

TOPICS ON ELEMENTARY PARTICLE
SYMMETRY

by

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ABSTRACT

The first part of the thesis deals with the relations among meson-baryon scattering amplitudes obtained under various collinear symmetries. The auxiliary group and its role in constructing relativistic field theories that incorporate internal symmetries is developed in the introductory chapter following Feldmann and Matthews. The collinear groups, in particular $S(U(3) \otimes U(3))$ is discussed in detail. Various predictions among the meson-baryon scattering amplitudes are worked out under this group and also under $SU(6)_W$ and compared with experimental data. These processes are also considered under the collinear groups $SU(4)_W$ and $S(U(2) \otimes U(2))_{I,Y}$. These smaller groups are used to compare the effects of symmetry-breaking in $SU(3)$ and $SU(4)$.

The second part examines the decomposition of the unitary irreducible representations of $SL(2, c)$ restricted to $U(1) \wedge T(2)$ and $SU(1, 1)$. Integral representations for unitary matrices that allow to pass from one set of basis

vectors to another (labelled by the corresponding Casimir operators of the subgroups $SU(2)$, $SU(1,1)$ and $U(1) \wedge T(2)$) are derived using the theory of Induced Representation.

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PREFACE

The work presented in this thesis has been carried out under the supervision of Professor P.T. Matthews at the Imperial College of Science and Technology between October 1964 and October 1967. The material contained herein is original except where stated and has not been submitted to this or any other University for any other degree. It consists of two distinct parts: (i) Baryon-meson scattering in some collinear groups and (ii) Decomposition of the unitary irreducible representations of $SL(2, \mathbb{C})$ restricted to $U(1) \times T(2)$ and $SU(1, 1)$.

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PART ONE

BARYON-MESON SCATTERING IN SOME
COLLINEAR GROUPS

Introduction:

Once the internal and space-time symmetries of the elementary particles were found to produce modest success in the static $SU(6)^{(1)}$, it was natural to attempt to combine the two symmetries relativistically. As it was, all the possibilities of this program were soon carried out resulting in various versions of relativistic $SU(6)$.

The group $SL(6,c)$ as the smallest relativistic generalisation for $SU(6)$ was first put forward by Rühl, Fulton and Wess, and Charap and Matthews⁽²⁾. These authors completed the covariant version simply by combining the Lorentz invariance group $SL(2,c)$ with the internal symmetry group $SU(3)$. The whole theory was then required to obey invariance under the larger group $SL(6,c)$.

However, in the aforementioned $SL(6,c)$ theory, one has to introduce space reflection invariance from

outside. This has been rendered unnecessary in the formulation of the more restrictive $U(6,6)$ theory of Salam, Delbourgo and Strathdee, and Sakita and Wali⁽³⁾. This extension has been achieved by using the $U(2,2)$ group as the space-time symmetry group instead of the Lorentz group. The initial success of this version of relativistic $SU(6)$ was quite astounding in that it could ascribe an explanation to the equality of the electric and magnetic form factors of proton. Another advantage scored over $SL(6,c)$ was in assigning a definite value to this proton magnetic moment which agrees fairly well with experiment.

It was soon realised that these symmetries are not compatible in general with the Unitarity conditions⁽⁴⁾. Besides, ignoring the unitarity problem, calculations of scattering processes with $U(6,6)$ invariant amplitudes yield results which are in violent disagreement with experiment.⁽⁵⁾

We shall examine here some consequences in the case of the meson-baryon scattering processes of the

$SL(6,c)$ and its subgroups. It is convenient to build up the relativistic $SU(6)$ in the way Feldman and Matthews⁽⁶⁾ have done by using the concept of auxiliary group and index invariance. This formalism, together, with the different subgroups of $U(6,6)$ and $SL(6,c)$, forms the content of the first chapter.

In the second chapter, we consider the group $U(3) \otimes U(3)$ ⁽⁷⁾ (which is the collinear subgroup of $SL(6,c)$) and work out the predictions for the meson-baryon scattering processes in this group.⁽⁸⁾ Comparison is made with the results obtained from the corresponding $U(6,6)$ subgroup.

Chapter three examines how far the very successful Johnson-Treiman⁽⁹⁾ relations survive in the hierarchy of subgroups. This has been done using the collinear subgroups of various invariance groups that can be constructed on the basis of Feldman-Matthews method.

CHAPTER I

The Auxiliary group

The use of the auxiliary group as described by Feldman and Matthews⁽⁶⁾ has proved a considerable boon in the understanding and study of the symmetries of high energy physics. They showed how the need for an auxiliary group and the corresponding auxiliary operators arose in a natural and elegant way.

Any physical particle state should belong to the unitary representations of the inhomogeneous Lorentz group (better known as the Poincare group). Under the transformation of this group the one-particle states or the operators which create them from the vacuum transform in a rather complicated way because the spin undergoes a Wigner rotation $R(\Lambda P) = A(\Lambda P)\Lambda^{-1}(P)$. (Here Λ is the Lorentz transformation, P is the momentum and $A(P)$ is such that $A(P)P = \hat{P}$ where $\hat{P}$ is the normal four-vector for a little group of the Poincare group). This rotation depends upon the parameters of the Lorentz transformation and on the momenta of the

particle before and after transformation i.e. P and ΛP . Auxiliary operators are constructed from the creation and annihilation operators so that these may have very simple transformation properties under the Poincare group. These operators can then be made to be the finite-dimensional representations of some auxiliary group which can be any group provided it contains the homogeneous Lorentz group as a subgroup. The only requirement on the representations is that they should include the required spin of the physical particle state in their decomposition. It is in this capacity that the finite non-unitary representations of the Lorentz group enter the theory. Thus the homogeneous Lorentz group serves as the smallest auxiliary group.

When local fields are constructed from the auxiliary operators and are required to satisfy causal commutation relations, these fields also introduce antiparticles and CTP invariance. S-matrix elements formed out of these local fields by saturating auxiliary suffices ensure crossing symmetry and CTP invariance. This

has been discussed by Wienberg⁽¹⁰⁾ in great detail.

These fields can be made to accommodate larger multiplets by a corresponding extension of the homogeneous Lorentz group. S-matrix elements constructed from the new fields will then remain invariant under the index transformation of the extended auxiliary group.

We examine below in detail these ideas about the auxiliary groups and the corresponding operators with the homogeneous Lorentz group, U(2,2) group and their extension with internal symmetry groups as the auxiliary groups.

The Poincare group:

The Poincare group is the group of all transformations $L = (\Lambda, a)$ composed of a matrix Λ and a translation 'a' acting on the elements of a real, four dimensional vector space,

$$x \xrightarrow{L} x' = \Lambda x + a \quad (1)$$

We shall consider only those transformations where Λ is such that

$$\det \Lambda = +1, \quad \Lambda^0_0 \geq 1$$

which alone form the proper orthochronous Lorentz group.

The infinitesimal generators of the group have the following commutation relations

$$\begin{aligned} [J_{\mu\nu}, J_{\lambda\rho}] &= i(g_{\mu\lambda}g_{\rho\nu} - g_{\mu\rho}g_{\lambda\nu} + g_{\nu\lambda}g_{\mu\rho} - g_{\nu\rho}g_{\mu\lambda}) \\ [P_\lambda, P_\mu] &= 0, \quad [P_\lambda, J_{\mu\nu}] = i(g_{\lambda\mu}P_\nu - g_{\lambda\nu}P_\mu) \end{aligned} \quad (2)$$

The basic spinor representation of the generators $J_{\mu\nu}$ is given by the six 4x4 matrices

$$\frac{1}{2} \sigma_{\mu\nu} = \frac{i}{4} [\gamma_\mu, \gamma_\nu] \quad (3)$$

where the four matrices γ_μ are the Dirac matrices satisfying

$$[\gamma_\mu, \gamma_\nu] = 2g_{\mu\nu} \quad \text{with metric } (1, -1, -1, -1). \quad (3a)$$

Since γ_0 is hermitian and γ_i ($i=1,2,3$) are antihermitian, σ_{ij} are hermitian and σ_{i0} antihermitian.

The irreducible representation can be partially labelled by the value of the Casimir operator

$$\langle P^2 \rangle = m^2$$

Furthermore, the sign of p_0 , the eigenvalue of the operator P_0 , may also be specified, since it cannot be reversed by any operator of the group. It is found that depending upon these two specifications we may have disjoint classes of four-momenta. But the physical states must have

$$\langle P^2 \rangle = m^2 \geq 0 \text{ and } P_0 \geq 0 \quad (4)$$

Thus a partial labelling is given by equation (4) which shows that we have selected those states corresponding to a given mass. Clearly the manifold of states satisfying the above specifications is infinite; it is called an orbit. Corresponding to this orbit, we take the representative element or point to be the zero-momentum state,

$$P_\mu = \hat{P} = m_\mu = (m, \underline{0}), \quad m > 0 \quad (5)$$

Any other point on the orbit can be reached by a Lorentz transformation, such that

$$A(p)P = \hat{P} \quad (6)$$

that is, for each state P_μ there is an operator that

brings it to the zero-momentum state. For each Lorentz transformation Λ and each four-vector P , there is the Wigner's rotation $R(\Lambda, P)$, defined as

$$R(\Lambda P) = A(\Lambda P) \Lambda A^{-1}(\Lambda P) \quad (7)$$

such that

$$R(\Lambda P) \hat{P} = \hat{P} \quad (8)$$

This shows that $R(\Lambda P)$ are the set of all Lorentz transformations which leave the four-vector $\hat{P}$ invariant. This, by definition, is the little group for $\hat{P}$. It can be shown that the little group is the same for all points on an orbit⁽¹¹⁾. For the physical state defined by equation (4), the little group is the group of three-dimensional rotation, $O(3)$. This result can be easily detained in the following way:

By using the commutation relations (2), the effect of a Lorentz transformation on any four-vector may be expressed in terms of the corresponding spinor matrix, defined by $\not{P} = P_\mu \gamma_\mu$. For an infinitesimal Lorentz

transformation on $\not{A}$, we obtain

$$\begin{aligned}\delta \not{A} &= \frac{i}{4} \varepsilon^{\mu\nu} [\sigma_{\mu\nu}, \not{A}] \\ &= \frac{i}{4} m \varepsilon^{\mu\nu} [\sigma_{\mu\nu}, \gamma_0]\end{aligned}$$

We find that under transformations generated by the infinitesimal generators σ_{ij} the above equation remains invariant. These precisely generate the compact subgroup $O(3)$ which is thus the little group of the Poincare group for points on physical orbit.

Thus, we may further specify the physical state by specifying the value of the Casimir operator $|J|^2$ for a representation of the little group and that of $|J_3|$ for the state in the representation. For one particle state these specifications are just the spin s and spin-component s_z . This completely determines the one particle physical state so far as its space-time properties are concerned. Thus we have the notation

$$|m^2, s; \underline{P}, s_z\rangle = |P, s\rangle$$

for a single particle state, where m^2 , s determine the

irreducible representation and $\underline{P}$, s_z fix a state in this representation. The right hand expression is just an abbreviation for the same state; for example, for the state at rest we write

$$|m^2, s; 0, s_z\rangle = |m, s\rangle \quad (9)$$

To consider the effect of a general Poincare transformation on a single particle state, we note that such a transformation parametrised by $\Lambda^{\mu\nu}$ and a_μ is

$$U(\Lambda, a) = \exp[iP_\mu a_\mu + \frac{1}{2} \eta^{\mu\nu} J_{\mu\nu}] \quad (10)$$

It is also convenient to use the boost operator $A^{-1}(P)$ defined in equation (6) which in spinor representation takes a particularly simple form

$$|P, s\rangle = N \exp[-i\underline{\varepsilon}(P) \cdot \underline{k}] |m, s\rangle \quad (11)$$

where $\underline{k} = J_{0i}$ and $\underline{\varepsilon}(P)$ is the direction of $\underline{P}$ and satisfies

$$|\cosh \varepsilon| = \frac{P_0}{m}, \quad |\sinh \varepsilon| = \frac{P}{m} \quad (12)$$

and N is a normalisation factor. Thus, a pure Lorentz

transformation on a general one-particle state is

$$\begin{aligned} U(\Lambda P) |P, s\rangle &= e^{-i\eta \underline{k}} |P, s\rangle \\ &= N e^{-i\underline{\varepsilon}' \cdot \underline{k}} (e^{i\underline{\varepsilon}' \cdot \underline{k}} e^{-i \underline{k} \cdot \underline{\varepsilon}} e^{-i\underline{\varepsilon} \cdot \underline{k}}) |m, s\rangle \quad (13) \end{aligned}$$

where $\underline{\varepsilon}(P') = \underline{\varepsilon}'$.

We notice that $e^{i\underline{\varepsilon}' \cdot \underline{k}} e^{-i\eta \underline{k}} e^{-i\underline{\varepsilon} \cdot \underline{k}} = R(\Lambda P)$ is the Wigner's rotation, as follows from definition. Because we have here

$$\begin{aligned} R(\Lambda P) |m, s\rangle &= e^{i\underline{\varepsilon}' \cdot \underline{k}} e^{-i\eta \underline{k}} e^{-i\underline{\varepsilon} \cdot \underline{k}} |m, s\rangle \\ &= e^{i\underline{\varepsilon}' \cdot \underline{k}} e^{-i\eta \underline{k}} |P, s\rangle \\ &= e^{i\underline{\varepsilon}' \cdot \underline{k}} U(\Lambda P) |P, s\rangle \\ &= |m, s\rangle \end{aligned}$$

Therefore, this is the little group mentioned above and is unitary. We may write

$$R(\Lambda P) |m, s\rangle = |ms'\rangle \langle ms' | D(P) |ms\rangle \quad (14)$$

where we have dropped Λ in $R(\Lambda P)$ and written $R(\Lambda P) = D(P)$.

Equations (13) and (14) give

$$U(\Lambda P) |Ps\rangle = |P's'\rangle \langle ms' | D(P) |m, s\rangle \quad (15)$$

Under a translation a_μ ,

$$\begin{aligned} |P, s\rangle &\rightarrow \exp(iP_\mu a^\mu) |P, s\rangle \\ &= \exp(iP_\mu a^\mu) |P, s\rangle \end{aligned} \quad (16)$$

These transformation properties can also be shown by taking a non-degenerate vacuum state $|0\rangle$ such that

$$\begin{aligned} a^\dagger(P)|0\rangle &= |P, s\rangle \\ \therefore U(\Lambda) a^\dagger(P, s) U^{-1}(\Lambda) &= a^\dagger(P', s') \langle ms' | D(P) | ms \rangle \end{aligned} \quad (17)$$

and since $D(P)$ is unitary

$$U(\Lambda) a(P, s) U^{-1}(\Lambda) = \langle m, s | D^{-1}(P) | ms' \rangle a(P', s') \quad (18)$$

Under a displacement a_μ ,

$$U(a) a(P, s) U^{-1}(a) = e^{-iP_\mu a^\mu} a(P, s) \quad (19)$$

and
$$U(a) a^\dagger(P, s) U^{-1}(a) = e^{iP_\mu a^\mu} a^\dagger(P, s) \quad (20)$$

These equations show clearly the transformation properties of the one-particle states. One can obtain the general form of the interaction Lagrangian allowed by Poincare invariance, in principle, by forming invariants

from the creation and annihilation operators of the various particles of the theory. But, as we have seen the matrices $D(P)$ has complicated dependence not only on the parameters $\eta_{\mu\nu}$, but also on the initial and final momenta of the operator being transformed. For this reason we introduce the auxiliary operators, which have very simple Lorentz transformation properties. To this end, we notice that the exponential factors appearing in the definition for $D(P)$ involve the generators K_i of the pure Lorentz transformation so that these generators are to be included in any auxiliary group we wish to use. This shows that the simplest auxiliary group is the homogeneous Lorentz group and the most convenient representations are the finite-dimensional non-unitary ones, $|\alpha\rangle$ which contain spin s of the particle. This is because spin is generated by the compact subgroup $O(3)$ of the Lorentz group and both the unitary representations $D^{(J)}$ of $O(3)$ and the non-unitary finite dimensional representations $D^{(j,j)}$ of the Lorentz group are defined over it. Over this common

domain of definition, the $D^{(j)}$ are equivalent to certain of the $D^{(j,j)}$. For example the representation $D^{(j,0)}$ is equivalent to (can be chosen to be equal to) the representation $D^{(j)}$ where they are both evaluated at the same pure rotation R .

$$D^{(j,0)}(R) = D^{(j)}(R)$$

Of course, for some Lorentz transformation Λ other than pure space-rotation, $D^{(j,0)}(\Lambda)$ is no longer a unitary matrix. Using the spinor representation in which $D_{\alpha\beta}^{(j,0)}(A(p)) = \langle \alpha | e^{i\underline{\varepsilon} \cdot \underline{K}} | \beta \rangle$, $D_{\alpha\beta}^{(j,0)}(\Lambda) = \langle \alpha | e^{i\underline{\varepsilon} \cdot \underline{K}} | \beta \rangle$, we can write

$$\begin{aligned} \langle ms' | D^{-1}(p) | m, s \rangle &= \langle ms' | \alpha \rangle \langle \alpha | e^{i\underline{\varepsilon} \cdot \underline{K}} | \beta \rangle \langle \beta | e^{i\underline{\varepsilon} \cdot \underline{K}} | \gamma \rangle \\ &\quad \langle \gamma | e^{-i\underline{\varepsilon} \cdot \underline{K}} | \delta \rangle \langle \delta | m, s \rangle . \end{aligned}$$

We may then define the function

$$\begin{aligned} A_{\alpha}(p) &= \langle \alpha | e^{-i\underline{\varepsilon} \cdot \underline{K}} | \delta \rangle \langle \delta | m, s \rangle a(p, s) \\ &= U_{\alpha(p)}^s a(p, s) . \end{aligned} \tag{21}$$

$$\begin{aligned} \therefore U(\Lambda) A_{\alpha(p)}' U^{-1}(\Lambda) &= \langle \alpha | e^{-i\underline{\varepsilon} \cdot \underline{K}} | \beta \rangle \langle \beta | m, s \rangle U(\Lambda) a(p, s) U^{-1}(\Lambda) \\ &= \langle \alpha | e^{-i\underline{\varepsilon} \cdot \underline{K}} | \beta \rangle \langle \beta | ms \rangle \langle ms | D^{-1}(p) | ms' \rangle a(p', s') . \\ &= \delta_{\alpha\gamma} \langle \gamma | e^{i\underline{\varepsilon} \cdot \underline{K}} | \beta \rangle A_{\beta(p')}^{s'} \\ &\equiv S_{\alpha}^{\beta} A_{\beta(p')}^{s'} \end{aligned} \tag{22}$$

So the new function transform with the finite non-unitary matrix $S_{\alpha}^{\beta}(D(j,0))$ as used above) which is parametrised only by the pure Lorentz transformation parameters, decoupled from the momentum P . This has the desired simplicity. The non-unitary states $|\alpha\rangle$ have no direct physical significance. The link between the physical states $|m,s\rangle$ and the non-unitary auxiliary states $|\alpha\rangle$ is given by the constant spinor $\langle\alpha|m,s\rangle$. It is however important that in the construction of $A_{\alpha}(P)$ the representation $|\alpha\rangle$ should contain the spin s , as has been noted earlier. If $|\alpha\rangle$ contains several spins, this can be used in putting particles of different spin in the same multiplet.

The auxiliary operator dual to $A_{\alpha}(P)$ is

$$A^{\alpha}(p) = a^+(ps) \langle m,s | \beta \rangle \langle \beta | e^{i\underline{\varepsilon} \cdot \underline{K}} | \alpha \rangle \quad (23)$$

with the transformation properties

$$\begin{aligned} U(\Lambda) A^{\alpha}(p) U^{-1}(\Lambda) &= A^{\beta}(p') \langle \beta | e^{-i\underline{\eta} \cdot \underline{K}} | \alpha \rangle \\ &= A^{\beta}(p') (S^{-1})_{\beta}^{\alpha} \end{aligned} \quad (24)$$

so that

$$A^\alpha(p) A_\alpha(p')$$

is a scalar. However, because α -representation is not unitary, $A^\alpha(P)$ is not the hermitian conjugate of A_α . To satisfy reality requirements it is necessary to find the transformation properties of $(A_\alpha)^+$ for each case separately.

The transformation properties of these operators under translation follows from the definition, equations (19) and (20).

$$\begin{aligned} A_\alpha(p) &\longrightarrow e^{-iP_\mu a^\mu} A_\alpha(p) \\ A^\alpha(p) &\longrightarrow e^{iP_\mu a^\mu} A^\alpha(p) \end{aligned} \quad (25)$$

With these auxiliary operators one can construct Poincare invariants. All one has to do is to take product of the appropriate operators and saturate the indices. For example

$$T = \int A^\alpha(p_1) A^\beta(p_2) \cdots A_\alpha(p_3) A_\beta(p_4) \times \frac{\delta(p_1+p_2+\dots-p_3-p_4)}{d^4p_1 \dots d^4p_4}$$

is invariant under the Poincare group. The δ function is necessary to remove the phase factors that arise

because of translations. The construction of these invariants does not require any antiparticle which, however, has to be introduced in connection with the local fields.

These local fields $\psi_\alpha(x)$ satisfy the causal commutation (or anti commutation) relations,

$$[\psi_\alpha(x), \psi_\beta(y)] = 0 \quad \text{if } (x-y)^2 < 0. \quad (25)$$

which is the causality relation.

$\psi_\alpha(x)$ can be obtained from $A_\alpha(P)$ by taking Fourier transform, but it will be necessary to have terms with positive and negative frequencies, corresponding to creation and annihilation operators. Operators with positive frequencies can be introduced through the B matrices defined below.

For any pure rotation D there exists a matrix B such that

$$\langle ms | D | m, s' \rangle = \langle m, s | B^{-1} D B | m, s \rangle \quad (27)$$

This shows that alternately we can write

$$U(\Lambda) a^+_{(p,s)} U^{-1}(\Lambda) = \langle m,s | B^{-1} D B | m,s' \rangle a^+(p's') \quad (28)$$

and consequently an alternate auxiliary field can be defined;

$$\tilde{A}_\alpha(p) = \langle \alpha | e^{-i \underline{\epsilon} \cdot \underline{K}} | \beta \rangle \langle \beta | B | m,s \rangle a^+(p,s) \quad (29)$$

which under pure Lorentz transformation transforms as

$$U(\Lambda) \tilde{A}_\alpha(p) U^{-1}(\Lambda) = \langle \alpha | e^{i \underline{\eta} \cdot \underline{K}} | \beta \rangle \tilde{A}_\beta(p') \quad (30)$$

This is the same as for $A_\alpha(p)$. However, under translation

$$\tilde{A}_\alpha(p) \longrightarrow e^{i p_\mu a^\mu} \tilde{A}_\alpha(p) \quad (31)$$

which is opposite to that for $A_\alpha(p)$, equation (25).

Then we can define a local field $\psi_\alpha(x)$

$$\psi_\alpha(x) = \int (A_{\alpha(p)} e^{-i p \cdot x} + \tilde{B}(p) e^{i p \cdot x}) \theta(p_0) 2\pi \delta(p^2 - \bar{m}^2) \frac{d^4 p}{(2\pi)^4} \quad (32)$$

where a new particle of mass $\bar{m}$ and spin $\bar{s}$ with the

auxiliary field

$$\begin{aligned}\tilde{B}_{\alpha(p)} &= \langle a | e^{-i \underline{E} \cdot \underline{K}} | \beta \rangle \langle \beta | B | \bar{m} \bar{s} \rangle b^+(p, s) \\ &\equiv \tilde{V}_{\alpha(p)} b^+(p, \bar{s})\end{aligned}\quad (33)$$

has been introduced. The variable $\bar{s}$ sums over the same range of values as s but the matrix $\langle \alpha | m s \rangle$ may be different from $\langle \beta | \bar{m} \bar{s} \rangle$.

If we now construct commutator (and anticommutator) using the above definition for the local fields, we

have

$$\begin{aligned}[\psi_{\alpha}(x), \psi_{\beta}(y)]_{\pm} &= \int \left\{ U_{\alpha(p)}^s (U_{\beta(p)})^{\mp} e^{-ip(x-y)} \mp \tilde{V}_{\alpha(p)}^{\bar{s}} (\tilde{V}_{\beta(p)}^{\bar{s}})^{\mp} \right. \\ &\quad \left. e^{ip(x-y)} \right\} \theta(p_0) 2\pi \delta(p^2 - m^2) \frac{d^4 p}{(2\pi)^4}\end{aligned}\quad (34)$$

For causality relation (26) to be satisfied, we must

have

$$\begin{aligned}[\psi_{\alpha}(x), \psi_{\beta(x)}^+]_{\pm} &= f_{\alpha}^{\beta}(\delta) \int (e^{-ip(x-y)} - e^{ip(x-y)}) \theta(p_0) \\ &\quad \times 2\pi \delta(p^2 - m^2) \frac{d^4 p}{(2\pi)^4} \\ &\equiv f_{\alpha}^{\beta}(\delta) \Delta(x-y).\end{aligned}\quad (35)$$

The important point to notice is the minus sign appearing in between the two exponential factors. One term

comes from a-particles and the other from b-particles, and so it is necessary to include the latter to satisfy the causality condition. This also leads to the relation between spin and statistics, and to the relative parities of particle and antiparticle. This is illustrated below by considering the cases for zero and $\frac{1}{2}$ spin particles.

For zero spin, things are quite simple, as $|P,s\rangle$ is the trivial representation of the little group and $|\alpha\rangle$ is the trivial representation of the auxiliary group i.e. the homogeneous Lorentz group in this case. Thus, we have

$$u = \tilde{v} = 1 \quad \text{and} \quad A(P) = a(P), \quad \tilde{B}(P) = b^+(P)$$

Assuming Bose statistics (and hence commutation relations for the operators a and b), we find

$$[\phi(x), \phi(y)] = \Delta(x-y)$$

which is the causality-relation (26), thus trivially satisfied. Although for causality relation to be satisfied we do not need to know about the parity

explicitly, we may make the natural assumption for the local fields under which

$$R\phi(x)R^{-1} = \pm \phi(x_0, -\underline{x})$$

where R is the parity-operator. This condition then requires that particles and antiparticles of zero-spin must have the same intrinsic parity.

Spin- $\frac{1}{2}$ particle case:

For the spin- $\frac{1}{2}$ case we use the reducible representation $(\frac{1}{2}, 0) + (0, \frac{1}{2})$ of the auxiliary group. If we use the Dirac matrices (3a), then from the definition of $U_{\alpha}^s(p)$ and $\tilde{V}_{\alpha}^s(p)$ (equations (21) and (33)) we get

$$\begin{aligned} U_{\alpha}^s(p) &= (e^{-i\varepsilon_I \sigma_0 I/2})_{\alpha}^{\delta} \langle \delta | m, s \rangle \\ U_{\alpha}^s(p) &= \langle m, s | \delta \rangle (e^{-i\varepsilon_I \sigma_0 I/2})_{\beta}^{\alpha} \\ \tilde{V}_{\alpha}^{\bar{s}}(p) &= (e^{-i\varepsilon_I \sigma_0 I/2})_{\alpha}^{\delta} \langle \delta | B | \bar{m} \bar{s} \rangle \\ \tilde{V}_{\alpha}^{\bar{s}}(p) &= \langle \bar{m} \bar{s} | B | \delta \rangle (e^{-i\varepsilon_I \sigma_0 I/2})_{\delta}^{\alpha} \end{aligned} \quad (36)$$

Also equations (3) and (12) give

$$e^{-i\varepsilon_I \sigma_0 I} = \left(\frac{\gamma_{\mu}^P}{m} \right) \gamma_0 \quad (37)$$

The γ_0 matrix has the properties of the parity operators. It is necessary to assume anti-commutation relations so that causality relations may be satisfied. Depending on what conditions are placed on the constant spinors $\langle \alpha | m, s \rangle$ and $\langle \alpha | \bar{m}, \bar{s} \rangle$, the spin sums UU^* and $\bar{V}\bar{V}^*$ appearing in the causality relations (34), will be different.

Thus, if there are as many s as there are α , then

$$\langle \alpha | m, s \rangle = \delta_{\alpha}^s$$

and we have parity-doubling,

$$\gamma_0 | m, S_{1,2} \rangle = + | m, S_{1,2} \rangle$$

$$\gamma_0 | m, S_{3,4} \rangle = - | m, S_{3,4} \rangle$$

and similarly for the antiparticles. Using (36) and (37) the spin sum is carried out and the causality relation becomes

$$\left\{ \psi_{\alpha}(x), \psi_{\beta}^+(y) \right\}_+ = \left(\frac{i\gamma_{\mu} \partial_{\mu} \cdot \gamma_0}{m} \right)_{\alpha}^{\beta} \Delta(x-y) .$$

There is no equation of motion for $\psi_{\alpha}(n)$ since there are no restrictions on $A_{\alpha}(p)$ and $\bar{B}_{\alpha}(p)$. Equations of motion appear only when the number of s is not equal to α . As far as causality is concerned, there is no

necessity for the antiparticles to appear.

If we restrict the parity of the particles by the relation

$$(\gamma_0)_\alpha^\beta \langle \beta | m, s \rangle = \langle \alpha | m, s \rangle \quad (38a)$$

we find that only when the antiparticles take opposite parity

$$(\gamma_0)_\alpha^\beta \langle \beta | \bar{m}, \bar{s} \rangle = -\langle \alpha | \bar{m}, \bar{s} \rangle \quad (38b)$$

the causality relations are satisfied. With these restrictions on the particle and antiparticle states, the sums appearing in the causality relation (34) are

$$(UU^\#) = e^{-i\varepsilon_1 \sigma_{0i}/2} \left(\frac{1+\gamma_0}{2} \right) e^{-i\varepsilon_1 \sigma_{0i}/2} = \left(\frac{\gamma_\mu p^\mu + m}{2} \right) \gamma_0$$

$$(\bar{V}V^\#) = e^{-i\varepsilon_1 \sigma_{0i}/2} \left(\frac{1-\gamma_0}{2} \right) e^{-i\varepsilon_1 \sigma_{0i}/2} = \left(\frac{\gamma_\mu p^\mu - m}{2} \right) \gamma_0$$

The causality relation is then

$$\left\{ \psi_\alpha(x), \psi_\beta^\dagger(y) \right\} = \left\{ (\gamma^\delta + m) \gamma_0 \right\}_\alpha^\beta \Delta(x-y)$$

This shows that the equation of motion of the field is

$$(\gamma_\mu \partial^\mu + m) \psi(x) = 0 \quad (39)$$

From the above analysis, it is particularly clear that the equations of motion such as (39) are just the boosts of the restrictions imposed by parity to reduce the number of independent components of the auxiliary operators $A_{\alpha(p)}$ to the number of independent operators $a(p,s)$.

U(2,2) as the auxiliary group:

In this section, we shall extend the auxiliary group to U(2,2) and examine how local fields which combine particles of different spin may be constructed under this group.

U(2,2) is the group of transformations whose infinitesimal generators are $F^r (r=1,2,\dots,16)$. In the basic 4x4 representation $|\alpha\rangle$, (quark representation)

$$\langle \alpha | F^r | \beta \rangle = \frac{1}{2} (F^r)_{\alpha}^{\beta} \quad (40)$$

where $F^r = 1, i\gamma_5, \gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu}$.
 $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3, \gamma_5^2 = 1,$

These generators satisfy

$$\gamma_0 (F^r)^+ \gamma_0 = F^r \quad (41)$$

As in the Lorentz group case, we can define an auxiliary operator

$$A_{\alpha}(p) = \langle \alpha | e^{-i\underline{\varepsilon} \cdot \underline{K}} | \beta \rangle \langle \beta | m, s \rangle a(p, s) \quad (42)$$

where $|\alpha\rangle$ is now a representation of $U(2,2)$. Under a $U(2,2)$ index transformation, we get

$$A_{\alpha}(p) \rightarrow \langle \alpha | e^{i\underline{\eta} \cdot \underline{R}} e^{i\underline{p} \cdot \underline{K}} | \beta \rangle A_{\beta}(p) = T_{\alpha}^{\beta} A_{\beta}(p) \quad (42)$$

where $\eta_{\mathbf{r}}$ are the parameters of the group. The auxiliary operators dual to $A_{\alpha}(p)$ is $A_{(p)}^{\alpha}$ defined by

$$A_{(p)}^{\alpha} = a_{(p,s)}^{+} \langle m, s | \beta \rangle \langle \beta | e^{i\underline{\varepsilon} \cdot \underline{K}} | \alpha \rangle \quad (43)$$

which transform under $U(2,2)$ as

$$A_{(p)}^{\alpha} \longrightarrow A_{(p)}^{\beta} (T^{-1})_{\beta}^{\alpha} \quad (44)$$

In this basic 4×4 representation $A_{(p)}^{\alpha}$ has the same transformation properties as $A_{\beta}^{\alpha}(\gamma_0)_{\beta}^{\alpha}$, which is verified from the definition of the $U(2,2)$ generators (41). All other auxiliary operators are to be constructed from these tensors.

We construct local fields from the auxiliary operators. For spin $\frac{1}{2}$ particles, the lowest representation of $U(2,2)$ which contains spin $\frac{1}{2}$ particles is the basic representation. Reduction under the Lorentz group gives $(\frac{1}{2}, 0) + (0, \frac{1}{2})$ representation. Construction of such fields follows exactly the same procedure as for the Lorentz group.

In the case of integral spin, we find that fields of spin 0 and 1 can be constructed from auxiliary operators like

$$A_{\alpha}^{\beta}(p) = (U_{(p)}^s)_{\alpha}^{\beta} a(p, s) \quad (45)$$

where

$$\begin{aligned} (U_{(p)}^s)_{\alpha}^{\beta} &= \langle \alpha^{\beta} | e^{-i\varepsilon \cdot \underline{K}} | \alpha'^{\beta'} \rangle \langle \alpha'^{\beta'} | m, s \rangle \\ &= (e^{-i\varepsilon_i \sigma_{oi}/2})_{\alpha}^{\alpha'} \langle \alpha'^{\beta'} | m, s \rangle (e^{i\varepsilon_i \sigma_{oi}/2})_{\beta'}^{\beta} \end{aligned} \quad (46)$$

From this operator we can construct the local fields $\phi_{\alpha}^{\beta}(x)$ as in the previous case

$$\phi_{\alpha}^{\beta}(x) = \int (A_{\alpha(p)}^{\beta} e^{-ip \cdot x} + \tilde{B}_{\alpha(p)}^{\beta} e^{ip \cdot x}) 2\pi \theta(P_0) \delta(p^2 - m^2) \frac{d^4 p}{(2\pi)^4} \quad (47)$$

where again we have used a field for another particle of mass m , with auxiliary field

$$\begin{aligned}\tilde{B}_\alpha^\beta(p) &= \langle \alpha^\beta | e^{-i\underline{\varepsilon} \cdot \underline{K}} | \alpha'^\beta \rangle \langle \alpha' | B_{m,s} b^+(p, \bar{s}) \\ &= (\tilde{V}_{(p)}^{\bar{s}})_\alpha^\beta b^+(p, \bar{s}).\end{aligned}\quad (48)$$

Assuming the operators $a(p,s)$, $b(p,s)$ etc., to be Bose operators so that they obey commutation relations, we have

$$\begin{aligned}[\phi_\alpha^\beta(x), \phi_\pi^{\rho+}(y)] &= \int \left\{ (U_{(p)}^s)_\alpha^\beta (U_{(p)}^s)_\pi^\rho e^{-ip(x-y)} - \right. \\ &\quad \left. (\tilde{V}_{(p)}^s)_\alpha^\beta (\tilde{V}_{(p)}^{\bar{s}})_\pi^\rho e^{ip(x-y)} - 2\pi\Theta(P_0)\delta(p^2-m^2)\frac{d^4p}{(2\pi)^4} \right\} \quad (49)\end{aligned}$$

Causality relations demand it to have the following form

$$[\phi_\alpha^\beta(x), \phi_\pi^{\rho+}(y)] = f_{\alpha\rho}^{\beta\pi}(\partial) \Delta(x-y).$$

This requires the summation of UU^{**} and VV^{**} . As in the spin $\frac{1}{2}$ case the summation depends on the choice of the projection operators,

$$\begin{aligned}O_{\alpha\beta}^{\pi\rho} &= \langle \alpha^\beta | m, s \rangle \langle m, s | \rho_\pi \rangle \\ O_{\alpha\beta}^{\pi\rho} &= \langle \alpha^\beta | B | m, \bar{s} \rangle \langle m, \bar{s} | \rho_\pi \rangle\end{aligned}$$

If S has as many values as α, β we then have

$$O_{\alpha\beta}^{\pi\rho} = \tilde{O}_{\alpha\beta}^{\pi\rho} = \delta_{\alpha}^{\pi} \delta_{\rho}^{\beta}$$

$$\therefore (U_{(p)}^S)_{\alpha}^{\beta} (U_{(p)}^{\pi\pi})_{\pi}^{\rho} = (\tilde{V}_{(p)}^{\bar{S}})_{\alpha}^{\beta} (\tilde{V}_{(p)}^{\pi\pi})_{\pi}^{\rho} =$$

$$(\frac{\gamma_{\mu} p^{\mu} \gamma_0}{m})_{\alpha}^{\beta} (\frac{\gamma_0 \gamma_{\mu} p^{\mu}}{m})_{\pi}^{\rho}$$

and

$$[\phi_{\alpha}^{\beta}(x), \phi_{\pi}^{\rho+}(y)] = (\frac{i\gamma_{\mu} \partial^{\mu} \gamma_0}{m})_{\alpha}^{\pi} (\gamma_0 \frac{i\gamma_{\mu} \partial^{\mu}}{m})_{\rho}^{\beta} \Delta(x-y)$$

This will mean a multiplet with sixteen particles and sixteen antiparticles, with no equation of motion, as there is no restriction on the operators.

Various restrictions can be imposed on the field through parity. Parity-operator in this representation is

$$\langle_{\alpha}^{\beta} | R |_{\pi}^{\rho} \rangle = (\gamma_0)_{\alpha}^{\pi} (\gamma_0)_{\rho}^{\beta} \quad (50)$$

We may require a state $|m, s\rangle$ to have a definite parity, say positive or negative. This restriction due to parity will result in an equation of motion for the fields.

This has been shown below by taking the most restrictive parity requirement, where over and above the field having a definite parity, we require each quark-component

to have a definite parity. each basic representation is subjected to a definite parity. This restriction will lead to the equations of motion for the fields which are the Bargman-Wigner equations introduced by Salam, Delbourgo and Strathdee⁽³⁾ and discussed in great detail by Charap, Matthews and Streater⁽¹²⁾.

We specify a positive or negative quark-parity for the state $|m,s\rangle$ by

$$(\gamma_0)_\alpha^\pi \langle_\pi^\beta | m,s\rangle = \pm \langle_\alpha^\beta | m,s\rangle \quad (51)$$

The antiquark-parity is positive or negative according as

$$\langle_\alpha^\rho | m,s\rangle (\gamma_0)_\rho^\beta = \pm \langle_\alpha^\beta | m,s\rangle \quad (52)$$

Let the quark-parity be positive, so that

$$(\gamma_0)_\alpha^\pi \langle_\pi^\beta | m,s\rangle = \langle_\alpha^\beta | m,s\rangle \quad (53)$$

$$\therefore \langle_\alpha^\beta | m,s\rangle = \frac{1}{2}(1+\gamma_0)_\alpha^\pi \langle_\pi^\beta | m,s\rangle$$

Further, if we require the parity of the state $|m,s\rangle$

to be negative, that is if

$$(\gamma_0)_\alpha^\pi \langle_\pi^\rho | m,s\rangle (\gamma_0)_\rho^\beta = - \langle_\alpha^\beta | m,s\rangle$$

then, the antiquark parity has to be negative

$$\begin{aligned} \langle_{\alpha}^{\rho} | m, s \rangle (\gamma_0)_{\rho}^{\beta} &= - \langle_{\alpha}^{\beta} | m, s \rangle \\ \therefore \langle_{\alpha}^{\beta} | m, s \rangle &= \frac{1}{2} (1 - \gamma_0)_{\rho}^{\beta} \langle_{\alpha}^{\rho} | m, s \rangle \end{aligned} \quad (54)$$

$$\begin{aligned} \text{Then, } O_{\alpha\rho}^{\pi\beta} &= \langle_{\alpha}^{\beta} | m, s \rangle \langle m, s | \rho_{\pi} \rangle \\ &= \frac{1}{2} (1 + \gamma_0)_{\alpha}^{\pi} \langle_{\pi}^{\beta} | m, s \rangle \langle m, s | \rho_{\pi} \rangle \\ &= \frac{1}{4} (1 + \gamma_0)_{\alpha}^{\pi} (1 - \gamma_0)_{\rho}^{\beta} \langle_{\pi}^{\rho} | m, s \rangle \langle m, s | \rho_{\pi} \rangle \\ &= \frac{1}{4} (1 + \gamma_0)_{\alpha}^{\pi} (1 - \gamma_0)_{\rho}^{\beta} \end{aligned}$$

To satisfy the causality relations, the quark-parity for $|m, \bar{s}\rangle$ must be taken opposite to that for $|m, s\rangle$.

This leads to

$$\begin{aligned} \tilde{O}_{\alpha\rho}^{\pi\beta} &= \frac{1}{4} (1 - \gamma_0)_{\alpha}^{\pi} (1 + \gamma_0)_{\rho}^{\beta} \\ \therefore [\not{\epsilon}_{\alpha}^{\beta}(x), \not{\epsilon}_{\pi}^{\rho+}(y)] &= \frac{1}{4m^2} [((i\gamma_{\mu}\partial^{\mu} + m)\gamma_0)_{\alpha}^{\pi} (\gamma_0(i\gamma_{\mu}\partial^{\mu} - m))_{\rho}^{\beta}] \Delta(x-y) \end{aligned}$$

The equations of motion for the fields are

$$\begin{aligned} (i\gamma_{\mu}\partial^{\mu} - m)_{\alpha}^{\pi} \not{\epsilon}_{\pi}^{\rho}(x) &= 0 \\ \not{\epsilon}_{\alpha}^{\rho}(x) (i\gamma_{\mu}\partial^{\mu} + m)_{\rho}^{\beta} &= 0 \end{aligned} \quad (55)$$

which are boosts of the parity-restrictions, equations (53) and (54).

These fields ϕ_{α}^{β} contain particle operators for spin-zero and spin-one particles. The spinors arising in its decomposition - equation (49), into positive and negative frequency parts obey the equations

$$\begin{aligned} (\gamma_{\mu} P^{\mu} - m)_{\alpha}^{\pi} U_{\pi}^{\beta}(p) &= U_{\alpha}^{\rho}(p) (\gamma_{\mu} P^{\mu} + m)_{\rho}^{\beta} = 0 \\ (\gamma_{\mu} P^{\mu} + m)_{\alpha}^{\pi} v_{\pi}^{\beta}(p) &= v_{\alpha}^{\rho}(p) (\gamma_{\mu} P^{\mu} - m)_{\rho}^{\beta} = 0 \end{aligned} \quad (56)$$

By this extension of the auxiliary group it becomes possible now to incorporate in a single causal field several irreducible representations of the Poincare group. This is naturally expected as the auxiliary group used here contains the homogeneous Lorentz group as one of its subgroups. These extended particle-multiplets will form irreducible representations of some extended little group. The little group can be determined in the following way:

This will be the group of all transformations that leave invariant the form $ma_{(0,s)}^{+} a_{(0,s)}$ for all single

particle operators. This denotes the projection of the free-particle energy-momentum operator at rest.

When expressed in terms of the auxiliary operator, this takes the form

$$m(A_{\alpha}(0))^+ \langle \alpha | m, s \rangle \langle m, s | \beta \rangle A_{\beta}(0) = m(A_{\alpha}(0))^+ O_{\alpha}^{\beta} A_{\beta}(0) \quad (57)$$

Since under the little group transformations the momentum remains invariant, it will be a sub-group of the auxiliary group that leaves the operator (57) invariant. In any representation $|\alpha\rangle$, these are elements T which satisfy

$$(T^+ O T)_{\alpha}^{\beta} = O_{\alpha}^{\beta}$$

In terms of the regular representations Γ

$$(\Gamma^+ O \Gamma)_{\alpha}^{\beta} = O_{\alpha}^{\beta}$$

$$\text{or, } (O \Gamma)_{\alpha}^{\beta} = (\Gamma^+ O)_{\alpha}^{\beta} \quad (58)$$

Since O_{α}^{β} always contains the unit matrix, equation (58) implies that

$$\Gamma = \Gamma^+ \quad (59)$$

Equation (59) allows only the hermitian generators to form the subgroup, so that the subgroup can be no larger than the maximal compact subgroup. However, the little group is further restricted by the choice of O_{α}^{β} . For the choices of O_{α}^{β} , $O_{\alpha\rho}^{\pi\beta}$ made so far for particles of spin $\frac{1}{2}$, 0 and 1 cases, the generators are found to be those of $U(2) \otimes U(2)$. As has been shown by Feldman and Matthews,⁽⁶⁾ if more restrictions are imposed on the fields, the little group may turn out to be $U(2)$.

Extended Auxiliary Group including Internal Symmetry:

The inclusion of internal symmetry in a large auxiliary group is very simple. If in addition to the Poincare invariance we assume that the single particle states transform as representations of an internal symmetry group S (for example $SU(3)$ or isospin group $SU(2)_I$), we know that the little group will be $SU(2) \otimes S$ for these states. $SU(2)$ refers here to the spin group. Then we introduce the auxiliary fields

which are the representations of some auxiliary group incorporating the internal symmetry group S mentioned above. From the properties of the auxiliary group we know that it must contain the Lorentz group and the little group as its subgroups. Naturally, the smallest possible auxiliary group is then $SL(2, c) \otimes S$. The auxiliary fields are then $\phi_{\alpha s}$ where α specifies a representation of the Lorentz group and s a representation of S . The auxiliary group transformations are index transformations and the indices of the two groups are used independently. Interactions invariant under the Poincare group and the internal symmetry group can be constructed by saturating the two indices independently.

Now, this auxiliary group can be enlarged to $SL(2, C)$ (we have used this only symbolically, bearing in mind that the internal groups we shall use are $su(n)$ type) under the transformations of which the space-time symmetry group and the internal symmetry group are interrelated. This means that the Lorentz group index α and the internal symmetry index s now no longer act

independently. Corresponding to such auxiliary groups, auxiliary fields are $\phi_{\alpha s} = \phi_A$ where A denotes a representation of $SL(2s, c)$.

Analogously larger auxiliary groups can be constructed incorporating s with $U(2, 2)$. The auxiliary group is then $U(2, 2) \otimes S$ and the basic representation is $\psi_{\alpha, s}$ where α is a representation in $U(2, 2)$ and s is the same as above. In the index transformations, the two indices are independent so that for an interaction to be invariant under the group, the two indices are saturated independently. If parity restrictions are no stronger than those for $U(2, 2)$, the little group will be $U(2) \otimes U(2) \otimes S$. It is again a simple matter to extend this group to the larger $U(2S, 2S)$ auxiliary group, having some additional transformations which relate the two subgroups, namely $U(2, 2)$ and s . The basic quark representation is $\psi_x = \psi_{\alpha s}$ where α is coupled to S and x is a representation of the larger group $U(2S, 2S)$.

With this knowledge about the extended auxiliary groups we may now examine the extent to which these can be used in constructing theories incorporating

higher symmetries. For this discussion, we can safely exclude the internal symmetry group. The implications of extending the auxiliary group to $U(2,2)$ are that we can construct multiplets such as ϕ_α^β , $\phi_{\alpha\beta\gamma}$ which are representations of $U(2,2)$ and which combine different irreducible representations of the Poincare group. These fields also carry the desired properties of the local fields, such as CTP invariance, crossing symmetry, existence of the antiparticles etc.

The interaction Lagrangian has to be invariant under the index transformation of $U(2,2)$ and is constructed by saturating all the indices of the group in the product of the appropriate fields. This restriction to index invariance leads to a restricted subclass of Poincare invariants. Thus, a term like

$$\psi^\alpha(q)(\gamma_\mu)_\alpha^\beta \psi_\beta(t) P_\mu$$

which is Poincare invariant, or even

$$\psi^\alpha(\gamma_\mu)_\alpha^\beta \psi_\beta \chi^\delta(\gamma^\mu)_\delta^\epsilon \chi_\epsilon$$

which are index invariant under the homogeneous Lorentz group, are excluded.

The allowed index $U(2,2)$ invariant interaction is invariant under the little group transformations in the static limit. It is in this sense that the extended auxiliary groups $SL(6,c)$ or $U(6,6)$ are the covariant versions of the non-relativistic $SU(6)$ or $U(6) \otimes U(6)$ respectively. However, in spite of such restrictions on the interaction Lagrangian, it is not possible to obtain S-matrix elements having this $U(2,2)$ index invariance. This is because these S-matrix elements necessarily contain factors of the form $P_\mu \gamma^\mu$ which destroys the generalised invariance leaving only Poincare invariance. Thus, for theories with auxiliary groups such as $SL(6,c)$ or $U(6,6)$ the only surviving invariance is that of $P \otimes SU(3)$.

This is as expected and shows that theories built with the auxiliary operators do not violate the Poincare theorem.⁽¹³⁾ According to this theorem, it is not possible to extend the invariance of a theory

with respect to a group larger than $P \otimes SU(3)$, which contains this outer product group as subgroup without extending the energy-momentum vector to more than four components.

Theories with a large number of energy-momentum vector components have been considered by various authors.⁽¹⁹⁾ For example, in the case of $IU(2,2)$ the inhomogeneous $U(2,2)$ - the energy-momentum operator has sixteen components. Auxiliary operators can then be constructed in the same way as was done for the Poincare group, and finally, to make contact with the physical world, all the P and x are restricted to run only over the real four values. This guides us essentially to the same kind of theories discussed above. The only difference is that such a theory leads to the interesting possibility of the so-called irregular or derivative coupling⁽¹⁵⁾ in the interaction Lagrangian.

It is befitting to conclude this section with two theorems established by Feldman and Matthews⁽¹⁶⁾ regarding unitarity in such index invariant theories.

These are:-

a. Unitarity Theorem! Invariance of the S-matrix with respect to the index transformations of the auxiliary group is incompatible with the unitarity of the S-matrix, unless all the auxiliary operators are unitary representations of the auxiliary group, and there is one-to-one correspondence between the components of the auxiliary representation and the states of the corresponding physical multiplet:-

b. The Causality Theorem: A local field constructed from the auxiliary operators satisfying the Unitarity theorem cannot satisfy a causal commutation relation.

U(6,6), SL(6,c) and the hierarchy of subgroups:

We shall discuss briefly these two relativistic versions of the SU(6) group with some of their important representations. Also we shall consider the hierarchy of subgroups under which collinear and coplanar processes remain invariant.

(i) $U(6, 6)$

$U(6, 6)$ is the group whose infinitesimal generators F_x ($x=1; 144$) are obtained from the outer product of the sixteen basic generators $\frac{1}{2}f^R$ of $U(2, 2)$ with the nine basic generators λ_i of $U(3)$. These generators, $F_x = i\gamma$ act on the fundamental 12-component (quark) spinor: which transforms under the group as

$$\psi_A \rightarrow S_A^B \psi_B \quad A = i\alpha, \quad i = 1, 2, 3, \quad \alpha = 1, 2, 3, 4 \quad (60)$$

where $s = \exp(\frac{i}{2} \theta_x \epsilon^x)$

summation is indicated for the saturated indices and ϵ^x are real parameters. This leaves $\bar{\psi}\psi = \psi^\dagger \gamma_0 \psi$ invariant and this satisfies

$$\gamma_0 F_x^\dagger \gamma_0 = F_x \quad (61)$$

Here, the adjoint (antiquark) representation is seen to be

$$\bar{\psi}^A = \bar{\psi}^{i\alpha} = (\psi_{i\beta})^\dagger (\gamma_0)^\alpha_\beta \quad (62)$$

The group is non-compact and so all the unitary representations are infinite-dimensional. Finite-dimensional

but non-unitary representations of $U(6,6)$ may be built up from the product of these quarks and antiquarks.

Then multispinors transform according to the following general rule

$$\psi_{AB\dots}^{CD\dots} \longrightarrow S_A^{A'} S_B^{B'} \dots (S^{-1})_C^C (S^{-1})_D^D \dots \psi_{A'B'\dots}^{C'D'\dots} \quad (63)$$

where S is given by equation (50) above.

To associate physical states with an irreducible finite-dimensional representation of the group $U(6,6)$, the usual procedure is to break up the indefinite metric into a set of subspace each of which is definite and invariant under the little group $v(6) \times v(6)$. This is the maximal compact subgroup for which

$$S \gamma_0 S^{-1} = \gamma_0 \quad \text{i.e.} \quad S^{-1} = S^+ \quad (64)$$

However to obtain spin specification in the little group, it is essential to use generalised Bargmann-Wigner equations after reducing $U(6,6)$ to $SU(3) \otimes SL(2, c)$ by means of the antisymmetric matrix $(C^{-1})^{\alpha\beta}$. The two most important multispinors are for mesons and baryons.

For mesons, it is the 143 regular representation of $U(6,6)$ ϕ_A^β which undergoes the following reduction on applying Bargmann-Wigner equations,

$$\phi_A^B = \phi_{\alpha i}^{\beta j} = \frac{1}{2\mu} (\not{X} + \mu) (\gamma_\mu \phi_{\mu i}^j - \gamma_5 \phi_{5 i}^j)^\beta_\alpha \quad (65)$$

where $\phi_{\mu i}^j$ and $\phi_{5 i}^j$ correspond to the 1^- and 0^- parts of the multiplet respectively and μ is the degenerate multiplet-mass. From each can be separated an $Sv(3)$ octet and singlet part,

$$\begin{aligned} \phi_{\mu i}^j &= \phi_\mu(1) + \phi_\mu(8) \\ \phi_{5 i}^j &= \phi_5(1) + \phi_5(8) \end{aligned} \quad (66)$$

The baryon representation is obtained from the fully symmetric 364 representation of $U(6,6)$. After reducing with Bargmann-Wigner equations, the baryon multiplet comes out as

$$\begin{aligned} \psi_{ABC} &= \frac{1}{2m} \sqrt{\frac{3}{2}} ((\not{X} + m) \gamma_\mu C)_{\alpha\beta} D_{\mu\gamma}(ijk) \\ &+ \frac{1}{2m} \sqrt{\frac{1}{6}} [((\not{X} + m) \gamma_5 C)_{\alpha\beta} \epsilon_{ij\mu} N_\gamma{}^\mu_k + \text{cycl}] \end{aligned} \quad (67)$$

Here D denotes the spin $\frac{3}{2}$ decuplet, N the spin $\frac{1}{2}$ octet, and m is the degenerate mass of the multiplet.

We have seen above that $U(6) \otimes U(6)$ is the subgroup whose representations stand for particles at rest. This is obtained by restricting the $U(6,6)$ transformations to the following condition,

$$\gamma_0 F_A = \Gamma_A \gamma_0$$

that is, all the transformations of $U(6,6)$ which commute with γ_0 form the little group $U(6) \otimes U(6)$.

For collinear processes characterised by two independent momenta we can find a group of transformations which have these two momenta invariant. In the frame in which each vector can be expressed in the form $P = P_0 \gamma_0 + P_3 \gamma_3$, the generators are all the generators of $U(6,6)$ which commute with γ_0 and γ_3 and clearly, these are

$$\frac{1}{2} \sqrt{\frac{3}{2}} \lambda_1 \otimes (1, \gamma_5 \gamma_1, \gamma_5 \gamma_2, \sigma_{12}) \quad (68)$$

This is the outer product of $U(1)$ and $SU(6)_w$ defined by Lipkin and Meshkov⁽¹⁷⁾.

Coplanar processes are characterised by three independent momenta where each vector can be written as

$$P = P_0 \gamma_0 + P_1 \gamma_1 + P_2 \gamma_2$$

The transformations which leave this invariant are to commute with $\gamma_0, \gamma_1, \gamma_2$ and the subgroup which is formed out of such generators turns out to be $U(3) \otimes U(3)$ with the generators $\frac{1}{2}(1 \pm \gamma_5 \gamma_3) \otimes \frac{1}{2} \frac{3}{2} \lambda_i$. This hierarchy of subgroups of $U(6,6)$ are also called the 'lesser' and 'least' groups, respectively; however, they are more popularly known as 'collinear' and 'coplanar' groups.

$SU(6)_W$:

Since we shall be dealing with this important collinear subgroup, it is worthwhile to discuss it in a little greater detail. This is an $SU(6)$ group where the ordinary $SU(2)_\sigma$ spin group is replaced by the $SU(2)_W$ W-spin group, consisting of generators $\sigma_{12}, \gamma_0 \sigma_{23}, \gamma_0 \sigma_{31}$. These operators satisfy $SU(2)$ commutation relations and represent a 'conserved spin' in any theory which is invariant under $U(6,6)$. From the definition

these generators commute with the generators of the Lorentz group in the z-direction as such a particle moving with any momentum in the z-direction will have the same W-spin as at rest. Thus it is a good symmetry for all collinear processes i.e. such as two-body decays of resonance, forward and backward scattering, photo-production in the forward-direction, radiative decay into a particle and a photon etc.

The W-spin and $SU(6)_W$ classification for mesons and baryons can be obtained from the following transformation properties of quarks and anti-quarks :

$$\begin{aligned} W_{x,y,z} \psi_A &= \sigma_{x,y,z} \psi_A \\ W_{x,y} \bar{\psi}^A &= -\sigma_{x,y} \bar{\psi}^A \\ W_z \bar{\psi}^A &= \sigma_z \bar{\psi}^A \end{aligned} \quad (69)$$

where $W_{x,y,z}$ refer to the three components of the W-spin generators, $\gamma_0 \sigma_x / 2$, $\gamma_0 \sigma_y / 2$, $\gamma_0 \sigma_z / 2$ respectively. The W-spin content remains the same for

all momenta. Taking the rest-state, we get $\gamma_0 = +1$ in the above equations. Considering baryons, which are obtained from three quarks, we find that the W-spin is the same as the ordinary spin for baryons as well as for anti-baryons. Next considering four meson states constituted from quark anti-quark pair, we find that the two-vector meson states $V_{\pm 1}$ with spin projections ± 1 on the z-axis have $W_z = \pm 1$ and therefore, $W = 1$. However the vector-meson V_0 with spin projection zero along the z-direction is the W-spin singlet (with $W = 0$) state and the pseudoscalar meson P_0 is the $W_z = 0$ component of the W-spin triplet. This spin W-spin flip occurs because of the negative sign appearing in the equation (69). Therefore, the W-spin triplet is (V_1, P_0, V_{-1}) and the W-spin singlet is V_0 . With this, the 35 meson representation of $SU(6)_W$ can be decomposed into its $SU(3) \otimes SU(2)_W$ content in the following way

$$\phi_B^A = \phi_{bj}^{ai} = (\sigma_z P_0 + 1 \quad V_0 + W_+ V_1 + W_- V_{-1})_{bj}^{ai} \quad (70)$$

$i, j = 1, 2, 3$
 $a, b = 1, 2.$

with

$$w_{\pm} = \mp \frac{1}{2} (w_x \pm iw_y)$$

and the 56 baryon representation has got the usual $SU(6) \rightarrow SU(3) \otimes SU(2)$ decomposition, namely

$$\begin{aligned} \psi_{ABC} = D_{abc} ijk + \frac{1}{3\sqrt{2}} (\epsilon_{ijl} \epsilon_{ab} N_{kc}^l + \epsilon_{jkl} \epsilon_{bc} N_{ia}^l \\ + \epsilon_{kil} \epsilon_{ca} N_{jb}^l) \quad (71) \end{aligned}$$

where $D_{abc,ijk}$ represent the decuplet and N_i^j , the baryon-octet.

SL(6,C)

SL(6,C) is the smallest group that contains SU(6) as a subgroup (or the little group) and permits a covariant generalization for this rest-symmetry group. The seventy generators of the group are

$$\lambda^i, \gamma_5 \lambda^i, \bar{\sigma} \lambda^i, \gamma_5 \bar{\sigma} \lambda^i \quad i = 0, 1, \dots, 8.$$

since $\gamma_5 \sigma_i = \sigma_{0i}$, together with σ_{ij} , $\gamma_5 \sigma_i$ form the generators $\sigma_{\mu\nu}$ of SL(2,C).

In this formalism, discussed in detail by Charap and Matthews, and Rühl, by means of the unitary trick the non-unitary representations are obtained as those of a chiral $U(6) \times U(6)$ having the generators

$$\frac{1}{2}(1+i\gamma_5)\lambda^i, \frac{1}{2}(1+i\gamma_5)\bar{\sigma}\lambda^i.$$

We generalise the two-component spinors of the homogeneous Lorentz group to six component quarks by defining

$$U_A = U_{\alpha a} \quad \alpha = 1, 2, 3, \quad a = 1, 2$$

where α labels the $SU(3)$ properties of the quarks.

Similarly we have

$$U_A^+ = U_{\dot{a}}^{+\alpha}, \quad V^{\dot{A}} = V_{\alpha}^{\dot{A}}, \quad V^{+A} = V^{+\alpha A}$$

The particle multiplets are then constructed from these entities. For example, the meson field can be described by the four different spinors $\phi^{\dot{A}\dot{B}}, \phi_{\dot{A}\dot{B}}, \phi_B^A, \phi_{\dot{B}}^{\dot{A}}$.

Bargmann-Wigner equations however, relate these fields

such that we have

$$\begin{aligned} \mu \phi^{\dot{A}B} &= \tilde{P}^{\dot{A}A} \phi_A^B \\ \mu \phi_{AB} &= P_{\dot{B}\dot{B}} \phi_A^{\dot{B}} \\ \mu^2 \phi_B^{\dot{A}} &= \tilde{P}^{\dot{A}A} P_{\dot{B}\dot{B}} \phi_A^{\dot{B}} \end{aligned} \quad (72)$$

where μ is the degenerate mass of the multiplet. The decomposition into the subgroup $SL(2, C) \otimes SU(3)^{(2)}$ can be written as

$$\phi^{\dot{A}B} = \frac{1}{2} \tilde{\sigma}^{\dot{a}b} \left[(\lambda)_{\alpha}^{\beta} (\underline{m}^{\mu}) + \sqrt{\frac{2}{3}} \delta_{\alpha}^{\beta} m^{\mu} \right] \quad (73a)$$

The fields $\underline{m}^{\mu}$ give a covariant combination of octets of particles of spin 1 and spin zero; similarly m^{μ} is an $SU(3)$ singlet of spin 1 and spin zero.

The parity-conjugate representation of the above is defined similarly by

$$Q_{AB} = \frac{1}{2} \sigma_{\mu\dot{a}\dot{b}} \left(\gamma_{\alpha}^{\beta} \underline{\tilde{m}}^{\mu} + \sqrt{\frac{2}{3}} \gamma_{\alpha}^{\beta} m^{\mu} \right) \quad (73b)$$

in terms of fields $\underline{\tilde{m}}^{\mu}$ and $\tilde{m}^{\mu}$.

The two representations ϕ_B^A and $\phi_{\dot{B}}^{\dot{A}}$ can be decomposed respectively in terms of self-dual and anti-

self-dual tensor fields. However, as has been pointed out above, all the representations are related through the Bargmann-Wigner equations.

Baryons are described in terms of the representations $\psi_{ABC}, \psi_{AB}^{\dot{C}}, \psi_A^{\dot{B}\dot{C}}, \psi^{\dot{A}\dot{B}\dot{C}}$. Again, these are related by the Bargmann-Wigner equations in the following way

$$\begin{aligned}\psi_{AB}^{\dot{C}} &= \frac{1}{m} P^{\dot{C}C} \psi_{ABC} \\ \psi_A^{\dot{B}\dot{C}} &= \frac{1}{m^2} P^{\dot{C}C} \tilde{P}^{\dot{B}B} \psi_{ABC} \\ \psi^{\dot{A}\dot{B}\dot{C}} &= \frac{1}{m^3} P^{\dot{C}C} \tilde{P}^{\dot{B}B} P^{\dot{A}A} \psi_{ABC}\end{aligned}$$

where m is the degenerate mass of the multiplet. These correctly reproduces the 56 representation when restricted to the static $SU(6)$ subgroup.

As in $U(6,6)$ we can now consider the chain of subgroups of $SL(6,6)$. For collinear processes, where there is one independent momentum vector, we can define the collinear subgroup as the group under which the momentum vector remains invariant. This leads to the $U(3) \times U(3)$ collinear group described in detail in the

next chapter. For coplanar processes the subgroup turns out to be $U(3)$.

Thus, we have the following hierarchy of subgroups for $U(6,6)$ and $SL(6,C)$ on restricting their invariance to rest-symmetry, collinear and coplanar processes respectively.

$$SL(6,6) \longrightarrow SU(6) \longrightarrow S(U(3) \times U(3)) \longrightarrow SU(3)$$

$$U(6,6) \longrightarrow U(6) \times U(6) \longrightarrow SU(6)_w \longrightarrow U(3) \times U(3).$$

CHAPTER II

S(U(3) $\otimes$ U(3)) and meson-baryon scattering

Group theory is useful in scattering processes because the assumption that the S-matrix is invariant under some group of transformations forces a number of physically distinct scattering amplitudes to be related to each other. This is evident even in such a group as the isospin group SU(2). For example, pion-nucleon scattering

$$N + \pi \longrightarrow N' + \pi'$$

would be described by eight independent amplitudes if there was no invariance group but only two if the S-matrix is isospin invariant. This is because N and π can combine only into isospin $I = \frac{1}{2}$ and $I = \frac{3}{2}$ states. Therefore there are six relations among the eight different scattering amplitudes. It is evident that if we enlarge the group, more and more amplitudes will be related.

This chapter begins with a discussion of pseudo-scalar meson-baryon scattering in the $SU(3)$ group. It then describes in detail the collinear $S(U(3) \otimes U(3))$ group and some calculations of meson-baryon scattering and baryon-resonance production in this group. The corresponding calculations in $SU(6)_w$ are briefly discussed. Comparison of these predictions with the available experimental data and a short discussion conclude the chapter.

To illustrate how group theory works in such a calculation, consider the process

$$B + M \rightarrow B' + M' \quad (1)$$

where B and B' belong to an irreducible representation D , and M and M' to the representation D' of the group considered. In order to find the independent amplitudes invariant under the group transformations, one has to find the irreducible representations from the direct-product $D \otimes D'$. Since the amplitude is invariant only those irreducible representations that occur on

both sides contribute and determine the number of invariant amplitudes. For pion-nucleon scattering

$$N + \pi \rightarrow N' + \pi'$$

$$I = \begin{matrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 1 \end{matrix}$$

$$D^{\frac{1}{2}} \otimes D^1 = D^{\frac{1}{2}} \oplus D^{\frac{3}{2}}.$$

There are only two possible invariant amplitudes:

$$I_{\frac{1}{2}} = \left\langle I = \frac{1}{2} \left| T \right| I = \frac{1}{2} \right\rangle$$

$$I_{\frac{3}{2}} = \left\langle I = \frac{3}{2} \left| T \right| I = \frac{3}{2} \right\rangle$$

Extending the group to SU(3), we find that mesons and baryons belong to the representation 8. We then have the following direct product decomposition into irreducible representations:

$$\underline{8} \otimes \underline{8} = 1 \oplus \underline{8}_d \oplus \underline{8}_f \oplus 10 \oplus \overline{10} \oplus 27 \quad (2)$$

where $\underline{8}_d$, $\underline{8}_f$ are the symmetric and antisymmetric combinations, and 10 and 10^* are conjugate representations.

Since the same direct-product appears on both sides we

have eight independent, SU(3) invariant amplitudes.

This number is reduced to seven because time-reversal invariance implies that $A_{cf} \rightarrow sd = A_{cd} \rightarrow sf$.

To determine the invariant amplitudes for processes which involve a baryon-resonance, for example

$$B + Z \rightarrow B' + Z' \quad (3)$$

we note that Z' occurs in the 10 representation of SU(3) and also that

$$\underline{8} \times \underline{10} = \underline{8} \oplus \underline{10} \oplus \underline{27} \oplus \underline{35} \quad (4)$$

Then from (2) and (4) we find that there will be four independent, invariant amplitudes to describe all the processes of the type (3). The scattering amplitude is given by:

$$\langle B' M' | T | B M \rangle = \sum_{\substack{\nu_B, \nu_M \\ \nu_{B'}, \nu_{M'}}} \begin{pmatrix} \lambda_B & \lambda_M & \lambda_{Z'} \\ \nu_B & \nu_M & \nu_{Z'} \end{pmatrix} (\delta_{if} T_{if}) \begin{pmatrix} \lambda_{B'} & \lambda_{M'} & \lambda_{Z'} \\ \nu_{B'} & \nu_{M'} & \nu_{Z'} \end{pmatrix}$$

where $T_{if} = \langle \lambda_i \| T \| \lambda_f \rangle$ and the SU(3) Clebsch-Gordon Coefficients are such that

$$\phi_{\nu_i}^{(\lambda_i)\gamma} = \sum_Y \begin{pmatrix} \lambda_B & \lambda_M & (\lambda_i)\gamma \\ \nu_B & \nu_M & \nu_i \end{pmatrix} \phi_{\nu_B}^{\lambda_B} \phi_{\nu_M}^{\lambda_M}$$

λ, ν, γ denote collectively the eigenvalues of different SU(3) operators, as defined by de Swart⁽¹²⁾ and ϕ defines completely a state in a SU(3) representation.

This way of calculating the scattering amplitudes is cumbersome (and much more so for the higher groups we shall be considering) as it means using the SU(3) Clebsch-Gordon coefficients explicitly. We may avoid this difficulty by using tensor representations and constructing the invariant couplings of the four representations. For meson-baryon scattering (1) let us represent the four octets by four 3x3 matrices, $A_i (i = 1, 2, 3, 4)$. The possible invariants are of two types,

$$T_r(A_1 A_2 A_3 A_4)$$

six of which are independent and

$$T_r(A_1 A_2) \quad T_r(A_3 A_4)$$

of which three are independent. Thus we have nine invariants in all. Also there is one linear relation among these nine invariants which comes from Burgoyne's Identity⁽¹⁹⁾;

$$\sum_{\text{6 different permutations}} T_2(A_1 A_2 A_3 A_4) = \sum_{\text{3 different permutations}} T_2(A_1 A_2) T_1(A_3 A_4) \quad (5)$$

6 different permutations 3 different permutations.

The nine invariants with one linear relations (5) are thus reduced to eight independent invariants, giving the same result as obtained from the direct product decomposition in equation (2).

Besides these two conventional methods for obtaining invariant amplitudes, there are of course, numerous short cuts to obtain relations among various amplitudes, particularly in SU(3). These methods are useful, simple and even elegant, but unfortunately, are not applicable in general groups. In the calculations to follow, we shall always use the second method to obtain invariant amplitudes.

$S(U(3) \otimes U(3))$:

This is the collinear subgroup of the group $SL(6, \mathbb{C})$. As in $U(6, 6)$, collinear processes are characterised by two independent momenta and all particles can be made to move in the z-direction in a particular Lorentz frame. In this frame, the momentum-vector can be written in the form

$$\hat{P} = P_0 Y_0 + P_3 Y_3$$

so that Y_1, Y_2 never occur. Therefore, under all transformations which commute with Y_0 and Y_3 , P will remain invariant. This subgroup is thus a good symmetry for these processes. This is the $S(U(3) \otimes U(3))$ subgroup defined by the generators $\lambda_i \left(\frac{1+\gamma_5}{2} \right)$ which commute with all Lorentz transformations in the z-direction and therefore gives a momentum-independent classification of all particle-states with momentum in the z direction only. This subgroup has been used to obtain many predictions for collinear processes such as form-factors vertex functions⁽²⁰⁾, forward and

backward scattering⁽²⁾ and two-body decays including electromagnetic and weak decays⁽²¹⁾.

It will be useful to introduce the triplet of quarks which correspond to the fundamental representation $(\underline{3})$ of $SU(3)$ in obtaining the representations for physical states, although as in all higher symmetry-schemes, physical quarks need not exist. Choosing the z-direction as the quantisation-axis for the spin of the quarks, the two $U(3)$ subgroups transform the $S_z = \pm \frac{1}{2}$ quark-states in the following way:

$$q_{\alpha}^1 \equiv q_{\alpha} \in (\underline{3}, 1) \quad S_z = +\frac{1}{2} \quad (6)$$

$$q_2 \equiv q^{\alpha} \in (1, \bar{3}) \quad S_z = -\frac{1}{2}$$

In the contravariant representations, S_z changes sign:

$$q^1 \equiv q^{\alpha} \in (\bar{3}, 1) \quad S_z = -\frac{1}{2}$$

$$q^2 \equiv q^{\bar{\alpha}} \in (1, \bar{3}) \quad S_z = +\frac{1}{2} \quad (7)$$

That is, q_{α} , $q_{\bar{\alpha}}$ define the quark-states with $S_z = +\frac{1}{2}$ and $S_z = -\frac{1}{2}$ respectively and $q^{\bar{\alpha}}$ and q^{α} represent the corresponding anti-quark states. The representations

(3,1) and (1,3) are related through parity-transformations. Under parity-transformation, the momentum will change sign, but since momentum does not occur explicitly in this formalism, we can define an extended parity operator

$$P' = (\exp i\pi_x).P = R_x(\pi).P \quad (8)$$

where P is the normal parity operator and $R_x(\pi)$ defines a rotation through π about the x-axis. P' leaves momenta invariant but changes the sign of S_z . Therefore,

$$(3,1) \xrightarrow{P'} (1,3) \quad \text{i.e.} \quad q_\alpha \xrightarrow{P'} q_{\bar{\alpha}} \quad (9)$$

Since charge-conjugation effects only the SU(3) properties, we have under charge conjugation,

$$(3,1) \xrightarrow{C} (\bar{3},1) \quad \text{i.e.} \quad q_\alpha \xrightarrow{C} q^\alpha \quad (10)$$

$$(1,3) \xrightarrow{C} (1,\bar{3}) \quad \text{i.e.} \quad q_{\bar{\alpha}} \xrightarrow{C} q^{\bar{\alpha}}$$

The 35 representations of SU(6) for mesons splits under $S(U(3) \otimes U(3))$ in the following way.

$$15_A^B = 15_{\beta\beta}^{a\alpha} = 15_{\beta_1}^{\alpha_1} \oplus 15_{\beta_2}^{\alpha_2} \oplus 15_{\beta_3}^{\alpha_3} \oplus 15_{\beta_4}^{\alpha_4} \quad (11a)$$

$(\alpha, \beta = 1, 2, 3)$
 $(\alpha, \beta = 1, 2)$

where

$$\begin{aligned}
 M_{\beta_1}^{\alpha_1} &= M_{\beta}^{\alpha} + \frac{1}{\sqrt{6}} \delta_{\beta}^{\alpha} M \\
 M_{\beta_2}^{\alpha_2} &= M_{\beta}^{\bar{\alpha}} + \frac{1}{\sqrt{6}} \delta_{\beta}^{\bar{\alpha}} M \\
 M_{\beta_2}^{\alpha_1} &= M_{\beta}^{\alpha} \\
 M_{\beta_1}^{\alpha_2} &= M_{\beta}^{\bar{\alpha}}
 \end{aligned} \tag{11b}$$

This represents the decomposition,

$$\underline{35} = (8,1) \oplus (1,8) \oplus (3,3) \oplus (\bar{3},3) \tag{12}$$

Reduction under SU(3) leads to the following SU(3) matrices

$$\begin{aligned}
 M_{\beta}^{\alpha} &= \frac{1}{\sqrt{2}} P_{\beta}^{\alpha} + V^{(0)\alpha} S_1 \\
 M_{\beta}^{\bar{\alpha}} &= \frac{1}{\sqrt{2}} P_{\beta}^{\alpha} + V^{(0)\alpha} S_2^2 \\
 M_{\beta}^{\alpha} &= V_{\beta}^{(-)\alpha} S_1^1 + \frac{1}{\sqrt{3}} \delta_{\beta}^{\alpha} V^{(1)} S_2^1 \\
 M_{\beta}^{\bar{\alpha}} &= V_{\beta}^{(+)\alpha} S_1^2 + \frac{1}{\sqrt{3}} \delta_{\beta}^{\alpha} V^{(1)} S_1^2
 \end{aligned} \tag{11c}$$

$$\text{and } M = \frac{1}{\sqrt{2}} S_1^1 V^{(1)} = -\frac{1}{\sqrt{2}} S_2^2 V^{(1)}$$

Here $(V^{\pm})$ is the vector octet with $S_z = \pm 1$, P is the pseudoscalar meson octet and S stands for various spin-states.

Under $S(U(3) \otimes U(3))$ the 56 baryon representation of $SU(6)$ split in the following way:

$$\underline{56} = (10, 1) \oplus (1, 10) \oplus (6, 3) \oplus (3, 6). \quad (13)$$

$$\text{or } B_{ABC} = B_{\alpha\beta\gamma} \oplus B_{\bar{\alpha}\bar{\beta}\bar{\gamma}} \oplus B_{\alpha\beta\bar{\gamma}} \oplus B_{\alpha\bar{\beta}\bar{\gamma}} \quad (14a)$$

where under $SU(3)$ these tensors reduce to decuplet D and octet N baryons:

$$\begin{aligned} B_{\alpha\beta\gamma} &= D_{\alpha\beta\gamma} (S)_{111} \\ B_{\bar{\alpha}\bar{\beta}\bar{\gamma}} &= D_{\alpha\beta\gamma} (S)_{222} \\ B_{\alpha\beta\bar{\gamma}} &= \sqrt{3} D_{\alpha\beta\gamma} (S)_{112} + \frac{1}{\sqrt{6}} (N_{\alpha}^{\delta} \in_{\delta\beta\gamma} + N_{\beta}^{\delta} \in_{\delta\alpha\gamma}) S_1 \\ B_{\alpha\bar{\beta}\bar{\gamma}} &= \sqrt{3} D_{\alpha\beta\gamma} (S)_{122} + \frac{1}{\sqrt{6}} (N_{\beta}^{\delta} \in_{\delta\alpha\gamma} + N_{\gamma}^{\delta} \in_{\delta\alpha\beta}) S_2 \end{aligned} \quad (14b)$$

Higher representations of $SU(6)$ can be analysed in a similar way under $S(U(3) \otimes U(3))$, but in our calculations only these two will be used.

Meson-baryon Scattering in $S(U(3) \otimes U(3))$:

In order to obtain the invariant amplitudes for pseudoscalar-meson baryon scattering we shall follow the second procedure described in section 1. Using (11b)

and (14b) it is easy to write down the most general expression which contains these scattering processes. Thus we write the following $S(U(3) \otimes U(3))$ invariant expression for the scattering amplitude:

$$\begin{aligned}
 A = & A_1 B^{\alpha\beta\bar{\gamma}} B_{\alpha\beta\bar{\gamma}} M_{\rho}^{\delta} M_{\delta}^{\rho} + A_2 B^{\alpha\beta\bar{\gamma}} B_{\alpha\beta\bar{\gamma}} M_{\bar{\rho}}^{\delta} M_{\delta}^{\bar{\rho}} \\
 & + A_3 B^{\alpha\beta\bar{\gamma}} B_{\delta\rho\bar{\gamma}} M_{\alpha}^{\delta} M_{\beta}^{\rho} \\
 & + A_4 B^{\alpha\beta\bar{\gamma}} B_{\delta\beta\bar{\gamma}} M_{\rho}^{\delta} M_{\alpha}^{\rho} + A_5 B^{\alpha\beta\bar{\gamma}} B_{\delta\beta\bar{\gamma}} M_{\rho}^{\delta} M_{\alpha}^{\rho} \\
 & + A_6 B^{\alpha\beta\bar{\gamma}} B_{\alpha\beta\bar{\delta}} M_{\bar{\rho}}^{\delta} M_{\bar{\gamma}}^{\bar{\rho}} + A_7 B^{\alpha\beta\bar{\gamma}} B_{\alpha\beta\bar{\delta}} M_{\bar{\rho}}^{\delta} M_{\bar{\gamma}}^{\bar{\rho}} \\
 & + A_8 B^{\alpha\beta\bar{\gamma}} B_{\alpha\delta\bar{\rho}} M_{\beta}^{\delta} M_{\bar{\gamma}}^{\bar{\rho}} + A_9 B^{\alpha\beta\bar{\gamma}} B_{\alpha\delta\bar{\rho}} M_{\beta}^{\delta} M_{\bar{\gamma}}^{\bar{\rho}} \quad (15)
 \end{aligned}$$

In writing (15) we note that the contribution with opposite baryon helicity-states may be obtained by a parity-transformation defined in equation (9), so that such terms are not included. The different A_i 's are not necessarily independent. This can be seen by considering the direct product of $S(U(3) \otimes U(3))$ representations of the baryons and pseudo-scalar mesons. It decomposes into the following irreducible representations:

$$\begin{aligned}
\psi_{\alpha\beta\bar{\gamma}} \otimes M_{\epsilon}^{\delta} \oplus \psi_{\alpha\bar{\beta}\bar{\gamma}} \otimes M_{\bar{\epsilon}}^{\delta} &\equiv (6,3) \otimes (8,1) \oplus (3,6) \otimes (8,1) \\
&= (24,3) \oplus (\bar{3},3) \oplus (6,3) \oplus (15,3) \oplus (3,6) \oplus (15,6) \oplus (\bar{6},6).
\end{aligned}
\tag{16}$$

Thus there are only seven invariant amplitudes, so that there must be two linear relations among the invariant amplitudes of equation (15). Again time-reversal invariance implies that

$$A\{(6,3) \longrightarrow (3,6)\} = A\{(3,6) \longrightarrow (6,3)\}$$

which reduces the number of independent $S(U(3) \otimes U(3))$ invariant amplitudes to six.

The same invariant amplitudes are also suitable for describing processes which involve baryon resonances with $S_z = \pm \frac{1}{2}$ and vector mesons with $S_z = 0$. We shall use this to find relations for those processes where only this type of baryon-resonances and vector-meson states occur. While this enables us to obtain more

relations than from $SU(3)$, these are fewer than from $SU(6)_w$ because, we cannot relate these processes with those which involve baryon-resonance states with $S_z = \pm \frac{3}{2}$ and vector-meson states with $S_z = \pm 1$.

The calculation is easily performed by using the expression for $B_{\alpha\beta\tilde{\gamma}}$, M_α^β etc., obtained from equations (11c) and (14b). Since we shall consider baryon resonance production processes which involve one baryon decuplet in the final state, and pseudo-scalar meson-baryon scattering, it is convenient to break up (15) into two distinct parts. Thus using (14b) for the outgoing baryon state $\bar{B}^{\alpha\beta\tilde{\gamma}}$ in (15) we obtain,

$$A = \left[\sqrt{3} D^{\alpha\beta\gamma} + \frac{1}{\sqrt{6}} (N_\delta^\alpha \epsilon^{\delta\beta\gamma} + N_\delta^\beta \epsilon^{\delta\alpha\gamma}) \right] [A_1 B_{\alpha\beta\tilde{\gamma}} \bar{M}_\rho^\delta M_\delta^\rho + \\ + A_2 B_{\alpha\beta\tilde{\gamma}} \bar{M}_\rho^\delta M_\delta^\rho + \dots] \quad (17)$$

where we have dropped the spin wave functions S_1 , S_{112} etc. Since we always have a baryon-octet in the initial state, we need retain only the octet part for $B_{\alpha\beta\tilde{\gamma}}$ and only the pseudo-scalar octet parts for M_β^α and $\bar{M}_\beta^\alpha$. This results in the following amplitude for the Baryon-

resonance production processes:

$$\begin{aligned}
 A = & \frac{1}{\sqrt{8}} \bar{D}^{\alpha\beta\gamma} [(A_1 + A_2) (N_{\alpha}^{\mu} \epsilon_{\mu\beta\gamma} + N_{\beta}^{\gamma} \epsilon_{\mu\alpha\gamma}) T_r(\bar{P}P) + A_3 (N_{\delta}^{\mu} \epsilon_{\delta\rho\gamma} + \\
 & + N_{\rho}^{\mu} \epsilon_{\mu\delta\gamma}) \bar{P}_{\alpha}^{\delta} P_{\beta}^{\rho} + A_4 (N_{\delta}^{\gamma} \epsilon_{\mu\beta\gamma} + N_{\beta}^{\mu} \epsilon_{\mu\delta\gamma}) \bar{P}_{\rho}^{\delta} P_{\alpha}^{\rho} + A_5 (N_{\delta}^{\mu} \epsilon_{\mu\beta\gamma} + \\
 & + N_{\beta}^{\mu} \epsilon_{\mu\delta\gamma}) \bar{P}_{\rho}^{\delta} P_{\alpha}^{\rho} + A_6 (N_{\alpha}^{\mu} \epsilon_{\mu\beta\delta} + N_{\beta}^{\mu} \epsilon_{\mu\alpha\delta}) \bar{P}_{\rho}^{\delta} P_{\gamma}^{\rho} + A_7 (N_{\alpha}^{\mu} \epsilon_{\mu\beta\delta} + \\
 & + N_{\beta}^{\mu} \epsilon_{\mu\alpha\delta}) \bar{P}_{\rho}^{\delta} P_{\gamma}^{\rho} + A_8 (N_{\alpha}^{\mu} \epsilon_{\mu\delta\rho} + N_{\delta}^{\mu} \epsilon_{\mu\alpha\rho}) \bar{P}_{\beta}^{\delta} P_{\gamma}^{\rho} + A_9 (N_{\alpha}^{\mu} \epsilon_{\mu\delta\rho} + \\
 & + N_{\delta}^{\mu} \epsilon_{\mu\alpha\rho}) \bar{P}_{\beta}^{\delta} P_{\gamma}^{\rho}] \quad (18)
 \end{aligned}$$

To find the amplitude for the meson-baryon scattering in a convenient form, we use the properties of the anisymmetric tensor:

$$\begin{aligned}
 \epsilon_{\alpha\beta\gamma} \epsilon^{\alpha\beta\gamma} &= 3! ; \quad \epsilon_{\alpha\beta\gamma} \epsilon^{\alpha\beta\mu} = 2! \delta_{\gamma}^{\mu} ; \\
 \epsilon_{\alpha\beta\gamma} \epsilon^{\alpha\mu\nu} &= \delta_{\beta}^{\mu} \delta_{\gamma}^{\nu} - \delta_{\gamma}^{\mu} \delta_{\beta}^{\nu} ; \\
 \epsilon_{\alpha\beta\gamma} \epsilon^{\mu\nu\rho} &= \delta_{\alpha}^{\mu} \delta_{\beta}^{\nu} \delta_{\gamma}^{\rho} - \delta_{\alpha}^{\mu} \delta_{\gamma}^{\nu} \delta_{\beta}^{\rho} + \delta_{\gamma}^{\mu} \delta_{\alpha}^{\nu} \delta_{\beta}^{\rho} - \delta_{\gamma}^{\mu} \delta_{\beta}^{\nu} \delta_{\alpha}^{\rho} \\
 &+ \delta_{\beta}^{\mu} \delta_{\gamma}^{\nu} \delta_{\alpha}^{\rho} - \delta_{\beta}^{\mu} \delta_{\alpha}^{\nu} \delta_{\gamma}^{\rho} \quad (19)
 \end{aligned}$$

whence, using (16), (14b) and (11c), we get the following amplitudes:

$$\begin{aligned}
 A = & \alpha_1 T_r(\bar{N}N) T_r(\bar{P}P) + \alpha_2 T_r(\bar{N}\bar{P}) T_r(NP) + \alpha_3 T_r(\bar{N}P) T_r(N\bar{P}) \\
 & + \alpha_4 T_r(\bar{N}N\bar{P}\bar{P}) + \alpha_5 T_r(\bar{N}N\bar{P}P) + \alpha_6 T_r(\bar{N}P\bar{P}N) + \alpha_7 T_r(\bar{N}P\bar{P}\bar{P}) \\
 & + \alpha_8 T_r(\bar{N}\bar{P}N\bar{P}) + \alpha_9 T_r(\bar{N}\bar{P}PN). \quad (20)
 \end{aligned}$$

Here the α 's are linear combinations of the A's,

$$\begin{aligned}
 \alpha_1 &= \frac{1}{12} [16(A_1+A_2) + A_3+A_4+A_5+4A_7-A_8-A_9+4A_6] \\
 \alpha_2 &= \frac{1}{12} [-A_3+A_8+A_9] = \alpha_3 \\
 \alpha_4 &= \frac{1}{12} [A_3+4A_5-2A_7-A_8-A_9] \\
 \alpha_5 &= \frac{1}{12} [A_3+4A_4-2A_6-A_8-A_9] \\
 \alpha_6 &= \frac{1}{12} [-A_5-4A_7+A_8+A_9] \\
 \alpha_7 &= \frac{1}{12} [-A_3-3A_8] \\
 \alpha_8 &= \frac{1}{12} [-A_3-3A_9] \\
 \alpha_9 &= \frac{1}{12} [-A_1-4A_6+A_8+A_4].
 \end{aligned} \tag{21}$$

We notice that $\alpha_2 = \alpha_3$ identically, and from Burgoyne's Identity, equation (5) we know that there are only eight independent α 's. These two linear relations among the α 's reduce the number of independent invariant amplitudes to seven, which agrees with the number obtained from the direct-product decomposition in (16). If we apply time reversal invariance, we get $\alpha_7 = \alpha_8$, which gives us $A_8 = A_9$. (22).

This further reduces the number of independent invariant amplitudes to six.

To calculate the traces in (20) explicitly, we use the following SU(3) matrices representing the baryon octet pseudoscalar meson-octet and the anti-baryon octet:

$$P = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & k^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & k^0 \\ k^- & \bar{k}^0 & \frac{2\eta}{\sqrt{6}} \end{pmatrix} \quad B = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & \frac{2\Lambda}{\sqrt{6}} \end{pmatrix}$$

and

$$\bar{B} = \begin{pmatrix} \frac{\bar{\Sigma}^0}{\sqrt{2}} + \frac{\bar{\Lambda}^0}{\sqrt{6}} & \bar{\Sigma}^- & \bar{\Xi}^- \\ \bar{\Sigma}^+ & -\frac{\bar{\Sigma}^0}{\sqrt{2}} + \frac{\bar{\Lambda}}{\sqrt{6}} & \bar{\Xi}^0 \\ \bar{p} & \bar{n} & -\frac{2\bar{\Lambda}}{\sqrt{6}} \end{pmatrix} \quad (23)$$

We shall also need the SU(3) assignments for the baryon decuplet,

$$\begin{aligned}
D_{111} &= N^{*++} & D_{112} &= N^{*+}/\sqrt{3} & D_{122} &= N^{*0}/\sqrt{3} & D_{222} &= N^{*-} \\
D_{113} &= \frac{Y^{*+}}{\sqrt{3}} & D_{123} &= \frac{Y^{*0}}{\sqrt{6}} & D_{223} &= \frac{Y^{*-}}{\sqrt{3}} \\
D_{133} &= \frac{\Xi^{*0}}{\sqrt{3}} & D_{233} &= \frac{\Xi^{*-}}{\sqrt{3}} & & & & (24) \\
D_{33} &= \Omega^-
\end{aligned}$$

We have calculated all the pseudoscalar meson-baryon scattering amplitudes with P as the target particle. These are tabulated in Table I in terms of the invariant amplitudes A_1 , A_2 etc.

In Table II all the allowed baryon-resonance production processes with a proton as the target particle are tabulated. We note here that only $S_z = \pm \frac{1}{2}$ decuplet-states contribute to these processes. Since all these amplitudes are obtained in terms of six independent invariant amplitudes, we would naturally get a large number of linear relations among them. We discuss some of these linear relations with respect to experimental results and compare these with the predictions obtained from $SU(6)_w^{(22)}$ in the next section.

Table I: Amplitudes for $MB \rightarrow M'B'$ in $S[U(3) \otimes U(3)]$.

PROCESSES	AMPLITUDES $\times \sqrt{24}$								
	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}
$K^0 P \rightarrow \bar{K}^0 P$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$	$\frac{5}{\sqrt{6}}$	$\frac{4}{\sqrt{6}}$	$\frac{2}{\sqrt{6}}$	$\frac{2}{\sqrt{6}}$	$\frac{2}{\sqrt{6}}$	$\frac{16}{\sqrt{6}}$	$\frac{16}{\sqrt{6}}$
$\rightarrow K^+ \Xi^0$	$-\frac{2}{\sqrt{6}}$					$\frac{1}{\sqrt{6}}$	$\frac{2}{\sqrt{6}}$		
$\rightarrow \pi^0 \Sigma^+$	$\frac{1}{\sqrt{3}}$		$\frac{2}{\sqrt{3}}$		$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{12}}$	$\frac{1}{\sqrt{3}}$		
$\rightarrow \eta \Sigma^+$	$-\frac{1}{3}$	$-\frac{4}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{6}$	$-\frac{1}{3}$		
$\rightarrow \pi^+ \Sigma^0$	$\frac{1}{\sqrt{3}}$		$\frac{2}{\sqrt{3}}$		$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{12}}$	$\frac{1}{\sqrt{3}}$		
$\rightarrow \pi^+ \Lambda^0$			$\frac{1}{3}$		$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{3}$		
$K^- P \rightarrow K^- P$			$\frac{5}{\sqrt{6}}$		$\frac{2}{\sqrt{6}}$			$\frac{16}{\sqrt{6}}$	$\frac{16}{\sqrt{6}}$
$\rightarrow \pi^+ \Xi^0$	$\frac{2}{\sqrt{6}}$					$\frac{1}{\sqrt{6}}$	$\frac{2}{\sqrt{6}}$		
$\rightarrow K^0 \Xi^0$	$\frac{2}{\sqrt{6}}$					$\frac{2}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$		
$\rightarrow K^+ \Xi^-$	$\frac{4}{\sqrt{6}}$					$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$		
$\rightarrow \pi^- +$			$\frac{4}{\sqrt{6}}$		$\frac{2}{\sqrt{6}}$				
$\rightarrow \pi^0 \Sigma^0$	$\frac{1}{\sqrt{6}}$		$\frac{2}{\sqrt{6}}$		$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{24}}$	$-\frac{1}{\sqrt{6}}$		
$\rightarrow \pi^0 \Lambda^0$			$\frac{1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{8}}$	$\frac{1}{\sqrt{2}}$		

Table I: (Cont.)

PROCESSES	AMPLITUDES X $\sqrt{24}$								
	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_1	A_2
$\rightarrow \bar{K}^0 n$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$		$\frac{4}{\sqrt{6}}$		$\frac{2}{\sqrt{6}}$	$\frac{2}{\sqrt{6}}$		
$\rightarrow \eta \Sigma^0$	$\frac{1}{3\sqrt{2}}$	$\frac{4}{3\sqrt{2}}$	$\frac{2}{3\sqrt{2}}$	$\frac{2}{3\sqrt{2}}$	$\frac{1}{3\sqrt{2}}$	$\frac{1}{6\sqrt{2}}$	$\frac{1}{3\sqrt{2}}$		
$\rightarrow \eta \Lambda^0$	$\frac{2}{\sqrt{6}}$	$\frac{2}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$	$\frac{2}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{24}}$	$\frac{1}{\sqrt{6}}$		
$\pi^- P \rightarrow \pi^- P$			$\frac{1}{\sqrt{6}}$		$\frac{4}{\sqrt{6}}$			$\frac{16}{\sqrt{6}}$	$\frac{16}{\sqrt{6}}$
$\rightarrow \pi^0 n$		$\frac{1}{\sqrt{12}}$	$\frac{1}{\sqrt{12}}$	$\frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$				
$\rightarrow \eta n$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{12}}$	$\frac{1}{\sqrt{12}}$	$\frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	$\frac{4}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$		
$\rightarrow K^0 \Sigma^0$	$\frac{1}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$		$\frac{1}{\sqrt{3}}$		$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{12}}$		
$\rightarrow K^0 \Lambda^0$		1		1		-1	$-\frac{1}{2}$		
$\rightarrow K^+ \Sigma^-$	$-\frac{2}{\sqrt{6}}$					$-\frac{2}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$		
$\pi^+ P \rightarrow \pi^+ P$		$\frac{1}{\sqrt{6}}$		$\frac{4}{\sqrt{6}}$				$\frac{16}{\sqrt{6}}$	$\frac{16}{\sqrt{6}}$
$\rightarrow K^+ \Sigma^+$		$\frac{4}{\sqrt{6}}$		$\frac{2}{\sqrt{6}}$					
$K^0 P \rightarrow K^0 P$	$\frac{1}{\sqrt{6}}$	$\frac{5}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$	$\frac{2}{\sqrt{6}}$	$\frac{4}{\sqrt{6}}$	$\frac{2}{\sqrt{6}}$	$\frac{2}{\sqrt{6}}$	$\frac{16}{\sqrt{6}}$	$\frac{16}{\sqrt{6}}$
$K^+ P \rightarrow K^+ P$		$\frac{5}{\sqrt{6}}$		$\frac{2}{\sqrt{6}}$				$\frac{16}{\sqrt{6}}$	$\frac{16}{\sqrt{6}}$

Table II: Amplitudes for processes $MB \rightarrow M'B^\pi$ in $S[U(3) \otimes U(3)]$

PROCESSES:	AMPLITUDES X $\sqrt{8}$								
	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}
$\bar{K}^0 P \rightarrow \bar{K}^0 N^{++}$			-1		2				
$\rightarrow K^- N^{++}$			$-\sqrt{3}$		$2\sqrt{3}$				
$\rightarrow K^+ \Xi^{+0}$	-1					2	-1		
$\rightarrow \pi^0 Y^{++}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$		$-\sqrt{2}$		$\sqrt{2}$	$\frac{1}{\sqrt{2}}$		
$\rightarrow Y^{++}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$	$\frac{2}{\sqrt{6}}$	$\frac{2}{\sqrt{6}}$	$\frac{4}{\sqrt{6}}$	$\frac{2}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$		
$\rightarrow \pi^+ Y^{+0}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$		$-\sqrt{2}$		$\sqrt{2}$	$\frac{1}{\sqrt{2}}$		
$K^- P \rightarrow K^0 \Xi^{+0}$	-1					-1	2		
$\rightarrow \pi^- Y^{++}$	-1	-1		2		-1	2		
$\rightarrow \pi^+ Y^{+-}$	2					-1	-1		
$\rightarrow \pi^0 Y^{+0}$	$\frac{3}{2}$	$\frac{1}{2}$		-1			$-\frac{3}{2}$		
$\rightarrow K^+ \Xi^{+-}$	2					-1	-1		
$K^- P \rightarrow K^- N^{++}$			1		-2				
$\rightarrow \bar{K}^0 N^{+0}$			1		-2				
$\rightarrow \eta Y^{+0}$	1	1	1	-2	-2	-2	1		
$\pi^- P \rightarrow \pi^- N^{++}$	-1	-1	1	2	-2	-1	2		

Table II (Cont.)

PROCESSES	AMPLITUDES X $\sqrt{8}$								
	A_3	A_4	A_5	A_6	A_7	A_8	A_9	1	A_2
$\rightarrow \pi^0 N^{*0}$	$\frac{3}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\sqrt{2}$	$\sqrt{2}$		$\frac{3}{\sqrt{2}}$		
$\rightarrow k^+ Y^{*-}$	2					-1	-1		
$\rightarrow k^0 Y^{*0}$	$\frac{1}{\sqrt{2}}$		$\frac{1}{\sqrt{2}}$		$-\sqrt{2}$	$\frac{1}{\sqrt{2}}$	$\sqrt{2}$		
$\rightarrow \pi^+ N^{*-}$	$2\sqrt{3}$					$-\sqrt{3}$	$-\sqrt{3}$		
$\rightarrow N^{*0}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$	$-\frac{2}{\sqrt{6}}$	$\frac{2}{\sqrt{6}}$	$\frac{2}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$		
$\pi^+ P \rightarrow k^+ Y^{*+}$	-1		-1		2	2	-1		
$\rightarrow \pi^+ N^{*+}$	-1	1	-1	-2	2	-2	-1		
$\rightarrow \pi^0 N^{*++}$	$-\frac{\sqrt{1}}{\sqrt{2}}$	$\frac{\sqrt{3}}{\sqrt{2}}$	$-\frac{\sqrt{3}}{\sqrt{2}}$	$-\sqrt{6}$	$\sqrt{6}$	$\sqrt{6}$	$-\frac{\sqrt{3}}{\sqrt{2}}$		
$\rightarrow \eta N^{*++}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\frac{1}{\sqrt{2}}$		
$k^+ P \rightarrow k^0 N^{*++}$		$-\sqrt{3}$		$2\sqrt{3}$					
$\rightarrow k^+ N^{*+}$		1		-2					
$k^0 P \rightarrow k^+ N^{*0}$		1		-2					

$SU(6)_w$ Results for Meson-baryon Scattering and Baryon-resonance production processes:

A large number of predictions were made about meson-baryon scattering amplitudes in the collinear $SU(6)_w$ subgroup of $U(6,6)$ by Carter et al⁽²²⁾. The calculation was performed using $SU(6)$ Clebsch-Gordan Coefficients. We here show how these results can be obtained in a much simpler way by constructing $SU(6)_w$ invariant amplitudes from the tensor representation of the group and list some of the results.

We note that the mesons belong to the 35 representation and the baryons to the 56 representation in $SU(6)_w$. For meson-baryon scattering the direct product decomposition gives

$$\underline{35} \otimes \underline{56} = \underline{56} \oplus \underline{70} \oplus \underline{700} \oplus \underline{1134} \quad (25)$$

Therefore, there are only four invariant amplitudes to describe all processes involving any of the 35 mesons and any of the 56 baryons. It is evident that there will be a lot more relations among these scatter-

ing amplitudes than obtained from $S(U(3) \otimes U(3))$ or $SU(3)$. The most general amplitude invariant under $SU(6)_w$ can be written as

$$A = \psi^{ABC} \left\{ a \psi_{ABC} M_E^D M_D^E + \psi_{ABD} (b M_E^D M_C^E + c M_C^E M_E^D) + d \psi_{ADE} M_A^D M_B^E \right\} \quad (26)$$

where ψ_{ABC} , M_B^A are the baryon and meson representations in $SU(6)_w$ respectively, introduced in equations (71) and (70) of chapter one. Using these equations to obtain $SU(3)$ matrices equation (26) becomes equation (26) for meson-baryon processes with

$$\begin{aligned} \alpha_1 &= a + 6b + 6c + 2d \\ \alpha_2 &= \alpha_3 = -3d \\ \alpha_4 &= 6b + 3d \\ \alpha_5 &= 6c + 3d \\ \alpha_6 &= -6b - 2d \\ \alpha_7 &= \alpha_8 = 2d \\ \alpha_9 &= -6c - 2d \end{aligned} \quad (27)$$

Thus under $SU(6)_w$ we get the additional relations,

$$\begin{aligned} 2\alpha_2 &= 2\alpha_3 = -3\alpha_7 = -3\alpha_8 \\ 2(\alpha_4 + \alpha_5) &= 2(\alpha_4 + \alpha_6) = \alpha_8 \end{aligned} \quad (28)$$

Although there seems to be too many relations among the $SU(3)$ invariant amplitudes, we find that these five relations also contain the condition imposed by Burgoyne's Identity, equation (5) so that the total number of $SU(6)_w$ invariant independent amplitudes are correctly reproduced. From these also follow the relations

$$\begin{aligned} A_3 &= -A_8 \\ A_5 &= 2A_7 \\ A_4 &= 2A_6 \end{aligned} \tag{27}$$

among the $S(U(3) \otimes U(3))$ invariant amplitudes, so that on imposing these relations in the $S(U(3) \otimes U(3))$ amplitudes, we obtain the $SU(6)_w$ results.

The baryon resonance production processes in $SU(6)_w$ are of particular interest, as all such processes are described in terms of one single amplitude. This can be seen from (26) using equations (70) and (71) in chapter one, which gives the amplitude

$$A = d \bar{\psi}^{abc} (P_b^d P_c^e) (N_d^g \epsilon_{gea} - N_e^g \epsilon_{gad}) \tag{30}$$

This is also obtained by Carter et al⁽²²⁾. This result can be seen more easily by examining the coupling in the crossed channel where the processes appear as baryon-antibaryon annihilation into two mesons; that is,

$$B\bar{B} \rightarrow M\bar{M} \quad (31)$$

$$\text{or} \quad \underline{56} \otimes \underline{\overline{56}} \rightarrow \underline{35} \otimes \underline{35}$$

Using the direct-product decompositions

$$\underline{56} \otimes \underline{\overline{56}} = 1 \oplus 35 \oplus 405 \oplus 2695$$

$$\underline{35} \otimes \underline{35} = 1 \oplus 35_F \oplus 35_D \oplus 405 \oplus 180 \oplus 280 \oplus \overline{280}$$

We see that there are four $SU(6)_W$ invariant amplitudes denoted by $\underline{1}$, 35_F , 35_D and 405. For the processes (listed in Table III) the only allowed channel is 405, because the total quantum numbers $Y=2$, $T=\frac{3}{2}$, $w=2$ belonging to processes (31) do not appear in the $\underline{1}$ or 35 channels. The total W-spin of the two mesons must be either $W=0$ or $w=2$ and $B\bar{B}^*$ can only have $w=1$ or 2, so that only the $w=2$ channel is allowed. The use of the crossed channel has nothing to do with crossing

symmetry. Nor are we relating a physical process in one channel to another physical process in another channel. We are only expressing the scattering amplitudes of the direct channel in terms of a different set of invariants which are defined by the irreducible representations of $SU(6)_w$ contained in the decomposition of the direct-product of one incoming and one outgoing particle at a time.

Amplitudes for these processes are given in Table III. These are also obtainable by imposing (29) on the results of Table II.

Table III Amplitudes for $MB \rightarrow MB^*$ in $SU(6)_w$

PROCESSES AMPLITUDE = $\sqrt{8} \times d$ PROCESSES AMPLITUDE = $\sqrt{8} \times d$

$K^- P \rightarrow K_{\equiv}^{*0}$	-1	$\pi^+ P \rightarrow K Y^{*+}$	-1
$\rightarrow \pi \bar{Y}^{*+}$	-1	$\rightarrow \pi^+ N^{*+}$	-1
$\rightarrow \pi^+ Y^{*-}$	2	$\rightarrow \pi^0 N^{*++}$	$-\sqrt{\frac{3}{2}}$
$\rightarrow \pi^0 Y^{*0}$	$\frac{3}{2}$	$\rightarrow \eta N^{*++}$	$-\sqrt{\frac{1}{2}}$
$\rightarrow K^+_{\equiv}^{*-}$	2	$\bar{K}^0 P \rightarrow \eta Y^{*+}$	$-\frac{1}{\sqrt{6}}$
$\rightarrow K^- N^{*+}$	0	$\rightarrow \eta^0 Y^{*+}$	$-\frac{1}{\sqrt{2}}$
$\rightarrow \eta Y^{*0}$	$\frac{1}{\sqrt{12}}$	$\rightarrow \eta^+ Y^{*0}$	$\frac{1}{\sqrt{2}}$

Table III (cont.)

PROCESSES	AMPLITUDE= $\sqrt{8} \times d$	PROCESSES	AMPLITUDE= $\sqrt{8} \times d$
		$\rightarrow K_{\Xi}^{+*0}$	-1
$\bar{r}P \rightarrow \bar{\pi}N^{*+}$	-1	$\rightarrow \bar{K}^0 N^{*+}$	0
$\rightarrow \pi^0 N^{*0}$	$\frac{3}{\sqrt{2}}$	$\rightarrow K^- N^{*++}$	0
$\rightarrow \eta N^{*0}$	$\frac{1}{\sqrt{6}}$	$K^+ P \rightarrow K^+ N^{*+}$	0
$\rightarrow K^+ Y^{*-}$	2	$\rightarrow K^0 N^{*++}$	0
$\rightarrow K^0 Y^{*0}$	$\frac{1}{\sqrt{2}}$	$K^0 P \rightarrow K^+ N^{*0}$	0
$\rightarrow \pi^+ N^{*-}$	$2/\sqrt{3}$	$\rightarrow K^+ N^{*0}$	0

Comparison of $S(U(3) \otimes U(3))$ amplitudes with $SU(6)_w$ results and experiment:

From Tables I and II we can find a large number of relations among various processes. We shall enumerate here some of those results which can be obtained neither in $SU(3)$ nor in isospin invariance. We shall also compare the relations with those obtainable from $SU(6)_w$ and see how these two sets of results compare with experiment. Since both groups $(SU(6)_w$ and

$S(U(3) \otimes U(3))$ contain $SU(3)$ as the internal symmetry group, the effects of $SU(3)$ symmetry-breaking will be present in both to the same extent. The important thing is that this comparison may provide us with a knowledge, however insufficient, of the relative magnitude of discrepancies resulting from the extension of $SU(3)$ to $SL(6, \mathbb{C})$ and $U(6,6)$. From the scanty experimental data available at the moment. $S(U(3) \otimes U(3))$ results seem to fare much better in comparison with the $SU(6)_w$ results.

We begin with the most famous equation - the Johnson-Treiman relations⁽⁹⁾. In view of the importance of this result, we shall postpone a discussion of it until the next section; for the present we simply observe that the full Johnson-Treiman relations are not obtained in the group $S(U(3) \otimes U(3))$; instead, as can be seen from Table I, we have the sum-rule

$$\begin{aligned}
 (K^+P)_{e\ell} - (K^-P)_{e\ell} &= (K^0P)_{e\ell} - (\bar{K}^0P)_{e\ell} + \\
 &+ (\pi^+P)_{e\ell} - (\pi^-P)_{e\ell} .
 \end{aligned} \tag{32}$$

This has been derived in the context of full $SL(6, c)$ groups by Ruhl⁽¹⁵⁾ and in $S(U(3) \otimes U(3))$ by Volkov and Ruegg⁽⁷⁾. As has been noted by Ruhl, when the irregular couplings are neglected, we get certain relations among the invariant amplitudes. In terms of $S(U(3) \otimes U(3))$ invariant amplitudes described in(15), these relations are

$$A_4 - A_5 - 2A_6 + 2A_7 = 0$$

$$A_8 - A_9 = 0 \quad (33)$$

The second relation also follows from time-reversal invariance. It was noted by Ruhl that on imposing the restraints (33), the full Johnson-Treiman relations are recovered. We notice that the set of relations in (33) is less restrictive than those in (29) which give the full $SU(6)_w$ invariance.

The simple equalities of $S(U(3) \otimes U(3))$ scattering amplitudes which can be read from the Table I are

$$\begin{aligned}
\frac{1}{2}(\bar{k}^- P \rightarrow k^+_{\Xi^-}) &= (\pi^- P \rightarrow k^+_{\Sigma^-}) \\
&= -(\bar{k}^0 P \rightarrow k^+_{\Xi^0}) \\
&= -(\bar{k}^- P \rightarrow k^0_{\Xi^0})
\end{aligned}$$

In terms of the differential cross-sections in the forward and backward directions, these become

$$\begin{aligned}
\frac{1}{4}\sigma(\bar{k}^- P \rightarrow k^+_{\Xi^-}) &= \sigma(\bar{k}^- P \rightarrow k^0_{\Xi^0}) \\
&= \sigma(\pi^- P \rightarrow k^+_{\Sigma^-}) \\
&= \sigma(\bar{k}^0 P \rightarrow k^+_{\Xi^0})
\end{aligned} \tag{34}$$

Naturally these can also be obtained in $SU(6)_w$. Since two of these equalities relate amplitudes which involve essentially the same masses, they should provide a good test. A rough estimate of the experimental results shows that for backward scattering the relation

$$\frac{1}{4}\sigma(\bar{k}^- P \rightarrow k^0_{\Xi^0}) = \sigma(\bar{k}^- P \rightarrow k^+_{\Xi^-}) \quad (35)$$

is fairly well-satisfied, but there is a large discrepancy in the forward direction.

Similar relations can also be obtained for the

following differential cross-sections in the backward and forward directions

$$\begin{aligned}\sigma(\bar{K}^0 P \rightarrow \eta^0 \Sigma^+) &= 2\sigma(K^- P \rightarrow \eta \Sigma^0) \\ \sigma(\bar{K}^0 P \rightarrow \pi^0 \Sigma^+) &= \sigma(\bar{K}^0 P \rightarrow \pi^+ \Sigma^0)\end{aligned}\quad (36)$$

Besides these direct relations, we have numerous triangular relations - relations among three scattering amplitudes. These equalities for scattering amplitudes may be translated into inequalities between experimentally measurable quantities; for example, the relation

$$2(K^- P \rightarrow \Sigma^0 \pi^0) = (K^- P \rightarrow \pi^- \Sigma^+) + (\bar{K}^0 P \rightarrow K^+ \Xi^0)$$

yields the inequality

$$2R(K^- P | \Sigma^0 \pi^0) < R(K^- P | \pi^- \Sigma^+) + R(\bar{K}^0 P | K^+ \Xi^0) \quad (37)$$

where

$$R(ab|cd) = [\sigma(ab \rightarrow cd)]^{1/2} \quad (38)$$

We may also have inequalities involving more than three processes such as the following quadrangular inequality' involving four scattering processes:

$$R(\pi^+P|k^+\Sigma^-) + R(\pi^+P|k^+\Sigma^+) \geq \frac{3}{2}R(\bar{K}^0P|\eta\Sigma^+) + \frac{3}{2}R(\bar{K}^0P|\pi^+\Sigma^0) \quad (39)$$

However, these are not very useful so far as experimental verification is concerned..

From the Table II we find the following relations for the scattering amplitudes:

$$(i) \quad \sqrt{3}(K^-P \rightarrow k^-N^{*+}) : (\bar{K}^0P \rightarrow k^0N^{*+}) : (\bar{K}^0P \rightarrow \bar{K}^0N^{*+}) = 1:1:1$$

(40)

$$(ii) \quad \sqrt{3}(K^0P \rightarrow k^+N^{*0}) : (K^+P \rightarrow k^+N^{*+}) : (K^+P \rightarrow K^0N^{*++}) = 1:1:1$$

$SU(6)_W$ predicts these two sets of amplitudes to be identically zero. For the last process in the second set, although there is a fall towards zero at forward and backward angles, the cross-section is non-vanishing. (22)

The relations between reactions involving different charge states for the same initial and final isospin multiplets present no difficulty in making comparison with experimental data. As has been pointed out by Jackson⁽²³⁾, even in such cases $SU(6)_W$ results generally show discrepancies of factors of 10 to 40 in

the forward direction. All such relations are absent in $S(U(3) \otimes U(3))$. For example, corresponding to the $SU(6)_W$ result

$$(k^-P \rightarrow \pi^- Y^{*+}) : (k^-P \rightarrow \pi^+ Y^{*-}) : (k^-P \rightarrow \pi^0 Y^{*0}) \\ = 4:16:9$$

which shows a violent disagreement with experiment, the $S(U(3) \otimes U(3))$ relation

$$2(k^-P \rightarrow \pi^0 Y^{*0}) = (k^-P \rightarrow \pi^- Y^{*+}) + (k^-P \rightarrow \pi^+ Y^{*-})$$

is just the one obtained from isospin-invariance.

This agrees well with experiment.

Olsson⁽²⁴⁾ has shown that a good experimental check has been obtained corresponding to the $SU(6)_W$ result that all processes

$$\pi + N \rightarrow \pi + \Delta$$

where Δ is N^* , are proportional to a single quantity.

Since in general, we have two isospin amplitudes

$T = \frac{1}{2}$, $T = \frac{3}{2}$ to describe these processes, this implies

that there exists a ratio between these two isospin amplitudes. For example, using

$$(\pi^+P \rightarrow \pi^+N^{*-+}) : (\pi^-P \rightarrow \pi^+N^{*-}) = -\sqrt{3} : 2$$

from $SU(6)_W$ and

$$(\pi^+P \rightarrow \pi^+N^{*-+}) = \sqrt{\frac{2}{5}} A_3$$

$$(\pi^-P \rightarrow \pi^+N^{*-}) = -\sqrt{\frac{1}{5}} [\sqrt{\frac{2}{5}} A_3 + A_1]$$

we find that $A_1/A_3 = \sqrt{10}$.

Olsson has found this to be true from a study of the scattering lengths. Jackson⁽²³⁾ has argued that predictions from $SU(6)_W$ may account only for the non-peripheral part of the scattering cross-sections. Since N^* production processes are predominantly S-wave, peripheral contributions are unimportant which explains the agreement obtained by Olsson.

A few more inequalities are obtained from the following equality relations among the scattering amplitudes:

$$\begin{aligned}
(\pi^+ P \rightarrow k^+ Y^{*+}) &= (\bar{K}^0 \rightarrow k^+ \Xi^{*0}) + (k^- P \rightarrow k^- N^{*+}) \\
(\bar{K}^0 P \rightarrow k^+ \Xi^{*0}) &= (k^- P \rightarrow \pi^- Y^{*+}) + (k^0 P \rightarrow k^+ N^{*0}) \\
\sqrt{2}(\pi^- P \rightarrow k^0 Y^{*0}) &= (\bar{K}^0 P \rightarrow k^+ \Xi^{*0}) + (k^- P \rightarrow k^- N^{*+})
\end{aligned}
\tag{41}$$

Adequate data to test these predictions are not available. In $SU(6)_W$ all these processes have a common amplitude multiplied by some numerical factors (Table III).

We end this section by referring to an independent quark model⁽²⁵⁾ which was proposed in order to find forward scattering amplitudes and hence total cross-sections. This model was extended to include small momentum transfers and inelastic scattering processes⁽²⁶⁾. The main assumption of the model is that of additivity which states that all scattering amplitudes for any meson-baryon or baryon-baryon reactions are simply given by the sum of all possible quark-quark and quark-antiquark reaction amplitudes. It is usual to assign the meson and baryons to a known multiplet in $SU(3)$ or $SU(6)$. This of course follows from the assumption

that any 0^- or 1^- meson is a quark-antiquark pair and any of the octet or decuplet baryons consists of three quarks. It is usual to make additional assumptions of invariance under $SU(3)$ or some other group to obtain many of the predictions from higher symmetries, such as $SU(6)_W$, $S(U(3)\otimes(3))$ etc.. Some new relations are also found which do not follow from any of the higher symmetries. However, no rigorous justification of the use of the model has yet been found. The model seems to work without anybody knowing why!

Johnson-Treiman Relations:

One of the early successes of $SU(6)$ symmetry was the prediction of the following relations between the total cross sections of the proton-pseudoscalar meson elastic scattering

$$\begin{aligned} \frac{1}{2}[\sigma(k^+P) - \sigma(k^-P)] &= [\sigma(k^0P) - \sigma(\bar{K}^0P)] \\ &= [\sigma(\pi^+P) - \sigma(\pi^-P)] \end{aligned} \quad (42)$$

where $\sigma(mp)$ is the total cross-section for the pseudo-scalar meson proton elastic scattering. The derivation

of the result follows from the fact that the direct-product decomposition of the 35 and the 56 dimensional representations yields four irreducible representations, namely

$$\underline{35} \oplus \underline{56} = 56 \oplus 70 \oplus 700 \oplus 1134$$

of SU(6), so that all meson-baryon scattering amplitudes can be expressed in terms of four invariants. The six scattering amplitudes become connected by the relations

$$\begin{aligned} \frac{1}{2}(f(k^+P) - f(k^-P)) &= (f(k^0P) - f(\bar{k}^0P)) \\ &= (f(\pi^+P) - f(\pi^-P)) \end{aligned} \tag{43}$$

where $f(p,m)$ is the amplitude for proton-meson elastic scattering. The application of the optical theorem then implies the corresponding relations (42) for the total cross-sections.

A very simple way of deriving this result is to note that meson-baryon scattering can also be described by the same coupling as the annihilation of baryon-antibaryon producing two mesons. This is because the one process is the crossed channel of the other. It

turns out that the amplitudes corresponding to the antisymmetric two-meson states are just the difference between the amplitudes for the two different meson-baryon reactions. For example,

$$\langle \text{P}\bar{\text{P}} | (\pi^+\pi^- - \pi^-\pi^+) \rangle = \langle \text{P}\bar{\text{P}} | \pi^+\pi^- \rangle - \langle \text{P}\bar{\text{P}} | \pi^-\pi^+ \rangle$$

becomes in the crossed-channel

$$\langle \text{P}\pi^+ | \text{P}\pi^+ \rangle = \langle \text{P}\pi^- | \text{P}\pi^- \rangle$$

The Johnson-Treiman reactions (42) involve just such processes. In the annihilation channel, the direct-product decompositions of $\underline{56} \otimes \underline{\bar{56}}$ and $\underline{35} \otimes \underline{35}$ show that there are four allowed couplings, namely 1, 35_{F} , 35_{D} and 405. Only the 35_{F} coupling contains an antisymmetric two-meson state. Thus each of the amplitude-differences in (43) are expressible in terms of this antisymmetric coupling⁽²⁷⁾.

The same result (42) has also been derived in different models. One of them uses the exchange of the octet vector-meson Regge trajectories⁽²⁸⁾ where the

trajectories in the pure F coupling to the baryon are retained. It is also shown that ϕ - ω mixing of singlet vector-meson etc. do not effect the result.

The relations as derived by Johnson and Treiman were subjected to the usual ambiguities associated with applying SU(6) to a non-static situation. It was shown by Blackenbeckler et al⁽⁵⁾ that they remain valid in U(6,6) if the amplitudes are restricted to the four regular amplitudes constructed from the fields by saturating their U(6,6) indices. Charap and Matthews⁽¹⁵⁾ demonstrated that even on inclusion of the irregular amplitudes, constructed from the fields and momentum spinors, these relations remained valid. The same is true also for SU(6)_v, where it was shown by Lipkin and Harari⁽²⁹⁾, That inclusion of some non-collinear meson exchange graphs with SU(6)_v invariant vertices does not invalidate the result.

As has been pointed out before, only the sum of the two relations of (42) i.e.

$$\sigma(k^+P) - \sigma(k^-P) = \sigma(k^0P) - \sigma(\bar{k}^0P) + \sigma(\pi^+P) - \sigma(\pi^-P) \quad (44)$$

is obtained from $SL(6, \mathbb{C})$. Both these relations are obtained if only regular amplitudes are allowed. This is also true for $S(U(3) \otimes U(3))$. However, it is worth observing that this linear relation (44) is better satisfied than the full Johnson-Treiman relations⁽²⁹⁾.

The relations (42) and (44) are also obtained in the quark model⁽²⁵⁾. For the full Johnson-Treiman relations, one has to use invariance under $SU(3)$ besides the quark model, while the sum-rule (44) is obtained simply from the quark model without any additional invariance assumption. Lipkin⁽³⁰⁾ has shown that the assumptions in the derivation of the Johnson-Treiman relations in quark model are less restrictive than the full $SU(6)_W$ group. This agrees with our finding that these relations are obtained in $S(U(3) \otimes U(3))$ when we use only the regular couplings (compare equations (29) which come from $SU(6)_W$ invariance and (33), obtained on restricting the amplitudes to the regular ones).

In view of all these different derivations, it is clearly not possible to claim that the agreement of the Johnson-Treiman relations with experiment is good evidence for validity of any of these models or symmetry-groups.

DISCUSSION:

The use of higher-symmetry groups in meson-baryon scattering processes is a knotty problem. All higher symmetries which have been considered contain $SU(3)$, which is notoriously bad for scattering processes. We can only hope that at very high energy where the mass-differences among the particles contained in a single multiplet have little effect, the $SU(3)$ relations may agree reasonably well. Also we do not know at what energies experimental comparisons should be made. It is fashionable to make comparisons using an ansatz given by Meshkov, Snow and Yodh⁽³¹⁾ where a comparison of squares of the invariant amplitudes is made as a function of Q value in the final state. The results

obtained from SU(3) alone are poor, but consistent - about 20-30 percent out.

When collinear groups are considered, things become more complicated. Under invariance of such a group, we obviously have collinearity of the momenta of the particles before and after the scattering, but what about the intermediate states? Not all the intermediate states will be collinear. The predictions as such involve assuming that only the collinear intermediate states are contributing to the scattering amplitudes; contributions from all symmetry-breaking non-collinear intermediate states being neglected. In physical terms, it excludes the modifications introduced by the existence of competing channels and the resultant inelasticities of the low partial waves.

The two important results derivable from $S(U(3) \otimes U(3))$ and comparable with existing experimental data are the sum-rule for the Johnson-Treiman relations (44) and the relation

$$\sigma(K^- p \rightarrow K^+ \Xi^-) = 4\sigma(K^- p \rightarrow K^0 \Xi^0) \quad (45)$$

The former is extremely well satisfied whereas the latter is satisfied only in the backward direction, showing a large discrepancy in the forward direction. This latter relation also follows from $SU(6)_W$. The predictions about the polarisations of resonance in the forward and backward directions made in $SU(6)_W$ are also in complete disagreement with experiments whereas no such predictions can be made in $S(U(3) \otimes U(3))$. Thus, we believe that as far as four-point functions are concerned, $S(U(3) \otimes U(3))$ is the better collinear symmetry of the two.

CHAPTER III

Collinear Subgroups of $SL(4,c)_{I,Y}$ and $U(4,4)_{I,Y}$

In the previous chapter we have found that the predictions for meson-baryon scattering processes and baryon-resonance production processes from $S(U(3) \otimes U(3))$ agree a little better with experiment than those from $SU(6)_w$. The same results are obtained from $SL(6,)$ and $U(6,6)$ respectively. We can attribute the better agreement of $SL(6,c)$ predictions to the fact that we have used the Lorentz group extended by $SU(3)$ as the auxiliary group (and hence as the relativistic group in the sense discussed in chapter I). This leads us to believe that the Lorentz group is the correct space-time symmetry as most of the unwanted results of $U(6,6)$ are eliminated here. But unfortunately, not all the bad predictions of $U(6,6)$ are eliminated; we still have one relation (equation 45 in Chapter II) (2.45) which is badly violated in the forward direction. Besides this one, we have the bad $SU(3)$ predictions still hanging around. Can

we get rid of them by using some smaller invariance groups? Certainly strong interaction obeys isospin symmetry and hypercharge conservation very accurately. It remains to be seen how this fares when combined with any of the space-time symmetries. Since the magnitude of $SU(3)$ breaking is fairly well-known, it will also allow us to compare the magnitude of space-time symmetry-breaking effects with those of $SU(3)$.

In this chapter, we shall consider the internal symmetry group $SU(2)_I \otimes U(2)_Y$, abbreviated as $SU(2)_{I,Y}$ and its relativistic generalisations using the space-time symmetry groups $SL(2,c)$ and $U(2,2)$ respectively. Results of some scattering-processes and baryon-resonance production processes considered in the last chapter are deduced in these groups, using only the collinear subgroup in both cases.

$$\begin{aligned} & \underline{\text{Auxiliary groups } SL(2,c) \otimes SU(2)_{I,Y}, SL(4,c)_{I,Y},} \\ & \underline{U(2,2) \otimes SU(2)_{I,Y} \text{ and } U(4,4)_{I,Y}.} \end{aligned} \quad (32)$$

From the discussion of chapter I on auxiliary

group, it is clear that one can trivially extend these auxiliary groups, if besides Poincare invariance, invariance of some other group is also required. The group $SU(2)_{I,Y}$ has the generators $\bar{T}, Y = \sqrt{\frac{2}{3}}(\lambda_0 - \sqrt{2} \lambda_8)$ where λ_0, λ_8 are the two $SU(3)$ generators and $\bar{T}$ generate the isospin group $SU(2)_I$. The explicit expression for Y shows that $SU(2)_{I,Y}$ is a subgroup of $SU(3)$; but in what follows we have nothing to do with $SU(3)$ and we can avoid λ_0, λ_8 in the expression for Y . We can then have the auxiliary groups:

(i) $SL(2, c) \otimes SU(2)_{I,Y}$ with the generators $\sigma_{\mu\nu}, \bar{T}, Y$. The basic representations is $\phi_{\alpha, s}$ where α, s are the $SL(2, c)$ and $SU(2)_{I,Y}$ indices respectively. Under the transformations of the group, the indices transform independently. The little group, representations of which define a physical state, is $SU(2)_\sigma \otimes SU(2)_{I,Y}$; the collinear group is $U(1) \otimes U(1) \otimes SU(2)_{I,Y}$.

(ii) $U(2, 2) \otimes SU(2)_{I,Y}$ with the generators $Y_R, \bar{T}, Y$. The basic representation is $\phi_{\alpha, s}$ where α is now the $U(2, 2)$ index. The two indices transform independently

under transformations of the auxiliary group. The little group is $U(2) \otimes U(2) \otimes SU(2)_{I,Y}$ and the collinear group, $SU(2)_W \otimes SU(2)_{I,Y}$ where $SU(2)_W$ is the W-spin group defined already.

(iii) $SL(4,c)_{I,Y}$ which is the abbreviation for $SL(4,c)_I \otimes SL(2,c)_Y$. It is obtained by enlarging (i) above and the basic representation is $\phi_A = \phi_{\alpha,s}$ such that the indices α,s are no longer independent of each other but inter-related by means of the additional transformations generated by $\phi_{\mu} \otimes \bar{\tau}$ and $\phi_{\nu} \otimes \gamma$. The little group is $SU(4)_{I,Y} \equiv SU(4)_I \otimes SU(2)_Y$ with the generators $\bar{\tau} \otimes \bar{\sigma}, \bar{\tau}, \bar{\sigma}, \bar{\sigma} \otimes \gamma$ where $\bar{\sigma}$ generate the ordinary spin-group. We recognise $SU(4)_I$ as the supermultiplet group of Wigner and $SU(2)_Y$ as the strange quark-spin introduced by Lipkin⁽²⁷⁾. The collinear subgroup is then $S(U(2) \otimes U(2))_{I,Y}$.

(iv) finally $U(4,4)_{I,Y} \equiv U(4,4)_I \otimes U(2,2)_Y$ with the generators $\gamma_R, \gamma_R \bar{\tau}; Y \gamma_R$. It takes account of the badly broken $SU(3)$ symmetry and generalises the ordinary space-time symmetry $SL(2,c)$ group to $U(2,2)$. Besides

the generators of the group (ii) it contains the additional generators $\gamma_R \bar{\gamma}$ which interrelates the two symmetry groups, $U(2,2)$ and $SU(2)_{I,Y}$. As a result the basic representation is $\phi_A = \phi_{\alpha_S}$. The little group is $(U(4) \otimes U(4))_{I,Y}$ and the collinear group is $(SU(4)_W)_{I,Y} = (SU(4)_{W,I} \otimes SU(2)_{W,Y})$. For brevity we shall refer to this group simply as $SU(4)_W$ from now on. These groups, little groups and collinear groups are summarised in Table V.

Table V

Relativistic groups	Little groups	Collinear groups
(i) $SL(2,c) \otimes SU(2)_{I,Y}$	$SU(2)_\sigma \otimes SU(2)_{I,Y}$	$U(1) \otimes SU(1) \otimes U(2)_{I,Y}$
(ii) $U(2,2) \otimes SU(2)_{I,Y}$	$U(2) \otimes U(2) \otimes SU(2)_{I,Y}$	$SU(2)_W \otimes SU(2)_{I,Y}$
(iii) $SL(4,c)_{I,Y}$	$SU(4)_{I,Y}$	$S(U(2) \otimes U(2))_{I,Y}$
(iv) $U(4,4)_{I,Y}$	$(U(4) \otimes U(4))_{I,Y}$	$(SU(4)_W)_{I,Y}$

We shall calculate some of the scattering processes considered earlier using the two groups $SL(4,c)_{I,Y}$ and $U(4,4)_{I,Y}$ out of the groups listed in Table V.

We shall use only the collinear subgroups, although the physical particle states are first identified as the representations of the corresponding little group.

However, we shall note the following points regarding $SU(4)_{I,Y}$ whose relativistic generalisations are $SL(4,c)_{I,Y}$ and $U(6,6)_{I,Y}$.

1) The proton-neutron magnetic moment ratio which is the first big success for the static $SU(6)$ theory⁽³³⁾ is also derivable in this group⁽³⁴⁾.

2) The definite value for the proton-magnetic moment that was first derived in the $U(6,6)$ theory is not due to the extension of the internal symmetry group to $SU(3)$; this result follows from the extension of the Lorentz group to the $U(2,2)$ group. The same definite value for the proton magnetic moment has also been obtained by Macfarlane et al⁽³⁵⁾ using the extension of $SU(4)_I$ to $U(4,4)_I$.

3) Referring to the $SU(6)$ group, there are two ways of breaking this symmetry to give smaller symmetry-groups. One way is

$$SU(6) \supset SU(2)_J \times SU(3) \supset SU(2)_J \times SU(2)_I \times U(1)_Y$$

Here the $SU(3)$ multiplets are decoupled according to spin first, and then each $SU(3)$ multiplet is broken according to isospin $\times$ hypercharge. An alternative way of breaking $SU(6)$ symmetry is:

$$SU(6) \supset S(4)_{\frac{1}{2}} \times SU(2)_S \times U(1)_Y \supset SU(2)_I \times SU(2)_N \times \\ \times SU(2)_S \times U(1)_Y$$

In this the total spin J is decoupled first into two parts, $\underline{J} = \underline{N} + \underline{S}$, N from the nonstrange particles and S from the strange particles.

One very interesting experimental question will be to find which of the symmetries $SU(2) \times SU(3)$ and $SU(4) \times SU(2) \times U(1)$ is more closely realised in nature. No clear preference has yet been found from experiment.

Despite $SU(3)$ normally being considered a better symmetry than $SU(4)$ Dyson⁽³⁶⁾ has argued to the contrary with the following evidence.

i) The scattering amplitudes for the baryon-resonance production processes are related in $SU(3)$ as

which is satisfied by neither ω nor ϕ . However, the SU(4) mass formula

$$p + \phi = 2k^*$$

is very well-satisfied. The squares of the masses are used in each case.

As will be shown below, there is a reaction derivable from $SU(4)_{I,Y}$ which agrees fairly well with experiment in the backward direction, but is badly violated in the forward direction. However, these experimental results need confirmation. Good agreement in the forward direction would be a strong piece of evidence for SU(4).

It was pointed out by Delbourgo et al⁽³²⁾ that this same question would occur in the relativistic theories as a question of the superiority of the reduction

$$U(6,6) \rightarrow U(2,2) \otimes SU(3)$$

versus

$$U(6,6) \rightarrow U(4,4) \otimes U(2,2)$$

in $U(6,6)$ and of the reduction

$$SL(6,c) \rightarrow SL(2,c) \otimes SU(3)$$

versus $SL(6,c) \rightarrow SL(4,c)_I \otimes SL(2,c)_Y$

in $SL(6,c)$. The calculations for meson-baryon scattering below will be carried out according to the second method of reduction in each case, without however, assuming invariance under the larger relativistic group.

$SL(4,c)_{I,Y}$ and its subgroups:

As has been noted in the beginning of the chapter, $SL(4,c)_{I,Y}$ affords a relativistic generalisation of the little group $SU(4)_{I,Y}$. Gursay, Radicati and Pais⁽³⁸⁾ call this group $W(Y) \otimes SU(4)(T) \times SU(2)(X)$ and Lipkin⁽²⁷⁾ $SU(4)_{P_n} \otimes SU(2)_\lambda$. These authors have discussed in some detail some of the physically interesting representations. Reduction of a large number of $SU(6)$ multiplets under this group has been obtained by Delbourgo et al⁽³²⁾. We can write down the following decomposition for some well-known $SU(6)$ multiplets:

$$\underline{35} - (4, 2, 1) \oplus (15, 1, 0) \oplus (1, 3, 0) \oplus (1, 1, 0) \oplus (\bar{4}, 2, -1) \quad (2)$$

$$\underline{56} - (20, 1, 1) \oplus (10, 2, 0) \oplus (4, 3, -1) \oplus (1, 4, -2) \quad (3)$$

$$\underline{70} - (20, \bar{1}, 1) \oplus (10, 2, 0) \oplus (6, 2, 0) \oplus (4, 1, -1) \oplus (4, 3, -1) \oplus (1, 2, -2) \quad (4)$$

The third entry in each bracket refers to the hypercharge of the $SU(4)_{I,Y}$ multiplet. $20, 20'$ come from the completely symmetric and mixed symmetric quark multiplets respectively.

As in the $S(U(3) \oplus U(3))$ case, we shall explicitly make use of the quark representation, and denote the quark and anti-quark states by

$$q_{a\alpha} = \begin{pmatrix} P \\ n \end{pmatrix} \text{ and } q_{\bar{\alpha}} = \lambda \text{ for the marks}$$

$$q^{a\alpha} = (\bar{p}\bar{n}) \text{ and } q^{\bar{\alpha}} = \bar{\lambda} \text{ for the anti-quarks.}$$

where, $a = 1, 2$ is the isospin index and $\alpha = 1, 2$ refers to a quark or anti-quark state with $s_z = +\frac{1}{2}, -\frac{1}{2}$ respectively. With this specification, all the particle states can be obtained. Thus, for (k, k^*) , we have the tensor

$$M_{a\alpha}^{\bar{\beta}} = M_{a1}^1 \oplus M_{a1}^2 \oplus M_{a2}^1 \oplus M_{a2}^2 \quad (5)$$

where, for example, $M_{a_1}^1$ is the k_a^* with $s_z = +1$,
 $(M_{a_1}^2 \oplus M_{a_2}^1)$ represents the physical k_a and k_a^* with $s_z =$
 and so on. For π, ρ, ω the $SU(4)_{IY}$ tensor has the
 following decomposition under $U(2) \otimes U(2)$:

$$M_{b\beta}^{a\alpha} = M_{b1}^{a1} \oplus M_{b2}^{a2} \oplus M_{b2}^{a1} \oplus M_{b1}^{a2} \quad (a, b = 1, 2) \quad (6)$$

As an example for the baryons, we take the $(20, 1)$
 representation for N, N^* which decomposes under $U(2) \otimes U(2)$

as

$$B_{ABC} = B_{\alpha a, \beta b, \gamma c} \rightarrow B_{abc} S_{111} \oplus B_{abc} S_{112} \oplus B_{abc} S_{122} \\ \oplus B_{abc} S_{222}$$

where

$$B_{abc} S_{112} = \sqrt{3} N_{abc}^* + \frac{1}{2} (N_a \in_{bc} + N_b \in_{ca}) S_1 \quad (7)$$

$$B_{abc} S_{122} = \sqrt{3} N_{abc}^* + \frac{1}{2} (N_b \in_{ca} + N_c \in_{ab}) S_2$$

N, N^* are the physical particles and S refers to the
 various space wave functions. These two $S(U(2) \otimes U(2))$
 representations are related through parity-transformation.

The amplitudes invariant under $S(U(2) \otimes U(2))$ are
 constructed from these representations, from which the
 scattering-amplitudes for various processes are ob-

tained. It is easy to see that whereas we needed only six independent, invariant amplitudes in $S(U(3) \otimes U(3))$ to describe the processes in Table I and II, we shall require a much larger number of amplitudes now. Besides obtaining the isospin results, relations for processes involving particles belonging to the same $SU(4)_{I,Y}$ representations are obtained which are the same as in $S(U(3) \otimes U(3))$. Thus the following $S(U(3) \otimes U(3))$ results survive:

$$(i) \quad 2(k^- P \rightarrow k_{\underline{0}}^0) = (k^- P \rightarrow k_{\underline{0}}^+)$$

$$(ii) \quad (\bar{k}^0 P \rightarrow \bar{k}^0 N^{*++}) : (\bar{k}^0 P \rightarrow k^- N^{*++}) : (k^- P \rightarrow k^- N^{*+}) : \\ (k^- P \rightarrow k^- N^{*0}) = 1 : -\sqrt{3} : 1 : 1$$

$$(iii) \quad (k^- P \rightarrow k_{\underline{0}}^{*0}) : (k^- P \rightarrow k_{\underline{0}}^{*-}) : (\bar{k}^0 P \rightarrow k_{\underline{0}}^{*0}) \\ = 1 : 2 : 1 \quad (8)$$

$$(iv) \quad (k^+ P \rightarrow k^0 N^{*++}) : (k^+ P \rightarrow k^+ N^{*+}) : (k^0 P \rightarrow k_{\underline{0}}^{*0}) \\ = -\sqrt{3} : 1 : 1$$

None of the triangular relations obtained in $S(U(3) \otimes U(3))$ survive. The first relation (i) which was claimed to

be a purely $SL(6, c)$ (and hence $S(U(3) \otimes U(3))$) result,^(7,8) is also obtained from this smaller group. However, we have seen that neither of the Johnson-Treiman relations (2.40) are left in this group, not even the sum-rule (2.42); this is because $k, \bar{k}, \pi$ all belong to different representations of $S(U(2) \otimes U(2))$.

This collinear group has also been studied by Buccella and Gatto⁽²¹⁾. These authors have deduced many implications of this collinear symmetry for various processes such as collinear photoproduction, nucleon-antinucleon annihilation, non-leptonic decays, magnetic moments of various baryons etc. In the last example, the predictions derived from this group require a very accurate experimental data to distinguish them from those obtained from $SU(3)$.

$U(4,4)_{I,Y}$ and its subgroups:

We shall conclude this chapter with a consideration of the group $U(4,4)_{I,Y} = U(4,4)_I \otimes U(2,2)_Y$, first obtained by Macfarlane et al⁽³⁵⁾ who have established

a definite value for proton magnetic moment in the basis of this group alone. Their result clearly showed that it was by extending the space-time symmetry from the Lorentz group to $U(2,2)$ which fixes this value. Since there will be no more symmetry-breaking in the internal symmetry part, it will be very interesting to see how any predictions from this group compare with experiment. Comparison with results obtained in the previous section will also provide the magnitude of the symmetry-breaking caused by the extension of the space-time symmetry alone, Nwachuku's results⁽³⁹⁾ for $p\bar{p}$ annihilation obtained in this scheme are found to be highly satisfactory.

Some predictions from the collinear subgroup of $SU(6)_w$ for collinear photoproduction of mesons, nucleon and hyperon electromagnetic form-factors, weak non-leptonic decays etc. are studied by Duccella⁽⁴⁰⁾. Below, we obtain the predictions concerning the scattering-amplitudes listed in Tables I and II in an attempt to see if the badly-violated predictions from $SU(6)_w$

could be avoided.

The first task in the extension of Macfarlan's⁽³⁵⁾ $U(4,4)$ is to construct the tensor representations for mesons and baryons. In the reduction

$$\begin{aligned} U(6,6) &\rightarrow U(4,4)_I \otimes U(2,2)_Y \\ &= U(4,4)_{I,Y} \end{aligned}$$

the $U(4,4)_{I,Y}$ tensors are obtained in the obvious way from the $U(6,6)$ tensors ϕ_Q^P and ψ_{PQR} . For example

$$\phi_P^Q = \phi_1 \begin{pmatrix} \beta b \\ \alpha a 0 \end{pmatrix} \oplus \phi_2 \begin{pmatrix} \beta b \\ \alpha 3 \end{pmatrix} \oplus \phi_3 \begin{pmatrix} \beta 3' \\ \alpha a \end{pmatrix} \oplus \phi_3 \begin{pmatrix} \beta 3' \\ \alpha 3 0 \end{pmatrix} \quad \begin{matrix} (9) \\ (9) \end{matrix}$$

and

$$\begin{aligned} \psi_{PQR} = & \psi_1(\alpha\beta\gamma, abc) \oplus \psi_2(\alpha\beta\gamma, ab3+c\gamma\alpha \text{ in } abc) \\ & + \psi_3(\alpha\beta\gamma, a33+c\gamma\alpha \text{ in } a, b, c) + \psi_4(\alpha\beta\gamma, 333) \end{aligned} \quad (10)$$

where P, Q, R are the $U(6,6)$ indices, $\alpha, \beta, \gamma (=1, 2, 3, 4)$, the $U(2,2)$ indices and $a, b, c (=1, 2)$, the $SU(2)_I$ indices and the meaning of ϕ_1, ψ_1 , etc. is made clear below. Finally, with the reduction

$$U(4,4)_{I,Y} \rightarrow U(2,2) \otimes SU(2)_{I,Y}$$

we obtain the following decomposition of the $U(6,6)$ tensors:

$$\begin{aligned} \underline{144} \rightarrow (15,3) \oplus (15,1) \oplus (1,3) \oplus (1,1) \oplus (1,2) \oplus (15,2) \oplus (1,\bar{2}) \\ \oplus (15,\bar{2}) \oplus (15,1) \oplus (1,1). \end{aligned} \quad (11)$$

$$\begin{aligned} \underline{364} \rightarrow (20,4) \oplus (20',2) \oplus (20,3) \oplus (20',3+1) \oplus (20+20',2) \\ \oplus (20,1) \oplus (\bar{4},1) \end{aligned} \quad (12)$$

Contact with space-time symmetry is achieved by the restriction of $U(2,2)$ to $SL(2,c)$ and by imposing on the fields ψ , ϕ , etc. the Bargmann-Wigner equations. This finally gives us the physical states obtained in (2) and (3) corresponding to (11) and (12) respectively. Now the meaning of ψ , ϕ , etc. is clear, because we find that

$$\begin{aligned} \psi_1 \supset (N, N^*), \psi_2 \supset (, , Y^*), \psi_3 \supset (\Xi, \Xi^*), \psi_4 \supset \Omega \\ \phi_1 \supset (\pi, \rho, \omega), \phi_2 \supset (k, k^*), \phi_2 \supset (\bar{k}, \bar{k}^*), \phi_3 \supset (\phi, X) \end{aligned}$$

We shall not, however, use the full $U(4,4)_{I,Y}$ group for the calculation of the scattering amplitudes, instead as in $U(6,6)$, we shall only use the collinear subgroup $SU(4)_V = SU(4)_I \otimes SU(2)_Y$. The $SU(4)_V$ tensor

representations for the mesons and baryons can be directly obtained from the above reductions i.e. in terms of the reduction of $SU(6)_W \rightarrow SU(4)_W$, given by equation (2) and (3), respectively.

Tensor representations for some of these $SU(4)_W$ multiplets are:

$$\begin{aligned}
 (20,1) &= \psi_{1,ABC} \equiv \psi_{1,(ai,bj,ck)} \\
 &\rightarrow N^*(abc,ijk) + \frac{1}{2\sqrt{3}} (\epsilon_{ab} \epsilon_{ij} N_{c,k} + \\
 &\quad + \epsilon_{bc} \epsilon_{jk} N_{a,i} + \epsilon_{ca} \epsilon_{ki} N_{b,j}) \quad (13)
 \end{aligned}$$

$$\begin{aligned}
 (10,2) &= \psi_{2,AB\bar{C}} \equiv \psi_{2,(ai,bj,c3)} \\
 &\rightarrow Y^*(ab,ijk) + \frac{1}{\sqrt{6}} (\epsilon_{ij} Y_{(ab),k} + \epsilon_{jk} Y_{(ab),i} + \\
 &\quad + \epsilon_{ki} Y_{(ab),j}) \quad (14)
 \end{aligned}$$

$$(15,1) = \phi_{1,A}^B \equiv \phi_{1,ia}^{jb} \rightarrow (\sigma_z P_0 + \sigma^+ V_+ + \sigma^- V_- + 1 \cdot V_0)_{i,a}^{j,b} \quad (15)$$

$$(4,2) = \phi_{2,A}^{\bar{B}} \equiv \phi_{2ia}^j \rightarrow (\sigma_z P_0 + \sigma^+ V_+ + \sigma^- V_- + 1 \cdot V_0)_{i,a}^j \quad (16)$$

where P_0 represents the pseudoscalar mesons and $V_{\pm}, V_0$ refer to the $s_z = \pm 1, 0$ states of the vector-mesons respectively and A, B, C and $\bar{A}, \bar{B}$ are respectively $(SU(4)_I)_W$

and $(SU(2)_Y)_W$ indices. The tensor representations for the remaining $SU(4)_W$ states can be similarly written.

The single expression in equation 26 of chapter two contains all the amplitudes for the processes we have considered in $SU(6)_W$, but in $SU(4)_W$ we shall have to find the expressions for each of the following types of process independently and the invariant amplitudes will be different in each case:

$$\begin{aligned} \pi N &\rightarrow \pi N, \pi N \rightarrow k \Sigma, k N \rightarrow k N \\ \bar{K} N &\rightarrow k \Xi, \bar{K} N \rightarrow \pi \Sigma \end{aligned} \quad (17)$$

where, for example; π refers to all the particles of the $SU(4)_W$ multiple which contain N .

The invariant amplitudes are written down using the $SU(4)_W$ tensor representations. For example, for $(\pi N \rightarrow \pi N)$, we have the invariant expression

$$\begin{aligned} A = & a_1 \bar{\psi}_1^{ABC} \psi_{1ABC} \phi_1^E \phi_{1E}^D + a_2 \bar{\psi}_1^{ABC} \psi_{1ABD} \phi_{1E}^D \phi_{1E}^E \\ & + a_3 \bar{\psi}_1^{ABC} \psi_{1ABD} \phi_{1E}^E \phi_{1E}^D + a_4 \bar{\psi}_1^{ABC} \psi_{1ADE} \phi_{1E}^E \phi_{1E}^D \end{aligned} \quad (18)$$

while for $\pi N \rightarrow k \Sigma$, the amplitudes are obtained from

$$A = b_1 \bar{\psi}_1^{ABC} \psi_{2AB\bar{D}} \phi_{1E}^E \phi_{2E}^{\bar{D}} + b_2 \bar{\psi}_1^{ABC} \psi_{2AED} \phi_{1E}^E \phi_{2E}^{\bar{D}} \quad (19)$$

Similar expression can be written for each of the remaining processes in (17). Using expressions (13), (15) for ψ_1, ϕ_1 etc., scattering amplitudes for different processes can be easily calculated. We have found that all the $SU(6)_W$ relations among processes which are described by the same type in (17) are retained in $SU(4)_W$. For example, the processes

$$\begin{aligned} \pi^- P \rightarrow \pi^- N^{*++}, \quad \pi^- P \rightarrow \pi^0 N^{*0}, \quad \pi P \rightarrow \pi^+ N^{*-} \\ \pi^- P \rightarrow \rho^+ N^{*-}, \quad \pi^+ P \rightarrow \pi^+ N^{*+}, \quad \pi^+ P \rightarrow \pi^0 N^{*++} \end{aligned}$$

are contained in $\pi N \rightarrow \pi N$; therefore, the same amplitudes will describe all the processes, and the $SU(6)_W$ result is reproduced, namely

$$\begin{aligned} (\pi^- P \rightarrow \pi^0 N^{*0}) : (\pi^- P \rightarrow \pi^+ N^{*-}) : (\pi^- P \rightarrow \rho^+ N^{*-}) : (\pi^+ P \rightarrow \pi^0 N^{*++}) \\ = \sqrt{3} : \sqrt{8} : \sqrt{14} : 1 \end{aligned}$$

Thus, some of the notorious $SU(6)_W$ results are still true in $SU(4)_W$, namely,

$$\begin{aligned} \text{i) the amplitudes of} \\ (k^+ P \rightarrow k^0 N^{*++}), (k^0 P \rightarrow k^+ N^{*0}), (k^- P \rightarrow k^- N^{*+}), \\ (\bar{k}^0 P \rightarrow \bar{k}^- N^{*+}) \end{aligned} \quad (20)$$

are identically zero

$$\text{ii) } (k^- p \rightarrow \bar{\pi}^- Y^{*+}) : (k^- p \rightarrow \bar{\pi}^+ Y^{*-}) : (k^- p \rightarrow \bar{\pi}^0 Y^{*0}) = 2:4:3 \quad (21)$$

$$\text{iii) } (k^- p \rightarrow k_{\Xi}^{+-}) : (k^- p \rightarrow k_{\Xi}^{00}) : (k^- n \rightarrow k_{\Xi}^{0-}) = 2:1:1 \quad (22)$$

The result (21) is extremely bad, as has been discussed in the $SU(6)_W$ case, and the other two results are also not in agreement with experiment. Also, as $\bar{\pi}$, k , $\bar{k}$ belong to different $SU(4)_W$ representations we lose both the Johnson-Treiman relations.

Predictions of the polarisation and alignment of particles produced in the forward and backward directions were made in $SU(6)_W$ which are wholly contradictory to experimental results⁽²³⁾. The same bad predictions about polarisation also follow from $SU(4)_W$.

CONCLUSION:

Comparison of the above results from $SU(4)_W$ and $(U(2) \otimes U(2))_{1,Y}$ with $SU(6)_W$ thus shows that not only are we unsuccessful in eliminating the bad $SU(6)_W$ results from $SU(4)_W$, but we have also lost both the

Johnson-Treiman relations in each group. The relation (22) is very important for the group $(U(2) \otimes U(2))_{1,Y}$ as it is the minimum group that can predict this result. Also this result is not obtained in $SU(3)$; therefore agreement with experiment would clearly show that symmetry-breaking in $SU(4)$ is much smaller than that in $SU(3)$.

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PART TWO

DECOMPOSITION OF THE UNITARY IRREDUCIBLE
REPRESENTATIONS OF $SL(2, \mathbb{C})$ RESTRICTED
TO $U(1) \wedge T(2)$ AND $SU(1, 1)$

Introduction:

The development of harmonic analysis in recent times as applied to the scattering amplitude has been drawing the attention of many physicists. The discovery of the relationship between the complex angular momentum and the representation of the three-dimensional Lorentz group $O(2,1)^{(1,2)}$ is one of its very interesting by-products. Another important result to follow from this new direction of study is the group theoretical proof⁽³⁾ of the conjecture of Freedman and Wong⁽⁴⁾ regarding the daughter trajectories associated with each Regge trajectory at zero-squared momentum transfer.

The application of harmonic analysis to the scattering amplitude involves two steps. Firstly the amplitude has to be represented as a function of the elements of a group and then this function is to be expanded in terms of the unitary irreducible representation of the group. Partial wave analysis is an example of harmonic analysis where the group applied is the three dimensional rotation group (or the two dimensional

unitary unimodular group which is homomorphic to it). By noticing that this is the little group of the Lorentz group $SL(2, c)$ under which all P with $P^2 = m^2 > 0$ remain invariant, one can also use the other little groups equally well to perform the 'crossed partial wave analysis'. Applying instead the representations of $O(2, 1)$ Toller and Sertorio, Joos, Boyce and Hadjiannou⁽⁵⁾ have been able to establish the aforementioned relationship to the complex angular momentum. This group, however, is the little group of $SL(2, c)$ having all the Lorentz transformations that leave P with $P^2 = m^2 < 0$ invariant. The great merit of this expansion is that it circumvents the usual cumbersome procedure for passing to complex J .

Taking the scattering of two particles, it is convenient to consider separately the cases of equal and unequal mass.

In the equal mass case, the four-momentum transfer Q is spacelike or vanishes, so that in the former case one can use the $O(2, 1)$ or equivalently $SU(1, 1)$ which

is homomorphic to it) group representations to expand the scattering amplitude. If $Q = 0$, the little group is the Lorentz group itself and naturally, the expansion of the scattering amplitude is in terms of the matrix elements of the principal unitary representations of this group. The high energy asymptotic behaviour of the amplitude can then be obtained in terms of the Lorentz poles in the complex σ -plane, where σ is an eigenvalue of a Casimir operator of the group. Also it is observed that if the Regge hypothesis of poles in the complex J plane is carried to the complex σ -plane, to one pole in σ -plane, there corresponds a host of integrally-spaced poles in the complex J -plane. This result was anticipated by Gribov and Volkov⁽⁶⁾ who had found that when the momentum transfer vanishes, the various Regge-pole contributions can no more be independent on each other but must be arranged in families of integrally spaced poles.

It is pertinent to remark here that instead of using the $O(3,1)$ group one can also work with the four

dimensional rotation group by analytic continuation to imaginary values of the energy variable. This enables one to get over the difficulties connected with the non-compactness of the $O(3,1)$ group. But its analytic continuation does not lead naturally to the above result⁽⁷⁾.

In the unequal mass case, the four-momentum transfer can never vanish. Here Q may either be space-like, time-like or light-like. The corresponding little group for light-like four-momentum transfer is the two-dimensional Euclidean group $U(1)\wedge T(2)$. This case is of interest in the consideration of the forward scattering of unequal masses and of all backward scattering.

The considerations above point to the fact that the various little group representations are related to each other. This is the aim of this part of this thesis to find such relations between $SU(1,1)$ and $U(1)\wedge T(2)$, and $SU(2)$ and $U(1)\wedge T(2)$. The method applied is the one used by Toller and Sciarrino⁽³⁾ in finding the

corresponding result for $SU(1,1)$ and the Lorentz group.

Basically this problem can be viewed in the following way: the conventional method for constructing the basis vectors of the homogeneous Lorentz group is to diagonalise its $SU(2)$ subgroup and to designate the states $|j_o \sigma j_m\rangle$ where j_o, σ are Casimir labels of the Lorentz group. Of course, one may equally well diagonalise the $SU(1,1)$ or $U(1) \wedge T(2)$ which are contained in the Lorentz group in which case the states could be equivalently denoted by $j_o k m$ or $|j_o \sigma \ell m\rangle$ respectively. These new state vectors now have the connotations:

$$(J_3^2 - k_1^2 - k_2^2) |j_o \sigma k m\rangle = k(k+1) |j_o \sigma k m\rangle$$

$$\text{or } (L_1^2 + L_2^2) |j_o \sigma \ell m\rangle = \ell^2 |j_o \sigma \ell m\rangle$$

where $L_1 = k_1 + J_2$, $L_2 = J_1 - k_2$, J, K are the usual Lorentz group generators and m is the J_3 eigenvalue.

All these representations are on a par with one another, though depending on circumstances one set of basis vectors may be more useful than another. If we

then find the unitary transformation functions

$$\langle jm|km\rangle, \quad \langle jm|\ell m\rangle, \quad \langle km|\ell m\rangle$$

then it becomes very easy to pass from one set of state vectors to another. The results of the decomposition carried out here exactly give these transformation matrices.

In the first two sections we briefly describe the unitary irreducible representations of the homogenous Lorentz group and of the subgroups $SU(1,1)$ and $U(1)\wedge T(2)$.

In the third section we perform the decomposition of the irreducible unitary representation of $SL(2,c)$ restricted to the subgroup $U(1)\wedge T(2)$ into a direct integral of its unitary irreducible representations. We note that this result is also obtained by Ström⁽⁵⁾ using a differential method for infinitesimal generators and by Delbourgo et al⁽⁸⁾ by a simpler method.

In the next section the same method is invoked to find the remaining relation between the subgroups $SU(1,1)$ and $U(1)\wedge T(2)$. The results are the same as

obtained in the last reference. In the final section, for completeness' sake, we briefly state the result obtained by Toller and Sciarrino⁽³⁾ of the decomposition of the Lorectz group restricted to $SU(1,1)$ and thus find the transformation matrix $\langle jm|km \rangle$.

2. Irreducible Representations of $SL(2, c)$:

$SL(2, c)$ is the group of unimodular complex 2×2 matrices of the form

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad a_{11}a_{22} - a_{12}a_{21} = 1 \quad (1)$$

The unitary (and also the non-unitary) irreducible representation of this group have been investigated and classified by Gelfand and Naimark⁽⁹⁾. The method used by these authors for the construction of these representations has been generalised by Mackey⁽¹⁰⁾ in the theory of induced representation.

The infinitesimal generators of the group are

$$\underline{J} = (J_{23}, J_{31}, J_{12})$$

$$\underline{K} = (J_{01}, J_{02}, J_{03})$$

with the commutation relations,

$$\begin{aligned} [J_i, J_j] &= i \epsilon_{ijk} J_k \\ [J_i, K_j] &= i \epsilon_{ijk} K_k \\ [K_i, K_j] &= i \epsilon_{ijk} J_k \end{aligned} \quad (2)$$

and the Casimir invariants

$$\frac{1}{2} J_{\mu\nu} J^{\mu\nu} = j_0^2 - \sigma^2 + 1, \quad \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} J^{\mu\nu} J^{\lambda\rho} = 2ij_0\sigma \quad (3)$$

It is possible to label a complete set of basis vectors $|j_0\sigma, jm\rangle$ with the two additional quantum numbers obtained from diagonalising the SU(2) subgroup.

It is convenient to parametrise the finite Lorentz transformation in the form:

$$a = U(\phi, \theta, \frac{\psi}{2}; \zeta; \frac{1}{2}\psi, \theta', \phi') = U(\phi, \theta, \frac{\psi}{2}) a_z(\zeta) U(\frac{\psi}{2}, \theta', \phi') \quad (4)$$

because upon taking representatives in the basis

$|j_0\sigma; jm\rangle$, we get

$$D_{jm, j'm'}^{j_0\sigma}(a) = \sum_{m''} D_{mm''}^j(\phi, \theta, \psi) d_{jm'' j'}^{j_0}(\zeta) D_{m''m}^{j'}(\phi, \theta', \psi') \quad (5)$$

where $D_{mm'}^j(\phi, \theta, \psi)$ is the well-known SU(2) representation function and $d_{jm j'}^{j_0}(\zeta) = \langle j_0\sigma, jm | e^{-i\zeta k_3} | j_0\sigma, j'm' \rangle$

$$= \int_{SU(2)} \overline{D_{j_0 m}^j(u)} (D_{(a_z(\zeta))}^{j_0} D_{j_0 m'}^{j'}(u)) du \quad (6)$$

is the SL(2, c) representation functions which has been calculated by Ström⁽¹¹⁾.

The unitary irreducible representations of $SL(2, \mathbb{C})$ fall into two categories in each of which each representation of $SU(2)$ occurs only once:

a) Principal series:

$$\operatorname{Re}(\sigma) = 0 \quad \text{and} \quad -\infty < \operatorname{Im} \sigma < \infty$$

$$j \gg j_0 = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$$

$$\text{and} \quad m = -j, -j+1, \dots, j$$

b) Supplementary series:

$$\operatorname{Im} \sigma = 0, \quad 0 < \operatorname{Re} \sigma < 1, \quad j_0 = 0 \quad \text{with}$$

$$j = 0, 1, 2, \dots$$

$$m = -j, -j+1, \dots, j.$$

In what follows we shall consider the principal series of representations only, as they alone permit the construction of a Hilbert space.

Unitary Irreducible Representations of $SU(1,1)$:

This is the subgroup of $SL(2, \mathbb{C})$ formed by all complex matrices of the form:

$$v = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \quad |\alpha|^2 - |\beta|^2 = 1 \quad (7)$$

The infinitesimal generators of the group are J_3 , k_1 and k_2 that satisfy the commutation relations

$$[K_1, K_2] = -iJ_3, \quad [J_3, K_1] = iK_2, \quad [J_3, K_2] = -iK_1 \quad (8)$$

Unitary representations have J_3 , K_1 , K_2 hermitian. The Casimir operator is $K^2 = J_3^2 - K_1^2 - K_2^2 = k(1-k)$ and the basis vectors are labelled by $|km\rangle$ where m refers to the J_3 eigenvalue. A convenient parametrisation of the group element is:

$$v = U_Z(\mu) a_X(\zeta) U_Z(\nu), \quad 0 \leq \mu < 4\pi, \quad 0 < \nu < 2\pi$$

$$0 \leq \zeta < \infty \dots \quad (9)$$

The unitary irreducible representations have been obtained by Bargmann⁽¹²⁾. The representation matrices have the form

$$D_{mm}^{\wedge}(U_Z(\mu) a_X(\zeta) U_Z(\nu)) = \exp(-im\mu - im\nu) d_{mm}^{\wedge}(\zeta) \quad (10)$$

where the subscript $\wedge$ stands for the parameters which label the equivalence classes of irreducible represen-

tations. The Bargmann classification of the unitary irreducible representations fall into the following classes with the values of Λ shown explicitly:

a) Discrete classes: Λ stands for $K_{\pm}$ with

$$k = \frac{1}{2}, 1, \frac{3}{2}, \dots$$

D^+ has $m = k, k+1, \dots$ and

D^- has $m = -k, -k-1, -k-2 \dots$

b) Continuous classes:

The parameters are ϵ, k where ϵ is either 0 or $\frac{1}{2}$ according as the representations are integral or half-integral type. There are again two subclasses of representation; principal series of representation correspond to complex values of the parameter k with $\text{Re } k = -\frac{1}{2}$ and $m = 0, \pm 1, \pm 2 \dots$ or $= \pm \frac{1}{2}, \pm \frac{3}{2}, \dots$ and the complementary series representations have

$$-\frac{1}{2} < \text{Re } k < 0, \quad \text{Im } k = 0$$

and m taking all integer values, $m=0, \pm 1, \pm 2 \dots$

The preceding discussion of the irreducible representations implicitly assumes that $SU(1,1)$ has only

one and two-valued representations up to a factor.

The functions $d_{mm}^k(\zeta)$ defined in equation (10) for the principal series are just the continuation $\cos \theta \rightarrow \cosh \zeta$ of the ordinary $SU(2)$ representations and can be expressed in terms of the hypergeometric series in the form:

$$d_{mm}^k(\zeta) = \left(\frac{\Gamma(k-m'+1) \Gamma(k+m+1)}{\Gamma(k+m'+1) \Gamma(k-m+1)} \right)^{\frac{1}{2}} (\cosh \frac{\zeta}{2})^{2k+m'-m} (\sinh \frac{\zeta}{2})^{m-m'}$$

$${}_2F_1(m-k, -m'-k; m-m'+1; \tanh^2 \frac{\zeta}{2}) / (m-m'+1)$$

$$m \geq m' \quad (11)$$

and has the following symmetry relations,

$$d_{mm}^k(\zeta) = (-1)^{m-m'} d_{-m, -m'}^k(\zeta); \quad d_{mm}^k(\zeta) = (-1)^{m-m'} d_{m', m}^k(\zeta)$$

$$(12)$$

For the discrete series, this function is

$$d_{mm}^{k+}(\zeta) = \left(\frac{\Gamma(m-k+1) \Gamma(m+k+1)}{\Gamma(m'-k) \Gamma(m'+k+1)} \right)^{\frac{1}{2}} (\cosh \zeta/2)^{-2k+m'-m} (\sinh \zeta/2)^{m-m'}$$

$${}_2F_1(m-k, -m'-k; m-m'+1; \tanh^2 \frac{\zeta}{2}) / \Gamma(m-m'+1) \quad (13)$$

and

$$d_{mm'}^{k-}(\zeta) = d_{-m'-m}^{k+}(\zeta) \text{ and } d_{mm'}^{k+}(\zeta) = d_{-m'-m}^{-k-1+}(\zeta) \text{ etc.}$$

Unitary irreducible Representations of the Euclidean

Group:

$U(1) \ltimes T(2)$ is a three-parameter group consisting of all the distance preserving transformations (reflections excluded) of a plane into itself. In fact it is the semi-direct product of the abelian subgroup of translations in two-dimensions and the rotation subgroup (in one-dimension)⁽¹³⁾.

The infinitesimal operators of this group are J_3 , $L_1 = K_1 + J_2$ and $L_2 = K_2 - J_1$. The commutation relations between these operators are

$$[L_1, L_2] = 0, [J_3, L_1] = iL_2, [J_3, L_2] = -iL_1 \quad (14)$$

The elements of this group can be parametrised in the following unique way:

$$t = U_z(\mu) t(r) U_z(\nu), \quad 0 \leq \mu < 4\pi, \quad 0 \leq \nu < 2\pi, \quad (15)$$

$$0 \leq r < \infty$$

The Casimir operator is $L_1^2 + L_2^2 = \ell^2$ except when $\ell=0$ and the states are diagonalised with respect to the J_3 eigenvalue. Thus, there are two classes of representation of this group; these are,

a) Discrete series:

Here $\ell=0$, also $L_1=L_2=0$. The representations are denoted by O_s and consists of the matrices $e^{is\theta}$ of rotation in x-y plane through an angle θ , $s=m$ can be zero, integer or half integer, negative or positive.

b) Principal series:

$$0 < \ell < \infty, \quad \text{Im} \ell = 0 \quad \text{and} \\ m = \begin{cases} 0, \pm 1, \pm 2, \dots \\ \pm \frac{1}{2}, \pm \frac{3}{2}, \dots \end{cases}$$

In the former case, the representation is single valued and in the latter, two-valued. These two are the only series representations of interest.

The representation functions $d_{mm}^{\ell}(r)$ are just the Bessel's functions i.e., $d_{mm}^{\ell}(r) = J_{m-m'}(\ell r)$. Also the following connection with the representation functions of $SU(2)$ and $SU(1,1)$ is worth noticing:

$$\lim_{j \rightarrow \infty} d_{mm}^j, \left(\frac{\ell r}{j} \right) = J_{m-m'}, (\ell r)$$

The symmetry properties are those of the Bessel functions, namely

$$d_{mm}^{\ell}, (r) = d_{-m, -m}^{\ell}, (r) = (-1)^{m-m'} d_{m', m}^{\ell}, (r). \quad (16)$$

3. Decomposition of $SL(2, c)$ restricted to $U(1) \wedge T(2)$.

This section will be devoted to decomposing the irreducible unitary representations $D_{j_0 \ell}^{j_0 \ell'}$ of $SL(2, c)$ restricted to $U(1) \wedge T(2)$ into a direct integral containing the irreducible unitary representations D^{ℓ} of $U(1) \wedge T(2)$ described in the previous section. The problem involves two stages: firstly, Mackey's⁽¹⁰⁾ general method of decomposing the restrictions of a representation is applied to the $SL(2, c)$ unitary representation which is restricted to those of $U(1) \wedge T(2)$. These reducible representations are then decomposed into a direct integral of irreducible unitary representations of $U(1) \wedge T(2)$ by the application of the Plancherel formula. Finally the matrix elements $D_{jmj', m}^{j_0 \ell'}$ (a) are

explicitly calculated in terms of the matrix elements $D_{mm}^{\Lambda}(t)$ of the representation functions of the Euclidean group.

The induced representations are to be constructed by means of the subgroup K which consists of the complex matrices of the form:

$$k = \begin{pmatrix} \lambda^{-1} & \mu \\ 0 & \lambda \end{pmatrix} \quad (17)$$

with the following one-dimensional unitary representations;

$$L^{j_0 \sigma}(k) = |\lambda(k)|^{2(\sigma+j_0)} [\lambda(k)]^{-2j_0}. \quad (18)$$

where $\lambda(k)$ is an element of the matrix K ; j_0 is integer or half-integer and σ is a pure imaginary parameter,

Now we shall have to determine the double-cosets of $SL(2, c)$ with respect to its subgroups K and $U(1) \wedge T(2)$ in order to perform the first step, namely, to decompose the restrictions of $SL(2, c)$ representations in reducible representations of $U(1) \wedge T(2)$. To do this, we note that every element $a \in SL(2, c)$ can be written in the form

$$\begin{aligned} a &= kU & \text{where } k \in K, & & U \in SU(2) \\ &= kU(\theta) U_z(\nu) \end{aligned} \quad (19)$$

Writing,

$$U_Y(\theta) = K, z, \quad K \in K, \quad z = \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \quad (20)$$

gives

$$r = \tan^2 \theta / 2 \quad \text{for } 0 \leq \theta < \pi, \quad 0 < r < \infty \quad (21)$$

This shows that every element $a \in SL(2, c)$ can be uniquely written as

$$a = kt \quad \text{where } t \in U(1) \wedge T(2) \quad (22)$$

so that there is only a single double coset of $SL(2, c)$ with respect to the subgroups $U(1) \wedge T(2)$ and K .

This means that for every function $f(a)$ defined over the Hilbert space of $SL(2, c)$ representation, we get a function $f(t)$ defined over the subgroup $U(1) \wedge T(2)$. These functions obey the covariance relation

$$f(a) = f(kt) = L^{j_0 \sigma}(\mathbf{k}) f(t) \quad (23)$$

This relation follows from the representation property of the subgroup K .

The scalar product is then

$$\begin{aligned}
\int_{SU(2)} |f(u)|^2 dU &= \frac{1}{4\pi} \int_0^{2\pi} dv \int_0^\pi |f(U_Y(\theta)U_Z(v))|^2 \sin\theta d\theta \\
&= \frac{1}{4\pi} \int_0^{2\pi} dv \int_0^\infty |f(t)|^2 \frac{24r \cdot dr}{(1+r^2)^2} \\
&= \int_{U(1) \wedge T(2)} |F(t)|^2 dt
\end{aligned} \tag{24}$$

where

$$F(t) = \frac{\sqrt{2}f(t)}{1+r^2} \tag{25}$$

This shows that the representation operators operate on the Hilbert space H in the following manner⁽¹⁴⁾

$$[D(t)f](t') = \left| \frac{P(t'(t')^{-1})}{P(t't(t't)^{-1})} \right|_0^2 f(t't) \tag{26}$$

$$\text{where we have } \left| P(t'(t')^{-1}) \right|_0^2 = 1+r^2 \tag{27}$$

which follows from relations (20). This gives,

$$(D(t)F)(t') = F(t't) \tag{28}$$

Equation (28) clearly defines the Hilbert space of the regular representation of the Euclidean group.

Therefore, we can get a decomposition in direct integral of irreducible representations of the Euclidean group from the regular representation so obtained. This decomposition is achieved by means of the Plancherel formula

$$\int_{U(1) \wedge T(2)} |f(t)|^2 dt = \int \sum_{mm'} |F_{mm'}^\ell|^2 d\rho \quad (29)$$

where $d\rho$ is the Plancherel measure, and

$$\begin{aligned} F_{mm'}^\ell &= \int_{U(1) \wedge T(2)} F(t) D_{mm'}^\ell(t^{-1}) dt \\ &= \frac{1}{8\pi^2} \int_0^{4\pi} d\mu \int_0^{2\pi} d\nu \int_0^\infty r \cdot dr \frac{\sqrt{2}f(t)}{1+r^2} D_{mm'}^\ell(t^{-1}) \\ &= \frac{1}{8\pi^2} \int_0^{4\pi} d\mu \int_0^{2\pi} d\nu \int_0^\infty \sqrt{2} \cdot r \cdot dr (1+r^2)^{\sigma-1} f(u) D_{mm'}^\ell(t^{-1}) \\ &= \frac{1}{8\pi^2} \int_0^{4\pi} d\mu \int_0^{2\pi} d\nu \int_0^\infty \sqrt{2} \cdot r \cdot dr (1+r^2)^{\sigma-1} \exp(i\mu(-j_0+m) \\ &\quad + i\nu(m'-m)) d_{j_0 m}^j(\theta) J_{m-m'}^\ell(kr) \end{aligned}$$

Using

$$f(u) = f(kt) = L^{j_0 \sigma}(k) f(t) = (1+r^2)^{-\sigma} f(t) \quad \text{and}$$

$$f(u) = \phi_{j_0 m}^j(u) = \exp(-ij_0 \mu - im\nu) d_{j_0 m}^j(\theta).$$

This yields,

$$F_{mm}^{\ell} = \delta_{mm} d_{mj}^{j_0 \sigma} \quad (30)$$

where

$$d_{mj}^{j_0 \sigma} = \int_0^\infty r^2 r dr (1+r^2)^{\sigma-1} d_{j_0 m}^j(\theta) J_{m-j_0}^{\ell}(r) \quad (31)$$

This can be expressed as a series of Meijer's G-functions as has been shown in references (8). The result is

$$\begin{aligned} d_{mj}^{j_0 \sigma} &= (-1)^{j-j_0} ((j-j_0)! (j+j_0)! (j-m)! (j+m)!)^{\frac{1}{2}} \left(\frac{1}{2}\right)^{-m-j_0} \\ &\times \sum_{k=0}^{j-m} \frac{(-1)^k}{k! (j-m-k)! (j_0+m+k)! (j-j_0-k)!} \\ &G_{13}^{21} \left(\begin{matrix} 2 \\ 4 \end{matrix} \middle| \begin{matrix} k+m+j_0-\ell \\ k+m+j_0-\sigma \end{matrix} \begin{matrix} m \\ j_0 \end{matrix} \right) \end{aligned} \quad (32)$$

using the following representation for the SU(2) representation function

$$\begin{aligned} d_{j_0 m}^j &= (-1)^{j-j_0} ((j-j_0)! (j+j_0)! (j-m)! (j+m)!)^{\frac{1}{2}} \times \\ &\sum_{k=0}^{j-m} (-1)^k \left(\cos \frac{\theta}{2}\right)^{k+\frac{m+j_0}{2}} \\ &\times (\sin \theta/2)^{j+k-\frac{m+j_0}{2}} (k! (j-m-k)! (j_0+m+k)! (j-j_0-k)!)^{-1} \end{aligned} \quad (33)$$

Equation (31) is the transformation matrix denoted by

$$\langle \ell m | j m \rangle.$$

For $t=0$, this also gives the orthonormality condition for the $d_{mj}^{j_0 \sigma}$:

$$\int d\varphi \overline{d_{mj}^{j_0 \sigma}} d_{mj}^{j_0 \sigma} = d_{jj}, \quad (34)$$

Using the following symmetry properties for the $SU(2)$ representation functions

$$\begin{aligned} d_{-m' -m}^j(\theta) &= (-1)^{m-m'} d_{m' m}^j(\theta) \\ d_{-mm'}^j(\theta) &= (-1)^{-m' -m} d_{m -m'}^j(\theta) \end{aligned} \quad (35)$$

in conjunction with these for the Bessel functions (16), we arrive at the symmetry-relation for $d_{mj}^{j_0 \sigma}$:

$$d_{-mj}^{j_0 \sigma} = d_{mj}^{-j_0 \sigma} \quad (36)$$

4. The Transformation matrix $\langle \ell m | k m \rangle$.

In this section we are finding an integral expression for the matrix $\langle \ell m | k m \rangle$ that relates the unitary irreducible representations of the two subgroups $U(1) \wedge T(2)$ and $SU(1,1)$. The procedure is completely analogous to the one just outlined for the matrix $\langle \ell m | j m \rangle$. But however, there is one important modi-

fication to be made which we shall explain as we come across it.

We have noted before that every element $a \in SL(2, c)$ can be written down in the following unique way

$$a = Kt, \text{ where } k \in K, \quad t \in U(1) \wedge T(2)$$

With the parametrisation used for the $SU(1,1)$ and $U(1) \wedge T(2)$ groups, we find that

$$t(r) = kb^+ a_x(\tau) \quad \text{with } r = \tanh \frac{\tau}{2}. \quad \text{where}$$

$$0 \leq r < \infty \quad \text{and } 0 \leq \tau < \infty \quad (37)$$

$$b^+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (38)$$

It is clear in this case that while τ covers its full range, r can range only over $0 \leq r \leq 1$. Therefore, to cover the full range of r we must introduce the following additional relation

$$t(r) = kb^- a_x(\tau) \quad \text{with } r = \coth \tau/2 \quad (39)$$

where

$$b^- = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (40)$$

These two equations show that the range of r is now

fully covered by ξ . As has been shown by Delbourgo et al⁽⁸⁾, this is just the additional rotation $e^{-i\pi J_2}$.

Equations (37) and (39) can be used to form the double cosets of $SL(2, c)$ with respect to the subgroups K and $SU(1, 1)$, and we find that each $a \in SL(2, c)$ can be written in one of the following two forms:

$$(i) \quad a = kb^+v \quad \text{and} \quad (ii) \quad a = kb^-v \quad (41)$$

These double cosets are disjoint and contain all the elements of $SL(2, c)$.

This decomposition into double cosets allows us to obtain restrictions on the representations of $SL(2, c)$ in pairs of reducible representations of $SU(1, 1)$. This necessitates decomposing the Hilbert spaces H of functions $f(a)$ defined on $SL(2, c)$ into two Hilbert spaces H^+ and H^- of functions $f^\pm(v)$ defined on $SU(1, 1)$. The procedure is completely analogous to the previous case, except that for each $f(a)$ of $SL(2, c)$ instead of one $f(t)$ of $U(1) \wedge T(2)$, now we obtain two $F(v)$ defined on two Hilbert spaces H^+ and H^- of $SU(1, 1)$. Thus H^+ and H^- are formed by functions $f^\pm(v)$ defined by

$$f_{\pm}^+(v) = f(b_{\pm}^+ v). \quad (42)$$

Using the change of variables of equations (37) and (39) we get,

$$\begin{aligned} U(1) \wedge T(2) \int |F(t)|^2 dt &= \frac{1}{2\pi} \int_0^{2\pi} dv \int_0^{\infty} |f((r)U_Z(v))|^2 r \cdot dr \cdot \left(\frac{r/2}{1+r^2}\right)^2 \\ &= \frac{1}{2\pi} \sum_{\rho} \int_0^{2\pi} dv \int_0^{\infty} |f^{\rho}(a_x(\zeta)U_Z(v))|^2 \left(\frac{r/2}{1+r^2}\right)^2 \sinh \zeta \cdot d\zeta \\ &\quad \left(\frac{1+r^2}{2\cosh \zeta}\right)^2 \\ &= \sum_{\rho} \int_{SU(1,1)} |F_{\rho}^{\pm}(v)|^2 dv \end{aligned} \quad (43)$$

$$\text{where } dv = (4\pi)^{-2} \sinh \zeta d\zeta \quad \text{and } F_{\rho}^{\pm}(v) = \frac{r/2}{\cosh \zeta} f^{\rho}(a_x(\zeta)U_Z(v)) \quad (44)$$

and ρ stands for $\pm$.

This defines the following decomposition

$$H = H^+ \oplus H^-$$

where the functions defining the Hilbert spaces $H^{\pm}$ are with norms

$$\int |F_{\rho}^{\pm}(v)|^2 dv = \|F_{\rho}^{\pm}\|^2$$

The representation functions $O_{(v)}^{+j_0}$ are such that

$$(O_{(v)}^{+j_0} F^{\pm})(v') = F^{\pm}(vv') \quad (45)$$

which is the regular representation of the $SU(1,1)$ group. Then application of the Plancherel formula yields the following decomposition into a direct integral of the irreducible $SU(1,1)$ representations.

$$\int_{SU(1,1)} |F^{\pm}(v)|^2 dv = \int \sum_m^{\Lambda_{\pm}^{+j_0}} |F_m^{\pm}|^2 d\Lambda \quad (46)$$

which, in terms of the Hilbert spaces, gives $H^{\pm j_0} = \int_{\Lambda_{\pm}^{+j_0}} H^{\Lambda_{\pm}} d\Lambda$ (47). Here $d\Lambda$ is the Plancherel measure and $H^{\Lambda_{\pm}}$ denotes the

Hilbert spaces of the irreducible representations of $SU(1,1)$ composed of functions $F_m^{\Lambda_{\pm}}$, defined by

$$F_m^{\Lambda_{\pm}}, \delta_{m \pm M} = \int D_{m, m}^{\Lambda_{\pm}}(v^{-1}) F^{\pm}(v) dv \quad (48)$$

Now,

$$F^{\pm}(v) = \left(\frac{\sqrt{2}}{\cosh \zeta} \right) f^{\pm}(U_Z(\mu) a_X(\zeta) U_Z(v)) \quad \text{using (44)}$$

$$v = \left(\frac{\sqrt{2}}{\cosh \zeta} \right) f(b^{\pm} U_Z(\mu) a_X(\zeta) U_Z(v))$$

$$= \left(\frac{\sqrt{2}}{\cosh} \right) f(k^{-1} U_Z(t) U_Z(v))$$

$$= \left(\frac{\sqrt{2}}{\cosh} \right) \left(\frac{\cosh}{1+r^2} \right) f(t)$$

Cont; -

$$= \left(\frac{\cosh \zeta}{1+r^2} \right)^{\sigma-1} F(t).$$

$$= \exp(\bar{+} i j_0 \mu - i m \nu) J_{j_0 - m}(\ell r) \left(\frac{\cosh \zeta}{1+r^2} \right)^{\sigma-1} \quad (49)$$

where r is used only symbolically as it enables us to write two equations in one and has to be replaced by $r = (\tanh \zeta / 2)^{\pm 1}$ for $F_m^{\pm}$ respectively.

Therefore using equations (48) and (49), we obtain

$$\begin{aligned} F_m^{\pm} &= \frac{1}{16\pi^2} \int_0^{4\pi} d\mu \int_0^{2\pi} d\nu \int_0^\infty \sinh \zeta \, d\zeta \exp(i m' \mu + i m \nu \mp i j_0 \mu - i m \nu) \\ &\quad d_{m'm}^{\pm}(\zeta) \left(\frac{\cosh \zeta}{1+r^2} \right)^{\sigma-1} J_{j_0 - m}(\ell r). \\ &= \delta_{mm'} \int_0^\infty \frac{1}{2} \sinh \zeta \, d\zeta (-1)^{m \mp j_0} d_{m \pm j_0}^{\pm}(\zeta) J_{j_0 - m}(\ell r) \left(\frac{\cosh \zeta}{1+r^2} \right)^{\sigma-1} \\ &= \delta_{mm'} d_{km}^{j_0 \sigma \pm} \dots \end{aligned}$$

where

$$d_{km}^{j_0 \sigma +} = \frac{1}{2} \int_0^\infty \sinh \zeta \, d\zeta (-1)^{m - j_0} d_{mj_0}^k(\zeta) J_{j_0 - m}(\ell \tanh \zeta / 2) (\cosh \zeta / 2)^{\sigma-1} \quad (50)$$

and

$$d_{km}^{j_0 \sigma -} = \frac{1}{2} \int_0^\infty \sinh \zeta \, d\zeta (-1)^{m + j_0} d_{m-j_0}^k(\zeta) J_{j_0 - m}(\ell \coth \zeta / 2) (\sinh \zeta / 2)^{\sigma-1} \quad (51)$$

Apart from a normalisation constant, these matrices are the same as those given by Delbourgo et al⁽⁸⁾. It has been shown that these matrices can also be expressed in terms of Meijer's G-functions. This is achieved here by using the series expansion for the hypergeometric function occurring in the expression for $d_{mj_0}^k$ in equation (11) and expressing the Bessel functions in terms of Meijer's G-function with the help of equation (5.6.3)⁽¹⁵⁾. The integrals thus obtained are evaluated by changing the variables to $(\tanh \frac{2\xi}{2})^{\pm}$ and using equations (20.5.1) and (20.5.2)⁽¹⁶⁾ respectively. The results come out as,

$$d_{km\ell}^{j_0\sigma+} = \Delta_{k,s,m,j_0} \left(\frac{2}{\ell}\right)^{m+j_0} \Gamma(-\sigma-k) G_{13}''\left(\frac{\ell^2}{4} \middle| j_0-s, j_0-m, j_0+\sigma+s\right) \quad (52)$$

$$d_{km\ell}^{j_0\sigma-} = \Delta_{k,s,m,-j_0} \left(\frac{2}{\ell}\right)^{m+j_0} \Gamma(-\sigma-k) F_{13}^{20}\left(\frac{\ell^2}{4} \middle| s-k+j_0+m, s+\sigma+j_0+m, j_0-m\right) \quad (53)$$

where

$$\Delta_{k,s,m,m'} = (-1)^{m-m'} \sum_{s=0}^{\infty} \frac{\Gamma(m-k+s) \Gamma(s-m'-k) \Gamma(m-m'+1)}{\Gamma(m-k) \Gamma(-m'-k) \Gamma(s+m-m'+1) s! (m-m'+1)!} \left(\frac{\Gamma(k-m'+1) \Gamma(k+m'+1)}{\Gamma(k+m'+1) \Gamma(k-m'+1)} \right)^{\frac{1}{2}} \quad (54)$$

5. Transformation matrix $\langle km | jm \rangle$.

For completeness' sake we include this where we calculate the transformation matrices $\langle km | jm \rangle$ which relates the $SU(2)$ basis vectors to the $SU(1,1)$ basis vectors. This was first calculated by Toller and Sciarrino⁽³⁾ using the method presented in the previous two sections.

We enumerate briefly the decomposition of unitary irreducible representations of $SL(2, c)$ (with the $SU(2)$ subgroup diagonalised) restricted to $SU(1,1)$ into a direct integral containing the unitary representations of $SU(1,1)$. This will lead to the transformation matrix $\langle km | jm \rangle$. This involves decomposition of the restrictions of the $SL(2, c)$ representations into reducible representations of $SU(1,1)$ in pairs; thus decomposing the Hilbert space of $SL(2, c)$ functions into two Hilbert spaces, because, in a search for the double-cosets of $SL(2, c)$ with respect to the subgroups $SU(1,1)$ and K , we find that there are three disjoint double-cosets one of which has vanishing invariant

measure. This can be shown easily as follows:

We find that an element $a \in SL(2, c)$ can be uniquely written as $A = k.u$, $k \in K$, $u \in U$ and with the parametrisation for U and V given in section 1, we notice that $U_Y(0) = kb^+ a_x(\zeta)$ for $\tan \frac{\theta}{2} = \tanh \frac{\zeta}{2}$

$$0 \leq \theta < \pi/2, \quad 0 \leq \zeta < \infty, \quad (55)$$

We observe that again this is not sufficient to cover the whole range of θ , so that we shall have to introduce the additional rotation $e^{-i\pi/2}$ whereby,

$$U_Y(\theta) = kb^- a_x(\zeta) \quad \text{with}$$

$$\tan \frac{\theta}{2} = \coth \frac{\zeta}{2}, \quad \frac{\pi}{2} < \theta < \pi, \quad 0 \leq \zeta < \infty \quad (56)$$

b^+ and b^- are the matrices given in section 4.

This allows us to write

$$\left. \begin{aligned} a &= kb^+ a_x(\zeta) U_Z(\nu) \\ a &= kb^- a_x(\zeta) U_Z(\nu) \\ a &= k U_Y(\pi/2) U_Z(\nu) \end{aligned} \right\} \quad (57)$$

With the realisation that $a_x(\zeta) U_Z(\nu) = v \in SU(1,1)$,

the above equation defines the three double-cosets.

Because the last one has vanishing invariant measure, we only deal with the first two.

This allows them to obtain

$$\int_{\text{SU}(2)} |f(u)|^2 du = \sum_{\epsilon} \int_{\text{SU}(1,1)} |F_{(v)}^{\epsilon}|^2 dv \quad \text{where } F_{(v)}^{\epsilon} = \frac{f^{\epsilon}(v)}{\cosh \zeta},$$

$\epsilon = \pm$

which gives the two Hilbert spaces H^+ and H^- for the $\text{SU}(1,1)$ functions $F_{(v)}^{\pm}$ i.e. $H = H^+ \oplus H^-$.

The representations that act in these Hilbert spaces are $O^{\pm}$ respectively such that

$$(O^{\pm}(v) F_{(v')}^{\pm}) = F_{(vv')}^{\pm}$$

This being the definition of the regular representations of $\text{SU}(1,1)$, allows the application of the Plancherel formula for obtaining a direct integral of the irreducible representations of $\text{SU}(1,1)$. Thus,

$$\int_{\text{SU}(1,1)} |F_{(v)}^{\pm}|^2 dv = \int_{\Omega_{\pm}} \sum_m |F_{\frac{1}{m}}^{\pm}|^2 d\wedge$$

where

$$F_{\frac{1}{m}}^{\pm} \delta_{\pm j_0, m} = \int_{\text{SU}(1,1)} D_{mm}^{\wedge}(v^{-1}) F_{(v)}^{\pm} dv.$$

This leads to the result

$$\begin{aligned}
 F_{m'}^{\wedge+} &= (2j+1)^{\frac{1}{2}} \int_0^{4\pi} d\mu \int_0^{2\pi} d\nu \int_0^\infty (\cosh \zeta)^{\sigma-1} d_{j_0 m}^j(\theta^\pm) d_{m' \pm j_0}^{\wedge}(\zeta) \\
 &\quad (-1)^{m' \pm j_0} \exp(i(m'-m)\nu) (4\pi)^{-2} d\zeta. \\
 &= \int_{mm'} d_{(\wedge, m, j)}^{j \sigma \pm}
 \end{aligned}$$

where

$$d_{(\wedge, m, j)}^{j \sigma \pm} = (2j+1)^{\frac{1}{2}} \int_0^\infty (\cosh \zeta)^{\sigma-1} d_{j_0 m}^j(\theta^\pm) d_{m \pm j_0}^{\wedge}(\zeta)^{\frac{1}{2}} \sinh \zeta d\zeta. \quad (58)$$

Here $\wedge$ stands for the parameters of the $SU(1,1)$ representations. This integral is the transformation matrix $\langle km | jm \rangle$ that can change the $SU(2)$ basis vectors in the usual designation of $SL(2, c)$ representations into $SU(1,1)$ basis vectors. Analytic properties of this function has been discussed in great detail by Toller and Sciarrino⁽³⁾. This also is expressible in terms of a series of Meijer's G-function and has also been obtained by Ström and Delbourgo et al⁽⁸⁾ using different methods.

This result is important because, with the asymptotic behaviour of this function, Toller and Sciarrino⁽³⁾ has been able to decompose a single pole in the Lorentz group space into several integrally-spaced poles in the complex angular momentum plane, which is the space of the $SU(1,1)$ subgroup. This is not surprising because in the zero-momentum transfer limit the little group enlarges to the whole Lorentz group, so that one would expect from the analyticity of the scattering amplitude a connection between the representations of the two groups.

6. Conclusion:

We have obtained in this part the decomposition of the principal series representation of $SL(2, c)$ as a direct integral over unitary representations of the subgroups $SU(1, 1)$ and $U(1) \wedge T(2)$. As has been noted before, these equations stand also for the integral representations of the transformation matrices $\langle km | jm \rangle$, $\langle \ell m | jm \rangle$ and $\langle km | \ell m \rangle$. Since these matrices allow

us to go from one set of basis vectors (of a little group which has been diagonalised in the corresponding $SL(2, c)$ representation) to another, it is believed that these matrices may prove useful in the generalized partial wave analyses for establishing relations between the various possible harmonic analyses and their continuations into the complex Casimir plane.

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Barion Resonance Decays in Broken $U_3 \times U_3$ (*).

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This note presents a new sum-rule (eq. (4)) for baryon resonance decays. It assumes an interaction which breaks SU_3 to first order within the context of the symmetry scheme of $U_3 \times U_3$.

$U_3 \times U_3$ is the well-known collinear subgroup of $SL_{6,0}$ (1) and as such has generators $\frac{1}{2}\lambda i(1 \pm \sigma_z)$ (2). The predictions of this scheme are less restrictive than those of $SU_{6,0}$, this being the corresponding collinear subgroup for $U_{6,0}$ (3).

The relations obtained under exact SU_3 symmetry are

$$(1) \quad (\mathcal{N}^* \rightarrow \mathcal{N}\pi) = -\sqrt{2} (Y^* \rightarrow \Lambda\pi) = \sqrt{3} (Y^* \rightarrow \Sigma\pi) = \sqrt{2} (\Xi^* \rightarrow \Xi\pi),$$

and to first order, the symmetry-breaking perturbation gives (4)

$$(2) \quad (\mathcal{N}^* \rightarrow \mathcal{N}\pi) + \sqrt{2} (\Xi^* \rightarrow \Xi\pi) = (-3/\sqrt{2}) (Y^* \rightarrow \Lambda\pi) - \frac{\sqrt{3}}{2} (Y^* \rightarrow \Sigma\pi).$$

The symmetry-breaking is described by a spurion which transforms as a member of $(8, 1) + (1, 8)$ multiplet under the $U_3 \times U_3$ symmetry and as an $I=0$, $Y=0$ member of an octet under SU_3 . There is only one invariant coupling for the baryon-resonance decays in exact $U_3 \times U_3$ (5):

$$\bar{D}^{ab\bar{c}} B_{a\bar{d}\bar{c}} M_b^d + \bar{D}^{ab\bar{c}} B_{a\bar{b}\bar{d}} \bar{M}_{\bar{c}}^{\bar{d}}.$$

(*) The research reported in this document has been sponsored in part by the Air Force Office of Scientific Research OAR through the European Office Aerospace Research U. S. Air Force.

(1) W. RÜHL: *Nuovo Cimento*, **37**, 301 (1965); J. M. CHARAP and P. T. MATTHEWS: *Proc. Roy. Soc.*, **A 289**, 300 (1965); T. FULTON and J. WESS: *Phys. Lett.*, **14**, 334 (1965).

(2) H. RUEGG and D. V. VOLKOV: to be published in *Nuovo Cimento*; F. BUCCELLA and R. GATTO: *Nuovo Cimento*, **40**, 684 (1965).

(3) Exhaustive references for $\tilde{U}_{12}$ and SU_6 are to be found in A. SALAM, R. DELBOURGO, M. A. RASHID and J. STRATHDEE in *High Energy Physics and Elementary Particles* published by IAEA, Vienna.

(4) V. GUPTA and V. SINGH: *Phys. Rev.*, **135**, B 1442 (1964); M. KONUMA and Y. TOMOZAWA: *Phys. Lett.*, **10**, 347 (1964); C. BECCHI, E. EBERLE and G. MORPURGO: *Phys. Rev.*, **136**, B 808 (1964).

(5) The notation used has been explicitly given by RUEGG and VOLKOV in ref. (2).

Because of the assumed behaviour of the symmetry-breaking interaction, only the $(6, 3) + (3, 6)$ members of the decuplet contribute to the amplitude. By introducing the spurion S , the $U_3 \times U_3$ -invariant amplitude is constructed for the process:

$$S + D \rightarrow B + M.$$

The four decay processes considered here are found to be expressible in terms of two independent linear combinations of the invariant amplitudes ⁽⁶⁾. There then follows the relations:

$$(3) \quad Y^* \rightarrow \Lambda \pi = -(\Xi^* \rightarrow \Xi \pi)$$

and

$$(4) \quad 2(N^* \rightarrow N \pi) + \sqrt{2}(Y^* \rightarrow \Lambda \pi) = \sqrt{3}(Y^* \rightarrow \Sigma \pi).$$

Equation (3) is similar to that obtained under exact SU_3 symmetry and when combined with eq. (4), it produces the well-known SU_3 -breaking relation (2). It is interesting to note that similar calculations carried out by introducing broken SU_3 into $\bar{U}_{12}$, SU_{6w} or SU_{6s} have produced different results, but in each case one of the three SU_3 -symmetric results of eq. (1) is retained ⁽⁷⁾.

For these decay processes, the reduced widths Λ and the total decay widths are related by

$$\Gamma = |\Lambda|^2 p^3 \frac{M_B}{M_{B^*}},$$

where M_B , M_{B^*} are the masses of the baryons and decaying resonances respectively, Γ is the decay width and P , the final-state momentum. With the experimental data available at present ⁽⁸⁾, eq. (3) shows a slight improvement on previous results ⁽⁷⁾. A triangular relation for the reduced widths is then obtained:

$$(5) \quad 2\Lambda(N^* \rightarrow N \pi) = \sqrt{2} \Lambda(Y^* \rightarrow \Lambda \pi) + \sqrt{3} \Lambda(Y^* \rightarrow \Sigma \pi).$$

Present data give a value of 0.72 for the l.h.s. of this expression and at most (0.61 ± 0.06) for the r.h.s.

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⁽⁶⁾ These are similar to the one constructed for meson-baryon scattering in ref. ⁽¹⁾.

⁽⁷⁾ V. DE ALFARO and Y. TOMOZAWA: *Phys. Rev.*, **138**, B 1193 (1965); R. FERRARI and M. KONUMA: *Phys. Rev. Lett.*, **14**, 378 (1965); H. HARARI, D. HORN, M. KUGLER, H. J. LIPKIN and S. MESHKOV: *Phys. Rev.*, **140**, B 1003 (1965); P. BABU: *Nuovo Cimento*, **40**, 261 (1965).

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Representation Mixing in $\tilde{U}_{12}$ (*).

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Summary. — Some consequences of the assumption that the physical baryons belong to some mixed representations of $U_{6,6}$ are studied in the context of the 3-point, electromagnetic and weak vertices and the 4-point Johnson-Treiman relations.

Introduction.

The work of ADLER and WEISBERGER has prompted a number of authors to look at some possible sort of representation mixing in $U_3 \otimes U_3$ and SU_{6W} . The various investigations ⁽¹⁻³⁾ so far, have taken place in the context of the algebra-of-currents component. In what follows, we shall look at the problem in $\tilde{U}_{12}$ symmetry. We assume that the physical baryons belong to some mixed states of the 20^+ , 56^+ , and 70^+ irreducible representations of ⁽⁴⁾ $U_6 \otimes U_6$, each

(*) The research reported in this document has been sponsored in part by the Air Force Office of Scientific Research OAR through the European Office Aerospace Research, United States Air Force.

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(1) R. GATTO, L. MAIANI and G. PREPARATA: *Phys. Rev. Lett.*, **16**, 377 (1966).
R. GATTO and L. MAIANI: preprint, Florence.

(2) H. HARARI: *Phys. Rev. Lett.*, **16**, 964 (1966).

(3) N. CABIBBO and H. RUEGG: preprint CERN, Geneva.

(4) This assumption breaks $U_6 \otimes U_6$ symmetry since the physical baryon tensor is no longer irreducible in $U_6 \otimes U_6$ space. By 20^+ we mean $(20, L=0)^+$. We restrict our considerations to $L=0$ positive-parity states, since the procedure for constructing positive-parity baryon states with odd integer l is not yet clearly understood in U_{12} terms. It may require the inclusion of quarks of the second type, *viz.* (1, 6).

of which contains the octet spin $\frac{1}{2}^+$ baryons. We investigate the consequence of this assumption on the weak and electromagnetic 3-point vertex ($\bar{B}BM$) and on the Johnson-Tireman relations for the four-point vertex ($\bar{B}BMM$). We shall merely discuss our results as the technique for the calculations is well known.

1. - The 3-point vertex ($\bar{B}BM$).

We assume that the physical baryon wave function $|B\rangle$ can be written as

$$(1) \quad |B\rangle = \alpha|20\rangle + \beta|56\rangle + \gamma|70\rangle,$$

where α, β, γ are mixing parameters and

$$\alpha^2 + \beta^2 + \gamma^2 = 1.$$

The contributions to the F/D and G_A/G_V ratios from the various irreducible representations contained in $|B\rangle$ are given in Table I. From this Table we see

TABLE I. - The different F/D and G_A/G_V ratios for the irreducible representations considered. The calculations are based on the expression for the multiplets given by DELBOURGO *et al.* ⁽⁵⁾. It is worth pointing out that the magnetic moment part of the vector current has the same F/D ratio as the pseudoscalar and axial vector currents.

Current for each I.R.	F/D for magnetic moment part of vector current (γ_μ)	F/D for charge part of vector current (P_μ)	F/D for pseudo-scalar and axial vector current $\gamma_5, i\gamma_\mu\gamma_5$	G_A/G_V
$\langle 56 J 56\rangle$	$3D + 2F$	$3F$	$3D + 2F$	$5/3$
$\langle 20 J 20\rangle$	$-D$	F	$-D$	-1
$\langle 70 J 70\rangle_I$		F	$-D$	-1
$\langle 70 J 70\rangle_{II}$	$\frac{1}{2}(D + F)$	F	$\frac{1}{2}(D + F)$	1
$\langle 70 J 56\rangle$	$-(3D + 5F)$	$D + \frac{1}{3}F$	$-(3D + 5F)$	$-$
$\langle 70 J 20\rangle$	$D - F$	$-(3D + F)$	$D - F$	$-$

⁽⁵⁾ DELBOURGO, *et al.*: *The $\tilde{U}_{12}$ Symmetry, in High-Energy Physics and Elementary Particles*, I.A.E.A., Vienna.

that the charge part of the vector current ⁽⁶⁾ for the cross terms $\langle 70|J|56 \rangle$ and $\langle 70|J|20 \rangle$ are not pure F . Thus the experimental and theoretical requirement that the charge of the neutron be zero (*i.e.* $\mu_s(n)=0$), when applied to these terms gives either

$$(2) \quad \gamma = 0,$$

which corresponds to a mixing of the 56^+ and the 20^+ only, or

$$(3) \quad \alpha = k\beta, \quad k \text{ constant},$$

which corresponds to a mixing of the 3 irreducible representations.

Calculations with (2) or (3) yield identical result, to wit:

$$(4) \quad \frac{G_A}{G_V} = \frac{D+F}{3F-D}.$$

This is in agreement with the $U_3 \otimes U_3$ results obtained by CABIBBO and RUEGG ⁽³⁾ from a consideration of the mixing of the $(56, L=0)^+$ and the $(20, L=0)^+$ and of the $(56, L=0)^+$ and the $(70, L=0)^-$ in the context of current algebra.

We notice that the inclusion of the 70 representation does not give any new result. However, it is satisfying that with a mixture of 3 representations, we do come out with a definite prediction. This is not always possible as pointed out by CABIBBO and RUEGG ^(3,7).

The prediction (4) is bad compared with recent experiments and with the predictions of HARARI ⁽²⁾ and GATTO *et al.* ⁽¹⁾. They obtained a value

$$\frac{G_A}{G_V} = \frac{D+F}{3(D-F)},$$

by considering $(56, L=0)^+$ and $(20, L=1)^-$. This is in excellent agreement with experimental value of 1.2 for G_A/G_V and 1.75 for D/F . If in (4) we use $D/F=1.75$, we get $G_A/G_V=2.2$. Alternatively, if $G_A/G_V=1.2$, $D/F=1.18$. However, if $D/F=\frac{3}{2}$ which corresponds to $\mu(p)/\mu(n)=-\frac{3}{2}$, a good experimental result, then (4) gives $G_A/G_V=-\frac{5}{3}$, the old SU_6 result.

Thus if we assume (1), then the requirements that

$$\begin{aligned} \text{i)} \quad & \mu_s(n) = 0, \\ \text{ii)} \quad & \frac{\mu(p)}{\mu(n)} = -\frac{3}{2}, \end{aligned}$$

⁽⁶⁾ In current algebra approach there are no cross terms as the current generator G takes an irreducible representation R into itself, *i.e.* $G|R\rangle = |R\rangle$.

⁽⁷⁾ We have a definite prediction here probably because we restrict ourselves to $L=0$, and are assuming a higher symmetry for the 3-point M -function.

automatically fix the ratio G_A/G_V to $-\frac{5}{3}$. No more, no less. So far, it appears that only the mixture of the $(56, L=0)^+$ with the $(20, L=1)^-$ can really depress $\frac{5}{3}$ to the experimental value 1.2 ⁽¹⁾.

2. - The 4-point vertex ($\bar{B}BMM$).

With the assumption (1) we shall consider the Johnson-Treiman type of reactions for the 4-point vertex. The number of amplitudes are greatly reduced for such processes when the crossed channel is used. As pointed out by LIPKIN ⁽⁸⁾, the difference relations for two different meson-baryon scattering reactions are expressible in the crossed channel in terms of an antisymmetric two-meson coupling. To make the calculations simpler we use the collinear SU_{6w} subgroup. The contributions to the amplitudes come from the following SU_{6w} scalars:

$$(\alpha^2 \bar{20} \otimes 20 + \beta^2 \bar{56} \otimes 56 + \gamma^2 \bar{70} \otimes 70_i + \gamma'^2 \bar{70} \otimes 70_{ii} + \\ + \alpha\gamma \bar{20} \otimes 70 + \beta\gamma \bar{56} \otimes 70) \times (35(K) \otimes 35'(-K) - 35(-K) \otimes 35'(K)),$$

written apart from some coupling constants. Direct calculations show that the first four amplitudes, containing no cross terms, separately satisfy the Johnson-Treiman relations. KURIYAN and SUDARSHAN ⁽⁹⁾ have also obtained similar results using noninvariance group. The last two amplitudes, containing cross terms, do not, however, give the Johnson-Treiman relations. Thus if we choose our parameters (5) in such a way that the cross terms drop out, the Johnson-Treiman relations are reproduced unambiguously. In fact, such a choice of parameters will give relations of the type (2) and (3). It is also worth observing that since only the 70^+ representation introduces the cross terms in the amplitude, the Johnson-Treiman relations are perfectly satisfied by a mixing of the 56^+ and the 20^+ irreducible representations only.

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⁽⁸⁾ H. LIPKIN: *Now we are SU_6* , in *High-Energy Physics and Elementary Particles*, I.A.E.A., Vienna.

⁽⁹⁾ J. G. KURIYAN and E. C. G. SUDARSHAN: *Phys. Rev. Lett.*, **16**, 825 (1966).

RIASSUNTO (*)

Nel quadro dei vertici debole ed elettromagnetico a tre punti e delle relazioni di Johnson-Treiman a quattro punti, si studiano alcune conseguenze dell'ipotesi che i barioni fisici appartengano a qualche rappresentazione mista di $U_{6,6}$.

(*) Traduzione a cura della Redazione.

Смешанное представление в $\tilde{U}_{12}$.

Резюме (*). — Исследуются некоторые последствия предположения, что физические барионы принадлежат к некоторым смешанным представлениям $U_{6,6}$, в связи с 3-точечными электромагнитными и слабыми вершинами, и с четырёх точечными соотношениями Джонсона-Треймана.

(*) Переведено редакцией.