



Boundary Element Plate Formulations for Dynamic Fracture and Ultrasonic Guided-Wave Structural Health Monitoring

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Declaration of Originality

The work for this thesis is my own work based on my personal research except where states otherwise by reference. This work has not been submitted before for any degree or other qualification.

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Dedication

I would like to dedicate this thesis to my beloved wife Chen, who has always been there for me.

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Abstract

In this thesis, new boundary element formulations based on first-order plate theories are developed for dynamic fracture problems and Lamb-wave based structural health monitoring. The boundary integral equations are formulated in the complex frequency domain (Laplace-transform domain), which avoids domain integrals due to the inertial terms. In these plate formulations, only line elements are required for the discretization of membrane and bending problems. The dual boundary element method (DBEM) provides a general and computationally efficient way to model cracks.

A new boundary element formulation for the Kane-Mindlin theory, a first-order theory for extensional motion of plates, is proposed for modelling the fundamental symmetric (S0) Lamb wave mode. The formulation is demonstrated to enable efficient and accurate modelling of the S0 mode over a wide frequency range. New fundamental solutions for such formulation are derived for the first time.

The above-mentioned formulation is extended to a dual boundary element formulation to solve dynamic crack problems. The traction boundary integral equation utilized for this dual formulation is derived for the first time as well as their corresponding fundamental solutions. An out-of-plane fracture mode and the influence of plate thickness on dynamic stress intensity factors are analysed using such formulation, and the studies of the S0-mode wave scattering by a crack are also carried out.

The cracked plate structures subjected to dynamic bending loads are investigated using a

new Laplace-transform dual boundary element formulation which is developed for the Mindlin theory, a first-order shear deformable plate theory. The singular behaviours of the Laplace-transform fundamental solutions are re-examined and corrected. The dynamic fracture analysis of Mindlin plates is carried out using this formulation, which also allows to analyse the interaction between the fundamental antisymmetric Lamb wave (A0 mode) and the crack. The original fundamental solutions are modified slightly to represent the dispersion characteristic of the A0 mode more accurately.

Piezoelectric actuators and sensors are incorporated into the above-mentioned boundary element plate formulations to simulate the Lamb wave excitation and sensing respectively, which provides an alternative numerical tool for the Lamb-wave based structural health monitoring (SHM). A new actuator model, called an equivalent pin-force model, is proposed, which makes the numerical implementation efficient and simple. This BEM model is able to provide data accurately and efficiently as input for damage detection algorithms to localize crack in a plate. Three-dimensional finite element simulation and experiments are carried out to validate the BEM-simulated results.

Finally, spectral elements are introduced into a two-dimensional Laplace-transform BEM to solve high-frequency elastodynamic problems. It is shown that the computational efficiency can be improved significantly using the spectral elements, which indicates that it is well worth introducing the spectral elements into the proposed boundary element plate formulations for ultrasonic guided-wave SHM applications in the future.

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Chapter 1

Introduction

Fracture control and health monitoring are two key aspects of structural integrity assessment in modern industrial practice. This is particularly true for structures subjected to dynamic loadings, with the dynamic stress intensity factor being the key parameter for life assessment. Structural health monitoring is an essential part of integrity assessment of in-service structures to ensure flaws are detected and maintenance is carried out before they reach critical size.

1.1 Structural health monitoring

Currently, structural health monitoring (SHM) is receiving increasing interest, which is aimed at detecting damage in in-service engineering structures in real time. According to Ref. [5], the damage defined here is a broader concept, referring to the structural variations which have an adverse effect on the performance of system. It can be caused by material degradation, as well as the changes of geometric properties, boundary conditions and component connectivity. When the damage occurs in the structures, their properties, such as mechanical stiffness, thermal and electrical conductivities, could be changed. Such property changes are able to be utilized for SHM applications.

SHM is closely related to the conventional non-destructive evaluation (NDE), but it is worth

paying more attention to their difference. According to Ref. [6], NDE technologies assess structural integrity by direct measurements and previous measurement data are not important. By contrast, in SHM, the damage detection process is conducted through the interpretation of the historical data, and measurements should be performed frequently.

With the advancement of sensor technology, real time and in-situ measurements are available. For example, piezoelectric wafer sensors, which are lightweight and inexpensive, are able to be deployed in the structure permanently, allowing to continuously monitor the structure [7]. Moreover, they can be used as actuators and provide active measurements.

The damage detection techniques based on ultrasonic guided waves hold great potential for the health monitoring of structures because these short-wavelength waves are highly sensitive to small-sized damage and able to travel over long distances. Among different types of guided waves, the applications of Lamb waves to SHM are especially popular and have received considerable attention [8]. Lamb-wave based SHM has been carried out on structures made of isotropic materials, such as metals [9], and anisotropic materials, such as composites [10], and on structures with different geometric features, such as single plates [11], pipe-like structures [12] and stiffened panels [13]. Different types of damage, such as crack damage [14], corrosion damage [15], and impact damage [16], are able to be detected using Lamb waves.

Based on Lamb waves, there are many algorithms for damage detection and they were summarized comprehensively in the book [8]. Some typical algorithms are the triangulation based on the time-of-flight of Lamb waves [17], time reversal techniques [18], migration techniques [19], tomography methods [9], probability-based imaging [20], phased-array beamforming [21] and artificial intelligence [22].

Although the development of Lamb-wave based SHM technologies relies on extensive experimental analysis, numerical analysis also plays a particularly important role for several reasons. Firstly, it provides a deep understanding of Lamb-wave behaviours, such as wave scattering by the damage. Secondly, it offers a cost-efficient way to validate and improve damage detection

algorithms. Thirdly, sensor network optimization can be achieved based on numerical simulation results. Lastly, data-driven methods for SHM can be strongly supported by the abundant data which are obtained by numerical modelling.

Such numerical simulation is often carried out using the finite element method (FEM) because of its easy accessibility brought by existing commercial software, such as ANSYS and ABAQUS [23–25]. However, this method is computationally very expensive for modelling ultrasonic Lamb waves. In SHM, Lamb waves are often excited at very high working frequency (kHz or MHz ranges), and thus very small element size (mm range) and time increment step (ns range) are required in the simulation. Furthermore, when it comes to the modelling of a structure with damage, since finer mesh is used in the damaged zone, the computational cost becomes higher and sometimes it is computationally prohibitive.

In order to overcome the above-mentioned drawbacks of FEM, alternative numerical methods have been proposed. Compared with the standard FEM, higher convergence rate can be achieved by using higher-order finite element methods [26], such as a spectral element method (SEM) [27], in which higher-order shape functions are used. Besides, a local interaction simulation approach (LISA) was applied to solve the Lamb-wave problems [28, 29]. Although more degrees of freedom are required by this method, its inherent property enables massive parallelization using graphics processing units (GPUs), which improves the efficiency significantly. In contrast to domain-discretized methods, boundary element methods have enjoyed some popularity because the meshing is only limited to the domain boundaries with the help of fundamental solutions, and as a result significant reduction in the degrees of freedom can be achieved in the modelling of Lamb waves [30–32].

1.2 Fracture and failure

Fracture mechanics is a subject of engineering science, which is aimed at helping us understand how structural failure happens and how to prevent this failure. Fracture mechanics can be

categorized into two groups: linear elastic fracture mechanics (LEFM) and nonlinear fracture mechanics (NLFM). LEFM, which is based on linear elastic mechanics, is fairly well established. It is suitable for brittle fracture, in which the plastic-deformation zone around the crack tip is very small. The nonlinearities due to geometric, material and time-history dependence are considered in NLFM [33].

In LEFM, the most important parameter is the so-called stress intensity factor (SIF), which is commonly denoted by K . Regardless of the geometrical configuration of the cracked structure, the stress field in the vicinity of a crack tip is dominated by a singular term $K/\sqrt{r}$, where r is the distance between a field point and the crack tip [34]. The strength of such singularity is determined solely by SIF. Thus, SIF governs the crack behaviours, such as crack growth. For a cracked structure, generally speaking, SIF is a function of the crack parameters, loading and geometrical configuration [35].

There is a critical SIF, which is known as the fracture toughness denoted by K_c . When $K \geq K_c$, the crack growth becomes unstable and structural failure is caused. One of the fundamental assumptions in LEFM is that K_c is a material property and does not depend on structural size and geometry. Thus, the fracture toughness K_c can be measured in the laboratory and then applied to engineering structures. Such failure criterion can be easily implemented for a given cracked structure once its SIF, K , is obtained.

When inertial effects cannot be disregarded, such as the structures subjected to impact loading, SIF becomes dependent on time, which is called the dynamic stress intensity factor (DSIF). The DSIF could be several times higher than the SIF for the corresponding static problem because of the interaction between the excited elastic waves and the crack tip [36]. In order to describe the crack behaviours accurately in this case, dynamic fracture analysis should be conducted.

SIF is able to be determined using a variety of methods [37]. The SIF for some common structures of simple geometry had been summarized in reference books and we can get it

in a minute. When the structure becomes complex, its SIF can be estimated by experimental methods of compliance, photoelasticity, caustics or strain gage [38]. However, such experiments are usually very expensive and time-consuming. Moreover, it is difficult to measure DSIF directly in the required short time period experimentally [39]. Continuous improvements in computer technology allow us to compute both SIF and DSIF in an efficient and economical manner by using numerical methods.

Some representational numerical methods in fracture mechanics are finite element method (FEM), boundary element method (BEM) and meshless method. Each of them has its own advantages and disadvantages, but more importantly, they are not independent from each other. The advances in one method sometimes can be adopted to make progress in other methods. For example, in order to reduce nodal density around the crack tip in the element-free Galerkin method (a meshless method), Fleming et al. [40] introduced enriched basis functions consisting of the leading terms of the near-tip asymptotic displacement field into the approximate expression of displacement. These enrichment functions were later incorporated into FEM based on a partition-of-unity technique by Belytschko and Black [41], which is the starting point of the well-known eXtended FEM (XFEM). In 2011, Simpson and Trevelyan [42] applied such enrichment technique to BEM and proposed an enriched BEM, which improved the accuracy in evaluating SIF.

1.3 Boundary element method

The boundary element method (BEM) is a numerical method which solves partial differential equations by reformulating these equations as boundary integral equations (BIEs), and it has been applied to a wide range of scientific and engineering problems, such as elasticity, fluid mechanics, acoustics, heat transfer and electromagnetics [35, 43, 44]. At an early stage, BIEs were first used to solve potential problems [45]. In 1976, Lachat and Watson [46] adopted the basic concepts, such as the shape functions, Gaussian quadrature and isoparametric elements, from the finite element method to solve BIEs numerically. Since that time, BEM has become

widely accepted as a new effective numerical method and gained in popularity.

Compared with domain-based numerical methods, BEM, as a boundary-based method, has four main advantages [47]. First of all, only domain boundaries need to be discretized and thus the dimension of problems is reduced by one, which results in a smaller system of linear algebraic equations. Furthermore, when it comes to infinite-domain problems, domain truncation can be avoided because the boundary conditions at infinity are fulfilled automatically. Thirdly, since boundary element meshing process is implemented in a more efficient way without considering the domain discretization, labour cost requirements become lower. Lastly, when the problem involves moving boundaries, it is easier to achieve mesh adjustment in BEM.

Among many engineering applications, BEM has been demonstrated to be very efficient and accurate for fracture problems, such as the evaluation of stress intensity factors and the prediction of crack growth [48]. The most commonly used BIE for elasticity is the displacement BIE, which can be derived by using Betti's reciprocal theorem [35]. However, the displacement BIE is not sufficient when it is directly applied to the crack modelling. This is because the co-planar crack surfaces have the same boundary element mesh and the boundary nodes on the opposite surfaces share the same spatial location. The collocation carried out on the two coincident nodes results in the same equation, and as a result the coefficient matrix becomes singular.

In early applications, only crack problems with symmetric properties can be solved based solely on the displacement BIE because after inserting symmetrical boundary conditions only one crack surface is needed to be modelled [49]. In order to deal with general crack problems, Blandford et al. [50] divided the cracked body into several subdomains along the crack and thus each crack surface belongs to different subdomains. This special treatment resolved the singular-matrix issue, but it was not efficient because extra boundary elements were required to model the artificial boundaries.

Although the above-mentioned multi-domain method is effective, the numerically beautiful

and efficient properties of the single-domain methods remain attractive for researchers. There were several successful trials regarding the single-domain methods by using the indirect BEM, including the displacement discontinuity [51] and fictitious stress methods [52]. However, it is difficult to apply these methods to general crack problems. A real breakthrough which makes the single-domain method generally available is usually attributed to Aliabadi and his co-workers, who have given the effective numerical implementation of the dual boundary element method (DBEM) for solving 2D [53] and 3D [54] fracture problems. In DBEM, displacement BIE is applied to one crack surface, while the opposite crack surface is modelled using the traction BIE which can be derived from the displacement BIE by incorporating the stress-displacement relation.

Apart from the static fracture problems, DBEM is also suitable for dynamic fracture problems. There are three basic methods to conduct the elastodynamic analysis, the time domain method (TDM) [55], the Laplace transform method (LTM) [4] and the dual reciprocity method (DRM) [56]. The BIEs based on TDM and LTM rely on the dynamic fundamental solutions in the time domain and Laplace frequency domain respectively, while DRM requires static fundamental solutions. In DRM, some fixed points which can be placed in the domain or on the boundary are introduced as reference points to construct radial basis functions, which are used to approximate the acceleration term. The radial basis function can be treated as the applied load to the system, and the particular solutions to such partial differential system help to transform the domain integrals involving the inertial term to boundary integrals.

The comparisons of the above-mentioned three methods combined with DBEM for dynamic fracture analysis were carried out in Refs. [57, 58]. Among these three methods, LTM is of great interest because it requires the least memory and leads to the most accurate results. The numerical implementation of the Laplace-transform DBEM for 2D [4] and 3D [59] theories of elasticity have been conducted successfully. However, when it comes to dynamic crack problems in plates of finite thickness, the dual boundary element formulations for plate theories, such as the Kane-Mindlin (extensional motion) [60] and Mindlin (bending motion) [61] theories, are

more suitable taking both accuracy and efficiency into account, and they will be developed in this thesis.

BEM is also well-suited for elastic wave scattering problems, including crack scattering, because the Sommerfeld radiation condition can be rigorously satisfied and the truncation of the infinite medium is unnecessary. BEM has been applied to solve both 2D and 3D scattering problems in the isotropic medium [62–64]. In addition, BEM is suitable to the scattering analysis in anisotropic [65, 66] and inhomogeneous [67] solids. Zhao and Rose [31] incorporated Lamb wave normal mode expansions into the 2D BEM formulation for plane strain condition and used it to investigate the scattering of Lamb waves by internal defect.

When it comes to the piezoelectric solids, which are widely used as both sensors and actuators for structural health monitoring purposes, their transient analysis is also able to be conducted using BEM. Daros and Antes [68] proposed the dynamic boundary element formulation for the transversely isotropic piezoelectric material. The 3D boundary element model for general piezoelectric structures was developed in Ref. [69]. Alaimo et al. [70] applied the 2D boundary element formulation to analyse the dynamic behaviour of the coupled system consisting of a piezoelectric patch and a host plate which are bonded via adhesive.

1.4 Overview of the thesis

Although the dynamic boundary element formulations for 3D theory of elasticity have existed for many years [59] and recently been applied to guided-wave structural health monitoring applications for 3D structures [71], these formulations are not suitable for solving the elastodynamic problems of thin-walled structures, especially for the modelling of ultrasonic Lamb waves, because they suffer from the numerical difficulties in accurately computing the nearly singular integrals in the boundary integral equations. Thus, it is more appropriate to solve these problems using the boundary element formulations based on first-order plate theories, which are not only more time-efficient, but also can resolve such numerical difficulties. These

dynamic boundary element plate formulations will be developed in this thesis for isotropic materials and they will be extended to dual boundary element formulations to solve corresponding dynamic crack problems.

The motion of the plate with a through-thickness crack can be categorized into two groups, symmetric motion and antisymmetric motion with respect to the plate mid-plane. In the framework of the first-order theories, the symmetric and antisymmetric motions are governed by the Kane-Mindlin and Mindlin plate theories, respectively. There are two main applications of the dual boundary element formulations for these plate theories, dynamic fracture and Lamb-wave scattering by cracks. Furthermore, after incorporating piezoelectric actuators and sensors into these formulations, they allow to carry out numerical simulation for the ultrasonic Lamb-wave structural health monitoring (SHM). There are eight chapters in this thesis, including current introduction chapter and original contributions presented in Chapters 3 through 7.

Chapter 2 outlines the fundamental concepts, such as the Lamb waves and fracture, associated with the studies in this thesis. In addition, taking the dual boundary element method for 2D dynamic crack problems as an example, its numerical implementation is presented in detail.

In Chapter 3, the dynamic fundamental solutions for the Kane-Mindlin (K-M) theory are derived in the Laplace domain. Then a new boundary element formulation is proposed based on these fundamental solutions. The ability of this new formulation to model high-frequency S0 Lamb wave mode is shown. This study is extended to a dual boundary element method which enables us to solve the dynamic crack problems in the K-M plates, which is given in Chapter 4. When it comes to the dynamic crack problems in plates under bending, revisiting Laplace-transform fundamental solutions for the Mindlin theory in the open literature, the corresponding dual boundary element method is developed in Chapter 5.

The application of the dual boundary element formulations in Chapters 4 and 5 to Lamb-wave SHM is presented in Chapter 6. Piezoelectric lead-zirconate-titanate (PZT) discs are modelled to actuate the Lamb waves and sense the plate responses. In order to integrate the

PZT actuator into the boundary integral equations in an efficient manner, a new actuator model is proposed, which helps to reduce the 2D domain integral due to the actuator to 1D line integral. The proposed boundary element model is validated by 3D finite element simulation and experiments. Based on the data provided by this model, a damage detection algorithm is evaluated in terms of the crack localization in an aluminium plate.

In the conventional boundary element method (BEM), linear or quadratic elements are commonly used for meshing. When it comes to the dynamic problems in the high frequency range, these low-order elements are computationally inefficient. It is anticipated that the introduction of high-order spectral elements into BEM helps to reduce the total number of degrees of freedom required because of their high interpolation accuracy. In Chapter 7, the advantage of the spectral BEM in terms of the computational efficiency is demonstrated by solving 2D elastodynamic problems.

Finally, several important conclusions are summarised in Chapter 8, as well as some recommendations for future studies.

1.5 Author's published works

- Jun Li, Zahra Sharif Khodaei, and M. H. Aliabadi. Boundary element modelling of ultrasonic Lamb waves for structural health monitoring. *Smart Materials and Structures*, 2020.
- Jun Li, Zahra Sharif Khodaei, and M. H. Aliabadi. Dynamic fracture analysis of Kane–Mindlin plates using the dual boundary element method. *Engineering Analysis with Boundary Elements*, 106:217–227, 2019.
- Jun Li, Zahra Sharif Khodaei, and M. H. Aliabadi. Modelling of the high-frequency fundamental symmetric lamb wave using a new boundary element formulation. *International Journal of Mechanical Sciences*, 155:235–247, 2019.

- Jun Li, Zahra Sharif Khodaei, and M. H. Aliabadi. Dynamic dual boundary element analyses for cracked Mindlin plates. *International Journal of Solids and Structures*, 152-153:248–260, 2018.
- Jun Li, Zahra Sharif Khodaei, and M. H. Aliabadi. Spectral BEM for the analysis of wave propagation and fracture mechanics. *Journal of Multiscale Modelling*, 8(03n04):1740007, 2017.
- Jun Li, Zahra Sharif Khodaei, and M. H. Aliabadi. Dynamic fracture analysis of plates loaded in tension and bending using the dual boundary element method. *Key Engineering Materials*, 827:440–445, 2019.
- Jun Li, Zahra Sharif Khodaei, and M. H. Aliabadi. A boundary element model for structural health monitoring based on the S0 lamb wave mode. *Key Engineering Materials*, 774:625–631, 2018.

Chapter 2

Fundamental concepts

2.1 Introduction

In this chapter, the fundamental concepts related to structural health monitoring, fracture mechanics and boundary element method will be shown.

Throughout this chapter, Greek indices (α, β, γ) take the values 1 to 2, while Latin indices (i, j, k) run from 1 to 3; repeated indices are summed (summation convention) except where otherwise indicated. The partial derivative of any function f with respect to the time t is denoted by $\partial f / \partial t = f_{,t}$, while its partial derivatives with respect to Cartesian coordinates x_i are represented by $\partial f / \partial x_i = f_{,i}$. The Dirac delta function is denoted by δ without any subscripts or superscripts, while δ_{ij} involving index is the Kronecker delta defined as:

$$\delta_{ij} = \begin{cases} 1 & \text{when } i = j, \\ 0 & \text{when } i \neq j. \end{cases} \quad (2.1)$$

2.2 Ultrasonic guided waves

When an infinite homogeneous isotropic body is considered, only three types of bulk waves, longitudinal (L) wave, shear vertical (SV) wave and shear horizontal (SH) wave, propagate in the body. If the body has two plane surfaces (boundaries), such as plates, the bulk waves, which cannot propagate freely, are reflected back and forth between the two plane surfaces. This means that the propagation of the bulk waves are guided by the two boundaries of the body. The resulting waves comprising such reflected bulk waves are called guided waves. The propagation of the guided waves relies on the boundaries.

Ultrasonic guided waves refer to the guided waves with frequency higher than 20 kHz. Such high-frequency waves have very short wavelengths, which are very sensitive to small-sized damage, such as cracks, and as a result they are widely utilized for structural health monitoring.

2.2.1 Phase velocity and group velocity

First a one-dimensional wave propagating along x -axis is considered and its displacement u can be expressed as follows:

$$u(x, t) = A \cos(kx - \omega t) \quad (2.2)$$

in which A is the amplitude; k and ω denote wave number and frequency respectively. The argument, $kx - \omega t$, represents the phase of the wave. The phase velocity is the velocity of the constant phase propagating along the x -axis. Thus, we have:

$$\frac{d}{dt}(kx - \omega t) = k \frac{dx}{dt} - \omega = kc_p - \omega = 0 \quad (2.3)$$

and the phase velocity is given by:

$$c_p = \frac{\omega}{k} \quad (2.4)$$

Next, a wave consisting of two one-dimensional waves which have very close frequencies and

wave numbers, as well as the same amplitude, has the following expression:

$$\begin{aligned} u(x, t) &= A \cos(kx - \omega t) + A \cos[(k + \Delta k)x - (\omega + \Delta\omega)t] \\ &= 2A \cos\left(\frac{2k + \Delta k}{2}x - \frac{2\omega + \Delta\omega}{2}t\right) \cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right) \end{aligned} \quad (2.5)$$

in which Δk and $\Delta\omega$ denote the differences of the wave number and frequency between such two wave components, respectively. When such differences tend to zero, we obtain:

$$\lim_{\Delta k \rightarrow 0, \Delta\omega \rightarrow 0} u(x, t) = 2A \cos(kx - \omega t) \cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right) \quad (2.6)$$

It can be seen from Eq. (2.6) that the carrier wave with the wavelength k is modulated by the wave with a smaller wavelength Δk . The modulated wave forms the envelope of such wave group. The propagating velocity (phase velocity) of such envelope is called the group velocity, which is given by:

$$c_g = \lim_{\Delta k \rightarrow 0, \Delta\omega \rightarrow 0} \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk} \quad (2.7)$$

2.2.2 Wave dispersion

Wave dispersion is a very important phenomenon because it governs the shape change of a wave packet propagating in a medium [72]. When waves of different frequencies propagate at different phase velocities, they are dispersive. As a result, a wave packet consisting of many different frequency components distorts in the process of propagation because some components travel fast while others travel slowly. If the waves are not dispersive, the phase velocity is equal to the group velocity and the wave packet travels in the medium without any deformation. In structural health monitoring, non-dispersive or less dispersive waves are preferred because they can travel over long distance and keep the shape, which facilitates the signal processing.

2.2.3 Lamb waves

A free plate with a thickness of h in plane strain state (see Fig. 2.1) is adopted to introduce Lamb waves. The x_1 -axis is located at the mid-plane of the plate and the normal to the plane is the x_3 -axis.

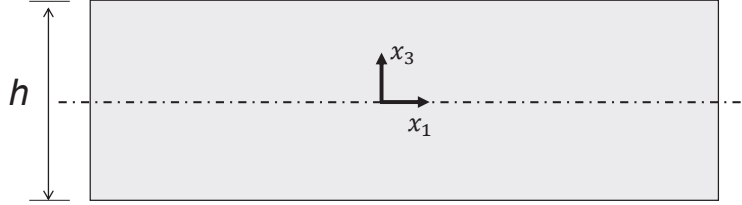


Figure 2.1: A free plate in plane strain state.

According to Helmholtz decomposition, the displacements u_1 and u_3 can be decomposed into two potentials ϕ_L and ϕ_S which governing the longitudinal wave and shear vertical wave respectively, and the displacements are given by:

$$\begin{aligned} u_1 &= \frac{\partial \phi_L}{\partial x_1} + \frac{\partial \phi_S}{\partial x_3} \\ u_3 &= \frac{\partial \phi_L}{\partial x_3} + \frac{\partial \phi_S}{\partial x_1} \end{aligned} \quad (2.8)$$

Substituting Eq. (2.8) into the stress-displacement relationship yields:

$$\begin{aligned} \sigma_{31} &= \mu \left(\frac{2\partial^2 \phi_L}{\partial x_1 \partial x_3} - \frac{\partial^2 \phi_S}{\partial x_1^2} + \frac{\partial^2 \phi_S}{\partial x_3^2} \right) \\ \sigma_{33} &= \lambda \nabla^2 \phi_L + 2\mu \left(\frac{\partial^2 \phi_L}{\partial x_3^2} - \frac{\partial^2 \phi_S}{\partial x_1 \partial x_3} \right) \end{aligned} \quad (2.9)$$

in which $\nabla^2(\cdot) = \frac{\partial^2(\cdot)}{\partial x_1^2} + \frac{\partial^2(\cdot)}{\partial x_3^2}$.

The two potentials satisfy wave equations:

$$\begin{aligned} \nabla^2 \phi_L &= \frac{1}{c_1^2} \frac{\partial^2 \phi_L}{\partial t^2} \\ \nabla^2 \phi_S &= \frac{1}{c_2^2} \frac{\partial^2 \phi_S}{\partial t^2} \end{aligned} \quad (2.10)$$

When harmonic waves are considered, the potentials can be written in the following form:

$$\begin{aligned}\phi_L &= \Phi_L(x_3) e^{i(kx_1 - \omega t)} \\ \phi_S &= \Phi_S(x_3) e^{i(kx_1 - \omega t)}\end{aligned}\tag{2.11}$$

in which Φ_L and Φ_S represent wave structures in the thickness direction.

Substituting Eq. (2.11) into Eq. (2.10) yields:

$$\begin{aligned}\frac{\partial^2 \Phi_L}{\partial x_3^2} + a^2 \Phi_L &= 0 \\ \frac{\partial^2 \Phi_S}{\partial x_3^2} + b^2 \Phi_S &= 0\end{aligned}\tag{2.12}$$

where $a^2 = \omega^2/c_1^2 - k^2$ and $b^2 = \omega^2/c_2^2 - k^2$. The solutions to Eq. (2.12) are:

$$\begin{aligned}\Phi_L &= C_L^A \sin(ax_3) + C_L^S \cos(ax_3) \\ \Phi_S &= C_S^S \sin(bx_3) + C_S^A \cos(bx_3)\end{aligned}\tag{2.13}$$

in which C_L^A , C_L^S , C_S^S and C_S^A are unknown coefficients. Inserting Eqs. (2.13) and (2.11) into Eq. (2.8), the displacements become (omitting the term $e^{i(kx_1 - \omega t)}$):

$$\begin{aligned}u_1 &= C_L^S \cos(ax_3) ki + bC_S^S \cos(bx_3) + C_L^A \sin(ax_3) ki - C_S^A b \sin(bx_3) \\ u_3 &= -C_L^S a \sin(ax_3) + C_S^S \sin(bx_3) ki + C_L^A a \cos(ax_3) + C_S^A \cos(bx_3) ki\end{aligned}\tag{2.14}$$

By observing this expression, the motions can be split into symmetric modes (cosines in u_1 and sines in u_3) and antisymmetric modes (sines in u_1 and cosines in u_3) with respect to $x_3=0$. Thus, the terms involving the coefficients with the superscript S and A denote the symmetric modes and antisymmetric modes, respectively.

When the symmetric modes are considered, substituting the symmetric terms in Eq. (2.13) into Eq. (2.9), the stress expressions for the symmetric modes can be obtained. After introducing traction-free boundaries conditions at $x_3 = \pm h/2$, we obtain two homogeneous equations for

the unknowns C_S^L and C_S^S . In order to get nontrivial solutions, the determinant of the system coefficient matrix should be equal to zero. Then, the characteristic equation for the symmetric modes is obtained and written as follows:

$$\frac{\tan(bh/2)}{\tan(ah/2)} = -\frac{4k^2ab}{(b^2 - k^2)^2} \quad (2.15)$$

Solving this equation numerically [1], the dispersion relation of the symmetric modes can be obtained.

When it comes to the antisymmetric modes, following the above-mentioned procedure, the corresponding characteristic equation is given by:

$$\frac{\tan(bh/2)}{\tan(ah/2)} = -\frac{(b^2 - k^2)^2}{4k^2ab} \quad (2.16)$$

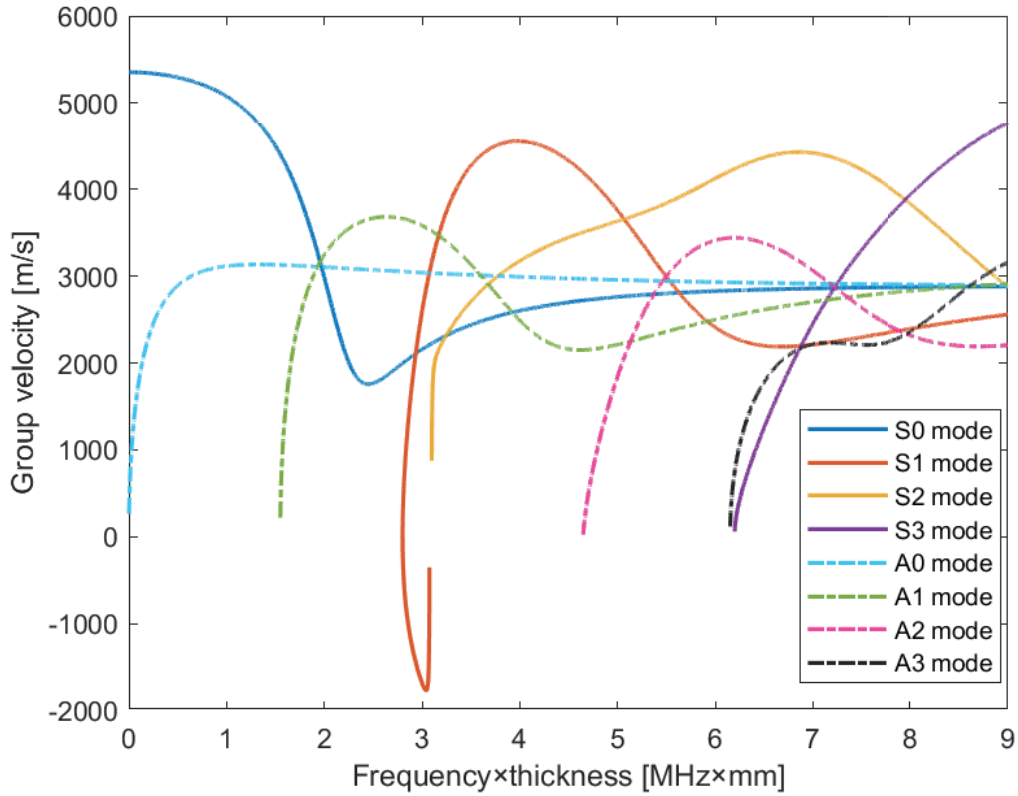


Figure 2.2: Group-velocity dispersion curves for Lamb waves in an aluminium plate.

For an aluminium plate, the group-velocity dispersion curves for the first four symmetric and antisymmetric modes are presented in Fig. 2.2.

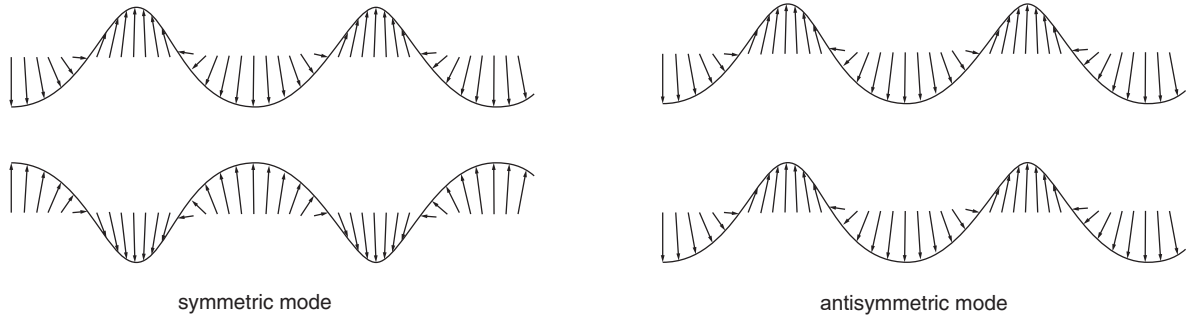


Figure 2.3: Lamb wave mode shape and the displacement vector field on the plate surfaces [1].

The mode shapes of the symmetric and antisymmetric Lamb waves are illustrated in Fig. 2.3. It can be seen that the symmetric and antisymmetric modes are similar to the compressional and flexural waves, respectively.

2.2.4 SH waves

As mentioned earlier, there exist three types of bulk waves, longitudinal (L) wave, shear vertical (SV) and shear horizontal (SH) waves, in an isotropic medium. Lamb waves are composed of the L wave and SV wave. Guided shear horizontal (SH) wave modes in a plate consist only the shear horizontal bulk wave. In the following section, the terms SH waves and SH modes refer to the guided SH waves, rather than the bulk SH waves.

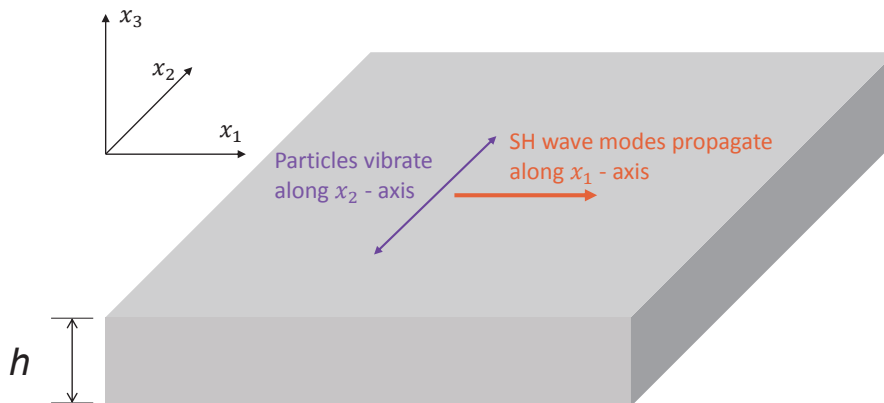


Figure 2.4: SH wave modes in a plate with a thickness of h .

As shown in Fig. 2.4, SH waves propagate along x_1 -axis and the particles in the plate only vibrate in the shear-horizontal direction (x_2 -axis). Under the plane strain condition, the governing equations for the SH modes are given by:

$$\begin{aligned}\nabla^2 u_2(x_1, x_3, t) &= \frac{1}{c_2^2} \frac{\partial^2 u_2(x_1, x_3, t)}{\partial t^2} \\ u_1 &= u_3 = 0\end{aligned}\tag{2.17}$$

Assuming a harmonic motion, the displacement u_2 can be written in the following form:

$$u_2 = \Phi_2(x_3) e^{i(kx_1 - \omega t)}\tag{2.18}$$

Substituting Eq. (2.18) into Eq. (2.17) yields:

$$\frac{\partial^2 \Phi_2}{\partial x_3^2} + \left(\frac{\omega^2}{c_2^2} - k^2 \right) \Phi_2 = 0\tag{2.19}$$

The solutions of Eq. (2.19) can be expressed as follows:

$$\Phi_2 = C_{\text{SH}}^{\text{A}} \sin(bx_3) + C_{\text{SH}}^{\text{S}} \cos(bx_3)\tag{2.20}$$

in which $b^2 = \omega^2/c_2^2 - k^2$; the unknown coefficients C_{SH}^{S} and C_{SH}^{A} can be determined using boundary conditions. Thus, disregarding the exponential term, the displacement u_2 becomes:

$$u_2 = C_{\text{SH}}^{\text{A}} \sin(bx_3) + C_{\text{SH}}^{\text{S}} \cos(bx_3)\tag{2.21}$$

After applying the traction-free boundary conditions:

$$\sigma_{32}|_{x_3=\pm h/2} = \mu \left. \frac{\partial u_2}{\partial x_3} \right|_{x_3=\pm h/2} = 0\tag{2.22}$$

to Eq. (2.21), we have the characteristic equation of the SH modes:

$$\sin(bh) = 0 \quad (2.23)$$

The solutions to Eq. (2.23) are obvious, from which the group velocities of the SH modes can be obtained as follows:

$$c_g^{\text{SH}} = c_2 \sqrt{1 - \left(\frac{n_{\text{SH}} \pi c_2}{\omega h} \right)^2}, \quad n_{\text{SH}} = 0, 1, 2, \dots \quad (2.24)$$

The group-velocity dispersion curves for the first six SH modes in an aluminium plate are shown in Fig. 2.5.

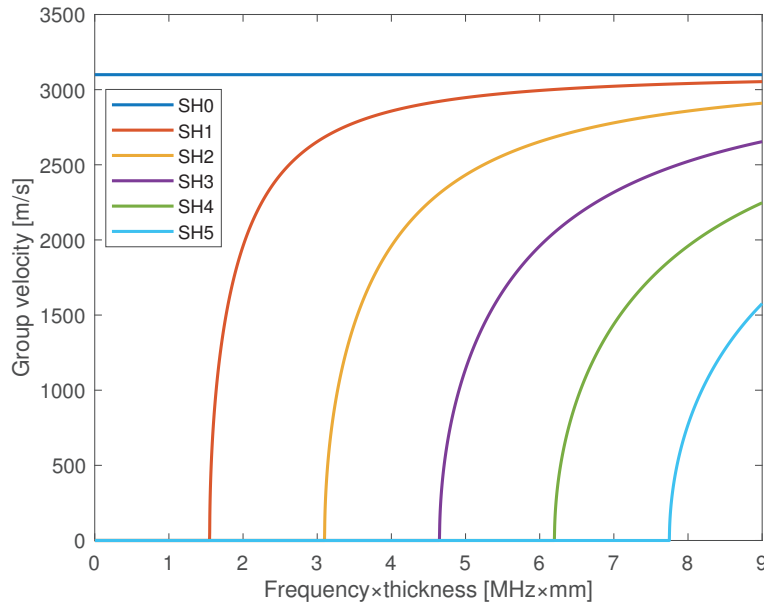


Figure 2.5: Group-velocity dispersion curves for SH modes in an aluminium plate.

2.2.5 Wave excitation signals

In SHM applications, in order to minimize the dispersion effect of the guided waves and facilitate signal processing, the excitation signal commonly consists of a sinusoidal tone burst signal

modulated by a Hann window:

$$V(t) = \frac{1}{2}V_m \sin(2\pi f_c t) \left[1 - \cos\left(\frac{2\pi f_c t}{N_w}\right) \right] H\left(\frac{N_w}{f_c} - t\right) \quad (2.25)$$

where f_c and N_w denote the central frequency and the number of cycles respectively; V_m represents the amplitude and H is the Heaviside function.

As a example, a five-cycle Hann-window-modulated sinusoidal tone burst with a central frequency of 300 kHz is illustrated in Fig. 2.6.

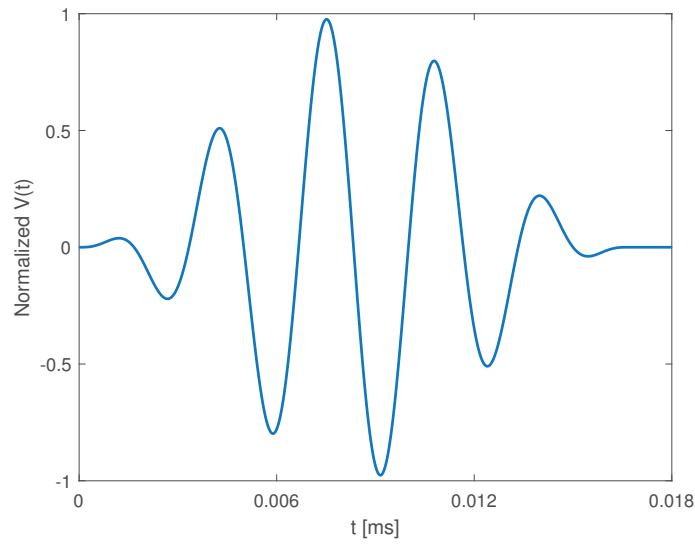


Figure 2.6: A five-cycle Hann-window-modulated sinusoidal tone burst with a central frequency of 300 kHz.

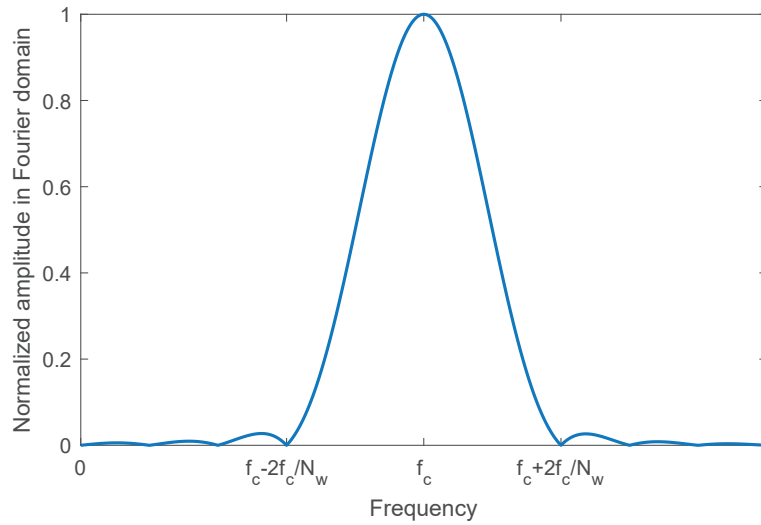


Figure 2.7: Frequency spectrum of the Hann-window-modulated sinusoidal tone burst.

Fig. 2.7 shows the frequency spectrum of the Hann-window-modulated sinusoidal tone burst. It can be seen that the energy of the signal is highly concentrated in the main lobe ranging from $f_c - 2f_c/N_w$ to $f_c + 2f_c/N_w$.

2.3 Fracture mechanics

Fracture mechanics focuses on the propagation of cracks and can be used to estimate residual life of cracked structures. In this thesis, the study is within the framework of linear elastic fracture mechanics (LEFM). LEFM is on the basis of linear theory of elasticity and applicable to brittle fractures.

2.3.1 Basic fracture modes

As shown in Fig. 2.8, there are three basic fracture modes, which correspond to three basic types of loading. For Mode I, the crack tends to open when the crack surfaces are subjected to tensile stress. The in-plane and out-of-plane shear stresses causes the Mode II and Mode III fractures, respectively. The combination of tensile and shear loads leads to mixed-mode fractures.

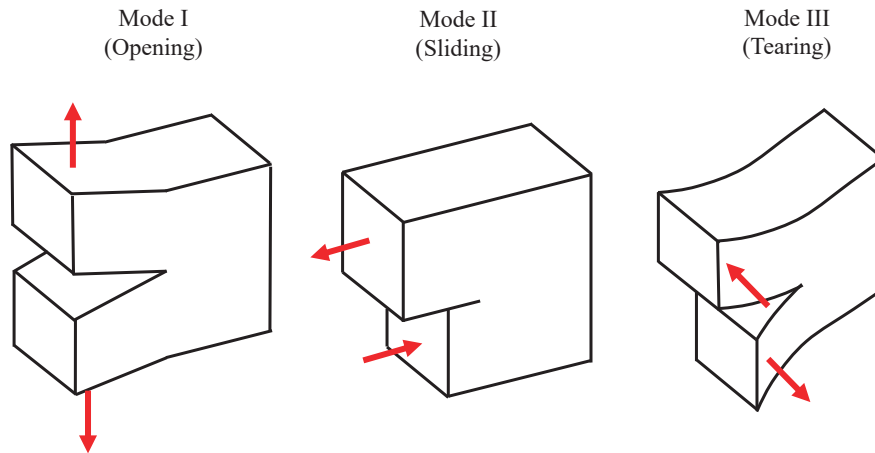


Figure 2.8: Three basic crack deformation modes.

2.3.2 Stress intensity factors

The definition of the cartesian and polar coordinate systems with their origin at the crack tip is illustrated in Fig. 2.9.

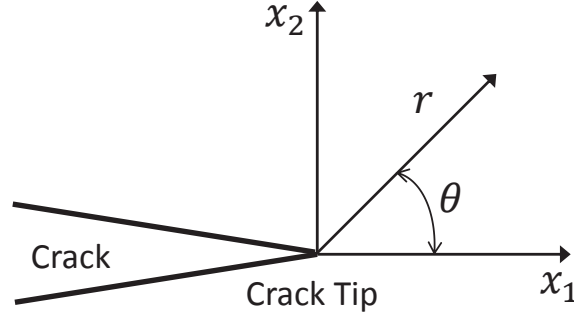


Figure 2.9: Crack-tip coordinate system (x_3 -axis is normal to the page).

In LEFM, the stress fields in the vicinity of a crack tip are considered to be solely determined by the stress intensity factors (SIFs). Such stresses for the three fracture modes can be expressed as follows [34]:

$$\begin{aligned}
 \sigma_{11}^{(I)} &= \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right] + \text{other terms} \\
 \sigma_{22}^{(I)} &= \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 + \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right] + \text{other terms} \\
 \sigma_{12}^{(I)} &= \frac{K_I}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right) + \text{other terms} \\
 \sigma_{11}^{(II)} &= -\frac{K_{II}}{\sqrt{2\pi r}} \sin\left(\frac{\theta}{2}\right) \left[2 + \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right) \right] + \text{other terms} \\
 \sigma_{22}^{(II)} &= \frac{K_{II}}{\sqrt{2\pi r}} \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right) + \text{other terms} \\
 \sigma_{12}^{(II)} &= \frac{K_{II}}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \right] + \text{other terms} \\
 \sigma_{31}^{(III)} &= -\frac{K_{III}}{\sqrt{2\pi r}} \sin\left(\frac{\theta}{2}\right) + \text{other terms} \\
 \sigma_{32}^{(III)} &= \frac{K_{III}}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) + \text{other terms}
 \end{aligned} \tag{2.26}$$

in which K_I , K_{II} and K_{III} denote the stress intensity factors for Mode I, Mode II and Mode III, respectively; other terms consist of the remaining terms, which are regular and do not contain

any singularities. As the coordinate r approaches zero, the singular terms controlled by SIFs dominate the stress fields and other terms can be disregarded. SIFs are of great importance in engineering applications because they can be used to predict crack propagation and estimate the residual strength of structures [35].

2.3.3 Dynamic fracture mechanics

Dynamic fracture mechanics is concerned with situations in which the inertial effects cannot be disregarded, such as structures subjected to impact loads. In dynamic cases, the stresses in the vicinity of a crack tip vary with time because of the propagation and reflection of elastic waves, and the time-dependent SIF is called the dynamic stress intensity factor (DSIF). The peak value of DSIF is higher than the corresponding static SIF, which accounts for the fact that the material is more likely to fail under impulsive dynamic loading than that under quasi-static loading [2]. For example, Fig. 2.10 illustrates the variation of the ratio between DSIF and the static SIF over time for a Mode I fracture problem. It can be seen that the peak DSIF value could be approximately twice higher than the static SIF value.

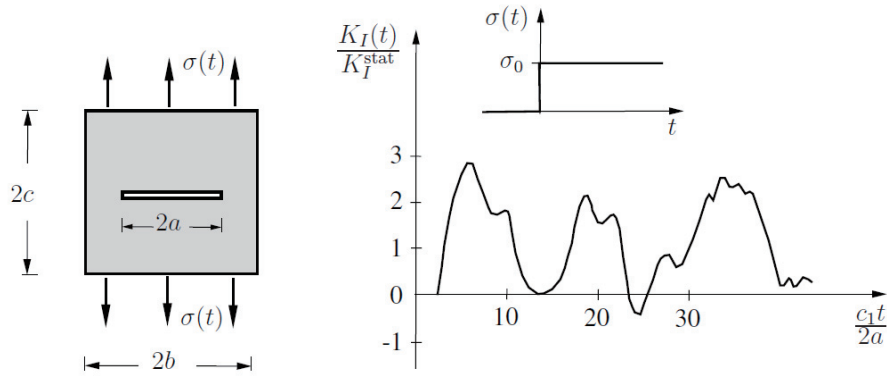


Figure 2.10: The ratio of dynamic stress intensity factor K_I to the static stress intensity factor K_I^{stat} for a rectangular plate with a central crack under impact tensile loading [2].

2.4 Boundary element method

In this section, the Laplace-transform BEM for two-dimensional (2D) problems of elastodynamics is taken as an example to introduce the main characteristics of BEM.

2.4.1 Elastodynamic equations

The equations governing the displacements u_α in a 2D isotropic body under the plane strain condition can be written as:

$$c_2^2 u_{\alpha,\beta\beta} + (c_1^2 - c_2^2) u_{\beta,\beta\alpha} + F_\alpha = u_{\alpha,tt} \quad (2.27)$$

in which the dilatational and shear wave speeds are given by $c_1^2 = (\lambda + 2\mu)/\rho$ and $c_2^2 = \mu/\rho$ respectively; ρ is the density; E and ν in the Lamé constants $\lambda = E\nu/[(1+\nu)(1-2\nu)]$ and $\mu = E/[2(1+\nu)]$ denote Young's modulus and Poisson's ratio, respectively; F_α stands for body force. After replacing c_1^2 with $c_3^2 = E/[\rho(1-\nu^2)]$, the governing equations under the generalized plane stress condition are obtained readily.

2.4.2 Laplace transform method

In order to avoid the domain integrals due to the inertial term in the boundary element formulations, the elastodynamic problems can be solved using a Laplace transform method [4]. With the help of this method, the partial derivative with respect to time becomes a multiplication operation in the Laplace domain.

The Laplace transform with respect to time can be expressed as:

$$\mathcal{L}[f(t)] = \tilde{f}(p) = \int_0^\infty f(t) e^{-pt} dt \quad (2.28)$$

where p denotes the Laplace transform parameter (also known as the complex frequency).

Applying Eq. (2.28) to Eq. (2.27) yields:

$$c_2^2 \tilde{u}_{\alpha,\beta\beta} + (c_1^2 - c_2^2) \tilde{u}_{\beta,\beta\alpha} + \tilde{F}_\alpha = p^2 \tilde{u}_\alpha \quad (2.29)$$

2.4.3 Fundamental solutions

Fundamental solutions play a very important role in the BEM. When an unit point body force represented by the Dirac function $\delta(\mathbf{X} - \mathbf{X}')$ is applied at the point $\mathbf{X}'$ in an infinite body $\Omega(\mathbf{X})$, the solutions to the governing equations are defined as the fundamental solutions, which can be derived using the Galerkin vector approach [35, 73]. In this thesis, the fundamental solutions associated with the static and dynamic problems are called static fundamental solutions and dynamic fundamental solutions, respectively.

2.4.4 Displacement boundary integral equation

With the help of the fundamental solutions, the displacement at any internal (domain) point can be determined using the well-known Betti's reciprocal work theorem [35] as follows:

$$\begin{aligned} \tilde{u}_\alpha(\mathbf{X}', p) = & \int_{\Gamma} \tilde{U}_{\alpha\beta}(\mathbf{X}', \mathbf{x}, p) \tilde{t}_\beta(\mathbf{x}, p) d\Gamma - \int_{\Gamma} \tilde{T}_{\alpha\beta}(\mathbf{X}', \mathbf{x}, p) \tilde{u}_\beta(\mathbf{x}, p) d\Gamma \\ & + \int_{\Omega} \tilde{U}_{\alpha\beta}(\mathbf{X}', \mathbf{X}, p) \tilde{F}_\beta(\mathbf{X}, p) d\Omega \end{aligned} \quad (2.30)$$

in which the tractions are $\tilde{t}_\beta = \tilde{\sigma}_{\alpha\beta} n_\alpha$, consisting of stress components $\tilde{\sigma}_{\alpha\beta}$ and the components of the outward unit normal vector, n_α , to the boundary Γ ; $\tilde{U}_{\alpha\beta}$ and $\tilde{T}_{\alpha\beta}$ are displacement and traction dynamic fundamental solutions given in Ref. [4]; $\mathbf{x}$ and $\mathbf{X}$ denote the points on the boundary Γ and in the domain Ω , respectively; p stands for the Laplace transform parameter.

When the load point is moved from $\mathbf{X}'$ in the domain to $\mathbf{x}'$ on the boundary, Eq. (2.30) becomes:

$$\begin{aligned} c_{\alpha\beta}(\mathbf{x}') \tilde{u}_\alpha(\mathbf{x}', p) = & \int_{\Gamma} \tilde{U}_{\alpha\beta}(\mathbf{x}', \mathbf{x}, p) \tilde{t}_\beta(\mathbf{x}, p) d\Gamma - \oint_{\Gamma} \tilde{T}_{\alpha\beta}(\mathbf{x}', \mathbf{x}, p) \tilde{u}_\beta(\mathbf{x}, p) d\Gamma \\ & + \int_{\Omega} \tilde{U}_{\alpha\beta}(\mathbf{x}', \mathbf{X}, p) \tilde{F}_\beta(\mathbf{X}, p) d\Omega \end{aligned} \quad (2.31)$$

where $\oint$ denotes Cauchy principal value integral; $c_{\alpha\beta}$ is the coefficient depending on the local boundary geometry characteristic around the point $\mathbf{x}'$ and it is equal to $\delta_{\alpha\beta}/2$ when $\mathbf{x}'$ is located on a smooth boundary [35]. Such equation is known as the displacement boundary integral

equation (DBIE), which provides an alternative way to describe the state of a finite-sized body. When the domain-integral term involving the body force can be neglected or the body force distribution is known before the analysis, a truly boundary-only formulation is obtained. If such domain integral cannot be calculated before the analysis, some techniques, such as a dual reciprocity method [74], can be used to transform it to a boundary integral. Observing the DBIE, it can be found that it is the fundamental solutions that link the boundary state to the domain state.

2.4.5 Traction boundary integral equation

Apart from the DBIE, a traction boundary integral equation (TBIE) is another commonly used equation in the boundary element method.

Substituting Eq. (2.30) into stress-displacement relation, the boundary integral equation for computing internal stress is obtained as follows:

$$\begin{aligned} \tilde{\sigma}_{\alpha\beta}(\mathbf{X}', p) = & \int_{\Gamma} \tilde{D}_{\alpha\beta\gamma}(\mathbf{X}', \mathbf{x}, p) \tilde{t}_{\gamma}(\mathbf{x}, p) d\Gamma - \int_{\Gamma} \tilde{S}_{\alpha\beta\gamma}(\mathbf{X}', \mathbf{x}, p) \tilde{u}_{\gamma}(\mathbf{x}, p) d\Gamma \\ & + \int_{\Omega} \tilde{D}_{\alpha\beta\gamma}(\mathbf{X}', \mathbf{X}, p) \tilde{F}_{\gamma}(\mathbf{X}, p) d\Omega \end{aligned} \quad (2.32)$$

where $\tilde{D}_{\alpha\beta\gamma}$ and $\tilde{S}_{\alpha\beta\gamma}$ are fundamental solutions, which can be found in Ref. [4]. Moving the source point $\mathbf{X}'$ in the domain to the point $\mathbf{x}'$ on a smooth boundary and considering the definition of traction ($\tilde{t}_{\alpha} = \tilde{\sigma}_{\alpha\beta} n_{\beta}$) yield the TBIE :

$$\begin{aligned} \frac{1}{2} \tilde{t}_{\alpha}(\mathbf{x}', p) = & n_{\beta}(\mathbf{x}') \left[\oint_{\Gamma} \tilde{D}_{\alpha\beta\gamma}(\mathbf{x}', \mathbf{x}, p) \tilde{t}_{\gamma}(\mathbf{x}, p) d\Gamma - \oint_{\Gamma} \tilde{S}_{\alpha\beta\gamma}(\mathbf{x}', \mathbf{x}, p) \tilde{u}_{\gamma}(\mathbf{x}, p) d\Gamma \right. \\ & \left. + \int_{\Omega} \tilde{D}_{\alpha\beta\gamma}(\mathbf{x}', \mathbf{X}, p) \tilde{F}_{\gamma}(\mathbf{X}, p) d\Omega \right] \end{aligned} \quad (2.33)$$

where $\oint$ denotes Hadamard finite-part integral.

It should be noted that although both DBIE and TBIE can be used individually for a standard boundary element analysis. Since the fundamental solutions in DBIE is simpler and

fewer computational costs are required, DBIE, rather than the TBIE, is used in the boundary element modelling. However, it does not mean that TBIE has little value. TBIE plays a key role in modelling cracks, which will be shown in the next subsection regarding the dual boundary element method.

2.4.6 Dual boundary element method

As mentioned earlier, DBIE is preferred in the boundary element modelling. However, when it comes to fracture problems, two co-planar crack surfaces are treated as extra boundaries. Modelling such two surfaces only using DBIE leads to mathematical degeneration. It can be resolved by introducing artificial boundaries, alongside the crack surfaces, to divide the body into several parts. However, this multi-domain method is less efficient. The dual boundary element method (DBEM), which avoids the artificial boundaries and enables a single-domain solution, has been demonstrated to be very efficient and promising [53]. In DBEM, DBIE is used for one crack surface, while the opposite surface is treated using TBIE.

2.4.7 Boundary element discretization

When there are no cracks in the body, quadratic isotropic line elements are used to discretize the boundaries. In every element, the geometry (x_α), displacements ($\tilde{u}_\alpha$) and tractions ($\tilde{t}_\alpha$) are approximated using the shape functions (N_i) defined by Lagrange interpolation functions:

$$\begin{aligned} x_\alpha(\xi) &= \sum_{i=1}^3 N_i(\xi) x_\alpha^i \\ \tilde{u}_\alpha(\xi) &= \sum_{i=1}^3 N_i(\xi) \tilde{u}_\alpha^i \\ \tilde{t}_\alpha(\xi) &= \sum_{i=1}^3 N_i(\xi) \tilde{t}_\alpha^i \end{aligned} \tag{2.34}$$

in which x_α^i , $\tilde{u}_\alpha^i$ and $\tilde{t}_\alpha^i$ are nodal values and $\xi \in [-1, 1]$.

When cracks are involved in the body, the discretization strategy proposed in Ref. [53] is

adopted, which is shown in Fig. 2.11.

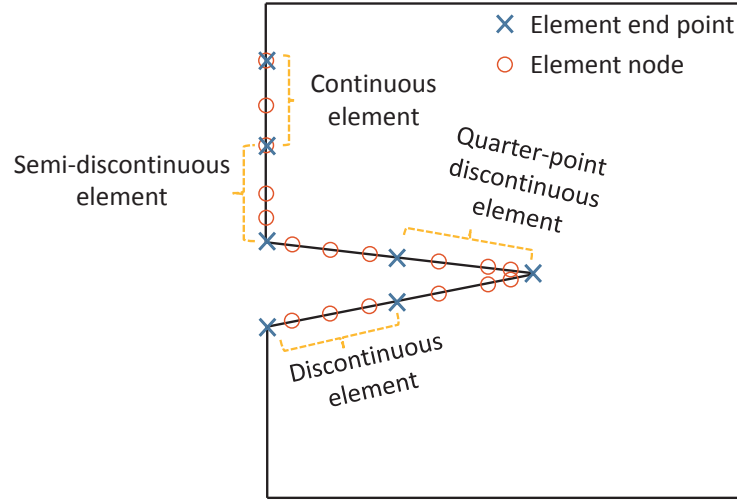


Figure 2.11: Boundary discretization strategy for a cracked body.

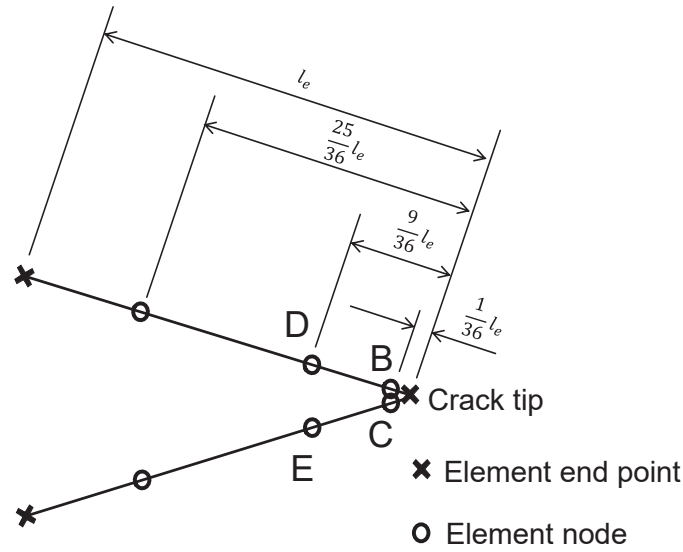


Figure 2.12: Quarter-point crack-tip element.

In order to fulfil the conditions for the existence of the Hadamard finite-part integrals in TBIE, the first-order derivatives of the displacements at crack element nodes should be continuous. Thus, discontinuous and semi-discontinuous quadratic boundary elements are used for the crack and crack-edge intersection, respectively. In these elements, the nodes at the end of an element are shifted one-sixth of element length towards the middle node. In addition,

since the displacement variation in the vicinity of the crack tip is of order $\sqrt{r_{\text{tip}}}$ (r_{tip} being the radial distance from the crack tip), the quarter-point discontinuous element [55] is used as the crack-tip element to represent such variation more accurately, which is illustrated in Fig. 2.12. It should be mentioned that in the above-mentioned special elements, only nodal positions in the elements are modified, while the shape functions are retained.

After all the boundaries, including cracks, are divided into boundary elements, DBIE is collocated on every node except for those on one crack surface collocating using TBIE. Regardless of the body force, when the collocation node is located at $\mathbf{x}^c$, DBIE and TBIE become:

$$\begin{aligned} c_{\alpha\beta}(\mathbf{x}^c) \tilde{u}_\beta(\mathbf{x}^c, p) &= \sum_{n=1}^{N_e} \sum_{i=1}^3 \left[\tilde{t}_\beta^{ni}(p) \int_{-1}^1 \tilde{U}_{\alpha\beta}(\mathbf{x}^c, \xi, p) N_i(\xi) J_n(\xi) d\xi \right. \\ &\quad \left. \tilde{u}_\beta^{ni}(p) \int_{-1}^1 \tilde{T}_{\alpha\beta}(\mathbf{x}^c, \xi, p) N_i(\xi) J_n(\xi) d\xi \right] \\ \frac{1}{2} \tilde{t}_\alpha(\mathbf{x}^c, p) &= \sum_{n=1}^{N_e} \sum_{i=1}^3 n_\beta(\mathbf{x}^c) \left[\tilde{t}_\gamma^{ni}(p) \int_{-1}^1 \tilde{D}_{\alpha\beta\gamma}(\mathbf{x}^c, \xi, p) N_i(\xi) J_n(\xi) d\xi \right. \\ &\quad \left. - \tilde{u}_\gamma^{ni}(p) \int_{-1}^1 \tilde{S}_{\alpha\beta\gamma}(\mathbf{x}^c, \xi, p) N_i(\xi) J_n(\xi) d\xi \right] \end{aligned} \quad (2.35)$$

where N_e denotes the total number of boundary elements; J_n represents the Jacobian for each element; the summation convention is applied only to β and γ . Such discretized integral equations can be assembled into a system equation and written in matrix form:

$$\tilde{\mathbf{H}}(p) \tilde{\mathbf{u}}(p) = \tilde{\mathbf{G}}(p) \tilde{\mathbf{t}}(p) \quad (2.36)$$

where $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{G}}$ are coefficient matrices involving the integral of fundamental solutions; $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{t}}$ are vectors consisting of nodal displacements and tractions respectively. After inserting boundary conditions, Eq. (2.36) can be rearranged as follows:

$$\tilde{\mathbf{A}}(p) \tilde{\mathbf{x}}(p) = \tilde{\mathbf{f}}(p) \quad (2.37)$$

in which $\tilde{\mathbf{A}}$ is the coefficient matrix and $\tilde{\mathbf{f}}$ is a known vector resulting from the boundary

conditions; unknown displacements and tractions are stored in $\tilde{\mathbf{x}}$. The LU decomposition approach is adopted in this thesis to solve Eq. (2.37).

2.4.8 Treatment of singularities

Since the dynamic fundamental solutions involve singular terms at $r = |\mathbf{x}^c - \mathbf{x}| = 0$, it is essential to understand the behaviour of these fundamental solutions in the neighbourhood of the collocation point. It has been found in Ref. [4] that the singular terms in the dynamic fundamental solutions have the same expression as those in the corresponding static fundamental solutions. Thus, the dynamic fundamental solutions can be split into two parts as follows:

$$\begin{aligned}\tilde{U}_{\alpha\beta}(\mathbf{x}^c, \mathbf{x}, p) &= U_{\alpha\beta}^{\text{static}}(\mathbf{x}^c, \mathbf{x}) + \tilde{U}_{\alpha\beta}^m(\mathbf{x}^c, \mathbf{x}, p) \\ \tilde{T}_{\alpha\beta}(\mathbf{x}^c, \mathbf{x}, p) &= T_{\alpha\beta}^{\text{static}}(\mathbf{x}^c, \mathbf{x}) + \tilde{T}_{\alpha\beta}^m(\mathbf{x}^c, \mathbf{x}, p) \\ \tilde{D}_{\alpha\beta\gamma}(\mathbf{x}^c, \mathbf{x}, p) &= D_{\alpha\beta\gamma}^{\text{static}}(\mathbf{x}^c, \mathbf{x}) + \tilde{D}_{\alpha\beta\gamma}^m(\mathbf{x}^c, \mathbf{x}, p) \\ \tilde{S}_{\alpha\beta\gamma}(\mathbf{x}^c, \mathbf{x}, p) &= S_{\alpha\beta\gamma}^{\text{static}}(\mathbf{x}^c, \mathbf{x}) + \tilde{S}_{\alpha\beta\gamma}^m(\mathbf{x}^c, \mathbf{x}, p)\end{aligned}\tag{2.38}$$

in which $U_{\alpha\beta}^{\text{static}}$, $T_{\alpha\beta}^{\text{static}}$, $D_{\alpha\beta\gamma}^{\text{static}}$ and $S_{\alpha\beta\gamma}^{\text{static}}$ are the static fundamental solutions, which does not involve the Laplace transform parameter; $\tilde{U}_{\alpha\beta}^m$, $\tilde{T}_{\alpha\beta}^m$, $\tilde{D}_{\alpha\beta\gamma}^m$ and $\tilde{S}_{\alpha\beta\gamma}^m$ are modified terms which do not contain any singularities in terms of the distance r , except for $\tilde{S}_{\alpha\beta\gamma}^m$ containing the weak singularities of order $\ln(r)$. Owing to this separation, the integrals involving the singular terms in the dynamic fundamental solutions can be computed using the techniques from the corresponding static case. There are three types of singularities in terms of the distance r in the static fundamental solutions: weak singularities of order $\ln(r)$, strong singularities of order $1/r$ and hypersingularities of order $1/r^2$; the textbook written by Aliabadi [35] has given a comprehensive review of the methods for computing the corresponding singular integrals.

2.4.9 Numerical inversion of Laplace transform

In the Laplace-transform boundary element method, the analyses are carried out at different complex frequencies. Then, the time-domain solutions are calculated using Durbin's inversion

formula [75]:

$$f(t) = 2\frac{e^{at}}{T} \left\{ \frac{1}{2} \operatorname{Re} [\tilde{f}(a)] + \sum_{l=1}^L \operatorname{Re} [\tilde{f}(p)] \cos \frac{2\pi l}{T} t - \sum_{l=1}^L \operatorname{Im} [\tilde{f}(p)] \sin \frac{2\pi l}{T} t \right\} \quad (2.39)$$

where T is the length of time period; the Laplace term $p = a + 2\pi li/T$ and satisfactory results can be achieved by setting the real part $a = 5/T$; L is the total number of Laplace terms used for the inversion. This formula is commonly used because it requires fewer Laplace terms. However, this formula has a drawback that the results towards the end of the time period T are not accurate. In order to obtain accurate results over the entire time period, another Durbin's formula, modified slightly from Eq. (2.39), can be used. Such variant is given by:

$$f(t) = \frac{e^{at}}{T} \left\{ \frac{1}{2} \operatorname{Re} [\tilde{f}(a)] + \sum_{l=1}^L \operatorname{Re} [\tilde{f}(p)] \cos \frac{\pi l}{T} t - \sum_{l=1}^L \operatorname{Im} [\tilde{f}(p)] \sin \frac{\pi l}{T} t \right\} \quad (2.40)$$

in which the Laplace term $p = a + \pi li/T$ and satisfactory results can be obtained when aT is equal to 3. The selection between these two formulas should be based on the requirements on both accuracy and efficiency.

2.5 Piezoelectricity

Piezoelectric lead-zirconate-titanate (PZT) transducers are widely used in the structural health monitoring applications because they have the ability to transform mechanical deformation to electric signal (direct piezoelectric effect), and vice versa (converse piezoelectric effect). Assuming PZT is polarized in the direction x_3 , the direct piezoelectric effect is governed by :

$$\begin{aligned} [D] &= [d] [\sigma] + [\varepsilon] [E] \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & d_{15} & 0 \\ 0 & 0 & 0 & d_{24} & 0 & 0 \\ d_{31} & d_{32} & d_{33} & 0 & 0 & 0 \end{bmatrix} [\sigma] + \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix} [E] \end{aligned} \quad (2.41)$$

in which $[\varepsilon]$ is the dielectric permittivity matrix linking the electric displacement field $[D] = [D_1 \ D_2 \ D_3]^T$ and the electric field $[E] = [E_1 \ E_2 \ E_3]^T$; $[\sigma] = [\sigma_{11} \ \sigma_{22} \ \sigma_{33} \ \sigma_{23} \ \sigma_{13} \ \sigma_{12}]^T$ denotes the stress vector; the matrix $[d]$ consists of the piezoelectric coefficients which represent the mechanical-electrical coupling effect.

The governing equation for the converse piezoelectric effect is given by:

$$\begin{aligned}
 [S] &= [C]^{-1} [\sigma] + [d]^T [E] \\
 &= \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}^{-1} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} + \begin{bmatrix} 0 & 0 & d_{31} \\ 0 & 0 & d_{32} \\ 0 & 0 & d_{33} \\ 0 & d_{24} & 0 \\ d_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \quad (2.42)
 \end{aligned}$$

in which $[C]$ is the elasticity matrix and $[S] = [S_{11} \ S_{22} \ S_{33} \ S_{23} \ S_{13} \ S_{12}]^T$ denotes the elastic strain.

2.6 Summary

In this chapter, the basic properties of guided waves, such as wave dispersion, were presented. The definition of stress intensity factor, the most fundamental parameter used in fracture mechanics, was also included. The 2D dual boundary element method in Laplace domain and its numerical implementation were introduced in detail, which is the foundation for the Laplace-transform boundary element plate formulations shown in the following chapters. Finally, the governing equations for both direct and converse piezoelectric effects were given, as well as the piezoelectric material's parameters.

Chapter 3

Boundary element method for Kane-Mindlin theory

3.1 Introduction

The boundary element method (BEM) for three-dimensional (3D) linear elasticity theory, theoretically speaking, is a versatile technique for a variety of linear elastodynamic problems. For example, Zou et al. [71] have applied the 3D BEM to the modelling of ultrasonic elastic waves in a 3D beam. However, such 3D BEM is not suitable for the modelling of Lamb waves propagating in thin plate structures because it is computationally expensive to accurately evaluate the nearly singular integrals in the boundary integral equation.

The BEM formulations for solving Lamb wave problems [30–32, 76], such as wave reflection and wave scattering, so far have been mainly based on the plane strain assumption. However, these two-dimensional BEM formulations are unable to consider the two in-plane dimensions. Thus, such formulations fail to handle real 3D Lamb wave problems.

In order to solve Lamb wave problems in a 3D manner, the BEM formulations based on plate theories should be used. In structural health monitoring, the excitation frequency is

commonly below the the cut-off frequency of the second antisymmetric Lamb mode (A1 mode) [17, 23, 77–80], at which only the fundamental symmetric Lamb mode (S0 mode), and the fundamental antisymmetric Lamb mode (A0 mode) are generated. The dispersion properties of these wave modes can be found in Subsection 2.2.3. Over such frequency range, first-order plate theories are sufficient for the modelling [81]. The A0 mode can be modelled using the existing boundary element formulation for the Mindlin plate theory [82] after modifying the fundamental solutions slightly to capture the dispersion behaviour accurately. When it comes to the S0 mode, the Kane-Mindlin (K-M) theory [60], a first-order plate theory, is applicable and it has been used in both analytical models [83, 84] and numerical analysis [85]. However, to the best of the author’s knowledge, no boundary element formulations for the K-M theory can be found in the literature.

In this chapter, the dynamic fundamental solutions for the K-M theory in the Laplace domain are first derived employing a Fourier transformation method [82]. On the basis of these newly derived fundamental solutions, a boundary element formulation for the analyses of dynamic K-M problems is first proposed. Since the dynamic problems are solved in the Laplace domain, the domain integrals resulting from the inertial term can be avoided and a boundary-only formulation is obtained. Hence, only plate boundaries need to be discretized using simple line elements and the number of degrees of freedom is significantly reduced. The K-M BEM results agree very well with analytical and finite-element results in three benchmark examples. In order to simulate the wave propagation of the S0 and A0 modes simultaneously, the boundary element formulations for the K-M theory and the Mindlin theory should be combined together.

3.2 Derivation of new fundamental solutions

Throughout this chapter, Greek indices (α, β, γ) take the values 1 to 2, while Latin indices (i, j, k) run from 1 to 3; repeated indices are summed (summation convention) except where otherwise indicated.

Considering a linear elastic isotropic plate, the sign conventions and coordinate system are illustrated in Fig. 3.1. The mid-plane of such plate is located at the $x_1 - x_2$ plane, and the x_3 -axis is the normal to the mid-plane.

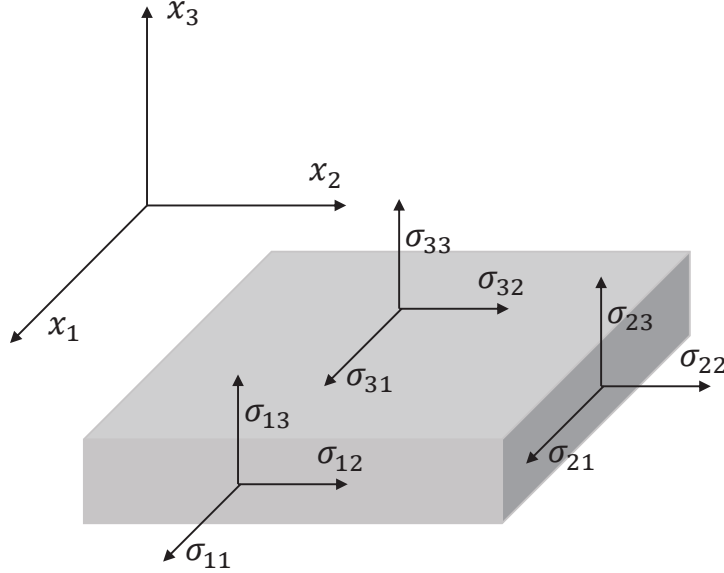


Figure 3.1: Sign conventions and coordinate system.

The Kane-Mindlin (K-M) theory is aimed at describing the plate motion which is symmetric with respect to the mid-plane. The plate displacements are approximated using the following expressions:

$$\begin{aligned}
 u_1(x_1, x_2, x_3, t) &= u_1^S(x_1, x_2, t) \\
 u_2(x_1, x_2, x_3, t) &= u_2^S(x_1, x_2, t) \\
 u_3(x_1, x_2, x_3, t) &= u_3^S(x_1, x_2, t)x_3
 \end{aligned} \tag{3.1}$$

In contrast to the generalised plane stress theory, the K-M theory takes the thickness-stretch and thickness-contraction effects (u_3^S) into consideration and as a result it can describe the symmetric motion of plates more accurately.

Based on the above assumption, the stress equilibrium equations in terms of stress resultants

can be written as follows:

$$\begin{aligned} N_{\alpha\beta,\beta} + F_\alpha^S &= \rho h u_{\alpha,tt}^S \\ R_{\alpha,\alpha} - N_{33} + F_3^S &= \frac{\rho h^3}{12} u_{3,tt}^S \end{aligned} \quad (3.2)$$

in which

$$\begin{aligned} N_{\alpha\beta} &= \int_{-h/2}^{h/2} \sigma_{\alpha\beta} dx_3 = h [\lambda (u_{\gamma,\gamma}^S + \kappa_S u_3^S) \delta_{\alpha\beta} + \mu (u_{\alpha,\beta}^S + u_{\beta,\alpha}^S)] \\ N_{33} &= \int_{-h/2}^{h/2} \sigma_{33} dx_3 = h [\kappa_S^2 (\lambda + 2\mu) u_3^S + \kappa_S \lambda u_{\gamma,\gamma}^S] \\ R_\alpha &= \int_{-h/2}^{h/2} \sigma_{\alpha 3} x_3 dx_3 = \frac{h^3 \mu}{12} u_{3,\alpha}^S \\ F_\alpha^S &= \sigma_{3\alpha}|_{x_3=h/2} - \sigma_{3\alpha}|_{x_3=-h/2} \\ F_3^S &= \frac{h}{2} (\sigma_{33}|_{x_3=h/2} + \sigma_{33}|_{x_3=-h/2}) \end{aligned} \quad (3.3)$$

$\lambda = E\nu/[(1+\nu)(1-2\nu)]$ and $\mu = E/[2(1+\nu)]$; the density, Young's modulus, Poisson's ratio and plate thickness are denoted by ρ , E , ν and h , respectively; κ_S is an adjustable shear factor which can be used to make dispersion curve modification for the waves governed by the K-M theory; $N_{\alpha\beta}$ and N_{33} are stress resultants; R_α are generalized shear forces which cause the pinching effect; F_α and F_3 represent applied generalized surface tractions.

The governing equations of motion are given by:

$$\begin{aligned} h\mu \nabla^2 u_\alpha^S + h(\lambda + \mu) u_{\beta,\beta\alpha}^S + \kappa_S h \lambda u_{3,\alpha}^S + F_\alpha^S &= \rho h u_{\alpha,tt}^S \\ \frac{h^3 \mu}{12} \nabla^2 u_3^S - \kappa_S h \lambda u_{\beta,\beta}^S - \kappa_S^2 h (\lambda + 2\mu) u_3^S + F_3^S &= \frac{\rho h^3}{12} u_{3,tt}^S \end{aligned} \quad (3.4)$$

Applying the Laplace transformation, Eq. (2.28), to the governing partial differential equations (3.4) yields:

$$\begin{aligned} h\mu \nabla^2 \tilde{u}_\alpha^S + h(\lambda + \mu) \tilde{u}_{\beta,\beta\alpha}^S + \kappa_S h \lambda \tilde{u}_{3,\alpha}^S + \tilde{F}_\alpha^S &= \rho h p^2 \tilde{u}_\alpha^S \\ \frac{h^3 \mu}{12} \nabla^2 \tilde{u}_3^S - \kappa_S h \lambda \tilde{u}_{\beta,\beta}^S - \kappa_S^2 h (\lambda + 2\mu) \tilde{u}_3^S + \tilde{F}_3^S &= \frac{\rho h^3 p^2}{12} \tilde{u}_3^S \end{aligned} \quad (3.5)$$

Eq. (3.5) can be decomposed into three Helmholtz-type differential equations:

$$\nabla^2 \tilde{\varphi}_i^S - (\zeta_i^S)^2 \tilde{\varphi}_i^S = 0, \quad i = 1, 2, 3. \quad (3.6)$$

where $\tilde{\varphi}_i^S$ denote three displacement potentials. The Laplace-domain wave numbers (ζ_i^S) are given by:

$$\begin{aligned} (\zeta_\alpha^S)^2 &= \frac{1}{2R_2} \left(\frac{p}{c_1} \right)^2 \left[R_1 + R_2 - (-1)^\alpha Q_S + \left(\frac{\omega_{cS}}{p} \right)^2 \right] \\ (\zeta_3^S)^2 &= \frac{p^2}{c_2^2} \end{aligned} \quad (3.7)$$

in which $R_\alpha = c_\alpha^2/c_3^2$, $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$, $c_2 = \sqrt{\mu/\rho}$, $c_3 = \sqrt{E/[\rho(1 - \nu^2)]}$, the cut-off frequency $\omega_{cS} = 2\sqrt{3}\kappa_S c_1/h$ and $Q_S = \sqrt{[R_1 + R_2 + (\omega_{cS}/p)^2]^2 - 4R_1 R_2 [1 + (\omega_{cS}/p)^2]}$.

The Laplace-transform displacements are expressed in terms of three independent potentials $\tilde{\varphi}_i^S$ as:

$$\begin{aligned} \tilde{u}_1^S &= \tilde{\varphi}_{1,1}^S + \tilde{\varphi}_{2,1}^S + \tilde{\varphi}_{3,2}^S \\ \tilde{u}_2^S &= \tilde{\varphi}_{1,2}^S + \tilde{\varphi}_{2,2}^S - \tilde{\varphi}_{3,1}^S \\ \tilde{u}_3^S &= e_1^S \tilde{\varphi}_1^S + e_2^S \tilde{\varphi}_2^S \end{aligned} \quad (3.8)$$

in which

$$e_\alpha^S = -\frac{1 - \nu}{\kappa_S \nu} \left[(\zeta_\alpha^S)^2 - \frac{1 - 2\nu}{2(1 - \nu)} (\zeta_3^S)^2 \right] \quad (3.9)$$

When the plate is in a fundamental state, it is subjected to a unit point force $\delta(x_1, x_2)$. First, such force is applied along the x_2 direction, i.e., $\tilde{F}_2^S = \delta(x_1, x_2)$, and the corresponding fundamental solutions will be derived. Only half of the plate ($x_2 \geq 0$) is considered here to analyse this problem conveniently. The displacements $\tilde{u}_i^S$ have the following symmetry

properties:

$$\begin{aligned}
\tilde{u}_1^S(x_1, x_2) &= -\tilde{u}_1^S(-x_1, x_2) \\
\tilde{u}_2^S(x_1, x_2) &= \tilde{u}_2^S(-x_1, x_2) \\
\tilde{u}_3^S(x_1, x_2) &= \tilde{u}_3^S(-x_1, x_2)
\end{aligned} \tag{3.10}$$

With the help of such symmetric properties, the general solutions of Eq. (3.6) can be obtained using the Fourier cosine and sine transforms and written as follows:

$$\begin{aligned}
\tilde{\varphi}_1^S &= \frac{2}{\pi} \int_0^\infty A_1 e^{-\beta_1 x_2} \cos(yx_1) dy \\
\tilde{\varphi}_2^S &= \frac{2}{\pi} \int_0^\infty A_2 e^{-\beta_2 x_2} \cos(yx_1) dy \\
\tilde{\varphi}_3^S &= \frac{2}{\pi} \int_0^\infty A_3 e^{-\beta_3 x_2} \sin(yx_1) dy
\end{aligned} \tag{3.11}$$

where $\beta_j^2 = y^2 + (\zeta_j^S)^2$ and A_i are unknown coefficients depending on boundary conditions.

Substituting Eq. (3.11) into Eq. (3.8) yields:

$$\tilde{u}_1^S = -\frac{2}{\pi} \int_0^\infty [yA_1 e^{-\beta_1 x_2} + yA_2 e^{-\beta_2 x_2} + \beta_3 A_3 e^{-\beta_3 x_2}] \sin(yx_1) dy \tag{3.12}$$

$$\tilde{u}_2^S = -\frac{2}{\pi} \int_0^\infty [\beta_1 A_1 e^{-\beta_1 x_2} + \beta_2 A_2 e^{-\beta_2 x_2} + yA_3 e^{-\beta_3 x_2}] \cos(yx_1) dy \tag{3.13}$$

$$\tilde{u}_3^S = \frac{2}{\pi} \int_0^\infty [e_1^S A_1 e^{-\beta_1 x_2} + e_2^S A_2 e^{-\beta_2 x_2}] \cos(yx_1) dy \tag{3.14}$$

Substituting Eqs. (3.12)-(3.14) into Eq. (3.3) gives:

$$\begin{aligned}
\tilde{N}_{11} &= -\frac{2h}{\pi} \int_0^\infty \{ [(\lambda + 2\mu) y^2 - \lambda (\kappa_S e_1^S + \beta_1^2)] A_1 e^{-\beta_1 x_2} \\
&\quad + [(\lambda + 2\mu) y^2 - \lambda (\kappa_S e_2^S + \beta_2^2)] A_2 e^{-\beta_2 x_2} + 2\mu y \beta_3 A_3 e^{-\beta_3 x_2} \} \cos(yx_1) dy
\end{aligned} \tag{3.15}$$

$$\begin{aligned} \tilde{N}_{22} = & \frac{2h}{\pi} \int_0^\infty \{ [(\lambda + 2\mu) \beta_1^2 + \lambda (\kappa_S e_1^S - y^2)] A_1 e^{-\beta_1 x_2} \\ & + [(\lambda + 2\mu) \beta_2^2 + \lambda (\kappa_S e_2^S - y^2)] A_2 e^{-\beta_2 x_2} + 2\mu y \beta_3 A_3 e^{-\beta_3 x_2} \} \cos(yx_1) dy \end{aligned} \quad (3.16)$$

$$\tilde{N}_{21} = \frac{2h\mu}{\pi} \int_0^\infty [2y\beta_1 A_1 e^{-\beta_1 x_2} + 2y\beta_2 A_2 e^{-\beta_2 x_2} + (y^2 + \beta_3^2) A_3 e^{-\beta_3 x_2}] \sin(yx_1) dy \quad (3.17)$$

$$\tilde{R}_1 = \frac{h^3\mu}{6\pi} \int_0^\infty [ye_1 A_1 e^{-\beta_1 x_2} + ye_2 A_2 e^{-\beta_2 x_2}] \sin(yx_1) dy \quad (3.18)$$

$$\tilde{R}_2 = -\frac{h^3\mu}{6\pi} \int_0^\infty [e_1 \beta_1 A_1 e^{-\beta_1 x_2} + e_2 \beta_2 A_2 e^{-\beta_2 x_2}] \cos(yx_1) dy \quad (3.19)$$

The boundary conditions along the axis x_1 have the following forms:

$$\tilde{u}_1^S = 0, \quad \tilde{u}_3^S = 0, \quad \tilde{N}_{22} = -\frac{1}{2} \delta(x_1, 0) \quad (3.20)$$

Taking Eqs. (3.12), (3.14) and (3.16) into account, the boundary conditions lead to the following equations:

$$yA_1 + yA_2 + \beta_3 A_3 = 0 \quad (3.21)$$

$$e_1^S A_1 + e_2^S A_2 = 0 \quad (3.22)$$

$$[(\lambda + 2\mu) \beta_1^2 + \lambda (\kappa_S e_1^S - y^2)] A_1 + [(\lambda + 2\mu) \beta_2^2 + \lambda (\kappa_S e_2^S - y^2)] A_2 + 2\mu y \beta_3 A_3 = -\frac{1}{4h} \quad (3.23)$$

Solving these equations, the coefficients A_i are obtained:

$$\begin{aligned} A_1 &= -\frac{e_2^S}{4h\mu(e_2^S - e_1^S)(\zeta_3^S)^2} \\ A_2 &= \frac{e_1^S}{4h\mu(e_2^S - e_1^S)(\zeta_3^S)^2} \\ A_3 &= \frac{y}{4h\mu\beta_3(\zeta_3^S)^2} \end{aligned} \quad (3.24)$$

Since the integral form of the zero-order modified Bessel function of the second kind is given by:

$$K_0(\zeta_i^S r) = \int_0^\infty \frac{\cos(yx_1)}{\beta_i} e^{-\beta_i x_2} dy \quad (3.25)$$

in which $r = \sqrt{x_1^2 + x_2^2}$, the fundamental solutions $\tilde{U}_{2i}^S$ can be written in the following forms:

$$\begin{aligned} \tilde{U}_{21}^S &= \tilde{u}_1^S = \frac{1}{2\pi h\mu(e_2^S - e_1^S)(\zeta_3^S)^2} \frac{\partial^2}{\partial x_1 \partial x_2} [e_2^S K_0(z_1^S) - e_1^S K_0(z_2^S) - (e_2^S - e_1^S) K_0(z_3^S)] \\ \tilde{U}_{22}^S &= \tilde{u}_2^S = \frac{1}{2\pi h\mu(e_2^S - e_1^S)(\zeta_3^S)^2} \frac{\partial^2}{\partial x_2^2} [e_2^S K_0(z_1^S) - e_1^S K_0(z_2^S) - (e_2^S - e_1^S) K_0(z_3^S)] \\ &\quad + \frac{1}{2\pi h\mu} K_0(z_3^S) \\ \tilde{U}_{23}^S &= \tilde{u}_3^S = \frac{6}{\pi h^3 \mu(e_2^S - e_1^S)} \frac{\partial}{\partial x_2} [K_0(z_1^S) - K_0(z_2^S)] \end{aligned} \quad (3.26)$$

where $z_i^S = \zeta_i^S r$. In a similar manner, the fundamental solutions $\tilde{U}_{1i}^S$ can be obtained easily using the above method.

When the unit point force $\tilde{F}_3^S = \delta(x_1, x_2)$ is applied on the plate, only half of the plate ($x_2 \geq 0$) is analysed due to symmetry properties. It can be verified easily that the displacements and stress resultants have the same expressions as those in Eqs. (3.12)-(3.19). The corresponding fundamental solutions can be obtained after considering the boundary conditions along the axis x_1 :

$$\tilde{u}_2^S = 0, \quad \tilde{N}_{21} = 0, \quad \tilde{R}_2 = -\frac{1}{2}\delta(x_1, 0) \quad (3.27)$$

The detailed expressions of the fundamental solutions $\tilde{U}_{ij}^S$ and $\tilde{T}_{ij}^S$ are listed in Appendix A.1.

3.3 New boundary element formulation

In this section, the boundary integral equation for Kane-Mindlin plate theory will be derived for the first time.

In the Laplace domain, the equilibrium equations (3.2) have the following forms:

$$\begin{aligned}\tilde{N}_{\alpha\beta,\beta} + \tilde{F}_\alpha^S &= \rho h p^2 \tilde{u}_\alpha^S \\ \tilde{R}_{\alpha,\alpha} - \tilde{N}_{33} + \tilde{F}_3^S &= \frac{\rho h^3 p^2}{12} \tilde{u}_3^S\end{aligned}\quad (3.28)$$

An integral identity based on the above equilibrium equations can be written as:

$$\int_{\Omega} \left[\left(\tilde{N}_{\alpha\beta,\beta} + \tilde{F}_\alpha^S - \rho h p^2 \tilde{u}_\alpha^S \right) \tilde{u}_\alpha^{S*} + \left(\tilde{R}_{\alpha,\alpha} - \tilde{N}_{33} + \tilde{F}_3^S - \frac{\rho h^3 p^2}{12} \tilde{u}_3^S \right) \tilde{u}_3^{S*} \right] d\Omega = 0 \quad (3.29)$$

where $\tilde{u}_\alpha^{S*}$ and $\tilde{u}_3^{S*}$ are Laplace-transform displacements in another independent self-equilibrated state; Ω denotes an arbitrary plate domain.

Repeated integration by parts (i.e. using the divergence theorem), and considering the relationship between the tractions and stress resultants $t_\alpha^S = N_{\alpha\beta} n_\beta$, $t_3^S = R_\beta n_\beta$, in which n_β are components of the outward unit normal vector to the plate boundary Γ , Eq. (3.29) becomes:

$$\begin{aligned}\int_{\Gamma} \tilde{t}_j^S \tilde{u}_j^{S*} d\Gamma + \int_{\Omega} \tilde{F}_j^S \tilde{u}_j^{S*} d\Omega &= \int_{\Omega} \tilde{R}_\alpha \tilde{u}_{3,\alpha}^{S*} d\Omega + \int_{\Omega} \tilde{N}_{\alpha\beta} \tilde{u}_{\alpha,\beta}^{S*} d\Omega + \int_{\Omega} \tilde{N}_{33} \tilde{u}_3^{S*} d\Omega \\ &+ \int_{\Omega} 2\rho h p^2 \tilde{u}_\alpha^S \tilde{u}_\alpha^{S*} d\Omega + \int_{\Omega} \frac{\rho h^3 p^2}{12} \tilde{u}_3^S \tilde{u}_3^{S*} d\Omega\end{aligned}\quad (3.30)$$

Substituting Eq. (3.3) into Eq. (3.30) yields:

$$\int_{\Gamma} \tilde{t}_j^{S*} \tilde{u}_j^S d\Gamma + \int_{\Omega} \tilde{F}_j^{S*} \tilde{u}_j^S d\Omega = \int_{\Gamma} \tilde{t}_j^S \tilde{u}_j^{S*} d\Gamma + \int_{\Omega} \tilde{F}_j^S \tilde{u}_j^{S*} d\Omega \quad (3.31)$$

which means Betti's reciprocal work theorem is still valid for the Kane-Mindlin plates.

When the state $(.)^*$ represents the fundamental state, Eq. (3.31) can be written as:

$$\begin{aligned} \tilde{u}_i^S(\mathbf{X}', p) &= \int_{\Gamma} \tilde{U}_{ij}^S(\mathbf{X}', \mathbf{x}, p) \tilde{t}_j^S(\mathbf{x}, p) d\Gamma - \int_{\Gamma} \tilde{T}_{ij}^S(\mathbf{X}', \mathbf{x}, p) \tilde{u}_j^S(\mathbf{x}, p) d\Gamma \\ &+ \int_{\Omega} \tilde{U}_{ij}^S(\mathbf{X}', \mathbf{X}, p) \tilde{F}_j^S(\mathbf{X}, p) d\Omega \end{aligned} \quad (3.32)$$

where $\mathbf{X}'$ is an arbitrary source point in the plate domain Ω ; $\mathbf{X}$ and $\mathbf{x}$ denote a field point in the domain Ω and on the boundary Γ , respectively.

Moving the source point $\mathbf{X}'$ to the point $\mathbf{x}'$ on the plate boundary Γ , the displacement boundary integral equation can be obtained as follows:

$$\begin{aligned} c_{ij}(\mathbf{x}') \tilde{u}_j^S(\mathbf{x}', p) &= \int_{\Gamma} \tilde{U}_{ij}^S(\mathbf{x}', \mathbf{x}, p) \tilde{t}_j^S(\mathbf{x}, p) d\Gamma - \oint_{\Gamma} \tilde{T}_{ij}^S(\mathbf{x}', \mathbf{x}, p) \tilde{u}_j^S(\mathbf{x}, p) d\Gamma \\ &+ \int_{\Omega} \tilde{U}_{ij}^S(\mathbf{x}', \mathbf{X}, p) \tilde{F}_j^S(\mathbf{X}, p) d\Omega \end{aligned} \quad (3.33)$$

where $\oint$ denotes Cauchy principal value integral; c_{ij} are coefficients which are determined by the geometry at the source point $\mathbf{x}'$ [35]. The corresponding boundary integral equation for static problems can be obtained easily by letting $p = 0$.

3.4 Treatment of singularities

When $\mathbf{x}$ approaches $\mathbf{x}'$, some of the transformed fundamental solutions $\tilde{U}_{ij}^S$ and $\tilde{T}_{ij}^S$ tend to become singular. In order to compute the integrals in Eq. (3.33), the types of singularities in such fundamental solutions should be understood firstly. As shown in Appendix C.1, the transformed fundamental solutions have the same singularities as those for the static problems, which means that such singularities do not contain the Laplace transform parameter p . Thus, it is convenient to split the transformed fundamental solutions into two parts:

$$\begin{aligned} \tilde{U}_{ij}^S(\mathbf{x}', \mathbf{x}, p) &= U_{ij}^{S|\text{static}}(\mathbf{x}', \mathbf{x}) + \tilde{U}_{ij}^{S|\text{lm}}(\mathbf{x}', \mathbf{x}, p) \\ \tilde{T}_{ij}^S(\mathbf{x}', \mathbf{x}, p) &= T_{ij}^{S|\text{static}}(\mathbf{x}', \mathbf{x}) + \tilde{T}_{ij}^{S|\text{lm}}(\mathbf{x}', \mathbf{x}, p) \end{aligned} \quad (3.34)$$

in which $U_{ij}^{S|\text{static}}$ and $T_{ij}^{S|\text{static}}$ are static fundamental solutions (see Appendix B.1), and there are no singularities in the modified terms $\tilde{U}_{ij}^{S|\text{m}}$ and $\tilde{T}_{ij}^{S|\text{m}}$. Owing to such separation, the integrals due to the singular components of $\tilde{U}_{ij}^S$ and $\tilde{T}_{ij}^S$ can be computed through the calculation of those involving the static fundamental solutions $U_{ij}^{S|\text{static}}$ and $T_{ij}^{S|\text{static}}$.

Two types of singularities, weak singularities of order $\ln(r)$ and strong singularities of order $1/r$, exist in the static fundamental solutions. The weak singular integrals can be tackled in several ways [35] including using the logarithmic weighted Gaussian quadrature or the bi-cubic transformation of variable technique proposed by Telles [86]. The strong singular integrals are evaluated indirectly through the rigid body motion. The plate can move freely along the direction x_1 or x_2 as a rigid body. In these cases, the plate is not subjected to any surface tractions, and the rigid body movements can be expressed as:

$$\begin{aligned} u_1^S &= \text{constant}, & u_2^S &= 0, & u_3^S &= 0 \\ u_2^S &= \text{constant}, & u_1^S &= 0, & u_3^S &= 0 \end{aligned} \quad (3.35)$$

It should be noted that the surface tractions are required to maintain the state in which the generalised displacement u_3^S is equal to a constant and the in-plane displacements u_3^α are zero. Thus, the rigid body motion cannot be used to compute the strong singular integrals involving $T_{33}^{S|\text{static}}$. However, such singular integrals can be estimated analytically.

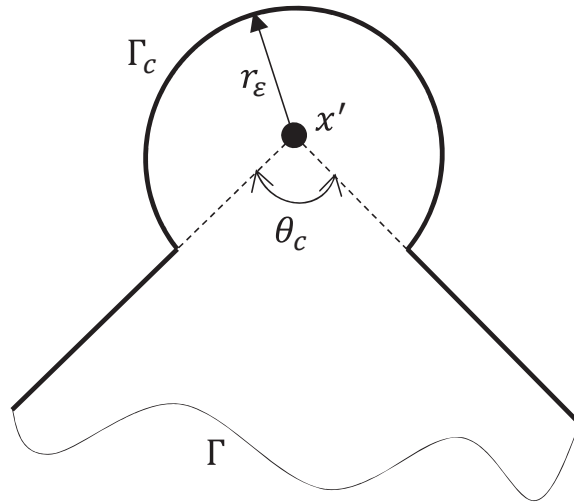


Figure 3.2: Treatment of the local boundary around the source point $\mathbf{x}'$.

Consider a source point $\mathbf{x}'$ at a boundary corner with an inner angle θ_c . As shown in Fig. 3.2, it is convenient to use an artificial arc Γ_c centred at the source point, with an infinitesimal radius r_ε , to represent the boundary around the point $\mathbf{x}'$. The singular integral involving $T_{33}^{S|\text{static}}$ is divided into two parts:

$$\begin{aligned} \int_{\Gamma} T_{33}^{S|\text{static}}(\mathbf{x}', \mathbf{x}) \tilde{u}_3^S(\mathbf{x}, p) d\Gamma &= \lim_{r_\varepsilon \rightarrow 0} \int_{\Gamma_c} T_{33}^{S|\text{static}}(\mathbf{x}', \mathbf{x}) \tilde{u}_3^S(\mathbf{x}, p) d\Gamma \\ &+ \lim_{r_\varepsilon \rightarrow 0} \int_{\Gamma - \Gamma_c} T_{33}^{S|\text{static}}(\mathbf{x}', \mathbf{x}) \tilde{u}_3^S(\mathbf{x}, p) d\Gamma \end{aligned} \quad (3.36)$$

in which the second integral on the right hand side can be treated as a Cauchy principal-value integral. Such principal-value integral is equivalent to the finite-part integral of the divergent integral. As explained in Ref. [35], the definitions of one-sided and two-sided finite-part integrals are used when the collocation is carried out at an end node and an inner node of an element, respectively. Based on these definitions, this integral can then be evaluated analytically.

Substituting Eq. (C.7) into Eq. (3.36), the first integral on the right hand side of Eq. (3.36) can be determined using the polar coordinate system as follows:

$$\begin{aligned} \lim_{r_\varepsilon \rightarrow 0} \int_{\Gamma_c} T_{33}^{S|\text{static}}(\mathbf{x}', \mathbf{x}) \tilde{u}_3^S(\mathbf{x}, p) d\Gamma &= \tilde{u}_3^S(\mathbf{x}', p) \lim_{r_\varepsilon \rightarrow 0} \int_{\Gamma_c} -\frac{1}{2\pi r_\varepsilon} d\Gamma \\ &= -\tilde{u}_3^S(\mathbf{x}', p) + \frac{\theta_c}{2\pi} \tilde{u}_3^S(\mathbf{x}', p) \end{aligned} \quad (3.37)$$

Thus, the displacement boundary integral equation regarding $T_{3j}^{S|\text{static}}$ in Eq. (3.33) can be rewritten as follows:

$$\begin{aligned} \frac{\theta_c}{2\pi} \tilde{u}_3^S(\mathbf{x}', p) &= \int_{\Gamma} \tilde{U}_{3j}^S(\mathbf{x}', \mathbf{x}, p) \tilde{t}_j(\mathbf{x}, p) d\Gamma - \oint_{\Gamma} \tilde{T}_{3j}^S(\mathbf{x}', \mathbf{x}, p) \tilde{u}_j^S(\mathbf{x}, p) d\Gamma \\ &+ \int_{\Omega} \tilde{U}_{3j}^S(\mathbf{x}', \mathbf{X}, p) \tilde{F}_j^S(\mathbf{X}, p) d\Omega \end{aligned} \quad (3.38)$$

3.5 Numerical implementation

The numerical implementation is the same as that in Subsection 2.4.7. Plate edges are discretized using quadratic isoparametric line elements. Durbin's formulas (see Subsection 2.4.9)

are adopted to invert the Laplace-transform solutions to time-domain results.

3.6 Examples

In the following examples, the centre of a plate is chosen as the coordinate origin. Since Kane-Mindlin plate elements are not provided in Abaqus, BEM results are validated using three-dimensional FEM simulations based on solid elements.

It is worth mentioning that the BEM based on the generalised plane stress theory [87] has been used to model the S0 mode. However, this 2D formulation is unable to capture the dispersive behaviour of the S0 mode and unable to analyse the responses of the S0 mode when the plate is subjected to out-of-plane loads. In order to demonstrate the wider applicability of the K-M theory, two-dimensional BEM analyses based on the generalised plane stress theory (see Section 2.4 in Chapter 2) are conducted for comparisons.

3.6.1 An infinite plate under in-plane loading

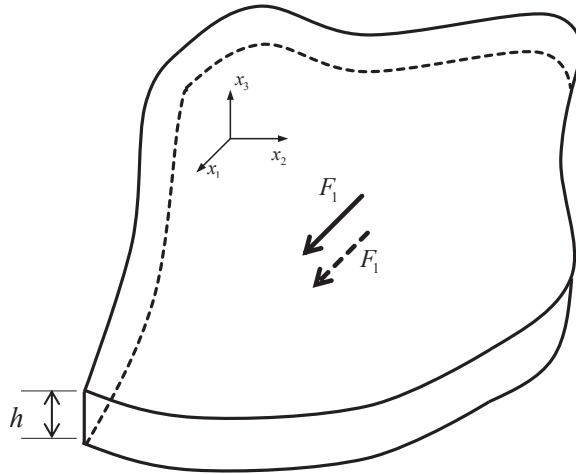


Figure 3.3: An infinite plate subjected to two point loads along the x_1 -axis on the top and bottom surfaces.

A 10-mm-thick infinite aluminium plate, which has the material properties $E = 72.7$ GPa, $\nu = 0.33$ and $\rho = 2700$ kg/m³, is considered here. As shown in Fig. 3.3, two point loads F_1

along the x_1 -axis are applied symmetrically on the surfaces of the plate at two points ($x_1 = 0$, $x_2 = 0$, $x_3 = \pm h/2$).

When such infinite plate is subjected to harmonic loads, Achenbach and Xu [88] have derived the Lamb wave responses analytically. When the Hann-window-modulated sinusoidal tone burst (see Eq. (2.25)) is adopted as the excitation signal, the responses in time domain are able to be computed by inverse Fourier transform of the product of the harmonic responses and the Fourier transform of the excitation signal [89]. Such analytical solutions can be used to examine the applicability of the Kane-Mindlin theory in terms of modelling Lamb waves. The BEM responses in time domain are obtained by numerical Laplace inversion of dynamic fundamental solutions in Eq. A.1.

Two cases are investigated here, in which the loads are in the forms of a 5-cycle Hann-window-modulated sinusoidal tone burst centred at 50 kHz and a 10-cycle Hann-window-modulated sinusoidal tone burst centred at 150 kHz, respectively. Numerical inversion of Laplace transform (Eq. 2.40) was conducted using 100 and 300 Laplace terms for these two cases, respectively. The displacement responses u_1 were normalized with respect to the peak amplitude of the S0 mode and measured at the point A ($x_1 = 1$ m, $x_2 = 0$, $x_3 = 5$ mm). Figs. 3.4a and 3.4b show such responses in the two cases, respectively. The responses were computed on the basis of the analytical solutions given by Achenbach and Xu [88], dynamic fundamental solutions for the Kane-Mindlin theory, and dynamic fundamental solutions for the generalised plane stress theory given in Ref. [4].

At a lower frequency (50 kHz), it can be seen from Fig. 3.4a that the S0 mode is nearly non-dispersive by comparing the shapes of the displacement signal and the excitation signal. Thus, both the K-M theory and the generalised plane stress theory are able to be used to model the S0 mode at such frequency. However, as the frequency increases, Fig. 3.4b shows that the generalised plane stress theory fails to represent the S0 mode accurately. This is because the dispersion of the S0 mode cannot be disregarded at higher frequencies. By contrast, the Kane-Mindlin theory still provides a satisfactory approximation for modelling the S0 mode at such

high frequency.

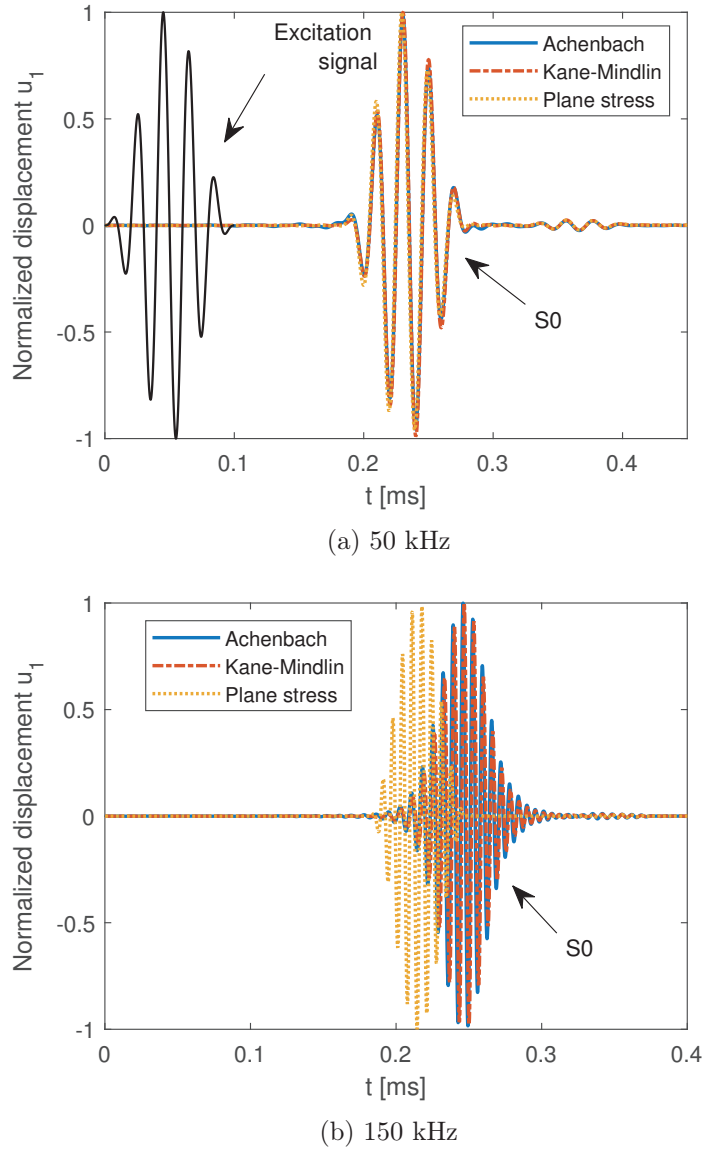


Figure 3.4: Comparison of normalized displacements u_1 obtained based on the analytical solutions given by Achenbach, fundamental solutions for the Kane-Mindlin theory and generalised-plane-stress theory at the point A of the plate subjected to in-plane point loads.

3.6.2 A square plate under out-of-plane loading

A square plate, which has the same material properties as those in the previous example, with four free edges (see Fig. 3.5) has an edge length ($2l$) of 1 m and a thickness (h) of 10 mm. This plate is subjected to vertical loads (F_3 and $-F_3$) which are applied at two points ($x_1 = 0$, $x_2 = 0$, $x_3 = \pm h/2$).

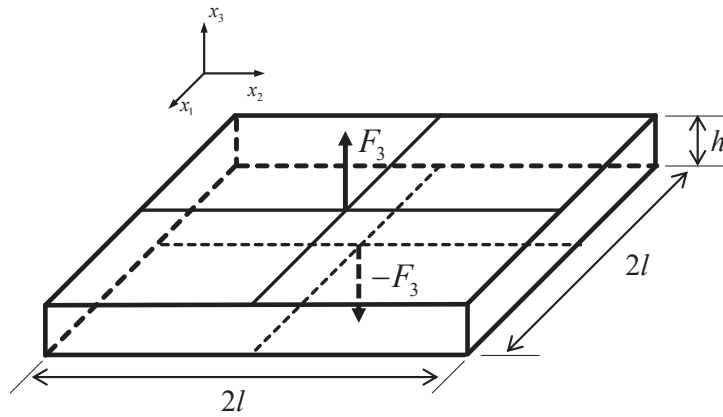


Figure 3.5: A square plate subjected to two out-of-plane point loads on the top and bottom surfaces.

A six-cycle Hann-windowed tone burst with a central frequency of 100 kHz was used as the excitation load, which only generated the S0 mode. In this case, the generalised plane stress theory fails to model the generation of the S0 mode since only in-plane loads are able to be considered in this theory. However, the K-M theory is still suitable for this case. In the K-M BEM simulation, the shear factor κ_S was set to 0.78 and 200 Laplace terms were used for numerically inverting the Laplace transform. Each plate edge was discretized using 50 quadratic boundary line elements, which resulted in 5.3 nodes per wavelength of the S0 mode at 100 kHz.

It should be noted that Achenbach and Xu [88] have also given the analytical solutions, as in the first example, for an infinite plate under such out-of-plane loads. Thus, these solutions can be used to validate the response of the direct arrival S0 mode obtained using the K-M BEM. The responses of the S0 mode reflected from the plate edges in the K-M BEM simulation were validated using Abaqus/Explicit. In the finite element analysis, due to symmetry, only one-eighth of the square plate was modelled using 333,666 linear brick elements (C3D8R), leading to 1,342,680 degrees of freedom, and there were 3 layers of elements across half-plate thickness. This finite element mesh was created using an approximate global size of 1.5 mm, which is equivalent to 36 nodes per wavelength of the S0 at 100 kHz.

Fig. 3.6 shows the displacement u_1 , which was normalized with respect to the peak amplitude

of the first S0 mode, obtained using the analytical solutions given by Achenbach and Xu [88], the K-M BEM and the 3D Abaqus/Explicit at the point A ($x_1 = 0.25$ m, $x_2 = 0$, $x_3 = 5$ mm) of the plate. When it comes to the first wave packet representing the direct arrival S0 mode, the K-M BEM signal agrees very well with both the analytical solution given by Achenbach and Xu [88] and the FEM result. In addition, the reflected S0 mode signal from the K-M boundary element analysis agrees very well with the FEM result. Since the analytical solutions were obtained based on the infinite-plate assumption, the reflected waves from plate edges could not be captured using these solutions and thus they were not observed in Fig. 3.6.

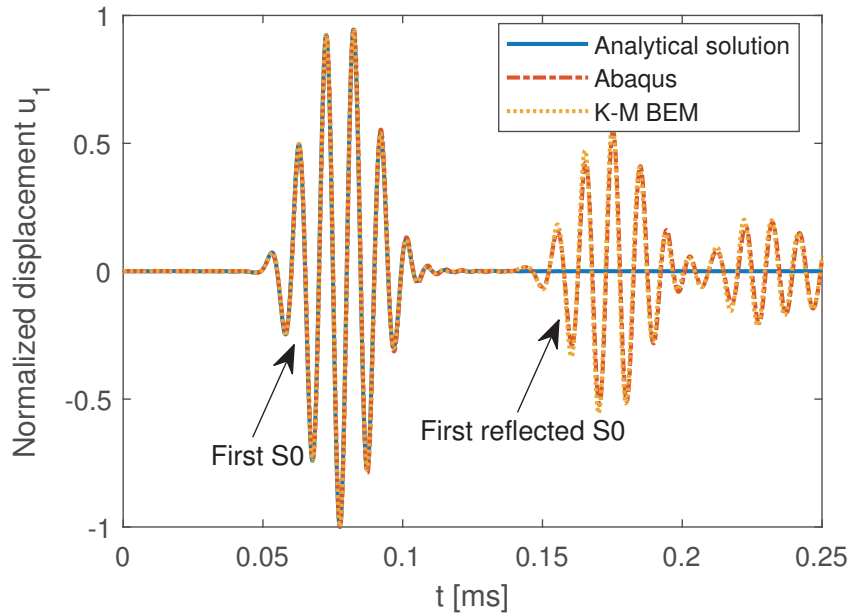


Figure 3.6: Comparison of normalized displacements u_1 at the point A of the square plate subjected to out-of-plane loads computed based on the analytical solutions given by Achenbach, using Abaqus/Explicit and the K-M BEM.

The above BEM-based signals can be used to estimate the group velocity of the S0 mode with the help of the Hilbert transform [26]. After the transformation, the envelope of the signals was plotted in Fig. 3.7. The envelope peaks of the first and the second wave packets appear at t_A and t_B , respectively. During the time period $t_B - t_A$, the S0 mode travels a distance d_{AB} of 0.5 m. Thus, the group velocity c^g of the S0 mode at 100 kHz can be estimated as 5219 m/s using the expression $c^g = d_{AB}/(t_B - t_A)$, and the corresponding analytical value is 5230 m/s.

The relative error of such velocity obtained using the K-M boundary element simulation is only 0.2%.

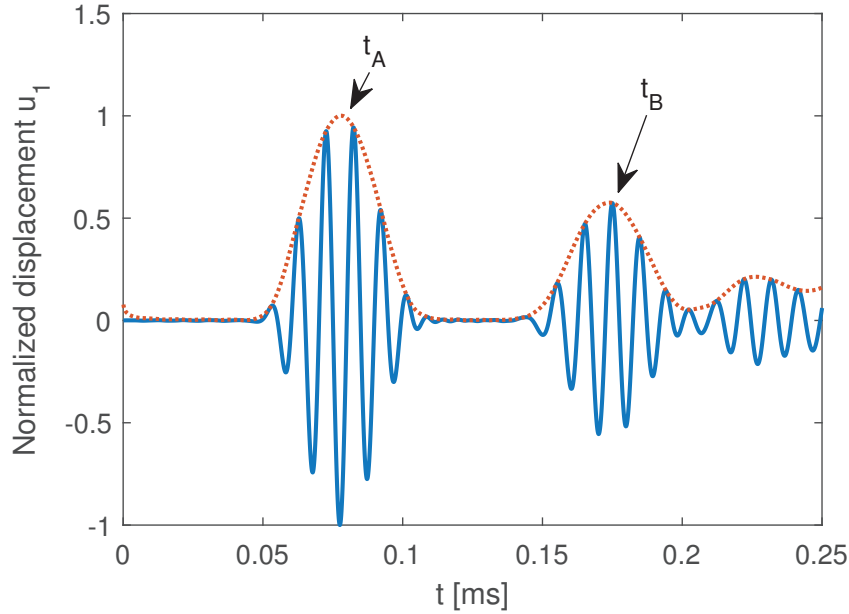


Figure 3.7: The enveloped of BEM-simulated normalized displacement signal u_1 at the point A of the square plate subjected to out-of-plane loads.

This numerical example has also been analysed by Zak [90] using a spectral element method, in which a linear displacement variation across the thickness (first-order approximation) was assumed. Such approximation is the same as that used in the K-M theory in terms of modelling the symmetric Lamb wave modes. In Ref. [90], 36-node spectral finite plate elements were used for the discretization, and the mesh consisted of 63,000 nodes, which resulted in 189,000 degrees of freedom and 13.3 nodes per wavelength of the S_0 mode at 100 kHz. On the basis of such mesh, the relative error of the group velocity of the S_0 mode was 0.3%. In the K-M BEM, only 400 nodes were used in the discretization of the plate edges and the total number of degrees of freedom was 1,200; the node density of 5.3 nodes per wavelength resulted in the relative error of 0.2%. Thus, compared with the spectral element method, modelling the S_0 mode using the K-M BEM leads to a significant reduction in the number of degrees of freedom.

3.6.3 A free plate with one circular actuation region

In structural health monitoring, both contact-type devices, such as piezoelectric patches, and non-contact type devices, such as the laser, can be used to actuate Lamb waves. When the device produces a circular actuation region on one surface of a plate, omni-directional Lamb waves are generated. In practice, since the actuation region is often very small compared with the plate, a line-force model can be used to represent the actuation effect [91]. In such model, the equivalent line traction applied on the plate surface is normal to the circumference of the circular region. Thus, in the boundary integral equation, the domain integrals associated with this equivalent traction reduce to line integrals.

Fig. 3.8 illustrates a square free plate with a circular actuation region. The plate has a thickness h of 2 mm and an edge length $2l$ of 0.6 m. The plate is made of aluminium with the properties $E = 69$ GPa, $\nu = 0.33$ and $\rho = 2700$ kg/m³. The actuation region with a radius R of 5 mm is centred at the origin (centre of the plate).

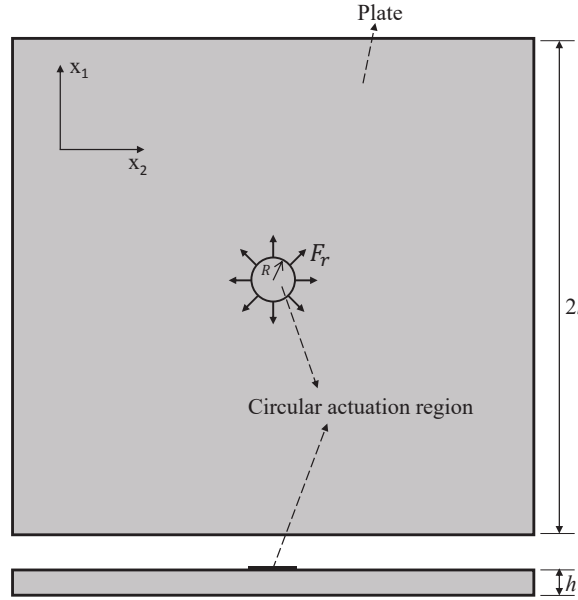


Figure 3.8: A square plate with a circular actuation region on the top surface.

It should be noted that this configuration can generate both symmetric and antisymmetric Lamb modes simultaneously. The line traction F_r is in the form of a five-cycle 300 kHz Hann-

windowed tone burst. At such frequency, only the S0 mode and the A0 mode are generated. This traction is decomposed into symmetric tractions and anti-symmetric tractions with respect to the middle plane of the plate, as shown in Fig. 3.9. The symmetric and anti-symmetric tractions excite the S0 mode and A0 mode, respectively. The proposed K-M boundary element formulation can be used to obtain the responses of the S0 mode, while those of the A0 mode are able to be analysed using the existing boundary element formulations based on the Mindlin plate bending theory [82]. In the Mindlin plate bending theory, the A0 mode is excited by the bending moment $M_r = F_r h/2$ due to the anti-symmetric tractions [92].

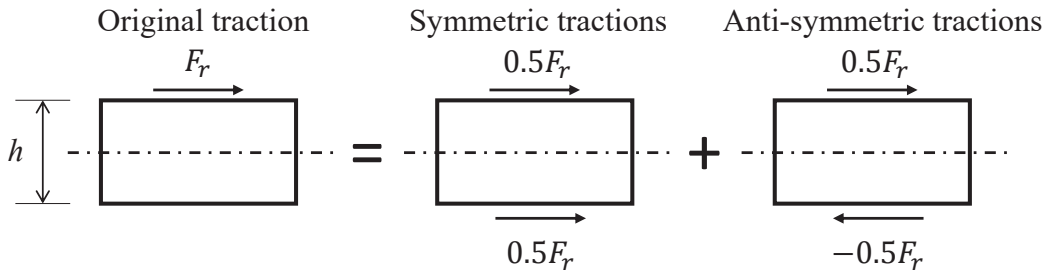


Figure 3.9: Decomposition of the original traction into symmetric loads and anti-symmetric loads.

The propagation and the reflection of the S0 mode and those of the A0 mode are decoupled for the simple single plate in this example. Thus, the dynamic responses of the S0 mode and the A0 mode can be obtained using the K-M BEM and the Mindlin-bending BEM, respectively. These two analyses were carried out separately and the dynamic response of the plate was obtained as the superposition of the responses of these two modes. However, when it comes to assembled plate structures, these two problems should be solved as a whole. The boundary element influence matrices for the K-M BEM formulation and the Mindlin-bending BEM formulation should be inserted into one global matrix, and subsequent calculations should be based on the manipulations of this global matrix. The detailed assembly method and the components of the global matrix are the same as those in the static boundary element analysis of assembled plates, which can be found in Refs. [35, 93, 94].

In the K-M BEM, 80 quadratic line elements were used to discretize each plate edge, which resulted in 1,920 degrees of freedom and 4.7 nodes per wavelength of the S0 mode at 300

kHz. The inversion of Laplace transform was carried out based on 200 Laplace terms. In this example, since the group velocity of the S0 mode (5268.5 m/s) is much higher than that of the A0 mode (2974.8 m/s) at 300 kHz, the reflected waves of the A0 mode from the plate edges cannot be observed in the time period of interest. Thus, when the BEM based on the Mindlin plate bending theory [82] was used to model the A0 mode, the plate edges can be disregarded and do not need to be discretized.

Raghavan and Cesnik [89] have given the analytical solutions for an infinite plate with this circular actuation region. Thus, their solutions can be used to validate the direct-arrival S0 and A0 modes in the BEM results. When it comes to the validation of the reflected wave signals from the plate edges, the Abaqus/Explicit solver was used. In the finite element simulation, due to symmetry, a quarter of the plate was modelled and the mesh consisting of 22,442,140 C3D8R brick elements was created on the basis of an approximate global element size of 0.2 mm. There were 10 layers of elements across the plate thickness and the total number of degrees of freedom was 74,158,128. The mesh resulted in 90 nodes per wavelength of the S0 mode and 51 nodes per wavelength of the A0 mode at 300 kHz.

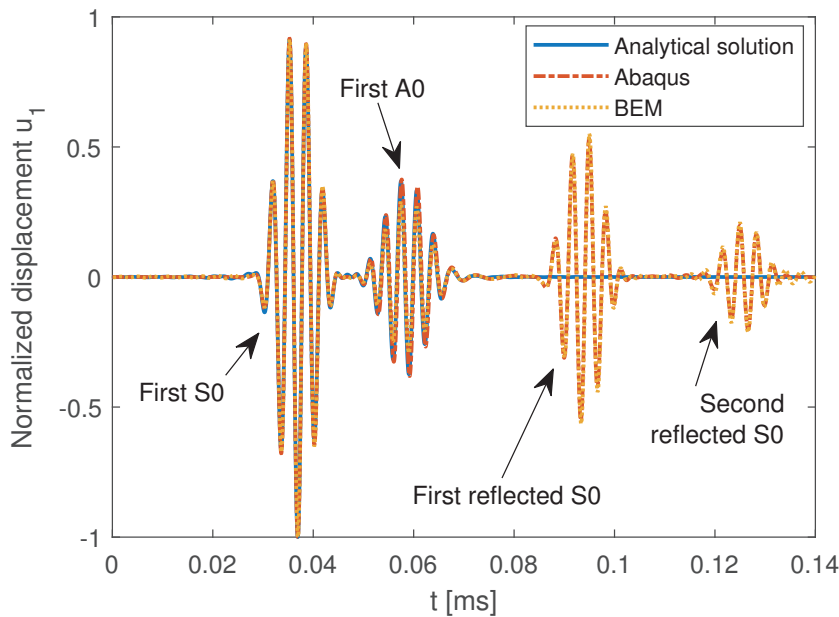


Figure 3.10: Comparison of normalized displacements u_1 at the point A of the plate with one circular actuation region obtained using the analytical solution, Abaqus/Explicit and the BEM.

The displacement u_1 was measured at the point A, which was located on the top surface of the plate ($x_3 = 1$ mm) and along the x_1 -axis at a distance 0.15 m from the origin, and normalized with respect to the peak amplitude of the first S0 mode. Fig. 3.10 presents the normalized displacements u_1 obtained using the analytical solution given by Raghavan and Cesnik [89], the 3D Abaqus/Explicit and the BEM. Since the analytical solution is on the basis of an infinite plate, boundary-reflected waves cannot be observed in Fig. 3.10. The first reflected S0 mode is caused due to the reflection of the S0 mode from one plate edge ($x_1 = 0.3$ m), while the reflected S0 waves from two plate edges ($x_2 = \pm 0.3$ m) result in the second reflected S0 mode. It should be noted that the amplitude of the A0 mode obtained using the Mindlin-bending BEM is underestimated, which has also been observed in Ref. [95]. The reason for this phenomenon is that the Mindlin bending theory is an approximate theory and it cannot represent all the features of the A0 mode accurately. Apart from this, the BEM results agree very well with both the analytical solution and the FEM results.

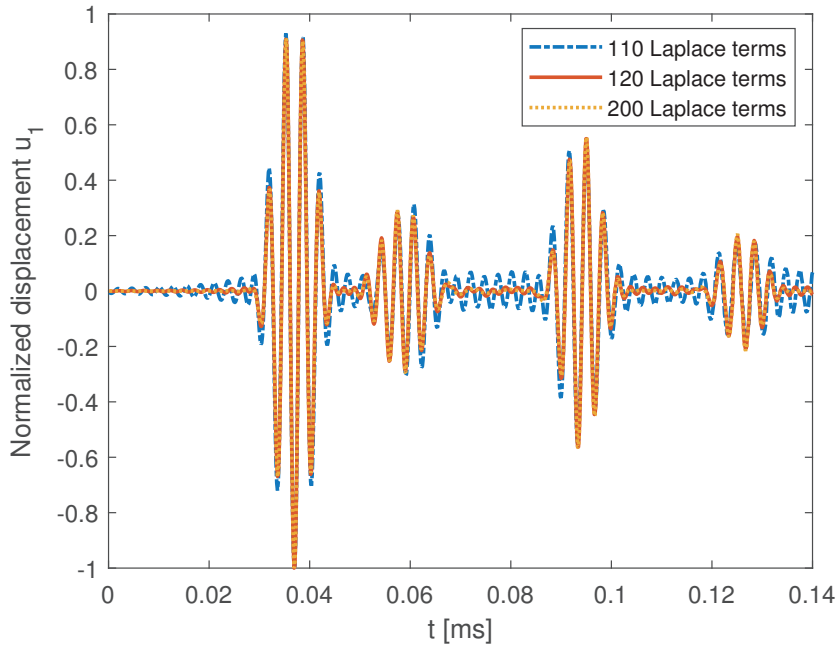


Figure 3.11: The influence of the number of Laplace terms on the BEM-based normalized displacement at the point A of the plate with one circular actuation region.

The number of Laplace terms is an important factor affecting the computational efficiency.

In the above examples, larger numbers of Laplace terms than strictly necessary were used to ensure the convergence. In order to increase the computational efficiency, fewer Laplace terms should be used. In the third example, the BEM results were obtained on the basis of 200 Laplace terms. However, it can be seen from Fig. 3.11 that 120 Laplace terms are enough to achieve convergence. Furthermore, the number of Laplace terms can be halved by using another numerical inversion formula (see Eq. (2.39)). However, this variant formula leads to inaccurate results toward the end of the time period. Thus, the selection between these two formulas should be based on the requirements on both accuracy and efficiency.

3.7 Discussion

In this chapter, the application of the BEM was extended to a new set of problems. Some comments should be made regarding the features of the proposed BEM formulation.

In contrast to the spectral plate finite element formulation [90], the main advantage of the proposed BEM formulation is that only plate edges are discretized and thus a significant reduction in the degrees of freedom can be achieved. In addition, a so-called dual boundary element method [53], which is more suitable for modelling cracked plates since crack surfaces are treated simply as extra plate boundaries, can be formulated using the proposed dynamic fundamental solutions and BEM formulation.

The BEM formulation presented in this chapter is suitable for isotropic materials; however by deriving additional fundamental solutions, it can be extended to anisotropic plate and shell structures. A well-known drawback of the BEM formulation compared to the FEM is the fully populated non-symmetric matrices leading to longer solution times for the same number of degrees of freedom. However, recent development of fast solvers [96] has significantly reduced the solution times.

3.8 Summary

In this chapter, dynamic fundamental solutions for the Kane-Mindlin plate theory are derived and presented in the Laplace domain for the first time. Based on these fundamental solutions, a new boundary element formulation is proposed for the dynamic analysis of Kane-Mindlin plates. This new formulation is capable of dealing with the thickness-stretch displacement of the S0 mode and out-of-plane loads, and accurately modelling the high-frequency S0 Lamb wave mode, which cannot be conducted using the conventional 2D BEM formulations based on the generalised plane stress theory. The results obtained using the proposed formulation agree very well with analytical results and those from three-dimensional finite element analyses. The proposed BEM formulation and the existing BEM formulations based on the Mindlin plate bending theory can be combined together for the modelling of the S0 mode and A0 mode simultaneously. This indicates that the BEM formulations on the basis of plate theories are very suitable for the numerical analyses of Lamb-wave based structural health monitoring problems.

Chapter 4

Dynamic dual boundary element analysis for cracked Kane-Mindlin plates

4.1 Introduction

In Chapter 3, the boundary element method (BEM) for Kane-Mindlin plates has been proposed. Such method will be extended to a corresponding dual boundary element method (DBEM) in this chapter to solve dynamic crack problems, such as dynamic fracture and Lamb wave scattering, in these plates.

Within the scope of linear elastic fracture mechanics, through-thickness crack problems, which are commonly encountered in plate structures, are three-dimensional (3D) in essence. The analysis should be carried out based on the 3D theory of elasticity to obtain comprehensive solutions to such problems. However, this theory is not popular in analytical studies due to mathematical difficulties and in numerical investigations due to high computational costs. Thus, two-dimensional elasticity theories are preferred in dealing with such problems.

The plane strain and generalized plane stress theories are two classical two-dimensional theories, but the limitations of these two theories are obvious. When it comes to the fracture

analysis of through-thickness crack problems, Sih [97] pointed out that the analysis under plane strain assumption was a limiting case for the 3D analysis of sufficiently thick plates, but the generalized plane stress solution was not a limit of the 3D solution for sufficiently thin plates. Furthermore, these two theories cannot take thickness into consideration for finite-thickness plates.

In fact, through-thickness crack problems are more complex than often thought. Although Mode II is the primary mode for the plate subjected to in-plane shear loading, a coupled out-of-plane mode is also generated due to the Poisson effect [98–101] and such coupled mode is sometimes called O-mode [101, 102]. Neither plane strain theory nor generalized plane stress theories can be used to describe this coupled mode.

The Kane-Mindlin theory (K-M theory) allows us to consider both the thickness effect and the O-mode, and it has been applied to through-thickness crack problems by many researchers [103–108]. In this first-order plate theory, a linear variation of the out-of-plane displacement across the thickness is assumed. It was shown that K-M solutions actually reflected an average through-thickness effect [101, 109].

Up to now, the K-M theory has been mainly applied to static fracture problems. Its application to dynamic fracture problems is rarely seen in open literature, and only Jin and Batra [105] conducted the dynamic fracture analysis of K-M plates. However, their study was limited to Mode I fracture problems in an infinite plate. In order to consider more complex fracture problems, such as the mixed mode fracture of finite plates, numerical methods should be relied on.

To the best of the author's knowledge, such dynamic fracture problems have not been solved with any numerical methods. In this chapter, these problems will be solved numerically for the first time by using the K-M DBEM. The stress intensity factors are evaluated based on the crack opening displacements and comparisons are made in three benchmark examples against solutions obtained using the Abaqus implicit solver.

Such DBEM is also suitable for modelling Lamb wave scattering by cracks because the boundary conditions for infinite plates are automatically satisfied and it is not necessary to create a large plate to avoid the contamination resulting from the boundary-reflected waves; thus, only crack surfaces are required to be discretized and the interaction between Lamb waves and cracks can be clearly observed. This study will also be carried out in this chapter.

4.2 Dual boundary integral equations

As mentioned in Subsection 2.4.6, Chapter 2, in order to model cracks based on DBEM, both displacement and traction integral equations should be used. The displacement boundary integral equation (3.33) has been given in Chapter 3. In this section, the traction boundary integral equation will be derived along with the associated dynamic fundamental solutions.

Assuming the collocation point $\mathbf{x}'$ is located on a smooth boundary, differentiating Eq. (3.33) with respect to $\mathbf{x}'$, and using the relations between stress resultants and displacements (see Eq. (3.3)), the traction boundary integral equation in the Laplace domain is obtained as follows:

$$\begin{aligned}
\frac{1}{2}\tilde{t}_\alpha^S(\mathbf{x}', p) &= n_\beta(\mathbf{x}') \left[\oint_\Gamma \tilde{D}_{\alpha\beta\gamma}^S(\mathbf{x}', \mathbf{x}, p) \tilde{t}_\gamma^S(\mathbf{x}, p) d\Gamma + \int_\Gamma \tilde{D}_{\alpha\beta 3}^S(\mathbf{x}', \mathbf{x}, p) \tilde{t}_3^S(\mathbf{x}, p) d\Gamma \right. \\
&\quad - \oint_\Gamma \tilde{S}_{\alpha\beta\gamma}^S(\mathbf{x}', \mathbf{x}, p) \tilde{u}_\gamma^S(\mathbf{x}, p) d\Gamma - \oint_\Gamma \tilde{S}_{\alpha\beta 3}^S(\mathbf{x}', \mathbf{x}, p) \tilde{u}_3^S(\mathbf{x}, p) d\Gamma \\
&\quad \left. + \int_\Omega \tilde{D}_{\alpha\beta k}^S(\mathbf{x}', \mathbf{X}, p) \tilde{F}_k^S(\mathbf{X}, p) d\Omega \right] \\
\frac{1}{2}\tilde{t}_3^S(\mathbf{x}', p) &= n_\beta(\mathbf{x}') \left[\int_\Gamma \tilde{D}_{3\beta\gamma}^S(\mathbf{x}', \mathbf{x}, p) \tilde{t}_\gamma^S(\mathbf{x}, p) d\Gamma + \oint_\Gamma \tilde{D}_{3\beta 3}^S(\mathbf{x}', \mathbf{x}, p) \tilde{t}_3^S(\mathbf{x}, p) d\Gamma \right. \\
&\quad - \oint_\Gamma \tilde{S}_{3\beta\gamma}^S(\mathbf{x}', \mathbf{x}, p) \tilde{u}_\gamma^S(\mathbf{x}, p) d\Gamma - \oint_\Gamma \tilde{S}_{3\beta 3}^S(\mathbf{x}', \mathbf{x}, p) \tilde{u}_3^S(\mathbf{x}, p) d\Gamma \\
&\quad \left. + \int_\Omega \tilde{D}_{3\beta k}^S(\mathbf{x}', \mathbf{X}, p) \tilde{F}_k^S(\mathbf{X}, p) d\Omega \right]
\end{aligned} \tag{4.1}$$

where $\oint$ stands for Hadamard finite-part integral and the dynamic fundamental solutions $\tilde{D}_{i\beta k}^S$,

$\tilde{S}_{i\beta k}^S$ can be obtained using the relationships as follows:

$$\begin{aligned}\tilde{X}_{\alpha\beta\gamma}^S &= h \left[\lambda \left(\tilde{Y}_{\eta\gamma,\eta}^S + \kappa_S \tilde{Y}_{3\gamma}^S \right) \delta_{\alpha\beta} + \mu \left(\tilde{Y}_{\alpha\gamma,\beta}^S + \tilde{Y}_{\beta\gamma,\alpha}^S \right) \right] \\ \tilde{X}_{\alpha\beta 3}^S &= h \left[\lambda \left(\tilde{Y}_{\gamma 3,\gamma}^S + \kappa_S \tilde{Y}_{33}^S \right) \delta_{\alpha\beta} + \mu \left(\tilde{Y}_{\alpha 3,\beta}^S + \tilde{Y}_{\beta 3,\alpha}^S \right) \right] \\ \tilde{X}_{3\beta\gamma}^S &= \frac{h^3 \mu}{12} \tilde{Y}_{3\gamma,\beta}^S \\ \tilde{X}_{3\beta 3}^S &= \frac{h^3 \mu}{12} \tilde{Y}_{33,\beta}^S\end{aligned}\tag{4.2}$$

in which $\tilde{X}^S$ denotes $\tilde{D}^S$ and $\tilde{S}^S$ when $\tilde{Y}^S$ stands for $\tilde{U}^S$ and $\tilde{T}^S$, respectively. The detailed expressions of the above fundamental solutions can be found in Appendix A.1.

4.3 Numerical implementation

The process of numerical implementation is the same as that introduced in Section 2.4, Chapter 2. The numerical inversion of Laplace transform is conducted using Eq. (2.39). It has been found in 2D isotropic problems, as shown in Subsection 2.4.8, that the dominant singularities in the dynamic fundamental solutions have the same expressions as those in the static fundamental solutions. This is also true for the K-M problems, as shown in Appendix C.1. Thus, in order to compute the singular integrals in Eq. (4.1), it is convenient to divide the dynamic fundamental solutions into two parts:

$$\begin{aligned}\tilde{D}_{k\beta j}^S(\mathbf{x}', \mathbf{x}, p) &= D_{k\beta j}^{S|\text{static}}(\mathbf{x}', \mathbf{x}) + \tilde{D}_{k\beta j}^{S|m}(\mathbf{x}', \mathbf{x}, p) \\ \tilde{S}_{k\beta j}^S(\mathbf{x}', \mathbf{x}, p) &= S_{k\beta j}^{S|\text{static}}(\mathbf{x}', \mathbf{x}) + \tilde{S}_{k\beta j}^{S|m}(\mathbf{x}', \mathbf{x}, p)\end{aligned}\tag{4.3}$$

in which $D_{k\beta j}^{S|\text{static}}$ and $S_{k\beta j}^{S|\text{static}}$ are the fundamental solutions for the static Kane-Mindlin plates and are listed in Appendix B.1; $\tilde{D}_{k\beta j}^{S|m}$ and $\tilde{S}_{k\beta j}^{S|m}$ are modified terms without any singularities, except for $\tilde{S}_{\alpha\beta\gamma}^{S|m}$ and $\tilde{S}_{3\beta 3}^{S|m}$ containing the weak singularities of order $\ln(r)$.

4.4 Estimation of dynamic stress intensity factors

As shown in Fig. 4.1, there are three symmetric fracture modes for the Kane-Mindlin (K-M) theory. The first two in-plane fracture modes, symmetric mode I and mode II, represent the Mode I and Mode II fractures shown in Fig. 2.8, respectively. It is worth noting that the fracture analyses conducted based on both the generalized plane stress theory and plane strain theory can be included in the K-M theory. Moreover, the K-M theory can consider the influence of plate thickness on the fractures. As for the symmetric mode III, such mode was considered to be similar to Mode III shown in Fig. 2.8 but symmetric with respect to the plate mid-plane [98]. However, the symmetric mode III is very different from Mode III because its existence depends on the symmetric mode II. It is coupled to the symmetric mode II and caused by the Poisson's effect. Thus, this symmetric mode is often called O-mode to distinguish it from Mode III [101].

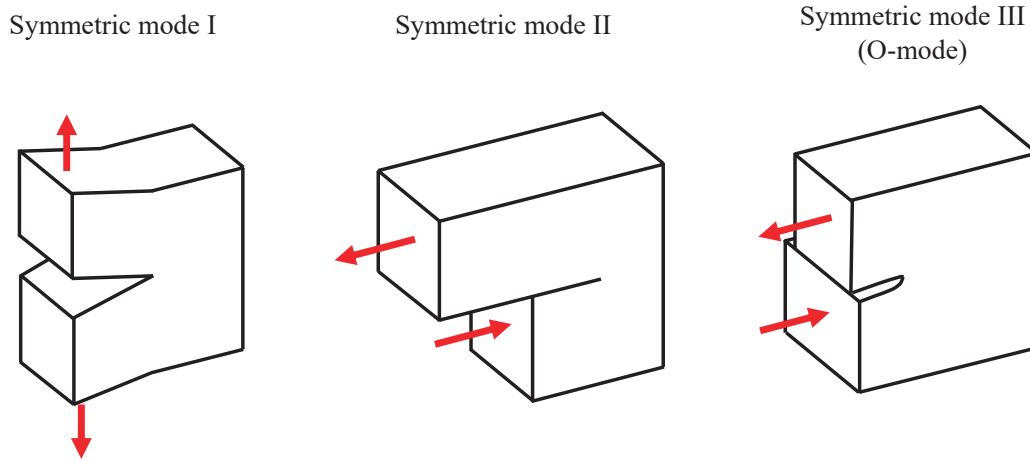


Figure 4.1: Symmetric fracture modes for the Kane-Mindlin theory.

In order to accurately capture the singular behaviour of stress fields in the vicinity of crack tips, the so-called discontinuous quarter-point elements (see Fig. 2.12) were used in this chapter. After the generalised displacements u_i^S in the K-M plates are obtained, the plate displacements u_i can be computed directly using Eq. (3.1). The crack opening displacement (COD) at the node pair B-C, Δu_i^{BC} , which is closest to the crack tip, was used for evaluating DSIFs. On the

basis of such COD, the DSIFs can be evaluated using the following expressions [35, 101]:

$$\begin{aligned} K_I &= \frac{E}{4(1-\nu^2)} \sqrt{\frac{\pi}{2R^{BC}}} \Delta u_2^{BC} \\ K_{II} &= \frac{E}{4(1-\nu^2)} \sqrt{\frac{\pi}{2R^{BC}}} \Delta u_1^{BC} \\ K_O &= \frac{E}{4(1+\nu)} \sqrt{\frac{\pi}{2R^{BC}}} \Delta u_3^{BC} \end{aligned} \quad (4.4)$$

where R^{BC} denotes the distance between the node pair B-C and the crack tip. In the discontinuous quarter-point element, R^{BC} is equal to $l_e/36$, in which l_e denotes the length of the crack-tip element. Thus, Eq. (4.4) becomes:

$$\begin{aligned} K_I &= \frac{3E}{2(1-\nu^2)} \sqrt{\frac{\pi}{2l_e}} \Delta u_2^{BC} \\ K_{II} &= \frac{3E}{2(1-\nu^2)} \sqrt{\frac{\pi}{2l_e}} \Delta u_1^{BC} \\ K_O &= \frac{3E}{2(1+\nu)} \sqrt{\frac{\pi}{2l_e}} \Delta u_3^{BC} \end{aligned} \quad (4.5)$$

In order to validate the BEM results, the DSIFs were also evaluated using the FEM, in which wedge quarter-point elements were used. Since these elements are not supported in the Abaqus explicit solver, the implicit solver was used. These elements are continuous elements, so R^{BC} is set to $l_e/4$ and the DSIFs can be evaluated as follows:

$$\begin{aligned} K_I &= \frac{E}{2(1-\nu^2)} \sqrt{\frac{\pi}{2l_e}} \Delta u_2^{BC} \\ K_{II} &= \frac{E}{2(1-\nu^2)} \sqrt{\frac{\pi}{2l_e}} \Delta u_1^{BC} \\ K_O &= \frac{E}{2(1+\nu)} \sqrt{\frac{\pi}{2l_e}} \Delta u_3^{BC} \end{aligned} \quad (4.6)$$

It has been pointed out in Refs. [101, 109] that the solutions based on the K-M theory agreed very well with the thickness-averaged results from the 3D FEM simulation. Thus, in the following numerical examples, the comparisons between the K-M DBEM and 3D FEM

results are based on the thickness-averaged DSIFs, $K_I^a = \frac{2}{h} \int_0^{h/2} K_I dx_3$, $K_{II}^a = \frac{2}{h} \int_0^{h/2} K_{II} dx_3$ and $K_O^a = \frac{2}{h} \int_0^{h/2} K_O dx_3$.

It should be mentioned here that the so-called corner singularity [98, 110–112], which is limited to a very small zone near the intersection between the crack front and free surface, is complex and cannot be described by using the inverse square-root singularity. However, since the corner singularity has been found to have a minor influence on the through-thickness averaged results [113] and the 3D FEM simulation aimed to validate the K-M based BEM results, there was no need to introduce such complex singularity into the finite element model in this chapter.

4.5 Examples

The FEM simulation was carried out using the Abaqus implicit solver. The cracked plates were discretized with the elements of type C3D20R, which can be modified to generate wedge quarter-point crack-tip elements easily. Since the field quantities near the intersection between the crack front and the free surface exhibit strong localized gradient along the x_3 -axis, the thickness of element layers was gradually decreased toward the free surface.

The time t is normalized with respect to l_c/c_2 , where c_2 is the shear wave velocity and l_c is a characteristic length, such as half crack length $a/2$. The DSIFs are normalized with respect to $\sigma_A \sqrt{\pi a/2}$, in which σ_A is the load amplitude.

4.5.1 A crack under opening loading

An infinite plate with a through-thickness crack, as shown in Fig. 4.2, is considered here. The plate thickness and crack length are denoted by h and a , respectively. The crack surfaces are subjected to uniform opening loading $F_N H(t)$, where F_N is the load amplitude and $H(t)$ is the Heaviside function. The material properties are the density, $\rho = 290 \text{ kg/m}^3$, Young's modulus, $E = 69 \text{ GPa}$, and the Poisson's ratio, $\nu = 0.3$.

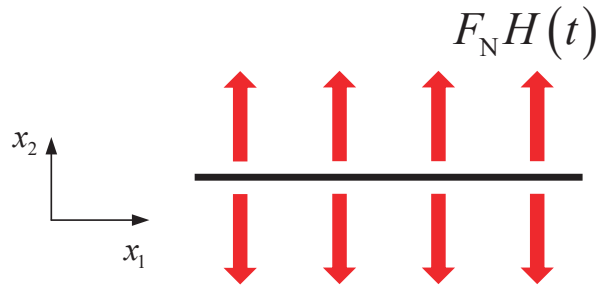


Figure 4.2: A crack subjected to opening loading.

Only the crack was required to be modelled in the DBEM analysis and each crack surface was discretized with 30 boundary elements. In the FEM simulation, a sufficiently large square plate with an edge length of $10a$ was created to approximate the infinite plate, which ensured that the DSIF during a specified time period was not influenced by the reflected waves from the plate edges. Due to the geometric and loading symmetry properties, only one-eighth of the plate was modelled in the finite element analysis.

It should be mentioned that the dynamic fracture analysis of this cracked infinite plate under opening loading was first carried out by Jin and Batra [105]. However, the DSIF obtained in their study seems inaccurate because their DSIF curves are very smooth and do not contain any kinks which are caused due to the propagation of Rayleigh-type surface waves along the crack surfaces.

First, a very thin plate ($h/a = 0.025$) was taken into consideration. Sih [97] pointed out that the generalized plane stress theory could not serve as a limiting thin-plate case for the three-dimensional analysis and it could only give approximate results for such thin plate. Thus, the fracture analysis of the thin plate should be carried out using higher-order plate theories, such as the K-M theory. In spite of this, for an infinitely thin plate ($h/a \rightarrow 0$), it has been found in Refs. [103, 105] that the Mode I stress intensity factor obtained using the K-M theory, K_I^{K-M} , is equal to $K_I^{PS}/\sqrt{1-\nu^2}$, in which K_I^{PS} is the solution based on the generalized plane stress theory. This relationship can be used to modify the DSIF K_I^{PS} obtained from the DBEM for the generalized plane stress problems (see Section 2.4 in Chapter 2), and then to validate

K_I^{K-M} . In the 3D FEM simulation, 204,880 elements were used in which there were ten layers of elements along the thickness direction and the radial size of the crack-tip singular elements was equal to $0.05\%a$. Fig. 4.3 compares the normalized DSIF K_I^a obtained using different numerical methods. The BEM results shown in this figure were obtained using 50 Laplace terms. It is evident that the DSIF obtained using the K-M based BEM is in excellent agreement with both the modified DSIF from the 2D boundary element analysis and the 3D FEM result.

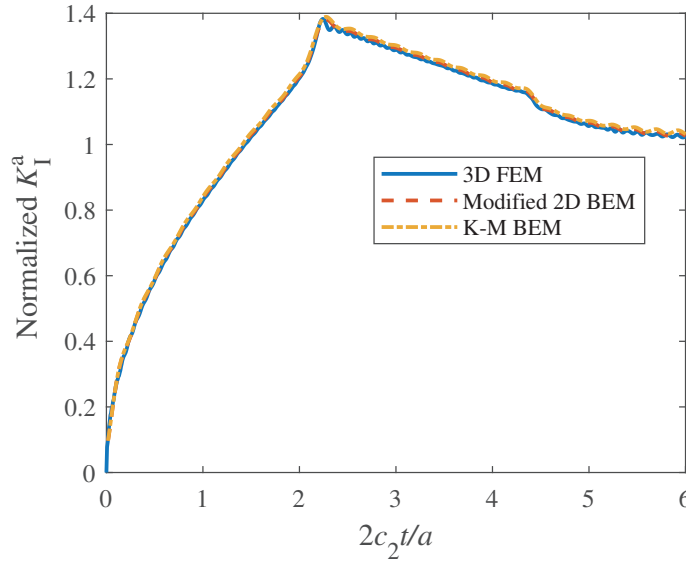


Figure 4.3: Normalized DSIFs computed based on K-M BEM and 3D FEM solutions, and modified 2D (generalized plane stress) BEM result for the crack under opening loading ($h/a = 0.025$).

The influence of the number of Laplace terms on the normalized DSIF K_I^a is shown in Fig. 4.4. It can be seen that the DSIF reaches the maximum value later when fewer Laplace terms are used. However, the maximum values of the DSIF obtained using 25 and 50 Laplace terms are very close (relative error $<1\%$). Thus, if the maximum value of the DSIF is the main focus, 25 Laplace terms can be used to increase the computational efficiency. If more details of the DSIF are required, 50 Laplace terms should be used.

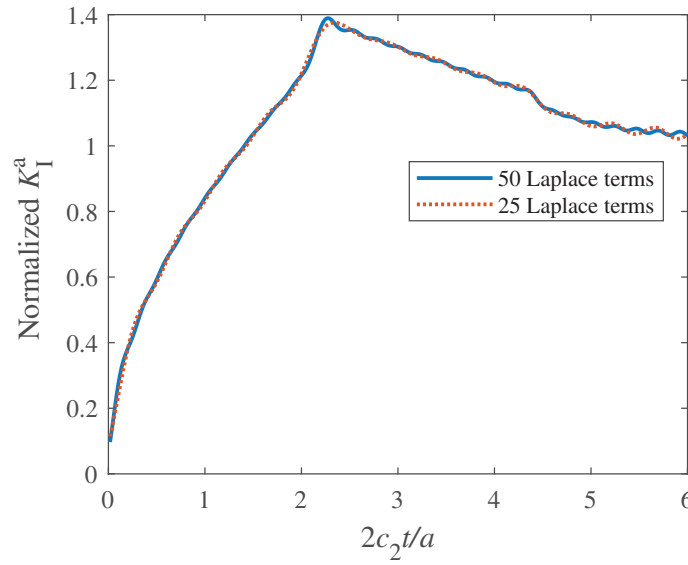


Figure 4.4: The influence of the number of Laplace terms on the normalized DSIF obtained using the K-M BEM for the crack under opening loading ($h/a = 0.025$).

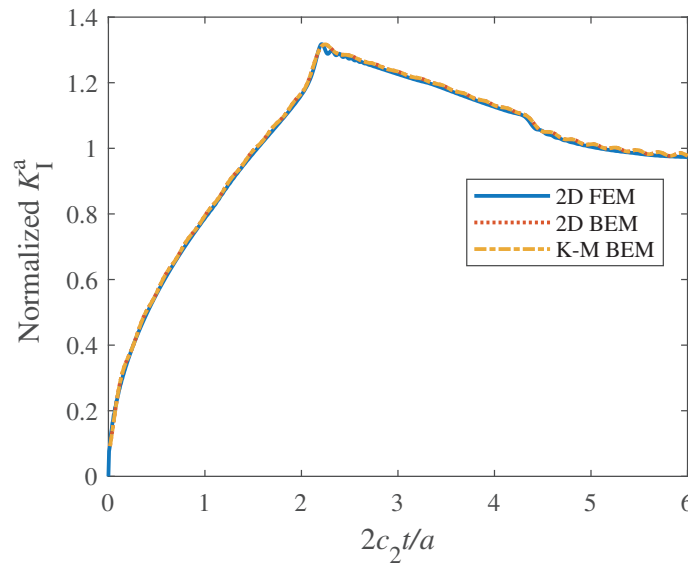


Figure 4.5: Normalized DSIF curves for the crack ($h/a = 100$) under opening loading obtained using the K-M BEM, 2D (plane strain) FEM and 2D (plane strain) BEM.

Next, a very thick plate ($h/a = 100$) was considered to approximate an infinitely thick plate. Sih [97] demonstrated that the plane-strain crack problem was exactly the limiting case for such thick plate. Thus, the finite element analysis using 58,000 plane-strain (CPE8R) elements, and the boundary element analysis for the plane strain problem (see Section 2.4 in Chapter 2), were

conducted to validate the K-M based BEM results. In the boundary element simulation, 50 Laplace terms were used. As shown in Fig. 4.5, the K-M based DSIF K_I^a is in excellent agreement with the plane-strain solutions. This conclusion is consistent with that given by Kotousov and Wang in the static fracture analysis of the K-M plates [106].

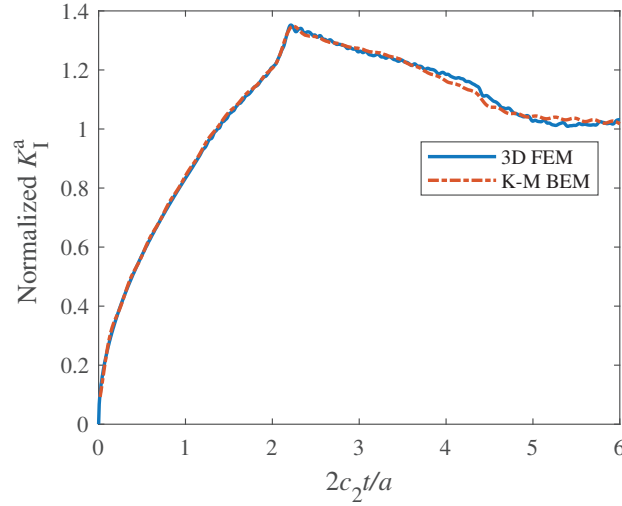
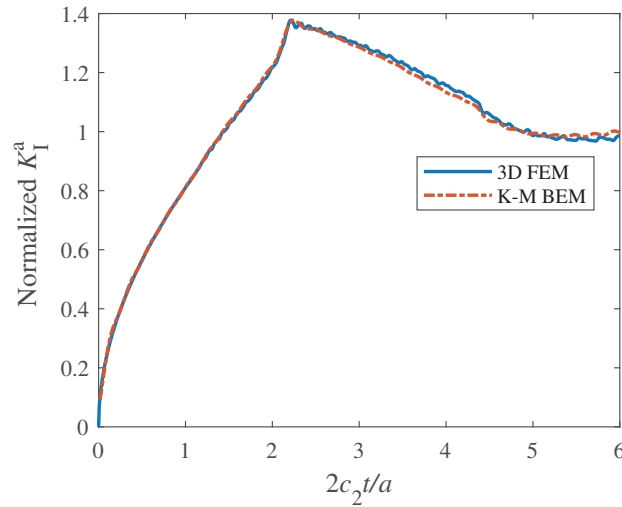
(a) $h/a = 1$ (b) $h/a = 2$

Figure 4.6: Comparison between the normalized DSIF curve computed using the K-M BEM and that obtained based on the 3D FEM for the crack under opening loading in moderately thick plates.

Finally, two moderately thick plates ($h/a = 1, 2$) were taken into consideration. The K-M BEM results were obtained using 50 Laplace terms. In the finite element analysis, these two plates were discretized using 191,200 and 267,680 elements with 25 and 35 layers of elements

along the thickness direction, respectively. In both cases, the radial sizes of the crack-tip wedge singular elements were the same and equal to $0.25\%a$. Figs. 4.6a and 4.6b compare the K-M based BEM DSIFs with those from the finite element simulation for the two moderately thick plates ($h/a = 1, 2$), respectively. It can be seen from these two figures that the BEM results agree well with the FEM solutions. The slight discrepancy results from the linear assumption for the displacement u_3 in the K-M theory. This assumption cannot represent the real variation accurately for moderately thick plates.

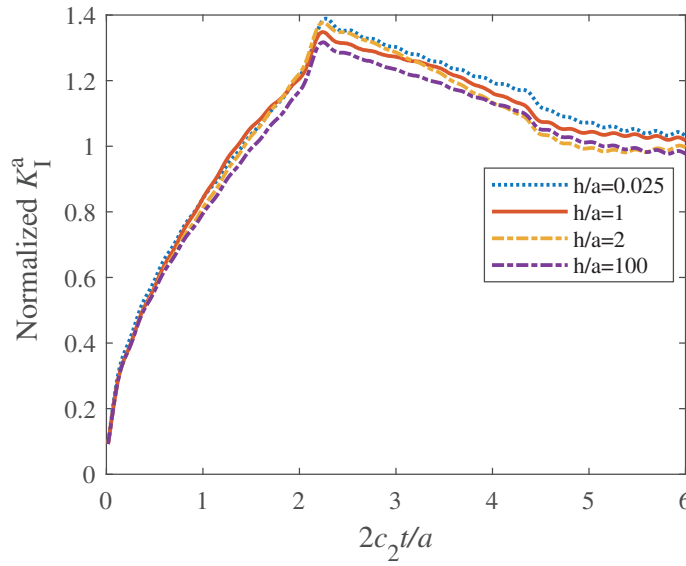


Figure 4.7: Influence of plate thickness on normalized DSIF for the crack under opening loading.

Since the K-M theory considers the plate thickness effect, this theory can be used to analyse the influence of the plate thickness on the DSIF K_I^a , which is shown in Fig. 4.7. Generally speaking, it can be seen that the plate thickness does not have a significant influence on the Mode I DSIF, especially on the value and location of the maximum DSIF. The DSIF for the infinitely thin plate is larger than that for the infinitely thick plate. However, the maximum value of the DSIF does not decrease monotonically with increasing plate thickness.

4.5.2 A crack under shear loading

Fig. 4.8 shows an infinite plate with a central crack loaded by shear loading $F_S H(t)$, where F_S is the load amplitude. As mentioned earlier, under such shear loading, Model II (primary mode) and O-mode (secondary mode) are coupled. Since the geometry and material properties are the same as those in the previous example, the same BEM models and FEM models, including the meshes, used in the first example were adopted in this example; and the only things updated were boundary conditions.

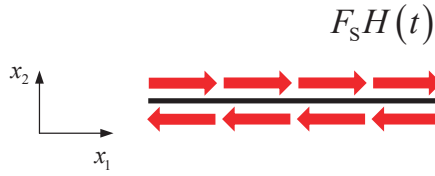


Figure 4.8: A crack subjected to shear loading.

As in the previous example, a very thin plate ($h/a = 0.025$) was considered first. Fig. 4.9 shows the normalized DSIFs K_{II}^a and K_O^a obtained using both the K-M based BEM and the 3D FEM. It can be seen that the BEM results agree very well with the FEM results. The maximum value of K_O^a is only around 15 percent of that value of K_{II}^a .

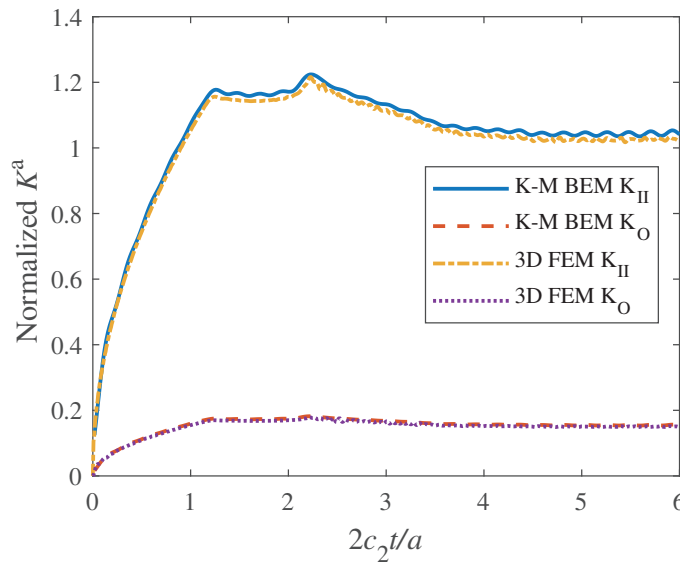


Figure 4.9: Comparison between normalized DSIFs estimated based on K-M BEM solutions and those from 3D FEM simulation for the crack under shear loading ($h/a = 0.025$).

Next, a very thick plate ($h/a = 100$) was analysed, for which the plane strain conditions prevail. It means that the coupled O-mode stress intensity factor should be negligible, which is reflected correctly in the K-M based BEM results, as shown in Fig. 4.10. The normalized DSIF K_{II}^a obtained using the K-M based BEM is in excellent agreement with both the BEM and FEM results on the basis of the plane strain theory.

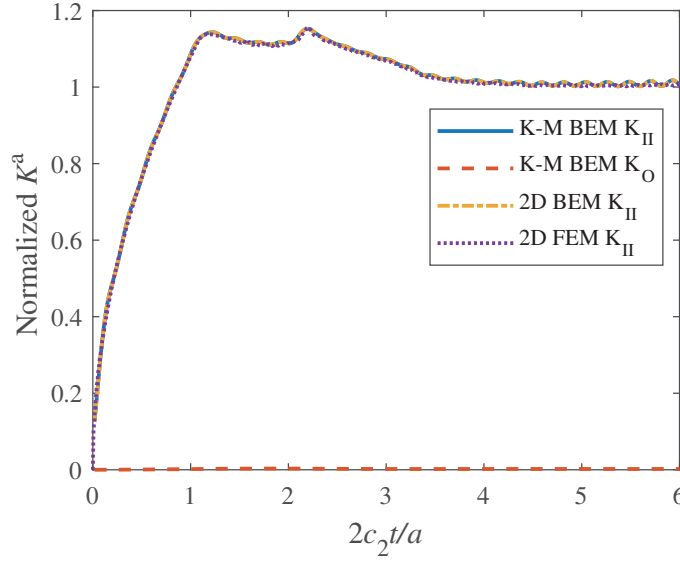


Figure 4.10: Comparison between normalized DSIFs estimated based on K-M BEM solutions and those from 2D plane-strain FEM and BEM simulations for the crack under shear loading ($h/a = 100$).

When it comes to moderately thick plates ($h/a = 1, 2$), the comparisons were carried out between the results from the K-M based boundary element analysis and the 3D finite element simulation. As shown in Figs. 4.11a and 4.11b, the DSIFs from the K-M based BEM match well with those from the 3D FEM. Furthermore, compared with the coupled DSIF K_O^a , better agreement is shown in terms of the DSIF K_{II}^a . As mentioned earlier, the limitation of the linear assumption in the K-M theory results in the discrepancy between the K-M results and the thickness-averaged 3D results.

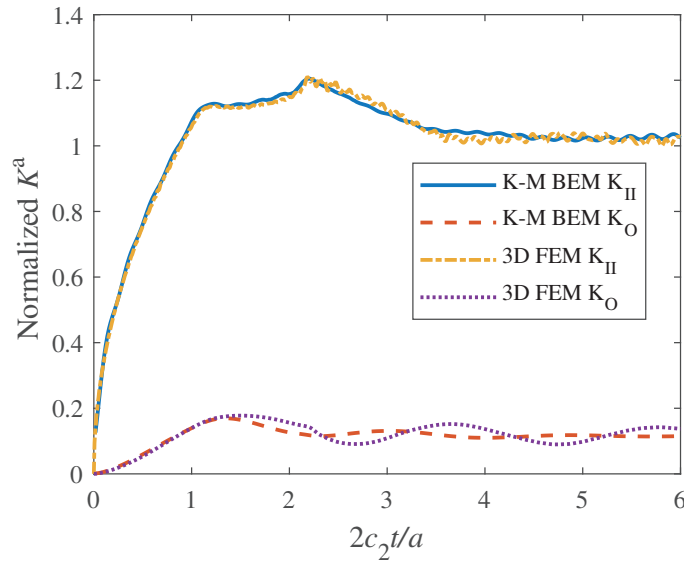
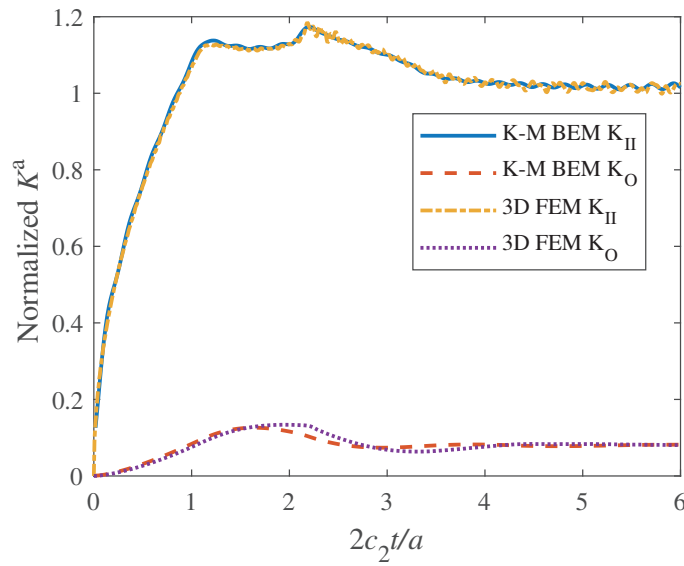
(a) $h/a = 1$ (b) $h/a = 2$

Figure 4.11: Comparison between normalized DSIFs estimated based on K-M BEM solutions and those from 3D FEM simulation for the crack under shear loading in moderately thick plates.

The influences of plate thickness on the Mode II and the O-mode DSIFs are presented in Fig. 4.12. It is shown that the plate thickness has a minor influence on the Mode II DSIF, but it affects the O-mode DSIF significantly. It seems that the maximum values of the normalized Mode II and O-mode DSIFs decrease as the plate becomes thicker.

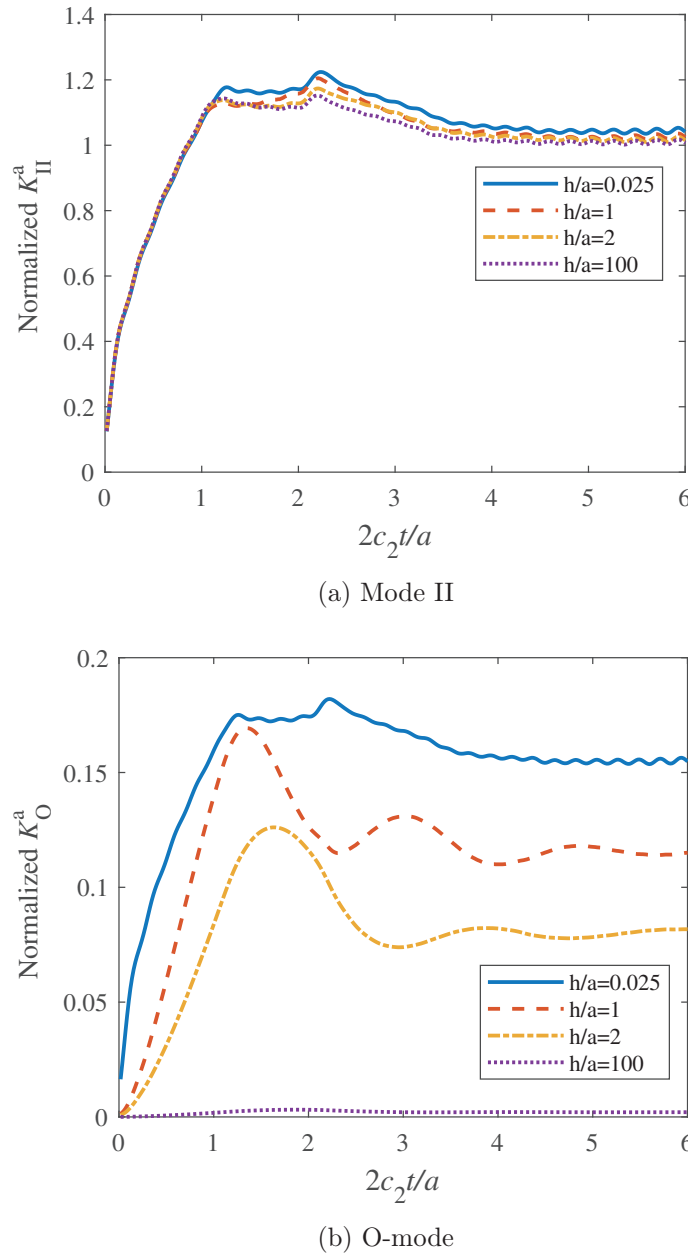


Figure 4.12: Influence of plate thickness on normalized DSIFs for the crack under shear loading.

4.5.3 A square plate with a slant central crack

In this example, a mixed-mode crack problem is addressed. A cracked square plate loaded in tension $\sigma_0 H(t)$ is illustrated in Fig. 4.13. The edge and crack lengths of this plate are $b = 15$ mm and $a = 4$ mm, respectively. The crack makes an angle of $\theta = \pi/4$ with the direction of the x_2 axis. This aluminium plate has a Young's modulus of 70 GPa, a Poisson's ratio of 0.33

and a density of 2700 kg/m^3 .

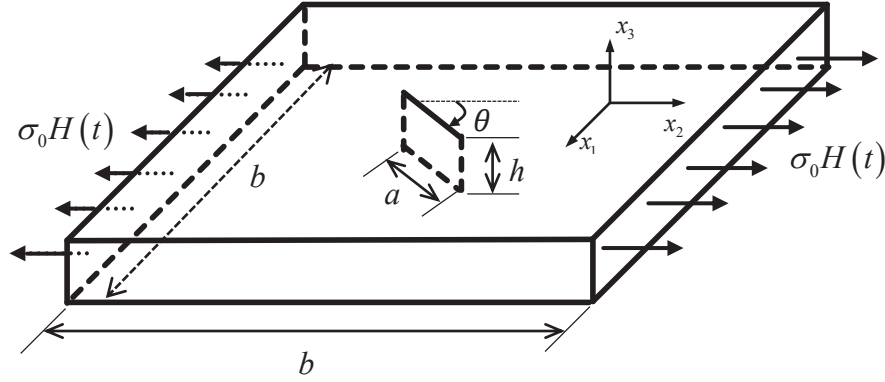


Figure 4.13: A square plate with a slant central crack.

In the following dual boundary element analyses, each plate edge and crack surface were discretized using 10 and 20 elements, respectively. The numerical inversion of Laplace transforms was conducted using 50 Laplace terms.

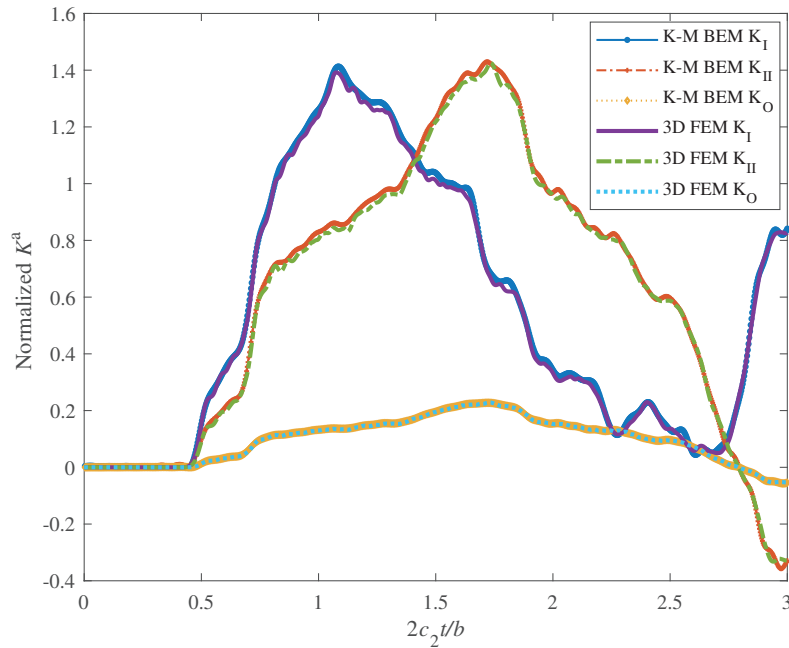


Figure 4.14: Comparison between normalized DSIFs estimated based on K-M BEM solutions and those from 3D FEM simulation for the square plate with a slant central crack ($h/a = 0.05$).

First, a thin plate ($h/a = 0.05$) was considered. Owing to symmetry conditions with respect to the mid-plane ($x_3 = 0$), in the 3D FEM simulation, only half plate ($x_3 \geq 0$) was modelled

with 392,480 elements, including 10 layers of elements along the x_3 direction and sufficiently small crack-tip elements with a radial size of $0.25\%a$. For this mixed-mode problem, all the three fracture modes, including the coupled O-mode, can be observed. Fig. 4.14 compares the DSIFs for these three fracture modes obtained from the K-M BEM and 3D FEM simulations. The comparison shows excellent agreement between the BEM and FEM results. In addition, it can be seen that the maximum values of Mode I and Mode II DSIFs are nearly the same, and the maximum value of the O-mode DSIF is around 16% of that of Mode II DSIF.

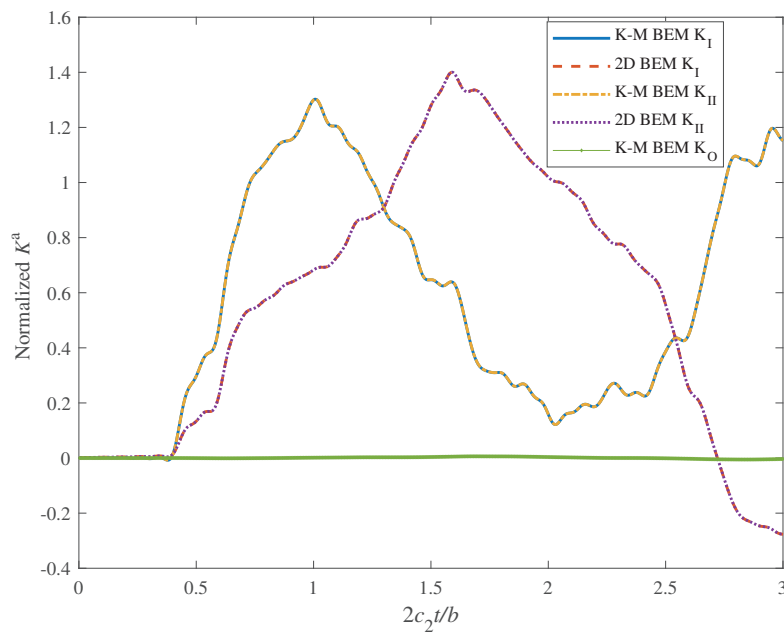
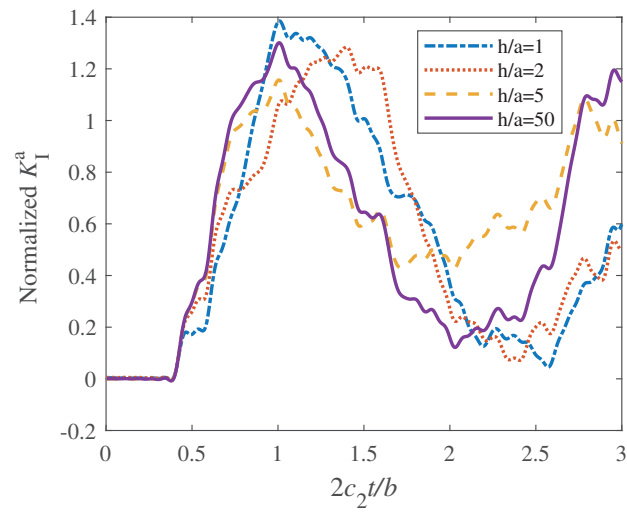
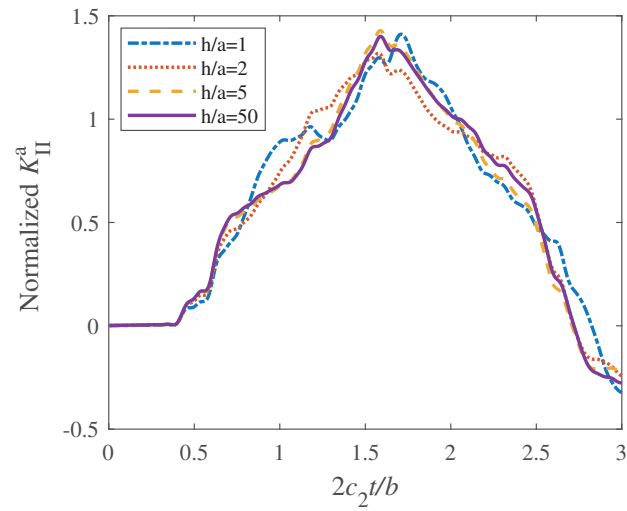


Figure 4.15: Comparison between normalized DSIFs from K-M BEM and 2D (plane strain) BEM simulations for the square plate with a slant central crack ($h/a = 100$).

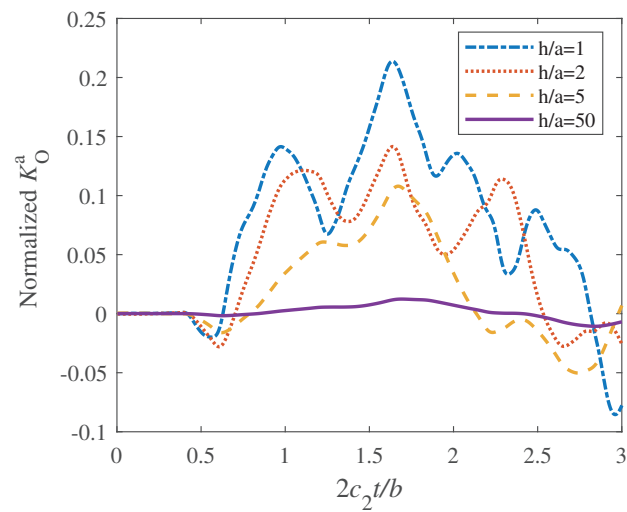
Next, a very thick plate ($h/a = 100$) was taken into account. For such a thick plate, plane strain conditions should prevail. Fig. 4.15 compares the DSIFs obtained using the K-M based BEM and the 2D BEM based on the plane strain theory. The excellent agreement in terms of K_I^a and K_{II}^a , as well as the negligible K_O^a , demonstrates that this limiting case (plane strain) can be represented accurately using the K-M theory.



(a) Mode I



(b) Mode II



(c) O-mode

Figure 4.16: Influence of plate thickness on normalized DSIFs for the square plate with a slant central crack.

Lastly, moderately thick plates were considered using the K-M based BEM. Fig. 4.16 presents the influence of the plate thickness on the DSIFs for all three fracture modes. Generally speaking, the plate thickness has a significant influence on the O-mode and Mode I DSIFs in terms of the maximum value, but its influence on Mode II DSIF is limited. There is no monotonic relationship between the plate thickness and the maximum values of Mode I and Mode II. It seems that the maximum value of the O-mode DSIF decreases with increasing plate thickness.

4.5.4 S0 mode wave scattering by a crack

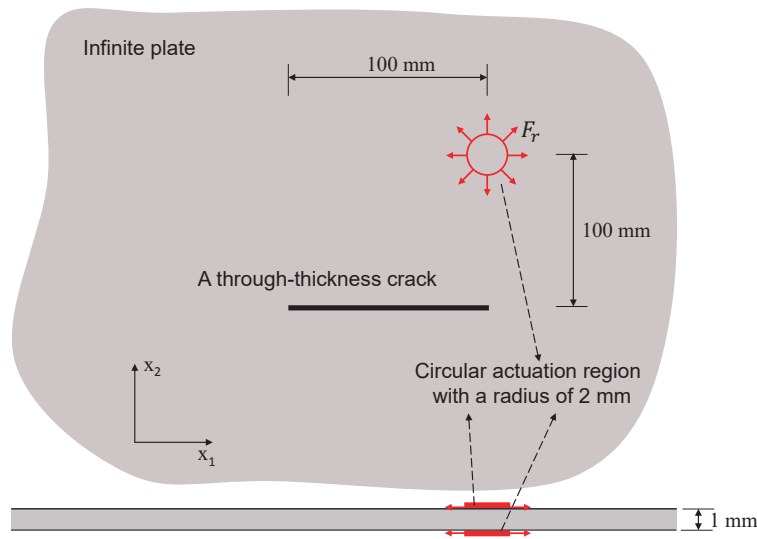


Figure 4.17: A through-thickness crack in an infinite plate with two circular actuation regions generating omnidirectional S0 mode.

Besides the dynamic fracture analysis, the proposed Kane-Mindlin DBEM enables us to solve the scattering of S0 mode Lamb wave by a through-thickness crack. Fig. 4.17 shows a through-thickness crack with a length of 100 mm in a 1-mm-thick infinite plate; in order to generate an omnidirectional S0 mode wave, surface tractions F_r (3-cycle Hann-window-modulated sinusoidal tone burst with a central frequency of 600 kHz) are applied on the circumference of two circular regions which are located on plate surfaces and symmetric with respect to the plate mid-plane. The plate has a density of 2700 kg/m³, a Poisson's ratio of 0.33 and a Young's modulus of 69 GPa. In the Kane-Mindlin DBEM, each crack surface was modelled using 90 boundary

elements and the time-domain results were computed based on 100 Laplace terms.

Fig. 4.18 presents the snapshots of displacement response of the cracked infinite plate after the S0 mode is excited by the applied tractions. At $t = 0.015$ ms, the initial wave packet of the S0 mode is propagating towards the crack. When it reaches the right crack tip, diffracted S0 mode and SH0 mode are generated, as well as diffracted Rayleigh-type waves propagating along crack surfaces at a speed slightly lower than that of the SH0 mode. When the initial S0 mode encounters the upper crack surface, reflected S0 and SH0 modes are generated. These diffracted and reflected waves can be seen clearly at $t = 0.029$ ms. SH0 mode waves are in-plane guided waves, so they do not have the out-of-plane displacement component and cannot be observed from the u_3 -displacement snapshot.

Subsequently, when the initial S0 mode arrives at the left crack tip, the second diffracted S0 and SH0 modes are generated and radiate into the plate medium centred at the left crack tip. In addition, diffracted Rayleigh-type crack-surface waves propagate toward the right crack tip. These second diffracted waves are presented in the snapshots at $t = 0.035$ ms.

It should be noted that when the diffracted S0 mode travels along the crack surface, it is accompanied by a so-called head wave, which ensures the traction-free boundary conditions are satisfied on the crack surface. The head wave originates from the crack surface and propagates into the media as fast as the SH0 mode. The head wave, as shown in the u_2 -displacement snapshot at $t = 0.048$ ms, can be seen after the diffracted S0 and SH0 modes propagate over a long distance and are separate sufficiently. However, the head waves are relatively weak, so they cannot be observed easily and clearly. Since the head wave is essentially a SH0 mode wave, it does not have the out-of-plane displacement component.

As discussed above, the S0 mode interacts with the crack in a very complex way, and the scattered wave is composed of all the reflected and diffracted waves.

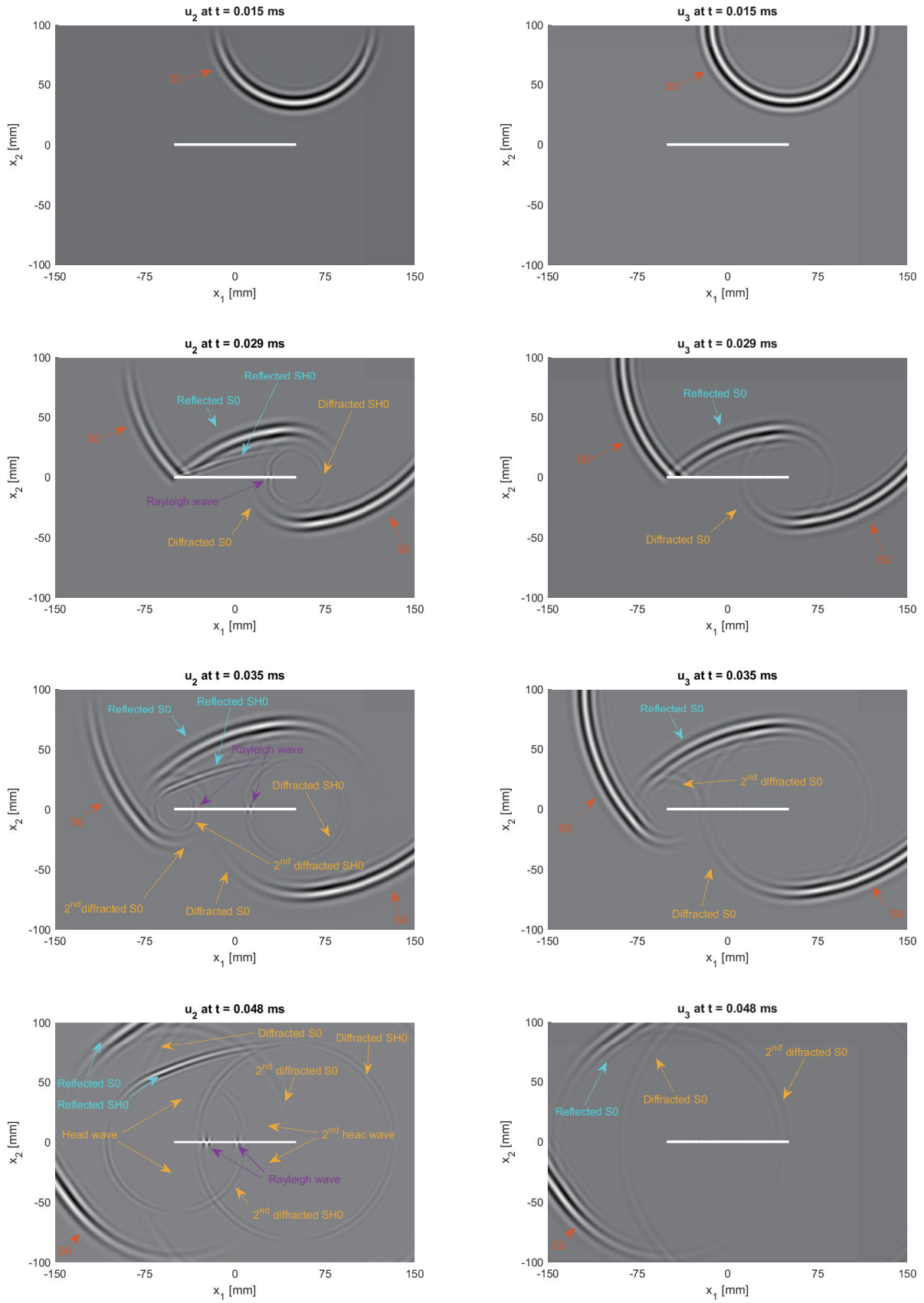


Figure 4.18: Displacement fields of a portion of the cracked infinite plate at the actuation frequency of 600 kHz.

4.6 Summary

In this chapter, a new dual boundary element formulation in the Laplace domain has been developed for the dynamic analyses of cracked Kane-Mindlin (K-M) plates. One application of such formulation is to estimate the dynamic stress intensity factors (DSIFs). It is found that the DSIFs based on the K-M theory represent average through-thickness stress intensity factors, which is consistent with the findings in the corresponding static problems, and the DBEM DSIFs have been shown to be in good agreement with the results from the 3D finite element calculations.

The proposed DBEM is capable of analysing the influence of plate thickness in the dynamic fracture analyses of plates. This is especially valuable for the numerical analyses of plates under in-plane shear loading and of mixed mode crack problems. For such cases, the classical two-dimensional theories of elasticity (plane strain and generalized plane stress) fail to consider the thickness effect and to capture the coupled O-mode, and the numerical simulations based on the three-dimensional elasticity theory are computationally inefficient. The results in this chapter reveal that compared with Mode I and Mode II DSIFs, plate thickness has a more significant influence on the coupled O-mode DSIF.

Another application of the K-M dual boundary element formulation is the wave-scattering analysis. The interaction between the S0 mode Lamb wave and a crack can be quite well described using such formulation.

Chapter 5

Dynamic dual boundary element analysis for cracked Mindlin plates

5.1 Introduction

The Laplace-transform boundary element method (BEM) for Mindlin plates under dynamic loads has been proposed by Wen and Aliabadi [82]. In this chapter, this BEM will be extended to a dual boundary element method (DBEM), which allows to solve dynamic crack problems including dynamic fracture and Lamb wave scattering in an efficient way.

When it comes to the dynamic fracture problems in plates under bending loads, the analyses are commonly based on the Kirchhoff-Love plate theory [114] and the Mindlin-Reissner plate theory [61]. According to Sih [99], the Kirchhoff-Love theory in fracture mechanics is insufficient, even for thin plates, because the theory does not satisfy all physical boundary conditions on crack surfaces. In addition, Sih noted that when the characteristic length of a region around a crack tip was comparable to the plate thickness, shear forces should not be disregarded in this region; and to obtain accurate stress intensity factors, the Mindlin-Reissner plate theory should be used.

Wen and Aliabadi [3] proposed a displacement discontinuity method to compute the dynamic stress intensity factors based on the Mindlin-Reissner plate theory. However, their method can only solve infinite-plate problems because it is unable to take plate boundaries into consideration. When the standard BEM is adopted to solve fracture problems in a plate of arbitrary geometry, artificial interfaces are required to divide the original domain into two regions and each region includes one crack surface, which is known as the multi-region method [50]. This method is computationally inefficient because the artificial boundaries lead to extra but unnecessary boundary elements. DBEM allows to solve crack problems in a single region and the extra interfaces can be avoided. In this method, one crack surface is modelled using the displacement boundary integral equation, while the opposite surface is modelled with the traction boundary integral equation. DBEM has been applied to static fracture analysis of Mindlin plates [115]. However, to the best of the author's knowledge, no DBEM formulation has been presented to solve dynamic problems in cracked Mindlin plates.

In this chapter, a dual boundary element formulation is presented for the first time for dynamic crack problems in shear deformable plates. The problem is formulated in the Laplace frequency domain and the fundamental solutions reported for the displacement and traction boundary integral equations by Wen and Aliabadi [82] are revisited. These fundamental solutions are rewritten in a compact form and their singular behaviours are re-examined and corrected. The stress resultant intensity factors for four benchmark problems are presented where comparisons are made against solutions obtained using ANSYS implicit FEM and other solutions reported in the literature.

It is worth mentioning that the proposed DBEM is also able to be used for the studies of A0-mode Lamb wave scattering by cracks, which is of great value for ultrasonic guided-wave structural health monitoring, and this ability is shown by one example in this chapter. In order to describe the dispersion properties of the A0 mode accurately, the fundamental solutions are modified to allow the adjustable shear factor.

5.2 Dual boundary integral equations

The plate motion which is antisymmetric with respect to the plate mid-plane is described in the Mindlin theory, and the displacements are approximated by the following expressions:

$$\begin{aligned} u_1(x_1, x_2, x_3, t) &= u_1^A(x_1, x_2, t)x_3 \\ u_2(x_1, x_2, x_3, t) &= u_2^A(x_1, x_2, t)x_3 \\ u_3(x_1, x_2, x_3, t) &= u_3^A(x_1, x_2, t) \end{aligned} \quad (5.1)$$

Since u_1^A and u_2^A are independent variables and not dependent on u_3^A , shear deformations can be taken into account in the Mindlin theory. Thus, the Mindlin theory provides a more accurate tool for solving the antisymmetric-motion problems than the classical plate theory (Kirchhoff-Love theory), in which there is only one independent displacement variable u_3^A .

The equilibrium equations for stress resultants are given by:

$$\begin{aligned} M_{\alpha\beta,\beta} - Q_\alpha + F_\alpha^A &= \frac{\rho h^3}{12} u_{\alpha,tt}^A \\ Q_{\alpha,\alpha} + F_3^A &= \rho h u_{3,tt}^A \end{aligned} \quad (5.2)$$

where the bending moments $M_{\alpha\beta}$, shear forces Q_α and domain forces F_j^A are defined as follows:

$$\begin{aligned} M_{\alpha\beta} &= \int_{-h/2}^{h/2} \sigma_{\alpha\beta} x_3 dx_3 = \frac{1-\nu}{2} D \left(u_{\alpha,\beta}^A + u_{\beta,\alpha}^A + \frac{2\nu}{1-\nu} u_{\gamma,\gamma}^A \delta_{\alpha\beta} \right) \\ Q_\alpha &= \int_{-h/2}^{h/2} \sigma_{\alpha 3} dx_3 = \kappa_A^2 \mu h (u_\alpha^A + u_{3,\alpha}^A) \\ F_\alpha^A &= \frac{h}{2} \left(\sigma_{3\alpha}|_{x_3=h/2} + \sigma_{3\alpha}|_{x_3=-h/2} \right) \\ F_3^A &= \sigma_{33}|_{x_3=h/2} - \sigma_{33}|_{x_3=-h/2} \end{aligned} \quad (5.3)$$

in which $D = Eh^3/[12(1-\nu^2)]$ and κ_A is an adjustable shear factor.

The generalized bending and shear tractions at a plate boundary point are defined as: $t_\alpha^A = M_{\alpha\beta} n_\beta$, $t_3^A = Q_\beta n_\beta$, in which n_β denotes the components of a unit outward normal vector to this point.

The equilibrium equations in terms of displacements can be written as follows:

$$\begin{aligned} \frac{D}{2} \left[(1 - \nu) \nabla^2 u_\alpha^A + (1 + \nu) u_{\beta, \beta \alpha}^A \right] - \kappa_A^2 \mu h (u_{3, \alpha}^A + u_\alpha^A) + F_\alpha^A &= \frac{\rho h^3}{12} u_{\alpha, tt}^A \\ \kappa_A^2 \mu h (\nabla^2 u_3^A + u_{\alpha, \alpha}^A) + F_3^A &= \rho h u_{3, tt}^A \end{aligned} \quad (5.4)$$

In the Laplace transform domain, the generalized displacements (rotations $\tilde{u}_\alpha^A$ and out-of-plane displacement $\tilde{u}_3^A$) at a field point $\mathbf{X}'$ in the plate domain Ω are governed by the following integral equation:

$$\begin{aligned} \tilde{u}_i^A(\mathbf{X}', p) &= \int_\Gamma \tilde{U}_{ij}^A(\mathbf{X}', \mathbf{x}, p) \tilde{t}_j^A(\mathbf{x}, p) d\Gamma - \int_\Gamma \tilde{T}_{ij}^A(\mathbf{X}', \mathbf{x}, p) \tilde{u}_j^A(\mathbf{x}, p) d\Gamma \\ &+ \int_\Omega \tilde{U}_{ij}^A(\mathbf{X}', \mathbf{X}, p) \tilde{F}_j^A(\mathbf{X}, p) d\Omega \end{aligned} \quad (5.5)$$

where $\mathbf{x}$ denotes any point on the boundary Γ ; $\tilde{U}_{ij}^A$ and $\tilde{T}_{ij}^A$ are dynamic fundamental solutions listed in Appendix A.2; p stands for the Laplace transform parameter and has been defined in Eq. (2.28).

When the domain point $\mathbf{X}'$ is moved to the boundary Γ , denoted by $\mathbf{x}'$, Eq. (5.5) becomes the displacement boundary integral equation (DBIE) and can be written as:

$$\begin{aligned} c_{ij}(\mathbf{x}') \tilde{u}_j^A(\mathbf{x}', p) &= \int_\Gamma \tilde{U}_{ij}^A(\mathbf{x}', \mathbf{x}, p) \tilde{t}_j^A(\mathbf{x}, p) d\Gamma - \oint_\Gamma \tilde{T}_{ij}^A(\mathbf{x}', \mathbf{x}, p) \tilde{u}_j^A(\mathbf{x}, p) d\Gamma \\ &+ \int_\Omega \tilde{U}_{ij}^A(\mathbf{x}', \mathbf{X}, p) \tilde{F}_j^A(\mathbf{X}, p) d\Omega \end{aligned} \quad (5.6)$$

where $\oint$ stands for Cauchy principal value integral and the free term c_{ij} is equal to $\delta_{ij}/2$ when $\mathbf{x}'$ is located on a smooth boundary.

In DBEM, DBIE is used for the modelling of one crack surface, while the other crack surface is modelled using the traction boundary integral equation (TBIE). Utilizing the DBIE and the stress-displacement relationship (see Eq. (5.3)), the TBIE for a smooth boundary is obtained

as follows:

$$\begin{aligned}
\frac{1}{2}\tilde{t}_\alpha^A(\mathbf{x}', p) &= n_\beta(\mathbf{x}') \left[\oint_\Gamma \tilde{D}_{\alpha\beta\gamma}^A(\mathbf{x}', \mathbf{x}, p) \tilde{t}_\gamma^A(\mathbf{x}, p) d\Gamma + \int_\Gamma \tilde{D}_{\alpha\beta 3}^A(\mathbf{x}', \mathbf{x}, p) \tilde{t}_3^A(\mathbf{x}, p) d\Gamma \right. \\
&\quad - \oint_\Gamma \tilde{S}_{\alpha\beta\gamma}^A(\mathbf{x}', \mathbf{x}, p) \tilde{u}_\gamma^A(\mathbf{x}, p) d\Gamma - \oint_\Gamma \tilde{S}_{\alpha\beta 3}^A(\mathbf{x}', \mathbf{x}, p) \tilde{u}_3^A(\mathbf{x}, p) d\Gamma \\
&\quad \left. + \int_\Omega \tilde{D}_{\alpha\beta k}^A(\mathbf{x}', \mathbf{X}, p) \tilde{F}_k^A(\mathbf{X}, p) d\Omega \right] \\
\frac{1}{2}\tilde{t}_3^A(\mathbf{x}', p) &= n_\beta(\mathbf{x}') \left[\int_\Gamma \tilde{D}_{3\beta\gamma}^A(\mathbf{x}', \mathbf{x}, p) \tilde{t}_\gamma^A(\mathbf{x}, p) d\Gamma + \oint_\Gamma \tilde{D}_{3\beta 3}^A(\mathbf{x}', \mathbf{x}, p) \tilde{t}_3^A(\mathbf{x}, p) d\Gamma \right. \\
&\quad - \oint_\Gamma \tilde{S}_{3\beta\gamma}^A(\mathbf{x}', \mathbf{x}, p) \tilde{u}_\gamma^A(\mathbf{x}, p) d\Gamma - \oint_\Gamma \tilde{S}_{3\beta 3}^A(\mathbf{x}', \mathbf{x}, p) \tilde{u}_3^A(\mathbf{x}, p) d\Gamma \\
&\quad \left. + \int_\Omega \tilde{D}_{3\beta k}^A(\mathbf{x}', \mathbf{X}, p) \tilde{F}_k^A(\mathbf{X}, p) d\Omega \right]
\end{aligned} \tag{5.7}$$

where $\oint$ stands for Hadamard finite-part integral and the dynamic fundamental solutions $\tilde{D}_{i\beta k}^A$, $\tilde{S}_{i\beta k}^A$ are given in Appendix A.2.

5.3 Numerical implementation

In the DBEM for the Mindlin theory, the discretization of plate boundaries including edges and crack surfaces is achieved based on 1D line elements, which are also used for the 2D DBEM. Thus, the discretization strategies introduced in Section 2.4, Chapter 2 for the 2D DBEM can be readily adopted here. What need to be paid more attention is the singular behaviour of the dynamic fundamental solutions for the Mindlin theory.

It is worth mentioning that the dynamic fundamental solutions $\tilde{U}_{ij}^A$ and $\tilde{T}_{ij}^A$ have been found to have the same singularities as their static counterparts in both isotropic problems [4, 59] and anisotropic problems [116, 117]. Thus, it is anticipated that this is also true for the dynamic problems in Mindlin plates. However, Ref. [82] showed that the dynamic fundamental solutions $\tilde{U}_{3\alpha}^A$ and $\tilde{T}_{3\alpha}^A$ did not have this behaviour and the coefficients of their singular terms included the Laplace transform parameter p . After manipulating these coefficients, it has been found that the Laplace parameter in the coefficients can be cancelled, and the kernels $\tilde{U}_{3\alpha}^A$ and $\tilde{T}_{3\alpha}^A$ still have the same singularities as those in the static kernels. Although this simplification does not

change the values of the kernels and the final DBEM results, it gives a correct understanding of the dynamic kernels. When it comes to the dynamic fundamental solutions $\tilde{D}_{k\beta j}^A$ and $\tilde{S}_{k\beta j}^A$, the dominant singular terms are also found to be the same as those in the static fundamental solutions. The detailed expressions of the above-mentioned singular terms can be found in Appendix C.2.

It should be mentioned that the dynamic fundamental solutions $\tilde{S}_{\alpha\beta\gamma}^A$ and $\tilde{S}_{3\beta 3}^A$ contain both hyper-singularities of order $1/r^2$ (dominant singularities) and weak singularities of order $\ln(r)$ (secondary singularities). The hyper-singularities are the same as those for static problems. The weak singularities involve the Laplace transform parameter p , so their expressions are different from those for static problems.

On the basis of the above discussion, all the dynamic fundamental solutions can be written in two parts:

$$\begin{aligned}\tilde{U}_{ij}^A(\mathbf{x}', \mathbf{x}, p) &= U_{ij}^{A|\text{static}}(\mathbf{x}', \mathbf{x}) + \tilde{U}_{ij}^{A|m}(\mathbf{x}', \mathbf{x}, p) \\ \tilde{T}_{ij}^A(\mathbf{x}', \mathbf{x}, p) &= T_{ij}^{A|\text{static}}(\mathbf{x}', \mathbf{x}) + \tilde{T}_{ij}^{A|m}(\mathbf{x}', \mathbf{x}, p) \\ \tilde{D}_{k\beta j}^A(\mathbf{x}', \mathbf{x}, p) &= D_{k\beta j}^{A|\text{static}}(\mathbf{x}', \mathbf{x}) + \tilde{D}_{k\beta j}^{A|m}(\mathbf{x}', \mathbf{x}, p) \\ \tilde{S}_{k\beta j}^A(\mathbf{x}', \mathbf{x}, p) &= S_{k\beta j}^{A|\text{static}}(\mathbf{x}', \mathbf{x}) + \tilde{S}_{k\beta j}^{A|m}(\mathbf{x}', \mathbf{x}, p)\end{aligned}\tag{5.8}$$

where $U_{ij}^{A|\text{static}}$, $T_{ij}^{A|\text{static}}$, $D_{k\beta j}^{A|\text{static}}$ and $S_{k\beta j}^{A|\text{static}}$ are the fundamental solutions for the static Mindlin theory (see Appendix B.2); $\tilde{U}_{ij}^{A|m}$, $\tilde{T}_{ij}^{A|m}$, $\tilde{D}_{k\beta j}^{A|m}$ and $\tilde{S}_{k\beta j}^{A|m}$ are modified terms without any singularities, except for $\tilde{S}_{\alpha\beta\gamma}^{A|m}$ and $\tilde{S}_{3\beta 3}^{A|m}$ containing the weak singularities of order $\ln(r)$.

The corresponding singular integrals can also be divided into two parts: static integrals including the static kernels and dynamic integrals including the modified terms. At non-crack boundaries, singularities only exist in static integrals and they can be determined using the generalized rigid body movements proposed by Rashed et al. [118]. At crack surfaces, singular static integrals are evaluated analytically, which has been introduced in detail in Ref. [53]. When it comes to the dynamic integrals, the weak singularities in $\tilde{S}_{\alpha\beta\gamma}^{A|m}$ and $\tilde{S}_{3\beta 3}^{A|m}$ can be

tackled using the bi-cubic transformation of variable technique proposed by Telles [86].

5.4 Estimation of dynamic stress intensity factors

Three antisymmetric fracture modes (see Fig. 5.1) are described in the Mindlin theory. The antisymmetric modes I and II contribute to Mode I and Mode II, respectively, but they are antisymmetric with respect to the plate mid-plane. The antisymmetric mode III can be linked to the basic fracture mode, Mode III, by assuming that the transverse shear stresses vary parabolically over the thickness.

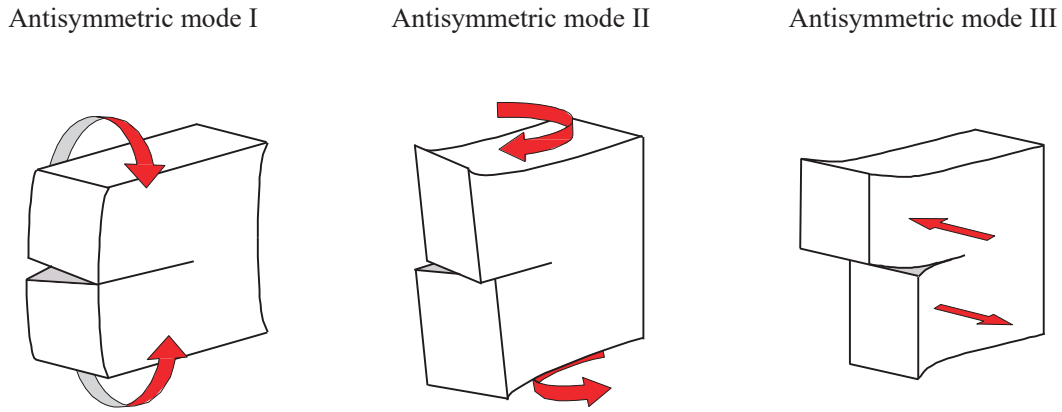


Figure 5.1: Antisymmetric fracture modes for the Mindlin theory.

After the stress resultant intensity factors K_I^A , K_{II}^A and K_{III}^A are obtained, the stress intensity factors K_I , K_{II} and K_{III} can be computed using the following relationship:

$$K_I = \frac{12x_3}{h^3} K_I^A, \quad K_{II} = \frac{12x_3}{h^3} K_{II}^A, \quad K_{III} = \frac{3}{2h} \left[1 - \left(\frac{2x_3}{h} \right)^2 \right] K_{III}^A \quad (5.9)$$

Thus, the dynamic stress resultant intensity factors are focused on in this chapter.

The crack-tip polar coordinate system shown in Fig. 2.9 is adopted here. According to Ref. [119], the dynamic stress resultant intensity factors can be estimated based on the generalized

crack opening displacement (COD) in the Mindlin plates as follows:

$$\begin{aligned}
K_I^A &= \frac{Eh^3}{96} \sqrt{\frac{2\pi}{r}} \Delta u_2^A(r) = \frac{Eh^3}{96} \sqrt{\frac{2\pi}{r}} [u_2^A(r, \theta = \pi) - u_2^A(r, \theta = -\pi)] \\
K_{II}^A &= \frac{Eh^3}{96} \sqrt{\frac{2\pi}{r}} \Delta u_1^A(r) = \frac{Eh^3}{96} \sqrt{\frac{2\pi}{r}} [u_1^A(r, \theta = \pi) - u_1^A(r, \theta = -\pi)] \\
K_{III}^A &= \frac{5Eh}{24(1+\nu)} \sqrt{\frac{\pi}{2r}} \Delta u_3^A(r) = \frac{5Eh}{24(1+\nu)} \sqrt{\frac{\pi}{2r}} [u_3^A(r, \theta = \pi) - u_3^A(r, \theta = -\pi)]
\end{aligned} \tag{5.10}$$

The discontinuous quarter-point element (see Fig. 2.12) was used as the crack-tip element in order to accurately capture the the singular fields in the vicinity of the crack tip. The dynamic stress resultant intensity factors can be evaluated on the basis of the generalized COD at the node pair closest to the crack tip ($r = l_e/36$ and l_e being the crack-tip element length):

$$\begin{aligned}
K_I^A &= \frac{Eh^3}{16} \sqrt{\frac{2\pi}{l_e}} \Delta u_2^A(r = l_e/36) \\
K_{II}^A &= \frac{Eh^3}{16} \sqrt{\frac{2\pi}{l_e}} \Delta u_1^A(r = l_e/36) \\
K_{III}^A &= \frac{5Eh}{8(1+\nu)} \sqrt{\frac{2\pi}{l_e}} \Delta u_3^A(r = l_e/36)
\end{aligned} \tag{5.11}$$

When it comes to the finite element analysis, which was carries out to validate the BEM results, continuous quarter-point crack-tip elements were used. For this type of elements, r is equal to $l_e/4$ and the dynamic stress resultant intensity factors can be computed as follows:

$$\begin{aligned}
K_I^A &= \frac{Eh^3}{48} \sqrt{\frac{2\pi}{l_e}} \Delta u_2^A(r = l_e/4) \\
K_{II}^A &= \frac{Eh^3}{48} \sqrt{\frac{2\pi}{l_e}} \Delta u_1^A(r = l_e/4) \\
K_{III}^A &= \frac{5Eh}{24(1+\nu)} \sqrt{\frac{2\pi}{l_e}} \Delta u_3^A(r = l_e/4)
\end{aligned} \tag{5.12}$$

5.5 Examples

In this section, the accuracy of the dual boundary element formulation was first validated in the case of a crack in an infinite plate under bending loads, followed by a crack in a finite-

sized plate under bending and a cantilever plate subjected to transverse shear loading. Next, a more complex fracture problem, mixed-mode fracture, was solved using this dual formulation. Finally, the ability of this formulation to model the interaction between the A0-mode Lamb wave and a crack was shown.

The software, ANSYS Mechanical, was used for FEM analysis. The finite element mesh consisted of 8-noded shell elements (Shell 281), which can be modified slightly to generate quarter-point crack-tip elements. In the following examples, 200 Laplace terms, which is more than the minimum number necessary to reach convergence over the time period of interest, were used for the numerical inversion of Laplace transforms (see Eq. (2.40)). The time t is normalized with respect to l_c/c_2 , where l_c is a characteristic length and c_2 is the shear wave velocity. The origin of the coordinate system is located at the centre of the plate.

5.5.1 An infinite plate with a central crack

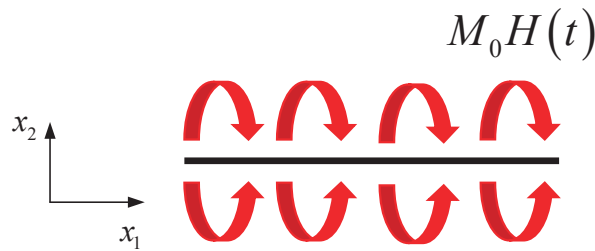


Figure 5.2: A crack subjected to bending load.

Fig. 5.2 shows a through-thickness crack in an infinite plate. The crack length and plate thickness are denoted by $2a$ and h , respectively. The crack surfaces are subjected to uniform bending moments $M_0H(t)$, where M_0 is the load amplitude and $H(t)$ denotes the Heaviside function. The material properties are the Young's modulus, $E = 69$ GPa, the Poisson's ratio, $\nu = 0.25$, and the material density, $\rho = 290$ kg/m³. In DBEM, since the non-reflecting boundary conditions are satisfied automatically, only the crack should be considered and each crack surface was discretized with 50 boundary elements.

This dynamic fracture problem has been solved by Wen et al. [3] using a displacement dis-

continuity method. Fig. 5.3 compares the normalized stress resultant intensity factor K_I^A/K_{01} , where K_{01} is equal to $M_0\sqrt{\pi a}$, computed by Wen et al. and that from the DBEM simulation. It is evident that such factor obtained using the proposed DBEM is in excellent agreement with existing results. It should be pointed out that Wen's method in Ref. [3] is only suitable for infinite plates, while the proposed DBEM is suitable for both infinite and finite plates.

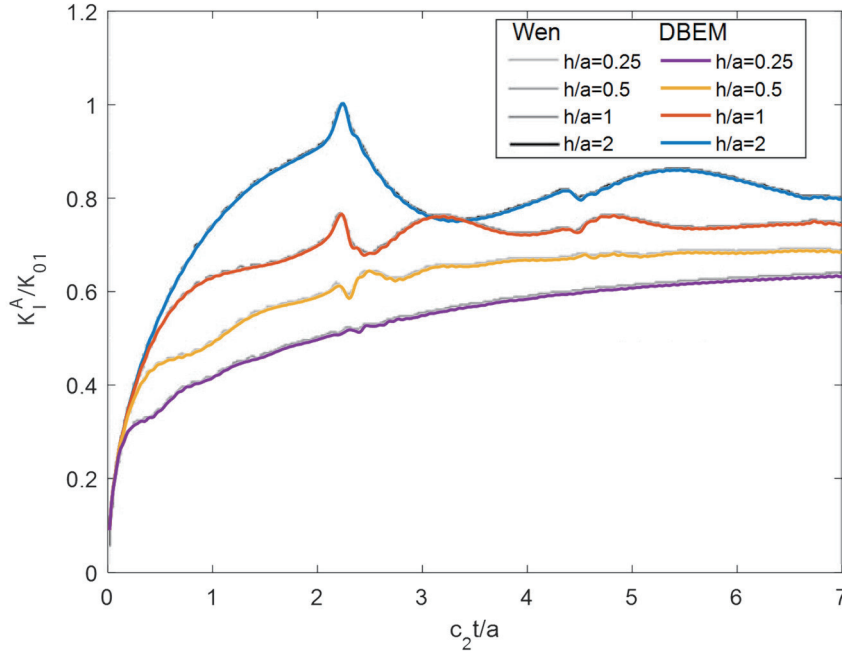


Figure 5.3: Comparison between the normalized dynamic stress resultant intensity factor obtained by DBEM and that published by Wen [3] for the infinite plate with a central crack.

5.5.2 A square plate with a central crack

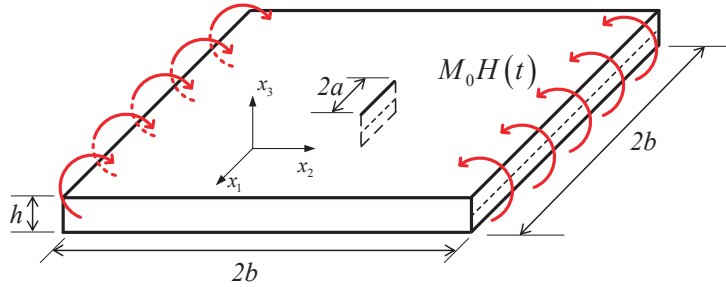


Figure 5.4: A square plate with a central crack.

A square plate with a central crack (Fig. 5.4) is loaded by edge bending moment $M_0 H(t)$. The material properties are: $E = 210$ GPa, $\nu = 0.3$ and $\rho = 290$ kg/m³. The plate dimensions are: edge length $2b = 2$ m, crack length $2a = 0.2$ m and plate thickness $h = 0.2$ m.

In the DBEM simulation, the plate boundaries were discretized with 220 boundary elements (40 elements for each edge and 30 elements for each crack surface). In the finite element analysis, in order to reduce computational burden, it was only necessary to use one quarter of the plate, modelled by 10,429 shell elements, because of the biaxial symmetry.

According to Thau et al. [120], the time evolution of the dynamic stress intensity factors depend on the interaction between elastic waves and the crack. In the Mindlin theory, three types of antisymmetric waves, slow wave (approximation to the A0 Lamb mode), fast wave (approximation to the A1 Lamb mode) and shear wave (approximation to the SH1 mode), can be described.

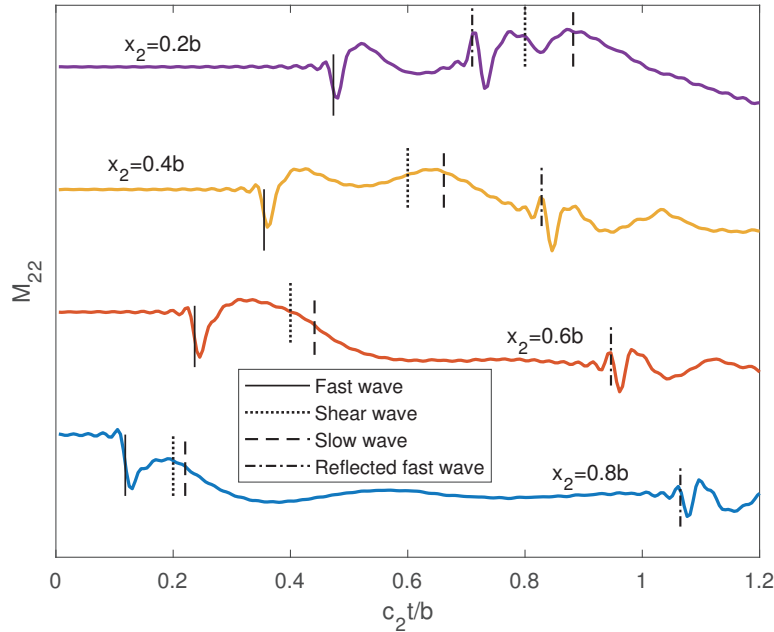


Figure 5.5: Moment M_{22} at four different points along the x_2 -axis.

Fig. 5.5 shows how the stress-resultant bending moments $M_{22} = \int_{-h/2}^{h/2} x_3 \sigma_{22} dx_3$ at four reference points along the x_2 -axis vary in a short time period, and this variation reflects the

propagation of the antisymmetric waves in the plate. After the bending moments are applied on the two edges, the three types of antisymmetric waves are generated and then propagate towards the crack. The fast wave reaches the crack surface first, and then is reflected and propagates along the opposite direction. These processes can be seen in Fig. 5.5.

The dynamic stress resultant intensity factor K_I^A which is normalized with respect to $K_{01} = M_0\sqrt{a}$ is presented in Fig. 5.6. It is clearly shown that the result from the boundary element analysis agree very well with the ANSYS result. The peak values obtained using the DBEM and the FEM are virtually identical.

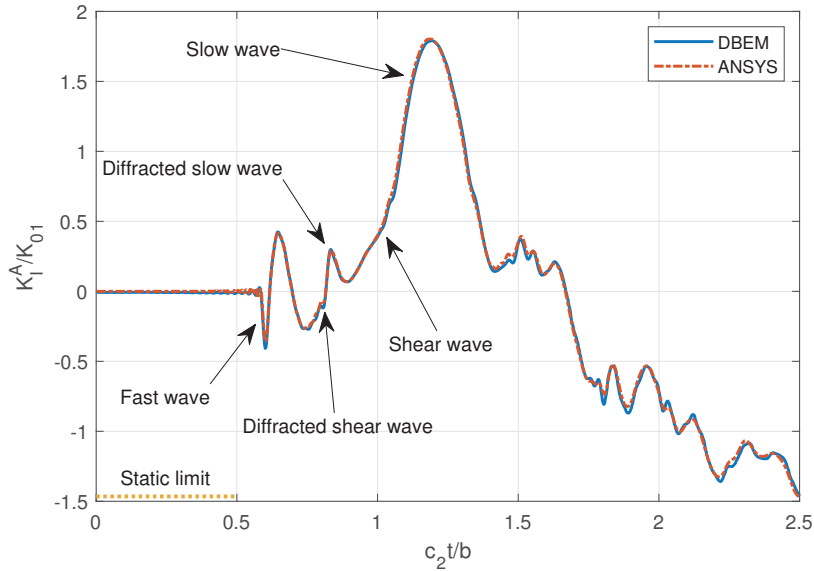


Figure 5.6: Normalized dynamic stress resultant intensity factor K_I^A/K_{01} for the square plate with a central crack.

Some kinks and wave peaks which can be seen in Fig. 5.6 are caused by the three antisymmetric waves originating from the plate edges where the bending moment is applied, and the diffracted waves resulting from wave-crack interaction. The effect of the slow wave on K_I^A is greater than that of the fast wave and the shear wave. Moreover, the peak value due to the slow wave is higher than the absolute value of the static limit. The effect of the shear wave is very weak, and can be disregarded.

When the fast wave reaches one crack tip, three types of diffracted waves, diffracted fast

wave, diffracted shear wave and diffracted slow wave, are generated. The stress resultant intensity factor measured at the other crack tip is affected after the diffracted waves travel to this tip. In comparison to the fast wave starting from the plate edges, the diffracted slow and shear waves have less influence on the stress resultant intensity factor. Moreover, the effect of the diffracted fast wave is negligible.

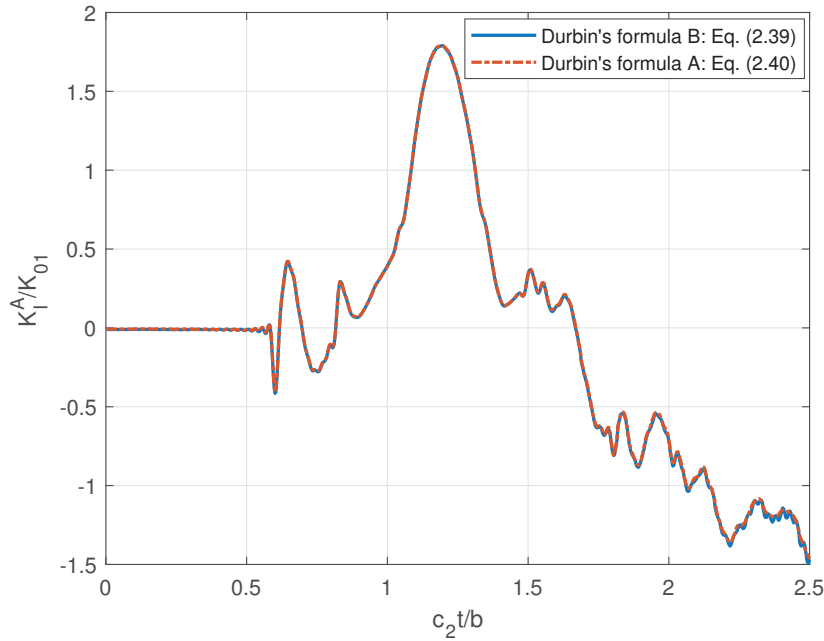


Figure 5.7: Normalized dynamic stress resultant intensity factor computed using different Durbin's Laplace-transform-inversion formulas.

It is worth mentioning that in order to ensure convergence, more Laplace terms and finer mesh were used in the boundary element simulation than strictly necessary. When large engineering structures are considered, in order to increase the time efficiency, the numerical inversion of Laplace transforms can be conducted using another formula (see Eq. (2.39)). Both formula A from Eq. (2.40) and formula B from Eq. (2.39) were proposed by Durbin [75]. The normalized dynamic stress resultant intensity factors K_I^A/K_{01} obtained using formula A and formula B are compared in Fig. 5.7. The advantage of formula B is that fewer Laplace terms are required, and therefore computational time can be saved. The curves in Fig. 5.7 show that the time-domain results based on formula B are as accurate as those from formula A, except for a short period

near the end time. In this example, the number of Laplace terms used in formula B is 100, which is only half the number of Laplace terms used in formula A.

The drawback of formula B is that the results near the end time are not accurate. It can be seen from Fig. 5.7 that the results obtained using formula B are oscillatory near the end time. This problem still occurs even though the number of Laplace terms increases. Thus, in order to compare the BEM results with the FEM results over the whole period, formula B is not used in this section. If the results toward the end of the time period are not of interest, formula B would be the better choice due to higher time efficiency.

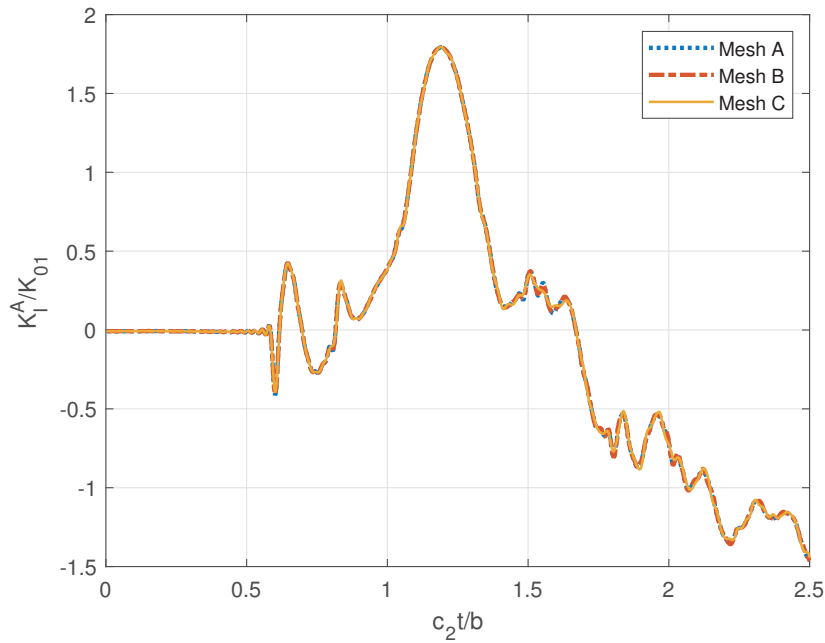


Figure 5.8: Normalized dynamic stress resultant intensity factor obtained using different boundary element meshes.

The number of boundary elements can also be reduced if some minor details are not necessary and can be disregarded. Fig. 5.8 shows the normalized dynamic stress resultant factor K_I^A/K_{01} obtained using three different meshes. The detailed mesh information is listed in Table 5.1. As can be seen in Fig. 5.8, although mesh C is insufficient to reach convergence over the entire time period, the major features can be captured based on it. Therefore, if some weak waves are not important, mesh C, a coarser mesh, can be used to further reduce computational time.

Table 5.1: The number of boundary elements for three different meshes.

	Mesh A	Mesh B	Mesh C
Number of elements on each edge	80	40	20
Number of elements on each crack surface	50	30	20

5.5.3 A cantilever rectangular plate with a central crack

Fig. 5.9 shows a cantilever rectangular plate of dimensions 2 m ($2b$) $\times$ 1 m ($2c$) $\times$ 0.1 m (h) with a 0.1 m ($2a$) long central through-thickness crack. A uniformly distributed shear force $Q_0 H(t)$, where Q_0 is the load amplitude, is applied on the plate edge opposite the fixed edge. The plate is made of aluminium, which has a Young's modulus of 70 GPa, a Poisson's ratio (ν) of 0.33 and a density of 2700 kg/m³.

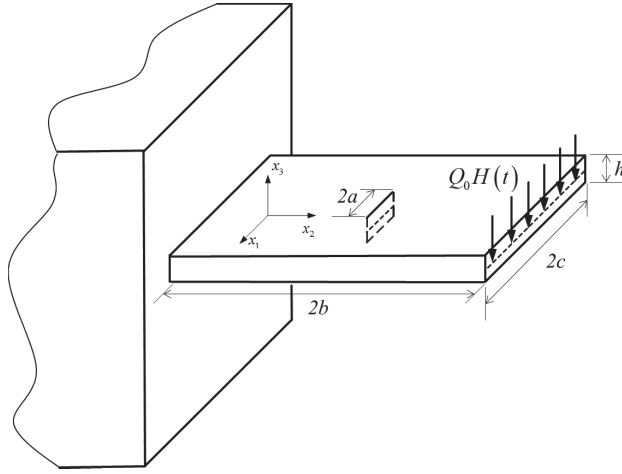


Figure 5.9: A cantilever rectangular plate with a central crack.

The number of boundary elements used for discretizing each crack surface, long edge and short edge of the plate were 20, 60 and 30, respectively. Due to geometric and loading symmetry, the FEM results were obtained based on half plate discretized with 10,320 shell elements.

Fig. 5.10 shows the time history of the shearing stress resultant $Q_2 = \int_{-h/2}^{h/2} \sigma_{23} dx_3$ at four reference points along the x_2 -axis, from which the propagation of the antisymmetric waves can be observed.

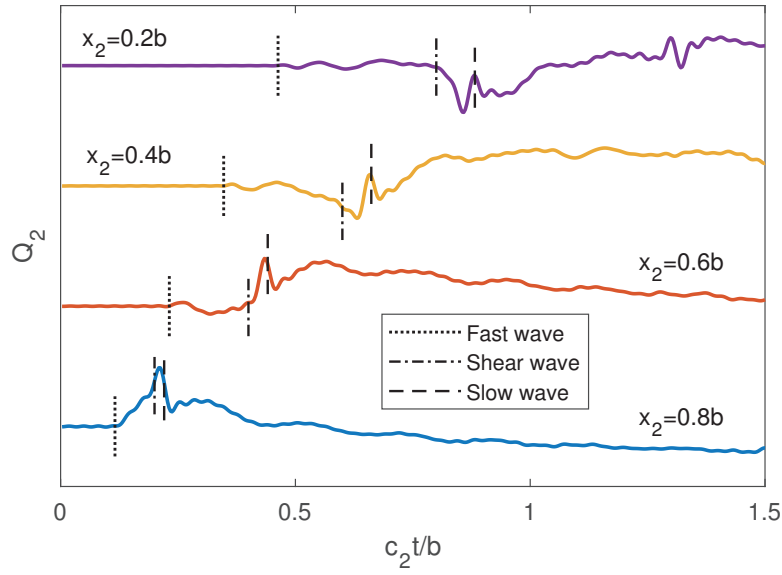


Figure 5.10: Shear force Q_2 at four different points along the x_2 -axis.

Figs. 5.11 and 5.12 present the normalized dynamic stress resultant intensity factors K_I^A/K_{01} and K_{III}^A/K_{03} respectively, where $K_{01} = 2Q_0c\sqrt{a}$ and $K_{03} = 2Q_0c\sqrt{10a}/[(1+\nu)h]$. It can be seen from these figures that the DBEM results agree very well with the FEM results. The effect of the fast wave on the DSIFs is the weakest among the three types of antisymmetric waves.

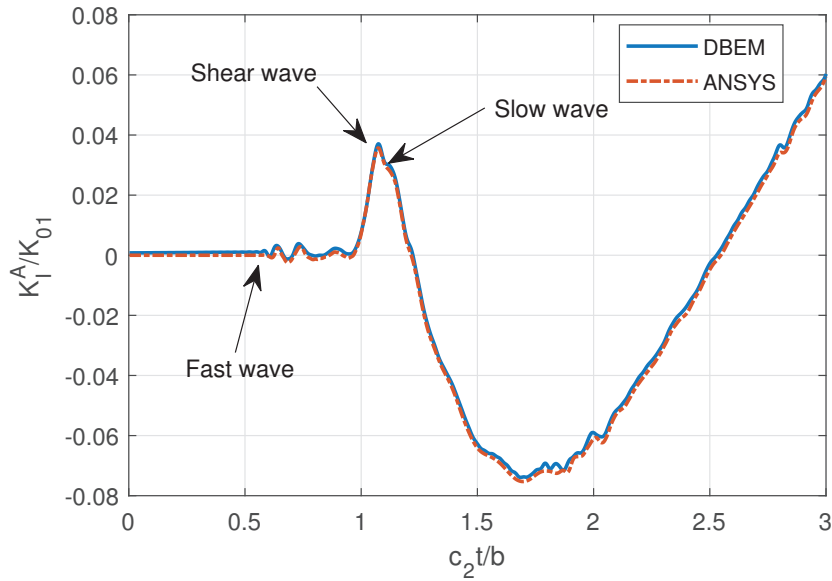


Figure 5.11: Normalized dynamic stress resultant intensity factor K_I^A/K_{01} for the cantilever plate with a central crack.

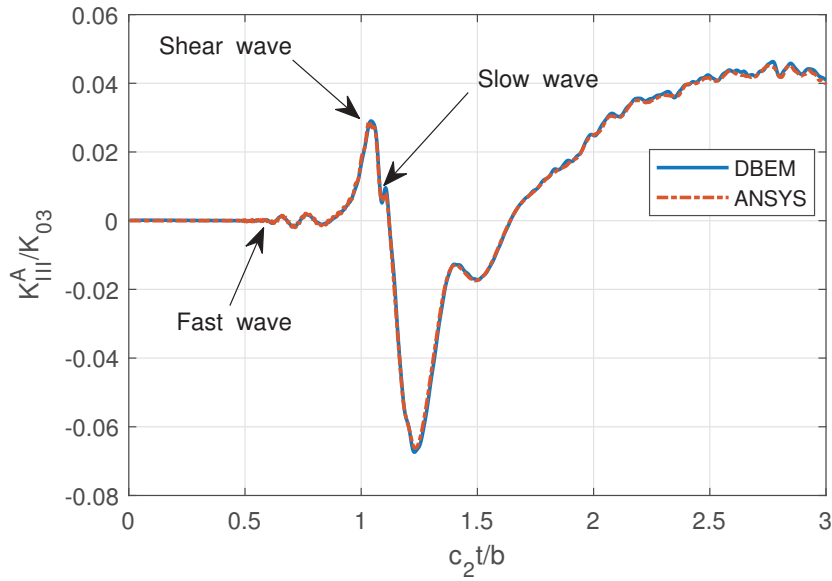


Figure 5.12: Normalized dynamic stress resultant intensity factor K_{III}^A/K_{03} for the cantilever plate with a central crack.

5.5.4 A square plate with a slant central crack

A mixed-mode crack problem is considered in this example. A cracked square plate loaded by edge bending $M_0 H(t)$, which has a length of $2b = 1$ m and a thickness $h = 0.1$ m, is shown in Fig. 5.13. The central crack has a length $2a = 0.1\sqrt{2}$ m and makes an angle of $\theta = \pi/4$ with the direction x_2 . The material properties are the same as those in the second example.

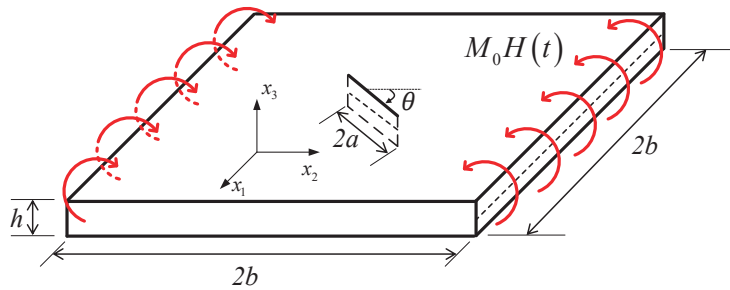


Figure 5.13: A square plate with a slant central crack.

In the dual boundary element analysis, 220 boundary elements were used, including 30 elements on each crack surface. When it comes to the finite element analysis, the whole plate was discretized with 9,977 shell elements.

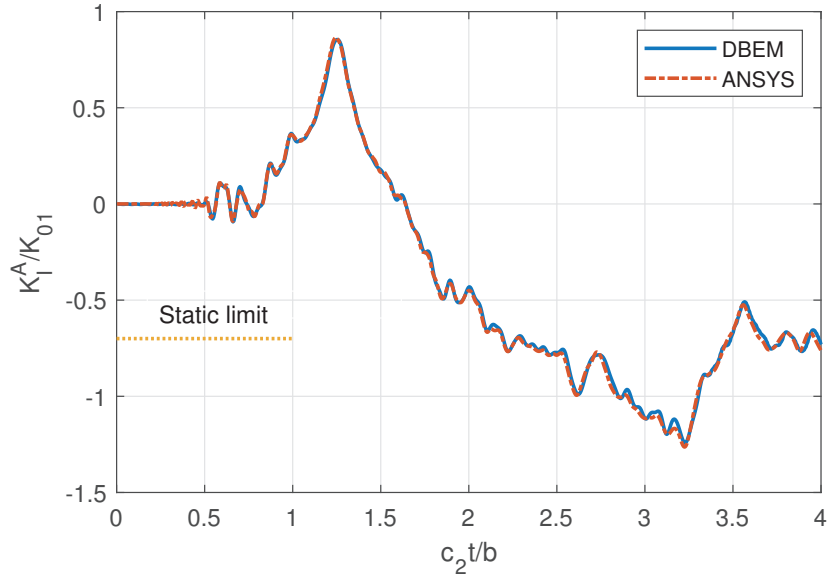


Figure 5.14: Normalized dynamic stress resultant intensity factor K_I^A/K_{01} for the square plate with a slant central crack.

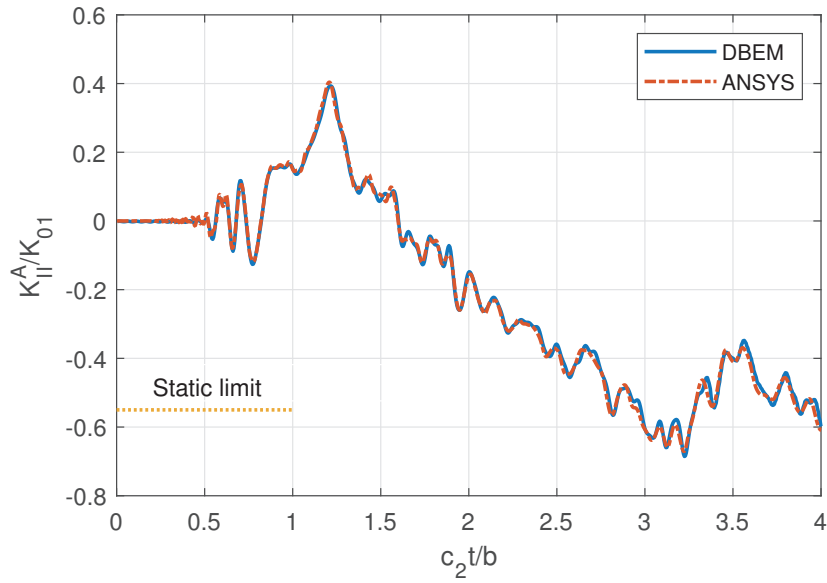


Figure 5.15: Normalized dynamic stress resultant intensity factor K_{II}^A/K_{01} for the square plate with a slant central crack.

The dynamic stress resultant intensity factors K_I^A and K_{II}^A are normalized with respect to $K_{01} = M_0\sqrt{a}$, while K_{III}^A is normalized with respect to $K_{03} = M_0\sqrt{10a}/[(1+\nu)h]$. Figs. 5.14 to 5.16 present the normalized dynamic stress resultant intensity factors K_I^A/K_{01} , K_{II}^A/K_{02} and

K_{III}^A/K_{03} , respectively. As shown in these figures, the DBEM results are in excellent agreement with the FEM results.

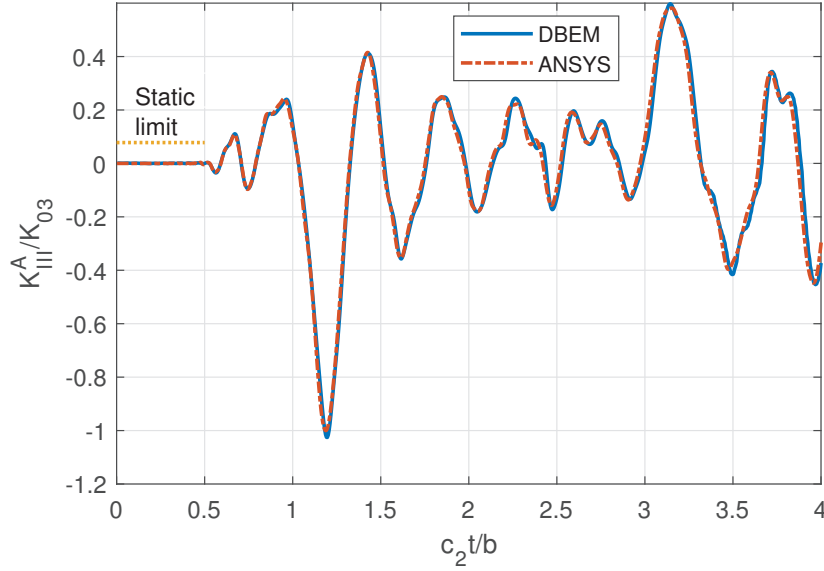


Figure 5.16: Normalized dynamic stress resultant intensity factor K_{III}^A/K_{03} for the square plate with a slant central crack.

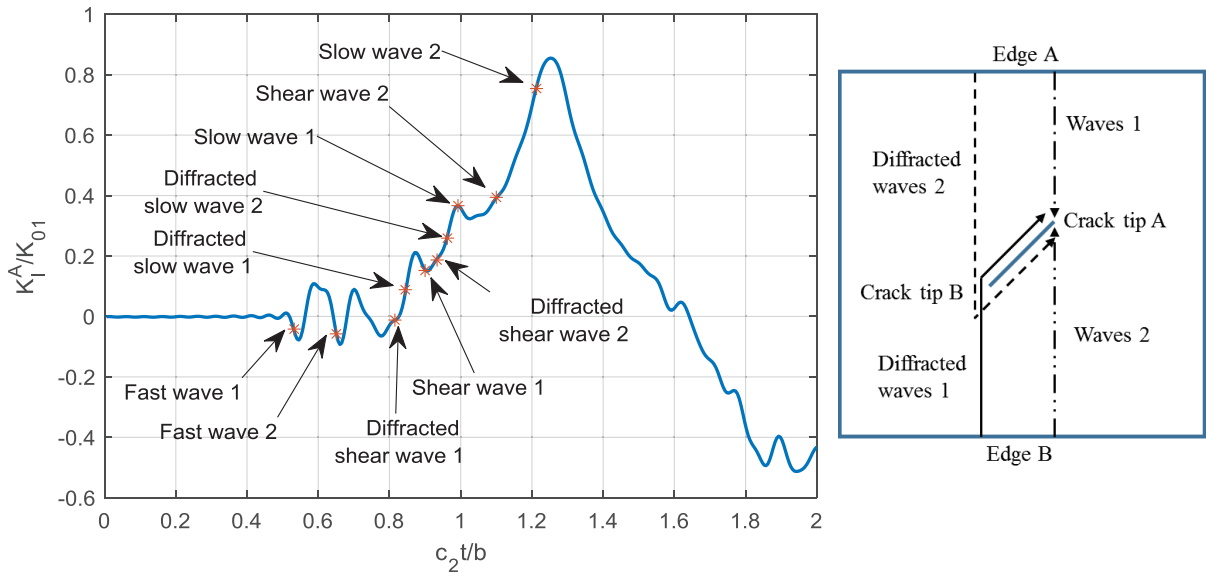


Figure 5.17: The effect of antisymmetric waves on the normalized dynamic stress resultant intensity factor K_I^A/K_{01} (Crack tip A).

Fig. 5.17 illustrates the effect of antisymmetric waves on the normalized dynamic stress resultant intensity factor K_I^A/K_{01} which is assessed at the crack tip A. Four groups of waves,

waves 1, waves 2, diffracted waves 1 and diffracted waves 2, can be observed at the initial stage. Waves 1 starting from the edge A contains fast wave 1, shear wave 1 and slow wave 1. Waves 2 is also composed of these three types of waves which start from the edge B. When the fast wave from the edge B reaches the crack tip B, diffracted waves 1 which includes diffracted shear wave 1 and diffracted slow wave 1 is generated and travels to the crack tip A. Similar to the diffracted waves 1, diffracted waves 2, including diffracted shear wave 2 and diffracted slow wave 2, is caused by the interaction between the fast wave from the edge A and the crack tip B.

The group velocities of the three antisymmetric wave packets (fast wave, shear wave and slow wave) generated by the impact load have been given in Ref. [82] and can be computed as follows: $c_{\text{fast}} = \sqrt{2/(1-\nu)}c_2$, $c_{\text{shear}} = c_2$ and $c_{\text{slow}} = \sqrt{\pi^2/12}c_2$. The arrival times of these waves at the crack tip A are able to be estimated by dividing the propagation distance of the waves by their group velocities and these time points are marked by red asterisks in Fig. 5.17.

5.5.5 A0 mode wave scattering by a crack

Since the A0 mode Lamb wave can be described quite well using the Mindlin theory, the proposed DBEM is also able to be used for the analysis of wave (A0 mode) scattering by the crack.

In order to observe the wave-scattering phenomenon clearly, an infinite plate is taken into consideration to avoid the crack-scattered waves being blurred by the reflected waves from plate edges. Fig. 5.18 shows a 100-mm-long through-thickness crack in a 2-mm-thick infinite aluminium plate, as well as the surface tractions which can excite omnidirectional A0 mode. Two circular actuation regions on the plate surfaces are symmetric with respect to the plate mid-plane. The line tractions F_r applied on the circumference of the regions are antisymmetric with respect to the mid-plane, which results in a pure bending effect ensuring that only antisymmetric Lamb waves, rather than the symmetric modes, are generated. Such tractions are in the form of a 5-cycle Hann-window-modulated sinusoidal tone burst with a central frequency of 500 kHz

(see Eq. (2.25)), at which only one type of the antisymmetric waves, A0 mode wave, is excited. The plate has a density of 2700 kg/m^3 , a Poisson's ratio of 0.33 and a Young's modulus of 69 GPa.

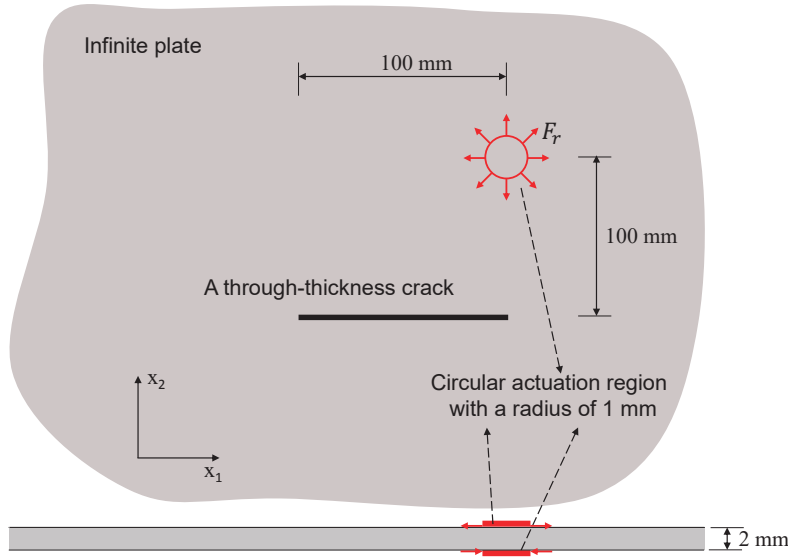


Figure 5.18: A through-thickness crack in an infinite plate with two circular actuation regions generating omnidirectional A0 mode.

For this cracked infinite plate, each crack surface was discretized with 60 boundary elements. In order to reduce the computational time, the numerical inversion of Laplace transforms was conducted by Durbin's formula B (Eq. (2.39)) and 70 Laplace terms were sufficient to provide satisfactory results.

The propagation of the excited A0 mode and its interaction with the crack are visualised in Fig. 5.19. It can be seen that the A0 mode arrives at the right crack tip first. At this time, a diffracted A0 mode is generated and radiates outward from this tip. Later, when the A0 mode reaches the upper crack surface, it is reflected back. Lastly, the A0 mode travels to the left crack tip, generating a second diffracted A0 mode wave.

The scattered wave of the A0 mode by a crack is composed of the above-mentioned reflected and diffracted A0 waves. Since the crack considered in this example is sufficiently long, these wave components can be distinguished and observed clearly and as a result the scattered wave

field has strong directionality. By contrast, the directionality becomes weaker for a shorter crack because the reflected and diffracted waves tend to overlap.

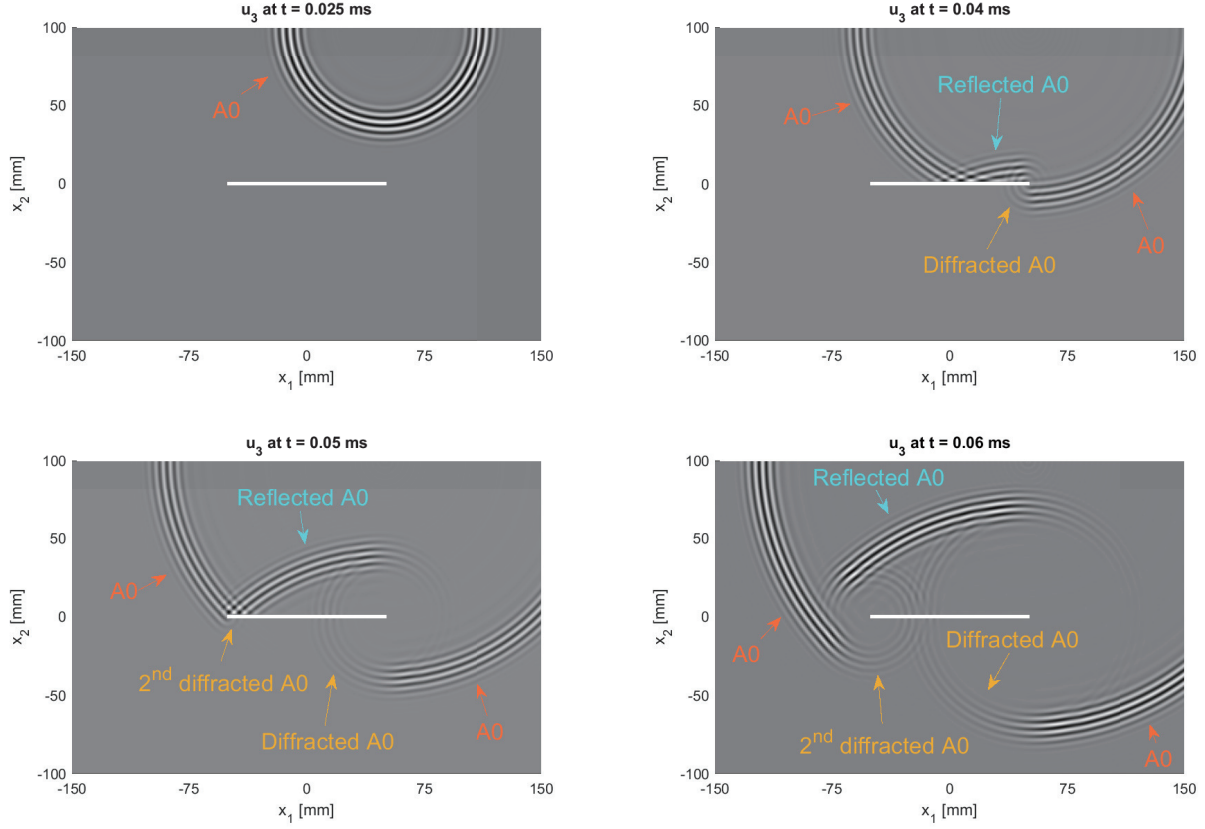


Figure 5.19: Displacement field of a portion of the cracked infinite plate at the actuation frequency of 500 kHz.

5.6 Summary

In this chapter, a new dual boundary element formulation in the Laplace domain has been developed for dynamic analyses of cracked Mindlin plates. Both bending loads and out-of-plane shear loads can be tackled in such formulation. The relationship, which was given in a published paper, between the singularities in the Laplace-transform fundamental solutions and those in the static fundamental solutions is re-examined and corrected.

When it comes to the dynamic fracture problems, crack opening displacements were used to estimate the dynamic stress resultant intensity factors and the results obtained using the

DBEM are in excellent agreement with published results and those computed from finite element analysis. The time evolution of the dynamic stress resultant intensity factors is determined by the three antisymmetric waves which are governed by the Mindlin theory, which was shown and explained in three examples.

The proposed DBEM formulation is also applicable to the studies of the A0-mode Lamb wave scattering by cracks. The wave-crack interaction process, including wave reflection and wave diffraction, was clearly described by one example.

Chapter 6

Boundary element modelling for Lamb-wave structural health monitoring

6.1 Introduction

Ultrasonic Lamb waves have been widely used for structural health monitoring (SHM) over recent decades. Small-sized transducers made of piezoelectric lead-zirconate-titanate (PZT), which are lightweight, inexpensive and could be bonded onto structural surfaces permanently, allow actuating and sensing Lamb waves conveniently [7].

The piezoelectric-mechanical coupled dynamic system can be numerically modelled by a variety of numerical methods. The boundary element method (BEM) has been demonstrated to be efficient and numerically stable for the health monitoring of three-dimensional structures using PZT transducers [71]. In contrast to domain-discretized methods, boundary element meshing is only limited to the domain boundaries with the help of fundamental solutions and as a result significant reduction in the degrees of freedom is achieved. This advantage has also been shown in the modelling of ultrasonic Lamb waves in thin plates [121], which is particularly attractive for high-frequency cases. Furthermore, when various damage scenarios need to be

simulated, only the damage boundary, rather than all domain boundaries, need to be remeshed, resulting in less data preparation.

Since the three-dimensional BEM used in Ref. [71] is computationally inefficient in modelling thin-walled structures, the BEM formulations for plate theories are preferred in terms of modelling Lamb waves, in which only plate edges are required to be meshed using 1D line elements. When it comes to the modelling of cracks, the well-known dual boundary element method (DBEM) [53] is more suitable and allows treating the crack surfaces as extra boundaries. The DBEM provides a general framework, which is applicable to both isotropic and anisotropic materials, and to both static and dynamic problems.

In order to incorporate the PZT transducers into the BEM model, a direct strategy is to model the transducers using a numerical method, such as a semi-analytical finite element method [71], and to solve the system in a fully coupled manner. However, when it comes to the boundary element plate formulations, this introduces additional plate-domain points to couple the transducers to the plate, which significantly deteriorate the efficiency of the boundary-element-only formulations. This drawback can be avoided by replacing the actuator by prescribed tractions obtained directly from the normal and shear stresses at the actuator-plate interface. The tractions are able to be calculated using analytical or numerical methods. When it comes to the analytical models, perhaps the simplest replacement is the pin-force model [89, 91], which considers that the shear traction is concentrated at the edge of the actuator assuming the perfect bonding between the actuator and the plate. Taking the effect of the adhesive layer into account, Crawley et al. [122] and Roy et al. [123] gave the analytical expression for the distributed shear traction generated by rectangular and circular actuators, respectively. However, they neglected the dynamics of the system, and therefore their models are not suitable for high-frequency excitations. Huang et al. [124] and Kapuria et al. [125] improved the analytical model for rectangular actuators by including the inertia term in the analysis. After the improvement, some limitations still remain. First, these dynamic models cannot be applied to circular actuators. Second, the normal traction at the interface, which

has been shown to be as important as the shear traction near the resonant frequencies of the actuator [126], was disregarded. Thus, in order to deal with general actuation scenarios, numerical methods, such as the FEM, should be relied on. Such distributed tractions can be directly inserted into the boundary element plate formulations. However, this direct method is not numerically efficient. In order to improve the efficiency, an equivalent pin-force model, in which the distributed tractions are simplified into line tractions, is proposed in this chapter. As for the PZT sensor, one common model [89, 123, 127, 128], in which the sensor is treated as a capacitor, can be easily integrated into the boundary element plate formulations.

In this chapter, the Laplace-transform dual boundary element formulations based on Kane-Mindlin (see Chapter 4) and Mindlin (see Chapter 5) plate theories are used to model the wave behaviours of the S0 and A0 Lamb modes in an isotropic cracked plate below the cut-off frequency of the A1 mode, respectively. An equivalent pin-force model is newly proposed to represent the PZT actuator, and an existing sensor model is implemented numerically. The accuracy of the proposed BEM-based model is validated using 3D FEM and experimental results. In addition, this model provides data for a damage detection algorithm to localize the crack, and the accuracy of this data is checked against the experimental data. Finally, some parametric studies are carried out using this BEM model.

6.2 First-order plate theories

When it comes to plate structures, in which the thickness dimension is much smaller than the in-plane dimensions, the elastodynamic analysis can be conducted easily and efficiently by plate theories. Such theories are two-dimensional theories developed from three-dimensional elasticity theory based on certain kinematic assumptions with regard to the variation of displacements through the thickness. The displacements can be expanded into infinite series of powers in the thickness coordinate and different plate theories require different truncations of the series. The corresponding governing equations and boundary conditions can be derived based on the principle of virtual displacements.

In this chapter, the first-order plate theories are considered, in which the displacements are approximated by taking the constant and linear terms from their Taylor series about $x_3 = 0$ (plate mid-plane):

$$\begin{aligned} u_1(x_1, x_2, x_3, t) &= u_1^S(x_1, x_2, t) + u_1^A(x_1, x_2, t)x_3 \\ u_2(x_1, x_2, x_3, t) &= u_2^S(x_1, x_2, t) + u_2^A(x_1, x_2, t)x_3 \\ u_3(x_1, x_2, x_3, t) &= u_3^A(x_1, x_2, t) + u_3^S(x_1, x_2, t)x_3 \end{aligned} \quad (6.1)$$

in which the superscripts S and A denote the motions symmetric and antisymmetric with respect to the mid-plane of the plate, respectively; t represents time. The symmetric and antisymmetric waves are decoupled in an isotropic plate of constant thickness, and as a result they can be analysed separately. Kane-Mindlin [60] and Mindlin plate theories [61] are concerned with the symmetric motion and antisymmetric motion respectively, and their governing equations have been given in Eq. (3.4) and Eq. (5.4) respectively.

6.3 Wave dispersion characteristics

Assuming harmonic motions, in the absence of surface tractions, the governing equation (3.4) for the Kane-Mindlin (K-M) theory can be decoupled into three wave equations:

$$\nabla^2 \varphi_i^S + (k_i^S)^2 \varphi_i^S = 0, \quad i = 1, 2, 3, \quad (6.2)$$

in which φ_i^S are three independent potentials and k_i^S denote the corresponding wave numbers. The relationships between k_i^S and the frequency ω are given by:

$$\begin{aligned} k_\alpha^S &= \frac{\omega}{c_1} \sqrt{\frac{1}{2R_2^S} \left[R_1^S + R_2^S - (-1)^\alpha Q_S^f - \left(\frac{\omega_{cS}}{\omega} \right)^2 \right]} \\ k_3^S &= \frac{\omega}{c_2} \end{aligned} \quad (6.3)$$

where $R_\alpha^S = c_\alpha^2/c_3^2$, $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$, $c_2 = \sqrt{\mu/\rho}$, $c_3 = \sqrt{E/[\rho(1-\nu^2)]}$ and the cut-off frequency $\omega_{cS} = 2\sqrt{3}\kappa_S c_1/h$; $Q_S^f = \sqrt{[(R_1^S + R_2^S) - (\omega_{cS}/\omega)^2]^2 - 4R_1^S R_2^S [1 - (\omega_{cS}/\omega)^2]}$. The group velocity of the three waves can be computed using the following expressions:

$$\begin{aligned} c_\alpha^{g|S} &= -\frac{(-1)^\alpha c_3^2 \omega k_\alpha^S Q_S^f}{(c_1^2 + c_2^2)(k_\alpha^S)^2 - 2\omega^2 + \omega_{cS}^2} \\ c_3^{g|S} &= c_2 \end{aligned} \quad (6.4)$$

Similar to the equilibrium equations in the K-M theory, Eq. (5.4) for the Mindlin theory can also be decoupled into three wave equations governing three potentials φ_i^A for harmonic motions:

$$\nabla^2 \varphi_i^A + (k_i^A)^2 \varphi_i^A = 0, \quad i = 1, 2, 3, \quad (6.5)$$

The wave numbers k_i^A have the following expressions:

$$\begin{aligned} k_\alpha^A &= \frac{\omega}{\omega_{cA}} \sqrt{\frac{1}{2} [R_1^A + R_2^A - (-1)^\alpha Q_A^f]} \\ k_3^A &= \sqrt{\kappa_A^2 R_2^A \left[\left(\frac{\omega}{\omega_{cA}} \right)^2 - 1 \right]} \end{aligned} \quad (6.6)$$

in which $Q_A^f = \sqrt{(R_1^A - R_2^A)^2 + 4R_1^A R_2^A (\omega_{cA}/\omega)^2}$, $R_1^A = \kappa_A^2 \mu h/D$, $R_2^A = 12/h^2$ and the cut-off frequency $\omega_{cA} = 2\sqrt{3}\kappa_A c_2/h$. The group velocity of the three antisymmetric waves can be computed as follows:

$$\begin{aligned} c_\alpha^{g|A} &= \frac{2\omega_{cA} \sqrt{2 [R_1^A + R_2^A - (-1)^\alpha Q_A^f]}}{2(R_1^A + R_2^A) - (-1)^\alpha [Q_A^f + (R_1^A - R_2^A)^2/Q_A^f]} \\ c_3^{g|A} &= c_2 \sqrt{1 - 3 \left(\frac{2\kappa_A c_2}{\omega h} \right)^2} \end{aligned} \quad (6.7)$$

It is shown from Eqs. (6.2) and (6.5) that three symmetric and three antisymmetric waves can be described in the K-M theory and Mindlin theory, respectively.

In the K-M theory, the three symmetric waves with wave numbers k_1^S , k_2^S and k_3^S are similar to the S0, S1 and SH0 modes, respectively. The expression of wave number k_3^S is exactly the same as that of the SH0 mode, so the SH0 mode can be represented accurately. In order to achieve the optimal approximation to the S0 mode, the dispersion curve of the first wave with wave number k_1^S can be adjusted by modifying the shear factor κ_S . The second wave with wave number k_2^S provides a poor approximation to the S1 mode.

When it comes to the Mindlin theory, the three antisymmetric waves with wave numbers k_1^A , k_2^A and k_3^A can be used to model the A0, A1 and SH1 modes, respectively. A1 and SH1 modes have the same cut-off frequency, which can be captured accurately by the Mindlin theory when the shear factor κ_A is equal to $\pi/2\sqrt{3}$. Under such factor, the wave with wave number k_3^A has the same dispersion curve as that of the SH1 mode. In order to model the A0 mode accurately over a wider frequency range, the shear factor κ_A should be adjusted.

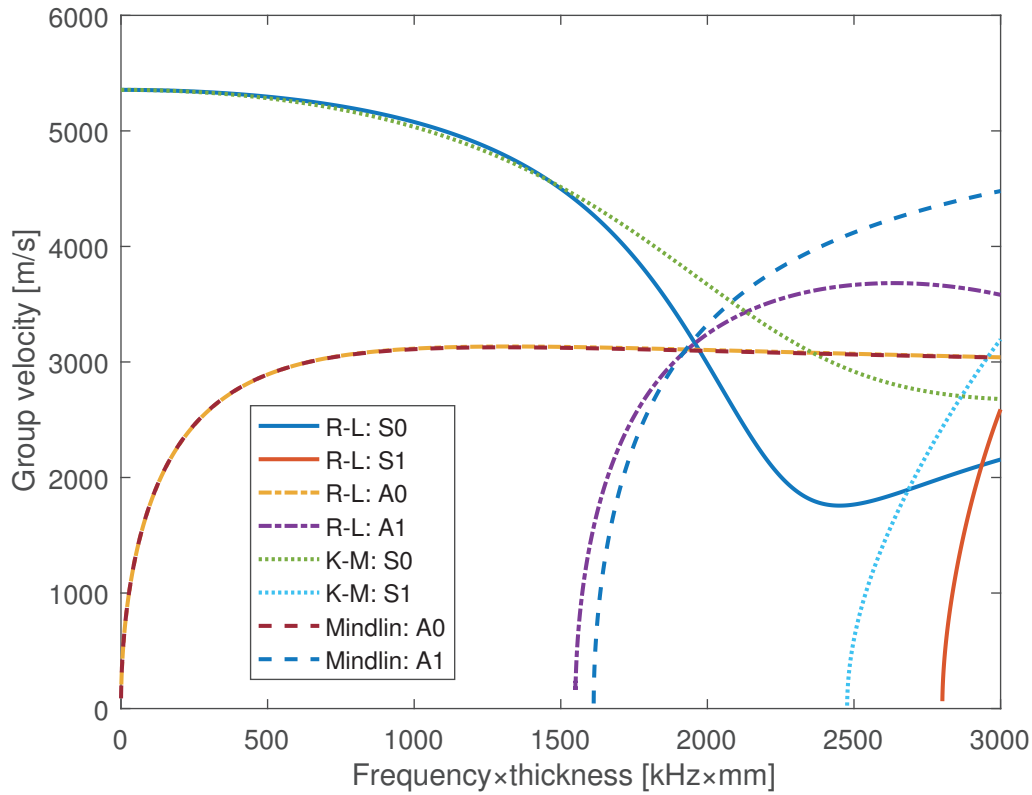


Figure 6.1: Comparison of group-velocity dispersion curves for the aluminium plate obtained by solving the Rayleigh-Lamb equation (R-L) and those based on Kane-Mindlin (K-M) and Mindlin plate theories.

In most SHM applications, only S0 and A0 modes are excited and the operating frequency is below the cut-off frequency of the A1 mode [95, 129]. If this frequency range is required, it has been found that the satisfactory approximation can be achieved when $\kappa_S = 0.73$ and $\kappa_A = 0.94$ for the aluminium plate. The exact dispersion relations of Lamb waves can be calculated numerically from the Rayleigh-Lamb frequency equations (2.15) and (2.16). The group-velocity dispersion curves of the Lamb wave modes (S0, S1, A0 and A1) on the basis of the plate theories and those obtained by solving the Rayleigh-Lamb equation are compared in Fig. 6.1. As can be seen on this figure, the K-M and Mindlin plate theories can be used to model the S0 and A0 modes with excellent accuracy at frequencies below the cut-off frequency of the A1 mode.

6.4 Boundary integral equations

The Laplace-transform dual boundary integral equations for the Kane-Mindlin plate (Eqs. (3.33) and (4.1)) and Mindlin plate (Eqs. (5.6) and (5.7)) are adopted here, and the discretization strategy for cracked plates is the same as that introduced in Subsection 2.4.7. After collocating at every nodal point and introducing boundary conditions, the discretized boundary integral equations can be written in matrix form as:

$$\begin{bmatrix} \tilde{\mathbf{A}}^S & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{A}}^A \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}}^S \\ \tilde{\mathbf{x}}^A \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{f}}^S \\ \tilde{\mathbf{f}}^A \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{F}}^S \\ \tilde{\mathbf{F}}^A \end{bmatrix} \quad (6.8)$$

in which $\tilde{\mathbf{A}}^Y$ is the coefficient matrix (the superscript Y denotes S and A for the symmetric and antisymmetric motions respectively); $\tilde{\mathbf{x}}^Y$ is comprised of unknown nodal displacements and tractions; $\tilde{\mathbf{f}}^Y$ is the known vector obtained by using the boundary conditions; $\tilde{\mathbf{F}}^Y$ is due to the effect of the generalized surface tractions (domain loads). The time-domain solutions are computed using the numerical inversion of Laplace transforms, which is given in Eq. (2.39).

6.5 Modelling of piezoelectric transducers

6.5.1 Actuator model

A new PZT actuator model, named an equivalent pin-force model, is developed here. First, the tractions applied to the host plate by the PZT actuator are obtained from the finite element simulation. Then, such distributed tractions are projected to the excited S0 and A0 modes using an integral-transform technique. Lastly, an equivalent pin-force, which leads to the same projection, is calculated.

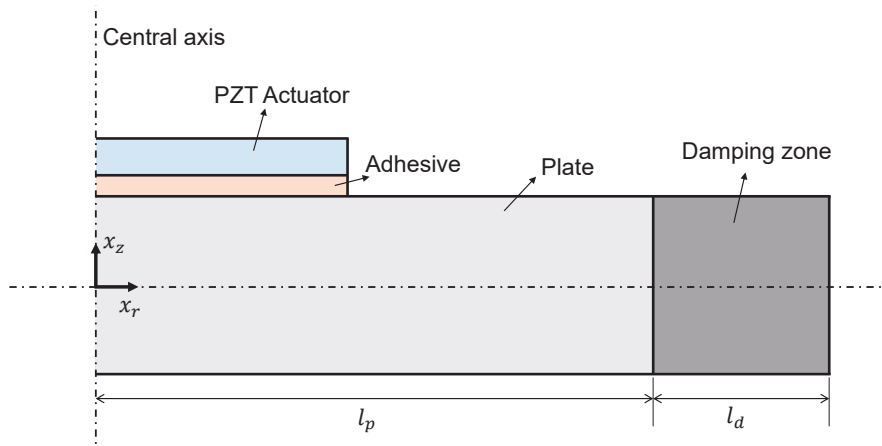


Figure 6.2: Actuation model used in ABAQUS.

The commercial finite element software ABAQUS is used to get the applied tractions in the frequency domain, and the time-domain results are obtained through the inverse Fourier transform. Since the PZT disc is polarized along the thickness direction and the material properties of the plate and adhesive layer are isotropic, the actuation problem can be solved using a 2D axisymmetric model (see Fig. 6.2). In the finite element analysis, only a small portion of the host plate is modelled, including an artificial damping zone at the end of the plate section to avoid wave reflections. The damping coefficients in this zone can be determined using the method given in Ref. [130].

The distributed tractions applied to the plate due to the actuator are equal to the normal and shear stresses at the adhesive-plate interface. In the equivalent pin-force model, the distributed

shear traction, $\tau_r(r, \omega)$, and normal traction, $\tau_z(r, \omega)$, are replaced by simple concentrated line tractions at $r = R$, $\tau_{r|EPF}(R, \omega)\delta(r - R)$ and $\tau_{z|EPF}(R, \omega)\delta(r - R)$, in which δ denotes the Dirac delta function. Thanks to this simplification, the corresponding 2D surface integral resulting from the actuator in the boundary integral equations is reduced to 1D line integral.

When the plate is subjected to the distributed tractions τ_r and τ_z below the cut-off frequency of the A1 mode, only S0 and A0 modes are generated. According to Quaegebeur et al. [126], the analytical solutions for the displacements of the S0 mode, u_r^S , and the A0 mode, u_r^A , at $x_3 = h/2$ in the frequency domain can be written as follows:

$$u_r^Y(r, \omega) = \frac{\pi i H_1^{(2)}(k^Y r)}{2\mu D_{,k}^Y(k^Y, \omega)} [N^Y(k^Y, \omega) \hat{\tau}_r^Y(k^Y, \omega) + M^Y(k^Y, \omega) \bar{\tau}_z^Y(k^Y, \omega)] \quad (6.9)$$

where Y denotes S and A for the S0 mode and A0 mode, respectively; ω and k^Y represent frequency and wavenumber respectively; i denotes the imaginary unit; $H_1^{(2)}$ is the Hankel function of the second kind of order one; $\hat{\tau}_r^Y$ and $\bar{\tau}_z^Y$ are the projections of the distributed tractions on the S0 and A0 modes, which are computed by the Hankel transforms of order one and zero (with first- and zeroth- order Bessel functions J_1 and J_0):

$$\begin{aligned} \hat{\tau}_r^Y(k^Y, \omega) &= \int_0^\infty r J_1(k^Y r) \tau_r(r, \omega) dr \\ \bar{\tau}_z^Y(k^Y, \omega) &= \int_0^\infty r J_0(k^Y r) \tau_z(r, \omega) dr \end{aligned} \quad (6.10)$$

the other terms involving the wavenumber are given by:

$$\begin{aligned} N^S(k^Y, \omega) &= k^Y b [(k^Y)^2 + b^2] c_a c_b \\ M^S(k^Y, \omega) &= (k^Y)^2 \{ [(k^Y)^2 - b^2] c_a s_b + 2ab c_b s_a \} \\ D^S(k^Y, \omega) &= [(k^Y)^2 - b^2]^2 c_a s_b + 4(k^Y)^2 ab s_a c_b \\ N^A(k^Y, \omega) &= -kb [(k^Y)^2 + b^2] s_a s_b \\ M^A(k^Y, \omega) &= (k^Y)^2 \{ [(k^Y)^2 - b^2] s_a c_b + 2ab s_b c_a \} \\ D^A(k^Y, \omega) &= [(k^Y)^2 - b^2]^2 s_a c_b + 4(k^Y)^2 ab c_a s_b \end{aligned} \quad (6.11)$$

in which $a^2 = \omega^2/c_1^2 - (k^Y)^2$, $b^2 = \omega^2/c_2^2 - (k^Y)^2$, $c_a = \cos(ha/2)$, $c_b = \cos(hb/2)$, $s_a = \sin(ha/2)$ and $s_b = \sin(hb/2)$.

It can be seen from Eqs. (6.9) - (6.11) that the influence of the distributed tractions on the Lamb wave response is determined solely by the projections $\hat{\tau}_r^Y$ and $\bar{\tau}_z^Y$. In other words, two different sets of tractions lead to the same response of a specific Lamb wave mode, provided that they have the same projections on this mode. Thus, if the projections of the equivalent line tractions are equal to those of the distributed tractions, these two sets of tractions are able to be interchangeable.

Assuming the equivalent line tractions $\tau_{r|EPF}^Y$ and $\tau_{z|EPF}^Y$ are applied at $r = R$, the projections on the S0 and A0 modes are given by :

$$\begin{aligned}\hat{\tau}_{r|EPF}^Y(k^Y, \omega) &= \int_0^\infty r J_1(k^Y r) \tau_{r|EPF}^Y(R, \omega) \delta(r - R) dr = \tau_{r|EPF}^Y(R, \omega) R J_1(k^Y R) \\ \bar{\tau}_{z|EPF}^Y(k^Y, \omega) &= \int_0^\infty r J_0(k^Y r) \tau_{z|EPF}^Y(R, \omega) \delta(r - R) dr = \tau_{z|EPF}^Y(R, \omega) R J_0(k^Y R)\end{aligned}\quad (6.12)$$

By letting the right-hand side of Eq. (6.12) equal to the right-hand side of Eq. (6.10), the frequency components of the unknown equivalent line tractions can be obtained as follows:

$$\begin{aligned}\tau_{r|EPF}^Y(R, \omega) &= \frac{\int_0^\infty r J_1(k^Y r) \tau_r(r, \omega) dr}{R J_1(k^Y R)} \\ \tau_{z|EPF}^Y(R, \omega) &= \frac{\int_0^\infty r J_0(k^Y r) \tau_z(r, \omega) dr}{R J_0(k^Y R)}\end{aligned}\quad (6.13)$$

If the frequency spectrum of the excitation voltage $V(t)$ is denoted by $V(\omega)$, the time-domain equivalent line tractions can be computed by taking the inverse Fourier transform of $V(\omega)\tau_{r|EPF}^Y$ and $V(\omega)\tau_{z|EPF}^Y$. After substituting the equivalent line tractions into Eqs. (3.3) and (5.3), the generalized domain loads on the Kane-Mindlin and Mindlin plates due to the PZT actuator are obtained. These loads are able to be integrated into the boundary integral equations and the resulting line integral is estimated using the composite trapezoidal rule, which is the same as the rule for θ in Eq. (6.15). Finally, the actuation effect of the PZT actuator is represented by the domain-load term $\tilde{\mathbf{F}}^Y$ in Eq. (6.8).

It should be noted that, unlike the conventional pin-force model (see Ref. [89]), the proposed equivalent line tractions are not necessarily applied at the circumference of the actuator disc, and furthermore the radius R to which the equivalent line tractions are applied is not unique. For different R , the amplitudes of such tractions which are determined by Eq. (6.13) are different. Since these different equivalent line tractions lead to the same projections, they have the same actuation effect on the corresponding Lamb wave mode. When the numerical difficulty, caused by the zeros of the Bessel functions in the denominators of Eq. (6.13), is encountered, it can be remedied by choosing R which is smaller than the radius of the actual actuator disc.

6.5.2 Sensor model

One common PZT sensor model [89, 123, 127, 128] is adopted in this chapter. In such model, the output voltage of the sensor, which is directly proportional to the total in-plane strains in the sensing area, is estimated using the following equation:

$$\tilde{V}(p) = \frac{d_{31}h_{pzt}E_{pzt}}{A_{pzt}[(1 - \nu_{pzt})\varepsilon_{33} - 2d_{31}^2E_{pzt}]} \int_A \tilde{u}_{\alpha,\alpha}(x_1, x_2, h/2, p) dA \quad (6.14)$$

in which E_{pzt} and ν_{pzt} denote the in-plane Young's modulus and Poisson's ratio of the PZT; h_{pzt} and A_{pzt} are sensor thickness and the area of the sensor bottom planar surface respectively; d_{31} and ε_{33} are the piezoelectric coefficient and out-of-plane dielectric permittivity respectively; $\tilde{u}_{\alpha,\alpha}$ represents the total in-plane strains at the sensor-plate interface.

Since the integration domain for the sensor disc is circular, the integral in Eq. (6.14) is computed numerically in polar coordinates by using the quadrature rule suggested in Ref. [131]. A composite trapezoidal rule is applied for θ , in which the interval $[0, 2\pi]$ is subdivided into N equal intervals, and the m -point Gauss quadrature rule is applied for r . Thus, the integral can be estimated as follows:

$$\int_A \tilde{u}_{\alpha,\alpha} dA = \int_0^{2\pi} \int_0^{R_s} \tilde{u}_{\alpha,\alpha} r dr d\theta = \frac{2\pi}{N} \sum_{m=1}^M \sum_{n=1}^N w_m \tilde{u}_{\alpha,\alpha} \left(r_m, \frac{2\pi n}{N} \right) r_m \quad (6.15)$$

where R_s denotes the radius of the PZT sensor; w_m and r_m are the weights and abscissas of Gaussian quadrature over the domain $[0, R_s]$, respectively.

Differentiating Eq. (6.1) with respect to x_α in the Laplace domain, the total in-plane strains at the sensor-plate interface can be calculated as follows:

$$\tilde{u}_{\alpha,\alpha}(x_1, x_2, h/2, p) = \tilde{u}_{\alpha,\alpha}^S(x_1, x_2, p) + \tilde{u}_{\alpha,\alpha}^A(x_1, x_2, p)h/2 \quad (6.16)$$

After obtaining all nodal displacements and tractions by solving Eq. (6.8), the strain components for the symmetric and antisymmetric motions are computed directly using the following integral equations:

$$\begin{aligned} \tilde{u}_{\alpha,\alpha}^Y(\mathbf{X}', p) = & \int_{\Gamma} \tilde{U}_j^Y(\mathbf{X}', \mathbf{x}, p) \tilde{t}_j^Y(\mathbf{x}, p) d\Gamma - \int_{\Gamma} \tilde{T}_j^Y(\mathbf{X}', \mathbf{x}, p) \tilde{u}_j^Y(\mathbf{x}, p) d\Gamma \\ & + \int_{\Omega} \tilde{U}_j^Y(\mathbf{X}', \mathbf{X}, p) \tilde{F}_j^Y(\mathbf{X}, p) d\Omega \end{aligned} \quad (6.17)$$

in which Y denotes S for the symmetric motion and A for the antisymmetric motion; $\tilde{U}_j^Y$ and $\tilde{T}_j^Y$ are fundamental solutions listed in Appendices A.1 and A.2.

6.6 Numerical simulation and experimental validation

6.6.1 Specimen description

A 2-mm-thick square plate is considered here. The plate is made of aluminium which has a Young's modulus of 69 GPa, a Poisson's ratio of 0.33 and a density of 2700 kg/m³. There is a horizontal through-thickness crack located at the center of the plate. Four PZT disc transducers, 0.5 mm thick, are attached to the top surface of the plate. The plate dimensions and layout of PZT transducers are illustrated in Fig. 6.3.

Fig. 6.4 shows the specimen for experiments. A through-thickness slit of 0.4 mm width is machined with the help of electrical discharge machining (EDM) to represent the crack. The

PZTs are made of PIC 255 produced by PI Ceramic GmbH. The material properties of PIC 255 are given in Table 6.1. The PZT discs were bonded to the plate using thermoplastic adhesive films.

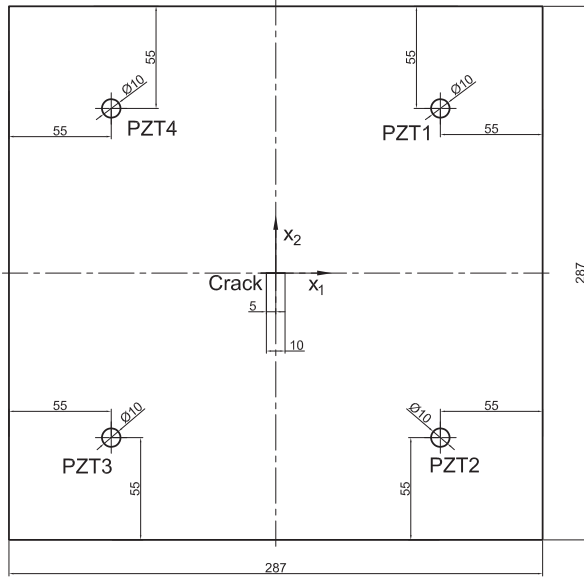


Figure 6.3: Schematic illustration of the cracked plate (Unit: mm).

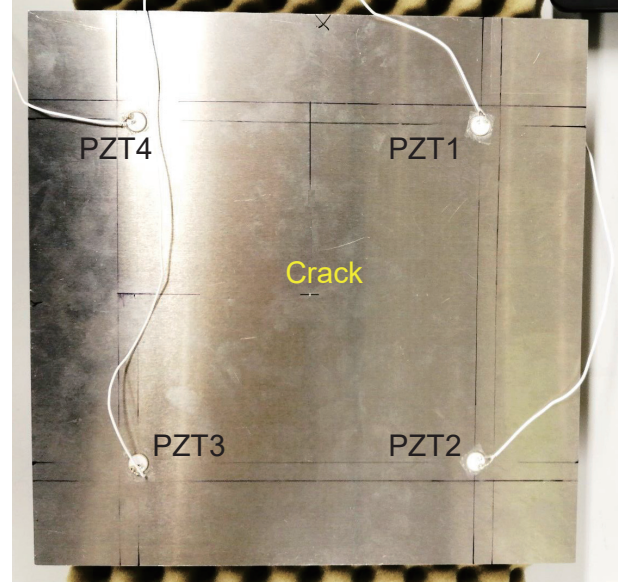


Figure 6.4: The cracked plate specimen used in the experiments.

Table 6.1: Material properties of the PZT transducers.

Elastic (GPa)								
C11	C12	C13	C22	C23	C33	C44	C55	C66
123	76.7	70.3	123	70.3	97.1	22.3	22.3	23.2

Piezoelectric ($\times 10^{-12}$ m/V)					Dielectric (nF/m)			Density (kg/m ³)
d31	d32	d33	d15	d24	ϵ_{11}	ϵ_{22}	ϵ_{33}	
-174	-174	394	535	535	14.6	14.6	15.5	

6.6.2 Actuation signal and frequency selection

A sinusoidal tone burst modulated by Hann window (see Eq. (2.25)) is chosen as the excitation signal because its limited bandwidth can alleviate the dispersion effect of Lamb waves. In the

SHM applications, it is desirable to generate a pure Lamb wave mode to simplify the signal analysis. For the disc actuators used in this chapter, both the S0 and A0 modes can be excited. The A0 and S0 modes dominate the sensor response around 50 kHz and 300 kHz, respectively. Since the A0 mode is highly dispersive at 50 kHz, the actuator is excited by a Hann-windowed tone burst centred at 300 kHz with 5 cycles.

6.6.3 Experimental setup

The experimental setup shown in Fig. 6.5 is mainly composed of a LabVIEW-based control panel, a waveform generator card and a data acquisition card. An arbitrary wave generator National Instruments (NI) PXI-5412 was used to generate the actuation signal. The response signals of the PZT sensors were recorded with a NI PXI-5105 digitizer card at the sampling frequency of 60 MHz.

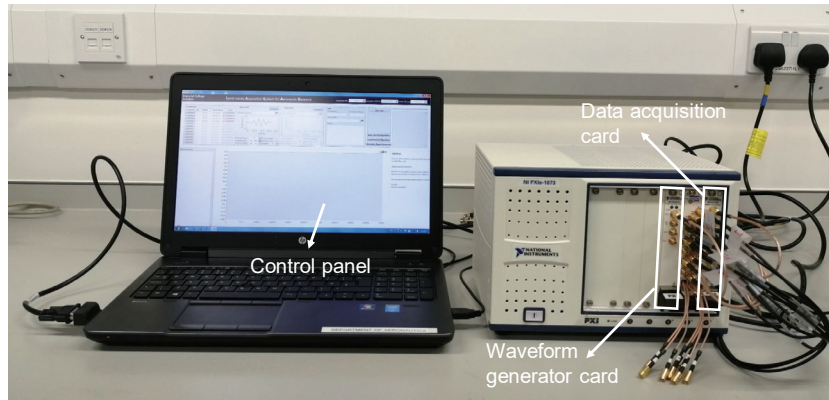


Figure 6.5: Experimental setup.

6.6.4 Sensor and plate responses

The proposed BEM model is used to simulate the signals generated by the PZT sensors. The BEM-simulated signals are compared with those obtained from 3D finite element modelling as well as experimental measurements.

It is worth mentioning that the thermoplastic adhesive film should be included in the numerical simulation for accurate modelling when the excitation frequency is around the local

resonant frequency of the coupled system consisting of the PZT disc, adhesive film and host plate. Willberg et al. [132] have found that the adhesive film is able to shift this resonance frequency and it has a great influence on the system response amplitude at the frequencies near the resonance. Hence, although the film does not change the group velocity of the Lamb-wave packet propagating in the host plate, it affects the shape of the wave packet.

In order to take the adhesive film into consideration in the numerical modelling, the material properties and thickness of the cured thermoplastic film are essential. However, it is difficult to accurately measure these parameters. Instead, they can be determined by calibrating the numerical model with experimental data. Kapuria et al. [125] pointed out that the film stiffness, rather than its density, dominates the stress-transfer effect between the PZT disc and the host plate. Thus, in the calibration process, the stiffness is adjustable while the density remains the same. Furthermore, according to Ref. [126], the stiffness is not sensitive to the Poisson's ratio, and the thickness and Young's modulus of the film contribute the most to the stiffness. Since the thickness is easier to measure than the Young's modulus, the Young's modulus is recommended to be the sole parameter for the calibration. The value of the calibrated Young's modulus is able to be determined by comparing the shape of the direct-arrival wave packet in the simulated sensor signal with that in the experimental signal. The density and Poisson's ratio can be found in the data sheet provided by the manufacturers. The errors due to these two parameters, as well as the measurement error of the thickness, are not important because the calibrated Young's modulus can compensate for them.

After the calibration, the final value of the Young's modulus for the numerical simulation does not necessarily represent the actual Young's modulus of the adhesive film. It is more appropriate to treat it as a generalised elastic modulus which reflects the adhesive effect on the system response, and this value could vary in different numerical models. In the 3D FEM simulation, both actuator adhesive and sensor adhesive are modelled, while only actuator adhesive is taken into account in the BEM model. Thus, the Young's modulus used in the BEM, which should include the contribution from the sensor adhesive, is different from that input to the

3D finite element simulation. In the following numerical simulations, the adhesive film has a density of 1200 kg/m^3 , a Poisson's ratio of 0.44 and a thickness of $20 \text{ }\mu\text{m}$. The Young's moduli are 1.2 Gpa and 2 Gpa for the BEM and 3D FEM models, respectively.

The equivalent pin-force model in the boundary element analysis requires the tractions which are applied to the host plate by the PZT actuator. These tractions are able to be obtained using a 2D axisymmetric analysis conducted with ABAQUS. As shown in Fig. 6.2, it is only necessary to model a small portion of the plate ($l_p = l_d = 50 \text{ mm}$). In the damping zone, the mass proportional damping coefficient has the value $\alpha_R(x_r) = 11 \times 10^6 \times [(x_r - l_p)/l_d]^3$. The plate and the adhesive layer were discretized with 5000 and 200 elements (CAX8R), respectively, while 75 piezoelectric elements (CAX8RE) were used to model the actuator. The connection of these three components was achieved by tie constraints.

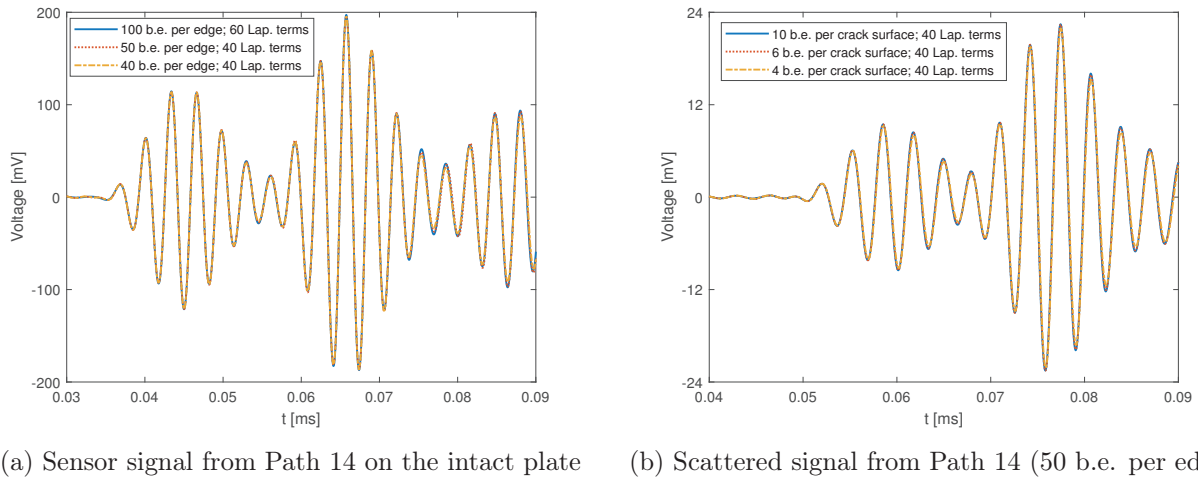


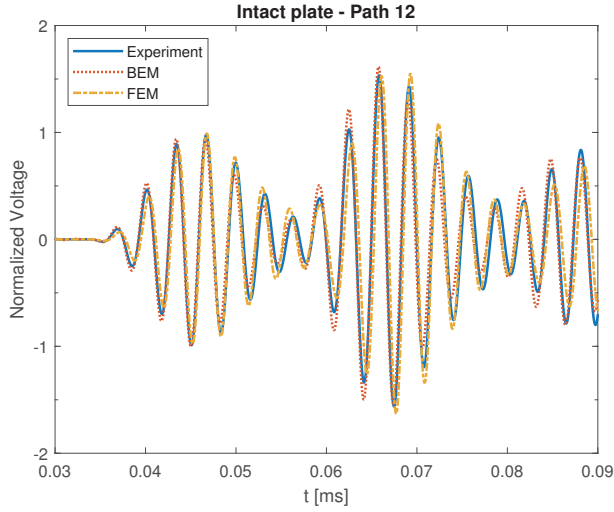
Figure 6.6: The influence of boundary element (b.e.) mesh and the number of Laplace (Lap.) terms on the simulated sensor signals (actuation frequency: 300 kHz).

The convergence properties of the Laplace-transform boundary element method are shown in Fig. 6.6. The term 'Path 14' refers to that PZT1 (the first number '1') is the actuator and PZT4 (the second number '4') is the sensor. The scattered signal is obtained by subtracting the response of the sensor on the intact plate from that on the cracked plate. Satisfactory results can be obtained when each plate edge was divided into 50 boundary elements and the crack was discretized using 12 elements (6 elements along each crack surface). In addition, 40 Laplace

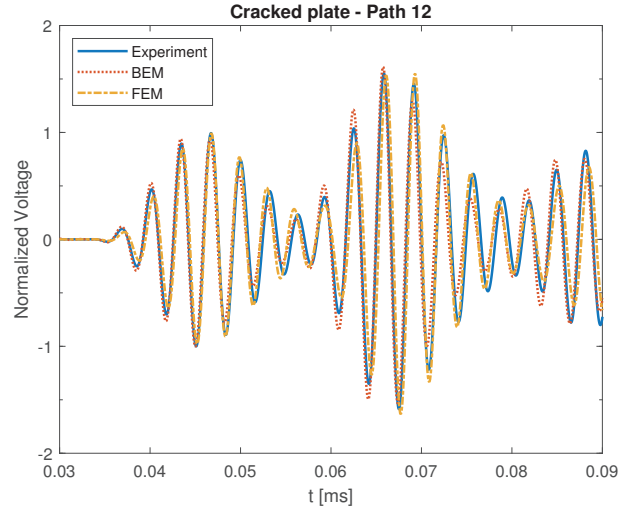
terms are sufficient for the convergence. Thus, the BEM solutions based on these parameters are used in the following analysis.

The proposed BEM model has low memory requirements. The amount of memory required to store the global coefficient matrix (double precision) in the discretized boundary integral equation (6.8) is 88 Mb and 105 Mb for the intact-plate and crack-plate cases, respectively. Furthermore, if the sparsity of this matrix is utilized, such memory requirement can be halved. When it comes to large-scale problems, such as assembled plate structures, the parallel implementation of this BEM can be achieved in a simple way. Since both the creation and the solution of Eq. (6.8) for every Laplace term are independent, these computational tasks for different Laplace terms are able to be conducted simultaneously on multiple CPUs. Alternatively, a fast solver based on hierarchical matrices [96] can be used to reduce the computational costs for solving the large-scale problems.

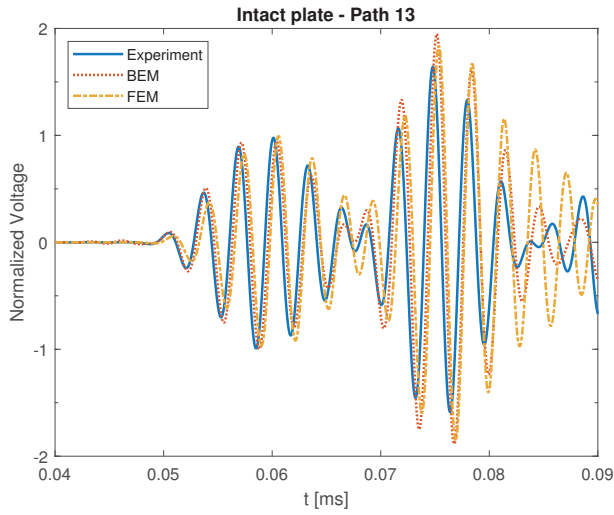
The 3D finite element simulation was based on the ABAQUS implicit solver, which provides the piezoelectric elements. Tie constraints were used to connect different components. The PZT actuator was discretized with 192 piezoelectric elements (C3D20RE), and this mesh was also applied to the PZT sensor. The adhesive layer was modelled using 9,904 linear elements (C3D8R). The group velocities of the S0 and A0 modes at 300 kHz are 5268.5 m/s and 2974.8 m/s, respectively. In order to capture the behaviours of these short-wavelength waves accurately, the plate should be modelled using sufficiently fine FE mesh. The intact and cracked plate were discretized using 164,738 and 179,800 quadratic hexahedral elements of type C3D20R, respectively, and such mesh density ensures that there are 37 nodes per wavelength of the S0 mode and 21 nodes per wavelength of A0 mode at 300 kHz. The finite element meshes for the intact plate and cracked plate lead to 2,731,968 and 2,980,428 degrees of freedom respectively. By contrast, the degrees of freedom required for the boundary element discretization of the intact plate and cracked plate are 2,400 and 2,616 respectively. It is shown clearly that the proposed BEM model is able to reduce the total degrees of freedom significantly.



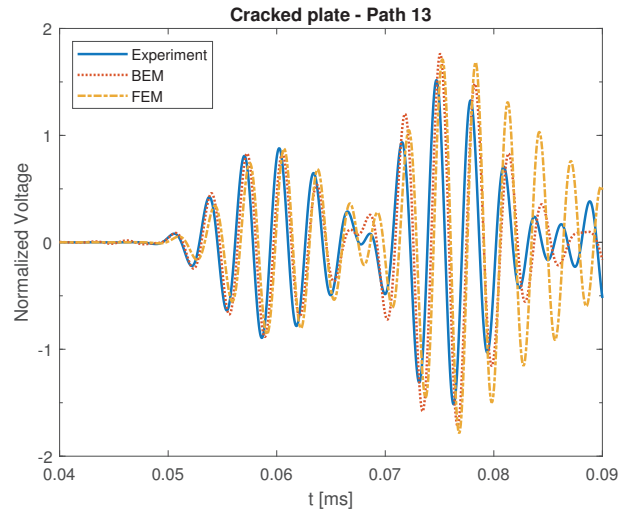
(a) Sensor signal (PZT2) from the intact plate



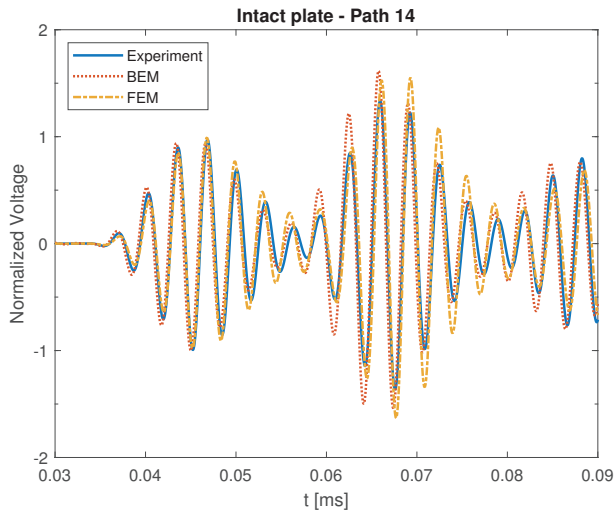
(b) Sensor signal (PZT2) from the cracked plate



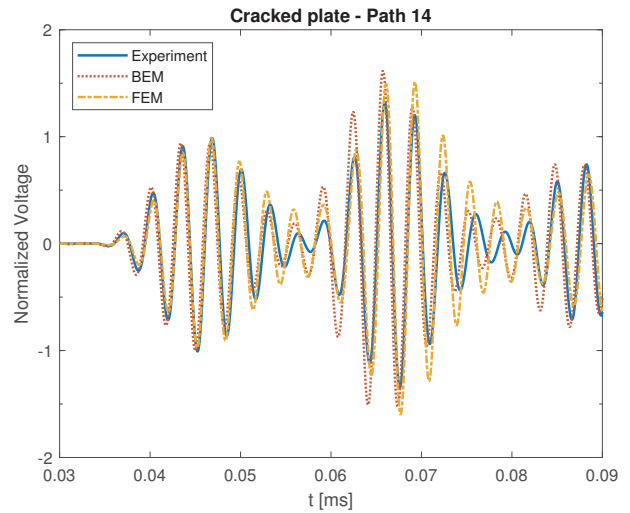
(c) Sensor signal (PZT3) from the intact plate



(d) Sensor signal (PZT3) from the cracked plate



(e) Sensor signal (PZT4) from the intact plate



(f) Sensor signal (PZT4) from the cracked plate

Figure 6.7: Comparison between sensor signals obtained from BEM, FEM and experiment (actuation frequency: 300 kHz).

In SHM, sensor signals from the intact plate are often used as the baseline for damage detection. Thus, the peak amplitude of the first wave packet in these signals is adopted to normalize the sensor responses from both the intact and cracked plate. When PZT1 acts as an actuator, the normalized responses of the sensors (PZT2, PZT3 and PZT4) on the intact and cracked plate are presented in Fig. 6.7. It can be seen that the BEM results agree very well with both the experimental and FEM signals.

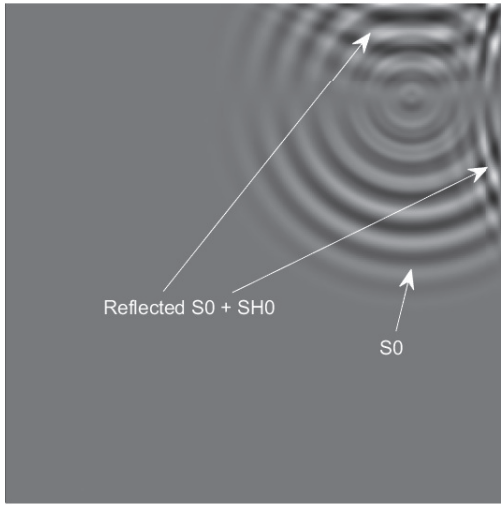
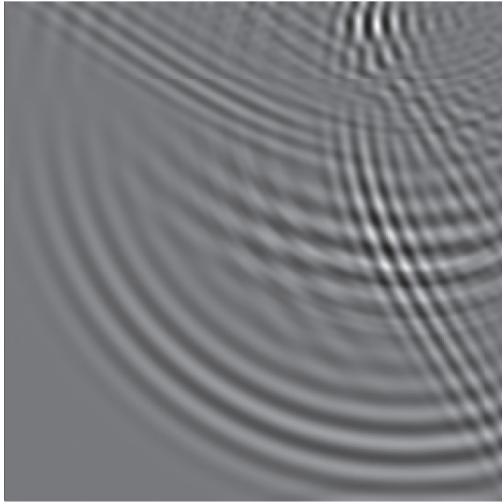
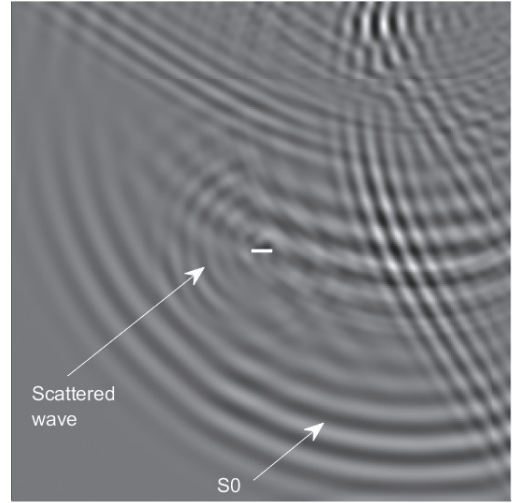
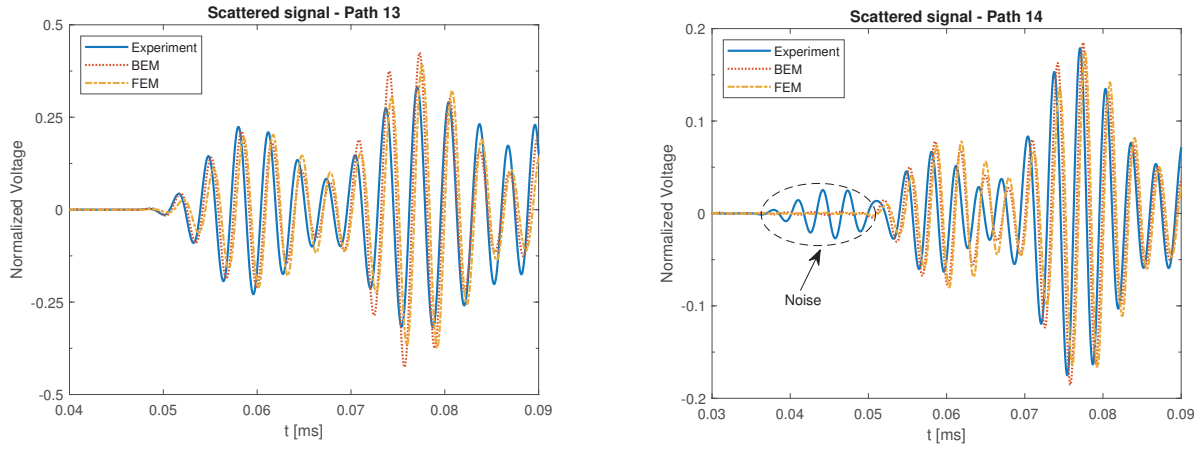
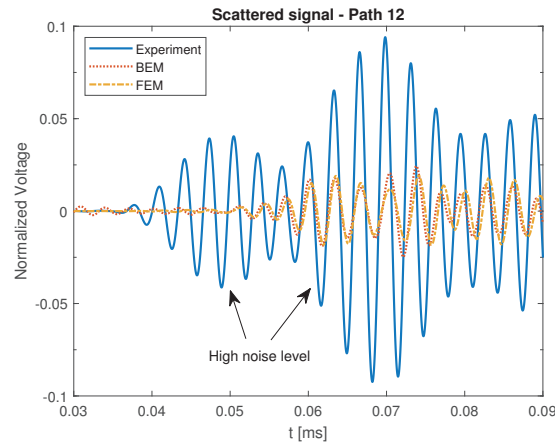
(a) Intact plate at $t = 25 \mu\text{s}$ (b) Cracked plate at $t = 25 \mu\text{s}$ (c) Intact plate at $t = 50 \mu\text{s}$ (d) Cracked plate at $t = 50 \mu\text{s}$

Figure 6.8: Displacement field u_2 at the top surface of the plate obtained using BEM (actuation frequency: 300 kHz).

The waves sensed by the sensors on the intact plate are comprised of two parts, the direct-arrival waves excited by the actuator and the reflected waves from plate edges. When it comes to the cracked plate, in addition to these two components, the sensor responses include the scattered waves caused by the crack. The snapshots of the displacement field (u_2) for the intact and cracked plate over two time frames are illustrated in Fig. 6.8. It is shown that the excited S0 mode reaches the plate edges which are close to the actuator first, followed by its interaction with the crack. The scattered waves by the crack can be observed by comparing Figs. 6.8c and 6.8d.



(a) Scattered signal computed based on PZT3 signals (b) Scattered signal computed based on PZT4 signals



(c) Scattered signal computed based on PZT2 signals

Figure 6.9: Comparison between scattered signals obtained from BEM, FEM and experiment (actuation frequency: 300 kHz).

Fig. 6.9 compares the normalized scattered signals computed based on BEM, FEM and experimental data. When it comes to PZT3 and PZT4 signals, it can be seen that the BEM results are in good agreement with both FEM and experimental results, except for some noise in the experimental data (see Fig. 6.9b). This noise (the first wave packet) is mainly due to the reflected S0 mode wave from the plate edge on the right-hand side. When the measurements were taken on the intact plate and the cracked plate, it was difficult to ensure that the boundary conditions in these two cases were the same, which could cause such signal difference. Apart from the boundary-condition difference, the noise could also originate from the environmental change and electronic devices. When the crack-scattered signal is weaker than the noise level, it is prone to be contaminated by the noise and thus cannot be identified accurately. This happens to the sensing path 12 (see Fig. 6.9c), in which the amplitude of the scattered wave is only around half of the noise amplitude. Such poor-quality signal could make some crack-detection algorithms, such as a method of triangulation [17], ineffective, in which the first wave packet in the scattered signal is considered as the scattered wave directly from the crack. In order to improve the detection accuracy of these algorithms, higher sensing amplitude of the scattered wave is required and BEM simulations can be conducted to provide guidance for the layout of PZT sensor network.

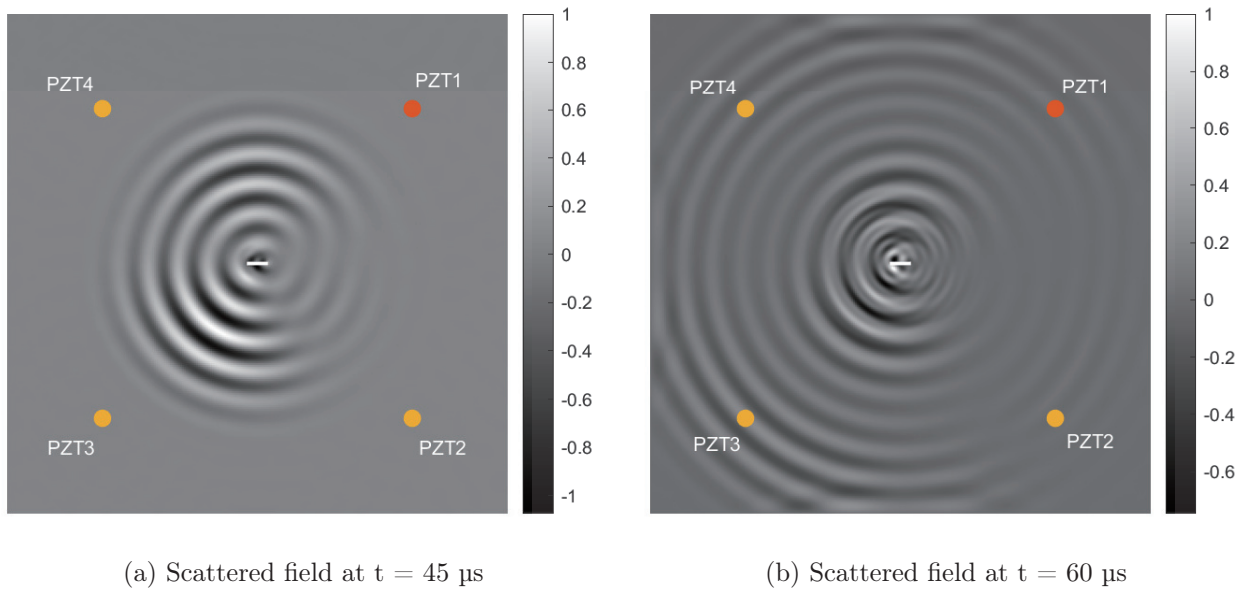


Figure 6.10: Scattered wave field ($u_{\alpha,\alpha}$) at the plate top surface obtained using BEM at 300 kHz (the red circle is the actuator and the yellow circles denote the sensors) (normalized scales).

The snapshots of the crack-scattered wave field (total strain $u_{\alpha,\alpha}$ on the plate surface $x_3 = h/2$) at $t = 45 \mu\text{s}$ and $60 \mu\text{s}$ are illustrated in Fig. 6.10. It can be seen that the scattered wave does not reach the sensors at $t = 45 \mu\text{s}$, and thus the first wave packets from the experimental signals in Figs. 6.9b and 6.9c do not represent the crack-scattered wave but are caused by the noise. This scattered wave is being sensed by the sensors at $t = 60 \mu\text{s}$, which corresponds to the first wave packets in the BEM-simulated results shown in Fig. 6.9.

6.6.5 Crack detection

The proposed BEM model can be used for many SHM applications, such as the evaluation of damage detection algorithms. In this section, a reconstruction algorithm for probabilistic inspection of defects (RAPID), which was proposed to detect small defects by Zhao et al. [133], is utilized alongside the proposed BEM model for crack localization. The aim of this section merely is to assess the application of the simulated signals (in particular the damage scattered signals) as input to damage detection algorithms, which can then be used for development purposes.

In the RAPID algorithm, the difference between the baseline sensor signal X and any monitored signal Y is described using a correlation coefficient $\rho^{\text{cor}} = \frac{C_{XY}}{\sigma_X \sigma_Y}$, in which C_{XY} is the covariance of the baseline data set X and the monitored data set Y , and σ_X and σ_Y denote the standard deviations of X and Y . Such correlation coefficient is linked to the crack probability at any position $\mathbf{x} = (x_1, x_2)$ in the plate domain by the following expression:

$$P(x_1, x_2) = \sum_{n=1}^{N_{\text{PZT}}} \sum_{m=1}^{N_{\text{PZT}}-1} (1 - \rho_{nm}^{\text{cor}}) \left[\frac{\beta_{\text{scale}} - R_{nm}(x_1, x_2)}{\beta_{\text{scale}} - 1} \right] \quad (6.18)$$

in which the PZT network consists of N_{PZT} transducers and an adjustable scaling parameter $\beta_{\text{scale}} > 1$ is used to control the size of effective crack-probability area around every sensing

path ($\beta_{\text{scale}} = 1.02$ in this chapter); R_{nm} is given by:

$$R_{nm}(x_1, x_2) = \begin{cases} R_{nm}^D(x_1, x_2) & \text{when } R_{nm}^D(x_1, x_2) < \beta_{\text{scale}} \\ \beta_{\text{scale}} & \text{when } R_{nm}^D(x_1, x_2) \geq \beta_{\text{scale}} \end{cases} \quad (6.19)$$

where $R_{nm}^D = (|\mathbf{x} - \mathbf{x}_n| + |\mathbf{x} - \mathbf{x}_m|) / (|\mathbf{x}_n - \mathbf{x}_m|)$ representing the ratio of the sum of the distance between the point $\mathbf{x}$ and an actuator center $\mathbf{x}_n$ and the distance between $\mathbf{x}$ and a sensor center $\mathbf{x}_m$, to the distance between the actuator and the sensor.

The crack distribution probability image is plotted using the probability function $P(x_1, x_2)$ and the pixel with maximum probability indicates the estimated crack location. Fig. 6.11 shows the constructed images on the basis of simulated and experimental data. It is shown that the RAPID algorithm succeeds in estimating the crack location using the data from both BEM simulation and the experiment, and these two images are nearly the same. This indicates that this BEM model can be used to evaluate a damage detection algorithm and check its applicability before conducting experiments, which helps to improve the algorithm and optimize sensor network layouts.

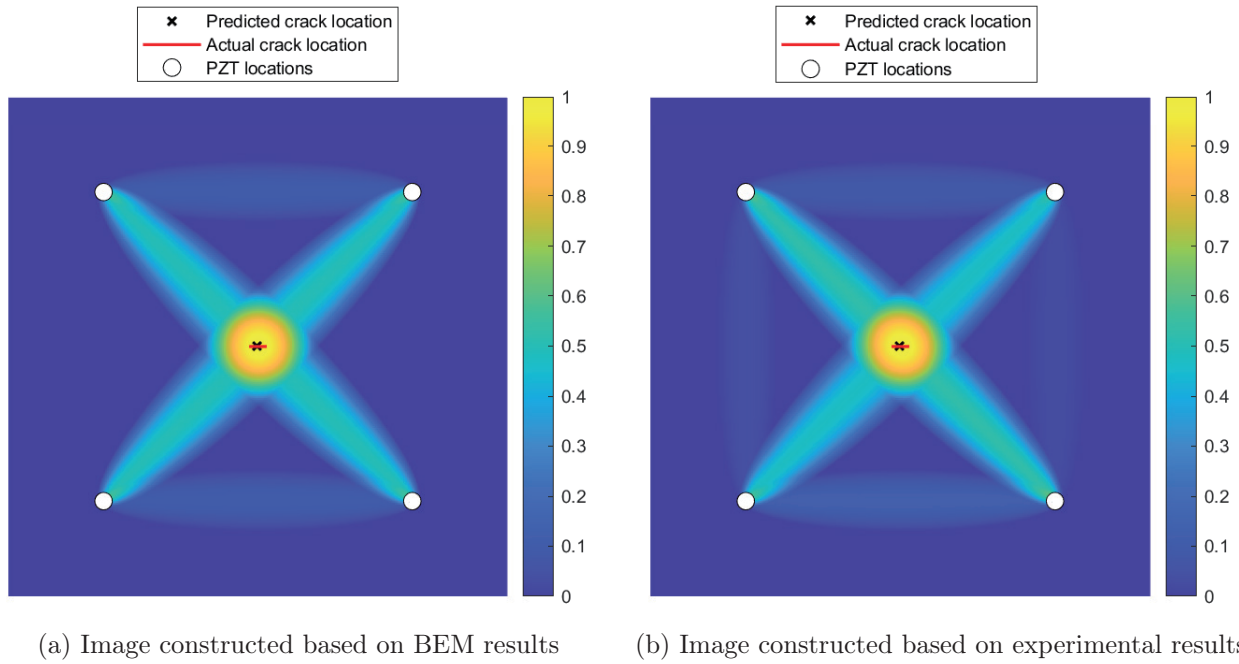


Figure 6.11: Crack detection using the RAPID algorithm (10-mm long crack centred at the plate centre).

6.6.6 Parametric study

The scattered field signal due to the crack can be fully estimated using the proposed BEM model. In order to quantify the strength of the scattered signal ($\widehat{R}_{\text{scatt}}$) received by the sensors, the following definition is used:

$$\widehat{R}_{\text{scatt}} = \frac{\widehat{V}_{\text{scatt}}^{\text{peak}}}{\widehat{V}_{\text{direct}}^{\text{peak}}} \quad (6.20)$$

in which $\widehat{V}_{\text{scatt}}^{\text{peak}}$ represents the peak value of the envelop of the scattered signal and $\widehat{V}_{\text{direct}}^{\text{peak}}$ denotes the peak value of the signal envelop due to the direct-arrival wave packet. The envelope of the sensor signal is calculated as follows:

$$\widehat{V}(t) = |V(t) + i\mathcal{H}[V(t)]| \quad (6.21)$$

where $\mathcal{H}$ denotes the Hilbert transform and i is the imaginary unit.

First, one actuator is supposed to be positioned along a quarter circle (125 mm radius) which is centred at the crack center and it accommodates different incident angles (θ_{inc}) ranging from 0° to 90° with 15° angular spacing. Seventy-two artificial sensors are evenly distributed along the the circle centred at the crack center with a radius of 100 mm. The angular distribution diagrams of the scattered-signal strength under different incident angles are illustrated in Fig. 6.12. Each diagram focuses on one incident angle and the angular distributions under other incident angles are shown with a transparent effect to facilitate comparison. Generally speaking, it can be seen that the scattered signals become stronger as the incident angle increases. In addition, the strength of the transmitted signals is higher than that of the reflected signals, which confirms the basic assumption in the RAPID algorithm [133].

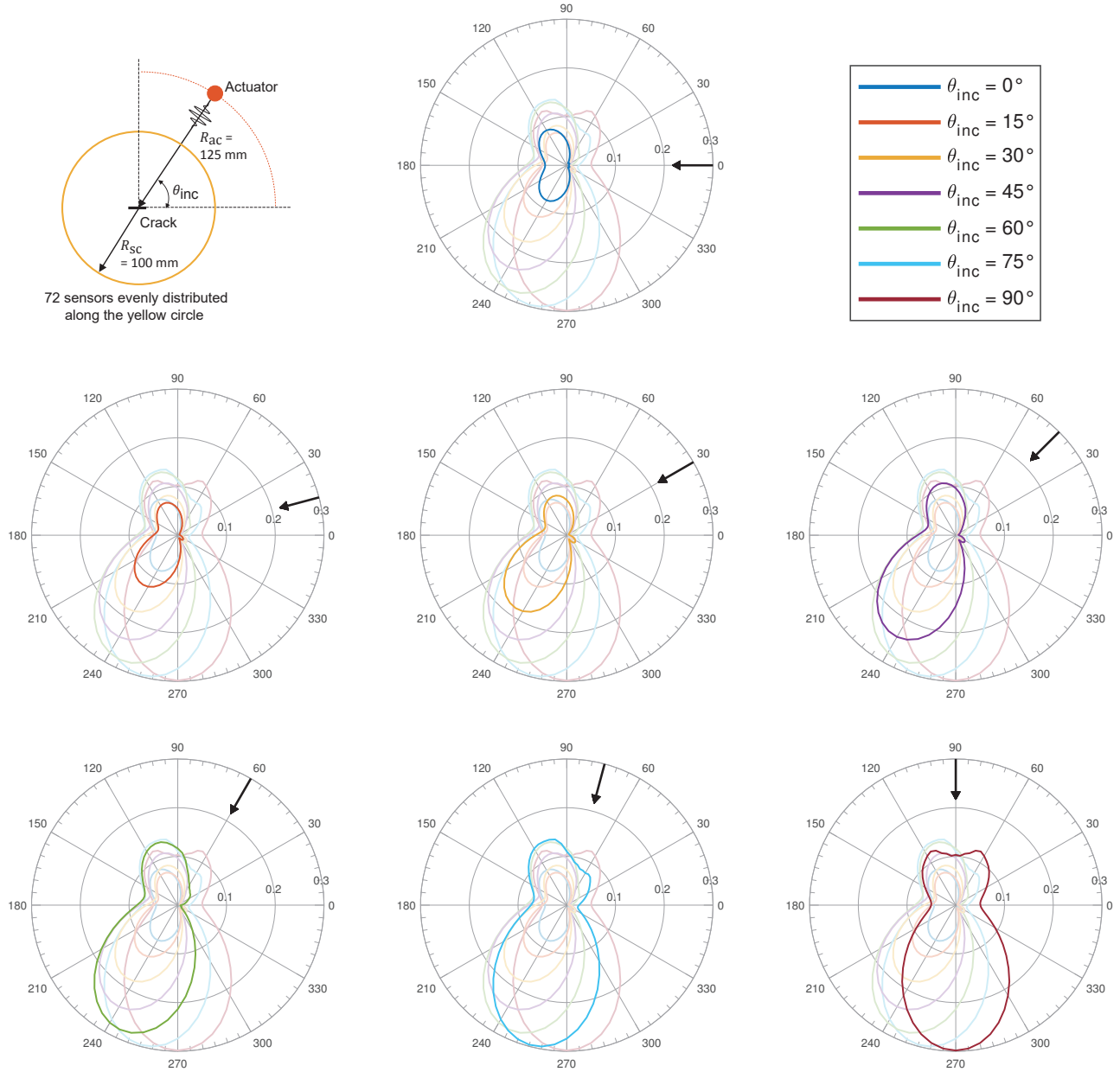


Figure 6.12: Influence of the incident angle on the scattered-signal strength received by the sensors evenly distributed along the circle centred at the crack center (actuation frequency: 300 kHz).

Next, the distance between the sensor and the crack center decreases from 100 mm to 75 mm, and then to 50 mm. Such decrease causes the variation of the angular distribution of the scattered-signal intensity, which is shown in Fig. 6.13. The scattered signals become stronger when the sensors are placed closer to the crack. This indicates that in order to minimize noise interference in experiments and real SHM applications, the covered area by the sensor network can be shrunk to increase the intensity of the scattered signals. When the noise strength is much

higher than the scattering intensity, the ‘scattered’ signals obtained from the measurement data include more noise than the scattered-wave information and as a result the RAPID algorithm could fail to localize the defect. This indication is especially important for the damage detection algorithms which depend highly on the time-of-flight of the scattered wave, such as the method of triangulation, because the time-of-flight obtained from the noise-dominated ‘scattered’ signal could deviate quite a lot from the actual value.

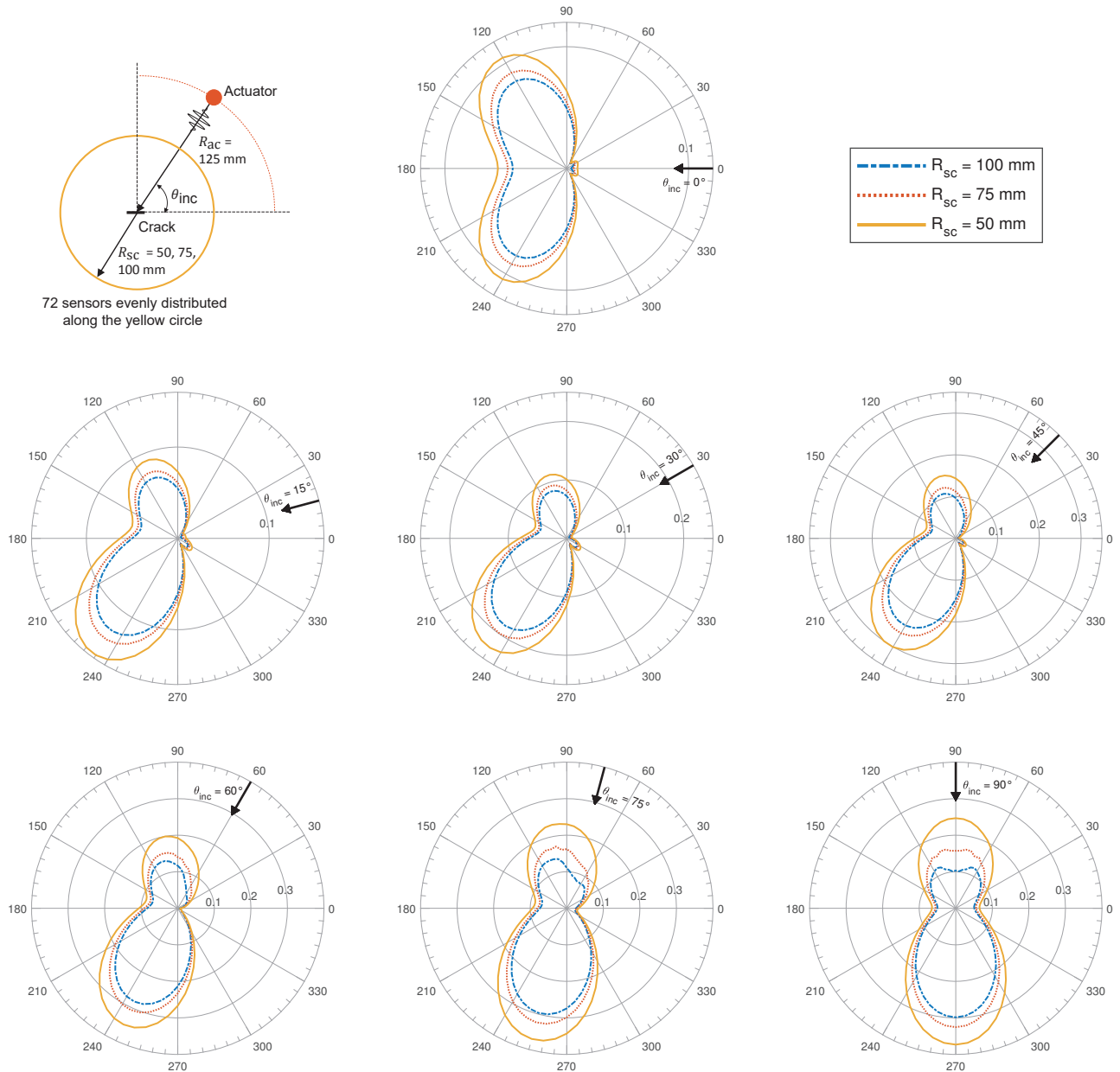


Figure 6.13: Influence of the distance between the crack center and the sensor on scattered-signal strength received by the sensors (actuation frequency: 300 kHz).

6.7 Summary

In this chapter, the boundary element formulations based on Kane-Mindlin and Mindlin plate theories have been first applied to the numerical modelling of Lamb-wave structural health monitoring (SHM) of plates. Both S0 and A0 modes can be described well using such formulations below the cut-off frequency of the A1 mode. The cracks are modelled using the dual boundary element method for these two plate theories.

Thin piezoelectric discs which are attached to the plate are used as both actuators and sensors. In order to incorporate the piezoelectric actuators into the BEM formulations efficiently and simply, an equivalent pin-force model, a new actuator model, has been proposed. The piezoelectric sensors are treated as capacitors and the output voltage is linked to the total in-plane strains within the sensing region.

The sensor output voltages obtained based on the boundary element simulations were shown to be in excellent agreement with the 3D FEM solutions and experimental results. In addition, some parametric studies were conducted using the BEM model, which showed that in order to increase the intensity of the sensor signals due to the scattered waves by a crack and mitigate noise interference in real SHM applications, sensors should be placed closer to the crack.

The proposed BEM model is very promising in modelling Lamb waves and wave-crack interaction because of the advantages of boundary-only discretization and low labor-cost requirement in mesh generation. This model is especially suitable for providing a large amount of data for a plate with many different crack parameters, such as length, orientation and location, because only the crack need to be remeshed and only coefficients associated with the crack in the global system coefficient matrix need to be updated.

Chapter 7

Two-dimensional spectral boundary element method

7.1 Introduction

When the boundary element model proposed in Chapter 6 is applied to large plate structures and high-frequency structural health monitoring applications, a large number of quadratic boundary elements which have low interpolation accuracy are required, leading to high computational costs. This chapter aims to explore the potential of high-order elements in increasing the computational efficiency.

Ultrasonic elastic waves are widely used in many engineering fields, such as non-destructive testing (NDT) and structural health monitoring (SHM). Numerical modelling plays an important role in these fields because it helps us to understand the behaviours of these waves in both intact and damaged structures. The finite element method (FEM) is an established numerical method for the modelling of wave propagation. However, this method is time-consuming for high-frequency elastic waves because their short wavelength results in very small element size. In order to overcome this problem, alternative numerical methods, such as the spectral element

method (SEM) and the boundary element method (BEM), can be used.

SEM is very similar to FEM, but the key difference is the element type used in the discretization. In SEM, the meshing is carried out with spectral elements in which nodal locations are determined by the zeros or extremes of high-order orthogonal polynomials, such as Chebyshev orthogonal polynomials. These high-order elements result in higher interpolation accuracy. Moreover, after adopting an integration rule which is based on the same family of orthogonal polynomials, the mass matrix can be diagonalized [134]. The elastic wave problems have been solved by using 2D SEM [135], plate SEM [90] and 3D SEM [136].

BEM is also an effective alternative to FEM for modelling ultrasonic elastic waves. The efficiency of BEM can be greater, due to the fact that only boundaries, rather than the entire domain of the structure, is discretized. Therefore, the total number of degrees of freedom is considerably reduced. This advantage becomes more significant when modelling high-frequency wave propagation in large-scale structures. Furthermore, the dual boundary element method (DBEM) provides a natural and simple way to model cracks by treating them as additional boundaries, and as a result it is very suitable to solve wave-crack interaction problems. A 3D BEM has been developed by Zou et al. [71] to model ultrasonic waves in beam structures and to provide data for damage detection.

On the basis of the successful applications of both SEM and BEM, it is anticipated that the introduction of spectral elements into BEM should make BEM more computationally efficient. Zou and Aliabadi [137] examined this and proposed a 3D spectral BEM. They compared the convergence speed of the spectral BEM with that of the conventional BEM in the Laplace domain and found that the computational efficiency of the spectral BEM was greater. Then, they [138] applied the spectral BEM to the modelling of ultrasonic elastic waves in a damaged 3D piezoelectric smart beam structure.

In the 3D spectral BEM, 2D spectral elements are used. However, these 2D elements are not suitable to the boundary element methods in which line elements are used for the discretization.

Thus, in this chapter, for the first time, the performance of 1D spectral elements is evaluated by introducing them into the 2D Laplace-transform BEM to solve elastodynamic problems. In addition, 1D discontinuous spectral elements are incorporated into the 2D DBEM to model cracks. The proposed 2D spectral BEM and the conventional 2D BEM are compared in terms of computational efficiency in three examples. It is worth mentioning that these 1D spectral elements are also applicable to the boundary element plate formulations.

7.2 Spectral boundary element discretization

The Laplace-transform dual boundary element formulation for the 2D elastodynamic problems, which has been introduced in Chapter 2, is adopted in this chapter. In the spectral BEM, high-order spectral elements are used for discretizing displacements and tractions, whereas low-order elements are used for representing geometric boundaries.

The nodes are unequally spaced in a spectral element, which is different from the elements in the conventional BEM. The locations of nodes in a spectral element are determined by the zeros or extrema of orthogonal polynomials. There are three basic types of spectral elements: Gauss-Legendre elements, Lobatto elements, and Chebyshev elements [137]. In an n -node Gauss-Legendre element, the intrinsic coordinates of nodes are the roots of n -th Legendre polynomials:

$$P_n(\xi) = 2^n \sum_{k=0}^n \xi^k \binom{n}{k} \binom{\frac{n+k-1}{2}}{n} \quad (7.1)$$

which consists of simple monomials and the multiplicative formula of the binomial coefficient. The intrinsic coordinates of nodes in a Lobatto element are given by solving the following equation:

$$(1 - \xi^2)P'_{n-2}(\xi) = 0 \quad (7.2)$$

where $P'_{n-2}(\xi)$ is the first derivative of the Legendre polynomials $P_{n-2}(\xi)$. In a Chebyshev

spectral element, the intrinsic coordinates of nodes can be written explicitly:

$$\xi_k = -\cos \left[\frac{\pi (k-1)}{n-1} \right], \quad k = 1, 2, \dots, n \quad (7.3)$$

After the locations of nodes are obtained, the shape functions of spectral elements can be derived from Lagrange interpolating polynomials and are expressed as follows:

$$N_m(\xi) = \prod_{j=1, j \neq m}^n \frac{\xi - \xi_j}{\xi_m - \xi_j}, \quad m = 1, 2, \dots, n \quad (7.4)$$

The displacement and traction fields are interpolated with high-order shape functions in the following manner:

$$\begin{aligned} \tilde{u}_\alpha(\xi) &= \sum_{m=1}^M N_m(\xi) \tilde{u}_\alpha^m \\ \tilde{t}_\alpha(\xi) &= \sum_{m=1}^M N_m(\xi) \tilde{t}_\alpha^m \end{aligned} \quad (7.5)$$

where the shape functions N_m are polynomials of degree $M-1$; $\tilde{u}_\alpha^m$ and $\tilde{t}_\alpha^m$ are nodal displacements and tractions respectively.

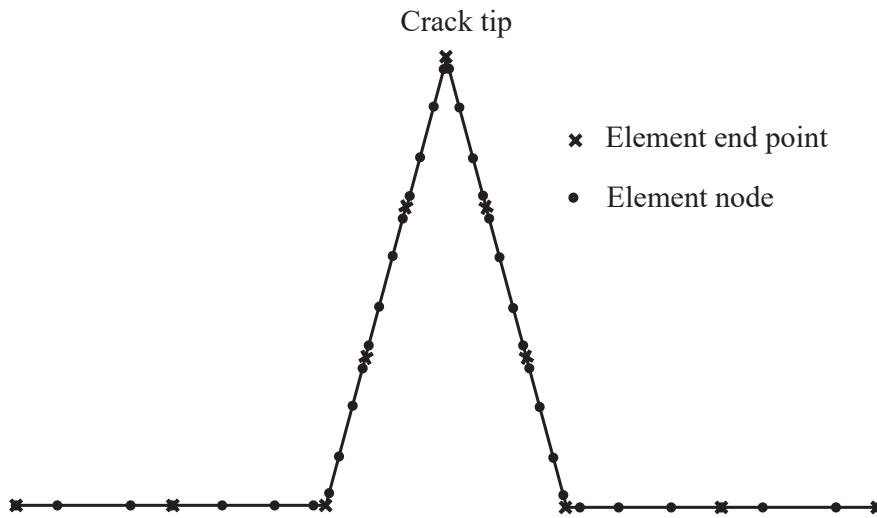


Figure 7.1: Boundary discretization using 4-node spectral elements.

Spectral elements with 4 nodes, as an example, are chosen to illustrate how to represent field variables on cracked boundaries based on spectral elements, which is shown in Fig. 7.1. The crack surfaces are discretized using Gauss-Legendre elements, which are discontinuous spectral elements. Continuous spectral elements, such as the Lobatto elements, are used for all non-crack boundaries except for the intersection between a crack and an edge. The intersection is modelled with a semi-discontinuous element. There are two steps to form a semi-discontinuous element. Firstly, the intrinsic coordinates of the semi-discontinuous element are set equal to those of a Gauss-Legendre element. Secondly, the node which is furthest from the intersection point is shifted to the end point of the element. This means that this shifted node is shared between the semi-discontinuous element and its adjacent continuous element.

As mentioned in Subsection 2.4.7, the so-called quarter-point discontinuous element are used to represent the field variables more accurately in the vicinity of the crack tip. When it comes to the discontinuous spectral element, it can be modified to create the corresponding quarter-point spectral element using the following method. The local coordinates of the nodes in a discontinuous spectral element are denoted by ξ_k . The quarter-point discontinuous spectral element has the same local coordinates ξ_k and associated shape functions, but nodal locations on the crack boundary are redefined based on the following expression [55]:

$$\begin{aligned} r_k &= \frac{l_e}{4}(1 - \xi_k)^2, & \text{for the crack tip at } \xi = 1 \\ r_k &= \frac{l_e}{4}(1 + \xi_k)^2, & \text{for the crack tip at } \xi = -1 \end{aligned} \tag{7.6}$$

where r_k denotes the distance between the node and the crack tip and l_e is the element length.

In the following sections, quarter-point elements refer to quarter-point discontinuous spectral element for short. Figs. 7.2a and 7.2b present the quarter-point elements modified from the quadratic element and the 4-node Gauss-Legendre element, respectively. In these two figures, r represents the distance from the crack tip. It can be seen from these figures that the nodes in quarter-point elements are concentrated toward the crack tip.

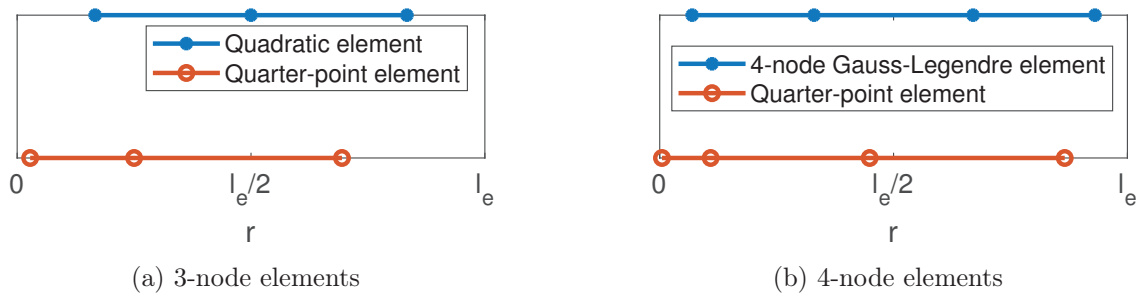


Figure 7.2: Discontinuous element and its corresponding quarter-point element.

When the geometry boundaries are simple and not highly curved, their approximation can be achieved by using low-order elements, such as 2-node linear elements or quadratic elements. Thus, the geometry in a low-order element is interpolated using the following expressions:

$$x_{\alpha}(\xi) = \sum_{g=1}^G N_g(\xi) x_{\alpha}^g \quad (7.7)$$

in which x_{α}^g are nodal coordinates in the global system and G is the number of nodes in the element.

As introduced in Subsection 2.4.8, the dynamic fundamental solutions can be written into two parts: the static fundamental solutions, and the modified terms. The element integrations of these two parts are carried out separately. It is worth mentioning that the modified terms include the Laplace-transform parameter. When the imaginary part of the Laplace-transform parameter is sufficiently large, the modified terms become highly oscillatory. Thus, more Gauss points are required to obtain accurate integrals involving the oscillatory terms. Furthermore, the element integrations involving the modified terms require quite large number of integration points when the element involves the collocation point. In order to get accurate results with fewer Gauss points, the integration path is split at the collocation point and the integration is applied over each side of the collocation point.

After getting the unknown displacements and tractions in the Laplace domain, numerical inversion of Laplace transforms is conducted to compute the time-domain results. Although

both Formula (2.39) and Formula (2.40) can be used for the numerical inversion, Formula (2.40) is adopted in this chapter for two reasons. First, since Formula (2.39) provides inaccurate results near the end of a time period, it is difficult to compare the BEM results with those from other methods in the entire time domain. In addition, the imaginary part of the Laplace parameter in Formula (2.39) is much larger, which increases the tendency of the modified terms in the dynamic fundamental functions to be highly oscillatory and as a result more Gauss points are required for the numerical integration. This issue becomes more serious for higher-order spectral elements because of their longer element length.

7.3 The assessment of dynamic stress intensity factors

In this chapter, the dynamic stress intensity factors (DSIFs) were estimated using the method given in Ref. [55]. The implementation of this method is introduced by taking the quarter-point element based on the 4-node Lobatto element as an example (as shown in Fig. 7.3).

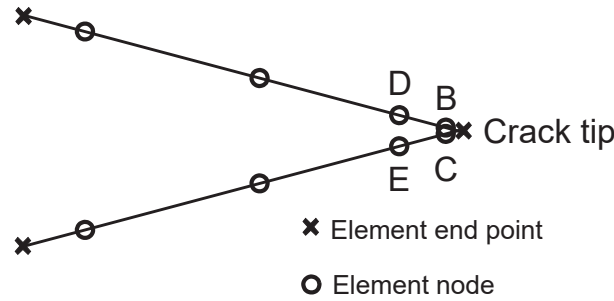


Figure 7.3: Quarter-point elements adjacent to a crack tip.

According to Ref. [55], after minimizing the sum of squared differences between the analytical and numerical values of crack opening displacements at node pairs BC and DE, the DSIFs can be estimated using the following expressions:

$$\begin{aligned} K_I &= \frac{\sqrt{2\pi\mu}}{\kappa+1} \frac{\Delta u_2^{BC} \sqrt{R^{BC}} + \Delta u_2^{DE} \sqrt{R^{DE}}}{R^{BC} + R^{DE}} \\ K_{II} &= \frac{\sqrt{2\pi\mu}}{\kappa+1} \frac{\Delta u_1^{BC} \sqrt{R^{BC}} + \Delta u_1^{DE} \sqrt{R^{DE}}}{R^{BC} + R^{DE}} \end{aligned} \quad (7.8)$$

where μ denotes the shear modulus; $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3 - \nu)/(1 + \nu)$ for generalized plane stress; ν is Poisson's ratio; the superscripts BC and DE denote the node pair; R is the distance between the node pair and the crack tip; Δu_1 and Δu_2 are the crack opening displacements along and normal to the crack respectively.

7.4 Examples

The comparisons between the computational efficiency of the spectral BEM and that of the standard BEM are carried out in three examples. Fracture problems are considered in the last two examples and the DSIFs are normalized with respect to $K_{00} = \sigma_0 \sqrt{\pi a}$, where σ_0 is the load amplitude and a denotes the crack length.

7.4.1 A square strip subjected to uniaxial compression

Fig. 7.4 shows a diagram of a square strip with the bottom fixed in the state of plane strain. This strip is subjected to uniform sinusoidal uniaxial compression σ_p with an amplitude of 1 GPa on the top side. The material properties are the Young's modulus ($E = 70$ GPa) and the material density ($\rho = 2700$ kg/m³). The Poisson's ratio is set to $\nu = 0$, which guarantees that the strip only has deformation along the compression direction.

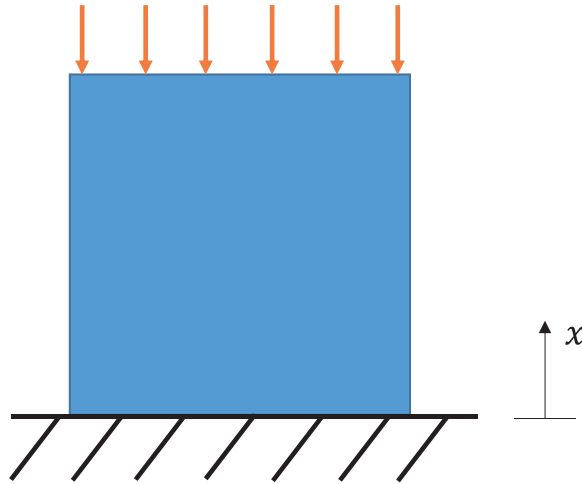
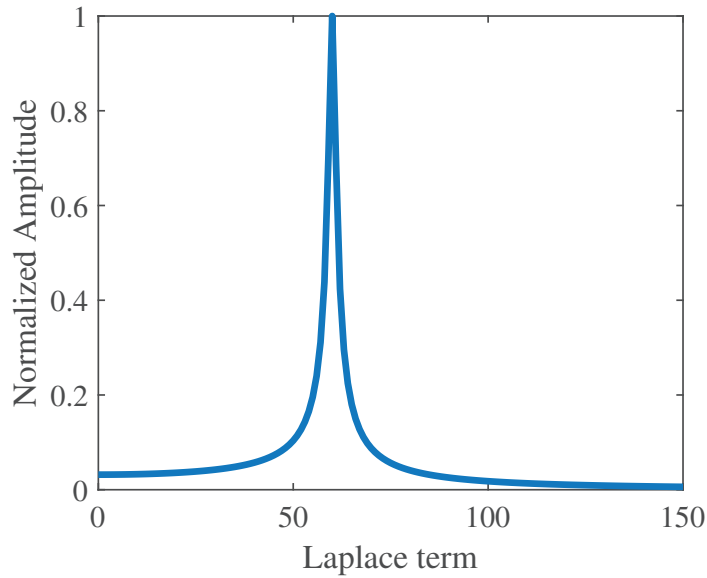


Figure 7.4: A square strip subjected to uniaxial compression.

Table 7.1: The side length and load frequency in the three models.

	Side length l_{st} ($\lambda_L = 50.92$ mm)	Load frequency (kHz)
Model A	$8\lambda_L$	100
Model B	$20\lambda_L$	100
Model C	$8\lambda_L$	300

Three models with different strip sizes and load frequencies are used in this example to compare the time efficiency of the spectral BEM and that of the standard BEM. The parameters of the three models are listed in Table 7.1, where λ_L denotes the wavelength of the longitudinal wave (wave velocity $c_1 = \sqrt{E/\rho}$) at 100 kHz.

Figure 7.5: The amplitude spectrum of the 100 kHz sinusoidal load ($T = 300$ μs) in the Laplace domain.

According to Ref. [137], the number of Laplace terms used for the numerical inversion of Laplace transforms can be chosen based on the amplitude spectrum of the load in the Laplace domain. For the sinusoidal load σ_p , the amplitude spectrum is determined by the frequency and the length of time period T . In Model A, T is equal to 300 μs and the amplitude spectrum is shown in Fig. 7.5. The normalized amplitude becomes less than 1% when the number of Laplace

terms exceeds 123, and as a result 150 Laplace terms are sufficient to provide satisfactory results for Model A. Similarly, 280 and 320 Laplace terms are used for Model B ($T = 800 \mu\text{s}$) and Model C ($T = 300 \mu\text{s}$), respectively.

The analytical expression of the strip displacement is given by [139]:

$$u_a(x, t) = \frac{1}{\rho c_1} \sum_{n=1}^{\infty} (-1)^{n-1} \left\{ H[g_1(x, t)] \int_0^{g_1(x, t)} \sigma_p(\tau) d\tau - H[g_2(x, t)] \int_0^{g_2(x, t)} \sigma_p(\tau) d\tau \right\} \quad (7.9)$$

where $g_1(x, t) = t - [(2n - 1) l_{\text{st}} - x] / c_1$ and $g_2(x, t) = t - [(2n - 1) l_{\text{st}} + x] / c_1$; H denotes Heaviside function. Fig. 7.6 shows the analytical solution of the displacement $u_a(l_{\text{st}}, t)$ on the top side of the strip in Model A.

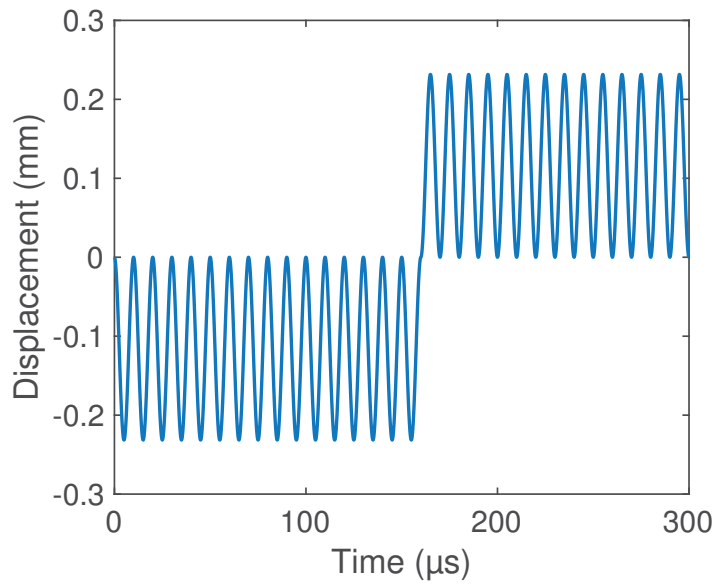


Figure 7.6: The analytical displacement on the top side of the strip in Model A.

When it come to the boundary element simulation, the convergence criterion is defined based on the relative error of the displacement on the top side of the strip. As shown in Fig. 7.6, there are many wave peaks and troughs. The relative errors are calculated at all the peaks and troughs over the entire time period. Among all of these relative errors, there is a maximum relative error for every node on the top side. If all the maximum relative errors with regard to all the top-side nodes are less than 1%, convergence is considered to be achieved and the

associated computational costs can be obtained.

In the spectral BEM, boundary displacements and tractions were discretized using Lobatto spectral elements with 4, 5, and 6 nodes, while linear elements were used to represent geometric boundaries. Three-noded quadratic isoparametric elements were used in the BEM. The local coordinates of the nodes in these elements are illustrated in Fig. 7.7. Each side of the square strip was discretized using the same number of boundary elements.

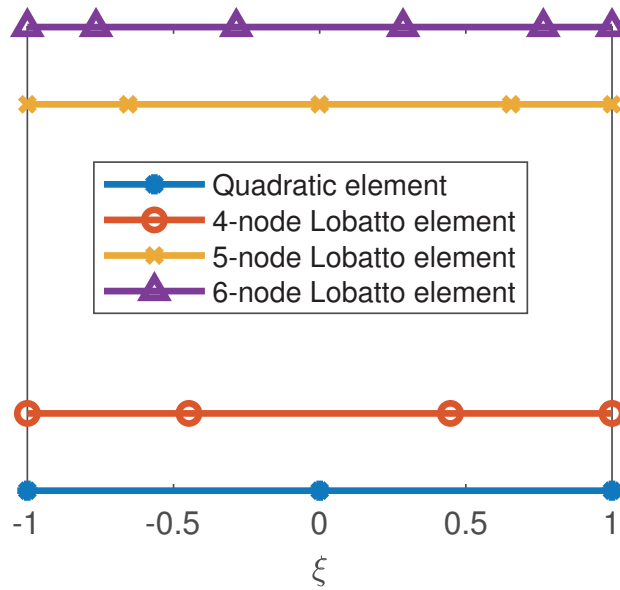


Figure 7.7: Nodal abscissas in the quadratic element and Lobatto elements.

Table 7.2: Comparison of computational costs among four boundary element types at convergence (Model A).

Element type	Number of nodes per wavelength	Number of Gauss points	Total number of nodes	Reduction in total number of nodes	Reduction in CPU time
3-node	7.4	4	232	/	/
4-node	6.5	5	204	12.1%	30.8%
5-node	6.1	6	192	17.2%	37.7%
6-node	6.4	10	200	13.8%	9.7%

Table 7.2 lists the computational costs which are strictly necessary to reach convergence with regard to four types of boundary elements. Fig. 7.8 shows that all the convergent results from the boundary element simulations agree very well with the analytic solution. It can be seen from Table 7.2 that spectral elements help to reduce the computational costs because fewer nodes are required. In addition, the number of nodes per wavelength varies slightly as the order of the spectral element increases. This is because the structure is not sufficiently large and the wave frequency is not sufficiently high.

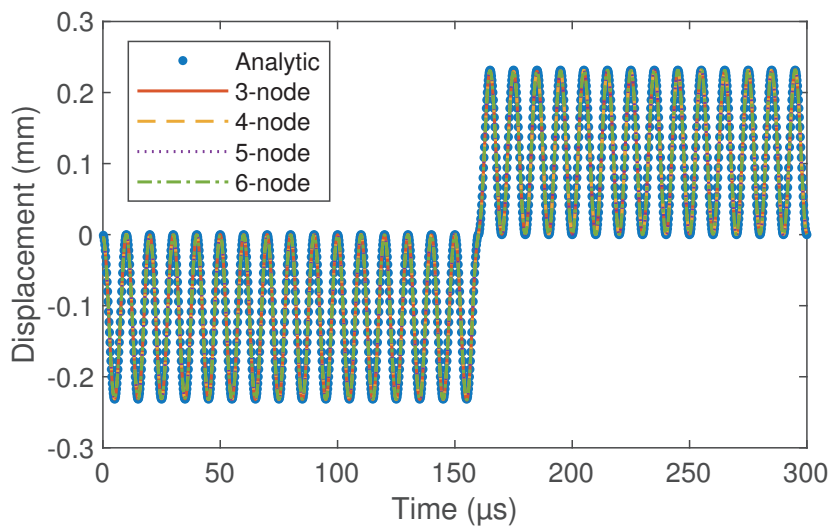


Figure 7.8: Displacement at the top side of the strip obtained based on different types of boundary elements and the analytical solution (Model A).

It should be noted that the number of Gauss integration points increases as the order of boundary element increases. The increasing order of shape functions is one reason. In addition, the Laplace-transform fundamental solutions tend to become highly oscillatory when the product of the element length and the imaginary part of the Laplace transform term increases. As the element order increases, the element becomes longer because fewer elements are required for the discretization. Thus, for longer elements, more Gauss integration points are required to deal with the integrals involving such oscillatory fundamental solutions.

As can be seen in Table 7.2, although the use of 6-node spectral elements leads to fewer nodes in total than the use of quadratic elements, the CPU time is not significantly saved.

This is because the number of Gauss integration points listed here is used for all the regular integrals. However, when the imaginary part of the Laplace term is smaller, the number of Gauss integration points can be reduced. Hence, if the number of Gauss integration points are set to be adaptive, the computational time can be reduced further for high-order spectral elements.

Table 7.3: Comparison of computational costs among four boundary element types at convergence (Model B).

Element type	Number of nodes per wavelength	Number of Gauss points	Total number of nodes	Reduction in total number of nodes	Reduction in CPU time
3-node	8.5	4	672	/	/
4-node	7.1	4	564	16.1%	44.2%
5-node	6.3	6	496	26.2%	60.1%
6-node	5.8	10	460	31.5%	67.9%

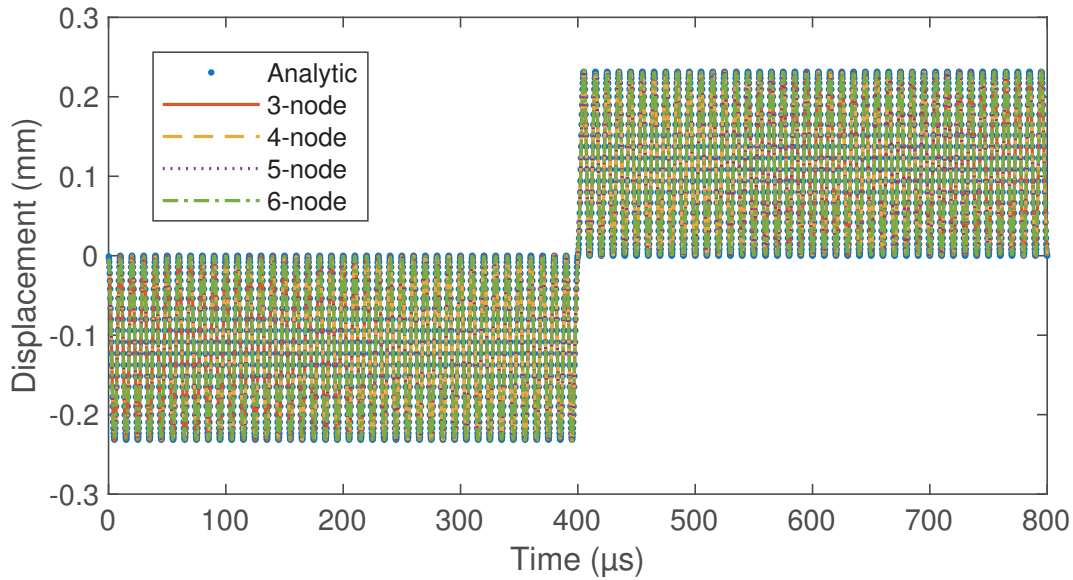


Figure 7.9: Displacement at the top side of the strip obtained based on different types of boundary elements and the analytical solution (Model B).

For Model B, the computational costs with regard to the four types of boundary elements

are given in Table 7.3. Fig. 7.9 shows the convergent numerical results which are in excellent agreement with the results from the analytic solution. It can be seen from Table 7.3 that the spectral BEM is more time-efficient than the standard BEM. Moreover, the boundary-element mesh consisting of the 6-node Lobatto elements leads to the fewest nodes and shortest CPU time.

The reductions in both total node number and CPU time resulting from the spectral BEM are extracted from Table 7.2 and Table 7.3 and then summarized in Table 7.4. As can be seen in Table 7.4, the percentage reductions in both the total number of nodes and CPU time increase when the side length of strip increases. This demonstrates that the spectral BEM becomes more time-efficient in modelling larger structures

Table 7.4: The influence of strip side length l_{st} on the computational efficiency of the spectral BEM at 100 kHz.

Element type	Reduction in total number of nodes		Reduction in CPU time	
	$l_{\text{st}} = 8\lambda_L$	$l_{\text{st}} = 20\lambda_L$	$l_{\text{st}} = 8\lambda_L$	$l_{\text{st}} = 20\lambda_L$
3-node	/	/	/	/
4-node	12.1%	16.1%	30.8%	44.2%
5-node	17.2%	26.2%	37.7%	60.1%
6-node	13.8%	31.5%	9.7%	67.9%

Table 7.5: Comparison of computational costs among four boundary element types at convergence (Model C).

Element type	Number of nodes per wavelength	Number of Gauss points	Total number of nodes	Reduction in total number of nodes	Reduction in CPU time
3-node	11.9	4	1136	/	/
4-node	7.8	4	744	34.5%	75.5%
5-node	7.0	6	672	40.9%	81.9%
6-node	6.5	6	620	45.4%	86.3%

When it comes to Model C, the load frequency is 300 kHz and the computational expenses for the four types of boundary elements are listed in Table 7.5. It can be seen that more reduction in computational costs can be achieved when higher-order spectral boundary elements are used. The convergent numerical results and the analytic solution are in excellent agreement, as shown in Fig. 7.10.

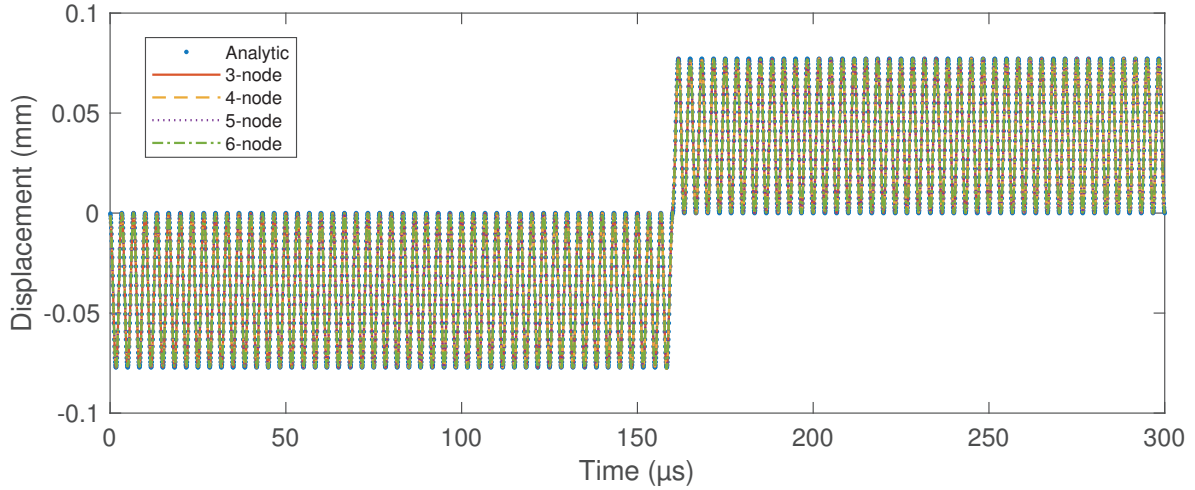


Figure 7.10: Displacement at the top side of the strip obtained based on different types of boundary elements and the analytical solution (Model C).

Table 7.6: The influence of load frequency on the computational efficiency of the spectral BEM.

Element type	Reduction in total number of nodes		Reduction in CPU time	
	$f = 100$ kHz	$f = 300$ kHz	$f = 100$ kHz	$f = 300$ kHz
3-node	/	/	/	/
4-node	12.1%	34.5%	30.8%	75.5%
5-node	17.2%	40.9%	37.7%	81.9%
6-node	13.8%	45.4%	9.7%	86.3%

Table 7.6 compares the improvements in computational efficiency due to the spectral BEM at different load frequencies. The data in Table 7.6 are extracted from Table 7.2 and Table 7.5. It can be seen from Table 7.6 that as the load frequency increases, the percentage reductions in both the total number of nodes and CPU time increase. It is demonstrated that the spectral

BEM is more time-efficient in modelling higher-frequency wave propagation.

Zou and Aliabadi [137] found that 36-node spectral boundary elements were more time-efficient than 25-node and 16-node spectral boundary elements in modelling three-dimensional structures. These three types of 2D spectral boundary elements can be treated as the extensions of 6-node, 5-node, and 4-node 1D spectral boundary elements from 2D spectral BEM to 3D spectral BEM, respectively. Thus, it seems that 6-node Lobatto elements should be the most time-efficient in modelling 2D structures among the three types of 1D spectral boundary elements. However, this is not true when the strip is small and the load frequency is low. There are three reasons for this disagreement.

First of all, the differences in terms of the number of nodes per wavelength among 4-node, 5-node and 6-node spectral elements are quite smaller than those among the corresponding spectral elements in the 3D spectral BEM. This means that when a small number of elements is used in the 2-D spectral BEM, the benefit of the 6-node spectral elements in the reduction of total number of nodes cannot be fully and explicitly realised. For example, Table 7.2 shows that when the side length of the strip l_{st} is equal to $8\lambda_L$, the 6-node Lobatto elements lead to a larger number of nodes per wavelength (6.4) than the 5-node Lobatto elements (6.1). Assuming that the number of nodes per wavelength related to the 6-node Lobatto elements is reduced to 5.9, the total number of nodes required for discretizing each side of the square strip should be 47.2 (5.9×8). Then the number of the 6-node Lobatto elements should be set to 9.24. However, the element number must be an integer, so at least 10 Lobatto elements should be used. Using this element number, the number of nodes per wavelength becomes 6.4 again, which is the same as the figure in Table 7.2.

Secondly, the conclusion in Ref. [137] was obtained in the Laplace domain, whereas in this chapter the comparisons were carried out in the time domain. Thus, errors from the numerical inversion of Laplace transforms should not be ignored. Lastly, Zou [137] did not focus more on the accuracy of results in the time domain, so fewer Laplace terms were required. As mentioned above, however, if more Laplace terms are used, higher-order spectral elements tend to require

more integration points to deal with the oscillatory integrals. As shown in Table 7.2, the number of Gauss points associated with the 6-node Lobatto elements is twice that with the 4-node Lobatto elements. As a result, although the 6-node elements lead to fewer nodes, the 4-node elements lead to the shortest CPU time.

In general, the results in this example indicate that 6-node spectral elements are not necessarily the most efficient for 2-D wave propagation problems. The 6-node spectral elements seem to be more suitable for modelling large structures under high-frequency load, but the integration should be checked carefully to determine if this kind of elements can lead to the shortest CPU time.

7.4.2 A plate with a central crack subjected to impact tensile load

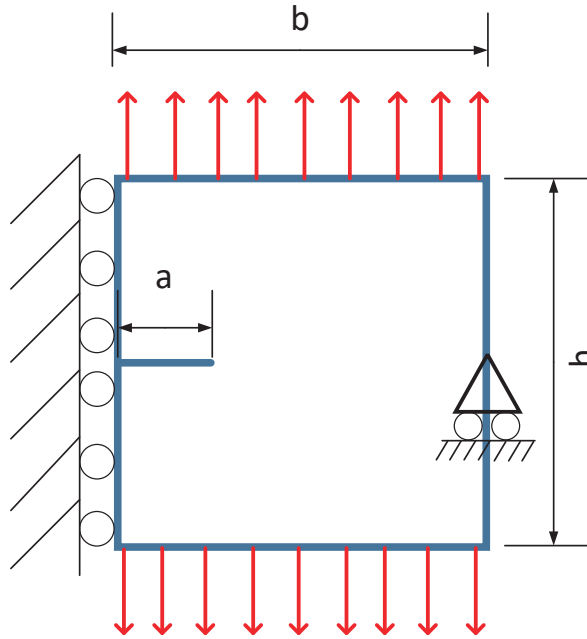


Figure 7.11: A plate with a central crack subjected to impact tensile load (half model).

A plate with a central crack in the state of plane strain (see Fig. 7.11) is considered in this example, and it is only necessary to model half plate owing to the symmetry property. Roller constraints are imposed along the symmetric axis. The middle point on the right side is constrained along the load direction. This model is adopted from Ref. [4], which aims to

validate the spectral DBEM codes used for this chapter and to examine the advantage of spectral elements in modelling cracks. It has a length of $b = 52$ mm and a width of $h = 40$ mm. The crack with a half-length of $a = 12$ mm is in the middle of the plate and parallel to the top side, which leads to a pure opening-mode fracture problem. The top and bottom sides are subjected to the impact tensile load $\sigma_t = \sigma_0 H(t)$, where σ_0 denotes the amplitude and $H(t)$ is the Heaviside function. The material properties are the Young's modulus, $E = 75.6$ GPa, the material density, $\rho = 2450$ kg/m³ and the Poisson's ratio, $\nu = 0.286$.

In the spectral BEM, spectral elements are used for discretizing field variables; non-crack boundaries and crack surfaces are modelled with the Lobatto elements and Gauss-Legendre elements, respectively. Linear elements are used to represent geometric boundaries. It should be mentioned that the orders of Lobatto elements and Gauss-Legendre elements are the same in this example. In the standard BEM, quadratic continuous and discontinuous elements are used to discretize non-crack boundaries and crack surfaces, respectively. The difference in terms of the nodal abscissas between the Gauss-Legendre elements and the quadratic discontinuous element is illustrated in Fig. 7.12.

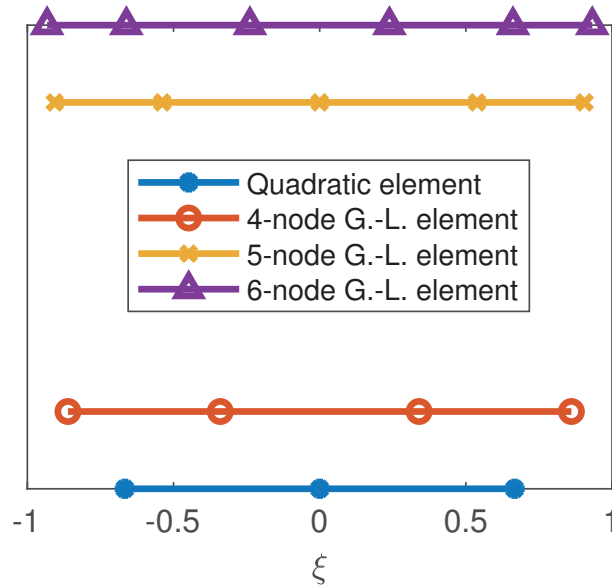


Figure 7.12: Nodal abscissas in the discontinuous quadratic element and Gauss-Legendre elements.

According to Fedelinski et al. [4], 50 Laplace terms were enough to give satisfactory results

in the time domain. Thus, this number of Laplace terms was also used here. Fig. 7.13 shows the convergent normalized DSIF K_I/K_{00} adopted from Fedelinski's paper [4] based on the standard BEM, in which 112 elements and 16 elements were used to discretize non-crack boundaries and each crack surface, respectively. This DSIF curve is considered to be accurate and chosen as the baseline to carry out convergence studies next. For every type of the spectral elements, the relative error was calculated at each discretized time point at which the normalized DSIF is larger than 1. If all of the relative errors at the selected time points are less than 1%, convergence is considered to be achieved.

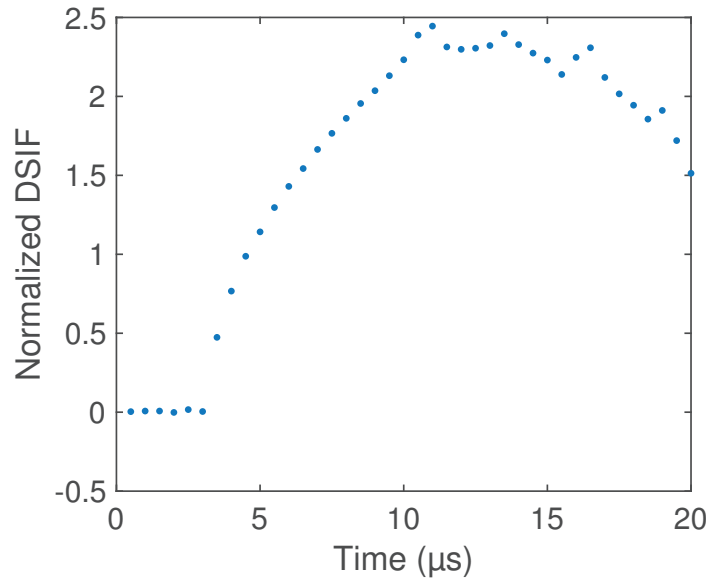


Figure 7.13: Convergent normalized DSIF adopted from Ref. [4] for the plate with a central crack subjected to impact tensile load.

The normalized DSIFs obtained using the spectral BEM of different orders and the standard BEM are presented in Fig. 7.14, which shows clearly that the spectral BEM solution is in excellent agreement with the standard BEM solution. The numbers of boundary elements required for convergence in the spectral BEM and the standard BEM are listed in Table 7.7. In order to focus only on the discontinuous elements, a large number of boundary elements were used for non-crack boundaries. In addition, the number of nodes on non-crack boundaries with regard to the spectral BEM was set to be comparable to, but no more than that with regard to the standard BEM. It can be seen from Table 7.7 that fewer nodes are required

for modelling the crack with the spectral elements than those with the quadratic elements. Moreover, as the element order increases, the total number of nodes required for discretizing the crack surfaces decreases. These results indicate that even though the load is not harmonic, the use of discontinuous spectral elements leads to less computational cost than the use of the discontinuous quadratic elements in modelling cracks.

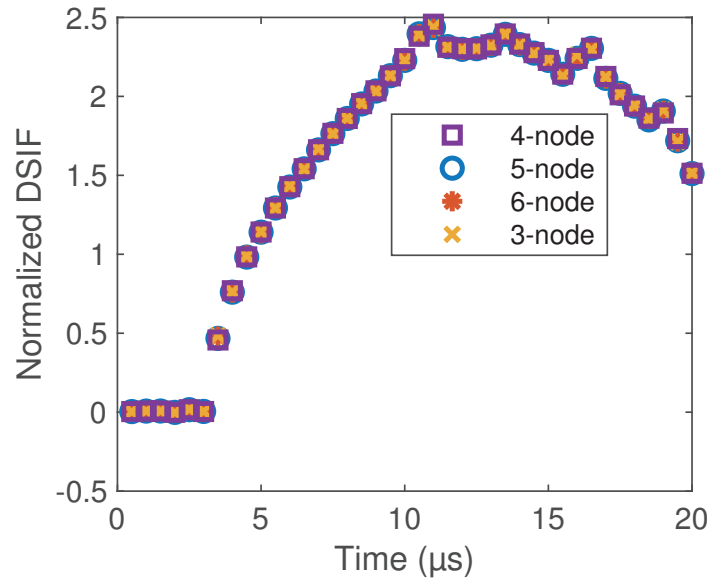


Figure 7.14: Normalized DSIF for the plate with a central crack subjected to impact tensile load computed based on boundary elements with different numbers of nodes.

Table 7.7: Comparison of the computational costs required for modelling the plate with a central crack subjected to impact tensile load using the spectral BEM and standard BEM.

Element type	Number of nodes on non-crack boundaries	Number of elements on each crack surface	Number of nodes on each crack surface
3-node	225	11	33
4-node	223	6	24
5-node	225	4	20
6-node	201	3	18

7.4.3 A central cracked plate subjected to cyclic tensile load

The model, including the material, in the previous example (see Fig. 7.11) is adopted in this example, except for its size and the load. The longitudinal wavelength at 100 kHz is 632.75 mm. The side length ($b = h$) of the plate and the crack length are set to 10 times the wavelength and twice the wavelength, respectively.

The load $\sigma_t = \sigma_0 [\sin(\omega t) + H(t)]$ is composed of a 100 kHz sinusoidal load and an impact tensile load, where σ_0 denotes the amplitude and $H(t)$ is the Heaviside function. The time-domain load normalized with respect to the amplitude σ_0 and its amplitude spectrum in the Laplace domain are shown in Fig. 7.15. When 130 Laplace terms are used for the numerical inversion of Laplace transforms, satisfactory time-domain results can be obtained.

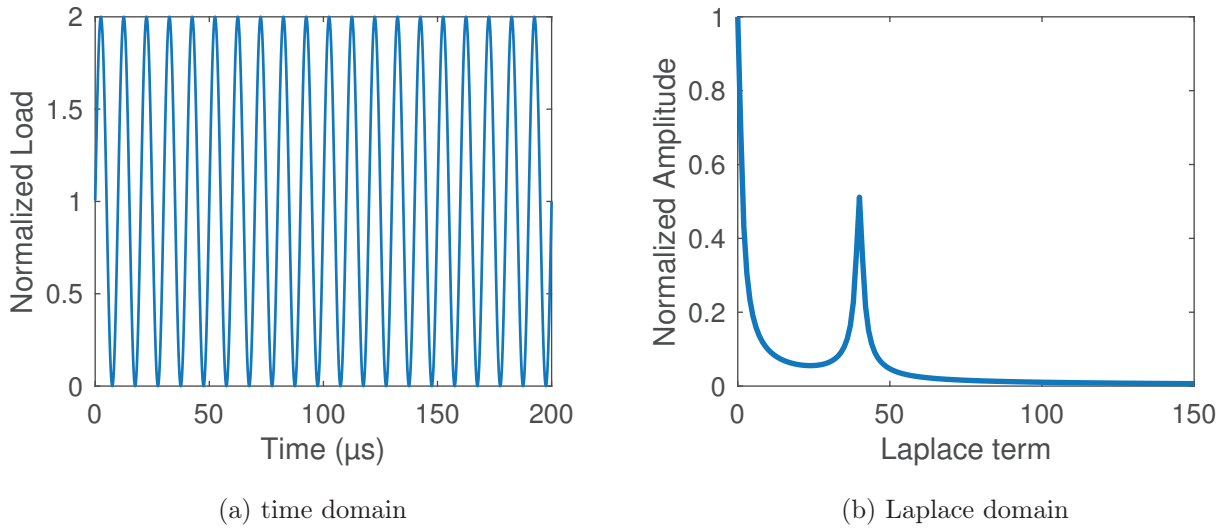


Figure 7.15: The time-domain load applied on the central cracked plate and its amplitude spectrum in the Laplace domain.

Fig. 7.16a presents the baseline normalized DSIF, which is obtained based on the standard BEM with much more elements than strictly necessary. To get this baseline, each side of the plate and each surface of the crack were discretized with as many as 80 elements and 50 elements respectively, leading to 941 nodes in total. For the boundary element analysis using different types of elements, the relative error was calculated at the time points when wave peaks appear. Convergence is considered to be achieved when all the relative errors are less than 1%.

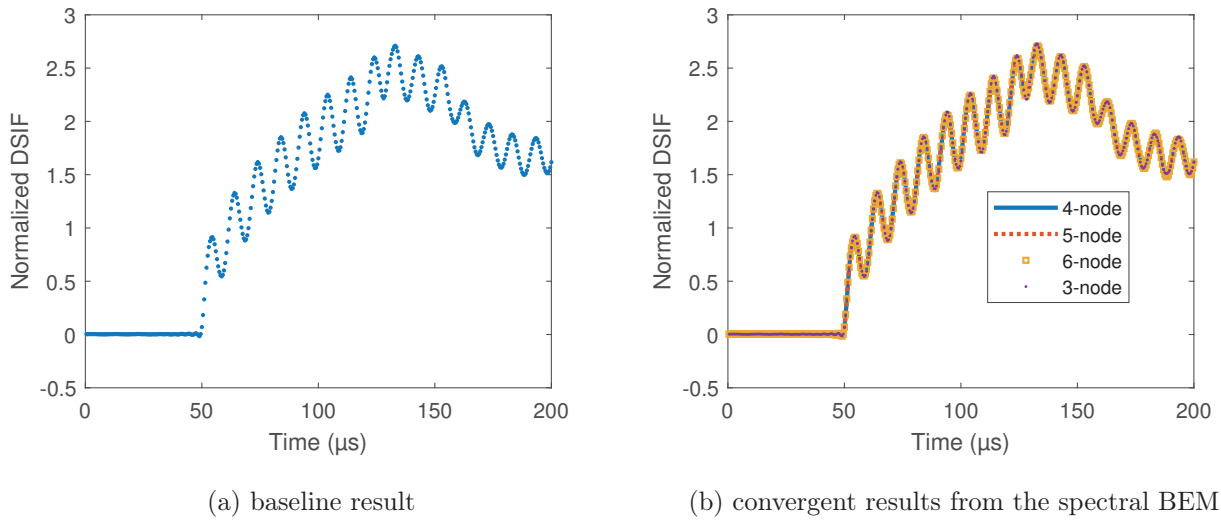


Figure 7.16: Normalized DSIF for the central cracked plate subjected to cyclic tensile load.

Table 7.8: Comparison of the computational costs required for modelling the central cracked plate subjected to cyclic tensile load using the spectral BEM and standard BEM.

Element type	Number of Gauss points	Total number of nodes	Reduction in total number of nodes	Reduction in CPU time
3-node	10	387	/	/
4-node	12	270	30.2%	29.3%
5-node	12	255	34.1%	36.4%
6-node	16	235	39.3%	25.9%

On the basis of the above-mentioned convergence criterion, the smallest number of nodes and the smallest number of Gauss integration points can be determined for every type of boundary elements, and the corresponding convergent results are presented in Fig. 7.16b. Table 7.8 lists the necessary computational costs used in the cases with different types of elements. It can be seen clearly that fewer nodes are required with the spectral elements than those with the quadratic elements. As the order of element increases, the total number of nodes becomes smaller. Although the integrals associated with the spectral elements require more Gauss integration points, the spectral BEM is still more time-efficient than the standard BEM.

As mentioned in the first example, CPU time can be further reduced if the number of integration points is chosen adaptively. In the case with the 6-node spectral elements, the total number of nodes is the smallest, but the number of integration points is the largest. Thus, the amount of CPU time in this case is not the shortest. More studies should be carried out to check if the adaptive integration enables the spectral BEM with 6-node elements to be the most time-efficient.

It is worth mentioning that the convergence studies in terms of the integration are essential for the problems of high-frequency wave propagation. As the frequency increases, more Laplace terms should be used to provide a satisfactory accuracy in the time domain. However, this means the integrands become to oscillate more rapidly and more integration points are required to achieve convergence.

7.5 Summary

In this chapter, spectral boundary elements have been introduced for the first time into the two-dimensional Laplace-transform boundary element method (BEM) for the analyses of high-frequency wave propagation and dynamic fracture problems. In the spectral BEM, non-crack boundaries and crack surfaces are discretized using the continuous spectral elements, such as the Lobatto elements, and the discontinuous elements, such as the Gauss-Legendre elements, respectively.

Since the spectral elements have higher interpolation accuracy than the quadratic elements, fewer nodes and shorter CPU time are required to reach convergence using the spectral BEM. This advantage of the spectral BEM has been demonstrated in three examples involving both high-frequency wave propagation and dynamic fracture.

Generally speaking, when the structure size is sufficiently large and the wave frequency is sufficiently high, we can make full use of the higher interpolation accuracy of the higher-order spectral elements and as a result fewer nodes are required to reach convergence. However, this

is not necessarily true for small-sized structures and low-frequency cases.

When it comes to the boundary integrals, the integrand becomes highly oscillatory for higher-order spectral elements and more Gauss integration points should be used to obtain accurate numerical integration. Thus, higher-order spectral elements could lead to fewer nodes, but shorter CPU time is not necessarily achieved. In order to improve the time-efficiency of the higher-order spectral BEM, adaptive Gaussian quadrature could be used for the integrals involving different Laplace terms.

Chapter 8

Conclusions and recommendations for future research

8.1 Conclusions

In this thesis, new boundary element formulations for Kane-Mindlin (K-M) and Mindlin plate theories were developed. They can be applied to solve elastodynamic crack problems including dynamic fracture and Lamb-wave scattering by the crack. The later application can be further extended to allow the modelling for Lamb-wave based crack detection when piezoelectric actuators and sensors are incorporated into the formulations. The main advantage of these formulations is that only plate edges and crack surfaces are required to be discretized using 1D line elements. The dynamic problems were solved in the Laplace-transform domain to avoid computing the domain integrals due to inertial terms.

The formulations developed based on K-M and Mindlin plate theories were able to model the S0 and A0 Lamb-wave modes respectively below the cut-off frequency of the A1 mode with high accuracy and efficiency, as demonstrated in Chapter 3. The newly proposed formulation for the K-M theory was used to model the S0 mode, which allows capturing the dispersion

behaviour of this mode at high frequencies compared with that for the conventional 2D generalized plane stress theory. In addition, in order to describe the dispersion property of the A0 mode accurately, existing dynamic fundamental solutions for the Mindlin theory were slightly modified.

The above-mentioned formulations were extended to corresponding dual boundary element formulations, which allow to analyse the S0-mode and A0-mode scatterings by the crack. After incorporating the actuator and sensor made of piezoelectric lead-zirconate-titanate (PZT) into these formulations, this BEM-based model was used as an alternative numerical tool for Lamb-wave structural health monitoring. An equivalent pin-force model was proposed for the first time to represent the actuator in Chapter 6. In addition, the sensor was treated as a capacitor and the output voltage was proportional to the total in-plane strains at the sensor-plate interface. The simulated sensor responses were in excellent agreement with 3D finite element solutions and experiment results.

These dual formulations can also be used for the dynamic fracture analyses. The dynamic stress intensity factors were estimated based on the crack opening displacements. The dual formulation based on the Mindlin theory was applicable to the cracked plate subjected to bending loads. The dual formulation for the K-M theory can be treated as an improved version of the 2D dual formulation based on the generalised plane stress theory. It is able to take the plate-thickness effect into account and to capture an O-mode fracture, which is coupled with Mode II fracture due to Poisson's effect.

A spectral boundary element method (spectral BEM), in which boundaries were discretized using high-order spectral elements, was shown to be more time-efficient for solving 2D elastodynamic problems than the conventional BEM based on linear or quadratic elements. The spectral BEM became more efficient for large structures and high-frequency dynamic problems. As the order of spectral boundary elements increased, more Gauss integration points were required to compute the boundary integrals accurately. Thus, in the spectral BEM, although the boundary element meshing with higher-order spectral elements resulted in fewer nodes, shorter

CPU time was not necessarily achieved if the number of Gaussian integration points was fixed during the whole analysis.

8.2 Recommendations for future research

In this thesis, the dynamic crack problems in plates have been solved efficiently using the Laplace-transform DBEM for first-order plate theories. On the basis of these works, some suggestions for future research are given here.

First of all, the proposed dual boundary element formulations could be applied to assembled plate structures, such as an aircraft wing box. The compatibility and equilibrium equations along the plate/plate junctions are the same as those for static problems, which can be found in the open literature. Unlike single-plate case, this multi-domain problem leads to sparse coefficient matrices.

Secondly, the spectral elements could be introduced into the boundary element formulations for plate theories. In this thesis, the spectral BEM for 2D problems has been demonstrated to be more time-efficient than the conventional 2D BEM. Since 1D line elements are used in both the 2D BEM and the BEM based on plate theories, the author believes that the spectral BEM for plate theories should also inherit this benefit. This improvement is especially valuable for high-frequency Lamb wave problems in assembled plate structures.

Thirdly, the system consisting of the PZT transducers and the host plate could be solved in a coupled manner. In this thesis, such coupling system is treated using an uncoupled method and thus the electro-mechanical impedance responses of the system cannot be computed based on the proposed BEM model. In order to couple the transducers to the boundary element formulations efficiently and easily, the PZT transducers could be modelled using a semi-analytical finite element method, in which only in-plane discretization is required and the thickness variation is analytically represented. However, the ill-conditioned problem of such method at high frequencies should be solved first.

Fourthly, the boundary element formulations for higher-order plate theories could be developed. The first-order plate theories are only suitable for modelling the S0 and A0 modes below the cut-off frequency of the A1 mode. In order to apply BEM to higher frequency cases, the dynamic fundamental solutions for higher-order plate theories should be derived first.

Finally, the proposed boundary element formulations could be extended to shell structures and anisotropic plates, which is highly desired in engineering. The Laplace-transform boundary element formulations for shell structures and anisotropic plates allow to solve the ultrasonic guided wave problems in pipes and composites, respectively. However, new Laplace-transform fundamental solutions are required.

Bibliography

- [1] Joseph L Rose. *Ultrasonic guided waves in solid media*. Cambridge university press, 2014.
- [2] Dietmar Gross and Thomas Seelig. *Fracture mechanics: with an introduction to micromechanics*. Springer, 2017.
- [3] PH Wen, M. H. Aliabadi, and JZ Zhang. Displacement discontinuity method for fracture mechanics analysis of reissner plates: static and dynamic. *International journal of fracture*, 135(1):95–116, 2005.
- [4] P Fedelinski, M.H. Aliabadi, and DP Rooke. The laplace transform dbem for mixed-mode dynamic crack analysis. *Computers & structures*, 59(6):1021–1031, 1996.
- [5] Charles R Farrar and Keith Worden. An introduction to structural health monitoring. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 365(1851):303–315, 2007.
- [6] Fu-Kuo Chang. *Structural health monitoring 2000*. CRC Press, 1999.
- [7] Victor Giurgiutiu. *Structural health monitoring: with piezoelectric wafer active sensors*. Academic Press, 2007.
- [8] Zhongqing Su and Lin Ye. *Identification of damage using Lamb waves: from fundamentals to applications*, volume 48. Springer Science & Business Media, 2009.
- [9] Xiang Zhao, Roger L Royer, Steven E Owens, and Joseph L Rose. Ultrasonic lamb

- wave tomography in structural health monitoring. *Smart Materials and Structures*, 20(10):105002, 2011.
- [10] Rolf Lammering, Ulrich Gabbert, Michael Sinapius, Thomas Schuster, and Peter Wierach. *Lamb-Wave Based Structural Health Monitoring in Polymer Composites*. Springer, 2017.
- [11] Victor Giurgiutiu and JingJing Bao. Embedded-ultrasonics structural radar for in situ structural health monitoring of thin-wall structures. *Structural Health Monitoring*, 3(2):121–140, 2004.
- [12] Kevin R Leonard and Mark K Hinders. Lamb wave tomography of pipe-like structures. *Ultrasonics*, 43(7):574–583, 2005.
- [13] Zahra Sharif-Khodaei, Mazdak Ghajari, and MH Aliabadi. Determination of impact location on composite stiffened panels. *Smart Materials and Structures*, 21(10):105026, 2012.
- [14] Ye Lu, Lin Ye, and Zhongqing Su. Crack identification in aluminium plates using lamb wave signals of a pzt sensor network. *Smart Materials and Structures*, 15(3):839, 2006.
- [15] V. T. Rathod and D. Roy Mahapatra. Ultrasonic lamb wave based monitoring of corrosion type of damage in plate using a circular array of piezoelectric transducers. *NDT & E International*, 44(7):628–636, 2011.
- [16] Seth S Kessler, S Mark Spearing, and Constantinos Soutis. Damage detection in composite materials using lamb wave methods. *Smart materials and structures*, 11(2):269, 2002.
- [17] Fangxin Zou, Jing Rao, and MH Aliabadi. Highly accurate online characterisation of cracks in plate-like structures. *NDT & E International*, 94:1–12, 2018.
- [18] Ryan Watkins and Ratneshwar Jha. A modified time reversal method for lamb wave based diagnostics of composite structures. *Mechanical Systems and Signal Processing*, 31:345–354, 2012.

- [19] X Lin and FG Yuan. Experimental study applying a migration technique in structural health monitoring. *Structural health monitoring*, 4(4):341–353, 2005.
- [20] Jennifer E Michaels. Detection, localization and characterization of damage in plates with an in situ array of spatially distributed ultrasonic sensors. *Smart Materials and Structures*, 17(3):035035, 2008.
- [21] Lingyu Yu and Victor Giurgiutiu. In-situ optimized pwas phased arrays for lamb wave structural health monitoring. *Journal of mechanics of materials and structures*, 2(3):459–487, 2007.
- [22] Ye Lu, Lin Ye, Zhongqing Su, Limin Zhou, and Li Cheng. Artificial neural network (ann)-based crack identification in aluminum plates with lamb wave signals. *Journal of Intelligent Material Systems and Structures*, 20(1):39–49, 2009.
- [23] Ye Lu, Lin Ye, Zhongqing Su, and Chunhui Yang. Quantitative assessment of through-thickness crack size based on lamb wave scattering in aluminium plates. *NDT & E International*, 41(1):59–68, 2008.
- [24] Adam Stawiarski, Marek Barski, and Piotr Pajak. Fatigue crack detection and identification by the elastic wave propagation method. *Mechanical Systems and Signal Processing*, 89:119–130, 2017.
- [25] William S Ragland, Dayakar Penumadu, and Richard T Williams. Finite element modeling of a full-scale five-girder bridge for structural health monitoring. *Structural Health Monitoring*, 10(5):449–465, 2011.
- [26] C Willberg, S Duczek, JM Vivar Perez, D Schmicker, and U Gabbert. Comparison of different higher order finite element schemes for the simulation of lamb waves. *Computer methods in applied mechanics and engineering*, 241:246–261, 2012.
- [27] Wieslaw Ostachowicz, Pawel Kudela, Marek Krawczuk, and Arkadiusz Zak. *Guided waves*

- in structures for SHM: the time-domain spectral element method*. John Wiley & Sons, 2011.
- [28] BC Lee and WJ Staszewski. Modelling of lamb waves for damage detection in metallic structures: Part ii. wave interactions with damage. *Smart Materials and Structures*, 12(5):815, 2003.
- [29] Yanfeng Shen and Carlos ES Cesnik. Modeling of nonlinear interactions between guided waves and fatigue cracks using local interaction simulation approach. *Ultrasonics*, 74:106–123, 2017.
- [30] Younho Cho and Joseph L Rose. An elastodynamic hybrid boundary element study for elastic guided wave interactions with a surface breaking defect. *International Journal of Solids and Structures*, 37(30):4103–4124, 2000.
- [31] Xiaoliang “George” Zhao and Joseph L. Rose. Boundary element modeling for defect characterization potential in a wave guide. *International Journal of Solids and Structures*, 40(11):2645–2658, 2003.
- [32] Shen Wang, Songling Huang, and Wei Zhao. Simulation of lamb wave’s interactions with transverse internal defects in an elastic plate. *Ultrasonics*, 51(4):432–440, 2011.
- [33] John D. Clayton. *Nonlinear Fracture Mechanics*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2018.
- [34] Ted L Anderson. *Fracture mechanics: fundamentals and applications*. CRC press, 2005.
- [35] M. H. Aliabadi. *The Boundary Element Method: Applications in solids and structures*, volume 2. John Wiley and Sons, 2002.
- [36] Dietmar Gross and Thomas Seelig. *Dynamic fracture mechanics*, pages 221–242. Springer, 2018.
- [37] D. P. Rooke, F. I. Baratta, and D. J. Cartwright. Simple methods of determining stress intensity factors. *Engineering Fracture Mechanics*, 14(2):397–426, 1981.

- [38] Sanda Swamy, Manda Srikanth, Kondepudi Murthy, and Puthuveettin Robi. Determination of mode i stress intensity factors of complex configurations using strain gages. *Journal of Mechanics of Materials and Structures*, 3(7):1239–1255, 2008.
- [39] Lambert B Freund. *Dynamic fracture mechanics*. Cambridge university press, 1998.
- [40] M Fleming, YA Chu, B Moran, and T Belytschko. Enriched element-free galerkin methods for crack tip fields. *International journal for numerical methods in engineering*, 40(8):1483–1504, 1997.
- [41] Ted Belytschko and Tom Black. Elastic crack growth in finite elements with minimal remeshing. *International journal for numerical methods in engineering*, 45(5):601–620, 1999.
- [42] Robert Simpson and Jon Trevelyan. A partition of unity enriched dual boundary element method for accurate computations in fracture mechanics. *Computer Methods in Applied Mechanics and Engineering*, 200(1-4):1–10, 2011.
- [43] L. C. Wrobel. *The Boundary Element Method: Applications in thermo-fluids and acoustics*, volume 1. John Wiley and Sons, 2002.
- [44] CA Brebbia and R Magureanu. The boundary element method for electromagnetic problems. *Engineering analysis*, 4(4):178–185, 1987.
- [45] MA Jaswon. Integral equation methods in potential theory. i. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 275(1360):23–32, 1963.
- [46] JC Lachat and JO Watson. Effective numerical treatment of boundary integral equations: a formulation for three-dimensional elastostatics. *International Journal for Numerical Methods in Engineering*, 10(5):991–1005, 1976.
- [47] Alexander H-D Cheng and Daisy T Cheng. Heritage and early history of the boundary element method. *Engineering Analysis with Boundary Elements*, 29(3):268–302, 2005.

- [48] MH Aliabadi and CA Brebbia. Boundary element formulations in fracture mechanics: a review. *WIT Transactions on Engineering Sciences*, 13, 1996.
- [49] TAa Cruse and W Vanburen. Three-dimensional elastic stress analysis of a fracture specimen with an edge crack. *International Journal of Fracture Mechanics*, 7(1):1–15, 1971.
- [50] George E Blandford, Anthony R Inghraffa, and James A Liggett. Two-dimensional stress intensity factor computations using the boundary element method. *International Journal for Numerical Methods in Engineering*, 17(3):387–404, 1981.
- [51] Steven L Crouch. Solution of plane elasticity problems by the displacement discontinuity method. i. infinite body solution. *International Journal for Numerical Methods in Engineering*, 10(2):301–343, 1976.
- [52] Steven L Crouch and AM Starfield. *Boundary element methods in solid mechanics: with applications in rock mechanics and geological engineering*. Allen & Unwin, 1982.
- [53] A Portela, M.H. Aliabadi, and DP Rooke. The dual boundary element method: effective implementation for crack problems. *International Journal for Numerical Methods in Engineering*, 33(6):1269–1287, 1992.
- [54] Y Mi and M. H. Aliabadi. Dual boundary element method for three-dimensional fracture mechanics analysis. *Engineering Analysis with Boundary Elements*, 10(2):161–171, 1992.
- [55] P Fedelinski, M.H. Aliabadi, and DP Rooke. A single-region time domain bem for dynamic crack problems. *International Journal of Solids and Structures*, 32(24):3555–3571, 1995.
- [56] P Fedelinski, MH Aliabadi, and DP Rooke. The dual boundary element method in dynamic fracture mechanics. *Engineering Analysis with Boundary Elements*, 12(3):203–210, 1993.
- [57] P Fedelinski, M.H. Aliabadi, and DP Rooke. Boundary element formulations for the

- dynamic analysis of cracked structures. *Engineering analysis with boundary elements*, 17(1):45–56, 1996.
- [58] PH Wen, MH Aliabadi, and A Young. Dual boundary element methods for three-dimensional dynamic crack problems. *The Journal of Strain Analysis for Engineering Design*, 34(6):373–394, 1999.
- [59] PH Wen, MH Aliabadi, and DP Rooke. Cracks in three dimensions: A dynamic dual boundary element analysis. *Computer methods in applied mechanics and engineering*, 167(1):139–151, 1998.
- [60] T. R. Kane and R. D. Mindlin. High-frequency extensional vibrations of plates. *American Society of Mechanical Engineers – Transactions – Journal of Applied Mechanics*, 23(2):277–283, 1956.
- [61] RD Mindlin. Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates. *J. appl. Mech.*, 18:31, 1951.
- [62] Ch Zhang and Jan Drewes Achenbach. Numerical analysis of surface-wave scattering by the boundary element method. *Wave Motion*, 10(4):365–374, 1988.
- [63] S Hirose and Jan Drewes Achenbach. Time-domain boundary element analysis of elastic wave interaction with a crack. *International journal for numerical methods in engineering*, 28(3):629–644, 1989.
- [64] PJ Schafbuch, FJ Rizzo, and RB Thompson. Boundary element method solutions for elastic wave scattering in 3d. *International journal for numerical methods in engineering*, 36(3):437–455, 1993.
- [65] C-Y Wang, Jan Drewes Achenbach, and S Hirose. Two-dimensional time domain bem for scattering of elastic waves in solids of general anisotropy. *International Journal of Solids and Structures*, 33(26):3843–3864, 1996.

- [66] Akira Furukawa, Takahiro Saitoh, and Sohichi Hirose. Convolution quadrature time-domain boundary element method for 2-d and 3-d elastodynamic analyses in general anisotropic elastic solids. *Engineering Analysis with Boundary Elements*, 39:64–74, 2014.
- [67] GD Manolis. Elastic wave scattering around cavities in inhomogeneous continua by the bem. *Journal of Sound and Vibration*, 266(2):281–305, 2003.
- [68] CH Daros and H Antes. Dynamic fundamental solutions for transversely isotropic piezoelectric materials of crystal class 6 mm. *International journal of solids and structures*, 37(11):1639–1658, 2000.
- [69] M Kogl and L Gaul. A boundary element method for transient piezoelectric analysis. *Engineering Analysis with Boundary Elements*, 24(7-8):591–598, 2000.
- [70] A Alaimo, A Milazzo, and C Orlando. On the dynamic behavior of piezoelectric active repair by the boundary element method. *Journal of intelligent material systems and structures*, 22(18):2137–2146, 2011.
- [71] F Zou, I Benedetti, and MH Aliabadi. A boundary element model for structural health monitoring using piezoelectric transducers. *Smart Materials and Structures*, 23(1):015022, 2013.
- [72] Jan Achenbach. *Wave propagation in elastic solids*, volume 16. Elsevier, 2012.
- [73] Thomas A Cruse and Frank J Rizzo. A direct formulation and numerical solution of the general transient elastodynamic problem. i. *Journal of Mathematical Analysis and Applications*, 22(1):244–259, 1968.
- [74] Paul William Partridge and Carlos Alberto Brebbia. *Dual reciprocity boundary element method*. Springer Science & Business Media, 2012.
- [75] F Durbin. Numerical inversion of laplace transforms: an efficient improvement to dubner and abate’s method. *The Computer Journal*, 17(4):371–376, 1974.

- [76] Younho Cho and Joseph L Rose. A boundary element solution for a mode conversion study on the edge reflection of lamb waves. *The Journal of the Acoustical Society of America*, 99(4):2097–2109, 1996.
- [77] Ye Lu, Lin Ye, Zhongqing Su, and Nao Huang. Quantitative evaluation of crack orientation in aluminium plates based on lamb waves. *Smart Materials and Structures*, 16(5):1907, 2007.
- [78] Andrea Gianneo, Michele Carboni, and Marco Giglio. Feasibility study of a multi-parameter probability of detection formulation for a lamb waves-based structural health monitoring approach to light alloy aeronautical plates. *Structural Health Monitoring*, 16(2):225–249, 2017.
- [79] Lingyu Yu, Zhenhua Tian, and Cara AC Leckey. Crack imaging and quantification in aluminum plates with guided wave wavenumber analysis methods. *Ultrasonics*, 62:203–212, 2015.
- [80] Paul D Wilcox, MJS Lowe, and P Cawley. Mode and transducer selection for long range lamb wave inspection. *Journal of intelligent material systems and structures*, 12(8):553–565, 2001.
- [81] Zhang Hai-Yan, Yao Jie-Cong, and Ma Shi-Wei. Scattering of s0 lamb mode from a blind hole in a plate using mindlin/mindlin plate theory. *Chinese Physics Letters*, 31(3):034301, 2014.
- [82] PH Wen and M. H. Aliabadi. Boundary element frequency domain formulation for dynamic analysis of mindlin plates. *International journal for numerical methods in engineering*, 67(11):1617–1640, 2006.
- [83] JCP McKeon and MK Hinders. Lamb wave scattering from a through hole. *Journal of Sound and Vibration*, 224(5):843–862, 1999.

- [84] Chun H Wang and Fu-Kuo Chang. Scattering of plate waves by a cylindrical inhomogeneity. *Journal of Sound and Vibration*, 282(1-2):429–451, 2005.
- [85] Magdalena Rucka. Modelling of in-plane wave propagation in a plate using spectral element method and kane–mindlin theory with application to damage detection. *Archive of Applied Mechanics*, 81(12):1877–1888, 2011.
- [86] J. C. F. Telles. A self adaptive coordinate transformation for efficient numerical evaluation of general boundary element integrals. *International Journal for Numerical Methods in Engineering*, 24(5):959–973, 1987.
- [87] Jun Li, Zahra Sharif Khodaei, and M. H. Aliabadi. A boundary element model for structural health monitoring based on the s0 lamb wave mode. *Key Engineering Materials*, 774:625–631, 2018.
- [88] JD Achenbach and Y Xu. Wave motion in an isotropic elastic layer generated by a time-harmonic point load of arbitrary direction. *The Journal of the Acoustical Society of America*, 106(1):83–90, 1999.
- [89] Ajay Raghavan and Carlos ES Cesnik. Finite-dimensional piezoelectric transducer modeling for guided wave based structural health monitoring. *Smart materials and structures*, 14(6):1448, 2005.
- [90] A Žak. A novel formulation of a spectral plate element for wave propagation in isotropic structures. *Finite Elements in Analysis and Design*, 45(10):650–658, 2009.
- [91] Victor Giurgiutiu. Tuned lamb wave excitation and detection with piezoelectric wafer active sensors for structural health monitoring. *Journal of intelligent material systems and structures*, 16(4):291–305, 2005.
- [92] Luyao Ge, Xinwei Wang, and Feng Wang. Accurate modeling of pzt-induced lamb wave propagation in structures by using a novel spectral finite element method. *Smart Materials and Structures*, 23(9):095018, 2014.

- [93] T. Dirgantara and M. H. Aliabadi. Boundary element analysis of assembled plate structures. *Communications in Numerical Methods in Engineering*, 17(10):749–760, 2001.
- [94] Luiz Carlos Facundo Sanches, Euclides Mesquita, Renato Pavanello, and Leandro Palermo Jr. Dynamic stationary response of reinforced plates by the boundary element method. *Mathematical Problems in Engineering*, 2007, 2007.
- [95] S Yang and FG Yuan. Transient wave propagation of isotropic plates using a higher-order plate theory. *International Journal of Solids and Structures*, 42(14):4115–4153, 2005.
- [96] I Benedetti, M.H. Aliabadi, and A Milazzo. A fast bem for the analysis of damaged structures with bonded piezoelectric sensors. *Computer Methods in Applied Mechanics and Engineering*, 199(9):490–501, 2010.
- [97] Go C Sih. A review of the three-dimensional stress problem for a cracked plate. *International Journal of Fracture Mechanics*, 7(1):39–61, 1971.
- [98] T Nakamura and DM Parks. Antisymmetrical 3-d stress field near the crack front of a thin elastic plate. *International Journal of Solids and Structures*, 25(12):1411–1426, 1989.
- [99] George C Sih. *Plates and shells with cracks: a collection of stress intensity factor solutions for cracks in plates and shells*, volume 3. Springer Science & Business Media, 2012.
- [100] Les P Pook, Filippo Berto, Alberto Campagnolo, and Paolo Lazzarin. Coupled fracture mode of a cracked disc under anti-plane loading. *Engineering Fracture Mechanics*, 128:22–36, 2014.
- [101] A Kotousov. Fracture in plates of finite thickness. *International Journal of Solids and Structures*, 44(25-26):8259–8273, 2007.
- [102] Z He, A Kotousov, and F Berto. Effect of vertex singularities on stress intensities near plate free surfaces. *Fatigue & Fracture of Engineering Materials & Structures*, 38(7):860–869, 2015.

- [103] Wei Yang and LB Freund. Transverse shear effects for through-cracks in an elastic plate. *International Journal of Solids and Structures*, 21(9):977–994, 1985.
- [104] Jin Zhi-He and Naotake Noda. A quasi-three-dimensional crack problem of an elastic plate subjected to in-plane shear loading. *Engineering fracture mechanics*, 49(5):773–779, 1994.
- [105] ZH Jin and RC Batra. Dynamic fracture of a kane-mindlin plate. *Theoretical and Applied Fracture Mechanics*, 26(3):199–209, 1997.
- [106] Andrei Kotousov and Chun Hui Wang. Fundamental solutions for the generalised plane strain theory. *International Journal of Engineering Science*, 40(15):1775–1790, 2002.
- [107] Filippo Berto, Paolo Lazzarin, and Chun Hui Wang. Three-dimensional linear elastic distributions of stress and strain energy density ahead of v-shaped notches in plates of arbitrary thickness. *International Journal of Fracture*, 127(3):265–282, 2004.
- [108] P Lazzarin and M Zappalorto. A three-dimensional stress field solution for pointed and sharply radiused v-notches in plates of finite thickness. *Fatigue & Fracture of Engineering Materials & Structures*, 35(12):1105–1119, 2012.
- [109] T Chang, W Guo, and HR Dong. Three-dimensional effects for through-thickness cylindrical inclusions in an elastic plate. *The Journal of Strain Analysis for Engineering Design*, 36(3):277–286, 2001.
- [110] Zdenek P Bazant and Luis F Estenssoro. Surface singularity and crack propagation. *International Journal of Solids and Structures*, 15(5):405–426, 1979.
- [111] T Nakamura and DM Parks. Three-dimensional stress field near the crack front of a thin elastic plate. *Journal of applied Mechanics*, 55(4):805–813, 1988.
- [112] JP Benthem. State of stress at the vertex of a quarter-infinite crack in a half-space. *International Journal of Solids and Structures*, 13(5):479–492, 1977.
- [113] Chongmin She and Wanlin Guo. The out-of-plane constraint of mixed-mode cracks in thin elastic plates. *International journal of solids and structures*, 44(9):3021–3034, 2007.

- [114] Augustus Edward Hough Love. The small free vibrations and deformation of a thin elastic shell. *Philosophical Transactions of the Royal Society of London. A*, 179:491–546, 1888.
- [115] JL Wearing and SY Ahmadi-Brooghani. The evaluation of stress intensity factors in plate bending problems using the dual boundary element method. *Engineering analysis with boundary elements*, 23(1):3–19, 1999.
- [116] C-Y Wang and Ch Zhang. 3-d and 2-d dynamic green's functions and time-domain bies for piezoelectric solids. *Engineering Analysis with Boundary Elements*, 29(5):454–465, 2005.
- [117] R Rojas-Díaz, A Sáez, F García-Sánchez, and Ch Zhang. Time-harmonic green's functions for anisotropic magnetoelectroelasticity. *International Journal of Solids and Structures*, 45(1):144–158, 2008.
- [118] YF Rashed, M. H. Aliabadi, and CA Brebbia. Hypersingular boundary element formulation for reissner plates. *International journal of solids and structures*, 35(18):2229–2249, 1998.
- [119] T Dirgantara and M.H. Aliabadi. Dual boundary element formulation for fracture mechanics analysis of shear deformable shells. *International Journal of Solids and Structures*, 38(44):7769–7800, 2001.
- [120] Stephen A Thau and Lu Tsin-Hwei. Transient stress intensity factors for a finite crack in an elastic solid caused by a dilatational wave. *International Journal of Solids and Structures*, 7(7):731–750, 1971.
- [121] Jun Li, Zahra Sharif Khodaei, and M. H. Aliabadi. Modelling of the high-frequency fundamental symmetric lamb wave using a new boundary element formulation. *International Journal of Mechanical Sciences*, 155:235–247, 2019.
- [122] Edward F Crawley and Javier De Luis. Use of piezoelectric actuators as elements of intelligent structures. *AIAA journal*, 25(10):1373–1385, 1987.

- [123] Surajit Roy, Kuldeep Lonkar, Vishnuvardhan Janapati, and Fu-Kuo Chang. A novel physics-based temperature compensation model for structural health monitoring using ultrasonic guided waves. *Structural Health Monitoring*, 13(3):321–342, 2014.
- [124] GL Huang, F Song, and R Reddy. Quantitative modeling of dynamic behavior of a piezoelectric wafer actuator bonded to an elastic plate for guided wave propagation. In *Health Monitoring of Structural and Biological Systems 2010*, volume 7650, page 76502I. International Society for Optics and Photonics, 2010.
- [125] Santosh Kapuria, Bhabagrahi Natha Sharma, and A Arockiarajan. Dynamic shear-lag model for stress transfer in piezoelectric transducer bonded to plate. *AIAA Journal*, 57(5):2123–2133, 2018.
- [126] N Quaegebeur, PC Ostiguy, and P Masson. Hybrid empirical/analytical modeling of guided wave generation by circular piezoceramics. *Smart Materials and Structures*, 24(3):035003, 2015.
- [127] Francesco Lanza di Scalea, Howard Matt, and Ivan Bartoli. The response of rectangular piezoelectric sensors to rayleigh and lamb ultrasonic waves. *The Journal of the Acoustical Society of America*, 121(1):175–187, 2007.
- [128] Hoon Sohn and Sang Jun Lee. Lamb wave tuning curve calibration for surface-bonded piezoelectric transducers. *Smart Materials and Structures*, 19(1):015007, 2009.
- [129] David N Alleyne and Peter Cawley. Optimization of lamb wave inspection techniques. *NDT & E International*, 25(1):11–22, 1992.
- [130] Prabhu Rajagopal, Mickael Drozd, Elizabeth A Skelton, Michael JS Lowe, and Richard V Craster. On the use of absorbing layers to simulate the propagation of elastic waves in unbounded isotropic media using commercially available finite element packages. *NDT & E International*, 51:30–40, 2012.
- [131] Pravin Madhavan. *Numerical integration in meshfree methods*. Thesis, 2010.

- [132] C Willberg, S Duczek, and U Gabbert. Increasing the scanning range of lamb wave based shm systems by optimizing the actuator–sensor design. *CEAS Aeronautical Journal*, 4(1):87–98, 2013.
- [133] Xiaoliang Zhao, Huidong Gao, Guangfan Zhang, Bulent Ayhan, Fei Yan, Chiman Kwan, and Joseph L Rose. Active health monitoring of an aircraft wing with embedded piezo-electric sensor/actuator network: I. defect detection, localization and growth monitoring. *Smart materials and structures*, 16(4):1208, 2007.
- [134] Paweł Kudela, Arkadiusz Żak, Marek Krawczuk, and Wiesław Ostachowicz. Modelling of wave propagation in composite plates using the time domain spectral element method. *Journal of Sound and Vibration*, 302(4–5):728–745, 2007.
- [135] W Dauksher and Ashley F Emery. The solution of elastostatic and elastodynamic problems with chebyshev spectral finite elements. *Computer methods in applied mechanics and engineering*, 188(1):217–233, 2000.
- [136] Haikuo Peng, Guang Meng, and Fucai Li. Modeling of wave propagation in plate structures using three-dimensional spectral element method for damage detection. *Journal of Sound and Vibration*, 320(4):942–954, 2009.
- [137] Fangxin Zou and MH Aliabadi. On modelling three-dimensional elastodynamic wave propagation with boundary spectral element method. *European Journal of Computational Mechanics*, 27(3):204–228, 2018.
- [138] Fangxin Zou and M. H. Aliabadi. On modelling three-dimensional piezoelectric smart structures with boundary spectral element method. *Smart Materials and Structures*, 26(5):055015, 2017.
- [139] AC Eringen and ES Suhubi. *Elstodynamics*, vol. ii (linear theory), 1975.

Appendix A

Dynamic fundamental solutions

A.1 Kane-Mindlin theory

The dynamic fundamental solutions for the symmetric case can be written as follows:

$$\tilde{U}_{\alpha\beta}^S = \frac{1}{2\pi h\mu (e_2^S - e_1^S) (\zeta_3^S)^2} (\psi^S \delta_{\alpha\beta} - \chi^S r_{,\alpha} r_{,\beta}) \quad (\text{A.1})$$

$$\tilde{U}_{\alpha 3}^S = -\tilde{U}_{3\alpha}^S = -\frac{6}{\pi h^3 \mu (e_2^S - e_1^S)} \eta^S r_{,\alpha} \quad (\text{A.2})$$

$$\tilde{U}_{33}^S = -\frac{6}{\pi h^3 \mu (e_2^S - e_1^S)} \xi^S \quad (\text{A.3})$$

$$\begin{aligned} \tilde{T}_{\alpha\beta}^S = & \frac{1}{2\pi (e_2^S - e_1^S) (\zeta_3^S)^2} \left[\left(\psi_{,r}^S - \frac{\chi^S}{r} \right) (\delta_{\alpha\beta} r_{,n} + n_{\alpha} r_{,\beta}) + 2 \left(2 \frac{\chi^S}{r} - \chi_{,r}^S \right) r_{,\alpha} r_{,\beta} r_{,n} \right. \\ & \left. + \frac{2\nu}{1-2\nu} \left(\psi_{,r}^S - \frac{1-\nu}{\nu} \frac{\chi^S}{r} - \chi_{,r}^S - \frac{12\kappa_S (\zeta_3^S)^2}{h^2} \eta^S \right) n_{\beta} r_{,\alpha} \right] \end{aligned} \quad (\text{A.4})$$

$$\tilde{T}_{\alpha 3}^S = \frac{1}{2\pi (e_2^S - e_1^S)} \left[\left(\frac{\eta^S}{r} - \eta_{,r}^S \right) r_{,\alpha} r_{,n} - \frac{\eta^S}{r} n_{\alpha} \right] \quad (\text{A.5})$$

$$\tilde{T}_{3\alpha}^S = \frac{12}{\pi h^2 (e_2^S - e_1^S)} \left[\frac{\nu}{1-2\nu} \left(\frac{1-\nu}{\nu} \frac{\eta^S}{r} + \eta_{,r}^S - \kappa_S \xi^S \right) n_\alpha - \left(\frac{\eta^S}{r} - \eta_{,r}^S \right) r_{,\alpha} r_{,n} \right] \quad (\text{A.6})$$

$$\tilde{T}_{33}^S = -\frac{1}{2\pi (e_2^S - e_1^S)} \xi_{,r}^S r_{,n} \quad (\text{A.7})$$

$$\begin{aligned} \tilde{D}_{\alpha\beta\gamma}^S = & -\frac{1}{2\pi (e_2^S - e_1^S) (\zeta_3^S)^2} \left[\left(\psi_{,r}^S - \frac{\chi^S}{r} \right) (\delta_{\alpha\gamma} r_{,\beta} + \delta_{\beta\gamma} r_{,\alpha}) + 2 \left(\frac{2\chi^S}{r} - \chi_{,r}^S \right) r_{,\alpha} r_{,\beta} r_{,\gamma} \right. \\ & \left. + \frac{2\nu}{1-2\nu} \left(\psi_{,r}^S - \frac{1-\nu}{\nu} \frac{\chi^S}{r} - \chi_{,r}^S - \frac{12\kappa_S (\zeta_3^S)^2}{h^2} \eta^S \right) \delta_{\alpha\beta} r_{,\gamma} \right] \end{aligned} \quad (\text{A.8})$$

$$\tilde{D}_{\alpha\beta 3}^S = \frac{12}{\pi h^2 (e_2^S - e_1^S)} \left[\frac{\nu}{1-2\nu} \left(\frac{1-\nu}{\nu} \frac{\eta^S}{r} + \eta_{,r}^S - \kappa_S \xi^S \right) \delta_{\alpha\beta} - \left(\frac{\eta^S}{r} - \eta_{,r}^S \right) r_{,\alpha} r_{,\beta} \right] \quad (\text{A.9})$$

$$\tilde{D}_{3\beta\gamma}^S = \frac{1}{2\pi (e_2^S - e_1^S)} \left[\left(\frac{\eta^S}{r} - \eta_{,r}^S \right) r_{,\beta} r_{,\gamma} - \frac{\eta^S}{r} \delta_{\beta\gamma} \right] \quad (\text{A.10})$$

$$\tilde{D}_{3\beta 3}^S = \frac{1}{2\pi (e_2^S - e_1^S)} \xi_{,r}^S r_{,\beta} \quad (\text{A.11})$$

$$\begin{aligned} \tilde{S}_{\alpha\beta 3}^S = & \frac{h\mu}{\pi (e_2^S - e_1^S)} \left[\left(\eta_{,rr}^S + \frac{3\eta^S}{r^2} - \frac{3\eta_{,r}^S}{r} \right) r_{,\alpha} r_{,\beta} r_{,n} - \left(\frac{\eta^S}{r^2} - \frac{\eta_{,r}^S}{r} \right) (r_{,\alpha} n_\beta + r_{,\beta} n_\alpha) \right. \\ & \left. - \frac{\nu}{1-2\nu} \left(\frac{1-\nu}{\nu} \left(\frac{\eta^S}{r^2} - \frac{\eta_{,r}^S}{r} \right) - \eta_{,rr}^S + \kappa_S \xi_{,r}^S \right) \delta_{\alpha\beta} r_{,n} \right] \end{aligned} \quad (\text{A.12})$$

$$\begin{aligned} \tilde{S}_{3\beta\gamma}^S = & -\frac{h\mu}{\pi (e_2^S - e_1^S)} \left[\left(\eta_{,rr}^S + \frac{3\eta^S}{r^2} - \frac{3\eta_{,r}^S}{r} \right) r_{,\beta} r_{,\gamma} r_{,n} - \left(\frac{\eta^S}{r^2} - \frac{\eta_{,r}^S}{r} \right) (\delta_{\gamma\beta} r_{,n} + r_{,\gamma} n_\beta) \right. \\ & \left. - \frac{\nu}{1-2\nu} \left(\frac{1-\nu}{\nu} \left(\frac{\eta^S}{r^2} - \frac{\eta_{,r}^S}{r} \right) - \eta_{,rr}^S + \kappa_S \xi_{,r}^S \right) r_{,\beta} n_\gamma \right] \end{aligned} \quad (\text{A.13})$$

$$\tilde{S}_{3\beta 3}^S = \frac{h^3 \mu}{24\pi (e_2^S - e_1^S)} \left[\left(\xi_{,rr}^S - \frac{\xi_{,r}^S}{r} \right) r_{,\beta} r_{,n} + \frac{\xi_{,r}^S}{r} n_{\beta} \right] \quad (\text{A.14})$$

$$\begin{aligned} \tilde{S}_{\alpha\beta\gamma}^S = & \frac{h\mu}{2\pi (e_2^S - e_1^S) (\zeta_3^S)^2} \left\{ 4 \left(\chi_{,rr}^S - 5 \frac{\chi_{,r}^S}{r} + 8 \frac{\chi^S}{r^2} \right) r_{,\alpha} r_{,\beta} r_{,\gamma} r_{,n} \right. \\ & - 2 \left(\frac{\psi_{,r}^S}{r} - \frac{\chi^S}{r^2} \right) (\delta_{\alpha\gamma} n_{\beta} + \delta_{\beta\gamma} n_{\alpha}) - \frac{4\nu}{1-2\nu} \left[\psi_{,rr}^S - \chi_{,rr}^S - \frac{\psi_{,r}^S}{r} \right. \\ & - \frac{1-2\nu}{\nu} \frac{\chi_{,r}^S}{r} + 2 \frac{1-\nu}{\nu} \frac{\chi^S}{r^2} + \frac{12\kappa_S (\zeta_3^S)^2}{h^2} \left(\frac{\eta^S}{r} - \eta_{,r}^S \right) \left. \left(\delta_{\alpha\beta} r_{,\gamma} r_{,n} + n_{\gamma} r_{,\alpha} r_{,\beta} \right) \right. \\ & - \left(\psi_{,rr}^S - \frac{\psi_{,r}^S}{r} - 3 \frac{\chi_{,r}^S}{r} + 6 \frac{\chi^S}{r^2} \right) [(\delta_{\beta\gamma} r_{,\alpha} + \delta_{\alpha\gamma} r_{,\beta}) r_{,n} + (n_{\alpha} r_{,\beta} + n_{\beta} r_{,\alpha}) r_{,\gamma}] \\ & + 4 \left(\frac{\nu}{1-2\nu} \right)^2 \left[\chi_{,rr}^S - \psi_{,rr}^S + 2 \frac{1-\nu}{\nu} \frac{\chi_{,r}^S}{r} - \frac{2-3\nu}{\nu} \frac{\psi_{,r}^S}{r} + \frac{1-2\nu}{\nu^2} \frac{\chi^S}{r^2} \right. \\ & \left. \left. + \frac{12\kappa_S (\zeta_3^S)^2}{h^2} \left(2 \left(\frac{1-\nu}{\nu} \frac{\eta^S}{r} + \eta_{,r}^S \right) - \kappa_S \xi^S \right) \right] \delta_{\alpha\beta} n_{\gamma} \right\} \end{aligned} \quad (\text{A.15})$$

where the normal derivative $r_{,n} = r_{,\alpha} n_{\alpha}$; the three Laplace-domain wave numbers can be expressed as:

$$\begin{aligned} (\zeta_{\alpha}^S)^2 &= \frac{1}{2R_2} \left(\frac{p}{c_1} \right)^2 \left[R_1 + R_2 - (-1)^{\alpha} Q_S + \left(\frac{\omega_{cS}}{p} \right)^2 \right] \\ (\zeta_3^S)^2 &= \frac{p^2}{c_2^2} \end{aligned} \quad (\text{A.16})$$

in which $R_{\alpha} = c_{\alpha}^2/c_3^2$, $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$, $c_2 = \sqrt{\mu/\rho}$, $c_3 = \sqrt{E/[\rho(1-\nu^2)]}$, the cut-off frequency $\omega_{cS} = 2\sqrt{3}\kappa_S c_1/h$, $Q_S = \sqrt{[R_1 + R_2 + (\omega_{cS}/p)^2]^2 - 4R_1 R_2 [1 + (\omega_{cS}/p)^2]}$ and

$$e_{\alpha}^S = -\frac{1-\nu}{\kappa_S \nu} \left[(\zeta_{\alpha}^S)^2 - \frac{1-2\nu}{2(1-\nu)} (\zeta_3^S)^2 \right] \quad (\text{A.17})$$

The fundamental solutions for calculating total in-plane strains are given by:

$$\tilde{U}_{\beta}^S = \frac{1}{2\pi h\mu (e_2^S - e_1^S) (\zeta_3^S)^2} \left(\chi_{,r}^S - \psi_{,r}^S + \frac{\chi^S}{r} \right) r_{,\beta} \quad (\text{A.18})$$

$$\tilde{U}_3^S = \frac{6}{\pi h^3 \mu (e_2^S - e_1^S)} \left(\frac{\eta^S}{r} + \eta_{,r}^S \right) \quad (\text{A.19})$$

$$\begin{aligned} \tilde{T}_\beta^S = & \frac{1}{\pi (e_2^S - e_1^S) (\zeta_3^S)^2} \left\{ \left(\chi_{,rr}^S - \psi_{,rr}^S + \frac{\psi_{,r}^S}{r} - 2 \frac{\chi^S}{r^2} \right) r_{,\beta} r_{,n} + \frac{\nu}{1 - 2\nu} \right. \\ & \times \left[\chi_{,rr}^S - \psi_{,rr}^S - \frac{1 - \nu}{\nu} \frac{\psi_{,r}^S}{r} + \frac{\chi_{,r}^S}{\nu r} + \frac{1 - 2\nu}{\nu} \frac{\chi^S}{r^2} + \frac{12\kappa_S (\zeta_3^S)^2}{h^2} \left(\eta_{,r}^S + \frac{\eta^S}{r} \right) \right] n_\beta \left. \right\} \end{aligned} \quad (\text{A.20})$$

$$\tilde{T}_3^S = \frac{1}{2\pi (e_2^S - e_1^S)} \left(\eta_{,rr}^S + \frac{\eta_{,r}^S}{r} - \frac{\eta^S}{r^2} \right) r_{,n} \quad (\text{A.21})$$

A.2 Mindlin theory

The dynamic fundamental solutions for the antisymmetric case are given by:

$$\tilde{U}_{\alpha\beta}^A = \frac{6}{\pi h^3 \mu (e_2^A - e_1^A) (\zeta_3^A)^2} (\psi^A \delta_{\alpha\beta} - \chi^A r_{,\alpha} r_{,\beta}) \quad (\text{A.22})$$

$$\tilde{U}_{\alpha 3}^A = -\tilde{U}_{3\alpha}^A = -\frac{6}{\pi h^3 \mu (e_2^A - e_1^A)} \eta^A r_{,\alpha} \quad (\text{A.23})$$

$$\tilde{U}_{33}^A = -\frac{1}{2\pi \kappa_A^2 \mu h (e_2^A - e_1^A)} \xi^A \quad (\text{A.24})$$

$$\begin{aligned} \tilde{T}_{\alpha\beta}^A = & \frac{1}{2\pi (e_2^A - e_1^A) (\zeta_3^A)^2} \left[\left(\psi_{,r}^A - \frac{\chi^A}{r} \right) (\delta_{\alpha\beta} r_{,n} + n_\alpha r_{,\beta}) + 2 \left(2 \frac{\chi^A}{r} - \chi_{,r}^A \right) r_{,\alpha} r_{,\beta} r_{,n} \right. \\ & \left. + \frac{2\nu}{1 - \nu} \left(\psi_{,r}^A - \frac{1}{\nu} \frac{\chi^A}{r} - \chi_{,r}^A \right) n_\beta r_{,\alpha} \right] \end{aligned} \quad (\text{A.25})$$

$$\tilde{T}_{\alpha 3}^A = \frac{6\kappa_A^2}{\pi h^2 (e_2^A - e_1^A)} \left[\left(\frac{\eta^A}{r} - \eta_{,r}^A - \frac{\chi^A}{(\zeta_3^A)^2} \right) r_{,\alpha} r_{,n} - \left(\frac{\eta^A}{r} - \frac{\psi^A}{(\zeta_3^A)^2} \right) n_\alpha \right] \quad (\text{A.26})$$

$$\tilde{T}_{3\alpha}^A = \frac{1}{\pi(e_2^A - e_1^A)} \left[\frac{1}{1-\nu} \left(\frac{\eta^A}{r} + \nu \eta_{,r}^A \right) n_\alpha - \left(\frac{\eta^A}{r} - \eta_{,r}^A \right) r_{,\alpha} r_{,n} \right] \quad (\text{A.27})$$

$$\tilde{T}_{33}^A = -\frac{1}{2\pi(e_2^A - e_1^A)} \left(\xi_{,r}^A - \frac{12\kappa_A^2}{h^2} \eta^A \right) r_{,n} \quad (\text{A.28})$$

$$\begin{aligned} \tilde{D}_{\alpha\beta\gamma}^A = & -\frac{1}{2\pi(e_2^A - e_1^A)(\zeta_3^A)^2} \left[\left(\psi_{,r}^A - \frac{\chi^A}{r} \right) (\delta_{\alpha\gamma} r_{,\beta} + \delta_{\beta\gamma} r_{,\alpha}) + 2 \left(2\frac{\chi^A}{r} - \chi_{,r}^A \right) r_{,\alpha} r_{,\beta} r_{,\gamma} \right. \\ & \left. + \frac{2\nu}{1-\nu} \left(\psi_{,r}^A - \frac{1}{\nu} \frac{\chi^A}{r} - \chi_{,r}^A \right) \delta_{\alpha\beta} r_{,\gamma} \right] \end{aligned} \quad (\text{A.29})$$

$$\tilde{D}_{\alpha\beta 3}^A = \frac{1}{\pi(e_2^A - e_1^A)} \left[\frac{1}{1-\nu} \left(\frac{\eta^A}{r} + \nu \eta_{,r}^A \right) \delta_{\alpha\beta} - \left(\frac{\eta^A}{r} - \eta_{,r}^A \right) r_{,\alpha} r_{,\beta} \right] \quad (\text{A.30})$$

$$\tilde{D}_{3\beta\gamma}^A = \frac{6\kappa_A^2}{\pi h^2(e_2^A - e_1^A)} \left[\left(\frac{\eta^A}{r} - \eta_{,r}^A - \frac{\chi^A}{(\zeta_3^A)^2} \right) r_{,\beta} r_{,\gamma} - \left(\frac{\eta^A}{r} - \frac{\psi^A}{(\zeta_3^A)^2} \right) \delta_{\beta\gamma} \right] \quad (\text{A.31})$$

$$\tilde{D}_{3\beta 3}^A = \frac{1}{2\pi(e_2^A - e_1^A)} \left(\xi_{,r}^A - \frac{12\kappa_A^2}{h^2} \eta^A \right) r_{,\beta} \quad (\text{A.32})$$

$$\begin{aligned} \tilde{S}_{\alpha\beta\gamma}^A = & \frac{h^3\mu}{24\pi(e_2^A - e_1^A)(\zeta_3^A)^2} \\ & \times \left\{ 4 \left(\chi_{,rr}^A - 5\frac{\chi_{,r}^A}{r} + 8\frac{\chi^A}{r^2} \right) r_{,\alpha} r_{,\beta} r_{,\gamma} r_{,n} - 2 \left(\frac{\psi_{,r}^A}{r} - \frac{\chi^A}{r^2} \right) (\delta_{\alpha\gamma} n_\beta + \delta_{\beta\gamma} n_\alpha) \right. \\ & - \frac{4\nu}{1-\nu} \left(\psi_{,rr}^A - \chi_{,rr}^A - \frac{\psi_{,r}^A}{r} - \frac{1-\nu}{\nu} \frac{\chi_{,r}^A}{r} + \frac{2}{\nu} \frac{\chi^A}{r^2} \right) (\delta_{\alpha\beta} r_{,\gamma} r_{,n} + n_\gamma r_{,\alpha} r_{,\beta}) \\ & - \left(\psi_{,rr}^A - \frac{\psi_{,r}^A}{r} - 3\frac{\chi_{,r}^A}{r} + 6\frac{\chi^A}{r^2} \right) [(\delta_{\beta\gamma} r_{,\alpha} + \delta_{\alpha\gamma} r_{,\beta}) r_{,n} + (n_\alpha r_{,\beta} + n_\beta r_{,\alpha}) r_{,\gamma}] \\ & \left. + 4 \left(\frac{\nu}{1-\nu} \right)^2 \left(\chi_{,rr}^A - \psi_{,rr}^A + \frac{2}{\nu} \frac{\chi_{,r}^A}{r} - \frac{2-\nu}{\nu} \frac{\psi_{,r}^A}{r} + \frac{1-\nu^2}{\nu^2} \frac{\chi^A}{r^2} \right) \delta_{\alpha\beta} n_\gamma \right\} \end{aligned} \quad (\text{A.33})$$

$$\begin{aligned}
\tilde{S}_{\alpha\beta 3}^A = & \frac{\kappa_A^2 h \mu}{2\pi (e_2^A - e_1^A) (\zeta_3^A)^2} \left\{ 2 \left[(\zeta_3^A)^2 \left(\eta_{,rr}^A + \frac{3\eta^A}{r^2} - \frac{3\eta_{,r}^A}{r} \right) + \chi_{,r}^A - 2\frac{\chi^A}{r} \right] r_{,\alpha} r_{,\beta} r_{,n} \right. \\
& - \frac{2\nu}{1-\nu} \left[\frac{(\zeta_3^A)^2}{\nu} \left(\frac{\eta^A}{r^2} - \frac{\eta_{,r}^A}{r} - \nu \eta_{,rr}^A \right) - \frac{1}{\nu} \frac{\chi^A}{r} - \chi_{,r}^A + \psi_{,r}^A \right] \delta_{\alpha\beta} r_{,n} \\
& \left. - \left[2(\zeta_3^A)^2 \left(\frac{\eta^A}{r^2} - \frac{\eta_{,r}^A}{r} \right) - \frac{\chi^A}{r} + \psi_{,r}^A \right] (r_{,\alpha} n_{\beta} + r_{,\beta} n_{\alpha}) \right\}
\end{aligned} \tag{A.34}$$

$$\begin{aligned}
\tilde{S}_{3\beta\gamma}^A = & -\frac{\kappa_A^2 \mu h}{2\pi (e_2^A - e_1^A) (\zeta_3^A)^2} \left\{ 2 \left[(\zeta_3^A)^2 \left(\eta_{,rr}^A + \frac{3\eta^A}{r^2} - \frac{3\eta_{,r}^A}{r} \right) + \chi_{,r}^A - 2\frac{\chi^A}{r} \right] r_{,\beta} r_{,\gamma} r_{,n} \right. \\
& - \frac{2\nu}{1-\nu} \left[\frac{(\zeta_3^A)^2}{\nu} \left(\frac{\eta^A}{r^2} - \frac{\eta_{,r}^A}{r} - \nu \eta_{,rr}^A \right) - \frac{1}{\nu} \frac{\chi^A}{r} - \chi_{,r}^A + \psi_{,r}^A \right] r_{,\beta} n_{\gamma} \\
& \left. - \left[2(\zeta_3^A)^2 \left(\frac{\eta^A}{r^2} - \frac{\eta_{,r}^A}{r} \right) - \frac{\chi^A}{r} + \psi_{,r}^A \right] (\delta_{\beta\gamma} r_{,n} + r_{,\gamma} n_{\beta}) \right\}
\end{aligned} \tag{A.35}$$

$$\begin{aligned}
\tilde{S}_{3\beta 3}^A = & \frac{\kappa_A^2 \mu}{2\pi h (e_2^A - e_1^A) (\zeta_3^A)^2} \left\{ \left[(\zeta_3^A)^2 \left(h^2 \frac{\xi_{,r}^A}{r} - 24\kappa_A^2 \frac{\eta^A}{r} \right) + 12\kappa_A^2 \psi^A \right] n_{\beta} \right. \\
& \left. + \left[(\zeta_3^A)^2 \left(h^2 \left(\xi_{,rr}^A - \frac{\xi_{,r}^A}{r} \right) + 24\kappa_A^2 \left(\frac{\eta^A}{r} - \eta_{,r}^A \right) \right) - 12\kappa_A^2 \chi^A \right] r_{,\beta} r_{,n} \right\}
\end{aligned} \tag{A.36}$$

where the three Laplace-domain wave numbers can be expressed as:

$$\begin{aligned}
(\zeta_{\alpha}^A)^2 &= \frac{1}{2} \left(\frac{p}{\omega_{cA}} \right)^2 [S_1 + S_2 - (-1)^{\alpha} Q_A] \\
(\zeta_3^A)^2 &= \kappa_A^2 S_2 \left[\left(\frac{p}{\omega_{cA}} \right)^2 + 1 \right]
\end{aligned} \tag{A.37}$$

in which $Q_A = \sqrt{(S_1 - S_2)^2 - 4S_1 S_2 (\omega_{cA}/p)^2}$, $S_1 = \kappa_A^2 \mu h / D$, $S_2 = 12/h^2$ and the cut-off frequency $\omega_{cA} = 2\sqrt{3}\kappa_A c_2 / h$; and

$$e_{\alpha}^A = \frac{2}{1-\nu} \left[(\zeta_{\alpha}^A)^2 - \frac{1-\nu}{2} (\zeta_3^A)^2 \right] \tag{A.38}$$

The fundamental solutions for calculating total strains can be written as:

$$\tilde{U}_\beta^A = \frac{6}{\pi h^3 \mu (e_2^A - e_1^A) (\zeta_3^A)^2} \left(\chi_{,r}^A - \psi_{,r}^A + \frac{\chi^A}{r} \right) r_{,\beta} \quad (\text{A.39})$$

$$\tilde{U}_3^A = \frac{6}{\pi h^3 \mu (e_2^A - e_1^A)} \left(\frac{\eta^A}{r} + \eta_{,r}^A \right) \quad (\text{A.40})$$

$$\begin{aligned} \tilde{T}_\beta^A = \frac{1}{\pi (e_2^A - e_1^A) (\zeta_3^A)^2} & \left[\left(\chi_{,rr}^A - \psi_{,rr}^A + \frac{\psi_{,r}^A}{r} - 2 \frac{\chi^A}{r^2} \right) r_{,\beta} r_{,n} \right. \\ & \left. + \frac{\nu}{1-\nu} \left(\chi_{,rr}^A - \psi_{,rr}^A - \frac{\psi_{,r}^A}{\nu r} + \frac{1+\nu}{\nu} \frac{\chi_{,r}^A}{r} + \frac{1-\nu}{\nu} \frac{\chi^A}{r^2} \right) n_\beta \right] \end{aligned} \quad (\text{A.41})$$

$$\tilde{T}_3^A = \frac{6\kappa_A^2}{\pi h^2 (e_2^A - e_1^A) (\zeta_3^A)^2} \left[(\zeta_3^A)^2 \left(\eta_{,rr}^A + \frac{\eta_{,r}^A}{r} - \frac{\eta^A}{r^2} \right) + \chi_{,r}^A - \psi_{,r}^A + \frac{\chi^A}{r} \right] r_{,n} \quad (\text{A.42})$$

In the above fundamental solutions, the functions ψ , χ , η and ξ , and their derivatives are given by:

$$\psi^Y = \frac{(e_2^Y - e_1^Y) \zeta_3^Y [K_1(z_3^Y) + z_3^Y K_0(z_3^Y)] - e_2^Y \zeta_1^Y K_1(z_1^Y) + e_1^Y \zeta_2^Y K_1(z_2^Y)}{r} \quad (\text{A.43})$$

$$\begin{aligned} \chi^Y = & \frac{(e_2^Y - e_1^Y) \zeta_3^Y [2K_1(z_3^Y) + z_3^Y K_0(z_3^Y)] - e_2^Y \zeta_1^Y [2K_1(z_1^Y) + z_1^Y K_0(z_1^Y)]}{r} \\ & + \frac{e_1^Y \zeta_2^Y [2K_1(z_2^Y) + z_2^Y K_0(z_2^Y)]}{r} \end{aligned} \quad (\text{A.44})$$

$$\eta^Y = \zeta_1^Y K_1(z_1^Y) - \zeta_2^Y K_1(z_2^Y) \quad (\text{A.45})$$

$$\xi^Y = e_1^Y K_0(z_1^Y) - e_2^Y K_0(z_2^Y) \quad (\text{A.46})$$

$$\psi_{,r}^Y = - \left[(e_2^Y - e_1^Y) (\zeta_3^Y)^3 K_1(z_3^Y) + \frac{\chi^Y}{r} \right] \quad (\text{A.47})$$

$$\chi_{,r}^Y = - \left[2 \frac{\chi^Y}{r} + (e_2^Y - e_1^Y) (\zeta_3^Y)^3 K_1(z_3^Y) - e_2^Y (\zeta_1^Y)^3 K_1(z_1^Y) + e_1^Y (\zeta_2^Y)^3 K_1(z_2^Y) \right] \quad (\text{A.48})$$

$$\eta_{,r}^Y = - \frac{\eta^Y}{r} - (\zeta_1^Y)^2 K_0(z_1^Y) + (\zeta_2^Y)^2 K_0(z_2^Y) \quad (\text{A.49})$$

$$\xi_{,r}^Y = -e_1^Y \zeta_1^Y K_1(z_1^Y) + e_2^Y \zeta_2^Y K_1(z_2^Y) \quad (\text{A.50})$$

$$\psi_{,rr}^Y = - \left[\frac{\psi_{,r}^Y}{r} + \frac{\chi_{,r}^Y}{r} - (e_2^Y - e_1^Y) (\zeta_3^Y)^4 K_0(z_3^Y) \right] \quad (\text{A.51})$$

$$\chi_{,rr}^Y = -3 \frac{\chi_{,r}^Y}{r} + (e_2^Y - e_1^Y) (\zeta_3^Y)^4 K_0(z_3^Y) - e_2^Y (\zeta_1^Y)^4 K_0(z_1^Y) + e_1^Y (\zeta_2^Y)^4 K_0(z_2^Y) \quad (\text{A.52})$$

$$\eta_{,rr}^Y = \frac{\eta^Y}{r^2} - \frac{\eta_{,r}^Y}{r} + (\zeta_1^Y)^3 K_1(z_1^Y) - (\zeta_2^Y)^3 K_1(z_2^Y) \quad (\text{A.53})$$

$$\xi_{,rr}^Y = e_1^Y (\zeta_1^Y)^2 K_0(z_1^Y) - e_2^Y (\zeta_2^Y)^2 K_0(z_2^Y) - \frac{\xi_{,r}^Y}{r} \quad (\text{A.54})$$

where $z_i^Y = \zeta_i^Y r$; K_0 and K_1 are the modified zeroth- and first-order Bessel functions of the second kind, respectively; the superscript Y denotes S and A for the symmetric and antisymmetric cases, respectively.

Appendix B

Static fundamental solutions

Two functions $A(z_A)$ and $B(z_A)$ are defined as follows: $A(z_A) = K_0(z_A) + \frac{2}{z_A} \left[K_1(z_A) - \frac{1}{z_A} \right]$ and $B(z_A) = K_0(z_A) + \frac{1}{z_A} \left[K_1(z_A) - \frac{1}{z_A} \right]$.

B.1 Kane-Mindlin theory

The fundamental solutions for static Kane-Mindlin plate problems can be obtained by modifying the fundamental solutions given in Ref. [106] and written as follows:

$$U_{\alpha\beta}^{S|\text{static}} = \frac{1}{8\pi h\mu(1-\nu)} (\psi\delta_{\alpha\beta} - \chi r_{,\alpha}r_{,\beta}) \quad (\text{B.1})$$

$$U_{\alpha 3}^{S|\text{static}} = -U_{3\alpha}^{S|\text{static}} = \frac{\nu}{4\pi h\kappa_S\mu} \left[\frac{1}{r} - \eta_S K_1(z_S) \right] r_{,\alpha} \quad (\text{B.2})$$

$$U_{33}^{S|\text{static}} = \frac{6}{\pi h^3\mu} K_0(z_S) \quad (\text{B.3})$$

$$T_{\alpha 3}^{S|\text{static}} = \frac{\nu h^2 \eta_S^2}{48\pi\kappa_S} \left[A(z_S) r_{,\alpha} r_{,n} - \frac{1}{z_S} \left(K_1(z_S) - \frac{1}{z_S} \right) n_\alpha \right] \quad (\text{B.4})$$

$$T_{3\alpha}^{S|\text{static}} = \frac{\nu\eta_S^2}{2\pi\kappa_S} [B(z_S) n_\alpha - A(z_S) r_{,\alpha} r_{,n}] \quad (\text{B.5})$$

$$T_{33}^{S|\text{static}} = -\frac{\eta_S}{2\pi} K_1(z_S) r_{,n} \quad (\text{B.6})$$

$$\begin{aligned} T_{\alpha\beta}^{S|\text{static}} = \frac{1}{8\pi(1-\nu)} \bigg\{ & \left(\psi_{,r} - \frac{\chi}{r} \right) (\delta_{\alpha\beta} r_{,n} + n_\alpha r_{,\beta}) - 2\frac{\chi}{r} (n_\beta r_{,\alpha} - 2r_{,\alpha} r_{,\beta} r_{,n}) \\ & - 2\chi_{,r} r_{,\alpha} r_{,\beta} r_{,n} + \frac{\lambda}{\mu} \left[\psi_{,r} - \frac{\chi}{r} - \chi_{,r} + 2\nu(1-\nu) \left(\frac{1}{r} - \eta_S K_1(z_S) \right) \right] r_{,\alpha} n_\beta \bigg\} \end{aligned} \quad (\text{B.7})$$

$$\begin{aligned} D_{\alpha\beta\gamma}^{S|\text{static}} = -\frac{1}{8\pi(1-\nu)} \bigg\{ & \left(\psi_{,r} - \frac{\chi}{r} \right) (\delta_{\alpha\gamma} r_{,\beta} + \delta_{\beta\gamma} r_{,\alpha}) - 2\frac{\chi}{r} (\delta_{\alpha\beta} r_{,\gamma} - 2r_{,\alpha} r_{,\beta} r_{,\gamma}) \\ & - 2\chi_{,r} r_{,\alpha} r_{,\beta} r_{,\gamma} + \frac{\lambda}{\mu} \left[\psi_{,r} - \frac{\chi}{r} - \chi_{,r} + 2\nu(1-\nu) \left(\frac{1}{r} - \eta_S K_1(z_S) \right) \right] \delta_{\alpha\beta} r_{,\gamma} \bigg\} \end{aligned} \quad (\text{B.8})$$

$$D_{\alpha\beta 3}^{S|\text{static}} = \frac{\nu\eta_S^2}{2\pi\kappa_S} [B(z_S) \delta_{\alpha\beta} - A(z_S) r_{,\beta} r_{,\alpha}] \quad (\text{B.9})$$

$$D_{3\beta\gamma}^{S|\text{static}} = \frac{\nu h^2 \eta_S^2}{48\pi\kappa_S} \left[A(z_S) r_{,\gamma} r_{,\beta} - \frac{1}{z_S} \left(K_1(z_S) - \frac{1}{z_S} \right) \delta_{\beta\gamma} \right] \quad (\text{B.10})$$

$$D_{3\beta 3}^{S|\text{static}} = \frac{\eta_S}{2\pi} K_1(z_S) r_{,\beta} \quad (\text{B.11})$$

$$\begin{aligned} S_{\alpha\beta 3}^{S|\text{static}} = \frac{\kappa_S \nu E h \eta_S}{2\pi(1-\nu^2)} \bigg\{ & \left[K_1(z_S) + \frac{4A(z_S)}{z_S} \right] r_{,\alpha} r_{,\beta} r_{,n} - \left[\frac{A(z_S)}{z_S} + K_1(z_S) \right] \delta_{\alpha\beta} r_{,n} \\ & - \frac{A(z_S)}{z_S} (n_\beta r_{,\alpha} + r_{,\beta} n_\alpha) \bigg\} \end{aligned} \quad (\text{B.12})$$

$$\begin{aligned} S_{3\beta\gamma}^{S|\text{static}} = -\frac{\kappa_S \nu E h \eta_S}{2\pi(1-\nu^2)} \bigg\{ & \left[K_1(z_S) + \frac{4}{z_S} A(z_S) \right] r_{,\beta} r_{,\gamma} r_{,n} - \left[\frac{A(z_S)}{z_S} + K_1(z_S) \right] r_{,\beta} n_\gamma \\ & - \frac{A(z_S)}{z_S} (n_\beta r_{,\gamma} + \delta_{\beta\gamma} r_{,n}) \bigg\} \end{aligned} \quad (\text{B.13})$$

$$S_{3\beta 3}^{S|\text{static}} = \frac{\mu h^3 \eta_S^2}{24\pi} \left[- \left(A(z_S) + \frac{2}{z_S^2} \right) r_{,\beta} r_{,n} + \frac{K_1(z_S)}{z_S} n_\beta \right] \quad (\text{B.14})$$

$$\begin{aligned} S_{\alpha\beta\gamma}^{S|\text{static}} = & \frac{h\mu}{8\pi(1-\nu)} \left\{ 4 \left(\chi_{,rr} - 5\frac{\chi_{,r}}{r} + 8\frac{\chi}{r^2} \right) r_{,\alpha} r_{,\beta} r_{,\gamma} r_{,n} \right. \\ & - 2 \left(\frac{\psi_{,r}}{r} - \frac{\chi}{r^2} \right) (\delta_{\alpha\gamma} n_\beta + \delta_{\beta\gamma} n_\alpha) + 2 \left[2\frac{\chi_{,r}}{r} - 4\frac{\chi}{r^2} \right. \\ & + \frac{\lambda}{\mu} \left(\chi_{,rr} - 2\frac{\chi}{r^2} - \psi_{,rr} + \frac{\psi_{,r}}{r} - 2\nu(1-\nu) \eta_S^2 A(z_S) \right) \left. \right] (\delta_{\alpha\beta} r_{,\gamma} r_{,n} + n_\gamma r_{,\alpha} r_{,\beta}) \\ & - \left(\psi_{,rr} - \frac{\psi_{,r}}{r} - 3\frac{\chi_{,r}}{r} + 6\frac{\chi}{r^2} \right) [(\delta_{\beta\gamma} r_{,\alpha} + \delta_{\alpha\gamma} r_{,\beta}) r_{,n} + (n_\alpha r_{,\beta} + n_\beta r_{,\alpha}) r_{,\gamma}] \\ & + \left[4\frac{\chi}{r^2} + 4\frac{\lambda}{\mu} \left(\frac{\chi_{,r}}{r} + \frac{\chi}{r^2} - \frac{\psi_{,r}}{r} - \frac{\nu(1-\nu)}{2r^2} (-2z_S^2 B(z_S) + 1 - z_S K_1(z_S)) \right) \right. \\ & + \left. \frac{\lambda^2}{\mu^2} \left(\chi_{,rr} + 2\frac{\chi_{,r}}{r} - \psi_{,rr} - \frac{\psi_{,r}}{r} - 2\nu(1-\nu) \eta_S^2 K_0(z_S) \right) \right] \delta_{\alpha\beta} n_\gamma \left. \right\} \end{aligned} \quad (\text{B.15})$$

in which $z_S = \eta_S r$, $\eta_S = \frac{\kappa_S}{h} \sqrt{\frac{6}{(1-\nu)}}$, and

$$\begin{aligned} \psi &= \frac{2\nu^2}{z_S} \left[K_1(z_S) - \frac{1}{z_S} \right] - (1-\nu)(3-\nu) \ln r \\ \chi &= 2\nu^2 A(z_S) + \nu^2 - 1 \\ \psi_{,r} &= -[\chi + 4(1-\nu)] \frac{1}{r} \\ \chi_{,r} &= -[\chi + 1 + \nu^2(z_S K_1(z_S) - 1)] \frac{2}{r} \\ \psi_{,rr} &= -\frac{\chi_{,r}}{r} + \frac{\chi + 4(1-\nu)}{r^2} \\ \chi_{,rr} &= -2 \left[\frac{\chi_{,r}}{r} - \nu^2 \eta_S^2 B(z_S) - \frac{\chi + 1}{r^2} \right] \end{aligned} \quad (\text{B.16})$$

B.2 Mindlin theory

The static fundamental solutions for the Mindlin theory have been given in the textbook [35], which are listed as follows:

$$\begin{aligned} U_{\alpha\beta}^{A|\text{static}} &= \frac{1}{8\pi D(1-\nu)} \{ [8B(z_A) - (1-\nu)(2 \ln z_A - 1)] \delta_{\alpha\beta} \\ &\quad - [8A(z_A) + 2(1-\nu)] \} r_{,\alpha} r_{,\beta} \end{aligned} \quad (\text{B.17})$$

$$U_{\alpha 3}^{A|\text{static}} = -U_{3\alpha}^{A|\text{static}} = \frac{1}{8\pi D} (2 \ln z_A - 1) r r_{,\alpha} \quad (\text{B.18})$$

$$U_{33}^{A|\text{static}} = \frac{1}{8\pi D (1 - \nu) \eta_A^2} \left[(1 - \nu) z_A^2 (\ln z_A - 1) - 8 \ln z_A \right] \quad (\text{B.19})$$

$$\begin{aligned} T_{\alpha\beta}^{A|\text{static}} = & -\frac{1}{4\pi r} \{ [4A(z_A) + 2z_A K_1(z_A) + 1 - \nu] (\delta_{\alpha\beta} r_{,n} + n_\alpha r_{,\beta}) \\ & + [4A(z_A) + 1 + \nu] r_{,\alpha} n_\beta - 2 [8A(z_A) + 2z_A K_1(z_A) + 1 - \nu] r_{,\alpha} r_{,\beta} r_{,n} \} \end{aligned} \quad (\text{B.20})$$

$$T_{\alpha 3}^{A|\text{static}} = \frac{\eta_A^2}{2\pi} [B(z_A) n_\alpha - A(z_A) r_{,\alpha} r_{,n}] \quad (\text{B.21})$$

$$T_{3\alpha}^{A|\text{static}} = -\frac{1 - \nu}{8\pi} \left[\left(2 \frac{1 + \nu}{1 - \nu} \ln z_A - 1 \right) n_\alpha + 2 r_{,\alpha} r_{,n} \right] \quad (\text{B.22})$$

$$T_{33}^{A|\text{static}} = -\frac{1}{2\pi r} r_{,n} \quad (\text{B.23})$$

$$\begin{aligned} D_{\alpha\beta\gamma}^{A|\text{static}} = & \frac{1}{4\pi r} \{ [4A(z_A) + 2z_A K_1(z_A) + 1 - \nu] (\delta_{\beta\gamma} r_{,\alpha} + \delta_{\alpha\gamma} r_{,\beta}) \\ & + [4A(z_A) + 1 + \nu] \delta_{\alpha\beta} r_{,\gamma} - 2 [8A(z_A) + 2z_A K_1(z_A) + 1 - \nu] r_{,\alpha} r_{,\beta} r_{,\gamma} \} \end{aligned} \quad (\text{B.24})$$

$$D_{\alpha\beta 3}^{A|\text{static}} = -\frac{1 - \nu}{8\pi} \left[\left(2 \frac{1 + \nu}{1 - \nu} \ln z_A - 1 \right) \delta_{\alpha\beta} + 2 r_{,\alpha} r_{,\beta} \right] \quad (\text{B.25})$$

$$D_{3\beta\gamma}^{A|\text{static}} = \frac{\eta_A^2}{2\pi} [B(z_A) \delta_{\beta\gamma} - A(z_A) r_{,\beta} r_{,\gamma}] \quad (\text{B.26})$$

$$D_{3\beta 3}^{A|\text{static}} = \frac{1}{2\pi r} r_{,\beta} \quad (\text{B.27})$$

$$S_{\alpha\beta 3}^{A|\text{static}} = \frac{D(1-\nu)\eta_A^2}{4\pi r} \{ [2A(z_A) + z_A K_1(z_A)] (r_{,\beta} n_\alpha + r_{,\alpha} n_\beta) - 2 [4A(z_A) + z_A K_1(z_A)] r_{,\alpha} r_{,\beta} r_{,n} + 2A(z_A) \delta_{\alpha\beta} r_{,n} \} \quad (\text{B.28})$$

$$S_{3\beta\gamma}^{A|\text{static}} = -\frac{D(1-\nu)\eta_A^2}{4\pi r} \{ [2A(z_A) + z_A K_1(z_A)] (n_\beta r_{,\gamma} + \delta_{\beta\gamma} r_{,n}) - 2 [4A(z_A) + z_A K_1(z_A)] r_{,\beta} r_{,\gamma} r_{,n} + 2A(z_A) r_{,\beta} n_\gamma \} \quad (\text{B.29})$$

$$S_{3\beta 3}^{A|\text{static}} = \frac{D(1-\nu)\eta_A^2}{4\pi r^2} \{ [z_A^2 B(z_A) + 1] n_\beta - [z_A^2 A(z_A) + 2] r_{,\beta} r_{,n} \} \quad (\text{B.30})$$

$$\begin{aligned} S_{\alpha\beta\gamma}^{A|\text{static}} = & \frac{D(1-\nu)}{4\pi r^2} \{ [4A(z_A) + 2z_A K_1(z_A) + 1 - \nu] (\delta_{\alpha\gamma} n_\beta + \delta_{\beta\gamma} n_\alpha) \\ & + [4A(z_A) + 1 + 3\nu] \delta_{\alpha\beta} n_\gamma - [16A(z_A) + 6z_A K_1(z_A) + z_A^2 K_0(z_A) + 2 - 2\nu] \\ & \times [(\delta_{\beta\gamma} r_{,\alpha} + \delta_{\alpha\gamma} r_{,\beta}) r_{,n} + (n_\alpha r_{,\beta} + n_\beta r_{,\alpha}) r_{,\gamma}] \\ & + 4 [24A(z_A) + 8z_A K_1(z_A) + z_A^2 K_0(z_A) + 2 - 2\nu] r_{,\alpha} r_{,\beta} r_{,\gamma} r_{,n} \\ & - 2 [8A(z_A) + 2z_A K_1(z_A) + 1 + \nu] (\delta_{\alpha\beta} r_{,\gamma} r_{,n} + n_\gamma r_{,\alpha} r_{,\beta}) \} \end{aligned} \quad (\text{B.31})$$

where $\eta_A = \frac{2\sqrt{3}\kappa}{h}$ and $z_A = \eta_A r$.

Appendix C

Singularities in the fundamental solutions

C.1 Kane-Mindlin theory

When the distance r between the source point $\mathbf{x}'$ and the field point $\mathbf{x}$ approaches zero, the limits of the transformed fundamental solutions have the following forms:

$$\lim_{r \rightarrow 0} \tilde{U}_{\alpha\beta}^S = -\frac{3-4\nu}{8\pi h\mu(1-\nu)} \ln r \delta_{\alpha\beta} = \lim_{r \rightarrow 0} U_{\alpha\beta}^{S|\text{static}} \quad (\text{C.1})$$

$$\lim_{r \rightarrow 0} \tilde{U}_{\alpha 3}^S = -\lim_{r \rightarrow 0} \tilde{U}_{3\alpha}^S = 0 = \lim_{r \rightarrow 0} U_{\alpha 3}^{S|\text{static}} = -\lim_{r \rightarrow 0} U_{3\alpha}^{S|\text{static}} \quad (\text{C.2})$$

$$\lim_{r \rightarrow 0} \tilde{U}_{33}^S = -\frac{6}{\pi h^3 \mu} \ln r = \lim_{r \rightarrow 0} U_{33}^{S|\text{static}} \quad (\text{C.3})$$

$$\lim_{r \rightarrow 0} \tilde{T}_{\alpha\beta}^S = -\frac{1}{4\pi(1-\nu)r} [(1-2\nu)(\delta_{\alpha\beta}r_{,n} + n_\alpha r_{,\beta} - n_\beta r_{,\alpha}) + 2r_{,\alpha}r_{,\beta}r_{,n}] = \lim_{r \rightarrow 0} T_{\alpha\beta}^{S|\text{static}} \quad (\text{C.4})$$

$$\lim_{r \rightarrow 0} \tilde{T}_{\alpha 3}^S = -\frac{\kappa_S \nu}{4\pi(1-\nu)} n_\alpha \ln r = \lim_{r \rightarrow 0} T_{\alpha 3}^{S|\text{static}} \quad (\text{C.5})$$

$$\lim_{r \rightarrow 0} \tilde{T}_{3\alpha}^S = -\frac{6\kappa_S \nu}{\pi h^2 (1 - \nu)} n_\alpha \ln r = \lim_{r \rightarrow 0} T_{3\alpha}^{S|\text{static}} \quad (\text{C.6})$$

$$\lim_{r \rightarrow 0} \tilde{T}_{33}^S = -\frac{1}{2\pi r} r_{,n} = \lim_{r \rightarrow 0} T_{33}^{S|\text{static}} \quad (\text{C.7})$$

$$\lim_{r \rightarrow 0} \tilde{D}_{\alpha\beta\gamma}^S = \frac{1}{4\pi (1 - \nu) r} [(1 - 2\nu) (\delta_{\alpha\gamma} r_{,\beta} + \delta_{\beta\gamma} r_{,\alpha} - \delta_{\alpha\beta} r_{,\gamma}) + 2r_{,\alpha} r_{,\beta} r_{,\gamma}] = \lim_{r \rightarrow 0} D_{\alpha\beta\gamma}^{S|\text{static}} \quad (\text{C.8})$$

$$\lim_{r \rightarrow 0} \tilde{D}_{\alpha\beta 3}^S = -\frac{6\kappa_S \nu}{\pi h^2 (1 - \nu)} \delta_{\alpha\beta} \ln r = \lim_{r \rightarrow 0} D_{\alpha\beta 3}^{S|\text{static}} \quad (\text{C.9})$$

$$\lim_{r \rightarrow 0} \tilde{D}_{3\beta\gamma}^S = -\frac{\kappa_S \nu}{4\pi (1 - \nu)} \delta_{\beta\gamma} \ln r = \lim_{r \rightarrow 0} D_{3\beta\gamma}^{S|\text{static}} \quad (\text{C.10})$$

$$\lim_{r \rightarrow 0} \tilde{D}_{3\beta 3}^S = \frac{1}{2\pi r} r_{,\beta} = \lim_{r \rightarrow 0} D_{3\beta 3}^{S|\text{static}} \quad (\text{C.11})$$

$$\lim_{r \rightarrow 0} \tilde{S}_{3\beta\gamma}^S = \frac{\kappa_S \nu h \mu}{2\pi (1 - \nu) r} [(2r_{,\beta} r_{,\gamma} - \delta_{\gamma\beta}) r_{,n} - r_{,\gamma} n_\beta + r_{,\beta} n_\gamma] = \lim_{r \rightarrow 0} S_{3\beta\gamma}^{S|\text{static}} \quad (\text{C.12})$$

$$\lim_{r \rightarrow 0} \tilde{S}_{\alpha\beta 3}^S = \frac{\kappa_S \nu h \mu}{2\pi (1 - \nu) r} [-(2r_{,\alpha} r_{,\beta} + \delta_{\alpha\beta}) r_{,n} + r_{,\alpha} n_\beta + r_{,\beta} n_\alpha] = \lim_{r \rightarrow 0} S_{\alpha\beta 3}^{S|\text{static}} \quad (\text{C.13})$$

$$\begin{aligned} \lim_{r \rightarrow 0} \tilde{S}_{\alpha\beta\gamma}^S &= \frac{h\mu}{2\pi (1 - \nu) r^2} \{2[-4r_{,\alpha} r_{,\beta} r_{,\gamma} + \nu (\delta_{\beta\gamma} r_{,\alpha} + \delta_{\alpha\gamma} r_{,\beta}) + (1 - 2\nu) \delta_{\alpha\beta} r_{,\gamma}] r_{,n} \\ &\quad + 2\nu (n_\alpha r_{,\beta} + n_\beta r_{,\alpha}) r_{,\gamma} + (1 - 2\nu) (\delta_{\alpha\gamma} n_\beta + \delta_{\beta\gamma} n_\alpha + 2n_\gamma r_{,\alpha} r_{,\beta}) \\ &\quad - (1 - 4\nu) \delta_{\alpha\beta} n_\gamma\} = \lim_{r \rightarrow 0} S_{\alpha\beta\gamma}^{S|\text{static}} \end{aligned} \quad (\text{C.14})$$

$$\lim_{r \rightarrow 0} \tilde{S}_{3\beta 3}^S = \frac{h^3 \mu}{24\pi r^2} (n_\beta - 2r_{,\beta} r_{,n}) = \lim_{r \rightarrow 0} S_{3\beta 3}^{S|\text{static}} \quad (\text{C.15})$$

C.2 Mindlin theory

When the distance r between the source point $\mathbf{x}'$ and the field point $\mathbf{x}$ tends to zero, the limits of the transformed fundamental solutions are given by:

$$\lim_{r \rightarrow 0} \tilde{U}_{\alpha\beta}^A = \frac{\nu - 3}{4\pi D (1 - \nu)} \ln r \delta_{\alpha\beta} = \lim_{r \rightarrow 0} U_{\alpha\beta}^{A|\text{static}} \quad (\text{C.16})$$

$$\lim_{r \rightarrow 0} \tilde{U}_{\alpha 3}^A = -\lim_{r \rightarrow 0} \tilde{U}_{3\alpha}^A = 0 = \lim_{r \rightarrow 0} U_{\alpha 3}^{A|\text{static}} = -\lim_{r \rightarrow 0} U_{3\alpha}^{A|\text{static}} \quad (\text{C.17})$$

$$\lim_{r \rightarrow 0} \tilde{U}_{33}^A = -\frac{1}{2\pi\kappa_A^2\mu h} \ln r = \lim_{r \rightarrow 0} U_{33}^{A|\text{static}} \quad (\text{C.18})$$

$$\lim_{r \rightarrow 0} \tilde{T}_{\alpha\beta}^A = -\frac{1}{4\pi r} [(1 - \nu) (\delta_{\alpha\beta} r_{,n} + r_{,\beta} n_\alpha - r_{,\alpha} n_\beta) + 2(1 + \nu) r_{,\alpha} r_{,\beta} r_{,n}] = \lim_{r \rightarrow 0} T_{\alpha\beta}^{A|\text{static}} \quad (\text{C.19})$$

$$\lim_{r \rightarrow 0} \tilde{T}_{\alpha 3}^A = -\frac{3\kappa_A^2}{\pi h^2} n_\alpha \ln r = \lim_{r \rightarrow 0} T_{\alpha 3}^{A|\text{static}} \quad (\text{C.20})$$

$$\lim_{r \rightarrow 0} \tilde{T}_{3\alpha}^A = -\frac{1 + \nu}{4\pi} n_\alpha \ln r = \lim_{r \rightarrow 0} T_{3\alpha}^{A|\text{static}} \quad (\text{C.21})$$

$$\lim_{r \rightarrow 0} \tilde{T}_{33}^A = -\frac{1}{2\pi r} r_{,n} = \lim_{r \rightarrow 0} T_{33}^{A|\text{static}} \quad (\text{C.22})$$

$$\lim_{r \rightarrow 0} \tilde{D}_{\alpha\beta\gamma}^A = \frac{1}{4\pi r} [(1 - \nu) (\delta_{\beta\gamma} r_{,\alpha} + \delta_{\alpha\gamma} r_{,\beta} - \delta_{\alpha\beta} r_{,\gamma}) + 2(1 + \nu) r_{,\alpha} r_{,\beta} r_{,\gamma}] = \lim_{r \rightarrow 0} D_{\alpha\beta\gamma}^{A|\text{static}} \quad (\text{C.23})$$

$$\lim_{r \rightarrow 0} \tilde{D}_{\alpha\beta 3}^A = -\frac{1 + \nu}{4\pi} \delta_{\alpha\beta} \ln r = \lim_{r \rightarrow 0} D_{\alpha\beta 3}^{A|\text{static}} \quad (\text{C.24})$$

$$\lim_{r \rightarrow 0} \tilde{D}_{3\beta\gamma}^A = -\frac{3\kappa_A^2}{\pi h^2} \delta_{\gamma\beta} \ln r = \lim_{r \rightarrow 0} D_{3\beta\gamma}^{A|\text{static}} \quad (\text{C.25})$$

$$\lim_{r \rightarrow 0} \tilde{D}_{3\beta 3}^A = \frac{1}{2\pi r} r_{,\beta} = \lim_{r \rightarrow 0} D_{3\beta 3}^{A|\text{static}} \quad (\text{C.26})$$

$$\lim_{r \rightarrow 0} \tilde{S}_{3\beta\gamma}^A = -\frac{3\kappa_A^2 D (1 - \nu)}{\pi h^2 r} (2r_{,\gamma} r_{,\beta} r_{,n} - n_{\gamma} r_{,\beta}) = \lim_{r \rightarrow 0} S_{3\beta\gamma}^{A|\text{static}} \quad (\text{C.27})$$

$$\lim_{r \rightarrow 0} \tilde{S}_{\alpha\beta 3}^A = \frac{3\kappa_A^2 D (1 - \nu)}{\pi h^2 r} (2r_{,\alpha} r_{,\beta} - \delta_{\alpha\beta}) r_{,n} = \lim_{r \rightarrow 0} S_{\alpha\beta 3}^{A|\text{static}} \quad (\text{C.28})$$

$$\begin{aligned} \lim_{r \rightarrow 0} \tilde{S}_{\alpha\beta\gamma}^A &= \frac{D (1 - \nu)}{4\pi r^2} \{ 2[-4(1 + \nu) r_{,\alpha} r_{,\beta} r_{,\gamma} + \nu (\delta_{\alpha\gamma} r_{,\beta} + \delta_{\beta\gamma} r_{,\alpha}) + (1 - \nu) \delta_{\alpha\beta} r_{,\gamma}] r_{,n} \\ &\quad + (1 - \nu) (\delta_{\gamma\alpha} n_{\beta} + \delta_{\gamma\beta} n_{\alpha} + 2n_{\gamma} r_{,\alpha} r_{,\beta} - \delta_{\alpha\beta} n_{\gamma}) \\ &\quad + 2\nu [(n_{\alpha} r_{,\beta} + n_{\beta} r_{,\alpha}) r_{,\gamma} + \delta_{\alpha\beta} n_{\gamma}] \} = \lim_{r \rightarrow 0} S_{\alpha\beta\gamma}^{A|\text{static}} \end{aligned} \quad (\text{C.29})$$

$$\lim_{r \rightarrow 0} \tilde{S}_{3\beta 3}^A = \frac{3\kappa_A^2 D (1 - \nu)}{\pi h^2 r^2} (n_{\beta} - 2r_{,\beta} r_{,n}) = \lim_{r \rightarrow 0} S_{3\beta 3}^{A|\text{static}} \quad (\text{C.30})$$