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Robust dynamic state feedback for underactuated systems with linearly parameterized disturbances

Enrico Franco[1] | **Ferdinando Rodriguez y Baena**[1] | **Alessandro Astolfi**[2,3]

[1]Mechanical Engineering Department, Imperial College London, London, UK

[2]Electrical and Electronic Engineering Department, Imperial College London, London, UK

[3]Dipartimento di Ingegneria Civile e Ingegneria Informatica, Università di Roma "Tor Vergata" Rome, Rome, Italy

Correspondence
Enrico Franco, Mechanical Engineering Department, Imperial College London, Exhibition Road, London SW7 2AZ, UK.
Email: ef1311@imperial.ac.uk

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Summary

This article investigates the control problem for underactuated port-controlled Hamiltonian systems with multiple linearly parameterized additive disturbances including matched, unmatched, constant, and state-dependent components. The notion of algebraic solution of the matching equations is employed to design an extension of the *interconnection and damping assignment passivity-based control* methodology that does not rely on the solution of partial differential equations. The result is a dynamic state-feedback that includes a disturbance compensation term, where the unknown parameters are estimated adaptively. A simplified implementation of the proposed approach for underactuated mechanical systems is detailed. The effectiveness of the controller is demonstrated with numerical simulations for the magnetic-levitated-ball system and for the ball-on-beam system.

KEYWORDS

adaptive control, disturbance rejection, mechanical systems, passivity-based control, underactuated systems

1 | INTRODUCTION

Interconnection and damping assignment passivity-based control (IDA-PBC) is an established approach for underactuated systems in port-controlled Hamiltonian (PCH) form.[1] The rationale behind this paradigm consists in shaping the energy of the closed-loop system with an appropriate control action in order to stabilize an open-loop unstable equilibrium. Applications of IDA-PBC include mechanical and electromechanical systems,[2,3] power system, underwater vehicles, and aerial vehicles.[4] As one of its main drawbacks, IDA-PBC requires solving a set of partial differential equations (PDE), termed matching equations, which in general can be a challenging task. Considerable effort has been directed toward addressing this issue and notable results include the following works: a detailed analysis of the matching equations in Reference 5; a new parameterization leading to a simplification of the PDE for a class of mechanical systems in Reference 6; sufficient conditions for PDE solvability in Reference 7. More recently, in Reference 8 the closed-loop energy was permitted to depend on the control input warranting more flexibility in the definition of the PDE, while in Reference 9 the PDE were avoided altogether introducing the notion of algebraic solution. Finally, the solution of the PDE was obviated for a specific class of mechanical systems employing a design strategy based on immersion and invariance (I&I) in Reference 10, and a new energy shaping control in Reference 11.

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A further aspect of energy shaping control that has been attracting increasing attention is robustness to disturbances, which are common to most practical application. Notable results in this area include the study of viscous friction within the controlled Lagrangians formulation,[12,13] and of continuous and smooth physical dissipation within the IDA-PBC framework,[14,15] while friction according to the Dahl model affecting the actuated part of the state was considered in Reference 16. Stability and robustness of disturbed PCH systems with dissipation were investigated in Reference 17. The robust energy shaping control of fully actuated mechanical systems with disturbances was presented in Reference 18, while disturbance attenuation for discrete-time systems was studied in Reference 19. More recently, integral IDA-PBC designs that rely on the classical solution of the PDE were proposed in References 20-22 for a class of underactuated mechanical systems with constant matched disturbances (ie, those affecting the actuated part of the state), and in Reference 23 for bounded matched disturbances. The result in Reference 21 was extended to unmatched constant disturbances in Reference 24, which, however, is only applicable to mechanical systems with constant inertia matrix. The case of nonconstant inertia matrix and variable but bounded disturbances was considered in Reference 25. In addition, the adaptive compensation of constant disturbances within energy shaping control was attempted in References 26,27: while the former still requires solving the PDE, the latter is confined to a limited class of mechanical systems. Besides energy shaping, a sliding mode control for a class of underactuated systems with dry friction was presented in Reference 28, while an active disturbance rejection control was proposed for systems with bounded uncertainties and applied to the inertia-wheel-pendulum in Reference 29. In addition, damping injection was combined with a nonlinear observer for the control of single-input PCH systems with bounded lumped disturbances in Reference 30. Finally, a controller robust to model parameters was proposed in Reference 31 for underactuated surface vessels. In summary, the robust energy shaping control of underactuated PCH systems subject to multiple additive variable disturbances that obviates the solution of the PDE remains an open problem with high practical relevance.

In this work, the dynamic IDA-PBC[9] is redesigned in order to account for a class of linearly parameterized disturbances, while the unknown parameters are estimated adaptively from the open-loop dynamics employing the I&I framework.[32] As a result, a new dynamic state-feedback controller that includes a disturbance compensation term is presented and stability conditions are discussed. The proposed approach is detailed for underactuated mechanical systems introducing an algebraic solution of the potential-energy PDE, provided that the kinetic-energy PDE is solvable analytically. The effectiveness of the controller is demonstrated with numerical simulations on the magnetic-levitated-ball system and the ball-on-beam system.

The rest of the article is organized as follows: Section 2 briefly recalls the notion of algebraic solution of the matching equations for completeness; Section 3 introduces the new dynamic state-feedback and the disturbance compensation strategy; Section 4 details the result for underactuated mechanical systems; Section 5 presents simulation results for the magnetic-levitated-ball system and for the ball-on-beam system; Section 6 contains the concluding remarks.

2 | ALGEBRAIC SOLUTION OF THE MATCHING EQUATIONS

Consider the following class of PCH systems with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^m$, interconnection matrix $J = -J^T \in \mathbb{R}^{n\times n}$, damping matrix $D = D^T > 0 \in \mathbb{R}^{n\times n}$, and continuously differentiable Hamiltonian $H(x)$ with gradient vector $\nabla H(x)$:

$$\begin{aligned} \dot{x} &= (J(x) - D(x))\nabla H(x) + G(x)u \\ y &= G(x)^T \nabla H(x). \end{aligned} \tag{1}$$

The input matrix $G \in \mathbb{R}^{n\times m}$, with $\text{rank}(G) = m < n \; \forall \, x \in \mathbb{R}^n$, and the matrix functions J, D are assumed continuously differentiable. The matrix $G^\perp$ is such that $G^\perp G = 0$ and $\text{rank}(G^\perp) = n - m$. The IDA-PBC control law u aims to achieve the following closed-loop dynamics:

$$\begin{aligned} \dot{x} &= (J_d(x) - D_d(x))\nabla H_d(x) \\ y_d &= G(x)^T \nabla H_d(x). \end{aligned} \tag{2}$$

The term $J_d = -J_d^T \in \mathbb{R}^{n\times n}$ is the desired interconnection matrix, $D_d = D_d^T > 0 \in \mathbb{R}^{n\times n}$ is the desired damping matrix, while the closed-loop Hamiltonian H_d is continuously differentiable and has a strict-minimizer at the assignable equilibrium x^* hence $\nabla H_d(x^*) = 0$ and $\nabla^2 H_d(x^*) > 0$.

Definition 1. [9]: x^* is an open-loop assignable equilibrium for (1) if the following condition is satisfied: $G(x^*)^\perp (J(x^*) - D(x^*))\nabla H(x^*) = 0$.

Designing J_d, D_d, H_d and the corresponding control input u requires solving the following matching equations, which is in general a challenging task:

$$G^\perp((J - D)\nabla H(x) - (J_d - D_d)\nabla H_d(x)) = 0. \tag{3}$$

To define the algebraic solution of the matching equations (3), we assume without loss of generality that the Hamiltonian can be expressed as $H = d + L^T x + \frac{1}{2}x^T\overline{H}x + h(x)$ with $d \in \mathbb{R}$, $L \in \mathbb{R}^n, \overline{H} \in \mathbb{R}^{n\times n}$, and $h(x) : \mathbb{R}^n \to \mathbb{R}$, where $h(x)$ contains the nonlinear terms of H and is such that $h(x^*) = 0, \nabla h(x^*) = 0, \nabla^2 h(x^*) = 0$. Thus, $x^* = 0$ is assignable provided that $G(0)^\perp (J(0) - D(0))L = 0$.

Definition 2. [9]: the continuously differentiable matrix valued function $P \in \mathbb{R}^{n\times n}$ is an $\mathcal{X}$ algebraic solution of the matching Equation (3), where $\mathcal{X}$ is a nonempty open set containing x^*, if:

$$\begin{aligned} \overline{H} + \nabla(P(x)x) > 0 \;\; x = x^* \\ G^\perp((J - D)\nabla H(x) - (J_d - D_d)(\nabla H(x) - L + P(x)x)) = 0 \;\; \forall x \in \mathcal{X} \subseteq \mathbb{R}^n. \end{aligned} \tag{4}$$

In particular, the mapping $x \to P(x)$ might not be the gradient vector of any scalar function hence it is in general not a solution of the original PDE (3). Based on the above definition of P, the auxiliary energy function $H_d(x, \xi)$ is defined on an extended state space as:

$$H_d(x, \xi) = H(x) - (L^T x + d) + \frac{1}{2}x^T P(\xi)x + \frac{1}{2}\|x - \xi\|^2_{R,} \tag{5}$$

where $R = R^T > 0$ is a free matrix to be defined, $\|w\|^2_R = w^T Rw$ is the weighted Euclidean norm, and $\xi \in \mathbb{R}^n$. Assuming $x^* = 0$ without loss of generality, (5) can be expressed as:

$$H_d(x, \xi) = \frac{1}{2}\begin{bmatrix} x^T & \xi^T \end{bmatrix}\begin{bmatrix} \overline{H} + P(0) + R & -R \\ -R & R \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} + h_d(x, \xi), \tag{6}$$

where h_d contains nonlinear terms and is such that $h_d(0, 0) = 0, \nabla h_d(0, 0) = 0, \nabla^2 h_d(0, 0) \geq 0$. It follows from (4) that the matrix in (6) is positive definite hence $H_d(x, \xi)$ is locally positive definite around the desired equilibrium where it has a local minimizer. In addition, if $P = P^T > 0$ and $h(x) > 0 \, \forall x \neq x^* \in \mathbb{R}^n$ then (6) is globally positive definite. Defining the terms $\Phi(x, \xi)$ and $\Psi(x, \xi)$ so that $(P(x) - P(\xi))x = \Phi(x, \xi)(x - \xi)$ and $\Psi(x, \xi) = (1/2)(\partial P(\xi)x/\partial\xi)$ and computing the partial derivatives of $H_d(x, \xi)$ from (5) gives:

$$\begin{aligned} \nabla_x H_d(x, \xi) = \nabla H - L + P(\xi)x + R(x - \xi) \\ \nabla_\xi H_d(x, \xi) = \Psi(x, \xi)x - R(x - \xi). \end{aligned} \tag{7}$$

According to Reference [9, Proposition 5], system (1) in closed loop with the following dynamic state-feedback, where $G^\dagger = (G^T G)^{-1}G^T$,

$$\begin{aligned} \dot{\xi} = -K\nabla_\xi H_d(x, \xi), \\ u = G^\dagger(-(J - D)\nabla H + (J_d - D_d)(\nabla H - L + P(x)x)) + v, \end{aligned} \tag{8}$$

has a PCH structure (locally) and a locally stable equilibrium at $(x, \xi) = (0, 0)$ for some $K > \overline{K} = \overline{K}^T \geq 0$ and $v = 0$, provided that $\Phi(x, \xi) = \Phi(x, \xi)^{\mathrm{T}} > 0$ in some nonempty open neighborhood Ω of the origin, that $\Psi^{\mathrm{T}}R^{-1}\dot{\Phi}R^{-1}\Psi < 2(N^T + P)D_d(N + P)$ and that $R(t) = \Phi(x(t), \xi(t))$ for all $t \geq 0$. Local asymptotic stability is concluded if y_d is a detectable output of (1) in closed-loop with (8) and $v = -\kappa y_d$.

3 | ROBUST DYNAMIC STATE-FEEDBACK

In this section the dynamic state-feedback (8) is redesigned for the following PCH system with $\tau + 1$ linearly parameterized additive (matched and unmatched) disturbances $\overline{f_k}(x)\delta_k \in \mathbb{R}^n$:

$$\dot{x} = (J(x) - D(x))\nabla H(x) + G(x)u - \sum_{k=0}^{\tau} \overline{f_k}(x)\delta_k$$
$$y = G(x)^T \nabla H(x). \tag{9}$$

The following assumptions are introduced:

Assumption 1. $\overline{f_k}(x) = \text{diag}\{[f_{kj}(x)]\}$ are continuously differentiable known functions, where $\overline{f_k}(x) \in \mathbb{R}^{n\times n}$ has components $f_{kj}(x)$, with index k, $0 \le k \le \tau \in \mathbb{N}$, indicating the specific disturbance, and index j, $1 \le j \le n \in \mathbb{N}$, indicating the state. In addition, $\overline{f_k}(x)$ and their first derivatives are bounded, $\overline{f_0} = I^n$ and $\overline{f_k}(x^*) \equiv 0$ for $k > 0$, while $\delta_k \in \mathbb{R}^n$ are unknown constants.

Notably, the above assumption does not require the disturbances to vanish at $x = x^*$, but for simplicity combines the contributions of all $\tau + 1$ disturbances at $x = x^*$ in the term $\overline{f_0}\delta_0$.

Assumption 2. there exists an assignable open-loop equilibrium x^* of (9) such that:

$$G^{\perp}((J(x^*) - D(x^*))\nabla H(x^*) - \delta_0) = 0. \tag{10}$$

The equilibrium is unstable if $\nabla^2 H(x^*) \le 0$. The equilibrium $x^* = 0$ is assignable if $G^{\perp}\delta_0 = 0$.

The first step in the design of the new controller consists in estimating the unknown parameters δ_k using the I&I methodology,[33] as outlined in the following result.

Lemma 1. *Consider the open-loop dynamics (9) under Assumption 1 and define the vectors of estimation errors $z_k \in \mathbb{R}^n$ as follows:*

$$z_k = \overline{r}_k^{-1}(\widetilde{\delta}_k - \delta_k) = \overline{r}_k^{-1}(\widehat{\delta}_k + \beta_k(x, \widehat{x}) - \delta_k). \tag{11}$$

The terms $\widehat{\delta}_k \in \mathbb{R}^n$ are vectors of estimator states, $\beta_k \in \mathbb{R}^n$ are free functions, $\widetilde{\delta}_k = \widehat{\delta}_k + \beta_k(x, \widehat{x})$ are vectors of adaptive estimates, $\overline{r}_k = \text{diag}\{[r_{kj}]\} \in \mathbb{R}^{n\times n}$ are diagonal matrices of dynamic scaling factors $r_{kj} \in \mathbb{R}^+$. Finally, $\widehat{x} \in \mathbb{R}^n$ are observer states obtained from the following filter with $K_e \ge \left(\prod_{k=0}^{\tau} \overline{r}_k\right)^2 + \varepsilon\alpha\theta^2 I^n \sum_{k=0}^{\tau} (\lambda_{\max}(\overline{r}_k)\|\sigma_k\|)^2$, where $\lambda_{\max}\{\overline{r}_k\}$ indicates the maximum eigenvalue of $\overline{r}_k$, and σ_k are known functions, while ε, α, $\theta \in \mathbb{R}^+$ are positive constants:

$$\dot{\widehat{x}} = (J - D)\nabla H(x) + G(x)u - \sum_{k=0}^{\tau} \overline{f_k}(x)\widetilde{\delta}_k - K_e e. \tag{12}$$

Since $\overline{f_k}(x)$ are continuously differentiable by hypothesis, we can write $\overline{f_k}(\widehat{x})^T - \overline{f_k}(x)^T = e^T\sigma_k$ for some functions $\sigma_k(x, e) \in \mathbb{R}^n$, with $e = \widehat{x} - x, \in \mathbb{R}^n$. Together with (12) consider the following adaptation law with estimates $\widetilde{\delta}_k$, with $\theta > 1 + \tau$, and with $\alpha > 1/3$:

$$\begin{aligned}
\widetilde{\delta}_k &= \widehat{\delta}_k + \beta_k(x, \widehat{x}) \\
\dot{\widehat{\delta}}_k &= -\tfrac{\partial \beta_k(x,\widehat{x})}{\partial x}\left((J - D)\nabla H(x) + G(x)u - \sum_{k=0}^{\tau} \overline{f_k}(x)\widetilde{\delta}_k\right) - \tfrac{\partial \beta_k(x,\widehat{x})}{\partial \widehat{x}}\dot{\widehat{x}} \\
\beta_k(x, \widehat{x}) &= -\alpha\overline{f_k}(\widehat{x})^T x \\
\dot{\overline{r}}_k &= \alpha\theta^2 \overline{r}_k (e^T\sigma_k)^2.
\end{aligned} \tag{13}$$

Then e and $\sum_{k=0}^{\tau}(\overline{f_k}(x)z_k\overline{r}_k)$ are bounded and converge to zero. In addition, $\overline{r}_k$ are bounded $\forall k$.

Proof: Computing the time derivative of z_k we obtain from (11):

$$\dot{z}_k = \overline{r}_k^{-1}\left(\dot{\widehat{\delta}}_k + \frac{\partial \beta_k}{\partial x}\left((J - D)\nabla H(x) + G(x)u - \sum_{k=0}^{\tau} \overline{f_k}(x)(\widetilde{\delta}_k - \overline{r}_k z_k)\right) + \frac{\partial \beta_k}{\partial \widehat{x}}\dot{\widehat{x}} - \dot{\overline{r}}_k z_k\right). \tag{14}$$

Substituting (13) with $\overline{f}_k(\hat{x})^T = \overline{f}_k(x)^T + e^T\sigma_k$ yields:

$$\dot{z}_k = -\bar{r}_k^{-1}\alpha\overline{f}_k(x)^T\sum_{k=0}^{\tau}\overline{f}_k(x)\bar{r}_k z_k - \bar{r}_k^{-1}\alpha(e^T\sigma_k)\sum_{k=0}^{\tau}\overline{f}_k(x)\bar{r}_k z_k - \alpha\theta^2(e^T\sigma_k)^2 z_k. \tag{15}$$

Defining the first Lyapunov function candidate $W = \frac{1}{2}\sum_{k=0}^{\tau}\left(z_k^T z_k\left(\prod_{i=0}^{\tau}\bar{r}_{i\neq k}\right)^{-2}\right) \geq 0$ and computing its time derivative while substituting (15) gives:

$$\begin{aligned}\dot{W} = &-\alpha\left(\sum_{k=0}^{\tau}\frac{\overline{f}_k(x)z_k}{\prod_{i=0}^{\tau}\bar{r}_{i\neq k}}\right)^2 - \alpha\theta^2\sum_{k=0}^{\tau}\left(\frac{z_k^T z_k(e^T\sigma_k)^2}{\left(\prod_{i=0}^{\tau}\bar{r}_{i\neq k}\right)^2}\right)\\ &-\alpha\left(\sum_{k=0}^{\tau}\frac{\overline{f}_k(x)z_k}{\prod_{i=0}^{\tau}\bar{r}_{i\neq k}}\right)^T\sum_{k=0}^{\tau}\left(\frac{z_k e^T\sigma_k}{\prod_{i=0}^{\tau}\bar{r}_{i\neq k}}\right) - \alpha\theta^2\sum_{k=0}^{\tau}\sum_{i=0}^{\tau}\left(\frac{z_k^T z_k(e^T\sigma_{i\neq k})^2}{\left(\prod_{i=0}^{\tau}\bar{r}_{i\neq k}\right)^2}\right).\end{aligned} \tag{16}$$

Defining the second Lyapunov function candidate $W' = W + \frac{1}{2}e^T e \geq 0$ and computing its time derivative, where $\dot{e} = -\left(\sum_{k=0}^{\tau}(\overline{f}_k(x)z_k\bar{r}_k) + K_e e\right)$ from (9) and (12) gives:

$$\begin{aligned}\dot{W}' = &-\alpha\sum_{k=0}^{\tau}\left[\left(\sum_{k=0}^{\tau}\frac{\overline{f}_k(x)z_k}{\prod_{i=0}^{\tau}\bar{r}_{i\neq k}}\right)^T\left(\frac{z_k e^T\sigma_k}{\prod_{i=0}^{\tau}\bar{r}_{i\neq k}}\right)^T\right]\begin{bmatrix}\frac{1}{4\theta^2} & \frac{1}{2}\\ \frac{1}{2} & \theta^2\end{bmatrix}\begin{bmatrix}\sum_{k=0}^{\tau}\frac{\overline{f}_k(x)z_k}{\prod_{i=0}^{\tau}\bar{r}_{i\neq k}}\\ \frac{z_k e^T\sigma_k}{\prod_{i=0}^{\tau}\bar{r}_{i\neq k}}\end{bmatrix}\\ &-\left[\left(\sum_{k=0}^{\tau}\frac{\overline{f}_k(x)z_k}{\prod_{i=0}^{\tau}\bar{r}_{i\neq k}}\right)^T e^T\right]\begin{bmatrix}\left(1-\frac{\tau+1}{4\theta^2}\right)\alpha & \frac{1}{2}\prod_{k=0}^{\tau}\bar{r}_k\\ \frac{1}{2}\prod_{k=0}^{\tau}\bar{r}_k & K_e\end{bmatrix}\begin{bmatrix}\sum_{k=0}^{\tau}\frac{\overline{f}_k(x)z_k}{\prod_{i=0}^{\tau}\bar{r}_{i\neq k}}\\ e\end{bmatrix}\\ &-\alpha\theta^2\sum_{k=0}^{\tau}\sum_{i=0}^{\tau}\left(\frac{z_k^T z_k(e^T\sigma_{i\neq k})^2}{\left(\prod_{i=0}^{\tau}\bar{r}_{i\neq k}\right)^2}\right).\end{aligned} \tag{17}$$

Recalling that $K_e > \left(\prod_{k=0}^{\tau}\bar{r}_k\right)^2$, observing that $\left(1-\frac{\tau+1}{4\theta^2}\right) \geq 3/4$ for $\theta \geq \tau+1$ and employing a Schur complement argument in (17) confirms that $\dot{W}' \leq 0$ if $\alpha > 1/3$.

To prove boundedness of $\bar{r}_k$ we define the Lyapunov function candidate $U = W' + \frac{\varepsilon}{2}\sum_{k=0}^{\tau}\lambda_{\max}(\bar{r}_k)^2 \geq 0$. Computing its time derivative from (13) and (17) we obtain:

$$\dot{U} \leq -e^T\left(K_e - \left(\prod_{k=0}^{\tau}\bar{r}_k\right)^2\right)e + \varepsilon\alpha\theta^2\sum_{k=0}^{\tau}(\lambda_{\max}(\bar{r}_k)^2\|\sigma_k\|^2)\|e\|^2. \tag{18}$$

Substituting K_e in (18) confirms that $\dot{U} \leq 0$ and consequently $\bar{r}_k \in \mathcal{L}_\infty$. In addition, from (17) $\frac{z_k}{\prod_{i=0}^{\tau}\bar{r}_{i\neq k}} \in \mathcal{L}_\infty$ and $e \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, while $\dot{e} \in \mathcal{L}_\infty$, thus $\lim_{t\to\infty} e = 0$. Finally, $\sum_{k=0}^{\tau}\left(\frac{\overline{f}_k(x)z_k}{\prod_{i=0}^{\tau}\bar{r}_{i\neq k}}\right) \in \mathcal{L}_2$, and since $\overline{f}_k(x)$ and their first derivatives are bounded by hypothesis (ref. Assumption 1), it follows from Barbalat's Lemma that $\sum_{k=0}^{\tau}(\overline{f}_k(x)z_k\bar{r}_k)$ is bounded and converges to zero ■.

Notably, the adaptation law (12) and (13) estimates a total of $n\times(\tau+1)$ different parameters hence it extends the result of Reference 33, where only $\tau+1$ parameters would be estimated.

Remark 1. The functions $\sigma_k(x, e) \in \mathbb{R}^n$ in (12) and (13) are chosen depending on the functions $\overline{f}_k(x)$, which are known by hypothesis (ref. Assumption 1). For the examples in Section 5, $\overline{f}_k(x) = x$ thus $\sigma_k(x, e) \equiv 1$. In general, if $\overline{f}_k(x, e)$ is continuously differentiable, then we can write $\overline{f}_k(x, e) = \overline{f}_k(x, 0) + \sigma_k(x, e)e$ and $\widetilde{f}_k(x, e) = \overline{f}_k(x, e) - \overline{f}_k(x, 0)$, thus $\sigma_k(x, e)e = \int_0^1\left(\frac{d(\widetilde{f}_k(x,es))}{ds}\right)ds$.[34] For the constant component of the disturbances we have $\overline{f}_0(x) = \overline{f}_0(\hat{x}) = I^n$ and consequently $\beta_0 = -\alpha x$,

$\frac{\partial \beta_0}{\partial \hat{x}} = 0, \dot{\bar{r}}_0 = 0$ in (13). In general, if the functions $\overline{f_k}(x)$ are integrable in x, the adaptation law (12) and (13) can be simplified adopting constant $\bar{r}_k$ and omitting the filter (12) as:

$$\begin{aligned} \widetilde{\delta}_k &= \hat{\delta}_k + \beta_k(x) \\ \dot{\hat{\delta}}_k &= \int \left(-\frac{\partial \beta_k(x)}{\partial x} \left((J-D)\nabla H(x) + G(x)u - \sum_{k=0}^{\tau} \overline{f_k}(x)\widetilde{\delta}_k \right) \right) dt \\ \beta_k(x) &= -\alpha \int (\overline{f_k}(x)^T) dx. \end{aligned} \tag{19}$$

Defining the Lyapunov function candidate $W_0 = \frac{1}{2}\sum_{k=0}^{\tau} z_k^T z_k$ and computing its time derivative while substituting (19) gives in this case:

$$\dot{W}_0 = -\alpha \sum_{k=0}^{\tau} \left((\overline{f_k}(x) z_k)^T \sum_{k=0}^{\tau} \overline{f_k}(x) z_k \right) < 0, \tag{20}$$

which holds true for all $\alpha > 0$. Finally, since according to Assumption 1 the state-dependent component of the disturbance vanishes at the equilibrium, it follows from Lemma 1 that z_0 converges to zero. Consequently, the assignable open-loop equilibrium x^* computed with the estimates $\widetilde{\delta}_0$ in (10) converges to the correct value.

The second step in the design of the new robust dynamic state-feedback consists in redefining the partial derivatives of the auxiliary energy function (7) as follows, where we have set $R(t) = \Phi(x(t), \xi(t))$ for all $t \geq 0$ as in Reference [9, Proposition 5]:

$$\begin{aligned} \nabla_x H_d'(x,\xi) &= \nabla H - L + P(x)x + F_1(x) + F_0 \\ \nabla_\xi H_d'(x,\xi) &= \Psi(x,\xi)x - (P(x) - P(\xi))x. \end{aligned} \tag{21}$$

The constant term F_0 and the state-dependent term $F_1(x)$ are designed according to the following set of one matching equation and two strict-minimizer conditions:

$$\begin{aligned} G^\perp \left((J_d(x) - D_d(x))(F_1(x) + F_0) + \sum_{k=0}^{\tau} \overline{f_k}(x)\widetilde{\delta}_k \right) &= 0 \\ \nabla_x^2 H(x^*) + \nabla_x (P(x^*)x^*) + \nabla_x F_1(x^*) &> 0 \\ F_0 &= -(\nabla H(x^*) - L + P(x^*)x^* + F_1(x^*)). \end{aligned} \tag{22}$$

The first equation in (22) is of algebraic nature and ensures matching of the disturbances in (9) hence it can be interpreted as a disturbance-matching condition. In particular, subtracting the first equation in (22) from the matching equations for system (9) recovers (3). Consequently, with the proposed controller design, the algebraic solution P of the original matching Equation (3) is preserved in the presence of disturbances. The two remaining conditions enforce a strict-minimizer of $H_d'(x,\xi)$ in $x = x^*$ and in some $\xi = \xi^*$. It follows from (10) that, employing the parameterization $H = d + L^T x + \frac{1}{2}x^T \overline{H} x + h(x)$, the existence of an assignable open-loop-unstable equilibrium in case $G^\perp \delta_0 \neq 0$ requires $G(x^*)^\perp (J(x^*) - D(x^*))((L^T + \overline{H}x^* + \nabla h(x^*)) - \delta_0) = 0$ and $\overline{H} + \nabla^2 h(x^*) \leq 0$. Comparing (22) with (4) shows that the terms F_0 and $F_1(x)$ ensure the existence of a strict-minimizer in $x = x^*$ even if $\nabla h(x^*) \neq 0$ and $\nabla^2 h(x^*) \neq 0$. From (21), the auxiliary energy function in a nonempty set containing the equilibrium x^* can be approximated as:

$$H_d'(x,\xi) = H(x) - (L^T x + d) + \frac{1}{2}x^T P(\xi)x + \frac{1}{2}\|x - \xi\|_\Phi^2 + (x - x^*)^T (F_0 + F_1(x^*)) + C. \tag{23}$$

If there exist some $C_x, C_0, C_1 \in \mathbb{R}^+$ so that $\|F_0\| < C_0$, $\|F_1(x)\| < C_1$ for all $\|x - x^*\| < C_x$ then there exist some arbitrary constant $C \in \mathbb{R}^+$ so that (23) is locally positive definite.

Finally, the new control law is constructed augmenting the dynamic state-feedback (8) with a disturbance compensation term according to the following result which builds upon.[9]

Proposition 1. *Consider the PCH system (9) with disturbances under Assumptions 1 and 2. Define the algebraic solution P so that $\Phi(x,\xi)=\Phi(x,\xi)^T>0$ for all (x,ξ) in a nonempty set $\Omega\subseteq\mathbb{R}^n\times\mathbb{R}^n$ containing the equilibrium (x^*,ξ^*), and let $R(t)=\Phi(x(t),\xi(t))$ for all $t\geq 0$ so that $\Psi^T R^{-1}\dot{\Phi}R^{-1}\Psi<2(N^T+P)D_d(N+P)$, where $Nx=\nabla H-L$. Define the following control law, with $F_1(x)$, F_0 satisfying conditions (22), and with $\widetilde{\delta}_k$ estimated according to (12) and (13):*

$$\begin{aligned}\dot{\xi}&=-K\nabla_\xi H_d'(x,\xi)\\ u&=G^\dagger(-(J-D)\nabla H+(J_d-D_d)(\nabla H-L+P(x)x))+u^*\\ u^*&=G^\dagger\left(\sum_{k=0}^{\tau}\overline{f}_k(x)\widetilde{\delta}_k+(J_d-D_d)(F_1(x)+F_0)\right).\end{aligned}\tag{24}$$

Then the equilibrium (x^*,ξ^*) is locally stable for some $K=K^T\geq 0$ if $\alpha\geq\alpha_0\left(\prod_{k=0}^{\tau}\overline{r}_k\right)^2>I^n,\alpha_0 D_d>I^n,\alpha_0>1$. In addition, if $\nabla_x H_d',\nabla_x^2 H_d',J_d,D_d\in\mathcal{L}_\infty$, then the equilibrium is (locally) asymptotically stable.

Proof: Adding the term u^* to u in (24) and regrouping common factors results in:

$$u=G^\dagger\left(-(J-D)\nabla H+(J_d-D_d)(\nabla H-L+P(x)x+F_1(x)+F_0)+\sum_{k=0}^{\tau}\overline{f}_k(x)\widetilde{\delta}_k\right).\tag{25}$$

Substituting (25) and $\nabla_x H_d'$ from (21) into (9) results in the following closed-loop dynamics:

$$\dot{x}=(J_d-D_d)\nabla_x H_d'+\sum_{k=0}^{\tau}\overline{f}_k(x)\overline{r}_k z_k.\tag{26}$$

According to Assumptions 1 and 2, x^* is an assignable equilibrium of the open-loop system (9), while it follows from (22) that x^* is a strict-minimizer of H_d'.

To prove the stability claim, we define the Lyapunov function $W''=H_d'(x,\xi)+W'(z_k,e)$. Computing the time derivative of W'' and substituting (17),(21),(26) yields:

$$\begin{aligned}\dot{W}''(x,\xi)=&\nabla_x {H_d'}^T\left((J_d-D_d)\nabla_x H_d'+\sum_{k=0}^{\tau}\overline{f}_k(x)\overline{r}_k z_k\right)-\nabla_\xi {H_d'}^T K\nabla_\xi H_d'\\ &-\alpha\sum_{k=0}^{\tau}\left[\left(\sum_{k=0}^{\tau}\frac{\overline{f}_k(x)z_k}{\prod_{i=0}^{\tau}\overline{r}_{i\neq k}}\right)^T\left(\frac{z_k e^T\sigma_k}{\prod_{i=0}^{\tau}\overline{r}_{i\neq k}}\right)^T\right]\begin{bmatrix}\frac{1}{4\theta^2}&\frac{1}{2}\\ \frac{1}{2}&\theta^2\end{bmatrix}\begin{bmatrix}\sum_{k=0}^{\tau}\frac{\overline{f}_k(x)z_k}{\prod_{i=0}^{\tau}\overline{r}_{i\neq k}}\\ \frac{z_k e^T\sigma_k}{\prod_{i=0}^{\tau}\overline{r}_{i\neq k}}\end{bmatrix}\\ &-\left[\left(\sum_{k=0}^{\tau}\frac{\overline{f}_k(x)z_k}{\prod_{i=0}^{\tau}\overline{r}_{i\neq k}}\right)^T e^T\right]\begin{bmatrix}\left(1-\frac{\tau+1}{4\theta^2}\right)\alpha&\frac{1}{2}\prod_{k=0}^{\tau}\overline{r}_k\\ \frac{1}{2}\prod_{k=0}^{\tau}\overline{r}_k&K_e\end{bmatrix}\begin{bmatrix}\sum_{k=0}^{\tau}\frac{\overline{f}_k(x)z_k}{\prod_{i=0}^{\tau}\overline{r}_{i\neq k}}\\ e\end{bmatrix}\\ &-\alpha\theta^2\sum_{k=0}^{\tau}\sum_{i=0}^{\tau}\left(\frac{z_k^T z_k(e^T\sigma_{i\neq k})^2}{\left(\prod_{i=0}^{\tau}\overline{r}_{i\neq k}\right)^2}\right).\end{aligned}\tag{27}$$

Rearranging terms in (27) and recalling that $J_d=-J_d^T$ yields:

$$\begin{aligned}\dot{W}''\leq&-\alpha\sum_{k=0}^{\tau}\left[\left(\sum_{k=0}^{\tau}\frac{\overline{f}_k(x)z_k}{\prod_{i=0}^{\tau}\overline{r}_{i\neq k}}\right)^T\left(\frac{z_k e^T\sigma_k}{\prod_{i=0}^{\tau}\overline{r}_{i\neq k}}\right)^T\right]\begin{bmatrix}\frac{1}{4\theta^2}&\frac{1}{2}\\ \frac{1}{2}&\theta^2\end{bmatrix}\begin{bmatrix}\sum_{k=0}^{\tau}\frac{\overline{f}_k(x)z_k}{\prod_{i=0}^{\tau}\overline{r}_{i\neq k}}\\ \frac{z_k e^T\sigma_k}{\prod_{i=0}^{\tau}\overline{r}_{i\neq k}}\end{bmatrix}\\ &-\left[\left(\sum_{k=0}^{\tau}\frac{\overline{f}_k(x)z_k}{\prod_{i=0}^{\tau}\overline{r}_{i\neq k}}\right)^T e^T\right]\begin{bmatrix}\left(\frac{1}{2}-\frac{\tau+1}{4\theta^2}\right)\alpha&\frac{1}{2}\prod_{k=0}^{\tau}\overline{r}_k\\ \frac{1}{2}\prod_{k=0}^{\tau}\overline{r}_k&K_e\end{bmatrix}\begin{bmatrix}\sum_{k=0}^{\tau}\frac{\overline{f}_k(x)z_k}{\prod_{i=0}^{\tau}\overline{r}_{i\neq k}}\\ e\end{bmatrix}\end{aligned}$$

$$-\left[\left(\sum_{k=0}^{\tau}\frac{\overline{f}_k(x)z_k}{\prod_{i=0}^{\tau}\overline{r}_{i\neq k}}\right)^T \nabla_x H_d'^T\right]\begin{bmatrix}\left(\frac{1}{2}-\frac{\tau+1}{4\theta^2}\right)\alpha & -\frac{1}{2}\prod_{k=0}^{\tau}\overline{r}_k\\ -\frac{1}{2}\prod_{k=0}^{\tau}\overline{r}_k & D_d\end{bmatrix}\begin{bmatrix}\sum_{k=0}^{\tau}\frac{\overline{f}_k(x)z_k}{\prod_{i=0}^{\tau}\overline{r}_{i\neq k}}\\ \nabla_x H_d'\end{bmatrix} \tag{28}$$
$$-\alpha\theta^2\sum_{k=0}^{\tau}\sum_{i=0}^{\tau}\left(\frac{z_k^T z_k (e^T\sigma_{i\neq k})^2}{\left(\prod_{i=0}^{\tau}\overline{r}_{i\neq k}\right)^2}\right)-\nabla_\xi H_d'^T K\nabla_\xi H_d'.$$

Observing that $\left(\frac{1}{2}-\frac{\tau+1}{4\theta^2}\right)\geq\frac{1}{4}$ for $\theta\geq\tau+1$ and employing a Schur complement argument in (28) while recalling from Lemma 1 that $K_e>\left(\prod_{k=0}^{\tau}\overline{r}_k\right)^2$ confirms that $\dot{W}''\leq 0$ for some $K=K^T\geq 0$ provided that $\alpha\geq\alpha_0\left(\prod_{k=0}^{\tau}\overline{r}_k\right)^2>I^n, \alpha_0 D_d>I^n$. In particular, $\left(\prod_{k=0}^{\tau}\overline{r}_k\right)^2>I^n$ assigning the initial condition $\overline{r}_k(t=0)=I^n$ and considering that $\dot{\overline{r}}_k\geq 0$ in (13), thus $\alpha_0\left(\prod_{k=0}^{\tau}\overline{r}_k\right)^2>I^n$ for all $\alpha_0>1$. Consequently (x^*,ξ^*) is a locally stable equilibrium of the closed-loop system.

To prove the asymptotic claim we observe that $\nabla_x H_d'\in\mathcal{L}_2$, while computing its time derivative and substituting (26) gives $\frac{d}{dt}\nabla_x H_d'=\nabla_x^2 H_d'\left((J_d-D_d)\nabla_x H_d'+\sum_{k=0}^{\tau}\overline{f}_k(x)\overline{r}_k z_k\right)$. Thus, $\frac{d}{dt}\nabla_x H_d'\in\mathcal{L}_\infty$ provided that $\nabla_x H_d'$, $\nabla_x^2 H_d'$, J_d, $D_d\in\mathcal{L}_\infty$ hence $\nabla_x H_d'$ converges to zero. Finally, it follows from (22) that x converges to x^* ■.

Corollary 1. *Consider the PCH system (9) with disturbances under Assumptions 1 and 2. Define the algebraic solution P so that $\Phi(x,\xi)=\Phi(x,\xi)^T>0$ for all (x,ξ) in a nonempty set $\Omega\subseteq\mathbb{R}^n\times\mathbb{R}^n$ containing the equilibrium (x^*,ξ^*), and let $R(t)=\Phi(x(t),\xi(t))$ for all $t\geq 0$ so that $\Psi^T R^{-1}\dot{\Phi}R^{-1}\Psi<2(N^T+P)D_d(N+P)$, where $Nx=\nabla H-L$. Assume that the functions $\overline{f}_k(x)$ are integrable and consider the control law (24) with $F_1(x)$, F_0 satisfying conditions (22), while $\widetilde{\delta}_k$ are estimated adaptively with (19). Then the equilibrium (x^*,ξ^*) is locally stable for some constants $K=K^T\geq 0$ and some $\alpha>0$ such that $\alpha D_d>I^n/4$.*

Proof: Defining the Lyapunov function candidate $W_0''=H_d'(x,\xi)+W_0$ computing its time derivative and substituting (9), (20), (21), (24) gives:

$$\dot{W}_0''=\nabla_x H_d'^T\left((J_d-D_d)\nabla_x H_d'+\sum_{k=0}^{\tau}\overline{f}_k(x)z_k\right)-\nabla_\xi H_d'^T K\nabla_\xi H_d'$$
$$-\alpha\sum_{k=0}^{\tau}\left((\overline{f}_k(x)z_k)^T\sum_{k=0}^{\tau}\overline{f}_k(x)z_k\right). \tag{29}$$

Rearranging terms in (29) gives:

$$\dot{W}_0''=-\begin{bmatrix}\sum_{k=0}^{\tau}(\overline{f}_k(x)z_k)^T & \nabla_x H_d'^T\end{bmatrix}\begin{bmatrix}\alpha & -\frac{I^n}{2}\\ -\frac{I^n}{2} & D_d\end{bmatrix}\begin{bmatrix}\sum_{k=0}^{\tau}\overline{f}_k(x)z_k\\ \nabla_x H_d'\end{bmatrix}-\nabla_\xi H_d'^T K\nabla_\xi H_d'\leq 0, \tag{30}$$

which holds true for some $K=K^T\geq 0$ if $\alpha>0$, $\alpha D_d>I^n/4$ and concludes the proof ■.

Remark 2. Controller (24) is expressed in a modular form combining the dynamic state-feedback (8), which remains unchanged, and the disturbance compensation term u^*. Taking advantage of this structure, different definitions of algebraic solution of the matching equations can be employed. The proposed method is also applicable to the conventional IDA-PBC design that relies on the classical solution of the PDE, and it represents an extension of References 26,35 for multiple additive variable disturbances. In any case, the terms $F_1(x)$, F_0 contained in u^* are computed from (22), which does not include any PDE. Finally, compared with controller (8) an additional parameter α is introduced in (24) from (12) to (13). In this respect, different tuning requirements for α, D_d are presented in Proposition 1 and Corollary 1: $\alpha\geq\alpha_0\left(\prod_{k=0}^{\tau}\overline{r}_k\right)^2>I^n$, $\alpha_0>1$, $\alpha_0 D_d>I^n$ are necessary in the former case; less stringent requirements corresponding to $\alpha>0$, $\alpha D_d>I^n/4$ are specified if the functions $\overline{f}_k(x)$ are integrable. In addition, the strict-minimizer condition $\nabla_x^2 H_d(x^*,\xi^*)>0$ in (22) provides the opportunity to introduce further tuning parameters within $F_1(x)$ which can serve the purpose of shaping the transient performance of the closed-loop system (ref. Section 5).

4 | UNDERACTUATED MECHANICAL SYSTEMS

In this section, the definition of algebraic solution is applied to the potential-energy PDE for underactuated mechanical systems in PCH form with multiple linearly parameterized additive disturbances, provided that the kinetic-energy PDE is solvable analytically. The system dynamics in the presence of disturbances is described as:

$$\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0 & I^n\\ -I^n & -D\end{bmatrix}\begin{bmatrix}\nabla_q H\\ \nabla_p H\end{bmatrix} + \begin{bmatrix}0\\ G\end{bmatrix} u - \begin{bmatrix}0\\ \delta_0 + \bar{f}_q(q)\delta_1 + \bar{f}_p(p)\delta_2\end{bmatrix}, \tag{31}$$

where $q \in \mathbb{R}^n$ represent the generalized positions, $p = M\dot{q}$ are the momenta, $M = M^T > 0$ is the open-loop inertia matrix, $G \in \mathbb{R}^{n\times m}$ is the input matrix, and $D > 0$ is the physical damping matrix. The disturbances comprise a constant term $\delta_0 \in \mathbb{R}^n$ and state-dependent terms $\bar{f}_q(q)\delta_1, \bar{f}_p(p)\delta_2 \in \mathbb{R}^n$, where $\bar{f}_p, \bar{f}_q \in \mathbb{R}^{n\times n}$ are known matrix valued functions satisfying Assumptions 1 and 2. The open-loop Hamiltonian is $H(q,p) = \frac{1}{2}p^T M^{-1} p + V$, where the potential energy can be written as $V = d + L^T q + \frac{1}{2}q^T \overline{H}_q q + h(q)$. As a result $H = d + L^T q + h(q) + \frac{1}{2}q^T \overline{H}_q q + \frac{1}{2}p^T M^{-1} p$, which corresponds to the expression introduced in Section 2, where $h(q)$ only contains the nonlinear terms of V. The closed-loop dynamics in the absence of disturbances is typically expressed as:

$$\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0 & M^{-1}M_d\\ -M_d M^{-1} & J_2 - D_d - DM^{-1}M_d\end{bmatrix}\begin{bmatrix}\nabla_q H_d\\ \nabla_p H_d\end{bmatrix}, \tag{32}$$

where $M_d = M_d^T > 0$ is the closed-loop inertia matrix to be defined, while $J_2(q,p) = -J_2^T$ is a free matrix and $D_d = D_d^T > 0$ is the damping injection matrix. The closed-loop Hamiltonian is defined as $H_d = \frac{1}{2}p^T M_d^{-1} p + V_d$ with potential energy V_d and has a strict-minimizer in $(q,p) = (q_0^*, 0)$. In the presence of disturbances the system has an assignable open-loop equilibrium if the following equality is verified, where consistently with Assumption 2 we have $\bar{f}_p(0) \equiv 0, \bar{f}_q(q^*) \equiv 0$:

$$G^\perp(\nabla_q V(q^*) + \delta_0) = 0. \tag{33}$$

In particular, if $G^\perp \delta_0 = 0$ then the open-loop equilibrium $q^* = q_0^*$ is assignable and can be stabilized with an appropriate control action. The matching equations corresponding to (3) are termed potential-energy PDE and kinetic-energy PDE and are defined as follows for system (31):

$$\begin{aligned} G^\perp(\nabla_q V(q) - M_d M^{-1}\nabla_q V_d(q)) &= 0\\ G^\perp(\nabla_q(p^T M^{-1} p) - M_d M^{-1}\nabla_q(p^T M_d^{-1} p) + 2(J_2 - D_d)M_d^{-1} p) &= 0. \end{aligned} \tag{34}$$

Typically $D_d = Gk_v G^T$ with the tuning parameter $k_v > 0$, while the matrix J_2 can be defined as $J_2 = \frac{1}{2}\sum_{i=1}^n U_i p_i$ where $U_i(q) = -U_i^T$ are free matrices.[1] This design choice allows reducing the number of kinetic-energy PDE to $s(s+1)(s+2)/6$, where $s = n - m > 0$ is the degree of underactuation.[36] In the particular case of constant inertia matrix M, the kinetic-energy PDE can be solved with a constant M_d and $J_2 = 0$. The following result is, however, not limited to this case, provided that the kinetic-energy PDE is solvable analytically, since the algebraic solution is only defined for the potential-energy PDE (see Section 5.2).

Applying Definition 2 to the disturbance-free system (31), the continuously differentiable matrix valued function $P(q) \in \mathbb{R}^{n\times n}$ is an $\mathcal{X}$ algebraic solution of the potential-energy PDE (34) if:

$$\begin{aligned} \nabla_q^2 V + \nabla_q(P(q)q) > 0 \;\; q = q^*\\ G^\perp(\nabla_q V(q) - M_d M^{-1}(\nabla_q V(q) - L + P(q)q)) = 0 \;\; \forall q \in \mathcal{X} \subseteq \mathbb{R}^n. \end{aligned} \tag{35}$$

In general $P(q)$ might not be a solution of the original PDE (34). Similarly to (5), we introduce the auxiliary energy function $H_d(q,p,\xi)$ defined on an extended state space with $\xi \in \mathbb{R}^n$ as: $H_d(q,p,\xi) = V - (d + L^T q) + \frac{1}{2}q^T P(\xi)q + \frac{1}{2}\|q - \xi\|_R^2 + \frac{1}{2}p^T M_d^{-1} p$. Defining the terms $\Phi(q,\xi)$ and $\Psi(q,\xi)$ so that $(P(q) - P(\xi))q = \Phi(q,\xi)(q - \xi)$ and $\Psi(q,\xi) = (1/2)(\partial P(\xi)q/\partial\xi)$ and computing the partial derivatives of $H_d(q,p,\xi)$ gives:

$$\nabla_p H_d(q,p,\xi) = M_d^{-1} p$$

$$\nabla_q H_d(q,p,\xi) = \frac{1}{2}\nabla_q(p^T M_d^{-1} p) + \nabla_q V - L + P(\xi)q + R(q-\xi)$$
$$\nabla_\xi H_d(q,p,\xi) = \Psi(q,\xi)q - R(q-\xi). \tag{36}$$

Drawing a parallel to controller (8), let us consider an algebraic solution P of the potential-energy PDE (34) such that $\Phi(q,\xi) = \Phi(q,\xi)^{\mathrm{T}} > 0$ in some nonempty open neighborhood Ω of the equilibrium, let $R(t) = \Phi(q(t),\xi(t))$ for all $t \geq 0$, and $\Psi^{\mathrm{T}} R^{-1}\dot{\Phi}R^{-1}\Psi < 2(N^T + P)D_d'(N+P)$, where $Nq = \nabla_q V - L$ and $D_d' = (D_d + DM^{-1}M_d)$. Define the following dynamic state-feedback control law for the disturbance-free version of system (31):

$$\dot{\xi} = -K\nabla_\xi H_d(q,p,\xi)$$
$$u = G^\dagger\left(\nabla_q H - M_d M^{-1}\left(\frac{1}{2}\nabla_q(p^T M_d^{-1}p) + \nabla_q V - L + P(q)q\right) + (J_2 - D_d)M_d^{-1}p\right) + v. \tag{37}$$

Then, according to Reference [9, Proposition 5] the disturbance-free system (31) in closed-loop with (37) has a locally stable equilibrium at $(q,p,\xi) = (q^*,0,0)$ for some $K = K^T \geq 0$ and $v = 0$ provided that $D_d' > 0$.

To account for the disturbances in the controller design we redefine the partial derivatives of the auxiliary energy function in (36) as follows, where $R(t) = \Phi(x(t),\xi(t))$ for all $t \geq 0$:

$$\nabla_p H_d'(q,p,\xi) = M_d^{-1}p$$
$$\nabla_q H_d'(q,p,\xi) = \frac{1}{2}\nabla_q(p^T M_d^{-1}p) + \nabla_q V - L + P(q)q + F_1(q,p) + F_0$$
$$\nabla_\xi H_d'(q,p,\xi) = (1/2)(\partial P(\xi)q/\partial\xi)q - (P(q) - P(\xi))q. \tag{38}$$

The constant term F_0 and the state-dependent term $F_1(q,p)$ should verify the following conditions:

$$G^\perp(M_d M^{-1}(F_1(q,p) + F_0) - (\widetilde{\delta}_0 + \bar{f}_q(q)\widetilde{\delta}_1 + \bar{f}_p(p)\widetilde{\delta}_2)) = 0$$
$$\nabla_q^2 V(q^*) + \nabla_q(P(q)\mathrm{q}) + \nabla_q F_1(q^*,0) > 0$$
$$F_0 = -(\nabla_q V(q^*) - L + P(q^*)q^* + F_1(q^*,0)). \tag{39}$$

From (39), the auxiliary closed-loop energy function in a nonempty set containing the equilibrium $(q^*,0,\xi^*)$ can be approximated as: $H_d'(q,p,\xi) = H_d(q,p,\xi) + (q-q^*)^T(F_0 + F_1(q^*,0)) + C$.

Finally, the robust dynamic state-feedback controller for underactuated mechanical systems is constructed augmenting the dynamic state-feedback (37) with a disturbance compensation term as stated in the following result.

Corollary 2. *Consider the underactuated mechanical system (31) with disturbances under Assumptions 1 and 2. Define the algebraic solution P of the potential-energy PDE as in (35) so that $\Phi(q,\xi) = \Phi(q,\xi)^T > 0$ for all (q,ξ) in a nonempty set $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$ containing the equilibrium (q^*,ξ^*), and let $\Psi^T R^{-1}\dot{\Phi}R^{-1}\Psi < 2(N^T + P)D_d'(N+P)$, where $R(t) = \Phi(q(t),\xi(t))$ for all $t \geq 0$ and $Nq = \nabla_q V - L$. Consider the control law $u' = u + u^*$ with u defined in (37), $F_1(q,p)$, F_0 defined according to (39), $\widetilde{\delta}_0, \widetilde{\delta}_1, \widetilde{\delta}_2$ estimated adaptively with (12)-(13), and u^* defined as follows:*

$$u^* = G^\dagger(\widetilde{\delta}_0 + \bar{f}_q(q)\widetilde{\delta}_1 + \bar{f}_p(p)\widetilde{\delta}_2 - M_d\, M^{-1}(F_1(q,p) + F_0)). \tag{40}$$

Then the assignable equilibrium $(q,p,\xi) = (q^*,0,\xi^*)$ with q^* defined in (33) is locally stable for some $K = K^T \geq 0$ if $\alpha = \alpha_0(\bar{r}_0\bar{r}_1\bar{r}_2)^2 > I^n, \alpha_0 > 1, \alpha_0 D_d' > I^n$. In addition, if $\bar{f}_q(q), \bar{f}_p(p)$ are integrable, the equilibrium $(q,p,\xi) = (q^*,0,\xi^*)$ is locally stable if $\alpha > 0$ and $\alpha D_d' > I^n/4$. The proof follows closely that of Proposition 1 and Corollary 1 employing the Lyapunov function candidate $W'' = H_d'(q,p,\xi) + \frac{1}{2}\left(\frac{z_0^T z_0}{(\bar{r}_1\bar{r}_2)^2} + \frac{z_1^T z_1}{(\bar{r}_0\bar{r}_2)^2} + \frac{z_2^T z_2}{(\bar{r}_0\bar{r}_1)^2}\right) + \frac{1}{2}e^T e$, thus it is omitted for brevity ■.

Remark 3. The algebraic solution (35) only applies to the potential-energy PDE and not to the kinetic-energy PDE, which should be solved in the traditional way. If an analytical solution of the kinetic-energy PDE exists, the result presented in this section is also applicable to systems with nonconstant inertia matrix M and nonconstant input matrix G (see Section 5.2). Limiting the algebraic solution to the potential-energy PDE simplifies the control law (37) and preserves the mechanical structure of the closed-loop system (32) on the extended state space. Notably, while the kinetic-energy PDE can be reduced to ordinary differential equations or to algebraic equations for different classes of systems under some

assumptions,[4,37] this is not the case for the potential-energy PDE. Finally, the disturbance compensation controller (40) can in principle be used with different definitions of algebraic solution besides the one proposed in (35). A promising approach in this respect would be to employ directly Definition 2 with the system state $x = \begin{bmatrix} q & p \end{bmatrix}$ and the closed-loop dynamics (2), consequently not preserving the mechanical structure of (31) in closed-loop. Employing this method could allow defining an algebraic solution also for the kinetic-energy PDE, which is among the goals of our future work.

Remark 4. Fulfilling the condition $\alpha_0 D'_d \geq I^n$ (or $\alpha D'_d > I^n/4$) required by Corollary 2 is typically a challenging task for underactuated mechanical systems since it might not be possible to assign a globally positive definite matrix D'_d in some cases.[15] Since the result of Corollary 2 is local, a sufficient condition is that $D'_d \geq 0$ globally and that $D'_d > 0$ in a nonempty set containing the equilibrium $(q, p) = (q^*, 0)$. This condition can be fulfilled in the presence of physical damping provided that locally $DM^{-1}M_d > 0$. Local stability of the assignable equilibrium could then be concluded with the following argument[34]: (i) the assignable equilibrium is (locally) asymptotically stable for system (31) in closed-loop with controller (37) and (40) if the combined estimation error $(z_0 + \bar{f}_q z_1 \bar{r}_1 + \bar{f}_p z_2 \bar{r}_2)$ is zero; ii) the combined estimation error is bounded and converges to zero asymptotically if $\alpha > 1/3$, provided that $\bar{f}_q, \bar{f}_p$ and their first derivatives are bounded (ref. Lemma 1).

5 | EXAMPLES: MAGNETIC-LEVITATED-BALL AND BALL-ON-BEAM

In this section, the effectiveness of the proposed approach is demonstrated with numerical simulations on two systems.

5.1 | Magnetic-levitated-ball system

The magnetic-levitated-ball system[9] consists of an iron ball of mass m suspended in the vertical magnetic field generated by a single electromagnet. The vertical position of the ball is indicated with $\vartheta \in (-\infty; 1)$, while $\imath$ is the current flowing through the coil of resistance $\varkappa$ and inductance $\eta(\vartheta) = \frac{\hbar}{1-\vartheta}$. The parameter $\hbar$ is a positive constant that depends on the number of coil turns. The unsaturated flux generated by the electromagnet is $\lambda = \eta(\vartheta)\imath$ and the induced force on the ball is $F \propto \left(\frac{\partial \eta}{\partial \vartheta}\right) \imath^2$. The system dynamics is expressed in PCH form with state $x = [\lambda, \vartheta, m\dot{\vartheta}]$, control input $u = \dot{\imath}$, and Hamiltonian $H = \frac{1}{2\hbar}(1 - x_2)x_1^2 + \frac{1}{2m}x_3^2 + mgx_2$ as:

$$\dot{x} = \begin{bmatrix} -\varkappa & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \nabla H(x) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u - \begin{bmatrix} \delta_{01} + \delta_{11} x_1 \\ 0 \\ \delta_{02} + \delta_{12} x_3 \end{bmatrix}. \tag{41}$$

The terms δ_{01}, $\delta_{11}x_1$ represent the matched disturbances on the current, while the terms δ_{02}, $\delta_{12}x_3$ represent the unmatched disturbances and correspond, respectively, to a constant force acting on the ball and to viscous friction. The relationship between ball position and velocity is typically unaffected by external disturbances hence the second element of the disturbance vector is null. The control aim for the disturbance-free version of system (41) is to stabilize the desired equilibrium $x_d = (\sqrt{2\hbar mg}, x_{2d}, 0)$ representing a constant position of the ball. This can be achieved with controller (8) as shown in Reference 9 where P, J_d, D_d, and $R = \Phi$ are defined in Appendix A1, while c is a tuning parameter and g is the gravity constant. Defining $\tilde{x} = x - x_d$ and $h(\tilde{x}) = -\frac{1}{2\hbar}\tilde{x}_1^2\tilde{x}_2$ yields

$$\nabla h(\tilde{x}) = -\frac{1}{2\hbar}\begin{bmatrix} 2\tilde{x}_1\tilde{x}_2 \\ \tilde{x}_1^2 \\ 0 \end{bmatrix}, \nabla^2 h(\tilde{x}) = -\frac{1}{\hbar}\begin{bmatrix} \tilde{x}_2 & \tilde{x}_1 & 0 \\ \tilde{x}_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

thus $\nabla h(0) = 0$ and $\nabla^2 h(0) = 0$. In the presence of disturbances, the assignable open-loop equilibrium for system (41) is computed from (10) resulting in $\tilde{x}^* = (-x_{1d} + \sqrt{2\hbar(mg + \tilde{\delta}_{02})}, 0, 0)$ which depends on the estimated value of the constant unmatched disturbance $\tilde{\delta}_{02}$ and admits a solution if $\tilde{\delta}_{02} > -mg$ (ref. Assumption 2). The unknown parameters are estimated using the adaptation law with dynamic scaling (12) and (13) resulting in: $\beta_{01} = -\alpha x_1; \beta_{02} =$

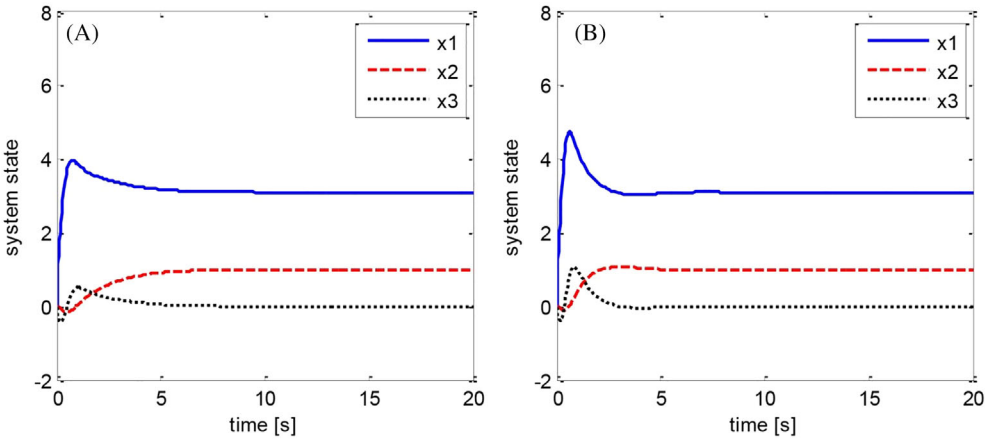


Figure 1 Magnetic-levitated-ball system (41) with controller (24): A, time history of the system state with $\gamma = 0$; B, and with $\gamma = 1.5$. The desired position of the ball is $x_{2d} = 1$. In comparison, the final position of the ball employing controller (8) with $u^* = 0$ is $x_2 = -1.65$ [Colour figure can be viewed at wileyonlinelibrary.com]

$-\alpha x_3$; $\beta_{11} = -\alpha x_1 \hat{x}_1$; $\beta_{12} = -\alpha x_3 \hat{x}_3$. Finally, the control law (24) is constructed with F_0, F_1 defined according to (22), which yields:

$$F_0 = -\widetilde{x}_1^* \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, F_1 = \begin{bmatrix} \gamma \widetilde{x}_2 - \frac{2\kappa(\widetilde{\delta}_{12} x_3 + \widetilde{\delta}_{02})}{\widetilde{x}_1 + 2x_{1d}} \\ \gamma \widetilde{x}_2 (c\widetilde{x}_1^2 + 2) + \frac{2\kappa(\widetilde{\delta}_{12} x_3 + \widetilde{\delta}_{02})(c\widetilde{x}_1^2 + 1)}{\widetilde{x}_1 + 2x_{1d}} - \widetilde{x}_1^* c \widetilde{x}_1^2 \\ \gamma \widetilde{x}_2 \end{bmatrix} \tag{42}$$

.

In particular, the third element of F_1 is freely chosen as $\gamma \widetilde{x}_2$ introducing the tuning parameter γ which appears in the strict-minimizer condition for H'_d:

$$\begin{aligned} \nabla_x^2 H'_d(\widetilde{x}^*) &= (1+\gamma)(4 + 5c(\widetilde{x}_1^*)^2 + \epsilon/(\widetilde{x}_1^* + 2x_{1d})^2) > 0, \\ \epsilon &= 2\kappa \widetilde{\delta}_{02}(c(\widetilde{x}_1^*)^2 + 4 - 4cx_{1d}\widetilde{x}_1^*) + 4c\widetilde{\delta}_{12}\kappa \widetilde{x}_1^*((\widetilde{x}_1^*)^2 + 2\widetilde{x}_1^* x_{1d} - 2\kappa \widetilde{\delta}_{02}). \end{aligned} \tag{43}$$

The above inequality holds true for some $\widetilde{\delta}_{02}, \widetilde{\delta}_{12}$ if $\gamma > -1$. While a constant γ is considered here for demonstrative purposes, any function $\gamma(\widetilde{x}) > -1$ that satisfies (22) can in principle be employed allowing further design freedom.

The numerical simulations employ the following parameters: $\varkappa = 1$; $m = 1$; $\kappa = 1$; $g = 9.81$; $x_{2d} = 1$; $c = 0.1$; $K \equiv 0$; $\kappa = 0$; $\alpha = 2$; $\theta = 1$; $\gamma = 0$ and alternatively $\gamma = 1.5$. In addition, $K_e = 1.5(\bar{r}_0 \bar{r}_1)^2$, with $\bar{r}_0 = I^2$ and $\bar{r}_1 = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$. The initial conditions were set to $x = \hat{x} = \xi \equiv 0$; $r_1 = r_2 = 1$ and the disturbances were defined as follows: $\delta_{01} = 1$; $\delta_{02} = -5$; $\delta_{11} = 3$; $\delta_{12} = 5$. The time history of the system states with $\gamma = 1.5$ and $\gamma = 0$ is depicted in Figure 1 confirming that the ball position reaches the desired value $x_{2d} = 1$ in spite of the disturbances. In addition, larger values of γ result in higher responsiveness in accordance with (43), while the additional states ξ remain at zero with the chosen tuning parameters. For comparison purposes, employing controller (8) without disturbance compensation results in the ball position settling at $x_2 = -1.65$ corresponding to a large error. Finally, the estimation errors $z_I = \widetilde{\delta}_{01} - \delta_{01} + (\widetilde{\delta}_{11} - \delta_{11})x_1$ and $z_{II} = \widetilde{\delta}_{02} - \delta_{02} + (\widetilde{\delta}_{12} - \delta_{12})x_3$ are bounded and converge to zero asymptotically, while the scaling factors r_1, r_2 are bounded (see Figure 2). The time histories of the control input and of the Lyapunov function W'' are depicted in Figure 3.

5.2 | Ball-on-beam system

The ball-on-beam system consists of a beam of length $2L$ hinged at the mid-point and actuated with a torque u, and of a ball with point mass that is free to move above the beam. Making simplifying assumptions on the masses of ball and beam as in Reference 1, the open-loop dynamics in the presence of disturbances consisting of constant forces δ_{01}, δ_{02} and viscous friction δ_{11}, δ_{12} becomes:

$$\ddot{q}_1 + g \sin(q_2) - q_1 \dot{q}_2^2 = -(\delta_{01} + \delta_{11} \dot{q}_1)$$

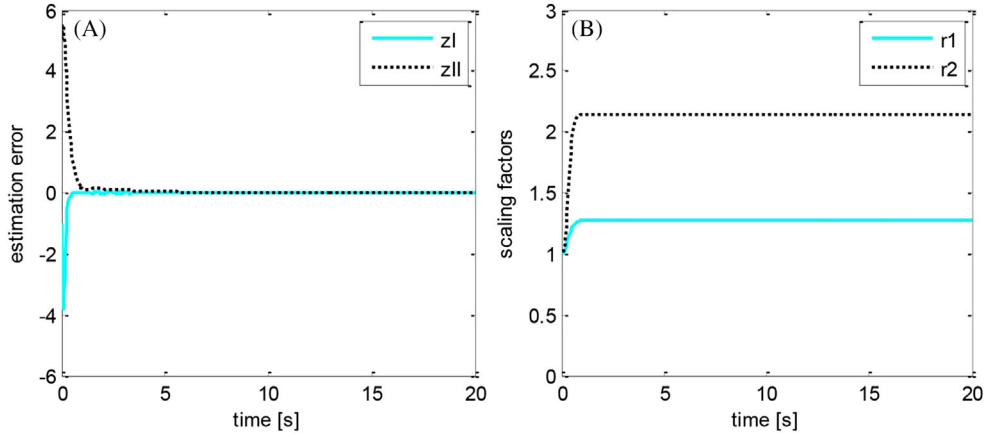


Figure 2 Adaptation law with dynamic scaling (12)-(13) and $\gamma = 0$: A, cumulative estimation errors z_I, z_{II}; B, scaling factors r_1, r_2 [Colour figure can be viewed at wileyonlinelibrary.com]

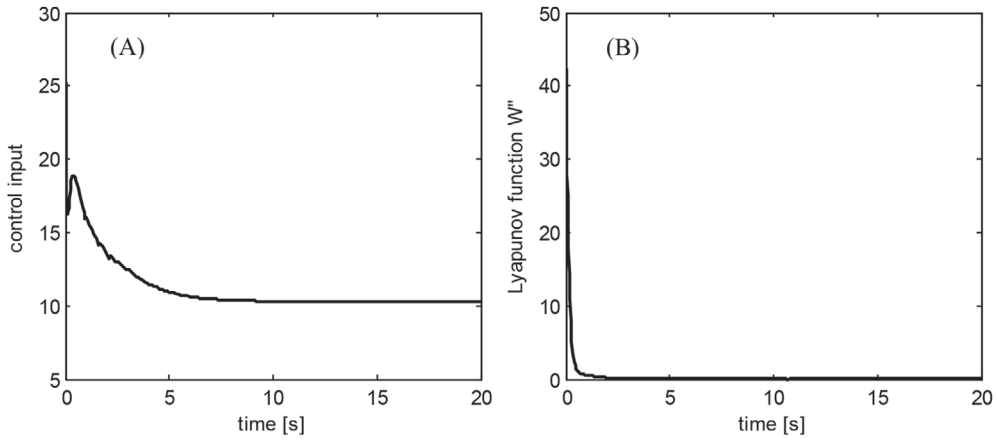


Figure 3 Magnetic-levitated-ball (41) with controller (24) and $\gamma = 0$: A, control input; B, Lyapunov function W''

$$(L^2 + q_1^2)\ddot{q_2} + gq_1\cos(q_2) + 2q_1\dot{q}_1\dot{q}_2 = u - (\delta_{02} + \delta_{12}\dot{q}_2) \tag{44}$$

The position $q_1 \in (-L; L)$ is the distance of the ball from the midpoint of the beam, $q_2 \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ is the inclination of the beam from the horizontal, and g is the gravity constant. To express (44) in the PCH form (31) we define the open-loop inertia matrix $M = \begin{bmatrix} 1 & 0 \\ 0 & L^2 + q_1^2 \end{bmatrix}$, which is not constant, the input matrix $G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and the potential energy $V = gq_1\sin(q_2)$. The latter can be parameterized as $V = d + L^T q + \frac{1}{2}q^T\overline{H}_q q + h(q)$ where d=0, $L^T = 0$, $\overline{H}_q = g\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and the nonlinear term $h(q) = gq_1(\sin(q_2) - q_2)$, which yields

$$\nabla h(q) = g\begin{bmatrix} \sin(q_2) - q_2 \\ q_1(\cos(q_2) - 1) \end{bmatrix}, \nabla^2 h(q) = g\begin{bmatrix} 0 & \cos(q_2) - 1 \\ \cos(q_2) - 1 & -q_1\sin(q_2) \end{bmatrix}.$$

The control aim for the disturbance-free version of system (44) is to stabilize the equilibrium $(q_1, q_2) = (0, 0)$ corresponding to the horizontal beam with the ball at its midpoint. Note thus that $\nabla h(0, 0) = 0$ and $\nabla^2 h(0, 0) = 0$. In the presence of disturbances, the open-loop assignable equilibrium is computed according to (33) as $q^* = \left(0, \sin^{-1}\left(-\frac{\widetilde{\delta}_{01}}{g}\right)\right)$ and exists if $|\widetilde{\delta}_{01}| < g$. In this case, since V contains a cross term in q_1, q_2, the unmatched disturbances affect the equilibrium of the actuated position q_2. The dynamic state-feedback controller (37) and the disturbance compensation (40) are employed, while the unknown parameters are estimated using the adaptation law with dynamic scaling (12) and (13) resulting in: $\beta_{01} = -\alpha\dot{q}_1; \beta_{02} = -\alpha\dot{q}_2(L^2 + q_1^2)$; $\beta_{11} = -\alpha\dot{q}_1\hat{\dot{q}}_1; \beta_{12} = -\alpha\dot{q}_2\hat{\dot{q}}_2(L^2 + q_1^2)$. The closed-loop inertia matrix computed solving the kinetic-energy PDE as in Reference 1 and the algebraic solution of the potential-energy PDE are reported in Appendix A2. The terms F_0, $F_1(q)$ are computed according to (39):

$$F_0 = \begin{bmatrix} \frac{K_p q_2^* \sqrt{2}}{2L} \\ 0 \end{bmatrix}, F_1 = \begin{bmatrix} \gamma q_1 \\ \widetilde{\delta}_{11}\dot{q}_1 + \widetilde{\delta}_{01} - \left(\gamma q_1 + \frac{\sqrt{2}K_p q_2^*}{2L}\right)\sqrt{2(L^2+q_1^2)} \end{bmatrix}. \tag{45}$$

In particular, the first element of F_1 can be freely chosen and is taken proportional to q_1 introducing the tuning parameter γ which provides a handle on the transient performance of the closed-loop system and appears in the following strict-minimizer condition:

$$\nabla_q^2 H_d'(q^*) = g(\gamma + K_p/(2L^2))\cos(q_2^*) > 0 \tag{46}$$

The above inequality holds true for all $\gamma > -K_p/(2L^2)$, since $|q_2^*| < \pi/2$ by hypothesis.

The numerical simulations employ the following parameters: $L = 0.5$; $g = 9.81$; $K_p = 1$; $K_v = 1$; $K \equiv 0$; $\kappa = 0$; $\alpha = 25$; $\theta = 1$; $\gamma = 1$ and alternatively $\gamma = -1$. In addition, $K_e = 2(\bar{r}_0\bar{r}_1)^2$, with $\bar{r}_0 = I^2$ and $\bar{r}_1 = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$. The initial conditions were set to $q = (0.5,0)$; $\xi \equiv 0$; $p \equiv 0$; $\widehat{q} = q$; $r_1 = r_2 = 1$ corresponding to the horizontal beam with the ball positioned at its endpoint, while the disturbances were defined as follows: $\delta_{01} = 2$; $\delta_{02} = 5$; $\delta_{11} = 15$; $\delta_{12} = 10$. Employing the IDA-PBC[1] without disturbance compensation the ball moves away from the desired equilibrium and beyond the limit length of the beam. Conversely, with controller (37) and (40) the position converges to the assignable equilibrium $q^* = (0, -0.205)$ at a faster rate for larger values of γ (see Figure 4).

The additional states ξ remain at zero with the chosen tuning parameters. In addition, the cumulative estimation errors $z_I = \widetilde{\delta}_{01} - \delta_{01} + (\widetilde{\delta}_{11} - \delta_{11})\dot{q}_1$ and $z_{II} = \widetilde{\delta}_{02} - \delta_{02} + (\widetilde{\delta}_{12} - \delta_{12})\dot{q}_2$ are bounded and converge to zero asymptotically, while the scaling factors r_1, r_2 are bounded (see Figure 5). Finally, the time histories of the control input and of the Lyapunov function W'' are depicted in Figure 6.

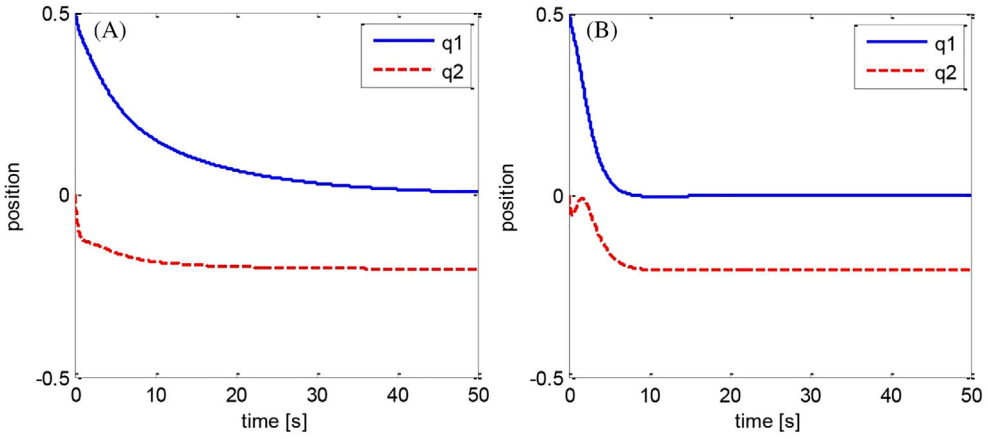


Figure 4 Ball-on-beam system (44) with controller (37) and (40): A, time history of the position with $\gamma = -1$; B, and with $\gamma = 1$. The assignable equilibrium is $q^* = (0, -0.205)$ [Colour figure can be viewed at wileyonlinelibrary.com]

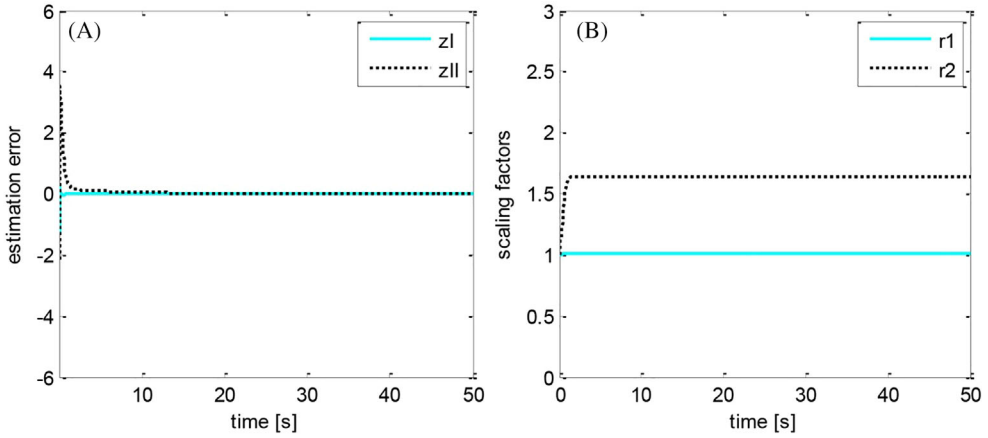


Figure 5 Adaptation law with dynamic scaling (12)-(13) and $\gamma = -1$: A, cumulative estimation errors z_I, z_{II}; B, scaling factors r_1, r_2 [Colour figure can be viewed at wileyonlinelibrary.com]

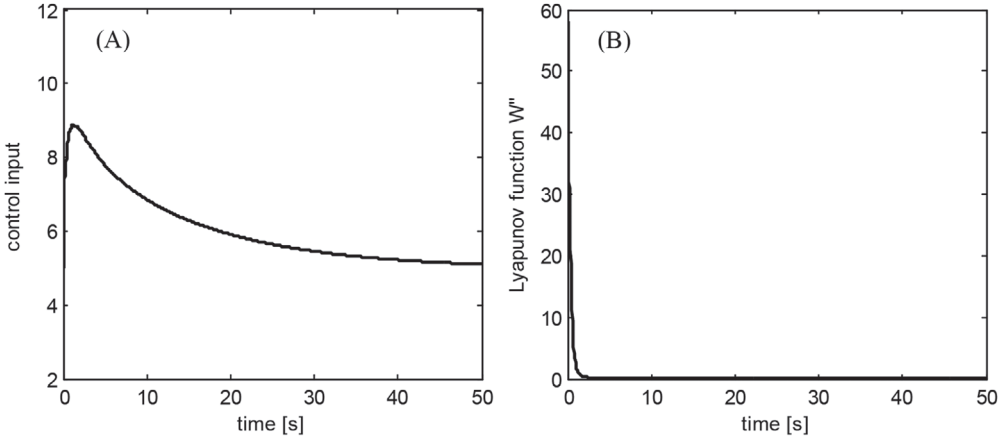


Figure 6 Ball-on-beam system (44): A, control input with $\gamma = -1$; B, Lyapunov function W''

6 | CONCLUSIONS

This work investigated the control problem for underactuated PCH systems with multiple additive linearly parameterized disturbances including matched, unmatched, constant, and state-dependent components. The definition of algebraic solution of the matching equations was employed to construct a new dynamic state-feedback controller which includes a disturbance compensation term, while an adaptation law was designed to estimate the unknown parameters. The proposed approach was detailed for underactuated mechanical systems employing an algebraic solution of the potential-energy PDE. The effectiveness of the controller was demonstrated with numerical simulations of the magnetic-levitated-ball system and of the ball-on-beam system. For the latter, an algebraic solution of the potential-energy PDE was provided. As part of our future work we aim to extend the definition of algebraic solution for mechanical systems to the kinetic-energy PDE, and to validate the results with experiments.

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ORCID

Enrico Franco https://orcid.org/0000-0001-9991-7377

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SUPPORTING INFORMATION

Additional supporting information may be found online in the Supporting Information section at the end of this article.

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APPENDIX

A.1 Magnetic-levitated-ball system

Algebraic solution of the matching equations for the magnetic-levitated-ball system,[9] where: $\widetilde{x} = x - x_d$; $x_d = \left[\sqrt{2\ell mg}\ x_{2d}\ 0\right]$; $(J-D) = \begin{bmatrix} -\gamma & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$; and c is a tuning parameter.

$$
\begin{aligned}
P &= \begin{bmatrix} 2 & 1+\frac{\widetilde{x}_1+x_{1d}}{\ell} & 1 \\ 1+\frac{\widetilde{x}_1+2x_{1d}}{2\ell} & c\widetilde{x}_1^2+2 & 1 \\ 1 & 1 & 2-\frac{1}{m} \end{bmatrix} \\
D_d &= \begin{bmatrix} c\widetilde{x}_1^2 & 0 & 0 \\ 0 & \frac{1}{m(3c\widetilde{x}_1^2+4)} & \frac{3m\widetilde{x}_1-2\ell+6mx_{1d}}{12m\ell(3c\widetilde{x}_1^2+4)} - \frac{1}{3m} \\ 1 & \frac{3m\widetilde{x}_1-2\ell+6mx_{1d}}{12m\ell(3c\widetilde{x}_1^2+4)} - \frac{1}{3m} & \frac{(\widetilde{x}_1+2x_{1d})(c\widetilde{x}_1^2+1)}{2\ell(3c\widetilde{x}_1^2+4)} \end{bmatrix} \\
J_d &= \begin{bmatrix} 0 & \frac{c\widetilde{x}_1^2+1}{m(3c\widetilde{x}_1^2+4)} & -\frac{(\widetilde{x}_1+2x_{1d})(2c\widetilde{x}_1^2+3)}{2\ell(3c\widetilde{x}_1^2+4)} \\ -\frac{c\widetilde{x}_1^2+1}{m(3c\widetilde{x}_1^2+4)} & 0 & \frac{3m\widetilde{x}_1+2\ell+6mx_{1d}}{12m\ell(3c\widetilde{x}_1^2+4)} + \frac{1}{3m} \\ \frac{(\widetilde{x}_1+2x_{1d})(2c\widetilde{x}_1^2+3)}{2\ell(3c\widetilde{x}_1^2+4)} & -\frac{3m\widetilde{x}_1+2\ell+6mx_{1d}}{12m\ell(3c\widetilde{x}_1^2+4)} - \frac{1}{3m} & 0 \end{bmatrix} \\
R &= \Phi = \begin{bmatrix} \frac{\widetilde{x}_2}{\ell} & 0 & 0 \\ 0 & \left(\frac{\xi_3-\widetilde{x}_3}{\xi_2-\widetilde{x}_2}\right)^2 + \left(\frac{\widetilde{x}_1}{2\ell} + c\widetilde{x}_2(\xi_1+\widetilde{x}_1)\right)\left(\frac{\xi_1-\widetilde{x}_1}{\xi_2-\widetilde{x}_2}\right) & -\frac{\xi_3-\widetilde{x}_3}{\xi_2-\widetilde{x}_2} \\ 0 & -\frac{\xi_3-\widetilde{x}_3}{\xi_2-\widetilde{x}_2} & 1 \end{bmatrix}.
\end{aligned}
\tag{A1}
$$

A.2 Ball-on-beam system

Algebraic solution of the potential-energy PDE for the ball-on-beam system. The matrix P and R were approximated with Taylor series for numerical simulation purposes. Notably $P \neq P^T$ thus it is not the gradient of any scalar function. The closed-loop inertia matrix M_d, the free matrix J_2, and the damping matrix D_d are defined as in Reference 1. The positive constants K_p, K_v are tuning parameters. Note that in this case the physical damping $D > 0$ is present in the form of viscous friction thus $D_d' > 0$ locally.

$$
\begin{aligned}
P &= \begin{bmatrix} \frac{K_p}{2L\sqrt{L^2+q_1^2}} & g\left(\frac{q_2^2}{6}-1\right) - \frac{K_p\sqrt{2}}{2\sqrt{L^2+q_1^2}} \\ g\left(\frac{q_2^2}{2}-1\right) - \frac{K_p\sqrt{2}}{2L} & g\left(1-\frac{q_2^2}{6}\right) + K_p \end{bmatrix} \\
J_2 &= \begin{bmatrix} 0 & q_1(p_1 - p_2\sqrt{2/(L^2+q_1^2)}) \\ -q_1(p_1 - p_2\sqrt{2/(L^2+q_1^2)}) & 0 \end{bmatrix} \\
M_d &= (L^2+q_1^2)\begin{bmatrix} \sqrt{2/(L^2+q_1^2)} & 1 \\ 1 & \sqrt{2(L^2+q_1^2)} \end{bmatrix} \\
D_d &= \begin{bmatrix} 0 & 0 \\ 0 & K_v \end{bmatrix} \\
R &= \Phi = \begin{bmatrix} \frac{K_p(\xi_1+q_1)(\sqrt{2}Lq_2-q_1)}{2L((L^2+\xi_1^2)\sqrt{(L^2+q_1^2)}+(L^2+q_1^2)\sqrt{(L^2+\xi_1^2)})} & 0 \\ 0 & \frac{g}{2}q_1(q_2+\xi_2) \end{bmatrix}.
\end{aligned}
\tag{A2}
$$