 \quad W^3_2 = 0 \quad W^3_3 = r \sin\theta$$

(3.1.2)

It can be observed that the following conditions hold

$$W^\mu_\nu W^{\nu\mu} = 1 \quad W^\mu_\nu W^{\nu\mu} = W^\mu_\nu W^{\nu\mu} = W^\mu_\nu W^{\nu\mu} = 0$$

(3.1.3)

and $W^\mu_\nu W^{\nu\mu} = 0$ for $a \neq b$. So taking

$$\eta_\nu = (a W^0_\nu + b W^1_\nu + c W^2_\nu + d W^3_\nu)$$

we find that the constraint that η_ν is time-like or null reduces to $\eta_\nu \eta^\nu = a^2 - b^2 - c^2 - d^2 \geq 0$ With these choices the weak energy condition takes the form

$$\begin{aligned} & T_{\nu}^{\nu} (a w_{\mu}^0 + b w_{\mu}^1 + c w_{\mu}^2 + d w_{\mu}^3) (a w^{\mu 0} + b w^{\mu 1} + c w^{\mu 2} + d w^{\mu 3}) \\ & \geq 0 \end{aligned} \quad (3.1.4)$$

which, upon using $T_0^0 = T_1^1$ and $T_2^2 = T_3^3$ can be rewritten as

$$(a^2 - b^2) T_0^0 - (c^2 + d^2) T_2^2 \geq 0 \quad (3.1.1)$$

There are two possibilities which we shall study in turn. The first is, $a \neq 0$ $b = c = d = 0$. This gives $\eta_{\mu} \eta^{\mu} \geq 0$. Hence η_{μ} is time-like, in this case the weak energy condition will be $a^2 T_0^0 \geq 0$, which implies that $T_0^0 \geq 0$. The second case is the general one, where $a \neq 0$, $b \neq 0$, $c \neq 0$ and $d \neq 0$. Here (3.1.1) can be written as

$$\frac{(a^2 - b^2)}{(c^2 + d^2)} T_0^0 - T_2^2 \geq 0 \quad (3.1.5)$$

If η_{μ} is null, then $a^2 - b^2 = c^2 + d^2$, and thus from (3.1.2) we obtain the result $T_0^0 - T_2^2 > 0$. If η_{μ} is time-like we have $\frac{a^2 - b^2}{c^2 + d^2} > 1$. With this, (3.1.5) can be rewritten as $\delta T_0^0 + (T_0^0 - T_2^2) \geq 0$ with $\delta > 0$. If the condition $T_0^0 - T_2^2 > 0$ is satisfied, then so will be the above condition. These results lead to the conclusion that for the energy-momentum tensors we are considering the weak-energy condition takes the form

$$T_0^0 > 0 \quad \text{and} \quad T_0^0 - T_2^2 > 0 \quad (3.1.6)$$

There is a stronger energy condition which is sometimes required for a space-time to be considered as physical. The strong energy-condition can be written as follows

$$- R_{\nu}^{\nu} \eta_{\mu} \eta^{\mu} \geq 0 \quad \text{using the same approach as in the case of the weak-energy condition. We obtain the following}$$

constraints on $R^\sim_\nu$, $-R^0_0 \geq 0$ and $R^2_2 - R^0_0 \geq 0$

These can be rewritten using the results $R^0_0 = T^2_2$ and $R^2_2 = T^0_0$, which follow from the calculations of $R^\sim_\nu$ and $T^\sim_\nu$. The strong-energy condition then takes the form:

$$-T^2_2 > 0 \text{ and } T^0_0 - T^2_2 > 0 \quad (3.1.7)$$

3.2 Examples of one-kink Metrics

We shall now investigate the kink metrics associated with specific choices for $\nu(r)$. The first example of a $\nu(r)$ will lead to a one kink metric. We will take as our choice $\nu(r) = r \sin^2 \alpha = \frac{2ar^2}{r^2+a^2}$. From this we obtain that $\sin \alpha = \sqrt{\frac{2ar}{r^2+a^2}}$ and $\cos \alpha = \frac{(r-a)}{\sqrt{r^2+a^2}}$. We notice how natural this choice for $\nu(r)$ is. With the half line $0 < r < \infty$ being projected onto a circle of radius a , the kink metric associated with $\nu(r)$ is

$$ds^2 = \left(1 - \frac{4ar}{r^2+a^2}\right) (dt^2 - dr^2) + 4 \frac{\sqrt{2ar}}{r^2+a^2} (r-a) dr dt - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

with the corresponding anti-

kink metric

$$ds^2 = \left(1 - \frac{4ar}{r^2+a^2}\right) (dt^2 - dr^2) - 4 \frac{\sqrt{2ar}}{r^2+a^2} (r-a) dr dt - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

We have before us what looks like a problem. We ask the rhetorical question, "Why not send a to zero?", as, when $a = 0$ both the kink and anti-kink metrics reduce to $ds^2 =$

$(dt^2 - dr^2) - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$ which is the Minkowski metric in a spherical-polar coordinate system. So it seems we

have constructed a continuous deformation between different kink sectors. We can consider this more carefully by looking at the function $\cos \alpha = \frac{cr-a}{\sqrt{r^2+a^2}}$. If a is strictly zero everywhere, then $\cos \alpha \equiv 1$. However, if a is arbitrarily small but non-zero, then as $r \rightarrow 0$ $\cos \alpha$ tends to -1 , which leads to a contradiction. The process of going from an arbitrarily small a to $a = 0$ is not in fact a continuous deformation. This is an example of a case which is studied in considerable detail in ref. (6), that for a parametrised family of kink metrics $g(\lambda)$, the process of going from an arbitrarily small λ to $\lambda = 0$ is not continuous.

For this parametrised family of kink metrics it is possible to find the zeroes of g_{00} in terms of the radial coordinate instead of α . We do this by solving the equation $g_{00} = 1 - \frac{4ar^2}{r^2+a^2} = 0$. There are two solutions at $r_{\pm} = a(2 \pm \sqrt{3})$. It has already been noted that all kinks have one or more regions of anomalous behaviour. For the above family there is one region bounded by the larger zero of g_{00} , i.e. by $r = a(2 + \sqrt{3})$. It can be seen that this region becomes more and more localized as a gets smaller. To take this result further we look again at the explicit form of $\nu(r)$ which is $\nu(r) = r \sin^2 \alpha = \frac{2ar^2}{r^2+a^2}$. The "mass" of the kink will be equal to $\nu(\infty) = \lim_{R \rightarrow \infty} \nu(R) = 2a$. This gives us a direct relationship between the mass of the kink and the extent of the anomalous region. A very massive kink would contain a large region of space-time in which there could occur phenomena that would be considered unphysical.

We shall now investigate the properties of the energy-momentum tensor. From $\nu(r) = \frac{2ar^2}{r^2+a^2}$ it follows immediately that $\frac{d\nu}{dr} = \frac{4a^3r}{(r^2+a^2)^2}$ and

$$\frac{d^2 \nu}{dr^2} = \frac{4a^3 (a^2 - 3r^2)}{(r^2 + a^2)^3}$$

which gives

$$T_0^0 = T_1^1 = \frac{2}{r^2} \frac{d\nu}{dr} = \frac{8a^3}{r} \frac{1}{(r^2 + a^2)^2}$$

and

$$T_2^2 = T_3^3 = \frac{1}{r} \frac{d^2 \nu}{dr^2} = \frac{4a^3}{r} \frac{(a^2 - 3r^2)}{(r^2 + a^2)^3}$$

(3.2.1)

From the above we find that the condition $T_0^0 \geq 0$ is satisfied everywhere, so we are furnished with an example where an $\alpha(r)$ can be chosen for a one-kink metric, so that $\nu(r)$ is a monotonically increasing function. Looking at the second condition on T_ν^μ which is that $T_0^0 - T_2^2 > 0$, we find, on substituting in the explicit forms for T_0^0 and T_2^2 , this condition will read

$$\begin{aligned} T_0^0 - T_2^2 &= \frac{4a^3}{r} \frac{1}{(r^2 + a^2)^3} [2(r^2 + a^2) - (a^2 - 3r^2)] \\ &= \frac{4a^3}{r} \frac{1}{(r^2 + a^2)^3} [5r^2 + a^2] \end{aligned}$$

which is greater than

zero everywhere, hence the weak energy condition is satisfied.

However, if we turn to the strong energy condition, we find that, while the constraint $T_0^0 - T_2^2 \geq 0$ is satisfied, the second constraint, that $T_2^2 \leq 0$ everywhere, is not, as

$$T_2^2 = \frac{4a^3}{r} \frac{(a^2 - 3r^2)}{(r^2 + a^2)^3} \quad \text{is positive for } r < \frac{a}{\sqrt{3}}$$

It would be of interest now to study a more complicated

one-kink metric, which can be done by choosing $\nu(r) = \left(\frac{3}{2 + \frac{1}{r}}\right)^{3/2}$

This choice gives us, $\sin \alpha = \frac{1}{r} \left(\frac{3}{2 + \frac{1}{r}}\right)^{3/2}$

and $\cos \alpha = \pm \sqrt{1 - \frac{1}{r^2} \left(\frac{3}{2 + \frac{1}{r}}\right)^3}$, where we take the

positive branch of this function for $r > r_0$, r_0 being defined

by $g_{00}(r_0) = 0$, with the negative branch being used for $r < r_0$.

The kink-metric corresponding to this choice of $\nu(r)$ is

$$\begin{aligned}
 ds^2 = & \left[1 - \frac{2}{r^2} \left(\frac{3}{2+\frac{1}{r}} \right)^3 \right] (dt^2 - dr^2) \\
 & - \frac{2}{r} \left(\frac{3}{2+\frac{1}{r}} \right)^{3/2} \sqrt{1 - \frac{1}{r} \left(\frac{3}{2+\frac{1}{r}} \right)^3} dr dt \\
 & - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)
 \end{aligned} \tag{3.2.2}$$

With these results we find that $T_0^0 > 0$ and that also $T_0^0 - T_2^2 = \frac{624}{r^4} \left(\frac{1}{2+\frac{1}{r}} \right)^5 > 0$ everywhere, so again the weak energy condition is satisfied. If we look at the strong energy condition we have $T_2^2 > 0$ for $r < \frac{1}{2}$, so the condition that $T_2^2 \leq 0$ is not satisfied everywhere.

At the present time the question of the existence of a one-kink metric which satisfies the strong-energy condition is open.

3.3 A one-parameter family of 2-kinks

We now turn our attention to examples of two-kink metrics.

For one of our examples of a 1-kink metric we chose $\nu(r) = r \sin^2 \alpha = \frac{2ar^2}{r^2+a^2}$. There is a natural choice of a 2-kink metric associated with this, which is where $\nu^2(r) = r \sin^2 2\alpha$. By using certain trigonometrical relationships, we obtain

$\nu^2(r) = \frac{8ar^2(r-a)^2}{(r^2+a^2)^2}$. This leads to the following 2-kink metric

$$\begin{aligned}
 ds^2 = & \left(1 - \frac{16ar}{r^2+a^2} + \frac{32a^2r^2}{(r^2+a^2)^2} \right) (dt^2 - dr^2) \\
 & + 8\sqrt{2} \frac{ar(r-a)}{r^2+a^2} \left[1 - \frac{4ar}{r^2+a^2} \right] dr dt
 \end{aligned}$$

$$- r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \quad (3.3.1)$$

For this metric we would expect four zeroes of g_{00} which we find at

$$r_{\pm} = \left[-\frac{2}{(1+\frac{1}{\sqrt{2}})} \pm \sqrt{\frac{4}{(1+\frac{1}{\sqrt{2}})^2} - 1} \right] \quad \text{and}$$

$$r'_{\pm} = \left[-\frac{2}{(1-\frac{1}{\sqrt{2}})} \pm \sqrt{\frac{4}{(1-\frac{1}{\sqrt{2}})^2} - 1} \right] \quad (3.3.2)$$

The next step is to calculate the energy-momentum tensor. Using the above form for $\nu^2(r)$ we find that

$$\frac{d\nu^2}{dr} = 16 \frac{ar(r-a)}{(r^2+a^2)^3} [2a^2r + ar^2 - a^3]$$

and

$$\frac{d^2\nu^2}{dr^2} = \frac{16a^3}{(r^2+a^2)^4} [-2r^5 - 3ar^4 + 16a^2r^3 - 2a^3r^2 - 6a^4r + a^5]$$

which leads to

$$T^0_0 = T^1_1 = 32 \frac{a}{r} \frac{(r-a)}{(r^2+a^2)^3} [2a^2r + ar^2 - a^3]$$

$$\text{and } T^2_2 = T^3_3 =$$

$$\frac{16a^2}{(r^2+a^2)^4} [-2r^5 - 3ar^4 + 16a^2r^3 - 2a^3r^2 - 6a^4r + a^5]$$

$$(3.3.3)$$

The weak energy condition fails as $T_0^0 < 0$ in the region $(-1 + \sqrt{2})a < r < a$. This agrees with the general statement made in the last chapter, that for N-kinks with $N > 1$, the energy-momentum tensor is such that T_0^0 cannot be greater than zero everywhere.

We will now check the mass formula of the last chapter. The mass of the one-kink metric is

$$\lim_{r \rightarrow \infty} \mu^1(r) = \lim_{r \rightarrow \infty} \left[\frac{2ar^2}{r^2 + a^2} \right] = 2a$$

and the mass of the 2-kink metric is

$$\lim_{r \rightarrow \infty} \mu^2(r) = \lim_{r \rightarrow \infty} \left[\frac{8ar^2(r-a)^2}{(r^2 + a^2)^2} \right] = 8a = (2)^2 2a$$

Hence the mass formula $\mu^2(\infty) = (2)^2 \mu^1(\infty)$ is verified.

3.4 A two parameter family of 2-kinks

The two kinks we have so far studied form a one parameter family of metrics. We shall now investigate a two parameter family which generalizes the above.

Given 2 one-kink families generated by

$$\mu(r) = r \sin^2 \alpha = \frac{2ar^2}{r^2 + a^2} \quad \text{and}$$

$$\mu^1(r) = r \sin^2 \alpha' = \frac{2br^2}{r^2 + b^2} \quad (3.4.1)$$

there is a natural family of 2-kink metrics associated with these generated by $\mu^2(r) = r \sin^2 (\alpha + \alpha')$

The explicit form for the 2-kink metric is

$$\begin{aligned}
ds^2 = & \left[\left(1 - \frac{4ar}{r^2+a^2} \right) \left(1 - \frac{4br}{r^2+b^2} - \frac{8r\sqrt{ab}(r-a)(r-b)}{(r^2+a^2)(r^2+b^2)} \right) \right] (dt^2 - dr^2) \\
& + 2 \left[\left(1 - \frac{4br}{r^2+b^2} \right) \frac{2\sqrt{2ar}(r-a)}{r^2+a^2} + \left(1 - \frac{4ar}{r^2+a^2} \right) \frac{2\sqrt{2br}(r-b)}{r^2+b^2} \right] dr dt \\
& - r^2 (d\theta^2 + \sin^2\theta d\phi^2) \tag{3.4.2}
\end{aligned}$$

The energy-momentum tensor has a very complicated form which we will not commit to print. However, we can show quite easily that the weak-energy condition is not satisfied.

Consider $\nu^2(r)$ this function is zero at $r = 0$ and $r = \sqrt{ab}$, hence $\nu^2(r)$ is not a monotonically increasing function. This implies that $\frac{d\nu^2}{dr}$ changes sign somewhere, hence $T_{\theta\theta} = \frac{2}{r^2} \frac{d\nu^2}{dr}$ cannot be positive everywhere and the weak energy condition fails.

CHAPTER 4

KINKS ON $R \times S^1 \times S^1 \times S^1$ 4.1 The classification of kinks on $M = R \times S^1 \times S^1 \times S^1$

In the previous chapters we have investigated kink metrics on R^4 . In this, the final chapter, we take as our underlying manifold the torus $R \times S^1 \times S^1 \times S^1$. This will lead to some interesting new results as well as showing that some of the results already derived for R^4 carry over to $M = R \times S^1 \times S^1 \times S^1$.

From the general classification scheme set up in the first chapter we know that kinks on $R \times S^1 \times S^1 \times S^1$ are classified by

$$[R \times S^1 \times S^1 \times S^1, T_{2L}(R \times S^1 \times S^1 \times S^1)] =$$

$$[R \times S^1 \times S^1 \times S^1, S_{4,3}] = [R \times S^1 \times S^1 \times S^1, R P^3]$$

$$= \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad (4.1.1)$$

where we remember that the homotopy classes we use are pointed, in this case an arbitrary base point is mapped into

$$|| \gamma_{null} || \in S_{4,3}$$

The step of going from the homotopy classes of cross-sections of the bundle $S_{4,3} \longrightarrow T_{2L}(R \times S^1 \times S^1 \times S^1)$



$$R \times S^1 \times S^1 \times S^1$$

(4.1.2)

to homotopy classes of maps of $R \times S^1 \times S^1 \times S^1$ into $S_{4,3}$ is meaningful as the above manifold is parallelizable, hence

there exists a global frame w^a , with this $g \in \bar{T}_2(M)$ can be written in the form $g(p) = \sum g_{ab}(p) w^a w^b$ where $\|g_{ab}\|_p \in S_{4,3}$ This holds at every point of the manifold.

The frame we will use is dx^μ which gives us

$$g = \sum g_{\mu\nu} dx^\mu dx^\nu \quad \text{This needs to be justified}$$

as dx^μ is not strictly a globally defined frame. To show such a step is permissible, we proceed in the following way: choosing $dx^0 = dt$, $dx^1 = d\theta_1$, $dx^2 = d\theta_2$, $dx^3 = d\theta_3$ (noting that we have chosen all the radii of the torus to be unity), then we have a well defined global frame in the regions

$$0 \leq \theta_1 < 2\pi \quad 0 \leq \theta_2 < 2\pi \quad 0 \leq \theta_3 < 2\pi$$

$-\infty \leq t \leq \infty$ This means dx^μ is well defined on a dense subset of $R \times S^1 \times S^1 \times S^1$, hence the map $\|g_{\mu\nu}\|: M \rightarrow S_{4,3}$ is defined on this subset. If we also use the fact that

$\|g_{\mu\nu}\|$ is a continuous function on this dense subset, then when we continue to the whole manifold, the continuation of $\|g_{\mu\nu}\|$ i.e. $\|\tilde{g}_{\mu\nu}\|: R \times S^1 \times S^1 \times S^1 \rightarrow S_{4,3}$ will also be well defined and continuous.

For the metrics on R^4 (with asymptotic boundary conditions) we devised a program by which we could work out to which kink sector any given metric belongs. For the case where $M = R \times S^1 \times S^1 \times S^1$, any attempt to develop a similar program would be very difficult, the problems lying mainly in the Z_2 part of the classification.

The precise step at which the argument used for $M = R \times S^3$ would break down for $M = R \times S^1 \times S^1 \times S^1$ is where we equated SO_4 with $SU(2) \times SU(2)$. Strictly, it should have been

$SO_4 \simeq SU(2) \times SU(2) / Z_2$ For $M = R \times S^3$ this was permissible as the classification depended on the third homotopy group of SO_4 , and it is true that

$\pi_3(SO_4) = \pi_3(SU(2) \times SU(2))$ However, for $M = R \times S^1 \times S^1 \times S^1$ the classification also depends upon $\pi_1(SO_4)$ which is not the same as $\pi_1(SU(2) \times SU(2))$ and hence our approach to the analysis of the maps into SO_4 does not hold in this case.

Due to this and that, for $M = R \times S^1 \times S^1 \times S^1$ we are interested in the $\mathbb{Z}_2$ component of the classification, we follow a different line of logic. Consider the family of metrics $ds^2 = (\cos 2N\alpha (dt^2 - R_1 d\theta_1^2) + 2\sin 2N\alpha R_1 d\theta_1 dt - R_2^2 d\theta_2^2 - R_3^2 d\theta_3^2)$ where α is a function of θ_1 , only with $\alpha(0) = 0$ and $\alpha(2\pi) = \pi$. We make the claim that for even N these metrics are deformable into $ds^2 = (dt^2 - R_1^2 d\theta_1^2) - R_2^2 d\theta_2^2 - R_3^2 d\theta_3^2$ which is the flat metric on $R \times S^1 \times S^1 \times S^1$, while, for odd N no such continuous deformation exists.

The proof of this statement is as follows. Writing the metrics in tetrad notation gives us

$$ds^2 = (W^0)^2 - (W^1)^2 - (W^2)^2 - (W^3)^2$$

where

$$W^0 = \cos N\alpha dt + \sin N\alpha R_1 d\theta_1$$

$$W^1 = \cos N\alpha R_1 d\theta_1 - \sin N\alpha dt$$

$$W^2 = R_2 d\theta_2$$

$$W^3 = R_3 d\theta_3$$

(4.1.4)

We then consider maps of S^1 into the space of tetrads. The explicit circle we choose is obtained from $R \times S^1 \times S^1 \times S^1$ by fixing t, θ_2, θ_3 and letting θ_1 vary between 0 and 2π . At $\theta_1 = 0$ we have $W^0 = dt, W^1 = R_1 d\theta_1, W^2 = R_2 d\theta_2$ and $W^3 = R_3 d\theta_3$ while at $\theta_1 = 2\pi$ we obtain the following: $W^0 = (-1)^N dt, W^1 = (-1)^N R_1 d\theta_1, W^2 = R_2 d\theta_2$ and $W^3 = R_3 d\theta_3$. For even N

these two sets of tetrads are the same, while for N odd they are different. This implies for N even it is possible to contract the circle into a point without breaking the continuity of the map of S^1 into the space of tetrads, while for odd N this is impossible, so completing the proof.

4.2 The light-cone structure

One of the most interesting features of this family of metrics is its light-cone structure. Restricting ourselves to the $\Theta_1 - t$ plane and also setting $R_1 = 1$ and $\Theta_1 = \Theta$ we obtain the light-cone structure equation $ds^2 =$

$$\cos 2N\alpha (dt^2 - d\Theta^2) + 2\sin 2N\alpha d\Theta dt = 0$$

The two solutions give us the branches of the light-cone. We find these at the angles $\alpha_1 = N\alpha + \frac{\pi}{4}$ and $\alpha_2 = N\alpha - \frac{\pi}{4}$ radians to the t -axis. Hence, we have that the branches are at an angle of $\pi/2$ radians to each other and the central time-like direction rotates the angle of rotation measured from $\Theta = 0$ is $N\alpha$ rads.

We have already commented on the fact that for this family of metrics, if N is even, then the metrics are deformable into the flat metric on the turns, while for N -odd the resulting metrics are not. We shall now see how this is reflected in the light-cone structure. The angle of rotation is $N\alpha$, as Θ varies from $\Theta = 0$ to $\Theta = 2\pi$ which means that the range of $N\alpha$ is $0 \leq N\alpha < N\pi$. For N even the central time-like direction would have rotated from the positive t -axis back to the positive t -axis. Hence the space-time is time orientable. However, for N odd it rotates from the positive t -axis at $\Theta = 0$ to the negative t -axis at $\Theta = 2\pi$. This means that the kink part of the above family of space-times is not time-orientable. This

result agrees with a statement of Geroch, that for a given manifold a non time orientable space-time cannot be deformed continuously into a time-orientable space-time.

4.3 Geometrical quantities associated with a family of kink metrics on $M = R \times S^1 \times S^1 \times S^1$

Following the same pattern as for kinks on R^4 , we now calculate certain geometrical quantities related to the Z_2 kinks on $R \times S^1 \times S^1 \times S^1$. These are the curvature, energy-momentum and Weyl tensors.

We carry out this program using the tetrad formalism. The family of metrics of interest is $ds^2 =$

$$(-1 + 2 \cos^2 N\alpha) dt^2 - (1 - 2 \sin^2 N\alpha) R_1^2 d\theta_1^2 + 2 R_1 \sin 2N\alpha d\theta_1 dt - R_2^2 d\theta_2^2 - R_3^2 d\theta_3^2$$

In tetrad notation this will read as

$$ds^2 = (W^0)^2 - (W^1)^2 - (W^2)^2 - (W^3)^2 \quad \text{where}$$

$$W^0 = \cos N\alpha dt + \sin N\alpha R_1 d\theta_1$$

$$W^1 = \cos N\alpha R_1 d\theta_1 - \sin N\alpha dt$$

$$W^2 = R_2 d\theta_2$$

$$W^3 = R_3 d\theta_3 \quad (4.2.1)$$

Using Cartan's first structure equation $dW^i = -W_p^i \wedge W^p$ we obtain the following

$$W_1^0 = -\frac{N\alpha'}{R_1} \sin N\alpha W^0 - \frac{N\alpha'}{R_1} \cos N\alpha W^1$$

$$W_0^1 = -\frac{N\alpha'}{R_1} \sin N\alpha W^0 - \frac{N\alpha'}{R_1} \cos N\alpha W^1 \quad (4.2.2)$$

with all the other W_p^i being identically zero. From these and the second Cartan structure equation,

$$dW_L^i + W_P^i \wedge W_L^P = \frac{1}{2} R_{LS}^i W^S \wedge W^L$$

we obtain

$$R'_{010} = R^0_{110} = \frac{1}{2R_{1,2}} \frac{d^2}{d\theta_{1,2}} [\cos 2N\alpha] \quad (4.2.3)$$

with all the other components of the curvature tensor being zero. To obtain the Ricci tensor, we carry out the following contraction, $R_{15} = R_{15k}^k$ to yield

$$\begin{aligned} R_{00} &= -\frac{1}{2R_{1,2}} \frac{d^2}{d\theta_{1,2}} [\cos 2N\alpha] \\ R_{11} &= \frac{1}{2R_{1,2}} \frac{d^2}{d\theta_{1,2}} [\cos 2N\alpha] \end{aligned} \quad (4.2.4)$$

$R_{22} = R_{33} = 0$, and all the R_J^i with $i \neq J$ being zero. A further contraction $R = R_i^i$ gives us that

$$R = -\frac{1}{R_{1,2}} \frac{d^2}{d\theta_{1,2}} [\cos 2N\alpha]$$

By using $G_J^i = R_J^i - \frac{1}{2} \delta_J^i R$, the Einstein's tensor is obtained:

$$G_0^0 = 0$$

$$G_1^1 = 0$$

$$G_2^2 = -\frac{1}{2R_{1,2}} \frac{d^2}{d\theta_{1,2}} [\cos 2N\alpha]$$

$$G_3^3 = -\frac{1}{2R_{1,2}} \frac{d^2}{d\theta_{1,2}} [\cos 2N\alpha] \quad (4.2.5)$$

By using the definition $\beta = \cos 2N\alpha$ and the Einstein's equation $T_\nu^\nu = -G_\nu^\nu$, the energy-momentum tensor follows easily.

$$T_0^0 = T_1^1 = 0 \text{ and } T_2^2 = T_3^3 = \frac{1}{2R_1^2} \frac{d^2\beta}{d\theta_1^2} \quad (4.2.6)$$

with the curvature scalar R being equal to $\frac{1}{R_1^2} \frac{d^2\beta}{d\theta_1^2}$

The calculation of the Weyl tensor is via

$$\begin{aligned} C_{abcd} = & R_{abcd} + \frac{1}{2} (g_{ac} R_{bd} + g_{bd} R_{ac} \\ & - g_{ab} R_{cd} - g_{cd} R_{ab}) + \\ & \frac{R}{6} (g_{ac} g_{bd} - g_{ad} g_{bc}) \end{aligned} \quad (4.2.7)$$

giving

$$C_{0101} = R/6$$

$$C_{0202} = -R/12$$

$$C_{0303} = -R/12$$

$$C_{1212} = R/12$$

$$C_{1313} = R/12$$

$$C_{2323} = R/6 \quad (4.2.8)$$

with all the other components of the Weyl tensor being identically zero.

For the above family of kink metrics on $R \times S^1 \times S^1 \times S^1$ the proofs that the energy-momentum tensor and its trace cannot vanish everywhere both follow from proving that $R \not\equiv 0$. This proof is as follows. If $R = 0$ everywhere, then it would follow that $\frac{d^2\beta}{d\theta_1^2} = 0$ giving $\beta = C\theta_1 + d$. This implies that $\cos 2N\alpha = C\theta_1 + d$ which is impossible as $C\theta_1 + d$ is a monotonically increasing or decreasing function in the range $0 < \theta_1 < 2\pi$ where $\cos 2N\alpha$ is not.

4.4 Examples of kink metrics on $M = R \times S^1 \times S^1 \times S^1$

We now investigate certain explicit examples of Z_2

kink metrics on $R \times S^1 \times S^1 \times S^1$. The first is where we choose $\alpha(\theta_1) = \frac{\theta_1}{2}$. This gives the family of metrics

$$ds^2 = \cos N\theta_1 (dt^2 - R_1^2 d\theta_1^2) + 2\sin N\theta_1 R_1 d\theta_1 dt - R_2^2 d\theta_2^2 - R_3^2 d\theta_3^2 \quad (4.4.1)$$

As before, we are interested in the zeroes of g_{00} , which are the solutions to the equation $\cos N\theta_1 = 0$. These occur at

$\theta_1 = \frac{\pi}{2N}, \frac{3\pi}{2N}, \dots, \frac{(4N-1)\pi}{2N}$. Each pair of zeroes of g_{00} separate regions where $g_{00} > 0$ and $g_{00} < 0$. Taking the case where $N = 1$, then from above we have two zeroes of g_{00} , one at $\theta_1 = \frac{\pi}{2}$, the other at $\theta_1 = \frac{3\pi}{2}$. In the range

$\frac{3\pi}{2} < \theta_1 < \frac{\pi}{2}$ g_{00} is greater than zero, otherwise it is less than zero.

For the general case we find that there are $2N$ zeroes of g_{00} separating $2N$ regions of alternating sign for g_{00} . There are two main differences between the above and the similar example for $M = R^4$. The first is for the N -kink on R^4 . There were only $2N-1$ regions of alternating sign for g_{00} . The second difference is that for the case of R^4 the number of zeroes of g_{00} was twice the kink number, while for $M = R \times S^1 \times S^1 \times S^1$ we take the number of zeroes of g_{00} , divide it by 2, and if the number is odd we have a metric which is not deformable into the flat metric on $R \times S^1 \times S^1 \times S^1$ and if it is even, then it is continuously deformable into the flat metric.

Following the general procedure for studying examples of kink metrics, we now check if the weak energy condition is satisfied. As before, this condition takes the form, $T_0^0 > 0$ and $T_0^0 - T_2^2 > 0$. As $T_0^0 = 0$ the first part of the condition is automatically satisfied. However,

$$T_3^0 - T_2^2 = - \frac{1}{2R_1^2} \frac{d^2}{d\theta_1^2} [\cos N\theta_1] = \frac{N^2}{R_1^2} \cos N\theta_1,$$

which is clearly not greater or equal to zero everywhere, so the weak energy condition fails. We note that this is true for the above metrics, whether or not they are deformable into the flat metric on $R \times S^1 \times S^1 \times S^1$.

The energy-momentum tensor for these metrics has some other interesting features; the first is that T_2^2 has its own zeroes and these occur for the same values of θ_1 as the zeroes of g_{00} . Secondly, T_2^2 and T_3^3 are strictly bounded, the value of this bound increasing with increasing N .

For our second example, we investigate a family of metrics, which are labelled by two integers (N, m) . To do this we choose $\alpha = \frac{\theta_1}{2} \left(\frac{\theta_1}{2\pi}\right)^m$. We note that $\alpha(0) = 0$ and $\alpha(2\pi) = \pi$. This gives us the following metrics:

$$\begin{aligned} ds^2 = & \cos N \left[\frac{\theta_1}{2} \left(\frac{\theta_1}{2\pi}\right)^m \right] (dt^2 - R_1^2 d\theta_1^2) \\ & - 2 \sin N \left[\frac{\theta_1}{2} \left(\frac{\theta_1}{2\pi}\right)^m \right] R_1 d\theta_1 dt \\ & - R_2^2 d\theta_2^2 - R_3^2 d\theta_3^2 \end{aligned}$$

(4.4.2)

We will only study the case where $N = 1$. Then

$$g_{00} = \cos \left[\frac{\theta_1}{2} \left(\frac{\theta_1}{2\pi}\right)^m \right]$$

has two zeroes; the first at $\theta_1 = (2^{m+1})^{1/(m+1)} \pi$, the second is at $\theta_1 = (2^{m+1})^{1/(m+1)} (3)^{1/(m+1)} \pi$. For $M = 1$ these will become $\theta_1 = \pi$ and $\theta_1 = \sqrt{3} \pi$. For the region

$\pi < \Theta_1 < \sqrt{3} \pi$ we find that $g_{00} < 0$. So in this example the anomalous region with $g_{00} < 0$ occupies less of the circle $0 < \Theta_1 < 2\pi$. Hence if we consider the extent of the kink to be this region, the kink is more localized in this example than for the earlier example.

For the case where $\alpha = \frac{\Theta_1}{2}$ we found that the weak-energy condition failed. Will this change if $\alpha = \frac{\Theta_1}{2} \left(\frac{\Theta_1}{2\pi} \right)$ Checking the two constraints, we find that $T_0^0 = 0$, so $T_0^0 > 0$ is satisfied, the other condition is $T_0^0 - T_2^2 > 0$, this fails as $T_0^0 - T_2^2 = \frac{1}{2\pi} \frac{1}{R_1^2} \left[5 \sin \frac{\Theta_1^2}{2\pi} + \frac{\Theta_1^2}{\pi} \cos \frac{\Theta_1^2}{2\pi} \right]$ which is less than zero for some regions of Θ_1 . We will not carry out an analysis of cases where $M > 1$, but if we did we would find, firstly, that the region with $g_{00} < 0$ would occupy a decreasing part of the circle $0 < \Theta_1 < 2\pi$ with increasing M , and secondly, the weak energy condition fails for all $M > 0$.

Appendix

Mathematical Definitions

We present here a collection of mathematical references which will be useful in studying this thesis. For the most part it is based upon references (12,13,14,15).

1) A map φ of an open set $O \subset \mathbb{R}^n$ to an open set $O' \subset \mathbb{R}^m$ is said to be of class c^r if the coordinates $(x'^1, x'^2, \dots, x'^m)$ of the image point $\varphi(p)$ in O' are r -times continuously differentiable functions of the coordinates $(x^1, x^2, \dots, x^n)$ of p in O . If a map is c^r for all $r > 0$, then it is said to be c^∞ . By a c^0 map we mean a continuous map.

2) A c^r n -dimensional manifold M , is a set M together with a c^r atlas $\{U_\alpha, \varphi_\alpha\}$ that is to say, a collection of charts $(U_\alpha, \varphi_\alpha)$, where U_α are subsets of M and φ_α are one-one maps of the corresponding U_α to open sets in $\mathbb{R}^n$, such that

1) The U_α cover M , i.e. $M = \bigcup U_\alpha$

2) If $U_\alpha \cap U_\beta$ is non-empty, then the map

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \rightarrow \varphi_\alpha(U_\alpha \cap U_\beta)$$

is a c^r map of an open subset of $\mathbb{R}^n$ to an open subset of $\mathbb{R}^n$.

3) In the thesis there occur many examples of lie groups, for instance $GL(4, \mathbb{R})$, SO_4 , $SU(2)$ etc.

Let G be a set endowed with a group structure and a structure of a differential manifold. These two structures are said to be compatible if the mappings $(x,y) \rightarrow xy$ of $G \times G$ into G and $x \rightarrow x^{-1}$ of G into G are of class c^∞ . A group endowed with a structure of a differential manifold which is compatible with the group structure is called a lie group.

4) The real projective space $\mathbb{R}P^3$ is defined to be $(\mathbb{R}^4 - 0)/\sim$

where s is the equivalence relation defined by $x \sim y \Leftrightarrow x = ry$ for some real number r . It is homeomorphic to S^3 / τ where

$x \tau y \Leftrightarrow x = \pm y$ i.e. Rp^3 can be obtained from S^3 by identifying antipodal points.

We also note the Rp^3 can be viewed as a quotient space in the following way. $Rp^3 = SO(4)/O_3$

5) A differential fibration (or simply a fibration) is by definition a triple $\lambda = (X, B, \pi)$ in which X and B are differential manifolds and π is a C^∞ mapping of X into B which is surjective and satisfies the following condition of local triviality, that is, for each $b \in B$ there exists an open neighbourhood U of b in B , with a differential manifold F and a diffeomorphism $\varphi: U \times F \rightarrow \pi^{-1}(U)$ such that $\pi(\varphi(y, t)) = y$ for all $y \in U$ and $t \in F$.

The manifold X is called ^{the} total space of the fibration the manifold B its base space, and the mapping π its projection. For each $b \in B$ the inverse image $X_b = \pi^{-1}(b)$ is a closed submanifold of X called the fibre of λ over b .

If all the fibres are diffeomorphic to the same manifold F , then X is said to be a fibre bundle of fibre type F . We often use the notation

$$\begin{array}{ccc} F & \longrightarrow & X \\ & & \downarrow \pi \\ & & B \end{array}$$

for a fibre bundle.

6) Let (E, B, π) be a differential fibration, such that for each $b \in B$ the fibre $E_b = \pi^{-1}(b)$ is endowed with a structure of a finite dimensional vector space. Then E endowed with the structure defined by the fibration (E, B, π) and the vector space structure on the fibres E_b , is said to be a real

vector bundle if the following condition is satisfied. For each $b \in B$ there exists an open neighbourhood u of b in B , a finite-dimension real vector space F , and a diffeomorphism

$$\varphi: u \times F \rightarrow \pi^{-1}(u) \quad \text{such that } \pi(\varphi(y, t)) = y$$

for $y \in u$ and $t \in F$, and such that for each point $y \in u$ the partial mapping $\varphi(y, \cdot)$ is an $\mathbb{R}$ -linear bijection of the vector space F onto the vector space E_y .

An important example of a vector bundle is the tangent bundle which is defined as follows. Let M be a differential manifold and let $T(M)$ be the union of tangent spaces $T_x(M)$ as x runs through M . Let $\Theta_M: T(M) \rightarrow M$ be the mapping which associates with each tangent vector $h_x \in T_x(M)$ the point $x \in M$ (the "origin" of h_x). Then there exists a unique structure of a differential manifold on $T(M)$ such that $T = (T(M), \Theta_M, 0_M)$ is a fibration, the tangent spaces $T_x(M)$ being the fibres, with the following condition being satisfied. For each chart $c = (u, \varphi)$ on M the mapping $\psi_c: (x, h) \rightarrow (c(x), \varphi(h))$ of $U \times \mathbb{R}^n$ onto $\Theta_M^{-1}(u)$ is a diffeomorphism. The tangent spaces $T_x(M)$ are vector spaces and the fibration T defines on $T(M)$ a structure of a real vector bundle, this vector bundle is called the tangent vector bundle or more briefly, the tangent bundle.

The cotangent bundle $T^*(M)$ can be defined in a similar manner, where the fibres are the cotangent spaces $T^*(M)$, this being also a real vector bundle.

7) We now define what is meant by two maps being homotopic or continuously deformable into each other.

Given two maps $f, g: Y \rightarrow X$ such that $f|_A = g|_A$ where $A \subset Y$ we say $f \simeq g \text{ rel } A$ if there is a map $F: Y \times I \rightarrow X$ satisfying

$$1) \quad F(y, 0) = f(y) \text{ all } y \in Y$$

- 2) $F(y, 1) = g(y)$ all $y \in Y$
 3) $F(y, t) = f(y) = g(y)$ all $y \in A$

F is called a homotopy from f to g , and if such a map exists the f is homotopic to g .

- 8) The relation $f \sim g$ rel A of the last section is an equivalence on the space of maps of $Y \rightarrow X$ such that the image of A under any of these maps is the same. We shall call this subset of X , A' .

Any equivalence relation on a space defines a set of equivalence classes, for the space of maps we are considering we shall denote the equivalence classes by $[Y, A, X, A']$. We will only be interested in the case where A is a single point in Y , known as the base point and denoted by y_0 . The image of this point in X shall be x_0 .

Of particular interest is the case where $Y = S^n$. Here we have the equivalence classes $[S^n, y_0, M, m_0]$ where m is the manifold into which S^n is mapped. A special notation is used for the above which is $\pi_n(M, m_0)$. These are known as the pointed homotopy groups of M .

We state without proof that the $\pi_n(M, m_0)$ are indeed groups, but note in general that the classes $[Y, A, X, A']$ are not groups.

- 9) A subspace A of a topological space X is a retract of X if there exists a map $r: X \rightarrow A$ (called a retraction), such that $r(a) = a$ for all $a \in A$: if $i: A \rightarrow X$ denotes the inclusion map, then r is a deformation retraction (and A is a deformation retract of X) if $i \circ r \simeq 1_X$. If also $i \circ r \simeq 1_X$ rel A , then r is a strong deformation retraction, and A is a strong deformation retract of X .

- 10) We now define the long exact homotopy sequence for the bundle (E, B, π) with fibre F .

Firstly we define what we mean by an exact sequence, given

any triple (x, y, z) such that $x \xrightarrow{f} y \xrightarrow{g} z$, then this sequence is exact if $\ker g = \operatorname{Im} f$

An infinite sequence $\cdots \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n \rightarrow \cdots$ is known as a long exact sequence if any pair of adjacent maps are exact in the sense of the above.

Now consider the bundle (E, B, π) with fibre F . We will assume that F and B have base points e_0 and b_0 respectively, and that F is identified with $\pi^{-1}(b_0) \subset E$ via a map $i: F \rightarrow E$ giving the diagram

$$\begin{array}{ccc} F & \xrightarrow{i} & E \\ & & \downarrow \pi \\ & & B \end{array}$$

With $\pi \circ i = b_0$ (the constant map which takes every element of F into b_0).

Then we have the long exact homotopy sequence,

$$\cdots \rightarrow \pi_n(B) \rightarrow \pi_{n-1}(F) \rightarrow \pi_{n-1}(E) \rightarrow \pi_{n-1}(B) \rightarrow \cdots$$

This sequence is of particular use when the homotopy groups of one of the spaces vanish, for instance in the case where the bundle is

$$\begin{array}{ccc} C^7 & \longrightarrow & S_{4,3} \\ & & \downarrow \\ & & \mathbb{R}P^3 \end{array}$$

which was considered in the thesis, $\pi_n(C^7) = 0$ for all n . Then the above sequence splits up into a number of short exact sequences $0 \rightarrow \pi_n(S_{4,3}) \rightarrow \pi_n(\mathbb{R}P^3) \rightarrow 0$

This implies that $\pi_n(S_{4,3}) = \pi_n(\mathbb{R}P^3)$ for all n . When such a condition holds the spaces are called weakly homotopically equivalent.

11) We now define singular homology modules. This is done in a number of steps.

Firstly we take a countable infinite product R^∞ of copies of R and consider the vectors.

$$E_0 = (0, 0, \dots, 0, \dots)$$

$$E_1 = (1, 0, \dots, 0, \dots)$$

$$E_2 = (0, 1, \dots, 0, \dots) \quad \text{etc}$$

We identify R^n with the subspace having all components of the n -th equal to 0. We let, for any $q > 0$, Δ_q denote the q dimension geometric simplex spanned by $E_0 \dots E_q$ called the standard (geometrical) q simplex. Thus Δ_0 is a point, Δ_1 the unit interval, Δ_2 a triangle (including its interior) etc.

If $p_0, \dots, p_q$ are points in some affine space $(E_0, \dots, E_q)$ will denote the restriction Δ_q of the unique affine map $R^q \rightarrow E$ taking E_0 into $p_0, \dots, E_q$ into p_q . Thus $(E_0 \dots E_q)$ is the identity map of which will be denoted S_q .

Given a space X , a singular q -simplex in X is a map. Thus for $q = 0$ it can be identified with a point in X , for $q = 1$ it is a path in X etc. The affine map $(p_0 \dots p_q)$ is a singular q -simplex in the affine space.

We add and subtract these singular q -simplexes in a purely formal manner. We also give ourselves a commutative ring R and we will multiply by scalars from R . More precisely define $S_q(X)$ to be the free R -module generated by all the singular q simplexes. The elements of $S_q(X)$ are formal linear combinations $\sum \alpha_\sigma \sigma$ where σ runs through singular q -simplexes, and the coefficients α_σ are from R . These sums are called singular q -chains.

For $q > 0$ we define $f_i^q : \Delta_{q-1} \rightarrow \Delta_q$ for $0 < i < q$ to be the affine map $(E_0, \dots, \hat{E}_i, \dots, E_q)$ where $\hat{E}_i$ means "omit E_i ". In other words, $f_i^q(E_j) = \begin{cases} E_j & j < i \\ E_{j+1} & j \geq i \end{cases}$

For an arbitrary singular q -simplex σ in a space X define the i -th face $\sigma^{(i)}$ of σ to be the singular $(q-1)$ -simplex

$\sigma \circ F_i^1$. Thus F_i^1 is the i -th face of Δ_q , and if

$$\sigma = (\rho_0 \dots \rho_q), \quad X \text{ affine}, \quad \sigma^{(i)} = (\rho_0 \dots \hat{\rho}_i \dots \rho_q)$$

Thus F_i^1 maps Δ_{q-1} homeomorphically and affinely onto the set theoretic face of Δ_q opposite to the vertex E_i .

We now define the boundary of a singular q -simplex σ to be the singular $(q-1)$ chain $\partial(\sigma) = \sum_{i=0}^q (-1)^i \sigma^{(i)}$

In the special case $\sigma = (\rho_0 \dots \rho_q)$

$$\partial(\sigma) = \sum_{i=0}^q (-1)^i (\rho_0 \dots \hat{\rho}_i \dots \rho_q)$$

we extend ∂ to a module homomorphism $S_q(X) \rightarrow S_{q-1}(X)$

by linearity, thus $\partial(\sum \alpha_i \sigma_i) = \sum \alpha_i \partial(\sigma_i)$. For

$q = 0$ the boundary of a 0-chain is defined to be zero. It can be proven that $\partial\partial = 0$.

A singular q -chain such that $\partial(c) = 0$ is called a cycle; if $c = \partial(c')$ for some $(q+1)$ chain c' is called a boundary. Two q -chains which differ by a boundary are called homologous, written as $c_1 \sim c_2$. From $\partial\partial = 0$ the boundaries form a submodule of the module of cycles, the quotient module Z_q/B_q is called the q -th singular homology module of X denoted by $H_q(X, R)$ or simply $H_q(X)$ when the reference to R is understood.

12) We now define singular cohomology modules.

The module $S^q(X)$ of all singular cochains on X is

$$\text{Hom}_R(S_q(X), R)$$

Thus a singular cochain of dimension q is a R -linear homomorphism $c : S_q(X) \rightarrow R$. If we denote the value of this homomorphism on a chain σ as $[\sigma, c]$ then we have the identities

$$[\sigma, c_1 + c_2] = [\sigma, c_1] + [\sigma, c_2]$$

$$[\sigma_1 + \sigma_2, c] = [\sigma_1, c] + [\sigma_2, c]$$

$$\text{and } [\alpha, \nu c] = \nu [\alpha, c] = [\alpha, \nu c]$$

so that $[\cdot, \cdot]$ is a bilinear pairing. We note that a q -cochain is uniquely determined by its values on the singular q -simplexes, and those values can be assigned arbitrarily. Thus $S^q(x)$ is isomorphic to the direct product of as many copies of R as there are singular q -simplexes in x .

Since S_q is a function from topological spaces to R -modules and $\text{Hom}_R(\cdot, R)$ is a contrafunctor on the category of R -modules, the composed function S^q is a contrafunctor from topological spaces to R -modules. More explicitly, if $f: x \rightarrow y$ is any map then $S^q(f): S^q(y) \rightarrow S^q(x)$ is defined by the formula $[\alpha, S^q(f)c] = [S_q(f)\alpha, c]$ for any q -chain α , q -cochain c . In the case where α is a singular q -simplex σ the formula becomes

$$[\sigma, S^q(f)c] = [f \circ \sigma, c]$$

In other words, $S^q(f)$ is the transpose ${}^t S_q(f)$ of $S_q(f)$. There is a unique homomorphism

$$\delta: S^q(x) \rightarrow S^{q+1}(x)$$

satisfying $[\partial \alpha, c] = [\alpha, \delta c]$ for all $(q+1)$ -chains α and q -cochains c . If $f: x \rightarrow y$ is any map, then

$$\delta S^q(f) = S^{q+1}(f) \delta \quad \text{Moreover,}$$

$\delta \delta = 0$. We call δ the coboundary operator.

We define the modules of cocycles and coboundaries by

$$Z^q(x) = \ker \delta: S^q(x) \rightarrow S^{q+1}(x)$$

$$B^q(x) = \text{image } \delta: S^{q-1}(x) \rightarrow S^q(x)$$

and the cohomology module $H^q(x; R)$ by

$$H^q(x, R) = Z^q(x) / B^q(x)$$

If $f: x \rightarrow y$ is any map then $S^q(f)$ respects cocycles and coboundaries, hence it induces by passage to the quotient, a

homomorphism $H^q(f) : H^q(Y) \rightarrow H^q(X)$

This makes H^q into a contrafunctor

$$H^q(gf) = H^q(f) H^q(g)$$

so that the cohomology modules are topological invariants.

REFERENCES

- 1) N. Steenrod "The Topology of Fibre Bundles", Princeton University Press (1951)
- 2) C. Chevalley "Theory of Lie Groups", Princeton University Press (1949)
- 3) Shastri A.R., Williams J.G. and Zvengrowski , 1980, Int. J. Theor. Phys. 19, 1-22
- 4) Finkelstein D. and Rubinstein J., J. Math. Phys. 9, 1762 (1968)
- 5) Finkelstein D. and Misner C.W., Ann. Phys. (N.Y.) 6, 230
- 6) Geroch R., Commun. Math. Phys., Vol. 13, No.3 (1969)
- 7) Finkelstein D. and McCollum G., J. Math. Phys., Vol. 16, No. 11 (1975)
- 8) Williams J.G. and Zia R.K.P., J. Phys. A. Nucl. Gen. Vol. 6 (January 1973)
- 9) Williams J.G. J. Phys. A. Nucl. Gen. Vol. 7, No. 15 (1974)
- 10) Williams J.G., J. Math. Phys., Vol. 11, No. 8 (1970)
- 11) Williams J.G., J. Math. Phys. Vol. 12, No. 2 (1971).
- 12) S.W. Hawking and G.F.R. Ellis. The Large Scale Structure of Space-time. Cambridge Monographs on Mathematical Physics
- 13) J. Dieudonne. Treatise on Analysis. Academic Press (1972)
- 14) M.J. Greenberg . Lectures on Algebraic Topology. Mathematics Lecture Note Series (1967)
- 15) C.R.F. Maunder. Algebraic Topology. Cambridge University Press. (1986)