INSURANCE , STORAGE AND POINT PROCESSES : AN APPROACH VIA PIECEWISE DETERMINISTIC MARKOV PROCESSES

by

Angelos Dassios

Thesis submitted for the

degree of

Doctor of Philosophy in the University of London

April 1987

ABSTRACT

Piecewise deterministic processes theory is a powerful mathematical tool for treating probabilistic models analytically. It is due to and developed by Professor M.H.A. Davis (1984). Virtually all non diffusion models can be encompassed within this framework.

In this thesis we apply piecewise deterministic processes theory in order to treat analytically various models from insurance risk, queuing, storage, dam and point processes theories.

In insurance risk theory we examine various possible capital surplus models. The questions asked are associated with probabilities of ruin for both the finite and infinite horizon cases. Also simple optimization problems are dealt with.

A new approach to queuing, storage and dam theories is provided. The main processes examined are the virtual waiting time of the G/G/1 queue, the shot noise process, the $M/G/\infty$ queue and various generalizations and modifications of them. Of interest are stationary distributions and hitting times.

Double stochastic point processes are another field where piecewise deterministic processes theory is seen to be useful. As "intensities" we consider processes having been examined in the thesis. Of interest are the joint distribution of the values of a doubly stochastic point process N_t at times $t_1, t_2, \ldots\ldots\ldots, t_n$ and/or the asymptotic behaviour of N_t. Furthermore we look at the distribution of point interarrival time intervals. Possible applications are also considered.

The methods used are based on the infinitessimal generator. We obtain appropriate integro-differential equations; we solve them and we find suitable martingales that lead us to the desirable results.

ACKNOWLEDGMENTS

I would like to thank my supervisor Dr. R. Grübel for his help. I am also deeply grateful to Dr. Paul Embrechts for his invaluable cooperation during the early stages of the preparation of this thesis. Special thanks to my fiancée for her immeasurable support.

CONTENTS

	Page
INTRODUCTION	1
CHAPTER 1	8
1.1 The Andersen model	8
1.2 Exponentially increasing income	13
1.3 Generalizations	19
1.4 Borrowing-Absolute ruin	23
1.5 The classical model with investment	27
1.6 Continuous path models	30
CHAPTER 2 QUEUES & RELATED MODELS	36
2.1 The G/G/1 queue	36
2.2 General decay rule	46
2.3 The shot-noise process	56
2.4 Time dependent parameters	66
2.5 Shot noise process with a renewal process as underlying jump arrival process	69
2.6 Continuous path models	77
2.7 The M/G/∞ queue	82
CHAPTER 3 DOUBLY STOCHASTIC POINT PROCESSES AGGREGATED PROCESSES	87
3.1 General	87
3.2 A piecewise continuous intensity	91
3.3 The virtual waiting time of the M/G/1 queue as intensity	100
3.4 Shot noise intensity	105
3.5 Shot noise intensity with generally distributed jump interarrival time intervals	113
3.6 Insurance applications	116
3.7 "Multiple" shot noise processes	120

CHAPTER 4 126

4.1 Simple dividend problems 126

4.2 An alternative model 134

4.3 The Tapiero Zuckerman model 138

4.4 An alternative model 146

LIST OF REFERENCES 153

INTRODUCTION

The mathematical analysis of continuous time stochastic processes has been an object of study for applied probabilists for many years. Martingale methods are a particularly powerful mathematical tool employed by them to analyze fairly sophisticated models. Enormous success has been achieved over the years as far as diffusion process models are concerned. The development of stochastic (Ito) calculus has advanced the subject very much (see Chung & Williams (1983)). However, their application is restricted to diffusion processes. An important "unifying" in the same sense attempt to embrace apparently all non diffusion models and to develop a mathematical tool to treat them systematically is the paper by M.H.A.Davis (1984) on piecewise deterministic Markov processes.

Essentially a piecewise deterministic process is a process which includes no diffusion element. So it is a fairly general concept. The paper by Davis formulates and defines such a process and an operator associated with it its generator which will help us to obtain martingales and thus prove results about specific models. Yet, the paper does not only provide us with a tool to treat apparently different models systematically but with a unifying approach to them as well. This thesis is on how to exploit this tool and this approach. By using the theory provided by Davis we will illustrate how to treat models from diverse fields such as risk, dam and queuing theory. This is by no means the first attempt for a unifying approach. N.U.Prabhu in his book (1980) provides a unifying approach based on *Wiener-Hopf factorization* techniques mainly. As Prabhu points out in his introduction these models have similar mathematical processes and gives them the common title "storage processes". This thesis will also make these similarities to seem apparent (e.g. the duality of the models in sections 1.1 and 2.1; these are the Andersen surplus process in risk theory and the virtual waiting time of the G/G/1 queue). Having said that, it should be stressed that these similarities are far from making the models identical as the questions

to be asked for different models will be of a different nature.

In the first chapter we will apply piecewise deterministic processes in insurance mathematics, examining the problem of *ruin* for models which are either new or not very thoroughly examined in the literature. The use of martingale methods in risk theory has been very sparse over the years. Gerber is probably pioneering their use in his book (1979; chapters 3 and 9) and a paper on a quite sophisticated model formulated in a time series framework (1982). The almost total absence of the use of martingale methods in the literature is rather surprising as the idea of "fair game" involved in the concept of a martingale is the "bread and butter" of risk theory (Gerber (1979)p.34). Chapter 1 of this thesis might be seen as a continuation or rather an extension of my M.Sc. thesis (1984); see also Dassios & Embrechts (1987). Yet, the models examined are far more sophisticated. Most of the results found are new. Recently, work is being done by Delbaen & Haezendonck (see 1986 & 1987).

The relationship to my M.Sc. thesis stops there. In chapter 2 we examine how piecewise deterministic process theory can be applied to answering questions about queues and storage processes. A few already known results are proved again in a quicker way and many new results are obtained. An important process examined is the shot noise process (see also Harrison & Resnick (1972) on which a lot of work has been done so we will also look at various generalizations which are not very often visited in the literature, at least not with the aim to find explicit results.

In chapter 3 we move to doubly stochastic (or conditionally Poisson) point processes. These are "Poisson" processes whose intensity is stochastic and in particular part of a piecewise deterministic Markov process. The need for the development of sophisticated point process models is apparent in many fields such as risk theory (see Seal (1983)) and rainfall models (see Cox & Isham (1986)). The assumption that claims (in insurance) and rains (in rainfall models) occur in terms of a (even non homogeneous) Poisson process is often inadequate

when faced with actual data. So alternative point processes have to be proposed and perhaps doubly stochastic point processes might prove themselves to be suitable. We will mostly use the processes of chapter 2, which will have been analyzed to some degree by then, as intensity processes. We will also propose possible applications, one of them in risk theory, obtaining again relevant results.

In the last chapter we will apply piecewise deterministic processes theory in optimization problems; the theory then appears to be particularly powerful. We will concentrate on insurance mathematics models proposed by others and we will treat them more in depth. However, we will avoid employing dynamic programming techniques; we will concentrate on evaluating appropriate quantities which will enable us to choose the optimum strategy. Additionally we will propose and analyze slightly modified alternatives for which more explicit results can be obtained.

We will now give a brief summary of Davis' paper so as to state the version of the theory we will apply in this thesis.

A piecewise deterministic process is a generalization of the concept of a *piecewise linear process* which is due to Gnedenko & Kovalenko (1966). We will give a definition adopted by Vermes (1980). A piecewise linear process is a process with two components (n_t,ζ_t). n_t is integer valued and ζ_t takes values from an interval $[a_n,b_n]$ when $n_t=n$ (b_n may be $+\infty$). Motion starts at (n,ζ) and as long as $n_t=n$, $\zeta_t=\zeta+t$ up to the first jump at $t=T_1$. Jumps occur spontaneously in terms of a Poisson process with rate $\lambda(n_t,\zeta_t)$. Also a jump occurs when ζ_t hits the end point b_{n_t} of its interval. The location of the process at T_i which denotes the time of the i^{th} jump is selected by a probability distribution $Q(\cdot,(n_{T_i-},\zeta_{T_i-}))$ so the process has right continuous paths. After a jump, motion resumes as before.

Davis generalizes this concept by assuming that $n_t \in K$ (a countable set), defining $d:K\to N$ to be a known function and setting M_n to be an open subset of $\mathbb{R}^{d(n)}$ for all $n\in K$.

Also define $E=\{(n,\zeta):n\in K,\zeta\in M_n\}$ and $\boldsymbol{E}$ to be a class of measurable subsets of E consisting of all $(n,L_n)=\{(n,\zeta):\zeta\in L_n\}$ such that L_n is M_n-Borel measurable. Thus $(E,\boldsymbol{E})$ is a Borel space. The process $x_t=(n_t,\zeta_t)$ is now determined by

i) Vector fields X_n, $n\in K$ such that for each $\zeta\in M_n$ there exists a unique integral curve $\phi_n(t,\zeta)$ satisfying

$$\frac{d}{dt}f(\zeta_t) = X_n f(\zeta_t)$$

where f is a suitable (smooth) function from $\mathbb{R}^{d(n)}$ to $\mathbb{R}$. Furthermore the integral curve has to be defined for all t (no explosions).

ii) A measurable function $\lambda:E\to\mathbb{R}^+$ (jump intensity).

iii) A transition measure $Q:(\boldsymbol{E}\times E\cup\Gamma^*)\to[0,1]$.

∂M_n is the boundary of M_n and $\partial^* M_n$ the boundary points through which integral curves of X_n exit from M_n. Γ^* is defined as the union of all $\partial^* M_n$ ($n\in K$) and define also $t^*(n,\zeta)=\inf\{t:\phi_n(t,\zeta)\in\partial^* M_n\}$. Denote $X_n f(n,\cdot)(\zeta)$ by $Xf(n,\zeta)$ and (n,ζ) by x for convenience.

We suppose that for all $x\in E$ there exists $\epsilon>0$ such that the function $s\to\lambda(n,\phi_n(s,\zeta))$ is integrable for $s\in[0,\epsilon)$.

A transition measure Q(L,x) is a measurable function of x for all given $L\in\boldsymbol{E}$ and defined for each $x\in E\cup\Gamma^*$ as being a probability measure on $(E,\boldsymbol{E})$.

The process evolves as follows. Define a survivor function $\mathbf{F}(t)$ by

$$\mathbf{F}(t) = \begin{cases} \exp[-\int_0^t \lambda(n,\phi_n(s,\zeta))ds] & t<t^*(x) \\ 0 & t\geq t^*(x) \end{cases}$$

Select a random variable T_1 such that $Pr(T_1>t)=\mathbf{F}(t)$ and an independent E-valued random variable (N,Z) with distribution $Q(\cdot,\phi_n(T_1,\zeta))$. The trajectory of $x_t=(n_t,\zeta_t)$ is given by

$$(n_t,\zeta_t) = \begin{cases} (n,\phi_n(t,\zeta)) & t<T_1 \\ (N,Z) & t=T_1 \end{cases}$$

We then start from x_{T_1} and select the next inter-jump time T_2-T_1 and the next post-jump location x_{T_i} similarly and so on.

We also make one assumption. That is

$$E\{\Sigma_i I(T_i \leq t)\} < \infty \quad \text{for all } t$$

This implies $T_i \to \infty$ a.s. as $i \to \infty$.

Now look at the associated jump process. We define a filtration $\mathcal{F}_t = \sigma\{x_s, s \leq t\}$ $p(t,L) = \sum_{T_i \leq t} I(x_{T_i} \in L)$ and $p_t^* = \sum_{T_i \leq t} I(x_{T_i} \in \Gamma^*)$. Also define the process

$$\tilde{p}(t,L) = \int_0^t Q(L,x_s)\lambda(x_s)ds + \int_0^t Q(L,x_s)dp_s^*$$

where the second integral is defined in a Riemann-Stieltjes fashion (see Chung & Williams (1983)p.24-25).

Then $\tilde{p}(t,L)$ is the *compensator* of $p(t,L)$; that is $p(t,L)-\tilde{p}(t,L)$ is an $\mathcal{F}_t$-martingale (see proposition 4.3 of Davis (1984)).

Davis then proceeds to define and describe the properties of the <u>generator</u> (an extended version) of a piecewise deterministic Markov process. We will not give any proofs of results as all can be found in Davis, paper where also extensions covering *local martingales* and associated concepts are developed. We will just give the results (slightly modified as a little weaker versions) that are going to be applied in this thesis.

The <u>generator</u> A of the process is an operator of functions $f: E \cup \Gamma^* \to \mathbb{R}$ and its *domain* contains functions f satisfying the following requirements (see theorem 5.5 of the paper)

(i) For all $(n,\zeta) \in E \cup \Gamma^*$ the function $t \to f(n,\phi_n(t,\zeta))$ is absolutely continuous for $t \in [0,t^*(n,\zeta))$

(ii) $f(x) = \int_E f(y)Q(dy,x)$ for $x \in \Gamma^*$

(iii) $E\{\sum_{T_i \leq t} |f(x_{T_i}) - f(x_{T_i-})|\} < \infty$ (the jumps in f should not be too large!)

Condition (iii) is weaker than the original condition given in theorem 5.5 of Davis (1984) and it is trivially satisfied if f is bounded in view of the assumption made before.

For the functions we use in this thesis the first requirement is usually

trivially satisfied and we will omit the relevant investigation. The second requirement is always investigated when not meaningless (it is meaningless when Γ^* is empty). The third requirement is trivially satisfied in most cases either because the chosen function is bounded or because it is otherwise clear that the expectation is finite and in such cases we will omit the relevant investigation. However, when things are not clear then we will explain why the chosen function satisfies (iii).

Now, the most important fact is that if f belongs to the domain of A, then

$$f(x_t) - \int_o^t Af(x_s)ds$$

is an $\mathcal{F}_t$-martingale.

The generator A acting on a function f belonging to its domain is given by

$$Af(x) = Xf(x) + \lambda(x)\{\int_E [f(y)-f(x)]Q(dy,x)\}$$

and if we want to include time t explicitly as a component of our process, then the augmented generator is given by

$$\frac{\partial}{\partial t}f(x,t) + Xf(x,t) + \lambda(x,t)\{\int_E [f(y,t)-f(x,t)]Q(dy,(x,t))\}$$

If we manage to find a function f such that Af=0 then $f(x_t)$ is an $\mathcal{F}_t$-martingale.

In all applications we will use as $\mathcal{F}_t$ the *natural* filtration of the probability space each time and we will use the term **martingale** refering to a martingale with respect to the natural filtration.

By now, it should be clear that solving the equation Af=0 for various circumstances will be important in our thesis. We will solve the equation for many apparently different models. In most situations this will be the appropriate equation to solve. However, in some models this could be inadequate and we will then resort to other equations (e.g. Af(x)=h(x) where h is a known function).

In order to examine some processes we might have to include other processes in the formulation of our model so as to either make the model Markov or directly involve quantities of interest in our calculations.

A summary of our technique is the following:

1) We first formulate our model as a piecewise deterministic Markov process including additional variables as appropriate.

2) We then state the equation $Af=0$ (or any other suitable equation) and look for solutions.

3) We find the solutions that will better suit our needs (it is by no means necessary to find all solutions; other solutions either lead nowhere near what we seek to find or they lead to the same results).

4) We use the martingale or martingales found to establish results. Of great importance for this purpose is Doob's optional stopping theorem.

We conclude our introduction by stating a version of this important theorem. It is taken from Karlin & Taylor (1975) (p.320)

Theorem *Let* $\{X_t,\ t\geq 0\}$ *be a martingale (with respect to a suitable filtration) and* τ *a stopping time. If* $\Pr(\tau<\infty) = 1$ *and* $E\{\sup|X_t|\}<\infty$ *(uniform integrability) then* $E\{X_0\}=E\{X_\tau\}$ *and in case* X_0 *is given deterministically* $X_0=E\{X_\tau\}$. Pr *denotes the restriction of our probability measure on* $\mathcal{F}_0$ *and* E *denotes* $E\{\cdot\,|\mathcal{F}_0\}$.

A stopping time (or optional time or Markov time by some authors) is a map from Ω to $[0,\infty)$ such that the event $\{\tau\geq t\}$ is $\mathcal{F}_t$-measurable for all t. An equivalent statement is that $\{\tau<t\}$ is $\mathcal{F}_t$-measurable.

1.1 THE ANDERSEN MODEL

We are visiting in this section the Andersen model which is well known in insurance mathematics . It is due to Sparre Andersen (1957). This model has also been examined by Thorin, using a different approach than the one we will adopt (see Thorin (1982) and all references to the same author given therein). We assume a company receives premium income at a constant rate c. Claims arrive in terms of a renewal counting process N_t; that is the claim interarrival time intervals are independent identically distributed random variables. P(y) is assumed to be their distribution function ($y \in \mathbb{R}^+$). The claim sizes are independent identically distributed random variables Y_i (i=1,2,.....) which are independent of the claim arrival process as well. Their distribution function is G(y), ($y \in \mathbb{R}^+$), and we also assume that

$$\phi(\nu) = \hat{g}(\nu) = \int_{\mathbb{R}^+} e^{\nu y} dG(y)$$

exists on an interval $(0,\alpha)$ ($\alpha>0$). The surplus process is

$$\zeta_t = \zeta_0 + ct - \sum_{i=1}^{N_t} Y_i$$

where ζ_0 is the initial capital of the company. We also assume that

$$c\int_{\mathbb{R}^+} y dP(y) > \int_{\mathbb{R}^+} y dG(y) \quad \text{(net profit condition)}$$

When $P(y) = 1-e^{-\lambda y}$ (Poisson claim arrival process) we have the classical model.

We will also include in our piecewise deterministic process u_t, the time elapsed since the last claim arrived (so u_t increases with unit rate till a claim arrives; then it is set back to 0). We will also assume that P(y) posseses a derivative (density) p(y) ($y \in \mathbb{R}^+$).

Consider the piecewise deterministic process (ζ_t, u_t, t). Its generator acting on a function $f(\zeta,u,t)$ belonging to its domain is given by

$$Af(\zeta,u,t) = c\frac{\partial f}{\partial \zeta} + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial u} + \frac{p(u)}{\mathbf{P}(u)}\left(\int_{\mathbb{R}^+} f(\zeta-y,0,t)dG(y) - f(\zeta,u,t)\right) \qquad (1.1.1)$$

$$(\mathbf{P}(u)=1-P(u))$$

We will also define $\tau \equiv \inf(t : \zeta_t<0)$. If $\zeta_t \geq 0$ for all $t>0$ then $\tau\equiv\infty$. Our

aim is to find expressions involving $\Pr(\tau \leq t)$ (finite horizon ruin probability) and $\Pr(\tau < \infty)$ (infinite horizon problem). We start with a technical lemma.

Lemma 1.1.1 *Let $P(y)$, $G(y)$ be probability measures ($y \in \mathbb{R}^+$), and*

$$\phi(\nu) = \hat{g}(-\nu) = \int_{\mathbb{R}^+} e^{\nu y} dG(y), \quad \hat{p}(\nu) = \int_{\mathbb{R}^+} e^{-\nu y} dP(y)$$

Assume also $\exists\ \alpha \in \mathbb{R}^+ \cup \{\infty\}$, s.t. $\varphi(y) \to \infty$ as $y \uparrow \alpha$

then for all $\theta > 0$ there exists a unique $\nu_\theta > 0$ such that

$$\phi(\nu)\hat{p}(\theta+c\nu) = 1 \tag{1.1.2}$$

is satisfied.

Furthermore if $c\int_{\mathbb{R}^+} y dP(y) > \int_{\mathbb{R}^+} y dG(y)$ (the net profit condition), then for $\theta > 0$ there exists a unique $\nu_0 > 0$ that satisfies (1.1.2) as well ($\phi(\nu_0)\hat{p}(c\nu_0) = 1$).

Proof

$\phi(\nu)\hat{p}(\theta+c\nu)$ is a convex function of ν since its second derivative is

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} (x-cy)^2 e^{\nu x-(\theta+c\nu)y} dG(x)dP(y) > 0$$

so since for $\nu=0$ $\phi(\nu)\hat{p}(\theta+c\nu)=\hat{p}(\theta)<1$, there exists a unique $\nu_\theta>0$ such that (1.1.2) is satisfied.

For $\theta=0$ $\phi(\nu)\hat{p}(\theta+c\nu)=0$ at $\nu=0$ but the first derivative at 0 is negative by the net profit condition, so there again exists a unique ν_0 satisfying (1.1.2).

□

We will now find a martingale in order to solve the problem of ruin.

Theorem 1.1.2 *Let ζ_t, u_t as defined; then*

$$\exp(-\nu_\theta \zeta_t)\exp(-\theta t)\frac{\exp[(\theta+c\nu)u_t]}{P(u_t)}\int_{u_t}^{\infty} \exp[-(\theta+c\nu_\theta)x]p(x)dx \tag{1.1.3}$$

is a martingale for all $\theta \geq 0$, where ν_θ is defined by (1.1.2)

Proof

To find a martingale of the form $f(\zeta_t, u_t, t)$ we have to solve

$$c\frac{\partial f}{\partial \zeta} + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial u} + \frac{p(u)}{P(u)}\left(\int_{\mathbb{R}^+} f(\zeta-y,0,t)dG(y) - f(\zeta,u,t)\right) = 0 \tag{1.1.4}$$

trying a solution of the form $e^{-\theta t}e^{-\nu\zeta}h(u)$ we get

$$-\theta h(u)-c\nu h(u)+h'(u)+\frac{p(u)}{\mathbf{P}(u)}(\phi(\nu)h(0)-h(u))=0 \qquad (1.1.5)$$

and solving (1.1.5)

$$h(u)=\frac{h(0)\exp[(\theta+c\nu)u]}{\mathbf{P}(u)}\left[\phi(\nu)\int_u^\infty p(x)\exp[-(\theta+c\nu)x]dx+[1-\phi(\nu)\hat{p}(\theta+c\nu)]\right] \qquad (1.1.6)$$

Now for $e^{-\nu\zeta}e^{-\theta t}h(u)$ to belong to the domain of the generator we should make the second term in the brackets in (1.1.6) to vanish, as it will otherwise violate the third requirement for f to belong to the domain of the generator (see introduction). For that purpose we choose $\nu=\nu_\theta$;then

$$h(u)=\frac{h(0)\exp[(\theta+c\nu_\theta)u]}{\mathbf{P}(u)}\phi(\nu_\theta)\int_u^\infty p(x)\exp[-(\theta+c\nu_\theta)x]dx \qquad (1.1.7)$$

$h(u)\leq h(0)\phi(\nu)$ so it is bounded ; therefore in order to verify that

$$E\left[\sum_{T_i\leq t}|f(\mathbf{x}_{T_i})-f(\mathbf{x}_{T_{i^-}})|\right]<\infty$$

for all $t\geq 0$, it suffices to show that if Y has distribution function G(y) , then $E[\exp(\nu_\theta Y)]<\infty$ which is true since $\phi(\nu_\theta)$ exists. The other requirements are trivially satisfied, so the theorem has been proved

□

So we can now apply the martingale we found. $\tau\wedge t \equiv \min(\tau,t)$ is a stopping time so applying Doob's optional stopping theorem, (notice that $u_\tau=0$)

$$\exp(-\nu_\theta\zeta_o)\frac{\exp[(\theta+c\nu_\theta)u_o]}{\mathbf{P}(u_o)}\phi(\nu_\theta)\int_{u_o}^\infty p(x)\exp[-(\theta+c\nu_\theta)x]dx=$$

$$E[\exp(-\nu_\theta\zeta_\tau-\theta\tau)\,|\,u_o,\tau\leq t]\Pr(\tau\leq t\,|\,u_o)+$$

$$E[\exp(-\nu_\theta\zeta_t)\exp(-\theta t)\phi(\nu_\theta)\frac{\exp[(\theta+c\nu)u_t]}{\mathbf{P}(u_t)}\int_{u_t}^\infty\exp[-(\theta+c\nu_\theta)x]\;p(x)dx\,|\,\tau>t,u_o]\Pr(\tau>t\,|\,u_o) \qquad (1.1.8)$$

Setting $\theta=0$ in (1.1.8) we get

$$\exp(-\nu_o\zeta_o)\,\phi(\nu_o)\int_{u_o}^\infty p(x)\exp[-c\nu_o x]dx\;/\mathbf{P}(u_o)=$$

$$E[\exp(-\nu_o\zeta_\tau)\,|\,\tau\leq t,u_o]\Pr(\tau\leq t\,|\,u_o)\;+$$

$$E\{\exp[-\nu_0\zeta_t]\ \phi(\nu_0)\int_{u_t}^{\infty}p(x)\exp[-c\nu_0 x]dx\ /\mathbf{P}(u_t)\,|\,\tau>t,u_0\}\Pr(\tau>t\,|\,u_0) \qquad (1.1.9)$$

Letting $t \to \infty$, $\zeta_t \to \infty$ a.s. because of the net profit condition (a direct application of the strong law of large numbers). So $\zeta_t \to \infty$ on $\{\tau=\infty\}$ save on a set of zero probability and (1.1.9) yields

$$\exp(-\nu_0\zeta_0)\ \phi(\nu_0)\int_{u_0}^{\infty}p(x)\exp[-c\nu_0 x]dx\ /\mathbf{P}(u_0) = E[\exp(-\nu_0\zeta_\tau)\,|\,\tau<\infty,u_0]\Pr(\tau<\infty\,|\,u_0) \qquad (1.1.10)$$

and thus

$$\Pr(\tau<\infty\,|\,u_0) = \exp(-\nu_0\zeta_0)\ \phi(\nu_0)\int_{u_0}^{\infty}p(x)\exp[-c\nu_0 x]dx\ /\{\mathbf{P}(u_0)\,E[\exp(-\nu_0\zeta_\tau)\,|\,\tau<\infty,u_0]\} \qquad (1.1.11)$$

(1.1.11) is the generalization of the formula for the classical model (see Gerber (1979) p.134, Dassios (1984) p.17)

From (1.1.8) we also get

$$\exp(-\nu_\theta\zeta_0)\frac{\exp[(\theta+c\nu_\theta)u_0]}{\mathbf{P}(u_0)}\ \phi(\nu_\theta)\int_{u_0}^{\infty}p(x)\exp[-(\theta+c\nu_\theta)x]dx \geq$$

$$E[\exp(-\nu_\theta\zeta_\tau-\theta\tau)\,|\,u_0,\tau\leq t]\Pr(\tau\leq t\,|\,u_0) \qquad \geq$$

$$\exp(-\theta t)E\{\exp(-\nu_\theta\zeta_\tau)\,|\,\tau\leq t,u_0\}\Pr(\tau\leq t\,|\,u_0) \qquad (1.1.12)$$

for all $\theta \geq 0$. So from (1.1.12) we get

$$\Pr(\tau\leq t\,|\,u_0) \leq \inf_{\theta\geq 0}\{\exp(\theta t-\nu_\theta\zeta_0)\frac{\exp[(\theta+c\nu_\theta)u_0]}{\mathbf{P}(u_0)}\ \phi(\nu_\theta)\int_{u_0}^{\infty}p(x)\exp[-(\theta+c\nu_\theta)x]dx\} \qquad (1.1.13)$$

(1.1.13) is rather sharp as the second term in (1.1.8) decreases exponentially with t. A sharp inequality might be better than many "explicit" formulas involving inversions of complicated Laplace transforms. This inequality is known to perform particularly well in the classical case at least for which it has also been obtained by Gerber (1979) (p.139).

All results and derivations up to this point can also be found in Dassios (1984) and Dassios & Embrechts (1987); in the latter for a slightly more general model. The reason for including them in this thesis is that the method used in the proof of theorem 1.1.2 and the idea of adding a supplementary variable (in this case the time elapsed since the last claim) will be used extensively throughout this thesis. The idea of adding a supplementary variable in order to

make the process Markov is not new. For a reference see Cox (1955).

We will also look at a new alternative method to solve the problem. We replace u_t, the time elapsed since the last claim, with v_t, the time needed for the next claim to arrive. v_t decreases with unit rate till it reaches 0; at this time a claim is assumed to occur, ζ_t moves to $\zeta_{t-}+Y_i$ and the post-jump location of v_t is a random variable independent of the past whose distribution function is P(y). There is no need for P(y) to posses a density any more and we drop this assumption. The generator of the piece wise deterministic process (ζ_t, v_t, t) acting on a function $f(\zeta,v,t)$ belonging to its domain is given by

$$Af(\zeta,v,t) = c\frac{\partial f}{\partial \zeta} + \frac{\partial f}{\partial t} - \frac{\partial f}{\partial v} \qquad (1.1.14)$$

The set of boundary points that can be reached via deterministic motion is $\mathbb{R}\times\{0\}\times\mathbb{R}^+$ and for f to belong to the domain of the generator the following requirement has to be fulfilled (requirement iii in the introduction):

$$f(\zeta,0,t) = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f(\zeta-y,x,t)dG(y)dP(x) \qquad (1.1.15)$$

We can now prove the following martingale theorem

Theorem 1.1.3 *Let* ζ_t, v_t *as defined,* $\theta\geq 0$ *and* ν_θ *as defined by lemma 1.1.1,then* $\exp(-\theta t)\exp(-\nu_\theta\zeta_t)\exp[-(\theta+c\nu_\theta)v_t]$ *is a martingale*

Proof

We try a solution of the form $\exp[-\theta t-\nu\zeta-(\theta+c\nu)v]$ which clearly satisfies Af=0; it remains to be seen for which ν,θ (1.1.15) is satisfied as well. Substituting into (1.1.15) we get

$$\exp(-\theta t)\exp(-\nu_\theta\zeta)=\exp(-\theta t)\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\exp[-\nu_\theta(\zeta-y)]\exp[-(\theta+c\nu_\theta)y]dG(y)dP(x) \qquad (1.1.16)$$

leading to $1 = \phi(\nu)\hat{p}(\theta+c\nu)$, so $\nu=\nu_\theta$

$\exp(-\theta t)\exp(-\nu_\theta\zeta)\exp[-(\theta+c\nu_\theta)v]$ trivially satisfies the other requirements for it to belong to the domain of the generator by applying the same argument as in theorem 1.1.2 □

The martingale obtained in theorem 1.1.3 will lead to results equivalent to those already obtained as $E\{\exp[-(\theta+c\nu)v_t]\,|\,u_t\}=E\{\exp[-(\theta+c\nu)(X-u_t)]\,|\,X>u_t\}$ where X is a random variable with distribution P(y),so

$$E\{\exp[-(\theta+c\nu)v_t]\,|\,u_t\} = \frac{\exp[(\theta+c\nu)u_t]}{P(u_t)}\int_{u_t}^{\infty}\exp[-(\theta+c\nu_\theta)x]dP(x) \qquad (1.1.17)$$

An advantage of using v_t instead of u_t is that we do not have to assume a density for P(y). On the other hand while in most applications u_t is observed, v_t is not, making our process somewhat artificial. From a mathematical point of view using v_t is usually easier; yet there might be situations where the equations involved are impossible to solve. In specific situations we will use either u_t or v_t depending on the individual circumstances of the problem.

1.2 EXPONENTIALLY INCREASING INCOME

We are now going to examine an insurance model with non linear income of a specific form. We are assuming that the funds available to the company are invested yielding a rate of income $\delta\zeta_t$ where ζ_t represents the funds available at time t. No premiums are paid. So ζ_t increases deterministically with rate $\delta\zeta_t$; jumps (claims) occur with Poisson rate λ and the post-jump location is $\zeta_{t-}-Y_i$ if $Y_i \leq \zeta_{t-}$ and the process moves to the absorbing state 0 if $Y_i > \zeta_{t-}$. Y_i has as before distribution function G(y) supported on $\mathbb{R}^+$. Y_i's are independent between them and of the claim arrival process as well. The generator of the process (ζ_t,t) acting on a function $f(\zeta,t)$ belonging to its domain is given by

$$A_1 f = \frac{\partial f}{\partial t} + \delta\zeta\frac{\partial f}{\partial \zeta} + \lambda\{\int_{(0,\zeta)} f(\zeta-y,t)dG(y)+f(0)\mathbf{G}(\zeta)-f(\zeta,t)\} \qquad (1.2.1)$$

$$\zeta>0 \qquad \mathbf{G}(y)=1-G(y)$$

A_1 acting on $f(\zeta)$ (a function of ζ only) gives

$$A_1 f(\zeta) = \delta\zeta f'(\zeta) + \lambda\{\int_{(0,\zeta)} f(\zeta-y)dG(y) + f(0)\mathbf{G}(\zeta) - f(\zeta)\} \qquad (1.2.2)$$

Notice that we are not required to have $f(0+) = f(0)$ $\zeta>0$. We will now find a martingale.

Theorem 1.2.1 *Let ζ_t as defined,*

$$\hat{g}(\xi)=\int_{\mathbb{R}^+} e^{-\xi y}dG(y) \ , \ \hat{\mathbf{G}}(y)= \frac{1-\hat{g}(\xi)}{\xi}$$

and G^* a probability measure such that

$$\int_{\mathbb{R}^+} e^{-\xi y}dG^*(y) = \exp[-(\lambda/\delta)\int_0^\xi \hat{\mathbf{G}}(u)du]$$

then $G^*(\zeta_t)$ is a martingale

Proof

To find a martingale we have to solve

$$\delta\zeta f'(\zeta) + \lambda\{\int_{(0,\zeta)} f(\zeta-y)dG(y) + f(0)\mathbf{G}(\zeta) - f(\zeta)\} = 0 \qquad (1.2.3)$$

Applying Laplace transforms on (1.2.3), we get

$$-\delta\frac{d}{d\xi}[\hat{\xi f(\xi)}] + \lambda[\hat{f(\xi)g(\xi)}+f(0)\hat{\mathbf{G}(\xi)}-\hat{f(\xi)}] = 0 \qquad (1.2.4)$$

which can also been written as

$$\frac{d}{d\xi}\{\exp[(\lambda/\delta)\int_0^\xi \hat{\mathbf{G}}(u)du\,\hat{\xi f(\xi)}\} = (\lambda/\delta)f(0)\hat{\mathbf{G}(\xi)}\exp[(\lambda/\delta)\int_0^\xi \hat{\mathbf{G}}(u)du] \qquad (1.2.5)$$

and therefore

$$\hat{\xi f(\xi)}-f(0) = K\exp[-(\lambda/\delta)\int_0^\xi \hat{\mathbf{G}}(u)du] \qquad (1.2.6)$$

where K is an arbitrary constant, so

$$f(\zeta) = f(0) + KG^*(\zeta) \qquad (1.2.7)$$

is the solution of (1.2.3) and $f(\zeta_t)$ is a martingale and in particular $G^*(\zeta_t)$ is a martingale. (As it is bounded it clearly belongs to the domain of the generator)

□

$\exp[-(\lambda/\delta)\int_0^\xi \hat{\mathbf{G}}(u)du]$ is the Laplace transform of a probability measure. (see Feller (1969) p.441). So G^* depends on λ and δ via (λ/δ) only and if (λ/δ) is

fairly large (and in practice it is) we can with some justification pretend that it is an integer and interpret G^* as the distribution function of a sum of identically distributed random variables. Furthermore the Laplace transform of all infinitely divisible probability measures can be written in this form (see Feller (1969)p.450).

Now to use the martingale we found we must first prove a technical lemma about the behaviour of ζ_t as $t \to \infty$. We define the process $x_t = \zeta_0 - \zeta_t e^{-\delta t}$. x_t is the discounted sum of claims up to time t; that is $\Sigma\, Y_i \exp(-\delta T_i)$ for all i such that $T_i \leq t$, where T_i is the time of the i^{th} claim. So the events $\{\zeta_s > 0 \text{ all } s \in [0,t]\}$ and $\{x_t < \zeta_0\}$ are identical.

We define $\tau \equiv \inf\{\, t : \zeta_t = 0 \,\}$ ($\tau = \infty$ if $\zeta_t > 0$ for all t), so $\{\tau > t\}$ is equivalent to $\{x_t < \zeta_0\}$

Lemma 1.2.2 *$\zeta_t \to \infty$ on $\{\tau = \infty\}$ save on a set of zero probability*

Proof

$\{\tau = \infty\}$ and $\{x_t < \zeta_0$ all t $\}$ are equivalent. But $\{x_t < \zeta_0$ all t$\}$ implies $\sup x_t \leq \zeta_0$. Since x_t is a random sum whose terms have distributions with no point masses, (because of the T_i's), $\{\sup x_t = 0\}$ has zero probability and given $\tau = \infty$ $\sup x_t < \zeta_0$ with probability one. This means there exists $\epsilon > 0$ such that $\sup x_t \leq \zeta_0 - \epsilon$, so $x_t \leq \zeta_0 - \epsilon$ for all t , so $\zeta_0 - \zeta_t e^{-\delta t} \leq \zeta_0 - \epsilon$ for all t and $\zeta_t \geq \epsilon e^{\delta t}$,so $\zeta_t \to \infty$ with probability one.

□

We are now ready to use the martingale of theorem 1.1.1

Theorem 1.2.3 *Let G^* , ζ_t , τ as defined, then*

$$\Pr(\tau < \infty) = G^*(\zeta_0) \tag{1.2.8}$$

Proof

Since $G^*(\zeta_t)$ is a martingale ,

$$G^*(\zeta_0) = E\{\, G^*(\zeta_{\tau \wedge t})\} = 0 \cdot \Pr(\tau \leq t) + E\{\, G^*(\zeta_t) \,|\, \tau > t\} \Pr(\tau > t) \tag{1.2.9}$$

Letting $t \to \infty$ and in view of lemma 1.2.2

$$G^*(\zeta_0) = \Pr(\tau = \infty) \lim_{\zeta \to \infty} G^*(\zeta) = \Pr(\tau = \infty) \tag{1.2.10}$$

and the theorem is proved. □

This solves the infinite horizon problem. In particular when claims are exponentially distributed ($dG(y)=\alpha e^{-\alpha y}$), we have

$$\int_{\mathbb{R}^+} e^{-\xi y} dG^*(y) = (\alpha/\alpha+\xi)^{\lambda/\delta} \text{ , so } \Pr(\tau=\infty) = \int_0^{\zeta_0} \frac{\alpha^{\lambda/\delta} y^{(\lambda/\delta)-1} e^{-\alpha y}}{\Gamma(\lambda/\delta)} dy$$

(Gamma (λ/δ) distribution function)

Now to look at the finite horizon problem we will examine the process ζ_t^* which behaves as ζ_t up to the point it reaches 0 (if ever). Then ζ_t does not stop but it continues evolving taking negative values.

The generator of (ζ_t , t) acting on a function $f(\zeta,t)$ belonging to its domain is given by

$$A_2 f = \frac{\partial f}{\partial t} + \delta\zeta \frac{\partial f}{\partial \zeta} + \lambda\{\int_{\mathbb{R}^+} f(\zeta-y,t)dG(y) \cdots - f(\zeta,t)\} \qquad (1.2.11)$$

We now define $x_t^* = \zeta_0 - \zeta_t^* e^{-\delta t}$ $(\zeta_0^* = \zeta_0)$ as we did for ζ_t. The generator of (x_t^*, t) acting on $f(x,t)$ is

$$A_3 f = \frac{\partial f}{\partial t} + \lambda\{\int_{\mathbb{R}^+} f(x+ye^{-\delta t},t) \overbrace{dG(y)} - f(x,t)\} \qquad (1.2.12)$$

Theorem 1.2.4 *For all* $\nu > 0$,

$$\exp(-\nu x_t^*)\exp\{\lambda\int_0^t [1-\hat{g}(\nu e^{-\delta s})ds\} \quad \textit{is a martingale}$$

Proof

To find a martingale we have to solve

$$\frac{\partial f}{\partial t} + \lambda\{\int_{\mathbb{R}^+} f(x+ye^{-\delta t},t) \overbrace{dG(y)} - f(x,t)\} = 0 \qquad (1.2.13)$$

We will try a solution of the form $e^{-\nu x}h(t)$ $(\nu \geq 0)$. (1.2.13) can be written as

$$e^{-\nu x}h'(t) + \lambda e^{-\nu x}\{\int_{\mathbb{R}^+} \exp(-\nu y e^{-\delta t})dG(y) - 1\}h(t) = 0 \qquad (1.2.14)$$

which is equivalent to

$$h'(t) + \lambda h(t)[\hat{g}(\nu e^{-\delta t}) - 1] = 0 \qquad (1.2.15)$$

whose solution is

$$h(t) = K\exp\{\lambda\int_0^t [1-\hat{g}(\nu e^{-\delta s})ds\} \qquad (1.2.16)$$

K is an arbitrary constant

Since $|e^{-\nu x}| \leq 1$, our solution clearly belongs to the domain of the generator and the theorem has been proved.

□

We will now use this martingale to solve the finite horizon problem

Theorem 1.2.5 *Let* ζ_t^* , τ *as defined and define a measure* G_t^* *such that*

$$\int_{\mathbb{R}^+} e^{-\nu y} dG_t^*(y) = \exp[-(\lambda/\delta)\int_{\nu e^{-\delta t}}^{\nu} \hat{G}(u)du] \tag{1.2.17}$$

then

$$Pr(\tau > t) = \int_{[0,\zeta_o)} dG_t^*(y) \tag{1.2.18}$$

Proof

As we saw before $\{\tau>t\}$ and $\{x_t^*<\zeta_o\}$ are equivalent. We define the probability measure G_t^* by

$$Pr(\ x_t^*<\zeta_o) = \ Pr(\tau > t) = \int_{[0,\zeta_o)} dG_t^*(y) \tag{1.2.19}$$

By the martingale in the previous theorem,

$$E\{\exp(-\nu x_t^*)\exp\{\lambda\int_o^t [1-\hat{g}(\nu e^{-\delta s})ds\}\} = 1 \tag{1.2.20}$$

that is

$$E\{\exp(-\nu x_t^*)\} = \int_{\mathbb{R}^+} e^{-\nu y} dG_t^*(y) = \exp\{-\lambda\int_o^t [1-\hat{g}(\nu e^{-\delta s})ds\} = \exp[-(\lambda/\delta)\int_{\nu e^{-\delta t}}^{\nu} \hat{G}(u)du] \tag{1.2.21}$$

□

Clearly the mass of G_t^* at zero has measure $e^{-\lambda t}$. As $t \to \infty$ we can see by (1.2.21) that G_t^* tends in distribution to G^*, the solutionof the infinite horizon problem.

The formulas obtained in this section for the ruin probabilities have also been obtained by Segerdhal (1959) (see also Gerber(1980) p.135) although by a different method. This model is included in the present thesis as a simple illustration of a method that is going to be used for more complicated models.The results themselves are going to be applied as well.

As an example consider the case where $dG(y)=\alpha e^{-\alpha y}$ (exponentially distributed

claim sizes).

$$\int_{\mathbb{R}^+} e^{-\nu y} dG_t^*(y) = [(\alpha+\nu e^{-\delta t})/(\alpha+\nu)]^{\lambda/\delta} = [e^{-\delta t} + (1-e^{-\delta t})\alpha/(\alpha+\nu)]^{\lambda/\delta} \qquad (1.2.22)$$

This can be interpreted as follows. We pretend λ/δ is an integer. Since in most applications it will be large this is fairly reasonable. Suppose $X_1, X_2, \ldots\ldots, X_{\lambda/\delta}$ are independent identically distributed random variables with exponential (α) distribution. Also define

$$X_i^* = \begin{cases} 0 \text{ with probability } e^{-\delta t} \\ X_i \text{ with probability } 1-e^{-\delta t} \end{cases} \quad i=1,\ldots\ldots,\lambda/\delta \qquad (1.2.23)$$

then G_t^* is the distribution of $X_1+X_2+\ldots\ldots\ldots\ldots+X_{\lambda/\delta}$

Now to calculate the ruin probability in general we can again pretend λ/δ is an integer. Then

$$\exp[-(\lambda/\delta)\int_{\nu e^{-\delta t}}^{\nu} \hat{G}(u)du] = \{\exp[-\int_{\nu e^{-\delta t}}^{\nu} \hat{G}(u)du]\}^{\lambda/\delta} \qquad (1.2.24)$$

$\{\exp[-\int_{\nu e^{-\delta t}}^{\nu} \hat{G}(u)du]\}$ is the Laplace transform of a probability measure with a mass at 0 and $G_t^*(y)$ is the distribution function of $\sum_1^{\lambda/\delta} Z_i$ with Z_i being independent identically distributed random variables with

$$E\{\exp(-\nu Z_i)\} = \exp[-\int_{\nu e^{-\delta t}}^{\nu} \hat{G}(u)du] \qquad (1.2.25)$$

From (1.2.25) we can see that

$$E(Z_i) = \mu_1(1-e^{-\delta t}) \qquad (1.2.26)$$

and

$$Var(Z_i) = \mu_2(1-e^{-2\delta t})/2 \qquad (1.2.27)$$

μ_1, μ_2 being the first two moments of the distribution G assuming they exist.

So as a first approximation we can use a central limit theorem approximation

$$Pr(\tau < t) \simeq \Phi\{[\zeta_0-(\lambda/\delta)\mu_1(1-e^{-\delta t})] / [(\lambda/\delta)\mu_2(1-e^{-2\delta t})]^{1/2}\} \qquad (1.2.28)$$

where Φ is the standard normal distribution function

1.3. GENERALIZATIONS

I

We will now generalize the model in section 1.2 by allowing its parameters to be time dependent. Consider the process ζ_t^* of section 1.2. The deterministic motion will not be an exponential increase any more, but ζ_t (corresponding to ζ_t^*) will be increasing with variable rate $\delta(t)\zeta_t$. $\delta(t)$ is a positive Riemann integrable function of t. Also the claim arrival rate is not constant but it varies with time; it is $\lambda(t)$ a non negative Riemann integrable function of t. The claim sizes depend on time as well. They have distribution function G(y;t), which is a distribution function for all t and Riemann integrable with respect to t for all y ($y\in\mathbb{R}^+$). Furthermore we will also make the assumption that $\lambda(t)$ is bounded on all intervals $[0,t_o)$; that is no explosions are allowed.

The generator of the generalized process (ζ_t , t) acting on a function $f(\zeta,t)$ belonging to its domain is given by

$$A_1 f = \frac{\partial f}{\partial t} + \delta(t)\zeta \frac{\partial f}{\partial \zeta} + \lambda(t)\{ \int_{\mathbb{R}^+} f(\zeta-y,t)dG(y;t) - f(\zeta,t)\} \qquad (1.3.1)$$

We will use a transformed process as we did in the previous section. We define

$$x_t = \zeta_o - \zeta_t \exp\{-\int_o^t \delta(s)ds\} \qquad (1.3.2)$$

So the generator of (x_t , t) acting on f(x,t) is given by

$$A_2 f = \frac{\partial f}{\partial t} + \lambda(t)\{ \int_{\mathbb{R}^+} f(x+y\exp\{-\int_o^t \delta(s)ds\},t)dG(y;t) - f(x,t) \} \qquad (1.3.3)$$

Theorem 1.3.1 *Let* x_t *as defined, then*

$$\exp(-\nu x_t)\exp\{ \int_o^t \lambda(u)[1-\hat{g}(\nu\exp\{-\int_o^u \delta(s)ds\};u)]du\} \quad \textit{is a martingale for all } \nu>0$$

($\hat{g}(u,t)$ *is the L.T of* G(y,t))

Proof

In order to find a martingale we have to solve

$$\frac{\partial f}{\partial t} + \lambda(t)\{ \int_{\mathbb{R}^+} f(x+y\exp\{-\int_o^t \delta(s)ds\},t)dG(y;t) - f(x,t) \} = 0 \qquad (1.3.4)$$

We will try a solution of the form $e^{-\nu x}h(t)$ as in the previous section.

Substituting we get

$$h'(t) + \lambda(t)h(t)[\hat{g}(\nu\exp\{-\int_0^t \delta(s)ds\};t)] - 1] = 0 \qquad (1.3.5)$$

Solving (1.3.5)

$$h(t) = K\exp\{\int_0^t \lambda(u)[1-\hat{g}(\nu\exp\{-\int_0^u \delta(s)ds\};u)]du\} \qquad (1.3.6)$$

where K is an arbitrary constant. Therefore

$$\exp(-\nu x_t)\exp\{\int_0^t \lambda(u)[1-\hat{g}(\nu\exp\{-\int_0^u \delta(s)ds\};u)]du\} \qquad (1.3.7)$$

is a martingale

□

As in theorem 1.2.5 we have (by exactly the same derivation)

$$\Pr(\tau > t) = \int_{[0,\zeta_0)} dG_t^*(y) \qquad (1.3.8)$$

where

$$\int_{\mathbb{R}^+} e^{-\nu y} dG_t^*(y) = \exp\{-\int_0^t \lambda(u)[1-\hat{g}(\nu\exp\{-\int_0^u \delta(s)ds\};u)]du\} \qquad (1.3.9)$$

To find $\Pr(\tau=\infty)$ we just have to let $t \to \infty$.

In practice most examples involve functions of a periodic nature thus incorporating a seasonal component which is important in insurance (see Beard et.al. (1984)p.20-21 for a discussion). One should compare these results with section 2.4 later on.

II

Another generalization we can make to the model of section 1.2 is to relax the assumption that the claim arrival process is Poisson. We will now assume that it is a general renewal process. We define our model as follows. It consists of ζ_t and v_t the time needed for the next claim to occur. ζ_t is as in section 1.2 with the exception of the claim interarrival time intervals which are independent identically distributed random variables, but their distribution function is not exponential any more. It is assumed to be P(y) (y>0). So v_t decreases with unit rate till it reaches 0 and a claim arrives, while v_t takes a new value with distribution function P(y).

The generator of the process (ζ_t, v_t, t) acting on $f(\zeta,v,t)$ is given by

$$Af(\zeta,u,t) = \delta\zeta \frac{\partial f}{\partial \zeta} + \frac{\partial f}{\partial t} - \frac{\partial f}{\partial v} \tag{1.3.10}$$

For f to belong to the domain of the generator the following condition has to be satisfied as well

$$f(\zeta,0,t) = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f(\zeta-y,v,t)dG(y)dP(v) \tag{1.3.11}$$

We are going to set

$$x_t = \zeta_o - \zeta_t e^{-\delta t} \tag{1.3.12}$$

as before. The generator of (x_t, v_t, t) acting on $f(x,v,t)$ is thus given by

$$A^* f = \frac{\partial f}{\partial t} - \frac{\partial f}{\partial v} \tag{1.3.13}$$

and for f to belong to its domain the following has to be satisfied.

$$f(x,0,t) = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f(x+ye^{-\delta t},v,t)dG(y)dP(v) \tag{1.3.14}$$

To find a martingale we now have to solve

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial v} = 0 \tag{1.3.15}$$

subject to (1.3.14)

(1.3.15) is satisfied by functions of the form $k(x,t+v)$ and substituting into (1.3.14) we get

$$k(x,t) = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} k(x+ye^{-\delta t},t+v)\, dG(y)dP(v) \tag{1.3.16}$$

In particular if we want to obtain a martingale of the form $\exp(-\nu x_t)h(t+u_t)$ we should solve

$$h(t) = \hat{g}(\nu e^{-\delta t})\int_{\mathbb{R}^+} h(t+v)dP(v) \tag{1.3.17}$$

Let $r = \nu e^{-\delta t}$ and $m(r) = h(t)$. Then (1.3.17) becomes

$$m(r) = \hat{g}(r)\int_{\mathbb{R}^+} m(re^{-\delta v})dP(v) \tag{1.3.18}$$

(1.3.18) can be solved directly in various special cases. The case $dP(v)=\lambda e^{-\lambda v}$ corresponds to the process in section 1.2. Another case is when dP(v) is degenerate, at T say. Then (1.3.18) becomes

$$m(r) = \hat{g}(r)m(re^{-\delta T}) \tag{1.3.19}$$

whose solution is

$$m(r) = \prod_{o}^{\infty} \hat{g}(re^{-i\delta T}) \tag{1.3.20}$$

which implies that

$$h(t) = \prod_{o}^{\infty} \hat{g}(\nu e^{-\delta(t+iT)}) \tag{1.3.21}$$

Since $\exp(-\nu x_t)h(t+v_t)$ is a martingale

$$E\{\exp(-\nu x_t) \prod_{o}^{\infty} \hat{g}\{\nu\exp[-\delta(t+iT+v_t)]\}\} = \prod_{o}^{\infty} \hat{g}(\nu e^{-\delta(i+1)T}) \tag{1.3.22}$$

(we are assuming without any real loss of generality that $v_o=T$). v_t is deterministic and equal to $T[t/T]+T-t$ ($[t/T]$ is the integer part of t/T); therefore

$$E\{\exp(-\nu x_t)\}\prod_{o}^{\infty} \hat{g}\{\nu\exp\{-\delta(iT+T+[t/T]\}\} = \prod_{1}^{\infty} \hat{g}(\nu e^{-\delta iT}) \tag{1.3.23}$$

and hence

$$E\{\exp(-\nu x_t)\} = \begin{cases} \prod_{1}^{[t/T]} \hat{g}(\nu e^{-\delta iT}) & t \geq T \\ 1 & 0 \leq t \leq T \end{cases} \tag{1.3.24}$$

To solve (1.3.18) in general we will try to find a series solution of the form

$$m(r) = \sum_{o}^{\infty} \frac{M_n}{n!} r^n$$

By differentiating (1.3.18) n times (assuming it can be differentiated; that is the n^{th} moment of G exists) we get the recursion

$$M_n = \sum_{o}^{n} \binom{n}{i} \mu_{n-i} M_i \hat{p}(i\delta) \tag{1.3.25}$$

where μ_j is the j^{th} moment of G. This leads to

$$M_n = \left\{ \sum_{o}^{n-1} \binom{n}{i} \mu_{n-i} M_i \hat{p}(i\delta) \right\} / \{1-\hat{p}(n\delta)\} \tag{1.3.26}$$

This means that if all moments of G exist

$$E\{\exp(-\nu x_t) \sum_{o}^{\infty} \frac{M_n}{n!} \{\nu\exp\{-\delta(t+v_t)\}\}^n\} = \sum_{o}^{\infty} \frac{M_n}{n!} (\nu\exp(-\delta u_o)^n \tag{1.3.27}$$

since $\exp(-\nu x_t)m(\nu e^{-\delta t})$ is a martingale.By (1.3.27) , we have

$$\lim_{t\to\infty} E\{\exp(-\nu x_t)\} = \sum_{0}^{\infty} \frac{M_n}{n!}(\nu\exp(-\delta u_0)^n \qquad (1.3.28)$$

So

$$Pr(\tau < \infty) = \int_{[\zeta_0,\infty)} dQ(y)$$

where $M_n\exp(-n\delta u_0)$ are the moments of Q . Compare these results with those in section 2.5 later on.

1.4. BORROWING -- ABSOLUTE RUIN

The importance of the model in section 1.2 is not so much its practical value as is its relation to other more complicated models to which results obtained in section 1.2 can be applied.

We will now describe such a process. We assume the classical surplus process in insurance with the added option of borrowing. This problem has been solved for exponentially distributed claim sizes (see Dassios (1984)p.94 and Dassios & Embrechts (1987)). The process evolves as follows. When $\zeta_t > 0$ we have deterministic motion of ζ_t with constant rate c and claims occur in terms of a Poisson process with rate λ. Claim sizes are independent identically distributed random variables with distribution function G(y) (y>0). The net profit condition $c>\lambda\mu_1$ where μ_1 is the first moment of G holds as well. When ζ_t becomes negative we do not assume ruin occurs but the company is allowed to borrow money instead, paying interest. Suppose the force of interest is δ; that is the rate of change of ζ_t below 0 is $c+\delta\zeta_t$. Clearly, if ζ_t falls as low as $-(c/\delta)$ or less, there is no positive income any more and no way back for the company. We call this event ***absolute ruin.*** We aim to evaluate the infinite horizon probability that this happens.

Our piecewise deterministic process consists of ζ_t and n_t. n_t indicates whether ζ_t is above or below 0 or even below $-(c/\delta)$. When ζ_t is negative but larger than $-(c/\delta)$, $n_t=1$; when ζ_t is positive or 0 it is $n_t=2$. When ζ_t falls to $-(c/\delta)$ or below, $n_t=0$ and this is an absorbing state. The generator of the process $(n_t\ ,\ \zeta_t)$ acting on a function $f_n(\zeta)$ (n=0,1,2) is given by

$$Af = \begin{cases} (c+\delta\zeta)f_1'(\zeta) + \lambda\{\int_{(0,\zeta+(c/\delta))} f_1(\zeta-y)dG(y)+f_0(\cdot)\mathbf{G}(\zeta+(c/\delta))-f_1(\zeta)\} & -(c/\delta)<\zeta\leq 0 \\ cf_2'(\zeta) + \lambda\{\int_{(0,\zeta]} f_2(\zeta-y)dG(y)+ \int_{(\zeta,\zeta+(c/\delta))} f_1(\zeta-y)dG(y)+f_0(\cdot)\mathbf{G}(\zeta+(c/\delta))-f_2(\zeta)\} & \zeta\geq 0 \end{cases} \tag{1.4.1}$$

We should also have the continuity requirement

$$\lim_{\zeta\uparrow 0} f_1(\zeta)\equiv f_1(0) = f_2(0) \tag{1.4.2}$$

We will also set $f_0(\cdot) = 0$

Theorem 1.4.1 *Let* ζ_t *as defined and*

$$f(\zeta) = \begin{cases} f_2(\zeta) & \zeta\geq 0 \\ f_1(\zeta) & -(c/\delta)\leq\zeta\leq 0 \\ 0 & \zeta<-(c/\delta) \end{cases} \tag{1.4.3}$$

where $f_1(\zeta) = h(\zeta+(c/\delta))$ *and*

$$\hat{h}(\xi) = \int_{\mathbb{R}^+} e^{-\xi y}h(y)dy = \frac{1}{\xi}\exp\{-(\lambda/\delta)\int_0^{\xi}\hat{\mathbf{G}}(u)du\} \tag{1.4.4}$$

and

$$\hat{f}_2(\xi) = \int_{\mathbb{R}^+} e^{-\xi y}f_2(y)dy = \frac{1}{\xi}\cdot\frac{h(c/\delta)-(\lambda/c)\int_0^{c/\delta}h((c/\delta)-y)\int_{\mathbb{R}^+}e^{-\xi x}dG(y+x)dy}{1-(\lambda/c)\hat{\mathbf{G}}(\xi)} \tag{1.4.5}$$

then $f(\zeta_t)$ *is a martingale*

Proof

In order to find a martingale we should solve

$$(c+\delta\zeta)f_1'(\zeta) + \lambda\{\int_{(0,\zeta+(c/\delta))} f_1(\zeta-y)dG(y)-f_1(\zeta)\} = 0 \tag{1.4.6}$$

$$cf_2'(\zeta) + \lambda\{\int_{(0,\zeta]} f_2(\zeta-y)dG(y)+ \int_{(\zeta,\zeta+(c/\delta))} f_1(\zeta-y)dG(y)-f_2(\zeta)\} = 0 \tag{1.4.7}$$

subject to (1.4.2)

To simplify things we set $z=\zeta+(c/\delta)$ and $h(z)=f_1(\zeta)$. (1.4.6) then becomes

$$\delta z h'(z) + \lambda\{ \int_{(0,z)} h(z-y)dG(y) - h(z)\} = 0 \qquad (1.4.8)$$

This is (1.2.3) of section 1.2 and its solution is h(z) where

$$\hat{h}(\xi) = \int_{\mathbb{R}^+} e^{-\xi y} h(y)dy = K\frac{1}{\xi}\exp[-(\lambda/\delta)\int_0^{\xi} \hat{G}(u)du] \qquad (1.4.9)$$

K is an arbitrary constant and without loss of generality we take K =1.

We can now concentrate on solving (1.4.7) which is very much like solving the classical problem. (1.4.7) can be rewritten as

$$cf_2'(\zeta) + \lambda\{\int_{(0,\zeta]} f_2(\zeta-y)dG(y) + \int_{(\zeta,\zeta+(c/\delta))} h(\zeta+(c/\delta)-y)dG(y) - f_2(\zeta)\} = 0 \qquad (1.4.10)$$

Taking Laplace transforms

$$c\xi\hat{f}_2(\xi) - cf_2(0) - \lambda\xi\hat{f}_2(\xi)\hat{G}(\xi) + \lambda\int_{\mathbb{R}^+} e^{-\xi\zeta}\int_{(\zeta,\zeta+(c/\delta))} h(\zeta+(c/\delta)-y)dG(y)d\zeta = 0 \qquad (1.4.11)$$

Changing the order of integration

$$\int_{\mathbb{R}^+} e^{-\xi\zeta}\int_{(\zeta,\zeta+(c/\delta))} h(\zeta+(c/\delta)-y)dG(y)d\zeta = \int_0^{c/\delta} h((c/\delta)-y)\int_{\mathbb{R}^+} e^{-\xi x}dG(y+x)dy \qquad (1.4.12)$$

and since $f_2(0) = h(c/\delta)$ by (1.4.2),

$$\hat{f}_2(\xi) = \int_{\mathbb{R}^+} e^{-\xi y} f_2(y)dy = \frac{1}{\xi}\cdot\frac{h(c/\delta) - (\lambda/c)\int_0^{c/\delta} h((c/\delta)-y)\int_{\mathbb{R}^+} e^{-\xi x}dG(y+x)dy}{1 - (\lambda/c)\hat{G}(\xi)} \qquad (1.4.13)$$

which is (1.4.5) indeed

□

Theorem 1.4.2. *Let* $\tau=\inf\{t:\zeta_t \leq -(c/\delta)\}$ ($\tau=\infty$ *if* $\zeta_t > -(c/\delta)$ *all* t) *and* $\psi(\zeta_0) = \Pr(\tau<\infty)$ *be the probability of ruin with initial capital* ζ_0, *then*

$$\hat{\psi}(\xi) = \int_{\mathbb{R}^+} e^{-\xi y}\psi(y)dy = \qquad (1.4.14)$$

$$= \frac{1}{\xi}\cdot\left[1 - \frac{h(c/\delta) - (\lambda/c)\int_0^{c/\delta} h((c/\delta)-y)\int_{\mathbb{R}^+} e^{-\xi x}dG(y+x)dy}{\int_0^{(c/\delta)}[1-(\delta y/c)h'(y)]dy\,[\,1 - (\lambda/c)\hat{G}(\xi)]/[1-(\lambda\mu_1/c)]}\right]$$

and if $-(c/\delta)<\zeta_0$, *we then have*

$$\psi(\zeta_0) = 1 - \{\, h(\zeta_0+(c/\delta))(1-\lambda\mu_1/c) \,/\, \int_0^{(c/\delta)}[1-(\delta y/c)h'(y)]dy \,\} \qquad (1.4.15)$$

h *is given by theorem 1.4.1 and* μ_1 *is the first moment of* G (*recall that* $c>\lambda\mu_1$)

Proof

We must first prove that $\zeta \to \infty$ on $\{\tau=\infty\}$ save on a set of zero probability. Indeed given that $\tau=\infty$ because of lemma 1.2.2 in section 1.2 there exists t^* such that $\zeta_{t^*} > 0$ for all $t > t^*$ and then ζ_t behaves like the classical surplus process (see section 1.1) and $\zeta_t \to \infty$ save on a set of zero probability.

We also need to find $\lim_{\zeta\to\infty} f_2(\zeta)$

$$\lim_{\zeta\to\infty} f_2(\zeta) = \lim_{\xi\downarrow 0} \xi\hat{f}_2(\xi) = \frac{h(c/\delta)-(\lambda/c)\int_0^{c/\delta} h((c/\delta)-y)\mathbf{G}(y)dy}{1 - (\lambda\mu_1/c)} \qquad (1.4.16)$$

Using (1.4.8) we can see that

$$\int_0^{c/\delta} h((c/\delta)-y)\mathbf{G}(y)dy = (\delta/\lambda)\int_0^{c/\delta} yh'(y)dy \qquad (1.4.17)$$

(from section 1.2 we know that $h'(y)$ exists)

(1.4.16) & (1.4.17) give us

$$\lim_{\zeta\to\infty} f_2(\zeta) = \frac{h(c/\delta)-(\delta/c)\int_0^{c/\delta} yh'(y)dy}{1 - (\lambda\mu_1/c)} = \frac{\int_0^{(c/\delta)}[1-(\delta y/c)h'(y)]dy}{1 - (\lambda\mu_1/c)} \qquad (1.4.18)$$

By the usual reasoning $f_2(\zeta_0) = E\{f_2(\zeta_{\tau\wedge t})\} = E\{f_2(\zeta_t)|\tau>t\}Pr(\tau>t)$ and letting $t \to \infty$

$$1 - \psi(\zeta_0) = Pr(\tau=\infty) = f_2(\zeta_0) \,/\, \lim_{\zeta\to\infty} f_2(\zeta) \qquad (1.4.19)$$

Now, if $\hat{\psi}(\xi)$ is the Laplace transform of $\psi(\zeta)$, by (1.4.5) & (1.4.18) , (1.4.14) follows.

If, however, we start with negative capital $\zeta_0 < 0$ (but larger than $-(c/\delta)$),then by (1.4.3) and (1.4.18) , (1.4.15) follows

□

Furthermore using (1.4.16) we can obtain the asymptotic behaviour of $\psi(\zeta_0)$ for large ζ_0 in the same line of argument as Cramer (1930) did for the classical model (see also Buhlman (1970) p.149-151 and Feller (1969) p.377). Assume G is such that there exists $\alpha > 0$ so that

$$\int_{\mathbb{R}^+} e^{\nu y} dG(y) < \infty \qquad \nu \in [0,\alpha)$$

Then since $c>\lambda\mu_1$ there exists $\rho > 0$ such that $(\lambda/c)\hat{\mathbf{G}}(-\rho) = 1$ and in line with Cramer's argument $\psi(\zeta_0) \sim L\exp(-\rho\zeta_0)$. Of course L is not the same as the

constant evaluated by Cramer for the classical model; indeed it is much smaller.

We let $\psi(\zeta_0) = k(\zeta_0)\exp(-\rho\zeta_0)$, then $\hat{k}(\xi) = \hat{\psi}(\xi-\rho)$ and therefore

$$\lim_{\zeta\to\infty} k(\zeta) = \lim_{\xi\downarrow 0} \xi\hat{k}(\xi) =$$

$$\lim_{\xi\downarrow -\rho}[(\xi+\rho)/\xi]\left[1 - \frac{h(c/\delta)-(\lambda/c)\int_0^{c/\delta} h((c/\delta)-y)\int_{\mathbb{R}^+} e^{-\xi x}dG(y+x)dy}{\int_0^{(c/\delta)}[1-(\delta y/c)h'(y)]dy\,[\,1 - (\lambda/c)\hat{G}(\xi)]/[1-(\lambda\mu_1/c)]}\right] \tag{1.4.20}$$

and since

$$\lim_{\xi\downarrow 0}[1-(\lambda/c)\hat{G}(\xi-\rho)] \;/\; \xi = -(\lambda/c)\hat{G}'(-\rho) \tag{1.4.21}$$

$$L = \lim_{\zeta\to\infty} k(\zeta) = (1/\rho)\left[1 - \frac{h(c/\delta)-(\lambda/c)\int_0^{c/\delta} h((c/\delta)-y)\int_{\mathbb{R}^+} e^{-\xi x}dG(y+x)dy}{(\lambda/c)\int_0^{(c/\delta)}[1-(\delta y/c)h'(y)]dy\int_0^{\infty} ye^{\rho y}G(y)dy\,/\,[\,1-(\lambda\mu_1/c)]}\right] \tag{1.4.22}$$

1.5 THE CLASSICAL MODEL WITH INVESTMENT

We will now consider a model mixing the characteristics of the classical model and the one described in section 1.2 . We assume that the company receives an income due to premiums with a constant rate c. In addition to that the funds available to the company are again invested yielding a rate of income $\delta\zeta_t$. So the total income rate is $c+\delta\zeta_t$. The process of the claims is as before; Poisson arrivals with rate λ and independent identically distributed claim sizes with distribution function G(y) (y>0). When ζ_t goes negative then we assume it reaches the absorbing state 0 and it ceases to evolve. The generator of ζ_t acting on a function $f(\zeta)$ belonging to its domain is given by

$$Af = (c+\delta\zeta)f'(\zeta) + \lambda\{\int_{(0,\zeta)} f(\zeta-y)dG(y) + f(0)G(\zeta) - f(\zeta)\} \qquad \zeta>0 \tag{1.5.1}$$

(note that $f(0+) \neq f(0)$ in general)

Now before actually solving the problem of ruin as in the previous sections we can first try to find a shortcut. Indeed $z_t = \zeta_t+(c/\delta)$ is the same as the process in section 1.2 as long as $z_t \geq (c/\delta)$. Recall that $f(z_t)$ is a martingale where

$$f(z) = \int_{(0,z)} dG^*(y) \qquad \text{and} \qquad \int_{\mathbb{R}^+} e^{-\xi y} dG^*(y) = \exp\{-(\lambda/\delta)\int_0^\xi \hat{G}(u)du\}$$

Now let $\tau^* \equiv \inf\{t: \zeta_t < 0\} = \inf\{t: z_t < c/\delta\}$, then applying the usual argument

$$f(z_0) = E\{f(z_{\tau^*}) \mid \tau^*<\infty\}\Pr(\tau^*<\infty) + \Pr(\tau^*=\infty) \tag{1.5.2}$$

($\lim f(z)=1$ and $z_t \to \infty$ by lemma 1.1.2)

So

$$\Pr(\tau^* < \infty) = \frac{1-f(z_0)}{1-E\{f(z_{\tau^*}) \mid \tau^*<\infty\}} \tag{1.5.3}$$

and therefore

$$\Pr(\tau^* < \infty) = \frac{1-f(\zeta_0+(c/\delta))}{1-E\{f((c/\delta)+\zeta_{\tau^*} \mid \tau^*<\infty\}} \tag{1.5.4}$$

Usually the size of one particular claim would be small compared to c/δ, so applying a no overshoot approximation the probability of ruin in the infinite horizon is approximately

$$\int_{(\zeta_0+(c/\delta),\infty)} dG^*(y) \quad / \quad \int_{((c/\delta),\infty)} dG^*(y) \tag{1.5.5}$$

This is more or less satisfactory for large ζ_0. However, it is not applicable for small ζ_0. For example for $\zeta_0=0$ (1.5.5) gives the value 1 which is useless result (although it might be very near to 1). So there is some scope in trying to find an exact formula.

Theorem 1.5.1 *Let* $f(\zeta)$ *be such that*

$$\hat{f}(\xi) = \int_{\mathbb{R}^+} e^{-\xi x} f(x)dx = (1/\xi)\int_0^\infty (c/\delta)\exp\{-(c/\delta)x\}\exp\{(\lambda/\delta)\int_0^x \hat{G}(u+\xi)du\}dx \tag{1.5.6}$$

then $f(\zeta_t)$ *is a martingale*

Proof

To find a martingale we have to solve

$$(c+\delta\zeta)f'(\zeta) + \lambda\{\int_{(0,\zeta)} f(\zeta-y)dG(y) + f(0)\mathbf{G}(\zeta) - f(\zeta)\} = 0 \qquad (1.5.7)$$

Without loss of generality we set f(0)=0 and we try

$$f(\zeta) = \int_0^{\zeta} h(y)dy + f(0+)$$

Integrating by parts

$$\int_{(0,\zeta)} f(\zeta-y)dG(y) = f(0+)\mathbf{G}(\zeta) - \int_0^{\zeta} h(\zeta-y)\mathbf{G}(y)dy \qquad (1.5.8)$$

so we have to solve

$$(c+\delta\zeta)h(\zeta) + \lambda\{f(0+)\mathbf{G}(\zeta) - \int_0^{\zeta} h(\zeta-y)\mathbf{G}(y)dy\} = 0 \qquad (1.5.9)$$

Taking Laplace transforms

$$c\hat{h}(\xi) - \delta\frac{d}{d\xi}[\hat{h}(\xi)] = \lambda\hat{h}(\xi)\,\hat{\mathbf{G}}(\xi) - \lambda f(0+)\,\hat{\mathbf{G}}(\xi) \qquad (1.5.10)$$

The only solution of (1.5.10) that can be a Laplace transform is

$$\hat{h}(\xi) = f(0+)\exp\{(c\xi/\delta) - (\lambda/\delta)\int_0^{\xi}\hat{\mathbf{G}}(u)du\}\int_{\xi}^{\infty}(\lambda/\delta)\,\exp\{-(c/\delta)x\}\exp\{(\lambda/\delta)\int_0^{x}\hat{\mathbf{G}}(u)du\}\,\hat{\mathbf{G}}(x)dx \qquad (1.5.11)$$

which can be rewritten as (we also integrate by parts)

$$\hat{h}(\xi) = f(0+)\int_0^{\infty}(c/\delta)\exp\{-(c/\delta)x\}\exp\{(\lambda/\delta)\int_0^{x}\hat{\mathbf{G}}(u+\xi)du\}dx \qquad (1.5.12)$$

so

$$\hat{f}(\xi) = f(0+)(1/\xi)\int_0^{\infty}(c/\delta)\exp\{-(c/\delta)x\}\exp\{(\lambda/\delta)\int_0^{x}\hat{\mathbf{G}}(u+\xi)du\}dx \qquad (1.5.13)$$

f clearly belongs to the domain of the generator since it is continuous and

$$E\{f(\zeta_t)-f(\zeta_{t-})\} = \int_0^{\zeta_{t-}} f(\zeta_{t-}-y)dG(y) - f(\zeta_{t-})$$

at all claim times which is finite by (1.5.7)

□

So

$$\lim_{\zeta\to\infty} f(\zeta) = f(0+)+\int_0^{\infty} h'(y)dy = \int_0^{\infty}(c/\delta)\exp\{-(c/\delta)x\}\exp\{(\lambda/\delta)\int_0^{x}\hat{\mathbf{G}}(u)du\}dx \qquad (1.5.14)$$

and reasoning as before

$$\psi(\zeta_0) = 1 - f(\zeta_0)/\lim_{\zeta\to\infty} f(\zeta) \qquad (1.5.15)$$

and in particular

$$\psi(0) = 1 - \left[\int_0^\infty (c/\delta)\exp\{-(c/\delta)x\}\exp\{(\lambda/\delta)\int_0^x \hat{G}(u)du\}dx\right]^{-1} \qquad (1.5.16)$$

(1.5.16) can also be used as a correction factor in approximation (1.5.5) for numerical calculations.

When claims are exponentially distributed an explicit and direct formula can be obtained (see Dassios (1984)p.35)

1.6 CONTINUOUS PATH MODELS

In this section we will deal with insurance arrangments where the claims made are not to be paid instantaneously but in a continuous rate over a certain time period. For example suppose a person is insured against illness paying premiums with a constant rate and when ill receiving benefit with a constant rate as well.

Very little work has been done on models of this kind. It can usually be found in insurance literature concerning specific examples such as the one mentioned later in this section.

Suppose the income rate of the company is c and when paying it pays out benefits with rate a , making its income c-a (which has to be negative, to make sense for the insured). Time spans without illness and durations of illnesses are independent and independent between them random variables with distribution functions $P_1(y)$ and $P_2(y)$ $(y>0)$ respectively. P_2 is such that there exists $\alpha>0$ s.t.

$$\int_{\mathbb{R}^+} e^{\rho y} dP_2(y) < \infty \qquad \text{for} \qquad \rho \in [0,\alpha)$$

Our piecewise deterministic process consists of ζ_t (capital surplus), n_t (state of illness ($n_t = 2$) or non illness ($n_t = 1$)), v_t the time needed for a transition from one state to another to occur and finally time t. Again we will use the time needed for the next jump rather than the time elapsed since the last to cater for point masses in P_1 and P_2. The generator of the process acting on a function $f(n,\zeta,v,t)$ belonging to its domain is given by

$$Af = \begin{cases} c\dfrac{\partial f(1,\zeta,v,t)}{\partial \zeta} - \dfrac{\partial f(1,\zeta,v,t)}{\partial v} + \dfrac{\partial f(1,\zeta,v,t)}{\partial t} \\ \\ (c-a)\dfrac{\partial f(2,\zeta,v,t)}{\partial \zeta} - \dfrac{\partial f(2,\zeta,v,t)}{\partial v} + \dfrac{\partial f(2,\zeta,v,t)}{\partial t} \end{cases} \tag{1.6.1}$$

In addition, for f to belong to the domain of the generator the following conditions have to be satisfied

$$f(1,\zeta,0,t) = \int_{\mathbb{R}^+} f(2,\zeta,v,t) dP_2(v) \tag{1.6.2}$$

$$f(2,\zeta,0,t) = \int_{\mathbb{R}^+} f(1,\zeta,v,t) dP_1(v) \tag{1.6.3}$$

We start by proving an analytical lemma

Lemma 1.6.1 *For all* $\theta > 0$ *ther exists a unique* $\nu_\theta > 0$ *such that*

$$\hat{p}_1(c\nu+\theta)\hat{p}_2(\theta-(a-c)\nu) = 1 \tag{1.6.4}$$

is satisfied, where

$$\hat{p}_i(\nu) = \int_{\mathbb{R}^+} e^{-\nu y} dP_i(y) \qquad i=1,2$$

Furthermore, provided

$$c\int_{\mathbb{R}^+} y dP_1(y) > (a-c)\int_{\mathbb{R}^+} y dP_2(y) \quad \textit{(net profit condition)} \tag{1.6.5}$$

there exists $\nu_0 > 0$ *satisfying (1.6.4); that is*

$$\hat{p}_1(c\nu_0)\hat{p}_2(-(a-c)\nu_0) = 1 \tag{1.6.6}$$

Proof

$\hat{p}_1(c\nu+\theta)\hat{p}_2(\theta-(a-c)\nu)$ is a convex function of ν since its second derivative with

respect to ν is

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}[cx-(a-c)y]^2 e^{-(c\nu+\theta)x-[\theta-(a-c)\nu]y}dP_1(x)dP_2(y) > 0$$

and since $\hat{p}_1(c\nu+\theta)\hat{p}_2(\theta-(a-c)\nu) = \hat{p}_1(\theta)\hat{p}_2(\theta) < 1$ at $\nu = 0$ the first part of the lemma follows. By (1.6.5) the first derivative is negative at $\nu = 0$ is negative and the rest of the lemma follows as well.

□

Theorem 1.6.2 *Let* f_θ *be such that*

$$f_\theta(n,\zeta,v,t) = \begin{cases} \exp(-\nu_\theta\zeta_t)\exp(-\theta t)\exp\{-(c\nu_\theta+\theta)v\} & n=1 \\ \exp(-\nu_\theta\zeta_t)\exp(-\theta t)\exp\{[(a-c)\nu_\theta-\theta]v\}\hat{p}_1(c\nu_\theta+\theta) & n=2 \end{cases} \tag{1.6.7}$$

then $f_\theta(n_t,\zeta_t,v_t,t)$ *is a martingale for all* $\theta \geq 0$

Proof

In order to find a martingale we should solve

$$c\frac{\partial f(1,\zeta,v,t)}{\partial \zeta} - \frac{\partial f(1,\zeta,v,t)}{\partial v} + \frac{\partial f(1,\zeta,v,t)}{\partial t} = 0 \tag{1.6.8}$$

$$(c-a)\frac{\partial f(2,\zeta,v,t)}{\partial \zeta} - \frac{\partial f(2,\zeta,v,t)}{\partial v} + \frac{\partial f(2,\zeta,v,t)}{\partial t} = 0 \tag{1.6.9}$$

subject to (1.6.2) and (1.6.3)

We try a solution of the form $e^{-\theta t}e^{-\nu\zeta}h_n(v)$. (1.6.8) gives $h_1'(v)=-(c\nu+\theta)h_1(v)$ and therefore

$$h_1(v) = K_1 e^{-(c\nu+\theta)v} \tag{1.6.10}$$

(K_1 a constant)

Similarly by (1.6.9)

$$h_2'(v) = K_2 e^{-[(c-a)\nu+\theta]} \tag{1.6.11}$$

(K_2 a constant)

Now (1.6.2) gives

$$K_1 = K_2\hat{p}_2(\theta-(a-c)\nu) \tag{1.6.12}$$

and (1.6.3) gives

$$K_2 = K_1\hat{p}_1(c\nu+\theta) \tag{1.6.13}$$

(1.6.12) & (1.6.13) combine to give us (1.6.4), so we choose $\nu = \nu_\theta$ and therefore $f_\theta(n_t,\zeta_t,v_t,t)$ is a martingale since it satisfies (1.6.8),(1.6.9),(1.6.2) and (1.6.3). By

reasoning as in theorem 1.1.2 of section 1.1, it belongs to the domain of the generator.

□

If ζ_o, v_o are the initial values of ζ_t, v_t and $n_o=1$, letting $\tau \equiv \inf\{t:\zeta_t<0\}$

$$\exp(-\nu_\theta \zeta_o)E\{\exp[-(c\nu_\theta+\theta)v_o]\} = E\{e^{-\theta\tau}\exp[\,[(a-c)\nu_\theta-\theta]v_\tau\,]\hat{p}_1(c\nu_\theta+\theta)\,|\,\tau\leq t\}\Pr(\tau\leq t) +$$
$$e^{-\theta t}E\{\exp(-\nu_\theta\zeta_\tau)\exp[-(2-n_t)(c\nu_\theta+\theta)v_t][\exp[[(a-c)\nu_\theta-\theta]v_t]\hat{p}_1(c\nu_\theta+\theta)]^{n_t-1}\,|\,\tau>t\}\Pr(\tau>t) \quad (1.6.14)$$

Letting $t \to \infty$ in (1.6.15) (for $\theta=0$ we need that $\zeta_t \to \infty$ a.s. , but this is once again ensured by the net profit condition and the strong law of large numbers)

$$E\{e^{-\theta\tau}\exp[[(a-c)\nu_\theta-\theta]v_\tau]\,|\,\tau<\infty\}\Pr(\tau<\infty) = \exp(-\nu_\theta\zeta_o)E\{\exp[-(c\nu_\theta+\theta)v_o]\} \;/\; \hat{p}_1(c\nu_\theta+\theta) \quad (1.6.15)$$

and letting $\theta = 0$ in (1.6.15)

$$\Pr(\,\tau < \infty\,) = \exp(-\nu_0\zeta_o)E\{\exp(-c\nu_0 v_o)\} \;/\; E\{\exp[(a-c)\nu_0 v_\tau]\,|\,\tau<\infty\}\hat{p}_1(c\nu_0) \quad (1.6.16)$$

Furthermore we can find as in section 1.1 an inequality for the finite horizon problem. Leaving out the second term in the right hand side of (1.6.14)

$$\begin{aligned}\exp(-\nu_\theta\zeta_o)E\{\exp[-(c\nu_\theta+\theta)v_o]\} &\geq E\{e^{-\theta\tau}\exp[\,[(a-c)\nu_\theta-\theta]v_\tau\,]\hat{p}_1(c\nu_\theta+\theta)\,|\,\tau\leq t\}\Pr(\tau\leq t)\\ &\geq e^{-\theta t}E\{\exp[\,[(a-c)\nu_\theta-\theta]v_\tau\,]\hat{p}_1(c\nu_\theta+\theta)\,|\,\tau\leq t\}\Pr(\tau\leq t)\\ &\geq e^{-\theta t}\,\hat{p}_1(c\nu_\theta+\theta)\,\Pr(\tau\leq t) \quad (1.6.17)\end{aligned}$$

and from (1.6.17) we get

$$P(\tau \leq t) \leq \inf_{\theta\geq 0}\{e^{\theta t}\exp(-\nu_\theta\zeta_o)E\{\exp[-(c\nu_\theta+\theta)v_o]\} \;/\; \hat{p}_1(c\nu_\theta+\theta)\} \quad (1.6.18)$$

To illustrate these results we will consider the following example (Ramsay (1984)). When the insured person is healthy illness occurs with a Poisson rate λ; the person gets healthy again with a Poisson rate μ. However, he does not receive benefit immediately after he gets ill but only after a fixed time T has elapsed and he has not recovered. When he does recover, benefit ceases. In this example we have $dP_2(v) = \mu e^{-\mu v}$ and P_1 is such that

$$\hat{p}_1(\xi) = e^{-(\xi+\mu)T}[\lambda/(\lambda+\xi)] \;/\; [\;1 - [\lambda/(\lambda+\xi)](1-e^{-\mu T})\;] \quad (1.6.19)$$

(1.6.11) gives (after some algebraic manipulations)

$$(c\nu_\theta+\theta+\lambda e^{-\mu T})[\mu+\theta-(a-c)\nu_\theta] = \lambda\mu\exp[-(c\nu_\theta+\theta+\mu)T] \tag{1.6.20}$$

and for $\theta = 0$

$$(c\nu_0+\lambda e^{-\mu T})[\mu-(a-c)\nu_0] = \lambda\mu\exp[-(c\nu_0+\mu)T] \tag{1.6.21}$$

Now, we assume that the person is healthy initially and that u_0 is exponential (λ). Because of the memoryless property of the exponential v_τ is independent of τ and exponentially (μ) distributed. Hence (1.6.16) gives

$$Pr(\tau < \infty) = [\lambda/(\lambda+c\nu_0)]\exp(-\nu_0\zeta_0) \tag{1.6.22}$$

and the relevant inequality for the finite horizon problem follows too (we omit details).

We now consider other income rates dependent on ζ_t. Consider a process with two states 1 and 2. The process changes from 1 to 2 with Poisson rate λ and from 2 to 1 with rate μ. Income rate is $c_1(\zeta_t)$ when at 1 and $c_2(\zeta_t)$ when at 2. c_1 and c_2 are Riemann integrable functions (there is no requirement for them to be either positive or negative; they can even change signs). The generator of the piecewise deterministic process (n_t,ζ_t) acting on a function $f_n(\zeta)$ is given by

$$Af = \begin{cases} c_1(\zeta)f_1'(\zeta) + \lambda\{f_2(\zeta)-f_1(\zeta)\} & n=1 \\ c_2(\zeta)f_2'(\zeta) + \mu\{f_1(\zeta)-f_2(\zeta)\} & n=2 \end{cases} \tag{1.6.23}$$

and in order to find a martingale we should solve

$$c_1(\zeta)f_1'(\zeta) + \lambda\{f_2(\zeta)-f_1(\zeta)\} = 0 \tag{1.6.24}$$

$$c_2(\zeta)f_2'(\zeta) + \mu\{f_1(\zeta)-f_2(\zeta)\} = 0 \tag{1.6.25}$$

The solution of the system (1.6.24), (1.6.25) is given by

$$f_1(\zeta) = K\int_\epsilon^\zeta (\lambda/c_1(x))\exp\{\lambda\int_{\epsilon_1}^x dy/c_1(y) + \mu\int_{\epsilon_2}^x dy/c_2(y)\}dx \tag{1.6.26}$$

$$f_1(\zeta) = -K\int_\epsilon^\zeta (\mu/c_2(x))\exp\{\lambda\int_{\epsilon_1}^x dy/c_1(y) + \mu\int_{\epsilon_2}^x dy/c_2(y)\}dx \tag{1.6.27}$$

K, ϵ, ϵ_1, ϵ_2 are arbitrary constants.

An application can be $c_1(\zeta) = c+\delta\zeta$, $c_2(\zeta) = c+\delta\zeta-a$; that is the model at the beginning of the section with P_1,P_2 exponential and interest included. One should observe that if ζ_t reaches $(a-c)/\delta$, ruin can not occur any more. So we see that the probability of ruin is just the probability that ζ_t reaches 0 before $(a-c)/\delta$. By (1.6.26) and (1.6.27), $f(n_t,\zeta_t)$ is a martingale where $f(n,\zeta)=f_n(\zeta)$ and

$$f_1(\zeta) = \int_0^{\zeta}(\lambda/\delta)[(c/\delta)+x]^{(\lambda/\delta)-1}\{[(a-c)/\delta]-x\}^{(\mu/\delta)}dx \tag{1.6.28}$$

$$f_2(\zeta) = \int_0^{\zeta}(\mu/\delta)[(c/\delta)+x]^{(\lambda/\delta)}\{[(a-c)/\delta]-x\}^{(\mu/\delta)-1}dx \tag{1.6.29}$$

$$\text{Let } \tau^* = \inf\{t: \zeta_t=0 \text{ or } \zeta_t=(a-c)/\delta\} \tag{1.6.30}$$

Clearly τ^* is a stopping time so

$$f_1(\zeta_0) = E\{f(n_{\tau^*},\zeta_{\tau^*})\} \tag{1.6.31}$$

(we are assuming $n_0 = 1$)

If $\zeta_{\tau^*} = 0$ then $n_{\tau^*} = 2$ and if $\zeta_{\tau^*} = (a-c)/\delta$ then $n_{\tau^*} = 1$

Therefore, by (1.6.31)

$$f_1(\zeta_0) = f_2(0)\Pr(\zeta_{\tau^*}=0)+f_1((a-c)/\delta)\Pr(\zeta_{\tau^*}=(a-c)/\delta) = f_1((a-c)/\delta)\Pr(\zeta_{\tau^*}=(a-c)/\delta) \tag{1.6.32}$$

so the probability of ruin is

$$1 - \Pr(\zeta_{\tau^*}=(a-c)/\delta) = 1 -[f_1(\zeta_0)/f_1((a-c)/\delta)] \tag{1.6.33}$$

which is

$$\frac{\int_0^{\zeta_0}(\lambda/\delta)[(c/\delta)+x]^{(\lambda/\delta)-1}\{[(a-c)/\delta]-x\}^{(\mu/\delta)}dx}{\int_0^{(a-c)/\delta}(\lambda/\delta)[(c/\delta)+x]^{(\lambda/\delta)-1}\{[(a-c)/\delta]-x\}^{(\mu/\delta)}dx} \tag{1.6.34}$$

CHAPTER 2

QUEUES & RELATED MODELS

Queues and related models are of supreme importance in the theory of stochastic processes as well as in practical applications. Applications include a wide range of fields such as hydrology (e.g rainfall models, see Cox & Isham (1986)) to computer networks. Inventories and storage systems are models related to queues too. The *shot noise process* and some variations of it will also be dealt with in this chapter. Also there is an apparent duality with insurance processes, as it will be evident by the properties of the processes and the results found. Martingale methods have been used in association with these processes (e.g Rosenkrantz (1981)) but have so far been applied to diffusion models mainly. The piecewise deterministic processes approach seems well equiped to be used in queues and storage models not containing a diffusion element. So we are going to examine the properties of these processes in a new way, finding some new results in the process of doing so.

2.1 THE G/G/1 QUEUE

A complicated model known for a long time is the G/G/1 queue. It consists of one server and customers arrive in terms of a renewal process and have service requirements which are independent identically distributed random variables. the model is so complicated that we can not even find a simple expression for the expected busy period in terms of the system parameters. Spectral theory is usually the way to treat this process analytically (see Kleinrock (1975) p.273)

We will provide an alternative in this section. Furthermore we will see various different ways to use piecewise deterministic processes theory which are going to be useful in analyzing other models later on. The time a customer that will just arrive will have to wait till starting to be served is called the *virtual waiting time.* We define the virtual waiting time process x_t as follows. As long as $x_t>0$ the process decreases linearly with unit rate and at times T_i (i=1,2,.....) x_t jumps to $x_t=x_{t-}+Y_i$. $T_{i+1}-T_i$ are independent identically distributed random variables with distribution function P(y) (y>0) and the Y_i's are independent identically distributed random variables and independent of the arrival process as well. Their distribution function is G(y) (y>0). Furthermore the first moments of P and G exist, are ρ_1 and μ_1 respectively and

$$\mu_1 < \rho_1 \tag{2.1.1}$$

When x_t reaches 0 (and it does so infinitely often because of (2.1.1)) x_t ceases to decrease and remains at 0 till the next jump occurs (queue idle).

We define the piecewise deterministic process (n_t,x_t,v_t,t) as follows. n_t is an index process taking the values 0 and 1. As long as $n_t=1$ x_t and v_t (the time needed for the next jump to occur) decrease linearly with unit rate. When v_t reaches 0, n_t stays at 1, x_t moves to $x_{t-}+Y_i$ where Y_i has distribution function G(y) and the post jump location of v_t has distribution function P(y). When x_t reaches 0, n_t moves to 0 and as long as $n_t=0$, $x_t=0$ while v_t decreases as before; when v_t reaches 0 n_t moves back to 1, x_t to a post jump location with distribution function G(y) and v_t the same with P(y). So v_t is the remaining lifetime process associated with the renewal process of arrivals and n_t indicates whether the queue is busy(1) or idle(0). The generator of the piecewise deterministic process (n_t,x_t,v_t,t) acting on a function f(n,x,v,t) belonging to its domain is given by

$$Af(n,x,v,t) = \begin{cases} \frac{\partial}{\partial t}f(1,x,v,t) - \frac{\partial}{\partial x}f(1,x,v,t) - \frac{\partial}{\partial v}f(1,x,v,t) & n=1 \\ \frac{\partial}{\partial t}f(0,0,v,t) - \frac{\partial}{\partial v}f(0,0,v,t) & n=0 \end{cases} \tag{2.1.2}$$

For f to belong to the domain of A the following have to be satisfied

$$f(1,0,v,t) = f(0,0,v,t) \tag{2.1.3}$$

$$f(i,x,0,t) = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f(1,x+y,v,t)dG(y)dP(v) \qquad (i=0,1) \tag{2.1.4}$$

The somewhat complicated state space is $\{0\}\times\mathbb{R}^+ \cup \{1\}\times\mathbb{R}^+\times\mathbb{R}^+$.

Using (2.1.3) we can suppress dependence on n and write f(1,x,v,t) as f(x,v,t) and f(0,0,v,t) as f(0,v,t) with

$$A^*f(x,v,t) = \begin{cases} \dfrac{\partial f}{\partial t} - \dfrac{\partial f}{\partial x} - \dfrac{\partial f}{\partial v} & x>0 \\ \dfrac{\partial f}{\partial t} - \dfrac{\partial f}{\partial v} & x=0 \end{cases} \tag{2.1.5}$$

and the condition to be observed is

$$f(x,0,t) = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f(x+y,v,t)dG(y)dP(v) \tag{2.1.6}$$

We will now set on to prove results about the G/G/1 queue.

Theorem 2.1.1 *The long run average proportion of time the queue is idle is* $1-(\mu_1/\rho_1)$. *That is*

$$\lim_{t\to\infty} \frac{\int_0^t \Pr(x_s=0)ds}{t} = 1 - (\mu_1/\rho_1)$$

Proof

Consider the process

$$z_t = \int_0^t I(x_s=0)ds$$

(I is the indicator function)

then the augmented generator B acting on a function f(z,x,u,t) is given by

$$Bf(z,x,v,t) = \begin{cases} \dfrac{\partial f}{\partial t} - \dfrac{\partial f}{\partial x} - \dfrac{\partial f}{\partial v} & x>0 \\ \dfrac{\partial f}{\partial z} + \dfrac{\partial f}{\partial t} - \dfrac{\partial f}{\partial v} & x=0 \end{cases} \tag{2.1.7}$$

and (2.1.6) becomes

$$f(z,x,0,t) = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f(z,x+y,v,t)dG(y)dP(v) \tag{2.1.8}$$

So, in order to find a martingale we should solve $Bf(z,x,v,t) = 0$ (2.1.9)

subject to (2.1.8).

We try a solution of the form z-Kt+h(x,v); (2.1.9) becomes

$$-K - \frac{\partial h}{\partial x} - \frac{\partial h}{\partial v} = 0 \qquad (2.1.10)$$

$$1 - K - \frac{\partial h}{\partial v}\Big|_{x=0} = 0 \qquad (2.1.11)$$

(2.1.8) becomes

$$h(x,0) = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} h(x+y,v)dG(y)dP(v) \qquad (2.1.12)$$

$h(x,v) = -x+(1-K)v$ satisfies (2.1.10) & (2.1.11) and then (2.1.12) is satisfied if and only if

$$-x = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+}[-x-y+(1-K)v]dG(y)dP(v) \qquad (2.1.13)$$

which is equivalent to

$$-\mu_1 + (1-K)\rho_1 = 0 \qquad (2.1.14)$$

that is

$$K = 1 - (\mu_1/\rho_1) \qquad (2.1.15)$$

Thus

$z_t - [1-(\mu_1/\rho_1)]t - x_t + (\mu_1/\rho_1)v_t$

is a martingale and therefore

$$E\{z_t\} - [1-(\mu_1/\rho_1)]t - E\{x_t\} + E\{(\mu_1/\rho_1)v_t\} = - E\{x_0\} + E\{(\mu_1/\rho_1)v_0\} \qquad (2.1.16)$$

and dividing by t and letting $t\to\infty$ ($E\{v_t\}\leq E\{v_T\}=\rho_1$ where T is the time of the last jump).

$$\lim_{t\to\infty} \frac{E\{z_t\}}{t} = 1 - (\mu_1/\rho_1) \qquad (2.1.17)$$

But

$$E\{z_t\} = E\{\int_0^t I(x_s=0)ds\} = \int_0^t E\{I(x_s=0)\}ds = \int_0^t Pr(x_s=0)ds$$

and therefore

$$\lim_{t\to\infty} \frac{\int_0^t Pr(x_s=0)ds}{t} = 1 - (\mu_1/\rho_1) \qquad (2.1.18)$$

□

So we have obtained a well known result (see for example Cohen (1969) p.290-291) in a new way, via piecewise deterministic processes. Note that theorem 2.1.1 gives us the asymptotic (stationary) probability that the queue is idle, should such a limit exist.

We will now try to find an expression related to the Laplace transform of the distribution of x_t.

Theorem 2.1.2 *Suppose there exists* $\lambda>0$ *such that*

$$\int_{\mathbb{R}^+} e^{\nu y} dP(y) < \infty \quad \text{for } \nu \in [0,\lambda)$$

Also suppose $\nu>0$ *and* $\beta(\nu)$ *is defined by* $\hat{g}(\nu)\hat{p}(-\beta(\nu)) = 1$ ($\hat{g}$, $\hat{p}$ *are the Laplace transforms of* G, P *respectively). Then*

$$E\{\exp(-\nu x_t)\exp[\beta(\nu)v_t]\} - E\{\exp(-\nu x_0)\exp[\beta(\nu)v_0]\} =$$

$$\int_0^t [\nu-\beta(\nu)]\, E\{\exp(-\nu x_s)\exp[\beta(\nu)v_s] I(x_s>0)\}ds - \int_0^t \beta(\nu) E\{\exp[\beta(\nu)v_s] I(x_s=0)\}ds \qquad (2.1.19)$$

Proof

$e^{-\nu x} e^{\beta(\nu)v}$ satisfies (2.1.6) so it belongs to the domain of the generator A* given by (2.1.5). Applying A* we get

$$A^*[e^{-\nu x} e^{\beta(\nu)v}] = \begin{cases} [\nu-\beta(\nu)] e^{-\nu x} e^{\beta(\nu)v} & x>0 \\ -\beta(\nu) e^{\beta(\nu)v} & x=0 \end{cases} \qquad (2.1.20)$$

and since for f(x,v) belonging to the domain of A*

$$f(x_t,v_t) - \int_0^t A^* f(x_s,v_s)ds$$

is a martingale,so

$$\exp(-\nu x_t)\exp[\beta(\nu)v_t] - \int_0^t [\nu-\beta(\nu)]\exp(-\nu x_s)\exp[\beta(\nu)v_s] I(x_s>0)ds + \int_0^t \beta(\nu)\exp[\beta(\nu)v_s] I(x_s=0)ds \qquad (2.1.21)$$

is a martingale.

Applying expectations to (2.1.21) and because the integrants of the integrals are non negative, expectations and integral signs are interchangeable and (2.1.19) is obtained.

□

In theorem 2.1.2 we obtained an equation relating aspects of the joint distribution of x_t and v_t (and n_t for that matter). In specific examples this could lead to more simple and explicit results.

An example could be the M/G/1 queue (see Cohen (1969) p.249-258). That is $dP(v)=\lambda e^{-\lambda v}dv$. Theorem 2.1.2 can be applied as follows. v_t and x_t are independent in this case and v_t is exponentially distributed with parameter λ;

also $\beta(\nu) = \lambda[1-\hat{g}(\nu)]$ and thus

$$E\{\exp[\beta(\nu)v_t]\} = \lambda/[\lambda-\beta(\nu)] = 1/\hat{g}(\nu)$$

so (2.1.19) gives

$$E\{\exp(-\nu x_t)\}-E\{\exp(-\nu x_0)\} = \int_0^t\{\nu-\lambda[1-\hat{g}(\nu)]\}E\{\exp(-\nu x_s)I(x_s>0)\}ds-\int_0^t\lambda[1-\hat{g}(\nu)]Pr(x_s=0)ds \tag{2.1.22}$$

Now, $E\{\exp(-\nu x_t)\}$ is differentiable with respect to t since if $x_t>\epsilon$

$$E\{\exp(-\nu x_{t+\epsilon})|x_t\} = \exp[-\nu(x_t-\epsilon)]e^{-\lambda\epsilon} + (1-e^{-\lambda\epsilon})E\{e^{-\nu Y}\}$$

where Y is an appropriate random variable, and for $x_t<\epsilon$

$$E\{\exp(-\nu x_{t+\epsilon})|x_t\} = (1-e^{-\lambda\epsilon})E\{e^{-\nu Y}\}$$

By letting $\epsilon\downarrow 0$ we see that a right derivative exists (similarly for the left derivative). So differentiating (2.1.22)

$$\frac{d}{dt}E\{\exp(-\nu x_t)I(x_t>0)\}+\frac{d}{dt}Pr(x_t=0) = \{\nu-\lambda[1-\hat{g}(\nu)]\}E\{\exp(-\nu x_t)I(x_t>0)\}- \lambda[1-\hat{g}(\nu)]Pr(x_t=0) \tag{2.1.23}$$

which is a rather more explicit form and if x_t is stationary; that is the derivatives in (2.1.23) are zero, we see that the stationary distribution is such that

$$E\{\exp(-\nu x_t)I(x_t>0)\} = \frac{\lambda[1-\hat{g}(\nu)]}{1-\lambda[1-\hat{g}(\nu)]}Pr(x_t=0) \tag{2.1.24}$$

and setting $\nu=0$ in (2.1.24)

$$Pr(x_t>0) = \frac{\lambda\mu_1}{1-\lambda\mu_1}Pr(x_t=0) \tag{2.1.25}$$

so

$$Pr(x_t=0) = 1 - \lambda\mu_1 \tag{2.1.26}$$

and

$$E\{\exp(-\nu x_t)I(x_t>0)\} = \frac{\lambda[1-g(\nu)]}{1-\lambda[1-\hat{g}(\nu)]}(1-\lambda\mu_1) \tag{2.1.27}$$

We will now proceed to examine other aspects of the process. Once again, we start with a technical lemma.

Lemma 2.1.3 *Assume that there exist* $\alpha>0$ *and* $\lambda>0$ *such that*

$\int_{\mathbb{R}^+} e^{\nu y} dP(y) < \infty$ *for* $\nu \in [0,\lambda)$ *and* $\int_{\mathbb{R}^+} e^{\nu y} dG(y) < \infty$ *for* $\nu \in [0,\alpha)$

Then for $\theta \geq 0$ *the equation*

$$\hat{g}(\nu)\hat{p}(\theta-\nu) = 1 \tag{2.1.28}$$

has exactly two real roots ν_{θ_1} *and* ν_{θ_2} *such that* $\nu_{\theta_1} \geq 0$ *and* $\nu_{\theta_2} < 0$

Proof

At $\nu=0$, $\hat{g}(\nu)\hat{p}(\theta-\nu)=\hat{p}(\theta)<1$ if $\theta>0$; so for $\theta>0$ it is sufficient to show that it is convex. Indeed its second derivative is

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} (v-y)^2 e^{-(\theta-\nu)v-\nu y} dP(v)dG(y) > 0$$

Even for $\theta=0$, it is convex. $\nu=0$ is one root and since its first derivative at 0 is $\rho_1-\mu_1$ which is positive, the other root is negative.

□

Theorem 2.1.4 *Let* G(y) *and* P(y) *as in lemma 2.1.3. Then*

$\nu_{\theta_1} e^{-\theta t}\exp(-\nu_{\theta_2} x_t)\exp[(\nu_{\theta_2}-\theta)v_t] - \nu_{\theta_2} e^{-\theta t}\exp(-\nu_{\theta_1} x_t)\exp[(\nu_{\theta_1}-\theta)v_t]$

is a martingale.

Proof

A function of the form $e^{-\theta t} e^{-\nu x} e^{\beta v}$ satisfies

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial x} - \frac{\partial f}{\partial v} = 0 \tag{2.1.29}$$

if and only if $\nu-\beta-\theta=0$; that is $\beta=\nu-\theta$. Also (2.1.6) is satisfied if (2.1.28) is satisfied and so ν can only take the values ν_{θ_1} and ν_{θ_2}. So (2.1.6) and (2.1.29) are satisfied by any linear combination of $e^{-\theta t}\exp(-\nu_{\theta_1} x)\exp[(\nu_{\theta_1}-\theta)v]$ and $e^{-\theta t}\exp(-\nu_{\theta_2} x)\exp[(\nu_{\theta_2}-\theta)v]$ and for

$$\frac{\partial f}{\partial t}\Big|_{x=0} - \frac{\partial f}{\partial v}\Big|_{x=0} = 0$$

to be satisfied , we should take

$L\{\nu_{\theta_1} e^{-\theta t}\exp(-\nu_{\theta_2} x)\exp[(\nu_{\theta_2}-\theta)v] + \nu_{\theta_2} e^{-\theta t}\exp(-\nu_{\theta_1} x)\exp[(\nu_{\theta_1}-\theta)v]\}$

where L is an arbitrary constant and the theorem has been proved.

□

So this leads to the following result concerning the time needed for x_t to cross a fixed level b.

Corollary 2.1.5 *Suppose* G(y) *and* P(y) *are as in lemma 2.1.3. Assume* $x_o<b$ *where* b *is a fixed level and define* $\tau_b \equiv \inf\{t: x_t \geq b\}$. *Then*

$$E\{\exp(-\theta\tau_b)[-(\nu_{\theta_2}/\hat{g}(\nu_{\theta_2}))\exp(-\nu_{\theta_1}x_{\tau_b})+(\nu_{\theta_1}/\hat{g}(\nu_{\theta_1}))\exp(-\nu_{\theta_2}x_{\tau_b})]\} =$$

$$E\{-\nu_{\theta_2}\exp(-\nu_{\theta_1}x_o)\exp[(\nu_{\theta_1}-\theta)v_o] +\nu_{\theta_1}\exp(-\nu_{\theta_2}x_o)\exp[(\nu_{\theta_2}-\theta)v_o]\} \quad (2.1.30)$$

Proof

Clearly $Pr(\tau_b<\infty) = 1$ (consider the first time a jump of size larger than b occurs; if this is not feasible consider the first time a sequence of jumps with "net" size larger than b occurs; as "net" size we understand the sum of their sizes minus any deterministic decrease that took place between them). So τ_b is a stopping time and applying Doob's optional stopping theorem (2.1.30) follows in view of the fact that at $t=\tau_b$ v_t has distribution function P(v) and is independent of τ_b and the then value of x_t, so its moment generating function valued at ν-θ is $\hat{p}(\theta-\nu)=1/\hat{g}(\nu)$.

□

This is a new result in the sense that it can not be found in the standard literature. It is a generalization of the result by Kennedy (1984) for the M/G/1 queue.

Using (2.1.30) we can find an approximate expression for the Laplace transform of the distribution of τ_b by applying a *no overshoot* approximation as Kennedy does.

$$E\{\exp(-\theta\tau_b)\} \simeq \frac{E\{-\nu_{\theta_2}\exp(-\nu_{\theta_1}x_o)\exp[(\nu_{\theta_1}-\theta)v_o] +\nu_{\theta_1}\exp(-\nu_{\theta_2}x_o)\exp[(\nu_{\theta_2}-\theta)v_o]\}}{[-(\nu_{\theta_2}/\hat{g}(\nu_{\theta_2}))\exp(-\nu_{\theta_1}b)+ (\nu_{\theta_1}/\hat{g}(\nu_{\theta_1}))\exp(-\nu_{\theta_2}b)]} \quad (2.1.31)$$

A case where (2.1.30) leads to an exact formula is the G/M/1 queue. In this case $dG(y) = \alpha e^{-\alpha y}dy$; at $t=\tau_b$ x_t-b has exponential (α) distribution and is independent of τ_b because of the memoryless property of the exponential. All these lead to the expression

$$E\{\exp(-\theta\tau_b)\} = \frac{E\{-\nu_{\theta_2}\exp(-\nu_{\theta_1}x_o)\exp[(\nu_{\theta_1}-\theta)v_o] + \nu_{\theta_1}\exp(-\nu_{\theta_2}x_o)\exp[(\nu_{\theta_2}-\theta)v_o]\}}{\{-[\nu_{\theta_2}(\alpha+\nu_{\theta_2})/(\alpha+\nu_{\theta_1})]\exp(-\nu_{\theta_1}b)+[\nu_{\theta_1}(\alpha+\nu_{\theta_1})/(\alpha+\nu_{\theta_2})]\exp(-\nu_{\theta_2}b)\}} \tag{2.1.32}$$

To find an expression for the Laplace transform of the distribution of the time needed for x_t to cross a fixed level b starting from $x_o>b$, is a rather easier task since we can ignore the behaviour of the process below b and in particular at 0. Consider the process x_t^* which behaves exactly as x_t till it reaches 0 for the first time and then continues to decrease with unit rate till the next jump occurs etc. In fact $-x_t^*$ is the insurance surplus process in section 1.1. Clearly for $0\leq t\leq\tau^b$ ($\tau^b\equiv\inf\{t: x_t=b\}$) we have $x_t^*=x_t$. The generator of (x_t^*,v_t,t) acting on a function f(x,v,t) is given by

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial x} - \frac{\partial f}{\partial v} \tag{2.1.34}$$

with the condition for f to belong to the domain of the generator

$$f(x,0,t) = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f(x+y,v,t)dG(y)dP(v) \tag{2.1.35}$$

but with no conditions at x=0 this time. So going back to the proof of theorem 2.1.4 $e^{-\theta t}\exp(-\nu_{\theta_1}x)\exp[(\nu_{\theta_1}-\theta)v]$ can be admitted as a solution of

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial x} - \frac{\partial f}{\partial v} = 0$$

satisfying (2.1.35) where ν_{θ_1} is the non negative root of $\hat{p}(\theta-\nu)\hat{g}(\nu) = 1$, so

$$e^{-\theta t}\exp(-\nu_{\theta_1}x_t^*)\exp[(\nu_{\theta_1}-\theta)v_t] \tag{2.1.36}$$

is a martingale and applying Doob's optional stopping theorem

$$E\{\exp[-\theta(\tau^b\wedge t)]\exp(-\nu_{\theta_1}x^*_{\tau^b\wedge t})\exp[(\nu_{\theta_1}-\theta)v_{\tau^b\wedge t}]\} = E\{\exp(-\nu_{\theta_1}x_o^*)\exp[(\nu_{\theta_1}-\theta)v_o]\} \tag{2.1.37}$$

This leads to

<u>Theorem 2.1.6</u> *Let* $x_o>b$ *and* τ^b *as defined, then*

$$E\{\exp(-\theta\tau^b)\exp[(\nu_{\theta_1}-\theta)v_{\tau^b}]\} = \exp(\nu_{\theta_1}b)E\{\exp(-\nu_{\theta_1}x_o^*)\exp[(\nu_{\theta_1}-\theta)v_o]\} \tag{2.1.38}$$

<u>Proof</u>

(2.1.38) is a direct consequence of (2.1.37). Indeed, (2.1.37) gives

$E\{\exp(-\theta\tau^b)\exp(-\nu_{\theta_1}x^*_{\tau^b})\exp[(\nu_{\theta_1}-\theta)v_{\tau^b}] \mid \tau^b \leq t\}Pr(\tau^b \leq t) +$

$E\{e^{-\theta t}\exp(-\nu_{\theta_1}x^*_t)\exp[(\nu_{\theta_1}-\theta)v_t] \mid \tau^b > t\}Pr(\tau^b > t) = E\{\exp(-\nu_{\theta_1}x^*_0)\exp[(\nu_{\theta_1}-\theta)v_0]\}$

and letting $t\to\infty$ and in view of the fact that $x^*_{\tau^b} =$, (2.1.38) follows.

□

An example where (2.1.31) leads to an exact result is the M/G/1 queue where $dP(v)=\lambda e^{-\lambda v}dv$; then v_{τ^b} is exponentially (λ) distributed and therefore

$$E\{\exp(-\theta\tau^b)\} = [(\lambda+\theta-\nu_{\theta_1})/\lambda]\exp(\nu_{\theta_1}b)E\{\exp(-\nu_{\theta_1}x^*_0)\exp[(\nu_{\theta_1}-\theta)v_0]\} \tag{2.1.39}$$

Finally we will generalize theorem 2.1.1 and find the asymptotic behaviour of the higher cumulants of

$$z_t = \int_0^t I(x_s=0)ds$$

We have already seen that

$$\lim_{t\to\infty}\frac{E\{z_t\}}{t} = 1 - (\mu_1/\rho_1)$$

by theorem 2.1.1. We extend it as follows

Theorem 2.1.7 *Let* $\gamma \in [0,\alpha)$ $(\alpha>0)$ *and* θ *be the unique positive solution of*

$$\hat{g}(-\gamma)\hat{p}(\gamma-\theta) = 1 \tag{2.1.40}$$

(Of course we assume $\hat{g}(-\gamma)$ *exists on* $[0,\alpha)$ *). Then*

$$\exp(-\gamma z_t)e^{\theta t}\exp(\gamma x_t)\exp[-(\gamma-\theta)v_t] \tag{2.1.41}$$

is a martingale.

Proof

Recall that the generator of (z_t,x_t,v_t,t) is given by

$$Bf(z,x,v,t) = \begin{cases} \dfrac{\partial f}{\partial t} - \dfrac{\partial f}{\partial x} - \dfrac{\partial f}{\partial v} & x>0 \\ \dfrac{\partial f}{\partial z} + \dfrac{\partial f}{\partial t} - \dfrac{\partial f}{\partial v} & x=0 \end{cases} \tag{2.1.42}$$

with f being continuous at x=0 and the requirement

$$f(z,x,0,t) = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f(z,x+y,v,t)dG(y)dP(v) \tag{2.1.43}$$

$e^{-\gamma z}e^{\theta t}e^{\gamma x}e^{-(\gamma-\theta)v}$ satisfies Bf = 0 and is continuous at x=0. (2.1.43) has to be satisfied as well, so we get

$$e^{-\gamma z}e^{\theta t}e^{\gamma x} = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} e^{-\gamma z}e^{\theta t}e^{\gamma(x+y)}e^{-(\gamma-\theta)v}dG(y)dP(v) \qquad (2.1.44)$$

which is equivalent to (2.1.40) and the theorem has been proved.

□

By theorem 2.17 we have that

$$E\{\exp(-\gamma z_t)\exp(\gamma x_t)\exp[-(\gamma-\theta)v_t]\} = e^{-\theta t}E\{\exp(-\gamma z_o)\exp(\gamma x_o)\exp[-(\gamma-\theta)v_o]\} \qquad (2.1.45)$$

and asymptotically

$$-\log\{E\{\exp(-\gamma z_t)\exp(\gamma x_t)\exp[-(\gamma-\theta)v_t]\}\} \sim \theta t \qquad (2.1.46)$$

Now, notice that $\exp(\gamma x_t)\exp[(\theta-\gamma)v_t]$ has finite expectation which does not grow with t, so (2.1.46) leads to

$$-\log\{E\{\exp(-\gamma z_t)\}\} \sim \theta t \quad \text{w.r.t } \gamma \qquad (2.1.47)$$

By differentiating n times and setting γ=0 we get the asymptotic behaviour of the n^{th} cummulant of z_t. We thus see that all cummulants grow asymptotically linearly with t. Furthermore the coefficients give the moments of the asymptotic distribution of x_t should it exist.

The importance of theorem 2.1.7 will be evident in chapters 3 and 4 when this line of thinking will be adopted when analyzing various different models.

2.2 GENERAL DECAY RULE

An important process described in the paper by Cox & Isham (1986) is a process decaying with a general rate and making additive positive exponentially distributed jumps at Poisson times. Such models are of use in hydrology, computer networks and storage models (an example is modelling the level of water in dams

where the supply from the dam is a rather complicated function of the level of water in it).

The piecewise deterministic process x_t evolves as follows. x_t decreases with rate -c(x), where c(x) is a non negative continuously differentiable function of x (x>0). Jumps occur in terms of a Poisson process with rate λ and their sizes are independent identically distributed random variables with probability density g(y). Now x_t can reach 0 if and only if

$$\lim_{x \downarrow 0} \int_x^M dy/c(y) < \infty$$

(This should be clear by looking at the equation $\frac{dx}{dt} = -c(x)$).
Should this happen we assume that x_t remains at 0 till the next jump occurs. Of course if x_t can not reach 0 this assumption is not needed. Such processes are also called storage processes. For a different treatment see Harrison & Resnick (1972 & 1978). The generator of (x_t,t) acting on a function f(x,t) belonging to its domain is given by

$$Af(x,t) = \begin{cases} \frac{\partial f}{\partial t} - c(x)\frac{\partial f}{\partial x} + \lambda[\int_0^\infty f(x+y,t)g(y)dy - f(x,t)] & x>0 \\ \\ \frac{\partial f}{\partial t} + \lambda[\int_0^\infty f(y,t)g(y)dy - f(0,t)] & x=0 \end{cases} \qquad (2.2.1)$$

(notice that the second leg is optional).

Formally we should have defined n_t as in the previous section indicating whether x=0 or x>0; however we supress dependence on n_t aswe did then.

The usual approach of applied probabilists to this and other models is via the Kolmogorov forward equations. The approach via piecewise deterministic processes has its own merits as we shall see. It could be quite illustrative to derive a form of the forward equation from the generator of the process. Clearly the method used in the proof can be applied to several other different processes. We will for that purpose restrict ourselves to the case where x_t can not reach 0 so as to avoid unnecessary complications.

Proposition 2.2.1 *Let* x_t *as defined and let*

$$\lim_{x \downarrow 0} \int_x^M dy/c(y) = \infty$$

If $p(x,t)$ *is the distribution density of* x_t *at time* t *and* $c(x)p(x,t)$ *is assumed to be differentiable with respect to* x, *then*

$$p(x,t)-p(x,0) = \int_0^t \{\frac{\partial}{\partial x}[c(x)p(x,s)]+\lambda[\int_0^x p(x-y,s)g(y)dy-p(x,s)]\}ds \quad \textit{almost everywhere} \tag{2.2.2}$$

Proof

Consider a class of functions B such that each member $f:\mathbb{R}^+\to\mathbb{R}^+$ belongs to the domain of the generator A and such that if

$$\int_{\mathbb{R}^+} p_1(x)f(x)dx = \int_{\mathbb{R}^+} p_2(x)f(x)dx \quad \text{for all } f\in B$$

then $p_1(x)=p_2(x)$ almost everywhere (such a class can be $f(x)=e^{-\nu x}$ $(\nu\geq 0)$). Now

$$Af(x) = c(x)\frac{df}{dx} + \lambda[\int_0^\infty f(x+y)g(y)dy-f(x)] \tag{2.2.3}$$

and since

$$f(x_t) - \int_0^t Af(x_s,s)ds \quad \text{is a martingale}$$

$$E\{f(x_t)\}-E\{f(x_0)\} = -E\{\int_0^t c(x_s)f'(x_s)ds\}+\lambda E\{\int_0^t\int_0^\infty f(x_s+y)g(y)dyds\}-\lambda E\{\int_0^t f(x_s)ds \tag{2.2.4}$$

Since the integrands are non negative, expectations and integration signs are interchangeable, so (2.2.4) yields

$$\int_{\mathbb{R}^+} f(x)p(x,s)dx - \int_{\mathbb{R}^+} f(x)p(x,0)dx =$$

$$\int_0^t \{-\int_{\mathbb{R}^+} c(x)f'(x)p(x,s)dx+\lambda\int_{\mathbb{R}^+}\int_0^\infty f(x+y)g(y)dyp(x,s)dx - \lambda\int_{\mathbb{R}^+} f(x)p(x,s)dx\}ds \tag{2.2.5}$$

But

$$-\int_{\mathbb{R}^+} c(x)f'(x)p(x,s)dx = \int_{\mathbb{R}^+} f(x)\frac{\partial}{\partial x}[c(x)p(x,s)]dx \quad \text{and}$$

$$\int_{\mathbb{R}^+}\int_0^\infty f(x+y)g(y)dyp(x,s)dx = \int_{\mathbb{R}^+} f(x)\int_0^x p(x-y,s)g(y)dydx$$

So (2.2.5) gives

$$\int_{\mathbb{R}^+} f(x)p(x,s)dx - \int_{\mathbb{R}^+} f(x)p(x,0)dx =$$

$$\int_{\mathbb{R}^+} f(x)\int_0^t \{\frac{\partial}{\partial x}[c(x)p(x,s)]+\lambda[\int_0^x p(x-y,s)g(y)dy-p(x,s)]\}dsdx \qquad (2.2.6)$$

and since $f\in B$ this leads to (2.2.2).

□

Differentiating (2.2.2) with respect to time gives the forward equation in its derivative form.

Since p(x,t) is a density

$$\int_0^\infty p(x,t)dx = 1 \qquad (2.2.7)$$

has to be observed for all t. Also the solution of

$$\frac{d}{dx}[c(x)p(x)]+\lambda\int_0^x p(x-y)g(y)dy-\lambda p(x) = 0 \qquad (2.2.8)$$

subject to (2.2.7) gives the stationary distribution (should it exist).

Suppose we do not allow the process to exceed the value M. That is at a jump time T_i, x_t becomes $(x_{t-}+Y_i)\wedge M$, so the generator becomes

$$Af(x) = c(x)\frac{df}{dx} + \lambda[\int_0^{M-x} f(x+y,t)g(y)dy+f(M)\mathbf{G}(M-x)-f(x)] \qquad (2.2.9)$$

Yet, proceeding as in the proof of proposition 2.2.1 it can be shown that (2.2.6) still holds but now with the condition

$$\int_0^M p(x,t)dx = 1 \qquad (2.2.10)$$

instead of (2.2.7). This has the following implication. The stationary distribution in this case will be the restriction on (0,M) of the stationary distribution in proposition 2.2.1. This situation occurs in dams with finite capacity.

We will now concentrate on an important problem. We want to find aspects of the distributions of τ_b and τ^b where τ_b is the time needed to cross a fixed level b starting from below b and τ^b from above exactly as in the previous section. To simplify matters we assume the jumps are exponentially distributed ($g(y)=\alpha e^{-\alpha y}$). Trying to find an explicit expression for their Laplace transforms will not lead us anywhere since we should then try to find a martingale of the form $e^{-\theta t}f(x_t)$ which involves solving

$$-\theta f(x)-c(x)f'(x)+\lambda[\int_0^\infty f(x+y)\alpha e^{-\alpha y}dy-f(x)] = 0 \qquad (2.2.11)$$

The solutions of (2.2.11) can not be written in a simple form for a general c(x) (although it can for various special cases as we shall see in subsequent sections). We can of course find series solutions to (2.2.11) but this will amount to the same mathematics that are going to follow coupled with several more technical problems. The alternative we will follow is to try to determine the moments of the distributions of τ_b and τ^b.

We start with τ_b. First of all we have to be sure that $Pr(\tau_b<\infty)=1$. This is clearly true since in the worst case we will just have to wait for the first time a jump larger than b occurs. We now define the process x_t^* as equal to x_t up to time τ_b; then x_t^* moves to the absorbing state w and then even time stops so the generator at w has no terms and we can assign any function as f(w,t). The generator of x_t^* is given by

$$Af(x,t) = \begin{cases} \dfrac{\partial f}{\partial t} - c(x)\dfrac{\partial f}{\partial x} + \lambda[\int_0^{b-x} f(x+y,t)\alpha e^{-\alpha y}dy+f(w,t)e^{-\alpha(b-x)}-f(x,t] & 0<x\leq b \\ \dfrac{\partial f}{\partial t} + \lambda[\int_0^{b-x} f(y,t)\alpha e^{-\alpha y}dy+f(w,t)e^{-\alpha b}-f(0,t)] & x=0 \end{cases} \qquad (2.2.12)$$

(once again the second leg is optional because 0 may or may not be attained).

We now proceed to find moments of τ_b.

Theorem 2.2.2 *Consider the family of functions* $q_i(x)$ i=0,1,2,..... *defined on* [0,b] *as follows*

$$q_0(x) \equiv 1 \qquad (2.2.13)$$

$$q_i'(x) = \{\exp[\alpha x+\lambda\int_x^b dz/c(z)]/c(x)\}\cdot$$

$$\{ie^{-\alpha b}q_{i-1}(b)-i\int_x^b \exp[-\lambda\int_y^b dz/c(z)][e^{-\alpha y}q_{i-1}(y)]'dy-q_i(b)e^{-\alpha b}\} \qquad (2.2.14)$$

$$q_i(b) = (1/\lambda)\{iq_{i-1}(b)-ie^{\alpha b}\int_0^b \exp[-\lambda\int_y^b dz/c(z)][e^{-\alpha y}q_{i-1}(y)]'dy\} \qquad (2.2.15)$$

Then

$$f(x_t^*,t) = \begin{cases} \sum_0^m \binom{m}{i} t^{m-i}q_i(x_t^*) & 0\leq x_t^*\leq b \\ & \textit{is a martingale} \\ t^m & x_t^*=w \end{cases}$$

Proof

For $f(x_t^*,t)$ to be a martingale we should have

$$\frac{\partial}{\partial t}\Big[\sum_0^m \binom{m}{i} t^{m-i} q_i(x)\Big] - c(x)\frac{\partial}{\partial x}\Big[\sum_0^m \binom{m}{i} t^{m-i} q_i(x)\Big] +$$

$$\lambda\Big\{\sum_0^m \binom{m}{i} t^{m-i}\int_0^{b-x} q_i(x+y)\alpha e^{-\alpha y}dy + t^m e^{-\alpha(b-x)} - \sum_0^m \binom{m}{i} t^{m-i} q_i(x)\Big\} = 0 \qquad (2.2.16)$$

but

$$\frac{\partial}{\partial t}\Big[\sum_0^m \binom{m}{i} t^{m-i} q_i(x)\Big] = \sum_0^m \binom{m}{i}(m-i)t^{m-i-1} q_i(x) = \sum_1^m \binom{m}{i} t^{m-i} i q_{i-1}(x) \qquad (2.2.17)$$

So looking at the coefficient of t^i in (2.2.16) (i=1,....,m-1 ; the coefficient of t^m is trivially $q_0(x)=1$).

$$iq_{i-1}(x) - c(x)q_i'(x) + \lambda\Big\{\int_0^{b-x} q_i(x+y)\alpha e^{-\alpha y}dy - q_i(x)\Big\} = 0 \qquad (2.2.18)$$

which is equivalent to

$$iq_{i-1}(x) - c(x)q_i'(x) + \lambda e^{\alpha x}\int_x^b q_i'(y)e^{-\alpha y}dy - \lambda e^{\alpha x} q_i(b)e^{-\alpha b} = 0 \qquad (2.2.19)$$

and to

$$\Big[\exp[-\lambda\int_x^b dz/c(z)]\int_x^b q_i'(y)e^{-\alpha y}dy\Big]' + iq_{i-1}(x)e^{-\alpha x}\exp[-\lambda\int_x^b dz/c(z)]/c(x) =$$

$$\lambda q_i(b)e^{-\alpha b}\exp[-\lambda\int_x^b dz/c(z)]/c(x) \qquad (2.2.20)$$

Integrating (2.2.20)

$$\int_x^b q_i'(y)e^{-\alpha y}dy = i\exp[\lambda\int_x^b dz/c(z)]\int_x^b\{q_{i-1}(y)e^{-\alpha y}\exp[-\lambda\int_y^b dz/c(z)]/c(y)\}dy -$$

$$\lambda q_i(b)e^{-\alpha b}\int_x^b\{\exp[-\lambda\int_y^b dz/c(z)]/c(y)\}dy \qquad (2.2.21)$$

Differentiating (2.2.21) and integrating by parts we get (2.2.14)

Also if x_t can reach 0 we need

$$iq_{i-1}(0) + \lambda\{\int_0^b q_i(y)\alpha e^{-\alpha y}dy - q_i(0)\} = 0 \qquad (2.2.22)$$

which by (2.2.18) is equivalent to

$$\lim_{x\downarrow 0} c(x)q_i'(x) = 0 \qquad (2.2.23)$$

and therefore (2.2.14) gives (2.2.15)

In case x_t can not reach 0 we still require (2.2.23) and thus (2.2.15) to hold, otherwise by (2.2.14)

$$\int_x^b q_i'(y)dy$$

clearly diverges as $x \downarrow 0$ and not only it is unbounded but it does not belong to the domain of the generator either. If (2.2.23) and (2.2.15) hold, then if we assume $q_{i-1}(x)$ is bounded,

$$\left|\int_x^b q_i'(y)dy\right| \leq K\left|\int_0^b \int_0^x \exp[-\lambda\int_y^b dz/c(z)]dy\{\exp[\lambda\int_x^b dz/c(z)]/c(x)\}dx\right| =$$

$$(K/\lambda)\int_0^b \{1 - \exp[-\lambda\int_y^b dz/c(z)]\}dy$$

which is bounded and since $q_0(x)\equiv 1$ is trivially bounded, by induction $q_i(x)$ is bounded for all i and f belongs to the domain of the generator.

□

We now proceed to utilise theorem 2.2.2

Theorem 2.2.3 *Let* $q_i(x)$ *be as in theorem 2.2.2, then*

$$E\{(\tau_b)^m | x_o\} = q_m(x_o) \qquad (2.2.24)$$

Proof

By theorem 2.2.2

$$f(x_t^*,t) = \begin{cases} \sum_0^m \binom{m}{i} t^{m-i} q_i(x_t^*) & 0 \leq x_t^* \leq b \\ \\ t^m & x_t^* = w \end{cases}$$

is a martingale and $\{\tau_b = t\}$ means $x_t^* = w$ for the first timeand by Doob's optional stopping theorem and since τ_b is a stopping time (2.2.24) follows immediately.

□

Using theorem 2.2.2 one can find the functions $q_i(x)$ recursively and by theorem 2.2.3 the moments of the distribution of τ_b.

For example if we assume we start from $x_o = b$ (after x_t leaves [0,b] b is the point of re-entry should this occur), we have

$$E\{\tau_b | x_o = b\} = q_1(b) = (1/\lambda)\{1 + e^{\alpha b}\int_0^b \alpha e^{-\alpha y}\exp[-\lambda\int_y^b dz/c(z)]dy\} \qquad (2.2.25)$$

(we omit the calculations).

Now, to determine the moments of the distribution of τ^b (we start from $x_o>b$), we first have to determine the conditions under which $Pr(\tau^b<\infty)=1$, otherwise b is a transient state and of course all moments of τ^b are infinite. We examine the behaviour of x_t till (if ever) it reaches b. So the behaviour when $x_t<b$ (and in particular when $x_t=0$) is irrelevant and we assume that the generator of the process is given by

$$\frac{\partial f}{\partial t} - c(x)\frac{\partial f}{\partial x} + \lambda[\int_o^\infty f(x+y,t)\alpha e^{-\alpha y}dy - f(x,t)] \qquad (2.2.26)$$

Lemma 2.2.4 *Suppose* $M > b$ and $M > x_o > b$. *Let* $\tau_M^b \equiv \inf\{t:x_t>M \text{ or } x_t=b\}$, *then*

$$Pr(x_{\tau_M^b}=b) = \frac{\{\exp[\alpha x_o - \lambda\int_b^{x_o}dz/c(z)] + \int_{x_o}^M \alpha e^{\alpha y}\exp[-\lambda\int_b^y dz/c(z)]dy\}}{e^{\alpha b} + \int_b^M \alpha e^{\alpha y}\exp[-\lambda\int_b^y dz/c(z)]dy\}} \qquad (2.2.27)$$

Proof

Consider the process y_t which agrees with x_t as long as $b\leq x_t\leq M$. When $x_t>M$ y_t moves to an absorbing state w as before. Suppose $f(w,t)=0$ and if we want to find a martingale as before we have to solve

$$-c(x)f'(x) + \lambda[\int_o^{M-x} f(x+y)\alpha e^{-\alpha y}dy - f(x)] = 0 \qquad (2.2.28)$$

Solving (2.2.28)

$$f(x) = f(M) + f(M)e^{-\alpha M}\int_x^M \{\lambda e^{\alpha y}\exp[\lambda\int_y^M dz/c(z)]/c(y)\}dy \qquad (2.2.29)$$

and then

$$f(y_t) = \begin{cases} f(y_t) & b\leq y_t\leq M \\ 0 & y_t=w \end{cases}$$

is a martingale and

$$f(x_o) = Pr(x_{\tau_M^b}=b)f(b) \qquad (2.2.30)$$

therefore

$$Pr(x_{\tau_M^b}=b) = \frac{f(x_o)}{f(b)} = \frac{1 + e^{-\alpha M}\int_{x_o}^M \{\lambda e^{\alpha y}\exp[\lambda\int_y^M dz/c(z)]/c(y)\}dy}{1 + e^{-\alpha M}\int_b^M \{\lambda e^{\alpha y}\exp[\lambda\int_y^M dz/c(z)]/c(y)\}dy} =$$

$$\frac{e^{-\alpha M}\exp[\alpha x_0-\lambda\int_{x_0}^{M}dz/c(z)] + e^{-\alpha M}\int_{x_0}^{M}\alpha e^{\alpha y}\exp[\lambda\int_{y}^{M}dz/c(z)]dy}{e^{-\alpha M}\exp[\alpha b -\lambda\int_{b}^{M}dz/c(z)] + e^{-\alpha M}\int_{b}^{M}\alpha e^{\alpha y}\exp[\lambda\int_{y}^{M}dz/c(z)]dy} =$$

$$\frac{\exp[\alpha x_0-\lambda\int_{b}^{x_0}dz/c(z)]+\int_{x_0}^{M}\alpha e^{\alpha y}\exp[-\lambda\int_{b}^{y}dz/c(z)]dy}{e^{\alpha b}+\int_{b}^{M}\alpha e^{\alpha y}\exp[-\lambda\int_{b}^{y}dz/c(z)]dy\}}$$

which is (2.2.27)

□

Now, by letting $M\to\infty$ in (2.2.27) we conclude that (we omit details)

$$\Pr(\tau^b<\infty) = 1 \text{ if and only if } \int_{x}^{\infty}\alpha e^{\alpha y}\exp[-\lambda\int_{b}^{y}dz/c(z)]dy\} \text{ is divergent}$$

Furthermore if the integral converges

$$\Pr(\tau^b<\infty) = \frac{\exp[\alpha x_0-\lambda\int_{b}^{x_0}dz/c(z)]+\int_{x_0}^{M}\alpha e^{\alpha y}\exp[-\lambda\int_{b}^{y}dz/c(z)]dy}{e^{\alpha b}+\int_{b}^{M}\alpha e^{\alpha y}\exp[-\lambda\int_{b}^{y}dz/c(z)]dy\}} \qquad (2.2.31)$$

We will now try to find the moments of the distribution of τ^b when $\Pr(\tau^b<\infty) = 1$ (a similar approach can be used to find conditional on $\tau^b<\infty$ moments when $\Pr(\tau^b<\infty) < 1$).

As $\Pr(\tau^b<\infty) = 1$ is not enough for the moments to exist, we have to see when b is a null recurrent state. Trying to find a martingale of the form $t+f(y_t)$ we have to solve

$$1 - c(x)f'(x) + \lambda[\int_{0}^{M-x}f(x+y)\alpha e^{-\alpha y}dy-f(x)] = 0 \qquad (2.2.32)$$

and assuming $f(M)=1$ we get

$$f'(x) = \{\exp[\alpha x -\lambda\int_{b}^{x}dz/c(z)]/c(x)\}\int_{x}^{M}\alpha e^{-\alpha y}\exp[\lambda\int_{b}^{y}dz/c(z)]dy$$

This implies that for $E\{\tau^b\} < \infty$ we need to have

$$\int_{x}^{\infty}\alpha e^{-\alpha y}\exp[\lambda\int_{b}^{y}dz/c(z)]dy$$

being convergent. We once again omit details.

Simlarly to theorem 2.2.3 we prove

Theorem 2.2.5. *Suppose*

$$\int_b^\infty \alpha e^{-\alpha y} \exp[\lambda \int_b^y dz/c(z)]dy \tag{2.2.33}$$

is convergent and consider functions $r_i(x)$ *(*$i=0,1,.....$*)* ($x \in [b,\infty)$) *defined as follows*

$$r_0(x) \equiv 1 \tag{2.2.34}$$

$$r_i'(x) = i\lambda\{ \exp[\alpha x - \lambda \int_b^x dz/c(z)]/c(x)\} \int_x^\infty \{r_{i-1}(y) e^{-\alpha y} \exp[\lambda \int_b^y dz/c(z)]/c(y)\}dy + i\,[r_{i-1}(x)/c(x)] \tag{2.2.35}$$

$$r_i(b) = 0 \quad (i \neq 0) \tag{2.2.36}$$

Then

$$E\{ (\tau^b)^m \mid x_o\} = r_m(x_o) \tag{2.2.37}$$

Proof

We are examining properties of x_t above b so we are not interested in its behaviour below b. The generator is given by

$$A^{**}f = \frac{\partial f}{\partial t} - c(x)\frac{\partial f}{\partial x} + \lambda[\int_o^\infty f(x+y,t)\alpha e^{-\alpha y}\, dy - f(x,t)]$$

Now

$$f(x_t,t) = \sum_o^m \binom{m}{i} t^{m-i} r_i(x_t)$$

is a martingale. We can see this as in theorem 2.2.2 by substituting and verifying that $A^{**}f(x,t) = 0$. The condition that $r_i(b)=0$ for $i\neq 0$ is not necessary but it makes things easier when applying Doob's theorem. Since the integral (2.2.33) is convergent the process is positive recurrent so τ^b is a stopping time and (2.2.37) follows by Doob's optional stopping theorem.

□

In particular

$$E\{\tau^b \mid x_o\} = \int_b^{x_o} \lambda\{ \exp[\alpha x - \lambda \int_b^x dz/c(z)]/c(x)\} \int_x^\infty \{e^{-\alpha y} \exp[\lambda \int_b^y dz/c(z)]/c(y)\}dydx + \int_b^{x_o} \frac{dx}{c(x)} =$$

$$= \int_b^{x_o} \{ \exp[\alpha x - \lambda \int_b^x dz/c(z)]/c(x)\} \int_x^\infty \alpha \{e^{-\alpha y} \exp[\lambda \int_b^y dz/c(z)]\}dydx \tag{2.2.38}$$

If we assume that $x_o - b$ has exponential (α) distribution (the first point of re-entry of x_t to the set (b,∞) has such a distribution because of the memoryless property of the exponential distribution),

$$E\{\tau^b\} = \int_0^\infty \alpha e^{-\alpha\zeta} \int_b^{b+\zeta} \{\exp[\alpha x - \lambda \int_b^x dz/c(z)]/c(x)\} \int_x^\infty \alpha \{e^{-\alpha y} \exp[\lambda \int_b^y dz/c(z)]\} dy dx d\zeta =$$

$$= (1/\lambda) e^{\alpha b} \int_b^\infty \alpha \{e^{-\alpha y} \exp[\lambda \int_b^y dz/c(z)]\} dy - (1/\lambda) \qquad (2.2.39)$$

If we view the process as an alternating renewal process going above and below b, then (see Cox & Miller (1965) p.350-356) we can deduce that when x_t is stationary its distribution function is $P(b) = 1 - \mathbf{P}(b)$ where $\mathbf{P}(b)$ is equal to

$$\frac{(1/\lambda) e^{\alpha b} \int_b^\infty \alpha \{e^{-\alpha y} \exp[\lambda \int_b^y dz/c(z)]\} dy - (1/\lambda)}{(1/\lambda)\{1 + e^{\alpha b} \int_0^b \alpha e^{-\alpha y} \exp[-\lambda \int_y^b dz/c(z)] dy\} + (1/\lambda) e^{\alpha b} \int_b^\infty \alpha \{e^{-\alpha y} \exp[\lambda \int_b^y dz/c(z)]\} dy - (1/\lambda)}$$

and therefore

$$\mathbf{P}(b) = \frac{\int_b^\infty \alpha \{e^{-\alpha y} \exp[\lambda \int_b^y dz/c(z)]\} dy - e^{-\alpha b}}{\int_0^b \alpha e^{-\alpha y} \exp[-\lambda \int_y^b dz/c(z)] dy + \int_b^\infty \alpha \{e^{-\alpha y} \exp[\lambda \int_b^y dz/c(z)]\} dy} \qquad (2.2.40)$$

which can also be derived from the forward equation (see Cox & Isham (1986)) By letting $b \downarrow 0$ we see that if $\int_b^M dy/c(y)$ is divergent $\mathbf{P}(0)=1$, while if it converges $\mathbf{P}(0)<1$.

2.3 THE SHOT-NOISE PROCESS

A famous particular example of the process described in the last section and of particular interest in many diverse fields is the *shot-noise process* where the decay is exponential ($c(x)=\delta x$, where δ is a constant). We will have generally distributed jump sizes with distribution function $G(y)$ $(y>0)$. From the last section it should be clear that the process can not ever reach 0. The generator of the process

(x_t,t) acting on a function $f(x,t)$ belonging to its domain is given by

$$Af(x,t) = \frac{\partial f}{\partial t} - \delta x\frac{\partial f}{\partial x} + \lambda[\int_{\mathbb{R}^+} f(x+y,t)dG(y)-f(x,t)] \qquad (2.3.1)$$

For this process we can find the distribution of x_t at any given time t given the distribution of x_o. We start with a related theorem.

Theorem 2.3.1 *Let* $z_t = x_t e^{\delta t}$. *Then*

$$\exp(-\nu z_t)\exp\{\lambda\int_o^t[1-\hat{g}(\nu e^{\delta s})]ds\} \qquad (\hat{g}(\nu)=\int_{\mathbb{R}^+} e^{-\nu y}dG(y))$$

is a martingale for all $\nu\geq 0$.

Proof

The generator of (z_t,t) acting on a function $f(z,t)$ is given by

$$A^*f(z,t) = \frac{\partial f}{\partial t} + \lambda[\int_{\mathbb{R}^+} f(z+ye^{\delta t},t)dG(y)-f(z,t)] \qquad (2.3.2)$$

So in order to find a martingale we should solve

$$\frac{\partial f}{\partial t} + \lambda[\int_{\mathbb{R}^+} f(z+ye^{\delta t},t)dG(y)-f(z,t)] = 0 \qquad (2.3.3)$$

and trying a solution of the form $e^{-\nu z}h(t)$ we get

$$h'(t) + \lambda[\hat{g}(\nu e^{\delta t})-1]h(t) = 0 \qquad (2.3.4)$$

yielding

$$h(t) = K\exp\{\lambda\int_o^t[1-\hat{g}(\nu e^{\delta s})]ds\} \qquad (2.3.5)$$

and since $e^{-\nu z}$ is bounded it belongs to the domain of the generator and the theorem has been proved.

□

We will now use the martingale found.

Theorem 2.3.2 *Let* x_t *as defined. Then*

$$E\{\exp(-\nu x_t)\,|\,x_o\} = \exp(-\nu x_o e^{-\delta t})\exp\{-\lambda\int_o^t[1-\hat{g}(\nu e^{-\delta s})]ds\} \qquad (2.3.6)$$

Proof

From theorem 2.3.1 for a fixed time t^* and a fixed constant $\nu^*\geq 0$ we have

$$E\{\exp(-\nu^* z_{t^*})\,|\,x_o\} = \exp(-\nu^* z_o)\,\exp\{-\lambda\int_o^{t^*}[1-\hat{g}(\nu^* e^{\delta s})]ds\} \qquad (2.3.7)$$

and since $z_t = x_t e^{\delta t}$

$$E\{\exp(-\nu^* e^{\delta t^*} x_{t^*}) \mid x_0\} = \exp(-\nu^* x_0)\, \exp\{-\lambda \int_0^{t^*} [1-\hat{g}(\nu^* e^{\delta s})]ds\}$$

and setting $\nu = \nu^* e^{\delta t^*}$ we have

$$E\{\exp(-\nu x_{t^*}) \mid x_0\} = \exp(-\nu e^{-\delta t^*} x_0) \exp\{-\lambda \int_0^{t^*} [1-\hat{g}(\nu e^{-\delta(t-s)})]ds\} =$$

$$= \exp(-\nu e^{-\delta t^*} x_0) \exp\{-\lambda \int_0^{t^*} [1-\hat{g}(\nu e^{-\delta s})]ds\} \qquad (2.3.8)$$

Since (2.3.8) holds for an arbitrary fixed t^*, it holds for all $t \geq 0$ and the theorem is proved.

□

Also the asymptotic (stationary) distribution can be obtained from theorem 2.3.2

Corollary 2.3.3 *The asymptotic distribution of* x_t *has Laplace transform*

$$\exp\{-\lambda \int_0^{\infty} [1-\hat{g}(\nu e^{-\delta s})]ds\}$$

which can also be written as

$$\exp\{-(\lambda/\delta) \int_0^{\nu} \hat{\mathbf{G}}(u)du\} \qquad (\hat{\mathbf{G}}(u) = [1-\hat{g}(u)]/u)$$

Proof

We let $t \to \infty$ in (2.3.6) and the corollary follows immediately.

□

One should notice the analogy of theorem 2.3.1, the Laplace transform found in theorem 2.3.2 and corollary 2.3.3 with results in section 1.2. Although the process in section 1.2 is transient and here we have a recurrent process, there are many common features and this is not a coincidence (z_t here and x_t of section 1.2 are essentially the same process with δ positive and negative respectively).

If we cut the process at M as in the previous section (see the notes after theorem 2.2.1) then the generator is given by

$$\frac{\partial f}{\partial t} - \delta x \frac{\partial f}{\partial x} + \lambda [\int_{(0,M-x)} f(x+y,t)dG(y) + f(M,t)\mathbf{G}(M-x) - f(x,t) \qquad (2.3.9)$$

As we have said in the previous section the stationary distribution function is

$G^*_M(y)$, which is such that

$$G^*_M(y) = G^*(y) \,/\, \int_{[0,M]} dG^*(y) \quad (0 \leq y \leq M \text{ and } \hat{g}^*(\nu) = \exp\{-(\lambda/\delta)\int_0^{\nu} \mathbf{G}(u)du\})$$

Using the martingale found in theorem 2.2.1 we can find the second and higher order properties of x_t. We proceed as follows.

Lemma 2.3.4 *Let* $t_1, t_2, \ldots\ldots, t_n$ *be fixed times and* $\nu_1^*, \nu_2^*, \ldots\ldots, \nu_n^*$ *fixed positive constants. Then*

$$E\{\prod_1^n \exp(-\nu_i^* z_{t+t_i}) \,|\, z_o\} = \exp(-\rho_1 z_o)\exp\{-\lambda\int_o^{t+t_1}[1-\hat{g}(\rho_1 e^{\delta s})]ds\}\prod_2^n \exp\{-\lambda\int_{t+t_{i-1}}^{t+t_i}[1-\hat{g}(\rho_i e^{\delta s})]ds\} \tag{2.3.10}$$

where

$$\rho_i = \sum_i^n \nu_j^*$$

Proof

We proceed by induction. (2.3.10) is clearly true for n=1 by theorem 2.3.1. Suppose it is true for n=k; then

$$E\{\prod_1^{k+1} \exp(-\nu_i^* z_{t+t_i}) \,|\, z_o\} = E\{\exp(-\nu_{k+1}^* z_{t+t_{k+1}}) \prod_1^k \exp(-\nu_i^* z_{t+t_i}) \,|\, z_o\} =$$

$$E\{E\{\exp(-\nu_{k+1}^* z_{t+t_{k+1}}) \,|\, z_{t+t_k}\} \prod_1^k \exp(-\nu_i^* z_{t+t_i}) \,|\, z_o\} \tag{2.3.11}$$

but by theorem 2.3.1 we have

$$E\{\exp(-\nu_{k+1}^* z_{t+t_{k+1}}) \,|\, z_{t+t_k}\} = \exp(-\nu_{k+1}^* z_{t+t_k}) \exp\{-\lambda\int_{t+t_{k-1}}^{t+t_k}[1-\hat{g}(\nu_{k+1}^* e^{\delta s})]ds\}$$

and since (2.3.10) is true for n=k using it with $\nu_{k+1}^*+\nu_k^*$ in place of ν_k^*, (2.3.10) follows for n=k+1.

□

We proceed to find the joint Laplace transform of x_t at various points in time. What we need to do is to set $z_t = x_t e^{\delta t}$ and $\nu_i^* = \nu_i \exp[-\delta(t+t_i)]$ in lemma 2.3.4

Theorem 2.3.5

$$E\{\prod_1^n \exp(-\nu_i x_{t+t_i}) \mid x_0\} = \exp(-\xi_1 e^{\delta t} x_0)\exp\{-\lambda\int_0^{t+t_1}[1-\hat{g}(\xi_1 \exp(\delta t_1)e^{-\delta s})]ds\} \cdot$$

$$\prod_2^n \exp\{-\lambda\int_0^{t_i-t_{i-1}}[1-\hat{g}(\xi_i \exp(\delta t_i)e^{-\delta s})]ds\} \qquad (2.3.12)$$

where

$$\xi_i = \sum_i^n \nu_j \exp(-\delta t_j) \qquad (2.3.13)$$

Proof

We substitute $\nu_i^* = \nu_i \exp[-\delta(t+t_i)]$ in (2.3.10) and (2.3.12) follows if we observe that

$$\int_0^{t+t_1}[1-\hat{g}(\xi_1 e^{-\delta(t-s)})]ds = \int_0^{t+t_1}[1-\hat{g}(\xi_1 \exp(\delta t_1)e^{-\delta s})]ds$$

and

$$\int_{t+t_{i-1}}^{t+t_i}[1-\hat{g}(\xi_i e^{-\delta(t-s)})]ds = \int_0^{t_i-t_{i-1}}[1-\hat{g}(\xi_i \exp(\delta t_i)e^{-\delta s})]ds$$

□

We can also get the asymptotic behaviour of x_t.

Corollary 2.3.6

$$\lim_{t\to\infty} E\{\prod_1^n \exp(-\nu_i x_{t+t_i}) \mid x_0\} = \exp(-\xi_1 e^{\delta t} x_0)\exp\{-\lambda\int_0^{\infty}[1-\hat{g}(\xi_1 \exp(\delta t_1)e^{-\delta s})]ds\} \cdot$$

$$\prod_2^n \exp\{-\lambda\int_0^{t_i-t_{i-1}}[1-\hat{g}(\xi_i \exp(\delta t_i)e^{-\delta s})]ds\} \qquad (2.3.14)$$

where ξ_i are as in theorem 2.3.5.

Proof

We let $t\to\infty$ in (2.3.12) and (2.3.14) follows immediately.

□

Another important problem is to find aspects of the distribution of the time needed for x_t to reach a fixed level b starting from above ($x_0>b$). Again it should be noted that we are interested in the behaviour of x_t above b only.

Theorem 2.3.7 *Let* $\tau^b \equiv \inf\{t: x_t=b\}$ $(x_o>b)$. *Then for all* $\theta>0$

$$E\{e^{-\theta\tau^b}|x_o\} = \frac{\int_o^\infty \exp(-x_o u)\exp[(\lambda/\delta)\int_o^u \hat{G}(v)dv]u^{(\theta/\delta)-1}du}{\int_o^\infty \exp(-bu)\exp[(\lambda/\delta)\int_o^u \hat{G}(v)dv]u^{(\theta/\delta)-1}du} \tag{2.3.15}$$

Proof

To find a martingale of the form $e^{-\theta t}f(x_t)$ we have to solve (we apply the generator)

$$-\theta f(x)-\delta x f'(x)+\lambda[\int_{\mathbb{R}^+} f(x+y)dG(y)-f(x)] = 0 \tag{2.3.16}$$

Trying a solution of the form $\int_o^\infty e^{-xu}h(u)du$ we get

$$-\theta\int_o^\infty e^{-xu}h(u)du+\delta x\int_o^\infty e^{-xu}uh(u)du+\lambda[\int_o^\infty e^{-xu}h(u)\hat{g}(u)du-\int_o^\infty e^{-xu}h(u)du] = 0 \tag{2.3.17}$$

Integrating by parts

$$x\int_o^\infty e^{-xu}uh(u)du = \int_o^\infty e^{-xu}[uh(u)]'du \tag{2.3.18}$$

(2.3.17) and (2.3.18) give us

$$-\theta h(u)+\delta[uh(u)]'-\lambda[1-\hat{g}(u)]h(u) = 0 \tag{2.3.19}$$

and solving (2.3.19)

$$h(u) = K\exp[(\lambda/\delta)\int_o^u \hat{G}(v)dv]u^{(\theta/\delta)-1} \tag{2.3.20}$$

so

$$e^{-\theta t}\int_o^\infty \exp(-x_t u)\exp[(\lambda/\delta)\int_o^u \hat{G}(v)dv]u^{(\theta/\delta)-1}du$$

is a martingale. (This function belongs to the domain of the generator of the process evolving up to the time $x_t=b$ since it is then bounded).

Applying Doob's theorem (2.3.15) follows.

□

We can find all moments of τ^b by succesively differentiating (2.3.15) with respect to θ. First notice that by integrating by parts

$$E\{e^{-\theta\tau^b}|x_o\} = \frac{\int_o^\infty \{\exp(-x_o u)\exp[(\lambda/\delta)\int_o^u \hat{G}(v)dv]\}' u^{(\theta/\delta)}du}{\int_o^\infty \{\exp(-bu)\exp[(\lambda/\delta)\int_o^u \hat{G}(v)dv]\}' u^{(\theta/\delta)}du} \tag{2.3.21}$$

By differentiating (2.3.21) and letting $\theta \downarrow 0$, we get

$$E\{e^{-\theta\tau^b}|x_0\} = -\frac{\int_0^\infty \{\exp(-x_0u)\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]\}'(1/\delta)\log u\,du}{\int_0^\infty \{\exp(-bu)\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]\}'du} +$$

$$\frac{\int_0^\infty \{\exp(-x_0u)\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]\}'du \int_0^\infty \{\exp(-bu)\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]\}'(1/\delta)\log u\,du}{\left\{\int_0^\infty \{\exp(-bu)\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]\}'du\right\}^2} =$$

$$\int_0^\infty \left\{\{\exp(-bu)\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]\}' - \{\exp(-x_0u)\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]\}'\right\}(1/\delta)\log u\,du =$$

$$(1/\delta)\int_0^\infty (1/u)[\exp(-bu)-\exp(-x_0u)]\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]du \qquad (2.3.22)$$

It is much more difficult to solve the problem of finding the distribution of the time needed for x_t to cross b starting from below b. Complicated formulas involving the distribution of the overshoot will be involved. However, when jumps are exponentially distributed, the approach of the previous section is applicable and we just have to substitute $c(x)=\delta x$ in theorem 2.2.5.

An application is the following. Consider the process x_t decaying with rate $c+\delta x_t$ instead of δx_t; then the problem of finding the distribution of the time needed for x_t to reach b starting from $x_0>b$ is solved by looking at the process $x_t^*=(c/\delta)+x_t$ which has rate of decay δx_t^* as before. b corresponds to $(c/\delta)+b$, so now

$$E\{e^{-\theta\tau^b}|x_0\} = \frac{\int_0^\infty \exp[-[x_0+(c/\delta)]u]\ \exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]u^{(\theta/\delta)-1}du}{\int_0^\infty \exp[-[b+(c/\delta)]u]\ \exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]u^{(\theta/\delta)-1}du} \qquad (2.3.23)$$

So if x_t models the level of a river with δx_t being the rate of water lost due to evaporation and c due to other causes and the jumps represent rains adding water to the river then the Laplace transform of the distribution of the period the river has water is (notice that the distribution function of x_0 is G(y); the first rain after an empty river)

$$\frac{\int_0^\infty \exp[-(c/\delta)u]\hat{g}(u)\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]u^{(\theta/\delta)-1}du}{\int_0^\infty \exp[-(c/\delta)u]\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]u^{(\theta/\delta)-1}du} \tag{2.3.24}$$

and the mean is

$$(1/\delta)\int_0^\infty \exp[-(c/\delta)u]\hat{\bar{G}}(u)\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]du \tag{2.3.25}$$

(calculations are omitted).

If the process is like the ones in section 2.2; that is at 0 (empty river) no deterministic change occurs, the expected time the river will take to have water again is $1/\lambda$, so if we view the process as an alternating renewal process the probability that the river is empty is

$$\frac{(1/\lambda)}{(1/\lambda) + (1/\delta)\int_0^\infty \exp[-(c/\delta)u]\hat{\bar{G}}(u)\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]du} \tag{2.3.26}$$

which after integrating by parts simplifies to

$$\frac{1}{\int_0^\infty (c/\delta)\exp[-(c/\delta)u]\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]du} \tag{2.3.27}$$

Furthermore we can find the stationary distribution of x_t in this case.

Theorem 2.3.8 *Let* x_t *as described (decay with rate* $c+\delta x_t$*) and* $x_t^*=x_t+(c/\delta)$ *(when* x_t *reaches* c/δ *it remains there till a jump occurs). Then*

$$\int_0^\nu (1/u)[1-\exp(-x_t^* u)]\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]du - \delta\int_0^t \exp(-\nu x_s^*)I(x_s^*>c/\delta)\exp[(\lambda/\delta)\int_0^\nu \hat{G}(v)dv]ds +$$

$$\delta\int_0^t I(x_s^*>c/\delta)ds - \lambda\int_0^t\int_0^\nu \exp[-(c/\delta)u]\hat{\bar{G}}(u)\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]du\,I(x_s^*=c/\delta)ds \tag{2.3.28}$$

is a martingale.

Proof

The generator of x_t^* acting on a function $f(x)$ is given by

$$A^*f(x) = \begin{cases} \delta x f'(x)+\lambda[\int_{\mathbb{R}^+} f(x+y)dG(y)-f(x)] & x>c/\delta \\ \lambda[\int_{\mathbb{R}^+} f(x+y)dG(y)-f(x)] & x=c/\delta \end{cases} \tag{2.3.29}$$

If we apply it on

$$f(x) = \int_o^{\nu}(1/u)[1-\exp(-xu)]\ \exp[(\lambda/\delta)\int_o^u \hat{\mathbf{G}}(v)dv]du$$

we get

$$-\delta x f'(x)I(x>c/\delta) + \lambda[\int_{\mathbb{R}^+} f(x+y)dG(y)-f(x)] =$$

$$-\delta x f'(x)I(x>c/\delta) + \lambda \int_{\mathbb{R}^+} f'(x+y)\mathbf{G}(y)dy =$$

$$-\delta\int_o^{\nu} xe^{-xu}\exp[(\lambda/\delta)\int_o^u \hat{\mathbf{G}}(v)dv]duI(x>c/\delta) + \lambda\int_o^{\nu} e^{-xu}\exp[(\lambda/\delta)\int_o^u \hat{\mathbf{G}}(v)dv]\hat{\mathbf{G}}(u)du =$$

$$-\delta e^{-x\nu}\exp[(\lambda/\delta)\int_o^{\nu}\hat{\mathbf{G}}(v)dv]I(x>c/\delta)-\delta I(x>c/\delta)- \lambda\int_o^{\nu}e^{-xu}\exp[(\lambda/\delta)\int_o^u \hat{\mathbf{G}}(v)dv]\hat{\mathbf{G}}(u)duI(x>c/\delta)$$

$$+ \lambda\int_o^{\nu} e^{-xu}\exp[(\lambda/\delta)\int_o^u \hat{\mathbf{G}}(v)dv]\hat{\mathbf{G}}(u)du =$$

$$-\delta e^{-x\nu}\exp[(\lambda/\delta)\int_o^{\nu}\hat{\mathbf{G}}(v)dv]I(x>c/\delta)-\delta I(x>c/\delta)+\lambda\int_o^{\nu}e^{-xu}\exp[(\lambda/\delta)\int_o^u \hat{\mathbf{G}}(v)dv]\hat{\mathbf{G}}(u)duI(x=c/\delta)$$

and since

$$f(x_t^*) - \int_o^t A^*f(x_s^*)ds$$

is a martingale (2.3.28) is a martingale.

□

The martingale in theorem 2.3.8 provides us with the equation

$$E\{\int_o^{\nu}(1/u)[1-\exp(-x_t^*u)]\exp[(\lambda/\delta)\int_o^u \hat{\mathbf{G}}(v)dv]du\} -$$

$$E\{\int_o^{\nu}(1/u)[1-\exp(-x_o^*u)]\exp[(\lambda/\delta)\int_o^u \hat{\mathbf{G}}(v)dv]du\} =$$

$$\delta\int_o^t E\{\exp(-\nu x_s^*)I(x_s^*>c/\delta)\}\exp[(\lambda/\delta)\int_o^{\nu}\hat{\mathbf{G}}(v)dv]ds - \delta\int_o^t Pr(x_s>c/\delta)ds +$$

$$\lambda\int_o^t\int_o^{\nu}\exp[-(c/\delta)u]\hat{\mathbf{G}}(u)\exp[(\lambda/\delta)\int_o^u \hat{\mathbf{G}}(v)dv]duPr(x_s^*=c/\delta)ds \qquad (2.3.30)$$

From (2.3.30) we conclude as with the forward equation (see proposition 2.2.1) that if x_t^* is stationary its distribution should obey the following

$$E\{\exp(-\nu x_t^*)I(x_t^*>c/\delta)\}\exp[(\lambda/\delta)\int_o^{\nu}\hat{\mathbf{G}}(v)dv] - Pr(x_t>c/\delta) +$$

$$(\lambda/\delta)\int_o^{\nu}\exp[-(c/\delta)u]\hat{\mathbf{G}}(u)\exp[(\lambda/\delta)\int_o^u \hat{\mathbf{G}}(v)dv]duPr(x_t^*=c/\delta) = 0 \qquad (2.3.31)$$

Integrating by parts

$$E\{\exp(-\nu x_t^*)I(x_t^*>c/\delta)\}\exp[(\lambda/\delta)\int_0^\nu \hat{G}(v)dv] =$$

$$1 - \{e^{-c\nu/\delta}\exp[(\lambda/\delta)\int_0^\nu \hat{G}(v)dv] + \int_0^\nu (c/\delta)\exp[-(c/\delta)u]\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]du\}\Pr(x_t^*=c/\delta) \tag{2.3.32}$$

Letting $t\to\infty$ in (2.3.32)

$$\Pr(x_t^*=c/\delta) = \frac{1}{\int_0^\infty (c/\delta)\exp[-(c/\delta)u]\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]du} \tag{2.3.33}$$

(cf (2.3.27)) and (2.3.32) & (2.3.33) give us

$$E\{\exp(-\nu x_t^*)I(x_t^*>c/\delta)\}\exp[(\lambda/\delta)\int_0^\nu \hat{G}(v)dv] =$$

$$1 - \frac{e^{-c\nu/\delta}\exp[(\lambda/\delta)\int_0^\nu \hat{G}(v)dv] + \int_0^\nu (c/\delta)\exp[-(c/\delta)u]\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]du}{\int_0^\infty (c/\delta)\exp[-(c/\delta)u]\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]du} =$$

$$= \frac{-e^{-c\nu/\delta}\exp[(\lambda/\delta)\int_0^\nu \hat{G}(v)dv] + \int_\nu^\infty (c/\delta)\exp[-(c/\delta)u]\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]du}{\int_0^\infty (c/\delta)\exp[-(c/\delta)u]\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]du}$$

and integrating by parts

$$E\{\exp(-\nu x_t^*)I(x_t^*>c/\delta)\} = \exp[-(\lambda/\delta)\int_0^\nu \hat{G}(v)dv]\cdot\frac{\frac{\lambda}{\delta}\int_\nu^\infty \exp[-(c/\delta)u]\hat{G}(u)\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]du}{\int_0^\infty (c/\delta)\exp[-(c/\delta)u]\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]du} \tag{2.3.34}$$

(2.3.34) gives the distribution of x_t^* when it is stationary. Then x_t is stationary too and we have

$$E\{\exp(-\nu x_t)I(x_t>0)\} = e^{c\nu/\delta}\exp[-(\lambda/\delta)\int_0^\nu \hat{G}(v)dv]\cdot\frac{\frac{\lambda}{\delta}\int_\nu^\infty \exp[-(c/\delta)u]\hat{G}(u)\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]du}{\int_0^\infty (c/\delta)\exp[-(c/\delta)u]\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]du} \tag{2.3.35}$$

Finally one should observe that from (2.3.34) and (2.3.33) we have

$$E\{\exp(-\nu x_t^*)\mid x_t^*>c/\delta\} = \exp[-(\lambda/\delta)\int_0^\nu \hat{G}(v)dv]\cdot\frac{\int_\nu^\infty \exp[-(c/\delta)u]\hat{G}(u)\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]du}{\int_0^\infty (c/\delta)\exp[-(c/\delta)u]\exp[(\lambda/\delta)\int_0^u \hat{G}(v)dv]du} \tag{2.3.36}$$

which is not in general the restriction of a distribution with Laplace transform

$\exp\{-(\lambda/\delta)\int_0^{\nu}\hat{G}(u)du\}$ as when we cut the process from above (see the notes after corollary 2.3.3).

2.4 TIME DEPENDENT PARAMETERS

We are now going to generalise the shot noise process described in the previous section by allowing all its parameters to depend on time. So we assume $\delta(t)$, $\lambda(t)$, $G(y;t)$ are all Riemann integrable functions of t and they are positive. Furthermore $\lambda(t)$ is bounded on all intervals $[0,t)$ $(t>0)$ (no explosions allowed). $\delta(t)x_t$ is the rate of decay, $\lambda(t)$ the rate of jump arrivals and $G(y;t)$ $(y>0)$ a distribution function for all t is the distribution function of their sizes. So the generator of (x_t,t) acting on a function $f(x,t)$ belonging to its domain is given by

$$Af(x,t) = \frac{\partial f}{\partial t} - \delta(t)x\frac{\partial f}{\partial x} + \lambda(t)\left[\int_{\mathbb{R}^+} f(x+y,t)dG(y;t) - f(x,t)\right] \qquad (2.4.1)$$

We will find an expression for the Laplace transform of the distribution of x_t at time t.

Lemma 2.4.1 *Let*

$$z_t = x_t\exp\left[\int_0^t \delta(u)du\right],$$

then for all $\nu^* \geq 0$

$$\exp(-\nu^* z_t)\exp\left\{\int_0^t \lambda(s)\left[1-\hat{g}\left(\nu^*\exp\left[\int_0^s \delta(u)du\right];s\right)\right]ds\right\}$$

is a martingale, where

$$\hat{g}(\nu;s) = \int_{\mathbb{R}^+} e^{-\nu y}dG(y;s)$$

Proof

The generator of (z_t,t) is given by

$$A^*f(z,t) = \frac{\partial f}{\partial t} + \lambda(t)\,[\int_{\mathbb{R}^+} f(x+y\exp[\int_0^t \delta(u)du],t)dG(y;t)-f(x,t)] \qquad (2.4.2)$$

Substituting $e^{-\nu^* z}h(t)$ in $A^*f=0$ we get

$$h'(t) + \lambda(t)[\hat{g}(\nu^*\exp[\int_0^t \delta(u)du];t)-1]h(t) = 0 \qquad (2.4.3)$$

and solving (2.4.3) we get

$$h(t) = K\exp\{\int_0^t \lambda(s)[1-\hat{g}(\nu^*\exp[\int_0^s \delta(u)du];s)]ds\}$$

and the result follows.

□

Theorem 2.4.2

$$E\{\exp(-\nu x_t)\,|\,x_0\} = \exp[-\nu x_0\exp(-\int_0^s \delta(u)du)]\exp\{\int_0^t \lambda(s)[1-\hat{g}(\nu^*\exp[\int_0^s \delta(u)du];s)]ds\} \qquad (2.4.4)$$

for $\nu\geq 0$.

Proof

We fix t and set

$$\nu^* = \nu\exp(-\int_0^s \delta(u)du)$$

in analogy to what we did in the proof of theorem 2.3.2 and the result follows.

□

We can try to obtain information about the asymptotic behaviour of x_t by letting $t\to\infty$. Of course x_t can be transient or recurrent and it may have or it may have not an asymptotic distribution.

We can also obtain second and higher moment properties as in theorem 2.3.5 by using the martingale in theorem 2.4.1 and working in exactly the same way. We omit details.

An example could be when $G(y;t)\equiv G(y)$, $\delta(t)\equiv\delta$, but $\lambda(t)=\lambda_0+\lambda_1\cos(2\pi t/T)$ $(0<\lambda_1<\lambda_0)$. This might model the level of a dam whose outflow rate is δx_t and inflow is in terms of jumps (due to rains) whose rate of arrival varies periodically. When rain is at its peak (t=kT, k an integer) we say we have

"winter" and when at its lowest (t=(2k+1)T/2) we call the event "summer".

By theorem 2.4.2

$$E\{\exp(-\nu x_t)\,|\,x_o\} = \exp[-\nu x_o e^{-\delta t}]\exp\{\int_o^t [\lambda_o+\lambda_1\cos(2\pi s/T)][1-\hat{g}(\nu e^{-\delta(t-s)}]ds\} \qquad (2.4.5)$$

so we can evaluate $E\{x_t\}$

$$E\{x_t\} = E\{x_o\}e^{-\delta t} + \int_o^t [\lambda_o+\lambda_1\cos(2\pi s/T)]\mu_1 e^{-\delta(t-s)}ds =$$

$$E\{x_o\}e^{-\delta t} + (\lambda_o\mu_1/\delta)(1-e^{-\delta t}) + \lambda_1\mu_1 e^{-\delta t}\int_o^t e^{\delta s}\cos(2\pi s/T)ds \qquad (2.4.6)$$

But integrating by parts

$$\int_o^t e^{\delta s}\cos(2\pi s/T)ds = (1/\delta)e^{\delta t}\cos(2\pi t/T) - 1 + \int_o^t (1/\delta)(2\pi/T)e^{\delta s}\sin(2\pi s/T)ds$$

and

$$\int_o^t (1/\delta)(2\pi/T)e^{\delta s}\sin(2\pi s/T)ds = (1/\delta)(2\pi/T)e^{\delta t}\sin(2\pi t/T) - (2\pi/\delta T)^2\int_o^t e^{\delta s}\cos(2\pi s/T)ds$$

and therefore

$$E\{x_t\} = E\{x_o\}e^{-\delta t} + (\lambda_o\mu_1/\delta)(1-e^{-\delta t}) + (\lambda_1\mu_1/\delta)\frac{\cos(2\pi t/T)+(2\pi/T\delta)\sin(2\pi t/T)-e^{-\delta t}}{1+(2\pi/\delta T)^2} \qquad (2.4.7)$$

Writing $t=kT+t^*$ $(0\leq t^*<T)$, letting $k\to\infty$ and denoting

$$E\{\hat{x}_{t^*}\} = \lim_{k\to\infty} E\{x_{kT+t^*}\}$$

(the expected level of the dam at the "season" t^* when a steady state has been reached), we have

$$E\{\hat{x}_{t^*}\} = (\lambda_o\mu_1/\delta) + (\lambda_1\mu_1/\delta)\frac{\cos(2\pi t^*/T)+(2\pi/T\delta)\sin(2\pi t^*/T)}{1+(2\pi/\delta T)^2} \qquad (2.4.8)$$

which can also be rewritten as

$$E\{\hat{x}_{t^*}\} = (\lambda_o\mu_1/\delta) + (\lambda_1\mu_1/\delta)\frac{\cos[(2\pi t^*/T)-\phi]}{\{1+(2\pi/\delta T)^2\}^{1/2}} \qquad (2.4.9)$$

where $\tan\phi = 2\pi/\delta T$.

So the expected level at t^* varies periodically as $\lambda(t)$ does. However, there is a phase difference between them with the maximum (minimum) expected level

being attained some time after "winter" (between 0 and T/4; that is between "winter" and "spring"). The phase difference is a decaying function of δ. That is the fastest the decay the more "domination" $\lambda(t)$ has on the model.

2.5 SHOT-NOISE PROCESS WITH A RENEWAL PROCESS AS UNDERLYING JUMP ARRIVAL PROCESS

In this section we examine the shot-noise process of section 2.3 with rate of decay δx_t and jump sizes independent identically distributed random variables which are independent of the jump size variables as well and their distribution function P(u) posseses a derivative (density) p(u). Also ρ_1 its first moment exists. The restriction that P(u) posseses a density is by no means necessary but we include it since this time we will include in our process as our supplementary variable the time elapsed since the last jump rather than the time needed for the next. This is because the mathematics involved will be much simpler and more elegant. The generator of the process (x_t,u_t,t) acting on a function f(x,u,t) belonging to its domain is given by (u_t denotes the time elapsed since the last jump)

$$Af(x,u,t) = \frac{\partial f}{\partial t} + \delta x\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} + \frac{p(u)}{\mathbf{P}(u)}[\int_{\mathbb{R}^+} f(x+y,0,t)dG(y)-f(x,u,t)] \qquad (2.5.1)$$

We will first state a useful proposition which is a well known result in renewal theory.

Proposition 2.5.1 *The asymptotic and stationary distribution of* u_t *has density* $\mathbf{P}(u)/\rho_1$

(For proof see Cox & Miller (1965) p.347)

We will now proceed to find the moments of x_t as we did in sections 1.3 and 2.1. We start by the first two moments.

Theorem 2.5.2 *Suppose the first two moments of* G *exist and are* μ_1 *and* μ_2, *then*

a)

$$e^{\delta t}[x_t - \mu_1 \exp(-\delta u_t)/\mathbf{P}(u_t)] \tag{2.5.2}$$

is a martingale, and

b)

$$e^{2\delta t}[x_t^2 - 2\mu_1 x_t \exp(-\delta u_t)/\mathbf{P}(u_t) - (\mu_2 - 2\mu_1^2)\exp(-2\delta u_t)/\mathbf{P}(u_t) \tag{2.5.3}$$

is a martingale too.

Proof

a)

Trying to find a martingale of the form $x_t e^{\delta t} + h(u,t)$ we substitute in the equation $Af=0$ and get

$$-\delta x e^{\delta t} + \delta x e^{\delta t} + \frac{\partial h}{\partial t} + \frac{\partial h}{\partial u} + \frac{p(u)}{\mathbf{P}(u)}[x e^{\delta t} + \mu_1 e^{\delta t} - x e^{\delta t} + h(0,t) - h(u,t)] = 0 \tag{2.5.4}$$

and simplifying (2.5.4)

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial u} + \frac{p(u)}{\mathbf{P}(u)}[\mu_1 e^{\delta t} + h(0,t) - h(u,t)] = 0 \tag{2.5.5}$$

and trying a solution of the form $e^{\delta t}k(u)$

$$\delta k(u) + k'(u) + \frac{p(u)}{\mathbf{P}(u)}[k(0) - k(u) + \mu_1] = 0 \tag{2.5.6}$$

which is equivalent to

$$[k(u)\mathbf{P}(u)]' + \delta \mathbf{P}(u)k(u) + p(u)(k(0) + \mu_1) = 0 \tag{2.5.7}$$

and

$$k(u) = -\mu_1 e^{-\delta u}/\mathbf{P}(u) \tag{2.5.8}$$

is the only solution that belongs to the domain of the generator since all the

others involve $[\mathbf{P}(u)]^{-1}$(as before) but in this case

$$E\{e^{-\delta T}/\mathbf{P}(T)\} = \int_{\mathbb{R}^+}\{e^{-\delta u}p(u)/\mathbf{P}(u)\}du < \infty$$

and (a) follows.

b)

Along the same lines we try a solution of the form

$$x^2e^{2\delta t}+2xe^{\delta t}h_1(u,t)+h_2(u,t)$$

Substituting in Af=0

$$2xe^{\delta t}[\frac{\partial h_1}{\partial u}+\frac{\partial h_1}{\partial t}]+\frac{\partial h_2}{\partial u}+\frac{\partial h_2}{\partial t}+$$

$$\frac{p(u)}{\mathbf{P}(u)}[2xe^{\delta t}h_1(0,t)+2\mu_1e^{\delta t}h_1(0,t)+2x\mu_1e^{2\delta t}+\mu_2e^{2\delta t}-2xe^{\delta t}h_1(u,t)-h_2(u,t)] = 0$$

which implies

$$2\{\frac{\partial h_1}{\partial t}+\frac{\partial h_1}{\partial u}+\frac{p(u)}{\mathbf{P}(u)}[\mu_1e^{\delta t}+h_1(0,t)-h_1(u,t)]\} = 0$$

which is (2.5.5) and therefore

$$k_1(u,t) = -e^{\delta t}\ \mu_1e^{-\delta u}/\mathbf{P}(u)$$

and

$$\frac{\partial h_2}{\partial t}+\frac{\partial h_2}{\partial u}+\frac{p(u)}{\mathbf{P}(u)}[-2\mu_1^2e^{2\delta t}+\mu_2e^{2\delta t}+h_2(0,t)-h_2(u,t)] = 0 \qquad (2.5.9)$$

and solving (2.5.9) exactly as (2.5.5) we get

$$h_2(u,t) = -(\mu_2-2\mu_1^2)e^{2\delta u}e^{-2\delta t}/\mathbf{P}(u)$$

and (b) follows.

□

So we conclude the following.

<u>Corollary 2.5.3</u> *Let* G *possess at least two moments* μ_1 *and* μ_2. *Let also* $\hat{p}(\theta)$ *denote the* Laplace *transform of* p(y). *Then*

$$\lim_{t\to\infty}E\{(x_t)^i\exp(i\delta u_t)\} \text{ and } \lim_{t\to\infty}E\{(x_t)^i\}$$

exist and if we denote them by L_i *and* M_i *respectively, we have*

a)

$$L_1 = \mu_1/[1-\hat{p}(\delta)] \tag{2.5.10}$$

$$M_1 = \mu_1/\rho_1\delta \tag{2.5.11}$$

b)

$$L_2 = \{\mu_2/2 + \mu_1^2\hat{p}(\delta)/[1-\hat{p}(\delta)]\} \;/\; [1-\hat{p}(2\delta)] \tag{2.5.12}$$

$$M_2 = \{\mu_2/2 + \mu_1^2\hat{p}(\delta)/[1-\hat{p}(\delta)]\} \;/\; \rho_1\delta \tag{2.5.13}$$

Proof

By (2.5.2) we have

$$E\{x_t\} - E\{\mu_1\exp(-\delta u_t)/\mathbf{P}(u_t)\} = e^{-\delta t}\{E\{x_0\} - E\{\mu_1\exp(-\delta u_0)/\mathbf{P}(u_0)\}\}$$

and by proposition 2.5.1

$$\lim_{t\to\infty} E\{\exp(-\delta u_t)/\mathbf{P}(u_t)\} = \int_0^\infty \frac{e^{-\delta u}}{\mathbf{P}(u)}\cdot\frac{\mathbf{P}(u)}{\rho_1}\,du = 1/\delta\rho_1$$

so (2.5.11) follows immediately by letting $t\to\infty$. Furthermore this shows the existence of M_1.

We also have that

$$E\{x_t\} = E\{x_t\exp(\delta u_t)\exp(-\delta u_t)\}$$

but $x_t\exp(\delta u_t)$ and u_t are independent by the structure of the process. $x_t\exp(\delta u_t)$ denotes the value of the process justafter the last jump which is independent of u_t because the jump interarrival time intervals are independent and independent of the sizes of the jumps. Therefore

$$E\{x_t\} = E\{x_t\exp(\delta u_t)\}E\{\exp(-\delta u_t)\}$$

and

$$L_1[1-\hat{p}(\delta)]/\rho_1\delta = M_1$$

so (2.5.10) follows.

By (2.5.3) we have that

$$E\{x_t^2\} - 2\mu_1 E\{x_t \exp(-\delta u_t)/\mathbf{P}(u_t)\} - (\mu_2 - 2\mu_1^2)E\{\exp(-2u_t)/\mathbf{P}(u_t)\} =$$

$$e^{-2\delta t}\{E\{x_0^2\} - 2\mu_1 E\{x_0 \exp(-\delta u_0)/\mathbf{P}(u_0)\} - (\mu_2 - 2\mu_1^2)E\{\exp(-2u_0)/\mathbf{P}(u_0)\}\}$$

Letting $t \to \infty$, M_2 exists and

$$M_2 = \lim_{t\to\infty}\{2\mu_1 E\{x_t \exp(\delta u_t)\}E\{\exp(-2\delta u_t)/\mathbf{P}(u_t)\} + (\mu_2 - 2\mu_1^2)E\{\exp(-2u_t)/\mathbf{P}(u_t)\}\} =$$

$$= 2\mu_1^2/2\rho_1\delta[1-\hat{p}(\delta)] + (\mu_2 - 2\mu_1^2)/2\rho_1\delta = \{\mu_2/2 + \mu_1^2\hat{p}(\delta)/[1-\hat{p}(\delta)]\} / \rho_1\delta$$

which is (2.5.13) and since

$$E\{x_t^2 \exp(2\delta u_t)\}E\{\exp(-2\delta u_t)\} = E\{x_t^2\}$$

$$L_2[1-\hat{p}(2\delta)]/\rho_1\delta = M_2$$

and (2.5.12) follows.

□

We will now generalize theorem 2.5.2 aiming at finding asymptotically all moments of x_t.

Theorem 2.5.4 *Let* A_i $i=0,1,2,\ldots\ldots$ *be constants such that* $A_0 = 1$ *and*

$$\sum_0^k \binom{k}{r} A_r \mu_{k-r} = 0 \qquad k=1,2,\ldots\ldots,m \tag{2.5.14}$$

(We are assuming that the first m *moments of* G *exist), then*

$$(x_t)^m e^{m\delta t} + \{e^{m\delta t}/\mathbf{P}(u_t)\}\sum_0^{m-1}\binom{m}{i}(x_t)^i \exp[-(m-i)\delta u_t]A_{m-i} \tag{2.5.15}$$

is a martingale.

Proof

For $f(x_t,u_t,t)$ to be a martingale we need

$$-\delta x\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial u} + \frac{p(u)}{\mathbf{P}(u)}[\int_{\mathbb{R}^+} f(x+y,0,t)dG(y) - f(x,u,t)] = 0 \tag{2.5.16}$$

We set $h(x,u,t)=\mathbf{P}(u)f(x,u,t)$ and (2.5.16) can be written as

$$-\delta x\frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} + \frac{\partial h}{\partial u} + p(u)\int_{\mathbb{R}^+} h(x+y,0,t)dG(y) = 0 \tag{2.5.17}$$

Substituting

$$h(x,u,t) = \mathbf{P}(u)x^m e^{m\delta t} + e^{m\delta t}\sum_0^{m-1}\binom{m}{i}x^i e^{-(m-i)\delta u}A_{m-i}$$

we get

$$p(u)\sum_{o}^{m-1}\binom{m}{j}x^j\mu_{m-j} + p(u)\sum_{o}^{m-1}\binom{m}{i}A_{m-i}\sum_{o}^{i}\binom{i}{j}x^j\mu_{i-j} = 0 \qquad (2.5.18)$$

All other terms cancel with each other. Notice that

$$(-\delta x\frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \frac{\partial}{\partial u})(x^i e^{m\delta t}e^{-(m-i)\delta u}) = 0$$

But

$$\sum_{o}^{m-1}\binom{m}{i}A_{m-i}\sum_{o}^{i}\binom{i}{j}x^j\mu_{i-j} = \sum_{o}^{m-1}x^j\sum_{j}^{m-1}\binom{m}{i}\binom{i}{j}\mu_{i-j}A_{m-i}$$

and so (2.5.18) is equivalent to

$$\sum_{o}^{m-1}\binom{m}{j}x^j\mu_{m-j} + \sum_{o}^{m-1}x^j\sum_{j}^{m-1}\binom{m}{i}\binom{i}{j}\mu_{i-j}A_{m-i} = 0 \qquad (2.5.19)$$

Equating coefficients of x^j in (2.5.19) we get

$$\mu_{m-j}\binom{m}{j} + \sum_{j}^{m-1}\binom{m}{i}\binom{i}{j}\mu_{i-j}A_{m-i} = 0$$

and setting k=m-j

$$\mu_k\binom{m}{k} + \sum_{m-k}^{m-1}\binom{m}{i}\binom{i}{m-k}\mu_{i-m+k}A_{m-i} = 0$$

so

$$\mu_k\binom{m}{k} + \sum_{1}^{k}\binom{m}{m-r}\binom{m-r}{m-k}\mu_{k-r}A_r \qquad (2.5.20)$$

but

$$\binom{m}{m-r}\binom{m-r}{m-k} = \binom{m}{k}\binom{k}{r}$$

and (2.5.20) gives

$$\mu_k\binom{m}{k} + \sum_{1}^{k}\binom{m}{k}\binom{k}{r}\mu_{k-r}A_r = 0$$

which leads to (2.5.14).

□

Corollary 2.5.5

$$L_m = \lim_{t\to\infty}E\{(x_t)^m\exp(\delta u_t)\}$$

exists for all m=0,1,2,..... *and*

$$L_m + [1-\hat{p}(m\delta)]^{-1}\sum_{o}^{m-1}\binom{m}{i}L_iA_{m-i} = 0 \qquad m=0,1,2,..... \tag{2.5.22}$$

assuming m *moments of* G *exist.*

Proof

By theorem 2.5.4

$$E\{(x_t)^m\exp(m\delta u_t)\}E\{\exp(-m\delta u_t)\} + E\{\exp(-m\delta u_t)/\mathbf{P}(u_t)\}\sum_{o}^{m-1}\binom{m}{i}A_{m-i}E\{(x_t)^i\exp(i\delta u_t)\} =$$

$$e^{-m\delta t}\{E\{(x_o)^m\} + E\{\exp(-m\delta u_o)/\mathbf{P}(u_o)\}\sum_{o}^{m-1}\binom{m}{i}A_{m-i}E\{(x_o)^i\exp(i\delta u_o)\}\} \tag{2.5.23}$$

Letting $t\to\infty$ and using proposition 2.5.1.

$$L_m[1-\hat{p}(m\delta)]/m\delta\rho_1 + (1/m\delta\rho_1)\sum_{o}^{m-1}\binom{m}{i}L_iA_{m-i} = 0$$

leading to (2.5.22).

Existence of L_m can be shown by induction. By letting $t\to\infty$ in (2.5.23) then if $L_1,L_2,.....L_{r-1}$ exist then L_r exists as well and since L_1,L_2 exist by theorem 2.5.3 L_m exists for all m as soon as m moments of G exist.

□

Furthermore we can find a recursive relationship involving μ_i's.

Theorem 2.5.6 *Suppose that* m *moments of* G *exist. Let* L_m *as before, then*

$$L_m = [1-\hat{p}(m\delta)]^{-1}\sum_{o}^{m-1}\binom{m}{i}L_i\hat{p}(i\delta)\mu_{m-i} \tag{2.5.24}$$

Proof

(2.5.22) can be rewritten as

$$L_m\hat{p}(m\delta) = \sum_{o}^{m}\binom{m}{i}L_iA_{m-i} \tag{2.5.25}$$

Consider the expansions

$$\sum_{o}^{\infty}\nu^iL_i/i!\ ,\quad \sum_{o}^{\infty}\nu^iL_i\hat{p}(i\delta)/i!\quad \text{and}\quad \sum_{o}^{\infty}\nu^iA_i/i!$$

By (2.5.25)

$$\left[\sum_{o}^{\infty}\nu^iL_i/i!\right]\left[\sum_{o}^{\infty}\nu^iA_i/i!\right] = \sum_{o}^{\infty}\nu^iL_i\hat{p}(i\delta)/i! \tag{2.5.26}$$

also by (2.5.14) (the definition of the A_i's)

$$\sum_{o}^{k} \binom{k}{r} A_r \mu_{k-r} = 0$$

so in the product

$$\left[\sum_{o}^{\infty} \nu^i A_i/i!\right]\left[\sum_{o}^{\infty} \nu^i \mu_i/i!\right]$$

the only non zero term is $\mu_o A_o = 1$ so

$$\left[\sum_{o}^{\infty} \nu^i A_i/i!\right]\left[\sum_{o}^{\infty} \nu^i \mu_i/i!\right] = 1 \qquad (2.5.27)$$

(2.5.26) & (2.5.27) give us

$$\left[\sum_{o}^{\infty} \nu^i L_i \hat{p}(i\delta)/i!\right]\left[\sum_{o}^{\infty} \nu^i \mu_i/i!\right] = \sum_{o}^{\infty} \nu^i L_i/i! \qquad (2.5.28)$$

and by looking at the coefficient of ν^m we get

$$L_m = \sum_{o}^{m} \binom{m}{i} L_i \hat{p}(i\delta)\mu_{m-i} \qquad (2.5.29)$$

which leads to (2.5.24).

□

One should compare (2.5.24) with (1.3.25) in section 1.3. The fact that the formulas are the same is no coincidence. The structure of the models is similar.

Finally we find the asymptotic limits of all the moments of x_t.

<u>Corollary 2..5.7</u> *Let* M_m *denote the* m^{th} *asymptotic moment of* x_t *(we assume* m *moments of* G *exist). Then* M_m *exists and*

$$M_m = (1/m) \sum_{o}^{m-1} \binom{m}{i} M_i \hat{p}(i\delta) i \mu_{m-i}/[1-\hat{p}(i\delta)] \qquad (2.5.30)$$

<u>Proof</u>

Existence follows immediately from the existence of L_m. Furthermore since

$$E\{(x_t)^i\} = E\{(x_t)^i \exp(i\delta u_t)\}E\{\exp(-i\delta u_t)\}$$

we have that

$$M_i = L_i[1-\hat{p}(i\delta)]/i\delta\rho_1$$

We substitute in (2.5.24) and (2.5.30) follows immediately.

□

An example is when $dG(y) = \alpha e^{-\alpha y} dy$. We then have exponentially distributed

jump sizes. We can now find M_i explicitly. By theorem 2.5.6

$$L_m = \sum_{i}^{m} \binom{m}{i} L_i \hat{p}(i\delta)(m-i)!\alpha^{i-m}$$

and looking at

$$\{\sum_{0}^{\infty} L_i \nu^i \hat{p}(i\delta)/i!\} \cdot \{\alpha/(\alpha-\nu)\} = \sum_{0}^{\infty} L_i \nu^i/i!$$

leading to

$$\alpha \sum_{0}^{\infty} L_i \nu^i \hat{p}(i\delta)/i! = \alpha \sum_{0}^{\infty} L_i \nu^i/i! - \sum_{1}^{\infty} L_{i-1} \nu^i/(i-1)!$$

so $iL_{i-1} = \alpha L_i[1-\hat{p}(i\delta)]$ and therefore

$$L_i = i! \; / \; \{\alpha^i \prod_{1}^{i} [1-\hat{p}(j\delta)]\} \tag{2.5.31}$$

so

$$M_i = (i-1)! \; / \{\delta\rho_1 \alpha^i \prod_{1}^{j-1} [1-\hat{p}(j\delta)]\} \tag{2.5.32}$$

2.6 CONTINUOUS PATH MODELS

I.

Suppose that in the shot noise process described in section 2.3 jumps do not occur instantaneously. Instead, the process enters a state where it increases deterministically for a while. For example when modeling levels of dams or rivers, rains might not be considered as instant events but as having a duration and an intensity. In particular consider the following model. It consists of n_t and x_t. n_t takes the values 1 and 2 and transitions from 1 to 2 occur with Poisson rate λ_1 and from 2 to 1 with Poisson rate λ_2. While n_t=1 x_t *increases* with constant rate c; and when n_t=2 x_t *decreases* with rate δx_t (exponential decay).

The generator of the process (n_t,x_t,t) acting on a function $f_n(x,t)$ belonging to its domain is given by

$$Af_n(x,t) = \begin{cases} c\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial t} + \lambda_1[f_2(x,t)-f_1(x,t)] \\ -\delta x\frac{\partial f_2}{\partial x} + \frac{\partial f_2}{\partial t} + \lambda_2[f_1(x,t)-f_2(x,t)] \end{cases} \tag{2.6.1}$$

We will now try to find the stationary distribution of the process.

Theorem 2.6.1 *Let x_t as described. Furthermore suppose x_t is stationary as well. Then the density of x_t is*

$$\frac{y^{(\lambda_2/\delta)-1}\exp(-\lambda_1 y/c)[(c/\delta)+y]}{\int_0^\infty y^{(\lambda_2/\delta)-1}\exp(-\lambda_1 y/c)[(c/\delta)+y]dy}$$

(a mixture of Gamma(λ_2/δ) and Gamma$((\lambda_2/\delta)+1)$ distributions.

Proof

We will try to find a function $f_n(x)$ such that $Af_n(x) = e^{-\nu x}-K$ where K is a suitable constant. So we have to solve

$$cf_1'(x) + \lambda_1[f_2(x)-f_1(x)] = e^{-\nu x} - K \tag{2.6.2}$$

$$-\delta x f_2'(x) + \lambda_2[f_1(x)-f_2(x)] = e^{-\nu x} - K \tag{2.6.3}$$

Multiplying (2.6.2) by λ_2, (2.6.3) by λ_1 and adding them together.

$$\lambda_2 cf_1'(x) - \lambda_1\delta x f_2'(x) = (e^{-\nu x}-K)(\lambda_1+\lambda_2) \tag{2.6.4}$$

Differentiating (2.6.2) we bet

$$cf_1''(x) + \lambda_1[f_2'(x)-f_1'(x)] = -\nu e^{-\nu x} \tag{2.6.5}$$

and by (2.6.4)

$$cf_1''(x) - \lambda_1 f_1'(x) + (\lambda_2 c/\delta x)f_1'(x) = (e^{-\nu x}-K)(\lambda_1+\lambda_2)/\delta x \tag{2.6.6}$$

and solving (2.6.6)

$$f_1'(x) = (1/c)x^{-(\lambda_2/\delta)}\exp(\lambda_1 x/c)\{\int_0^x\{(e^{-\nu y}-K)(\lambda_1+\lambda_2)(1/\delta y)-\nu e^{-\nu y}\}y^{\lambda_2/\delta}\exp(-\lambda_1 y/c)dy+Q\}$$

where Q is a constant. Now if Q=0 f_1' will be bounded so we choose it this way. Therefore

$$f_1'(x) = (1/c)x^{-(\lambda_2/\delta)}\exp(\lambda_1 x/c)\int_0^x\{(e^{-\nu y}-K)(\lambda_1+\lambda_2)(1/\delta y)-\nu e^{-\nu y}\}y^{\lambda_2/\delta}\exp(-\lambda_1 y/c)dy \tag{2.6.7}$$

We will also choose f_1' so that f_1 belongs to the domain of the generator. For this a sufficient condition is

$$\lim_{x\to\infty} f_1'(x) = 0$$

so we should have

$$\int_0^\infty \{(e^{-\nu y}-K)(\lambda_1+\lambda_2)(1/\delta y)-\nu e^{-\nu y}\} y^{\lambda_2/\delta} \exp(-\lambda_1 y/c)dy = 0 \qquad (2.6.8)$$

but integrating by parts we have that

$$\int_0^\infty \nu e^{-\nu y} y^{\lambda_2/\delta} \exp(-\lambda_1 y/c)dy = \int_0^\infty \{(e^{-\nu y}-K)\{(\lambda_2/\delta y)+(\lambda_1/c)\}\, y^{\lambda_2/\delta} \exp(-\lambda_1 y/c)dy$$

and so (2.6.8) becomes

$$\int_0^\infty \{(e^{-\nu y}-K)\{(\lambda_1/\delta y)+(\lambda_1/c)\}\, y^{\lambda_2/\delta} \exp(-\lambda_1 y/c)dy = 0$$

and therefore

$$K = \frac{\int_0^\infty e^{-\nu y}\, y^{(\lambda_2/\delta)-1} \exp(-\lambda_1 y/c)[(c/\delta)+y]dy}{\int_0^\infty y^{(\lambda_2/\delta)-1} \exp(-\lambda_1 y/c)[(c/\delta)+y]dy} \qquad (2.6.9)$$

So now f_1 is found by integrating f_1' and adding an arbitrary constant and f_2 is given by (2.6.2)

We have that $Af_n(x) = e^{-\nu x}-K$ and therefore

$$f_{n_t}(x_t) - \int_0^t (\exp(-\nu x_s)-K)ds$$

is a martingale giving

$$E\{f_{n_t}(x_t)\} - E\{f_{n_0}(x_0)\} = \int_0^t \{E\{\exp(-\nu x_s)\}-K\}ds$$

so if the process is stationary $E\{\exp(-\nu x_t)\} = K$ and by (2.6.9) and the uniqueness of the Laplace transform the result follows.

□

Of course if we let $\lambda_1\to\infty$, $c\to\infty$ with λ_1/c remaining constant we get the distribution of a stationary shot noise process with exponentially distributed jump sizes.

II.

We will now consider the process in I with a slight modification. The rate of income of x_t when $n_t=1$ is now $c-\delta x_t$ instead of c. When $n_t=2$, x_t still decreases with rate δx_t. This is a little bit more realistic when modeling water levels

since the outflow does not cease when it rains. Furthermore (this can turn out to be an advantage or a disadvantage of the model depending on the circumstances) the level of water clearly can not reach and certainly can not exceed c/δ. n_t goes from 1 to 2 with Poisson rate λ_1 and vice versa with λ_2 sa before. The generator of the process acting on a function $f_n(x,t)$ is given by

$$Af_n(x,t) = \begin{cases} (c-\delta x)\dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_1}{\partial t} + \lambda_1[f_2(x,t)-f_1(x,t)] \\ -\delta x\dfrac{\partial f_2}{\partial x} + \dfrac{\partial f_2}{\partial t} + \lambda_2[f_1(x,t)-f_2(x,t)] \end{cases} \tag{2.6.10}$$

We will try to find the distribution of the process when stationary as we did before.

Theorem 2.6.2 *Let* (x_t,n_t) *be as described and let them be stationary as well. Then the distribution of* x_t *has density*

$$\frac{y^{(\lambda_2/\delta)-1}[(c/\delta)-y]^{(\lambda_1/\delta)-1}}{\int_0^{c/\delta} y^{(\lambda_2/\delta)-1}[(c/\delta)-y]^{(\lambda_1/\delta)-1}dy} \qquad (0<y<c/\delta)$$

(a Beta distribution).

Proof

As in theorem 2.6.1 we are going to try to find $f_n(x)$ such that $Af_n(x)= e^{-\nu x}-K$. So we have to solve

$$(c-\delta x)f_1'(x) + \lambda_1[f_2(x)-f_1(x)] = e^{-\nu x}-K \tag{2.6.11}$$

$$-\delta x f_2'(x) + \lambda_2[f_1(x)-f_2(x)] = e^{-\nu x}-K \tag{2.6.12}$$

Multiplying (2.6.11) by λ_2, (2.6.12) by λ_1 and adding them together.

$$\lambda_2(c-\delta x)f_1'(x) - \lambda_1\delta x f_2'(x) = (\lambda_1+\lambda_2)(e^{-\nu x}-K) \tag{2.6.13}$$

Differentiating (2.6.12)

$$-\delta[xf_2'(x)]' - \lambda_2 f_2'(x) + \lambda_2 f_1'(x) = -\nu e^{-\nu x}$$

and with (2.6.13)

$$-\delta[xf_2'(x)]' - (\lambda_2/x)xf_2'(x) + [\lambda_1\delta/(c-\delta x)]xf_2'(x) + [(\lambda_1+\lambda_2)/(c-\delta x)](e^{-\nu x}-K) = -\nu e^{-\nu x} \tag{2.6.14}$$

For f to belong to the domain of the generator we choose the solution with

$$\lim_{x\downarrow 0} x^{(\lambda_2/\delta)+1}f_2'(x) = 0$$

that is

$$f'_2(x)=x^{-(\lambda_2/\delta)-1}[(c/\delta)-x]^{-\lambda_1/\delta}\{\int_0^x\{[(\lambda_1+\lambda_2)(e^{-\nu y}-K)/[(c/\delta)-y]]+\nu e^{-\nu y}\}y^{(\lambda_2/\delta)}[(c/\delta)-y]^{(\lambda_1/\delta)} \tag{2.6.15}$$

For f to belong to the domain of the generator we also choose

$$\lim_{x\to(c/\delta)} [(c/\delta)-x]^{(\lambda_1/\delta)}f'_2(x) = 0$$

that is

$$\int_0^{(c/\delta)}(\lambda_1+\lambda_2)(e^{-\nu y}-K)y^{(\lambda_2/\delta)}[(c/\delta)-y]^{(\lambda_1/\delta)-1}dy=\int_0^{(c/\delta)}\nu e^{-\nu y}y^{(\lambda_2/\delta)}[(c/\delta)-y]^{(\lambda_1/\delta)}\,dy \tag{2.6.16}$$

Integrating by parts the second integral as in the proof of theorem 2.6.1 we get

$$\int_0^{(c/\delta)}\{[(\lambda_1+\lambda_2)/(c-\delta y)]+[\lambda_2/\delta y]-[\lambda_1/(c-\delta y)]\}\{e^{-\nu y}-K\}dy = 0 \tag{2.6.17}$$

which means that

$$K = \frac{\int_0^{c/\delta} e^{-\nu y}\, y^{(\lambda_2/\delta)-1}[(c/\delta)-y]^{(\lambda_1/\delta)-1}dy}{\int_0^{c/\delta} y^{(\lambda_2/\delta)-1}[(c/\delta)-y]^{(\lambda_1/\delta)-1}dy} \tag{2.6.18}$$

f'_1 is given by (2.6.13) and f_1 by integrating and adding an arbitrary constant. f_2 is given by (2.6.12). By our choice of f_1 and f_2, f belongs to the domain of the generator, so $Af_n(x) = e^{-\nu x}-K$ and therefore

$$f_{n_t}(x_t) - \int_0^t(\exp(-\nu x_s)-K)ds$$

is a martingale giving

$$E\{f_{n_t}(x_t)\} - E\{f_{n_0}(x_0)\} = \int_0^t\{E\{\exp(-\nu x_s)\}-K\}ds$$

so when the process is stationary $E\{\exp(-\nu x_t)\} = K$ and by (2.6.18) and the uniqueness of Laplace transforms the theorem is proved.

□

We can also generalize the models in I and II by allowing as rates of increase in state 1 and decrease in state 2 general Riemann integrable functions $c_1(x)$ and $c_2(x)$. We proceed as in theorems 2.6.1 and 2.6.2 and produce similar results.

Furthermore we can allow N possible states where N is a fixed integer with change rates $c_i(x)$ (i=1,2,.....,N) which can be positive or negative or indeed change signs should we so wish. The generator will have the form

$$Af_i(x,t) = \frac{\partial f_i}{\partial t} + c_i(x)\frac{\partial f_i}{\partial x} + \sum_{i\neq j} \lambda_{ij}[f_j(x,t)-f_i(x,t)] \quad (i=1,2,...,N) \tag{2.6.19}$$

In this case we can proceed as before and we will prove the same theorems. We will eventually have to solve

$$c_i(x)f_i'(x) + \sum_{i\neq j} \lambda_{ij}[f_j(x)-f_i(x)] = h_i(x) \qquad (i=1,2,....,N) \tag{2.6.20}$$

where $h_i(x)$ are known functions. We omit details as solving the system (2.6.20) may be quite cumbersome.

2.7 THE M/G/∞ QUEUE

We will now consider a queuing model with an infinite number of servers available, where the customers arrive in terms of a Poisson process with service time requirements being independent identically distributed random variables with distribution function G(y) supported on $\mathbb{R}^+$. Service for each customer commences as soon as he arrives. The M/G/∞ queue can be used in approximating queuing systems with large numbers of servers available and the other parameters of the model (drawn from everyday experience) making the event of having a customer waiting for a server highly unlikely. Usually queues with infinite numbers of servers are used in association with computer or other network models. Using the piecewise deterministic processes theory approach we are going to determine the joint distribution of various quantities associated with the process. These are the following; the busy period τ of the process, that is the time interval between the first time the queue ceases to be idle and the first time it becomes idle again; the number of customers N served during that period and W the total work done

during that same period. We define $W=Y_1+Y_2+.....+Y_N$ (a random sum) where the Y_i's represent the time requirements of the customers served during a busy period. The Y_i's are neither independent of N nor independent between them conditionally on any value of N. So the "naive" argument that W is a sum of independent identically distributed random variables and therefore $E\{e^{-\nu W}\}= E\{[\hat{g}(\nu)]^N\}$ is not true.

Our piecewise deterministic process consists of t (time), x_t the work still to be done for the existing customer with the highest unfinished service requirement (clearly the event that the queue is idle is equivalent to $x_t=0$), n_t the number of customers having arrived up to time t and w_t where

$$w_t = \sum_{1}^{n_t} Y_i$$

Y_i being the service requirements of the customers (who may or may not have left the system). x_t decreases linearly with unit rate, while n_t and w_t remain constant. Jumps occur with rate λ. At $t=T_i$ the time of the i^{th} jump x_t moves to $x_t=\max(x_{t-},Y_i)$, n_t moves to $n_t=n_{t-}+1$ and w_t to $w_t=w_{t-}+Y_i$. The generator of the process (x_t,w_t,t,n_t) acting on a function f(x,w,t,n) is given by

$$Af(x,w,t,n) = \frac{\partial f}{\partial t} - \frac{\partial f}{\partial x} + \lambda[\int_o^x f(x,w+y,t,n+1)dG(y)+\int_x^\infty f(y,w+y,t,n+1)dG(y)-f(x,w,t,n)]$$

(We denote $\int_{(0,x]}$ by $\int_o^x$ and $\int_{(x,\infty)}$ by $\int_x^\infty$) (2.7.1)

We will concentrate on the behaviour of the process till x_t reaches 0. (That is why we have laid no specific assumptions about what happens when x_t reaches 0). The fact that x_t as defined may turn negative does not matter for our calculations. We start by finding a suitable martingale.

Lemma 2.7.1 *Let* x_t, n_t *and* w_t *as defined. Consider constants* $\beta\geq 0$, $\nu\geq 0$, $0\leq\theta\leq 1$. *Then*

$$e^{-\beta t}\exp(-\nu w_t)\theta^{n_t}\int_{x_t}^\infty e^{-(\beta+\lambda)y}\exp[\lambda\theta\int_o^y\int_o^u e^{-\nu v}dG(v)du]dy$$

is a martingale.

Proof

For $f(x_t,w_t,t,n_t)$ to be a martingale we have to solve $Af(x,w,t,n)=0$. We

substitute a function of the form $e^{-\beta t}e^{-\nu w}\theta^{n}h(x)$ in the equation and we get

$$-\beta h(x) - h'(x) + \lambda[\theta h(x)\int_0^x e^{-\nu y}dG(y)+\theta\int_x^\infty h(y)e^{-\nu y}dG(y)-h(x)] = 0 \qquad (2.7.2)$$

and integrating by parts this becomes

$$-\beta h(x) - h'(x) - \lambda\theta\int_x^\infty\int_0^y e^{-\nu u}dG(u)h'(y)dy - \lambda h(x) = 0 \qquad (2.7.3)$$

Differentiating (2.7.3)

$$-\beta h'(x) - h''(x) + \lambda\theta h'(x)\int_0^x e^{-\nu u}dG(u) - \lambda h'(x) = 0 \qquad (2.7.4)$$

and solving (2.7.4)

$$h'(x) = -Ke^{-(\beta+\lambda)x}\exp[\lambda\theta\int_0^x\int_0^u e^{-\nu v}dG(v)du] \qquad (2.7.5)$$

where K is an arbitrary constant and

$$h(x) = K\int_x^\infty e^{-(\beta+\lambda)y}\exp[\lambda\theta\int_0^y\int_0^u e^{-\nu v}dG(v)du]dy \qquad (2.7.6)$$

This is bounded so our function belongs to the domain of the generator and the lemma is proved.

□

Corollary 2.7.2 *Let* τ_o *be the first time* x_t *reaches* 0. *That is* $\tau_o \equiv \inf\{t: x_t=0\}$, *then for* $\beta\geq 0$, $\nu\geq 0$, $0\leq\theta\leq 1$,

$$E\{\exp(-\beta\tau_o)\exp(-\nu w_{\tau_o})\theta^{n_{\tau_o}} \mid w_o, n_o\} =$$

$$E\{\exp(-\nu w_o)\theta^{n_o}\}\cdot\frac{\int_{x_o}^\infty e^{-(\beta+\lambda)y}\exp[\lambda\theta\int_0^y\int_0^u e^{-\nu v}dG(v)du]dy}{\int_0^\infty e^{-(\beta+\lambda)y}\exp[\lambda\theta\int_0^y\int_0^u e^{-\nu v}dG(v)du]dy} \qquad (2.7.7)$$

Proof

It is immediate by Doob's optional stopping theorem and the martingale of theorem 2.7.1.

□

Theorem 2.7.3 *Let* τ *be the length of the busy period of the* M/G/∞ *queue,* N *the customers served during that period and* W *the total work done. The joint Laplace transform-probability generating function of their distribution is given by*

$$E\{e^{-\beta\tau}e^{-\nu W}\theta^N\} = \frac{\theta\int_0^\infty\int_0^y e^{-\nu v}dG(v)e^{-(\beta+\lambda)y}\exp[\lambda\theta\int_0^y\int_0^u e^{-\nu v}dG(v)du]dy}{\int_0^\infty e^{-(\beta+\lambda)y}\exp[\lambda\theta\int_0^y\int_0^u e^{-\nu v}dG(v)du]dy} \qquad (2.7.8)$$

Proof

We start operating the queue the time the first customer arrives; so we set $x_0=w_0=Y_1$ where Y_1 has distribution function G(y) and $n_0=1$. Then $(\tau_0,w_{\tau_0},n_{\tau_0})$ becomes (τ,W,N) so using corollary 2.7.2

$$E\{e^{-\beta\tau}e^{-\nu W}\theta^N\} = \frac{\theta\int_0^\infty e^{-\nu x}\int_x^\infty e^{-(\beta+\lambda)y}\exp[\lambda\theta\int_0^y\int_0^u e^{-\nu v}dG(v)du]dy}{\int_0^\infty e^{-(\beta+\lambda)y}\exp[\lambda\theta\int_0^y\int_0^u e^{-\nu v}dG(v)du]dy} \qquad (2.7.9)$$

and by changing the order of integration in the numerator the result follows.

□

By suitably differentiating (2.7.8) we can get any joint moments of τ, N, W we wish. For example one can find $E\{W\} = \mu_1 e^{\lambda\mu_1}$ and $E\{N\} = e^{\lambda\mu_1}$ implying that the "naive" approach for W mentioned at the beginning of this section is correct for the first moment. However, when looking at the second moment we get

$$E\{W^2\} = e^{\lambda\mu_1}\{\mu_2 + 2\lambda\mu_1\mu_2 + 2\lambda^2\int_0^\infty\exp(\lambda\int_y^\infty \mathbf{G}(u)du)y\mathbf{G}(y)dy\} \qquad (2.7.10)$$

($\mathbf{G}(y)=1-G(y)$)

while a "naive" approach would have yielded

$$e^{\lambda\mu_1}\{\mu_2 + 2\lambda\mu_1^3 + 2\lambda^2\int_0^\infty\exp(\lambda\int_y^\infty \mathbf{G}(u)du)y\mathbf{G}(y)dy\}$$

underestimating the variance by a term propotional to $\mu_2-\mu_1^2$, the variance of Y_i.

Not surprisingly many relevant results are to be found in the literature. Takacs (1962) has given a method to evaluate moments of τ. Shanbhag (1966) went even further and evaluated the joint Laplace transform probability generating function of a slightly more general model where customers arrive in batches. That is at each arrival time point T_i, r_i customers arrive with probability p_r having service requirements $Y_i^{(1)},Y_i^{(2)},.....,Y_i^{(r_i)}$ (independent). Our method can easily be modified to accomodate this generalization. Furthermore we do not need $Y_i^{(1)},.....,Y_i^{(r_i)}$ to be independent. The only thing we need to know

is the joint distribution of $Y_i^{(1)}+Y_i^{(2)}+.....+Y_i^{(r_i)}$ and $\max(Y_i^{(1)},.....,Y_i^{(r_i)})$ conditional on r_i.

Formula (2.7.8) can be used to evaluate the variance of τ. In terms of numerical approximations Ramalhoto (1984) and Sathe (1985) found bounds for this variance in terms of the first moments of G. It might be the case that the methods applied by them can be modified to yield similar bounds for higher moments of the distribution of τ.

CHAPTER 3

DOUBLY STOCHASTIC POINT PROCESSES

AGGREGATED PROCESSES

3.1 GENERAL

Many actuaries have questioned in the past the validity of the Poisson assumption about the claim arrival process in insurance models. Other processes have indeed been used; one example is the Andersen model we encountered in section 1.1 or the Polya process (see Beard et al p.38). But there is no clear natural reason to assume any particular process (see Seal (1983) for a discussion of the Poisson assumption). So there is much scope about developing other models to use as claim arrival processes. Many claims are associated with natural phenomena (fires, floods). There is of course a seasonal element then. We have already seen such a model in section 1.3 (see also Dassios (1984), Dassios & Embrechts (1987); for aspects of seasonality see Beard et al p.32). There is a complicated stochastic background as well. For example fires occur randomly but fires are less catastrophic when there is a large rainfall and more catastrophic when strong winds prevail; events which are of a stochastic nature themselves. Here comes the importance of more elaborate stochastic models such as doubly stochastic point processes.

Many results about point processes can be calculated with the aid of piecewise deterministic processes theory and we will set on this task in this chapter. We will not concentrate on insurance aspects, however, with the exception of the penultimate section where insurance applications will be considered.

Doubly stochastic or conditional Poisson point processes can be viewed as two step randomization procedures. A process λ_t is used to generate another process N_t by acting as its intensity. That is N_t is a Poisson process conditional on λ_t (if λ_t is deterministic then N_t is a Poisson process). The term "doubly stochastic" was introduced by Cox (1955a). Examples which in practice could model various forms

of clustering will be examined in this chapter. Most of the intensities that will be considered will be processes examined in the previous chapter.

We will now define the piecewise deterministic process $(N_t,\lambda_t,\mathbf{u}_t,t)$ $(\mathbf{u}_t=u_{1t},\ldots,u_{n,t}$; we allow for any necessary additional variables so as to make the process Markov). We will state what form the components which determine the piecewise deterministic process have (see introduction).

Deterministic motion is given by the vector fields:

0 (for N_t; N_t is piecewise constant), $c_0(\lambda,\mathbf{u},t)\frac{\partial}{\partial\lambda}$, $c_1(\lambda,\mathbf{u},t)\frac{\partial}{\partial u_1}$,....., $c_n(\lambda,\mathbf{u},t)\frac{\partial}{\partial u_n}$, $\frac{\partial}{\partial t}$
Jumps occur with rate $\lambda+\rho(\lambda,\mathbf{u},t)$. The transition measure is as follows. With probability $\lambda/[\lambda+\rho(\lambda,\mathbf{u},t)]$ N_t moves to N_t+1 with $\lambda_t,\mathbf{u}_t,t$ not changing value. Also with probability $\rho(\lambda,\mathbf{u},t)/[\lambda+\rho(\lambda,\mathbf{u},t)]$ N_t remains where it is and $\lambda_t,\mathbf{u}_t$ change value according to a measure $Q(\cdot;\lambda,\mathbf{u},t)$. These determine the piecewise deterministic process $(N_t,\lambda_t,\mathbf{u}_t,t)$. The generator A of the process acting on a function $f(N,\lambda,\mathbf{u},t)$ belonging to its domain is given by

$$Af(N,\lambda,\mathbf{u},t) = c_0(\lambda,\mathbf{u},t)\frac{\partial}{\partial\lambda} + c_1(\lambda,\mathbf{u},t)\frac{\partial}{\partial u_1} + \ldots\ldots\ldots + c_n(\lambda,\mathbf{u},t)\frac{\partial}{\partial u_n} + \frac{\partial}{\partial t} +$$
$$[\lambda+\rho(\lambda,\mathbf{u},t)]\left\{\frac{\lambda}{\lambda+\rho(\lambda,\mathbf{u},t)}[f(N+1,\lambda,\mathbf{u},t)-f(N,\lambda,\mathbf{u},t)] + \frac{\rho(\lambda,\mathbf{u},t)}{\lambda+\rho(\lambda,\mathbf{u},t)}[\int f(N,y,\mathbf{v},t)dQ(y,\mathbf{v};\lambda,\mathbf{u},t) - f(N,\lambda,\mathbf{u},t)]\right. \tag{3.1.1}$$

Simplifying

$$Af(N,\lambda,\mathbf{u},t) = c_0(\lambda,\mathbf{u},t)\frac{\partial}{\partial\lambda} + c_1(\lambda,\mathbf{u},t)\frac{\partial}{\partial u_1} + \ldots\ldots\ldots + c_n(\lambda,\mathbf{u},t)\frac{\partial}{\partial u_n} + \frac{\partial}{\partial t} +$$
$$\lambda[f(N+1,\lambda,\mathbf{u},t)-f(N,\lambda,\mathbf{u},t)] + \rho(\lambda,\mathbf{u},t)\,[\int f(N,y,\mathbf{v},t)dQ(y,\mathbf{v};\lambda,\mathbf{u},t) - f(N,\lambda,\mathbf{u},t)] \tag{3.1.2}$$

We denote

$$A_\lambda f(N,\lambda,\mathbf{u},t) = c_0(\lambda,\mathbf{u},t)\frac{\partial}{\partial\lambda} + c_1(\lambda,\mathbf{u},t)\frac{\partial}{\partial u_1} + \ldots\ldots\ldots + c_n(\lambda,\mathbf{u},t)\frac{\partial}{\partial u_n} + \frac{\partial}{\partial t} +$$
$$\rho(\lambda,\mathbf{u},t)\,[\int f(N,y,\mathbf{v},t)dQ(y,\mathbf{v};\lambda,\mathbf{u},t) - f(N,\lambda,\mathbf{u},t)] \tag{3.1.3}$$

So we can write

$$Af(N,\lambda,\mathbf{u},t) = \lambda[f(N+1,\lambda,\mathbf{u},t)-f(N,\lambda,\mathbf{u},t)] + A_\lambda f(N,\lambda,\mathbf{u},t) \tag{3.1.4}$$

and if we consider a function $f(\lambda,\mathbf{u},t)$, then

$$Af(\lambda,\mathbf{u},t) = A_\lambda f(\lambda,\mathbf{u},t) \tag{3.1.5}$$

We can see that $(\lambda_t,\mathbf{u}_t,t)$ could have been constructed as a piecewise deterministic process (with generator A_λ acting only on functions $f(\lambda,\mathbf{u},t)$ belonging to its domain) by itself. However, we just consider it as part of the piecewise deterministic process $(N_t,\lambda_t,\mathbf{u}_t,t)$ and the filtration used will be the natural filtration $\sigma\{N_s,\lambda_s,\mathbf{u}_s;\ 0\leq s\leq t\}$.

In this chapter we will also consider the process

$$x_t = \int_0^t \lambda_s ds \qquad \textit{(the aggregated process)} \tag{3.1.6}$$

Since $Q,c_0,..,c_n$ and ρ do not depend on the first component of the state space it can be shown that conditionally on $\{\lambda_s:s\geq 0\}$ the process $\{N_t:t\geq 0\}$ is a non-homogeneous Poisson process with intensity $t\to\lambda_t$, in particular

$$E\{\theta^{N_{t_2}-N_{t_1}} \mid \lambda_s;\ 0\leq s\leq t\} = \exp[-(1-\theta)\int_{t_1}^{t_2}\lambda_s ds] = \exp[-(1-\theta)(x_{t_2}-x_{t_1})] \tag{3.1.7}$$

This suggests the equivalence of the problem of finding the distribution of N_t to the problem of finding the distribution of x_t. Other quantities of interest in this chapter will be higher order properties of N_t and x_t as well as point interarrival time intervals of N_t.

We close this introductory section with a lemma that will be useful when trying to find the distribution of point interarrival time intervals.

<u>Lemma 3.1.1</u> *Let* $(N_t,\lambda_t,\mathbf{u}_t,t)$ *as defined. Suppose that* $f(\lambda,\mathbf{u},t)$ *belongs to the domain of the generator* A *of the process* $(N_t,\lambda_t,\mathbf{u}_t,t)$ *and furthermore*

$$\lim_{t\to\infty} E\{f(\lambda_t,\mathbf{u}_t,t)\exp(-\int_0^t \lambda_s ds) \mid \lambda_0,\mathbf{u}_0\} = 0 \tag{3.1.8}$$

If $h(\lambda,\mathbf{u},t)$ *is such that*

$$\lambda[h(\lambda,\mathbf{u},t)-f(\lambda,\mathbf{u},t)] + A_\lambda f(\lambda,\mathbf{u},t) = 0 \tag{3.1.9}$$

then if $\tau_1=\inf\{t:N_t=1 \mid N_0=0\}$

$$E\{h(\lambda_{T_1},u_{T_1},T_1)\,|\,\lambda_o,u_o\} = f(\lambda_o,u_o,0) \tag{3.1.10}$$

Proof

By (3.1.9)

$$f(\lambda_t,u_t,t) + \int_o^t \lambda_s[h(\lambda_s,u_s,s)-f(\lambda_s,u_s,s)]ds \tag{3.1.11}$$

is a martingale (always with respect to the filtration $\sigma\{N_s,\lambda_s,u_s:0\leq s\leq t\}$) and $T_1\wedge t$ is a stopping time. Therefore by Doob's optional stopping theorem

$$E\{f(\lambda_{T_1\wedge t},u_{T_1\wedge t},T_1\wedge t)\,|\,\lambda_o,u_o\} + E\{\int_o^{T_1\wedge t} \lambda_s[h(\lambda_s,u_s,s)-f(\lambda_s,u_s,s)]ds\,|\,\lambda_o,u_o\} = f(\lambda_o,u_o,0) \tag{3.1.12}$$

We now condition on the realisation λ_v,u_v $0\leq v\leq t$.

$$E\{f(\lambda_{T_1\wedge t},u_{T_1\wedge t},T_1\wedge t)\,|\,\lambda_o,u_o\} = \int_\Omega E\{f(\lambda_{T_1\wedge t},u_{T_1\wedge t},T_1\wedge t)\,|\,\lambda_v,u_v,0\leq v\leq t\}dP(\lambda_v,u_v) \tag{3.1.13}$$

and

$$E\{\int_o^{T_1\wedge t} \lambda_s[h(\lambda_s,u_s,s)-f(\lambda_s,u_s,s)]ds\,|\,\lambda_o,u_o\} =$$

$$\int_\Omega E\{\int_o^{T_1\wedge t} \lambda_s[h(\lambda_s,u_s,s)-f(\lambda_s,u_s,s)]ds\,|\,\lambda_v,u_v,0\leq v\leq t\}dP(\lambda_v,u_v) \tag{3.1.14}$$

where $dP(\lambda_v,u_v)$ is the probability differential of a particular realisation (in actual fact λ_v,u_v represents a set of realisations with N_v $0\leq v\leq t$ taking any values) in Ω the set of all possible realisations.

N_t is Poisson conditionally on λ_r $0\leq r\leq t$, so $T_1\wedge t$ is distributed with density

$$\lambda_r\exp(-\int_o^r \lambda_s ds) \text{ on } (0,t)$$

and a mass

$$\exp(-\int_o^t \lambda_s ds) \quad \text{at } t$$

Therefore we have

$$E\{f(\lambda_{T_1\wedge t},u_{T_1\wedge t},T_1\wedge t)\,|\,\lambda_v,u_v,0\leq v\leq t\}=f(\lambda_t,u_t,t)\exp(-\int_o^t \lambda_s ds)+\int_o^t f(\lambda_r,u_r,r)\lambda_r\exp(-\int_o^r \lambda_s ds)dr \tag{3.1.15}$$

and

$$E\{\int_0^{\tau_1 \wedge t} \lambda_s[h(\lambda_s,u_s,s)-f(\lambda_s,u_s,s)]ds \mid \lambda_v,u_v,0\leq v\leq t\} =$$

$$\int_0^t \lambda_s[h(\lambda_s,u_s,s)-f(\lambda_s,u_s,s)]ds\exp(-\int_0^t \lambda_s ds)+\int_0^t\int_0^r \lambda_s[h(\lambda_s,u_s,s)-f(\lambda_s,u_s,s)]ds\,\lambda_r \exp(-\int_0^r \lambda_s ds)dr \tag{3.1.16}$$

and integrating by parts

$$E\{\int_0^{\tau_1 \wedge t} \lambda_s[h(\lambda_s,u_s,s)-f(\lambda_s,u_s,s)]ds \mid \lambda_v,u_v,0\leq v\leq t\} =$$

$$\int_0^t \lambda_r[h(\lambda_r,u_r,r)-f(\lambda_r,u_r,r)]\exp(-\int_0^r \lambda_s ds)dr \tag{3.1.17}$$

Adding together (3.1.15) & (3.1.17) and in view of (3.1.13) and (3.1.14), (3.1.12) becomes

$$E\{f(\lambda_t,u_t,t)\exp(-\int_0^t \lambda_s ds) \mid \lambda_0,u_0\}+\int_\Omega\int_0^t h(\lambda_r,u_r,r)\lambda_r \exp(-\int_0^r \lambda_s ds)drdP(\lambda_r,u_r) = f(\lambda_0,u_0,0) \tag{3.1.18}$$

Letting $t\to\infty$ and noticing that the first term tends to 0 by (3.1.8) we get (3.1.10).

□

3.2 A PIECEWISE CONSTANT DENSITY

I.

We will now examine a process N_t whose underlying density process λ_t evolves as follows. λ_t is constant until a jump occurs and then λ_t moves to a

new value whose distribution function is G(y) ($y \in \mathbb{R}^+$) and is independent of its previous value and the time needed for this change to occur. Jump interarrival time intervals are independent identically distributed random variables with distribution function P(v) ($v \in \mathbb{R}^+$) and such that there exists $\alpha>0$ so that

$$\int_{\mathbb{R}^+} e^{\beta v} dP(v) < \infty \quad \text{for } \beta \in [0,\alpha)$$

We are going to investigate the behaviour of N_t when (λ_t, u_t) is stationary. For this purpose we will examine the aggregated process $x_t = \int_0^t \lambda_s ds$ which is an equivalent problem as we have seen in the previous section. We will take u_t the time elapsed since the last change of value of λ_t as our supplementary variable. It will then be easy to manipulate expectations when exploiting the properties of the martingales obtained. We have of course to assume that P(u) possesses a density p(u).

Our piecewise deterministic process consists of u_t, λ_t, x_t and t. The generator of the process (x_t, λ_t, u_t, t) acting on a function $f(x,\lambda,u,t)$ belonging to its domain is given by

$$Af(x,\lambda,u,t) = \lambda \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} + \frac{\partial f}{\partial t} + \frac{p(u)}{\mathbf{P}(u)} \underset{(\mathbf{P}(u)=1-P(u))}{\left[\int_{\mathbb{R}^+} f(x,y,0,t)dG(y) - f(x,\lambda,u,t)\right]} \qquad (3.2.1)$$

We will now find a martingale. We start with a definition of a function.

Definition 3.2.1 *Let* P(u) *and* G(u) *as described and let*

$$\hat{g}(\nu) = \int_{\mathbb{R}^+} e^{-\nu y} dG(y)$$

We define the function $\beta(\nu):[0,\infty) \to [0,\infty)$ *by the relationship*

$$\int_0^\infty \exp[\beta(\nu)z] p(z) \hat{g}(\nu z) dz = 1 \qquad (3.2.2)$$

$\beta(\nu)$ is uniquely defined since $\int_0^\infty e^{\beta z} p(z) \hat{g}(\nu z) dz$ is a monotone function of β for all ν. It is not feasible to express explicitly $\beta(\nu)$ in terms of ν even for the most simple non trivial situations. However it is easy to find derivatives of $\beta(\nu)$ evaluated at $\nu=0$ which in fact are the quantities of interest as we shall see later. For example differentiating (3.2.2) once we get

$$\int_0^\infty \beta'(\nu)z\exp[\beta(\nu)z]p(z)\hat{g}(\nu z)dz + \int_0^\infty \exp[\beta(\nu)z]p(z)\hat{g}'(\nu z)zdz = 0 \qquad (3.2.3)$$

and setting $\nu=0$ in (3.2.3) we get $\beta'(0)\rho_1-\mu_1\rho_1$ (we are assuming μ_1,ρ_1 are the first moments of G,P respectively and that they exist), so therefore

$$\beta'(0) = \mu_1 \qquad (3.2.4)$$

Differentiating (3.2.3) and setting $\nu=0$ we get

$$\beta''(0) = -(\rho_2/\rho_1)(\mu_2-\mu_1^2) \qquad (3.2.5)$$

(μ_2,ρ_2 are the second moments of G,P assuming they exist).

We now proceed to find a martingale.

Theorem 3.2.2 *Let* x_t,λ_t,u_t *as defined. Then*

$$\{\exp(-\nu x_t)\exp[-\beta(\nu)u_t]\exp(\nu\lambda_t u_t)/\mathbf{P}(u_t)\}\int_{u_t}^\infty \exp[\beta(\nu)z]\exp(-\nu\lambda_t z)p(z)dz\exp[\beta(\nu)t] \qquad (3.2.6)$$

is a martingale.

Proof

We will try to find a martingale of the form $\exp(-\nu x_t)e^{\beta t}h(\lambda_t,u_t)$. Substituting into $Af(x,\lambda,u,t) = 0$ we get

$$\beta h(\lambda,u) - \lambda\nu h(\lambda,u) + \frac{\partial h}{\partial u} + \frac{p(u)}{\mathbf{P}(u)}[\int_{\mathbb{R}^+}h(y,0)dG(y)-h(\lambda,u)] = 0 \qquad (3.2.7)$$

which is equivalent to

$$\frac{\partial}{\partial u}[e^{\beta u}e^{-\lambda\nu u}\mathbf{P}(u)h(\lambda,u)] = -e^{\beta u}e^{-\lambda\nu u}p(u)\int_{\mathbb{R}^+}h(y,0)dG(y) \qquad (3.2.8)$$

which leads to

$$h(\lambda,u) = \int_{\mathbb{R}^+}h(y,0)dG(y)\{e^{-\beta u}e^{\lambda\nu u}/\mathbf{P}(u)\}\int_u^\infty e^{\beta z}e^{-\lambda\nu z}p(z)dz + K(\lambda)e^{-\beta u}e^{\lambda\nu u}/\mathbf{P}(u) \qquad (3.2.9)$$

where $K(\lambda)$ is an arbitrary function of λ. However, while the first term belongs to the domain of generator since it is bounded the expectation of the second term conditional on all values of λ_t greater than β/ν is infinite so our solution belongs to the domain of the generator if and only if $K(\lambda)\equiv 0$. Therefore

$$h(\lambda,u) = \int_{\mathbb{R}^+}h(y,0)dG(y)\{e^{-\beta u}e^{\lambda\nu u}/\mathbf{P}(u)\}\int_u^\infty e^{\beta z}e^{-\lambda\nu z}p(z)dz \qquad (3.2.10)$$

Setting $u=0$ in (3.2.10)

$$h(\lambda,0) = \int_{\mathbb{R}^+}h(y,0)dG(y)\int_0^\infty e^{\beta z}e^{-\lambda\nu z}p(z)dz \qquad (3.2.11)$$

so

$$\int_{\mathbb{R}^+} h(y,0)dG(y) = \int_{\mathbb{R}^+} h(y,0)dG(y) \int_{\mathbb{R}^+} \int_0^\infty e^{\beta z} e^{-y\nu z} p(z)dz dG(y) \tag{3.2.12}$$

so we need to have

$$\int_0^\infty \exp[\beta(\nu)z]p(z)\hat{g}(\nu z)dz = 1 \tag{3.2.13}$$

and therefore $\beta = \beta(\nu)$ and by (3.2.10) we coclude that the required solution of $Af=0$ is

$$Le^{-\nu x}\exp[\beta(\nu)t]\{\exp[-\beta(\nu)u]e^{\lambda\nu u}/\mathbf{P}(u)\}\int_u^\infty \exp[\beta(\nu)z]e^{-\lambda\nu z}p(z)dz \tag{3.2.14}$$

where L is an arbitrary constant and the theorem is proved.

□

One can of course in the same way prove the analogue for N_t, that is

$$\theta^{N_t}\{ \exp[-\beta(1-\theta)u_t]\exp[(1-\theta)\lambda_t u_t]/\mathbf{P}(u_t)\}\int_{u_t}^\infty \exp[\beta(1-\theta)z]\exp[-\lambda_t(1-\theta)z]p(z)dz \exp[\beta(1-\theta)t] \tag{3.2.15}$$

is a martingale.

So we can now apply theorem 3.2.2 to find the behaviour of x_t when λ_t and u_t are in the stationary state. It is clear that when stationary, λ_t has distribution function G(y) and is independent of u_t which has probability density $\mathbf{P}(u)/\rho_1$ (see proposition 2.5.1). So we can now prove the following theorem.

Theorem 3.2.3 *Let λ_t, u_t be stationary. Then*

$$E\{\exp(-\nu x_t)\} = \frac{\int_0^\infty \hat{g}(\nu z)\mathbf{P}(z)dz \int_0^\infty \beta(\nu)\, e\, \exp[\beta(\nu)z]\, \hat{g}(\nu z)\mathbf{P}(z)dz}{\rho_1\{1 - \int_0^\infty \hat{g}(\nu z)\mathbf{P}(z)dz\}} \exp[-\beta(\nu)t] \tag{3.2.16}$$

Proof

By theorem 3.2.2 and recalling that $x_0=0$

$$E\{\{\exp(-\nu x_t)\exp[-\beta(\nu)u_t]\exp(\nu\lambda_t u_t)/\mathbf{P}(u_t)\}\int_{u_t}^\infty \exp[\beta(\nu)z]\exp(-\nu\lambda_t z)p(z)dz\} =$$

$$\exp[-\beta(\nu)t]E\{\{\exp[-\beta(\nu)u_o]\exp(\nu\lambda_t u_o)/\mathbf{P}(u_o)\}\int_{u_o}^\infty \exp[\beta(\nu)z]\exp(-\nu\lambda_o z)p(z)dz\} \tag{3.2.17}$$

Now, by the structure of the process (λ_t, u_t) is independent of $x_t - \lambda_t u_t$ (the position of x_t at the time of the last change of value of λ_t) so (3.2.17) can be written as

$$E\{\exp[-\nu(x_t-\lambda_t u_t)]\}E\{\{\exp[-\beta(\nu)u_t]/\mathbf{P}(u_t)\}\int_{u_t}^{\infty}\exp[\beta(\nu)z]\exp(-\nu\lambda_t z)p(z)dz\} =$$

$$\exp[-\beta(\nu)t]E\{\{\exp[-\beta(\nu)u_o]\exp(\nu\lambda_t u_o)/\mathbf{P}(u_o)\}\int_{u_o}^{\infty}\exp[\beta(\nu)z]\exp(-\nu\lambda_o z)p(z)dz\} \qquad (3.2.18)$$

When stationary, u_t and λ_t are independent and distributed with density $\mathbf{P}(u)/\rho_1$ and distribution function G(y) respectively. The same is true for λ_o, u_o. So

$$E\{\{\exp[-\beta(\nu)u_t]/\mathbf{P}(u_t)\}\int_{u_t}^{\infty}\exp[\beta(\nu)z]\exp(-\nu\lambda_t z)p(z)dz\} =$$

$$\int_{\mathbb{R}^+}\int_o^{\infty}\{\exp[-\beta(\nu)u]/\mathbf{P}(u)\}\int_u^{\infty}\exp[\beta(\nu)z]e^{-\nu yz}p(z)dz\{\mathbf{P}(u)/\rho_1\}dudG(y) =$$

$$(1/\rho_1)\int_o^{\infty}\exp[-\beta(\nu)u]\int_u^{\infty}\exp[\beta(\nu)z]e^{-\nu yz}p(z)dzdu = (1/\rho_1)\int_o^{\infty}\frac{\exp[\beta(\nu)z]-1}{\beta(\nu)}\hat{g}(\nu z)p(z)dz =$$

$$[1/\rho_1\beta(\nu)]\{1 - \int_o^{\infty}\hat{g}(\nu z)p(z)dz\} \qquad (3.2.19)$$

Also

$$E\{\exp[-\beta(\nu)u_o]\exp(\nu\lambda_t u_o)/\mathbf{P}(u_o)\}\int_{u_o}^{\infty}\exp[\beta(\nu)z]\exp(-\nu\lambda_o z)p(z)dz\} =$$

$$\int_{\mathbb{R}^+}\int_o^{\infty}\{\exp[-\beta(\nu)u]e^{\nu yu}/\mathbf{P}(u)\}\int_u^{\infty}\exp[\beta(\nu)z]e^{-\nu yz}p(z)dz\{\mathbf{P}(u)/\rho_1\}dudG(y) =$$

$$(1/\rho_1)\int_{\mathbb{R}^+}\int_o^{\infty}\int_o^{\infty}\exp[\beta(\nu)z]e^{-\nu yz}p(z+u)dzdudG(y) = (1/\rho_1)\int_o^{\infty}\int_o^{\infty}\exp[\beta(\nu)z]\hat{g}(\nu z)p(z+u)dzdu$$

$$= (1/\rho_1)\int_o^{\infty}\exp[\beta(\nu)z]\hat{g}(\nu z)\mathbf{P}(z)dz \qquad (3.2.20)$$

(3.2.18), (3.2.19) & (3.2.20) combine to give

$$E\{\exp[-\nu(x_t-\lambda_t u_t)]\} = \frac{\int_o^{\infty}\beta(\nu)\exp[\beta(\nu)z]\hat{g}(\nu z)\mathbf{P}(z)dz}{1 - \int_o^{\infty}\hat{g}(\nu z)\mathbf{P}(z)dz} \qquad (3.2.21)$$

and because $\lambda_t u_t$ and $x_t-\lambda_t u_t$ are independent

$$E\{\exp(-\nu x_t)\} = E\{\exp[-\nu(x_t-\lambda_t u_t)]\}E\{\exp(-\nu\lambda_t u_t)\} \qquad (3.2.22)$$

But

$$E\{\exp(-\nu\lambda_t u_t)\} = \int_{\mathbb{R}^+}\int_o^{\infty}e^{-\nu yu}\{\mathbf{P}(u)/\rho_1\}dudG(y) = (1/\rho_1)\int_o^{\infty}\hat{g}(\nu z)\mathbf{P}(z)dz \qquad (3.2.23)$$

(3.2.21), (3.2.22) & (3.2.23) give us (3.2.16).

□

Formula (3.2.16) also gives us the limiting distribution of

$$\int_t^{t+s} \lambda_r dr \text{ (also of } (N_{t+s}-N_t) \text{ of course).}$$

If we take logarithms in (3.2.16)

$$-\log\{E\{\exp(-\nu x_t)\}\} = \beta(\nu)t + \log\left[\frac{\rho_1\{1 - \int_0^\infty \hat{g}(\nu z)\mathbf{P}(z)dz\}}{\int_0^\infty \hat{g}(\nu z)\mathbf{P}(z)dz \int_0^\infty \beta(\nu) \exp[\beta(\nu)z]\, \hat{g}(\nu z)\mathbf{P}(z)dz}\right] \quad (3.2.24)$$

and setting

$$\alpha(\nu) = \log\left[\frac{\rho_1\{1 - \int_0^\infty \hat{g}(\nu z)\mathbf{P}(z)dz\}}{\int_0^\infty \hat{g}(\nu z)\mathbf{P}(z)dz \int_0^\infty \beta(\nu) \exp[\beta(\nu)z]\, \hat{g}(\nu z)\mathbf{P}(z)dz}\right] \quad (3.2.25)$$

$$-\log\{E\{\exp(-\nu x_t)\}\} = \beta(\nu)t + \alpha(\nu) \quad (3.2.26)$$

which is an expression for the cumulant generating function and if we denote by $k_n(t)$ the n^{th} cumulant of x_t and by $\beta^{(n)}(\nu)$ and $\alpha^{(n)}(\nu)$ the n^{th} derivatives of $\beta(\nu)$ and $\alpha(\nu)$

$$k_n(t) = (-1)^{n-1}\{\beta^{(n)}(0)t + \alpha^{(n)}(0)\} \quad (3.2.27)$$

(we assume that n derivatives of α,β exist; that is n moments of P,G exist).

We already seen in the beginning of the section how easy is to calculate derivatives of $\beta(\nu)$. We have proved that the cumulants of x_t are linear when λ_t,u_t is stationary and we have found the appropriate coefficients. By (3.2.26) and (3.1.4) the analogue for N_t follows; that is

$$-\log\{E\{\theta^{N_t}\}\} = \beta(1-\theta)t + \alpha(1-\theta) \quad (3.2.28)$$

II.

We will also try to find the distribution of interarrival times of points of N_t. By τ_k we denote the length of the interval between the $(k-1)^{th}$ and the k^{th} point. We will not merely concentrate on finding the distribution of τ_1 (the

distribution of τ_1 given λ_0,v_0 is the same as the distribution of τ_k given the same values for $\lambda_{\tau_{k-1}},v_{\tau_{k-1}}$), but we will look at higher order properties as well. We will now include v_t the time needed for the next jump to occur in our process instead of u_t. So we do not need the assumption that P(y) has a density any more. Furthermore we will not need $\beta(v)$ also so we drop the assumption that $\int_{\mathbb{R}^+} e^{\beta v} dP(v)$ exists for $\beta>0$. The generator of the process (N_t,λ_t,v_t,t) acting on a function $f(N,\lambda,v,t)$ belonging to its domain is given by

$$Af(N,\lambda,v,t) = \frac{\partial f}{\partial t} - \frac{\partial f}{\partial v} + \lambda[f(N+1,\lambda,v,t)-f(N,\lambda,v,t)] \tag{3.2.29}$$

with the requirement for f to belong to the domain of the generator (boundary condition)

$$f(N,\lambda,0,t) = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f(N,y,v,t)dG(y)dP(v) \tag{3.2.30}$$

An implication of (3.2.30) is that at v=0 f has to be functionally independent of λ.

We will now prove an important theorem.

Theorem 3.2.4 *Let* $k(\lambda,v)$ *be a function such that*

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} k(y,u)dG(y)dP(v) < \infty$$

Let $N_0=0$ *and* $\tau_1 \equiv \inf\{t:N_t=1\}$. *Define also*

$$B(\beta;k) = \frac{\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} e^{-(\beta+y)v} y \int_0^v e^{(\beta+y)z} k(y,z)dzdP(u)dG(y)}{\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}[1-e^{-(\beta+y)u}]dP(u)dG(y)} \tag{3.2.31}$$

Then

$$E\{\exp(-\beta\tau_1)k(\lambda_{\tau_1},v_{\tau_1}) \mid \lambda_0,v_0\} = \{B(\beta;k)+\lambda_0\int_0^{v_0} k(\lambda_0,z)\exp[(\beta+\lambda_0)z]dz\}\exp[-(\beta+\lambda_0)v_0] \tag{3.2.32}$$

Proof

By lemma 3.1.2 if

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial v} + \lambda[e^{-\beta t}k(\lambda,v)-f(\lambda,v,t)] = 0 \tag{3.2.33}$$

and

$$f(\lambda,0,t) = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f(y,v,t)dG(y)dP(v) \tag{3.2.34}$$

then

$$E\{\exp(-\beta\tau_1)k(\lambda_{\tau_1},v_{\tau_1})\,|\,\lambda_o,v_o\} = f(\lambda_o,v_o,0) \tag{3.2.35}$$

So we need to solve (3.2.33) subject to (3.2.34). We try a solution of the form $e^{-\beta t}h(\lambda,v)$ and we get

$$-\beta h(\lambda,v) - \frac{\partial h}{\partial v} + \lambda[k(\lambda,v)-h(\lambda,v)] = 0 \tag{3.2.36}$$

and

$$h(\lambda,0) = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} h(y,v)dG(y)dP(v) \tag{3.2.37}$$

(3.2.37) implies

$$h(\lambda,0) = K \tag{3.2.38}$$

where K is an arbitrary constant. (3.2.36) can be rewritten as

$$\frac{\partial}{\partial u}[e^{(\beta+\lambda)v}h(\lambda,v)] = \lambda k(\lambda,v)e^{(\beta+\lambda)v} \tag{3.2.39}$$

so

$$h(\lambda,v) = e^{-(\beta+\lambda)v}\lambda\int_o^v k(\lambda,z)e^{(\beta+\lambda)z}dz + Ke^{-(\beta+\lambda)v} \tag{3.2.40}$$

and using (3.2.37)

$$K = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} e^{-(\beta+y)v}\{K+y\int_o^v k(y,z)e^{(\beta+y)z}dz\}dG(y)dP(v) \tag{3.2.41}$$

so

$$K = \frac{\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} e^{-(\beta+y)v}y\int_o^v e^{(\beta+y)z}k(y,z)dzdP(u)dG(y)}{\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}[1-e^{-(\beta+y)u}]dP(u)dG(y)} = B(\beta;k) \tag{3.2.42}$$

and hence

$$h(\lambda,v) = [B(\beta;k)+\lambda\int_o^v k(\lambda,z)e^{(\beta+\lambda)z}dz]e^{-(\beta+\lambda)v} \tag{3.2.43}$$

proving the theorem with (3.2.35)

□

So we can immediately conclude about the Laplace transform of the distribution of τ_1.

Corollary 3.2.5

$$E\{\exp(-\beta\tau_1)\,|\,\lambda_o,v_o\} = B(\beta;1)\exp[-(\beta+\lambda_o)v_o] + [\lambda_o/(\beta+\lambda_o)][1-\exp[-(\beta+\lambda_o)v_o]] \qquad (3.2.44)$$

where

$$B(\beta;1) = \frac{\int_{\mathbb{R}^+}[y/(\beta+y)][1-\hat{p}(\beta+y)]dG(y)}{\int_{\mathbb{R}^+}[1-\hat{p}(\beta+y)]dG(y)} \qquad (3.2.45)$$

Proof

We just set $k(\lambda,v)\equiv 1$ in theorem 3.2.4.

□

Using theorem 3.2.4 we can proceed further to higher order properties. We can find the joint Laplace transform of the distribution of $\tau_1,\tau_2,.....\tau_n$. We will demonstrate how to do that with τ_1 and τ_2. If we wish to include more τ_i's the same procedure as in the proof of the next theorem has to be repeated.

Theorem 3.2.6

$$E\{\exp(-\beta_1\tau_1-\beta_2\tau_2)\,|\,\lambda_o,v_o\} =$$

$$\{B(\beta_1;k_{\beta_2})+\lambda_o[B(\beta_2;1)-\frac{\lambda_o}{\beta+\lambda_o}][\exp[(\beta_1-\beta_2)v_o]-1]/[\beta_1-\beta_2] + \frac{\lambda_o^2[\exp[(\beta_1+\lambda_o)v_o]-1]}{(\beta_1+\lambda_o)(\beta_2+\lambda_o)}\}e^{-(\beta_1+\lambda_o)v_o} \qquad (3.2.46)$$

where $B(\beta_2;1)$ *is given by (3.2.45) and*

$$B(\beta_1;k_{\beta_2}) = \frac{\int_{\mathbb{R}^+}\{y[\,B(\beta_2;1)-\frac{y}{y+\beta_2}]\frac{\hat{p}(\beta_2+y)-\hat{p}(\beta_1+y)}{\beta_1-\beta_2} + \frac{y^2[1-\hat{p}(\beta_1+y)]}{(\beta_2+y)(\beta_1+y)}\}dG(y)}{\int_{\mathbb{R}^+}[1-\hat{p}(\beta_1+y)]dG(y)} \qquad (3.2.47)$$

Proof

Using corollary 3.2.5 and the fact that τ_1 given the values of λ_o,v_o has the same distribution as τ_2 given the same values for λ_{τ_1} and v_{τ_1} we get

$$E\{\exp(-\beta_2\tau_2)\,|\,\tau_1,\lambda_{\tau_1},v_{\tau_1}) = E\{\exp(-\beta_2\tau_2)\,|\,\lambda_{\tau_1},v_{\tau_1}) = k_{\beta_2}(\lambda_{\tau_1},v_{\tau_1}) \qquad (3.2.48)$$

where

$$k_{\beta_2}(\lambda_{\tau_1},v_{\tau_1}) = B(\beta_2;1)\exp[-(\beta_2+\lambda_{\tau_1})v_{\tau_1}]+[\lambda_{\tau_1}/(\beta_2+\lambda_{\tau_1})]\{1-\exp[-(\beta_2+\lambda_{\tau_1})v_{\tau_1}]\} \qquad (3.2.49)$$

We use theorem 3.2.4 with k_{β_2} in place of k. Then (3.3.31) gives

$$B(\beta\ ;k_{\beta_2}) = \frac{\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} e^{-(\beta+y)v} y \int_0^v e^{(\beta+y)z} k_{\beta_2}(y,z)\,dz\,dP(u)\,dG(y)}{\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}[1-e^{-(\beta+y)u}]\,dP(u)\,dG(y)} \tag{3.2.50}$$

But

$$\int_0^v e^{(\beta+y)z} k_{\beta_2}(y,z)\,dz = \int_0^v \left\{B(\beta_2;1)\exp[-(\beta_2+y)z] + [y/(\beta_2+y)][1-\exp[-(\beta_2+y)z]\right\}\exp[(\beta+y)z] =$$

$$\{B(\beta_2;1)-[y/(\beta_2+y)]\}\int_0^v \exp[(\beta-\beta_2)z]\,dz + [y/(\beta_2+y)]\int_0^v \exp[(\beta+y)z]\,dz =$$

$$\{B(\beta_2;1)-[y/(\beta_2+y)]\}\frac{\exp[(\beta-\beta_2)v]-1}{\beta-\beta_2} + [y/(\beta_2+y)]\frac{\exp[(\beta+y)v]-1}{\beta+y} \tag{3.2.51}$$

Substituting into (3.2.50), (3.2.47) follows and

$$E\{\exp(-\beta_1\tau_1-\beta_2\tau_2)\,|\,\lambda_0,v_0\} = E\{\exp(-\beta_1\tau_1)E\{\exp(-\beta_2\tau_2)\,|\,\lambda_{\tau_1},v_{\tau_1}\}\,|\,\lambda_0,v_0\} =$$

$$E\{\exp(-\beta_1\tau_1)k_{\beta_2}(\lambda_{\tau_1},v_{\tau_1})\,|\,\lambda_0,v_0\} \tag{3.2.52}$$

and by theorem 3.2.4, (3.2.49) follows.

□

As we have said the last step of the proof can be repeated so as to incorporate τ_3,τ_4 etc in our formula. Also one should be very careful when using corollary 3.2.5 or theorem 3.2.6 because some of the integrals might not exist when $\beta=0,\beta_1=0$ or $\beta_2=0$ and take limits where appropriate.

3.3 THE VIRTUAL WAITING TIME OF THE M/G/1 QUEUE AS INTENSITY

The approach to establishing the properties of the distribution of N_t or x_t

by finding a martingale of the form $\exp(-\nu x_t)\exp[(\beta(\nu)t]h(\lambda_t)$ might fail because we might not be able to solve the equation involved. So we need an alternative approach to give us the behaviour of some of the moments at least.

An example is the case where λ_t is set to be the virtual waiting time of the M/G/1 queue. We have already examined some of the properties of a more general process in the previous chapter. We assume λ_t decreases linearly, jumps arrive with rate ρ and G(y) is the distribution function of the jump sizes. When λ_t reaches 0 it remains there till a jump occurs. Such a process N_t with λ_t as intensity can be used to model some form of clustering. For example we can consider rains as consequent events of "storms" (represented by jumps) which fade out. Also x_t might be useful itself when applied to evaluating costs associated with networks involving the M/G/1 queue. We will examine x_t. The generator of the process (x_t,λ_t,t) acting on a function $f(x,\lambda,t)$ belonging to its domain is given by

$$Af(x,\lambda,t) = \begin{cases} \lambda\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} - \frac{\partial f}{\partial \lambda} + \rho[\int_{\mathbb{R}^+} f(x,\lambda+y,t)dG(y) - f(x,\lambda,t)] & \lambda>0 \\ \frac{\partial f}{\partial t} + \rho[\int_{\mathbb{R}^+} f(x,y,t)dG(y) - f(x,0,t) & \lambda=0 \end{cases} \qquad (3.3.1)$$

We also assume that μ_1 the first moment of G exists and $\rho\mu_1<1$ so as to make λ_t recurrent.

In order to find a martingale of the form $\exp(-\nu x_t)e^{\beta t}h(\lambda_t)$ we need to substitute in Af = 0 and get

$$(\beta-\lambda\nu)h(\lambda) - h'(\lambda) + \rho[\int_{\mathbb{R}^+} h(\lambda+y)dG(y) - h(\lambda)] = 0 \qquad (3.3.2)$$

$$\rho\int_{\mathbb{R}^+} h(y)dG(y) = (\beta+\rho)h(0) \qquad (3.3.3)$$

(3.3.3) is equivalent to $h'(0) = 0$ (3.3.4)

To solve (3.3.2) subject to (3.3.4) is not easy since there is no simple solution in general and no direct relationship between β and ν. A series solution approach will lead to the same mathematics that will follow adding some technical difficulties as well. We will examine the behaviour of moments as we have

already done before when facing similar problems.

We start with some heuristics. Assume there is a martingale of the form $\exp(-\nu x_t)\exp[\beta(\nu)t]e^{-a(\lambda_t;\nu)}$ (we write $h(\lambda)=e^{-a(\lambda;\nu)}$). By $a_i(\lambda;\nu)$ we denote the i^{th} derivative of $a(\lambda;\nu)$ with respect to ν. Differentiating $\exp(-\nu x_t)\exp[\beta(\nu)t]e^{-a(\lambda_t;\nu)}$ we get

$$\{-x_t + \beta'(\nu)t - a_1(\lambda_t;\nu)\}\exp(-\nu x_t)\exp[\beta(\nu)t]e^{-a(\lambda_t;\nu)} \tag{3.3.5}$$

and setting $\nu=0$ and assuming $\beta(0)=0$ and $a(\lambda;0)=0$ this becomes $-x_t+\beta'(0)t-a_1(\lambda_t;0)$ and it suggests to try to find a martingale of the form $x_t-K_1t+h_1(\lambda)$.

Theorem 3.3.1 *Let* x_t,λ_t *as defined. Suppose* G(y) *posseses at least two moments* μ_1 *and* μ_2*; then* $x_t - K_1t + h_1(\lambda)$ *is a martingale where*

$$K_1 = \rho\mu_2/[2(1-\rho\mu_1)] \tag{3.3.6}$$

$$h_1(\lambda) = \lambda^2/[2(1-\rho\mu_1)] \tag{3.3.7}$$

Proof

We substitute $x-K_1t+h_1(\lambda)$ into $Af=0$ and we get

$$\lambda - K_1 - h_1'(\lambda) + \rho[\int_{\mathbb{R}^+}h_1(\lambda+y)dG(y)-h_1(\lambda)] = 0 \tag{3.3.8}$$

and

$$h_1'(0) = 0 \tag{3.3.9}$$

Setting $h_1(\lambda)=A_1\lambda^2$ (so (3.3.9) is satisfied) (cf (3.3.4))

$$\lambda - K_1 - 2A_1\lambda + \rho(2A_1\lambda\mu_1+A_1\mu_2) = 0 \tag{3.3.10}$$

and equating coefficients in (3.3.10)

$$A_1 = 1/[2(1-\rho\mu_1)] \qquad K_1 = \rho\mu_2/[2(1-\rho\mu_1)] \tag{3.3.11}$$

$E\{x_t\}$ and $E\{\lambda_t^2\}$ are finite for all t so our function belongs to the domain of the generator.

□

Theorem 3.3.1 leads to the asymptotic behaviour of $E\{x_t\}$.

Corollary 3.3.2

$$\lim_{t\to\infty}\frac{E\{x_t\}}{t} = \rho\mu_2/[2(1-\rho\mu_1)] \tag{3.3.12}$$

Proof

By theorem (3.3.1)

$$E\{x_t\} = t\rho\mu_2/[2(1-\rho\mu_1)] - E\{x_0\} + E\{\lambda_t^2-\lambda_0^2\}/[2(1-\rho\mu_1)] \tag{3.3.13}$$

Dividing both sides of (3.3.13) by t and letting $t\to\infty$ (3.3.12) follows.

□

Of course it is immediate that

$$\lim_{t\to\infty}\frac{E\{N_t\}}{t} = \rho\mu_2/[2(1-\rho\mu_1)] \tag{3.3.14}$$

We now proceed to examine the behaviour of the second moment of x_t. By continuing our heuristics we see that the second derivative of $\exp(-\nu x_t)\exp[\beta(\nu)t]e^{-a(\lambda_t;\nu)}$ evaluated at $\nu=0$ is $\{-x_t+\beta'(0)t-a_1(\lambda_t;0)\}^2 + \beta''(0)t - a_2(\lambda_t;0)$ suggesting a martingale of the form $\{x_t-K_1t+h_1(\lambda_t)\}^2 - K_2t + h_2(\lambda_t)$ with K_1 and $h_1(\lambda)$ being the same as in theorem 3.3.1.

Theorem 3.3.3 *Let* x_t,λ_t *as defined and suppose* G(y) *posseses at least four moments* μ_1,μ_2,μ_3 *and* μ_4*; then* $\{x_t-K_1t+h_1(\lambda_t)\}^2 - K_2t + h_2(\lambda_t)$ *is a martingale, where*

$$K_1 = \rho\mu_2/[2(1-\rho\mu_1)] \tag{3.3.15}$$

$$h_1(\lambda) = \lambda^2/[2(1-\rho\mu_1)] \tag{3.3.16}$$

$$K_2 = \rho\left\{\frac{\mu_4}{4(1-\rho\mu_1)^2} + \frac{5\rho\mu_3\mu_2}{6(1-\rho\mu_1)^3} + \frac{\rho_2\mu_2^3}{2(1-\rho\mu_1)^4}\right\} \tag{3.3.17}$$

$$h_2(\lambda) = \frac{\rho\mu_2}{3(1-\rho\mu_1)^3}\lambda^3 + \left\{\frac{\rho\mu_3}{2(1-\rho\mu_1)^3} + \frac{\rho_2\mu_2^2}{2(1-\rho\mu_1)^4}\right\}\lambda^2 \tag{3.3.18}$$

Proof

Substituting $\{x-K_1t+h_1(\lambda)\}^2 - K_2t + h_2(\lambda)$ into $Af=0$ we get

$$2[\lambda-K_1-h_1'(\lambda)][x-K_1t+h_1(\lambda)]-K_2-h_2'(\lambda)+$$

$$\rho\{2(x-K_1t)[\textstyle\int_{\mathbb{R}^+}h_1(\lambda+y)dG(y)-h_1(\lambda)]-\int_{\mathbb{R}^+}h_1^2(\lambda+y)dG(y)-h_1^2(\lambda)+\int_{\mathbb{R}^+}h_2(\lambda+y)dG(y)-h_2(\lambda)\}=0 \tag{3.3.19}$$

as well as $h_2'(0)=0$ and $h_1'(0)=0$.

Using (3.3.8) from the previous theorem we obtain

$$-K_2 - h_2'(\lambda) + \rho\{\int_{\mathbb{R}^+}[h_1(\lambda+y)-h_1(\lambda)]^2 dG(y) + \int_{\mathbb{R}^+} h_2(\lambda+y)dG(y) - h_2(\lambda)\}=0 \qquad (3.3.20)$$

and since $h_1(\lambda) = \lambda^2/[2(1-\rho\mu_1)]$ we get

$$-K_2 - h_2'(\lambda) + \rho\{\frac{\mu_4+4\mu_3\lambda+4\mu_2\lambda^2}{4(1-\rho\mu_1)^2} + \int_{\mathbb{R}^+} h_2(\lambda+y)dG(y) - h_2(\lambda)\}=0 \qquad (3.3.21)$$

Trying $h_2(\lambda) = A_3\lambda^3+A_2\lambda^2$

$$-K_2-3A_3\lambda^2-2A_2\lambda+\rho\{\frac{\mu_4+4\mu_3\lambda+4\mu_2\lambda^2}{4(1-\rho\mu_1)^2} +A_3(3\lambda^2\mu_1+3\lambda\mu_2+\mu_3)+A_2(2\lambda\mu_1+\mu_2)\} = 0 \qquad (3.3.22)$$

Equating coefficients in (3.3.22)

$$A_3 = \frac{\rho\mu_2}{3(1-\rho\mu_1)^3} \qquad (3.3.23)$$

$$A_2 = \frac{\rho\mu_3}{2(1-\rho\mu_1)^3} + \frac{3A_3\rho\mu_2}{2(1-\rho\mu_1)} = \frac{\rho\mu_3}{2(1-\rho\mu_1)^3} + \frac{\rho^2\mu_2^2}{2(1-\rho\mu_1)^4} \qquad (3.3.24)$$

and

$$\begin{aligned} K_2 &= \rho\left\{\frac{\mu_4}{4(1-\rho\mu_1)^2} + A_3\mu_3 + A_2\mu_2 \right\} = \\ &= \rho\left\{\frac{\mu_4}{4(1-\rho\mu_1)^2} + \frac{\rho\mu_3\mu_2}{3(1-\rho\mu_1)^3} + \frac{\rho\mu_3\mu_2}{2(1-\rho\mu_1)^3} + \frac{\rho_2\mu_2^3}{2(1-\rho\mu_1)^4} \right\} = \\ &= \rho\left\{\frac{\mu_4}{4(1-\rho\mu_1)^2} + \frac{5\rho\mu_3\mu_2}{6(1-\rho\mu_1)^3} + \frac{\rho_2\mu_2^3}{2(1-\rho\mu_1)^4} \right\} \end{aligned} \qquad (3.3.25)$$

and the theorem has been proved.

□

So we then find the asymptotic behaviour of the variance of x_t.

Corollary 3.3.4

$$\lim_{t\to\infty} E\left\{\left[\frac{x_t-K_1t}{\sqrt{t}}\right]^2\right\} = K_2 \qquad (3.3.26)$$

Proof

We just have to use the martingale of theorem 3.3.3 and exactly as in corollary 3.3.2 we divide by t and let $t\to\infty$.

□

So the variance of x_t behaves asymptotically like K_2t; that is it is asymptotically linear. Using that $var(N_t) = var(x_t) + E\{N_t\}$ the variance of N_t behaves asymptotically like $(K_1+K_2)t$.

We can carry on our heuristics to establish the asymptotic behaviour of higher cummulants of x_t and N_t. For example looking at the third derivative of $\exp(-\nu x_t)\exp[\beta(\nu)t]e^{-a(\lambda_t;\nu)}$ at $\nu=0$ we get

$\{-x_t+\beta'(0)t-a_1(\lambda_t;0)\}^3+3\{\beta''(0)t+a_2(\lambda_t;0)\}\{-x_t+\beta'(0)t-a_1(\lambda_t;0)\}+\beta^{(3)}(0)t+a_3(\lambda_t;0)$

suggesting that we should try to find a martingale of the form

$\{x_t-K_1t+h_1(\lambda_t)\}^3 + 3\{x_t-K_1t+h_1(\lambda_t)\}\{-K_2t+h_2(\lambda_t)\} - K_3t + h_3(\lambda_t)$

where K_1,h_1,K_2,h_2 are already found and K_3,h_3 are to be found by finding the appropriate solution of $Af(x,\lambda,t)=0$.

3.4 SHOT NOISE INTENSITY

In this section we assume a shot noise process like the one examined in section 2.3 as our intensity process. So we consider a point process N_t which is conditionally Poisson with intensity λ_t. λ_t is a shot noise process with rate of decay $\delta\lambda_t$, rate of jump arrivals ρ and jump size distribution function G(y) (y>0). We will actually examine the aggregated process x_t and use relationship (3.1.4) to deduce results about N_t. The generator of the process (x_t,λ_t,t) acting on a function $f(x,\lambda,t)$ belonging to its domain is given by

$$Af(x,\lambda,t) = \frac{\partial f}{\partial t} + \lambda\frac{\partial f}{\partial x} - \delta\lambda\frac{\partial f}{\partial \lambda} + \rho[\int_{\mathbb{R}^+} f(x,\lambda+y,t)dG(y)-f(x,\lambda,t)] \qquad (3.4.1)$$

Also the generator of (N_t,λ_t,t) acting on $f(N,\lambda,t)$ is given by

$$A^*f(N,\lambda,t) = \frac{\partial f}{\partial t} + \lambda[f(N+1,\lambda,t)-f(N,\lambda,t)] - \delta\lambda\frac{\partial f}{\partial \lambda} + \rho[\int_{\mathbb{R}^+} f(N,\lambda+y,t)dG(y)-f(N,\lambda,t)] \qquad (3.4.2)$$

In this situation we can do much better than merely establishing the

asymptotic behaviour of x_t and N_t. We will be able to evaluate the Laplace transform of the distribution of x_t at time t. Afterwards we will look at higher order properties as well.

Theorem 3.4.1 *Let x_t and λ_t as defined above and evolving up to a fixed time t^*. Also, consider constants κ_1 and κ_2 such that $\kappa_1 \geq 0$ and $\kappa_2 \geq -\kappa_1 e^{-\delta t^*}$; then*

$$\exp(-\kappa_1 \delta x_t)\exp[-(\kappa_1+\kappa_2 e^{\delta t})\lambda_t]\exp\{\rho\int_0^t [1-\hat{g}(\kappa_1+\kappa_2 e^{\delta s})]ds\} \tag{3.4.3}$$

$$\left(\hat{g}(\nu) = \int_{\mathbb{R}^+} e^{-\nu y} dG(y) \right)$$

is a martingale.

Proof

Define $w_t=\delta x_t+\lambda_t$ and $z_t=\lambda_t e^{\delta t}$, then the generator of the process (w_t,z_t,t) acting on a function f(w,z,t) is given by

$$A_1 f(w,z,t) = \frac{\partial f}{\partial t} + \rho[\int_{\mathbb{R}^+} f(w+y,z+ye^{\delta t},t)dG(y) - f(w,z,t)] \tag{3.4.4}$$

and f(w,z,t) has to satisfy $A_1 f=0$ for $f(w_t,z_t,t)$ to be a martingale. Trying $\exp(-\kappa_1 w)\exp(-\kappa_2 z)h(t)$ we get the equation

$$h'(t) - \rho[1-\hat{g}(\kappa_1+\kappa_2 e^{\delta t})]h(t) = 0 \tag{3.4.5}$$

$\exp(-\kappa_1 w)\exp(-\kappa_2 z)h(t)$ belongs to the domain of the generator because of our choice of κ_1,κ_2 (the function is bounded for all $t \leq t^*$ and our process evolves up to time t^* only; we are not interested in what happens afterwards, to avoid complications we can assume the process ceases to evolve and remains where it is).

Solving (3.4.5)

$$h(t) = K\exp\{\rho\int_0^t [1-\hat{g}(\kappa_1+\kappa_2 e^{\delta s})]ds\} \tag{3.4.6}$$

where K is an arbitrary constant. Therefore

$$\exp(-\kappa_1 w_t)\exp(-\kappa_2 z_t)\exp\{\rho\int_0^t [1-\hat{g}(\kappa_1+\kappa_2 e^{\delta s})]ds\}$$

is a martingale and hence (3.4.3) is a martingale.

□

Corollary 3.4.2 *Let* $\nu_1 \geq 0$, $\nu_2 \geq 0$, $\nu \geq 0$, $0 \leq \theta \leq 1$ *and* t_1, t_2 *be fixed times. Then*

$$E\{\exp[-\nu_1(x_{t_2}-x_{t_1})]\exp(-\nu_2\lambda_{t_2})\,|\,x_{t_1},\lambda_{t_1}\} = E\{\exp[-\nu_1(x_{t_2}-x_{t_1})]\exp(-\nu_2\lambda_{t_2})\,|\,\lambda_{t_1}\} =$$

$$\exp\{-\lambda_{t_1}\{\frac{\nu_1}{\delta}+(\nu_2-\frac{\nu_1}{\delta})\exp[-\delta(t_2-t_1)]\}\}\exp\{-\rho\int_0^{t_2-t_1}[1-\hat{g}(\frac{\nu_1}{\delta}+(\nu_2-\frac{\nu_1}{\delta})e^{-\delta s})]ds\} \qquad (3.4.7)$$

and

$$E\{\theta^{N_{t_2}-N_{t_1}}\exp(-\nu\lambda_{t_2})\,|\,N_{t_1},\lambda_{t_1}\} = E\{\theta^{N_{t_2}-N_{t_1}}\exp(-\nu\lambda_{t_2})\,|\,\lambda_{t_1}\} =$$

$$\exp\{-\lambda_{t_1}\{\frac{1-\theta}{\delta}+(\nu-\frac{1-\theta}{\delta})\exp[-\delta(t_2-t_1)]\}\}\exp\{-\rho\int_0^{t_2-t_1}[1-\hat{g}(\frac{1-\theta}{\delta}+(\nu-\frac{1-\theta}{\delta})e^{-\delta s})]ds\} \qquad (3.4.8)$$

Proof

We set $\kappa_1 = \nu_1/\delta$, $\kappa_2 = [\nu_2-(\nu_1/\delta)]\exp(-\delta t_2)$, $t^* \geq t_2$ in theorem 3.4.1 and (3.4.7) follows immediately.

(3.4.8) follows by (3.4.7) and (3.1.4) of section 3.1.

□

Corollary 3.4.3 *Let* x_t, N_t, λ_t *as before; then*

$$E\{\exp[-\nu(x_{t_2}-x_{t_1})]\,|\,\lambda_{t_1}\} = \exp\{-\frac{\nu}{\delta}[1-\exp[-\delta(t_2-t_1)]\lambda_{t_1}\}\exp\{-\rho\int_0^{t_2-t_1}[1-\hat{g}(\frac{\nu}{\delta}(1-e^{-\delta s}))]ds\} \qquad (3.4.9)$$

$$E\{\theta^{N_{t_2}-N_{t_1}}\,|\,\lambda_{t_1}\} = \exp\{-\frac{1-\theta}{\delta}[1-\exp[-\delta(t_2-t_1)]\lambda_{t_1}\}\exp\{-\rho\int_0^{t_2-t_1}[1-\hat{g}(\frac{1-\theta}{\delta}(1-e^{-\delta s}))]ds\} \qquad (3.4.10)$$

Furthermore if λ_t *is stationary then*

$$E\{\exp[-\nu(x_{t_2}-x_{t_1})]\} = \exp\{-(\rho\nu/\delta)\int_0^{t_2-t_1}\hat{\mathbf{G}}(\frac{\nu}{\delta}(1-e^{-\delta s}))ds\} \qquad (3.4.11)$$

$$E\{\theta^{N_{t_2}-N_{t_1}}\} = \exp\{-(\rho/\delta)(1-\theta)\int_0^{t_2-t_1}\hat{\mathbf{G}}(\frac{1-\theta}{\delta}(1-e^{-\delta s}))ds\} \qquad (3.4.12)$$

where $\hat{\mathbf{G}}(u) = [1-\hat{g}(u)]/u$

Proof

Set $\nu_2=0$, $\nu=0$ in (3.4.7) and (3.4.8). (3.4.9) and (3.4.10) follow. When λ_t is stationary it has a distribution with Laplace transform

$$\exp[-(\rho/\delta)\int_0^{\nu}\hat{\mathbf{G}}(u)du]$$

(see corollary 2.3.3 section 2.3); therefore

$$E\{\exp[-\nu(x_{t_2}-x_{t_1})]\} = \exp\{-(\rho/\delta)\int_0^{\nu[1-\exp[-\delta(t_2-t_1)]}\hat{\mathbf{G}}(u)du\}\exp\{-\rho\int_0^{t_2-t_1}[1-\hat{g}(\frac{\nu}{\delta}(1-e^{-\delta s}))]ds\}$$

$$= \exp\{-\rho\int_0^{t_2-t_1}\{[1-\hat{g}(\frac{\nu}{\delta}(1-e^{-\delta s}))]/(1-e^{-\delta s})\}e^{-\delta s}ds\}\exp\{-\rho\int_0^{t_2-t_1}[1-\hat{g}(\frac{\nu}{\delta}(1-e^{-\delta s}))]ds\} =$$

$$= \exp\{-\rho\int_0^{t_2-t_1}\{[1-\hat{g}(\frac{\nu}{\delta}(1-e^{-\delta s}))]/(1-e^{-\delta s})\}ds\} = \exp\{-(\rho\nu/\delta)\int_0^{t_2-t_1}\hat{\mathbf{G}}(\frac{\nu}{\delta}(1-e^{-\delta s}))ds\} \quad (3.4.13)$$

Similarly (3.4.12) follows.

□

(3.4.11) & (3.4.12) give us the asymptotic behaviour of x_t (and therefore of N_t) if we let t_2-t_1 become large.

$$-\log E\{\exp(-\nu x_t)\} \sim \rho[1-\hat{g}(\nu/\delta)]t - (\rho\nu/\delta)\{\int_0^{\infty}\{\hat{\mathbf{G}}(\nu/\delta) - \hat{\mathbf{G}}(\frac{\nu}{\delta}(1-e^{-\delta s}))\}ds\} \quad (3.4.14)$$

we get the asymptotic behaviour of all cummulants of x_t. This implies the following very simple result. If μ_r the r^{th} moment of G exists and $k_r(t)$ denotes the r^{th} cummulant of x_t,

$$\lim_{t\to\infty} k_r(t) = \frac{\rho\mu_r}{\delta^r} \quad (3.4.15)$$

The analogue of (3.4.14) for N_t is

$$-\log E\{\theta^{N_t}\} \sim \rho[1-\hat{g}((1-\theta)/\delta)]t - (\rho/\delta)(1-\theta)\{\int_0^{\infty}\{\hat{\mathbf{G}}((1-\theta)/\delta) - \hat{\mathbf{G}}(\frac{1-\theta}{\delta}(1-e^{-\delta s}))\}ds\} \quad (3.4.16)$$

Using the important corollary 3.4.2 we can also look at higher order properties of x_t. We are going to find the joint Laplace transform of the distribution of $x_{t_2}-x_{t_1}$, $x_{t_3}-x_{t_2}$,.....,$x_{t_{n+1}}-x_{t_n}$ (n succesive increments).

Lemma 3.4.4 *Let* x_t,λ_t *as defined and* $\nu_1,\nu_2,.....,\nu_k$ *and* γ_{k+1} *be non negative constants; then*

$$E\{\prod_1^k \exp[-\nu_i(x_{t_{i+1}}-x_{t_i})]\exp(-\gamma_{k+1}\lambda_{t_{k+1}})\,|\,\lambda_{t_1}\} =$$

$$E\{\prod_1^{k-1}\exp[-\nu_i(x_{t_{i+1}}-x_{t_i})]\exp(-\gamma_k\lambda_{t_k})\,|\,\lambda_{t_1}\}\exp\{-\rho\int_0^{t_{k+1}-t_k}[1-\hat{g}(\frac{\nu_k}{\delta}+(\gamma_{k+1}-\frac{\nu_k}{\delta})e^{-\delta s})]ds\} \quad (3.4.17)$$

where

$$\gamma_k = \frac{\nu_k}{\delta} + (\gamma_{k+1}-\frac{\nu_k}{\delta})\exp[-\delta(x_{t_{k+1}}-x_{t_k})] \quad (3.4.18)$$

Proof

$$E\{\prod_1^k \exp[-\nu_i(x_{t_{i+1}}-x_{t_i})]\exp(-\gamma_{k+1}\lambda_{t_{k+1}})\,|\,\lambda_{t_1}\} =$$

$$E\{\prod_{1}^{k-1}\exp[-\nu_i(x_{t_{i+1}}-x_{t_i})]\,E\{\exp[-\nu_k(x_{t_{k+1}}-x_{t_k})]\exp(-\gamma_{k+1}\lambda_{t_{k+1}})\,|\,x_{t_1},x_{t_2},\ldots,x_{t_k},\lambda_{t_1},\ldots,\lambda_{t_k}\}\,|\,\lambda_{t_1}\} \tag{3.4.19}$$

but

$$E\{\exp[-\nu_k(x_{t_{k+1}}-x_{t_k})]\exp(-\gamma_{k+1}\lambda_{t_{k+1}})\,|\,x_{t_1},x_{t_2},\ldots,x_{t_k},\lambda_{t_1},\ldots,\lambda_{t_k}\} =$$

$$E\{\exp[-\nu_k(x_{t_{k+1}}-x_{t_k})]\exp(-\gamma_{k+1}\lambda_{t_{k+1}})\,|\,\lambda_{t_k}\} \tag{3.4.20}$$

and by theorem 3.4.2

$$E\{\exp[-\nu_k(x_{t_{k+1}}-x_{t_k})]\exp(-\gamma_{k+1}\lambda_{t_{k+1}})\,|\,\lambda_{t_k}\} =$$

$$\exp\left\{-\lambda_{t_k}\left\{\frac{\nu_k}{\delta}+(\gamma_{k+1}-\frac{\nu_k}{\delta})\exp[-\delta(x_{t_{k+1}}-x_{t_k})]\right\}\right\}\exp\{-\rho\int_0^{t_{k+1}-t_k}[1-\hat{g}(\frac{\nu_k}{\delta}+(\gamma_{k+1}-\frac{\nu_k}{\delta})e^{-\delta s})]ds\} \tag{3.4.21}$$

Define

$$\gamma_k = \frac{\nu_k}{\delta}+(\gamma_{k+1}-\frac{\nu_k}{\delta})\exp[-\delta(x_{t_{k+1}}-x_{t_k})]$$

and the lemma is proved.

□

Theorem 3.4.5 *Let* x_t,λ_t *be as defined. Let also* ν_k k=1,2,.....,n *be non negative constants. Then*

$$E\{\prod_{1}^{n}\exp[-\nu_i(x_{t_{i+1}}-x_{t_i})]\,|\,\lambda_{t_1}\} =$$

$$\exp\left\{-\lambda_{t_1}\sum_{1}^{n}\frac{\nu_i}{\delta}[\exp(-\delta t_i)-\exp(-\delta t_{i+1})]\exp(\delta t_1)\right\}\prod_{1}^{n}\exp\left\{\rho\int_0^{t_{k+1}-t_k}[1-\hat{g}(\frac{\nu_k}{\delta}+(\xi_{k+1}-\frac{\nu_k}{\delta})e^{-\delta s})]ds\right. \tag{3.4.22}$$

where $\xi_k = \exp(\delta t_k)\sum_{k}^{n}\frac{\nu_i}{\delta}\exp[-\delta(t_i-t_{i+1})]$ $\xi_{n+1}=0$ (k=1,2,...,n)

Proof

By succesively applying lemma 3.4.4 we get

$$E\{\prod_{1}^{n}\exp[-\nu_i(x_{t_{i+1}}-x_{t_i})]\,|\,\lambda_{t_1}\} = \exp(-\gamma_1\lambda_{t_1})\prod_{1}^{n}\exp\left\{\rho\int_0^{t_{k+1}-t_k}[1-\hat{g}(\frac{\nu_k}{\delta}+(\xi_{k+1}-\frac{\nu_k}{\delta})e^{-\delta s})]ds\right. \tag{3.4.23}$$

with

$$\gamma_{n+1}=0 \quad \text{and} \quad \gamma_k = \frac{\nu_k}{\delta}+(\gamma_{k+1}-\frac{\nu_k}{\delta})\exp[-\delta(x_{t_{k+1}}-x_{t_k})] \tag{3.4.24}$$

(3.4.24) can be rewritten as

$$\gamma_k \exp(-\delta t_k) = \frac{\nu_k}{\delta}[\exp(-\delta t_k)-\exp(-\delta t_{k+1})] + \gamma_{k+1}\exp(-\delta t_{k+1})$$

therefore

$$\gamma_k \exp(-\delta t_k) = \gamma_1 \exp(-\delta t_1) - \sum_1^{k-1} \frac{\nu_i}{\delta}[\exp(-\delta t_i)-\exp(-\delta t_{i+1})] \tag{3.4.25}$$

so

$$\gamma_{n+1} \exp(-\delta t_{n+1}) = \gamma_1 \exp(-\delta t_1) - \sum_1^{n-1} \frac{\nu_i}{\delta}[\exp(-\delta t_i)-\exp(-\delta t_{i+1})] \tag{3.4.26}$$

and since $\gamma_{n+1}=0$,

$$\gamma_1 = \exp(\delta t_1) \sum_1^{n-1} \frac{\nu_i}{\delta}[\exp(-\delta t_i)-\exp(-\delta t_{i+1})] \tag{3.4.27}$$

and

$$\gamma_k = \exp(\delta t_k) \sum_k^{n} \frac{\nu_i}{\delta}[\exp(-\delta t_i)-\exp(-\delta t_{i+1})] \tag{3.4.28}$$

Substituting (3.4.27) & (3.4.28) into (3.4.23) we get (3.4.22)

□

The problem of finding the distribution of $\tau_1 \equiv \inf\{t: N_t=1 \mid N_t=0\}$ can be easily solved since we have found the distribution of N_t. By corollary 3.4.3 and (3.4.10)

$$E\{\theta^{N_t} \mid \lambda_o, N_o=0\} = \exp\{-\frac{1-\theta}{\delta}[1-\exp(-\delta t)]\lambda_o\} \exp\{-\rho\int_o^t [1-\hat{g}(\frac{1-\theta}{\delta}(1-e^{-\delta s}))]ds\} \tag{3.4.29}$$

So setting $\theta=0$ we get the tail of the distribution of τ_1.

$$\Pr(\tau_1>t \mid \lambda_o, N_o=0) = \Pr(N_t=0 \mid \lambda_o, N_o=0) = \exp\{-(\lambda_o/\delta)(1-e^{-\delta t})\}\exp\{-\rho\int_o^t [1-\hat{g}(\frac{1-e^{-\delta s}}{\delta})]ds\} \tag{3.4.30}$$

However, if we want to find the distribution of the point interarrival time intervals of N_t when λ_t is stationary **we can not substitute the stationary distribution of λ_t for λ_o.** We have to assume that λ_o is a time at which a point of N_t has occured, and clearly if we denote by $\lambda_{(n)}$ the value of λ_t when N_t takes the value n for the first time when the process is stationary, the distribution of $\lambda_{(n)}$ <u>is not the same</u> as the distribution of λ_t and we have to find it.

<u>Theorem 3.4.6</u> *Let T_n be the time of the n^{th} point of N_t. Assume μ_1 is the first moment of G and that it exists. When the process λ_{T_i} is stationary*

$$E\{\exp(-\nu\lambda_{T_i}) = [\hat{\mathbf{G}}(\nu)/\mu_1]\exp\{-(\rho/\delta)\int_o^\nu \hat{\mathbf{G}}(u)du\} \tag{3.4.31}$$

Proof

We will use lemma 3.1.2 which implies that if $f(\lambda)$ and $h(\lambda)$ are such that

$$\lambda[h(\lambda)-f(\lambda)] - \delta\lambda f'(\lambda) + \rho[\int_{\mathbb{R}^+} f(\lambda+y)dG(y)-f(\lambda)] = 0 \tag{3.4.32}$$

and (3.1.9) is satisfied then by starting the process from T_i

$$E\{h(\lambda_{T_{i+1}})|\lambda_{T_i}\} = f(\lambda_{T_i}) \tag{3.4.33}$$

We will use

$$f(\lambda) = \left\{\lambda - \frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}\right\}e^{-\nu\lambda} \tag{3.4.34}$$

$f(\lambda)$ clearly satisfies (3.1.9) and substituting into (3.4.32)

$$\lambda[h(\lambda)-\lambda e^{-\nu\lambda}+\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}e^{-\nu\lambda}] + \delta\nu\lambda\left\{\lambda-\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}\right\}e^{-\nu\lambda} - \delta\lambda e^{-\nu\lambda} =$$

$$-\rho\left\{\left\{\lambda-\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}\right\}e^{-\nu\lambda}\hat{g}(\nu) - e^{-\nu\lambda}\hat{g}'(\nu) - \left\{\lambda-\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}\right\}e^{-\nu\lambda}\right\} \tag{3.4.35}$$

Notice that terms with $e^{-\nu\lambda}\hat{g}'(\nu)$ in the right hand side cancel each other and dividing by λ and simplifying

$$h(\lambda) = e^{-\nu\lambda}\left\{\lambda(1-\delta\nu) - (1-\delta\nu)\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)} + \delta + \rho[1-\hat{g}(\nu)]\right\} \tag{3.4.36}$$

and by (3.4.33)

$$E\{\lambda_{T_{i+1}}\exp(-\nu\lambda_{T_{i+1}})(1-\delta\nu) - (1-\delta\nu)\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}\exp(-\nu\lambda_{T_{i+1}}) + [\delta+\rho[1-\hat{g}(\nu)]]\exp(-\nu\lambda_{T_{i+1}})\} =$$

$$E\{\lambda_{T_i}\exp(-\nu\lambda_{T_i})\} - \frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}E\{\exp(-\nu\lambda_{T_i})\} \tag{3.4.37}$$

Assuming stationarity $\lambda_{T_{i+1}}$ and λ_{T_i} have the same distribution whose Laplace transform we denote by $H(\nu)$; by (3.4.37) we then have

$$-(1-\delta\nu)H'(\nu) - (1-\delta\nu)\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}H(\nu) + [\delta+\rho[1-\hat{g}(\nu)]]H(\nu) = -H'(\nu) - \frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}H(\nu) \tag{3.4.38}$$

and grouping similar terms together

$$H'(\nu) + \frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}H(\nu) + [\frac{1}{\nu}+(\rho/\delta)\frac{1-\hat{g}(\nu)}{\nu}]H(\nu) = 0 \tag{3.4.39}$$

So $H(\nu)$ is given by solving (3.4.39) subject to

$$H(0) = 1 \tag{3.4.40}$$

(it has to be the Laplace transform of a distribution); and we get

$$H(\nu) = A_1 \frac{1-\hat{g}(\nu)}{\nu} \exp\{-(\rho/\delta)\int_0^{\nu} \hat{G}(u)du\} \tag{3.4.41}$$

where A_1 is a constant and because of (3.4.40) $A_1 = 1/\mu_1$ and

$$H(\nu) = [\hat{G}(\nu)/\mu_1]\exp\{-(\rho/\delta)\int_0^{\nu} \hat{G}(u)du\} \tag{3.4.42}$$

□

We should now notice the interesting fact that this is the distribution of the sum of two independent random variables one having the the stationary distribution of λ_t as its distribution and the other having density $G(y)/\mu_1$.

We can now substitute in (3.4.30) and find the tail of the point interarrival time intervals when stationarity has been achieved and this is

$$\Pr(\tau>t) = [\hat{G}(\frac{1}{\delta} - \frac{1}{\delta}e^{-\delta t})/\mu_1]\exp\{-\rho\int_0^t \frac{1-\hat{g}(\frac{1}{\delta} - \frac{1}{\delta}e^{-\delta s})}{1-e^{-\delta s}} ds\} \tag{3.4.43}$$

Another interesting fact is that if we integrate (3.4.43) we get $E\{\tau\}=\delta/(\rho\mu_1)$ which means the expected point interarrival time is the inverse of the average intensity of the process.

The model so far described can be generalized so as to allow its parameters be time dependent; that is Riemann integrable functions of time as in section 2.4. Exactly the same method can be carried over to this model producing the generalizations of the results of this section (with the exception of the last theorem). Details are omitted as there will be no new ideas and formulas obtained are rather complicated.

3.5 SHOT NOISE INTENSITY WITH GENERALLY DISTRIBUTED JUMP INTERARRIVAL TIME INTERVALS

We can generalize the shot noise process of the previous section by assuming a general renewal process as its jump arrival process in the lines of section 2.5. We assume $\delta\lambda_t$ is the rate of decay of λ_t, jump sizes being distributed with distribution function G(y) (y>0). We also assume the jump interarrival time intervals are independent identically distributed random variables with density p(u) and that they are independent of the jump sizes as well, also that there exists a positive constant γ such that

$$\int_0^\infty e^{\beta y}p(y)dy < \infty \qquad \text{for } \beta \in [0,\gamma)$$

We define $x_t = \int_0^t \lambda_s ds$ and N_t as a doubly stochastic point process having λ_t as its intensity. We also include u_t the time elapsed since the last jump in our piecewise deterministic process. The generator of the process (x_t,λ_t,u_t,t) acting on a function $f(x,\lambda,u,t)$ belonging to its domain is given by

$$Af(x,\lambda,u,t) = \lambda\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial u} - \delta\lambda\frac{\partial f}{\partial \lambda} + \frac{p(u)}{\mathbf{P}(u)}[\int_{\mathbb{R}^+} f(x,\lambda+y,0,t)dG(y) - f(x,\lambda,u,t)] \qquad (3.5.1)$$

We will not be able to find explicit results for the distribution of x_t at a fixed time t. However we can investigate the asymptotic behaviour of x_t (and therefore deduce the asymptotic behaviour of N_t as well).

We start with a definition.

Definition 3.5.1 *Suppose* $\nu \geq 0$. *We define the function* $\beta:[0,\infty)\to[0,\infty)$ *as* $\beta(\nu)=\beta$ *where*

$$\hat{p}(-\beta)\hat{g}(\nu/\delta) = 1 \qquad (3.5.2)$$

$$\left(\ \hat{p}(\nu) = \int_0^\infty e^{-\nu y}p(y)dy \quad \textit{and} \quad \hat{g}(\nu) = \int_{\mathbb{R}^+} e^{-\nu y}dG(y)\ \right)$$

Theorem 3.5.2 *Let* x_t,λ_t,u_t *as defined; then for all* $\nu \geq 0$

$$\exp[-\nu(x_t+\frac{\lambda_t}{\delta})]\exp[\beta(\nu)t]\int_{u_t}^\infty \exp[\beta(\nu)v]p(v)dv\ /\{\exp[\beta(\nu)u_t]\mathbf{P}(u_t) \qquad (3.5.3)$$

is a martingale.

Proof

For $f(x_t,\lambda_t,u_t,t)$ to be a martingale we should have $Af(x,\lambda,u,t) = 0$. Substituting $\exp[-\nu[x+(\lambda/\delta)]]e^{\beta t}h(u)$ into $Af=0$ we get

$$\beta h(u) + h'(u) + \frac{p(u)}{\mathbf{P}(u)}[\hat{g}(\nu/\delta)h(0)-h(u)] \tag{3.5.4}$$

Its solution is

$$h(u) = \frac{h(0)\int_u^\infty e^{\beta v}p(v)dv\,\hat{g}(\nu/\delta)}{e^{\beta u}\mathbf{P}(u)} + \frac{h(0)}{e^{\beta u}\mathbf{P}(u)}[1-\hat{p}(-\beta)\hat{g}(\nu/\delta)] \tag{3.5.5}$$

If we choose $\beta=\beta(\nu)$ then $\hat{p}(-\beta)\hat{g}(\nu/\delta)=1$ and

$$h(u) = \frac{h(0)\int_u^\infty e^{\beta v}p(v)dv\,\hat{g}(\nu/\delta)}{e^{\beta u}\mathbf{P}(u)} \tag{3.5.6}$$

which is bounded and belongs to the domain of the generator.

(It is not the case that other solutions will not lead to martingales. As a matter of fact by choosing a different β we see that other solutions belong to the domain of the generator and they do lead to martingales. However, we chose the particular h(u) because it is bounded and it will be the most suitable for our aims).

Since h(u) satisfies (3.5.4) $\exp[-\nu(x+(\lambda/\delta)]\exp[\beta(\nu)t]h(u)$ satisfies $Af=0$ and the theorem is proved.

□

Theorem 3.5.3 *Let* x_t,λ_t,u_t *as defined; then for all* $\nu\geq 0$

$$E\{\exp[-\nu(x_t+\frac{\lambda_t}{\delta})]\} = \frac{\exp(-\nu\lambda_o/\delta)\exp[-\beta(\nu)t]\int_{u_o}^\infty \exp[\beta(\nu)(v-u_o)]p(v)dv/\mathbf{P}(u_o)}{E\{\int_{u_t}^\infty \exp[\beta(\nu)(v-u_t)]p(v)dv/\mathbf{P}(u_o)\,|\,u_o\}} \tag{3.5.7}$$

Proof

Because of the structure of the process we can see that

$$\lambda_t\exp(\delta u_t) \quad \text{and} \quad x_t - \lambda_t\frac{\exp(\delta u_t)-1}{\delta}$$

are independent of u_t (they are the positions of x_t and λ_t just after the last

jump in λ_t; see also section 2.5).

But $x_t - \lambda_t\frac{\exp(\delta u_t)-1}{\delta} + \lambda_t\frac{\exp(\delta u_t)}{\delta} = x_t + \frac{\lambda_t}{\delta}$ so $x_t + \frac{\lambda_t}{\delta}$ is independent of u_t as well.

Using the martingale of theorem 3.5.2

$$E\{ \exp[-\nu(x_t+\frac{\lambda_t}{\delta})]\exp[\beta(\nu)t]\int_{u_t}^{\infty}\exp[\beta(\nu)v]p(v)dv /\{\exp[\beta(\nu)u_t]\mathbf{P}(u_t)\,|\,\lambda_o,u_o\} =$$

$$\frac{\exp(-\nu\lambda_o/\delta)\int_{u_o}^{\infty} e^{\beta(\nu)v}p(v)dv}{\exp(\beta u_o)\mathbf{P}(u_o)} \qquad (3.5.8)$$

So

$$E\{ \exp[-\nu(x_t+\frac{\lambda_t}{\delta})]\,|\,\lambda_o,u_o\}e^{\beta(\nu)t}E\{\int_{u_t}^{\infty}\exp[\beta(\nu)v]p(v)dv /\{\exp[\beta(\nu)u_t]\mathbf{P}(u_t)\,|\,u_o\} =$$

$$\frac{\exp(-\nu\lambda_o/\delta)\int_{u_o}^{\infty} e^{\beta(\nu)v}p(v)dv}{\exp(\beta u_o)\mathbf{P}(u_o)} \qquad (3.5.9)$$

and (3.5.7) follows. □

So using theorem 3.6.3 we can deduce the asymptotic behaviour of the distribution of x_t. Clearly as $t\to\infty$ we have

$$-\log E\{\exp[-\nu(x_t+\frac{\lambda_t}{\delta})]\} \sim \beta(\nu)t \qquad (3.5.10)$$

and since as $t\to\infty$ the cummulants of λ_t are dominated by the cummulants of x_t we conclude that

$$\lim_{t\to\infty}\frac{k_r(t)}{t} = (-1)^{r-1}\beta^{(r)}(0) \qquad (3.5.11)$$

(by $k_r(t)$ we denote the r^{th} cummulant of x_t and by $\beta^{(r)}(\nu)$ the r^{th} derivative of $\beta(\nu)$, should they exist). We similarly get for N_t

$$-\log E\{\theta^{N_t}\exp[(1-\theta)\lambda_t/\delta]\} \sim \beta(1-\theta)t \qquad (3.5.12)$$

3.6 INSURANCE APPLICATIONS

In this and the next sections we will consider possible applications of doubly stochastic point processes with a shot noise process as intensity.

In this section we will use such a point process as our claim arrival process in a surplus process in risk theory. We consider the surplus process ζ_t with constant income rate c, claim sizes independent and identically distributed with distribution function H(y) (y>0). H(y) is such that there exists $\alpha_1>0$ so that

$$\int_{\mathbb{R}^+} e^{\nu y} dH(y) < \infty \quad \text{for } \nu \in [0,\alpha_1)$$

The claim arrival rate is λ_t where λ_t is **a shot noise process with a renewal process as its jump arrival process.** We assume λ_t decays with rate $\delta\lambda_t$; jump interarrival time intervals have density p(u) with no further assumptions imposed on it this time in contrast to the previous section. Jump sizes have distribution function G(y) which is such that there exists $\alpha_2>0$ so that

$$\int_{\mathbb{R}^+} e^{\nu y} dG(y) < \infty \quad \text{for } \nu \in [0,\alpha_2)$$

We also include u_t the time elapsed since the last jump in λ_t. The generator of the piecewise deterministic process $(\zeta_t,\lambda_t,u_t,t)$ acting on a function $f(\zeta,\lambda,u,t)$ belonging to its domain is given by

$$Af(\zeta,\lambda,u,t) = c\frac{\partial f}{\partial \zeta} + \frac{\partial f}{\partial t} + \lambda\left[\int_{\mathbb{R}^+} f(\zeta-y,\lambda,u,t)dH(y) - f(\zeta,\lambda,u,t)\right] +$$

$$\frac{\partial f}{\partial u} - \delta\lambda\frac{\partial f}{\partial \lambda} + \frac{p(u)}{\mathbf{P}(u)}\left[\int_{\mathbb{R}^+} f(\zeta,\lambda+y,u,t)dG(y) - f(\zeta,\lambda,u,t)\right] \qquad (3.6.1)$$

We will try to find a martingale as in section 1.1. We will first have to prove a relevant lemma.

Lemma 3.6.1 *Consider the equation*

$$\hat{p}(\theta+c\nu)\hat{g}\left(-\frac{\phi(\nu)-1}{\delta}\right) = 1 \qquad (3.6.2)$$

for fixed $\theta\geq 0$, *where*

$$\hat{p}(\theta) = \int_0^\infty e^{-\theta y} p(y)dy,\ \hat{g}(\nu) = \int_{\mathbb{R}^+} e^{-\nu y} dG(y) \text{ and } \phi(\nu) = \int_{\mathbb{R}^+} e^{\nu y} dH(y) \qquad (3.6.3)$$

then the following are true:

i) For all $\theta>0$ *there exists a unique positive* ν_θ *such that (3.6.2) is satisfied for* $\nu=\nu_\theta$.

ii) Given that

$$c > \eta_1\mu_1/\rho_1\delta \tag{3.6.4}$$

where

$$\eta_1 = \int_{\mathbb{R}^+} y dH(y) \ , \ \mu_1 = \int_{\mathbb{R}^+} y dG(y), \quad \rho_1 = \int_0^\infty yp(y)dy \tag{3.6.5}$$

then for $\theta=0$ *there exists a unique positive* ν_0, *so that (3.6.2) is satisfied for* $\nu=\nu_0$.

Proof

$\hat{p}(\theta+c\nu)\hat{g}(-\frac{\phi(\nu)-1}{\delta})$ is a convex function of ν for all $\theta\geq 0$. Indeed looking at its second derivative we see that it is

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\{[-cu+\frac{\phi'(\nu)}{\delta}y]^2+\frac{\phi''(\nu)}{\delta}\}e^{-(\theta+c\nu)u}\exp(\frac{\phi(\nu)-1}{\delta}y)p(u)dudG(y) \geq 0 \tag{3.6.6}$$

Also at $\nu=0$ $\hat{p}(\theta+c\nu)\hat{g}(-\frac{\phi(\nu)-1}{\delta}) = \hat{p}(\theta) < 1$ if $\theta>0$ so there exists a unique ν_θ which is positive and satisfies (3.6.2).

For $\theta=0$ we need the first derivative to be negative at $\nu=0$ for ν_0 to exist (uniqueness is guaranteed by convexity. Indeed the derivative at $\nu=0$ is $-c\rho_1+(\eta_1\mu_1/\delta)$ and this is negative by (3.6.4).

□

Theorem 3.6.2 *Let* ζ_t,λ_t,u_t *as defined. Suppose also* $c>\eta_1\mu_1/\rho_1\delta$ *(net profit condition; it is (3.6.4)). Let* $\theta\geq 0$ *and* ν_θ *be such that (3.6.2) is satisfied when* $\nu=\nu_\theta$. *Then*

$$\exp(-\nu_\theta\zeta_t)e^{-\theta t}\exp(\lambda_t\frac{\phi(\nu)-1}{\delta})\int_{u_t}^\infty \exp[-(\theta+c\nu_\theta)v]p(v)dv/\exp[-(\theta+c\nu_\theta)u_t]\mathbf{P}(u_t) \tag{3.6.7}$$

is a martingale.

Proof

In order to find a martingale we should solve $Af=0$. If we try

$$e^{-\nu\zeta}e^{-\theta t}\exp(\lambda\frac{\phi(\nu)-1}{\delta})h(u)$$

we get

$$h'(u) - (\theta+c\nu)h(u) + \frac{p(u)}{\mathbf{P}(u)}[\hat{g}(-\frac{\phi(\nu)-1}{\delta})h(0)-h(u)] = 0 \tag{3.6.8}$$

Solving (3.6.8)

$$h(u) = h(0)\frac{\int_u^\infty e^{-(\theta+c\nu)v}p(v)dv}{e^{-(\theta+c\nu)u}\mathbf{P}(u)}\hat{g}(-\frac{\phi(\nu)-1}{\delta}) + h(0)\frac{1 - \hat{p}(\theta+c\nu)\hat{g}(-\frac{\phi(\nu)-1}{\delta})}{e^{-(\theta+c\nu)u}\mathbf{P}(u)} \tag{3.6.9}$$

but for our function to belong to the domain of the generator we should make the second term which has infinite expectation to vanish so as to be left with the first which is bounded. For that purpose we should have

$$\hat{p}(\theta+c\nu)\hat{g}(-\frac{\phi(\nu)-1}{\delta}) = 1 \tag{3.6.10}$$

and therefore $\nu = \nu_\theta$ (3.6.11).

So, then

$$h(u) = h(0)\frac{\int_u^\infty e^{-(\theta+c\nu)v}p(v)dv}{e^{-(\theta+c\nu)u}\mathbf{P}(u)}\hat{g}(-\frac{\phi(\nu)-1}{\delta}) \tag{3.6.12}$$

and the theorem is proved.

□

We can now prove important inequalities about ruin probabilities with the help of this martingale.

Theorem 3.6.3 *Let* ζ_t,λ_t,u_t *as defined and* $c > \eta_1\mu_1/\rho_1\delta$. *Define* $\tau\equiv\inf\{t:\zeta_t<0\}$ *(the time of ruin); then*

$$\Pr(\tau\leq t \mid \lambda_o,\zeta_o,u_o) \leq \inf_{\theta>0}\left\{\frac{e^{\theta t}}{\hat{p}(\theta+c\nu_\theta)}\exp(-\nu_\theta\zeta_o)\exp(\lambda_o\frac{\phi(\nu_\theta)-1}{\delta})\frac{\int_{u_o}^\infty \exp[-(\theta+c\nu_\theta)v]p(v)dv}{\exp[-(\theta+c\nu_\theta)u_o]\mathbf{P}(u_o)}\right\} \tag{3.6.13}$$

$$\Pr(\tau<\infty \mid \lambda_o,\zeta_o,u_o) \leq \frac{1}{\hat{p}(c\nu_o)}\exp(-\nu_o\zeta_o)\exp(\lambda_o\frac{\phi(\nu_o)-1}{\delta})\frac{\int_{u_o}^\infty \exp(-c\nu_o v)p(v)dv}{\exp(-c\nu_o u_o)\mathbf{P}(u_o)} \tag{3.6.14}$$

Proof

(3.6.7) is a martingale and $\tau\wedge t$ a stopping time; so by Doob's optional stopping theorem we have

$$E\{\exp(-\nu_\theta \zeta_{\tau\wedge t})e^{-\theta(\tau\wedge t)}\exp(\frac{\phi(\nu_\theta)-1}{\delta}\lambda_{\tau\wedge t})\frac{\int_{u_{\tau\wedge t}}^{\infty}\exp[-(\theta+c\nu_\theta)v]p(v)dv}{\exp[-(\theta+c\nu_\theta)u_{\tau\wedge t}]\mathbf{P}(u_{\tau\wedge t})}|\zeta_o,\lambda_o,u_o\} =$$

$$\exp(-\nu_\theta\zeta_o)\exp(\frac{\phi(\nu_\theta)-1}{\delta}\lambda_o)\frac{\int_{u_o}^{\infty}\exp[-(\theta+c\nu_\theta)v]p(v)dv}{\exp[-(\theta+c\nu_\theta)u_o]\mathbf{P}(u_o)} \qquad (3.6.15)$$

so

$$E\{e^{-\theta\tau}\exp(-\nu_\theta\zeta_\tau)\exp(\frac{\phi(\nu_\theta)-1}{\delta}\lambda_\tau)\frac{\int_{u_\tau}^{\infty}\exp[-(\theta+c\nu_\theta)v]p(v)dv}{\exp[-(\theta+c\nu_\theta)u_\tau]\mathbf{P}(u_\tau)}|\zeta_o,\lambda_o,u_o,\tau\leq t\}\Pr(\tau\leq t) +$$

$$E\{e^{-\theta t}\exp(-\nu_\theta\zeta_t)\exp(\frac{\phi(\nu_\theta)-1}{\delta}\lambda_t)\frac{\int_{u_t}^{\infty}\exp[-(\theta+c\nu_\theta)v]p(v)dv}{\exp[-(\theta+c\nu_\theta)u_t]\mathbf{P}(u_t)}|\zeta_o,\lambda_o,u_o,\tau>t\}\Pr(\tau>t)$$

$$\exp(-\nu_\theta\zeta_o)\exp(\frac{\phi(\nu_\theta)-1}{\delta}\lambda_o)\frac{\int_{u_o}^{\infty}\exp[-(\theta+c\nu_\theta)v]p(v)dv}{\exp[-(\theta+c\nu_\theta)u_o]\mathbf{P}(u_o)} \qquad (3.6.16)$$

So we have

$$\exp(-\nu_\theta\zeta_o)\exp(\frac{\phi(\nu_\theta)-1}{\delta}\lambda_o)\frac{\int_{u_o}^{\infty}\exp[-(\theta+c\nu_\theta)v]p(v)dv}{\exp[-(\theta+c\nu_\theta)u_o]\mathbf{P}(u_o)} \geq$$

$$E\{e^{-\theta\tau}\exp(-\nu_\theta\zeta_\tau)\exp(\frac{\phi(\nu_\theta)-1}{\delta}\lambda_\tau)\frac{\int_{u_\tau}^{\infty}\exp[-(\theta+c\nu_\theta)v]p(v)dv}{\exp[-(\theta+c\nu_\theta)u_\tau]\mathbf{P}(u_\tau)}|\zeta_o,\lambda_o,u_o,\tau\leq t\}\Pr(\tau\leq t) \qquad (3.6.17)$$

Therefore

$$\exp(-\nu_\theta\zeta_o)\exp(\frac{\phi(\nu_\theta)-1}{\delta}\lambda_o)\frac{\int_{u_o}^{\infty}\exp[-(\theta+c\nu_\theta)v]p(v)dv}{\exp[-(\theta+c\nu_\theta)u_o]\mathbf{P}(u_o)} \geq$$

$$e^{-\theta t}E\{\exp(-\nu_\theta\zeta_\tau)\exp(\frac{\phi(\nu_\theta)-1}{\delta}\lambda_\tau)\frac{\int_{u_\tau}^{\infty}\exp[-(\theta+c\nu_\theta)v]p(v)dv}{\exp[-(\theta+c\nu_\theta)u_\tau]\mathbf{P}(u_\tau)}|\zeta_o,\lambda_o,u_o,\tau\leq t\}\Pr(\tau\leq t) \qquad (3.6.18)$$

but $\exp(-\nu_\theta\zeta_\tau)\geq 1$ a.s. , $\exp(\frac{\phi(\nu_\theta)-1}{\delta}\lambda_\tau)\geq 1$ a.s. and $\frac{\int_{u_\tau}^{\infty}\exp[-(\theta+c\nu_\theta)v]p(v)dv}{\exp[-(\theta+c\nu_\theta)u_\tau]\mathbf{P}(u_\tau)} \geq \hat{p}(\theta+c\nu_\theta)$ a.s.

(the last inequality follows from the monotonicity of this function). (So (3.6.18)

gives us

$$\Pr(\tau \leq t) \leq \frac{e^{\theta t}}{\hat{p}(\theta+c\nu_\theta)} \exp(-\nu_\theta \zeta_o) \exp\left(\lambda_o \frac{\phi(\nu_\theta)-1}{\delta}\right) \frac{\int_{u_o}^{\infty} \exp[-(\theta+c\nu_\theta)v] p(v)dv}{\exp[-(\theta+c\nu_\theta)u_o] \mathbf{P}(u_o)} \tag{3.6.19}$$

for all $\theta \geq 0$, so (3.6.13) follows.

Also setting $\theta=0$ in (3.6.19)

$$\Pr(\tau \leq t) \leq \frac{1}{\hat{p}(c\nu_o)} \exp(-\nu_o \zeta_o) \exp\left(\lambda_o \frac{\phi(\nu_o)-1}{\delta}\right) \frac{\int_{u_o}^{\infty} \exp(-c\nu_o v) p(v)dv}{\exp(-c\nu_o u_o) \mathbf{P}(u_o)} \tag{3.6.20}$$

(3.6.20) is true for all t so (3.6.14) follows.

□

It should be appropriate for us to repeate the comment we made in section 1.1 that the inequalities of theorem 3.6.3 are rather sharp since the second term in the left hand side of (3.6.16) tends to 0 with exponential rate (remember that $\zeta_t \to \infty$ a.s. as $t \to \infty$; see section 1.1)

3.7 "MULTIPLE" SHOT NOISE PROCESSES

We can investigate other possible applications of doubly stochastic point processes by trying to incorporate them in more elaborate and sophisticated models. For example using the shot noise process to model the level of water in dams and rivers is in practice quite inadequate. Rains do not occur according to a Poisson process and more complicated models might be needed. This is exactly what doubly stochastic point processes can provide us with. Leaving the seasonal element apart, it is much more realistic to assume there is some form of

clustering. We will assume the rain arrival rate (intensity) is a shot noise process itself. So we consider the following process. w_t the quantity of interest (the level of water, say) decays with rate γw_t (exponentially) and additive jumps occur whose sizes are independent with common distribution function H(y) (y>0). The intensity of jump arrivals is λ_t which is itself a shot noise process with rate of decay $\delta\lambda_t$, constant rate of jump arrivals ρ and jump sizes independent identically distributed random variables with distribution function G(y) (y>0). We call the process (w_t,λ_t) a "double" shot noise process. The generator of the process (w_t,λ_t,t) acting on a function $f(w,\lambda,t)$ belonging to its domain is given by

$$Af(w,\lambda,t) = \frac{\partial f}{\partial t} - \gamma w \frac{\partial f}{\partial w} + \lambda[\int_{\mathbb{R}^+} f(w+y,\lambda,t)dH(y)-f(w,\lambda,t)] -$$

$$-\delta\lambda\frac{\partial f}{\partial \lambda} + \rho[\int_{\mathbb{R}^+} f(w,\lambda+y,t)dG(y)-f(w,\lambda,t)] \qquad (3.7.1)$$

We will find the joint Laplace transform of the distribution of w_t,λ_t at a fixed time t_1 given its value at t=0. We define $\hat{g}(\nu),\hat{h}(\nu)$ as the Laplace transforms of the distributions with distribution functions G,H.

Theorem 3.7.1 *Let w_t,λ_t as defined and evolving up to time t^*. Let κ_1 and κ_2 be constants such that*

$$\kappa_1 \geq 0 \quad \textit{and} \quad \kappa_2 \geq \int_0^{t^*} e^{-\delta s}[1-\hat{h}(\kappa_1 e^{\gamma s})]ds \qquad (3.7.2)$$

then

$$\exp(-\kappa_1 e^{\gamma t}w_t)\exp\{\lambda_t e^{\delta t}[\kappa_2-\int_0^t e^{-\delta s}[1-\hat{h}(\kappa_1 e^{\gamma s})]ds]\}\cdot$$

$$\exp\{\rho\int_0^t [1-\hat{g}\left(\kappa_2 e^{\delta s}-e^{\delta s}\int_0^s e^{-\delta u}[1-\hat{h}(\kappa_1 e^{\gamma u})du\right]]ds\} \qquad (3.7.3)$$

is a martingale.

Proof

We will try to find a martingale of the form $\exp(-a_1(t)w_t)\exp(-a_2(t)\lambda_t)e^{m(t)}$ where $a_1(t)\geq 0$, $a_2(t)\geq 0$ for $t\geq t^*$ so the corresponding function belongs to the domain of the generator of the process (assuming it evolves up to time t^* only). Substituting into Af=0 we get

$$-wa_1'(t) - \lambda a_2'(t) + m'(t) + \gamma w a_1(t) + \lambda[\hat{h}(a_1(t))-1] + \delta\lambda a_2(t) + \rho[\hat{g}(a_2(t))-1] = 0 \tag{3.7.4}$$

Equating coefficients in (3.7.4)

$$a_1'(t) = \gamma a_1(t) \tag{3.7.5}$$

$$-a_2'(t) + \delta a_2(t) = 1 - \hat{h}(a_1(t)) \tag{3.7.6}$$

and

$$m'(t) = \rho[1 - \hat{g}(a_2(t))] \tag{3.7.7}$$

Solving (3.7.5), (3.7.6) & (3.7.7)

$$a_1(t) = \kappa_1 e^{\gamma t} \tag{3.7.8}$$

$$a_2(t) = \kappa_2 e^{\delta t} - e^{\delta t}\int_0^t e^{-\delta s}[1-\hat{h}(\kappa_1 e^{\gamma s})]ds] \tag{3.7.9}$$

$$m(t) = \kappa_3 + \rho\int_0^t [1-\hat{g}\left(\kappa_2 e^{\delta s} - e^{\delta s}\int_0^s e^{-\delta u}[1-\hat{h}(\kappa_1 e^{\gamma u})du\right)]ds \tag{3.7.10}$$

We choose κ_1,κ_2 satisfying (3.7.2) and $\kappa_3=0$ and it follows that (3.7.3) is indeed a martingale.

□

Theorem 3.7.2 *Let* w_t,λ_t *as defined and let* t_1 *be a fixed time, then*

$$E\{\exp(-\nu_1 w_{t_1})\exp(-\nu_2\lambda_{t_1})\,|\,w_0,\lambda_0\} = \exp(-\nu_1 e^{-\gamma t_1} w_0)\cdot$$
$$\exp\{-\lambda_0[\nu_2 e^{-\delta t_1} + \int_0^{t_1} e^{-\delta(t_1-s)}[1-\hat{h}(\nu_1 e^{-\gamma s})]ds]\}\cdot$$
$$\exp\{-\rho\int_0^{t_1}[1-\hat{g}\left(\nu_2 e^{-\delta s} + e^{-\delta s}\int_0^s e^{\delta v}[1-\hat{h}(\nu_1 e^{-\gamma v})]dv\right)]ds\} \tag{3.7.11}$$

and therefore

$$\lim_{t\to\infty} E\{\exp(-\nu_1 w_{t_1})\exp(-\nu_2\lambda_{t_1})\} = \exp\{-\rho\int_0^{\infty}[1-\hat{g}\left(\nu_2 e^{-\delta s} + e^{-\delta s}\int_0^s e^{\delta v}[1-\hat{h}(\nu_1 e^{-\gamma v})]dv\right)]ds\} \tag{3.7.12}$$

Proof

We choose $t^*\geq t_1$ and let $a_1(t) = \nu_1$ and $a_2(t) = \nu_2$. This means that we set

$$\kappa_1 = \nu_1 e^{-\gamma t_1} \tag{3.7.13}$$

and

$$\nu_2 = \kappa_2 e^{\delta t_1} - e^{\delta t_1}\int_0^{t_1} e^{-\delta s}[1-\hat{h}(\nu_1 e^{-\gamma(t_1-s)})]ds = \kappa_2 e^{\delta t_1} - \int_0^{t_1} e^{\delta s}[1-\hat{h}(\nu_1 e^{-\gamma s})]ds \quad (3.7.14)$$

and therefore

$$\kappa_2 = \nu_2 e^{-\delta t_1} + \int_0^{t_1} e^{-\delta(t_1-s)}[1-\hat{h}(\nu_1 e^{-\gamma s})]ds \quad (3.7.15)$$

Also

$$m(t_1) = \rho\int_0^{t_1}\{1-\hat{g}\left[\kappa_2 e^{\delta s} - e^{\delta s}\int_0^s e^{-\delta u}[1-\hat{h}(\kappa_1 e^{\gamma u})]du\right]\}ds =$$

$$\rho\int_0^{t_1}\{1-\hat{g}\left[\nu_2 e^{-\delta(t_1-s)} + e^{\delta s}\int_0^{t_1} e^{-\delta(t_1-u)}[1-\hat{h}(\nu_1 e^{-\gamma u})]du - e^{-\delta s}\int_0^s e^{-\delta u}[1-\hat{h}(\nu_1 e^{-\gamma(t_1-u)})]du\right]\}ds =$$

$$\rho\int_0^{t_1}\{1-\hat{g}\left[\nu_2 e^{-\delta(t_1-s)} + e^{\delta s}\int_s^{t_1} e^{-\delta u}[1-\hat{h}(\nu_1 e^{-\gamma(t_1-u)})]du\right]\}ds =$$

$$\rho\int_0^{t_1}[1-\hat{g}\left[\nu_2 e^{-\delta s} + e^{-\delta s}\int_0^s e^{\delta v}[1-\hat{h}(\nu_1 e^{-\gamma v})]dv\right]]ds\} \quad (3.7.16)$$

Using the martingale of theorem 3.7.1 we get (3.7.11). (3.7.12) is obtained by letting $t\to\infty$ in (3.7.11).

□

The "double" shot noise process concept can be generalized as follows. We consider the concept of a "multiple" shot noise process. It consists of n component processes $\lambda_{1,t},\lambda_{2,t},.....,\lambda_{n,t}$ and for i=1,2,.....,n-1 $\lambda_{i,t}$ decays with rate $\delta_i(t)\lambda_{i,t}$ (we allow time dependence on all parameters as in section 2.4) and additive jumps occur with rate $\lambda_{i+1,t}$ (ie each process acts as a jump intensity for the previous one). Jumps sizes are independent identically distributed random variables with distribution functions $G_i(y;t)$ (y>0). $\lambda_{n,t}$ decreases with rate $\delta_n(t)\lambda_{n,t}$ but its jump arrival rate is deterministic $\rho(t)$. Its jump sizes have distribution function $G_n(y;t)$. $\delta_i(t)$, $G_i(y;t)$, $\rho(t)$ are all Riemann integrable functions of t and for all t^*, $\rho(t)$ is bounded on $[0,t^*]$ (no explosions). We will also consider the process

$$x_t = \int_0^t \lambda_{1,s}ds$$

which can be useful either by itself or in evaluating aspects about a point process N_t which has $\lambda_{1,t}$ as its intensity. The generator of the process $(x_t,\lambda_{1,t},\lambda_{2,t},.....,\lambda_{n,t},t)$ acting on a function $f(x,\lambda_1,\lambda_2,.....,\lambda_n,t)$ belonging to its

domain is given by

$$A^*f(x,\lambda_1,\lambda_2,\ldots,\lambda_n,t) =$$

$$\lambda_1\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} - \sum_1^n \delta_i(t)\lambda_i\frac{\partial f}{\partial \lambda_i} + \sum_1^{n-1}\lambda_{i+1}[\int_{\mathbb{R}^+} f(x,\lambda_1,\lambda_2,\ldots,\lambda_i+y,\ldots,\lambda_n,t)dG_i(y;t) - f(x,\lambda_1,\ldots,\lambda_n t)] +$$

$$\rho(t)[\int_{\mathbb{R}^+} f(x,\lambda_1,\lambda_2,\ldots,\lambda_n+y,t)dG_n(y;t) - f(x,\lambda_1,\lambda_2,\ldots,\lambda_n,t)] \qquad (3.7.17)$$

We can now find a martingale.

Theorem 3.7.3 *Let* $x_t,\lambda_{1,t},\lambda_{2,t},\ldots,\lambda_{n,t}$ *as defined and evolving up to time* t^*. *Let also* ξ *be a positive constant. We define the following functions*

$$a_1(t) = \kappa_1\exp[\int_o^t \delta_1(u)du - \xi\int_o^t \exp[\int_s^t \delta_1(u)du]ds \qquad (3.7.18)$$

$$a_i(t) = \kappa_i\exp[\int_o^t \delta_i(u)du - \xi\int_o^t \exp[\int_s^t \delta_i(u)du][1-\hat{g}_{i-1}(a_{i-1}(s);s)]ds \qquad i=2,3,\ldots,n \qquad (3.7.19)$$

$$m(t) = \int_o^t \rho(s)[1-\hat{g}_n(a_n(s);s)]ds \qquad (3.7.20)$$

where $\kappa_1,\kappa_2,\ldots,\kappa_n$ *are constants such that* $a_i(t)\geq 0$ *on* $[0,t^*]$ *for all* i=1,2,..,n. *Then*

$$\exp(-\xi x_t)\prod_1^n \exp(-a_i(t)\lambda_{i,t})e^{m(t)} \qquad (3.7.21)$$

is a martingale.

Proof

In order to find a martingale we should solve $A^*f=0$. Substituting

$$e^{-\xi x}\prod_1^n \exp(-a_i(t)\lambda_i)e^{m(t)}$$

into $A^*f=0$ we get

$$-\lambda_1\xi - \sum_1^n \lambda_i a_i'(t) + m'(t) + \sum_1^n \delta_i(t)\lambda_i a_i(t) + \sum_1^{n-1}\lambda_{i+1}[\hat{g}_i(a_i(t);t)-1] + \rho[\hat{g}_n(a_n(t);t)-1] = 0 \qquad (3.7.22)$$

but

$$\sum_1^{n-1}\lambda_{i+1}[\hat{g}_i(a_i(t);t)-1] = \sum_2^n \lambda_i[\hat{g}_{i-1}(a_{i-1}(t);t)-1] \qquad (3.7.23)$$

so equating coefficients in (3.7.22) we get

$$-\xi - a_1'(t) + \delta_1(t)a_1(t) = 0 \qquad (3.7.24)$$

$$-a_i'(t) + \delta_i(t)a_i(t) + [\hat{g}_{i-1}(a_{i-1}(t);t)-1] \qquad i=2,3,\ldots,n \qquad (3.7.25)$$

$$m'(t) = \rho(t)[1-\hat{g}_n(a_n(t);t)] \qquad (3.7.26)$$

Solving (3.7.24), (3.7.25) & (3.7.26) we get (3.7.18), (3.7.19) & (3.7.20) and assuming $a_i(t)\geq 0$ for $t\in[0,t^*]$, our function belongs to the domain of the generator of the process that evolves up to t^* and satisfies $A^*f=0$ as well so the theorem has been proved.

□

We can now find the joint Laplace transform of the distribution of $x_t,\lambda_{1,t},\lambda_{2,t},.....,\lambda_{n,t}$ at a fixed time t_1.

Theorem 3.7.4 *Let* $x_t,\lambda_{1,t},\lambda_{2,t},.....,\lambda_{n,t}$ *as defined and* $\xi,\nu_1,\nu_2,.....,\nu_n$ *be non negative constants. Define a set of functions* $a_i^*(t)$ and constants $\kappa_1^*,\kappa_2^*,.....,\kappa_n^*$ *such that*

$$a_i^*(t) = a_i(t) \quad \textit{and} \quad \kappa_i^* = \kappa_i \qquad (3.7.27)$$

(a_i's and κ_i's of theorem 3.7.3) with

$$a_i^*(t_1) = \nu_i \qquad (3.7.28)$$

Then

$$E\{\exp(-\xi x_{t_1})\prod_1^n\exp(-\nu_i\lambda_{i,t_1}) \mid x_0,\lambda_{1,0},\lambda_{2,0},.....,\lambda_{n,0}\} =$$

$$\prod_1^n\exp(-\kappa_i^*\lambda_{i,0})\exp[-\int_0^{t_1}\rho(s)[1-\hat{g}_n(a_n^*(s);s)]ds \qquad (3.7.29)$$

Proof

It is a direct application of the martingale of theorem 3.7.3 We just set $\kappa_i=\kappa_i^*$ and (3.7.29) follows immediately.

□

Finally it should be noted that if one should like to find results about higher order properties of $x_t,\lambda_{1,t},\lambda_{2,t},.....,\lambda_{n,t}$ he could by adopting the procedure of section 3.4.

CHAPTER 4

In this chapter we will look at various optimization problems arising in insurance mathematics. The piecewise deterministic processes approach is very powerful when tackling optimization problems although not only then, as it should have been evident by now. We will restrict ourselves to simple applications and modifications of some well known insurance models. We will avoid the use of dynamic programming as it is very much outside the scope of this thesis. We will illustrate how a simple mathematical approach via piecewise deterministic processes theory can solve fairly general optimization problems. There is not much work that has been done on optimization problems in insurance mathematics and it is limited to very simple (and thus not particularly realistic) models. So there is much scope for the use of piecewise deterministic processes in this field as problems otherwise difficult to treat mathematically can be put within this framework. For non insurance applications one should look at Davis et.al.(1987).

4.1 SIMPLE DIVIDEND PROBLEMS

We will now consider a simple dividend payout scheme. We assume a classical surplus process ζ_t with the possibility for the company to pay out dividends to its shareholders at a rate $r(\zeta_t)$. Our objective function will be

$$X = \int_0^{\tau} e^{-\beta s} r(\zeta_s) ds \tag{4.1.1}$$

where $\tau \equiv \inf\{t:\zeta_t < 0\}$ (4.1.2)

(the time of ruin). Notice that X is a random variable. β is a discount factor and $r(\zeta)$ is a non negative Borel measurable function of ζ. X is **the total**

discounted dividend the company pays out. ζ_t increases deterministically with constant rate c. Claims arrive with Poisson rate λ and have independent sizes with common distribution function G(y). Most of the work done on the model is to find the strategy $r(\cdot)$ that maximizes E{X} (see Buhlman (1970)p.169, Gerber (1980) p.146, Dassios & Embrechts (1987)). The optimum strategy is then of the form $r(\zeta) = cI(\zeta=w)$ (I is the indicator function) assuming $\zeta_0<w$. These are called barrier strategies. The company does not pay out anything while its surplus is below w while when it reaches w it pays out its whole income as dividends (see Buhlman (1970) p.168-171 and related references given therein to see why they are optimal). We will restrict ourselves to such strategies not only because of their optimality properties in this specific problem but because they are easily realisable in practice too. We will not merely evaluate an expression for E{X} but for other moments of X as well. Notice that when a barrier strategy is adopted then ζ_t can not exceed w and ruin is certain, however late it occurs.

We will consider the following piecewise deterministic process. It consists of n_t, ζ_t, x_t and time t. n_t can take the values 0,1,2. 0 indicates that ruin has occured; 1 that the surplus is between 0 and w; and 2 that the surplus is at w and all income is being paid out as dividends. While $n_t=2$, x_t the total discounted dividends paid up to time t increases deterministically with rate $ce^{-\beta t}$. One should note that $X=x_T$. $n_t=0$ (ruin) occurs when ζ_t turns negative and it is an absorbing state; the process ceases to evolve thereafter. The generator of the process (n_t,ζ_t,x_t,t) acting on a function $f(n,\zeta,x,t)$ belonging to its domain is given by

$$Af=\begin{cases}\dfrac{\partial f}{\partial t}(1,\zeta,x,t)+c\dfrac{\partial f}{\partial \zeta}(1,\zeta,x,t)+\lambda\left[\int_{(0,\zeta]} f(1,\zeta-y,x,t)dG(y)+\int_{(\zeta,\infty)} f(0,\zeta-y,x,t)dG(y)-f(1.\zeta,x,t)\right] & n=1\\ \dfrac{\partial f}{\partial t}(2,w,x,t)+ce^{-\beta t}\dfrac{\partial f}{\partial x}(2,w,x,t)+\lambda\left[\int_{(0,w]} f(1,w-y,x,t)dG(y)+\int_{(w,\infty)} f(0,w-y,x,t)dG(y)-f(2,w,x,t)\right] & n=2\end{cases} \tag{4.1.3}$$

with the continuity condition

$$\lim_{\zeta\uparrow w} f(1,\zeta,x,t) = f(2,w,x,t) \tag{4.1.4}$$

So we can now set out to find expressions about $E\{x_T^m\}$ (=E{X}). As $n_t=0$ is

an absorbing state the generator then has no terms and we can set any function we like as f(0,ζ,x,t). We will use functions of the form V(x). So because of (4.1.4) we can suppress dependence on n.

Lemma 4.1.1 *Let* f(ζ,x,t) be such that

$$\frac{\partial f}{\partial t} + c\frac{\partial f}{\partial \zeta} + \lambda[\int_{(0,\zeta]} f(\zeta-y,x,t)dG(y) + x^m \mathbf{G}(\zeta) - f(\zeta,x,t)] = 0 \qquad \mathbf{G}(\zeta)=1-G(\zeta) \tag{4.1.5}$$

and

$$\frac{\partial f}{\partial \zeta}\Big|_{\zeta=w} = e^{-\beta t}\frac{\partial f}{\partial x}\Big|_{\zeta=w} \tag{4.1.6}$$

then

$$E\{x_\tau^m \mid \zeta_o\} = f(\zeta_o,0,0) \tag{4.1.7}$$

Proof

We set $f(0,\zeta,x,t) = x^m$ and we can immediately see that $f(1,\zeta,x,t)=f(\zeta,x,t)$ and $f(2,w,x,t)=f(w,x,t)$ satisfy Af=0 by looking at (4.1.3) and (4.1.4), so then

$$\begin{cases} f(x_t,\zeta_t,t) & n_t \neq 0 \\ x_t^m & n_t = 0 \end{cases} \tag{4.1.8}$$

is a martingale and since τ is a stopping time (remember $Pr(\tau<\infty)=1$) by Doob's optional stopping theorem we have

$$E\{x_\tau^m \mid \zeta_o\} = f(\zeta_o,x_o,0) \tag{4.1.9}$$

but $x_o = 0$ so (4.1.7) follows.

□

So we can now try to find $E\{x_\tau^m\}$.

Theorem 4.1.2 *Consider functions* $h_i(\zeta)$ *such that*

$$\hat{h}_i(\xi) = \int_o^\infty e^{-\xi\zeta} h_i(\zeta)d\zeta = \frac{1}{\xi - (\beta i/c) - (\lambda/c)[1-\hat{g}(\xi)]} \tag{4.1.10}$$

($\hat{g}(\xi)$ *is the Laplace transform of the distribution with distribution function* G(y)*) and constants* $K_i(w)$ *(dependent on* w*) such that*

$$K_i(w)h_i'(w) = iK_{i-1}(w)h_i(w) \tag{4.1.11}$$

$$K_1(w)h_1'(w) = 1 \tag{4.1.12}$$

then

$$E\{X^m|\zeta_o\} = E\{x_\tau^m|\zeta_o\} = K_m(w)h_m(\zeta_o) \tag{4.1.13}$$

Proof

We will consider a function of the form

$$f(\zeta,x,t) = \sum_o^m \binom{m}{i} x^{m-i}e^{-\beta it}f_i(\zeta) \tag{4.1.14}$$

Substituting into (4.1.5) we get

$$-\sum_o^m \binom{m}{i} x^{m-i}\beta i e^{-\beta it}f_i(\zeta)+c\sum_o^m \binom{m}{i} x^{m-i}e^{-\beta it}f_i'(\zeta)+$$

$$\lambda\sum_o^m \binom{m}{i} x^{m-i}e^{-\beta it}[\int_{(0,\zeta]} f_i(\zeta-y)dG(y)-f_i(\zeta)] + \lambda x^m \mathbf{G}(\zeta) = 0 \tag{4.1.15}$$

which is satisfied by $f_i(\zeta)$ such that

$$f_o(\zeta) \equiv 1 \qquad \text{(say)} \tag{4.1.16}$$

and

$$-\beta i f_i(\zeta) + cf_i'(\zeta) + \lambda[\int_{(0,\zeta]} f_i(\zeta-y)dG(y)-f_i(\zeta)] = 0 \tag{4.1.17}$$

Taking Laplace transforms in (4.1.17)

$$-\beta i\hat{f}_i(\xi) + c\xi\hat{f}_i(\xi) - cf_i(0) + \lambda\hat{f}_i(\xi)\hat{g}(\xi) - \lambda\hat{f}_i(\xi) = 0 \tag{4.1.18}$$

yielding

$$\hat{f}_i(\xi) = \frac{f_i(0)}{\xi - (\beta i/c) - (\lambda/c)[1-\hat{g}(\xi)]} \tag{4.1.19}$$

(4.1.6) gives

$$\begin{aligned}\sum_o^m \binom{m}{i} x^{m-i}e^{-\beta it}f_i'(w) &= e^{-\beta t}\sum_o^m \binom{m}{i}(m-i)x^{m-i-1}e^{-\beta it}f_i(w)\\ &= \sum_o^m \binom{m}{i-1}(m-i+1)x^{m-i}f_{i-1}(w)\\ &= \sum_o^m \binom{m}{i} x^{m-i}e^{-\beta it}if_{i-1}(w)\end{aligned} \tag{4.1.20}$$

(4.1.20) gives

$$f_i'(w) = if_{i-1}(w) \tag{4.1.21}$$

(4.1.16), (4.1.19) & (4.1.21) imply

$$f_i(\zeta) = K_i(w)h_i(\zeta) \tag{4.1.22}$$

and therefore $x^m + \sum_o^m \binom{m}{i} x^{m-i}e^{-\beta it}K_i(w)h_i(\zeta)$ satisfies (4.1.4) and by lemma 4.1.1

$$E\{X^m | \zeta_o\} = E\{x_\tau^m | \zeta_o\} = K_m(w)h_m(\zeta_o) \tag{4.1.23}$$

□

Let us now demonstrate how to use theorem 4.1.2. We will first examine the asymptotic behaviour of $h_i(\zeta)$.

Lemma 4.1.3 *Let* $h_i(\zeta)$ *be such that*

$$\hat{h}_i(\xi) = \int_o^\infty e^{-\xi\zeta} h_i(\zeta)d\zeta = \frac{1}{\xi - (\beta i/c) - (\lambda/c)[1-\hat{g}(\xi)]} \tag{4.1.24}$$

Suppose also that there exists $\gamma>0$ *such that*

$$\hat{g}(-\xi) = \int_{\mathbb{R}^+} e^{\xi y} dG(y) < \infty \quad \textit{for } \xi<\gamma.$$

Then for $\beta>0$ *there are two and only two real roots of the equation*

$$\xi - (\beta i/c) - (\lambda/c)[1-\hat{g}(\xi)] = 0 \tag{4.1.25}$$

ν_i *and* $-\rho_i$ *which are such that* ν_i *and* ρ_i *are positive. Furthermore the asymptotic behaviour of* $h_i(\zeta)$ *is given by*

$$h_i(\zeta) \sim \exp(\nu_i\zeta)\left[\frac{1}{1+(\lambda/c)\hat{g}'(\nu_i)} + \frac{1}{1+(\lambda/c)\hat{g}'(-\rho_i)}\exp[-(\nu_i+\rho_i)\zeta]\right] \tag{4.1.26}$$

Also when $G(y)=1-\alpha e^{-\alpha y}$ *and therefore* $\hat{g}(\xi)=\alpha/(\alpha+\xi)$ *(exponential claims), we in fact have*

$$h_i(\zeta) = \frac{\alpha+\nu_i}{\nu_i+\rho_i}\exp(\nu_i\zeta) - \frac{\alpha-\rho_i}{\nu_i+\rho_i}\exp(-\rho_i\zeta) \tag{4.1.27}$$

Proof

i)

When $\xi=0$ $\xi-(\beta i/c)-(\lambda/c)[1-\hat{g}(\xi)] = -\beta i/c < 0$ and $\xi-(\beta i/c)-(\lambda/c)[1-\hat{g}(\xi)]$ is convex as its second derivative is $(\lambda/c)\hat{g}''(\xi)>0$, so there are two and only two real roots of (4.1.25) one positive and one negative and we denote them by ν_i and $-\rho_i$.

ii)

Since ν_i is a root of (4.1.25),if we write $h_i(\zeta)=\exp(\nu_i\zeta)k_i(\zeta)$ then

$$\hat{k}_i(\xi) = \hat{h}_i(\xi+\nu_i) = \frac{1}{\xi+\nu_i-(\beta i/c)-(\lambda/c)[1-\hat{g}(\xi+\nu_i)]} = \frac{1}{\xi-(\lambda/c)[\hat{g}(\nu_i)-\hat{g}(\xi+\nu_i)]} \tag{4.1.28}$$

which is the Laplace transform of a bounded function. Also,

$$\lim_{\zeta\to\infty} k_i(\zeta) = \lim_{\xi\downarrow 0} \xi\hat{k}_i(\xi) = \lim_{\xi\downarrow 0} \frac{\xi}{\xi-(\lambda/c)[\hat{g}(\nu_i)-\hat{g}(\xi+\nu_i)]} = \frac{1}{1+(\lambda/c)\hat{g}'(\nu_i)} \tag{4.1.29}$$

We further define

$$a_i(\zeta) = k_i(\zeta) - \frac{1}{1+(\lambda/c)\hat{g}'(\nu_i)} \tag{4.1.30}$$

then

$$\hat{a}_i(\xi) = \frac{1}{\xi-(\lambda/c)[\hat{g}(\nu_i)-\hat{g}(\xi+\nu_i)]} - \frac{1}{1+(\lambda/c)\hat{g}'(\nu_i)}\cdot\frac{1}{\xi} \tag{4.1.31}$$

The only non zero real root of the denominator of the first term is $-(\nu_i+\rho_i)$, so letting $a_i(\zeta) = \exp[-(\nu_i+\rho_i)\zeta]b_i(\zeta)$

$$\hat{b}_i(\xi) = \frac{1}{\xi-\rho_i-(\beta i/c)-(\lambda/c)[1-\hat{g}(\xi-\rho_i)]} - \frac{1}{1+(\lambda/c)\hat{g}'(\nu_i)}\cdot\frac{1}{\xi-\nu_i-\rho_i} \tag{4.1.32}$$

and as with k_i

$$\lim_{\zeta\to\infty} b_i(\zeta) = \lim_{\xi\downarrow 0} \xi\hat{b}_i(\xi) = \frac{1}{1+(\lambda/c)\hat{g}'(-\rho_i)} \tag{4.1.33}$$

so

$$h_i(\zeta) = \exp(\nu_i\zeta)\left[\frac{1}{1+(\lambda/c)\hat{g}'(\nu_i)} + b_i(\zeta)\exp[-(\nu_i+\rho_i)\zeta]\right] \tag{4.1.34}$$

with

$$\lim_{\zeta\to\infty} b_i(\zeta) = \frac{1}{1+(\lambda/c)\hat{g}'(-\rho_i)} \tag{4.1.35}$$

iii)

When $\hat{g}(\xi) = \alpha/(\alpha+\xi)$

$$\hat{h}_i(\xi) = \frac{1}{\xi-(\beta i/c)-(\lambda/c)[\xi/(\alpha+\xi)]} = \frac{\alpha+\xi}{\xi^2+[\alpha-(\beta i+\lambda)/c]\xi-\alpha\beta i/c} = \frac{\alpha+\nu_i}{\nu_i+\rho_i}\cdot\frac{1}{\xi-\nu_i} - \frac{\alpha-\rho_i}{\nu_i+\rho_i}\cdot\frac{1}{\xi+\rho_i} \tag{4.1.36}$$

and inverting the Laplace transform

$$h_i(\zeta) = \frac{\alpha+\nu_i}{\nu_i+\rho_i}\exp(\nu_i\zeta) - \frac{\alpha-\rho_i}{\nu_i+\rho_i}\exp(-\rho_i\zeta) \tag{4.1.37}$$

□

It is easy to check that (4.1.27) is (4.1.26) with equality for exponential claims.

Now if we just want to maximize $E\{X|\zeta_0\}$ as in Buhlman (1970) (p.169) and Gerber (1980) (p.169) then $E\{X|\zeta_0\} = K_1(w)h_1(\zeta_0)$ ($K_1(w)=1/h_1'(w)$) and we just have to maximize $K_1(w)$ with respect to w since $E\{X|\zeta_0\}$ depends on w via $K_1(w)$ only. This is equivalent to minimizing $h_1'(w)$ with respect to w; hence we will have to solve the equation $h_1''(w)=0$ and assuming β is small then w* satisfying $h_1''(w)=0$ will be large (see Buhlman (1970) p.174) and by lemma 4.1.3

$$h_1(\zeta) \sim \frac{1}{1+(\lambda/c)\hat{g}'(\nu_1)}\exp(\nu_1\zeta) + \frac{1}{1+(\lambda/c)\hat{g}'(-\rho_1)}\exp(-\rho_1\zeta)$$

and therefore

$$h_1''(\zeta) \sim \frac{\nu_1^2}{1+(\lambda/c)\hat{g}'(\nu_1)}\exp(\nu_1\zeta) + \frac{\rho_1^2}{1+(\lambda/c)\hat{g}'(-\rho_1)}\exp(-\rho_1\zeta) \qquad (4.1.38)$$

and so an asymptotic expression for w* would be

$$\frac{\nu_1^2}{1+(\lambda/c)\hat{g}'(\nu_1)}\exp(\nu_1 w^*) - \frac{\rho_1^2}{1+(\lambda/c)\hat{g}'(-\rho_1)}\exp(-\rho_1 w^*) \sim 0$$

and so

$$w^* \sim \frac{1}{\nu_1+\rho_1}\log\left[-\frac{\rho_1^2[\,1+(\lambda/c)\hat{g}'(\nu_1)]}{\nu_1^2[\,1+(\lambda/c)\hat{g}'(-\rho_1)]}\right] \qquad (4.1.39)$$

In particular for exponential claims we need not worry if β is small or not since we then have an exact result.

$$h_1(\zeta) = \frac{\alpha+\nu_1}{\nu_1+\rho_1}\nu_1^2\exp(\nu_1\zeta) - \frac{\alpha-\rho_1}{\nu_1+\rho_1}\rho_1^2\exp(-\rho_1\zeta) \qquad (4.1.40)$$

and

$$w^* = \frac{1}{\nu_1+\rho_1}\log\left[\frac{(\alpha-\rho_1)\rho_1^2}{(\alpha+\nu_1)\nu_1^2}\right] \qquad (4.1.41)$$

(4.1.39) produces a good approximation if it yields a large enough w* (compared to μ_1 the first moment of G, say). (4.1.41) works if it produces a positive w*. Otherwise the best policy is to payout everything immediately.

Alternatively we may look at expectations of other functions of X since we can effectively evaluate or approximate expectations of functions of X in terms of its moments. For example another objective can be to maximize

$$E\{X|\zeta_0\} + (\theta/2)E\{X^2|\zeta_0\}$$

where θ is a small constant. When $\theta>0$ this is a "risk seeking" criterion while if $\theta<0$ a "risk averse". $E\{X|\zeta_0\}+(\theta/2)E\{X^2|\zeta_0\}$ depends on w via $K_1(w)$ and $(\theta/2)K_2(w)$ only so we can try to maximize $K_1(w)+(\theta^*/2)K_2(w)$ (θ^* will depend on ζ_0 but will again be small). When we have exponentially distributed claims this is to maximize

$$\frac{1}{\frac{\alpha+\nu_1}{\nu_1+\rho_1}\nu_1 \exp(\nu_1 w) + \frac{\alpha-\rho_1}{\nu_1+\rho_1}\rho_1 \exp(-\rho_1 w)} \cdot \left[1 - (\theta^*/2)\frac{\frac{\alpha+\nu_1}{\nu_1+\rho_1}\exp(\nu_1 w) - \frac{\alpha-\rho_1}{\nu_1+\rho_1}\exp(-\rho_1 w)}{\frac{\alpha+\nu_2}{\nu_2+\rho_2}\nu_2 \exp(\nu_2 w) + \frac{\alpha-\rho_2}{\nu_2+\rho_2}\rho_2 \exp(-\rho_2 w)}\right] \tag{4.1.42}$$

or in the case of generally distributed claims

$$\frac{1}{\frac{\nu_1}{1+(\lambda/c)\hat{g}'(\nu_1)}\exp(\nu_1 w) - \frac{\rho_1}{1+(\lambda/c)\hat{g}'(-\rho_1)}\exp(-\rho_1 w)} \cdot \left[1 - (\theta^*/2)\frac{\frac{1}{1+(\lambda/c)\hat{g}'(\nu_1)}\exp(\nu_1 w) + \frac{1}{1+(\lambda/c)\hat{g}'(-\rho_1)}\exp(-\rho_1 w)}{\frac{\nu_2}{1+(\lambda/c)\hat{g}'(\nu_2)}\exp(\nu_2 w) - \frac{\rho_2}{1+(\lambda/c)\hat{g}'(-\rho_2)}\exp(-\rho_2 w)}\right] \tag{4.1.43}$$

We should point out that we do not claim that barrier strategies are optimal in this case. We have just showed how to find the optimal barrier strategy. Barrier strategies are important because they are easily realisable even when they are not optimal.

Finally, it should be noted that we can generalize our model by allowing the

option of borrowing when ζ_t falls below 0 as in section 1.4. The mathematics are roughly the same as we can see in Dassios & Embrechts (1987).

4.2 AN ALTERNATIVE MODEL

We will now consider an alternative to the model proposed in the last section. Instead of discounting the dividends paid out by the company, we assume that the cash funds available to the company are continuously undervalued (due to inflation, operating costs and other factors) with a rate $\delta\zeta_t$ so that its net real income rate is $c-\delta\zeta_t$. ζ_t now denotes the company's surplus in **real** terms. Claims again arrive in terms of a Poisson process with parameter λ and their sizes have distribution function G(y) (y>0). Notice that the company's **real** surplus can not exceed c/δ and ruin is certain. We will again assume a barrier strategy. When the company's real surplus reaches a fixed level w, the company pays out dividends with a rate $c-\delta w$ (its whole income) and we can now define x_t as the total (not discounted) dividends paid out up to time t, or more formally

$$x_t = \int_0^t (c-\delta w) I(\zeta_s=w) ds$$

Our objective function is x_τ where $\tau \equiv \inf\{t:\zeta_t<0\}$ is the time of ruin.

We define the following piecewise deterministic process. It consists of n_t, ζ_t, x_t and time t. n_t can as in the previous section take the values 0 (ruin), 1 ($0 \leq \zeta_t < w$ and no dividends paid out) and 2 ($\zeta_t=w$; dividends being paid out). 0 is an absorbing state as in the previous section and the generator of the process

(n_t,ζ_t,x_t,t) acting on a function $f(n,\zeta,x,t)$ belonging to its domain is given by

$$Af(n,\zeta,w,t) =$$

$$\begin{cases} \frac{\partial f}{\partial t}(1,\zeta,x,t)+(c-\delta\zeta)\frac{\partial f}{\partial \zeta}(1,\zeta,x,t)+\lambda[\int_{(0,\zeta]} f(1,\zeta-y,x,t)dG(y)+\int_{(\zeta,\infty)} f(0,\zeta-y,x,t)dG(y)-f(1.\zeta,x,t)] \\ \frac{\partial f}{\partial t}(2,w,x,t)+(c-\delta w)\frac{\partial f}{\partial \zeta}(2,w,x,t)+\lambda[\int_{(0,w]} f(1,w-y,x,t)dG(y)+\int_{(w,\infty)} f(0,w-y,x,t)dG(y)-f(2,w,x,t)] \end{cases} \tag{4.2.1}$$

with the continuity condition

$$\lim_{\zeta \uparrow w} f(1,\zeta,x,t) = f(2,x,w,t) \tag{4.2.2}$$

We will now try to find $E\{\exp(-\theta x_\tau)|\zeta_o\}$. We will prove a lemma similar to (but not exactly the analogue to) lemma 4.1.1.

Lemma 4.2.1 *Let* $f(\zeta)$ *be such that*

$$(c-\delta\zeta)f'(\zeta) + \lambda[\int_{(0,\zeta]} f(\zeta-y)dG(y)+\mathbf{G}(\zeta)-f(\zeta)] = 0 \tag{4.2.3}$$

and

$$f'(w) = -\theta f(w) \tag{4.2.4}$$

Then

$$E\{\exp(-\theta x_\tau)|\zeta_o\} = f(\zeta_o) \tag{4.2.5}$$

Proof

Set $f(0,\zeta,x,t) = e^{-\theta x}$, $f(1,\zeta,x,t) = f(\zeta)e^{-\theta x}$, $f(2,w,x,t) = f(w)e^{-\theta x}$ and substitute into the equation $Af=0$; by the first leg of the generator we get

$$(c-\delta\zeta)f'(\zeta) + \lambda[\int_{(0,\zeta]} f(\zeta-y)dG(y)+\mathbf{G}(\zeta)-f(\zeta)] = 0 \tag{4.2.6}$$

Furthermore by the second leg of the generator and (4.2.2) we get $f'(w)=-\theta f(w)$, so then

$$\begin{cases} \exp(-\theta x_t)f(\zeta_t) & n_t \neq 0 \\ \exp(-\theta x_t) & n_t = 0 \end{cases} \tag{4.2.7}$$

is a martingale and since τ is a stopping time we use Doob's optional stopping theorem and get

$$E\{\exp(-\theta x_\tau)|\zeta_o\} = f(\zeta_o) \tag{4.2.8}$$

□

This leads to the following theorem.

Theorem 4.2.2 *Suppose* $h(\zeta)$ *is a function of* ζ *(*$\zeta\in[0,w]$*) belonging to the domain of the generator* A *and*

$$(c-\delta\zeta)h'(\zeta) + \lambda[\int_{(0,\zeta]} h(\zeta-y)dG(y)-h(\zeta)] = 0 \tag{4.2.9}$$

then

$$E\{\exp(-\theta x_T)\,|\,\zeta_0\} = 1 - \frac{\theta h(\zeta_0)}{h'(w)+\theta h(w)} \tag{4.2.10}$$

which means that $\Pr(x_T=0\,|\,\zeta_0\} = 1 - h(\zeta_0)/h(w)$ *and conditionally on* $x_T>0$ *the distribution of* x_T *is exponential with parameter* $h'(w)/h(w)$ *(mean* $h(w)/h'(w)$*).*

Proof

Set

$$f(\zeta) = 1 - \frac{\theta h(\zeta)}{h'(w)+\theta h(w)} \tag{4.2.11}$$

then by substituting into (4.2.6) we see that (4.2.6) and (4.2.9) are equivalent; therefore by (4.2.5)

$$E\{\exp(-\theta x_T)\,|\,\zeta_0\} = f(\zeta_0) = 1 - \frac{\theta h(\zeta_0)}{h'(w)+\theta h(w)} = [1-\frac{h(\zeta_0)}{h(w)}] + \frac{h(\zeta_0)}{h(w)}\cdot\frac{h'(w)/h(w)}{[h'(w)/h(w)]+\theta} \tag{4.2.12}$$

□

The simplest possible example is when the claims are exponentially distributed; that is $dG(y)=\alpha e^{-\alpha y}dy$. We can then find $E\{\exp(-\theta x_T)\,|\,\zeta_0\}$ explicitly. (4.2.9) can be written as

$$(c-\delta\zeta)h'(\zeta) + \lambda[\int_0^{\zeta} h(\zeta-y)\alpha e^{-\alpha y}dy-h(\zeta)] = 0 \tag{4.2.13}$$

Applying the operator $D+\alpha I$ on (4.2.13)

$$(c-\delta\zeta)h''(\zeta) - \delta h'(\zeta) + \alpha(c-\delta\zeta)h'(\zeta) - \lambda h'(\zeta) = 0 \tag{4.2.14}$$

whose solution is

$$h'(\zeta) = Ke^{-\alpha\zeta}[1-\frac{\delta\zeta}{c}]^{-(\lambda/\delta)-1} \tag{4.2.15}$$

where K is an arbitrary constant; setting $\zeta=0$ in (4.2.13) $h(0)=Kc/\lambda$ so

$$h(\zeta) = Kc/\lambda + K\int_o^{\zeta} e^{-\alpha y}[1-\frac{\delta y}{c}]^{-(\lambda/\delta)-1} dy \qquad (4.2.16)$$

and therefore

$$E\{\exp(-\theta x_T)|\zeta_o\} = 1 - \frac{\theta\{c/\lambda + \int_o^{\zeta_o} e^{-\alpha y}[1-\frac{\delta y}{c}]^{-(\lambda/\delta)-1} dy\}}{e^{-\alpha w}[1-\frac{\delta w}{c}]^{-(\lambda/\delta)-1} + \theta\{c/\lambda + \int_o^{w} e^{-\alpha y}[1-\frac{\delta y}{c}]^{-(\lambda/\delta)-1} dy\}} \qquad (4.2.17)$$

Suppose now that we want to find w* so as to maximize

$$E\{L(x_T)|\zeta_o\} = \begin{cases} \frac{1-E\{\exp(-\theta x_T)|\zeta_o\}}{\theta} & \theta \neq 0 \\ E\{x_T|\zeta_o\} & \theta = 0 \end{cases} \qquad (4.2.18)$$

When $\theta>0$ this is a *risk averse* criterion, when $\theta<0$ a *risk seeking* and when $\theta=0$ a *risk neutral*. We have that

$$E\{L(x_T)|\zeta_o\} = \frac{c/\lambda + \int_o^{\zeta_o} e^{-\alpha y}[1-\frac{\delta y}{c}]^{-(\lambda/\delta)-1} dy}{e^{-\alpha w}[1-\frac{\delta w}{c}]^{-(\lambda/\delta)-1} + \theta\{c/\lambda + \int_o^{w} e^{-\alpha y}[1-\frac{\delta y}{c}]^{-(\lambda/\delta)-1} dy\}} \qquad (4.2.19)$$

So maximizing $E\{L(x_T)|\zeta_o\}$ is equivalent to minimizinng

$$e^{-\alpha w}[1-\frac{\delta w}{c}]^{-(\lambda/\delta)-1} + \theta\{c/\lambda + \int_o^{w} e^{-\alpha y}[1-\frac{\delta y}{c}]^{-(\lambda/\delta)-1} dy\} \qquad (4.2.20)$$

whose first derivative is

$$-\alpha e^{-\alpha w}[1-\frac{\delta w}{c}]^{-(\lambda/\delta)-1} + \frac{(\lambda/\delta)+1}{c/\delta} e^{-\alpha w}[1-\frac{\delta w}{c}]^{-(\lambda/\delta)-2} + \theta e^{-\alpha w}[1-\frac{\delta w}{c}]^{-(\lambda/\delta)-1} \qquad (4.2.21)$$

and w* is given by

$$-\alpha e^{-\alpha w^*}[1-\frac{\delta w^*}{c}]^{-(\lambda/\delta)-1} + \frac{(\lambda/\delta)+1}{c/\delta} e^{-\alpha w^*}[1-\frac{\delta w^*}{c}]^{-(\lambda/\delta)-2} + \theta e^{-\alpha w^*}[1-\frac{\delta w^*}{c}]^{-(\lambda/\delta)-1} = 0 \qquad (4.2.22)$$

which is equivalent to

$$-\alpha + \frac{(\lambda/\delta)+1}{c/\delta}[1-\frac{\delta w^*}{c}]^{-1} + \theta = 0 \qquad (4.2.23)$$

so

$$w^* = (c/\delta) - \frac{\alpha - \theta}{(\lambda/\delta)+1} \qquad (4.2.22)$$

Of course if this turns out to be negative, then the company is better off by

investing everything immediately.

The advantages of this model compared with the one in the previous section are obvious from a calculations point of view. In the previous section we merely found how to calculate the moments of the objective function while here we proved that our objective function is actually exponentially distributed and the only thing remaining to be done is to determine the parameter of the distribution either analytically (e.g when the claims are exponentially distributed) or otherwise if the claim size distribution is complicated (simulation techniques).

4.3 THE TAPIERO ZUCKERMAN MODEL

In this section we will deal with a more elaborate model for investment. It is described and dealt with by Tapiero & Zuckerman (1983). We will treat this model with a piecewise deterministic processes approach.

The model consists of two processes; ζ_t is the classical surplus process increasing with constant rate c. Claims arrive with rate λ and have sizes distributed with distribution function G(y) (y>0). We adopt a barrier strategy in the lines of the previous sections. When ζ_t reaches w it ceases to increase as its income is invested. However not the whole of it since an investment cost is levied leaving the rate of income invested as ac (a<1). So the rate of increase of x_t which is the other process of our model is $acI(\zeta_t=w)-r\zeta_t$; $r\zeta_t$ is an opportunity cost (due to inflation etc; r is a constant). An additional feature is that the company is not ruined when a claim larger than its available surplus arrives. In

that case the surplus of the company falls to 0 and the remaining part of the claim is paid by disinvesting. However, disinvestment costs occur as well and therefore the amount subtracted from x_t is $b(Y-\zeta_{t-})$ where Y is the size of the claim ($b>1$). x_t is the total money invested up to time t minus all costs.

The generator of the process (x_t,ζ_t,t) acting on a function $f(x,\zeta,t)$ belonging to its domain is given by

$$Af(x,\zeta,t) = \frac{\partial f}{\partial t} - r\zeta\frac{\partial f}{\partial x} + cI(\zeta<w)\frac{\partial f}{\partial \zeta} + acI(\zeta=w)\frac{\partial f}{\partial x}$$

$$+ \lambda[\int_{(0,\zeta]} f(x,\zeta-y,t)dG(y)+\int_{(\zeta,\infty)} f(x-b(y-\zeta),0,t)dG(y)-f(x,\zeta,t)] \qquad (4.3.1)$$

f has to be continuous with respect to ζ even at w. We have suppressed dependence on the indicating process n_t as in the previous sections.

The objective is to find w so as to maximize

$$\lim_{t\to\infty} \frac{E\{x_t\}}{t}$$

which is the long run average value of x_t the total investment minus the costs.

In the paper by Tapiero & Zuckerman this problem is solved for the case of exponentially distributed claim sizes and they propose a diffusion approximation to be applied otherwise. We will solve the problem for exponentially distributed claim sizes as well but substantially reducing the relevant calculations. We will also propose an asymptotic approximation for other claim size distributions. Furthermore we will look at higher order moments of x_t (eg the behaviour of $E\{x_t^2\}$). In the next section we will also propose an alternative model.

<u>Theorem 4.3.1</u> *Let* x_t,ζ_t *as defined and assume that* μ_1 *the first moment of* G *exists. If* $h_1(\zeta)$ *is a function of* ζ *($\zeta\geq 0$) such that*

$$\hat{h}_1(\xi) = \int_{\mathbb{R}^+} e^{-\xi\zeta}h_1(\zeta)d\zeta = \frac{\frac{r}{c\xi^2} + \frac{K_1}{c\xi} + \frac{\lambda b}{c}\cdot\frac{\mu_1-\hat{\mathbf{G}}(\xi)}{\xi}}{1 - (\lambda/c)\hat{\mathbf{G}}(\xi)} \qquad (4.3.2)$$

$$\hat{\mathbf{G}}(\xi) = \int_{\mathbb{R}^+} e^{-\xi\zeta}\mathbf{G}(\zeta)d\zeta \quad \mathbf{G}(\zeta) = 1-G(\zeta)$$

and K_1 *is a constant chosen so that*

$$h_1(w) = a \qquad (4.3.3)$$

then

$$x_t - K_1 t + \int_0^{\zeta_t} h_1(y)dy \tag{4.3.4}$$

is a martingale.

Proof

For $f(x_t,\zeta_t,t)$ to be a martingale we should have

$$Af(x,\zeta,t) = 0 \tag{4.3.5}$$

Continuity of f at $\zeta=w$ means that as in the previous section

$$a \frac{\partial f}{\partial x}\Big|_{\zeta=w} = \frac{\partial f}{\partial \zeta}\Big|_{\zeta=w} \tag{4.3.6}$$

Trying a function of the form $x-K_1t+f_1(\zeta)$ we get

$$-r\zeta-K_1+cf_1'(\zeta)+\lambda[\int_{(0,\zeta]} f_1(\zeta-y)dG(y)+f_1(0)\mathbf{G}(\zeta)-b\int_{(\zeta,\infty)}(y-\zeta)dG(y)-f_1(\zeta)] = 0 \tag{4.3.7}$$

with

$$f_1'(w) = a \tag{4.3.8}$$

Notice that $\int_{(\zeta,\infty)}(y-\zeta)dG(y) = \int_\zeta^\infty \mathbf{G}(y)dy$ and applying Laplace transforms on (4.3.7)

$$-\frac{r}{\xi^2}-\frac{K_1}{\xi}+c(\xi\hat{f}_1(\xi)-f_1(0))+\lambda[\hat{f}_1(\xi)\hat{g}(\xi)+f_1(0)\hat{\mathbf{G}}(\xi)-\frac{\mu_1-\hat{\mathbf{G}}(\xi)}{\xi}-\hat{f}_1(\xi)] = 0 \tag{4.3.9}$$

which gives

$$\xi\hat{f}_1(\xi) - f_1(0) = \frac{\frac{r}{c\xi^2}+\frac{K_1}{c\xi}+\frac{\lambda b}{c}\cdot\frac{\mu_1-\hat{\mathbf{G}}(\xi)}{\xi}}{1-(\lambda/c)\hat{\mathbf{G}}(\xi)} \tag{4.3.10}$$

which is equivalent to

$$f_1'(\zeta) = h_1(\zeta) \tag{4.3.11}$$

and (4.3.8) to

$$h_1(w) = a \tag{4.3.12}$$

proving the theorem.

□

This leads to the following corollary.

Corollary 4.3.2

$$\lim_{t\to\infty} \frac{E\{x_t\}}{t} = K_1 \tag{4.3.13}$$

Proof

$$x_t - K_1 t + \int_0^{\zeta_t} h_1(y)dy$$

is a martingale, so

$$E\{x_t\} = K_1 t + E\{\int_0^{\zeta_t} h_1(y)dy\} - \int_0^{\zeta_0} h_1(y)dy \tag{4.3.14}$$

But $\int_0^{\zeta} h_1(y)dy$ is bounded since it is continuous and defined on the closed interval [0,w], so dividing both sides of (4.3.14) by t and letting $t\to\infty$ (4.3.13) follows.

□

We will later illustrate how to use theorem 4.3.1 and corollary 4.3.2.

Another question arising, not considered by Tapiero & Zuckerman is the behaviour of higher moments of x_t. One could try to find a martingale of the form $\exp(-\nu x_t)e^{\theta t}f(\zeta_t;\nu)$ which will involve solving

$$(r\nu\zeta+\theta)f(\zeta)+cf'(\zeta)+\lambda[\int_{(0,\zeta]} f(\zeta-y)dG(y)+f(0)\int_{(\zeta,\infty)} e^{\nu(y-\zeta)}dG(y)-f(\zeta)]=0 \tag{4.3.15}$$

which is not easy to solve even when G is exponential.

Yet, we can adopt the procedure of finding moments one by one as in section 3.3 for example. Looking at the problem heuristically and differentiating $e^{-\nu x}e^{\theta t}f(\zeta;\nu)$ twice we get the form $[x_t-K_1t+f_1(\zeta_t)]^2-K_2t+f_2(\zeta_t)$. An equivalent form is $(x_t-K_1t)^2+2(x_t-K_1t)f_1(\zeta_t)-K_2t+f_2(\zeta_t)$ so we will try to find a martingale of this form.

Theorem 4.3.3 *Suppose* μ_2 *the second moment of* G *exists and let* $h_2(\zeta)$ $(\zeta>0)$ *be such that its Laplace transform is*

$$\hat{h}_2(\xi) = \frac{-\frac{2r\hat{h}_1'(\xi)}{c\xi} + \frac{2r\hat{h}_1(\xi)}{c\xi^2} + \frac{2K_1\hat{h}_1(\xi)}{c\xi} + \frac{K_2}{c\xi} - \frac{\lambda b^2}{c}\cdot\frac{(\mu_2\xi/2)-\mu_1+\hat{G}(\xi)}{\xi^2}}{1-(\lambda/c)\hat{G}(\xi)} \tag{4.3.16}$$

where $K_1, h_1(\zeta)$ *are given in theorem 4.3.1 and* K_2 *is a constant chosen so that*

$$h_2(w) = a\int_0^w h_1(y)dy \tag{4.3.17}$$

then

$$(x_t - K_1 t)^2 + 2(x_t - K_1 t)\int_0^{\zeta_t} h_1(y)dy - K_2 t + \int_0^{\zeta_t} h_2(y)dy \tag{4.3.18}$$

is a martingale.

Proof

Substituting

$$(x-K_1t)^2+2(x-K_1t)\int_0^{\zeta} h_1(y)dy-K_2t+f_2(\zeta)$$

into Af=0 we get (after cancelling out similar terms)

$$-2K_1\int_0^{\zeta} h_1(y)dy-2r\zeta\int_0^{\zeta} h_1(y)dy-K_2+cf_2'(\zeta)$$

$$+\lambda[\int_{(0,\zeta]} f_2(\zeta-y)dG(y)+f_2(0)\mathbf{G}(\zeta)+b^2\int_{(\zeta,\infty)}(y-\zeta)^2dG(y)-f_2(\zeta)] = 0 \tag{4.3.19}$$

Noticing that $\int_{(\zeta,\infty)}(y-\zeta)^2dG(y) = \int_\zeta^\infty\int_y^\infty \mathbf{G}(u)dudy$ and taking Laplace transforms as in theorem 4.3.1 we get

$$-2K_1\frac{\hat{h}_1(\xi)}{\xi} - 2r\left[\frac{\hat{h}_1(\xi)}{\xi}\right]' - \frac{K_2}{\xi} + c[\xi\hat{f}_2(\xi)-f_2(0)] +$$

$$\lambda\hat{f}_2(\xi)\hat{g}(\xi)-\lambda f_2(0)\mathbf{G}(\xi)-\lambda\hat{f}_2(\xi)+\lambda b^2\frac{(\mu_2\xi/2)-\mu_1+\hat{\mathbf{G}}(\xi)}{\xi^2} = 0 \tag{4.3.20}$$

giving

$$\xi\hat{f}_2(\xi) - f_2(0) = \frac{-\frac{2r\hat{h}_1'(\xi)}{c\xi} + \frac{2r\hat{h}_1(\xi)}{c\xi^2} + \frac{2K_1\hat{h}_1(\xi)}{c\xi} + \frac{K_2}{c\xi} - \frac{\lambda b^2}{c}\cdot\frac{(\mu_2\xi/2)-\mu_1+\hat{\mathbf{G}}(\xi)}{\xi^2}}{1 - (\lambda/c)\hat{\mathbf{G}}(\xi)} \tag{4.3.21}$$

and therefore

$$f_2'(\zeta) = h_2(\zeta) \tag{4.3.22}$$

and (4.3.6) yields

$$f_2'(w) = a\int_0^w f_1(y)dy \tag{4.3.23}$$

thus proving the theorem.

□

Corollary 4.3.4 *Let* K_1 *and* K_2 *as in theorems 4.3.1 and 4.3.3; then*

$$\lim_{t\to\infty} \frac{E\{(x_t-K_1t)^2\}}{t} = K_2 \tag{4.3.24}$$

Proof

Since (4.3.18) is a martingale

$$E\{(x_t-K_1t)^2\} + 2E\{(x_t-K_1t)\int_0^{\zeta_t} h_1(y)dy\} - K_2t + E\{\int_{\zeta_0}^{\zeta_t} h_2(y)dy\} \tag{4.3.25}$$

Dividing by t we get

$$E\{\frac{(x_t-K_1t+\int_0^{\zeta_t} h_1(y)dy)^2}{t}\} - K_2 + E\{\frac{\int_0^{\zeta_t} h_2(y)dy-(\int_0^{\zeta_t} h_1(y)dy)^2}{t}\} = \frac{\int_0^{\zeta_0} h_2(y)dy}{t} \tag{4.3.26}$$

and since $\int_0^{\zeta} h_1(y)dy$ and $\int_0^{\zeta} h_2(y)dy$ are bounded (continuous functons defined on a closed interval)

$$\lim_{t\to\infty} E\{\frac{(x_t-K_1t+\int_0^{\zeta_t} h_1(y)dy)^2}{t}\} = K_2 \tag{4.3.27}$$

but

$$E\{\frac{(x_t-K_1t+\int_0^{\zeta_t} h_1(y)dy)^2}{t}\} = E\{\left[\frac{x_t-K_1t}{\sqrt{t}} + \frac{\int_0^{\zeta_t} h_1(y)dy}{\sqrt{t}}\right]^2\} \tag{4.3.28}$$

and since $\int_0^{\zeta} h_1(y)dy$ is bounded $\frac{\int_0^{\zeta_t} h_1(y)dy}{\sqrt{t}} \to 0$ a.s as $t\to\infty$, therefore

$$E\{\frac{(x_t-K_1t+\int_0^{\zeta_t} h_1(y)dy)^2}{t}\} \to \frac{E\{(x_t-K_1t)^2\}}{t} \tag{4.3.29}$$

and (4.3.24) follows.

□

This procedure can be followed to investigate the asymptotic behaviour of higher moments of x_t.

We will now show how to use these results. From theorem 4.3.1 it is clear that in order to find K_1 we have to invert the Laplace transform $h_1(\zeta)$ and solve $h_1(w)=a$. For simple claim size distributions such as the exponential this is

straight forward as we shall see. For a more general distribution we will look for an asymptotic formula.

Lemma 4.3.5 *Let* $h_1(\zeta)$ *be such that*

$$\hat{h}_1(\xi) = \frac{\frac{r}{c\xi^2} + \frac{K_1}{c\xi} + \frac{\lambda b}{c}\cdot\frac{\mu_1 - \hat{G}(\xi)}{\xi}}{1 - (\lambda/c)\hat{G}(\xi)} \tag{4.3.30}$$

Suppose also that $G(y)$ *is such that there exists* $\gamma>0$ *so that*

$$\int_{\mathbb{R}^+} e^{\nu y} dG(y) < \infty \quad \text{for} \quad \nu \in [0,\gamma)$$

Furthermore we assume $c>\lambda\mu_1$ *(the net profit condition). Then the asymptotic behaviour of* $h_1(\zeta)$ *is given by*

$$h_1(\zeta) \sim \frac{r}{c-\lambda\mu_1}\zeta + \frac{K_1}{c-\lambda\mu_1} - \frac{\lambda\mu_2}{2}\cdot\frac{r}{(c-\lambda\mu_1)^2} + \exp(-\nu_0\zeta)\frac{\frac{r}{c\nu_0^2} - \frac{K_1}{c\nu_0} + (\lambda b/c)\frac{\hat{G}(-\nu_0)-\mu_1}{\nu_0}}{-(\lambda/c)\hat{G}'(-\nu_0)} \tag{4.3.31}$$

where ν_0 *is the unique positive root of* $(\lambda/c)\hat{G}(-\nu) = 1$.

Also, when $dG(y)=\alpha e^{-\alpha y}$ *(exponential); that is* $\hat{g}(\xi) = \alpha/(\alpha+\xi)$, *we actually have that*

$$h_1(\zeta) = \frac{r\alpha}{c\alpha-\lambda}\zeta + \frac{K_1\alpha - r\lambda/(c\alpha-\lambda)}{c\alpha-\lambda} + \exp[-(\alpha-\tfrac{\lambda}{c})\zeta]\frac{r\lambda - K_1\lambda[\alpha-(\lambda/c)] + (\lambda b/c\alpha)[\alpha-(\lambda/c)]^2}{(c\alpha-\lambda)^2} \tag{4.3.32}$$

Proof

i)

We have already seen in section 1.1 that when $c>\lambda\mu_1$, ν_0 as defined exists. We rewrite $\hat{h}_1(\xi)$ as follows.

$$\hat{h}_1(\xi) = \frac{r}{c-\lambda\mu_1}\cdot\frac{1}{\xi^2} + \left\{\frac{K_1}{c-\lambda\mu_1} - \frac{\lambda\mu_2}{2}\cdot\frac{r}{(c-\lambda\mu_1)^2}\right\}\cdot\frac{1}{\xi} + \frac{1}{\xi^2}\cdot\left\{\frac{r}{c-\lambda\hat{G}(\xi)} - \frac{r}{c-\lambda\mu_1} + \frac{r\lambda\mu_2\xi}{(c-\lambda\mu_1)^2}\right\} +$$

$$+ \frac{1}{\xi}\cdot\left\{\frac{K_1}{c-\lambda\hat{G}(\xi)} - \frac{K_1}{c-\lambda\mu_1}\right\} + \frac{\lambda b\cdot\frac{\mu_1-\hat{G}(\xi)}{\xi}}{c-\lambda\hat{G}(\xi)} \tag{4.3.33}$$

Now the last three terms are all Laplace transforms of functions tending to

zero and behave asymptotically as $L_i\exp(-\nu_0\zeta)$ (i=1,2,3) where L_i are suitable constants. We can determine L_i's as follows.

$$L_1 = \lim_{\xi \downarrow 0} \left\{ \frac{\xi}{(\xi-\nu_0)^2} \cdot \left\{ \frac{r}{c-\lambda\hat{G}(\xi-\nu_0)} - \frac{r}{c-\lambda\mu_1} + \frac{r\lambda\mu_2(\xi-\nu_0)}{(c-\lambda\mu_1)^2} \right\} \right\} = \frac{r/(c\nu_0^2)}{-(\lambda/c)\hat{G}'(-\nu_0)} \tag{4.3.34}$$

$$L_2 = \lim_{\xi \downarrow 0} \left\{ \frac{\xi}{(\xi-\nu_0)} \cdot \left\{ \frac{K_1}{c-\lambda\hat{G}(\xi-\nu_0)} - \frac{K_1}{c-\lambda\mu_1} \right\} \right\} = \frac{-K_1/c\nu_0}{-(\lambda/c)\hat{G}'(-\nu_0)} \tag{4.3.35}$$

$$L_3 = \lim_{\xi \downarrow 0} \left\{ \xi \cdot \frac{\lambda b \cdot \dfrac{\mu_1 - \hat{G}(\xi-\nu_0)}{\xi-\nu_0}}{c-\lambda\hat{G}(\xi-\nu_0)} \right\} = \frac{(\lambda b/c)\dfrac{\hat{G}(-\nu_0)-\mu_1}{\nu_0}}{-(\lambda/c)\hat{G}'(-\nu_0)} \tag{4.3.36}$$

Also the first term in (4.3.33) is the Laplace transform of

$$\frac{r}{c-\lambda\mu_1}\zeta \tag{4.3.37}$$

and the second term of

$$\frac{K_1}{c-\lambda\mu_1} - \frac{\lambda\mu_2}{2} \cdot \frac{r}{(c-\lambda\mu_1)^2} \tag{4.3.38}$$

Adding (4.3.34), (4.3.35), (4.3.36), (4.3.37) & (4.3.38) together (4.3.31) follows.

ii)

If we take $\hat{g}(\xi)=\alpha/(\alpha+\xi)$ we have

$$\hat{h}_1(\xi) = \frac{r(\alpha+\xi) + K_1\xi(\alpha+\xi) + \lambda b\xi^2/\alpha}{\xi^2(c\alpha+c\xi-\lambda)} \tag{4.3.39}$$

and analyzing it into partial fractions

$$\hat{h}_1(\xi) = \frac{r\alpha}{c\alpha-\lambda}\cdot\frac{1}{\xi^2} + \frac{K_1\alpha - r\lambda/(c\alpha-\lambda)}{c\alpha-\lambda}\cdot\frac{1}{\xi} + \frac{r\lambda - K_1\lambda[\alpha-(\lambda/c)]+(\lambda b/c\alpha)[\alpha-(\lambda/c)]^2}{(c\alpha-\lambda)^2[\alpha+\xi-(\lambda/c)]} \tag{4.3.40}$$

and inverting (4.3.40), (4.3.32) follows.

□

One can easily check that the coefficients of (4.3.31) and (4.3.32) match.

In the case of exponentially distributed claim sizes we can solve $h_1(w)=a$ and get

$$K_1 = \frac{a(c\alpha-\lambda) - r\alpha + [r\lambda/(c\alpha-\lambda)]\exp[-[\alpha-(\lambda/c)]w] - (\lambda b/c\alpha)\exp[-[\alpha-(\lambda/c)]w]}{\alpha - (\lambda/c)\exp[-[\alpha-(\lambda/c)]w]} \qquad (4.3.41)$$

Therefore the optimum w* is found by maximizing (4.3.41) with respect to w.

When we have a general claim size distribution with the restriction of lemma 4.3.5 then we can use the asymptotic formula (4.3.31) as an approximation and get

$$K_1 \simeq \frac{a - \frac{r}{c-\lambda\mu_1}w + \frac{\lambda\mu_2}{2}\cdot\frac{r}{(c-\lambda\mu_1)^2} - \exp(-\nu_0\zeta)\dfrac{\frac{r}{c\nu_0^2} + (\lambda b/c)\frac{\hat{G}(-\nu_0)-\mu_1}{\nu_0}}{-(\lambda/c)\hat{G}'(-\nu_0)}}{\frac{1}{c-\lambda\mu_1} + \frac{1}{\lambda\nu_0\hat{G}'(-\nu_0)}} \qquad (4.3.42)$$

In this case we again find an approximation for w* by maximzing (4.3.42). However, this is reliable only if w* is large enough; that is large compared with μ_1.

In case we have different objectives (either risk seeking or risk averse) such as to maximize $K_1+\theta K_2$ where θ is a small constant we can then calculate K_2 or an asymptotic approximation for it. We will have to maximize $K_1+\theta K_2$ as we did with K_1 in order to find w*.

4.4 AN ALTERNATIVE MODEL

We will now reconsider the Tapiero Zuckerman model with an alteration. We are not going to assume that opportunity costs are subtracted from x_t the total amount invested minus the costs. We assume that these are subtracted from ζ_t, the cash surplus. So ζ_t increases with rate $(c-\delta\zeta_t)I(\zeta_t<w)$ and the rate of

investment is $a(c-\delta w)I(\zeta_t=w)$. A barrier strategy is assumed as before. It is clear that we should have $w < c/\delta$ since ζ_t can not exceed c/δ anyway.

The generator of the process (x_t,ζ_t,t) acting on a function $f(x,\zeta,t)$ belonging to its domain is given by

$$Af(x,\zeta,t) = \frac{\partial f}{\partial t} + (c-\delta\zeta)I(\zeta<w)\frac{\partial f}{\partial \zeta} + a(c-\delta w)I(\zeta=w)\frac{\partial f}{\partial x}$$

$$+ \lambda[\int_{(0,\zeta]} f(x,\zeta-y,t)dG(y)+\int_{(\zeta,\infty)} f(x-b(y-\zeta),0,t)dG(y)-f(x,\zeta,t)] \qquad (4.4.1)$$

(f should be continuous at $\zeta=w$).

We will now try to find $\lim_{t\to\infty}\frac{E\{x_t\}}{t}$, the long run average ammount invested as we did in the previous section.

Theorem 4.4.1 *Let x_t,ζ_t as defined. Let also $f_1(\zeta)$ be a function of ζ ($\zeta\in[0,w]$) such that*

$$-K_1 + (c-\delta\zeta)f_1'(\zeta) + \lambda[\int_{(0,\zeta]} f_1(\zeta-y)dG(y)+f_1(0)\mathbf{G}(\zeta)-b\int_{\zeta}^{\infty}\mathbf{G}(y)dy-f_1(\zeta)] = 0 \qquad (4.4.2)$$

and K_1 a constant such that

$$f_1'(w) = a \qquad (4.4.3)$$

then

$$x_t - K_1 t + f_1(\zeta_t) \qquad (4.4.4)$$

is a martingale.

Proof

In order to find a martingale we should solve $Af=0$. Applying the generator A on $x-K_1t+f_1(\zeta)$ we get

$$-K_1 + (c-\delta\zeta)f_1'(\zeta) + \lambda[\int_{(0,\zeta]} f_1(\zeta-y)dG(y)+f_1(0)\mathbf{G}(\zeta)-b\int_{\zeta}^{\infty}\mathbf{G}(y)dy-f_1(\zeta)] = 0 \quad (\zeta < w) \qquad (4.4.5)$$

$$-K_1 + (c-\delta w)a + \lambda[\int_{(0,\zeta]} f_1(\zeta-y)dG(y)+f_1(0)\mathbf{G}(\zeta)-b\int_{\zeta}^{\infty}\mathbf{G}(y)dy-f_1(\zeta)] = 0 \quad (\zeta = w) \qquad (4.4.6)$$

but in view of (4.4.5), (4.4.6) and (4.4.3) are equivalent, so f_1 as defined is a solution of the equation $Af=0$ and the theorem is proved.

□

Corollary 4.4.2

$$\lim_{t\to\infty}\frac{E\{x_t\}}{t} = K_1 \tag{4.4.7}$$

Proof

Since $x_t - K_1 t + f_1(\zeta_t)$ is a martingale we have

$$E\{x_t\} - K_1 t + f_1(\zeta_t) - f_1(\zeta_0) = 0 \tag{4.4.8}$$

f_1 is a continuous function on a closed interval so it is bounded and dividing (4.4.8) by t and letting $t\to\infty$, (4.4.7) follows.

□

We will now use theorem 4.4.1 and corollary 4.4.2 for the specific case of exponentially distributed claim sizes. In order to find $f_1(\zeta)$ and subsequently K_1 we have to solve equation (4.4.2) subject to (4.4.3). When $dG(y)=\alpha e^{-\alpha y}dy$ (4.4.2) can be written as

$$-K_1 + (c-\delta\zeta)f_1'(\zeta) + \lambda\left[\int_0^{\zeta} f_1(\zeta-y)\alpha e^{-\alpha y}dy + f_1(0)e^{-\alpha\zeta} - (b/\alpha)e^{-\alpha\zeta} - f_1(\zeta)\right] = 0 \tag{4.4.9}$$

Applying the operator $D+\alpha I$ on (4.4.9)

$$-\alpha K_1 + (c-\delta\zeta)f_1''(\zeta) - \delta f_1'(\zeta) + \alpha(c-\delta\zeta)f_1'(\zeta) - \lambda f_1'(\zeta) = 0 \tag{4.4.10}$$

From (4.4.10) we get

$$\left[e^{\alpha\zeta}[(c/\delta)-\zeta]^{(\lambda/\delta)+1} f_1'(\zeta)\right]' = (\alpha K_1/\delta)[(c/\delta)-\zeta]^{\lambda/\delta}e^{\alpha\zeta} \tag{4.4.11}$$

and integrating (4.4.11)

$$e^{\alpha\zeta}[(c/\delta)-\zeta]^{(\lambda/\delta)+1} f_1'(\zeta) = \int_0^{\zeta}(\alpha K_1/\delta)[(c/\delta)-y]^{\lambda/\delta}e^{\alpha y}dy + f_1'(0)(c/\delta)^{(\lambda/\delta)+1} \tag{4.4.12}$$

Setting $\zeta=0$ in (4.4.9) we get

$$f_1'(0) = (K_1/c)+(\lambda b/c\alpha) \tag{4.4.13}$$

and (4.4.12) can be rewritten as

$$e^{\alpha\zeta}[(c/\delta)-\zeta]^{(\lambda/\delta)+1} f_1'(\zeta) = \int_0^{\zeta}(\alpha K_1/\delta)[(c/\delta)-y]^{\lambda/\delta}e^{\alpha y}dy + [(K_1/c)+(\lambda b/c\alpha)](c/\delta)^{(\lambda/\delta)+1} \tag{4.4.14}$$

From (4.4.3) we should have $f_1'(w)=a$, so

$$ae^{\alpha w}[(c/\delta)-w]^{(\lambda/\delta)+1} = K_1\left[(\alpha/\delta)\int_0^{w}[(c/\delta)-y]^{\lambda/\delta}e^{\alpha y}dy + (1/c)(c/\delta)^{(\lambda/\delta)+1}\right] + \frac{\lambda b}{c\alpha}(c/\delta)^{(\lambda/\delta)+1} \tag{4.4.15}$$

giving us

$$K_1 = \frac{ae^{\alpha w}[(c/\delta)-w]^{(\lambda/\delta)+1} - \frac{\lambda b}{c\alpha}(c/\delta)^{(\lambda/\delta)+1}}{(\alpha/\delta)\int_0^w [(c/\delta)-y]^{\lambda/\delta} e^{\alpha y}dy + (1/c)(c/\delta)^{(\lambda/\delta)+1}} \qquad (4.4.16)$$

In order to maximize the long run average amount invested minus the costs we need to maximize (4.4.16) with respect to w.

If one wants to investigate the higher moments of x_t he has to repeat the procedure followed in section 4.3. However, in this section we will aim for something rather different.

Theorem 4.4.3 *Let* x_t, ζ_t *as defined and suppose* $f(\zeta)$ *is such that*

$$(c-\delta\zeta)f'(\zeta) + \lambda[\int_{(0,\zeta]} f(\zeta-y)dG(y) + f(0)\int_{(\zeta,\infty)} e^{\nu(y-\zeta)}dG(y) - f(\zeta)] = 0 \qquad (4.4.17)$$

(we assume that $\int_{(\zeta,\infty)} e^{\nu(y-\zeta)}dG(y)$ *exists)*

and suppose also that ν *is chosen so that*

$$f'(w) = -a\nu f(w) \qquad (4.4.18)$$

then

$$\exp(-\nu x_t)f(\zeta_t). \qquad (4.4.19)$$

is a martingale.

Proof

We substitute $e^{-\nu x}f(\zeta)$ into $Af=0$ and we get (4.4.17) and

$$-a\nu(c-\delta w) + \lambda[\int_{(0,w]} f(w-y)dG(y) + f(0)\int_{(w,\infty)} e^{\nu(y-w)}dG(y) - f(w)] = 0 \qquad (4.4.20)$$

but (4.4.17) and (4.4.20) give us (4.4.18), so $e^{-\nu x}f(\zeta)$ satisfies $Af=0$. It also belongs to the domain of the generator and therefore (4.4.19) is a martingale.

□

If for some particular values of w $\nu>0$, we can then use the concept of **ruin** for x_t. We will fix a level $-x^*$ ($x^*>0$) and we will look at the probability that $x_t \geq -x^*$ for all t. We define $\tau \equiv \inf\{t: x_t < -x^*\}$ with $\tau=\infty$ if $x_t \geq -x^*$ for all t. Since $\exp(-\nu x_t)$ is a martingale and $\tau \wedge t$ a stopping time we use Doob's optional stopping

theorem and we have

$$f(\zeta_o) = E\{\exp(-\nu x_{\tau\wedge t})f(\zeta_{\tau\wedge t})\} =$$
$$= E\{\exp(-\nu x_\tau)f(\zeta_\tau)\,|\,\tau\leq t\}\Pr(\tau\leq t) + E\{\exp(-\nu x_t)f(\zeta_t)\,|\,\tau>t\}\Pr(\tau>t) \tag{4.4.21}$$

and letting $t\to\infty$ in (4.4.21) since f is bounded and assuming that $x_t\to\infty$ a.s. as $t\to\infty$ (this is equivalent to assuming $K_1>0$).

$$\Pr(\tau<\infty)E\{\exp(-\nu x_\tau)f(\zeta_\tau)\,|\,\tau<\infty\} = f(\zeta_o) \tag{4.4.22}$$

or equivalently

$$\Pr(\tau<\infty) = \frac{e^{-\nu x^*}f(\zeta_o)}{E\{\exp[-\nu(x_\tau-x^*)]f(\zeta_\tau)\,|\,\tau<\infty\}} \tag{4.4.23}$$

We will now evaluate ν explicitly for the case of exponentially distributed claim sizes ($dG(y)=\alpha e^{-\alpha y}dy$). In this case (4.4.17) can be written as

$$(c-\delta\zeta)f'(\zeta) + \lambda[\int_o^\zeta f(\zeta-y)\alpha e^{-\alpha y}dy + f(0)\frac{\alpha}{\alpha-b\nu}e^{-\alpha\zeta} - f(\zeta)] = 0 \tag{4.4.24}$$

Applying the operator $D+\alpha I$ on (4.4.240 we get

$$(c-\delta\zeta)f''(\zeta) - \delta f'(\zeta) + \alpha(c-\delta\zeta)f'(\zeta) - \lambda f'(\zeta) = 0 \tag{4.4.25}$$

and solving (4.4.25)

$$f'(\zeta) = f'(0)(c/\delta)^{(\lambda/\delta)+1}e^{-\alpha\zeta}[(c/\delta)-\zeta]^{-(\lambda/\delta)-1} \tag{4.4.26}$$

Setting $\zeta=0$ in (4.4.24) we get

$$f'(0) = -\frac{\lambda b\nu}{\alpha-b\nu}f(0) \tag{4.4.27}$$

so

$$f'(\zeta) = -\frac{\lambda b\nu}{\alpha-b\nu}f(0)(c/\delta)^{(\lambda/\delta)+1}e^{-\alpha\zeta}[(c/\delta)-\zeta]^{-(\lambda/\delta)-1} \tag{4.4.28}$$

and therefore

$$f(\zeta) = f(0)\Big\{1 - \frac{1}{c}\cdot\frac{\lambda b\nu}{\alpha-b\nu}f(0)(c/\delta)^{(\lambda/\delta)+1}\int_o^\zeta e^{-\alpha y}[(c/\delta)-y]^{-(\lambda/\delta)-1}dy \tag{4.4.29}$$

and (4.4.18) gives

$$\frac{1}{c}\cdot\frac{\lambda b\nu}{\alpha-b\nu}e^{-\alpha w}[(c/\delta)-w]^{-(\lambda/\delta)-1} = a\nu(c/\delta)^{-(\lambda/\delta)-1} - \frac{\alpha\nu}{c}\cdot\frac{\lambda b\nu}{\alpha-b\nu}\int_0^w e^{-\alpha y}[(c/\delta)-y]^{-(\lambda/\delta)-1}dy \tag{4.4.30}$$

and assuming $\nu\neq 0$

$$\nu = \alpha\cdot\frac{a - (\lambda b/c\alpha)e^{-\alpha w}[1-(\delta w/c)]^{-(\lambda/\delta)-1}}{1 + (\lambda/c)\int_0^w e^{-\alpha y}[1-(\delta y/c)]^{-(\lambda/\delta)-1}} \tag{4.4.31}$$

We assume $\nu>0$; that is

$$a > (\lambda b/c\alpha)e^{-\alpha w}[1-(\delta w/c)]^{-(\lambda/\delta)-1} \tag{4.4.32}$$

$\exp(-\nu x_t)$ is a martingale where ν is given by (4.4.31) and f is given by (4.4.29). Then as in (4.4.23)

$$Pr(\tau<\infty) = \frac{e^{-\nu x^*}f(\zeta_0)}{E\{\exp[-\nu(x_\tau-x^*)]f(\zeta_\tau)\,|\,\tau<\infty\}} \tag{4.4.33}$$

Notice that now $-(x_\tau-x^*)$ (the *overshoot*) is exponentially distributed with parameter α/b because of the memoryless property of the exponential distribution. Therefore

$$E\{\exp[-\nu(x_\tau-x^*)]f(\zeta_\tau)\,|\,\tau<\infty\} = \frac{\alpha}{\alpha-b\nu}E\{f(\zeta_\tau)\,|\,\tau<\infty\} \tag{4.4.34}$$

We actually have $\zeta_\tau=0$. So

$$E\{\exp[-\nu(x_\tau-x^*)]f(\zeta_\tau)\,|\,\tau<\infty\} = \frac{\alpha}{\alpha-b\nu}f(0) \tag{4.4.35}$$

and hence

$$Pr(\tau<\infty) = e^{-\nu x^*}\cdot\frac{\alpha-b\nu}{\alpha}\cdot\frac{f(\zeta_0)}{f(0)} \tag{4.4.36}$$

(4.4.36) implies that we can use ν as a measure for the likelihood of ruin irrespective of x^* and ζ_0. So ν can be justifiably chosen as a criterion for optimality to compare barrier strategies. We should then choose w so as to maximize

$$\nu = \alpha\cdot\frac{a - (\lambda b/c\alpha)e^{-\alpha w}[1-(\delta w/c)]^{-(\lambda/\delta)-1}}{1 + (\lambda/c)\int_0^w e^{-\alpha y}[1-(\delta y/c)]^{-(\lambda/\delta)-1}} \tag{4.4.37}$$

It has to be stressed that that this is an extremely *risk averse* criterion as the company is only interested to minimize the chance that the level of x_t ever falls below a fixed level. It might be too unenterprising a criterion for the company to regard it as sensible. However, whatever the objectives of the company are it is of supreme importance to have an idea of how large the probability that everything goes entirely wrong is; that is the probability that x_t falls below a certain level.

LIST OF REFERENCES

1. Andersen,E-Sparre, (1957), *On the collective theory of risk in case of contagion between the claims.* Transactions: XVth international congress of actuaries, New York, II, 219-229.

2. Beard,R.E., Pentikainen,T.,Pesonen,E., (1984), *Risk Theory*, Chapman & Hall (3rd edition).

3. Bremaud,P., (1981), *Point Processes and Queues*, Springer-Verlag.

4. Buhlman,H., (1970), *Mathematical Methods in Risk Theory*, Springer-Verlag.

5. Chung,K.L., Williams,R.J., *Introduction to Stochastic integration*, Birkhauser, Boston.

6. Cohen,J.W., (1969), *Single Server Queue*, North Holland.

7. Cox,D.R., (1955), *The analysis of non-markovian stochastic processes by the inclusion of supplementary variables*, Proceedings, Cambridge Philosophical Society, 51, 433-441.

8. Cox,D.R., (1955a), *Some statistical methods connected with series of events*, J.R.Stat. Soc. B, 17, 129-164.

9. Cox,D.R., Miller,H.D., (1965), *The Theory of Stochastic Processes*, Chapman & Hall.

10. Cox,D.R., Isham,V., (1986), *The virtual waiting time and related processes*, Adv. Appl. Prob. 18, 558-573.

11. Cramer,H., (1930), *On the mathematical theory of risk*, Skand. Jubilee Volume, Stockholm.

12. Dassios,A., (1984), *Martingale Applications in Risk Theory*, Report submitted for M.Sc. & D.I.C., Imperial College, London.

13. Dassios,A., Embrechts,P., (1987), *Martingales and insurance risk*, to appear.

14. Davis,M.H.A., (1984), *Piecewise deterministic Markov processes: A general class of non diffusion stochastic models*, J. R. Stat. Soc. B, 46, 353-388.

15. Davis,M.H.A., Dempster,M.A.H., Sethi,S.P., Vermes,D., (1987), *Optimal capacity expansion under uncertainty*, Adv. Appl. Prob. 19, 156-176.

16. Delbaen,F., Haezendonck,J., (1986), *Martingales in Markov processes applied to risk theory,* Insurance: Mathematics and Economics, 5, 201-216.

17. Delbaen,F.,Haezenonck,J., (1987), *Classical risk theory in an economic environment,* to appear.

18. Feller,W., (1969), *An Introduction to Probability Theory and its Applications*, Vol. II, Wiley.

19. Gerber, H.U., (1969), (in German), *Entscheidungskriterien fur den zusammengesetzten,* Poison-Prozess M.S.V. 69, 185-228.

20. Gerber,H.U., (1979), *An Introduction to Mathematical Risk Theory*, Huebner Foundation Monographs.

21. Gerber,H.U., (1982), *Ruin theory in the linear model*, Insurance: Mathematics and Economics 1, 177-184.

22. Gnedenko,B.V., Kovalenko,I.I., (1966), (in Russian), *Introduction to the Theory of Mass Service*, Nauka, Moscow. [English translation (1968); Jerusalem].

23. Haris.on,J.M., Resnick,S.I., (1972), *On dams with additive inputs and general release rule*, J. Appl. Prob. 9, 472-492.

24. Haris. on,J.M., Resnick,S.I., (1978), *The recurrence classification of risk and storage processes*, Mathematics of Operational Research, Vol.3, N^{o}1.

25. Karlin,S., Taylor,H.M., (1975), *A First Course in Stochastic Processes*, New York: Academic Press.

26. Kennedy,D.P., (1984), Contribution to the discussion of [14], J. R. Stat. Soc. 46.

27. Kleinrock,L., (1975), *Queuing Theory,* Vol.I, Wiley.

28. Prabhu,N.U., (1980), *Stochastic Storage Processes: Queues, Insurance Risk and Dams,* Springer-Verlag.

29. Ramalhoto,M.S., (1984), *Bounds for the variance of the busy period of the $M/G/\infty$ queue,* Adv. Appl. Prob. 16, 929-932.

30. Ramsay,C.M., (1984), *The asymptotic ruin problem when the healthy and sick periods form an alternating renewal process,* Insurance: Mathematics and Economics 3, 139-143.

31. Rosenkrantz,W.A., (1981), *Some martingales associated with queuing and storage processes,* Z.Wahrscheinlichtheorie verw. Gebiete 58, 205-222.

32. Sathe, Y.S., (1985), *Improved bounds for the variance of the busy period of the $M/G/\infty$ queue,* Adv. Appl. Prob. 17, 913-914.

33. Seal,H.L., (1983), *The Poisson process: Its failure in risk theory,* Insurance: Mathematics and Economics 2, 287-288.

34. Segerdhal,C.O., (1959), *A survey of results in the collective theory of risk,* Harold Cramer Volume, Wiley, Stockholm, 276-299.

35. Shanbhag,P.N., (1966), *On infinite sever queues with batch arrivals,* J. Appl. Prob. 3, 274-279.

36. Takacs,L., (1962), *An Introduction to Queuing Theory,* Oxford University Press.

37. Tapiero,C., Zuckerman,D., (1983), *Optimal investment policy of an insurance firm with a compound Poisson claim process,* Insurance: Mathematics and Economics 2, 103-112.

38. Thorin,O., (1982), *Probabilities of ruin,* (Review), Skand. Actuarial Journal, 65-102.

39. Vermes,D., (1980), *Optimal dynamic control of a useful class of randomly jumping processes,* Report pp80-15, International Institute for Applied Systems Analysis, Laxenburg, Austria.