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Integral passivity-based control of underactuated mechanical systems with actuator dynamics and constant disturbances

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Abstract

This work investigates the energy shaping control of a class of underactuated mechanical systems with first-order actuator dynamics and subject to both matched and unmatched constant additive disturbances. To this end, a new non-linear control law which includes two independent integral actions is presented. The controller design is outlined for systems with first-order actuator dynamics, and also for systems with direct actuation. The effectiveness of the proposed approach is demonstrated with numerical simulations on an inertia wheel pendulum and on a ball-on-beam system, both actuated by electric DC motors and subject to constant disturbances.

KEYWORDS

actuator dynamics, constant disturbances, energy shaping, nonlinear systems, underactuated mechanical systems

1 | INTRODUCTION

Underactuation in mechanical systems can be due to actuator failure, to the characteristics of the system (e.g., high compliance in soft robotic manipulators¹), or it can result from design choices.² The control of underactuated mechanical systems has been approached with various methodologies, including sliding mode designs^{3,4} and energy shaping controllers, each having its respective advantages and drawbacks. In particular, the *Interconnection and damping assignment Passivity based control* (IDA-PBC) methodology⁵ involves designing the control action such that the closed-loop dynamics preserves the port-Hamiltonian structure and is characterized by a desired total energy. This approach offers a range of desirable features, including the physical interpretation of the control action in terms of system energy, which makes it well suited to multidomain systems involving energy exchanges between different components. Notable examples in this regard include magnetic levitation systems,⁶ suspension systems,⁷ and weakly-coupled electro-mechanical systems.⁸ In addition, IDA-PBC controllers do not rely on high gains or on switching control actions. In principle, the IDA-PBC methodology can be employed to account for any type of actuator dynamics. Notable results in this direction include IDA-PBC designs for mechanical systems that account for the first-order dynamics of electric motors in References 9 and 10. In addition, an energy shaping controller for a soft continuum manipulator actuated by electroactive polymers has been presented in Reference 11. More recently, an IDA-PBC design has been proposed for underactuated mechanical

Abbreviations: IDA-PBC, interconnection and damping assignment passivity-based control; PDEs, partial differential equations

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systems with fluidic actuation.^{12,13} Further works have investigated the effect of actuator saturation.¹⁴ In other cases, the actuator dynamics has been neglected for simplicity, considering that many actuators have higher bandwidth compared to the corresponding mechanical subsystems.¹⁵ In general, accounting for the actuator dynamics results in a system that is not input affine and where the disturbances affecting the mechanical subsystem may not be directly canceled by the control input. A number of research works have investigated the effect of disturbances within IDA-PBC, resulting in more sophisticated controllers that achieve robustification through integral actions¹⁶ or nonlinear observers.¹⁷ In particular, the *Integral IDA-PBC* methodology¹⁶ results in an extended closed-loop dynamics characterized by a port-Hamiltonian structure that preserves the energy-based interpretation and the corresponding stability properties. The initial formulation of Integral IDA-PBC was limited to systems with constant inertia matrix, constant input matrix, subject to constant matched disturbance, and it employed a burdensome change of coordinates.¹⁸ Subsequent works have refined this approach by avoiding the change of coordinates,¹⁹ and they have extended it to systems with nonconstant inertia matrix.¹⁹ Recent works have investigated the Integral IDA-PBC methodology for systems with physical damping²⁰ and with nonconstant disturbances,²¹ and have presented controllers for the specific case of power converters.²² In general however, most work on Integral IDA-PBC has so far been limited to systems with matched disturbances. Notable exceptions include,²³ which presented a result for fully actuated systems where unmatched disturbances represent a perturbation affecting the velocity measurement, and References 24 and 8, where unmatched disturbances were accounted for in the stability analysis but not in the design of the integral action. In addition, to the best of the authors' knowledge, the Integral IDA-PBC methodology has only been formalized for systems with direct actuation. In summary, the extension of the Integral IDA-PBC methodology to underactuated mechanical systems with actuator dynamics subject to both matched and unmatched disturbances is an open problem.

This work investigates the energy shaping control of a class of underactuated mechanical systems which are not in affine form due to the presence of first-order actuator dynamics, and which are subject to both matched and unmatched constant additive disturbances. To this end, a new control law is constructed by employing the IDA-PBC methodology⁵ with the concepts of integral action¹⁶ and of extended Hamiltonian.^{17,18} In summary, this work presents the following new results.

- A controller design procedure that accounts for generic first-order actuator dynamics thus extending our prior work.¹²
- An integral action that compensates the effect of constant matched disturbances for systems that are not in affine form.
- A second integral action that explicitly accounts for the effect of constant unmatched disturbances.
- A control law for underactuated mechanical systems with direct actuation subject to matched and unmatched disturbances.
- Simulation results for two motivating examples: an inertia wheel pendulum and a ball-on-beam system, both actuated by an electric DC motor and subject to matched and unmatched constant additive disturbances.

A brief overview of the Integral IDA-PBC methodology is provided in Section 2 for completeness. The controller design is then outlined in Section 3 for underactuated mechanical systems with actuator dynamics and for underactuated mechanical systems with direct actuation. The results of numerical simulations for both examples are presented in Section 4. Concluding remarks are contained in Section 5.

2 | PROBLEM FORMULATION

2.1 | Overview of Integral IDA-PBC

The dynamics of an underactuated mechanical system with n degrees-of-freedom (DOFs) and with the control input $u \in \mathbb{R}^m$ applied through the input matrix $G(q) : \mathbb{R}^n \mapsto \mathbb{R}^{n \times m}$, where $\text{rank}(G) = m < n$ for all $q \in \mathbb{R}^n$, subject to matched disturbances $\delta_0 \in \mathbb{R}^m$ and with physical damping $D = D^T \geq 0$, $D \in \mathbb{R}^{n \times n}$, is described in port-Hamiltonian form as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & -D \end{bmatrix} \begin{bmatrix} \nabla_q H \\ \nabla_p H \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} (u + v - \delta_0),$$

$$y = G^T \nabla_p H, \quad (1)$$

where $v \in \mathbb{R}^m$ is the auxiliary control input corresponding to the integral action. The mechanical energy $H(q, p) : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^+$

$$H(q, p) = \frac{1}{2}p^T M^{-1}p + \Omega, \quad (2)$$

is characterized by the inertia matrix $M(q) = M(q)^T > 0, M(q) : \mathbb{R}^n \mapsto \mathbb{R}^{n \times n}$, and the potential energy $\Omega(q) : \mathbb{R}^n \mapsto \mathbb{R}^+$. The system states are the position $q \in \mathbb{R}^n$ and the momenta $p = M\dot{q} \in \mathbb{R}^n$. The remaining terms in (1) are the identity matrix I , the vector of partial derivatives of H in q , $\nabla_q H$, and the vector of partial derivatives of H in p , $\nabla_p H$. The control aim is to stabilize the equilibrium $(q, p) = (q^*, 0)$. This is achieved in the absence of disturbances and physical damping (i.e., $\delta = 0, D = 0$) with $v = 0$ and the IDA-PBC control⁵

$$u = G^\dagger (\nabla_q H - M_d M^{-1} \nabla_q H_d + J_2 \nabla_p H_d) - k_v G^T \nabla_p H_d, \quad (3)$$

where $G^\dagger = (G^T G)^{-1} G^T$. The resulting closed-loop dynamics is

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & M^{-1} M_d \\ -M_d M^{-1} & J_2 - G k_v G^T \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix}, \quad (4)$$

where $H_d = \frac{1}{2}p^T M_d^{-1}p + \Omega_d$, $H_d(q, p) : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^+$. The design parameters in (4) are the potential energy $\Omega_d : \mathbb{R}^n \mapsto \mathbb{R}^+$, the inertia matrix $M_d = M_d^T > 0, M_d(q) : \mathbb{R}^n \mapsto \mathbb{R}^{n \times n}$, the matrix $J_2 = -J_2^T, J_2(q, p) : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^{n \times n}$, and the constant matrix $k_v = k_v^T > 0, k_v \in \mathbb{R}^{m \times m}$. To achieve the regulation goal, the potential energy Ω_d should admit a strict minimizer in q^* hence verifying the conditions $\nabla_q \Omega_d(q^*) = 0$ and $\nabla_q^2 \Omega_d(q^*) > 0$. In addition, M_d and Ω_d should verify for all $(q, p) \in \mathbb{R}^{2n}$ the partial differential equations (PDEs)

$$G^\perp (\nabla_q (p^T M^{-1}p) - M_d M^{-1} \nabla_q (p^T M_d^{-1}p) + 2J_2 M_d^{-1}p) = 0, \quad (5)$$

$$G^\perp (\nabla_q \Omega - M_d M^{-1} \nabla_q \Omega_d) = 0, \quad (6)$$

where $G^\perp$ is such that $G^\perp G = 0$ and $\text{rank}(G^\perp) = n - m$. Computing the time derivative of H_d along the trajectories of the closed-loop system (4) yields then

$$\dot{H}_d = -\nabla_p H_d^T G k_v G^T \nabla_p H_d \leq 0. \quad (7)$$

Thus the equilibrium $(q, p) = (q^*, 0)$ is asymptotically stable provided that the output $y = G^T \nabla_p H_d$ is detectable.⁵

If the system is subject to constant matched disturbances $\delta_0 \neq 0$, if the input matrix G and the matrix M_d are constant, and if M is independent of the unactuated coordinates, the auxiliary control v in its simplest form is designed as¹⁶

$$\begin{aligned} v &= -K_2 K_I K_2^{-T} G^T M^{-1} \nabla_q \Omega_d - k_v K_I \zeta, \\ \dot{\zeta} &= K_2^T G^T M^{-1} \nabla_q \Omega_d, \end{aligned} \quad (8)$$

where $K_I > 0$ is a tuning parameter, $K_2 = (G^T M^{-1} G)^{-1}$, and ζ is the integral state. The storage function W_d that characterizes the closed-loop system is defined as

$$W_d = \frac{1}{2}z_2^T M_d^{-1}z_2 + \Omega_d(z_1) + \frac{1}{2}(z_3 - \alpha)^T K_I (z_3 - \alpha), \quad (9)$$

where $z_1 = q$, $z_2 = p + G K_2 K_I (\zeta - \alpha)$, $z_3 = \zeta$, and $\alpha = \delta_0 (k_v K_I)^{-1}$. Computing the time derivative of (9) along the trajectories of the closed-loop system yields¹⁶

$$\dot{W}_d = -\nabla_{z_2} W_d^T G k_v G^T \nabla_{z_2} W_d \leq 0,$$

thus the equilibrium $(q, p, \zeta) = (q^*, 0, \alpha)$ is asymptotically stable provided that the output $y = G^T M_d^{-1} z_2$ is detectable. Further variations of the integral action (8) are detailed in Reference 16, which the interested reader is referred to. Subsequent versions of the Integral IDA-PBC formulations have omitted the coordinate transformation,¹⁹ and have extended the results to systems with nonconstant matrix M_d ,²⁵ and to nonconstant disturbances.²¹

2.2 | System model

This work considers a class of underactuated mechanical systems with first-order actuator dynamics and subject to constant additive disturbances, which expressed in port-Hamiltonian form yields

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ -I & 0 & F_{23} \\ 0 & -F_{23}^T & -F_{33} \end{bmatrix} \begin{bmatrix} \nabla_q W \\ \nabla_p W \\ \nabla_{x_1} W \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ G_0 u \end{bmatrix} - \begin{bmatrix} 0 \\ G\delta_0 + G^{\perp T} \delta_1 \\ 0 \end{bmatrix}. \quad (10)$$

The total energy is $W = H + \frac{1}{2} x_1^T K_1 x_1$, where x_1 is the actuator's state and $K_1 = K_1^T > 0$ is a constant matrix. The input matrix is $G_0 \in \mathbb{R}^{m \times m}$, F_{23} represents the coupling between the actuator and the mechanical subsystem, the matched disturbances are δ_0 , and the unmatched disturbances are δ_1 . The following assumptions define the class of systems considered in this work.

Assumption 1. The PDEs (5) and (6) for system (1) with direct actuation are solvable analytically, $q^* = \text{argmin}(\Omega_d)$, and the output $y = G^T M_d^{-1} p$ is detectable. In addition, the input matrix G is constant.

Assumption 2. The system dynamics (10) is characterized by $\text{rank}(F_{23}) = m$, $F_{33} = F_{33}^T > 0$, while G_0 is full-rank and invertible for all $(q, p, x_1) \in \mathbb{R}^{2n+m}$.

Assumption 3. All model parameters are exactly known and all system states are measurable. The matched disturbances $\delta_0 \in \mathbb{R}^m$ and the unmatched disturbances $\delta_1 \in \mathbb{R}^{n-m}$ are unknown but constant and bounded (i.e., with unknown bound). The prescribed equilibrium q^* is assignable, that is $G^\perp (\nabla_q \Omega + G^{\perp T} \delta_1) = 0$ at $q = q^*$.

3 | MAIN RESULT

The controller design is initially outlined for underactuated mechanical systems with first-order actuator dynamics and subsequently for underactuated mechanical systems with direct actuation.

3.1 | Systems with actuator dynamics

The control law is designed for system (10) to achieve the regulation goal $q = q^*$ following a similar procedure to Reference 12 but including two independent integral actions denoted by ζ_0 and ζ_1 . The closed-loop dynamics is thus

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{x}_1 \\ \dot{\zeta}_0 \\ \dot{\zeta}_1 \end{bmatrix} = \begin{bmatrix} 0 & S_{12} & S_{13} & S_{14} & S_{15} \\ -S_{12}^T & -S_{22} & S_{23} & S_{24} & S_{25} \\ -S_{13}^T & -S_{23}^T & -S_{33} & S_{34} & S_{35} \\ -S_{14}^T & -S_{24}^T & -S_{34}^T & -S_{44} & S_{45} \\ -S_{15}^T & -S_{25}^T & -S_{35}^T & -S_{45}^T & -S_{55} \end{bmatrix} \begin{bmatrix} \nabla_q W_d \\ \nabla_p W_d \\ \nabla_{x_1} W_d \\ \nabla_{\zeta_0} W_d \\ \nabla_{\zeta_1} W_d \end{bmatrix}. \quad (11)$$

The storage function W_d is defined as

$$W_d = H_d + \frac{1}{2} \zeta^T \zeta + \frac{1}{2} (\zeta_0 - G^T p - \alpha)^T K_{I_1} (\zeta_0 - G^T p - \alpha) + \frac{1}{2} (\zeta_1 - G^\perp p - \beta)^T K_{I_2} (\zeta_1 - G^\perp p - \beta) + K_0, \quad (12)$$

where $H_d = \Omega_d + \Lambda^T(q - q^*) + \frac{1}{2}p^T M_d^{-1}p$, with the constant unknown terms $\alpha = \delta_0(K_{I_1}(k_v G^T G + \Gamma_0))^{-1}$ and $\beta = \delta_1(K_{I_2}\Gamma_1)^{-1}$, and the positive tuning parameters $K_{I_1}, K_{I_2}, K_0, \Gamma_0, \Gamma_1$. In particular, H_d can be interpreted as an extended Hamiltonian with the vector Λ representing the closed-loop nonconservative forces,¹⁷ which is defined such that

$$G^\perp \left(\Gamma_1 G^{\perp T} K_{I_2} (\zeta_1 - G^\perp p) + G^{\perp T} \Gamma_1 \nabla_{\zeta_1} (\Lambda^T(q - q^*)) - M_d M^{-1} \nabla_q (\Lambda^T(q - q^*)) \right) = 0, \quad (13a)$$

$$\nabla_p (\Lambda^T(q - q^*)) + G^{\perp T} \nabla_{\zeta_1} (\Lambda^T(q - q^*)) = 0, \quad (13b)$$

$$\nabla_q \Omega_d + \nabla_q (\Lambda^T(q - q^*)) = 0 \quad q = q^*, \quad (13c)$$

$$\nabla_q^2 \Omega_d + \nabla_q^2 (\Lambda^T(q - q^*)) > 0 \quad q = q^*, \quad (13d)$$

where q^* is an assignable equilibrium of system (10) (see Assumption 3). Note that, since ζ_1 is only contained in the first term in parenthesis of (13a), that is $\eta = K_{I_2}(\zeta_1 - G^\perp p)$ for which $\nabla_p \eta = -G^{\perp T} \nabla_{\zeta_1} \eta$, it follows that Λ can be chosen such that $\nabla_p (\Lambda^T(q - q^*)) + G^{\perp T} \nabla_{\zeta_1} (\Lambda^T(q - q^*)) = 0$ (i.e., thus verifying (13b)). Note also that (13c) and (13d) ensure that $q = q^*$ is a strict minimizer of H_d .²⁶ The remaining term ς is defined as

$$\begin{aligned} \varsigma = & G^\dagger \left(-\nabla_q \left(\Omega + \frac{1}{2}p^T M^{-1}p \right) + M_d M^{-1} \nabla_q \left(\Omega_d + \Lambda^T(q - q^*) + \frac{1}{2}p^T M_d^{-1}p \right) + \mathcal{F}_{23} K_1 x_1 \right) \\ & + G^\dagger (G k_v G^T - J_2) M_d^{-1} p + k_v G^T \nabla_p (\Lambda^T(q - q^*)) - K_{I_1} (k_v G^T G + \Gamma_0) (\zeta_0 - G^T p). \end{aligned} \quad (14)$$

The terms S_{ij} are defined so that the open-loop dynamics (10) matches the extended closed-loop dynamics (11), that is

$$\begin{aligned} S_{12} &= M^{-1} M_d, \quad S_{13} = -(M^{-1} M_d \nabla_p \varsigma + M^{-1} M_d G \nabla_{\zeta_0} \varsigma + M^{-1} M_d G^{\perp T} \nabla_{\zeta_1} \varsigma) (\nabla_{x_1} \varsigma)^{-1}, \quad S_{33} = k_i, \\ S_{14} &= M^{-1} M_d G, \quad S_{15} = M^{-1} M_d G^{\perp T}, \quad S_{22} = G k_v G^T - J_2, \quad S_{24} = G \Gamma_0 + J_2 G, \quad S_{25} = G^{\perp T} \Gamma_1 + J_2 G^{\perp T}, \\ S_{23} &= G \left(1 + G^\dagger M^{-1} M_d \nabla_q \varsigma + (k_v G^T - G^\dagger J_2) \nabla_p \varsigma - (\Gamma_0 + G^\dagger J_2 G) \nabla_{\zeta_0} \varsigma - G^\dagger J_2 G^{\perp T} \nabla_{\zeta_1} \varsigma \right) (\nabla_{x_1} \varsigma)^{-1} \\ &+ G^{\perp T} \left(G^\otimes M^{-1} M_d \nabla_q \varsigma - G^\otimes J_2 \nabla_p \varsigma - G^\otimes J_2 G \nabla_{\zeta_0} \varsigma - (\Gamma_1 + G^\otimes J_2 G^{\perp T}) \nabla_{\zeta_1} \varsigma \right) (\nabla_{x_1} \varsigma)^{-1}. \\ S_{34} &= -S_{23}^T G, \quad S_{35} = -S_{23}^T G^{\perp T}, \quad S_{44} = \Gamma_0 G^T G - G^T J_2 G, \quad S_{45} = G^T J_2 G^{\perp T}, \quad S_{55} = \Gamma_1 G^\perp G^{\perp T} - G^\perp J_2 G^{\perp T}, \end{aligned} \quad (15)$$

where $G^\otimes = (G^\perp G^{\perp T})^{-1} G^\perp$. The control input is thus

$$\begin{aligned} u = & G_0^{-1} \left(\mathcal{F}_{33} K_1 x_1 + \mathcal{F}_{23}^T \nabla_p H - S_{13}^T \left(\nabla_q \left(\Omega_d + \Lambda^T(q - q^*) + \frac{1}{2}p^T M_d^{-1}p \right) + \nabla_q \varsigma^T \varsigma \right) - S_{23}^T (M_d^{-1} p + \nabla_p \varsigma^T \varsigma) \right) \\ & - G_0^{-1} \left(k_i (\nabla_{x_1} \varsigma^T \varsigma) + S_{23}^T G (\nabla_{\zeta_0} \varsigma^T \varsigma) + S_{23}^T G^{\perp T} \nabla_{\zeta_1} \varsigma^T \varsigma \right), \end{aligned} \quad (16)$$

where S_{13} and S_{23} are given in (15), ς is given in (14), and the integral states ζ_0 and ζ_1 are defined by the update laws

$$\dot{\zeta}_0 = -(M^{-1} M_d G)^T \left(\nabla_q \left(\Omega_d + \Lambda^T(q - q^*) + \frac{1}{2}p^T M_d^{-1}p \right) + \nabla_q \varsigma^T \varsigma \right) - (G \Gamma_0 + J_2 G)^T (M_d^{-1} p + \nabla_p \varsigma^T \varsigma) \quad (17a)$$

$$\begin{aligned} & - \Gamma_0 G^T \nabla_p (\Lambda^T(q - q^*)) + G^T S_{23} (\nabla_{x_1} \varsigma^T \varsigma) - (\Gamma_0 G^T G - G^T J_2 G) (\nabla_{\zeta_0} \varsigma^T \varsigma) + G^T J_2 G^{\perp T} (\nabla_{\zeta_1} \varsigma^T \varsigma), \\ \dot{\zeta}_1 = & - \left(M^{-1} M_d G^{\perp T} \right)^T \left(\nabla_q \left(\Omega_d + \Lambda^T(q - q^*) + \frac{1}{2}p^T M_d^{-1}p \right) + \nabla_q \varsigma^T \varsigma \right) + G^\perp S_{23} (\nabla_{x_1} \varsigma^T \varsigma) \\ & - \left(G^{\perp T} \Gamma_1 + J_2 G^{\perp T} \right)^T (M_d^{-1} p + \nabla_p \varsigma^T \varsigma) + G^\perp J_2 G (\nabla_{\zeta_0} \varsigma^T \varsigma) - (\Gamma_1 G^\perp G^{\perp T} - G^\perp J_2 G^{\perp T}) (\nabla_{\zeta_1} \varsigma^T \varsigma). \end{aligned} \quad (17b)$$

Proposition 1. The system (10) with Assumptions 1–3 in closed-loop with the control law (16) and the integral actions (17) yields (11) with the parameters (14) and (15), provided that the PDEs (13a) and (13b) are verified.

Proof. Computing the partial derivatives of W_d from (12) yields

$$\begin{aligned}\nabla_q W_d &= \nabla_q \left(\Omega_d + \frac{1}{2} p^T M_d^{-1} p + \Lambda^T (q - q^*) \right) + \nabla_q \zeta^T \zeta, \\ \nabla_p W_d &= M_d^{-1} p + \nabla_p \zeta^T \zeta + \nabla_p \left(\Lambda^T (q - q^*) \right) - K_{I_1} G (\zeta_0 - G^T p - \alpha) - K_{I_2} G^{\perp T} (\zeta_1 - G^\perp p - \beta), \\ \nabla_{x_1} W_d &= \nabla_{x_1} \zeta^T \zeta, \\ \nabla_{\zeta_0} W_d &= K_{I_1} (\zeta_0 - G^T p - \alpha) + \nabla_{\zeta_0} \zeta^T \zeta, \\ \nabla_{\zeta_1} W_d &= K_{I_2} (\zeta_1 - G^\perp p - \beta) + \nabla_{\zeta_1} \left(\Lambda^T (q - q^*) \right) + \nabla_{\zeta_1} \zeta^T \zeta.\end{aligned}\quad (18)$$

Equating the corresponding rows of (10) and of (11) yields the matching equations

$$M^{-1} p = S_{12} \nabla_p W_d + S_{13} \nabla_{x_1} W_d + S_{14} \nabla_{\zeta_0} W_d + S_{15} \nabla_{\zeta_1} W_d, \quad (19a)$$

$$-\nabla_q H + \mathcal{F}_{23} K_{I_1} x_1 - G \delta_0 - G^{\perp T} \delta_1 = -S_{12}^T \nabla_q W_d - S_{22} \nabla_p W_d + S_{23} \nabla_{x_1} W_d + S_{24} \nabla_{\zeta_0} W_d + S_{25} \nabla_{\zeta_1} W_d, \quad (19b)$$

$$-\mathcal{F}_{23}^T \nabla_p H - \mathcal{F}_{33} K_{I_1} x_1 + G_0 u = -S_{13}^T \nabla_q W_d - S_{23}^T \nabla_p W_d - S_{33} \nabla_{x_1} W_d + S_{34} \nabla_{\zeta_0} W_d + S_{35} \nabla_{\zeta_1} W_d, \quad (19c)$$

$$\dot{\zeta}_0 = -S_{14}^T \nabla_q W_d - S_{24}^T \nabla_p W_d - S_{34}^T \nabla_{x_1} W_d - S_{44} \nabla_{\zeta_0} W_d + S_{45} \nabla_{\zeta_1} W_d, \quad (19d)$$

$$\dot{\zeta}_1 = -S_{15}^T \nabla_q W_d - S_{25}^T \nabla_p W_d - S_{35}^T \nabla_{x_1} W_d - S_{45}^T \nabla_{\zeta_0} W_d - S_{55} \nabla_{\zeta_1} W_d. \quad (19e)$$

Step 1. Substituting $S_{12}, S_{13}, S_{14},$ and S_{15} from (15) and the partial derivatives of W_d from (18) into (19a) yields

$$\begin{aligned}M^{-1} p &= M^{-1} M_d \left(M_d^{-1} p + \nabla_p \zeta^T \zeta + \nabla_p \left(\Lambda^T (q - q^*) \right) - K_{I_1} G (\zeta_0 - G^T p - \alpha) - K_{I_2} G^{\perp T} (\zeta_1 - G^\perp p - \beta) \right) \\ &\quad - \left(M^{-1} M_d \nabla_p \zeta + M^{-1} M_d G \nabla_{\zeta_0} \zeta + M^{-1} M_d G^{\perp T} \nabla_{\zeta_1} \zeta \right) (\nabla_{x_1} \zeta)^{-1} (\nabla_{x_1} \zeta)^T \zeta \\ &\quad + M^{-1} M_d G \left(K_{I_1} (\zeta_0 - G^T p - \alpha) + \nabla_{\zeta_0} \zeta^T \zeta \right) + M^{-1} M_d G^\perp \left(K_{I_2} (\zeta_1 - G^\perp p - \beta) + \nabla_{\zeta_1} \left(\Lambda^T (q - q^*) \right) + \nabla_{\zeta_1} \zeta^T \zeta \right).\end{aligned}$$

Refactoring the previous expression yields (13b) (i.e., premultiplied by $M^{-1} M_d$, where $M = M^T > 0$ and $M_d = M_d^T > 0$), which is verified by hypothesis.

Step 2. Substituting $S_{12}, S_{22}, S_{23}, S_{24},$ and S_{25} from (15) and the partial derivatives of W_d from (18) into (19b) yields

$$\begin{aligned}-\nabla_q \left(\Omega + \frac{1}{2} p^T M^{-1} p \right) + \mathcal{F}_{23} K_{I_1} x_1 - G \delta_0 - G^{\perp T} \delta_1 &= -M_d M^{-1} \left(\nabla_q \left(\Omega_d + \frac{1}{2} p^T M_d^{-1} p + \Lambda^T (q - q^*) \right) + \nabla_q \zeta^T \zeta \right) \\ &\quad - (G k_v G^T - J_2) \left(M_d^{-1} p + \nabla_p \zeta^T \zeta + \nabla_p \left(\Lambda^T (q - q^*) \right) - K_{I_1} G (\zeta_0 - G^T p - \alpha) - K_{I_2} G^{\perp T} (\zeta_1 - G^\perp p - \beta) \right) \\ &\quad + G \left(1 + G^\dagger M^{-1} M_d \nabla_q \zeta + (k_v G^T - G^\dagger J_2) \nabla_p \zeta - (\Gamma_0 + G^\dagger J_2 G) \nabla_{\zeta_0} \zeta - G^\dagger J_2 G^{\perp T} \nabla_{\zeta_1} \zeta \right) (\nabla_{x_1} \zeta)^{-1} (\nabla_{x_1} \zeta)^T \zeta \\ &\quad + G^{\perp T} \left(G^\otimes M^{-1} M_d \nabla_q \zeta - G^\otimes J_2 \nabla_p \zeta - G^\otimes J_2 G \nabla_{\zeta_0} \zeta - (\Gamma_1 + G^\otimes J_2 G^{\perp T}) \nabla_{\zeta_1} \zeta \right) (\nabla_{x_1} \zeta)^{-1} (\nabla_{x_1} \zeta)^T \zeta \\ &\quad + (G \Gamma_0 + J_2 G) \left(K_{I_1} (\zeta_0 - G^T p - \alpha) + \nabla_{\zeta_0} \zeta^T \zeta \right) \\ &\quad + (G^{\perp T} \Gamma_1 + J_2 G^{\perp T}) \left(K_{I_2} (\zeta_1 - G^\perp p - \beta) + \nabla_{\zeta_1} \left(\Lambda^T (q - q^*) \right) + \nabla_{\zeta_1} \zeta^T \zeta \right).\end{aligned}$$

Substituting α and β in the former expression cancels the disturbances δ_0 and δ_1 , and refactoring terms yields

$$\begin{aligned}-\nabla_q \left(\Omega + \frac{1}{2} p^T M^{-1} p \right) + M_d M^{-1} \nabla_q \left(\Omega_d + \Lambda^T (q - q^*) + \frac{1}{2} p^T M_d^{-1} p \right) &+ (G k_v G^T - J_2) \left(M_d^{-1} p + \nabla_p \left(\Lambda^T (q - q^*) \right) \right) \\ &+ \mathcal{F}_{23} K_{I_1} x_1 - (G^{\perp T} \Gamma_1 + J_2 G^{\perp T}) \nabla_{\zeta_1} \left(\Lambda^T (q - q^*) \right) - G K_{I_1} (k_v G^T G + \Gamma_0) (\zeta_0 - G^T p) - \Gamma_1 G^{\perp T} K_{I_2} (\zeta_1 - G^\perp p) \\ &= -M_d M^{-1} \left(\nabla_q \zeta^T \zeta \right) - (G k_v G^T - J_2) \left(\nabla_p \zeta^T \zeta \right) + (G \Gamma_0 + J_2 G) \left(\nabla_{\zeta_0} \zeta^T \zeta \right) + (G^{\perp T} \Gamma_1 + J_2 G^{\perp T}) \left(\nabla_{\zeta_1} \zeta^T \zeta \right) \\ &\quad + G \left(1 + G^\dagger M^{-1} M_d \nabla_q \zeta + (k_v G^T - G^\dagger J_2) \nabla_p \zeta - (\Gamma_0 + G^\dagger J_2 G) \nabla_{\zeta_0} \zeta - G^\dagger J_2 G^{\perp T} \nabla_{\zeta_1} \zeta \right) (\nabla_{x_1} \zeta)^{-1} \nabla_{x_1} \zeta^T \zeta \\ &\quad + G^{\perp T} \left(G^\otimes M^{-1} M_d \nabla_q \zeta - G^\otimes J_2 \nabla_p \zeta - G^\otimes J_2 G \nabla_{\zeta_0} \zeta - (\Gamma_1 + G^\otimes J_2 G^{\perp T}) \nabla_{\zeta_1} \zeta \right) (\nabla_{x_1} \zeta)^{-1} \nabla_{x_1} \zeta^T \zeta.\end{aligned}$$

Multiplying the former expression by $G^\dagger$ yields the PDE (13b) (i.e., premultiplied by $G^\dagger J_2$) which is verified by hypothesis: the left side of the equal is the sum of ζ and (13b), while S_{23} cancels all terms in ζ . Multiplying it instead by $G^\perp$ yields the sum of the PDEs (5), (6), (13a), and (13b) (i.e., premultiplied by $G^\perp J_2$), which are all verified by hypothesis.

Step 3. Substituting S_{34} and S_{35} from (15) and the partial derivatives of W_d from (18) into (19c) yields

$$\begin{aligned} -\mathcal{F}_{23}^T \nabla_p H - \mathcal{F}_{33} K_1 x_1 + G_0 u = & -S_{13}^T \left(\nabla_q \left(\Omega_d + \Lambda^T(q - q^*) + \frac{1}{2} p^T M_d^{-1} p \right) + \nabla_q \zeta^T \zeta \right) \\ & - S_{23}^T (M_d^{-1} p + \nabla_p \zeta^T \zeta + \nabla_p (\Lambda^T(q - q^*)) - K_{I_1} G(\zeta_0 - G^T p - \alpha) - K_{I_2} G^{\perp T} (\zeta_1 - G^\perp p - \beta)) \\ & - k_i \nabla_{x_1} \zeta^T \zeta - S_{23}^T G (K_{I_1} (\zeta_0 - G^T p - \alpha) + \nabla_{\zeta_0} \zeta^T \zeta) - S_{23}^T G^{\perp T} (K_{I_2} (\zeta_1 - G^\perp p - \beta) + \nabla_{\zeta_1} (\Lambda^T(q - q^*)) + \nabla_{\zeta_1} \zeta^T \zeta). \end{aligned}$$

Refactoring the former expression cancels the unknown terms α and β , and substituting the control input (16) yields the PDE (13b) (i.e., premultiplied by S_{23}^T), which is verified by hypothesis.

Step 4. Substituting S_{14} , S_{24} , S_{34} , S_{44} and S_{45} from (15) and the partial derivatives of W_d from (18) into (19d) yields

$$\begin{aligned} \dot{\zeta}_0 = & -(M^{-1} M_d G)^T \left(\nabla_q \left(\Omega_d + \Lambda^T(q - q^*) + \frac{1}{2} p^T M_d^{-1} p \right) + \nabla_q \zeta^T \zeta \right) \\ & - (G \Gamma_0 + J_2 G)^T (M_d^{-1} p + \nabla_p \zeta^T \zeta + \nabla_p (\Lambda^T(q - q^*)) - K_{I_1} G(\zeta_0 - G^T p - \alpha) - K_{I_2} G^{\perp T} (\zeta_1 - G^\perp p - \beta)) \\ & + G^T S_{23} \nabla_{x_1} \zeta^T \zeta - (\Gamma_0 G^T G - G^T J_2 G) (K_{I_1} (\zeta_0 - G^T p - \alpha) + \nabla_{\zeta_0} \zeta^T \zeta) \\ & + G^T J_2 G^{\perp T} (K_{I_2} (\zeta_1 - G^\perp p - \beta) + \nabla_{\zeta_1} (\Lambda^T(q - q^*)) + \nabla_{\zeta_1} \zeta^T \zeta). \end{aligned}$$

Refactoring the former expression cancels the terms α and β , yielding the sum of the update law (17a) and the PDE (13b) (i.e., premultiplied by $G^T J_2$) which is verified by hypothesis.

Step 5. Substituting S_{15} , S_{25} , S_{35} , S_{45} , and S_{55} from (15) and the partial derivatives of W_d from (18) into (19e) yields

$$\begin{aligned} \dot{\zeta}_1 = & -(M^{-1} M_d G^{\perp T})^T \left(\nabla_q \left(\Omega_d + \Lambda^T(q - q^*) + \frac{1}{2} p^T M_d^{-1} p \right) + \nabla_q \zeta^T \zeta \right) \\ & - (G^{\perp T} \Gamma_1 + J_2 G^{\perp T})^T (M_d^{-1} p + \nabla_p \zeta^T \zeta + \nabla_p (\Lambda^T(q - q^*)) - K_{I_1} G(\zeta_0 - G^T p - \alpha) - K_{I_2} G^{\perp T} (\zeta_1 - G^\perp p - \beta)) \\ & + G^\perp S_{23} \nabla_{x_1} \zeta^T \zeta + G^\perp J_2 G (K_{I_1} (\zeta_0 - G^T p - \alpha) + \nabla_{\zeta_0} \zeta^T \zeta) \\ & - (\Gamma_1 G^\perp G^{\perp T} - G^\perp J_2 G^{\perp T}) (K_{I_2} (\zeta_1 - G^\perp p - \beta) + \nabla_{\zeta_1} (\Lambda^T(q - q^*)) + \nabla_{\zeta_1} \zeta^T \zeta). \end{aligned}$$

Refactoring the former expression cancels the terms α and β yielding the sum of the update law (17b) and the PDE (13b) (i.e., premultiplied by $(\Gamma_1 G^\perp - G^\perp J_2)$) which is verified by hypothesis. ■

Proposition 2. Consider system (10) with Assumptions 1–3 in closed-loop with the control law (16) where the integral actions are computed as in (17). Then the equilibrium point $(q, p, x_1, \zeta_0, \zeta_1) = (q^*, 0, x_1^*, \alpha, \beta)$ is asymptotically stable for all positive $k_v, k_i, K_{I_1}, K_{I_2}, \Gamma_0, \Gamma_1$, provided that $y = G^T M_d^{-1} p$ is detectable.

Proof. It follows from Assumption 3 that there exist a constant $K_0 > 0$ such that W_d in (12) is positive definite in a nonempty open set containing q^* . Recalling that $J_2^T = -J_2$ by design, it follows that $G^T J_2 G$ and $G^\perp J_2 G^{\perp T}$ are also skew symmetric. Computing the time derivative of (12) along the trajectories of the closed-loop system (11) yields then

$$\dot{W}_d = -\nabla_p W_d^T G k_v G^T \nabla_p W_d - \nabla_{x_1} W_d^T k_i \nabla_{x_1} W_d - \nabla_{\zeta_0} W_d^T \Gamma_0 G^T G \nabla_{\zeta_0} W_d - \nabla_{\zeta_1} W_d^T \Gamma_1 G^\perp G^{\perp T} \nabla_{\zeta_1} W_d \leq 0, \quad (20)$$

which is verified for all $k_v > 0, k_i > 0$, and $\Gamma_0 > 0, \Gamma_1 > 0$ hence all states are bounded and the equilibrium is stable. In particular, $\nabla_{x_1} W_d = \nabla_{x_1} \zeta^T \zeta$ where it follows from (14) that $\nabla_{x_1} \zeta = G^\dagger \mathcal{F}_{23} K_1 \neq 0$ according to Assumption 2, while taking Γ_0 and Γ_1 as diagonal matrices results in $\Gamma_0 G^T G$ and $\Gamma_1 G^\perp G^{\perp T}$ being symmetric. Recalling the expression of the partial derivatives of W_d from (18) it follows that $\zeta, (\zeta_0 - G^T p - \alpha) \in \mathcal{L}^2 \cap \mathcal{L}^\infty$ and $G^T \nabla_p W_d, G^{\perp T} \nabla_{\zeta_1} W_d \in \mathcal{L}^2$, while $p, (\zeta_1 - G^\perp p - \beta), \Lambda^T(q - q^*) \in \mathcal{L}^\infty$. Consequently

$K_{I_2}(\zeta_1 - G^\perp p - \beta) + \nabla_{\zeta_1}(\Lambda^T(q - q^*)) \in \mathcal{L}^2$ and $G^T M_d^{-1} p + G^T \nabla_p(\Lambda^T(q - q^*)) - G^T K_{I_2} G^{\perp T}(\zeta_1 - G^\perp p - \beta) \in \mathcal{L}^2$. Combining the former expressions and substituting the PDE (13b) yields thus $y = G^T M_d^{-1} p \in \mathcal{L}^2 \cap \mathcal{L}^\infty$. In addition, it follows from (11) that $\dot{p}, \dot{\zeta}, \dot{x}_1, \dot{\zeta}_0, \dot{\zeta}_1 \in \mathcal{L}^\infty$, while α and β are bounded according to Assumption 3. Therefore $y, \zeta, (\zeta_0 - G^T p - \alpha)$ and $K_{I_2}(\zeta_1 - G^\perp p - \beta) + \nabla_{\zeta_1}(\Lambda^T(q - q^*))$ converge to zero asymptotically according to Barbalat's lemma.²⁷ Employing a similar argument to Reference 5, it follows that the equilibrium is asymptotically stable provided that the output $y = G^T M_d^{-1} p$ is detectable (i.e., $y \rightarrow 0 \Rightarrow p \rightarrow 0$). In particular, computing $\dot{p}$ from (11) at $\dot{p} = p = \zeta = 0$ and at $\zeta_0 = \alpha$ and $\zeta_1 = \beta$ yields

$$M_d M^{-1} \nabla_q (\Omega_d + \Lambda^T(q - q^*)) + (G k_v G^T - J_2) \nabla_p (\Lambda^T(q - q^*)) - (G^{\perp T} \Gamma_1 + J_2 G^{\perp T}) \nabla_{\zeta_1} (\Lambda^T(q - q^*)) = 0,$$

where the last two terms vanish at $q = q^*$, thus yielding (13c) which is verified by hypothesis (i.e., $q = q^*$ is an extremum of W_d). In addition, it follows from (13d) that $q = q^*$ is a strict minimizer of W_d . Computing $\zeta = 0$ from (14) at $(q, p, x_1, \zeta_0, \zeta_1) = (q^*, 0, x_1^*, \alpha, \beta)$ yields $G^\dagger (-\nabla_q \Omega + F_{23} K_1 x_1^*) = 0$ which defines x_1^* . Thus the equilibrium point $(q, p, x_1, \zeta_0, \zeta_1) = (q^*, 0, x_1^*, \alpha, \beta)$ is asymptotically stable, concluding the proof. ■

Corollary 1. *If the system (10) has constant inertia matrix, the control law (16) and the integral actions (17) yield the closed-loop (11) where $J_2 = 0$, that is*

$$\begin{aligned} S_{22} &= G k_v G^T, \quad S_{24} = G \Gamma_0, \quad S_{25} = G^{\perp T} \Gamma_1, \quad S_{44} = \Gamma_0 G^T G, \quad S_{45} = 0, \quad S_{55} = \Gamma_1 G^\perp G^{\perp T}, \\ S_{23} &= G (1 + G^\dagger M_d M^{-1} \nabla_q \zeta + k_v G^T \nabla_p \zeta - \Gamma_0 \nabla_{\zeta_0} \zeta) (\nabla_{x_1} \zeta)^{-1} + G^{\perp T} (G^\otimes M_d M^{-1} \nabla_q \zeta - \Gamma_1 \nabla_{\zeta_1} \zeta) (\nabla_{x_1} \zeta)^{-1}. \end{aligned}$$

Proof. If M is constant, the PDEs (5) and (6) are solvable with a constant M_d and with $J_2 = 0$.⁵ Substituting $J_2 = 0$ in (15), (16), and (17) completes the proof. ■

Remark 1. The first integral action ζ_0 fully compensates the effect of the constant matched disturbances δ_0 even though the system (10) is not in affine form. Conversely, the presence of unmatched disturbances affects the system dynamics on the unactuated states, thus it cannot be fully compensated by the control action. In spite of this limitation, which is intrinsic to underactuated systems, it has been shown in References 26 and 17 that explicitly accounting for the unmatched disturbances in the control law can improve performance. To this end, the integral action ζ_1 computed with (17b) cancels the constant unmatched disturbances δ_1 (see Step 2 in Proposition 1). This is a key difference from prior works,^{8,24} where unmatched disturbances were accounted for in the stability analysis (i.e., by employing the input-to-state stability theory) but not in the design of the integral action. Differently from References 17 and 26 which employed the Immersion and Invariance methodology²⁸ to estimate the disturbances (i.e., from the open-loop dynamics), the proposed controller (16) employs the integral actions ζ_0 and ζ_1 thus resulting in the extended closed-loop dynamics (11) which is port-Hamiltonian. Consequently, the physical interpretation of the control action in terms of system energy is preserved. In addition, the stability conditions resulting from Proposition 2 (i.e., $k_v > 0, k_i > 0, K_{I_1} > 0, K_{I_2} > 0, \Gamma_0 > 0, \Gamma_1 > 0$) are less stringent than those in Reference 17 (i.e., $\alpha(G k_v G^T + D M^{-1} M_d) > I/4$, with $\alpha = \Gamma_0 = \Gamma_1$ and D representing the physical damping). On the other hand, the integral action ζ_1 introduces the additional PDEs (13a) and (13b), thus representing a complication compared to Reference 17. In summary, the solution of the PDEs (13a) and (13b) is system specific (see Section 4), similarly to the solution of the PDEs (5) and (6) that characterize the IDA-PBC methodology. The solvability of PDEs within IDA-PBC is a challenging task which has motivated a number of works including a method to simplify the PDEs for mechanical systems,²⁹ the study of kinetic energy shaping for mechanical systems,³⁰ the algebraic solutions of the PDEs in Reference 6, and the numerical solution of the PDEs with reinforcement learning in Reference 31. The study of this aspect will be part of future work.

Remark 2. Extending the controller design to systems with physical damping $D \neq 0$ requires modifying the term S_{22} such that $S_{22} = G k_v G^T - J_2 + D M^{-1} M_d$ (see Reference 32 for details), which in turn affects the remaining terms S_{ij} with $i, j \geq 2$. Extracting the terms dependent on δ_0, δ_1 and α, β from (19b) yields then

$$\begin{aligned} G \delta_0 &= ((G k_v G^T - J_2 + D M^{-1} M_d) G + S_{24}) K_{I_1} \alpha, \\ G^{\perp T} \delta_1 &= ((-J_2 + D M^{-1} M_d) G^{\perp T} + S_{25}) K_{I_2} \beta. \end{aligned}$$

Thus, defining $S_{24} = G\Gamma_0 + (J_2 - DM^{-1}M_d)G$ and $S_{25} = G^{\perp T}\Gamma_1 + (J_2 - DM^{-1}M_d)G^{\perp T}$ preserves the expression of α and β . The remaining terms in (15) preserve their structure, while J_2 is replaced by $J_2 - DM^{-1}M_d$. For instance, S_{23} yields

$$S_{23} = G \left(1 + G^\dagger S_{12} \nabla_q \zeta + G^\dagger S_{22} \nabla_p \zeta - (\Gamma_0 + G^\dagger (J_2 - DM^{-1}M_d)G) \nabla_{\zeta_0} \zeta - G^\dagger (J_2 - DM^{-1}M_d)G^{\perp T} \nabla_{\zeta_1} \zeta \right) (\nabla_{x_1} \zeta)^{-1} \\ + G^{\perp T} \left(G^\otimes S_{12} \nabla_q \zeta + G^\otimes S_{22} \nabla_p \zeta - G^\otimes (J_2 - DM^{-1}M_d)G \nabla_{\zeta_0} \zeta - (\Gamma_1 + G^\otimes (J_2 - DM^{-1}M_d)G^{\perp T}) \nabla_{\zeta_1} \zeta \right) (\nabla_{x_1} \zeta)^{-1}.$$

Similarly, ζ is obtained from (14) by replacing J_2 with $J_2 - DM^{-1}M_d$. Thus, the control law is still given by (16), and the time derivatives of the integral states are obtained by replacing J_2 with $J_2 - DM^{-1}M_d$ in (17). In such case however, the closed-loop system (11) is port-Hamiltonian if and only if $S_{24}^T G^{\perp T} = -(S_{25}^T G)^T$, that is $G^T (J_2 - DM^{-1}M_d)^T G^{\perp T} = -G^T (J_2 - DM^{-1}M_d)G^{\perp T}$. Employing the matrix decomposition $DM^{-1}M_d = D_S + D_A$, where $D_S = \frac{1}{2} ((DM^{-1}M_d) + (DM^{-1}M_d)^T)$ is the symmetric part and $D_A = \frac{1}{2} ((DM^{-1}M_d) - (DM^{-1}M_d)^T)$ is the antisymmetric part, that is $D_S = D_S^T$ and $D_A = -D_A^T$, the former equation yields $G^T (J_2 - D_A - D_S)^T G^{\perp T} = -G^T (J_2 - D_A - D_S)G^{\perp T}$ and is verified by all J_2 and D_A , and by some D_S such that $G^T D_S^T G^{\perp T} = -G^T D_S G^{\perp T}$ (i.e., this condition is verified for the examples in Section 4 if D_S is diagonal). Finally, $S_{44} = S_{24}^T G$ and $S_{55} = S_{25}^T G^{\perp T}$ contain the damping terms, therefore the stability conditions in Proposition 2 change to $Gk_v G^T + D_S > 0$, $\Gamma_0 G^T G - G^T D_S G > 0$ and $\Gamma_1 G^{\perp} G^{\perp T} - G^{\perp} D_S G^{\perp T} > 0$, which might be more stringent depending on D_S . This point highlights the limits imposed by pervasive damping to the Integral IDA-PBC methodology (see Reference 20 for details).

Remark 3. Consider the hypothetical case in which (13a) does not admit a global analytical solution but is only verified locally at the equilibrium. In such case, the controller (16) with the integral actions (17) can still yield boundedness of the closed-loop system trajectories. To highlight this point, we assume that (13a) yields a residual $G^{\perp} \Delta(q)$ that vanishes at the equilibrium (i.e., $\Delta(q^*) = 0$), which is then accounted for in the Lyapunov derivative $\dot{W}_d$, that is

$$\dot{W}_d \leq -\nabla_p W_d^T Gk_v G^T \nabla_p W_d - \nabla_{x_1} W_d^T k_i \nabla_{x_1} W_d + |\nabla_p W_d^T| |G^{\perp T} \Delta| - \nabla_{\zeta_0} W_d^T \Gamma_0 G^T G \nabla_{\zeta_0} W_d - \nabla_{\zeta_1} W_d^T \Gamma_1 G^{\perp} G^{\perp T} \nabla_{\zeta_1} W_d.$$

Introducing the Young's inequality $\nabla_p W_d^T G^{\perp T} \Delta \leq \frac{1}{4} \nabla_p W_d^T \nabla_p W_d + \Delta^T G^{\perp} G^{\perp T} \Delta$ and refactoring terms yields then

$$\dot{W}_d \leq -\nabla_p W_d^T \left(Gk_v G^T - \frac{I}{4} \right) \nabla_p W_d - \nabla_{x_1} W_d^T k_i \nabla_{x_1} W_d - \nabla_{\zeta_0} W_d^T \Gamma_0 G^T G \nabla_{\zeta_0} W_d \\ - \nabla_{\zeta_1} W_d^T \Gamma_1 G^{\perp} G^{\perp T} \nabla_{\zeta_1} W_d + \Delta^T G^{\perp} G^{\perp T} \Delta.$$

It follows that $\dot{W}_d \leq 0$ for $k_i > 0$, $\Gamma_0 > 0$ and for some k_v, Γ_1 such that $(Gk_v G^T - I/4) > 0$ and that $\nabla_{\zeta_1} W_d^T \Gamma_1 G^{\perp} G^{\perp T} \nabla_{\zeta_1} W_d > \Delta^T G^{\perp} G^{\perp T} \Delta$. Then the trajectories of the closed-loop system are bounded provided that $\Delta \in \mathcal{L}^2 \cap \mathcal{L}^\infty$ (see Lemma 5.3 in Reference 33). The inequality $\nabla_{\zeta_1} W_d^T \Gamma_1 G^{\perp} G^{\perp T} \nabla_{\zeta_1} W_d > \Delta^T G^{\perp} G^{\perp T} \Delta$ is verified in a nonempty set containing the equilibrium by a sufficiently large $\Gamma_1 > 0$ since $\Delta(q^*) = 0$. Verifying the inequality $(Gk_v G^T - I/4) > 0$ is typically more challenging since $Gk_v G^T$ is rank deficient, but it is achievable if physical damping $D \neq 0$ is present, such that $\left(Gk_v G^T + D_S - \frac{I}{4} \right) > 0$, where D_S has been defined in Remark 2. In comparison, if the unmatched disturbances δ_1 are disregarded by setting $\Gamma_1 = 0$, the Lyapunov derivative $\dot{W}_d$ yields instead

$$\dot{W}_d \leq -\nabla_p W_d^T \left(Gk_v G^T - \frac{I}{4} \right) \nabla_p W_d - \nabla_{x_1} W_d^T k_i \nabla_{x_1} W_d - \nabla_{\zeta_0} W_d^T \Gamma_0 G^T G \nabla_{\zeta_0} W_d + \delta_1^T G^{\perp} G^{\perp T} \delta_1.$$

As a result, $\dot{W}_d \leq 0$ and the trajectories of the closed-loop system are bounded only if $\nabla_{x_1} W_d^T k_i \nabla_{x_1} W_d > \delta_1^T G^{\perp} G^{\perp T} \delta_1$, which implies a more stringent condition on k_i since $\delta_1 \neq 0$. In conclusion, even in case the PDE (13a) only admits a local solution at the equilibrium, the controller (16) with the integral actions (17) can still yield improved performance in the presence of unmatched disturbances (see Section 4.2).

3.2 | Systems with direct actuation

In case the system (1) is directly actuated, the closed-loop dynamics is defined in a similar way to (11), that is

$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{\zeta}_0 \\ \dot{\zeta}_1 \end{bmatrix} = \begin{bmatrix} 0 & S_{12} & S_{13} & S_{14} \\ -S_{12}^T & -S_{22} & S_{23} & S_{24} \\ -S_{13}^T & -S_{23}^T & -S_{33} & S_{34} \\ -S_{14}^T & -S_{24}^T & -S_{34}^T & -S_{44} \end{bmatrix} \begin{bmatrix} \nabla_q W_d \\ \nabla_p W_d \\ \nabla_{\zeta_0} W_d \\ \nabla_{\zeta_1} W_d \end{bmatrix}, \quad (21)$$

where the elements S_{ij} of the above matrix are given by

$$\begin{aligned} S_{12} &= M^{-1}M_d, \quad S_{13} = M^{-1}M_dG, \quad S_{14} = M^{-1}M_dG^{\perp T}, \quad S_{22} = Gk_vG^T - J_2, \quad S_{23} = G\Gamma_0 + J_2G, \\ S_{24} &= G^{\perp T}\Gamma_1 + J_2G^{\perp T}, \quad S_{33} = G^T\Gamma_0G - G^TJ_2G, \quad S_{34} = G^TJ_2G^{\perp T}, \quad S_{44} = G^{\perp}\Gamma_1G^{\perp T} - G^{\perp}J_2G^{\perp T}. \end{aligned} \quad (22)$$

The storage function W_d is defined as

$$W_d = H_d + \frac{1}{2}(\zeta_0 - G^T p - \alpha)^T K_{I_1} (\zeta_0 - G^T p - \alpha) + \frac{1}{2}(\zeta_1 - G^{\perp} p - \beta)^T K_{I_2} (\zeta_1 - G^{\perp} p - \beta) + K_0, \quad (23)$$

where $H_d = \Omega_d + \Lambda^T(q - q^*) + \frac{1}{2}p^T M_d^{-1}p$, with $\alpha = \delta_0(K_{I_1}(k_v G^T G + \Gamma_0))^{-1}$ and $\beta = \delta_1(K_{I_2}\Gamma_1)^{-1}$. Also in this case, Ω_d should solve the PDE (6), M_d should solve the PDE (5) with a corresponding J_2 , and Λ should verify the conditions (13). The control input is given by

$$\begin{aligned} u &= G^{\dagger} \left(\nabla_q H - M_d M^{-1} \nabla_q \left(\Omega_d + \Lambda^T(q - q^*) + \frac{1}{2}p^T M_d^{-1}p \right) - (Gk_vG^T - J_2)M_d^{-1}p \right) \\ &\quad + K_{I_1} (k_v G^T G + \Gamma_0) (\zeta_0 - G^T p). \end{aligned} \quad (24)$$

The integral states are defined by the update laws

$$\dot{\zeta}_0 = -G^T M_d M^{-1} \nabla_q \left(\Omega_d + \Lambda^T(q - q^*) + \frac{1}{2}p^T M_d^{-1}p \right) - \Gamma_0 G^T \nabla_p \left(\Lambda^T(q - q^*) \right) - (\Gamma_0 G^T - G^T J_2) M_d^{-1} p, \quad (25a)$$

$$\dot{\zeta}_1 = -G^{\perp} M_d M^{-1} \nabla_q \left(\Omega_d + \Lambda^T(q - q^*) + \frac{1}{2}p^T M_d^{-1}p \right) - (\Gamma_1 G^{\perp} - G^{\perp} J_2) M_d^{-1} p. \quad (25b)$$

Proposition 3. *The system (1) with Assumptions 1 and 3 in closed-loop with the control law (24) and the integral actions (25) yields (21) with the parameters (22), provided that the PDEs (13a) and (13b) are verified.*

The proof is similar to that of Proposition 1 and is given in Appendix A.

Proposition 4. *Consider the system (1) with Assumptions 1 and 3 in closed-loop with the control law (24) where the integral actions are computed as in (25). Then the equilibrium point $(q, p, \zeta_0, \zeta_1) = (q^*, 0, \alpha, \beta)$ is asymptotically stable for all positive $k_v, K_{I_1}, K_{I_2}, \Gamma_0, \Gamma_1$, provided that the output $y = G^T M_d^{-1}p$ is detectable.*

The proof is similar to that of Proposition 2 and is given in Appendix A.

4 | SIMULATION RESULTS

4.1 | Inertia wheel pendulum

The inertia wheel pendulum system²⁹ is modified here by including a DC motor for the actuation (see Figure 1). The system consists of an unactuated planar inverted pendulum with an actuated wheel attached to its end. The system has two degrees-of-freedom: the angular position of the pendulum q_1 and the angular position of the wheel q_2 . The system

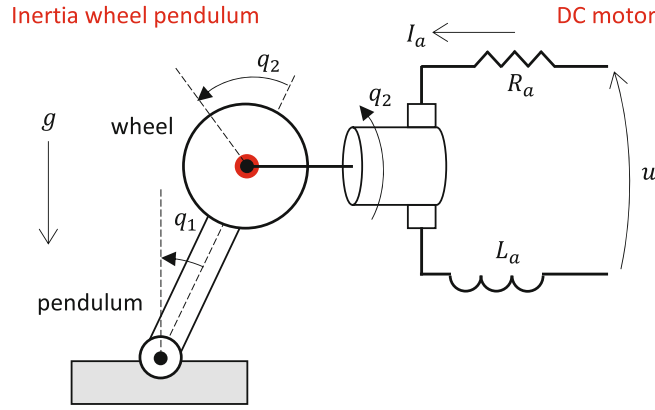


FIGURE 1 Schematic of the inertia wheel pendulum actuated by a DC motor, with key model parameters.

states are thus q and $p = M\dot{q}$. The dynamics of the mechanical subsystem in port-Hamiltonian form is given by (1) with total energy $H = \Omega + \frac{1}{2}p^T M^{-1}p$ where $\Omega = a_3(\cos(q_1) + 1)$, the input matrix is $G^T = [0 \ 1]$, while $M = \begin{bmatrix} a_1 + a_2 & a_2 \\ a_2 & a_2 \end{bmatrix}$ is the constant inertia matrix. The terms a_1, a_2, a_3 are constant parameters depending on the size of the pendulum and of the wheel. The control goal consists in reaching the prescribed position $(q_1, q_2) = (q_1^*, q_2^*)$, where $q_1^* = 0$ if and only if $\delta_1 = 0$. The IDA-PBC controller (3) for the system with direct actuation yields²⁴

$$u = \gamma_2 \sin(q_1) + k_p \gamma_3 (q_2 + \gamma_1 \epsilon (a_1 + a_2) q_1) - k_v \frac{(-m_2 p_1 + m_1 p_2)}{a_1 a_2 (m_1 m_3 - m_2^2)},$$

where $\gamma_1 = \frac{1}{a_2(m_2 - m_1)}$, $\gamma_2 = -a_3 \gamma_1 a_2 (m_2 - m_3)$, and $\gamma_3 = -\epsilon (a_1 + a_2) \gamma_1 a_2 (m_2 - m_3) - (-m_2 a_2 + m_3 (a_1 + a_2))$. The closed-loop dynamics of the mechanical subsystem is given by (4) where $\Omega_d = -a_3 \gamma_1 \cos(q_1) + \frac{1}{2} k_p (q_2 + \gamma_1 \epsilon (a_1 + a_2) q_1)^2$ and $M_d = a_1 a_2 \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix}$ with $m_2 = \frac{m_1 a_2}{a_1 + a_2} + \epsilon$. The tuning parameters are $k_p, k_v, m_1, m_3, \epsilon$. The complete system dynamics described with (10), including the DC motor driving the wheel, yields $F_{23} = \frac{K_e}{L_a} G$, $F_{33} = \frac{R_a}{L_a^2}$, and the total energy is $W = \Omega + \frac{1}{2}p^T M^{-1}p + \frac{1}{2}L_a I_a^2$. The system states are $q, p = M\dot{q}$, and the armature current $x_1 = I_a$, while K_e, L_a , and R_a are the torque constant of the motor, the inductance, and the armature resistance respectively. The inertia of the motor is assumed to be included in that of the wheel for simplicity. The control law is computed as in (16) and the integral states are computed as in (17). The assignable equilibrium in the presence of the constant unmatched disturbance δ_1 is $q_1^* = \arcsin(\delta_1/a_3)$, which is constant. The estimate of δ_1 computed with the integral action is $\tilde{\delta}_1 = \Gamma_1 K_{I_2} (\zeta_1 - p_1)$. The analytical solution of the PDEs (13a) and (13b) which verifies the minimizer conditions (13c) and (13d) is

$$\Lambda = \left[\frac{\Gamma_1 K_{I_2} \left(\zeta_1 - p_1 + \frac{\Gamma_1 (q_1 - q_1^*)}{2a_2(m_1 - m_2)} \right)}{a_2(m_1 - m_2)} - \frac{k_p q_1^* (a_1 m_2 - a_2 m_1)^2}{a_2^2(m_1 - m_2)^2}, \frac{k_p q_1^* (a_1 m_2 - a_2 m_1)}{a_2(m_1 - m_2)} \right].$$

Simulations have been conducted in MATLAB using an ODE23 solver with the model parameters $a_1 = 0.0124, a_2 = 0.0025, a_3 = 0.4446$, and $L_a = 1, R_a = 0.5, K_e = 0.1$ for illustrative purposes. The tuning parameters for the controller (16) have been set as $K_p = 1, k_v = 0.001, m_1 = 0.4, m_3 = 5, \epsilon = 1$ and $K_{I_1} = 10, K_{I_2} = 10, k_i = 2 \times 10^3, \Gamma_0 = 0.005, \Gamma_1 = 0.001$. The initial conditions are $(q_1, q_2, \dot{q}_1, \dot{q}_2, I_a, \zeta_0, \zeta_1) = (0.1, 0, 0.2, 0, 0, 0, 0)$. The simulation results in Figure 2 show the system response with the controller (16) in the presence of the unknown matched disturbance $\delta_0 = 0.01$ and the unmatched disturbance $\delta_1 = 0.001$. The controller (16) with the proposed tuning correctly achieves the regulation goal corresponding to the assignable equilibrium $q = q^*$, the integral states settle to constant values, and the total energy W_d converges to zero at the equilibrium. This is achieved thanks to the integral states ζ_0 and ζ_1 which compensate the effect of the disturbances on the wheel position, provided that $\Gamma_0 > 0$ and $\Gamma_1 > 0$. In comparison, setting $\Gamma_1 = 0$ the unmatched disturbances are neglected resulting in a noticeable steady state error on the wheel position. Setting $\Gamma_0 = \Gamma_1 = 0$, which corresponds to the

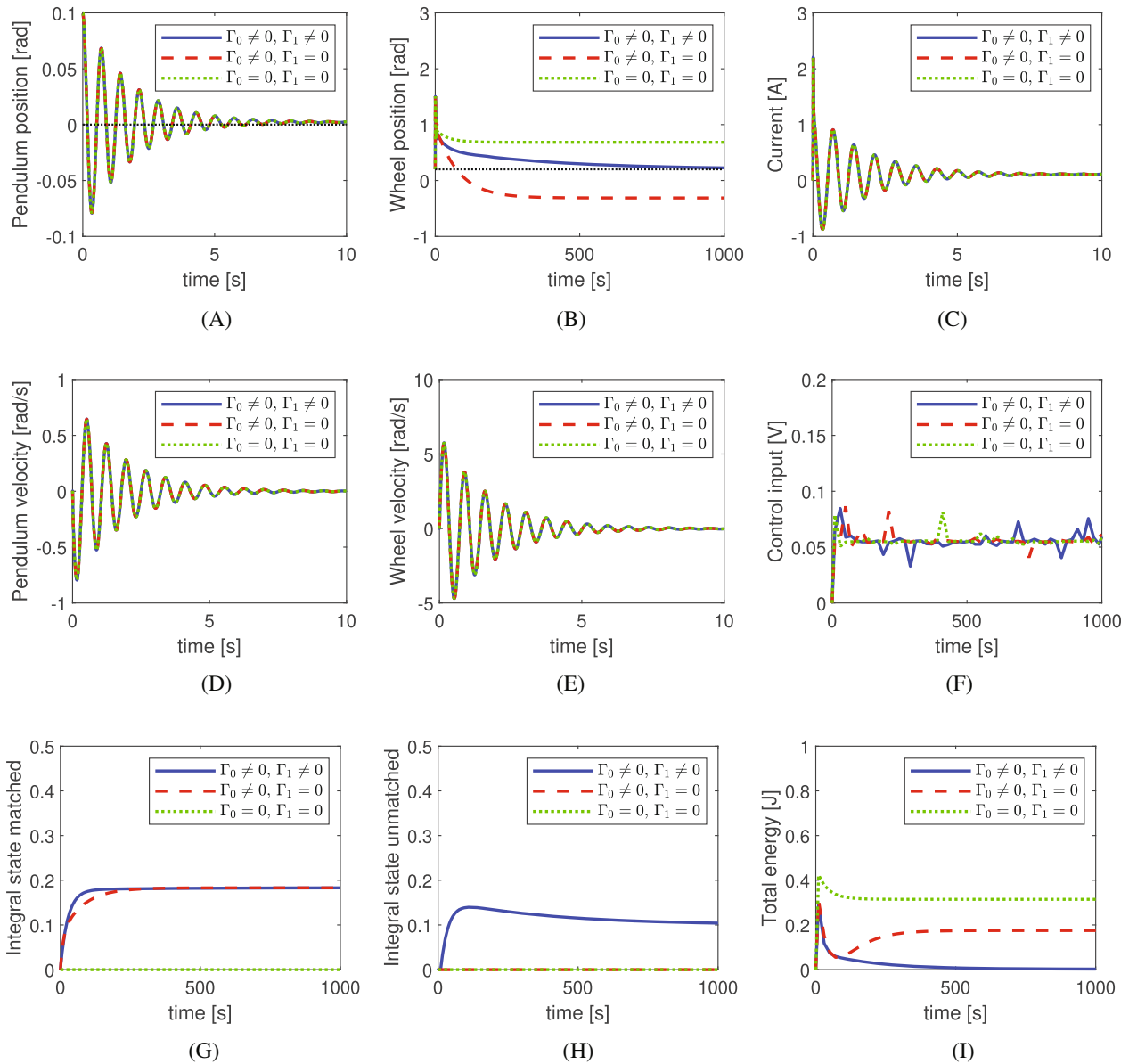


FIGURE 2 Simulation results for inertia wheel pendulum actuated by a DC motor with controller (16) and integral actions (17) subject to constant disturbances $\delta_0 = 0.01$ and $\delta_1 = 0.001$: (A) position q_1 ; (B) position q_2 ; (C) armature current I_a ; (D) velocity $\dot{q}_1$; (E) velocity $\dot{q}_2$; (F) control input u ; (G) integral state ζ_0 ; (H) integral state ζ_1 ; (I) total energy W_d .

controller (16) without integral actions, yields a position error of the opposite sign, since in this case the disturbances δ_0 and δ_1 have opposite effects on the system dynamics, while the total energy settles to an even larger value. The simulation results in Figure 3 show the system response with only the matched disturbance $\delta_0 = 0.01$, while $\delta_1 = 0$. In this case, the regulation goal is achieved with either $\Gamma_1 \neq 0$ or $\Gamma_1 = 0$, while $q_1^* = 0$. Note that the transient is slightly slower if $\Gamma_1 \neq 0$, which again is due to the fact that the unmatched disturbances and thus the integral state ζ_1 have an opposite effect to ζ_0 . Additional simulation results that illustrate the effect of the tuning parameters k_p , k_v , k_i , Γ_0 and Γ_1 on performance are included in Appendix B.

4.2 | Ball on beam

The ball-on-beam system is modified here by accounting for the dynamics of a DC motor driving the beam (see Figure 4). The equations of motion of the ball-on-beam with direct actuation, under some assumptions on the masses for simplicity,

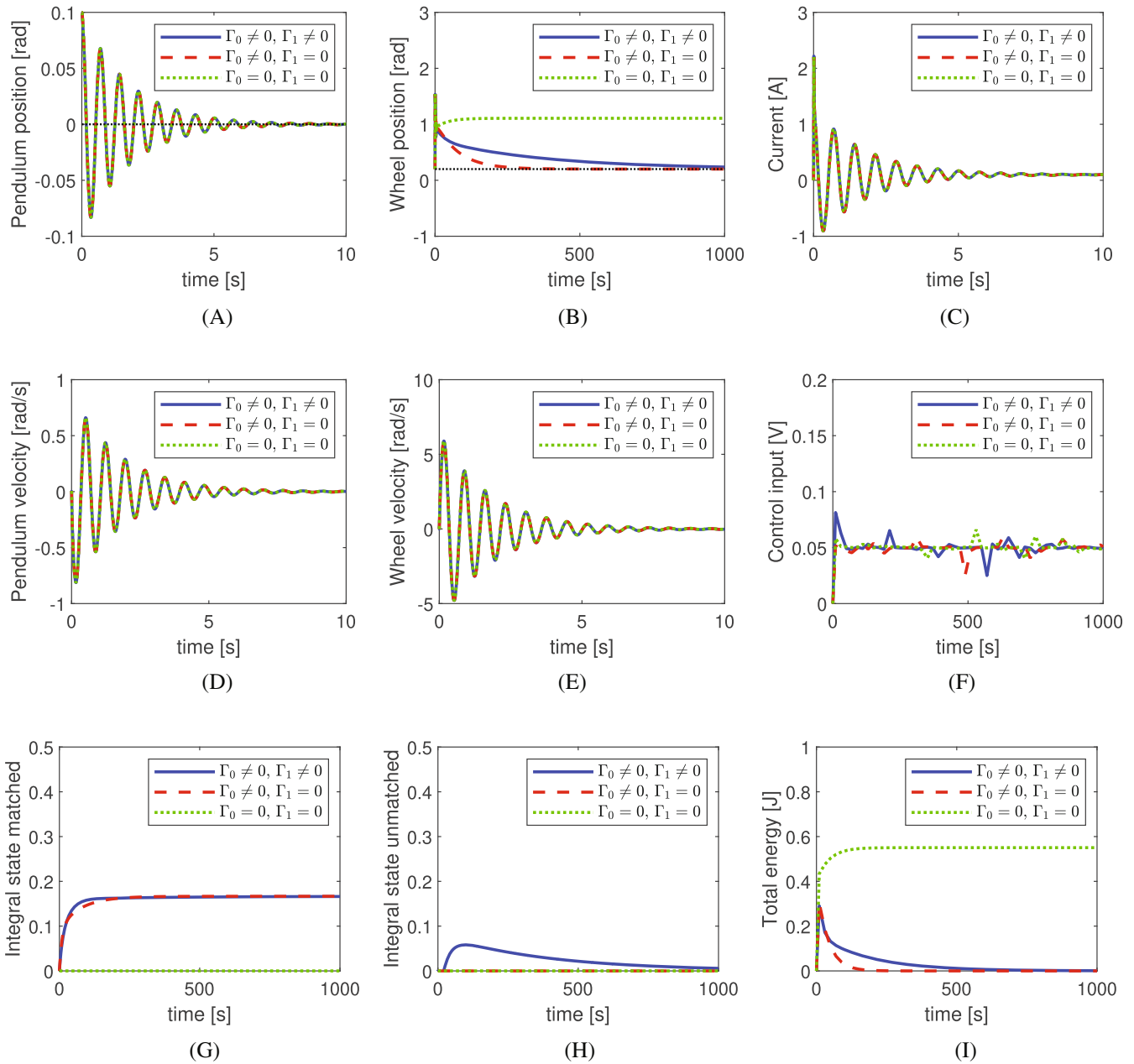


FIGURE 3 Simulation results for inertia wheel pendulum actuated by a DC motor with controller (16) and integral actions (17) subject to constant matched disturbances $\delta_0 = 0.01$: (A) position q_1 ; (B) position q_2 ; (C) armature current I_a ; (D) velocity $\dot{q}_1$; (E) velocity $\dot{q}_2$; (F) control input u ; (G) integral state ζ_0 ; (H) integral state ζ_1 ; (I) total energy W_d .

are given by (1) with the parameters $M = \begin{bmatrix} 1 & 0 \\ 0 & L^2 + q_1^2 \end{bmatrix}$, $G^T = [0 \quad 1]$, and $\Omega = gq_1 \sin(q_2)$, where q_1 is the position of the ball, q_2 is the inclination of the beam, L is half the length of the beam, and g is the gravity constant. The prescribed equilibrium in the absence of disturbances is $(q_1^*, q_2^*) = (0, 0)$. The IDA-PCB design (3) for the mechanical subsystem without disturbances yields⁵

$$M_d = (L^2 + q_1^2) \begin{bmatrix} \sqrt{\frac{2}{(L^2 + q_1^2)}} & 1 \\ 1 & \sqrt{2(L^2 + q_1^2)} \end{bmatrix}, J_2 = \left(p_1 - p_2 \sqrt{\frac{2}{(L^2 + q_1^2)}} \right) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$\Omega_d = g - g \cos(q_2) + \frac{k_p}{2} \left(q_2 - \frac{1}{\sqrt{2}} \operatorname{arcsinh} \left(\frac{q_1}{L} \right) \right)^2,$$

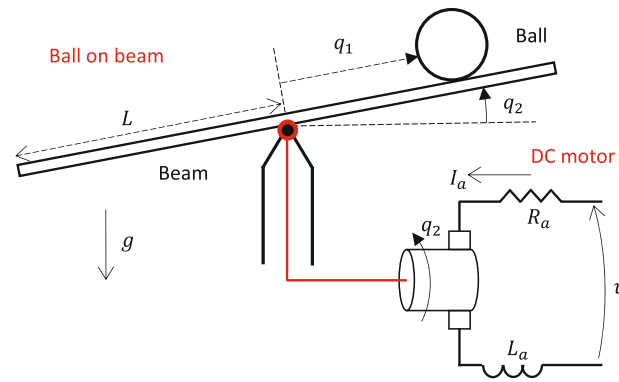


FIGURE 4 Schematic of the ball-on-beam actuated by a DC motor, with key model parameters.

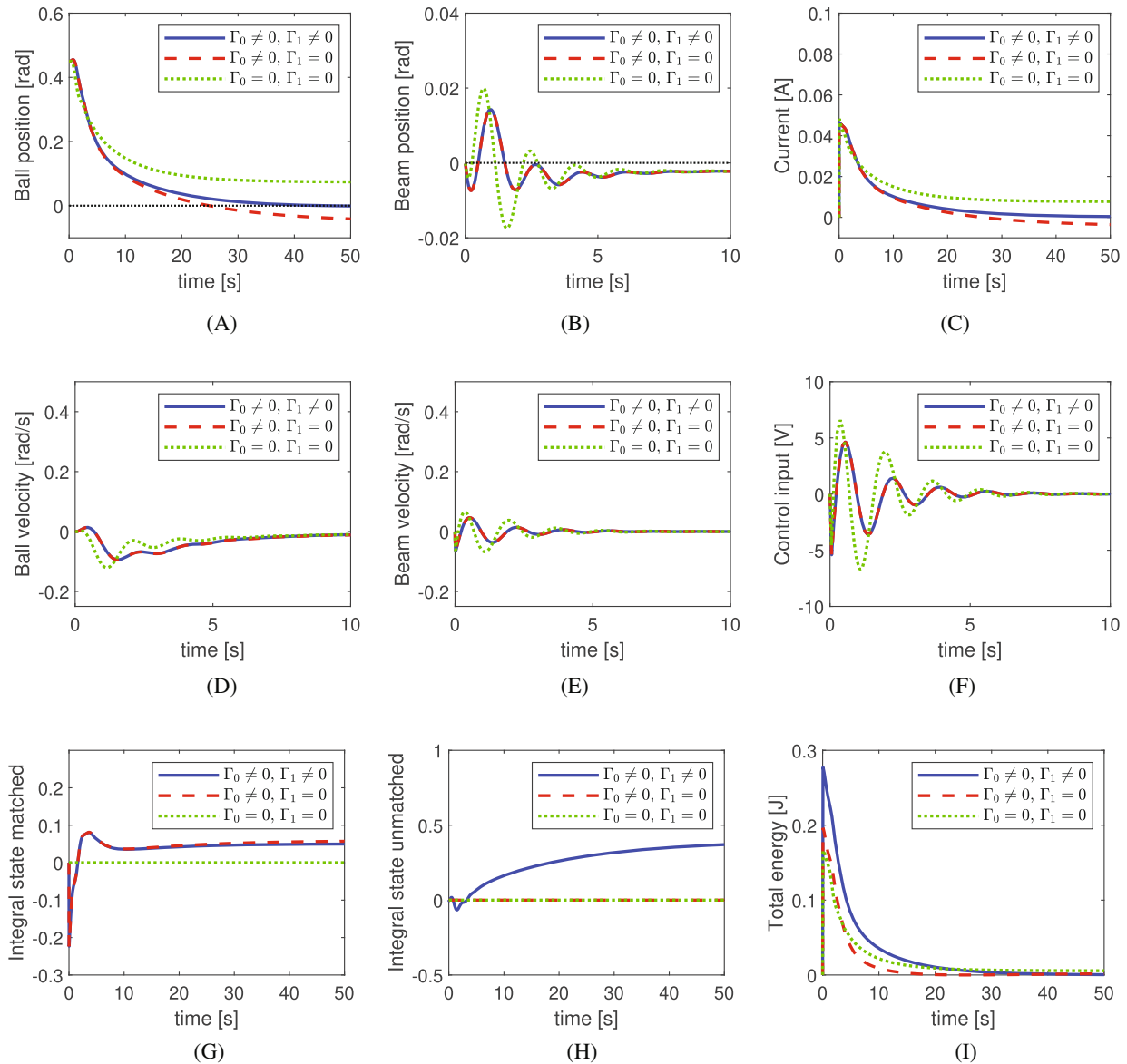


FIGURE 5 Simulation results for ball-on-beam actuated by a DC motor with controller (16) and integral actions (17) subject to constant matched disturbances $\delta_0 = 0.05$ and unmatched disturbances $\delta_1 = 0.02$: (A) position q_1 ; (B) position q_2 ; (C) armature current I_a ; (D) velocity $\dot{q}_1$; (E) velocity $\dot{q}_2$; (F) control input u ; (G) integral state ζ_0 ; (H) integral state ζ_1 ; (I) total energy W_d .

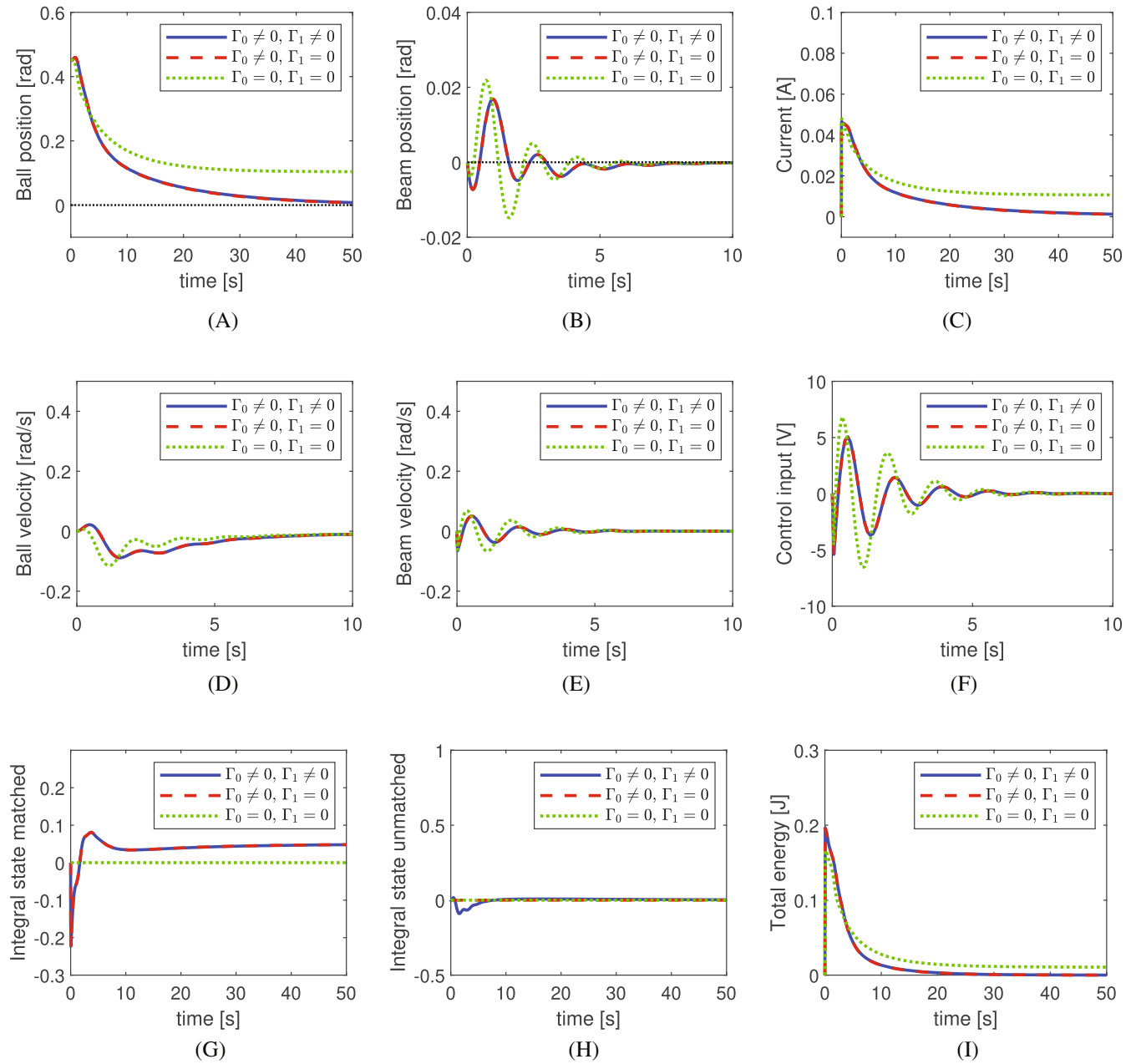


FIGURE 6 Simulation results for ball-on-beam actuated by a DC motor with controller (16) and integral actions (17) subject to constant matched disturbance $\delta_0 = 0.05$: (A) position q_1 ; (B) position q_2 ; (C) armature current I_a ; (D) velocity $\dot{q}_1$; (E) velocity $\dot{q}_2$; (F) control input u ; (G) integral state ζ_0 ; (H) integral state ζ_1 ; (I) total energy W_d .

where k_p and k_v are positive tuning parameters. The complete system dynamics described with (10), including the DC motor driving the beam, yields $F_{23} = \frac{K_s}{L_a}G$, $F_{33} = \frac{R_a}{L_a^2}$, with total energy $W = \Omega + \frac{1}{2}p^T M^{-1}p + \frac{1}{2}L_a I_a^2$. Similarly to the previous example, the system states are q , $p = M\dot{q}$, and $x_1 = I_a$, while K_e , L_a , and R_a are the torque constant of the motor, the inductance, and the armature resistance respectively. For simplicity, the inertia of the motor is assumed negligible compared to that of the beam. The control law is computed as in (16) and the integral states are computed as in (17). The assignable equilibrium in the presence of the constant unmatched disturbance δ_1 is $q_2^* = -\arcsin(\delta_1/g)$, which is constant. The estimate of δ_1 computed with the integral action is $\tilde{\delta}_1 = \Gamma_1 K_{I_2} (\zeta_1 - p_1)$. The analytical solution of the PDEs (13a) and (13b) which verifies the minimizer conditions (13c) and (13d) is thus

$$\Lambda = \left[\frac{\sqrt{2}k_p q_2^*}{2L} \right. \\ \left. \Gamma_1 K_{I_2} \left(\zeta_1 - p_1 + \frac{1}{2}\Gamma_1(q_2 - q_2^*) \right) - \frac{k_p q_2^*}{L} \sqrt{L^2 + q_1^2} \right],$$

which however yields the residual $\Delta = \frac{\sqrt{2}k_p q_1 q_2^*(q_2 - q_2^*)}{L}$ (i.e., the residual can be eliminated by employing an algebraic solution of the PDE as in our prior work¹⁷). In particular, $\Delta = 0$ at the prescribed equilibrium, and it vanishes for all q_2 if and only if $q_2^* = 0$, that is in the absence of unmatched disturbances. In addition, Δ can be made arbitrarily small by reducing k_p , which should however remain strictly positive to verify the minimizer condition (13d).

Simulations have been conducted in MATLAB with an ODE23 solver using the model parameters $L = 0.5, g = 9.81, R_a = 1, L_a = 1, K_e = 100$ with the initial conditions $(q_1, q_2, \dot{q}_1, \dot{q}_2, I_a, \zeta_0, \zeta_1) = (0.45, 0, 0, 0, 0, 0, 0)$. The tuning parameters have been set to $k_p = 1, k_v = 1, k_i = 0.01, \Gamma_0 = 0.01, \Gamma_1 = 0.05, K_{I_1} = 1, K_{I_2} = 1$ for illustrative purposes. The simulation results in Figure 5 show the system response with the controller (16) in the presence of the unknown matched disturbance $\delta_0 = 0.05$ and the unmatched disturbance $\delta_1 = 0.02$. The controller (16) with the proposed tuning correctly achieves the regulation goal corresponding to the assignable equilibrium $q = q^*$, the integral states settle to a constant value, and the total energy W_d converges to zero at the equilibrium. This is achieved thanks to the integral states ζ_0 and ζ_1 which compensate the effect of the disturbances on the ball position, provided that $\Gamma_0 > 0$ and $\Gamma_1 > 0$. In comparison, setting $\Gamma_1 = 0$ the unmatched disturbances are neglected resulting in a steady state error on the ball position, while setting $\Gamma_0 = \Gamma_1 = 0$ yields an even larger position error since all disturbances are neglected. The simulation results in Figure 6 show the system response with only the matched disturbance $\delta_0 = 0.05$, while $\delta_1 = 0$. In this case $q_2^* = 0$, and the regulation goal is achieved with either $\Gamma_1 \neq 0$ or $\Gamma_1 = 0$, which yield the same transient response. Note that employing the controller (3) with the same tuning parameters, which neglects the actuator dynamics results in instability even without disturbances. The simulation results for the ball-on-beam system with direct actuation employing the controller (24) and the integral actions (25) with the tuning parameters $\Gamma_0 = 0.1$ and $\Gamma_1 = 0.5$ is shown in Figure 7. Also in this case, the integral actions

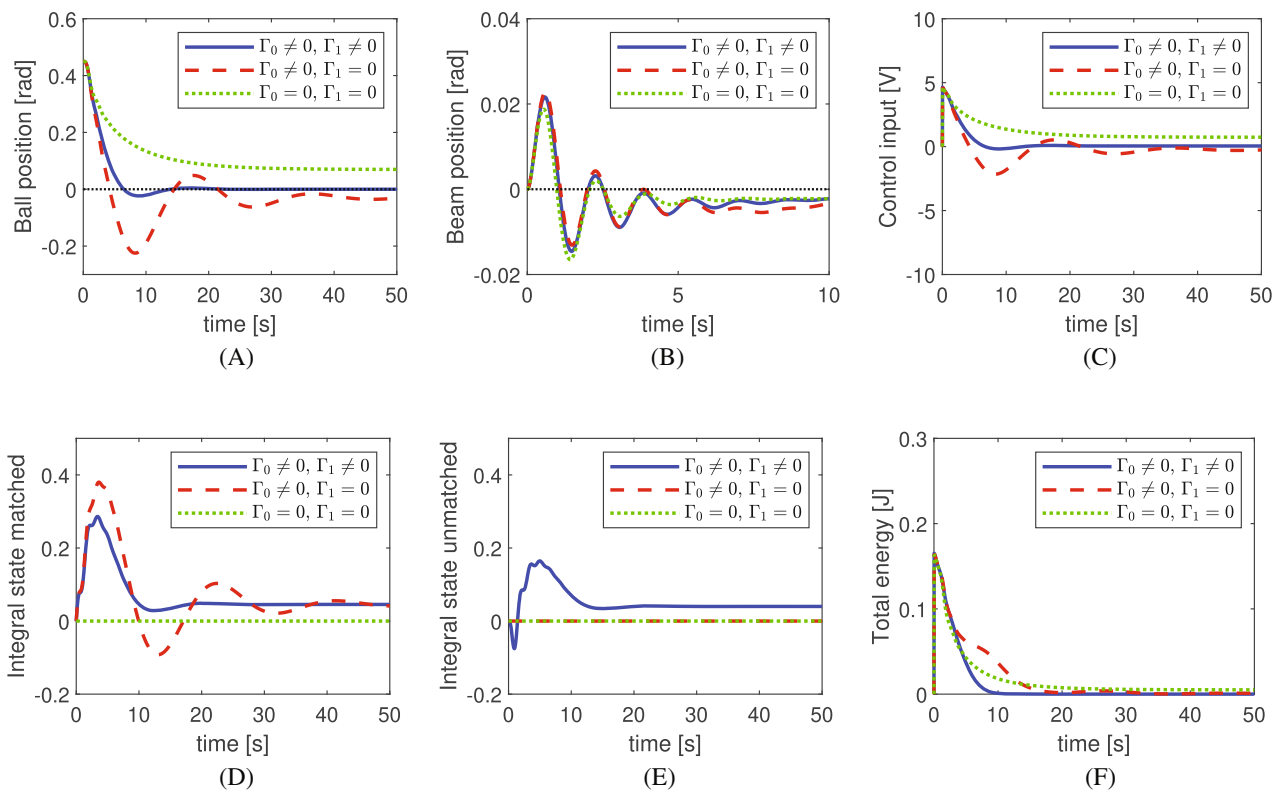


FIGURE 7 Simulation results for ball-on-beam with direct actuation with controller (24) and integral actions (25) subject to constant matched disturbance $\delta_0 = 0.05$ and unmatched disturbances $\delta_1 = 0.02$: (A) position q_1 ; (B) position q_2 ; (C) control input u ; (D) integral state ζ_0 ; (E) integral state ζ_1 ; (F) total energy W_d .

yield noticeable performance improvements in the presence of matched and unmatched disturbances, compared to setting either $\Gamma_1 = 0$ or $\Gamma_0 = \Gamma_1 = 0$.

5 | CONCLUSION

This work presented a new control law for a class of underactuated mechanical systems that are not in affine form due to the presence of first-order actuator dynamics, and that are subject to both matched and unmatched constant additive disturbances. The controller design builds upon the IDA-PBC methodology in a modular fashion, such that the PDEs that characterize potential energy shaping and kinetic energy shaping are preserved. Two independent integral states are introduced by employing the concept of extended Hamiltonian in order to compensate matched constant disturbances and to explicitly account for unmatched constant disturbances. Compared to the use of nonlinear observers that have been employed in previous works, the proposed approach yields less stringent stability conditions and preserves the interpretation of the control action in terms of system energy. Compared to robust controllers, the proposed approach does not require prior knowledge of the disturbance bounds. Nevertheless, the use of the integral actions results in additional PDEs which complicates the controller design. The simulation results for two examples, namely an inertia wheel pendulum and a ball-on-beam system, demonstrate the effectiveness of the controller. Although the unmatched disturbances cannot be canceled completely in underactuated mechanical systems, accounting for their effect with the proposed controller clearly improves performance.

The proposed work has nevertheless a few limitations. First of all, the solution of the additional PDEs is system specific, similarly to the PDEs characterizing the IDA-PBC methodology. In addition, the control formulation is specific to constant additive disturbances and to systems with constant input matrix. Future work will thus aim to remove these limitation.

CONFLICT OF INTEREST

The authors declared that they have no conflict of interest to this work.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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APPENDIX A

Proof of Proposition 3. Computing the partial derivatives of W_d from (23) yields

$$\begin{aligned}
 \nabla_q W_d &= \nabla_q \left(\Omega_d + \frac{1}{2} p^T M_d^{-1} p + \Lambda^T (q - q^*) \right), \\
 \nabla_p W_d &= M_d^{-1} p + \nabla_p \left(\Lambda^T (q - q^*) \right) - K_{I_1} G(\zeta_0 - G^T p - \alpha) - K_{I_2} G^{\perp T} (\zeta_1 - G^{\perp} p - \beta), \\
 \nabla_{\zeta_0} W_d &= K_{I_1} (\zeta_0 - G^T p - \alpha), \\
 \nabla_{\zeta_1} W_d &= K_{I_2} (\zeta_1 - G^{\perp} p - \beta) + \nabla_{\zeta_1} \left(\Lambda^T (q - q^*) \right).
 \end{aligned} \tag{A1}$$

Equating the corresponding rows of (1) and of (21) yields the matching equations

$$M^{-1}p = S_{12}\nabla_p W_d + S_{13}\nabla_{\zeta_0} W_d + S_{14}\nabla_{\zeta_1} W_d, \quad (\text{A2a})$$

$$-\nabla_q H + Gu - G\delta_0 - G^{\perp T} \delta_1 = -S_{12}^T \nabla_q W_d - S_{22}\nabla_p W_d + S_{23}\nabla_{\zeta_0} W_d + S_{24}\nabla_{\zeta_1} W_d, \quad (\text{A2b})$$

$$\dot{\zeta}_0 = -S_{13}^T \nabla_q W_d - S_{23}^T \nabla_p W_d - S_{33}\nabla_{\zeta_0} W_d + S_{34}\nabla_{\zeta_1} W_d, \quad (\text{A2c})$$

$$\dot{\zeta}_1 = -S_{14}^T \nabla_q W_d - S_{24}^T \nabla_p W_d - S_{34}^T \nabla_{\zeta_0} W_d - S_{44}\nabla_{\zeta_1} W_d. \quad (\text{A2d})$$

Step 1. Substituting S_{12}, S_{13}, S_{14} from (22) and the partial derivatives of W_d from (A1) into (A2a) yields

$$\begin{aligned} M^{-1}p &= M^{-1}M_d \left(M_d^{-1}p + \nabla_p (\Lambda^T(q - q^*)) - K_{I_1}G(\zeta_0 - G^T p - \alpha) - K_{I_2}G^{\perp T}(\zeta_1 - G^{\perp} p - \beta) \right) \\ &\quad + M^{-1}M_d G K_{I_1} (\zeta_0 - G^T p - \alpha) + M^{-1}M_d G^{\perp T} (K_{I_2} (\zeta_1 - G^{\perp} p - \beta) + \nabla_{\zeta_1} (\Lambda^T(q - q^*))). \end{aligned}$$

Refactoring the former expression cancels the terms in ζ_0, ζ_1 and the terms α, β yielding the identity, provided that the PDE (13b) is verified.

Step 2. Substituting S_{12}, S_{22}, S_{23} , and S_{24} from (22) and the partial derivatives of W_d from (A1) into (A2b) yields

$$\begin{aligned} -\nabla_q H + Gu - G\delta_0 - G^{\perp T} \delta_1 &= -M_d M^{-1} \nabla_q \left(\Omega_d + \frac{1}{2} p^T M_d^{-1} p + \Lambda^T(q - q^*) \right) \\ &\quad - (Gk_v G^T - J_2) (M_d^{-1} p + \nabla_p (\Lambda^T(q - q^*)) - K_{I_1}G(\zeta_0 - G^T p - \alpha) - K_{I_2}G^{\perp T}(\zeta_1 - G^{\perp} p - \beta)) \\ &\quad + (G\Gamma_0 + J_2 G) K_{I_1} (\zeta_0 - G^T p - \alpha) + (G^{\perp T} \Gamma_1 + J_2 G^{\perp T}) (K_{I_2} (\zeta_1 - G^{\perp} p - \beta) + \nabla_{\zeta_1} (\Lambda^T(q - q^*))). \end{aligned} \quad (\text{A3})$$

Substituting α and β in the previous expression cancels the disturbance δ_0 and δ_1 yielding

$$\begin{aligned} -\nabla_q H + Gu + M_d M^{-1} \nabla_q \left(\Omega_d + \Lambda^T(q - q^*) + \frac{1}{2} p^T M_d^{-1} p \right) &+ (Gk_v G^T - J_2) (M_d^{-1} p + \nabla_p (\Lambda^T(q - q^*))) \\ &- (G^{\perp T} \Gamma_1 + J_2 G^{\perp T}) \nabla_{\zeta_1} (\Lambda^T(q - q^*)) - G K_{I_1} (k_v G^T G + \Gamma_0) (\zeta_0 - G^T p) - \Gamma_1 G^{\perp T} K_{I_2} (\zeta_1 - G^{\perp} p) = 0. \end{aligned}$$

Multiplying the former expression by $G^\dagger$ and substituting the control law (24) yields the PDE (13b) (i.e., pre-multiplied by $G^\dagger J_2$), which is verified by hypothesis. Multiplying it instead by $G^\perp$ yields the sum of the PDEs (5), (6), (13a), and (13b) (i.e., pre-multiplied by $G^\perp J_2$), which are all verified by hypothesis.

Step 3. Substituting $S_{13}, S_{23}, S_{33}, S_{34}$ from (22) and the partial derivatives of W_d from (A1) into (A2c) yields

$$\begin{aligned} \dot{\zeta}_0 &= -G^T M_d M^{-1} \nabla_q \left(\Omega_d + \frac{1}{2} p^T M_d^{-1} p + \Lambda^T(q - q^*) \right) \\ &\quad - (\Gamma_0 G^T - G^T J_2) (M_d^{-1} p + \nabla_p (\Lambda^T(q - q^*)) - K_{I_1}G(\zeta_0 - G^T p - \alpha) - K_{I_2}G^{\perp T}(\zeta_1 - G^{\perp} p - \beta)) \\ &\quad - (G\Gamma_0 G^T - G^T J_2 G) K_{I_1} (\zeta_0 - G^T p - \alpha) + G^T J_2 G^{\perp T} (K_{I_2} (\zeta_1 - G^{\perp} p - \beta) + \nabla_{\zeta_1} (\Lambda^T(q - q^*))). \end{aligned}$$

Refactoring terms in the former expression cancels α and β yielding (25a), provided that (13b) is verified.

Step 4. Substituting $S_{14}, S_{24}, S_{34}, S_{44}$ from (22) and the partial derivatives of W_d from (A1) into (A2d) yields

$$\begin{aligned} \dot{\zeta}_1 &= -G^\perp M_d M^{-1} \nabla_q \left(\Omega_d + \frac{1}{2} p^T M_d^{-1} p + \Lambda^T(q - q^*) \right) \\ &\quad - (\Gamma_1 G^\perp - G^\perp J_2) (M_d^{-1} p + \nabla_p (\Lambda^T(q - q^*)) - K_{I_1}G(\zeta_0 - G^T p - \alpha) - K_{I_2}G^{\perp T}(\zeta_1 - G^{\perp} p - \beta)) \\ &\quad + G^\perp J_2 G K_{I_1} (\zeta_0 - G^T p - \alpha) - (G^\perp \Gamma_1 G^{\perp T} - G^\perp J_2 G^{\perp T}) (K_{I_2} (\zeta_1 - G^{\perp} p - \beta) + \nabla_{\zeta_1} (\Lambda^T(q - q^*))). \end{aligned}$$

Refactoring terms in the former expression cancels α and β yielding (25b), provided that (13b) is verified. ■

Proof of Proposition 4. Computing the time derivative of W_d in (23) along the trajectories of the closed-loop system (21) yields

$$\dot{W}_d = -\nabla_p W_d^T G k_v G^T \nabla_p W_d - \nabla_{\zeta_0} W_d^T \Gamma_0 G^T G \nabla_{\zeta_0} W_d - \nabla_{\zeta_1} W_d^T \Gamma_1 G^\perp G^{\perp T} \nabla_{\zeta_1} W_d \leq 0,$$

which is verified for all positive k_v, Γ_0, Γ_1 hence the equilibrium is stable. Substituting the partial derivatives of W_d from (A1), it follows that $(\zeta_0 - G^T p - \alpha) \in \mathcal{L}^2 \cap \mathcal{L}^\infty$ and $G^T \nabla_p W_d, \nabla_{\zeta_1} W_d \in \mathcal{L}^2$, while $p, (\zeta_1 - G^\perp p - \beta), \Lambda^T(q - q^*) \in \mathcal{L}^\infty$. Consequently $K_{I_2}(\zeta_1 - G^\perp p - \beta) + \nabla_{\zeta_1}(\Lambda^T(q - q^*)) \in \mathcal{L}^2$ and $G^T M_d^{-1} p + G^T \nabla_p(\Lambda^T(q - q^*)) - G^T K_{I_2} G^{\perp T}(\zeta_1 - G^\perp p - \beta) \in \mathcal{L}^2$. Combining the former expressions and substituting (13b) yields thus $y = G^T M_d^{-1} p \in \mathcal{L}^2 \cap \mathcal{L}^\infty$. In addition, it follows from (21) that $\dot{p}, \dot{\zeta}_0, \dot{\zeta}_1 \in \mathcal{L}^\infty$, while α and β are bounded according to Assumption 3. Therefore $y, (\zeta_0 - G^T p - \alpha)$ and $K_{I_2}(\zeta_1 - G^\perp p - \beta) + \nabla_{\zeta_1}(\Lambda^T(q - q^*))$ converge to zero asymptotically according to Barbalat's lemma.²⁷ Employing a similar argument to Proposition 2, it follows that the equilibrium is asymptotically stable provided that the output y is detectable⁵ ■

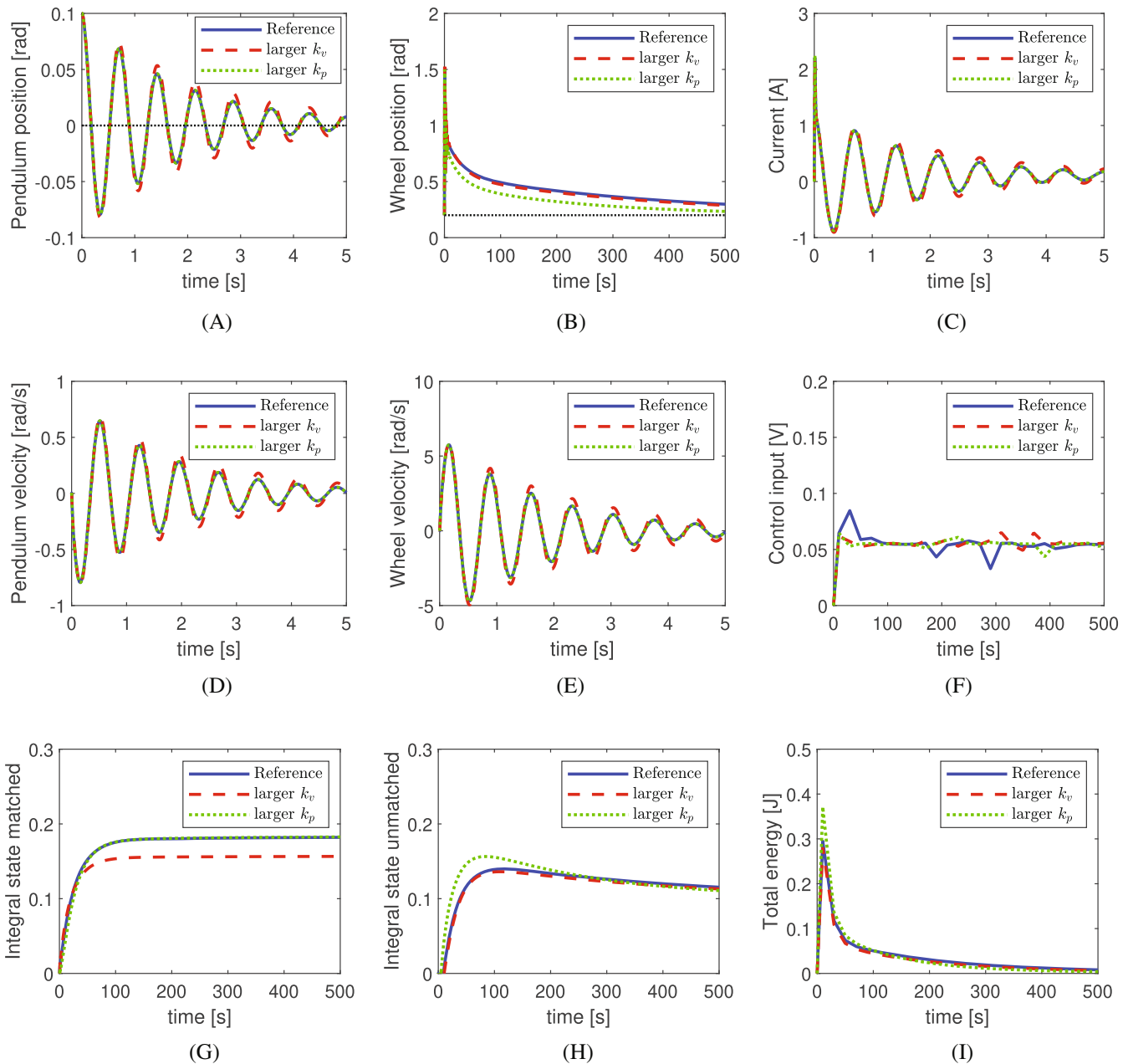


FIGURE B1 Simulation results for inertia wheel pendulum actuated by a DC motor with controller (16) and integral actions (17) subject to constant disturbances $\delta_0 = 0.01$ and $\delta_1 = 0.001$ employing different tuning parameters: (A) position q_1 ; (B) position q_2 ; (C) armature current I_a ; (D) velocity $\dot{q}_1$; (E) velocity $\dot{q}_2$; (F) control input u ; (G) integral state ζ_0 ; (H) integral state ζ_1 ; (I) total energy W_d .

APPENDIX B

Figures B1 and B2 illustrate the effect of the tuning parameters k_p, k_v, k_i, Γ_0 , and Γ_1 for the inertia wheel pendulum of Section 4.1. The reference condition corresponds to the tuning parameters $k_p = 1, k_v = 0.001, k_i = 2 \times 10^3, \Gamma_0 = 0.005$, and $\Gamma_1 = 0.001$ employed in Section 4.1. Figure B1 shows that increasing k_p (i.e., $k_p = 2$) yields a faster convergence of the wheel position to the prescribed equilibrium. This is expected, since k_p affects the minimizer condition (13d). Increasing k_v (i.e., $k_v = 0.002$) reduces the steady-state value of ζ_0 , according to the definition of α in Section 3.1 (i.e., $\alpha = \delta_0(K_I(k_v G^T G + \Gamma_0))^{-1}$).

Figure B2 shows that increasing k_i (i.e., $k_i = 4 \times 10^3$) reduces the amplitude of the oscillations. This occurs since k_i introduces a dissipative effect related to the actuator dynamics, which is in accordance with our prior work.¹² Increasing

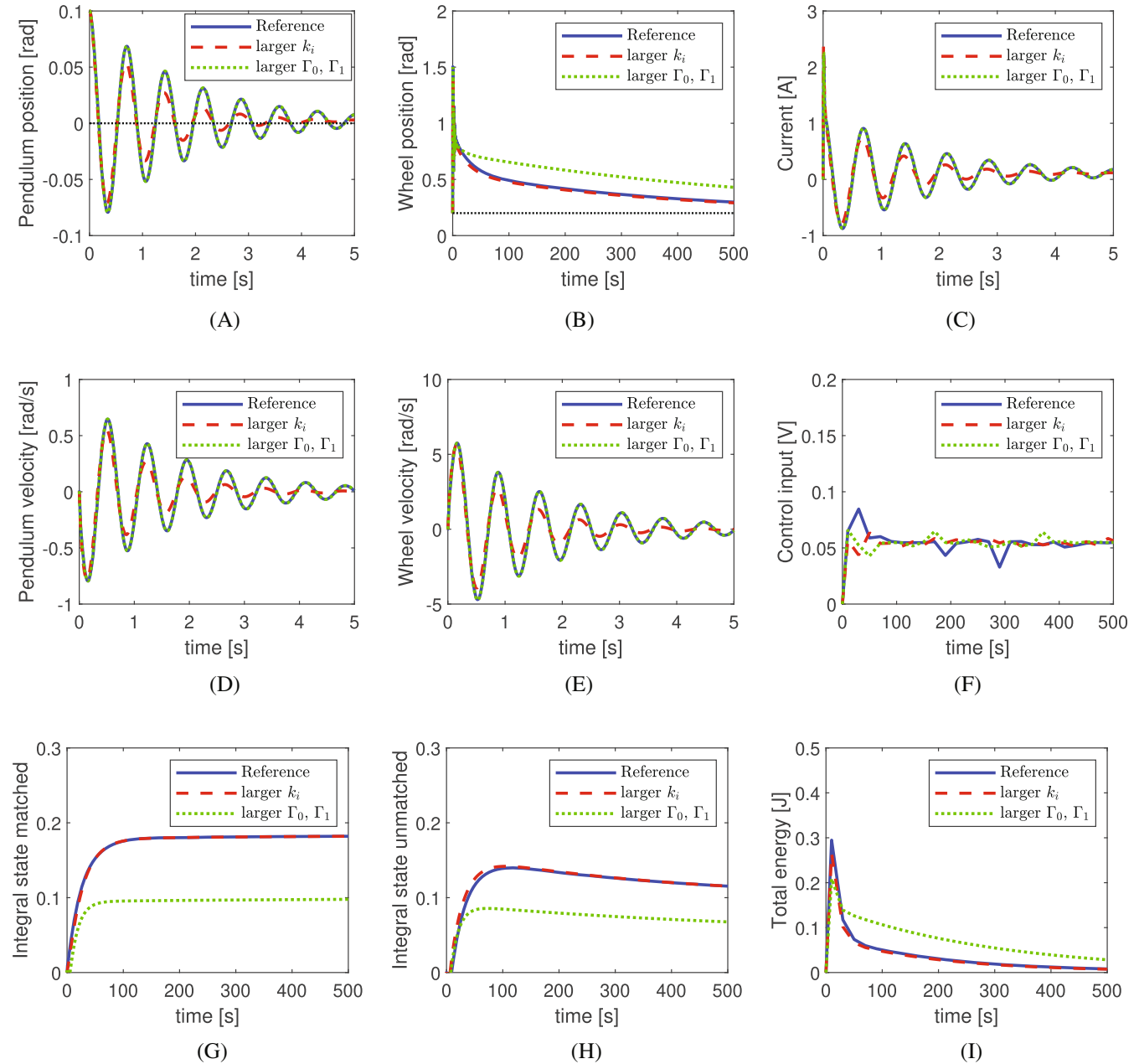


FIGURE B2 Simulation results for inertia wheel pendulum actuated by a DC motor with controller (16) and integral actions (17) subject to constant disturbances $\delta_0 = 0.01$ and $\delta_1 = 0.001$ employing different tuning parameters: (A) position q_1 ; (B) position q_2 ; (C) armature current I_a ; (D) velocity $\dot{q}_1$; (E) velocity $\dot{q}_2$; (F) control input u ; (G) integral state ζ_0 ; (H) integral state ζ_1 ; (I) total energy W_d .

Γ_0 and Γ_1 (i.e., $\Gamma_0 = 0.01, \Gamma_1 = 0.002$) results in the integral states ζ_0 and ζ_1 converging faster and to smaller values. This can be explained considering the effect of Γ_0 and Γ_1 on the time derivative $\dot{W}_d$ (i.e., see Proposition 2) and recalling the definition of α and β in Section 3.1 (i.e., $\alpha = \delta_0(K_{I_1}(k_v G^T G + \Gamma_0))^{-1}$ and $\beta = \delta_1(K_{I_2}\Gamma_1)^{-1}$). In this case, increasing Γ_0 and Γ_1 yields a slower convergence of the wheel position to the prescribed equilibrium, which is in agreement with the results in Section 4.1 (see Figure 3), and it is due to the fact that the unmatched disturbances and thus the integral state ζ_1 have an opposite effect to ζ_0 .