

Graphical Abstract

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Highlights

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Abstract

A partially turbulent flow continuously incorporates irrotational fluid into the turbulent region, a phenomenon known as entrainment. Although entrainment locally acts at viscous scales, the thin interface separating the turbulent from the irrotational region is extremely convoluted, and twisted in such a way that renders the global entrainment flux scale-independent. Despite turbulent entrainment being widely recognised as a multi-scale process, the theoretical basis for quantifying the entrainment flux at multi-scales is lacking. In this paper we derive an equation that allows us to quantify the local entrainment velocity at multi-scales. This is done by defining the local entrainment velocity as the propagation speed of an iso-surface of filtered enstrophy relative to the coarse-grained velocity field, and using the filtered enstrophy budget to split the total velocity into its individual components, i.e. viscous, inviscid, baroclinic and sub-filter. The equation is used to investigate the entrainment at multi-scales in simulated turbulent mixing layers, where turbulence is sustained by either a mean shear or an unstable buoyancy gradient (Rayleigh-Taylor turbulence).

Keywords: Turbulence, Entrainment, Turbulent/non-turbulent interface

1. Introduction

Turbulent flows are ubiquitous in nature and, although studied widely, they are far from being fully understood. Turbulence occurs in many natural and industrial phenomena spanning a huge range of scales, from small, such as suspension dynamics ($10^{-6} - 10^{-3} m$) [1, 2, 3] to extremely large such as in supernovae explosion ($10^6 m$) [4]. Despite turbulence being ubiquitous, it is however uncommon to observe flows that are everywhere turbulent. Indeed, in many situations, irrotational and swirling regions co-exist and are separated by a region over which the flow transitions from being irrotational to turbulent [5, 6].

Partially turbulent flows can be categorized as semi-bounded and unbounded. Examples of semi-bounded flows include boundary layers [7, 8, 9], gravity currents [10, 11, 12] and penetrative convection [13]. Unbounded flows includes free jets [14, 15], shear-driven mixing layers [16] and buoyancy-driven mixing layers [17]. **While the turbulent flow region is vortical and characterized by a large range of eddy sizes, from the viscous eddies up large-scale vortical structures and even superstructures having a size that is many times the characteristic length scale [18, 19], the surrounding ambient flow is irrotational.** A type of mixing layer of particular interest is the compound shear- and buoyancy-driven turbulent mixing layer, whose instability mechanism is often referred as Rayleigh-Taylor/Kelvin-Helmholtz (RT/KH) instability [20, 21, 22, 23]. For this type of flow, it has been shown through a phenomenological analysis and confirmed by direct numerical simulations (DNSs) that the turbulence is initially dominated by the mean shear (figure 1 a) and at later times transitions to a buoyancy dominated state (figure 1 b) [24].

For both shear- and buoyancy-driven mixing, the turbulent region grows at the expense of the non-turbulent flow region through the process of entrainment. Irrotational fluid is continuously incorporated within the turbulent region by crossing the turbulent/non-turbulent interface (TNTI), an unsteady convoluted surface bounding the turbulence (see figure 2 b). Corrsin and Kistler [5] were the first to conjecture that the transition from irrotational to turbulent occurs over a thin layer that is proportional to the smallest active scale of the flow, i.e. the Kolmogorov length $\eta = (\nu^3/\epsilon)^{1/4}$, where ν is the fluid viscosity and ϵ is the average turbulent dissipation rate. Corrsin and Kistler [5] also speculated that the TNTI propagates towards the irrotational region at the Kolmogorov velocity $u_\eta = \nu/\eta$ by entraining non-turbulent fluid

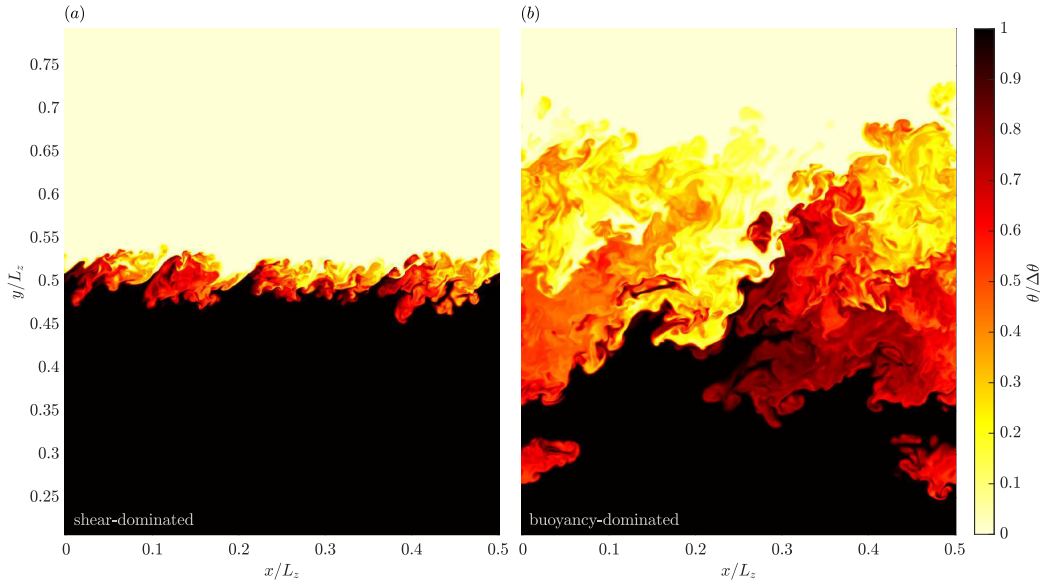


Figure 1: Detail of the scalar field for simulation *ST6* in the shear dominated phase (a) and RT phase (b). The figures represent a wall-normal slice of the scalar field. Lengths are made non-dimensional with the vertical box size, while the temperature field θ field is scaled with the initial temperature difference $\Delta\theta$.

which acquires vorticity through viscous diffusion. These hypotheses have been experimentally and numerically verified in a large number of studies [25]. Holzner and Lüthi [26] additionally showed that, although the TNTI propagates with a speed of the order of the Kolmogorov velocity, it is uncorrelated with the local dissipation field.

A notable exception for which the TNTI propagates at a velocity that is not only of viscous nature is the case of non-equilibrium turbulence, a special state in which the classical balance between production and dissipation $\epsilon = C_\epsilon u_L^3/L$ (L and u_L being the integral length and the large scale velocity fluctuations) with constant C_ϵ does not hold [27]. In this case, the TNTI has been observed to propagate at a speed scaling as ν/λ , instead of ν/η , where λ is the Taylor length [28]; since the Taylor length is also determined by the large scale velocity fluctuations, the TNTI propagation speed is thus also affected by inertial effects when the core turbulence is in a non-equilibrium state. Similarly, when conditioning the entrainment velocity to the proximity of large vortical structures, the entrainment velocity has been observed to increase [29, 30, 31].

Despite the TNTI local propagation speed being of the order of u_η in almost every case, the global growth rate of the turbulent region is observed to be independent of viscosity from a global viewpoint in the infinite Reynolds number limit (see all the examples of partially turbulent flows in [32]). This apparent inconsistency between the global and local viewpoint is reconciled through a multi-scale view of the entrainment process, as firstly conjectured by Sreenivasan et al. [33]. To better explain this concept, we define v_n^Δ as the local entrainment velocity at the scale Δ , and A^Δ as the TNTI area at the scale Δ . The total entrainment flux is the integral of the local entrainment velocity over the TNTI, or equivalently, for the mean-value theorem, the average local entrainment velocity at the TNTI location multiplied by the total TNTI area:

$$Q^\Delta = \int_{A^\Delta} v_n^\Delta dA^\Delta = \langle v_n^\Delta \rangle A^\Delta, \quad (1)$$

where the brackets indicate the average conditioned on the filtered TNTI.

The current understanding of multi-scale entrainment can be summarised by the following four assumptions: 1) K41 inertial scaling holds between the Kolmogorov and integral scales, namely: $\epsilon = \delta u^3/\Delta$ with $\eta < \Delta < L$; 2): the local entrainment velocity is of the order of the Kolmogorov velocity, namely $v_n \simeq \nu/\eta = u_\eta = (\nu\epsilon)^{1/4}$ (note that this is not true in non-equilibrium turbulence where $v_n \simeq \nu/\lambda$); 3) the TNTI behaves as a fractal object between η and L , namely $A^\Delta/A_0 = (\Delta/\eta)^{2-D}$, where D is its fractal dimension; 4) Q^Δ is independent of Δ . Assumption 1 is generally accepted in the infinite Reynolds number limit. Assumptions 2 and 3 have been widely observed in many studies, considering passive scalar [33], enstrophy [26, 13] or kinetic energy interfaces [34]. Equation (1), in addition to assumption 1, 2 and 3 allows to express the total entrainment flux as $Q/(A_0 u_L) \simeq Re^{3D/4-7/4}$. Assumption 4 leads to the celebrated fractal exponent $D = 7/3$ for the TNTI, that has been observed e.g. by de Silva et al. [34]. As a direct consequence, the local entrainment velocity at scale Δ scales as $v_n^\Delta \sim \Delta^{1/3}$. Note that conclusion and assumptions can be inverted at will, so that by assuming $v_n^\Delta \sim \Delta^{1/3}$ one can derive the scale independence of the entrainment flux. In later work, Meneveau and Sreenivasan [35] further developed these concepts generalizing the theory to the case of intermittent turbulence. Unfortunately, the intermittency correction still await either experimental or numerical confirmation.

Within the framework described above, the global entrainment flux is set by large scales while small and intermediate scales adapt in such a way that,

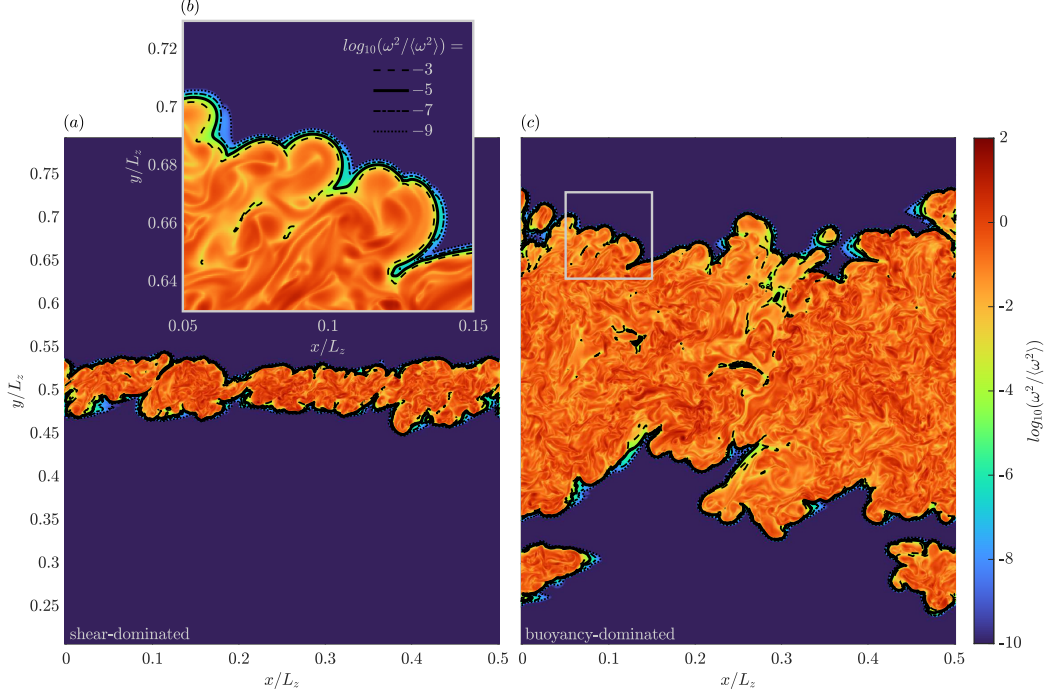


Figure 2: Wall-normal slice of the enstrophy field of simulation *ST6* in the shear dominated phase (a) and RT phase (c). The black lines demarcate the TNTI. Panel (c) shows a zoom on the TNTI in the RT phase, highlighting the offset between different enstrophy thresholds. Lengths are made non-dimensional with the vertical box size, while the enstrophy field ω^2 is scaled with the average value of the enstrophy in the co responding snapshot.

the higher the observation resolution, the larger the interface surface area and the smaller the local entrainment velocity, until the limit of Kolmogorov resolution. Mistry et al. [36] tested these hypotheses in an axisymmetric jet experimentally and drew a consistent picture in which $v_n^\Delta \sim \Delta^{1/3}$, $A^\Delta \sim \Delta^{-1/3}$ and Q^Δ is observed to be constant over two decades of Δ . In their work, to quantify the local entrainment velocity, the authors employed an interface tracking procedure based on the coarse-graining of a passive scalar field. The main limitations of the approach of [36] are two-fold. The first is related to the measuring procedure, namely laser induced fluorescence and particle image velocimetry, that allows to sample 2D slices only. When measuring the entrainment velocity by tracking the TNTI on a 2D slice, the motion in the third direction cannot be detected. This affects the estimate of the local entrainment velocity. Indeed, when a turbulent tongue moves in the out-of-

plane direction, it produces a large increment in the entrainment rate that is nothing but an artifact. The authors discussed this problem in Appendix A.2 of their manuscript. Another issue related to the 2D sampling is that filtering on 2D planes produces different results than filtering in 3D. The second limitation concerns the interface tracking procedure adopted to evaluate the entrainment rate at multi-scales. By inferring the propagation velocity directly from the movement of the filtered TNTI, one cannot distinguish between individual contributions to the total propagation velocity. In other words, one cannot determine how the relative strength between viscous, inviscid, baroclinic and sub-filter contributions affects the total entrainment velocity at multi-scales.

While it is now established that entrainment is a viscous process at small scales and inviscid at large scales, it is still unknown what is the relative strength of each individual contribution at multi-scales. Below, we derive an equation for the local entrainment velocity at multi-scales, by defining it as an iso-surface of filtered enstrophy relative to the filtered velocity field. After introducing the filtered enstrophy balance equation, this approach allows to quantify each single component contributing to the total entrainment flux at multi-scales (viscous, inviscid, baroclinic and sub-filter). We then analyze the TNTI and entrainment process in simulations of shear-driven and RT mixing layer. We choose RT/KH turbulence because this flow configuration allows to analyze simultaneously the effect of a mean shear and an unstable buoyancy gradient. Moreover, RT turbulence is completely new to any local entrainment analysis, and, because of the presence of an active scalar field, a non-zero baroclinic torque at the TNTI location could manifest.

The paper is organized as follows: in sec. 2 we clarify in which condition a RT/KH mixing layer manifests shear-dominated or RT turbulence. We then connect the definition of local and multi-scale entrainment rate for these two cases, and write a formula for the corresponding entrainment rates. In sec. 3 we derive the multi-scale local entrainment velocity equation. In sec. 4 we describe the numerical methods employed for simulating the flow and detecting the TNTI at multi scales. In sec. 5 we present and discuss the results of the analysis, and in sec. 6 we provide concluding remarks.

2. The global entrainment rate in shear and RT turbulence

In shear-driven turbulent mixing layers, a mean shear provides the energy supply at large scale for the growth of the mixing zone. Pope [32, chapter 11]

provides a full description of this type of flow, which is expected to expand linearly in time, because the large scale velocity u_L is equal to the mean shear velocity ΔU which is constant. More explicitly, a shear driven mixing layer grows as:

$$h(t) = S\Delta U t, \quad (2)$$

where S is a constant that has been measured both experimentally and numerically ranging from 0.06 to 0.11 [32].

RT turbulence arises at the interface between two fluids of different densities in the presence of an unstable buoyancy gradient. By balancing potential and kinetic energy one gets that the large scale velocity grows linearly in time as $u_L \sim \beta g \Delta \theta t$, where β is the thermal expansion coefficient, g is the gravitational acceleration and $\Delta \theta$ is the temperature difference. The mixing layer thickness grows as $h(t) \sim u_L t$, from which one obtains the well known quadratic expansion law [37, 38, 17]:

$$h(t) = \alpha \beta g \Delta \theta t^2. \quad (3)$$

α is a proportionality constant that has been measured both experimentally and numerically, and ranges from 0.03 to 0.07 [39, 4, 17].

When a mean shear is compounded with an unstable buoyancy gradient, a transition can occur from a shear-dominated to a buoyancy dominated turbulent state. This transition occurs at a time proportional to the cross-over time [24]:

$$t_c = \frac{\Delta U}{\beta g \Delta \theta}. \quad (4)$$

that is obtained by matching the shear mixing thickness scaling of equation 2 with the RT scaling of equation 3. Note that, for the transition from shear-dominated to RT turbulence to happen, the flow must be turbulent at early times, namely the shear Reynolds number at the time t_c must be much larger than the shear-driven transition-to-turbulence Reynolds number. More explicitly $Re_c = \Delta U h_c / \nu \gg 1$, where $h_c = h(t_c) = \Delta U t_c$. In conclusion, in a RT/KH flow, if $Re_c \gg 1$ we can observe shear-driven turbulence (linear growth) for $t \ll t_c$, and RT turbulence (quadratic growth) for $t \gg t_c$.

From the expansion laws of the shear-dominated and RT turbulence mixing layers, we can derive an expression for the global entrainment rate, namely the ratio between the entrainment flux and the large scale velocity. In order to connect the multi-scale entrainment process with the expansion laws of the

shear and RT turbulence, we consider the following relation connecting the entrainment flux at multi-scales with the expansion rate of the mixing zone:

$$Q^\Delta = \langle v_n^\Delta \rangle A^\Delta \simeq \frac{dh(t)}{dt} A_0 \quad (5)$$

This relation can be considered as a generalization of the global-local connection [40, 41], in which the entrainment flux at large scale $Q^L = \frac{dh(t)}{dt} A_0$ is hypothesized to be equal to the entrainment flux at dissipation scales $Q^\eta = A^\eta \langle v_n^\eta \rangle$. The entrainment constant E is defined as the ratio between the measured mixing layer expansion velocity $dh(t)/dt$ and the expected large velocity scale u_L , and can be obtained by using equation 5:

$$E = \frac{1}{u_L} \frac{dh(t)}{dt} = \frac{1}{u_L A_0} \frac{dV(t)}{dt} \simeq \frac{Q^\Delta}{u_L A_0}, \quad (6)$$

where $V(t)$ is the volume of fluid in the turbulent region.

Equation 6 in combination with 2 and 3 provides the entrainment constant E at multi-scales in shear-dominated turbulence:

$$S^\Delta = \frac{Q^\Delta}{2A_0 \Delta U} \quad (7)$$

and Rayleigh-Taylor turbulence:

$$\alpha^\Delta = \frac{Q^\Delta}{2A_0 \beta g \Delta \theta t}. \quad (8)$$

Note that, to derive equations 8 and 7, we opted for calculating u_L as the time derivative of the mixing layer thickness scalings (equations 3 and 2) and leaving out the constants S and α . In this way, we include the factor 2 coming from the derivation, rendering $S^\Delta \simeq S$ and $\alpha^\Delta \simeq \alpha$ theoretically equivalent to the constants reported in the literature, at least for large Δ . Indeed, in the literature S and α are mostly measured by a best-fitting of the mixing thickness directly. Moreover, the entrainment flux and constant are expected to be scale independent, so that we expect $S^\Delta \simeq S$ and $\alpha^\Delta \simeq \alpha$.

The peculiarity of RT turbulence is that, contrarily to other studied cases [36, 42, 43], RT turbulence is an accelerating flow, in which the entrainment velocity and flux grow linearly in time, and so must do the normalization factor to generate a time-independent entrainment rate. The key point is that equation 7 and 8 allows to quantify the flow constants S and α at multi-scales. In fact, these quantities are believed to be constant in a bulk perspective only, while here we are defining them at any scale.

3. Theory of entrainment at multi-scales

In this section we derive an equation for the entrainment velocity that is valid at multi-scales by defining the entrainment velocity as the propagation velocity of an iso-surface of filtered enstrophy relative to the coarse-grained fluid velocity field. The approach is conceptually similar to the "Large Eddy Simulation" approach developed by Philip et al. [44], who considered a TNTI defined based on the kinetic energy, and developed integral equation for the total fluxes.

We define a linear filter operator as the convolution between a smoothing kernel $G(\mathbf{x})$ and a scalar function $f(\mathbf{x})$:

$$\bar{f} = f \star G = \int_{R^3} G(\mathbf{x} + \mathbf{y}) f(\mathbf{y}) d^3\mathbf{y}. \quad (9)$$

The smoothing kernel can be, for instance, a normalized Gaussian or box function. We consider the enstrophy field ω^2 . The total derivative of an iso-surface of filtered enstrophy $\bar{\omega}^2$ with its own propagating velocity $\mathbf{u}_s^\Delta$ vanishes by definition (see [26] for a similar approach without filtering):

$$\frac{\partial \bar{\omega}^2}{\partial t} + \mathbf{u}_s^\Delta \cdot \nabla \bar{\omega}^2 = 0. \quad (10)$$

We define the entrainment velocity at scale Δ as the difference between the iso-surface propagation speed and the coarse grained fluid velocity:

$$\mathbf{u}_s^\Delta = \bar{\mathbf{u}} + v_n^\Delta \hat{\mathbf{n}}^\Delta \quad (11)$$

where $\bar{\mathbf{u}}$ is the coarse-grained velocity field, and $\hat{\mathbf{n}}^\Delta = \nabla (\bar{\omega}^2) / |\nabla \bar{\omega}^2|$ is the unit vector normal to the iso-surface of filtered enstrophy. We thus obtain:

$$\frac{\partial \bar{\omega}^2}{\partial t} + \bar{\mathbf{u}} \cdot \nabla \bar{\omega}^2 = -v_n^\Delta |\nabla \bar{\omega}^2| \quad (12)$$

We consider the enstrophy budget equation:

$$\frac{\partial \omega^2}{\partial t} + \mathbf{u} \cdot \nabla \omega^2 = 2\omega_i \omega_j S_{ij} + \nu \nabla^2 \omega^2 - 2\nu \frac{\partial \omega_i}{\partial x_j} \frac{\partial \omega_i}{\partial x_j} + 2\boldsymbol{\omega} \cdot (\nabla \times \mathbf{b}) \quad (13)$$

where S_{ij} is the strain rate tensor, ν is the fluid viscosity and $\mathbf{b}$ is the buoyancy force in the Boussinesq approximation. The latter is equal to $\beta g \theta \mathbf{e}_3$, where

β is the thermal expansion coefficient, g is the gravitational acceleration, θ is the temperature and $\mathbf{e}_3$ is the vertical unit vector. By filtering equation 13 and splitting the mean and residual of the non-linear advection term we obtain the filtered enstrophy budget equation:

$$\frac{\partial \overline{\omega^2}}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \overline{\omega^2} = 2\overline{\omega_i \omega_j S_{ij}} + \nu \nabla^2 \overline{\omega^2} - 2\nu \overline{\frac{\partial \omega_i}{\partial x_j} \frac{\partial \omega_i}{\partial x_j}} + 2\overline{\boldsymbol{\omega} \cdot (\nabla \times \mathbf{b})} - 2 \left(\overline{\mathbf{u} \cdot \nabla \omega^2} - \overline{\mathbf{u}} \cdot \nabla \overline{\omega^2} \right) \quad (14)$$

When the filter size Δ approaches zero, the sub-filter term vanishes and equation (13) is recovered. By combining equation (12) and (14) we derive an expression for the local entrainment velocity at scale Δ :

$$v_n^\Delta = \underbrace{-\frac{2\overline{\omega_i \omega_j S_{ij}}}{|\nabla \omega^2|}}_{\mathcal{P}} - \underbrace{\frac{\nu \nabla^2 \overline{\omega^2}}{|\nabla \omega^2|}}_{\mathcal{D}} + \underbrace{\frac{2\nu \overline{\frac{\partial \omega_i}{\partial x_j} \frac{\partial \omega_i}{\partial x_j}}}{|\nabla \omega^2|}}_{\epsilon} - \underbrace{\frac{2\overline{\boldsymbol{\omega} \cdot (\nabla \times \mathbf{b})}}{|\nabla \omega^2|}}_{\mathcal{B}} + \underbrace{\frac{\overline{\mathbf{u} \cdot \nabla \omega^2} - \overline{\mathbf{u}} \cdot \nabla \overline{\omega^2}}{|\nabla \omega^2|}}_{\mathcal{S}} \quad (15)$$

Equation (15) provides the local propagation velocity of an iso-surface of filtered enstrophy respect to the filtered velocity field, i.e. the local entrainment velocity at multi-scales. $\mathcal{P}$ represents the inviscid contribution to the total entrainment velocity, which originates from the enstrophy production term. The viscous contribution is composed of $\mathcal{D}$, which corresponds to the viscous diffusion of enstrophy, and ϵ , which is a purely negative term that corresponds to the viscous destruction of enstrophy. The term $\mathcal{B}$ is the buoyancy contribution which originates from the baroclinic torque present in the enstrophy balance equation. The latter contribution is the sub-filter term $\mathcal{S}$, which originates from the non-closure term of the filtered enstrophy budget equation.

Equation (15) has twofold utilities: first it allows to evaluate the local entrainment velocity at multi-scales from a single velocity field snapshot without requiring any interface tracking procedure, and, second, it provides each individual contribution to the local entrainment velocity at multi-scales.

4. Numerical methods

In this section we describe the numerical methods employed to i) simulate the turbulent mixing layer and ii) detect the TNTI at multi-scales.

label	$L_x L_y L_z$	$N_x N_y N_z$	Re_c	t_{end}/t_c
SL1	96^3	2160^3	1.06×10^3	19.63
SL2	96^3	2160^3	4.00×10^3	7.81
SL3	96^3	2160^3	7.99×10^3	5.21
ST4	$96^2 \times 192$	$1080^2 \times 2160$	6.25×10^4	2.30
ST5	$96^2 \times 192$	$1080^2 \times 2160$	6.95×10^4	5.99
ST6	$96^2 \times 192$	$1080^2 \times 2160$	1.04×10^5	1.60

Table 1: Simulations parameters: L_i and N_i are the size and the number of grid points along the i^{th} direction of the domain respectively; Re_c is the Reynolds number corresponding to the cross-over time t_c ; t_{end} is the total time of the simulation.

4.1. Direct numerical simulations

The present results are obtained by analyzing three-dimensional snapshots of the RT/KH turbulent mixing layer simulated by Brizzolara et al. [24]. The mixing layer is simulated by directly integrating the Navier-Stokes equations in the Boussinesq approximation in combination with an advection-diffusion equation for the temperature field. A fourth-order-accurate finite volume spatial discretization scheme and a third-order Adams-Bashforth scheme for time integration are employed [45, 46]. Periodic boundary conditions are imposed in the stream-wise and span-wise direction, while the upper and lower boundary are free-slip walls. The stream-wise velocity profile and the temperature field are both initialized with a step $u = \Delta U \text{sign}(z)/2$ [16] and $\theta = \Delta\theta/2 (\text{sign}(z) + 1)$, respectively. The Prandtl number $Pr = \nu/\kappa$ (where κ is the thermal diffusion coefficient) was fixed to unity in all the simulations. More details regarding the numerical methods can be found in [24]. Table 1 reports the simulations parameters, namely the number of grid points, the Reynolds number at the cross-over between shear-dominated and RT turbulence and the total duration of each simulation. It can be seen that simulations ST4, ST5 and ST6 have a sufficiently high Reynolds number to manifest a shear-dominated turbulent state. SL1, SL2 and SL3 possess a lower Re_c , meaning that the flow will be mostly of the RT type; nevertheless, the long duration of SL1 allows to reach a high Reynolds number in the RT phase.

4.2. Multi-scales TNTI detection from numerical data

Consistent with the theory described in sec. 3, the TNTI is identified by thresholding the instantaneous enstrophy field. Given the multi-scale nature of the analysis, the choice of an appropriate threshold for the TNTI identification is a delicate task, and different approaches can be followed. Mistry et al. [36] used a constant threshold on the passive scalar field at every scale to determine the TNTI position. In our case this would correspond to using the same value of the enstrophy threshold $\overline{\omega^2}_{thr}$ at every scale.

Krug et al. [47] analysed the TNTI surface area at multi-scales by imposing an appropriate threshold at full resolution and then filtering the resulting binary field. Following this approach, the vertical position of the mean interface, or equivalently the ratio between the turbulent and non-turbulent fluid volume, is the same at every scale.

It was not possible to adapt the approach of Krug et al. [47] in our case because we rely on a filtered enstrophy field. In order to preserve the vertical position of the mean interface, we first tried to adopt the following procedure: i) we fix an appropriate threshold at full resolution, ii) we calculate the corresponding turbulent fluid volume percentage $V_{turb}^{\%}$, iii) we filter the enstrophy field and iv) we define a threshold as the percentile of the filtered enstrophy corresponding to $100 - V_{turb}^{\%}$. By implementing this procedure, we observe that this approach preserves the mean interface vertical position equivalently to filtering the binary field. Nevertheless, the results on the entrainment velocity were inconsistent with the work of Mistry et al. [36]. Indeed, the entrainment velocity at the interface becomes quickly positive (detrainment), since the destruction term prevails on the others, making the viscous contribution dominating and positive. Holzner and van Reeuwijk [13] observed a similar trend by increasing the value of the enstrophy threshold, namely entering the turbulent region. It appears that attempting to fix the average interface position by adjusting the enstrophy threshold resulted in a drift deeper into the turbulent flow region. We thus preferred fixing the threshold at a constant value as in [36, 42], accepting that this does not perfectly preserve the mean interface position (see the iso-surfaces in figure 3). As we will show below, this did not result in an appreciable variation of the entrainment flux.

In our work we filtered the data with a 3D normalized Gaussian kernel. The fixed enstrophy threshold allows to effectively discriminate between the turbulent and the non-turbulent region. To confirm this assertion, we show in figure (3 a) the probability density function (pdf) of the logarithm of the

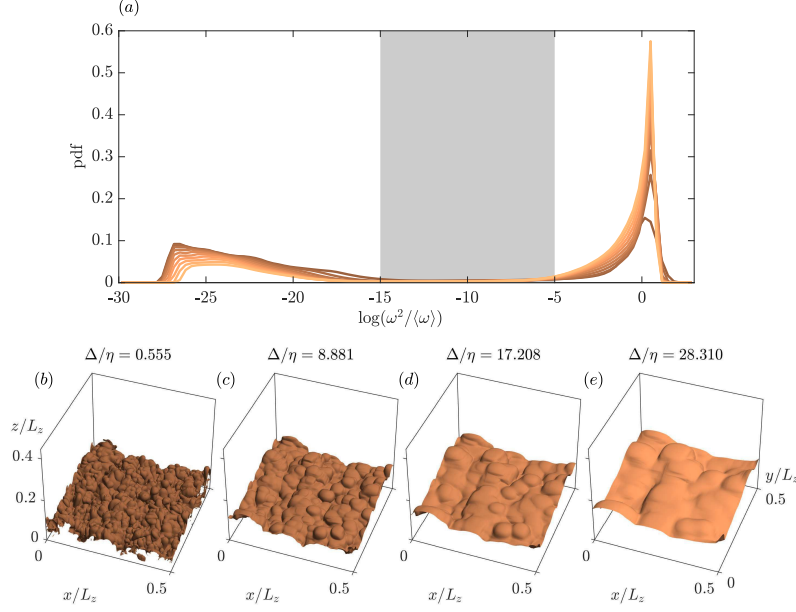


Figure 3: Probability density function of the logarithm of filtered enstrophy for increasing filter size (a). TNTIs obtained as iso-surfaces of filtered enstrophy for different filter size and the fixed value of the enstrophy threshold $\log(\omega_{thr}^2 / \langle \omega^2 \rangle) = -5$ for all the filter sizes Δ (b-e). The color change from brown to orange increasing the filter size.

enstrophy. The pdf possess two distinct peaks whose location is independent on the filter size, one in the turbulent region ($\log(\omega^2 / \langle \omega^2 \rangle) \simeq 1$), and one in the irrotational region ($\log(\omega^2 / \langle \omega^2 \rangle) \simeq -25$).

5. Results and discussion

In this section we present the analysis of the RT/KH mixing layer, both from a bulk perspective and the local entrainment process perspective, relying on the theory developed in sec 2 and sec. 3. We first analyze the bulk statistics of the flow, confirming the existence of a shear-dominated and a buoyancy dominated turbulent state. In addition, we analyze the internal flow structure of the mixing layer, proving that it is self-similar independently of the regime. We then analyze the local entrainment process at multi-scales, by making use of equation 15.

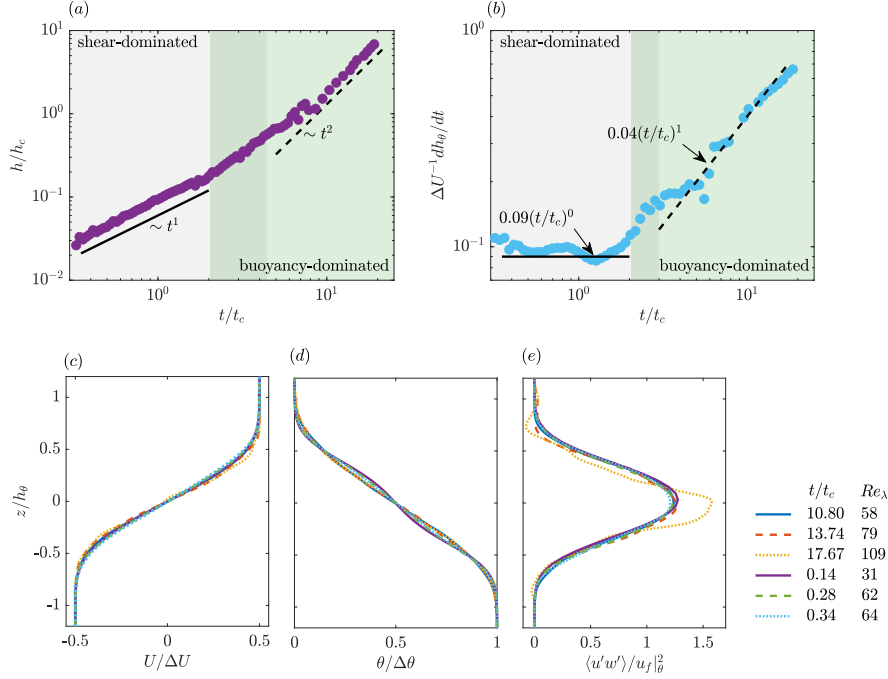


Figure 4: Bulk properties of the turbulent mixing layers. Temporal evolution of the normalized mixing layer thickness (a) and its time derivative (b), averaged over all the six simulations after rendering the quantities non-dimensional. Non-dimensional mean stream-wise velocity (c), temperature (d) and turbulent flux (e), averaged over horizontal planes for each time-step. Each line is color-coded with the non-dimensional time (darker lines corresponds to later times).

5.1. Flow self-similarity analysis

Both in shear-dominated and RT turbulence, mixing layers are expected to possess a self-similar structure, that is characterized by a set of characteristic scales. The integral length scale is identified with the mixing thickness $h(t)$, that can be estimated through the momentum or scalar thickness. In the case of RT/KH turbulence, we chose the integral temperature concentration thickness proposed by Vladimirova and Chertkov [48]:

$$h_\theta = \int_{-\infty}^{+\infty} 4\theta(1 - \theta)dz, \quad (16)$$

that is a valid proxy of the mixing thickness for the well-developed RT phase. Considering the shear-dominated phase, a momentum thickness defined

through the stream-wise velocity is the ideal quantity to represent the mixing thickness (see e.g. Rogers and Moser [16] for a definition of the momentum thickness). However, in the shear-dominated phase, the temperature is expected to behave as a passive scalar field transported by the shear velocity [24], reason for which the temperature thickness is expected to be a good proxy of the mixing thickness at early time [49].

For the stream-wise velocity and temperature profiles, we use as characteristic scales ΔU and $\Delta\theta$ respectively. Concerning the turbulent flux $\langle u'w' \rangle$ we define the instantaneous mean turbulent flux, that is the turbulent flux averaged within the mixing thickness h_θ :

$$u_f^2 = \frac{1}{h_\theta} \int_{-\infty}^{+\infty} \langle u'w' \rangle dz \quad (17)$$

Figure 4 shows the bulk statistics of the flow. Panel (a) shows the temporal evolution of the mixing layer thickness, that is in good approximation linear at early times (equation 2) and quadratic at late times (equation 3). In addition, panel (b) shows its derivative, compared with the expected scalings.

To test the self-similarity of the internal structure of the mixing layer in both phases, we report in figure 4 the normalized vertical profiles of the mean stream-wise velocity (c), temperature (d) and turbulent flux (d), averaged over horizontal planes. We report six different times, three in the shear dominated phase ($t < 2t_c$) and three in the buoyancy dominated phase ($t > 10t_c$), and consider only curves corresponding to developed turbulence, i.e. a value of the Taylor Reynolds number Re_λ averaged within the mixing layer larger than 30. The mean stream-wise velocity (4 (c)) is approximately self-similar in both regimes. The same applies for the scalar concentration profile shown in figure 4 (c), which differs only for the violet solid line, corresponding to the lowest Re_λ . The turbulent flux profile in figure 4 (d) is collapsed always on the same master curve, except for the largest time considered ($t = 17.64t_c$). The reason of the latter difference may be related to the fact that the simulation domain has an aspect ratio of one, and at late times the largest flow structures are influenced by the periodic stream-wise boundary conditions.

5.2. Local entrainment velocity and flux at multi-scale

We examine the scalings of the TNTI area, entrainment velocity and entrainment flux for two distinct three-dimensional snapshots, one belonging to the shear dominated (figure 6) and the other to the RT regime (figure 5).

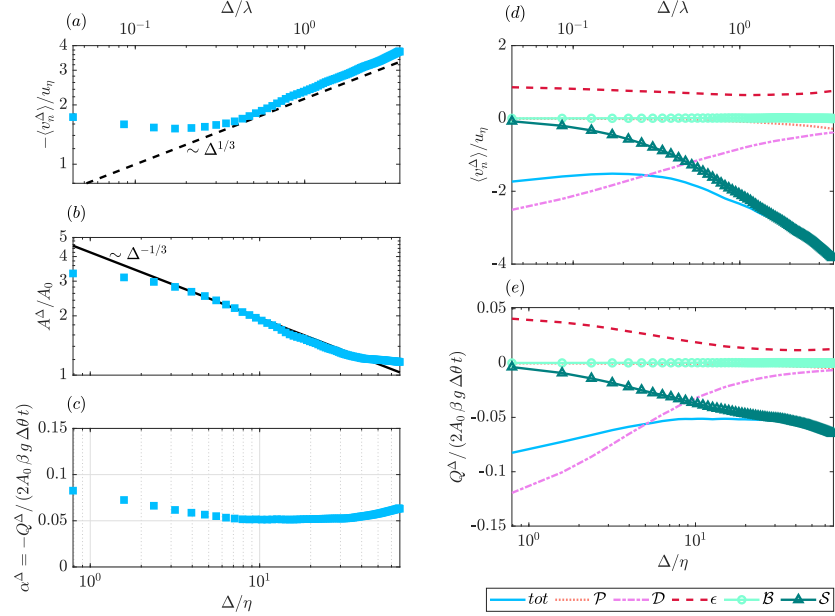


Figure 5: Scalings of some TNTI properties at time $t = 14.7 t_c$, when, accordingly to figure 4 (a) and (b), the turbulence is dominated by buoyancy. On the left hand side panel (a) shows the mean local entrainment velocity, (b) the TNTI surface area and (c) the total entrainment flux. On the right hand side, panel (d) shows each individual contributions to the local entrainment velocity and (e) the flux. The filter size Δ is normalized with the Kolmogorov length η (bottom axis) and the Taylor length $\lambda = \sqrt{15\nu/\epsilon u'}$ (top axis), where the dissipation rate ϵ and the root mean square velocity u' are averaged within the mixing zone. The results are obtained by considering a sub-volume of size $N_x \cdot N_y \cdot N_z = 512^3$ from simulation *SL1*. The Taylor Reynolds number is $Re_\lambda = u'\lambda/\nu \simeq 86$.

In the RT regime (figure 5), both the entrainment velocity (panel (a)) and surface area (panel (b)) scale consistently with the theoretical prediction. The resulting entrainment flux (panel (c)) is thus, in good approximation, scale-independent for more than one decade of scale separation. A slight deviation occurs at $\Delta \simeq 10\eta$. The non-dimensional entrainment flux provides a value of $\alpha \simeq 0.05$, that is consistent with the literature ($0.02 < \alpha < 0.07$) [17]. By analyzing the individual components of the entrainment velocity and flux (panel (d and e)) it can be seen that at full resolution the classical balance between viscous production and destruction holds, confirming a well-known feature of the entrainment process which is believed to be universal [26]. By increasing the filter size, the viscous production term becomes progressively

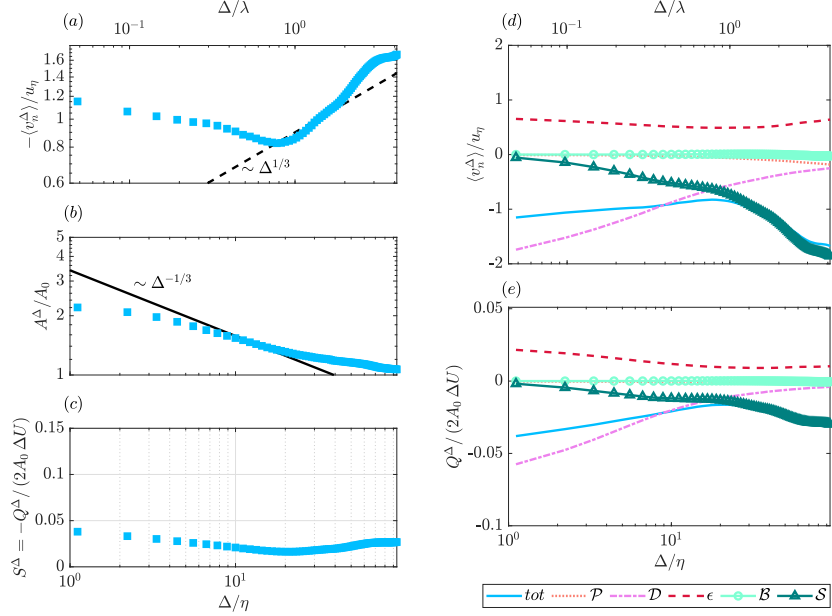


Figure 6: Scalings of some TNTI properties at time $t = 1.2t_c$, when, accordingly to figure 4 (a) and (b), the turbulence is dominated by shear. On the left hand side panel (a) shows the mean local entrainment velocity, (b) the TNTI surface area and (c) the total entrainment flux. On the right hand side, panel (d) shows each individual contributions to the local entrainment velocity and (e) the flux. The filter size Δ is normalized with the Kolmogorov length η (bottom axis) and the Taylor length $\lambda = \sqrt{15\nu/\epsilon u'}$ (top axis), where the dissipation rate ϵ and the root mean square velocity u' are averaged within the mixing zone. The results are obtained by considering a sub-volume of size $N_x \cdot N_y \cdot N_z = 512^3$ from simulation *ST5*. The Taylor Reynolds number is $Re_\lambda = u'\lambda/\nu \simeq 80$.

less prominent. At $\sim 6\eta$ the sub-filter term overcomes the viscous diffusion and balances viscous destruction. At larger scales, the entrainment flux is thus mainly carried by energy containing eddies through advective transport of enstrophy. This is reminiscent of the results of Philip et al. [44] who also observed a transition from viscous to sub-filter scale fluxes with increasing filter size. It must be noted that, despite the flow being driven by buoyancy, the baroclinic torque does not play an active role at any scale, resulting always negligible at the TNTI.

In the shear-driven turbulence regime (figure 6), the entrainment velocity and surface area (panels (a) and (b)) deviate significantly from the predicted scalings, except for a narrow range where agreement is observed. This be-

havior is due to the limited scale separation available at early times, when the flow is shear-dominated [24]. Nevertheless, the resulting entrainment flux (panel (c)) is approximately scale-independent for almost one decade of scale separation. In fact, despite the deviation from the expected inertial-range scaling prediction for v_n^Δ and A^Δ , the flow adapts to render the total entrainment flux almost scale-independent. After non-dimensionalization, the entrainment flux show a value of approximately $S \simeq 0.02$, that is consistent with the lower bound reported in the literature [16]. The individual components of the entrainment flux and velocity (panels (d) and (e)) show an almost indistinguishable behavior from the RT case. Indeed, while at the Kolmogorov scale viscous diffusion balances destruction, it becomes progressively less important as the filter size increases. At $\Delta \simeq 10\eta$ the sub-filter term dominates over viscous diffusion, and balances viscous destruction.

6. Concluding remarks

In this work we analyzed the entrainment process in direct numerical simulation of shear- and buoyancy-driven turbulence. This was done by developing a novel multi-scale equation for the entrainment velocity and flux that allows to quantify each individual contribution to the total entrainment (viscous, inviscid, baroclinic and sub-filter). The equation permits to quantify the shear-driven mixing layer constant S as well as the RT α coefficient from a single simulation snap-shot at multi-scales. Notably, these constants are almost scale-independent for the analyzed snapshots, and possess values that are consistent with the literature. The analysis of the individual contributions to the entrainment flux shows that, not only the baroclinic contributes marginally to the entrainment velocity at full resolution, but also at inertial range scales. While at full resolution local entrainment is brought about by viscous diffusion of enstrophy, at larger scales, the entrainment flux is thus mainly carried by energy containing eddies through advective transport of enstrophy.

Similarly to the TNTI, sharp boundaries also exist within the turbulent flow region, which are called turbulent-turbulent interfaces [50]. These separate regions of different turbulent intensities, for instance intense vortical structures including large super-structures [19] and their less turbulent surrounding flow. The theoretical framework introduced in this work can be applied to these internal interfaces to study how vortical structures entrain fluid at multi-scales. This would provide a better understanding on how large-scale coherent

vortices promote transport of mass, momentum and heat [18]

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References

- [1] Greg A Voth and Alfredo Soldati. Anisotropic particles in turbulence. *Annu. Rev. Fluid Mech*, 49(1):249–276, 2017.
- [2] Stefano Brizzolara, Marco Edoardo Rosti, Stefano Olivieri, Luca Brandt, Markus Holzner, and Andrea Mazzino. Fiber tracking velocimetry for two-point statistics of turbulence. *Physical Review X*, 11(3):031060, 2021.
- [3] François-Gaël Michalec, Itzhak Fouxon, Sami Souissi, and Markus Holzner. Zooplankton can actively adjust their motility to turbulent flow. *Proceedings of the National Academy of Sciences*, 114(52):E11199–E11207, 2017.
- [4] William H Cabot and Andrew W Cook. Reynolds number effects on rayleigh–taylor instability with possible implications for type ia supernovae. *Nature Physics*, 2(8):562–568, 2006.
- [5] Stanley Corrsin and Alan L Kistler. Free-stream boundaries of turbulent flows. Technical report, Johns Hopkins University, Baltimore, 1955.
- [6] Arkady Tsinober. *An informal conceptual introduction to turbulence*. Springer, 2009.
- [7] Guillem Borrell and Javier Jiménez. Properties of the turbulent/non-turbulent interface in boundary layers. *Journal of Fluid Mechanics*, 801: 554–596, 2016.
- [8] Ron Schnapp, Erez Shapira, David Peri, Yardena Bohbot-Raviv, Eyal Fattal, and Alex Liberzon. Extended 3d-ptv for direct measurements of

- lagrangian statistics of canopy turbulence in a wind tunnel. *Scientific reports*, 9(1):1–13, 2019.
- [9] Kapil Chauhan, Jimmy Philip, Charitha M De Silva, Nicholas Hutchins, and Ivan Marusic. The turbulent/non-turbulent interface and entrainment in a boundary layer. *Journal of Fluid Mechanics*, 742:119–151, 2014.
 - [10] Mark A Hallworth, Jeremy C Phillips, Herbert E Huppert, and R Stephen J Sparks. Entrainment in turbulent gravity currents. *Nature*, 362(6423):829–831, 1993.
 - [11] JS Turner. Turbulent entrainment: the development of the entrainment assumption, and its application to geophysical flows. *Journal of Fluid Mechanics*, 173:431–471, 1986.
 - [12] TH Ellison and JS Turner. Turbulent entrainment in stratified flows. *Journal of Fluid Mechanics*, 6(3):423–448, 1959.
 - [13] Markus Holzner and Maarten van Reeuwijk. The turbulent/nonturbulent interface in penetrative convection. *Journal of Turbulence*, 18(3):260–270, 2017.
 - [14] Maarten van Reeuwijk and Markus Holzner. The turbulence boundary of a temporal jet. *Journal of Fluid Mechanics*, 739:254–275, 2014.
 - [15] Marc Wolf, Markus Holzner, Beat Lüthi, Dominik Krug, Wolfgang Kinzelbach, and Arkady Tsinober. Effects of mean shear on the local turbulent entrainment process. *Journal of fluid mechanics*, 731:95–116, 2013.
 - [16] Michael M Rogers and Robert D Moser. Direct simulation of a self-similar turbulent mixing layer. *Physics of Fluids*, 6(2):903–923, 1994.
 - [17] Guido Boffetta and Andrea Mazzino. Incompressible rayleigh–taylor turbulence. *Annual Review of Fluid Mechanics*, 49:119–143, 2017.
 - [18] Ambrish Pandey, Janet D Scheel, and Jörg Schumacher. Turbulent superstructures in rayleigh–bénard convection. *Nature communications*, 9(1):2118, 2018.

- [19] N Hutchins and Ivan Marusic. Evidence of very long meandering features in the logarithmic region of turbulent boundary layers. *Journal of Fluid Mechanics*, 579:1–28, 2007.
- [20] Bhanesh Akula, Prasoon Suchandra, Mark Mikhaeil, and Devesh Ranjan. Dynamics of unstably stratified free shear flows: an experimental investigation of coupled kelvin–helmholtz and rayleigh–taylor instability. *Journal of Fluid Mechanics*, 816:619–660, 2017.
- [21] Dale M Snider and Malcolm J Andrews. Rayleigh–taylor and shear driven mixing with an unstable thermal stratification. *Physics of Fluids*, 6(10):3324–3334, 1994.
- [22] Aditi Sengupta, Bhavna Joshi, and Atul Kumar Verma. Thermally stratified free shear layers: Combined kelvin–helmholtz rayleigh–taylor instability. *Physics of Fluids*, 34(9):094113, 2022.
- [23] Aditi Sengupta and Bhavna Joshi. Effects of stabilizing and destabilizing thermal gradients on reversed shear-stratified flows: Combined kelvin–helmholtz rayleigh–taylor instability. *Physics of Fluids*, 2023.
- [24] Stefano Brizzolara, Jean-Paul Mollicone, Maarten van Reeuwijk, Andrea Mazzino, and Markus Holzner. Transition from shear-dominated to rayleigh–taylor turbulence. *Journal of Fluid Mechanics*, 924, 2021.
- [25] Carlos B da Silva, Julian CR Hunt, Ian Eames, and Jerry Westerweel. Interfacial layers between regions of different turbulence intensity. *Annu. Rev. Fluid Mech*, 46(1):567–590, 2014.
- [26] M Holzner and B Lüthi. Laminar superlayer at the turbulence boundary. *Physical review letters*, 106(13):134503, 2011.
- [27] Pedro Cardoso Valente and John Christos Vassilicos. Universal dissipation scaling for nonequilibrium turbulence. *Physical review letters*, 108(21):214503, 2012.
- [28] Gioacchino Cafiero and J Christos Vassilicos. Non-equilibrium scaling of the turbulent-nonturbulent interface speed in planar jets. *Physical Review Letters*, 125(17):174501, 2020.

- [29] Marius M Neamtu-Halic, Dominik Krug, George Haller, and Markus Holzner. Lagrangian coherent structures and entrainment near the turbulent/non-turbulent interface of a gravity current. *Journal of fluid mechanics*, 877:824–843, 2019.
- [30] Marius M Neamtu-Halic, Dominik Krug, Jean-Paul Mollicone, Maarten Van Reeuwijk, George Haller, and Markus Holzner. Connecting the time evolution of the turbulence interface to coherent structures. *Journal of fluid mechanics*, 898, 2020.
- [31] MM Neamtu-Halic, J-P Mollicone, M Van Reeuwijk, and M Holzner. Role of vortical structures for enstrophy and scalar transport in flows with and without stable stratification. *Journal of Turbulence*, 22(7): 393–412, 2021.
- [32] Stephen B Pope. Turbulent flows, 2001.
- [33] KR Sreenivasan, R Ramshankar, and CH Meneveau. Mixing, entrainment and fractal dimensions of surfaces in turbulent flows. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 421 (1860):79–108, 1989.
- [34] Charitha M de Silva, Jimmy Philip, Kapil Chauhan, Charles Meneveau, and Ivan Marusic. Multiscale geometry and scaling of the turbulent-nonturbulent interface in high reynolds number boundary layers. *Physical review letters*, 111(4):044501, 2013.
- [35] Charles Meneveau and KR Sreenivasan. Interface dimension in intermittent turbulence. *Physical Review A*, 41(4):2246, 1990.
- [36] Dhiren Mistry, Jimmy Philip, James R Dawson, and Ivan Marusic. Entrainment at multi-scales across the turbulent/non-turbulent interface in an axisymmetric jet. 2016.
- [37] E Fermi and J von Neumann. Taylor instability of incompressible liquids. part 1. taylor instability of an incompressible liquid. part 2. taylor instability at the boundary of two incompressible liquids. Technical report, Los Alamos National Lab.(LANL), Los Alamos, NM (United States), 1953.

- [38] JR Ristorcelli and TT Clark. Rayleigh–taylor turbulence: self-similar analysis and direct numerical simulations. *Journal of Fluid Mechanics*, 507:213–253, 2004.
- [39] Guy Dimonte, DL Youngs, A Dimitis, S Weber, M Marinak, S Wunsch, C Garasi, A Robinson, MJ Andrews, P Ramaprabhu, et al. A comparative study of the turbulent rayleigh–taylor instability using high-resolution three-dimensional numerical simulations: the alpha-group collaboration. *Physics of Fluids*, 16(5):1668–1693, 2004.
- [40] Maarten van Reeuwijk, J Christos Vassilicos, and John Craske. Unified description of turbulent entrainment. *Journal of Fluid Mechanics*, 908, 2021.
- [41] Dominik Krug, Daniel Chung, Jimmy Philip, and Ivan Marusic. Global and local aspects of entrainment in temporal plumes. *Journal of Fluid Mechanics*, 812:222–250, 2017.
- [42] Dhiren Mistry, James R Dawson, and Alan R Kerstein. The multi-scale geometry of the near field in an axisymmetric jet. *Journal of Fluid Mechanics*, 838:501–515, 2018.
- [43] A Cimarelli and G Boga. Numerical experiments on turbulent entrainment and mixing of scalars. *Journal of Fluid Mechanics*, 927, 2021.
- [44] Jimmy Philip, Charles Meneveau, Charitha M de Silva, and Ivan Marusic. Multiscale analysis of fluxes at the turbulent/non-turbulent interface in high reynolds number boundary layers. *Physics of Fluids*, 26(1):015105, 2014.
- [45] John Craske and Maarten van Reeuwijk. Energy dispersion in turbulent jets. part 1. direct simulation of steady and unsteady jets. *Journal of Fluid Mechanics*, 763:500–537, 2015.
- [46] RWCP Verstappen and AEP Veldman. Symmetry-preserving discretization of turbulent flow. *Journal of Computational Physics*, 187(1):343–368, 2003.
- [47] Dominik Krug, Markus Holzner, Ivan Marusic, and Maarten van Reeuwijk. Fractal scaling of the turbulence interface in gravity currents. *Journal of Fluid Mechanics*, 820, 2017.

- [48] Natalia Vladimirova and Michael Chertkov. Self-similarity and universality in rayleigh–taylor, boussinesq turbulence. *Physics of Fluids*, 21(1):015102, 2009.
- [49] J Westerweel, C Fukushima, Jakob Martin Pedersen, and JCR Hunt. Momentum and scalar transport at the turbulent/non-turbulent interface of a jet. *Journal of Fluid Mechanics*, 631:199–230, 2009.
- [50] Jerke Eisma, Jerry Westerweel, Gijs Ooms, and Gerrit E Elsinga. Interfaces and internal layers in a turbulent boundary layer. *Physics of Fluids*, 27(5):055103, 2015.