

Extended Magnetohydrodynamic Effects in Z-Pinch Plasmas

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Declaration

I hereby certify that the material of this thesis, which I now submit for the award of Doctor of Philosophy, is entirely my own work unless otherwise cited or acknowledged within the body of text.

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Abstract

Magnetic fields are central to magneto-inertial fusion (MIF) schemes that aim to achieve controlled nuclear fusion as well as in laboratory astrophysics experiments designed to explore fundamental physics. Accurate numerical modelling of the coupled behaviour of magnetic fields and dense plasmas is crucial for understanding results, interpreting diagnostics and designing future experiments. However, conventional resistive magnetohydrodynamic (MHD) models are inaccurate in low density plasmas such as at a vacuum-plasma interface. These regions are frequently encountered in pulsed power experiments. In this work, we establish which are the dominant physics effects in this regime in order to develop a more physical model of this region.

We show that the equations of Braginskii MHD in one dimensional planar geometry are amenable to the method of self-similar solutions and use this to develop a semi-analytic model of pressure equilibria in magnetised plasmas. This tool is useful for validating numerical MHD codes and it is used to provide a test problem for the Ohmic heating and Ettingshausen effects. The self-similar code is used to perform a parameter scan over plasma β . This shows that in the high β plasma characteristic of the stagnation region of MIF experiments such as MagLIF, the most relevant transport effects are magnetised thermal conduction and the Nernst effect. However, we find that as the β is reduced, the Ohmic heating and resistive diffusion terms become increasingly important. At a vacuum-plasma interface, the Ohmic heating becomes the dominant term in the energy balance. However, we show the Ettingshausen effect plays a key role in reducing the unphysically high electron temperatures in this region. This may be of use in resistive MHD codes which traditionally lack the Ettingshausen term, but also tend to overestimate the electron temperature at vacuum-plasma interfaces. We then use the self-similar code to design a potential experiment to measure the Nernst effect in a pulsed power plasma. We show that it is the dimensionless magnetic Lewis number which dictates the impact of the Nernst effect. The lower temperatures generated in pulsed power experiments when compared to laser platforms attempting to measure the same effect means that a measurement of the Nernst effect would be difficult in this regime.

To support theoretical analysis of the different physics effects at a vacuum-plasma interface, fully kinetic simulations of the edge of a z-pinch using the EPOCH particle-in-cell (PIC) code are then carried out. These simulations show how an initial MHD equilibrium undergoes a transient phase which leads to the formation of a long-lasting charge separation layer at the edge of the pinch. This layer has width comparable to the ion Larmor radius and the steady state electric field in this region is dominated by

the Hall term, indicating it is the dominant beyond-MHD effect at a vacuum-plasma interface. Through studying the evolution of the edge of the z-pinch in 2D, we show that macroscopically the Hall effect leads to shearing of plasma flows in the direction of current flow.

Based on the analysis in previous sections, we then implement the Hall effect within the Gorgon extended MHD code. The Hall term is notoriously difficult to handle numerically, so we develop a new semi-implicit method for it involving electron inertia. This is tested thoroughly by comparing to MHD test problems and theoretical dispersion relations for wave modes driven by the Hall effect. This new solver is then used to study the evolution of an $m = 0$ sausage instability in a z-pinch. We find that similarly to the PIC results, the Hall effect causes the lobes of the instability to shear in the direction of current flow by diverting magnetic field with the electron drift velocity. This has the effect of increasing the growth rate of the instability slightly compared to the resistive MHD case, though this does depend on the plasma parameters used.

Further work is needed to explore other beyond-MHD terms, such as anomalous resistivity and finite Larmor radius effects, and implement them within a fluid code to provide a more physical model of vacuum-plasma interfaces. The Hall solver developed as part of this work can be used in the future to study beyond-MHD effects in magneto-inertial fusion and laboratory astrophysics experiments.

Role of the Author

The research in this thesis depends on two existing numerical codes.

Chimera/Gorgon is a radiation-magnetohydrodynamic code developed at the Centre for Inertial Fusion Studies (CIFS) at Imperial College London [1]. This code is the cumulative work of many authors over two decades. The analysis of results used from this code was entirely the work of the author of this thesis. All simulations discussed in this thesis were carried out by the author of this thesis. The Ettingshausen effect was implemented and tested by the author of this thesis. During this thesis, a new magnetohydrodynamics solver including the Hall effect was developed. The development, implementation and debugging of this solver was the work of the author of this thesis. Subroutines for interpolation and boundary conditions written by Mr Aidan Boxall were used as part of this solver. The test problems used to validate this solver are mostly the work of other authors, who are referenced. The implementation of the initial conditions for the Hall drift wave, Whistler wave and Farmer drift wave test problems was carried out by Aidan Boxall. The analysis of the results from these test problems was carried out by the author of this thesis. Consultation and advice about implementation and usage of the Chimera/Gorgon code came from all members of CIFS.

The EPOCH code was developed at the University of Warwick [2]. This code is distributed open-source with a GNU General Public License v3.0. The source code was written entirely by authors at Warwick and was not edited by the author of this thesis. The set up of input decks for the problems of interest and the analysis of output data was fully the work of the author of this thesis.

Some of the research contained within this thesis has already been published by the author of this thesis. This publication can be found at reference [3]. It is indicated at the beginning of a chapter where content in that chapter comes from this publication. Permissions for any external copyrighted material used can be found in the Permissions section at the end of this thesis.

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1 Introduction

Scientists have pursued the goal of controlled nuclear fusion on Earth since the mid-1940s, when research into nuclear physics during the Manhattan project proved the potency of fission for energy. Requiring extreme temperatures and pressures to overcome the Coulomb barrier between charged nuclei, fusion fuel is naturally in the plasma state.

Nuclear fusion has a lot of features that make it ideal as an energy source. There are no harmful greenhouse gas emissions, making it a clean source of power that will not contribute to climate change [4]. There is no potential for reactor meltdown, making it safe compared to nuclear fission [5]. In addition, the deuterium required as fuel for many fusion schemes is abundant in seawater [6], making the energy supply effectively limitless.

Developing nuclear fusion may also prove to be essential for future energy security. The dwindling supply of fossil fuels and the damage they do to the climate when burned means that they will not be able to be used in a few decades time. Renewable sources are not as energy-dense as fusion [6] and are not suitable for all climates or parts of the world. At the same time, the worldwide population is growing and the global demand for energy is increasing prodigiously as the world develops. Nuclear fusion could be the key to address these problems.

However, despite 70 years of intensive research into plasma physics, controlled nuclear fusion on Earth still remains elusive. Confining hot charged particles for long enough to produce fusion gain is an as yet unsolved problem. Different approaches to this challenge have been tried across the world to mixed success. This thesis is motivated by recent renewed interest in magneto-inertial fusion schemes, which attempt to combine the benefits of the main other contrasting approaches to fusion by introducing magnetic fields to dense plasma. The complex behaviour of magnetised plasmas means that the success of these fusion schemes depends on numerical modelling to design and interpret experiments. Understanding the limits of current models and identifying the most important physics for magneto-inertial fusion experiments makes up the work in this thesis.

1.1 Approaches to Fusion

Nuclear fusion occurs when two nuclei combine, with the difference in mass between reactants and products being released as energy. This requires forcing charged nuclei together against their electrostatic repulsion. The extreme temperatures and pressures required to accomplish this are the main reason why controlled nuclear fusion is not yet

a reality on Earth.

For a fusion scheme to be viable, it needs to achieve gain. This is defined as [4]:

$$Q = \frac{\text{electrical energy produced by the reactor}}{\text{electrical energy needed to achieve fusion}}. \quad (1.1)$$

A value of $Q > 1$ means that a fusion reactor produces more electrical energy than is input. For economically viable fusion, $Q \gg 1$. As fusion has not yet been achieved, let alone optimised for reactors, more often scientific gain is discussed:

$$Q_s = \frac{\text{thermonuclear yield}}{\text{heating energy applied to plasma}}. \quad (1.2)$$

This is an easier metric to achieve because the efficiency of converting the thermonuclear yield into useful electric energy does not need to be considered. Historically, there are two main fusion schemes that take almost opposite approaches to solving this problem.

Magnetic confinement fusion (MCF) was the first explored approach, with examples including the JET tokamak in Oxfordshire [7] and the Wendelstein stellarator in Germany [8]. Large magnetic fields are used to confine comparatively low densities of plasma for long enough that sufficient fusion reactions to extract energy can occur by suppressing cross-field particle and energy losses. For tokamaks, these energy confinement times can be on the order of seconds [4]. Until 2021, the JET tokamak held the record for the closest a machine had come to scientific breakeven of $Q_s = 0.67$ [7]. However, plasmas are inherently unstable and the confined plasma tends to become turbulent. The effect of this turbulence is larger-than-expected particle and energy losses from the experiment, in effect reducing the energy confinement time [4]. Understanding and mitigating this magnetised turbulence has been a primary limitation for MCF schemes thusfar [7].

Inertial confinement fusion (ICF) is a much newer and very different approach, first proposed by Nuckolls in 1972 [9]. By causing a capsule of fusion fuel to rapidly implode, high plasma densities and temperatures are achieved in order to drive a self-sustaining fusion process where the alpha particles produced heat the plasma even more [10]. The reaction is naturally unstable and ends in the fuel disassembling itself under its own pressure, but successful gain will occur if enough energy can be produced by fusion reactions before this point. For positive gain fusion, the energy produced by the alpha particles and the implosion must be larger than the heat losses to radiation and thermal conduction [11]. In basic terms, this can be thought of in terms of power balance [12]:

$$\frac{dU}{dt} = P_\alpha + P_{pdV} - P_C - P_R, \quad (1.3)$$

where dU/dt is the rate of change of energy density of the fuel, P_α is the power produced due to alpha particle fusion products depositing their energy in the fuel, P_{pdV} is the power due to pdV heating due to the compression, P_C are the losses due to thermal conduction away from the hot plasma and P_R are the losses due to radiation. For gain, $dU/dt > 0$ integrated over the whole experiment [11]. The most successful ICF

schemes to date have been driven by lasers, primarily at the National Ignition Facility (NIF) in Livermore, California. In 2021, NIF broke JET's record by achieving scientific breakeven ($Q_s > 1$) [13] and produced evidence of a burning plasma dominated by alpha particle reactions for the first time [14]. In December 2022, improvements to the laser increased the yield, producing more energy out than was delivered in laser energy to the capsule [15]. To date, this is the closest to controlled fusion that has been achieved in a laboratory experiment.

1.2 Magneto-inertial Fusion

1.2.1 What is magneto-inertial fusion?

In this thesis, we are concerned with magneto-inertial fusion (MIF) which attempts to combine some of the benefits of the other approaches [16, 17]. In general, MIF involves introducing a magnetic field to a plasma that is then compressed by either lasers or magnetic fields [18]. As the plasma implodes, it also compresses the magnetic field which can reach magnitudes of 10^4T or more at stagnation [19]. MIF schemes resemble ICF in that the magnetic field magnitude is not usually large enough to confine the plasma directly as it does in MCF. Instead, the aim is for the inertia of the plasma to contain the system for long enough for sufficient fusion to occur for gain before the target disassembles. However, the presence of the magnetic field in principle makes the power balance of equation (1.3) easier to achieve. It does this by confining hot electrons in the magnetic field, preventing them from transporting energy across field lines and thus reducing thermal conduction losses [20]. It has been theorised that the magnetic field can also confine alpha particles, making them deposit more of their energy within the capsule before they escape, thus increasing P_α [20].

However, the magnetic field can also have deleterious effects. For example, the confinement of charged alpha particles by the field could prevent the efficiency by which they heat the capsule by limiting their energy deposition to a smaller region [21]. Understanding and mitigating the growth of hydrodynamic instabilities is of crucial implosion for ICF (and by extension MIF) schemes, as it reduces the efficiency of conversion of initial imploding capsule kinetic energy to fusion fuel thermal energy. The field changes the growth rate of these instabilities by altering the heat flow, suppressing some [22] and enhancing others [23–27]. The magnetic field can also introduce new instabilities (e.g. the magnetothermal instability in ref. [28]) by generating new components of heat flow that do not exist in an unmagnetised plasma [29, 30]. In spherical implosions, such as those carried out on the NIF, spherical symmetry is important to maintain because it leads to the maximum compression. The magnetic field introduces a preferential direction of heat flow, breaking this symmetry, and experiments have shown this leads to an implosion that is no longer spherical [31]. The fields can also complicate the interpretation of diagnostics, making experiments more difficult to understand [32–35].

The interplay of magnetic fields and plasma is complex and nonlinear. Depending on

its magnitude, the magnetic field can either directly impact the dynamics of the plasma or can affect it by altering the collisional phenomena that dictate the transport of heat and energy throughout the system. Designing an effective and robust MIF scheme requires an understanding of both the impact of the magnetic field on the plasma as well as how the plasma dynamics change the evolution of the field. This can only be done using numerical modelling, validated by experiments, to ensure that we understand the relevant parameter regimes. Conventionally, resistive magnetohydrodynamic (MHD) codes are used for this modelling. However, there are experimental results that indicate that this is not sufficient to explain all the observations that have been made. The various approaches used to model magnetised plasma, and in particular where they are insufficient and need to be extended to include additional physics, are the topic of this thesis. We review them in depth in chapter 2. The ultimate motivation of this work is that improved modelling of MIF schemes is crucial if they are ever to deliver a viable path to controlled nuclear fusion.

1.2.2 Examples of MIF Schemes

1.2.2.1 Z-Pinches

Z-pinches are the oldest and simplest nuclear fusion device [36]. An axial current passing through a azimuthally symmetric cylinder of plasma will produce an azimuthal B_θ magnetic field. This produces an inward confining Lorentz force on the plasma, counteracting the outward acting force due to the thermal pressure gradient. This device is known as a z-pinch.

In an ideal situation, this situation would be in steady state and fusion would occur in the confined plasma. However, in practice, z-pinches are infamously unstable to magnetohydrodynamic instabilities, causing them to break apart long before sufficient thermonuclear fusion reactions can occur. Early research into z-pinch devices realised this quickly [36] and the z-pinch was abandoned as a practical controlled nuclear fusion device. However, its use nowadays depends on its instability; the dense and hot plasma formed by an unstable z-pinch implosion is often used as an x-ray source [36]. Z-pinches are also the basis of many MIF schemes, such as the MagLIF scheme that we describe later in this chapter.

Z-pinches are of particular interest for this thesis because the edge of the pinch is an example of a vacuum-plasma interface. This is a region where the assumptions made in conventional models of magnetised plasma break down and additional physics emerges. We discuss this further in chapter 2.

1.2.2.2 Magnetised NIF Scheme

As discussed earlier, a major source of energy losses in an ICF implosion is thermal conduction away from the hot fuel. Recently, experiments on the NIF have explored using externally applied magnetic fields to mitigate this loss mechanism [31]. Unmagnetised NIF experiments have extremely stringent requirements to achieve ignition, such

as capsules manufactured with high precision so as not to provide any seeds for instabilities [37]. The purpose of magnetising the experiment is to mitigate these constraints, so fusion could be obtained more easily and reliably [38].

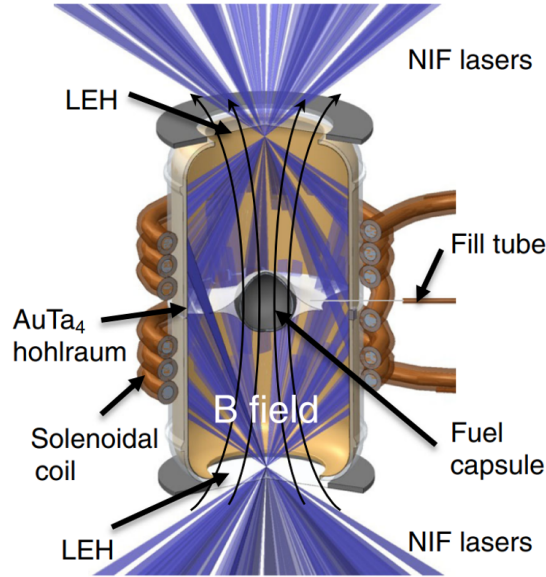


Figure 1.1: Image showing a schematic of the magnetised NIF indirect-drive ICF platform. A fusion capsule is contained in a gold-tantalum hohlraum, which is irradiated by lasers to produce x-rays. An external field coil wrapped around the hohlraum produces an axial magnetic field. Reproduced with permission from ref. [31].

Figure 1.1 shows a diagram of the target that is fired during these experiments. A fusion capsule, made up of deuterium fuel with a high-density-carbon (HDC) ablator, is present inside a gold-tantalum hohlraum. Rather than directly illuminating the capsule, which drives laser-plasma instabilities [39, 40], the laser is instead focused on the walls of the hohlraum. This heats the gold-tantalum, causing it to radiate x-rays, which then heat the surface of the capsule more uniformly [10]. This is an example of indirect-drive ICF. The HDC surface of the capsule heats and explodes, a process known as ablation, which causes the capsule to implode by the rocket effect. The innovation of the magnetised NIF campaign can be seen in the solenoidal coil that is wrapped around the hohlraum. A current is passed through this coil, generating an axial magnetic field through the capsule as shown by black arrows in the diagram. The interaction of the magnetic field with the hohlraum necessitated the use of a new material to make the hohlraum walls. The conventional gold hohlraums were too conductive, preventing the magnetic field produced by the coils from diffusing in, hence the use of the more resistive gold-tantalum alloy [41].

The thermal pressure of the plasma is so high in this experiment that the magnetic field effect is primarily in reducing the cross-field conduction losses [30]. However, this goes on to influence factors such as the shock structure in the implosion [42] which can substantially impact the shape of the capsule. Both experiments [31] and simulations

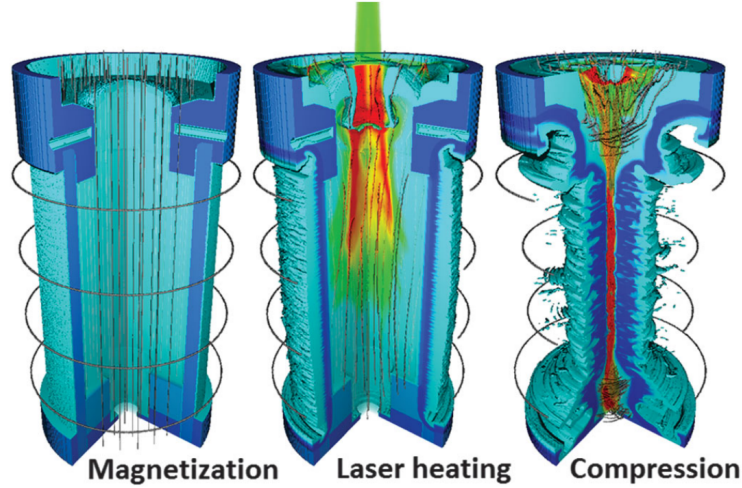


Figure 1.2: Simulation showing the stages of the MagLIF process. A cylindrical column of plasma (light blue) contained by a beryllium liner (dark blue) is axially premagnetised and preheated before being compressed by magnetic fields. Reproduced from ref. [47] with permission.

[38, 43, 44] showed that this effect led to a modest enhancement in ion temperature and yield, but a significant change in the shape of capsules that required retuning the laser drive. Understanding the impact of the magnetic field on the process, and by extension the evolution of the field, is crucial for effectively designing and interpreting experiments.

1.3 Magnetised Liner Inertial Fusion (MagLIF)

Controlled fusion using z-pinches, despite the aforementioned stability problems, is being pursued at Sandia National Laboratories using the Magnetised Liner Inertial Fusion (MagLIF) platform. This is the MIF scheme that we are most concerned with in this thesis. Here, rather than just thermally isolating the fuel as in magnetised ICF schemes, the magnetic field directly compresses the fusion fuel using magnetic pressure [45]. This magnetic drive is provided by the high current on pulsed power facilities such as the Z machine [46]. This differs from a normal z-pinch because the aim is not to remain in equilibrium, but to inertially compress fuel cylindrically to thermonuclear conditions.

An issue with trying to achieve nuclear fusion using z-pinches is that greater convergence is needed for a cylindrical implosion than a spherical implosion to reach the same stagnation densities and thus pressures [48]. Under this greater convergence, the plasma is more susceptible to instabilities. To address this problem, MagLIF differs from a conventional z-pinch in several ways. The fuel is contained within a dense cylindrical liner made of beryllium [48]. The liner improves the robustness of the implosion to Rayleigh-Taylor instabilities driven by magnetic compression [45]. At stagnation, it functions as a high density tamper to improve stability and to help to confine fusion products [47, 48]. Just before implosion, the fusion fuel is preheated using a pulse from

the Z beamlet laser [48]. This increases the temperature of the fuel to ion temperatures $T_i \sim 50 - 500\text{eV}$ [45], ionising the fuel. This is necessary to reach fusion temperatures when compressed. However, due to the slower compression in MagLIF compared to a NIF experiment, this heat will conduct out before stagnation so magnetic insulation is needed to contain the heat flows. This is done using external field coils that pre-magnetise the fuel with an axial B_z field of approximately 10T before it implodes [48]. The magnetic field can also act to improve the stability during the implosion [45]. The combination of these effects is that the required cylindrical convergence to reach fusion conditions is significantly reduced, meaning that a relatively rapid ($\sim 100\text{ns}$) implosion on a pulsed power facility, such as Z, is a practical route to fusion [48].

Figure 1.2 is a schematic showing the stages of the MagLIF implosion. The fuel is axially premagnetised and preheated by Z beamlet. When the Z machine discharges, large B_θ magnetic fields are formed around the liner due to the MA-scale axial currents. This azimuthal magnetic field then compresses the liner radially, forming a hot and dense core. The axial magnetic field is also compressed with the plasma, providing thermal insulation at stagnation. The implosion is subsonic, to prevent shock-driven instabilities such as Richtmeyer-Meshkov [49], and takes around 100ns. The cylinder is around 1cm tall and 2mm in radius [45]. Significantly longer liners and far higher (around two orders of magnitude more) laser preheat energy would be required for μs duration implosions [48].

Both the inner and outer surfaces of the imploding liner are subject to instability growth. At the outer surface, the magnetic field that is compressing the liner behaves as an infinitely light fluid pushing on a denser fluid, driving the magneto Rayleigh Taylor (MRT) instability. This can be seen in the last panel of figure 1.2. The inner surface can also be vulnerable to instabilities at high convergence ratios, leading to liner material mixing into the fuel. At the moment, unexplained mix of high Z materials into the fuel [50, 51] and growth of these instabilities are major limiting factors for MagLIF implosions [52].

Currently, MagLIF experiments have only been carried out with non-cryogenic deuterium fuel as the Z facility due to absence of a cryostat and difficulties in handling tritium. However, despite this, neutron yields of more than 10^{13} DD neutrons have been measured [52]. It is a significant technical challenge to engineer targets for cryogenic implosions, though this work is ongoing [53–55].

There have been proposals to build a next generation pulsed power facility to replace the Z machine, which has existed in its current form since 1996 [46]. This new machine would be significantly larger and, crucially, have a much larger peak current of more than 60MA. With currents this large, fusion yields of up to 100MJ could potentially be obtained [17, 45, 56]. Thoroughly understanding the physics that underlies MagLIF is crucial to making it work as a viable energy scheme as well as to scaling the platform to work effectively at higher peak currents [57, 58].

1.3.1 Magnetic fields in the MagLIF scheme

MagLIF is an integrated experiment with complex physics throughout. In this thesis, we are particularly concerned with the places in the MagLIF scheme where magnetic fields have a significant impact on the behaviour.

As stated before, the purpose of premagnetising the fuel with an axial field is to thermally insulate the fuel and reduce the convergence requirements for fusion conditions. In principle, this field will be compressed with the fuel, reaching a magnitude of up to 10^4T at stagnation [19, 45, 48]. However, the Nernst effect has been shown in simulation to reduce this flux compression [45, 48] and thus the efficiency of the thermal insulation, leading to greater than expected heat losses into the liner [59]. This occurs due to the laser preheat stage, which creates a radially outward temperature gradient that the Nernst effect advects magnetic fields down. This has been confirmed in experiments studying the scaling of yield with preheat energy [52]. Accurately calculating the magnetic field in the implosion requires including this effect, particularly if alpha particles are produced, as it is important to understand whether the fusion products will be confined.

The azimuthal magnetic field driving the target creates an MRT instability that also rips apart the liner [17]. It is believed that the electrothermal instability (ETI), where perturbations in the resistivity of the liner lead to variations in the density and temperature via Ohmic heating [60], is the seed for these MRT instabilities [17, 61]. Characterising the growth rate of the MRT and understanding how it interacts with the ETI is crucial for designing a liner that is robust to this instability, and thus also for efficient fusion.

The axial magnetic field is significantly smaller than the azimuthal magnetic field in MagLIF implosions [45]. However, x-ray radiographs taken of the liner as it implodes regularly show helical striations in density [62]. This can be seen in figure 1.3. This is surprising as the dominant B_θ field implies that the striations should appear horizontal from this line of sight. This structure was not seen if the liner was initially unmagnetised, implying it is related to the initial B_z field. Some of the current theories explaining the formation of this helical structure are related to the Hall effect [17]. It has been proposed that low density plasma from the transmission lines may be compressed by the Hall effect onto the liner, increasing the B_z field and leading to the observed helix [63]. Alternatively, recent work has shown that simply including the Hall effect within simulations can lead to the observation of currents going helical in liner implosions [64, 65] and in gas puff z-pinch [66], potentially driving the helical instability. Resistive MHD simulations that do not include the Hall effect do not reproduce this behaviour. It should be remarked that these are currently just theories, there is significant debate about the true cause of the helical perturbation. These results indicate that more detailed modelling of the magnetic field behaviour is needed to fully understand the experiment; conventional resistive MHD models are insufficient.

Looking further away from the load, accurately characterising the power flow through

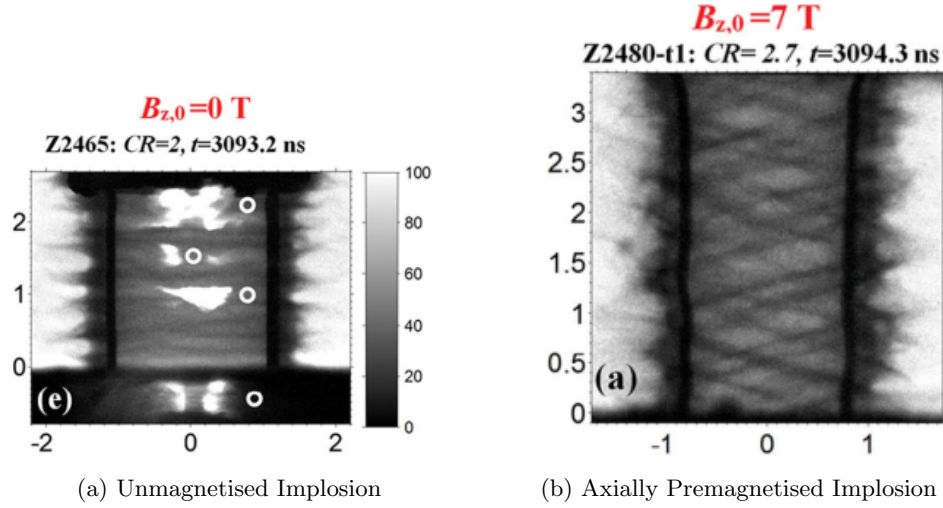


Figure 1.3: Radiographs taken of the MagLIF liner during implosion. In figure 1.3a, no initial axial field is applied and the striations are azimuthal. In figure 1.3b, an axial field of 7T is applied and the striations are helical. These radiographs were reproduced from ref. [62] with permission. The grayscale bar shows the transmission percentage.

the transmission lines is crucial for any pulsed power experiment. To understand this, we briefly summarise the flow of energy through a Z experiment. A more detailed description can be found in many references (e.g. [46, 67, 68]). A shot on the Z facility begins by charging the banks of capacitors that surround the load. There are 36 Marx generators around the perimeter, each composed of 60 $2.6\mu F$ capacitors. These can each be charged to 90kV, for a total of $> 22\text{MJ}$ energy stored in total [46]. This energy is then discharged over a few μs into the pulse forming section, which compresses the electrical energy in time to produce a peak electrical power of approximately 80TW in a 100ns long current pulse [46].

The currents on Z are so large that it would cause any dielectric material across the transmission lines to break down. For this reason, magnetically insulated transmission lines (MITLs) are used [68]. These are transmission lines under vacuum, which work by magnetising electrons in the vacuum so that if their Larmor radius is smaller than the electrode gap, then any electrons emitted by the cathode return to the cathode rather than reaching the anode and causing current loss. The process by which the electrodes break down to produce these electrons differ on the cathode and anode sides and is still not fully understood [68]. An important effect however is believed to be due to contaminants in the transmission lines that are desorbed due to the electric field stress [46, 69]. It is desirable to minimise current losses in these transmission lines, to maximise the electrical energy delivered to the target. However, this requires understanding the power flow through this region. The magnetic and electric fields are extremely high, due to the large current, and the plasma in the region is very low density. For these reasons, this process is very challenging to model. However, understanding the power flow is

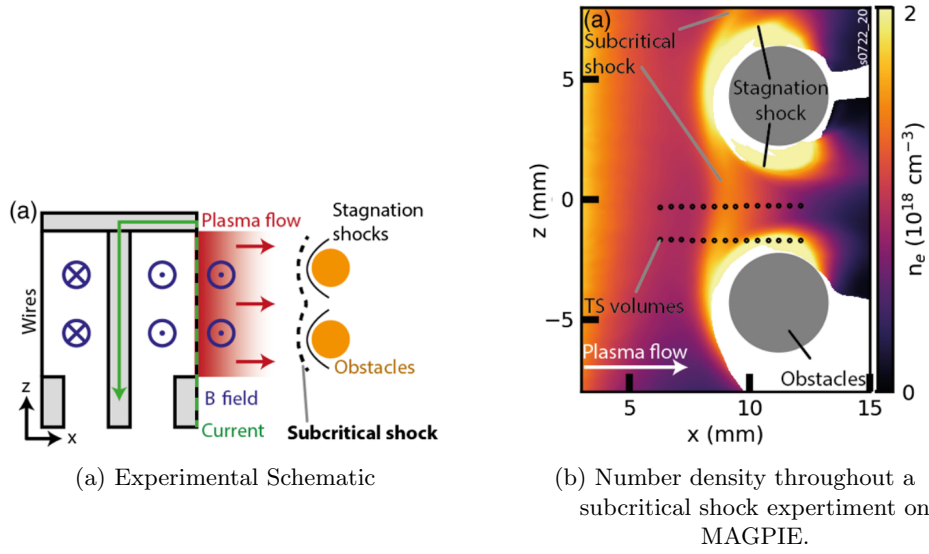


Figure 1.4: Schematic showing how plasma flow from an inverse wire array sets up shocks when collided into obstacles. Reproduced from ref. [72] with permission. Figure 1.4a shows a schematic of the experiment and figure 1.4b shows the number density in the experiment recreated from an interferogram.

particularly important for scaling Z to a larger facility, as the even higher electric field stresses could put power flow into a different regime [46, 70]. It has also been suggested that this low-density plasma desorbed from the electrodes could impact on the target, causing unexpected energy losses or instabilities [46, 63, 71]. Understanding the physics of magnetised low density plasmas is important for modelling power flow, and therefore also for achieving fusion on a pulsed power facility.

1.4 Examples from Laboratory Astrophysics Experiments

As well as in magneto-inertial fusion, magnetic fields often play a key role in laboratory astrophysics experiments. These are experiments designed to explore physics relevant to astrophysical phenomena by studying them scaled down to the laboratory [12]. They are regularly carried out at the MAGPIE pulsed power machine at Imperial College London, which is similar to the Z machine but smaller in scale [73]. Experiments have been carried out into rotating plasmas [74, 75], radiatively driven plasmas [76], reconnection [77–79], turbulence [80, 81], magnetised shocks [72, 82] and many other phenomena [83].

These experiments are commonly modelled using resistive MHD codes to help interpret the experiments and inform future designs. Understanding the behaviour of magnetised plasma is crucial to carrying out this process effectively. However, there are results from MAGPIE that indicate that purely resistive MHD modelling is insufficient to explain certain observations. In this thesis we are particularly concerned with the

breakdown of conventional plasma models in low density plasma, so here we will discuss the experimental results that show tension with simulations in this regime.

As an example, a recent study on MAGPIE looked at the structure of shocks in a magnetised plasma [72]. Figure 1.4a shows the experimental setup. A current passing through an inverse wire array heats the wire and produces a plasma that is accelerated outwards by the $\mathbf{j} \times \mathbf{B}$ Lorentz force. Obstacles are placed in this plasma flow, which sets up shocks around the obstacles. These shocks can be seen in figure 1.4b which shows the electron density throughout the domain reconstructed from an experimental interferogram. Notably, a subcritical shock forms which is a shock where there is a super-to-sub transition of the magnetosonic Mach number, but not the sonic Mach number which is above 1 everywhere [84]. This experiment was the first to measure a subcritical shock in the collisional regime in the laboratory and established that the shock width was determined by the resistive diffusion length scale in accordance with prior theoretical predictions [72]. Resistive MHD simulations were carried out to model the experiment. These were able to match the width of the shock, but only could predict the correct shock location after the magnetic field in simulations was artificially increased by a factor of approximately 2.5 above the experimentally measured value. It has been suggested that this may be due to two-fluid effects, specifically the Hall effect, being neglected in the simulations [72]. The shock width is comparable in size to the ion inertial length $\lambda_i = \frac{c}{\omega_{pi}}$ and early in time, this length scale is larger than the resistive diffusion length. This could mean that early in time, the Hall effect is important in the formation of the shock which the authors hypothesise could lead to the subcritical shock forming further upstream. A larger field is required in simulations that do not include this effect in order to push the shock downstream to the location that is observed in the experiment [72].

A different experiment motivated by accretion flows in astrophysics explored differentially rotating plasmas [74]. Figure 1.5 shows the set up for this experiment. Plasma flow is produced by passing large currents through wires laid out in a cylindrical geometry, as shown in panel b) of the figure. The return posts on the outside of the wires through which the current flows are rotated azimuthally slightly from the position of the wires. This means that the inward plasma flow from each wire is offcentre when they converge at the central axis, producing a rotating plasma column as seen in panel a) [74]. An axial jet is launched from the rotating plasma column, due to axial pressure gradients, as can be seen in panel d) of the figure. This jet is surrounded by a low density plasma halo that can be seen in this figure. An unexpected result, given that it is a hot rotating plasma, was that this jet remained highly collimated [75]. It is believed that the low density plasma halo plays a key role in maintaining this collimation. Within the halo, the plasma parameters vary strongly on scales on the order of the ion inertial length and the presence of a large magnetic field (with $|B| \sim 3\text{T}$) is inferred [74]. It is this additional magnetic field which tamps the expansion of the jet. Resistive MHD simulations do not show the same jet collimation [75]. These simulations do not

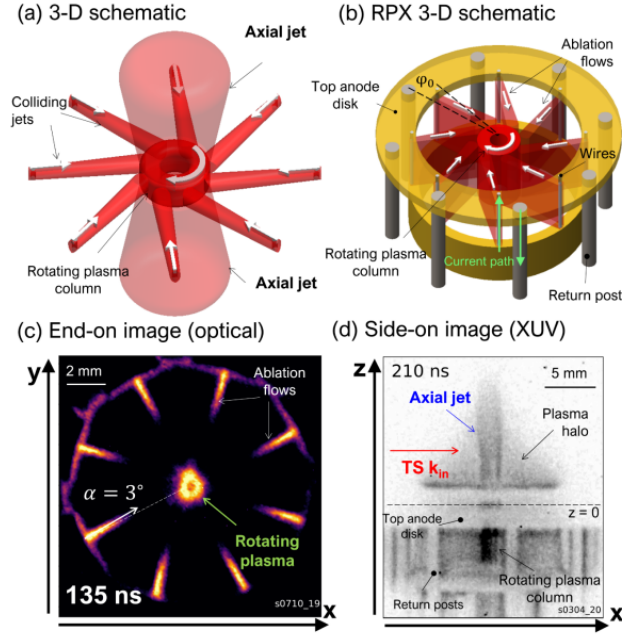


Figure 1.5: Figures showing the schematic of the rotating plasma experiment carried out on MAGPIE. Panel a) shows a schematic of the experiment and b) shows the experimental hardware. Panel c) shows optical self-emission imaging taken end-on. Panel d) is an extreme ultraviolet (XUV) image of the rotating plasma. This figure was reproduced from ref. [74] with permission.

include the Hall effect so would not accurately capture the behaviour of the plasma on ion inertial length scales.

As we highlighted at the beginning of this section, a vast range of laboratory astrophysics experiments are carried out on MAGPIE. We have only discussed a couple of recent experiments where extended MHD effects are believed to be relevant. There are many other potential phenomena that cannot be explained by purely resistive MHD in laboratory astrophysics experiments. For example, a key property of the Hall effect is that it breaks the symmetry of the resistive MHD system, so reversing the polarity of electrodes can qualitatively change results [83]. However, the Hall effect has not yet been directly measured in experiments in the pulsed power regime, meaning simulations of it are currently not benchmarked.

1.5 Thesis Aims

As we have laid out over the past few sections, magnetic fields play a central role in magneto-inertial fusion schemes and laboratory astrophysics experiments. Understanding their impact on plasma dynamics, heat flow and particle confinement as well as their transport in magnetised plasma is vital to successfully interpreting and designing future experiments. The nonlinear complex physics they introduce predicates the use of simulations. However, there is a wealth of experimental observations in magnetised plasma that cannot be well explained by resistive MHD, which is the conventionally used

approach. Alternative simulation paradigms, such as kinetic codes, are too expensive to routinely be used to model experiments. Therefore, the key aim of this thesis is to extend a resistive MHD code to include the crucial missing physics to simulate vacuum-plasma interfaces and low-density plasma. A key point is that all models used in plasma physics are approximations to some extent and it is not possible to construct a model that includes all the necessary physics to fully describe magnetised plasma. However, we aim to evaluate which are the dominant effects that are not included within current codes, discuss their effects and consider how a reduced model of them might be included within existing simulation tools. Having done this, the intention is to see whether this improved MHD model is better able to match experimental observations.

As part of this, it is necessary to study which magnetised transport effects are the most relevant for pulsed power driven MIF schemes and laboratory astrophysics. To do this, in chapter 4, we develop a semi-analytic tool to model the evolution of an interface between magnetised plasmas. This is useful both to rapidly scan over parameter space and establish which magnetised transport effects are dominant, but also to benchmark more detailed extended MHD codes. It is important to see how resistive MHD breaks down at a vacuum-plasma interface in order to determine which additional physics needs to be included in models. This is studied theoretically in chapter 2 and then with kinetic simulations in chapter 5. Finally, in chapter 6 we extend the Gorgon multi-physics code to account for the leading order corrections to resistive MHD and use it to study Hall physics instabilities and the evolution of an unstable z-pinch.

1.6 Thesis Outline

The thesis is split into the following chapters:

- **Chapter 2:** The key background theory that we rely on for the remainder of the thesis is outlined. How the kinetic description of a plasma can be reduced to a more tractable fluid model is shown. The closure of classical transport theory and its realm of validity is discussed. We then derive the different MHD limits of interest for this thesis. Finally, a simplified model of a vacuum-plasma interface is studied theoretically to elucidate the different physics effects that need to be considered.
- **Chapter 3:** The numerical tools and methods used for the results in this thesis are discussed. The Gorgon multiphysics code is introduced and the MHD algorithm in particular is described in detail. The various methods of simulating kinetic physics in plasmas are briefly reviewed and the particle-in-cell approach summarised.
- **Chapter 4:** A semi-analytic self-similar model to study the evolution of an interface between magnetised plasmas is developed. This method is used to perform a scan over plasma β , to show how the balance of magnetised transport terms changes in different parameter regimes. We then specialise to studying the key

transport phenomena at a vacuum-plasma interface. The self-similar model is used to establish what part of parameter space produces the largest observable Nernst effect in the design of a theorised pulsed power experiment.

- **Chapter 5:** The EPOCH PIC code is used to carry out fully kinetic simulations of the edge of a z-pinch as an example of a vacuum-plasma interface. The electric field and distribution function are compared to MHD results to highlight the dominant physics missing from the fluid approach for this problem.
 - **Chapter 6:** A new MHD solver that includes the Hall effect is developed for the Gorgon code. The theoretical basis for including electron inertia within the MHD update is laid out and the numerical implementation is described. It is then tested thoroughly by comparing to dispersion relations for wave modes driven by the Hall effect. The new solver is then used to study the evolution of an $m=0$ sausage instability in a z-pinch.
 - **Chapter 7:** The conclusions from the other chapters are summarised and potential avenues of future work are highlighted.
-

2 Theory

In this work, we carry out theoretical and computational studies of the behaviour of plasma at vacuum-plasma interfaces. We set this into context in this chapter by reviewing the fundamental plasma physics theory that we rely on throughout.

We begin by defining a plasma and highlighting why its behaviour differs from a neutral gas (§2.1). This leads to a discussion of the kinetic theory of plasmas, the most detailed model used to describe their behaviour (§2.2). We show how this theory can be reduced to more tractable fluid theories, such as classical transport theory (§2.2.5). In particular, we discuss the fluid theory of magnetised plasmas and how reduced models such as ideal and resistive magnetohydrodynamics (MHD) arise from it (§2.3). Finally, we discuss the breakdown of simple theories for plasmas at a vacuum-plasma interface (§2.4), motivating the work in this thesis.

2.1 Basic Plasma Physics

A plasma is an ionised gas that is quasineutral and exhibits collective behaviour [4]. It consists of ions, unbound electrons and neutral particles that are individually free to move. Charged particles interact via the long-range Coulomb potential, in contrast to the binary collisions of a neutral gas. This means that each particle interacts simultaneously with many other particles, driving a rich variety of collective phenomena such as waves and instabilities.

A key property of plasmas is that they exhibit quasineutrality. This means that on a large enough length scale L , the plasma has no net charge:

$$\sum_{\alpha} q_{\alpha} N_{\alpha} = 0, \quad (2.1)$$

where q_{α} is the charge of species α , N_{α} is the number of particles of that species and the sum is over all the species in a plasma. For a fully ionised plasma with just one ion species of charge $+Ze$, this implies:

$$n_e = Z n_i, \quad (2.2)$$

where n_e and n_i are the electron and ion number densities respectively. This property arises due to the free movement of charged particles. If a charge imbalance arises at some point in the plasma, this will set up an electric field due to the presence of a charge density. This will cause charged particles to redistribute due to the Coulomb force to

neutralise it. This charge-screening effect is known as Debye shielding and it defines the length scale above which a plasma can be considered quasineutral, known as the Debye length:

$$\lambda_D^2 = \frac{\epsilon_0 T_e}{n_e e^2}, \quad (2.3)$$

where T_e is the electron temperature in energy units. For $L \gg \lambda_D$, the plasma is quasi-neutral and there is no bare charge. This defines a spherical region of radius λ_D in which there are significant electric fields due to unscreened charge imbalances: the Debye sphere. For Debye screening to be valid, it is necessary to assume that the number of particles in the Debye sphere is large:

$$n_e \lambda_D^3 \gg 1. \quad (2.4)$$

This quantity is known as the plasma parameter. Any perturbations to the charge-screening will drive oscillations of the electrons that restore the system to quasi-neutrality, with a characteristic timescale defined by the plasma frequency:

$$\omega_{pe}^2 = \frac{n_e e^2}{m_e \epsilon_0}. \quad (2.5)$$

For timescales $\tau \gg \omega_{pe}^{-1}$, it can be assumed that the charges have redistributed and screened any electric fields.

2.2 Kinetic Theory of Plasmas

2.2.1 The distribution function

The most detailed description of a plasma would be to evolve every particle in the system according to the Lorentz force:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (2.6)$$

evolving the electric and magnetic fields self-consistently using Maxwell's equations. However, this would be intractable for any real problem and be difficult to interpret due to random fluctuations.

Instead, we work in terms of the distribution function $f(\mathbf{r}, \mathbf{v}, t)$ which resides in a 6 dimensional phase space. This is defined so that for a species α , $f_\alpha(\mathbf{r}, \mathbf{v}, t) d^3\mathbf{r} d^3\mathbf{v}$ is the number of particles in an infinitesimal volume $d^3\mathbf{r} d^3\mathbf{v}$ at a point $(\mathbf{r}, \mathbf{v})$ in phase space at time t . Explicitly, $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ is the number density of particles of species α at time t at a point defined by position vector $\mathbf{r}$ and velocity vector $\mathbf{v}$ in phase space. The evolution of this distribution function is given by the Boltzmann equation [85]:

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} f_\alpha + \frac{q_\alpha}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_\alpha = \left(\frac{\partial f_\alpha}{\partial t} \right)_{coll} \quad (2.7)$$

with each species with mass m_α and charge q_α satisfying its own Boltzmann equation. ∇_r and ∇_v are the gradients with respect to the position space coordinates and the velocity space coordinates respectively. The kinetic equation is a statement of the conservation of phase space density. The term on the right hand side of the equation gives the change in the distribution function due to collisions between particles. This is discussed later in section 2.2.4. Whilst the motion of charged particles generates randomly fluctuating microscopic electric and magnetic fields, Debye shielding and the averaging procedure used to obtain a distribution function allows us to separate these microscopic fields out into the collision term [86]. The electric and magnetic field that then appear in equation (2.7) are averaged macroscopic fields. These fields evolve by Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0} \quad (2.8)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2.9)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (2.10)$$

$$\nabla \times \mathbf{B} = \frac{1}{\mu_0 \epsilon_0} \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}, \quad (2.11)$$

where ρ_c and $\mathbf{j}$ are the charge and current density respectively.

Plasmas can have a wide range of distribution functions in different physical situations, but the most important is the Maxwellian distribution. For electrons, it can be written as:

$$f_m(v) = n_e \left(\frac{1}{2\pi v_{th}^2} \right)^{3/2} \exp \left(-\frac{v^2}{2v_{th}^2} \right), \quad (2.12)$$

where the standard deviation v_{th} is known as the thermal speed, defined by:

$$v_{th} = \sqrt{\frac{2T_e}{m_e}}. \quad (2.13)$$

This is important because the effect of collisions is to relax any distribution function towards a Maxwellian, meaning plasmas in thermal equilibrium have a Maxwellian distribution.

Solving the kinetic equation gives the most detailed description of a plasma and is crucial for predicting certain important phenomena such as Landau damping [4]. There has been extensive research into methods for solving the kinetic equation, which we discuss in section 3.4.1. However, it is a complex theory and difficult to solve for most problems. Therefore, often simplifications are made which are described in the next section.

2.2.2 Obtaining fluid equations from the kinetic equation

Kinetic theory is a microscopic theory, but often we are interested in the macroscopic behaviour of the plasma that it is describing. We only care in the evolution of the distribution function insofar as it tells us about the measurable observables of the plasma. We convert to this macroscopic theory by taking moments of the distribution function. For any function of velocity $\psi(\mathbf{v})$, its velocity space average is given by:

$$\langle \psi_\alpha \rangle = \frac{\iiint_v \psi f_\alpha d^3\mathbf{v}}{\iiint_v f_\alpha d^3\mathbf{v}} \quad (2.14)$$

This integral is over the whole velocity space, parametrised by the coordinates $\mathbf{v} = (v_1, v_2, v_3)$. $\psi(\mathbf{v})$ is some physical property of the particles in the plasma. By doing this integration, we are integrating out the velocity space components of the particles to understand how the quantity ψ varies in real (position) space. In performing this, we are assuming that $f_\alpha \rightarrow 0$ as $v \rightarrow \infty$ fast enough to give this integral a finite value.

Whilst in principle the choice of ψ is completely arbitrary, only a few functions have a useful physical interpretation. These are $\psi(\mathbf{v}) = (1, \mathbf{v}, v^2, w^2\mathbf{w}, \mathbf{w}\mathbf{w})$, which give rise to number density, fluid velocity, total energy density, heat flux and the pressure tensor respectively:

$$n_\alpha(\mathbf{r}, t) = \iiint_v f_\alpha(\mathbf{r}, \mathbf{v}, t) d^3\mathbf{v} \quad (2.15)$$

$$\mathbf{u}_\alpha(\mathbf{r}, t) = \langle \mathbf{v} \rangle = \frac{1}{n_\alpha} \iiint_v \mathbf{v} f_\alpha(\mathbf{r}, \mathbf{v}, t) d^3\mathbf{v} \quad (2.16)$$

$$U_\alpha(\mathbf{r}, t) = \frac{1}{2} m_\alpha n_\alpha \langle v^2 \rangle = \frac{1}{2} m_\alpha \iiint_v \mathbf{v} \cdot \mathbf{v} f_\alpha(\mathbf{r}, \mathbf{v}, t) d^3\mathbf{v} \quad (2.17)$$

$$\mathbf{q}_\alpha(\mathbf{r}, t) = \frac{1}{2} m_\alpha n_\alpha \langle w^2 \mathbf{w} \rangle = \frac{1}{2} m_\alpha \iiint_w (\mathbf{w} \cdot \mathbf{w}) \mathbf{w} f_\alpha(\mathbf{r}, \mathbf{v}, t) d^3\mathbf{w} \quad (2.18)$$

$$\underline{\pi}_\alpha(\mathbf{r}, t) = m_\alpha n_\alpha \langle \mathbf{w}\mathbf{w} \rangle = m_\alpha \iiint_w \mathbf{w}\mathbf{w} f_\alpha(\mathbf{r}, \mathbf{v}, t) d^3\mathbf{w}. \quad (2.19)$$

We have defined the vector $\mathbf{w}$ in these integrals, which is the deviation of the particle velocity from the mean fluid velocity: $\mathbf{w} = \mathbf{v} - \mathbf{u}_\alpha$. This is the velocity of the particle in the reference frame in which the fluid as a whole is at rest. We can also use these definitions to calculate the charge density and current density for Maxwell's equations:

$$\rho_{c,\alpha} = q_\alpha n_\alpha = q_\alpha \iiint_v f_\alpha d^3\mathbf{v} \quad (2.20)$$

$$\mathbf{j}_\alpha = q_\alpha n_\alpha \mathbf{u}_\alpha = q_\alpha \iiint_v \mathbf{v} f_\alpha d^3\mathbf{v}. \quad (2.21)$$

We can also take moments of the Vlasov equation itself by multiplying it by the function ψ and integrating over velocity space. In doing this, we can derive equations that describe the evolution in time of the velocity space averages $\langle \psi \rangle$. The details of

this are shown in Appendix B.1. The zeroth moment gives the continuity equation:

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{u}_\alpha) = 0, \quad (2.22)$$

where we have assumed that the collision term on the right hand side of equation (2.7) only changes the velocity of particles and there are no processes like recombination or ionisation that could change n_α . This equation describes how in some finite volume, the number of particles can only change due to a flux through the surface of that volume.

The first moment yields the momentum equation:

$$\frac{\partial(\rho_\alpha \mathbf{u}_\alpha)}{\partial t} + \nabla \cdot (\rho_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha + \underline{\pi}_\alpha) = n_\alpha q_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) + \mathbf{R}_\alpha. \quad (2.23)$$

The quantity $\rho_\alpha \mathbf{u}_\alpha$ is the momentum density of the fluid and $\rho_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha$ is its flux. It is useful to rewrite the pressure tensor $\underline{\pi}_\alpha$ as the combination of a scalar pressure p_α and an anisotropic stress tensor:

$$\pi_{\alpha,ij} = m_\alpha n_\alpha \langle w_i w_j \rangle = p_\alpha \delta_{ij} + \sigma_{\alpha,ij}, \quad (2.24)$$

where δ_{ij} is the Kronecker delta. The anisotropic stress is usually known as viscosity. In the work in this thesis, we assume that the plasma is inviscid and so we always drop the $\underline{\sigma}$ term. The right hand side of equation (2.23) is the source terms for momentum density. The first describes the effect of electromagnetic fields on the species. The final term is:

$$\mathbf{R}_\alpha = m_\alpha \int_v \mathbf{v} \left(\frac{\partial f_\alpha}{\partial t} \right)_{coll} d^3 \mathbf{v}, \quad (2.25)$$

which describes the change in momentum density due to collisions. As in general plasmas consist of more than one species, this term can be written as:

$$\mathbf{R}_\alpha = \sum_\beta \mathbf{R}_{\alpha\beta}, \quad (2.26)$$

where $\mathbf{R}_{\alpha\beta}$ is the change in momentum density to collisions between particles of species α and species β . The sum is over all species in the plasma, including electrons. Note that $\mathbf{R}_{\alpha\alpha} = 0$ since collisions conserve momentum and that $\mathbf{R}_{\alpha\beta} = -\mathbf{R}_{\beta\alpha}$.

The second moment of the kinetic equation is the energy equation, which can be rewritten in terms of the pressure p_α :

$$\frac{3}{2} \frac{\partial p_\alpha}{\partial t} + \nabla \cdot \left(\frac{3}{2} p_\alpha \mathbf{u}_\alpha \right) + p_\alpha \nabla \cdot \mathbf{u}_\alpha + \nabla \cdot \mathbf{q}_\alpha = Q_\alpha. \quad (2.27)$$

This is an equation for the energy balance of the plasma. The second term on the left hand side is the advection of energy density (pressure) with the fluid velocity and the third is the heating of the plasma by $p dV$ work. $\mathbf{q}_\alpha$ is the heat flux as defined above.

Q_α is the heat generated due to collisions between particles:

$$\begin{aligned} Q_\alpha &= \int_v \frac{m_\alpha v^2}{2} \left(\frac{\partial f_\alpha}{\partial t} \right)_{coll} d^3v - \mathbf{R}_\alpha \cdot \mathbf{u}_\alpha \\ &= \int_w \frac{m_\alpha^2 w^2}{2} \left(\frac{\partial f_\alpha}{\partial t} \right)_{coll} \end{aligned} \quad (2.28)$$

The scalar pressure p_α has an important relationship to the temperature, density and energy density. This relationship is known as the equation of state (EoS) and is a key part of plasma physics modelling. Advanced EoS that account for more realistic physics such as molecular effects and even quantum behaviours such as Fermi degeneracy can be developed [87]. For all of the work in this thesis however, we consider the case of an ideal plasma. This is a plasma whose EoS is just the ideal gas law, where $p_\alpha = n_\alpha T_\alpha$ for each species with T_α the temperature of that species. The pressure is proportional to the energy density in this case. This approximation is appropriate for low density and high temperature plasmas, which is true of most of our work. This is an assumption that could be revisited for the dense z-pinch simulations in chapter 6 however. Kinetic theory generally assumes this EoS and to our knowledge there are no kinetic codes that relax this assumption, though theoretical work does exist to do so (e.g. ref. [88, 89]).

2.2.3 Collisions

It is important to discuss collisions in a plasma since these have an important effect on the distribution function. As discussed before, collisions cause a distribution function to relax to a Maxwellian distribution. In this section, we discuss how collisions in a plasma differ from that of a neutral gas.

In a neutral gas, interactions between particles are only due to short-range forces which act at distances comparable to the size of the interacting particles. This means that collisions are binary and generally the particles deflect by an angle on the order of 90 degrees (a so-called large angle collision). In contrast in a plasma, the particles interact with all the other particles in the Debye sphere as well as the macroscopic electromagnetic fields. This leads to a charged particle deflecting through many simultaneous small angle collisions. We account for this mathematically by considering the cumulative effect of many small angle collisions as a single large angle collision, which we can then treat as binary collisions. For the neutral gas case, it is useful to define a collision time τ_{coll} and a mean free path λ_{mfp} which are respectively the time and distance travelled by the particle between collisions. For a plasma, τ_{coll} is instead defined as the total time over which the cumulative deflection to a particle's angle reaches 90° and the mean free path is defined as the distance travelled by the particle during this process.

Performing this analysis, it can be shown [86] that the collision time for electron-ion

collisions in a plasma is given by:

$$\tau_{ei} = \frac{3\sqrt{m_e}}{4\sqrt{2}\pi e^4 Z^2 n_i \ln \Lambda} T_e^{3/2}, \quad (2.29)$$

where T_e is the electron temperature in energy units, n_i is the ion density and $\ln \Lambda$ is the Coulomb logarithm. The Coulomb logarithm is defined:

$$\ln \Lambda = \ln \left(\frac{b_{max}}{b_{min}} \right), \quad (2.30)$$

where b_{max} and b_{min} are the maximum and minimum impact parameters for a collision. These are cut-offs introduced in the special case of Coulomb collisions in a plasma, to account for the screening of the potential by Debye shielding. Normally, we take $b_{max} = \lambda_D$, since particles do not interact with the electric fields of particles outside their Debye sphere. We take b_{min} to be the maximum of the de Broglie radius, since then classical Rutherford scattering is not valid, and of the impact parameter for large angle scattering b_0 [4]. For the analysis of classical transport theory to be valid, $\ln \Lambda \gg 1$. This is usually the case for most plasmas we consider, although it does break down for denser and colder plasmas.

The electron-electron collision time $\tau_{ee} \sim Z\tau_{ei}$. Ion collisions are dominated by ion-ion interactions, since electrons are too light to significantly deflect the ions [86]. The ion-ion collision time is then given by:

$$\begin{aligned} \tau_{ii} &\sim \frac{1}{Z^2} \sqrt{\frac{m_i}{m_e}} \left(\frac{T_i}{T_e} \right)^{3/2} \tau_{ei} \\ &= \frac{3\sqrt{m_i}}{4\sqrt{\pi}(Ze)^4} \frac{T_i^{3/2}}{n_i \ln \Lambda}, \end{aligned} \quad (2.31)$$

where T_i is the ion temperature. From now on, we use $\tau_e = \tau_{ei}$ and $\tau_i = \tau_{ii}$ for ease. This then defines electron and ion mean free paths:

$$\lambda_{mfp,e} = v_{th,e} \tau_e \quad (2.32)$$

$$\lambda_{mfp,i} = v_{th,i} \tau_i \quad (2.33)$$

In contrast to a neutral gas, particles in magnetised plasmas do not travel in straight lines between collisions. Particles gyrate around magnetic field lines with a gyrofrequency Ω_c , tracing a Larmor orbit of radius r_L . Each collision will shift the guiding centre of this orbit, causing the particle to jump to a new orbit. This behaviour defines an important quantity in magnetised plasmas, known as the electron Hall parameter:

$$\chi_e = \Omega_e \tau_e = \frac{e|B|}{m_e} \tau_e. \quad (2.34)$$

A similar quantity can be defined for ions. The Hall parameter is a measure of how many Larmor orbits a particle completes on average between collisions. It can also be

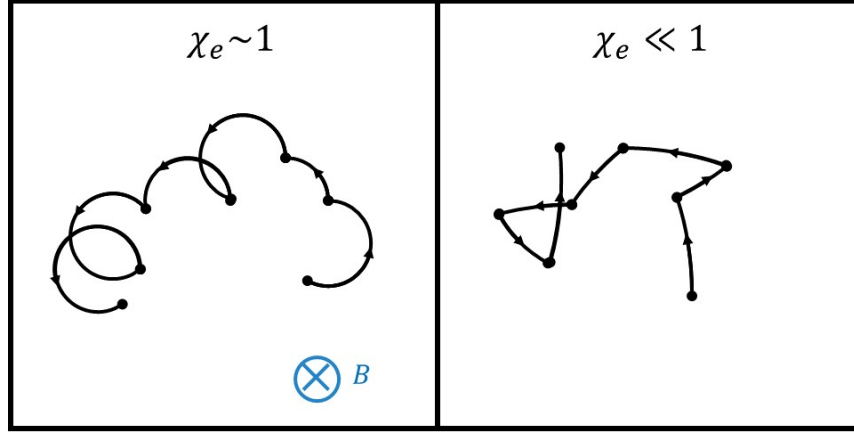


Figure 2.1: Diagram showing the behaviour of electrons in a magnetised plasma with $\chi_e = \Omega_e \tau_e \sim 1$ and in an unmagnetised plasma with $\chi_e \ll 1$. The arrows represent the direction of the electron velocity and the points represent its position at each collision.

viewed as the ratio of λ_{mfp} to r_L . This quantity defines the magnetisation of the plasma, so we describe a plasma with $\chi_e \gtrsim 1$ as magnetised. In an unmagnetised plasma, $\chi_e < 1$ and the particle does not complete a full Larmor orbit between collisions. In this case, the collisions rather than the Larmor orbits dictate the behaviour of the plasma. In contrast, in a magnetised plasma with $\chi_e \gtrsim 1$ the particle completes several Larmor orbits between collisions. This means that the magnetic field has an effect on the particle motion and thus the transport. These two cases are shown in figure 2.1.

2.2.4 The collision operator

All of the information about collisions in the plasma is contained in the collision operator on the right hand side of equation (2.7). For collisions between species α and species β , we can write the change in the distribution function of species α due to these collisions as:

$$\left(\frac{\partial f_\alpha}{\partial t} \right)_{coll} = C[f_\alpha, f_\beta] = C_{\alpha\beta}, \quad (2.35)$$

where C is a functional of the two distribution functions. The total effect of collisions on f_α can then be summed over all ion species: $C_\alpha = \sum_\beta C_{\alpha\beta}$. As stated before, the effect of collisions is to relax the distribution function towards a Maxwellian. The collision operator has the special property that $C[f_\alpha, f_\alpha] = 0$ if f_α is Maxwellian.

The choice of collision operator depends on the problem and in general they can be highly complicated [90]. The most simple is the Bhatnagar-Gross-Krook (BGK) operator [91]:

$$C_\alpha = -\frac{f_\alpha - f_M}{\tau_c}, \quad (2.36)$$

where f_α is the distribution function, f_M is a Maxwellian and τ_c is the collision time. Though simple, the BGK operator predicts the correct qualitative behaviour of causing f_α to relax to f_M on the collision timescale τ_c . However, for accurate predictions of a collisional plasma it is too approximate.

If the collisions in a plasma are dominated by small-angle collisions, as defined in the previous section, the collision operator will be in Fokker-Planck (FP) form [90]. The full derivation of this collision operator is not relevant to this thesis, but it can be found in many references (e.g. ref [90]). In one dimension in velocity space, the collision operator can be written as:

$$\begin{aligned} C[f] &= \left(\frac{\partial f}{\partial t} \right)_{coll} \\ &= -\frac{\partial}{\partial v} \left(\frac{\langle \Delta v \rangle}{\Delta t} f \right) + \frac{\partial^2}{\partial v^2} \left(\frac{\langle (\Delta v)^2 \rangle}{2\Delta t} f \right). \end{aligned} \quad (2.37)$$

Particle collisions are inherently random and the angle brackets $\langle \rangle$ denote the expectation value in the change of the variable due to collisions over a timestep Δt . In effect, the FP operator is a Taylor expansion of the effect of collisions on a particle's velocity distribution, in the limit of small angle scattering. The FP operator is accurate to $\mathcal{O}\left(\frac{1}{\ln \Lambda}\right)$ due to this approximation. The first term in the collision operator acts as a drag term, acting to slow down a particle. The second term is a diffusion term, representing that collisions cause a particle to take a random walk in velocity space. This diffusive term causes spreading out of the distribution function in velocity space. The combination of these two effects is to cause the distribution function to relax to a Maxwellian. When the kinetic equation is solved using a Fokker-Planck collision operator, it is known as the Vlasov-Fokker-Planck (VFP) equation. This represents a gold standard in modelling collisional plasma using kinetics [92].

2.2.5 Closure: classical transport theory

An important property of the system of moment equations is that they are not closed. Each equation depends on a higher order moment. In the set that we have written down, the evolution of the first order moment $\mathbf{u}$ depends on the second order moments p and $\underline{\pi}$, which in turn depend on the third order moment $\mathbf{q}$. This is an infinite set of equations that cannot be solved unless an approximation is made that cuts it off somewhere. This process is known as assuming a closure for the equations. The choice of where to cut off the set indicates the approximation that has to be made in fluid theory compared to kinetic theory. Classical transport theory uses kinetic theory to provide expressions for $\mathbf{q}_\alpha$, $\mathbf{R}_\alpha$ and Q_α in terms of the primary variables n , T and $\mathbf{v}$ [86]. The calculation is done using the Fokker-Planck collision operator of section 2.2.4.

This calculation was first done for an unmagnetised plasma by Spitzer *et al.* [93]. This derivation was generalised to the case of a arbitrarily magnetised plasma by Braginskii [86]. Later work by Epperlein-Haines corrected the work by Braginskii in the limit of

large magnetisation [94]. This remained the standard until recently, where work has again tried to improve the accuracy of the so-called transport coefficients obtained by classical transport theory [95–97]. The different calculations of transport coefficients are unlikely to materially affect the results of this thesis and we usually use either the Braginskii or Epperlein-Haines forms. This is because for the geometries and parameter regimes that we consider in this thesis, these produce virtually identical results to more recent work [97].

The actual transport calculation is complex and beyond the scope of this thesis. We sketch the derivation here and summarise the results. A detailed description of the calculation is given in Braginskii [86] or Epperlein-Haines [94] or in refs [85, 90].

The distribution function is expanded using a Chapman-Enskog expansion. This assumes that any changes take place on timescales much longer than the collision time τ_c and spatial scales much longer than the mean free path λ_{mfp} . This is parametrised by the Knudsen number:

$$\text{Kn} = \frac{\lambda_{mfp}}{L}, \quad (2.38)$$

where L is the length scale of interest. For example, for a shock this length scale is the shock width. For the Chapman-Enskog expansion, and thus classical transport theory, to be valid it is necessary that $\text{Kn} \ll 1$ [86]. First, we expand the distribution function in Cartesian tensors:

$$f(\mathbf{r}, \mathbf{v}, t) = f_0(\mathbf{r}, \mathbf{v}, t) + \mathbf{f}_1(\mathbf{r}, \mathbf{v}, t) \cdot \frac{\mathbf{v}}{v} + \underline{\underline{\mathbf{f}_2}}(\mathbf{r}, \mathbf{v}, t) : \frac{\mathbf{v}\mathbf{v}}{v^2} + \dots, \quad (2.39)$$

where f_0 is a Maxwellian and f_1 is a small perturbation to this. Assuming that $\text{Kn} \ll 1$ is equivalent to assuming that f_1 and f_2 are small perturbations to f_0 . This expanded distribution function is then substituted into the electron VFP equation which is linearised, including the collision operator. This linearised VFP equation is solved numerically and used to find expressions for the higher order moments in terms of n , T and $\mathbf{u}$. This approach, used by Epperlein-Haines [94] and later work [95–97], differs slightly from Braginskii’s original work which calculated the transport coefficients analytically by expanding the distribution function in orthogonal polynomials instead [86, 90].

These higher order moments are proportional to the non-equilibrium processes that perturb that distribution function, such as temperature gradients or electric fields. The constants of proportionality are functions of the basic variables $(n, t, \mathbf{u})$ known as transport coefficients. Due to the effect of the magnetic field on the behaviour of colliding particles, as described in section 2.2.3, the transport coefficients are a function of the Hall parameter $\chi_e = \omega_e \tau_e$. They are also anisotropic tensors in general, since the direction of motion of a particle relative to the magnetic field determines its behaviour between collisions. For example, the ion heat flux $\mathbf{q}_i$ can be written:

$$\mathbf{q}_i = \underline{\underline{\kappa}}^i \cdot \nabla T_i, \quad (2.40)$$

where $\underline{\underline{\kappa}}^i$ is the ion heat conductivity tensor and is an example of a transport coefficient.

For a general transport coefficient tensor $\underline{\underline{\psi}}$ (which could, for example, be a thermal conductivity), it can be expressed in terms of components relative to the magnetic field unit vector $\mathbf{b} = \mathbf{B}/|\mathbf{B}|$ and the driving term $\mathbf{s}$ (which could be a temperature gradient in the case of a heat flux) by:

$$\underline{\underline{\psi}} \cdot \mathbf{s} = \psi_{\parallel} \mathbf{b}(\mathbf{b} \cdot \mathbf{s}) + \psi_{\perp} \mathbf{b} \times (\mathbf{s} \times \mathbf{b}) + \psi_{\wedge} \mathbf{b} \times \mathbf{s}. \quad (2.41)$$

The first component is the projection of the transport flux in the direction of the magnetic field, the second is in the direction perpendicular to the magnetic field lines (i.e. in the $\mathbf{b} \times (\mathbf{s} \times \mathbf{b})$ direction) and the third is in the direction perpendicular to both $\mathbf{b}$ and the driving term $\mathbf{s}$.

The results of the classical transport theory calculation for the case of a simple plasma, which is a plasma only containing one ion species, are summarised in the next section.

2.2.6 Transport theory of a simple plasma

The classical work by Braginskii performs the calculation of the transport coefficients for a simple plasma, consisting of one ion species [86]. This has been generalised to multiple ion species in later work although the calculation is complex [98]. Starting from the general moment equations of section 2.2.2, we write equations for the electron and ion species:

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{u}_e) = 0, \quad (2.42)$$

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) = 0, \quad (2.43)$$

$$m_e n_e \left(\frac{\partial \mathbf{u}_e}{\partial t} + \mathbf{u}_e \cdot \nabla \mathbf{u}_e \right) = -\nabla p_e - e n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) + \mathbf{R}, \quad (2.44)$$

$$m_i n_i \left(\frac{\partial \mathbf{u}_i}{\partial t} + \mathbf{u}_i \cdot \nabla \mathbf{u}_i \right) = -\nabla p_i + Z e n_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) - \mathbf{R}, \quad (2.45)$$

$$\frac{3}{2} n_e \left(\frac{\partial T_e}{\partial t} + \mathbf{u}_e \cdot \nabla T_e \right) + n_e T_e \nabla \cdot \mathbf{u}_e + \nabla \cdot \mathbf{q}_e = Q_e + S_e, \quad (2.46)$$

$$\frac{3}{2} n_i \left(\frac{\partial T_i}{\partial t} + \mathbf{u}_i \cdot \nabla T_i \right) + n_i T_i \nabla \cdot \mathbf{u}_i + \nabla \cdot \mathbf{q}_i = Q_i + S_i. \quad (2.47)$$

We have used an ideal EoS, so that $p_{\alpha} = n_{\alpha} T_{\alpha}$ for each species. As we only have electrons and ions, we write the momentum transfer term as $\mathbf{R}$ which is the momentum transferred between electrons and ions as a result of collisions between the species. We have dropped the stress tensor components $\underline{\underline{\sigma}}_{\alpha}$ for both species as we are considering an inviscid plasma. S_{α} is any additional sources or sinks of energy for each species, such as laser absorption or radiative losses. The evolution of the EM fields is given by

Maxwell's equations:

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \quad (2.48)$$

$$\frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\mu_0 \mathbf{j}, \quad (2.49)$$

where $\mathbf{j} = Zen_i \mathbf{u}_i - en_e \mathbf{u}_e$ is the current density. In principle, the solution of these two equations always satisfies the other two Maxwell's equations for $\nabla \cdot \mathbf{B}$ and $\nabla \cdot \mathbf{E}$ though there are numerical considerations discussed in chapter 3.

Classical transport theory is used to find a closure for the transport terms, which we describe in the next sections.

2.2.6.1 The momentum term

The momentum transfer term $\mathbf{R} = \mathbf{R}_U + \mathbf{R}_T$ is made up of two components: a frictional component that accounts for the relative velocity $\mathbf{U} = \mathbf{u}_e - \mathbf{u}_i$ between electrons and ions, and a thermal component that arises due to an electron temperature gradient and the temperature dependence of the collision frequency.

We can write the current density of the plasma as: $\mathbf{j} = -en_e \mathbf{U}$ by quasineutrality. This makes it clear that the frictional force $\mathbf{R}_U$ arises due to a current in the plasma and collisions between the electrons and the ions. This term is what causes plasma to have a finite resistivity. Physically, this term represents the change in momentum due to drag between the species.

The thermal force comes about due to the temperature dependence of the collision frequency. To understand its physical origin, we can assume that electrons and ions are both on average at rest $\langle v_e \rangle = \langle v_i \rangle = 0$. This means that if we consider some surface in a homogeneous plasma, the flux of electrons per unit time left to right through the surface must be the same as right to left. This flux is of the order of $n_e v_e$, which will not be zero due to the thermal motion of particles. Due to electron and ion collisions, these fluxes experience frictional forces of order $m_e n_e v_e / \tau_e$ that are balanced [86]. However, if there is a temperature gradient in the plasma, then the electrons from the hotter region have a higher thermal velocity than from the colder region. Since $\tau_e \propto v_{th}^3$, this means that the hotter electrons are less collisional and therefore feel less frictional force. This leads to an imbalance in the frictional force in the plasma where there is a temperature gradient. The presence of a magnetic field gives rise to a different component of this force, shown in diagram 2.2. Consider electrons gyrating in the $x - y$ plane due to a magnetic field in the z direction, while there is a temperature gradient in the x direction. At some point $x = x_0$, electrons coming from the left have come from a region with lower temperature than the electrons on the right. Therefore, the electrons from the left are on average slower than those from the right. Due to the velocity dependence of the collision frequency, this causes an imbalance of the frictional forces due to collisions between electrons and ions at the surface at $x = x_0$. As clear from the diagram, this leads to a net component of the frictional force in the $\nabla T \times \mathbf{B}$ direction. This component is

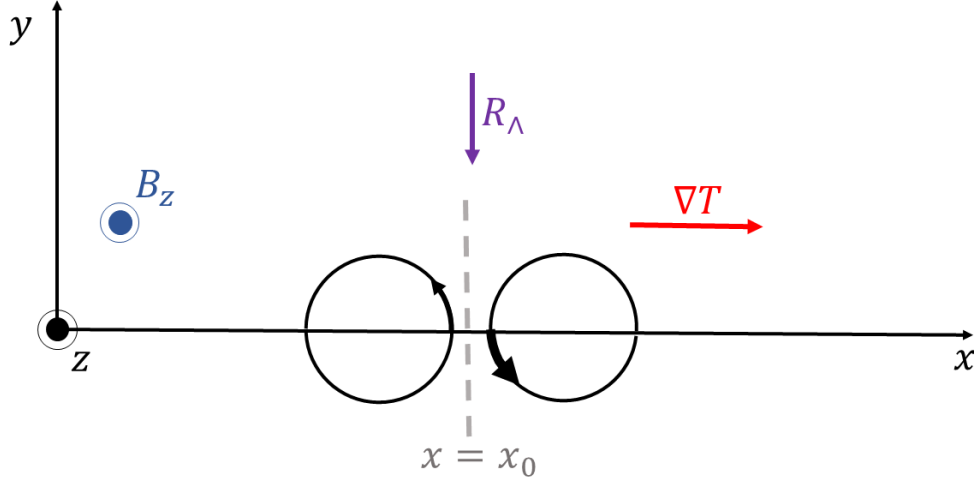


Figure 2.2: A diagram showing how a component of thermal force arises when there is a magnetic field $\mathbf{B}$ perpendicular to a temperature gradient ∇T . This component is known as the Nernst effect. This diagram was adapted from ref. [86].

known as the Nernst effect and it is particularly important for magnetised high energy density plasmas [29, 48]. It causes magnetic field to be advected down temperature gradients by an electron flow [30]. We discuss the effect of this term on a magnetised plasma in detail in chapter 4.

Using equation (2.41), the frictional force can be written as:

$$\begin{aligned}\mathbf{R}_U &= \underline{\underline{\alpha}} \cdot \mathbf{U} = \underline{\underline{\alpha}} \cdot (\mathbf{u}_e - \mathbf{u}_i) \\ &= -\alpha_{\parallel} \mathbf{U}_{\parallel} - \alpha_{\perp} \mathbf{U}_{\perp} + \alpha_{\wedge} (\mathbf{b} \times \mathbf{U}),\end{aligned}\quad (2.50)$$

where $\mathbf{U}_{\parallel} = \mathbf{b}(\mathbf{b} \cdot \mathbf{U})$ is the component of relative velocity along and $\mathbf{U}_{\perp}$ is the component of relative velocity perpendicular to the magnetic field. As alluded to, the $\underline{\underline{\alpha}}$ tensor is related to the resistivity of the plasma by $\underline{\underline{\eta}} = \frac{1}{e^2 n_e^2} \underline{\underline{\alpha}}$. Using similar notation, the thermal force can also be expanded:

$$\begin{aligned}\mathbf{R}_T &= \underline{\underline{\beta}}^{UT} \cdot \nabla T_e \\ &= -\beta_{\parallel}^{UT} \nabla_{\parallel} T_e - \beta_{\perp}^{UT} \nabla_{\perp} T_e - \beta_{\wedge}^{UT} (\mathbf{b} \times \nabla T_e).\end{aligned}\quad (2.51)$$

As discussed above, the $\beta_{\wedge}$ term is known as the Nernst effect.

The tensors $\underline{\underline{\alpha}}$ and $\underline{\underline{\beta}}$ are transport coefficients calculated from classical transport theory. They are functions of the lower order moments $(n, T, \mathbf{u})$ and a dimensionless part:

$$\underline{\underline{\alpha}} = \frac{m_e n_e}{\tau_e} \underline{\underline{\alpha}}^C \quad (2.52)$$

$$\underline{\underline{\beta}}^{UT} = n_e \underline{\underline{\beta}}^C, \quad (2.53)$$

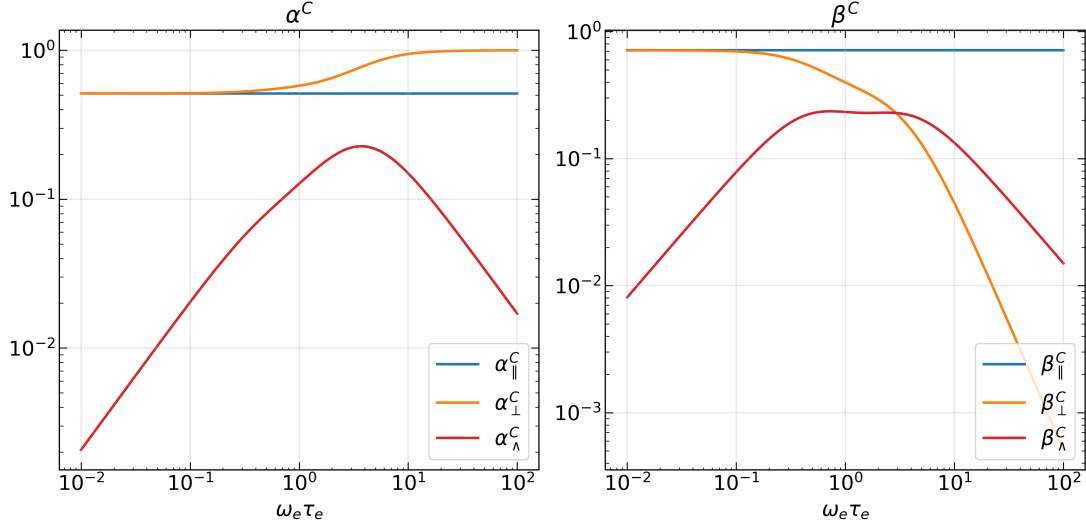


Figure 2.3: Plots showing the values of the components of the dimensionless $\underline{\alpha}^C$ and $\underline{\beta}^C$ tensors as a function of the Hall parameter $\omega_e \tau_e$. The fits are calculated for $Z = 1$.

where $\tau_e = \tau_{ei}$ is the electron-ion collision time and superscript C denotes the dimensionless part. The collisions in a plasma are affected by the charge of the ions Z and the magnetic field via the Hall parameter $\chi_e = \omega_e \tau_e$. The conventional approach used in classical transport theory is to use the linearised VFP equation to obtain f_1 and calculate the transport fluxes $\mathbf{R}$ from this for various different values of χ_e and Z . A polynomial $f(Z, \chi_e)$ is then fit to this data to describe the dimensionless part of the transport coefficients. Different methods of calculating the coefficients (e.g. Braginskii vs Epperlein-Haines) use different fits to these transport coefficients. Figure 2.3 shows the values of the components of $\underline{\alpha}^C$ and $\underline{\beta}^C$ as a function of $\omega_e \tau_e$ for the fits proposed by Braginskii for $Z = 1$. It is clear that the $\parallel$ and $\perp$ coefficients are equal for $\omega_e \tau_e \rightarrow 0$, and that the $\wedge$ components go to zero. This is because in an unmagnetised plasma, the particles travel as if there was no magnetic field and so there is no anisotropy. The $\wedge$ components also go to zero in the limit of high magnetisation $\omega_e \tau_e \rightarrow \infty$ as in this limit the particles are effectively fully tied to field lines.

2.2.6.2 Heat flux terms

The heat flux is closely related to the collisional force $\mathbf{R}$ from the previous section. If there is a collision that causes exchange in momentum between electrons and ions, that arises in the $\mathbf{R}$ term. Similarly, energy will also be exchanged in that collision which arises in the heat flux $\mathbf{q}$. Therefore, as with $\mathbf{R}$, the heat flux also has a part corresponding to the relative motion between electrons and ions and a thermal component: $\mathbf{q} = \mathbf{q}_U + \mathbf{q}_T$.

The ion heat flux only has a thermal component due to collisions between ions with

different temperatures:

$$\mathbf{q}_i = \frac{n_i T_i \tau_i}{m_i} \underline{\underline{\gamma}}^{i,C} \cdot \nabla T_i. \quad (2.54)$$

The electron heat flux can be written as:

$$\mathbf{q}_e = \mathbf{q}_U + \mathbf{q}_T, \quad (2.55)$$

where

$$\begin{aligned} \mathbf{q}_U &= \underline{\underline{\beta}}^{TU} \cdot \mathbf{U} \\ &= \beta_{\parallel}^{TU} \mathbf{U}_{\parallel} + \beta_{\perp}^{TU} \mathbf{U}_{\perp} + \beta_{\wedge}^{TU} (\mathbf{b} \times \mathbf{U}) \end{aligned} \quad (2.56)$$

$$\begin{aligned} \mathbf{q}_T &= \underline{\underline{\kappa}}^e \cdot \nabla T_e \\ &= -\kappa_{\parallel}^e \nabla_{\parallel} T_e - \kappa_{\perp}^e \nabla_{\perp} T_e - \kappa_{\wedge}^e (\mathbf{b} \times \nabla T_e). \end{aligned} \quad (2.57)$$

As before, the $\underline{\underline{\beta}}$ tensor and electron thermal conductivity tensor $\underline{\underline{\kappa}}^e$ can be expressed in terms of the lower order moments and a dimensionless function:

$$\underline{\underline{\beta}}^{TU} = T_e \underline{\underline{\beta}}^{UT} = n_e T_e \underline{\underline{\beta}}^C \quad (2.58)$$

$$\underline{\underline{\kappa}}^e = \frac{n_e T_e \tau_e}{m_e} \underline{\underline{\gamma}}^{e,C}. \quad (2.59)$$

Figure 2.4 shows the components of the $\underline{\underline{\gamma}}$ tensor plotted as a function of $\omega_e \tau_e$ for $Z = 1$ using Braginskii fits for both electrons and ions. For both species, $\gamma_{\perp}$ and $\gamma_{\wedge} \rightarrow 0$ as $\omega_e \tau_e \rightarrow \infty$. In this case, the charged particles are confined so tightly to magnetic field lines that they cannot carry any heat flux in a direction perpendicular to the field. It is this effect that motivates the use of magnetic fields in magneto-inertial fusion. Similarly to the $\underline{\underline{\alpha}}$ and $\underline{\underline{\beta}}$ tensors, the $\parallel$ and $\perp$ components are equal for $\omega_e \tau_e \rightarrow 0$ and the $\wedge$ components goes to 0. In this limit, the magnetic field has no effect so heat flow does not depend on the magnetic field direction. There is no $\wedge$ component as particles no longer have gyro-orbits.

The frictional component of the electron heat flux $\mathbf{q}_U$ is sometimes known as the intrinsic heat flux. It represents the flow of energy due to an electric current through the plasma. The $\parallel$ and $\perp$ terms are known as the heat current and the anisotropic heat current respectively. The $\beta_{\wedge}$ term is known as the Ettingshausen term and is particularly important for the work in this thesis. It describes a heat flow that travels in the $\mathbf{j} \times \mathbf{B}$ direction. Figure 2.5 shows a diagram explaining how this heat flux arises. We consider the gyration of electrons, shown by the circular arrows, due to a magnetic field in the z direction B_z . In the electron rest frame, the relative velocity between electrons and ions is given by $\mathbf{U} = -\mathbf{u}_i$. Due to this relative velocity, the friction force due to collisions with ions accelerates electrons for one half of their gyromotion and decelerates them for the other half of the gyration. In the diagram, this means that electrons are accelerated during their gyromotion below the x axis and decelerated

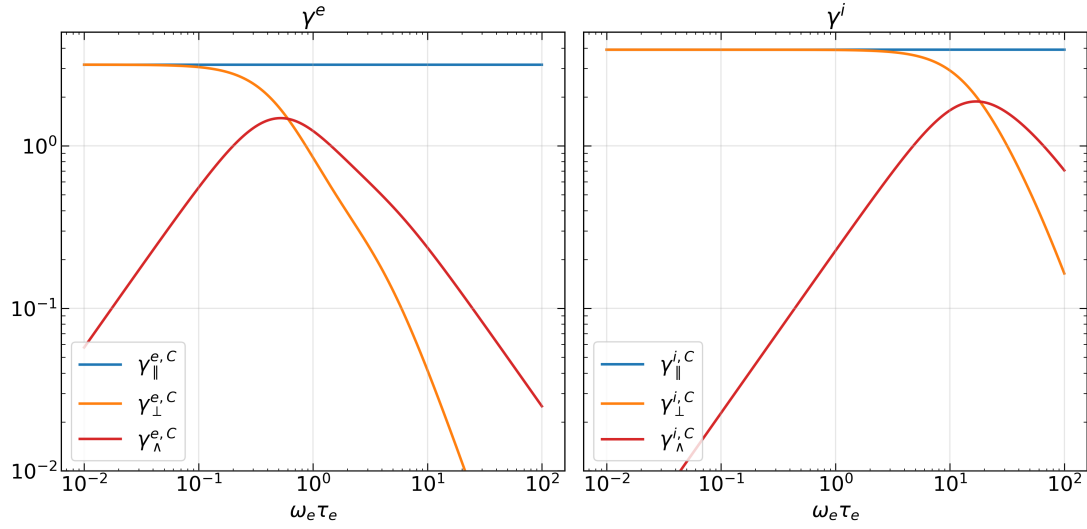


Figure 2.4: Plots showing the Braginskii fits to the components of the dimensionless γ tensor for electrons and ions. The components are plotted as a function of the electron Hall parameter, assuming $Z = 1$ and $T_i = T_e$.

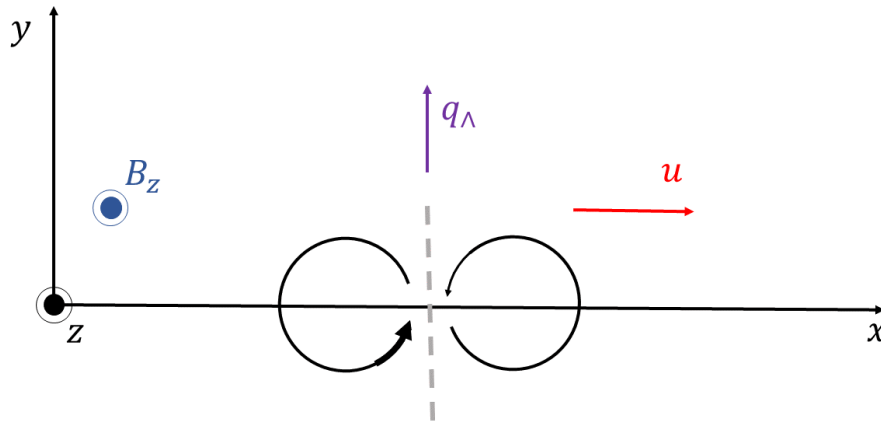


Figure 2.5: A diagram showing how the Etingshausen heat flux arises from a magnetic field B_z and a current due to a relative velocity between electrons and ions u . The heat flux is in the $\mathbf{j} \times \mathbf{B}$ direction. This diagram was adapted from ref. [86].

above. Where the arrows meet at the dotted line, an electron travelling downwards is at its minimum energy and an electron travelling upwards is at its maximum. This means that there is an imbalance between the energy flux going up towards the axis and the flux coming downwards.

The $\mathbf{q}_T$ term is known as the thermal heat flux. It is more familiar from neutral gases and describes the flow of heat due to a temperature gradient. The parallel component is the thermal conductivity that would be observed in an unmagnetised plasma and the perpendicular component describes how the magnetic field reduces heat flow across magnetic field lines. The $\wedge$ component is known as the Righi-Leduc term and arises due to the magnetic field deflecting heat flow along isotherms. It arises for a similar reason as the Ettingshausen effect shown in figure 2.5. The Righi-Leduc term is important for magnetised plasmas and has been shown to cause instabilities due to its interaction with the Nernst heat flow [24, 28].

2.2.6.3 Heat exchange term

The final term concerns the transfer of energy between electrons and ions due to collisions. The heat gained by the ions due to collisions with the electrons is given by:

$$Q_i = \frac{3m_e}{m_i} \frac{n_e}{\tau_e} (T_e - T_i). \quad (2.60)$$

This term is known as electron-ion energy equilibration. It comes about because the temperature dependence of the collision frequency means that if $T_e \neq T_i$, collisions will cause energy to be exchanged between the electrons and the ions. If $T_e = T_i$, then the electron and ion populations are in thermal equilibrium with each other and there is no net transfer of energy between the species. It can be shown that if $T_e \sim T_i$, then:

$$\tau_{ee}^E : \tau_{ii}^E : \tau_{ei}^E = 1 : \sqrt{\frac{m_i}{m_e}} : \frac{m_i}{m_e}, \quad (2.61)$$

where τ_{ee}^E and τ_{ii}^E are the electron-electron and ion-ion equilibration times respectively and τ_{ei}^E is the electron-ion energy equilibration timescale [86]. As $m_e \ll m_i$, this implies that a thermal equilibrium is established within each component before it is established between the components. This means that it is possible for the electron and ion temperatures to separate, whilst still remaining in thermal equilibrium within each component. This allows for the solution of separate electron and ion energy equations without violating the assumption of being close to thermodynamic equilibrium.

In general, as well as this direct energy transfer due to collisions, there is also the change in energy of the particles due to the effect of the frictional force $\mathbf{R}$. This is a factor $\sim \frac{m_e}{m_i}$ smaller for the ions than the electrons and thus negligible on the ion energy equation. However, for the electrons, the total energy generated in the electron

population due to collisions with ions can be written as:

$$\begin{aligned} Q_e &= -\mathbf{R} \cdot \mathbf{U} - Q_i \\ &= -\mathbf{R}_U \cdot \mathbf{U} - \mathbf{R}_T \cdot \mathbf{U} - \frac{3m_e}{m_i} \frac{n_e}{\tau_e} (T_e - T_i), \end{aligned} \quad (2.62)$$

where $\mathbf{U} = \mathbf{u}_e - \mathbf{u}_i$ is the relative velocity between electrons and ions, $\mathbf{R}_U$ is the frictional force defined in equation (2.50) and $\mathbf{R}_T$ is the thermal force defined in equation (2.51). Using the definition of the current density, the frictional term $-\mathbf{R}_U \cdot \mathbf{U}$ can be rewritten as $\frac{1}{en_e} \mathbf{R}_U \cdot \mathbf{j}$, which makes it clear that this is the energy gained in the electrons due to collisions when there is a current in the plasma. This is usually called the Joule or Ohmic heating [86].

There may also be additional energy source or sink terms due to fusion reactions, Brehmsstrahlung radiative losses or laser absorption which would be added to Q_e and Q_i . We do not discuss these effects any further in this thesis however.

2.2.7 Limits of validity

For any theory, it is important to consider in what limits it is valid. The model laid out in section 2.2.6 is limited by the assumptions made to obtain closure to the moment equations. It is only valid for timescales much longer than the collision time τ_e and for spatial scales much longer than the mean free path λ_{mfp} . Under these approximations, the distribution function will be close to Maxwellian and the Chapman-Enskog expansion is valid. There are many situations where this approximation is not appropriate in high energy density plasmas [99–101]. For example, shocks in ICF capsules are generally kinetic phenomena as the mean free path is long compared to the shock width due to the high plasma temperature. Similarly, it has been shown by experiment and theory that for Knudsen numbers even still as small $\text{Kn} \sim 0.1$, kinetic effects can become observable in heat flux [102, 103]. This is known as non-local heat flow. Many other kinetic effects can occur and finding reduced models to implement them in more computationally tractable hydrodynamic simulations is a rich topic of research currently in the field [99].

Another restriction arises from the collision operator. It is necessary to assume that the ion Larmor radius r_L is large compared to the Debye length λ_D . This is equivalent to stating that the plasma frequency ω_p is much larger than the gyrofrequency Ω_c . Otherwise, the magnetic field plays a role during the Coulomb interaction of charged particles. This is the regime of strongly magnetised plasma and an alteration to the collision operator is needed [104, 105]. We are not in this regime in this thesis, though in principle this effect could be accounted for with corrections to the transport coefficients.

2.3 Magnetohydrodynamics

2.3.1 Single Fluid MHD

The set of equations laid out in section 2.2.6 define two-fluid MHD. This is a fluid model of plasma that self-consistently evolves the electron and ion populations as separate fluids coupled through collisions and the electromagnetic fields. This allows for modelling effects associated with the finite electron mass, which become important at length scales comparable to the electron inertial length, also known as the skin depth, $\lambda_e = c/\omega_{pe}$. At even smaller length scales, charge separation becomes important and the plasma can no longer be considered quasineutral. This charge separation becomes relevant for timescales on the order of the electron plasma frequency $\tau \sim \omega_{pe}^{-1}$ and spatial scales on the order of the Debye length $L \sim \lambda_D$. For a typical wire-array z-pinch, these can be as small as $\tau \sim 10^{-15}$ s and $L \sim 10^{-9}$ m. These values are significantly smaller than the smallest timescales and spatial scales that could practically be observed in a pulsed power experiment.

It is convenient to consider these equations in a centre of mass frame, due to the disparate masses of the electrons and the ions. We can sum the electron and ion continuity equations to obtain:

$$\frac{\partial}{\partial t}(\rho_e + \rho_i) + \nabla \cdot (\rho_e \mathbf{u}_e + \rho_i \mathbf{u}_i) = 0. \quad (2.63)$$

We define a total mass density ρ and a centre of mass velocity $\mathbf{u}$:

$$\rho = \rho_e + \rho_i \approx \rho_i \quad (2.64)$$

$$\mathbf{u} = (\rho_e \mathbf{u}_e + \rho_i \mathbf{u}_i)/\rho \approx \mathbf{u}_i, \quad (2.65)$$

which are dominated by the ion mass. We can then rewrite the continuity equation for the plasma as a whole to be:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \quad (2.66)$$

In principle, we still need to account for the electron density as charge separation could arise. Defining the charge density as:

$$\rho_c = Zen_i - en_e, \quad (2.67)$$

we can use the electron and ion continuity equations to write an equation for its evolution as:

$$\frac{\partial \rho_c}{\partial t} + \nabla \cdot \mathbf{j} = 0. \quad (2.68)$$

However, as stated above, it is generally not practical to resolve the time and spatial scales of two-fluid physics in modelling pulsed power experiments. Therefore, it is

assumed from now on that the plasma is quasineutral on the scales that we care about and that therefore $\rho_e \ll Zen_i$. We can then sum the electron and ion momentum equations to obtain an equation for the evolution of the centre of mass velocity $\mathbf{u}$:

$$\begin{aligned} \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) &= -\nabla(p_e + p_i) + (Zen_i - en_e)\mathbf{E} + (Zen_i\mathbf{u}_i - en_e\mathbf{u}_e) \times \mathbf{B} \\ &= -\nabla p + \mathbf{j} \times \mathbf{B}, \end{aligned} \quad (2.69)$$

where we have defined the total pressure $p = p_e + p_i$ and recognised that $\mathbf{j} = Zen_i\mathbf{u}_i - en_e\mathbf{u}_e$. Note that due to quasineutrality, the electric field does not appear in the momentum equation and that $\mathbf{R}_e = -\mathbf{R}_i$ due to conservation of momentum in collisions. The $\mathbf{j} \times \mathbf{B}$ term is known as the Lorentz force and it represents the force on the plasma due to the magnetic field. By the argument made in section 2.2.6.3, it is consistent to retain separate electron and ion energy equations which we write as:

$$\frac{3}{2}n_e \left(\frac{\partial T_e}{\partial t} + \mathbf{u}_e \cdot \nabla T_e \right) + n_e T_e \nabla \cdot \mathbf{u}_e + \nabla \cdot \mathbf{q}_e = Q_e + S_e, \quad (2.70)$$

$$\frac{3}{2}n_i \left(\frac{\partial T_i}{\partial t} + \mathbf{u}_i \cdot \nabla T_i \right) + n_i T_i \nabla \cdot \mathbf{u}_i + \nabla \cdot \mathbf{q}_i = Q_i + S_i, \quad (2.71)$$

where all symbols are as defined in section 2.2.6. These equations define what is known as single-fluid MHD, since we assume that the electron and ion fluids do not evolve independently. This is by far the most common form of MHD approach used for magnetised plasma modelling.

We need to evolve these equations along with Maxwell's equations:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \quad (2.72)$$

$$\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \mu_0 \mathbf{j}. \quad (2.73)$$

It is worth noting that retaining the displacement current $\frac{\partial \mathbf{E}}{\partial t}$ whilst not accounting for charge separation is not consistent. By taking the divergence of equation (2.73) and using Gauss' law for electrostatics, we arrive at equation (2.68). This implies that allowing displacement current can lead to charge separation. A better way of stating this is that if $\nabla \cdot \mathbf{j} \neq 0$, then the plasma is not quasineutral. However, for cases where the electron and ion velocities are non-relativistic then the displacement current contribution to Ampère's Law will be negligible [4, 85, 148]. This is the limit of relevance for our work. We discuss the numerical implications of including the displacement current, even though it should be small, in detail in chapters 3 and 6.

An expression for the current density $\mathbf{j}$ is needed to close this system of equations. This is known as the generalised Ohm's law (GOL) and can be obtained from the

electron and ion momentum equations. It can be shown to be:

$$\frac{m_e}{e^2 n_e} \left(\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot \left[\mathbf{u} \mathbf{j} + \mathbf{j} \mathbf{u} - \frac{1}{en_e} \mathbf{j} \mathbf{j} \right] \right) = \mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{1}{en_e} \mathbf{j} \times \mathbf{B} - \frac{\mathbf{R}_e}{en_e} + \frac{1}{en_e} \nabla p_e, \quad (2.74)$$

where $\mathbf{u}$ is the centre of mass velocity. The full derivation is shown in Appendix B.2. We have used the notation $\mathbf{j} \mathbf{j} = \mathbf{j} \otimes \mathbf{j}$ to denote the outer product throughout this thesis. For the results in this chapter and in chapter 6, we consider just the resistive parts of the momentum transfer term $\mathbf{R}_e$. Other effects associated with the thermal force, such as the Nernst effect, arise through $\mathbf{R}_e$. In the simplest form, this can be written as $\frac{\mathbf{R}_e}{en_e} = \eta \mathbf{j}$ where η is the resistivity of the plasma. In general, the electron pressure p_e and η are tensors, but we have simplified the equation here by assuming the non-scalar components are small. The terms on the left hand side represent the contributions of electron inertia to the current density. On the right hand side, the $\mathbf{u} \times \mathbf{B}$ is the so-called frozen-in-flow term, the $\mathbf{j} \times \mathbf{B}$ term is the Hall effect, $\eta \mathbf{j}$ is resistive diffusion and ∇p_e is sometimes known as the Biermann battery effect. There is a huge amount of physics in this equation and in many situations we can work with a simpler form. We will discuss the various approximations that allow us to simplify Ohm's Law for the purpose of more practical use in the following sections. This defines the various single-fluid MHD approximations that can be used.

2.3.2 Ideal MHD

The most simplified approach is ideal MHD. In this, we neglect the electron inertia, Hall, resistive and Biermann terms. This is equivalent to assuming that the electrons have zero mass and follow the ion flow and that the plasma is infinitely conducting. This reduces the GOL to:

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0. \quad (2.75)$$

We also assume the speed of light is infinite, allowing us to ignore the displacement current in Maxwell's equations. This means that the current is defined by the magnetic field:

$$\mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{B}. \quad (2.76)$$

In this case, the GOL is used to calculate the electric field and Ampère's Law is used to calculate the current. Using equation (2.75) and Faraday's Law, the magnetic field is then evolved using the induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = \nabla \times (\mathbf{u} \times \mathbf{B}). \quad (2.77)$$

The plasma is then evolved using the single-fluid plasma equations. This induction equation has an important consequence, originally shown by Alfvén [106], known as the frozen-in-flow condition. This is that the magnetic field is carried with the velocity field $\mathbf{u}$. Stated differently, the magnetic flux through a surface moving with the velocity field is conserved.

The ideal MHD model is used extensively throughout plasma physics and astrophysics due to its simplicity and qualitative importance [4, 85, 107]. There are many magnetised phenomena, particularly in space physics, that can be well described by a frozen-in-flow picture of the magnetic field [108, 109]. In fact, much of the early work on the stability of z-pinchs in pulsed power used ideal MHD models [36] due to their analytic tractability. However, the assumptions of ideal MHD have a limited range of validity and the theory cannot predict many observed phenomena such as magnetic reconnection or resistive diffusion. These effects are frequently observed in pulsed power experiments and so for more accurate modelling of these plasmas, more detailed approaches are needed [83]. We go on to describe these in the subsequent sections.

2.3.3 Resistive MHD

Frequently, it is not the case that the plasma is perfectly conducting and we must account for the impact of collisions on the current density. In this limit, we cannot drop the resistivity from the GOL. It then becomes:

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \eta \mathbf{j}, \quad (2.78)$$

where η is the resistivity. Again, we obtain the current density from Ampère's Law: $\mathbf{j} = \nabla \times \mathbf{B} / \mu_0$. Forming the induction equation from Faraday's Law, it is now given by:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times \left(\frac{\eta}{\mu_0} \nabla \times \mathbf{B} \right). \quad (2.79)$$

In addition to the frozen-in-flow term, we have gained another term due to the damping of currents by collisions between electrons and ions. It can be shown that this term leads to diffusion of the magnetic field due to the resistivity, so it is normally called resistive diffusion. The MHD model in this limit is known as resistive MHD. The addition of resistive diffusion allows for magnetic reconnection as well as many other phenomena that are of importance in laboratory astrophysics such as resistive heating [72, 83, 110].

We can evaluate when resistive effects become important using the magnetic Reynold's number R_m , which is defined as the ratio of the frozen-in-flow to the resistive diffusion terms:

$$R_m = \frac{\text{Advection}}{\text{Diffusion}} = \frac{|\nabla \times (\mathbf{u} \times \mathbf{B})|}{|\nabla \times \eta \nabla \times \mathbf{B} / \mu_0|} \sim \frac{\mu_0 u L}{\eta}, \quad (2.80)$$

where L is the characteristic magnetic field length scale. R_m is analogous to the Reynold's number for a neutral fluid, with resistivity replacing the viscosity. In a

plasma with $R_m \gg 1$, the frozen-in-flow term dominates and ideal MHD is an appropriate model. For laboratory astrophysics experiments such as those on Magpie, R_m can vary widely from $10 - 10^3$ [83, 111]. In addition, the loads that are experimented on using the MAGPIE facility are resistively heated, so a resistive MHD model is needed to model them [110].

A large part of the reason why more detailed MHD models are rarely used is that for many situations the resistive MHD model is sufficient to understand experimental observations. Models in plasma physics are rarely quantitatively predictive and further refinements often drastically complicate the numerical solution of the equations. Therefore, this limit is the most commonly used MHD model for pulsed power modelling [83]. However, as we highlighted in chapter 1, resistive MHD cannot explain certain observations in laboratory astrophysics and MIF experiments. This is particularly the case in low density plasma, where the Hall effect becomes important. We discuss that limit in the next section.

2.3.4 Hall MHD

An important limit for this thesis is Hall MHD. In addition to resistive MHD, this retains the $\mathbf{j} \times \mathbf{B}$ term, which is known as the Hall term. This changes the GOL to:

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \eta \mathbf{j} + \frac{1}{en_e} \mathbf{j} \times \mathbf{B}, \quad (2.81)$$

and the induction equation to:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{u} \times \mathbf{B} - \frac{\mathbf{j}}{en_e} \times \mathbf{B} \right) - \nabla \times \left(\frac{\eta}{\mu_0} \nabla \times \mathbf{B} \right). \quad (2.82)$$

Again, the current density is calculated from Ampère's Law, $\mathbf{j} = \frac{\nabla \times \mathbf{B}}{\mu_0}$.

Similarly to with the magnetic Reynold's number, it is useful to consider when the Hall term will dominate over the ideal MHD and resistive diffusion terms. The Hall effect will become comparable to resistive diffusion when:

$$\begin{aligned} \left| \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{en_e \mu_0} \right| &\sim \left| \left(\frac{\eta}{\mu_0} \nabla \times \mathbf{B} \right) \right| \\ \Rightarrow \frac{B^2}{en \mu_0 L} &\sim \frac{\eta}{\mu_0} \frac{B}{L} \\ \Rightarrow \frac{B}{en} &\sim \frac{m_e}{\tau_e e^2 n} \\ \Rightarrow \Omega_e \tau_e &\sim 1, \end{aligned} \quad (2.83)$$

where Ω_e is the electron cyclotron frequency, τ_e is the electron-ion collision time and we have used an approximation for the resistivity from Braginskii [86]. In a highly collisional plasma, the transport of magnetic field is dominated by collisions between particles. In a magnetised plasma, the Hall effect is the more relevant term in magnetic

field transport. We can perform a similar comparison for the Hall and frozen-in-flow terms:

$$\begin{aligned}
 |\mathbf{u} \times \mathbf{B}| &\sim \left| \frac{\mathbf{j} \times \mathbf{B}}{en_e} \right| \\
 \implies uB &\sim \left| \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{en_e\mu_0} \right| \\
 \implies u &\sim \frac{B}{Len\mu_0} \\
 \implies \frac{u}{v_A} \frac{L}{\lambda_i} &\sim 1,
 \end{aligned} \tag{2.84}$$

where we have introduced the Alfvén speed $v_A = \frac{B}{\sqrt{\mu_0 m_i n}}$ and the ion inertial length $\lambda_i = \frac{c}{\omega_{pi}} = \sqrt{\frac{m_i}{\mu_0 e^2 n}}$. This means that the Hall effect becomes comparable to the frozen-in-flow term when the ion inertial length λ_i becomes comparable to the length scale L . Physically, λ_i is the smallest length scale on which ions can respond to changes in the plasma conditions. On length scales $L < \lambda_i$, the electrons and ions decouple from each other. It can be seen from the Hall MHD induction equation (eq. (2.82)) that the Hall effect acts as an additional advection of the magnetic field with a velocity $\mathbf{v}_H = -\frac{\mathbf{j}}{en_e} \approx \mathbf{u}_e$. Combining this with the scaling in equation (2.84), this shows that on scales shorter than λ_i , the magnetic field is no longer frozen with the ion flow $\mathbf{u}$ but becomes decoupled and moves with the electron flow velocity instead. Scaling (2.83) then describes how collisions break this new frozen-in-flow condition for electrons causing the magnetic field to no longer be advected with the electron flow. The Hall effect has many other important consequences that we discuss in detail in chapters 5 and 6.

As the ion inertial length is inversely proportional to $\sqrt{n_e}$, the Hall effect becomes more relevant as the plasma density decreases. This means that it is an important term at vacuum-plasma interfaces, where a dense plasma is confined by a strong magnetic field, such as at the edge of a z-pinch. We discuss this more in section 2.4. The Hall effect is challenging to implement numerically, particularly in pulsed power plasmas due to the vacuum regions and large density gradients. It is only recently that numerical methods to handle the Hall effect in this regime have been developed [112–115] and therefore there is limited work in using Hall MHD to study pulsed power plasmas, though there is a growing body of work mostly using the PERSEUS code [63–65, 114, 116–118]. The reasons why the Hall effect is difficult to solve numerically and a new method for implementing this term within the Gorgon code makes up the work in chapter 6.

2.3.5 Extensions to Hall MHD

It is possible to continue adding terms from the GOL to derive more detailed MHD descriptions beyond Hall MHD.

For the MIF schemes that we discussed in chapter 1, the fusion plasma is dense

and hot at stagnation. In this limit the plasma is relatively conductive and neither Hall MHD or resistive MHD are a good description [30]. Instead, the most important terms to include are the Nernst effect and the Biermann ∇p_e term [29, 30, 59]. The Biermann battery effect can generate magnetic fields due to the presence of non-parallel temperature and density gradients [119]. Even in the absence of a magnetic field, the large temperatures in an ICF implosion mean that significant self-generated fields can be produced by this effect [120]. These magnetic fields are advected by the frozen-in-flow and Nernst effects [30, 120], but due to the large plasma β in MIF experiments, they are mostly important in their magnetisation of the thermal conduction [29]. We discuss the Nernst term briefly in chapter 4, but we are mostly interested in lower density plasmas in this thesis where other effects are more important.

Another regime beyond MHD becomes relevant when the ion Larmor radius r_L becomes comparable to the scale length L . At this point, finite Larmor radius effects can become important and non-diagonal components of the ion pressure tensor are relevant. It has been suggested that these can have a stabilising effect on z-pinchs [36, 121].

When the electron inertial length $\lambda_e = \frac{c}{\omega_{pe}}$ becomes comparable to the length scale of the plasma, the electron inertial terms within the GOL become relevant. Recent work has suggested this regime is of relevance for low density plasmas in pulsed power, where the high current density can mean that the electron drift velocity approaches velocities nearing the speed of light [114]. Inertial effects for the electrons are necessary to bound these drift velocities.

Charge separation becomes dominant for $L \sim \lambda_D$, at which point we need to consider sheath physics and the plasma model completely breaks down. Finally, it is also worth remarking that we are still constrained by the assumption that the electron and ion populations are close to Maxwellian, quantified by the Knudsen number $\text{Kn} = \frac{\lambda_{mfp}}{L}$. As the density decreases, the mean free path λ_{mfp} increases. Therefore, at some density we need to consider kinetic effects and the fluid model will no longer be valid.

2.3.6 Anomalous resistivity

A particularly important example of beyond MHD physics for our specific case is anomalous resistivity. This is an effect that is theorised to occur in low-density current-carrying plasmas, such as those found in pulsed power. If the drift velocity v_e of current-carrying electrons exceeds some critical value, usually the sound speed c_s of the plasma, microinstabilities (i.e. instabilities of the distribution function in velocity space) will occur [122, 123]. The least restrictive instability criteria is for lower hybrid drift instabilities (LHDIs), so these are often considered the most relevant for z-pinch type plasmas [36]. This will set up a turbulent structure in the electromagnetic fields, which scatters electrons. On the macroscopic spatial scales considered in fluid models, this appears as additional collisionality of the electrons [123]. It can therefore be accounted for by adding an additional, so-called anomalous resistivity to the fluid model. These microinstabilities are regularly observed using kinetic simulations, but these are usually not

practical to use to study long timescales in pulsed power simulations. Therefore, the standard approach is to use a kinetic simulation to estimate the saturated value of the microinstabilities, relate this back to a resistivity and then implement this as a model within a fluid code since on fluid timescales the microinstabilities should saturate [122]. A review of various different models produced by this approach is found in ref [123]. Importantly, anomalous resistivity is a function of the electron drift velocity and also acts to reduce this quantity; making the mechanism self-limiting.

Anomalous resistivity is frequently added to models of pulsed power plasma because experimental observations suggest the plasma is more resistive than expected in high temperature regions, both through observation of increased magnetic field diffusion and faster magnetic reconnection [123–126]. Classical Spitzer resistivity decreases in high temperature plasmas, with $\eta_{Spitzer} \propto T_e^{-3/2}$, so it is unlikely that the Spitzer model can explain these measurements. In contrast, many anomalous resistivity models have the property that η_{Anom} increases in high temperature plasmas, providing a convenient explanation for the experimental observations. Anomalous resistivity remains somewhat contentious however, as it has not been directly measured in pulsed power plasmas, though results in space plasmas have indicated that it may be playing a role [124].

2.4 Physics of a Vacuum-Plasma Interface

There are many situations in pulsed power where accurate modelling of vacuum-plasma interfaces is crucial. A vacuum-plasma interface is where there is a transition from a region where the plasma pressure is dominant to a region where the magnetic pressure is. The high density plasma is well described as a fluid, whereas the behaviour of the low-density magnetic pressure dominated area is less clear. These boundary regions are therefore very difficult to model.

However, vacuum-plasma interfaces are ubiquitous in pulsed power experiments. For example, implosion experiments driven by magnetic fields, such as the liners on MagLIF, rely on a significantly higher magnetic pressure pushing on a dense plasma to drive the dynamics. Similarly, at the edge of a z-pinch there is a rapid transition from a dense plasma to a vacuum, making this region an example of a vacuum-plasma interface. Understanding the dynamics of these boundaries is important for accurately modelling pulsed power experiments and is a key aim of this thesis. In this section, we review the different theories for understanding this parameter regime.

It is not possible to construct a practical MHD model that is valid in all of the parameter regimes that are relevant for pulsed power experiments. However, the aim of this work is to improve the fidelity of Gorgon in modelling low density plasma as well as to understand better the physics of these vacuum-plasma interfaces. Therefore, we take the approach of adding terms to the MHD model to increase the detail of the physics it describes. We use the phrase “beyond-MHD physics” throughout this thesis to refer to both extended MHD and kinetic effects that are not included in conventional resistive MHD. An understanding of which terms are most relevant in this regime is

Condition	Regime	Explanation
$\lambda_i \sim L$	Hall physics	Electrons and ions decouple, Hall term becomes non-negligible
$r_{L,i} \sim L$	Finite Larmor radius effects	Ion Larmor radius is not negligibly small, non-diagonal components of the ion stress tensor $\underline{P}_i$
$\lambda_e \sim L$	Finite electron inertia	Electron mass can no longer be assumed to be zero, inertial terms need to be added to the GOL, two-fluid physics becomes relevant
$\lambda_D \sim L$	Sheath physics	Plasma can no longer be assumed quasineutral, non-neutral physics becomes important, two-fluid code needed
$\lambda_{mfp}^{\text{electron}} \sim L$	Electron kinetic regime	Electron Knudsen number is no longer small, electron distribution function will depart from Maxwellian and kinetic effects may become relevant
$\lambda_{mfp}^{\text{ion}} \sim L$	Ion kinetic regime	Ion Knudsen number is no longer small, ion distribution function will depart from Maxwellian and kinetic effects may become relevant
$v_e \sim c_s$	Anomalous resistivity	Usual condition for anomalous resistivity [36, 123], additional component added to resistivity due to electron scattering due to microinstabilities
$v_e \sim c$	Relativistic electrons	Electron population affected by relativistic effects, relativistic GOL needed
$v_A \sim c$	Relativistic MHD	Fully relativistic treatment of MHD needed

Table 2.1: Table summarising limits under which extensions to the MHD model become relevant. This list is not exhaustive, but these are the limits that we consider in this thesis. L is the length scale of the plasma, λ_i and λ_e are the ion and electron inertial lengths, λ_D is the Debye length, $r_{L,i}$ is the ion Larmor radius, λ_{mfp} is the thermal mean free path, $v_e = -j/en$ is the electron drift velocity, c_s is the speed of sound, c is the speed of light and v_A is the Alfvén speed.

thus necessary. This depends on the specific problem is being studied. Therefore, in lieu of showing integrated examples of where this beyond-MHD physics is believed to influence experimental results, in this section we use the edge of a z-pinch as an example of a vacuum-plasma interface to illustrate theoretically how we expect the assumptions made in MHD to break down in this regime. We use this to motivate the necessary features of an improved fluid model to study vacuum-plasma interfaces.

In table 2.1, we summarise the different conditions that may break down and what regime they physically correspond to. The list of effects here is not exhaustive, there is a rich amount of physics that becomes relevant as the density decreases at the vacuum-plasma interface, but these are the only conditions that we consider in this thesis.

2.4.1 Density profile of a z-pinch

In order to evaluate which effects may play a role at the edge of a dense z-pinch, we need theoretical profiles for the density, temperature and magnetic field. As we are trying to find an MHD equilibrium, we find the steady state of the plasma momentum equation (eq. (2.69)). We want a static equilibrium, so we also set $\mathbf{u} = 0$. Therefore, the equilibrium is defined by:

$$\nabla p = \mathbf{j} \times \mathbf{B}. \quad (2.85)$$

We assume that the z-pinch is azimuthally and axially symmetric, so this equation can be written in cylindrical geometry as:

$$\begin{aligned} \frac{\partial p}{\partial r} &= -j_z B_\theta \\ &= -\left(\frac{B_\theta}{\mu_0 r} + \frac{1}{\mu_0} \frac{\partial B_\theta}{\partial r} \right) B_\theta, \end{aligned} \quad (2.86)$$

where we have used $\nabla \times \mathbf{B}$ in cylindrical coordinates and assumed that $j_\theta = B_z = 0$. Depending on the choice of model used, this equation admits many different solutions and theoretical work defining different z-pinch equilibria is extensive [36]. We need to pick a profile for $j_z(r)$ in order to define the pressure profile. For simplicity, in this section we consider the case of a highly collisional and resistive plasma ($\eta \rightarrow \infty$), such that we can assume current density is uniformly distributed within the pinch. Therefore, $j_z = \frac{I}{\pi a^2}$, where a is the radius of the pinch and I is the current. This defines the magnetic field by Ampère's law:

$$B_\theta(r) = \begin{cases} \frac{\mu_0 I}{2\pi a^2} r, & \text{if } r \leq a \\ \frac{\mu_0 I}{2\pi r}, & \text{otherwise} \end{cases} \quad (2.87)$$

Integrating the force balance equation then gives the pressure profile:

$$p(r) = \begin{cases} \frac{\mu_0 I^2}{4\pi^2 a^2} \left(1 - \frac{r^2}{a^2}\right), & \text{if } r \leq a \\ 0, & \text{otherwise} \end{cases} \quad (2.88)$$

As the plasma is collisional, the heat conductivity is high and so the temperature can also be assumed constant inside the pinch with $T_e = T_i$. Its value can be found by the well-known Bennett relation:

$$(Z\bar{T}_e + \bar{T}_i) = \frac{\mu_0 I^2}{8\pi N}, \quad (2.89)$$

where $\bar{T}_e$ and $\bar{T}_i$ are the average electron and ion temperature respectively, I is the current in the pinch and $N = \int_0^a 2\pi n_i(r) dr$ is the ion line density [36]. Equation (2.89) is true irrespective of the current and density distribution within the pinch. We consider hydrogenic z-pinchs with $Z = 1$. This then defines a density profile by the ideal gas equation of state. In figure 2.6, we show the resulting density and magnetic field profiles for a z-pinch with typical values of $I = 10^5 \text{ A}$, $a = 300 \mu\text{m}$ and $N = 2 \times 10^{19} \text{ m}^{-1}$. The red dashed line represents the edge of the z-pinch. To the left of this line, the temperature and current density are uniform with values of 160 eV and $3.5 \times 10^{11} \text{ A/m}^2$. There is no plasma, and hence no temperature or density, for $r \geq a$. The density profile is parabolic, as expected from the pressure equation. It is worth noting that the parabolic z-pinch has a very hard cut-off in density, giving a vanishingly small density gradient scale length right at the edge of the pinch. In reality, the edge of the z-pinch is unlikely to be so well defined, since particles in the high energy tail of the distribution function will form a slightly extended low-density region. In fact, theoretical work even using an MHD analysis suggests that the density profile will “level out” at some low value [127, 128]. However, this subtlety does not significantly affect the analysis in this section.

Another important z-pinch profile is the Bennett profile [85, 129, 130]. This model instead assumes a constant electron drift velocity across the pinch, leading to different profiles for the plasma variables:

$$p(r) = \frac{\mu_0 I^2}{8\pi^2} \frac{a^2}{(r^2 + a^2)^2} \quad (2.90)$$

$$n(r) = \frac{p(r)}{2\bar{T}} \quad (2.91)$$

$$B_\theta(r) = \frac{\mu_0 I}{2\pi} \frac{r}{r^2 + a^2} \quad (2.92)$$

$$j_z(r) = \frac{I}{\pi} \frac{a^2}{(r^2 + a^2)^2} \quad (2.93)$$

where $\bar{T}$ is the temperature given by the Bennett relation. The Bennett profile has the property that $n(r) \neq 0$ for any r , but it still drops off rapidly with radius. The Bennett profile is the appropriate profile for a collisionless plasma [36]. We compare our theoretical analysis for both the parabolic and Bennett pinch profiles later.

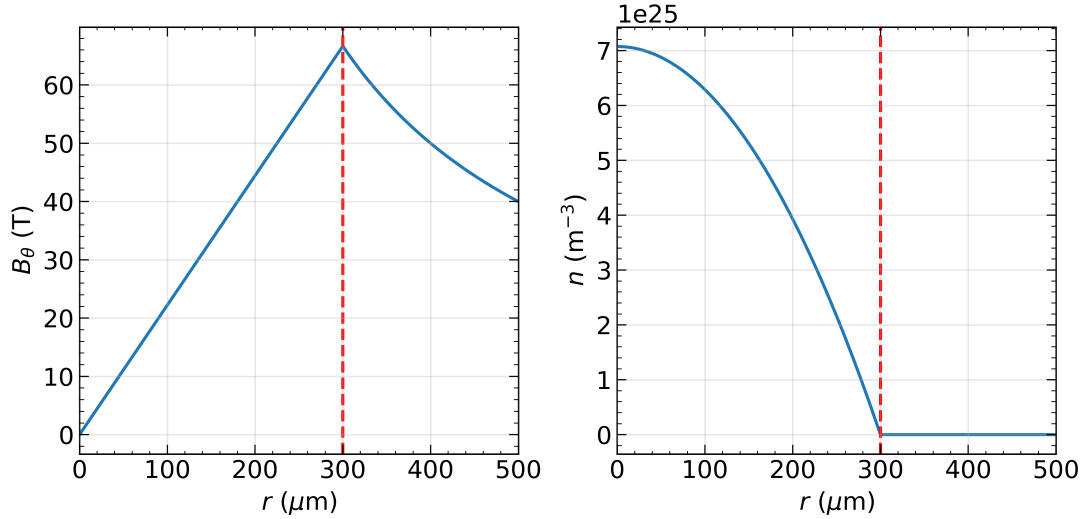


Figure 2.6: The theoretical magnetic field B_θ and density n profiles of a highly collisional z-pinch plasma. The red dashed line shows the edge of the z-pinch at $r = 300 \mu\text{m}$.

2.4.2 Analysis of terms at the edge of the pinch

We now evaluate the limits in table 2.1 for the z-pinch profiles in the previous section. As the characteristic length scale L , we use the density gradient length scale:

$$L_n = \left| \frac{d \ln n}{dr} \right|^{-1}, \quad (2.94)$$

and we calculate the electron drift velocity from the current density $|v_e| = \left| \frac{j}{en} \right|$. This assumes that $v_i \ll v_e$, which is reasonable for the fast current-carrying electrons at the edge of a z-pinch. Figure 2.7 shows these limits superimposed on the z-pinch density profile from figure 2.6 for comparison. The lines are drawn at the point where the assumption made in the MHD model begins to break down. The physical meaning of each of these limits is discussed in table 2.1. This plot illustrates that as we move out radially in the z-pinch and the density decreases, the approximations made in the MHD model progressively break down and theoretically we need to account for more physics. In fact, additional physics becomes relevant as high in density as $4 \times 10^{25} \text{m}^{-3}$, where the scale length becomes comparable to the ion inertial length. At slightly lower densities of $3 \times 10^{25} \text{m}^{-3}$, the electron drift velocity exceeds the sound speed, implying anomalous resistivity effects may occur. We find that $L_n \gg \lambda_D$ except for right at the edge of the pinch, so generally charge separation physics will only play a role at the lowest z-pinch densities. As noted in the previous section, there may be an extended low density region rather than the hard cut-off in density the parabolic z-pinch profile predicts. Charge separation physics will therefore likely play a role in this coronal region of the pinch. The plot indicates that relativistic effects are only likely to become relevant right at the edge region. Arguably the most important effects to account for within an improved

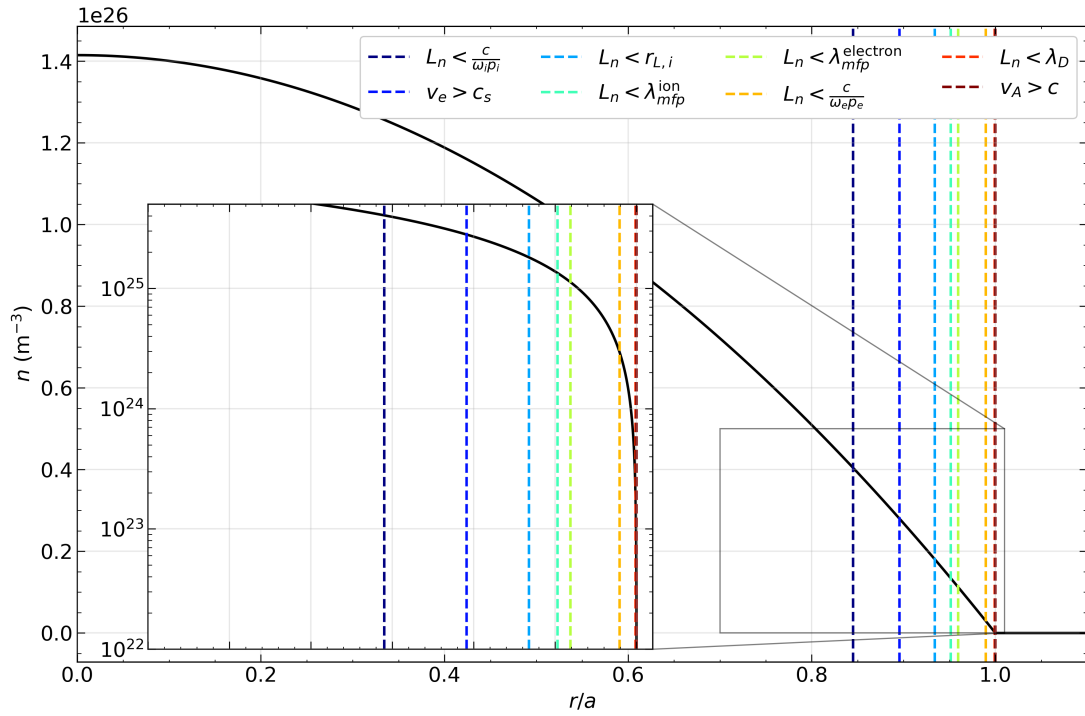


Figure 2.7: Plot showing how the MHD model breaks down at the edge of a dense z-pinch. The z-pinch number density $n = n_e = n_i$ is plotted in black. Superimposed are dashed lines showing when various limits assumed in the MHD model begin to break down. These limits are discussed in table 2.1. The length scale used for the calculation is the density gradient length scale L_n .

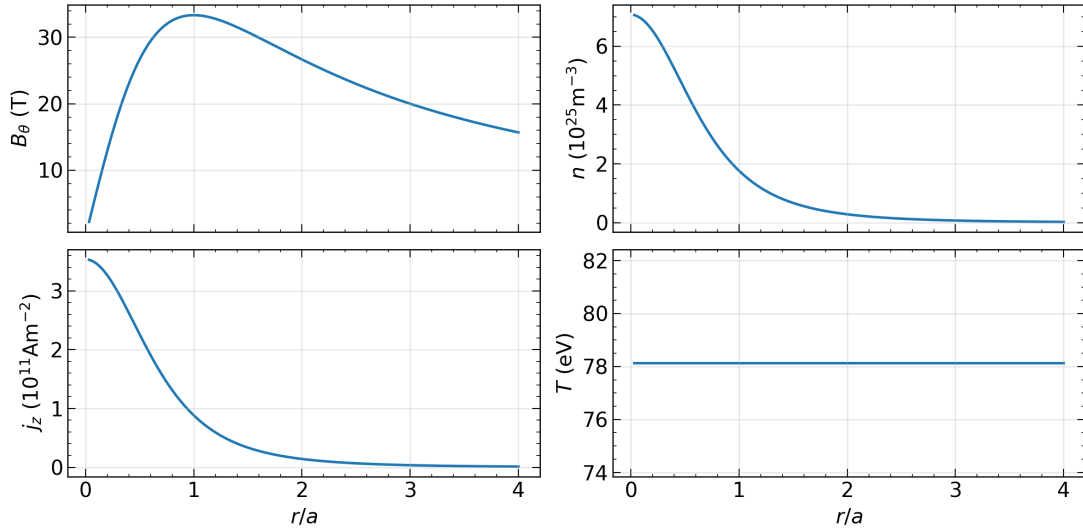


Figure 2.8: Profiles for azimuthal magnetic field B_θ , number density $n = n_e = n_i$, axial current density j_z and temperature $T = T_e = T_i$ for the Bennett z-pinch profile, discussed in section 2.4.1.

MHD model of the vacuum-plasma interface are those that come into play “first” on this plot. Therefore, it is the Hall effect and anomalous resistivity that are likely to play the largest observable role. Kinetic effects also likely play an important role, in fact occurring before electron inertial effects. These cannot be meaningfully accounted for within an MHD model, only their qualitative impacts assessed. This highlights that there is no “complete” model of the vacuum-plasma interface that accounts for all physics, assumptions have to be made. It is worth noting that the ion mean free path is longer than the electron mean free path for this problem because we have assumed $T_i \sim T_e$, which may not be true due to the large currents Ohmically heating the electron population. Also, the threshold $L \sim \lambda_{mfp}$ that is used for this calculation corresponds to $\text{Kn} \sim 1$. Theoretical work suggests that the distribution function begins to depart from Maxwellian at Knudsen numbers as modest as 0.1 [99]. Therefore, kinetic effects are likely to be even more important than is implied by this plot.

We have used the density gradient length scale for this analysis, which may not be the most suitable length scale since $\lim_{r \rightarrow a} L_n = 0$ for the parabolic profile. There is no correct value to use for the length scale, this analysis gives only a qualitative picture of which terms are relevant, but the same comparison can be performed with any length scale L . For example, using $L \sim a$, where a is the pinch radius, we find similar qualitative conclusions. Relativistic, non-neutral and electron inertial effects all become important very close to the edge of the pinch. The most important “first-order” terms to include are still the Hall effect and anomalous resistivity.

It is also worth considering what effect a different z-pinch profile has on this analysis. We consider the Bennett profile discussed in section 2.4.1 choosing the same I , line density N and pinch radius a values for consistency. This produces slightly different values for the peak density and magnetic field, but the values are within an order of magnitude

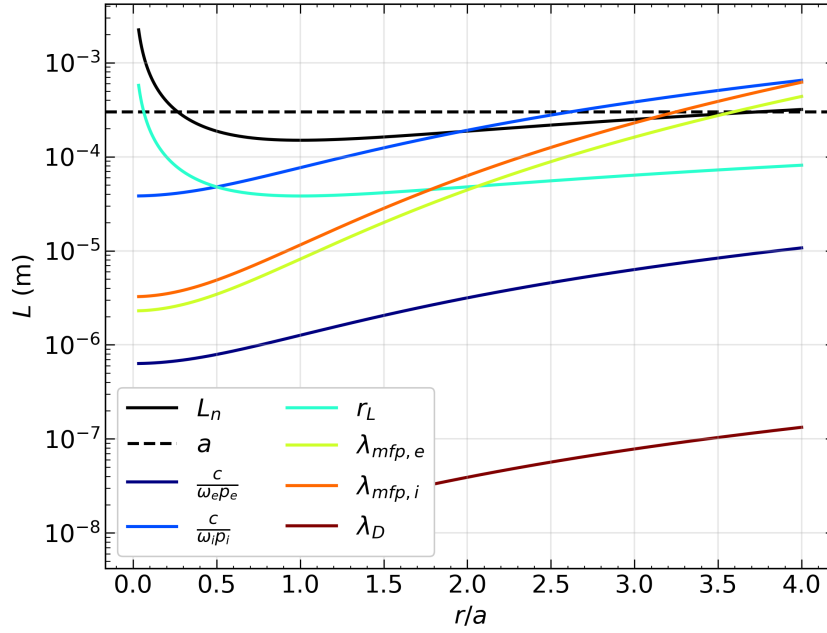


Figure 2.9: Plot showing how the density gradient length scale L_n varies with pinch radius a . The threshold length scales for beyond-MHD effects are shown as well. When L_n is smaller than one of these length scales, it becomes necessary to account for beyond-MHD physics. These length scales are discussed in table 2.1.

of those used for the parabolic pinch. The density, magnetic field, temperature and current density profiles are shown in figure 2.8. As discussed above, there is no hard “edge” to the profile as there is for the parabolic z-pinch; there is density as $r \rightarrow \infty$. However, most of the plasma is contained within one radius a . The temperature is the same as for the parabolic z-pinch, since the Bennett relation is independent of density profile. From the profiles for j_z and n , the axial velocity of electrons is constant, with value $|v_e| = 3.0 \times 10^4 \text{m/s}$. As the temperature is constant, the ion sound speed is also constant with value $c_s = 8 \times 10^4 \text{m/s}$. Therefore, $c_s > |v_e|$ everywhere and so we might not expect the anomalous resistivity to be relevant for this profile. This is just a consequence of the lower current density in the edge region for the Bennett profile. However estimates for when anomalous resistivity become important vary between different models, and microinstabilities are possible at nearly all electron drift velocities [123]. Therefore, given that v_e is of a comparable magnitude to c_s , we would likely still expect some anomalous resistivity to occur. In figure 2.9, we compare the density gradient length scale L_n and the pinch radius a to the different length scales for beyond-MHD effects as we did previously for the parabolic profile. Due to the much less steep density profile than the parabolic case, the length scale L is much larger than any of the thresholds for beyond-MHD effects until $r \sim 2a$. Qualitatively, the conclusions are similar to the parabolic z-pinch profile. The Hall effect is again the most important beyond-MHD term to account for, with $L_n \sim \lambda_i$ occurring at around $r \sim 2a$. Kinetic effects in the electron and ion population are also important, though they cannot be accounted for

within a fluid model. The Debye length is at least 4 orders of magnitude below L_n , so non-neutral effects are not likely to be relevant for the Bennett profile. Using the pinch radius a instead of L_n as the length scale for comparison again produces qualitatively similar conclusions.

The profiles that we have used so far are defined by a set of parameters: the current I , the line density N and the pinch radius a . We have picked parameters that are similar to those used in past theoretical work [122] as well as broadly comparable with the values that might be observed on the MAGPIE facility. It is possible by changing these values to obtain slightly different conclusions to those that we have drawn in this section. For example, for lower currents I the magnetic field and Bennett temperature are reduced without changing the density. This means that the thermal velocity of particles is reduced, changing the threshold at which kinetic and finite Larmor radius effects become important. In contrast, lowering the line density increases the temperature without changing the magnetic field. This causes kinetic and finite Larmor radius effects to become the most relevant beyond-MHD terms. It is not possible with this analysis to consider all possible z-pinch parameters so we have just focused on a set that is representative for our problem.

This analysis is limited in that we do not consider the dynamic evolution of the pinch. However, we would expect similar qualitative conclusions from a more detailed study given the effect of different profiles and pinch parameters. Ion kinetic effects are likely to be important at the edge of a z-pinch, but kinetic simulations are not routinely possible and we want to find possible extensions to a fluid model. Motivated by the importance of kinetic effects, we carry out fully kinetic simulations of the edge region of a z-pinch in chapter 5. Two-fluid and non-neutral effects become relevant at very low densities, so we do not evaluate their effects as a first approximation. The most important terms to account for (in order) are the Hall term, anomalous resistivity and finite Larmor radius effects. In chapter 6, we present a new numerical method to implement the Hall term within an MHD code. We use that new model to study the evolution of various Hall instabilities and the edge region of a z-pinch. That model could in principle be extended to account for finite Larmor radius and anomalous resistivity effects, and that is a potential avenue for future work.

3 Numerical Tools

There are few problems in plasma physics that can be solved analytically. Those that can be are usually toy problems to illustrate basic physics or reduced models that are only valid for a very limited part of parameter space. Modelling of high energy density systems cannot be practically done without computational approaches.

In this chapter, we briefly review some key numerical approaches that we rely on in this thesis. We discuss the Gorgon radiation magnetohydrodynamics code [1] which is used frequently for results throughout this thesis. We also highlight the EPOCH particle-in-cell code [2], which is used in chapter 5.

3.1 Numerical Methods

3.1.1 Finite volume methods

The key to solving partial differential equations numerically is discretisation. The spatial domain is divided up into zones, with variables having a different value in each zone. This allows spatial derivatives to be defined by gradients across the zones. Similarly, the equation is also discretised in time and only solved at discrete times. By doing this, the differential equation is reduced to algebraic relations that can be solved to advance the variables in time. Different choices of how to do this discretisation offer different advantages and challenges.

Many partial differential equations in plasma physics can be written as conservation systems. For example, the mass continuity equation for hydrodynamics can be written as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (3.1)$$

where ρ is the mass density and $\mathbf{u}$ is the fluid velocity. This equation describes how the mass density in an infinitesimally small fluid element can only be changed by a net flux inward or outward, here given by $\rho \mathbf{u}$. When solving equations that can be written like this, it is beneficial to design a numerical scheme that maintains this conservation property exactly. If we do not ensure this, numerical errors can erroneously cause changes in ρ without a corresponding flux, which will lead to unphysical results. In the Gorgon code used in this thesis, this is done using finite volume methods. Figure 3.1 illustrates how this can be done for equation (3.1) in the one-dimensional case where $\mathbf{u} = u_x \hat{x}$ only. The domain is divided into zones (or cells), each of which has its own value of ρ , with $\rho(x_i) = \rho_i$. Fluxes of ρ at the left and right hand sides of each cell, defined $F^{L,R}$

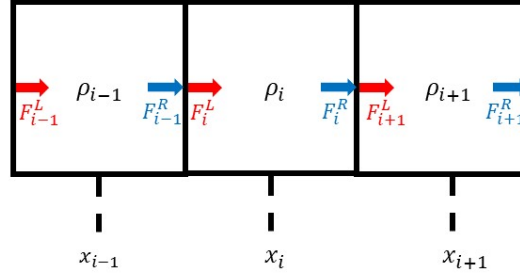


Figure 3.1: A diagram of a finite volume method

are calculated. The most basic scheme requires just the information from cells i and $i + 1$ to calculate the fluxes at the boundary between them. Higher spatial order can be obtained by including information from further away cells [131, 132]. The construction of these schemes can be complicated, particularly for coupled and multidimensional equations such as those of plasma physics, and is beyond the scope of this thesis [132]. Importantly, the fluxes are defined so that $F_i^R = F_{i+1}^L$. The value of ρ_i is then updated using:

$$\rho_i^{j+1} = \rho_i^j + \Delta t (F_i^L - F_i^R), \quad (3.2)$$

where Δt is the timestep and the superscript indicates the time level. Due to the definition of the fluxes, this ensures that any change to ρ_i (say due to an outgoing flux F_i^R) leads to the corresponding change in ρ_{i+1} . This ensures that across the whole domain, mass is conserved exactly.

3.1.2 Explicit vs implicit codes

In the previous section, we discussed the spatial discretisation of equations by the finite volume method. We must also discretise them in time. There are two main approaches for doing this: explicit and implicit methods.

The difference between these can be seen when solving the general ordinary differential equation:

$$\frac{dy}{dt} = f(t, y(t)), \quad (3.3)$$

for some arbitrary function f . An explicit method advances y only based on information that is known at the moment of the update. The first order Euler method is then given by:

$$\frac{y^{n+1} - y^n}{\Delta t} = f(t^n, y^n), \quad (3.4)$$

where $y^n = y(t^n)$ and Δt is the timestep. Since y^{n+1} only appears on the left hand side,

we can write a closed form expression for it:

$$y^{n+1} = y^n + \Delta t f(t^n, y^n). \quad (3.5)$$

In contrast, an implicit scheme advances y based on information at the future timestep. At first order, this can be written as:

$$\frac{y^{n+1} - y^n}{\Delta t} = f(t^{n+1}, y^{n+1}). \quad (3.6)$$

Note that f is now calculated using the values from the advanced time level. In general, obtaining y^{n+1} from this relationship requires solving an algebraic equation that is not always linear. Therefore, implicit methods are harder to implement than explicit methods. Partial differential equations introduce another challenge for implicit methods. To see this, consider the linear 1D advection equation:

$$\frac{\partial f}{\partial t} = -v \frac{\partial f}{\partial x}. \quad (3.7)$$

This describes the advection of some scalar f with velocity v (here assumed to be $v > 0$). An implicit scheme to update f can be written:

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} = -v \frac{f_i^{n+1} - f_{i-1}^{n+1}}{\Delta x}, \quad (3.8)$$

where we have defined $f_i^n = f(x_i, t^n)$ and we have used a first order accurate approximation for the spatial derivative. Note that both f_i^{n+1} and f_{i-1}^{n+1} appear on the right hand side. The implicit scheme couples together every cell in the domain, so solving for f^{n+1} requires solving for every cell simultaneously by inverting a matrix the size of the domain. In general, this inversion cannot be done analytically and iterative methods are needed. These can be inefficient and difficult to implement [133, 134]. 2D or 3D simulations often make use of parallel computing for efficiency. In the Gorgon and EPOCH codes, this is done by dividing the cells modelling the domains between different processors which can then all operate simultaneously. Iterative matrix inversions are very challenging to parallelise and so implicit methods do not scale well for parallel codes, such as those used in this thesis [135, 136].

The benefit of implicit approaches over explicit is their superior stability properties. Explicit methods set a maximum timestep that must be resolved or numerical errors can propagate and cause the solution to become numerically unstable. It can be shown that [131] for advective equations, such as equation (3.1) or (3.7), the timestep is given by:

$$\Delta t \leq \frac{\Delta x}{v}, \quad (3.9)$$

where Δx is the spatial size of the cell and v is the velocity. This is known as the Courant-Friedrichs-Lewy (CFL) condition. This easily generalises to higher dimensions.

Other kinds of PDE need different timestep requirements. For a multi-physics code that solves many different variables, these explicit timesteps can be very restrictive and require taking many very small Δt to advance to times of interest. This is a particular problem for stiff systems, where there is a large separation of scales between the explicit timestep and the timescale of interest. For example: in a typical simulation of a direct-drive experiment, the ratio of the explicit timestep required to stably resolve thermal conduction and the timescale of the implosion can be:

$$\frac{\tau_{Implosion}}{\Delta t_{TC}} \sim 10^4. \quad (3.10)$$

This means that 10^4 steps of thermal conduction are needed to model the implosion, even though almost all of the thermal conduction happens early in time. This severely reduces the efficiency of explicit codes. In contrast, implicit methods are often unconditionally stable. This means they will remain numerically stable even if Δt is much larger than the explicit timestep, though they can be inaccurate in this limit as a result. This means that it is possible to step over high frequencies that may not be of interest. In effect, implicit codes generally have a larger computational cost per timestep, but they need to take fewer timesteps than explicit codes. This tradeoff needs to be made when designing a code. The EPOCH and Gorgon codes used in this thesis exclusively use explicit differencing, as scaling to many processors is important for large simulations.

3.1.3 Operator splitting

Gorgon makes frequent use of operator splitting as a method to simplify solving PDEs. As an example, consider a basic advection-diffusion equation. A similar equation is obtained when solving resistive MHD:

$$\frac{\partial y}{\partial t} = D \frac{\partial^2 y}{\partial x^2} - v \frac{\partial y}{\partial x}, \quad (3.11)$$

where y is some scalar and D is a diffusion coefficient. Whilst in principle a scheme could be found that discretises everything and solves both the diffusive and advective parts simultaneously, it is far easier to split the update into separate stages. In this example, the explicit timestep for the diffusion term is usually more restrictive than that of the advective term (this can be a stiff system). Therefore, for efficiency, it is preferable to solve the two parts separately. First, a method is used to solve the advective term to update y to some intermediary value y' . y' can then be updated using the diffusive term to obtain the final value of y at the advanced timestep. This is particularly helpful for equations with many disparate source terms, such as the energy equation. This method introduces an accuracy error, but this is often worth the increased efficiency and ease of implementation. A particular advantage of operator splitting is that it allows for the use of a tailored numerical scheme for the different sub-problems, a property that we take advantage of in chapter 6.

3.2 The Gorgon Radiation Magnetohydrodynamics Code

Throughout this thesis, we make frequent use of the Gorgon code. Gorgon is a parallel 3D multiphysics code that was developed at Imperial College London to study wire-array z-pinchs [1], but has been adapted over the years to include a wide range of physics that makes it useful for many other high energy density systems. In this section, we briefly summarise the features of Gorgon. We discuss the MHD solver in the next section in more depth, because that is the most relevant feature for our work.

Gorgon solves the equations of single-fluid extended MHD, with a two-temperature approximation so that separate equations for the ion and electron energy can be evolved. The hydrodynamic equations are repeated here for ease of reference:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (3.12)$$

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) = -\nabla(p_e + p_i) + \mathbf{j} \times \mathbf{B} \quad (3.13)$$

$$\frac{\partial U_i}{\partial t} + \mathbf{u} \cdot \nabla U_i = -p_i \nabla \cdot \mathbf{u} - \nabla \cdot \mathbf{q}_i - Q_{ei} \quad (3.14)$$

$$\frac{\partial U_e}{\partial t} + \mathbf{u}_e \cdot \nabla U_e = -p_e \nabla \cdot \mathbf{u}_e - \nabla \cdot \mathbf{q}_e + Q_{ei} + Q_{Ohmic} + Q_R + \dots \quad (3.15)$$

The left hand side of each of these equations describes the advection of the quantity with the fluid flow. These terms are treated using the finite volume machinery described in section 3.1.1. A Lagrangian-remap scheme is used where the force and heating terms in the momentum and energy equations are used to accelerate and heat the plasma in each cell on the Eulerian grid. The advection part of the equations is handled by remapping the expanded fluid element calculated in the Lagrangian step back to the original Eulerian grid [137]. This is done in a flux conservative way, using a van Leer correction to increase the spatial accuracy of the scheme to second order [138, 139]. Gorgon uses a von Neumann artificial viscosity to assist in capturing shocks [137, 140]. The purpose of this term is to decelerate the plasma across a shock and reduce unphysical numerical oscillations that arise downstream of the shock.

Gorgon has the capability to do a radiation transport solution as well, which is coupled to the hydrodynamics by the Q_R term in the electron energy equation. We do not use this feature in this work, but the thesis of K. McGlinchey describes this in depth [140]. Atomic physics data is provided by the in-house SpK microphysics code, described in ref. [141]. The system is closed by using an equation of state (EoS) to relate the electron and ion energies to pressures, and assuming small departures from Maxwellian to obtain an expression for the heat flux $\mathbf{q}$. In this work, we only use an ideal gas EoS though tabulated ones are also possible. Braginskii expressions, corrected by Epperlein-Haines [94] and Sadler-Walsh [97], are used for the heat flux and other transport coefficients [86].

Gorgon is fully explicitly differenced and has shown good scaling to tens of thousands of cores. It is frequently the case that the stable timescales for some processes are

smaller than the hydrodynamic timestep that is used to advance the momentum equation. In particular, thermal conduction is a diffusive process and has a $\Delta t \propto \Delta x^2$ scaling, which means that the thermal conduction timestep is usually far smaller than the hydrodynamic scales. Therefore, the thermal conduction is subcycled for computational efficiency. This means that during each hydrodynamic timestep, multiple thermal conduction steps are taken. The heat flux terms are implemented using a corner-averaging scheme to minimise numerical diffusion, as developed by C. Walsh [30]. In chapter 4, we take an analogous approach to implement the Ettingshausen heat-flow.

3.3 The MHD solver in Gorgon

In this work, we are primarily concerned with MHD and so a more detailed description of the MHD solver used in Gorgon is necessary. The field solver in its current form was developed by Drs C. Jennings and C. Walsh and is described in detail in their theses [30, 136].

3.3.1 The vector potential and the staggered grid

A significant challenge in developing MHD solvers is to design a scheme that ensures the divergence free condition:

$$\nabla \cdot \mathbf{B} = 0. \quad (3.16)$$

Whilst theoretically this condition is automatically satisfied whilst solving the induction equation, numerical errors are unavoidable and will develop a finite divergence. If these errors are not controlled, unphysical behaviour and numerical instability can result [142]. There are various ways of ensuring the divergence of $\mathbf{B}$ condition is maintained. Divergence-cleaning methods remove any non-zero $\nabla \cdot \mathbf{B}$ from the system numerically. For example, the 8-wave method of Powell *et al.* adds additional unphysical advection terms to the induction equation to advect any $\nabla \cdot \mathbf{B}$ out of the system faster than the timescales of interest [143]. These methods are inefficient, difficult to implement and are by nature reactive. They dissipate any $\nabla \cdot \mathbf{B}$ errors that have arisen and an imperfect cleaning scheme may fail to prevent a finite divergence from building [144]. Gorgon instead uses a constrained-transport approach on a staggered grid, known as a Yee mesh [145]. Figure 3.2 is a diagram showing this grid. As in normal hydrodynamics, hydrodynamic variables such as ρ or $\mathbf{u}$ are located at cell centres. The magnetic field components are staggered from this, located on cell faces. The electric field components (and currents) are located on cell edges. This ensures that the curl of a magnetic field naturally locates to cell edges and the curl of an electric field locates to cell faces. This is shown in figure 3.3. For example, consider the x component of the curl of the electric

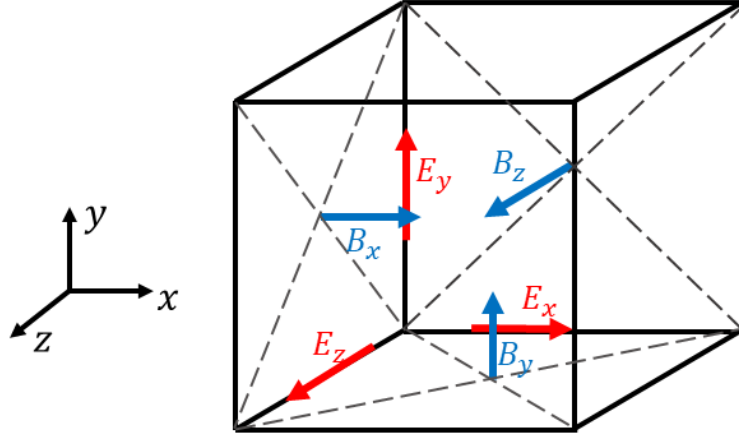


Figure 3.2: A diagram showing the location of electric and magnetic fields on the staggered Yee mesh. Hydrodynamic variables, such as ρ or T , are located in the cell centre. Electric fields are located on the cell edges and magnetic fields are located on the cell faces. This is so that $\nabla \times \mathbf{E}$ has values on cell faces and $\nabla \times \mathbf{B}$ has values on cell edges.

field:

$$\begin{aligned}
 (\nabla \times \mathbf{E})_x &= \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\
 &= \frac{E_z(x, y + \Delta y, z) - E_z(x, y, z)}{\Delta y} - \frac{E_y(x, y, z + \Delta z) - E_y(x, y, z)}{\Delta z}. \quad (3.17)
 \end{aligned}$$

Considering figure 3.3, it is clear that these finite difference equations produce a $(\nabla \times \mathbf{E})_x$ that is located on the x face of the cell: the same location as the B_x component of the magnetic field. The magnetic field is ultimately updated by Faraday's Law:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}. \quad (3.18)$$

Since $\nabla \cdot (\nabla \times \mathbf{E}) = 0$ by definition, this ensures that if $\mathbf{B}$ is only ever updated by the curl of $\mathbf{E}$, it cannot acquire any divergence errors. Therefore, if $\mathbf{B}$ starts with $\nabla \cdot \mathbf{B} = 0$, it will remain that way. The staggered grid also ensures that $\nabla \cdot \mathbf{B}$ is located at the cell centres.

Instead of evolving $\mathbf{B}$ as a primary variable, Gorgon uses the vector potential $\mathbf{A}$, defined:

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (3.19)$$

The vector potential resides in the same location as the electric field on the grid. The electric field can be written in terms of the vector potential and the scalar potential ϕ

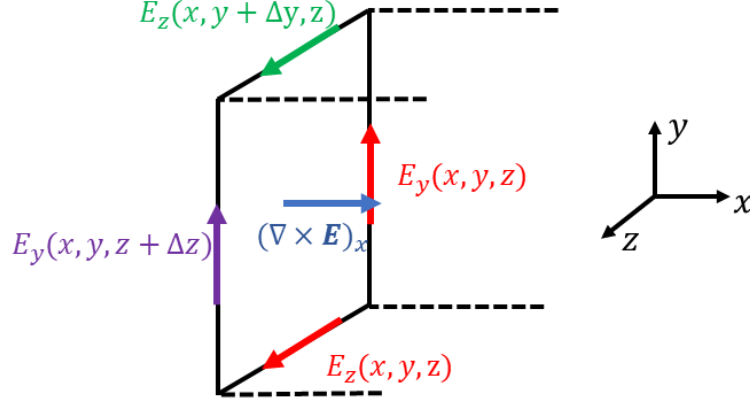


Figure 3.3: Diagram showing how the staggered position of variables on the Yee grid ensures that $\nabla \times \mathbf{E}$ is located at cell faces.

as:

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}. \quad (3.20)$$

Gorgon uses the Weyl gauge where $\phi = 0$, so Faraday's Law can be written as:

$$\frac{\partial \nabla \times \mathbf{A}}{\partial t} = -\nabla \times \mathbf{E}, \quad (3.21)$$

or

$$\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{E}. \quad (3.22)$$

The impact of this within the code is that $\mathbf{B}$ is never explicitly updated; it is only calculated from equation (3.19) after $\mathbf{A}$ is advanced using equation (3.22). It has been shown that this $(\mathbf{A}, \mathbf{E})$ MHD solver is completely consistent with a normal constrained transport $(\mathbf{B}, \mathbf{E})$ scheme [68].

3.3.2 Advective terms

It can be shown [29] that the induction equation for MHD (without displacement current) can be written as:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\eta \nabla \times \mathbf{B}) + \nabla \times (\mathbf{v}_B \times \mathbf{B}) + \nabla \times \frac{\nabla p_e}{en_e}. \quad (3.23)$$

This follows from substituting an expression for Ohm's law for the electric field into Faraday's equation. The first term on the right hand side represents resistive diffusion of the magnetic field due to a finite resistivity in the plasma. The second term represents advection of the magnetic field with the velocity field v_B . This velocity includes the hydrodynamic flow $\mathbf{u}$ which leads to the frozen-in-flow effect of ideal MHD. Past

work has shown that the Nernst effect, anisotropic resistivity and Hall effect can also be written as advection velocities that contribute to $\mathbf{v}_\mathbf{B}$ [29, 96, 97]. Therefore, within Gorgon these terms are implemented as advectons. As the third term does not depend on magnetic field, it acts as a source term for magnetic flux [146]. This term is known as the Biermann battery and has been theorised to have important astrophysical implications and consequences for ICF implosions [120]. In this work, we do not study this term at all and it is dropped from future equations. A detailed description of its implementation can be found in the thesis of C. Walsh [30]. The implementation of the resistive terms is described in the next section; here we discuss the advective terms.

An accurate (and conservative) advection of the magnetic field is extremely important as it leads to ideal MHD. The advective part of equation (3.23) can be written as:

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v}_\mathbf{B} \times \mathbf{B}) \\ &= -\nabla \times \mathbf{E}',\end{aligned}\tag{3.24}$$

where $\mathbf{E}' = -\mathbf{v}_\mathbf{B} \times \mathbf{B}$. Therefore, the advection term is just an electric field that can be used to update $\mathbf{B}$, and should thus be edge-centered on the grid. The electric field can be thought of as a flux that updates the magnetic field. The advection velocity $\mathbf{v}_\mathbf{B}$ is first calculated in the centre of each cell before being averaged to the correct edge of the Yee grid. Considering just the advection of B_z field in the $\hat{x}$ direction by a v_x , $\mathbf{E}' = v_x B_z \hat{\mathbf{y}}$. Therefore, v_x should for this case be averaged to the E_y edge. For conservative differencing, the flux is then calculated using upwinding. This is to account for the fact that physically the magnetic field downstream of the flow cannot affect the values upstream. This is illustrated in figure 3.4. In this case, this means that:

$$E'_y = \begin{cases} -\bar{v}_x B_1, & \text{if } \bar{v}_x > 0 \\ -\bar{v}_x B_2, & \text{if } \bar{v}_x < 0, \end{cases}\tag{3.25}$$

where $\bar{v}_x$ is the value of v_x averaged from 4 neighbouring cells to the E_y location and B_1 and B_2 correspond the values shown in figure 3.4. Having calculated these electric fields, they are used to update $\mathbf{A}$ by equation (3.22) which can in turn be used to update $\mathbf{B}$. It is again worth remarking that Gorgon is not a true Lagrangian remap MHD code since only the hydrodynamics are solved using a remap step, the MHD equations are evolved on the fixed Eulerian grid [139].

3.3.3 Vacuum field solver

The vacuum-plasma interface region is of primary interest for this thesis. Gorgon has several features that make it appropriate for studying these regions.

In Gorgon, the vacuum is treated as a region with constant density ρ_{vac} and constant resistivity η_{vac} . The vacuum does not take part in the hydrodynamics, to prevent numerical errors where the density might go negative, and has no energy or momentum

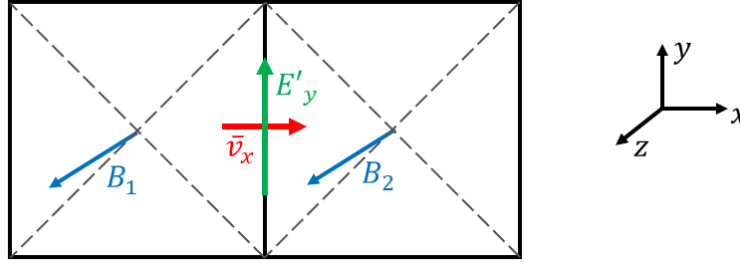


Figure 3.4: Diagram showing the x - y plane for the calculation of advective fluxes for updating magnetic fields. The x component of the advection velocity is averaged from 4 cell centres to the E_y edge. The electric field due to this advection is calculated from the upstream B_1 value if $\bar{v}_x$ is positive, or B_2 if $\bar{v}_x$ is negative.

associated with it. If any cells lose material such that their density would go below ρ_{vac} , their momentum and energy are set to zero. This can introduce small momentum or energy conservation errors into the system, but it is crucial for stability. The resistivity η_{vac} of the vacuum is set to an arbitrarily high value. This is because physically there are no currents or plasma in the vacuum and so it should be highly resistive. This approach is common in codes modelling pulsed power [147], however it introduces severe numerical problems. The large resistivity requires a very small explicit timestep, given by:

$$\Delta t \leq \frac{\Delta x^2}{2\eta}. \quad (3.26)$$

Note that the timestep scales quadratically with cell size, making it more punitive than the normal advective timesteps. Therefore, an explicit code needs to evolve very small timesteps to account for the resistivity in a vacuum region that ultimately bears very little relevance to the problem of interest. Implicit codes are an option, but the large resistivity makes the necessary matrix inversion very expensive to carry out [136] and the very steep gradient in resistivity at the vacuum-plasma boundary makes the matrix difficult to precondition (and thus even harder to invert) [114, 148]. Historically, this was a major limitation in modelling pulsed power experiments using resistive MHD [147] and so alternative approaches are needed.

As alluded to in the introduction, a significant part of the motivation of this thesis is to develop a more physical model for the vacuum regions in Gorgon. The vacuum is treated by the use of arbitrary switches and artificial floor values for variables to maintain stability. However, this leads to results that vary with different choices for these arbitrary parameters. In chapters 5 and 6, we work to develop a model of the vacuum that takes into account more physics to lead to a less artificial solution in this regime.

Gorgon handles this issue by reinstating the displacement current into Maxwell's equations when solving for the magnetic field. Then, the system of equations to be

solved becomes:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \quad (3.27)$$

$$\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \mu_0 \mathbf{j}, \quad (3.28)$$

rather than just the induction equation as before. The current term $\mathbf{j}$ within Ampère's Law is now obtained from the Generalised Ohm's Law (GOL). The displacement current acts when there is an inequality between $\mu_0 \mathbf{j}$ and $\nabla \times \mathbf{B}$ to bring them into alignment via a time-varying electric field [149]. However, it has been found that it can be used as a convenient numerical device for solving vacuum regions. This method was first introduced by Jennings [136]. From equation (3.23), the electric field can be written as:

$$E = \eta \mathbf{j} - \mathbf{v}_B \times \mathbf{B}, \quad (3.29)$$

where $\mathbf{j}$ is the current density. This is an expression for the GOL in the absence of a Hall effect, electron pressure and electron inertia terms. We operator split off the $\mathbf{v}_B \times \mathbf{B}$ term, advancing $\mathbf{B}$ using the advection machinery of the previous section. After that update, we need to solve the system:

$$\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{E} \quad (3.30)$$

$$\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \mu_0 \frac{\mathbf{E}}{\eta}, \quad (3.31)$$

where we have inserted the definition of $\mathbf{j} = \mathbf{E}/\eta$ into the last equation. We operator split the $\mathbf{E}$ and $\mathbf{A}$ updates, assuming that the $\nabla \times \mathbf{B}$ does not change significantly over a timestep, allowing us to solve the electric field update in equation (3.31) using an integrating factor:

$$\mathbf{E}(t + \Delta t) = \mathbf{E}(t) \exp\left(-\frac{\mu_0 c^2}{\eta} \Delta t\right) + \frac{\eta}{\mu_0} \nabla \times \mathbf{B} \left[\exp\left(-\frac{\mu_0 c^2}{\eta} \Delta t\right) - 1 \right]. \quad (3.32)$$

This is updated in a leapfrog fashion with the $\mathbf{A}$ field in order to obtain second order time accuracy. Therefore, the overall update can be written as:

$$\mathbf{A}^{n+1} = \mathbf{A}^n - \Delta t \left(\mathbf{E}^n \exp\left(-\frac{\mu_0 c^2}{\eta} \frac{\Delta t}{2}\right) + \frac{\eta}{\mu_0} \nabla \times \mathbf{B}^n \left[\exp\left(-\frac{\mu_0 c^2}{\eta} \frac{\Delta t}{2}\right) - 1 \right] \right) \quad (3.33)$$

$$\mathbf{E}^{n+1} = \mathbf{E}^n \exp\left(-\frac{\mu_0 c^2}{\eta} \Delta t\right) + \frac{\eta}{\mu_0} \nabla \times \mathbf{B}^{n+1} \left[\exp\left(-\frac{\mu_0 c^2}{\eta} \Delta t\right) - 1 \right], \quad (3.34)$$

where we have used the notation $\mathbf{A}^{n+1} = \mathbf{A}(t + \Delta t)$. Note that the electric field is updated by $\nabla \times \mathbf{B}(t + \Delta t) = \nabla \times \nabla \times \mathbf{A}(t + \Delta t)$ in the second equation, so this is a second order update. In reality, we have carried out a half timestep update of $\mathbf{E}$, used

this value to update $\mathbf{A}$ and then used this updated value of $\mathbf{A}$ to advance $\mathbf{E}$ from time t to $t + \Delta t$.

This has two key benefits. It can be shown that the explicit timestep requirement to evolve this system is $\Delta t \sim \frac{\Delta x}{c}$ rather than the very restrictive diffusive explicit timestep from earlier [136, 150]. This makes the inclusion of a vacuum region with $\eta \rightarrow \infty$ possible. Secondly, this method produces a smooth transition between the vacuum and plasma limits. In the vacuum limit, where $\eta \rightarrow \infty$, the update becomes

$$\mathbf{A}^{n+1} = \mathbf{A}^n - \Delta t \left[\mathbf{E}^n - c^2 \nabla \times \mathbf{B}^n \right] \quad (3.35)$$

$$\mathbf{E}^{n+1} = \mathbf{E}^n - \Delta t c^2 \nabla \times \mathbf{B}^{n+1} \quad (3.36)$$

whereas in the plasma limit with $\eta \rightarrow 0$, it is:

$$\mathbf{A}^{n+1} = \mathbf{A}^n + \Delta t \frac{\eta}{\mu_0} \nabla \times \mathbf{B}^n \quad (3.37)$$

$$\mathbf{E}^{n+1} = \frac{\eta}{\mu_0} \nabla \times \mathbf{B}^{n+1}. \quad (3.38)$$

In the vacuum limit, this describes an electromagnetic wave propagating at speed c . In the plasma limit, this update ensures that $\mathbf{j} = \nabla \times \mathbf{B} / \mu_0$, using $\mathbf{j} = \mathbf{E} / \eta$. Therefore, this appropriately relaxes to the case with no displacement current in a plasma.

We commented in section 2.3 that including the displacement current can lead to violations to quasineutrality when $\nabla \cdot \mathbf{j} \neq 0$. Even though Gorgon is not used for relativistic plasmas where the displacement current can be large, numerical roundoff can in principle lead to a finite displacement current and $\nabla \cdot \mathbf{j}$ errors. In practice, we find in Gorgon that the MHD solution scheme described in this section correctly relaxes to the case where the displacement current is zero in plasma cells. $\frac{\partial \mathbf{E}}{\partial t}$ is only non-zero in vacuum cells and vacuum-plasma boundary cells, where we would not expect a physical answer anyway due to the arbitrary cutoffs that are introduced to make the MHD solution tractable. Including the displacement current is crucial for an efficient explicit code however, for the numerical reasons we discussed in the previous paragraphs, so it is included even though it is theoretically not entirely consistent.

3.3.4 Boris correction and reduced speed of light

In the previous section, we discussed how the displacement current can be used to mitigate the resistive stability requirements of the vacuum region. The low densities also present a problem, since the Alfvén speed can become large in this region:

$$v_A = \frac{B}{\sqrt{\mu_0 \rho}}. \quad (3.39)$$

The Alfvén speed would physically be controlled by relativistic effects. At speeds approaching the speed of light, the inertia of the plasma increases and the Alfvén wave becomes an electromagnetic wave that does not introduce perturbations in the plasma.

Gorgon accounts for this physics using a correction originally proposed by Boris [151]. A maximum Alfvén speed s_{max} is chosen and the magnetic force ($\mathbf{j} \times \mathbf{B}$) terms in the momentum equation are multiplied by a factor:

$$\frac{1}{1 + \left(\frac{v_A}{s_{max}} \right)^2}, \quad (3.40)$$

which reduces the magnetic force to zero if $v_A \gg s_{max}$. This is commonly used in the field and approximates relativistic behaviour without having to solve relativistic MHD. The resemblance of this term to the Lorentz gamma factor is not coincidental, s_{max} in effect functions as an artificial speed of light to control Alfvén speeds.

The other approximation made is that the actual numerical value of the speed of light is irrelevant, as long as its magnitude is still much larger than the other speeds in the system. Therefore, it is common in the field to artificially reduce the value of the speed of light by an order of magnitude or more to mitigate the timestep requirements of vacuum simulations. Generally, when carrying out simulations we set c to be as low as possible whilst still being larger than the maximum wave speed in the system. Therefore, if we resolve $\Delta t = \frac{\Delta x}{c}$, we ensure all of the other waves are resolved as well.

3.3.5 Applied external currents

To model plasma driven by a pulsed power system, Gorgon allows for the application of an external current to a system. This external current can either be specified with some known profile or alternatively calculated using an external circuit model. That current is then applied to the plasma through boundary conditions on the magnetic field. This is illustrated in figure 3.5. Consider a plasma, shown in red, surrounded by a numerical vacuum region shown in pale blue. Beyond some radius far away from the plasma, indicated by the dotted blue line, the magnetic field is set to be azimuthal with a value of:

$$B_\theta(r, t) = \frac{\mu_0 I(t)}{2\pi r}, \quad (3.41)$$

where I is the external current to be applied and r is the radial distance from the centre of the box as indicated in the diagram. This is the magnetic field set up by an axial current through a conductor, so imposing it as a boundary condition simulates the effect of applying the external current I axially to the plasma. Any change of the magnetic field is transmitted to the plasma by electromagnetic waves across the vacuum region which we have by virtue of retaining the displacement current in Ampère's Law. This method successfully recreates the curl-free $1/r$ dependence of the magnetic field in the vacuum region as required by Maxwell's equations.

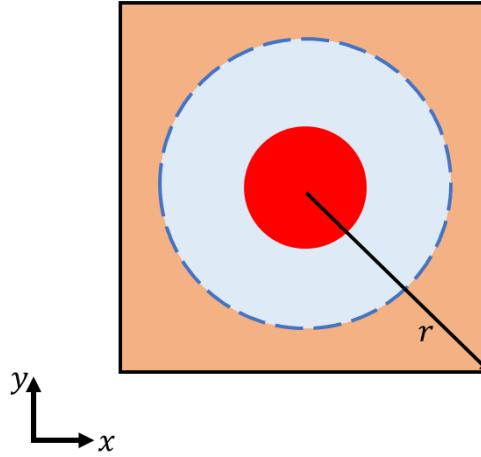


Figure 3.5: Diagram showing how external currents are applied in Gorgon. The numerical domain is shown by the black box. A plasma, shown in red, is surrounded by a numerical vacuum region shown in pale blue. A magnetic field boundary condition is applied in the orange region, simulating the effect of an external current.

3.4 The EPOCH Code

In this section, we outline methods used for modelling kinetic plasmas, focusing on the particle-in-cell approach. We discuss the EPOCH electromagnetic particle-in-cell code [2], which is used for simulations in chapter 5.

3.4.1 Modelling of kinetic plasmas

As discussed in chapter 2, if the mean free path of particles is longer than the length scale, the distribution function of the particles will no longer be well described by a Maxwellian. This happens in plasmas that are not highly collisional, such as the low density plasma at the edge of a z-pinch. In this case, fluid approaches such as MHD are a poor approximation. The plasma needs to be modelled by solving the kinetic equation for each species α :

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla_r f_\alpha + \frac{q_\alpha}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f_\alpha = \left(\frac{\partial f_\alpha}{\partial t} \right)_{\text{collision}}, \quad (3.42)$$

where $f(\mathbf{r}, \mathbf{v}, t)$ is the distribution function, ∇_r is the gradient in position space and ∇_v is the gradient in velocity space. This is coupled to the evolution of the electric and

magnetic fields by Maxwell's equations for $\mathbf{E}$ and $\mathbf{B}$:

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \sum_{\alpha} q_{\alpha} \iiint_{\mathbf{v}} f_{\alpha} d^3\mathbf{v} \quad (3.43)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (3.44)$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \quad (3.45)$$

$$\frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \mu_0 \sum_{\alpha} q_{\alpha} \iiint_{\mathbf{v}} \mathbf{v} f_{\alpha} d^3\mathbf{v}, \quad (3.46)$$

where the sum is over particle species α and the integral is over the whole velocity space. There are various approaches to solving these coupled equations.

Continuum methods directly discretise the phase space. This is used in codes such as iFP [152] and has recently been used to study Kelvin-Helmholtz instabilities in a low β plasma [153]. The issue with this approach is the high dimensionality of the phase space. To fully discretise the phase space, it would require discretising 6 dimensions and then advancing the equation in time, potentially for multiple species. There is also an issue of timescales since the kinetic equation describes physics on very short timescales such as the collision time τ_{coll} and the electron plasma period ω_{pe}^{-1} . To solve the kinetic equation using continuum methods for the timescales of interest in an experiment is still too computationally intensive for most approaches. A simplifying approach is to recognise that collisions drive the distribution function to be isotropic in velocity space. Therefore, rather than discretising velocity space it is possible to expand the distribution function in terms of angular basis functions like spherical harmonics [92]. This means that the equation only needs to be solved in one dimension in velocity space, although for a formally infinite series of terms that describe the velocity space anisotropy. However for moderately collisional plasmas, this series can be truncated making a solution more efficient. This is the approach used in the IMPACT [154] and OSHUN [155] codes. However, excessively large numbers of harmonics are needed for weakly collisional plasmas [92], making this approach impractical in that limit.

3.4.2 Particle-in-cell codes

The alternative approach to solving the kinetic equation is the particle-in-cell (PIC) method, first introduced for plasma physics in the 1950s [156]. The PIC approach has two main steps. Firstly, charged particles are moved freely in space under the influence of electromagnetic fields. The velocities and positions of these charged particles are then used to calculate the charge density and current to update Maxwell's equations, which are solved on a discrete grid. By moving charged particles freely in space, the PIC approach solves the kinetic equation in the Lagrangian frame with the phase space represented by finite-sized macroparticles. The benefit of this approach is that the particles do not directly interact with each other: all interactions occur through a finite grid, making the problem more tractable to calculate.

Given an initial charge and current density distribution, a finite difference time domain (FDTD) method can be used to advance the EM fields by solving Faraday's and Ampère's laws [2]. This is a leapfrog scheme using the same Yee grid as discussed in section 3.3.1 and is therefore second order in both space and time.

The EM fields are used to push the particles by solving the Lorentz force law for each particle:

$$\frac{d\mathbf{x}}{dt} = \mathbf{v} \quad (3.47)$$

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (3.48)$$

where q and m are the charge and mass of the particle and $\mathbf{x}$ and $\mathbf{v}$ are its position and velocity vector respectively. In many particle-in-cell codes, including EPOCH, the relativistic Lorentz force law is solved instead by solving for the particle's momentum instead of velocity. This system of equations is difficult to solve accurately as the scheme must ensure that the $\mathbf{v} \times \mathbf{B}$ term does no work on the particles. A very common method for doing this is the Boris rotation algorithm due to its accuracy when pushing a particle many times in a simulation [157, 158]. In this method, the motion is split into two steps: acceleration of the charged particle by $\mathbf{E}$ and rotation of its velocity vector around $\mathbf{B}$. Once the particles have been moved, the currents needed to evolve the electromagnetic fields can be calculated from their new positions and velocities. In EPOCH, this is done in such a way that the electric field update by the FDTD method always satisfies $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, where ρ is the charge density, without having to explicitly solve this equation using some sort of Poisson solver [2].

Even with the solution of Maxwell's equations on a grid, solving the particle push for every particle in a plasma would still be completely intractable. To get around this, in a PIC code each computational particle is a superparticle representing a large number of physical particles. The number of physical particles each superparticle represents is known as its weight. Density gradients can be incorporated either by having a varying number of superparticles or by varying the number of real particles each superparticle represents. As stated earlier, the particles travel freely in space whilst the fields are only defined on a Yee grid. Therefore, when pushing particles or depositing currents from them, it is necessary to interpolate the superparticle position back to the grid. To account for the fact that each superparticle represents many real particles, the superparticle is assumed to have a spatial distribution of real particles throughout its volume, known as its shape function [2]. The simplest case is to assume that real particles are evenly distributed within the superparticle volume $\Delta x \times \Delta y \times \Delta z$. However, this causes particle quantities to vary strongly cell-to-cell and thus simulations using it are noisy. The next simplest choice is to use a triangular shape function, with the peak of the triangle at the position of the superparticle and its width $2\Delta x$ (in 1D). There are also higher order shape functions but they are more computationally expensive. EM fields are interpolated to the particles by calculating the overlap of the particle shape func-

tion with the function defining the fields on the grid (which is constant in each cell). Particle quantities can be similarly interpolated to the grid using their shape functions. The choice of shape function has important consequences for the performance of PIC simulations. It is well documented that PIC codes can exhibit numerical self-heating due to the interpolation of electric fields to the particle positions when the Debye length λ_D of the plasma is under-resolved [2]. This is a consequence of the discrete grid for the fields and the shape functions for the particles. However, recent research has found that for higher order spatial functions (i.e. higher order than the basic case discussed here), the amount of numerical self-heating is negligibly small [159].

The PIC approach is effective for studying collisionless plasmas. However, adjustments to the algorithm can be made to allow for collisions between particles. In EPOCH, this is done using a binary collision algorithm. Particles in the same grid cell are randomly paired up and assumed to scatter by Coulomb scattering. This is performed after the particle push and does not change the particle positions, only scattering them in velocity space. However, including collisions in PIC codes reduces their efficiency. As particles now directly interact rather than just via the EM fields, this increases the computational expense of a simulation. Additionally, whilst this method of collisions is formally correct if each superparticle represents only one real particle, the accuracy is poor when the superparticles have a large weighting. This is because the superparticles are in effect a finite-sized cloud of charge. Colliding them with each other by treating them as hard balls that deflect in a binary collision clearly does not produce physically accurate answers. In fact, it has been found that in excess of 10^5 superparticles per computational cell is needed to well recreate Spitzer heat flux in test problems [2]. This means that whilst collisions can be included in PIC codes, this feature needs to be used carefully.

PIC codes are widely used in kinetic simulations of plasmas, due to their ease of implementation and wide range of validity [2]. Comparatively, continuum codes are much more specialised in application [92]. PIC codes are commonly used in modelling laser-plasma interactions, both in ICF [160] and in plasma accelerator schemes [161]. They are particularly crucial in this regime as the high temperatures and low densities tend to make the plasma highly kinetic. Within pulsed power, PIC codes are used extensively to study power flow [68, 162, 163] and have recently begun to be used for modelling of the dense plasma focus [164, 165].

3.4.3 The EPOCH particle-in-cell code

EPOCH is a fully electromagnetic particle-in-cell code, developed at the University of Warwick to study a wide range of kinetic physics [2].

EPOCH is explicitly differenced in time, meaning it must resolve both $\Delta x/c$ and the electron plasma period ω_{pe}^{-1} . However, the use of higher order shape functions means that it is often not necessary to resolve the Debye length, mitigating its computational expense. The code can be used in 1D, 2D or 3D Cartesian geometry and is parallelised

using domain-decomposition with MPI. It uses load balancing to balance the distribution of superparticles between processors for computational efficiency. EPOCH is fully relativistic, which is potentially important given the large electron drift velocities that can develop in the vacuum region that we are interested in.

EPOCH is a well-validated code that has been extensively tested in both the collisionless and collisional regime. It has been compared against both other PIC codes [166] and theory [2]. Therefore, we do not perform any novel validation of the code beyond testing that it has been built correctly and is producing consistent answers with results in the documentation [2].

In chapter 5, we discuss in more detail our rationale for using EPOCH to model low-density plasmas as well as the specific setup of the code for the problem that we are interested in. EPOCH has not widely been used for pulsed-power problems and we discuss the issues with this parameter regime in that chapter as well.

4 Self-similar solutions to understand magnetised plasma transport

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4.1 Motivation

As described in chapter 1, recently there has been a renewed interest in magneto-inertial fusion (MIF) on both MagLIF [48] and the magnetised NIF platform [31, 38]. Extended MHD simulations are the most common way of modelling these experiments. However, they can be highly computationally expensive and there is still a limited availability of codes with these capabilities. This makes carrying out parameter scans to understand relevant magnetised physics difficult. Additionally, the numerical methods to implement extended MHD within codes are still under research and need validation. In this chapter, we discuss a technique that allows for the rapid solution to arbitrary accuracy of the one dimensional extended MHD equations under certain assumptions. The resulting family of solutions can be used to perform rapid parameter scans to establish whether various effects are expected to have an impact in a particular experiment and to qualitatively understand their behaviour. Due to the high-resolution of these solutions, they can be used as test problems to establish whether multi-physics codes are correctly solving the underlying equations of extended MHD.

4.2 The method of self-similar solutions

Past work has shown that the equations of extended MHD in 1D planar geometry admit self-similar solutions [59]. This means that the solution does not depend independently on the coordinate x and time t , but rather some dimensionless combination of them $f(x, t)$. An example of this is shown by the linear heat equation:

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}, \quad (4.1)$$

where T is temperature, t is time, D is thermal diffusivity and x is the spatial coordinate. We can introduce normalised variables: $\bar{x} = x/L$, $\bar{t} = t/\tau$, $\bar{T} = T/T_0$, where L , τ and T_0 are suitable scales. Absorbing these scales into the diffusion coefficient, we can write

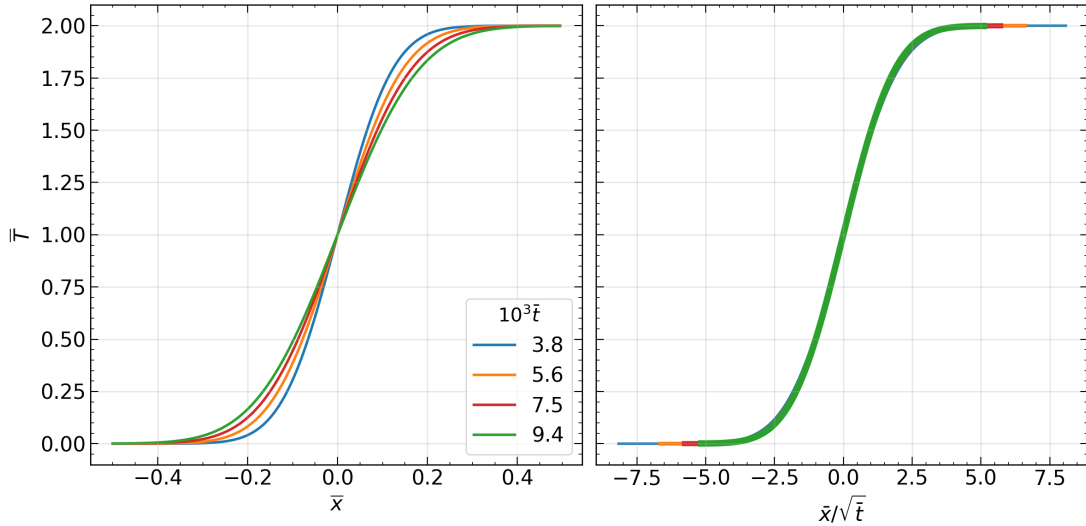


Figure 4.1: The numerical solution of the equation $\frac{\partial \bar{T}}{\partial \bar{t}} = \frac{\partial^2 \bar{T}}{\partial \bar{x}^2}$. (Left) The solution for $\bar{T}$ plotted as a function of $\bar{x}$ at several different times. (Right) The solution for $\bar{T}$ plotted as a function of $\bar{x}/\sqrt{\bar{t}}$ for the same time values as on the left. Note that the curve is now invariant.

down

$$\frac{\partial \bar{T}}{\partial \bar{t}} = \hat{D} \frac{\partial^2 \bar{T}}{\partial \bar{x}^2}, \quad (4.2)$$

where $\hat{D} = \frac{\tau}{L^2} D$ is dimensionless. As this is just an example, let us assume we have picked units so that $\hat{D} = 1$. We can solve this equation numerically, using some initial profile that we choose to be a tanh function. We discretise the equation using centered differencing in space and explicit forward Euler stepping in time. The numerical solution as a function of x and t is shown in figure 4.1. It is clear that the shape of $\bar{T}$ as a function of $\bar{x}$ varies with $\bar{t}$, with the normalised temperature relaxing over time. However, when plotted as a function of a new variable $\bar{x}/\sqrt{\bar{t}}$ it is clear that $\bar{T}$ is now invariant in time and all the curves superimpose. This is a powerful property because it in effect reduces the dimensionality of the problem and allows us to solve an ordinary differential equation in $\bar{x}/\sqrt{\bar{t}}$ for the evolution of T instead of a partial differential equation in both $\bar{x}$ and $\bar{t}$. This has an important fundamental interpretation as well: the self-similar solution is the asymptotic solution to which the system relaxes when there are no characteristic time or spatial scales [167]. This means that no matter the initial profile of T , given enough time the solution will still relax to the self-similar one if the boundary conditions are the same.

4.3 Past work on self-similar solutions to transport

A body of past work has studied the use of self-similar solutions to model the behaviour of magnetised plasmas. In this section, that past work is summarised and the novelty

of the work in this chapter is highlighted.

An important parameter to understand magnetised transport is the plasma β , the ratio of thermal to magnetic pressure:

$$\beta = \frac{p_{\text{thermal}}}{p_{\text{magnetic}}}. \quad (4.3)$$

β is important because it parametrisises which transport terms dominate in the magnetised limit. We discuss the reason for this in depth in section 4.5.3.

Past theoretical work has focused on magnetised transport in the high β limit, as this is of interest for the large temperatures and densities in ignition-scale MIF experiments. This reduces the pressure balance to purely thermal pressure and allows for several transport terms to be dropped simplifying the problem. The reason for this is also explained in section 4.5.3. Velikovich *et al.* used self-similar solutions in this limit to estimate magnetic flux and energy losses from hot D₂ plasma to a fixed, cold liner and found that the profiles are significantly affected by the inclusion of the Nernst effect [59]. A similar study by Garcia-Rubio *et al.*, also in the high β limit, included mass-ablation effects in this problem to estimate concentration gradient losses [168]. Both approaches were later generalised to the low β regime [169, 170]. However, both of these works maintained the constraint that the wall bounding the plasma was an infinitely dense and cold sink of magnetic flux. This means that it is not possible to study the interpenetration of plasma and magnetic field and significantly restricts the boundary conditions. This rigid wall constraint was relaxed in later work by Velikovich *et al.* in the high β limit [171], which allows for a finite density and temperature difference between the liner and the plasma. This allowed the authors to predict a previously unknown phenomena: the Nernst thermomagnetic wave. The lower β regime is however accessed in the pulsed power experiments that we are interested in for this thesis. Similarly, the extreme magnetic fields generated in magnetic flux compression schemes [19, 172] and in MIF mean that the magnetic pressure can be quite substantial. Furthermore, all magnetic confinement devices, including z-pinchs, are in a low β regime at the edge. Understanding the breakdown of MHD at a vacuum-plasma interface requires a model that works in this regime. The novelty of this work is in generalising the past work to an arbitrary β regime, whilst retaining the flexibility to have completely arbitrary boundary conditions.

4.4 Outline

In section 4.5, we show how the extended MHD partial differential equations from section 2.3 admit self-similar solutions under a series of assumptions. We then discuss the numerical method used to find these self-similar solutions in section 4.6. In section 4.7, we perform a parameter scan over the plasma β to show how different extended MHD terms dominate in different parameter regimes. We then discuss the role of the Ettingshausen effect at a vacuum-plasma interface in section 4.8. Finally, we show how

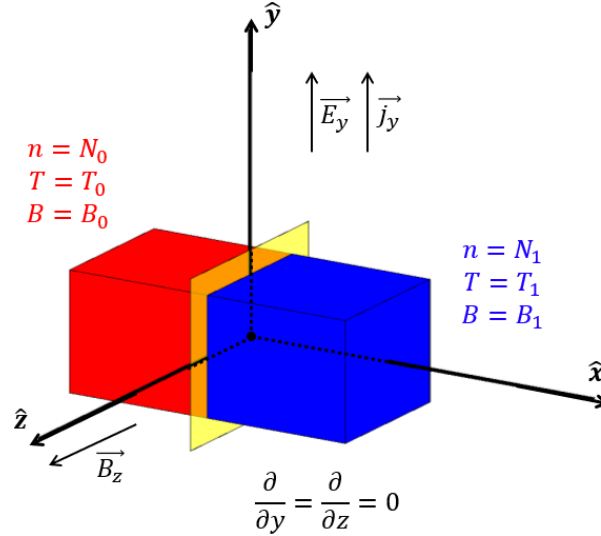


Figure 4.2: A diagram showing the geometry of the problem. The blue represents the values at $x = \infty$ and the red the values at $-\infty$. The yellow rectangle shows the position of the original interface at $x = 0$. Initially, the plasma density N , temperature T and magnetic field B are constant on either side of the interface but with an arbitrary discontinuity across of it.

our technique can be used to rapidly assess the role of different magnetised transport terms in section 4.9, to aid in the design of experiments to observe these phenomena and validate our models.

4.5 Derivation of a set of self-similar equations

We consider an interface between two plasmas with initially uniform but distinct densities and temperatures. We want to understand how the evolution of the plasma is affected by different magnetised transport terms. Solving a full MHD code would produce this behaviour, but at significant computational cost. In this section, we show that under a series of assumptions the equations of extended MHD admit self-similar solutions which allow us to cheaply explore the parameter space. We derive the equations in Gaussian CGS units, with temperature in energy units, for compatibility with past work. We continue to use CGS units throughout this chapter, unless otherwise specified. In all other chapters, we use SI units.

4.5.1 Assumptions and geometry

We model the problem in 1D planar geometry, as in past work [59, 169, 171]. This is an approximation to the cylindrical experiments that motivate the work, valid if the transport effects occur in a narrow region near the initial interface, which we have found to be true. This is also a necessary assumption to form self-similar solutions as 1D cylindrical geometry introduces an additional spatial scale, both the radius r

and the initial position of the interface. We assume that all quantities only vary in the $\hat{\mathbf{x}}$ direction and that we have a magnetic field in the $\hat{\mathbf{z}}$ direction. This means that we are only interested in the transverse electric field and current components E_y and j_y . This is shown in figure 4.2. We also assume quasineutrality and solely consider hydrogen plasma with $Z = 1$ and $A = 1$, though this work can easily be generalised to any species. We ignore radiation losses and take a single-temperature approximation, so that we can replace the separate ion and electron energy equations with a single expression for plasma temperature. This assumption is further explored in section 4.8. Crucially, we also assume that the phenomena in which we are interested are slow compared to the local sound speed. This means that any perturbations to the pressure in the system will be smoothed by sound waves on a shorter temporal scale than we are interested in. Therefore, this assumption is equivalent to requiring that our system remains isobaric at all times. Initially our plasma density, temperature and magnetic field are uniform on either side of the interface but we allow an arbitrary discontinuity across it. This set up is significantly more general than in past work which has either only studied a half-domain thus limiting the boundary conditions [59, 168–170, 173], or has only explored the high β regime [171].

4.5.2 Governing equations

The governing equations for our system are conservation equations for number density, momentum and energy. We solve these together with Faraday’s and Ampère’s laws for the electromagnetic fields. These equations were shown in section 2.3, here we derive them for our geometry.

In planar geometry, the continuity equation is given by:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu_x) = 0, \quad (4.4)$$

where n is the ion number density and u_x the centre of mass velocity. We note that due to quasineutrality we have $n = n_i = n_e$ for $Z = 1$. Due to our isobaric condition, we can replace the momentum equation with an equation for pressure balance:

$$2nT + \frac{B^2}{8\pi} = \text{constant}, \quad (4.5)$$

where the constant is determined by the boundary conditions. It is worth noting that there is no magnetic tension in this planar geometry. We are assuming that energy equilibration between electrons and ions occurs on a shorter timescale than we are interested in, so we can obtain a single energy balance equation by summing the electron and ion equations. In our geometry, the Righi-Leduc heat flow is along a direction of symmetry and therefore makes no contribution to the energy balance. Thus, the energy

equation is given by:

$$\frac{\partial}{\partial t}(3nT) + \frac{\partial}{\partial x}(3nTu_x) + 2nT\frac{\partial u_x}{\partial x} = \frac{\partial Q}{\partial x} + \frac{1}{4\pi} \frac{\partial B_z}{\partial x} E_y, \quad (4.6)$$

where

$$Q = \frac{nT\tau_e}{m_e} \left(\gamma_{\perp}^e + \sqrt{\frac{2m_e}{m_i}} \gamma_{\perp}^i \right) \frac{\partial T}{\partial x} + \frac{c\beta_{\wedge} T}{4\pi e} \frac{\partial B_z}{\partial x} \quad (4.7)$$

is the magnetised heat flux. In our geometry, we only have the perpendicular component of thermal conduction, for which we include both electron and ion components. The second term in Q is the Ettingshausen effect, which is heat flux driven by the current in the $\mathbf{j} \times \mathbf{B}$ direction and forms a large part of our work in this chapter. The second term on the right hand side of equation (4.6) is the Ohmic heating contribution, representing the conversion of magnetic energy into the thermal energy of the plasma. $\gamma_{\perp}^{i,e}$ and $\beta_{\wedge}$ are dimensionless fitting functions to the transport coefficients. In our work, we exclusively use the Braginskii parametrisation of these functions [86], given by

$$\beta_{\wedge} = \frac{\chi_e(\beta_1''\chi_e^2 + \beta_0'')}{\chi_e^4 + \delta_1\chi_e^2 + \delta_0}, \quad (4.8)$$

where $\beta_1'' = 1.5$, $\beta_0'' = 3.053$, $\delta_1 = 14.79$, $\delta_0 = 3.7703$ for the hydrogen plasma we are studying and χ_e is the electron Hall parameter. We also have:

$$\begin{aligned} \gamma_{\perp}^e &= \frac{\gamma_1'\chi_e^2 + \gamma_0'}{\chi_e^4 + \delta_1\chi_e^2 + \delta_0} \\ \gamma_{\perp}^i &= \frac{2\chi_i^2 + 2.645}{\chi_i^4 + 2.6\chi_i^2 + 0.677}, \end{aligned} \quad (4.9)$$

where $\chi_i = \omega_i\tau_i$ is the ion Hall parameter, $\gamma_1 = 4.664$ and $\gamma_0 = 11.92$. The disputed coefficient fits in the limits of $\chi_e \rightarrow 0$ and $\chi_e \rightarrow \infty$ [94–97] are not relevant to our geometry and we have found that using the most recent Sadler corrections [97] produces nearly identical results.

We use the generalised Ohm's law to obtain the electric field. We neglect electron inertia and viscosity effects, replacing the pressure tensor with a scalar electron pressure p_e . In our geometry, $\mathbf{u}$ only has an $\hat{x}$ component and $\mathbf{B}$ only has a $\hat{z}$ component. The electric field is only used to evolve the magnetic field through Faraday's law $\partial_t \mathbf{B} = -c\nabla \times \mathbf{E}^*$. Therefore, only the transverse $\hat{y}$ components of the electric field are relevant for our case. This means that we have no Hall or Biermann battery effects. Here we follow Braginskii's notation of writing the comoving electric field without asterisk as $\mathbf{E}$ and the electric field in the lab frame as $\mathbf{E}^*$. We can therefore write the generalised Ohm's law as:

$$E_y^* - \frac{1}{c} u_x B_z = E_y, \quad (4.10)$$

where E_y is the frictional component of the electric field (the rate of change of electron momentum due to collisions with ions). E_y is given by:

$$E_y = -\frac{cm_e\alpha_\perp}{4\pi e^2 n\tau_e} \frac{\partial B_z}{\partial x} - \frac{\beta_\perp}{e} \frac{\partial T}{\partial x}. \quad (4.11)$$

The first term is the electric field set up due to diffusion of charge carriers down magnetic field gradients, whilst the second is the electrothermal Nernst effect. $\alpha_\perp$ is a dimensionless fitting function given by

$$\alpha_\perp = 1 - \frac{\alpha'_1 \chi_e^2 + \alpha'_0}{\chi_e^4 + \delta_1 \chi_e^2 + \delta_0}, \quad (4.12)$$

where $\alpha_1 = 6.416$ and $\alpha_0 = 1.837$. By taking the curl of the electric field and substituting into the planar Faraday's law $\partial_t B_z = -\partial_x E_y^*$, we obtain the induction equation:

$$\frac{\partial B_z}{\partial t} + \frac{\partial}{\partial x}(u_x B_z) = -c \frac{\partial E_y}{\partial x}. \quad (4.13)$$

This system of equations (4.4) to (4.13) is completed with boundary conditions on all of the non-auxiliary variables n , T , B and u . In this work, we consider a system divided into two uniform half-spaces for $t \leq 0$. We make no assumptions about the values of the variables in those half spaces, leaving them completely general. This leads to the initial conditions:

$$\begin{aligned} T(x = -\infty, t = 0) &= T_0 & T(\infty, 0) &= T_1 \\ n(-\infty, 0) &= n_0 & n(\infty, 0) &= n_1 \\ B_z(-\infty, 0) &= B_0 & B_z(\infty, 0) &= B_1 \\ u_x(-\infty, 0) &= 0 & u_x(\infty, 0) &= 0. \end{aligned} \quad (4.14)$$

At $t = 0$, the interface at $x = 0$ is removed and the plasma in the half spaces is allowed to evolve. One should expect that far enough from the original interface, the plasma parameters are still given by equation (4.14). Therefore, these constraints serve as boundary conditions as well. The self-similar solutions represent the late-time asymptotic behaviour of this system and should not depend on its exact initial state [167]. However, the asymptotic behaviour does depend crucially on the system states at plus and minus infinity.

4.5.3 Dimensionless parameters

In section 4.3, we stated that β is an important parameter in understanding magnetised transport. Here we show why that is. This is an argument first discussed by Velikovich *et al.* [59]. We can define temperature and magnetic field gradient scale lengths

$$L_T = \left| \frac{1}{T} \frac{\partial T}{\partial x} \right|^{-1}, \quad L_B = \left| \frac{1}{B} \frac{\partial B}{\partial x} \right|^{-1}. \quad (4.15)$$

We can then compare the resistive diffusion and the Nernst terms in equation (4.11), defining the ratio:

$$\begin{aligned}
 R_1 &= \left| \frac{\frac{cm_e \alpha_\perp}{4\pi e^2 n \tau_e} \frac{\partial B}{\partial x}}{\frac{\beta_\perp}{e} \frac{\partial T}{\partial x}} \right| = \frac{\frac{cm_e}{4\pi e^2 n \tau_e} \frac{\alpha_\perp}{\beta_\perp} \frac{B L_B^{-1}}{T L_T^{-1}}}{\frac{1}{e}} \\
 &= \frac{cm_e}{4\pi e n \tau_e} \frac{B}{T} \frac{\alpha_\perp}{\beta_\perp} \frac{L_T}{L_B} \\
 &= \frac{cm_e}{2\pi e \tau_e} \frac{1}{2nT} \frac{B^2}{8\pi} \frac{\alpha_\perp}{B} \frac{L_T}{\beta_\perp L_B} \\
 &= \frac{4}{\omega_e \tau_e} \frac{1}{\beta} \frac{\alpha_\perp}{\beta_\perp} \frac{L_T}{L_B}, \tag{4.16}
 \end{aligned}$$

where we have used the definition of β and $\omega_e = \frac{eB}{m_e c}$ to simplify. Defining $\Delta_e = \chi_e^4 + \delta_1 \chi_e^2 + \delta_0$ (see e.g. Eq. (4.12)) and $\chi_e = \omega_e \tau_e$, we can rewrite this as:

$$R_1 = \frac{4}{\omega_e \tau_e} \frac{1}{\beta} \frac{\alpha_\perp}{\beta_\perp} \frac{L_T}{L_B} = \frac{1}{\beta} \frac{4}{\chi_e^2} \frac{\Delta_e}{(\beta_1'' \chi_e^2 + \beta_0'')} \left(1 - \frac{\alpha_1' \chi_e^2 + \alpha_0'}{\Delta_e} \right) \frac{L_T}{L_B}, \tag{4.17}$$

where it is important not to confuse the plasma β with the fitting coefficients β_1'' and β_0'' . Velikovich *et al.* [59] highlights that it is important to compare the thermal diffusivity, defined by

$$\chi_\parallel = \frac{T \tau_e}{m_e} \gamma_\perp^e (\omega_e \tau_e = 0), \tag{4.18}$$

where the last term means setting χ_e to 0 in equation (4.9), to the magnetic diffusivity, defined by

$$\nu_{m\parallel} = \frac{m_e c^2}{4\pi e^2 n \tau_e} \alpha_\perp (\omega_e \tau_e = 0). \tag{4.19}$$

They define a dimensionless ratio of these quantities which they denote the magnetic Lewis number, Lm [59]:

$$\begin{aligned}
 \text{Lm} &= \frac{\chi_\parallel}{\nu_{m\parallel}} \\
 &= A \beta \chi_e^2, \tag{4.20}
 \end{aligned}$$

where A is a constant formed from Braginskii fitting coefficients. In their work on fully ionised deuterium, $A = 0.529$ [59]. For all plasmas of interest in this thesis, $A \sim \mathcal{O}(1)$. We now take the limits of equation (4.17) with the magnetisation to obtain:

$$R_1 \sim \begin{cases} \frac{1}{\text{Lm}} \frac{L_T}{L_B}, & \chi_e \ll 1 \\ \frac{1}{\beta} \frac{L_T}{L_B}, & \chi_e \gg 1, \end{cases} \tag{4.21}$$

where we have used the definition of Lm and ignored the constants as they are $\mathcal{O}(1)$ and we are interested in the scaling. For a magnetised plasma with $\chi_e \gg 1$, we see that the

plasma β is the dimensionless parameter that determines whether the Nernst or resistive diffusion is the dominant term in the magnetic field transport. For example, we can quickly establish that for a low β , magnetised plasma we would expect resistive diffusion to be more important than the Nernst in shaping the magnetic field profile. In the unmagnetised limit, it is the Lewis number instead that determines this comparison. It should also be noted that the dimensionless numbers are not the only factor determining the magnetic field transport: the gradient scale lengths also play a role since this is gradient driven transport.

A similar comparison can be made for the Ettingshausen effect vs the electron thermal conduction in the heat flux:

$$\begin{aligned}
 R_2 &= \left| \frac{\frac{c\beta_\Lambda T}{4\pi e}}{\frac{nT\tau_e\gamma_\perp^e}{m_e}} \right| \\
 &= \frac{4}{\beta} \left[\frac{\beta_1''\chi_e^2 + \beta_0''}{\gamma_1'\chi_e^2 + \gamma_0'} \right] \frac{L_T}{L_B}
 \end{aligned} \tag{4.22}$$

Again, taking limits:

$$R_2 \sim \begin{cases} \frac{4\beta_0''}{\gamma_0'} \frac{1}{\beta} \frac{L_T}{L_B}, & \chi_e \ll 1 \\ \frac{4\beta_1''}{\gamma_1'} \frac{1}{\beta} \frac{L_T}{L_B}, & \chi_e \gg 1. \end{cases} \tag{4.23}$$

In this case, it is clear that the plasma β determines which term dominates the heat flux in both limits. For a low β plasma, the Ettingshausen effect dominates over the thermal conduction. It is worth noting however that it is difficult to find a reasonable set of plasma parameters that make both $\beta \ll 1$ and $\chi_e \ll 1$. It is not trivial to perform the same analysis to compare Ohmic heating with the heat conduction, but past work has indicated that Ohmic heating becomes more important in the low β limit [170]. Relations (4.21) and (4.23) show that the plasma β is an important quantity for determining which magnetised transport terms are dominant. Later in this chapter, we perform a parameter scan over plasma β to show the transition between the different regimes.

4.5.4 Conversion to self-similar form

The set of equations outlined in the previous section are amenable to the method of self-similar solutions. We define the self-similar variable, η :

$$\eta = \eta_0 \frac{x}{\sqrt{t}}, \tag{4.24}$$

where η_0 is an arbitrary normalisation coefficient which makes η dimensionless. Conventionally, η is used to denote resistivity, but within this chapter it is exclusively used as the self-similar variable as in past work. We can then introduce normalised self-similar

variables for the other quantities:

$$\begin{aligned} T(x, t) &= T_0 \theta(\eta) & n(x, t) &= n_0 \tilde{n}(\eta) \\ B_z(x, t) &= H_0 h(\eta) & Q(x, t) &= \frac{Q_0}{\sqrt{t}} q(\eta) \\ E_y(x, t) &= \frac{E_0}{\sqrt{t}} \epsilon(\eta) & u_x(x, t) &= \frac{u_0}{\sqrt{t}} v(\eta). \end{aligned} \quad (4.25)$$

n_0 and T_0 are the reference density and temperature respectively which are set based on boundary conditions. We define the other normalisation variables in terms of these so as to make the self-similar variables dimensionless:

$$\begin{aligned} H_0 &= \sqrt{16\pi n_0 T_0} & \eta_0 &= \frac{n_0 T_0}{Q_0} \\ u_0 &= \frac{1}{\eta_0} & E_0 &= \frac{\sqrt{8\pi} Q_0}{c H_0} \\ \tau_{e0} &= \frac{3\sqrt{m_e} T_0^{3/2}}{4\sqrt{2\pi} e^4 Z n_0 \ln \Lambda_0} & Q_0 &= \sqrt{(n_0 T_0)^2 \frac{T_0 \tau_{e0}}{m_e}} \\ & & \ln \Lambda_0 &= \ln \left(\frac{T_0}{e\hbar} \sqrt{\frac{3m_e}{\pi n_0}} \right). \end{aligned} \quad (4.26)$$

τ_{e0} is the electron-ion collision time calculated using the reference temperature and density and $\ln \Lambda_0$ is the reference Coulomb logarithm. We have found that using a different model for the Coulomb logarithm does not qualitatively change the solutions obtained, but the results in this chapter are obtained using the model given here.

We now perform a variable transformation, noting that

$$\frac{\partial}{\partial t} = -\frac{\eta}{2t} \frac{d}{d\eta} \quad \frac{\partial}{\partial x} = \frac{\eta_0}{\sqrt{t}} \frac{d}{d\eta} \quad (4.27)$$

Applying these results, we can transform the continuity equation (4.4) to obtain:

$$\frac{dv}{d\eta} = \frac{1}{\tilde{n}} \left(\frac{\eta}{2} - v \right) \frac{d\tilde{n}}{d\eta}. \quad (4.28)$$

We can transform the equations for the electric field (4.11) and heat flux (4.7) to obtain:

$$\frac{E_0}{\eta_0} \epsilon = -\frac{cm_e}{4\pi e^2 n_0 \tau_{e0}} \frac{\alpha_{\perp}}{\tilde{n} \hat{\tau}} H_0 \frac{dh}{d\eta} - \frac{\beta_{\wedge} T_0}{e} \frac{d\theta}{d\eta} \quad (4.29)$$

$$\frac{Q_0}{\eta_0} q = \frac{n_0 T_0^2 \tau_{e0}}{m_e} \tilde{n} \hat{\tau} \theta \gamma_{\perp} \frac{d\theta}{d\eta} + \frac{c T_0 H_0}{4\pi e} \beta_{\wedge} \theta \frac{dh}{d\eta}, \quad (4.30)$$

where we have combined the ion and electron heat conductivities to form a new variable $\gamma_{\perp} = \gamma_{\perp}^e + \sqrt{\frac{2m_e}{m_i}} \gamma_{\perp}^i$. We have also defined another new variable $\hat{\tau}$ which represents

the variation of the electron collision time τ_e with the self-similar variable η :

$$\hat{\tau}(\eta) = \frac{\theta^{1.5}}{\tilde{n}\tilde{\lambda}(\eta)}, \quad (4.31)$$

where $\tilde{\lambda}$ is the function denoting the evolution of the Coulomb logarithm with η :

$$\tilde{\lambda}(\eta) = 1 + \frac{1}{\ln \Lambda_0} \ln \left(\frac{\theta}{\sqrt{\tilde{n}}} \right). \quad (4.32)$$

Equations (4.29) and (4.30) are simultaneous equations for $dh/d\eta$ and $d\theta/d\eta$ that we can solve to show:

$$\frac{dh}{d\eta} = \left[\frac{\mathcal{B}\alpha_{\perp}}{\tilde{n}\hat{\tau}} - \frac{\mathcal{A}\mathcal{C}\beta_{\wedge}^2}{\tilde{n}\hat{\tau}\gamma_{\perp}} \right]^{-1} \left(\epsilon - \frac{\mathcal{C}\beta_{\wedge}q}{\tilde{n}\theta\hat{\tau}\gamma_{\perp}} \right), \quad (4.33)$$

$$\frac{d\theta}{d\eta} = \frac{1}{\tilde{n}\theta\hat{\tau}\gamma_{\perp}} \left(q - \mathcal{A}\beta_{\wedge}\theta \frac{dh}{d\eta} \right), \quad (4.34)$$

where $\mathcal{A} = cT_0H_o\eta_0/4\pi eQ_0$, $\mathcal{B} = cm_eH_o\eta_e/4\pi e^2n_0\tau_{e0}E_0$ and $\mathcal{C} = T_0\eta_0/eE_0$ are constant and dimensionless functions of the normalisation variables. The temperature balance equation (4.6) is used to give an equation for the divergence of the heat flux:

$$\begin{aligned} \frac{dq}{d\eta} = 3\tilde{n} \left(v - \frac{\eta}{2} \right) \frac{d\theta}{d\eta} + 2\tilde{n}\theta \frac{dv}{d\eta} \\ - \mathcal{A}\beta_{\wedge} \frac{dh}{d\eta} \frac{d\theta}{d\eta} - \frac{\mathcal{B}}{\sqrt{2\pi}} \frac{\alpha_{\perp}}{\tilde{n}\hat{\tau}} \left(\frac{dh}{d\eta} \right)^2. \end{aligned} \quad (4.35)$$

Finally, the induction equation (4.13) is used to obtain an expression for the derivative of the self-similar electric field

$$\frac{d\epsilon}{d\eta} = \mathcal{D} \left[-\frac{\eta}{2} \frac{dh}{d\eta} + v \frac{dh}{d\eta} + h \frac{dv}{d\eta} \right], \quad (4.36)$$

where $\mathcal{D} = H_0/eE_0\eta_0$. The density is not explicitly evolved, only being calculated from the normalised pressure balance

$$\tilde{n}\theta + h^2 = 1 + \frac{1}{\beta_0}, \quad (4.37)$$

where

$$\beta_0 = \frac{2n_0T_0}{B_0^2/8\pi} = \frac{H_0^2}{B_0^2} \quad (4.38)$$

is the ratio of the original thermal and magnetic pressures. It is possible to calculate a

derivative of this density for the continuity equation:

$$\frac{d\tilde{n}}{d\eta} = \frac{h^2 - \left(1 + \frac{1}{\beta_0}\right)}{\theta^2} \frac{d\theta}{d\eta} - \frac{2h}{\theta} \frac{dh}{d\eta} \quad (4.39)$$

It is interesting to note that the self-similar variable for the extended MHD equations is proportional to the self-similar variable for the idealised diffusion equation discussed in section 4.2. Equations that have this self-similar variable are known as having diffusive scaling [167].

Equation (4.33) shows that $\frac{dh}{d\eta}$ can be written only in terms of the set of self-similar variables $\{\theta, v, \epsilon, q, h\}$. All later equations can be written as functions of this derivative and this set. Therefore, we can write down our system of ordinary differential equations as:

$$\frac{d\mathbf{y}}{d\eta} = \mathbf{F}(\mathbf{y}), \quad (4.40)$$

where $\mathbf{y} = (h, \theta, \epsilon, q, v)^T$ and $\mathbf{F}$ is a nonlinear function.

Our initial conditions on the PDE system, given by equation (4.14) are instead converted to boundary conditions on the self-similar variables:

$$\begin{aligned} \theta(\eta = -\infty) &= 1 & \theta(\infty) &= \frac{T_1}{T_0} \\ \tilde{n}(-\infty) &= 1 & \tilde{n}(\infty) &= \frac{n_1}{n_0} \\ h(-\infty) &= \frac{B_0}{H_0} = \sqrt{1/\beta_0} & h(\infty) &= \frac{B_1}{H_0} \\ v(-\infty) &= 0 & v(\infty) &= 0. \end{aligned} \quad (4.41)$$

As we have 5 first order equations, described in equation (4.40), we can only enforce 5 of these 8 boundary conditions. The density $\tilde{n}$, temperature θ and magnetic field h are not independent variables due to the pressure balance, so it does not matter which one is chosen for the boundary conditions. In this work the density and magnetic field are both constrained at $\eta = -\infty$ and $\eta = \infty$ and the velocity is fixed at $\eta = -\infty$. This does mean that some of our solutions do not satisfy $v = 0$ at $\eta = \infty$. The physical meaning of this is that there needs to be some inward velocity (i.e. towards $\eta = 0$) to provide the magnetic flux that is removed by resistive diffusion, if we continue to assume pressure balance. This observation is in contrast to Velikovich *et al.*, who found that the self-similar velocity was equal to zero at both $\eta = \pm\infty$ [171]. This is a consequence of the high β assumption taken in that work simplifying the pressure balance and energy equations considerably. In the low β limit, the advection of magnetic pressure in the energy equation prevents the velocity being equal to zero on both sides in general [170]. We observe the same qualitative behaviour in our results, with an increasing $|v(\eta = \infty)|$ as the β is decreased.

4.5.5 Comparison between the Eulerian and Lagrangian approaches

In Garanin [173], the self-similar problem is solved in the vacuum limit using a Lagrangian approach. Here we show how it is related to the Eulerian approach that we have used in the previous sections. The governing equations are recast into Lagrangian form, where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u_x \frac{\partial}{\partial x}, \quad (4.42)$$

in our geometry. This form of the equations describes changes in a frame moving with the plasma at the fluid velocity u_x . Pressure balance remains the same as equation (4.5), but the continuity equation becomes:

$$\frac{dn}{dt} + n \frac{\partial u_x}{\partial x} = 0. \quad (4.43)$$

The energy equation is recast to:

$$\frac{d}{dt}(3nT) - 5T \frac{dn}{dt} = \frac{\partial Q}{\partial x} + \frac{1}{4\pi} \frac{\partial B_z}{\partial x} E_y, \quad (4.44)$$

where Q and E_y are defined in equations (4.7) and (4.11) respectively. Similarly, the induction equation becomes:

$$\frac{dB_z}{dt} - \frac{B_z}{n} \frac{dn}{dt} = -c \frac{\partial E_y}{\partial x}. \quad (4.45)$$

dn/dt can be calculated from the pressure balance equation. Since u_x is not a variable in equations (4.44) or (4.45), this means that the continuity equation is decoupled from the other equations and can be solved separately. Following Garanin, it is helpful to define a different self-similar variable ξ :

$$\xi = \xi_0 \frac{\int_0^x n(X', t) dX'}{\sqrt{t}}, \quad (4.46)$$

where ξ_0 is a normalisation constant. This obeys the relations:

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x} \frac{d}{d\xi} = \frac{\xi_0 n(x, t)}{\sqrt{t}} \frac{d}{d\xi} \quad (4.47)$$

$$\begin{aligned} \frac{\partial \xi}{\partial t} &= \xi_0 \frac{\int_0^x \frac{\partial n(X', t)}{\partial t} dX'}{\sqrt{t}} - \frac{\xi}{2t} \\ &= -\xi_0 \frac{\int_0^x \frac{\partial}{\partial X'} (nU) dX'}{\sqrt{t}} - \frac{\xi}{2t} \\ &= -\xi_0 \frac{n(x, t)U(x, t)}{\sqrt{t}} - \frac{\xi}{2t}, \end{aligned} \quad (4.48)$$

where in the second step of the latter we have used the continuity equation to evaluate $\frac{\partial N}{\partial t}$. Combining these results gives the following variable transformation:

$$\begin{aligned}\frac{d}{dt} &= \frac{d\xi}{dt} \frac{d}{d\xi} = \left(\frac{\partial \xi}{\partial t} + u_x \frac{\partial \xi}{\partial x} \right) \frac{d}{d\xi} \\ &= - \frac{\xi}{2t} \frac{d\xi}{dt}.\end{aligned}\tag{4.49}$$

This result is why ξ is a more convenient self-similar variable than η for the Lagrangian form of the equations, however the two formulations should describe the same physics and therefore give the same results.

We can form an equivalent set of self-similar equations using ξ . These equations are not shown here as the process is similar to as in section 4.5.4. As before, this system of ODEs can be written as

$$\frac{d\mathbf{y}}{d\xi} = \mathbf{F}(\mathbf{y}),\tag{4.50}$$

where $\mathbf{y} = (\theta, h, \epsilon, q)^T$. We now solve separately for v as it is decoupled from the other equations. It is helpful to note that

$$\begin{aligned}\frac{\partial \xi}{\partial x} &= \frac{\xi_0 n(x, t)}{\sqrt{t}} \\ \frac{\partial \xi}{\partial(\frac{x}{\sqrt{t}})} &= \xi_0 n_0 \tilde{n}(\eta),\end{aligned}\tag{4.51}$$

so

$$\begin{aligned}\frac{\partial \xi}{\partial(\frac{1}{\xi_0 n_0} \frac{x}{\sqrt{t}})} &= \tilde{n}(\eta) \\ \frac{d\xi}{d\eta} &= \tilde{n}.\end{aligned}\tag{4.52}$$

Therefore, instead of solving self-similar equations in ξ and then having to find a non-trivial transformation back to (x, t) , we can transform our self-similar equations in ξ to equations in η and solve those. We have found that we get identical solutions whether we use the Lagrangian or Eulerian approach, validating our numerical calculation. However, we have found that the Eulerian method led to improved stability in the numerical solution of the problem and so use that exclusively.

Now, we show that it is possible to transform between the self-similar variables η and ξ and thus that they are equivalent. If we assume

$$\xi = -\tilde{n}(v - \eta),\tag{4.53}$$

then

$$\begin{aligned}
 \frac{d\xi}{d\eta} &= -\frac{d\tilde{n}}{d\eta} \left(v - \eta \right) - \tilde{n} \left(\frac{dv}{d\eta} - 1 \right) \\
 &= -\frac{d}{d\eta}(\tilde{n}v) + \eta \frac{d\tilde{n}}{d\eta} + \tilde{n} \\
 &= \tilde{n},
 \end{aligned} \tag{4.54}$$

as expected from equation (4.52). Therefore, we can easily convert between either set of self-similar equations.

4.6 Numerical solution

Equation (4.40) constitutes a system of nonlinear ordinary differential equations that needs to be solved subject to the boundary conditions given by (4.41). Given that $\frac{d\mathbf{y}}{d\eta} \approx 0$ near $\eta = -\infty$, we have found that it is not possible to use the boundary conditions at $\eta = -\infty$ as an initial condition and advance $\mathbf{y}$ forwards. Instead, we must assume a value for $\mathbf{y}$ at some point $\eta \in (-\infty, \infty)$ and then calculate the solution to $\eta = \pm\infty$, checking that it matches the boundary conditions there. In this work, we solve this boundary value problem using a shooting method. We make a guess for the values of the variables $(\theta, h, \epsilon, q, v)$ at $\eta = 0$ and propagate the solution to $\eta = \pm\infty$ using an ODE solver from the `scipy` library. The difference between the values of the variables at $\pm\infty$ and the desired boundary conditions is then calculated. Minimising this quantity to a specific tolerance by adjusting the value of $\mathbf{y}$ at $\eta = 0$ completes the solution of this problem. We checked that all solutions had converged with respect to step size by limiting the maximum step taken by the solver to increasingly small values.

The `scipy` library is used when solving the ODE system as the methods are well-validated and very easily implemented. There are efficiency concerns with using `python` routines, but these methods are implemented in `C` or `Fortran` and so are still quick. In experimenting with different ODE solvers, we found that explicit methods such as adaptive RK4 were either unstable or far too computationally costly. This is due to the system of equations being stiff. Implicit methods give far better results; we found that first order accuracy was sufficient. Higher order methods did not improve the results enough to justify their extra computational cost.

The minimisation problem to satisfy the boundary conditions can either be solved as an optimisation problem or a root-finding problem. In the optimisation approach we are trying to minimise a cost function, here the absolute difference between the value of $\mathbf{y}$ at $\eta = \pm\infty$ and the boundary conditions. We found that a Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm from the `scipy` library correctly converged. Alternatively, we can find the roots of the function $\mathcal{F}$ which takes $(\theta, h, \epsilon, q, v)(\eta = 0)$ as inputs and outputs the difference between the boundary conditions and the values of $\mathbf{y}$ at $\pm\infty$. This is a highly nonlinear function and so an approximate Newton-Raphson method

was used to find the roots. In this method, we numerically estimate $\mathcal{F}'$ by evaluating both $\mathcal{F}(x+\delta)$ and $\mathcal{F}(x)$ for small δ . Whilst both methods produced correct convergence, we found that the root-finding method was both more reliable and more efficient for the problems that we tried.

The number of degrees of freedom and the nonlinearity of the equations mean that a shooting algorithm can be unstable. We have found that a poor guess for the value of the variables at the initial interface will usually lead to variables heading to 0 or ∞ , causing the optimisation to crash. Unfortunately, this is a known problem of shooting methods and it is not easy to prevent. An effective mitigating strategy is to slowly move in parameter space allowing the method to converge at each step. For example, if changing the density boundary condition from $\tilde{n}(\infty) = n_1$ to $\tilde{n}(\infty) = n_2$, it should first be changed from n_1 to $n_1 + \alpha(n_2 - n_1)$ where α is small. This process can then be repeated. The size of this required step depends on the problem, but we have found that it needs to be smaller for highly magnetised problems.

Although the ODE system is solved numerically, we refer to these solutions as semi-analytic as they are converged to arbitrary accuracy and do not have the same numerical trade-offs that a PDE code does.

4.7 Parameter scan over plasma β

As discussed in section 4.3, past work has focused on the high β regime where the Ettingshausen and Ohmic heating terms can be dropped from the energy equation and the resistive diffusion term has a negligible impact in the induction equation. In this section, we perform a parameter scan over plasma β to show how these neglected terms take effect and their qualitative roles as the β is reduced. However, it is not possible to fully characterise the parameter space in terms of just the β since as we will show the relative impact of different terms depends on the boundary conditions. Several sets of boundary conditions can have the same plasma β . Instead, in this section, we refer to different sets of boundary conditions which we summarise in table 4.1. These conditions are intended as an illustrative set, which vary in plasma β and plasma profile and indicate the expected behaviour in different situations. They are motivated by examples of MIF experiments. We use the set-up described in section 4.5.1.

To demonstrate the utility of self-similar solutions for verification of MHD codes, we compare our results with a code that solves the full MHD equations without the assumption of pressure balance. For this, we use the Gorgon radiation-MHD code, described in chapter 3. We use the same initial and boundary conditions in Gorgon as those used in the self-similar code. It is not possible to exactly satisfy pressure balance in the initial conditions in Gorgon, due to the staggered grid used in the implementation of the MHD algorithm. The initial discontinuity in thermal pressure leads to compressive sound waves that propagate outwards from the initial interface. The region of the profile satisfying the self-similar solution is left in the wake of these pressure waves, so the comparison needs to be made after enough time has passed for the waves to move

Case	n_∞ (cm ⁻³)	$n_{-\infty}$	B_∞ (T)	$B_{-\infty}$	T_∞ (eV)	$T_{-\infty}$	β_∞	$(\omega_e \tau_e)_\infty$
A	7.5×10^{23}	1.5×10^{23}	1000	1000	5000	1000	603	4.6
B	10^{20}	5×10^{20}	142	0	250	100	1.0	44.5
C	4.7×10^{20}	1.42×10^{22}	2500	0	160	533	0.01	101.3
D	10^{20}	2×10^{20}	150	0	250	250	1.0	44.5
E	4.7×10^{20}	4.78×10^{22}	2500	0	160	160	0.01	101.3

Table 4.1: Table of boundary conditions that will be discussed in section 4.7. The subscript refers to the value of the parameter at $x = \infty$ or $x = -\infty$. The densities are given in cm⁻³, the magnetic field in T and the temperatures in eV. The values of the plasma β and Hall parameter $\omega_e \tau_e$ at $x = \infty$ are given here for reference.

away from the region of interest. In testing, we have found that deviations from pressure balance in the wake of the waves are less than 1% and thus can be neglected.

4.7.1 High beta plasma

First, we consider the high β case. We use the “Nernst wave” boundary conditions of Velikovich *et al.* [171], marked as case A in table 4.1. These are motivated by measured experimental conditions in MagLIF during stagnation. We calculate self-similar profiles using these boundary conditions and then compare to profiles calculated using the same boundary conditions with the Gorgon code. We make the comparison at 4ns to allow the Gorgon profiles to settle to pressure balance. This comparison is shown in panels a-d of figure 4.3. The initially discontinuous plasma is set into motion by thermal conduction; the hot plasma on the right hand side cools as energy is carried down the temperature gradient and thus contracts to increase density and maintain pressure balance. Similarly, the cold plasma on the left hand side heats and expands. This compression of the hot plasma can be seen by the peak of the velocity profile, which is to the right of the position of the initial discontinuity (at $x = 0$). The magnetic field does not significantly contribute to the pressure in this situation. Instead, its evolution is solely dictated by the balance between the Nernst, resistive diffusion and frozen-in-flow terms. The left hand peak in the magnetic field is formed due to the Nernst effect advecting magnetic field down the temperature gradient - this is the Nernst wave described by Velikovich [171]. The right hand peak is due to the frozen-in-flow advection of the field by the plasma. In this situation, the resistive diffusion has a negligible effect. This can be seen from panels e and f of figure 4.3, which shows the contribution of the different transport terms to the energy and induction equations at 4ns. Looking at the balance between the Ettingshausen, Ohmic heating and thermal conduction in the energy equation, it is clear that the heat conduction is hugely dominant over the other two terms and it is valid to neglect them in this case. In the induction equation balance, the Nernst and frozen-in-flow terms are significantly larger than the resistive diffusion. It should be noted that they carry magnetic flux in opposite directions in this situation. It can be seen from the orange line that the left hand peak in the magnetic field is due to the

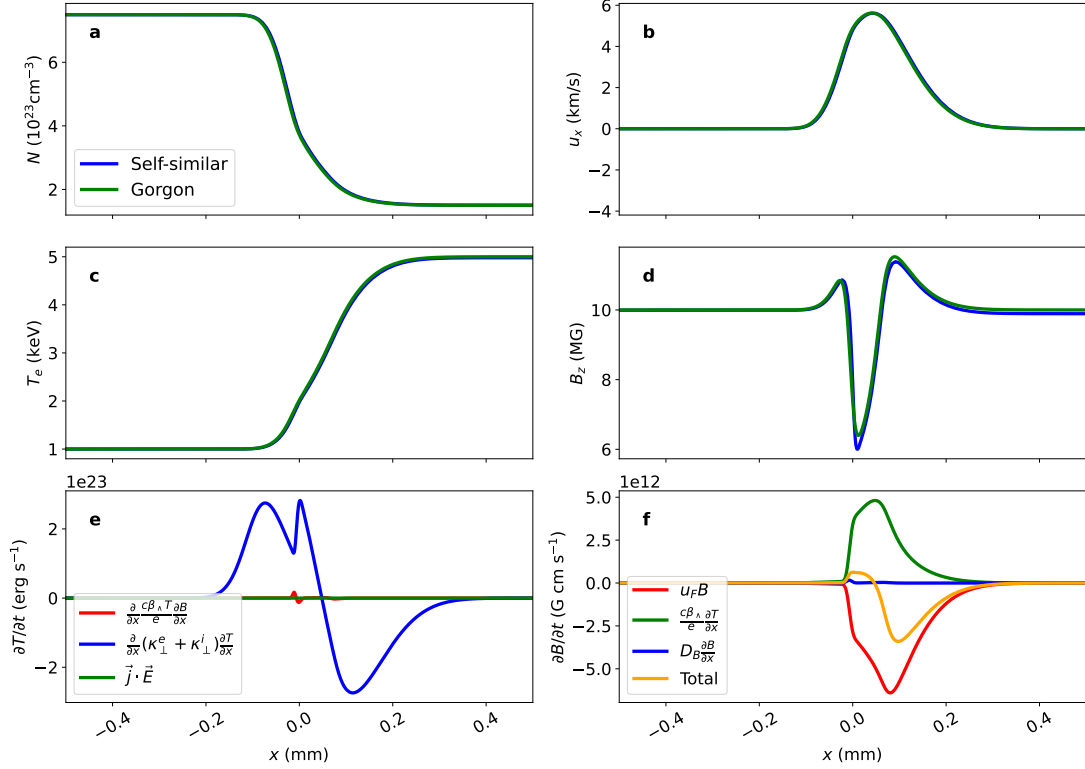


Figure 4.3: **a-d**: Plots comparing the profiles calculated for the high β boundary conditions (case A in table 4.1) for the self-similar code (in blue) to Gorgon (in green). The profiles are compared at 4ns. **e**: The contribution to $\partial T / \partial t$ of the different terms in the energy equation (the Ettingshausen in red, the Ohmic heating in green, the thermal conduction in blue) at the same time, calculated from the self-similar code. **f**: The contribution to $\partial B / \partial t$ of the different terms in the induction equation (the frozen-in-flow in red, the Nernst in green, the resistive diffusion in blue).

Nernst and the right due to the frozen-in-flow. Whilst these terms are calculated at just a time snapshot, the self-similarity of our solution means that the plasma profiles broaden over time and thus the relative role of these gradient-driven transport terms remains qualitatively the same at later times. These conclusions are consistent with previous theoretical work [59, 171].

The agreement between the self-similar model and Gorgon is good. There is some slight disagreement in the magnetic field profile. This is to be expected and is a consequence of the outward pressure waves as discussed in section 4.6. This is also seen in other studies [171]. Though we use Gorgon here as validation of our self-similar solution, this serves as a demonstration of how the self-similar code can be used as a test problem for MHD. Obtaining the same solution as the self-similar code requires correct implementations of frozen-in-flow, Nernst advection and thermal conduction - as well as normal hydrodynamic behaviour. Thus, obtaining agreement rigorously tests that the MHD code is solving the underlying equations correctly.

The initially uniform magnetic field in this case means that no steep magnetic field

gradients are set-up and thus no large currents are induced. This reduces the impact of the Ettingshausen, Ohmic and resistive diffusion effects. However, we have found that even if there is a large gradient in the magnetic field (with 0T on the left and 1000T on the right), the Nernst, frozen-in-flow and heat conduction terms still dominate the transport in the high β regime. The only way to increase the magnetic field gradient in our model is by increasing the field strength, we set the boundary conditions and the gradient is self-consistently calculated. Thus, increasing the field gradient is equivalent to decreasing the plasma β on one side of the domain. We explore the effect of this in the next section.

4.7.2 Moderate to low beta with a temperature gradient

We now present results for a β of order unity with a moderate temperature gradient, marked as case B in table 4.1. These parameters are indicative of what might be attained during the laser-preheat phase of MagLIF, although the field is significantly higher than in experiments. As before, we use our self-similar code to calculate profiles of density, temperature, magnetic field and velocity. The profiles of temperature and magnetic field are shown in panels a and b of 4.4. The competing magnetic transport processes are the Nernst effect and resistive diffusion carrying magnetic field down the temperature and magnetic field gradients respectively, from the right to the left. Both the Ettingshausen effect and the thermal conduction are carrying temperature in the same direction. As the magnetic field moves into the denser plasma on the left hand side, it expands to reduce density and maintain pressure balance. This expansion of the plasma compresses the magnetic field by the frozen-in-flow effect, leading to the peak at about 0.08mm. At the same time, the damping of currents in the low density plasma leads to Ohmic heating, increasing the temperature. However, it is difficult to decouple which term is dominating here. The impact of ignoring certain terms in the calculation is shown in the temperature profile in figure 4.4. It can be seen that neglecting the Ettingshausen or the Ohmic heating changes the profile only slightly. In panel b, we show the effect on the magnetic field profile of neglecting particular terms. Our solution method does not allow setting the resistive diffusion to zero, so we show its qualitative effect by reducing the resistive diffusion coefficient by a factor of 2 and calculating the profiles. It is clear that both the Nernst and resistive diffusion effects have some impact on the structure of the field profile. It is not the case that the Nernst effect is always dominant, as seen in past work [169]. We also calculate the contributions to the induction and energy balance equations in panels c and d in figure 4.4. Looking at the balance of terms in the energy equation, it can be seen that the transport terms have a similar peak magnitude. However, the Ettingshausen and Ohmic heating terms act near the initial interface whilst the heat conduction acts over a larger region of space. This explains why the small impact of the Ettingshausen and Ohmic heating effects is confined to a small region in the temperature profile in figure 4.4. It also seems that in this case, the Ettingshausen and Ohmic effects have opposing effects on the temperature

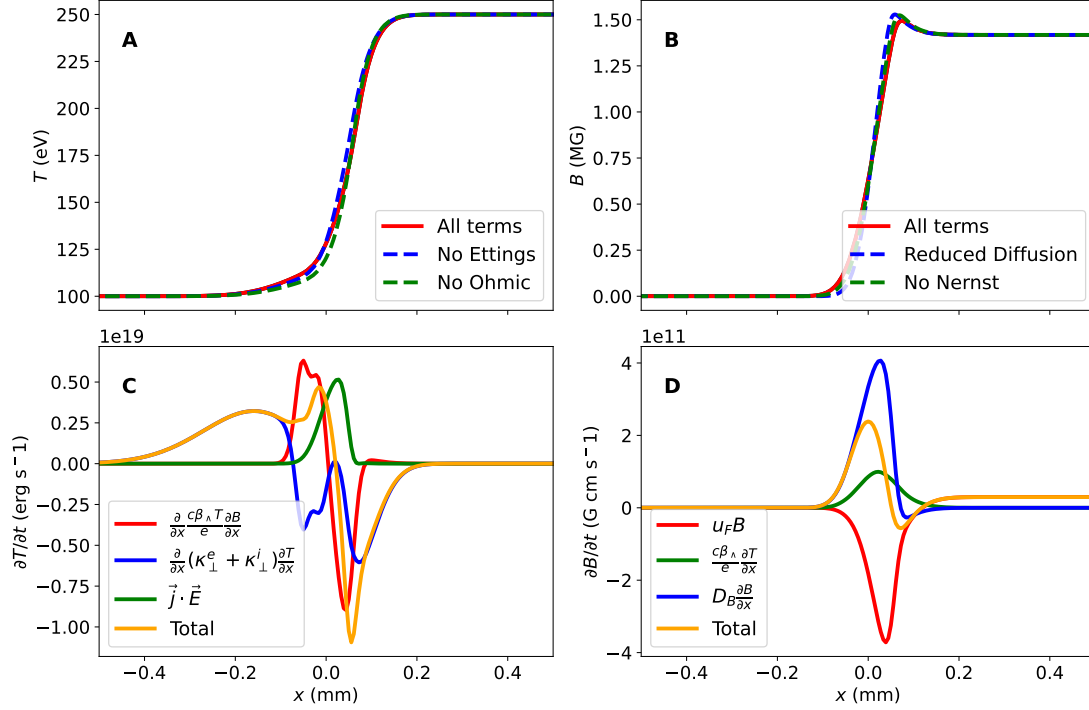


Figure 4.4: **a**: Profile of the temperature calculated at a time of 4ns from the self-similar code for the initial conditions of case B in table 4.1. The profile is shown when all terms are included in the calculation (in red), when the Ettingshausen is excluded from the calculation (in blue) and when the Ohmic heating is not included (in green). **b**: The magnetic field profile with all terms included (in red), with the resistive diffusion coefficient reduced (in blue) and with the Nernst excluded (in green). **c**: The contribution of different terms to the energy equation, calculated from the self-similar code at 4ns. **d**: The contribution of different terms to the induction equation.

(see panel c at $x \approx 0.07\text{mm}$). It should be noted that this is not generally the case and does depend on the magnetic field and temperature gradients; this is discussed at length in section 4.8. On the other hand, in the magnetic field balance the Nernst effect is significantly reduced compared to the resistive diffusion term, whilst the frozen-in-flow remains large. It can be seen from panel b that the Nernst effect essentially acts to increase the effective magnetic diffusivity of the plasma. This is observed in other work [59, 170].

We have tested the effect of having the temperature and magnetic field gradients in opposite directions, whilst maintaining the same unity plasma β and have found the same qualitative behaviour of the transport terms. In this case, the Nernst opposes the resistive diffusion (effectively reducing the diffusivity) but it is still smaller in magnitude. The heat conduction acts over a longer scale and again the Ettingshausen and Ohmic heating somewhat counteract each other.

We then calculate the self-similar profiles for a low β plasma, described by case C in table 4.1. In figure 4.5, we show the temperature and magnetic field profiles when particular terms are excluded from the calculation and the contribution of terms to the

induction and energy equations. Looking at the terms in the induction equation, it is clear that the resistive diffusion is the dominant effect in magnetic field evolution across the entire domain and the Nernst effect has little impact. This is clear from the magnetic field profile that shows the Nernst having no effect. As the magnetic field diffuses into the dense plasma, magnetic energy is converted to thermal by Ohmic heating and the Ettingshausen effect advects energy down the magnetic field gradient. Neglecting the Ettingshausen effect in the calculation causes a noticeable peak in the temperature profile due to the absence of this additional advection. The increased temperature and magnetic pressure causes the dense plasma to expand to maintain pressure balance. Neglecting the Ohmic heating leads to less ablation of the dense plasma by the magnetic field, due to the reduced temperature and thus thermal energy. The leftward shift of the temperature profiles when Ohmic heating is excluded demonstrates this. Panel c shows the contribution of terms to the energy equation. It is clear that all terms have a comparable magnitude, with Ettingshausen and Ohmic heating again counteracting each other. The thermal conduction is balanced by the Ettingshausen on the left hand side of the interface. It is interesting that the Nernst effect cannot be observed whilst the Ettingshausen can, given that they both depend on the β_Λ transport coefficient. However, this is consistent with the theory that the ratio of the magnitudes of the Ettingshausen to the heat conduction approximately varies inversely with plasma β , where the ratio of the Nernst to resistive diffusion varies directly [59]. Through comparison of figures 4.3 through 4.5, it is clear that the length scales of temperature and magnetic field also reduce as the plasma β is reduced. In particular, the temperature gradient scale length, L_T becomes comparable to L_B . It is not possible to decouple whether the increased effect of e.g. the Ettingshausen term is due to this change to the length scales or directly due to the plasma β . This does however validate our earlier assumption that the transport processes occur in a narrow region near the interface.

We have found that, despite the value of the plasma β , if the magnetic field is initially uniform (as in case A), the currents induced by the frozen-in-flow and Nernst advection of magnetic field are not sufficient to cause significant Ohmic heating or Ettingshausen advection of the temperature. We believe this is the reason why past studies that did not impose a magnetic field gradient (e.g. [169]) are unable to observe a large effect due to these terms. This further emphasises our argument that it is not possible to classify the dominance of these terms in dimensionless parameters: it is profile dependent. This can be seen most clearly from the unity β case shown in figure 4.4. Despite the Ettingshausen and heat conduction being of similar magnitudes, the Ettingshausen has almost no effect on the temperature profile as it acts over a much shorter length scale than the heat conduction. We can show qualitatively that reducing the plasma β reduces the temperature length scale to become comparable to that of magnetic field, but it clearly depends on the boundary conditions in a way that is difficult to determine a priori.

The benefit of our model is that it can recreate this low β regime that is inaccessible to other models [59, 168, 171], but also obtain consistent results in the high β limit.

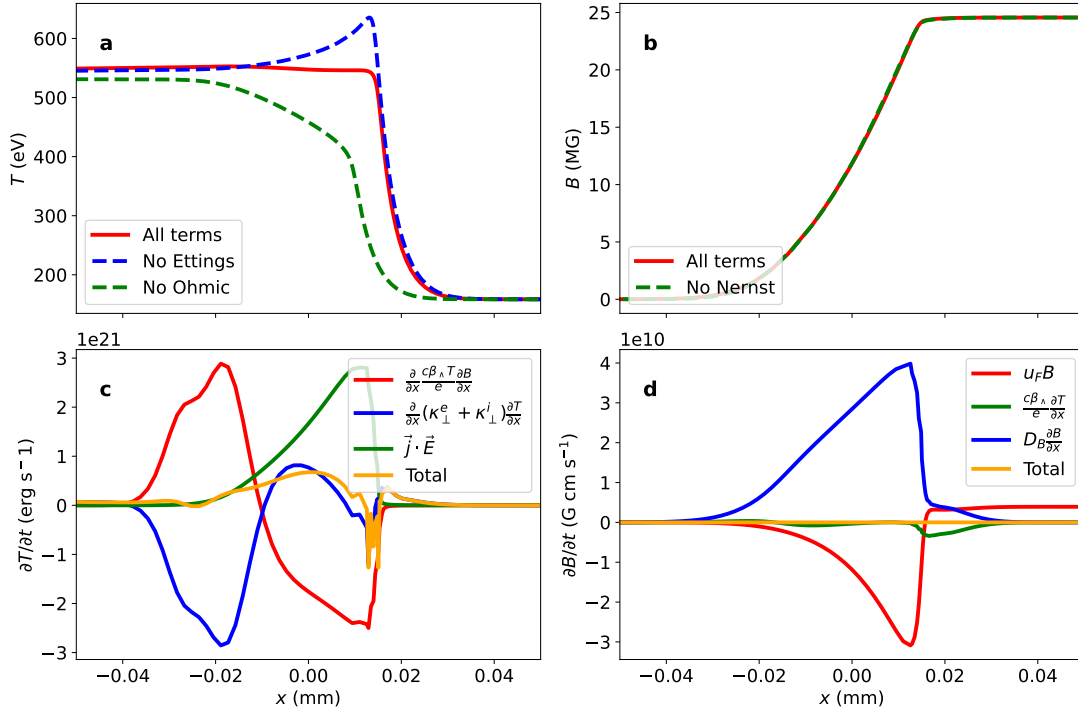


Figure 4.5: **a:** Profile of the temperature calculated at a time of 4ns from the self-similar code for the initial conditions of case C in table 4.1. The profile is shown when all terms are included in the calculation (in red), when the Ettingshausen is excluded from the calculation (in blue) and when the Ohmic heating is not included (in green). **b:** The magnetic field profile with all terms included (in red) and with Nernst excluded (in green). **c:** The contribution of different terms to the energy equation, calculated from the self-similar code at 4ns. **d:** The contribution of different terms to the induction equation.

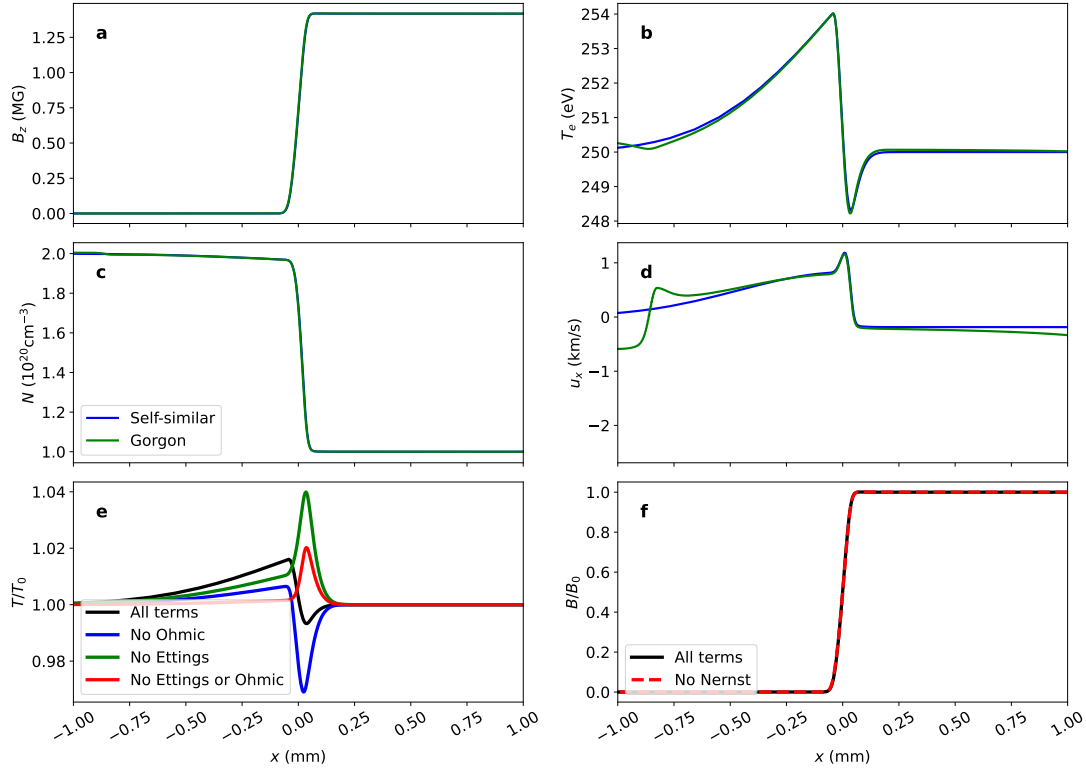


Figure 4.6: **a-d**: The profiles of density, temperature, velocity and magnetic field for the order unity β case, marked as case D in table 4.1. The results of the self-similar code are shown in blue, and the results using the MHD code Gorgon are shown in green. The comparison is done at a time of 3ns. **e**: Temperature profiles when terms are excluded from the calculation. **f**: Magnetic field profile when the Nernst effect is excluded from the calculation.

We have not thoroughly explored even smaller plasma β values, but we would expect the qualitative trend to continue - with the Ettingshausen and particularly the Ohmic heating terms increasing in importance.

4.7.3 A test problem for the Ettingshausen and Ohmic heating effects

In the preceding sections, we showed how the balance of the transport terms is related to the plasma β . Ohmic heating is important in pulsed power experiments due to the lower β in the surface plasma. Our analysis here implies that the Ettingshausen term may be important under these conditions as well. Given the wide variety of reduced MHD codes used to model these experiments [1, 114], we argue that a test problem for these terms would be a useful tool. That is the purpose of this section.

If we start with an initially constant temperature, this reduces the amount of thermal conduction and makes the role of the other transport terms more obvious. We use the boundary conditions of case D, given in table 4.1. These parameters are intended to be somewhat similar to the edge region of a z-pinch [128, 174]. The comparison between our self-similar code and Gorgon is shown in panels a-d in figure 4.6. In panels e and f,

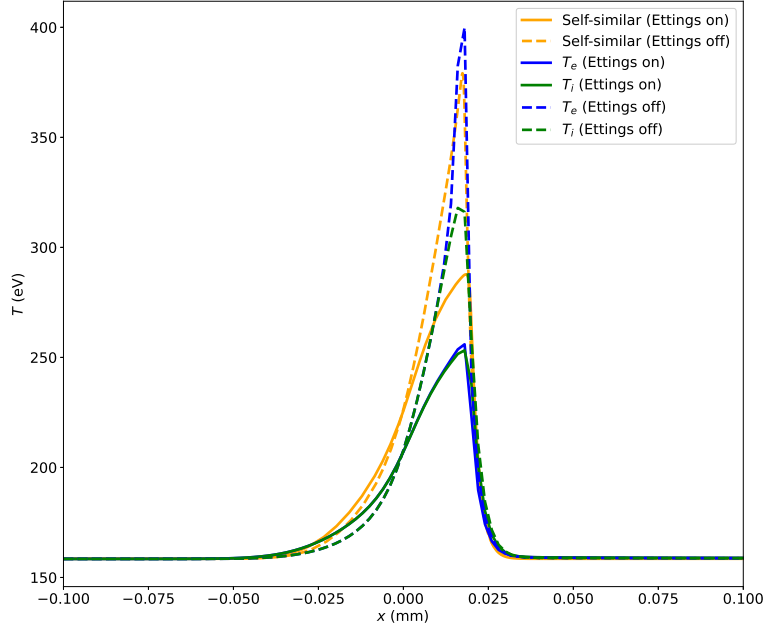


Figure 4.7: Plot showing the temperature profiles produced by Gorgon at 2.5ns for the magnetised interface problem, case E in table 4.1. The solid lines show the temperature calculated from Gorgon with all terms included, the dashed lines show the profiles from Gorgon with the Ettingshausen effect switched off. The orange lines show the solution calculated from the single-temperature self-similar code.

we show the impact on the temperature and magnetic field profiles of ignoring particular terms in the calculation. The magnetic field profile is primarily determined by resistive diffusion into the dense plasma. This leads to Ohmic heating of the plasma due to the induced current and finite resistivity. As seen from panel e of figure 4.6, Ohmic heating is responsible for the peak in the temperature profile. The same current drives advection of the temperature from right to left by the Ettingshausen effect. This leads to the formation of the trough in the temperature. Excluding both of these terms leads to a peak in the temperature due to pdV work, which can now dominate (see the red line in panel e of figure 4.6). The agreement between the self-similar code and Gorgon is good, with the exception of a pressure wave on the left hand side of the plot, as discussed in previous sections. This is therefore an effective test problem to test the implementation separately of the Ohmic heating and Ettingshausen terms.

4.8 Temperature separation at a vacuum-plasma interface

We have assumed a single plasma temperature in formulating our self-similar model. This is equivalent to assuming that the timescale for electron-ion energy equilibration is small compared to the timescales that we are interested in. In this section, we explore the consequences if that assumption is not made. It should be noted that as the electron-ion energy equilibration term does not scale diffusively, it is not possible to develop a

self-similar model that solves the two-temperature system of equations.

As electrons have a lower mass, the energy from Ohmic heating predominantly goes into the electron population. If a sufficiently large amount of energy goes into the electrons, we may therefore expect the electron and ion temperatures to separate. To model this effect, we solve the low β boundary condition problem (defined by case E in table 4.1) using Gorgon. This situation is chosen to be a proxy for a vacuum-plasma interface problem, and again is very similar to the edge region of a z pinch. The results of the Gorgon simulation are shown in figure 4.7. We compare these results to the self-similar code. The left-right asymmetry in the temperature profile is a consequence of the variation of the magnetisation. On the left, the plasma is unmagnetised, electron thermal conductivity dominates and the temperature profile is broad. On the right, the plasma is highly magnetised and conductivity is reduced. While the agreement between Gorgon and the self-similar solution is reasonable, it is challenging to get better agreement in this case as the small scale length of the temperature profile makes it difficult to get enough resolution in the region of interest whilst also leaving the simulation domain large enough for pressure waves to not affect the solution.

As predicted, the Ohmic heating causes the temperature to strongly peak. When the Ettingshausen effect is turned off (dashed lines), the electron and ion temperatures separate. The energy equilibration term is not sufficient to counteract the Ohmic heating. The magnitude of the Ohmic heating reduces later in time as the magnetic field gradient relaxes, but we have found that the temperature separation remains for 10s of nanoseconds. However, if the Ettingshausen effect is included in the calculation (solid lines), the temperatures do not significantly separate and the electron temperature is substantially reduced. This is because the current responsible for Ohmic heating also drives strong advection of the electron temperature through the Ettingshausen effect. Figure 4.8 shows the electron and ion temperatures when the equilibration term has been set to zero, i.e. with the temperatures decoupled from each other. It is clear that the ion profile is not directly affected by the Ettingshausen, but that the advection of the electron temperature by this effect is large. The Ettingshausen effect reduces the electron temperature sufficiently for the equilibration to bring it into balance with the ion temperature. It is likely that lower densities and higher magnetic fields would lead to stronger Ohmic heating and potentially temperature separation, but that even in this $\beta = 0.01$ situation, the Ettingshausen term is strongly mitigatory. This behaviour of the Ettingshausen effect has been observed before in z-pinch studies [174]. This suggests that in a situation where strong Ohmic heating is observed, neglecting the Ettingshausen effect could lead to overestimates of the electron temperature. In addition, this behaviour demonstrates that our self-similar solutions still produce quantitatively similar results to the two-temperature MHD code even despite the single-temperature assumption.

A recurring problem in simulations of pulsed-power experiments has been extremely high electron temperatures ($T_e > 1\text{keV}$) on vacuum-plasma interfaces (such as the surface of a z-pinch or an exploding wire). These temperatures are not routinely observed

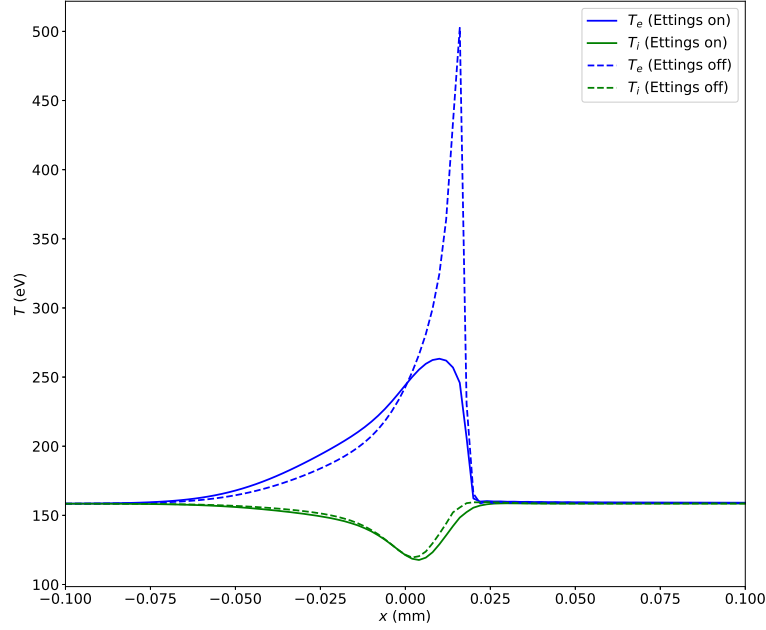


Figure 4.8: Plots of the magnetised interface problem from Gorgon at 2.5ns showing the electron temperature (in blue) and the ion temperature (in green) when the electron-ion equilibration term is set to zero. Solid lines include the Ettingshausen effect, dashed lines set it to zero.

in experiments and it is believed to be due to Ohmic heating due to unphysically large currents [63, 175]. Past work has indicated the Hall effect [63] or anomalous resistivity [123] are important phenomena for limiting the surface current and therefore the Ohmic heating. The Ettingshausen term is rarely included in extended MHD simulations. Figure 4.7 indicates that excluding it can lead to unphysical predictions of both T_e and T_i , demonstrating that an evaluation of all possible transport terms is important for accurate simulations of pulsed-power plasmas. The Ettingshausen effect will advect temperature in the $\mathbf{j} \times \mathbf{B}$ direction, causing it to be advected radially inwards in a z-pinch or exploding wire experiment. This may contribute to the observation of high surface temperatures in the literature.

4.9 A design of an experiment to measure the Nernst effect

In section 4.8, we discussed the role of magnetised transport terms in a low β plasma. This is relevant for laboratory astrophysics and for the edge regions of magneto-inertial fusion schemes where the density is significantly lower. However, the density and temperature in the fusion fuel at stagnation can be extremely high and therefore generally $\beta \gg 1$ in this region. For example, typical MagLIF conditions in the fusion fuel at stagnation can reach $B_z \approx 10^3 \text{T}$, $T_i \approx 1 \text{keV}$ and $n_i \approx 10^{29} \text{m}^{-3}$ [17, 52]; giving $\beta \approx 10^2$. As we discuss in section 4.7, in this limit the Nernst effect is an important term in magnetic field transport. The importance of this term has been extensively explored

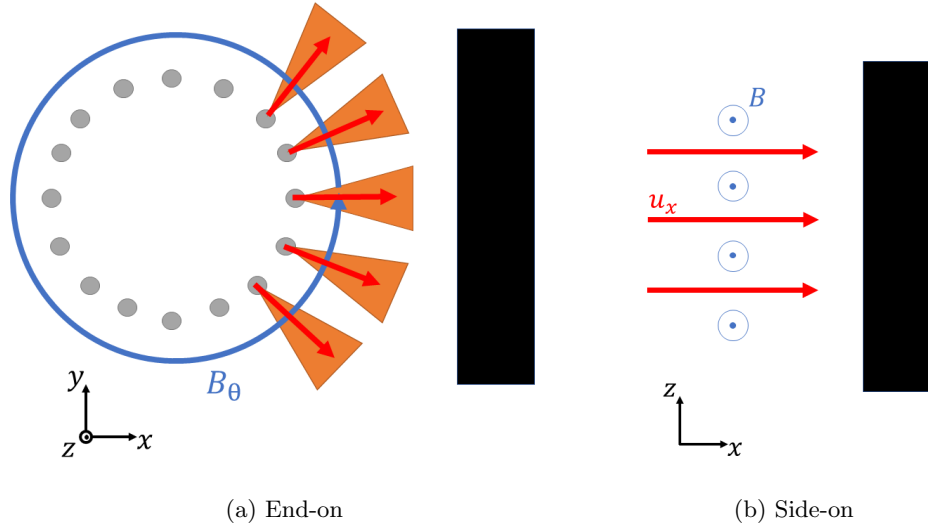


Figure 4.9: A sketch showing the geometry of the inverse wire array experiment considered in section 4.9. The wires are shown in grey. A current travels in the positive z direction up through the outer wires and downward through the central conductor. This generates an azimuthal magnetic field B_θ (shown in blue) and resistively heats the wires producing a plasma flow (shown in orange). This plasma flow is accelerated towards the target by the Lorentz force. Fig. 4.9a shows the end-on view looking down the wire array. Fig. 4.9b shows the view side-on as the flow moves towards the target in black.

for different MIF schemes in the literature [48, 52, 120, 176]. Despite this importance however, the Nernst effect is extremely challenging to measure. Some experiments using laser-plasma platforms have been proposed [29, 177] and the analysis of data from those experiments is still ongoing. There have been no pulsed power experiments that attempt to measure the Nernst term. This means that the extended MHD codes used to model it are largely unbenchmarked. In this section, we use our self-similar model to consider potential plasma parameters for a pulsed-power experiment to measure the Nernst effect.

Recent experiments on the MAGPIE pulsed power generator have studied reverse shocks generated when a plasma flow produced by an inverse wire array collides with a barrier [80, 81]. An axial current flows in the positive z direction through the wires and then returns downwards through the central conductor. This current induces an azimuthal magnetic field around the wire and resistively heats the wire to produce a plasma. This plasma is then accelerated by the Lorentz force towards a cold wall. Using wires of a different material changes the behaviour of the reverse shocks. When aluminium wires are used, the shock is clearly defined. When instead a tungsten wire array is used, there is no shock and instead a broad turbulent region is set up. It is believed this difference is due to increased radiative cooling for the higher Z tungsten [80]. A sketch of the set up is shown in figure 4.9. The wall acts as a sink for both heat and magnetic flux. Therefore, due to the large temperature gradient from the wire-array plasma to the cold barrier, it might be expected that the Nernst effect will cause

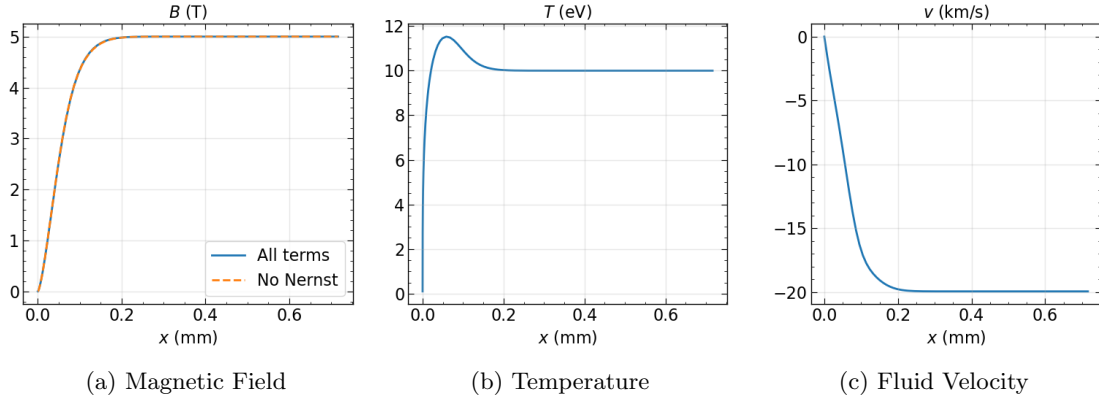


Figure 4.10: Profiles of magnetic field, temperature and fluid velocity obtained from the self-similar code at a time of 1ns for inflow plasma parameters given in eq. 4.57.

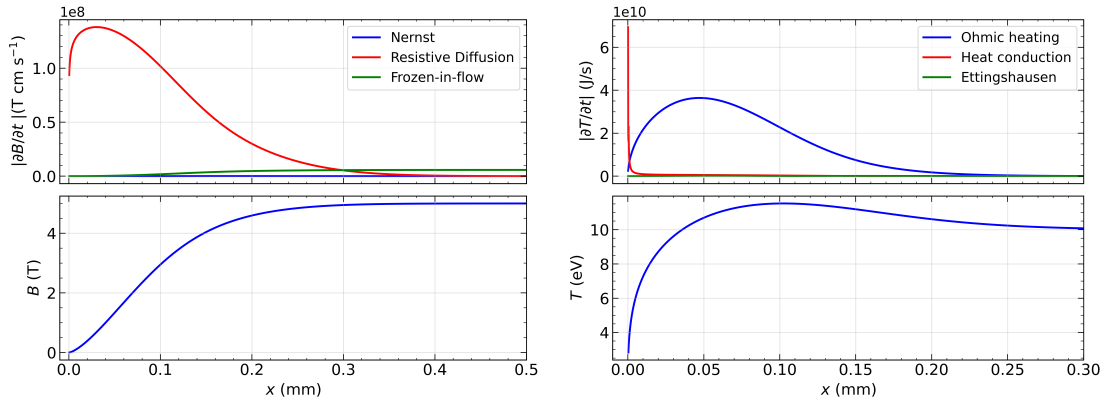
increased advection of magnetic field into the wall which may be able to be measured. We used our self-similar model to establish which plasma parameters would produce a measurable Nernst effect.

Initially, we modelled the wall as a plasma with a much higher density and a much lower temperature than the plasma inflow and used our self-similar model as laid out in section 4.6. We assumed that the wall and the inflow plasma were of the same species (with $Z = 8$ and $A = 27$ based on experimental parameters). This is not a good model for the wall, but we are only interested in the plasma profiles in the inflow plasma anyway. However, we found that the low wall temperature often caused the shooting method to produce negative temperature values inside the wall region, crashing the code and preventing convergences. We found that this was exacerbated by the density gradient: a larger ratio between the densities of the wall and inflow plasmas meant convergence was even harder to obtain. Therefore, we adapted our self-similar model for this problem. We instead imposed zero magnetic field and a low temperature in the wall as boundary conditions and only used the self-similar model to calculate the plasma profiles from $\eta = 0$ to $\eta = \infty$. In the notation of section 4.6, this meant that we only need to make estimates for ϵ and q at $\eta = 0$ as our initial guess for the shooting method. This drastically improved the stability of the self-similar code for this problem and allowed us to actually extract plasma profiles for the inflow plasma.

We usually assumed that the wall temperature T_{wall} was 0.01eV, which set the wall density n_{wall} by pressure balance to be

$$n_{wall} = \frac{\left(1 + \frac{1}{Z}\right)n_{in}T_{in} + B^2/2\mu_0}{\left(1 + \frac{1}{Z}\right)T_{wall}}, \quad (4.55)$$

where n_{in} and T_{in} are the electron density and temperature of the inflow plasma and B is the magnetic field. We found profiles converged with $T_{in}/T_{wall} \sim 10^3$. For all of this



(a) Contribution of terms in the induction equation (b) Contribution of terms in the energy equation

Figure 4.11: The contribution of different terms to the energy and induction equations, calculated from the profiles produced by the self-similar code at 1ns.

work, we assumed $Z = 4$ and $A = 27$ based on Thomson scattering data and results from the FLYCHK atomic physics code [178] that indicated this level of ionisation [80]. We took transport coefficients from Epperlein-Haines [94] and a model for the low temperature Coulomb logarithm from Braginskii [86]:

$$\ln \Lambda = 23.4 - 1.15 \log_{10} n + 3.45 \log_{10} T. \quad (4.56)$$

The inflow plasma is well characterised by experimental diagnostics [80], with

$$\begin{aligned} B &\approx 5\text{T} \\ T &\approx 10\text{eV} \\ n_e &\approx 10^{19}\text{cm}^{-3}. \end{aligned} \quad (4.57)$$

Figure 4.10 shows the profiles of magnetic field, temperature and fluid velocity obtained using our self-similar model for these inflow conditions. Plasma flows towards the wall at $x = 0$, as shown by the negative fluid velocity. The fluid velocity obtained from the self-similar model agrees reasonably well with experimental data showing an inflow velocity near the target of approximately 15km/s, validating our approach. Finite thermal conductivity and resistivity cause the temperature and magnetic field respectively to diffuse into the cold wall, setting up a very steep temperature gradient. The temperature gradient is steeper than the magnetic field gradient as the magnetic field magnetises the thermal conduction, making the conductivity small where B is large. The yellow line in Fig. 4.10a shows the magnetic field profile if the Nernst effect is excluded from the calculation. Clearly, the Nernst effect has no measurable impact on the magnetic field profile for these experimental parameters. This is shown in more depth in figure 4.11, which shows the contribution of different terms to the energy and induction equations. These contributions have been directly calculated from the profiles produced by the self-similar code. From figure 4.11a, it is clear that the resistive diffusion (shown in red)

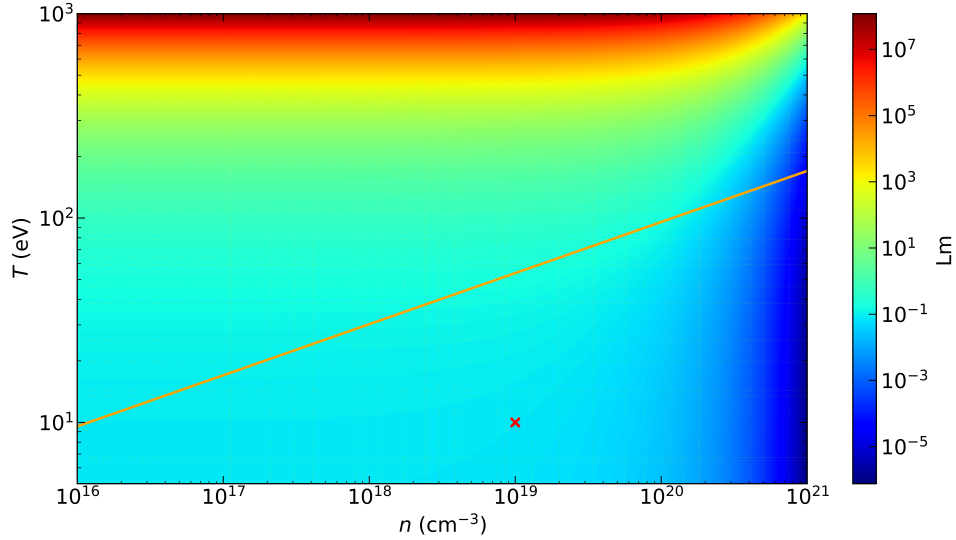


Figure 4.12: A heatmap showing how the Lewis number L_m varies as a function of density n and temperature T . The red cross shows the values for the inflow plasma given in equation (4.57). The orange line is the contour of $L_m = 1$.

is the key term determining the shape of the magnetic field profile near the wall. The frozen-in-flow begins to become larger away from the wall where the fluid velocity is high. From figure 4.11b, it is clear that the peak in the temperature profile at around $x = 0.1\text{mm}$ is due to the Ohmic heating. The thermal conduction becomes extremely large close to the wall where the temperature gradient becomes increasingly large. We find that artificially lowering the thermal conduction causes the peak in the temperature to appear higher and closer to the wall.

Thermal or magnetic diffusivity sets up a diffusion layer within which the diffusive process occurs. Comparing figures 4.10a and 4.10b, it is clear that the magnetic field gradient extends to about $x \approx 0.2\text{mm}$ and the temperature gradient to $x \approx 0.1\text{mm}$. Therefore, the magnetic diffusion layer is larger than the thermal diffusion layer. This causes resistive diffusion to dominate the Nernst effect in the magnetic field transport. Another way of stating this is that the magnetic Lewis number, defined in section 4.5.3, is small. In fact, for the parameters of the inflow plasma: $L_m \sim 10^{-3}$.

The magnetic Lewis number is independent of magnetic field and so we can plot out the parameter space of L_m in order to find density and temperature values for our inflow plasma that might produce a larger Nernst effect. This parameter space is shown in figure 4.12. The orange line in this figure is the contour of $L_m = 1$. The theory suggests that to observe a significant Nernst effect, we need to have $L_m \gg 1$. Clearly, we would need to decrease the density by at least 3 orders of magnitude to get a Lewis number of order unity which is not realistic on MAGPIE. It is significantly easier to increase L_m by increasing the temperature of the flow. This is challenging with the plasma flow from a wire array. Potentially, a viable source of high temperature plasma of a comparable density may be the outflows from a reconnection region. Experiments

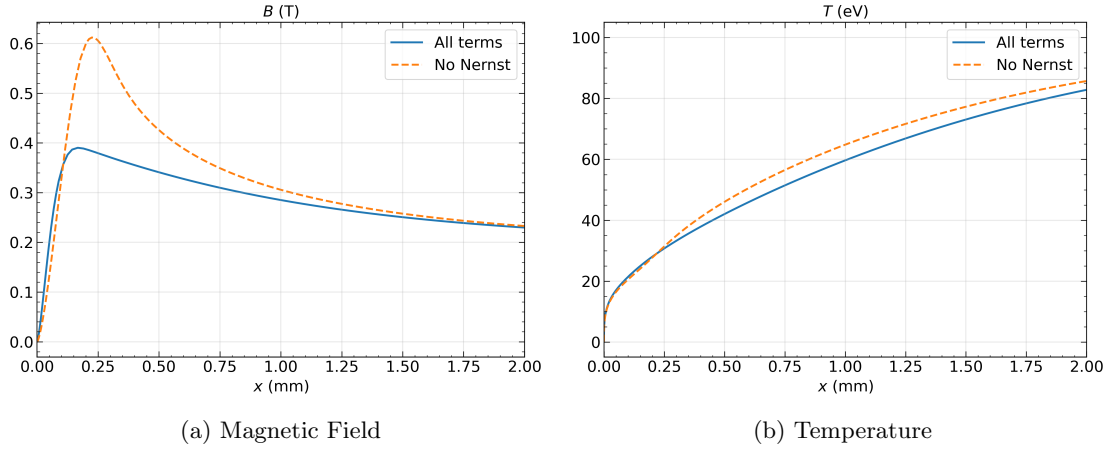


Figure 4.13: Plots showing how the magnetic field and temperature differ with the Nernst effect turned on and off, calculated from the self-similar code at 1ns.

are routinely done on MAGPIE where two separate wire arrays are set up and produce magnetised plasma flows towards each other. This leads to the formation of a magnetic reconnection region between the two wire arrays [78]. High temperature plasma is ejected from this reconnection region. We take rough experimental values from past MAGPIE experiments for this outflow plasma [77, 78]:

$$\begin{aligned}
 T &\approx 100\text{eV} \\
 n &\approx 10^{17}\text{cm}^{-3} \\
 B &\approx 0.2\text{T},
 \end{aligned} \tag{4.58}$$

and put these values into our self-similar model. This yields the profiles shown in figure 4.13. With these parameters, the Lewis number of the inflow plasma is approximately 120. As predicted, the Nernst effect now has a large impact on the magnetic field. When the Nernst effect is excluded from the calculation, a large peak in magnetic field develops at $x \approx 0.25\text{mm}$. The Nernst increases advection of magnetic field into the cold wall. It can also be seen that the width of the magnetic field diffusion layer is now much smaller than the width of the thermal diffusion layer. This means that the Nernst dominates over the resistive diffusion.

Whilst this does show that increasing L_m is an effective way to increase the role of the Nernst effect, this would likely be a challenging measurement to make. In an experiment, one would be attempting to measure the absence of the high peak. This would mean that the magnetic field would need to be measured very carefully and compared to the results from extended MHD simulations. This somewhat undermines the idea of using a measurement of Nernst to benchmark extended MHD. Additionally, as shown in the experiment, colliding plasma into an obstacle produces an extended stagnation region rather than the sharp boundary between hot plasma and cold wall

that the self-similar model predicts [80]. In other words, the hydrodynamics of how the plasma is slowed as it approaches the wall is relevant to the problem and the isobaric self-similar model cannot recreate this physics. The diagnostics would be challenging for this measurement as well. The high density of the stagnation layer would mean laser probe measurements, such as Faraday rotation, could not be used in this region. Lower density plasma would likely be needed. The laser driven platforms that are being explored currently to measure the Nernst effect are more promising as they naturally have higher plasma temperatures [29, 177], leading to an increased Nernst effect that is easier to measure with proton radiography.

However, the idea of increasing the Lewis number may be useful in designing experiments. The self-similar model allows us to make quick estimates of whether we expect extended MHD terms to have an impact, which can be useful for quickly looking around parameter space for designing experiments.

5 Fully kinetic simulations of a vacuum-plasma interface

The steep density gradients that define a vacuum-plasma interface mean that the single-fluid MHD models that are conventionally used in modelling pulsed power plasmas are no longer valid. The progressive breakdown of the assumptions behind MHD was discussed in section 2.4. However, more complete models such as two-fluid MHD or kinetic codes are usually computationally infeasible for realistic simulation of experiments. A reduced model that approximately accounts for the dominant beyond-MHD effects, but within a more computationally tractable single-fluid MHD framework, would be a useful tool for studying this regime. The development of such a code is the main aim of this thesis. In this chapter, we use a particle-in-cell code to explore the evolution of a z-pinch edge, as a proxy for a vacuum-plasma interface, and compare this to the MHD behaviour. The purpose of this is to evaluate using a detailed model which effects are the most important to account for in a reduced model. We go on to show how this affects the evolution of an $m = 0$ instability in a z-pinch. This indicates that these microphysics effects can lead to observable consequences in experimental-like conditions. PIC codes are not often used to model z-pinch and we conclude by discussing why they prove to be a challenging tool for this problem as well as highlighting the ideal features for a code describing this regime.

5.1 Motivation

5.1.1 Choice of model

In this section, we discuss why a PIC code was chosen as the more detailed model for a vacuum-plasma interface. In order to explore the breakdown of single-fluid MHD, the model should self-consistently include more physics but reduce to the single-fluid limit in the appropriate regime. This restricts the choice of code to either two-fluid MHD or a kinetic model. We chose not to use a two-fluid code as there are very few open-access codes available and we want to assess whether kinetic effects are relevant to the problem. It would be an interesting avenue of future work to explore how the Hall MHD limit emerges from a two-fluid code where charge separation is self-consistently accounted for. Within kinetic codes, there is a choice between particle-in-cell and continuum codes such as Vlasov-Fokker-Planck (VFP). The VFP codes to which we had access all exclusively solve the electron VFP equation and assume fluid ions. At a vacuum-plasma interface, the ion mean free path is usually longer than the electron

mean free path. Therefore, electron VFP is likely not useful for studying this regime. Additionally, as was highlighted in section 3.4.1, VFP codes usually do not directly discretise velocity space and instead decompose the distribution function into spherical harmonics. This makes them unsuitable for weakly collisional plasmas that may develop significant velocity space anisotropy. The vacuum-plasma interface can approach this limit, depending on the minimum density considered, and therefore PIC was chosen. Another aim of the work in this chapter was to establish whether these anisotropy effects are significant enough to preclude VFP being used for this problem.

5.1.2 Kinetic simulations of vacuum-plasma interfaces

Within this thesis, we are interested in the edge of a z-pinch as an example of a vacuum-plasma interface. We focus on this aspect within this chapter. However, there are extensive examples of using kinetic simulations to study other aspects of z-pinch plasmas in the literature. Continuum kinetic approaches have been used to study the stability of z-pinch modes to $m=0$ and $m=1$ modes with large Larmor radius effects [179]. Recently, new high order continuum kinetic methods have been developed to further study the kinetic stability of z-pinch modes and transport processes that could be influencing pulsed power systems [153, 180]. PIC codes have been used to evaluate the stability of the shear flow stabilised z-pinch concept [181] as well as to study the energy spectrum of neutrons produced by imploding z-pinch modes [182]. They are also routinely used to study dense plasma focus devices [164, 183, 184]. The application of PIC codes to these problems is mostly to study fast non-thermal ions and the production of neutrons. In this thesis, we are interested in the departure from MHD in the low-density vacuum-plasma interface regime which is quite distinct physics from what has been studied before.

The most relevant kinetic simulation work for our problem has been done using PIC codes to study power flow in pulsed power systems. The mechanism of how current is lost in transmission lines is not fully understood and there have been attempts to model it using both explicit PIC codes [68] and the implicit LSP, CHICAGO and EMPIRE PIC codes [118, 163]. The low density plasma produced by the ablation of electrodes in the transmission lines is in the vacuum-plasma interface regime in which we are interested. In another work, the LSP code was used to simulate anomalous resistivity effects in a low-density current-carrying plasma [123]. This was then used to evaluate different analytic anomalous resistivity models.

The past work suggests that PIC simulations can be used for the low-density plasma-vacuum interface regime, albeit with a high computational cost. However, there is a lack of study of the structure of the vacuum-plasma interface. Understanding how this region differs from its MHD description is the primary aim of this chapter.

5.2 Simulation set-up

In this section, we discuss how the PIC simulations were set up in order to study a vacuum-plasma interface. The initial and boundary conditions are discussed and specific unconventional choices made in the parameters of the PIC code are highlighted.

5.2.1 Vacuum-plasma interface conditions

The simulations are initialised in an MHD equilibrium to observe how the plasma departs from this solution in PIC. It would be interesting to see whether from different initial conditions a PIC simulation relaxes to a fluid equilibrium. This would likely take a significant amount of time and computational particles, in order to obtain macroscopic fluid-like behaviour, and therefore is beyond the scope of this work.

The edge of a z-pinch is an example of a vacuum-plasma interface and we use this as the MHD equilibrium we initialise from. However, the computational demands of a PIC simulation mean that it is not practical to model an entire z-pinch. The dense core of the pinch is highly collisional and is well described by conventional MHD. Therefore, in this work we study only a small region at the edge of the z-pinch where there is a steep density gradient to a vacuum outside the plasma. This is the region that MHD breaks down and where we are mostly likely to observe beyond MHD effects. In the following sections we describe how the simulation is set up and the caveats to this approach.

5.2.2 Z-pinch edge profiles

The z-pinch equilibrium profiles used as initial conditions were derived in section 2.4.1 in cylindrical geometry. In this work, we consider the parabolic profile, which assumes a constant current density throughout the pinch, and the Bennett profile, which instead assumes a constant electron drift velocity through the pinch. The parabolic profile is more suitable for collisional plasmas, where the finite resistivity allows the diffusion of magnetic field through the plasma. In contrast, the Bennett profiles are correct for the collisionless case.

The EPOCH code currently however only has Cartesian geometry. This means that at least 2D is needed to recreate these profiles in an r - θ geometry and studying any axial variations needs 3D. It is not possible to use cylindrical profiles in 1D Cartesian. PIC simulations are generally the most computationally expensive class of plasma simulation due to the large number of particles needed to smooth noise. This is exacerbated for our problem since the large density gradients necessitate even larger number of particles. Performing 2D and 3D simulations in a regime where PIC is not commonly used is not practical. Therefore, initially in simulations we approximate 1D cylindrical geometry with 1D Cartesian; effectively setting $x \sim r$. At the time that this work was done, there were no open-source PIC codes with cylindrical geometry that were suitable.

Whilst this is not the correct geometry for the z-pinch, it is suitable for qualitatively understanding some aspects of the physics at the vacuum-plasma interface. The differ-

ence between cylindrical and Cartesian geometry is most stark when convergent effects become important, such as when the z-pinch implodes, since this is an effect not captured in 1D Cartesian. Due to the expense of PIC codes, the system is never simulated for long enough for these effects to matter anyway. Additionally, using 1D Cartesian to represent a cylindrical geometry is a fair approximation if we only consider a small region at the edge of the cylinder. As discussed in the previous section, we only model the edge of the dense z-pinch and handle the central region using boundary conditions. Therefore, we expect that Cartesian geometry will produce qualitatively correct results. For a similar reason, we would not expect much impact due to curvature drift of charged particles as the magnetic field curvature is less important at large radii. However, cylindrical simulations are an important refinement that could be done to this work. To evaluate our argument, later in the chapter we compare our 1D results with simulations in the 2D r - θ plane to establish the impact of geometry on the conclusions.

The balance between thermal and magnetic pressures is different in Cartesian geometry compared to cylindrical, leading to different profiles for the pressure and field. In cylindrical geometry, the radial gradient of thermal pressure is balanced by magnetic tension and pressure. However, in 1D Cartesian geometry we can have no field line curvature. Therefore, the pressure balance is simply:

$$n_e T_e + n_i T_i = \frac{B^2}{2\mu_0}. \quad (5.1)$$

We now derive analogous profiles for the parabolic and Bennett profile in 1D Cartesian geometry. For the parabolic case, we assume that the density still varies parabolically with distance $n \propto [1 - (x/a)^2]$, where a is the radius of the pinch. We assume that the temperature is constant with a value T_0 that we choose. We set this temperature to be equal to a typical value obtained from the Bennett relation, usually around 50eV. This leads to pressure and density profiles given by:

$$n_e = n_i = n_0 \left[1 - \left(\frac{x}{a} \right)^2 \right] \quad (5.2)$$

$$p = p_e + p_i = 2n_0 T_0 \left[1 - \left(\frac{x}{a} \right)^2 \right], \quad (5.3)$$

where n_0 is the peak density on axis of the pinch. These profiles are only valid for $x \leq a$, n and p are set to some minimum value for $x > a$. Throughout this chapter, it is assumed the plasma is hydrogen with $Z = 1$ and $A = 1$. Using the pressure balance equation, this leads to a magnetic field profile given by:

$$B_z = \begin{cases} 2\sqrt{\mu_0 T_0 n_0} \frac{x}{a}, & \text{if } x \leq a \\ 2\sqrt{\mu_0 T_0 n_0}, & \text{otherwise} \end{cases} \quad (5.4)$$

Note that we choose the magnetic field to be in the z direction so as to simulate the geometry of an azimuthal magnetic field, which leads to a current density in the y

direction with profile:

$$j_y = \begin{cases} -\frac{2}{a} \sqrt{\frac{T_0 n_0}{\mu_0}}, & \text{if } x \leq a \\ 0, & \text{otherwise.} \end{cases} \quad (5.5)$$

Whilst not identical to the cylindrical case, these profiles are analogous. The density and thermal pressure decrease parabolically, forming a sharp edge to the pinch. The magnetic field increases linearly inside the pinch and is curl free outside; leading to a constant current density inside. Note that in 1D Cartesian geometry, a curl free magnetic field is constant rather than $\propto \frac{1}{r}$ as in cylindrical.

For the Bennett profile, we again assume a constant temperature T_e and T_i for the electrons and ions respectively, so then we can write the pressure balance equation as:

$$\nabla p = \mathbf{j} \times \mathbf{B} \quad (5.6)$$

$$(T_e + Z^{-1}T_i) \frac{dn(x)}{dx} = j_y B_z. \quad (5.7)$$

The current density can be written as both $j_y = -\frac{1}{\mu_0} \frac{\partial B_z}{\partial x}$ and as $j_y = -en(x)v_{ey}$, where v_{ey} is the electron drift velocity which is assumed to be constant for the Bennett profile. Using these, the pressure balance can be rewritten as:

$$-e\mu_0 v_{ey} n(x) = \frac{d}{dx} \left[\frac{(T_e + Z^{-1}T_i)}{en(x)v_{ey}} \frac{dn(x)}{dx} \right] \quad (5.8)$$

$$-\frac{e^2 \mu_0 v_{ey}^2}{(T_e + Z^{-1}T_i)} n(x) = \frac{d}{dx} \left[\frac{1}{n(x)} \frac{dn(x)}{dx} \right]. \quad (5.9)$$

We solve this nonlinear ODE subject to the same boundary conditions as used by Bennett [85]:

$$n(x=0) = n_0 \quad (5.10)$$

$$\left. \frac{dn}{dx} \right|_{x=0} = 0. \quad (5.11)$$

This gives for the density profile:

$$n(x) = n_0 \operatorname{sech}^2 \left(\frac{\sqrt{2\alpha n_0}}{2} x \right), \quad (5.12)$$

where $\alpha = \frac{e^2 \mu_0 v_{ey}^2}{(T_e + Z^{-1}T_i)}$. We then obtain the initial condition for the magnetic field from $B_z = -\frac{(T_e + Z^{-1}T_i)}{en(x)v_{ey}} \frac{dn(x)}{dx}$. We need to specify an initial temperature and electron drift velocity v_{ey} . We did this by substituting values until we produced profiles with boundary values similar to those of the parabolic profile. This is the generalisation of the Bennett pinch to 1D Cartesian geometry. We later compare the particle-in-cell

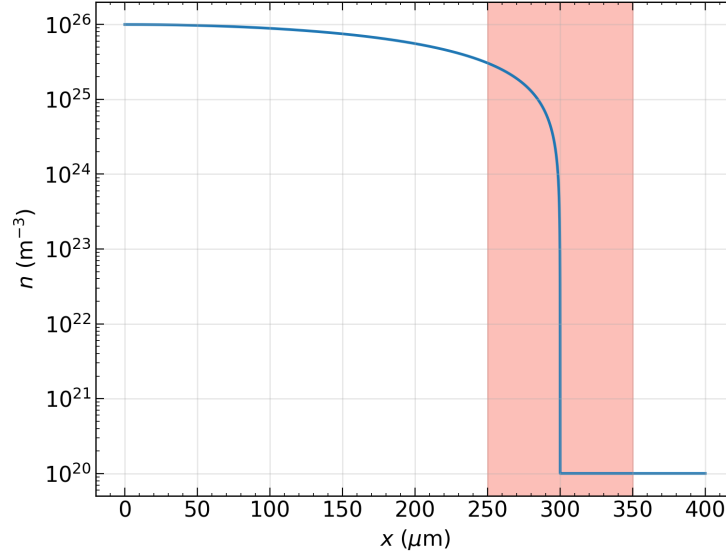


Figure 5.1: The Cartesian parabolic density profile that we use for 1D simulations, with peak density 10^{26}m^{-3} and pinch radius $300\mu\text{m}$. We set a floor density (here of 10^{20}m^{-3}) to control the size of the density gradient. In the PIC simulation, we only simulate the region highlighted in red.

results produced using both of these profiles.

5.2.3 Boundary conditions

Separate boundary conditions need to be chosen for both the particles and the EM fields in a PIC simulation.

In 1D simulations, we are attempting to model just the edge region of a z-pinch. In figure 5.1, we show an example of the 1D Cartesian parabolic density profile that is used to initialise some simulations. The red highlighted region indicates a typical region that is simulated using PIC. On the left hand side of this region, there is a dense collisional plasma and on the right hand side there is a very low density plasma or a vacuum. We use outflow conditions for both the particles and fields on the right hand side. We use reflective boundary conditions on the left hand side for both the particles and fields. Whilst this does produce a clear edge effect on the left hand boundary, we are interested primarily in the lower density side so this is less important. Physically, we are motivated by cylindrical geometry where a reflective boundary is used on axis in the radial direction.

In 2D simulations, we choose outflowing boundary conditions on all boundaries for both particles and fields. This means that any particles or EM waves that hit the boundary are transmitted out of the simulation (albeit with a small reflection for EM waves). This is because the outer region of the simulation should be a vacuum region, so outflow is a reasonably physical boundary condition. However, we usually also set the boundaries to be far away to not influence the behaviour.

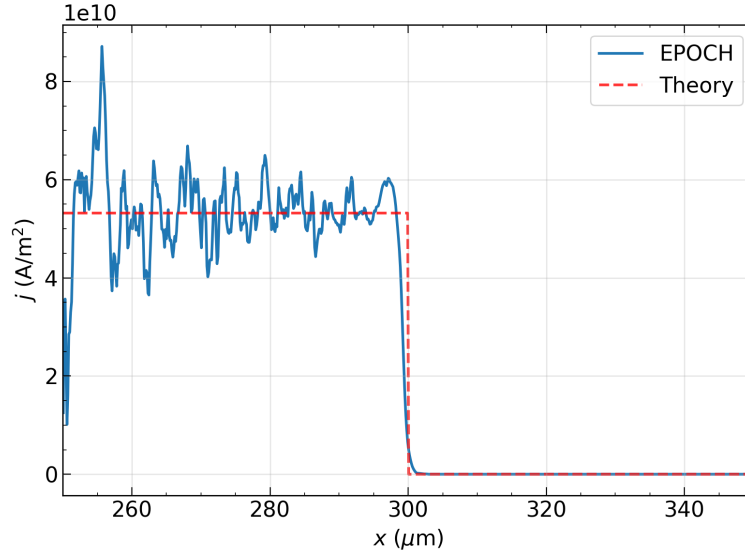


Figure 5.2: Plot showing the current density calculated self-consistently from the velocities of all PIC superparticles, shown in blue, compared to the theoretical current density shown in dashed red.

5.2.4 Current density

In order to produce physical results, we need to specify an initial current density within the simulation. Through our choice of initial conditions, we are assuming that the plasma has settled to an MHD equilibrium. This means that the current density should be related to the magnetic field by $\mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{B}$. If this is not the case, then there will be a finite displacement current which will drive these two quantities into equilibrium. Vasyliunas argues that this displacement current will either change $\mathbf{B}$ via EM waves or change $\mathbf{j}$ by exciting plasma oscillations [149]. The dominant effect is dictated by the shorter timescale of ω_{pe}^{-1} and L/c , where L is the lengthscale. Changing $\mathbf{j}$ is the desired effect, but certain parameter regimes lead to EM fields that are significantly different from our intended initial conditions. Therefore, it is necessary to apply an initial external current to the system to ensure that we initially satisfy Ampère's law.

EPOCH allows for the specification of an external current which is simply added to the total after the current due to particle motions is calculated: $\mathbf{j}_{\text{total}} = \sum_s q_s n_s \mathbf{v}_s + \mathbf{j}_{\text{external}}$, where the sum is over particle species s . This total current is then used in evolving the EM fields. In other words, this external current does not produce a drift in the velocity of charged particles, which is a feature we are interested in resolving. Instead, we initialised the current density by applying a drift to the electron population. The superparticles in a PIC code are normally randomly initialised from a Maxwellian. By applying a spatially dependent drift $v_{ey}(x) = -\frac{j_y(x)}{en(x)}$ to the mean value of the distribution function, we found that this approximately produced the desired current density. This comparison is shown in figure 5.2. The current density calculated self-consistently from the velocities of all superparticles in the simulation, shown in blue,

is compared to the theoretical current density calculated from the curl of the magnetic field, shown in red. The EPOCH current density approximately matches the theory, with the poor agreement being due to noise due to the finite number of particles in the simulation. A particular issue with our approach of defining j_{initial} is that near the edge of the pinch (here $300\mu\text{m}$), the number density is low but the required current density is constant; causing the electron drift velocity to diverge. We have found both that increasing the number of particles reduces the noise, as expected, and that the differences between theory and initial current density do not significantly affect the results.

5.2.5 PIC Algorithm

In this section, we justify some of the choices made in settings for the EPOCH solver.

The Debye length λ_D is inversely proportional to $\sqrt{n}$, meaning λ_D becomes small in high density regions. As discussed earlier in this thesis, PIC codes can suffer from unphysical numerical self-heating if the Debye length is not spatially resolved [2, 159]. Part of the rationale for only studying the edge region of the pinch is that this increases the size of the minimum λ_D that needs to be resolved. However, for large density gradients this is still a significant constraint on the feasibility of simulations. Therefore, we generally do not resolve the smallest Debye length. The Debye length of the low density plasma in which we are interested is naturally resolved. Recent research has found that using higher order shape functions for the superparticles and current smoothing mitigates numerical self-heating [2, 159]. We have not observed numerical heating in our work and found that resolving the Debye length does not significantly change the results.

Density gradients in a PIC code can be represented by either varying the number of superparticles in space whilst having all superparticles represent the same number of physical particles, or by varying the weighting of superparticles in space. The former of these produces far worse noise levels. For example, with a density gradient of a factor of 100; there needs to be 100 times as many particles in the high density region as the low density region. This means excessively large number of particles are needed to mitigate noise in the high density region, but then most of the computational resources are spent in the area that is not under study. However, using variable particle weighting can introduce unphysical behaviour into the simulation. For example, if a high weight particle moves into the low density region, this leads to spikes in the density. Numerical heating can also be observed when low weight and high weight particles interact. To avoid these errors, we do not use variable weighting and accept the higher computational cost of constant particle weights.

We do not use the collisions module in our simulations. This is because the collisions significantly worsened the runtime of simulations and since we do not have sufficient particles to resolve collisional transport anyway. The collisions module within EPOCH is only implemented for variable particle weighting, which we are not using. At the

Variable	Value
ω_{pe} (s^{-1})	5.97×10^{13}
ω_{pi} (s^{-1})	1.39×10^{12}
Ω_{ce} (s^{-1})	3.50×10^{12}
Ω_{ci} (s^{-1})	1.91×10^9

Table 5.1: Table summarising the key timescales for the discussion in this section.

edge of the z-pinch, collisions become less relevant anyway due to the lower density. However, the impact of resistivity and the departure from the collisional Ohm's law would be of real interest and further work using collisions within a PIC code to study it is an interesting future avenue of work.

We use the default triangle shape function throughout and do not use current smoothing, as we found that it seemed to introduce boundary condition problems in the “dense” plasma. For 1D simulations, we used 1000 cells and 1000 particles per cell (unless otherwise discussed). This is the largest number of particles that we could use without excessive runtimes. For the 2D simulations, we used 300 cells in both the x and y directions and used 500 particles per cell. This was due to their far greater computational cost.

5.3 One Dimensional Results

5.3.1 Departure from MHD Pressure Balance

First, we present results from 1D simulations. The purpose of these simulations was to study departures from MHD pressure balance. The key timescales to understand this process are the electron and ion plasma frequencies ω_{pe}, ω_{pi} and gyrofrequencies Ω_{ce}, Ω_{ci} . We calculate the plasma frequencies using the mean density across the pinch and the gyrofrequencies using the field at the vacuum plasma interface. The values of these parameters are shown in table 5.1. These simulations used $n_0 = 10^{25} m^{-3}$, $T_0 = 50 eV$ and a pinch radius $a = 300 \mu m$. We had to pick a cut-off density below which we would assume that the density was 0, to avoid modelling too many orders of magnitude in density with the PIC simulation. We found that our qualitative results were not affected by this cut off density, assuming that it was at least one order of magnitude lower than the highest density simulated.

The evolution of the electron and ion number density over time is shown in figure 5.3. The number densities shown on this plot are averaged over a window of ~ 10 electron plasma periods to reduce noise in the plot. It can be seen from the initial condition (top left plot) that initially there is no charge separation of electrons and ions and we resolve over an order of magnitude in density well. There is an extremely sharp drop off at the edge of the z-pinch due to the use of finite particles in PIC. We begin the problem in pressure balance, but this necessarily means that we do not have $\mathbf{E} = 0$. The reason for this can be seen from the generalised Ohm's law. We set $\mathbf{j} \times \mathbf{B} = \nabla(p_e + p_i)$ for

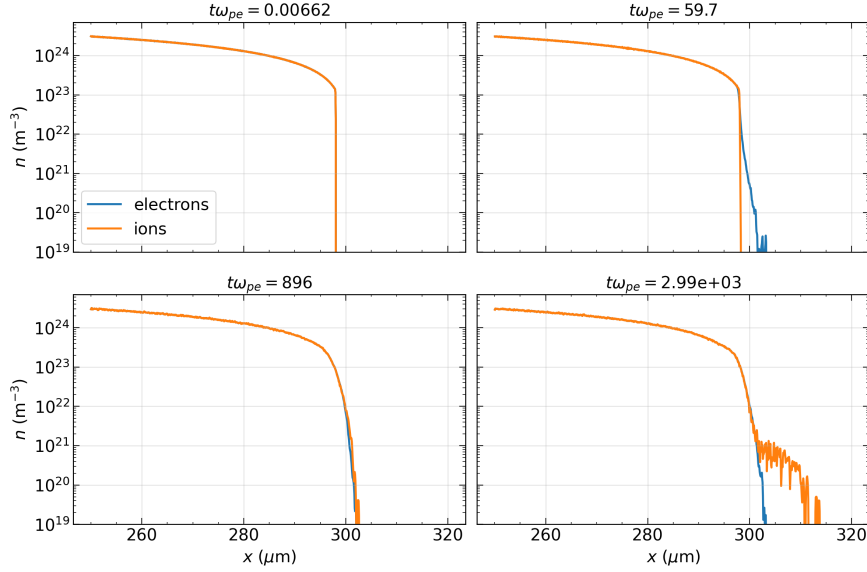


Figure 5.3: Plots showing how the number density of electrons and ions varies over time for a 1D simulation. The electron number density is shown in blue and the ion in orange. The number density is averaged over a period of ~ 10 plasma oscillations to produce the curve shown in each figure. The reference electron plasma frequency in these plots is calculated using the mean number density of $1.12 \times 10^{24} \text{ m}^{-3}$.

pressure balance, but $en_e \mathbf{E} = \mathbf{j} \times \mathbf{B} - \nabla p_e$ from the GOL (dropping inertial and resistive terms). Hence, we see an electric field build up in our simulations. This electric field corresponds to the charge separation of electrons and ions due to thermal expansion of electrons out into the vacuum. This process occurs on the timescale of the electron plasma frequency. On the timescale of the electron cyclotron frequency ($\Omega_{ce}^{-1} \approx 20\omega_{pe}^{-1}$ for these parameters), the electrons are deflected by the magnetic field. This limits the motion of the electrons into the vacuum region. This is shown in figure 5.4, which shows the mean velocity of electrons calculated from the PIC simulation at two times. In the left plot, the electrons have been accelerated and the velocity is positive at the surface of the pinch. In the lower density the electric field is higher so the highest electron velocity is right at the plasma-vacuum interface. However, after approximately an electron cyclotron period, some electrons have been deflected by the magnetic field and the average velocity is closer to 0. At the same time, the electric field produced by the acceleration of electrons into the vacuum acts to decelerate the electrons. It is the combination of this effect and the magnetic field that acts to limit the electron acceleration into the vacuum. We have found that if the magnetic field is set to zero that the electrons are not stopped as rapidly and travel far further into the vacuum. As well as decelerating the electrons, the charge separation electric field acts to accelerate the ions out into the vacuum region. The ion plasma period $\omega_{pi}^{-1} \approx 40\omega_{pe}^{-1}$ is comparable in order of magnitude to the electron cyclotron period. This means that the electrons begin to be deflected around the same time the ions are accelerated by the electric field.

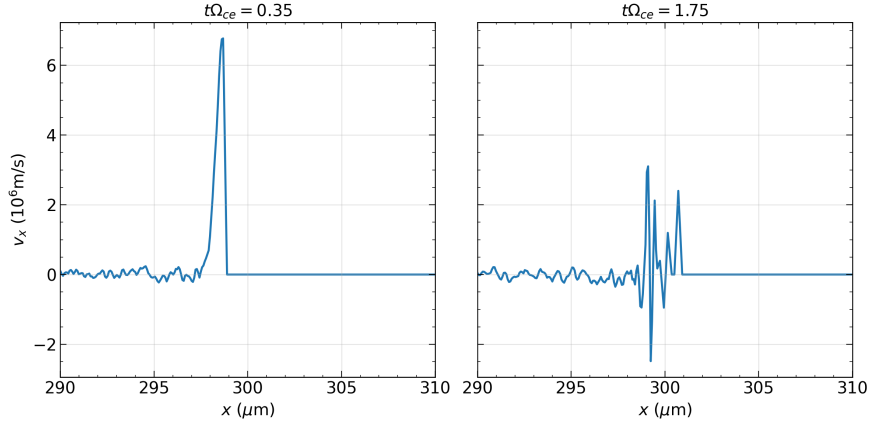


Figure 5.4: Plot showing the average electron velocity in each cell of the PIC simulation at two times. The reference Ω_{ce} used here is calculated using the value of the magnetic field at the vacuum-plasma interface, $|B| = 20\text{T}$.

The third panel in figure 5.3 shows that eventually the ions catch up with the electrons. However, due to their greater inertia they overshoot and travel further into the vacuum. The ion cyclotron period is far longer than that of the electrons, thus leading to a larger Larmor radius by a factor $\sim \sqrt{\frac{m_i}{m_e}}$. This can be seen in the final panel on figure 5.3, showing that the ions form an extended low density layer with charge separation from the electrons. This arises due to the difference in Larmor radius between electrons and ions and its width is comparable to the difference between electron and ion Larmor radii. This charge separation layer is relatively stable and remains in place until the end of the simulation. A similar layer has been observed in past work using kinetic simulations to study the boundaries of magnetospheres [185]. We initially began the PIC simulation in MHD pressure balance, but this is a single-fluid charge-neutral solution. In effect, in MHD we average over the timescales we have discussed in this section but explicitly modelling them using PIC shows the formation of a charge separation layer that lasts for macroscopically long times (a few Ω_{ci}). This same picture of a charge separation layer still occurs if we drastically increase the number of particles (therefore resolving a larger density gradient), or even if we use a different profile such as the Bennett profile. Figure 5.5 shows a 2D simulation done in the $x - y$ plane using Bennett profiles for the number density and magnetic field. The density and field values are comparable to those used for our 1D simulations. The figure shows the electron and ion number densities after a few thousand electron plasma periods (as in the final panel of figure 5.3). As in 1D, it is clear that the electrons have moved somewhat radially outward from their equilibrium position. The ions on the other hand have formed a significantly broader low density outer layer. This emphasises that the observation of how the system departs from MHD pressure balance was not a consequence of us carrying out the simulation in planar geometry or of the z-pinch edge profiles we chose.

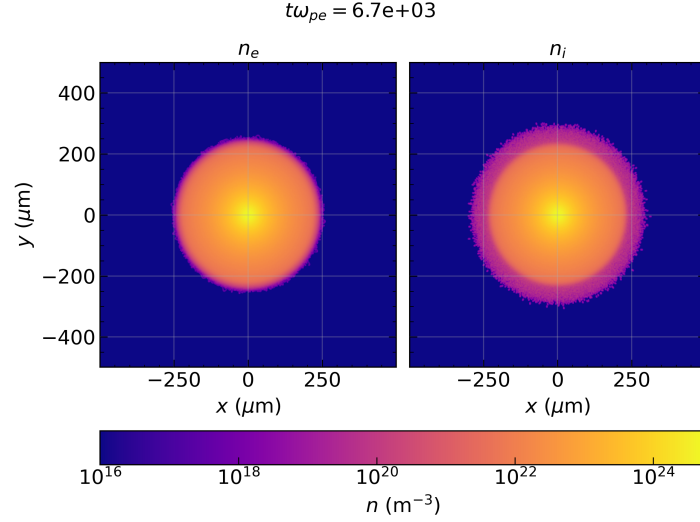


Figure 5.5: Plot showing the electron and ion number densities calculated from the PIC simulation after a few thousand electron plasma periods. Bennett profiles were used and the simulations were carried out in the 2D $x - y/r - \theta$ plane.

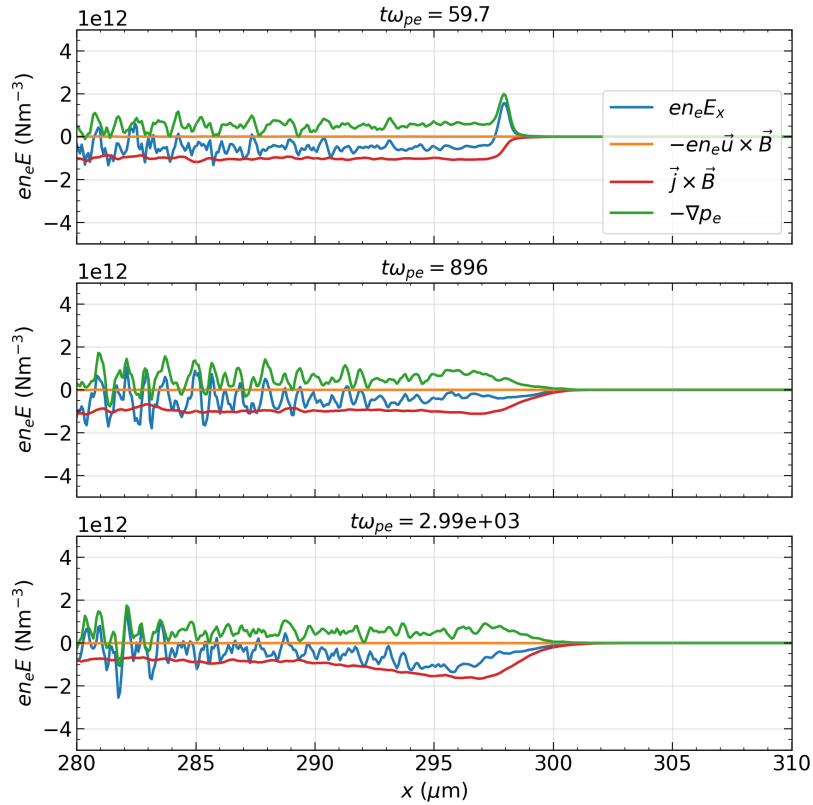


Figure 5.6: Plot showing the electric field force density in the PIC simulations over time. The times correspond to the same times as shown for the densities in figure 5.3. Only the ideal MHD, Hall and pressure gradient electric fields are included.

5.3.2 Electric Field

Whilst this formation of a charge separation layer at the vacuum-plasma interface is interesting, it cannot be included in a single fluid MHD simulation. We can determine which terms are relevant to include for a more physical model of the vacuum-plasma interface by looking at the electric field from the PIC simulation. We compare this to the generalised Ohm's law (GOL); dropping resistive, inertial and anisotropic electron pressure terms. This comparison is shown in figure 5.6. The blue line shows the electric field directly taken from the PIC simulation. The other lines are terms in the GOL calculated from the PIC simulation. The times in this figure correspond to the times where we showed the number density previously. At all times, the $\mathbf{u} \times \mathbf{B}$ electric field is negligible. This is unsurprising since the computational demands of PIC mean that we are restricted to only relatively early times and so there is not enough time for macroscopic flows to be set up, nor do we depart far enough from pressure balance for this to happen. The electric field noise is particularly dominated by the pressure gradient electric field due to small cell-to-cell differences in number densities leading to large oscillations in the pressure gradient. These oscillations average to zero however except in the edge region around $x = 280\mu\text{m}$ to $x = 300\mu\text{m}$ in which we are interested. In the first panel, the electric field is decelerating the electrons. Whilst there is a good match between the generalised Ohm's law and the electric field in the PIC simulation, it is worth remarking that $\nabla p_e = \nabla p_i$ for our parameters and that is what is actually determining the electric field. At $t\omega_{pe} = 896$, the electrons have stopped accelerating and the ions have “caught up” to them in the vacuum (as seen in figure 5.3). Therefore, there is little separation between electrons and ions and so the electric field is nearly zero. In the final panel of the figure, the ions have overshoot the electrons and are confined at a larger radius by the magnetic field. This sets up a relatively static charge separation electric field, which can be seen from the blue line. This lines up relatively well with the Hall electric field in the plot, which emphasises an important point. The charge separation set up by the disparity between electron and ion Larmor radii in a magnetised plasma is a manifestation of the Hall electric field. We cannot model charge separation within a single fluid MHD code, but we can include its macroscopic consequence by including the Hall electric field. If the green (pressure gradient) and red (Hall term) lines are summed, then they add closely to the electric field. This is shown in figure 5.7. This plot also illustrates that the dominant terms contributing to the electric field are the pressure gradient and the Hall term. This reinforces the theoretical analysis we did back in chapter 2, where we discussed the terms that are likely to be dominant at a plasma vacuum interface. At “first order”, the most important terms to include for a more physical model of this region are the Hall effect and electron pressure terms.

It should be remarked that PIC does not solve the generalised Ohm's Law to obtain the electric field. The electric field is evolved using Maxwell's equations, with the charge and current densities calculated from moving the computational particles. The

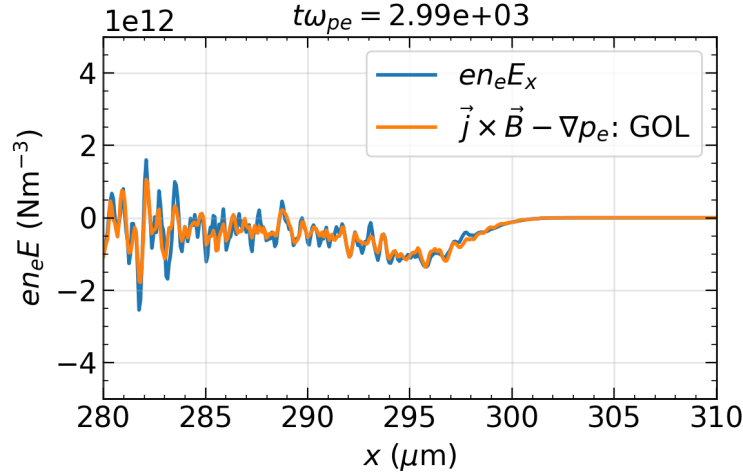


Figure 5.7: Plot comparing the electric field force density from a PIC simulation and the sum of the terms from the generalised Ohm’s Law. The agreement is good, showing that the Hall and electron pressure gradient dominate the electric field.

particles are only moved by the electric and magnetic fields through the Lorentz force $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. The GOL emerges from this microscopic physics on fluid scales. The fact that we can reasonably approximate the PIC electric field with the GOL validates our approach somewhat as it suggests that even in this vacuum-interface regime the GOL holds relatively true. This also suggests that it is still meaningful to explain the behaviour of the plasma in terms of fluid concepts such as the pressure gradient, while implicitly meaning that the movement of particles self-consistently sets up a charge separation and corresponding electric field that appears as an acceleration on fluid scales. Stated differently, the analysis in this section highlights that the GOL is still valid at the vacuum-plasma interface and adding terms to it is the most important “first order” effect.

5.3.3 Distribution Function

Figure 5.8 shows the distribution functions of the electrons and ions for the Bennett profile after a few ion plasma periods. The distribution is plotted in 2D space as a function of the x position and each of the velocity components. The other velocity directions are integrated out in each plot. The distribution function is indicated by the colour scale. It is normalised so that summing the distribution function across all cells gives the total number of real particles in the simulation. The electron velocities are larger than the ion velocities due to the higher electron thermal velocity. We find that the qualitative conclusions about the distribution function in this section are also true for simulations where a parabolic density profile was used instead of the Bennett profile shown here.

Initially, the distribution function is Maxwellian and uniform in each velocity dimen-

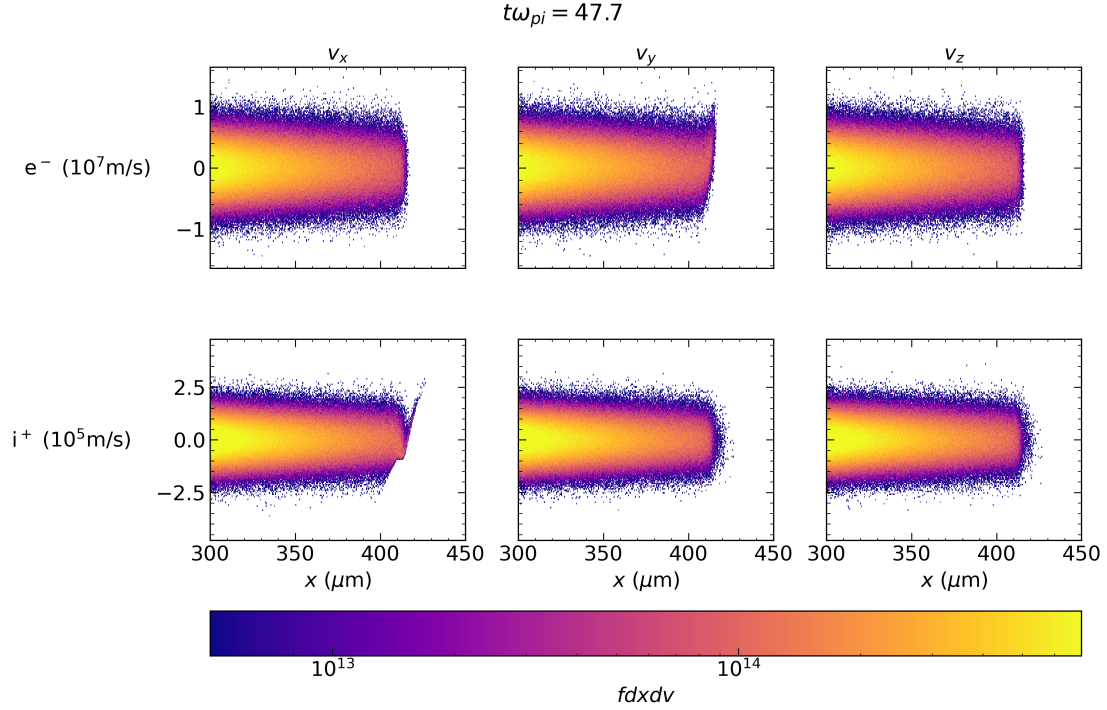


Figure 5.8: Figure showing the distribution functions for the electron and ion populations from an EPOCH simulation using the Cartesian Bennett pinch. In each plot, the distribution is shown as a function of the spatial coordinate x and a velocity coordinate. All other velocity coordinates are integrated out. The plot is shown at $t\omega_{pi} = 47.7$, using the same reference ion plasma frequency from table 5.1.

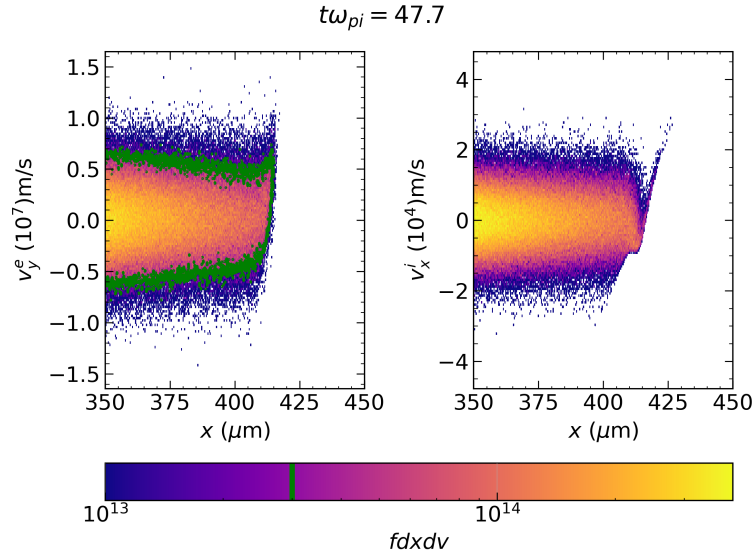


Figure 5.9: Figure showing the same distribution functions as in figure 5.8, but focusing on the anisotropic components. The electron $x - v_y^e$ distribution and the ion $x - v_x^i$ distribution are shown. The contour of 3×10^{13} is drawn in green on the electron distribution function to show the drift in the vacuum region. The ion distribution has a notch and low density population of faster ions.

sion. This can still be seen in the v_z components for both the electrons and the ions. The expansion of ions in the vacuum is visible from the slightly extended distribution compared to the electrons. There is a shift in the mean value of the $x - v_y$ distribution due to the drift velocity that provides the current for the Bennett profile. However, for the parameters that we have used, the electron thermal velocity v_{th} is much larger than the drift velocity v_{ey} , and so this is not easily visible. By $t\omega_{pi} = 48$, the electrons have expanded outwards from their initial profile and have been confined by the magnetic field. The charge separation electric field has begun to accelerate the ions. This can be seen from the slant of the $x - v_x$ profile for the ions, showing ions at the surface of the pinch are being accelerated. There is also a slope in the $x - v_y$ profile for the electrons. These distributions are shown in more detail in figure 5.9. Initially, the electrons have a constant drift v_{ey} in the $\hat{y}$ direction. As they expand outwards into the vacuum, a low density layer is formed (as seen earlier in figure 5.3). In this region, $\nabla \times \mathbf{B}$ is not equal to $\mu_0 \mathbf{j}$ as the electron density is lower and so the current density is also lower for the same drift velocity. This causes a finite displacement current which accelerates the electrons in the outer layer so that the current matches $\nabla \times \mathbf{B}$.

It can be seen that the ion distribution function in the charge separation layer (from about $400\mu\text{m}$) has some anisotropy. There is an asymmetry in the $x - v_x$ component due to the acceleration of ions by the charge separation electric field. In figure 5.10, the $x - p_x$ ion distribution function is compared against an equivalent Maxwellian distribution. This distribution is given by:

$$f_i(x, v_x) = \frac{n_i(x)}{\sqrt{\frac{2\pi T_i(x)}{m_i}}} \exp\left(-\frac{m}{2T_i(x)}(v_x - u_x(x))^2\right), \quad (5.13)$$

where v_x refers to the coordinate in phase space and u_x is the mean velocity in the x direction of all particles at that x position. The momentum is given by $p_x = m_i v_x$. When plotted, the values for (n_i, T_i, u_x) are taken directly from the PIC simulation. Lineout a) shows that at $x \approx 120\mu\text{m}$, the distribution function from EPOCH is well approximated by a Maxwellian. This is unsurprising as at this part of the plasma, the densities are high and the plasma can be modelled as a fluid. Due to the finite particle numbers, there is poor resolution at the tails of the distribution. The apparent narrowing of the distribution with x is a numerical effect of having fewer particles at larger values of x due to the density gradient. At lower density, the peak magnitude of $f dx dv$ is lower as $f \propto n$. However, there are also fewer computational particles at this point. As all superparticles have the same weighting in phase space, this means we have fewer superparticles in the tails at lower densities and so the distribution function appears to narrow. In the edge region, where there is finite charge separation, the EPOCH distribution function differs from Maxwellian. This can be seen at $x = 400\mu\text{m}$ in the colourmaps in figure 5.10 and in lineout b). There is a “notch” in the distribution due to the acceleration of ions out into the vacuum depleting the distribution of particles at negative p_x . It can be seen that there is an extended low density tail of fast particles

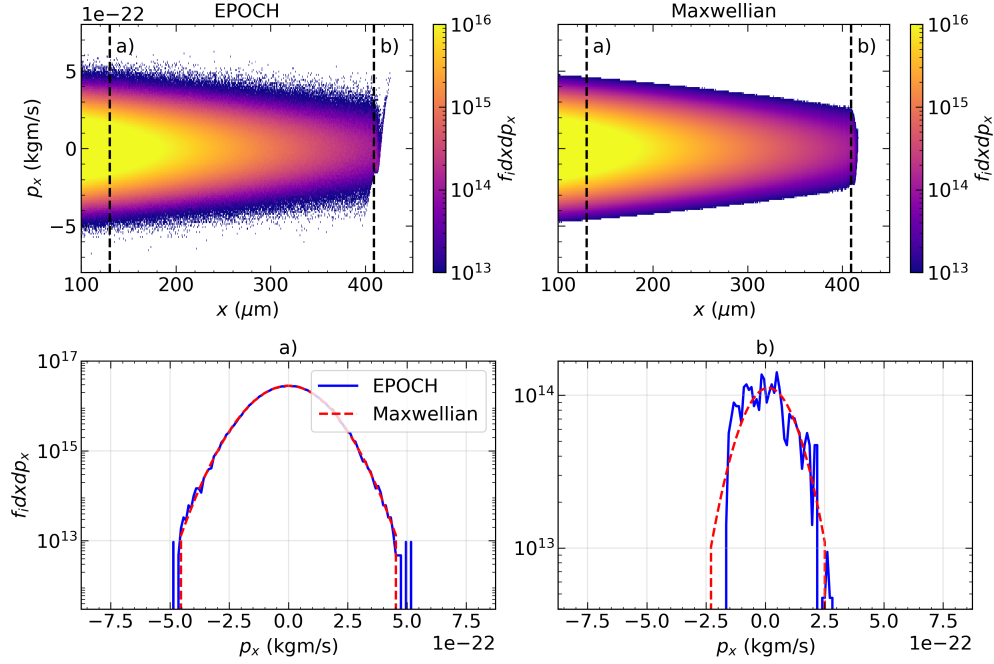


Figure 5.10: Plot comparing the distribution function taken from EPOCH to an equivalent Maxwellian. In the top row, the distribution functions are shown in $x - p_x$ space. Two lineouts at particular x positions are taken, marked a) and b). The comparison of the EPOCH distribution function along these lineouts to the Maxwellian is shown in the bottom row.

for the same reason. The EPOCH distribution is significantly noisier than it was in lineout a), due to the worse particle statistics. However, the qualitative behaviour of the distribution function remains the same if more particles are used in the simulation or if averaged over a different time window.

An issue with trying to extract the distribution function from a PIC code is that due to the density gradient, we have the worst particle statistics in the region that we are trying to study. In addition to the fewer particles in the tails, this makes it difficult to resolve how non-Maxwellian the distribution function is in the low density region. The simulation shown in figure 5.10 used 5×10^7 particles and needed to run for 96 hours on 256 cores on the ARCHER2 UK National Supercomputing Service. Extracting a smooth distribution would require significant computational resources, which is the reason continuum methods are sometimes preferred.

In figure 5.11, we show how the ion distribution function changes over time in the simulation. Due to the large numbers of particles required to resolve the distribution function well, it is not practical to run for more than a few ion cyclotron periods. Fewer particles were used in the simulation shown in these plots since with the number of particles used in figure 5.8 and figure 5.10, the runtime was excessive. The first row of figure 5.11, corresponding to a time of $t\omega_{ci} = 0.1$, is at the same time as in figure 5.10. There is an asymmetry in the $x - p_x$ plot due to the expansion of the ions into the vacuum region, though this is hardly visible due to the worse particle statistics.

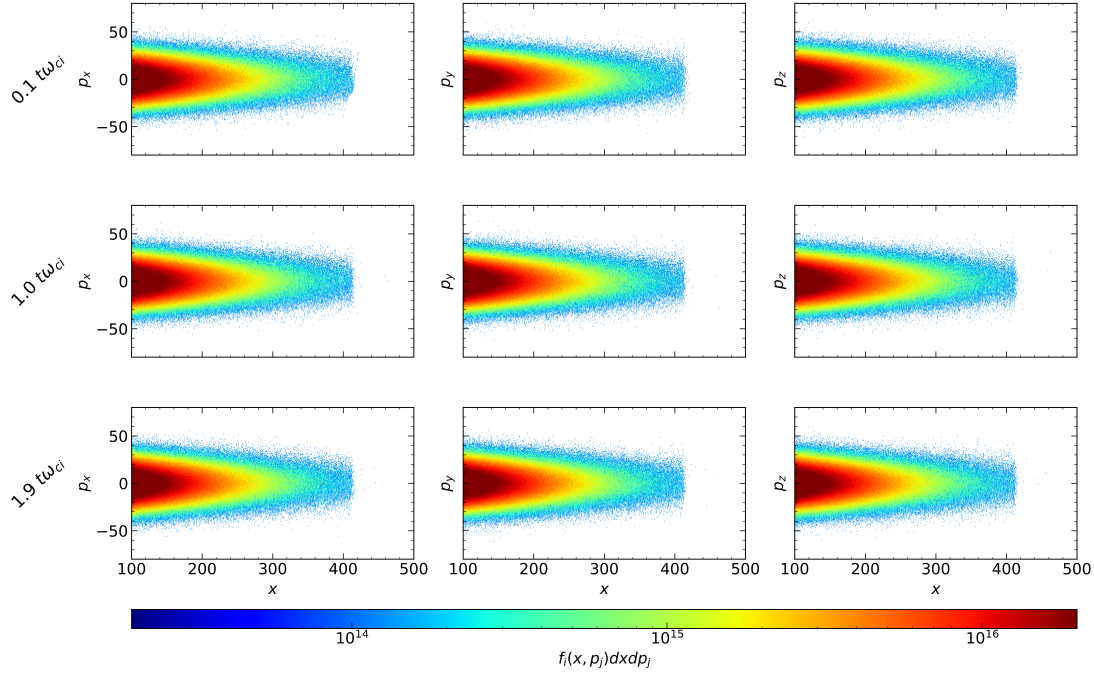


Figure 5.11: Plot showing the distribution function of the ions from an EPOCH simulation over time. In each plot, all directions that are not shown are integrated over. The distribution function is isotropic and does not change significantly over time.

By $t\omega_{ci} = 1$, this asymmetry is not present and all three components ($x - p_x$, $x - p_y$, $x - p_z$) are similar to each other and the distribution function is Maxwellian. The non-Maxwellian behaviour discussed for the previous figure is a transient effect driven by the acceleration of the ion population. After the fastest ions have escaped the domain and a stable charge separation layer is set up, the distribution function is Maxwellian (at least to the degree that it can be resolved in these PIC simulations). This is shown by the distribution function at $t\omega_{ci} = 2$ being virtually unchanged from $t\omega_{ci} = 1$. The same is true of the electron population, as should be expected as the electron mean free path is shorter than that of the ions and so it is unlikely they would be kinetic when the ions are not.

Part of the reasoning for using a PIC code rather than a VFP code was that the continuum method often decomposes the distribution function as spherical harmonics rather than discretising the velocity space. This can mean that large numbers of moments are needed for anisotropic distributions making VFP simulations expensive. We aimed to use PIC to assess whether this potential barrier to VFP simulations existed. Whilst there appears to be transient anisotropy when the charge separation layer is forming, at late times the distribution function appears to be isotropic and Maxwellian. This suggests VFP could be used. However, interpreting the distribution function is difficult due to the noise due to using a finite number of particles. With more particles, anisotropy might emerge and further study is needed.

5.3.4 Evolution of an MHD Equilibrium

It is important to emphasise how the evolution of the vacuum-plasma interface differed in the PIC simulation compared to the MHD prediction.

The simulation was initialised with an MHD equilibrium, defined by pressure balance: $\nabla p = \mathbf{j} \times \mathbf{B}$. As there is no charge separation in single-fluid MHD, initially $n_e(x) = n_i(x)$ everywhere. The current density is set so as to ensure there is no displacement current: $\mathbf{j} = \frac{\nabla \times \mathbf{B}}{\mu_0}$. We start with no electric field. As we do not perturb the interface, the MHD solution predicts that the profiles should remain static.

However, in reality, the electrons right at the interface move out into the vacuum region. Their motion sets up an electric field, which is initially dominated by the electron pressure gradient. The ensuing charge separation between the electrons and initially static ions slows the electrons (on a timescale on the order of the plasma period ω_{pe}^{-1}). They are confined in Larmor orbits by the magnetic field on a timescale of the order of the electron cyclotron period Ω_{ce}^{-1} . The ions are accelerated out into the vacuum by the charge separation electric field and they overshoot the electrons due to their higher mass. When the ions are eventually confined by the magnetic field, their large Larmor radii compared to the electrons means a charge separation layer with lengthscale on the order of difference between the ion and electron Larmor radius forms over a few ion cyclotron periods. This charge separation is a manifestation of the Hall electric field. During the formation of this charge separation layer, the ion distribution function is anisotropic and non-Maxwellian.

The system eventually reaches a steady state, with the dense plasma region unchanged from the initial MHD equilibrium. However, a low density charge separation layer with electric field dominated by the Hall term $\mathbf{E} = \frac{\mathbf{j} \times \mathbf{B}}{en_e}$ has formed. There is no longer the sharp cut-off between dense plasma and vacuum predicted by MHD. The distribution functions of the ions and electrons are Maxwellian and isotropic (at least to the extent that can be resolved in PIC). The plasma is still in pressure balance, but the equilibrium profiles differ from the initial single-fluid MHD solution in the lower densities (below $n \approx 5 \times 10^{22} \text{ m}^{-3}$ in our problem). This is due to the original motion of particles at the vacuum-plasma interface. The charge separation effect becomes important at even lower densities ($n \approx 10^{21} \text{ m}^{-3}$). Importantly, in this steady state solution, the electric field is well approximated by the generalised Ohm's law and $\mathbf{j} = \nabla \times \mathbf{B}$ everywhere, so there are no macroscopic displacement currents that would preclude an MHD solution. This result implies that Hall MHD is a meaningful limit and is crucial for understanding lower density plasma at a vacuum-plasma interface.

We have not considered kinetic effects within this argument, which we may expect to become important for the ions given their long mean free path. However, these effects cannot be included with an MHD code which is ultimately the aim of this work. Further study of these kinetic effects would be valuable future work. It is also worth remarking that the quality of the Hall MHD approximation decreases in the lower densities where charge separation effects become important. In other words, whilst Hall MHD allows

for more physical modelling of lower density plasma (as compared to resistive MHD), other beyond-MHD effects become dominant at some point. However, within the scope of this thesis, adding the Hall term to an MHD code is an important step to more physical modelling of vacuum-plasma interfaces.

5.4 2D results: the evolution of a Cartesian $m = 0$ mode

In this section, we study the evolution of a vacuum-plasma interface in 2D. To do this, we use the example of a $m=0$ “sausage” mode in a z-pinch. This is because ultimately we are interested in the breakdown of MHD at the edge of a z-pinch and this instability is dominant for z-pinches. We discuss this problem in significantly more detail when simulating these modes with Gorgon in chapter 6.

We are still restricted to Cartesian geometry, so we can only meaningfully study the edge region of the pinch. We model the pinch in 2D as if we were studying it in the $r - z$ domain. In Cartesian EPOCH, the radial direction is x and the axial direction is y . We assume that initially the temperature and magnetic field are uniform in the y -direction, with the same profile in the x direction as for the one dimensional case. We apply a small perturbation to the density profile, so the initial electron and ion number densities are both given by:

$$n(x, y) = n_0 \left[1 - \left(\frac{x}{a} \right)^2 \right] \left[1 + \Delta \sin \left(\frac{m\pi y}{L_y} \right) \right], \quad (5.14)$$

where n_0 is the peak density (here 10^{25}m^{-3}), a is the pinch radius ($300 \mu\text{m}$), $\Delta = 0.2$ is the initial perturbation size, $m = 5$ is the mode number and $L_y = 300 \mu\text{m}$ is the extent of the pinch in the axial y direction. This profile is still parabolic in the radial x direction, but now has an sinusoidal axial perturbation. This axial perturbation in the density leads to the formation of the sausage instability. The evolution of this instability is well understood in MHD, we intend to see how this differs in a PIC simulation. We are limited by computational constraints in 2D as to the number of particles we can afford in a simulation, meaning that our profiles are significantly noisier than in 1D. We have checked that increasing the number of particles does not qualitatively change the physics conclusions drawn in this section.

The evolution of the profiles in the x direction is similar to the one dimensional case, with the electrons initially accelerating out into the vacuum and then being overtaken by the ions. A similar electric field dominated by the Hall effect is established. However, there are non-trivial dynamics in the axial direction. Figure 5.12 shows the magnetic field and ion density over time at a few early time snapshots. The electron density has a similar evolution to the ions. Initially, the density is perturbed whilst the magnetic field is uniform in the axial direction. In the first set of panels, after a tenth of an ion cyclotron period, noticeable lobes have formed in the magnetic field. This is because particles move down the pressure gradient created by the density perturbation, away

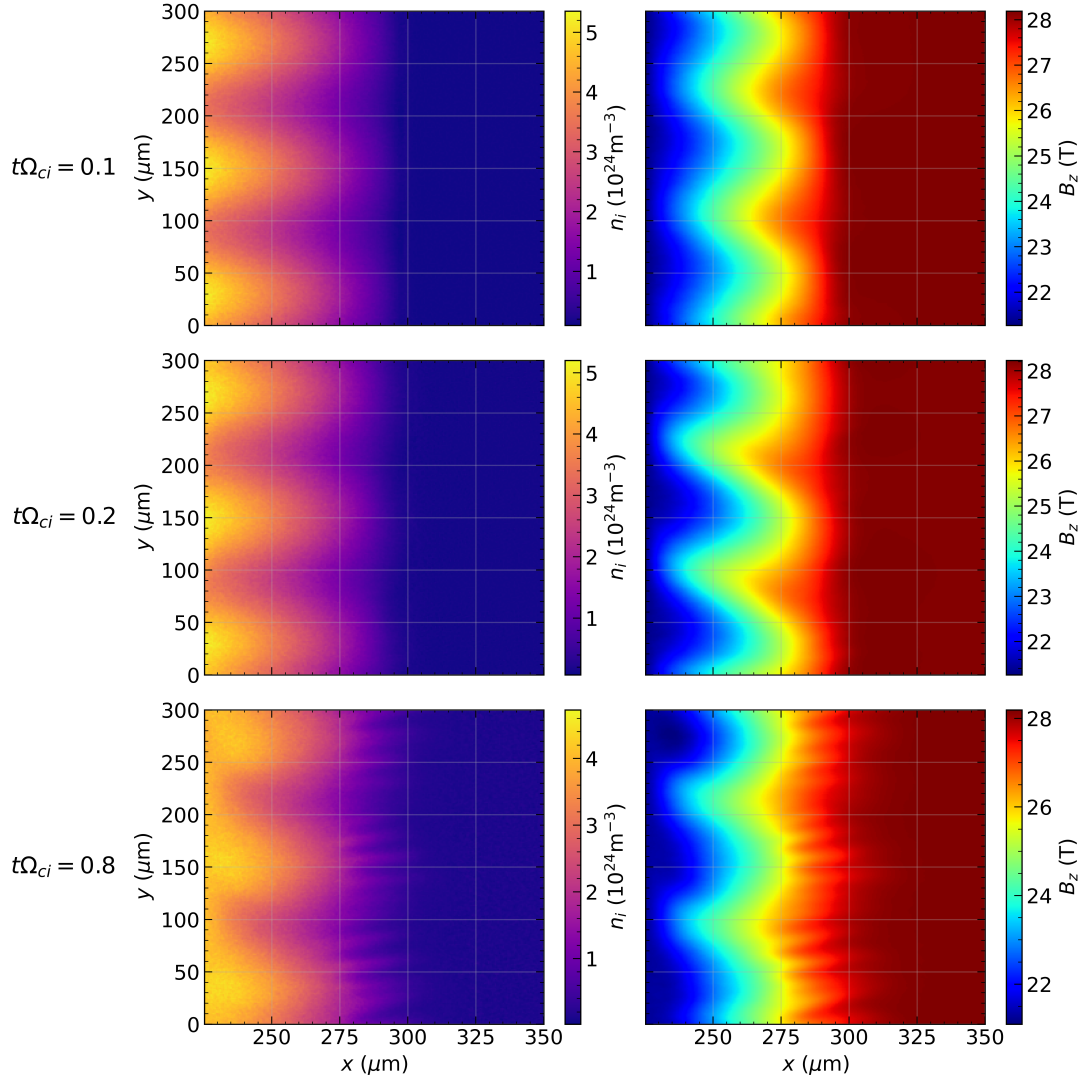


Figure 5.12: Plots showing the ion number density and magnetic field from the PIC simulation at early time. The reference Ω_{ci} used here is $2.7 \times 10^9 \text{s}^{-1}$. The current has a constant value in the negative y direction and the electron flow is in the positive x direction. The lobes in magnetic field bend downwards in the direction of current flow.

from the high density regions. This causes magnetic field to move with this plasma by the frozen-in-flow effect. This means the magnetic field is reduced where the density is higher and vice versa. There is some expansion of the ions into the vacuum region that we demonstrated while discussing the one dimensional behaviour, but the density profile does not change significantly on this timescale. In the second set of panels, at $t\Omega_{ci} = 0.2$, the density profile is still quite similar, but the magnetic field has visibly begun to shear downwards, which is the direction of the current density $\mathbf{j}$ for this setup. This is also in the direction of $\mathbf{B} \times \nabla n$, which indicates that the shearing is due to a Hall drift wave [186]. This is a characteristic wave mode of the Hall effect that arises in inhomogeneous plasmas; we discuss it in detail in chapter 6. Finally, in the last row of figure 5.12, the density itself begins to shear downwards. This is a consequence of the magnetic field shearing, but it takes $t \sim \mathcal{O}(\Omega_{ci}^{-1})$ for the ions to respond to the changes in the magnetic field so the evolution of the density profile occurs on a different timescale. At the very edge of the pinch, secondary instabilities have begun to form. This is more obvious in the field profile. This comes about due to small perturbations in the number of particles per cell leading to variations in the density, which then seeds instabilities. In section 6.6, we carry out Hall MHD simulations of the evolution of an $m=0$ mode using Gorgon. The qualitative evolution of the instability is extremely similar to the results of the PIC simulations in this chapter. This indicates that the Hall effect has a crucial impact on the evolution of this instability.

In figure 5.13, we show the magnetic field obtained from a Gorgon simulation of the same problem as has been modelled using PIC in this section. The case where the Hall effect is included in the calculation is compared with the case where the Hall effect is not included. Resistivity was also turned off in the Gorgon simulations so as to better match the PIC simulations which did not have collisions. The implementation of the Hall effect within Gorgon is that discussed in the next chapter. For stability reasons when the Hall effect is included, Gorgon is not able to simulate as large a density gradient as EPOCH, so the gradient is reduced and thus the magnetic fields are not as high in Gorgon as in the EPOCH simulation. Contours of the magnetic field are drawn on in black. In both cases, lobes form in the magnetic field. In the left hand “no Hall” case, the lobes in magnetic field do not shear. In contrast, in the case where the Hall effect is included in the calculation, the lobes shear downwards as is also seen in the EPOCH simulation. This illustrates a few important points. Firstly, very similar qualitative behaviour is observed in the field structure in both the Gorgon Hall case and in the EPOCH simulations. The only difference between the “no Hall” and “Hall” simulations is the Hall term. This therefore validates our assertion that it is the Hall effect that is driving this shearing behaviour. Secondly, the fact that we are able to qualitatively reproduce the results of the PIC simulation is an encouraging result for the new Hall solver that makes up the work in chapter 6. Our approach in this chapter has been to find which terms can be added to an MHD code to more physically model a vacuum-plasma interface. This comparison indicates that this approach is possible and that the Hall term is a crucial “first-order” term. Finally, in section 5.3, we discussed

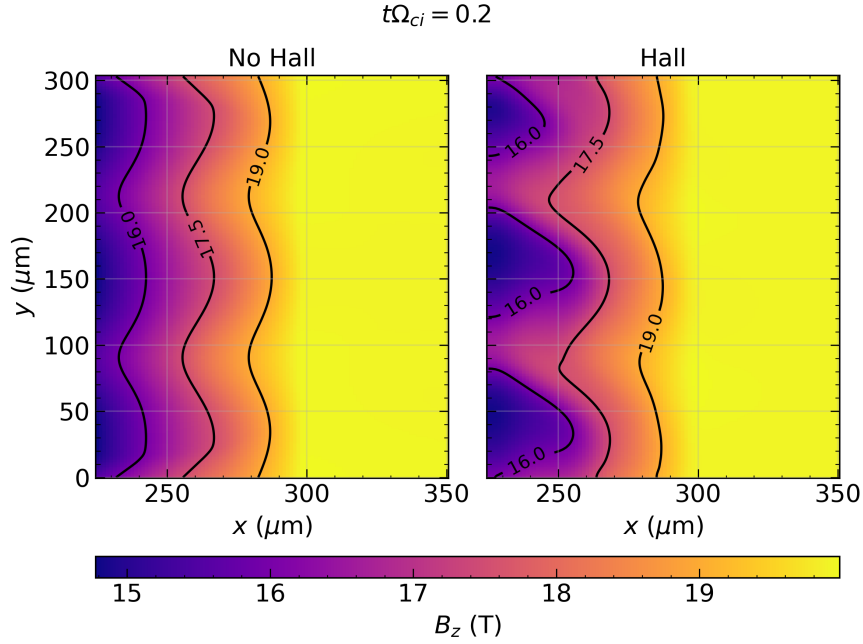


Figure 5.13: Plots showing the B_z magnetic field from a Gorgon simulation of the axial z-pinch problem. The case where the Hall effect is included in the calculation is compared to the case where it is not. Contours of particular magnetic field value are added to highlight the field structure.

the microscopic picture of the Hall effect as the charge separation electric field due to the difference between electron and ion Larmor radii. The similarity between figure 5.13 and figure 5.12 demonstrates that PIC is also able to recreate the macroscopic fluid-type behaviour of the Hall term as well.

We are limited by computational constraints in this problem however. In order to model the density gradient, we need a large number of particles. In this problem, 3×10^7 particles were used and we only resolve 1.5 orders of magnitude in density well. The evolution of the ions and thus the instability occurs on the timescale of the ion cyclotron period. With the simulation also being two dimensional, this means that these axial simulations are computationally expensive and we cannot afford to run them for more than a few Ω_{ci}^{-1} . More importantly, the geometry of this problem is challenging. We are approximating the edge of a cylinder using Cartesian geometry and modelling the inner part of the pinch using a reflective boundary condition. This is only a reasonable approximation for a region right at the edge of the cylinder. In the problem we have modelled in this section, the pinch will implode at the necks due to the instability. After a few ion cyclotron periods, we start to see an unphysical build up of density at the left hand side of our domain due to our resistive boundary condition. For a more realistic simulation at later times, we need to model more of the cylinder but then the 2D Cartesian approximation breaks down. A three dimensional Cartesian simulation would not be tractable. Therefore, we are restricted to only studying the early stages of this instability using PIC. A cylindrical version of EPOCH is in development. An

avenue of future work could be to redo these simulations in a 2D $r - z$ plane to study the evolution of this instability into more relevant fluid timescales.

5.5 Conclusions

5.5.1 Challenges of a particle-in-cell code

Particle-in-cell codes are the standard tool for simulating kinetic plasma physics due to their wide range of applicability [2]. However, they are more frequently used in situations involving laser plasmas or low density collisionless plasmas. As we highlighted in section 5.1.2, there has been limited use of PIC codes to study vacuum-plasma interfaces. In this chapter, we have shown how it is possible to extract some information about the impact of extended MHD terms from a PIC simulation, but the work is limited. In this section, we discuss those limitations. We conclude by highlighting the ideal features of a code for studying this regime.

Most of the challenges with using PIC for this problem arise due to the large density gradient. In the high density region, the Debye length is small and the plasma period is short. Resolving the Debye length therefore necessitates small cells, which leads to short timesteps from the CFL condition. The large numbers of particles in the high density region also makes including collisions computationally expensive. At the same time, the density gradient means that a vast number of particles are needed if the low density region is to be resolved well. This is exacerbated by requiring all of our particles to have a uniform computational weight. However, we found using variable computational weights introduced unphysical behaviour into simulations. This means that most of the computational resources are spent on simulating the high density part of the simulation when it is the low density that is of interest. The high number of particles to simulate the density gradient makes 3D simulations prohibitively expensive.

The other issue is the separation of time scales, although this is not a problem limited to PIC. The shortest physical time scale is the electron plasma period. However, to see macroscopic evolution, the simulation needs to run for many ion cyclotron periods. For our parameters, there are $\sim 10^4$ plasma periods for every cyclotron period. Thus simulations are relatively inefficient and take a long time to run. It is possible to decrease the cyclotron period by increasing the magnetic field, but this also means increasing the peak density (due to pressure balance), which worsens the issues discussed in the previous paragraph.

5.5.2 Ideal modelling tool for this problem

The issues with timescale separation could potentially be mitigated with an implicit PIC code, such as LSP or Chicago. This would allow for stepping over the faster electron timescales. However, in taking larger timesteps than these scales, the result could be inaccurate. Given that we are particularly interested in charge separation at the vacuum-plasma interface and that this process is governed by electron scales, care

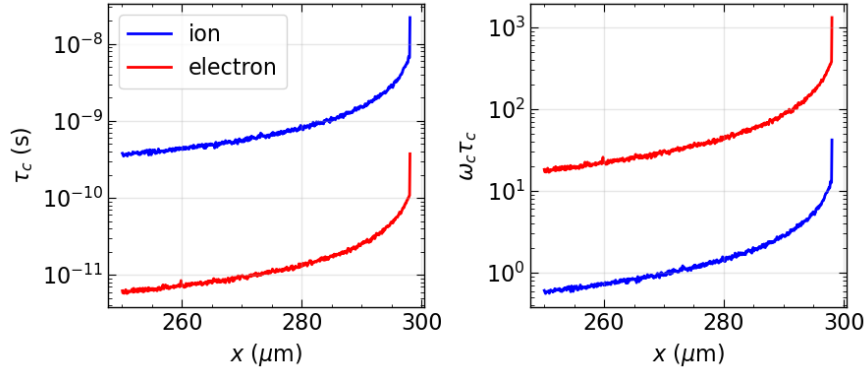


Figure 5.14: Plot showing the electron and ion collision times and Hall parameters for the 1D initial conditions in section 5.3. The electrons are shown in red and the ions are shown in blue.

would need to be taken that the implicit code was converged with timestep. Another potential issue is the high densities. Implicit PIC codes can display numerical cooling [187], particularly if Monte Carlo collisions are included [188]. New numerical schemes to handle this are only just emerging in the literature, so a standard implicit PIC code may not be sufficient.

By calculating the collision frequencies, it can be assumed that the electrons are collisional and ions are nearly collisionless. This is shown in figure 5.14 which shows the electron collision time τ_e , the ion collision time τ_i and the electron and ion Hall parameters χ_e and χ_i for the 1D results. Both the electrons and ions are magnetised at the edge of the pinch. The ion collision time is almost 2 orders of magnitude higher than the electron collision time however. This suggests that the ions are likely kinetic while the electrons are closer to fluid-like. This indicates that a hybrid PIC model, where the ions are treated kinetically and electrons are treated as a fluid, could be successful at simulating this regime. Whilst this would not improve the electron timescales at all, it would substantially reduce the computational overhead of simulations.

A different approach to handle the density gradient would be a continuum kinetic code. This would have the advantage of resolving the distribution function well, even in the low densities, without potentially introducing unphysical effects due to the noise of finite particles. For our parameter regime, it could be possible to only model the ions kinetically and assume a fluid solution for the electrons. The downside of this approach is that there are very few publically available continuum kinetic codes and none of them include kinetic ions, to our knowledge. The iFP code used to model dense ICF experiments would be ideal [152, 189], but it is a restricted access national lab code. The solution would likely be computationally expensive as well due to having to discretise the entirety of phase space. A spherical harmonic type expansion like that used in many electron VFP codes could be an option, but it would need to be assessed whether the ion distribution function is anisotropic [92]. This was an aim of our work in this chapter, but it was challenging to resolve the ion distribution function well enough.

A completely different approach could be a spatially coupled PIC scheme. The dense part of the z-pinch is collisional enough that a fluid solution is appropriate, whereas we know that the edge region requires a more complete description of the physics. A code that solves the fluid equations in some inner part of the radius, whilst solving a PIC solution at some outer part, would be able to describe both regimes. These two solutions could be coupled by means of boundary conditions. This has been done for astrophysical plasma simulations [190, 191] and an attempt made in the past for HEDP type plasmas [68]. However, the method is new and largely untested so it is not clear whether it would be effective for this problem.

However, for practical simulation of vacuum-plasma interfaces in experimental scenarios, kinetic methods are too computationally expensive to be used routinely. Whilst there are many interesting questions that can be answered using kinetic codes, and they are the only appropriate tool for certain cases, experimental modelling can only really be done with MHD. In this chapter, we have highlighted that a crucial extended MHD term for understanding the vacuum-plasma interface is the Hall term. In the next chapter, we develop a new numerical MHD method for implementing the Hall term within Gorgon and use it to study z-pinches.

6 A new implementation of the Hall effect in Gorgon

In chapter 2, we discussed the physics beyond MHD that becomes relevant at a vacuum-plasma interface. Through a basic qualitative argument, it was concluded that the most important terms to account for were the Hall effect and anomalous resistivity terms. The results in chapter 5 also suggest that the Hall term has a critical effect on the low density plasma at the edge of a z-pinch. However, it also emphasised that it is not practical to use PIC codes to model this beyond MHD behaviour. In order to interpret experimental results from machines such as Magpie, as well as design future platforms, a less computationally expensive fluid model is necessary. In this chapter, we discuss a new numerical method for implementing the Hall effect within an extended MHD code such as Gorgon.

The Hall effect is notorious for the numerical challenges it poses in fluid codes and we will highlight these, as well as discuss other approaches past work has taken to mitigate these difficulties. We describe in detail the approach that has been taken and show how it deals with these challenges. We then present a comprehensive suite of test problems to validate the code, as well as to highlight some of the unusual behaviour of the Hall term. We conclude by showing simulations of an equilibrium z-pinch with an $m = 0$ instability, discussing how the Hall term alters the growth of the instability qualitatively and quantitatively. We conclude by comparing the effect of the Hall term and electron inertia on the growth rate of the instability.

6.1 Motivation

6.1.1 Numerical challenges due to the Hall term

The Hall effect is notorious for the numerical problems that it introduces when included in fluid codes. This is mostly due to the timestepping requirements it imposes.

In explicit codes, we have to ensure that the timestep is set so that no information can cross more than 1 cell during each timestep. This means ensuring that we resolve the velocity of all wave modes.

The Hall term however introduces new modes to the MHD system. This can be seen from the induction equation for Hall MHD:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{u} \times \mathbf{B} - \frac{\mathbf{j}}{en_e} \times \mathbf{B} \right) - \nabla \times \left(\frac{\eta}{\mu_0} \nabla \times \mathbf{B} \right). \quad (6.1)$$

Ignoring the $\mathbf{u} \times \mathbf{B}$ and resistive contributions, we can use the identity $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) + (\nabla f) \times \mathbf{A}$ to rewrite this as [107]:

$$\frac{\partial \mathbf{B}}{\partial t} = -\frac{1}{en_e} \nabla \times (\mathbf{j} \times \mathbf{B}) + \frac{1}{en_e^2} \nabla n_e \times (\mathbf{j} \times \mathbf{B}). \quad (6.2)$$

This makes it clear that there are two modes associated with the Hall effect. The first term on the right hand side leads to the whistler mode, which is an electromagnetic wave that travels along the magnetic field. The second term is due to a finite density gradient in the plasma and leads to the Hall drift wave, which travels perpendicular to both the magnetic field and the density gradient. These are discussed in more depth and their dispersion relations derived when validating the new solver in section 6.5. These waves introduce new velocities that need to be resolved given by:

$$v_w = \frac{kv_A^2}{\Omega_i} \quad (6.3)$$

$$v_{HDW} = v_A \frac{\lambda_i}{L_n}, \quad (6.4)$$

where v_A is the Alfvén velocity, Ω_i is the ion cyclotron frequency, λ_i is the ion inertial length, $L_n = \left| \frac{\partial \ln n}{\partial x} \right|^{-1}$ is the density gradient length scale and k is the wavenumber [107, 112, 113]. v_{HDW} is the phase velocity of the Hall drift wave and v_w is that of the whistler wave. These can both be highly punitive for a simulation of a z-pinch. In both cases, the wave velocities regularly exceed the Alfvén velocity v_A , predicating smaller timesteps than a normal ideal MHD simulation [112, 113]. The speed of the Hall drift wave depends on the steepness of the density gradient. We showed in chapter 2 that L_n becomes far smaller than the ion inertial length λ_i at the edge of a z-pinch, causing v_{HDW} to grow. For the whistler wave, we see that $v_w \propto k$. This makes the wave dispersive and means that the whistler waves with shortest wavelength travel fastest. Therefore, increasing grid resolution quadratically lowers the timestep. This is known to generate high k noise in Hall MHD simulations which severely restricts the timestep unless a method to mitigate the noise is used [113].

Additionally, including the Hall effect within a simulation reduces the size of the length scales that need to be resolved. To observe the Hall effect, the cell size Δx well resolves the ion inertial length scale λ_i [186]. This is particularly problematic for pulsed power simulations that tend to have large density gradients and thus a wide range of ion inertial lengths. Using a wire array simulation as an example, using a peak density of 10^{26}m^{-3} and resolving densities as low as 10^{22}m^{-3} , the ratio of the highest to the lowest ion inertial lengths is 100. Therefore, the cell size needed to resolve the ion inertial length at the edge of the simulation is two orders of magnitude smaller than is needed at the core of the simulation. This drastically increases the number of cells, unless mesh refinement is also used. In combination with the steep timestepping requirements that the Hall term introduces, requiring small cell sizes severely limits the simulations that can be run.

These factors mean that explicit approaches are computationally expensive unless some sort of mechanism is used to control the wave speed. Some of these are discussed in the next section. An alternative approach is implicit approaches, which in principle allow for a larger timestep. However, this means that the wavenumbers that are not resolved by this timestep are damped unphysically which can qualitatively impact predictions [107]. In addition, the spatial derivatives of magnetic field in the induction equation mean that a matrix with a size on the order of the number of grid cells needs to be inverted for a numerical solution. The dispersive nature of the whistler mode makes this matrix difficult to precondition and therefore inversion can be highly inefficient. This means that implicit methods are often not much more efficient than an explicit code for this problem [135].

6.1.2 Past work

Other works have implemented the Hall term within MHD simulations, via a range of methods. We review the most relevant of those methods in this section.

The Naval Research Laboratory Hall MHD code VooDoo is used to study plasma opening switches, space physics and other devices where Hall physics can be relevant [192]. The term is included in the Hall MHD limit, where the speed of light is assumed to be infinite and electrons are massless. It is an explicit finite volume code that resolves all of the timescales that we discussed in the previous section. The issue of short wavelength noise that the Hall can introduce is mitigated by artificially smoothing the electric field using a 3-point average before it is used to update the magnetic field by the induction equation [192]. VooDoo does not use a vacuum region in the same way as in Gorgon. This fully explicit method was deemed too expensive for practical pulsed power simulations.

The Hall effect is of interest in space physics as well, though the regime is very different. Generally, the Hall MHD limit is used. The density gradients are less severe, so the Hall drift wave is of less importance and the whistler mode is the main concern. O’Sullivan and Downes introduced a new fully explicit approach that showed that dimensionally splitting the Hall effect can improve stability [193, 194]. In essence, the B_x field is advanced by a full timestep using the original B_y and B_z fields. The B_y is then updated to the next timestep using the original B_z and the updated B_x . Finally, the B_z field is updated using the updated B_y and B_x . In each timestep, the order of the updates is rotated so as not to introduce a preferential direction. The authors find that this leads to a less restrictive timestep for the whistler wave [193]. This method has gone on to be used in the Athena MHD code for astrophysics [195].

There have been several attempts to implement the Hall term using implicit approaches. Sheehey *et al.* simulated the formation of fibre z-pinches using a two-temperature Hall MHD code [196]. An alternating direction implicit method was used to solve the Hall terms. Chacon *et al.* developed a different fully implicit model for the Hall effect, showing that a careful choice of pre-conditioner can reduce the inefficiency

of inverting the matrix [135]. Whilst both of these approaches are well tested, fully implicit approaches are far from ideal because they severely restrict the scalability of the code on parallel cores. We therefore seek methods that do not have this issue.

More recent work has shown that including additional physics terms can mitigate the stability requirements of Hall MHD. This is similar to how including the displacement current changes resistive MHD from a stiff diffusion problem to an advection problem at the speed of light, which scales better with grid size [136]. Srinivasan *et al.* showed that in many cases it is computationally more efficient to use a two-fluid model than a Hall MHD model [112, 113]. This is because reinstating a finite electron mass bounds the whistler wave speed at the electron cyclotron frequency. We show this is also true for our model when validating the Hall solver in section 6.5. A similar approach whilst still using single-fluid Hall MHD is to introduce an artificial hyper-resistivity into the problem. This changes the generalised Ohm’s law for Hall MHD to:

$$en_e \mathbf{E} = -\mathbf{u} \times \mathbf{B} + \mathbf{j} \times \mathbf{B} + \eta \mathbf{j} + \nu \nabla^2 \mathbf{j}, \quad (6.5)$$

where ν is a hyper-resistivity parameter, usually held constant in space and time. Similarly to the anomalous resistivity discussed earlier, this term arises in principle from microinstabilities driving turbulence in the current [197, 198]. In practice, this term functions to dissipate small scale structure in the electric field due to its higher spatial derivative [199]. The parameter ν is arbitrarily chosen to effectively “cut off” the simulation at a particular length scale so further resolution is not needed and noise does not cause the simulation to fail [112]. Most explicit approaches have to apply a hyper-resistivity in order to make simulations tractable [200, 201].

Of particular interest for this thesis is the work of Seyler *et al.* on the PERSEUS extended MHD code [114]. This model extends Hall MHD by including the displacement current in Ampère’s law, as well as electron inertia within the generalised Ohm’s law. By including these terms, the timestepping requirements of the Hall wave modes and resistive MHD are mitigated. A semi-implicit relaxation method is used in the PERSEUS code, where the hyperbolic terms within the update are included explicitly and the source terms are implicit. By including the electron inertia terms within the solution, the Hall and resistive MHD terms become source terms in the evolution of the current density. This means that the terms can be time integrated point implicitly, making them viable to include in a massively parallel code. It is shown in ref. [114] and ref. [63] that the update scheme is a relaxation scheme, where the update reduces to the normal Hall MHD limit in the case where the electron inertial length is under-resolved by the grid scale. The PERSEUS model has been used for several different applications within pulsed power modelling including wire array z-pinchs [114, 148], power flow [117], cylindrical foil implosions [64] and modelling of axial instabilities in the MagLIF experiment [63, 65]. This relaxation method is now implemented in Sandia National Laboratories’ modelling suite, in the FLEXO code. A similar model has been implemented by Angus *et al.* and used to some success [150, 202]. Due to the point

implicit method allowing for a more tractable Hall effect that can still be scaled to many processors, we decided to adapt this approach for our Hall solver. In the remainder of this chapter, we describe the implementation of the model and highlight the novelty of our approach compared to PERSEUS.

6.2 Governing equations

The governing equations for the plasma in our model are those of single-fluid MHD, while the electromagnetic fields evolve by Maxwell's equations. It is assumed there is no charge separation, so $\nabla \cdot \mathbf{j} = 0$. As discussed before, this is valid for length scales $L \gg \lambda_D$, which is always true of the length scales that we are able to resolve. Finally, we retain the displacement current contribution in Ampère's law as it has beneficial effects on the timestep and is important for coupling the vacuum region to the plasma. This means that the governing equations for the electric and magnetic field are:

$$\frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \mu_0 \mathbf{j} \quad (6.6)$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}. \quad (6.7)$$

The effect of the evolution of the plasma on the electromagnetic fields is captured by the generalised Ohm's law, which was derived in chapter 2. It is given by:

$$\frac{m_e}{e^2 n_e} \left(\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot \left[\mathbf{u} \mathbf{j} + \mathbf{j} \mathbf{u} - \frac{1}{en_e} \mathbf{j} \mathbf{j} \right] \right) = \mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{1}{en_e} \mathbf{j} \times \mathbf{B} - \eta \mathbf{j} + \frac{1}{en_e} \nabla p_e. \quad (6.8)$$

Past work has found that the hyperbolic terms on the left hand side of this equation are not relevant for most pulsed power simulations [65, 117], so they are neglected for the results in this chapter. The ∇p_e term has also not yet been implemented, though we would expect it to have an effect that further work should quantify. It would be straightforward to add these terms to the model at some later point. This means that in the new solver, the evolution of the current density is given by:

$$\frac{\partial \mathbf{j}}{\partial t} = \frac{e^2 n_e}{m_e} \left[\mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{1}{en_e} (\mathbf{j} \times \mathbf{B}) - \eta \mathbf{j} \right]. \quad (6.9)$$

6.3 Numerical Model

In this section, we discuss the numerical model developed to solve the system of equations (6.6), (6.7) and (6.9). Due to the modular nature of Gorgon, this update to the MHD field solver can be considered separately to the various algorithms that update the hydrodynamics, which are unchanged. As generally the MHD phenomena happen on a shorter scale than the hydrodynamic phenomena, we assume all of the hydrodynamic variables ($n_{e,i}$, $T_{e,i}$, $\mathbf{u}_i$) are constant during the MHD update.

The new solver uses the same basic MHD model as Gorgon, described in section 3.3.1,

where the magnetic field is calculated from the vector potential using $\mathbf{B} = \nabla \times \mathbf{A}$ and the variables lie on a staggered Yee mesh to ensure $\nabla \cdot \mathbf{B} = 0$. The method that we will outline below is similar to the PERSEUS extended MHD code [114], but the novelty in this work is applying this algorithm to the staggered grid. This means that we do not need to use any divergence cleaning methods.

Each of the governing equations can be split into a hyperbolic part, which is the spatial derivative of a flux term, and a source part. For example, for Ampère's Law:

$$\begin{aligned} \left(\frac{\partial \mathbf{E}}{\partial t} \right)_{\text{hyperbolic}} &= c^2 \nabla \times \mathbf{B} \\ \left(\frac{\partial \mathbf{E}}{\partial t} \right)_{\text{source}} &= -c^2 \mu_0 \mathbf{j}. \end{aligned} \quad (6.10)$$

The source terms introduce restrictive timestep requirements that are challenging to resolve [134, 135, 148, 193]. However, to avoid complications due to inverting a large matrix, an explicit scheme is used for the hyperbolic part of the equations. This ensures that the implicit source term update is local to each cell in the domain.

We use operator splitting to split the hyperbolic and source updates. We have found that it is important to carry out the full update of $\mathbf{E}$ and $\mathbf{j}$, including both the implicit and explicit steps, before $\mathbf{B}$ is evolved. Otherwise, numerical instabilities are introduced due to varying $\mathbf{B}$ with an electric field that is still to change during the timestep.

In this section, we summarise the new electron inertia MHD solution. A detailed discretisation of the scheme is given in Appendix A.1. The first stage of the solve is to explicitly update the electric fields using:

$$\mathbf{E}' = \mathbf{E}^n + c^2 \Delta t \nabla \times \mathbf{B}^n, \quad (6.11)$$

where $\mathbf{B}^n = \nabla \times \mathbf{A}^n$ and the superscript refers to the time level of the quantity $\mathbf{B}^n = \mathbf{B}(t = t_n)$. A source term update is then carried out for just the current and electric field, since the magnetic field has no non-hyperbolic terms. We use a backward Euler scheme for this step:

$$\begin{aligned} \frac{\mathbf{E}^{n+1} - \mathbf{E}'}{\Delta t} &= -c^2 \mu_0 \mathbf{j}^{n+1} \\ \frac{\mathbf{j}^{n+1} - \mathbf{j}^n}{\Delta t} &= \frac{e^2 n_e}{m_e} \left[\mathbf{E}^{n+1} + \mathbf{u} \times \mathbf{B}^n - \frac{1}{en_e} (\mathbf{j}^{n+1} \times \mathbf{B}^n) - \eta \mathbf{j}^{n+1} \right]. \end{aligned} \quad (6.12)$$

The backward Euler scheme ensures that this source term update is unconditionally stable, albeit inaccurate if the maximum stable timestep is under-resolved. We also tried an implicit midpoint scheme [131] for this stage, but found that it was unstable, possibly due to averaging on an explicit step. This update process can be rewritten as:

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{j} \end{pmatrix}^n = \underline{\underline{\mathbf{M}}} \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{j} \end{pmatrix}^{n+1}, \quad (6.13)$$

where $\underline{\mathbf{M}}$ is a constant 6×6 matrix. This matrix can be inverted analytically (see Appendix A.1) to obtain values for $\mathbf{E}^{n+1}$ and $\mathbf{j}^{n+1}$. Finally, these updated values are used to update the magnetic field via the vector potential:

$$\begin{aligned}\mathbf{A}^{n+1} &= \mathbf{A}^n - \Delta t \mathbf{E}^{n+1} \\ \mathbf{B}^{n+1} &= \nabla \times \mathbf{A}^{n+1}\end{aligned}\tag{6.14}$$

Through experimentation, it was found that an update of first order accuracy in time was not sufficient as it led to excessive numerical diffusion. Therefore, for second order accuracy in time the algorithm in this section is used in a form of predictor-corrector scheme [131]. $\mathbf{E}$ and $\mathbf{j}$ are calculated at an intermediary timestep $t_n + \Delta t/2$ using the known values of $\mathbf{E}$, $\mathbf{j}$ and $\mathbf{A}$ at timestep t_n (still following the same hyperbolic then source update process). These intermediary values $\mathbf{E}^{n+1/2}$ and $\mathbf{j}^{n+1/2}$ are then used to update $\mathbf{A}$ from timestep n to timestep $n+1$. Finally, this updated value of $\mathbf{A}^{n+1}$ is used to update the original values of $\mathbf{E}$ and $\mathbf{j}$ from timestep n to timestep $n+1$ and the intermediary values are discarded.

The existing MHD solver in Gorgon does not include the effect of electron inertia. For comparison to results using that solver, we apply a constant multiplier to the electron mass in equation (6.13) to reduce it to arbitrarily small values. We also developed an implicit Hall solver that does not include electron inertia, which is described in Appendix A.2, but the electron inertia solver was used for the results in this chapter. The two models give identical answers if the electron inertia is artificially reduced.

6.3.1 Staggered Grid

The staggered Yee mesh ensures that the magnetic field remains divergence free, but it also complicates the implicit update. The update shown in equation (6.13) is only valid if all components of $\mathbf{E}$, $\mathbf{j}$ and $\underline{\mathbf{M}}$ are defined at the same location. However, on the Yee grid each electric field/current component is defined on a different edge, each magnetic field component on a different face, and hydrodynamic variables such as n_e are defined in cell centres. In order to make this update possible, the source term update is performed 3 times in each cell: once in every electric field location; i.e. once on each edge of each grid cell. During each update, we interpolate all the quantities that are not defined in that electric field location to the edge that we are updating and then perform the implicit update there. Figure 6.1 shows this interpolation process. We then only keep the updated values of $\mathbf{E}$ and $\mathbf{j}$ that are on their native location. For example when updating at the E_x location, we interpolate $(n_e, \eta, B_x, B_y, B_z, E_y, E_z, j_y, j_z)^n$ to that location and then only keep the values of E_x^{n+1} and j_x^{n+1} after the update. This does not introduce $\nabla \cdot \mathbf{B}$ errors since we are still only ever updating $\mathbf{B}$ by $\nabla \times \mathbf{E}$ on the Yee mesh.

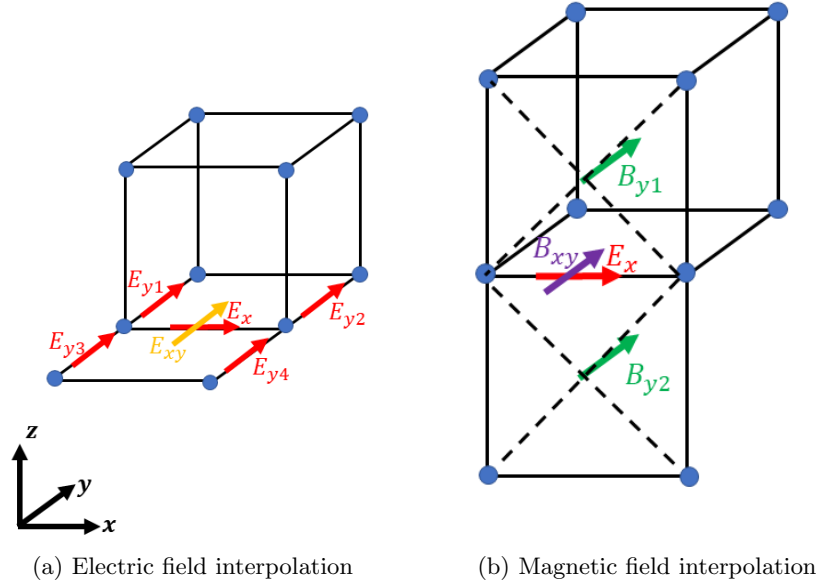


Figure 6.1: A diagram showing interpolation of variables to electric field locations for the implicit source term update. In 6.1a, E_y electric fields are interpolated from the y edge of the grid to the x edge to form the yellow interpolated variable. In 6.1b, B_y magnetic fields are interpolated from the y face to the x edge to form the purple interpolated magnetic fields. Similar interpolations are performed for cell-centred variables and at other electric field locations.

6.3.2 Vacuum regions

This interpolation scheme introduces a complication on vacuum-plasma interfaces. As discussed in chapter 3, in Gorgon the vacuum is modelled as a region with a floor density and arbitrarily high resistivity that does not take part in the hydrodynamics. There is no plasma in the vacuum, so there can be no currents. Therefore, we do not carry out the implicit source term update in vacuum cells; only the hyperbolic part of the update that describes the electromagnetic waves. Similarly, when performing the interpolation to grid edges, we do not include any vacuum cells in the averaging of the current j . As the vacuum number density and vacuum resistivity are arbitrary, it does not make sense to include n_e or η values from vacuum cells when interpolating them to grid edges. As $\mathbf{E}$ is calculated using these quantities, it was decided that electric fields should only use plasma values when averaging as well. This is not an issue in the vacuum region since no interpolation is required for the hyperbolic update. However, to allow electromagnetic fields in the vacuum to impact on the plasma itself, we do not distinguish between plasma and vacuum cells when interpolating magnetic fields; including both in the averaging. This averaging logic is summarised in table 6.1.

Several permutations of this vacuum-plasma averaging scheme were tried, but we found that the above scheme led to reasonable and continuous answers on the vacuum-plasma boundary and consistent answers with past Gorgon results. It should be noted

Variable	$\mathbf{B}$	$\mathbf{E}$	$\mathbf{j}$	$\mathbf{u} \times \mathbf{B}$	n_e	η
Grid Location	Face	Edge	Edge	Edge	Centre	Centre
Averaging Scheme	Both	Plasma	Plasma	Plasma	Plasma	Plasma

Table 6.1: Table summarising the averaging scheme for each variable in the implicit source update on the vacuum-plasma boundary. $\mathbf{B}$ is interpolated by averaging both plasma and vacuum cells, whilst all the other variables only use plasma cells.

that these averaging schemes are somewhat arbitrary. The vacuum region is unphysical; as discussed in chapter 2 and shown with PIC simulations in chapter 5, the MHD model breaks down in this region. Other physics which cannot be meaningfully included in an MHD code, such as kinetics or charge separation, becomes relevant. Therefore, an averaging scheme that does not produce artifacts on the vacuum-plasma interface is the best achievable solution.

6.3.3 Advective Term

It is important to treat the $\mathbf{u} \times \mathbf{B}$ term in Ohm's Law carefully because it is responsible for ideal MHD. As this term gives rise to advection of magnetic field, in Gorgon it is handled using the same upwinding scheme as the hydrodynamic advection. A van Leer correction is used for second order spatial accuracy [137, 138]. During the update, we calculate the $\mathbf{u} \times \mathbf{B}$ terms using the same upwinding and van Leer method as described in section 3.3. Each component of this quantity is located on a different edge of the cell, with $(\mathbf{u} \times \mathbf{B})_x$ on the same edge of E_x . As there are no source terms in Faraday's Law used to update $\mathbf{B}$, we know that $\partial/\partial t(\mathbf{u} \times \mathbf{B}) = 0$ during the source term update. Therefore, we can define a temporary variable $\mathbf{E}'' = \mathbf{E} + \mathbf{u} \times \mathbf{B}$ and advance this instead within the source term update. It was found that this was the most numerically stable way of implementing the advection term. This is discussed in more detail in Appendix A.1.

Additionally, any other extended MHD terms that can be rearranged as an advection velocity, such as the Nernst effect [97], can be included in the model in this way.

6.3.4 Timestepping

We are using an explicit scheme to evolve the hyperbolic quantities. This means that we need to resolve all the timescales that these terms introduce to prevent numerical instabilities. The hyperbolic terms in Faraday's and Ampère's law describe the propagation of electromagnetic waves, so we need to resolve the light waves that transit across the cell. This sets the maximum timestep to:

$$\Delta t_{\text{hyperbolic}} = \frac{\min(\Delta x, \Delta y, \Delta z)}{c}. \quad (6.15)$$

Whilst this can be very restrictive, it is commonplace within pulsed power modelling to reduce the value of c . As long as the speed of light remains the fastest velocity in the system, its magnitude does not matter. When deciding on a value of c , we calculate:

$$\max(v_A, c_s, v_m, |\mathbf{u}|), \quad (6.16)$$

where $v_A = B/\sqrt{\mu_0\rho}$ is the Alfvén velocity, c_s is the sound speed, $v_m = \sqrt{v_A^2 + c_s^2}$ is the magnetosonic speed and $|\mathbf{u}|$ is the fluid velocity. The numerical speed of light is then set to be larger than this quantity. This often allows for a reduction in c by a factor of 10 or more, making the timestep more reasonable for a simulation of a full experiment. In practice, this also sets a limit on the density of the vacuum region ρ_{vac} . For low ρ_{vac} , relativistic effects become important as v_A or v_e approach c . We are limited to simulating conditions where this is not true.

The source term update should remain stable despite the timestep used, so we therefore update the source terms with the same $\frac{\Delta x}{c}$ timestep that is used for the hyperbolic part. A risk of this is that phenomena which are not resolved by the timestep are damped unphysically, leading to reduced accuracy. This can be mitigated by ensuring results are converged with the timestep, to the limit that can be afforded computationally.

6.4 Limitations of the model

It is important to highlight approximations that have been made within the inertia solver and where they might break down. Future work could involve relaxing some of these approximations.

The most obvious approximation is the inclusion of displacement current without allowing charge separation. We have justified this in chapters 2 and 3. Physically, the displacement current should be small for a non-relativistic plasma. Numerically, including displacement current is beneficial for stability and the scheme described in this section relaxes to the zero displacement current case in plasma cells [114, 136, 148, 202]. However, the question of whether including the displacement current in a single-fluid MHD formulation leads to unphysical predictions could only be resolved by using two-fluid MHD, where charge separation is allowed to occur through evolving the electron and ion fluids separately. It would be important work to evaluate whether two-fluid effects are relevant given the recent focus on Hall physics used to explain many phenomena in pulsed power. This would also be useful to benchmark the different numerical methods used in Hall MHD codes to see whether they are physically valid. However, a two-fluid treatment would require spatially resolving the Debye length λ_D and the speed of light crossing time λ_D/c . For typical wire array stagnation densities of 10^{26}m^{-3} and temperatures of 100eV, this would require timesteps smaller than 10^{-17}s and cell sizes smaller than 5nm. For a 3D simulation of a wire-array that might evolve for microseconds over a spatial scale of millimetres, this would be completely computationally impractical for even the largest supercomputers. Investigating this problem is a potential avenue of

future work.

Another limitation is that we have neglected the hyperbolic terms within the generalised Ohm's Law. These should have the form:

$$\frac{\partial \mathbf{j}}{\partial t} = -\nabla \cdot \left(\mathbf{u}\mathbf{j} + \mathbf{j}\mathbf{u} - \frac{1}{en_e}\mathbf{j}\mathbf{j} \right), \quad (6.17)$$

The first two terms on the right hand side describe the evolution of the current density with the ion flow and the third term describes the advection of the current density by the electron drift velocity. These terms were not included because the spatial derivatives mean they cannot be treated point implicitly. Schemes have been derived for including these terms within the update (e.g. [114, 148, 202]). However, they only become dominant for length scales comparable to the electron inertial length $\lambda_e = \frac{c}{\omega_{pe}}$. We are not expecting our model to produce physical answers in this limit due to ignoring charge separation. Past work has also often neglected these hyperbolic terms (e.g. [63, 116, 117]) and found them to make minimal difference to calculations. However, it would be possible to include these terms using an explicit scheme in the future.

For very low densities, it is conceivable that j/en_e becomes comparable to the speed of light. At this point, relativistic effects will become important. In principle, these can be accounted for within the electron inertia solver. Equation (6.9) is derived from the momentum equation for the electron population. This can be corrected for relativistic effects. However, due to ease of implementation we have neglected this effect. We instead ensure that the minimum densities in simulations are such that the electron drift velocity does not become superluminal. It is worth noting that when $v_A \sim c$, the ion momentum equation would also need to be treated relativistically. Currently in Gorgon, we approximate this using a Boris correction [151], as described in section 3.3.4. It would not be practical to adapt the hydrodynamics in Gorgon to incorporate relativistic effects.

Finally, several MHD terms are yet to be implemented. In this work, we focus on the Hall term and the resistive diffusion term. However, it would be straightforward to include other magnetised transport terms such as the Nernst term or anisotropic resistivities by including them as additional advection velocities, which is the way that they are currently treated in Gorgon away from vacuum [29]. More interesting would be the gradient of electron pressure term. This term away from vacuum gives rise to the Biermann Battery effect [120], but it also varies inversely with electron density and thus becomes relevant in a similar part of parameter space as the Hall effect, so neglecting it may have effects on the simulations presented in this chapter. Additionally, past work has shown that finite Larmor radius effects can be important in z-pinch type plasmas [121, 196]. These effects become important when the spatial length scale is comparable to the ion Larmor radius. They add non-diagonal components to the ion pressure tensor. This has been shown to interact with the Hall effect in non-trivial ways, for example changing predictions of magneto-Rayleigh Taylor growth [203] or z-pinch stability [179, 204].

6.5 Validation of the inertia solver

When developing a new numerical method, it is crucial to check that it produces results that are physical and not an artifact of the implementation. This is done by test problems: simplified situations that test the model in various limits to check that it produces consistent answers or matches analytical theories. In this section, we discuss the set of test problems carried out to validate the new electron inertia field solver. Each test problem includes gradually more physics. It is hoped that other implementations of Hall physics can use the same system of problems as a thorough test framework in the future.

6.5.1 Semi-analytic resistive diffusion test

This test is designed to check that resistive diffusion is implemented correctly. We set the density and temperature to be constant and uniform with values $\rho = 1\text{kgm}^{-3}$ and $T_e = T_i = 100\text{eV}$. Hydrodynamics is not included in the solution and the fluid velocity is set to 0 everywhere. We assume the plasma is fully ionised and consider a deuterium plasma with $Z = 1$ and $A = 2$. The resistivity is assumed to be constant everywhere in the domain with a value of $\eta = 4\pi \times 10^{-6}\Omega\text{m}$. The domain is 1D along x with size 1mm. We checked convergence of our solution with cell size. The boundary conditions are transmissive. The magnetic field has a z component set to be:

$$B_z = \frac{B_0}{2}[\tanh(x/\Delta) + 1], \quad (6.18)$$

where $B_0 = 5\text{T}$ and $\Delta = 10^{-5}$. The current is set as $\mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{B} = -\frac{1}{\mu_0} \frac{\partial B_z}{\partial x} \hat{\mathbf{y}}$ and $\mathbf{E} = \eta \mathbf{j}$ since there is no fluid flow and the Hall term has no effect in this geometry. The evolution of B_z is then calculated over time.

Under these conditions, we would expect the evolution of the magnetic field to be given by the equation:

$$\frac{\partial B_z}{\partial t} = \frac{\eta}{\mu_0} \frac{\partial^2 B_z}{\partial x^2}. \quad (6.19)$$

This is a linear partial differential equation that can be solved with a standard method. We use a forward time centred space (FTCS) scheme from ref [131]. Figure 6.2 compares this known result with the profiles from the original Gorgon field solver and the new inertia solver. These profiles are shown at time 1ns. The magnetic field has relaxed from its initially step-like function due to the finite resistivity. It can be seen that the magnetic field profile is identical in the different cases, indicating resistive diffusion is implemented correctly. Also encouragingly, we find that the electric field is given by $\eta \mathbf{j}$ at all times, indicating that under these conditions the inertial model correctly approaches the limit of no displacement current. In figure 6.3, we compare profiles calculated with a first order and with a second order accurate in time scheme. It is clear that the first

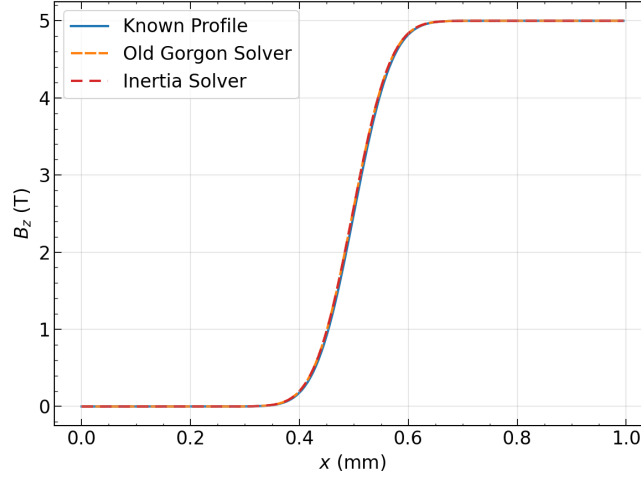


Figure 6.2: Profiles calculated for B_z at time $t = 1\text{ns}$ for the resistive diffusion test. The blue line shows the semi-analytic result, the orange line shows the result using the original Gorgon field solver and the red line shows the result from the new inertia field solver.

order scheme is more diffusive. This level of numerical diffusivity makes it impractical for use.

6.5.2 Basic advection test

This test is designed to check that the $\mathbf{u} \times \mathbf{B}$ term is implemented correctly. As discussed above, this term is important because it gives rise to ideal MHD. We again set the plasma density and temperature to be constant and uniform with values of $\rho = 1\text{kg/m}^3$ and $T = 100\text{eV}$. We consider fully ionised deuterium plasma with $Z = 1$ and $A = 2$. We assume that the resistivity $\eta = 10^{-8}\Omega\text{m}$ everywhere. The value is not important, but it is chosen so that the plasma is essentially fully conductive. There is no hydrodynamics in this test problem but we assume the plasma has a fluid velocity $\mathbf{u} = u_0\hat{\mathbf{y}}$, where $u_0 = 10^6\text{m/s}$. The domain is 2D with size $L_x = L_y = 4\text{mm}$. The boundary conditions are periodic in $\hat{\mathbf{y}}$ and transmissive in $\hat{\mathbf{x}}$. The initial magnetic field profile is in the $\hat{\mathbf{z}}$ direction with profile:

$$B_z = B_0 + \delta B \cos\left(\frac{2\pi my}{L_y}\right), \quad (6.20)$$

where $B_0 = 4\text{T}$, $\delta B/B_0 = 10^{-2}$ and $m = 2$ is the mode number. We found that changing the mode number of the perturbation did not affect the accuracy of the solver. As before, we set $\mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{B}$. The ion inertial length $\delta_i \sim 10^{-5}\text{m}$ for these initial conditions, so $L_y \gg \delta_i$, and the ratio of electron drift velocity to the fluid velocity $(\frac{j}{en_e})/u \sim 10^{-6}$. Therefore, it is safe to neglect the impact of the Hall term and the electric field will be given by $\mathbf{E} = -\mathbf{u} \times \mathbf{B}$.

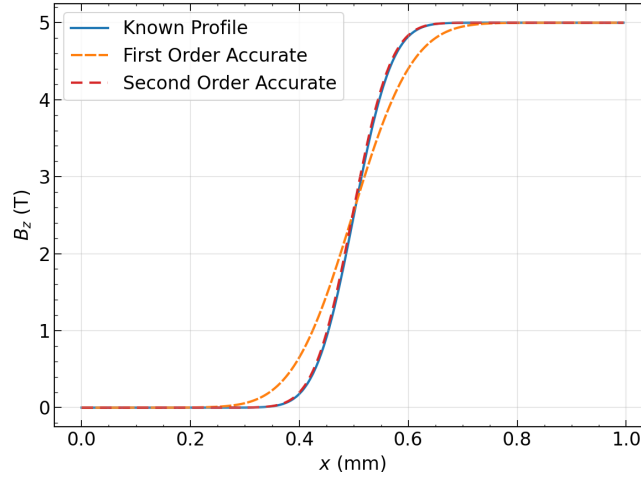


Figure 6.3: Profiles calculated for B_z at time $t = 1\text{ns}$ for the resistive diffusion test. The blue line shows the semi-analytic result, the orange line shows the result calculated using the inertia model with a first order accurate in time scheme, and the red line shows the profile calculated using a second order scheme.

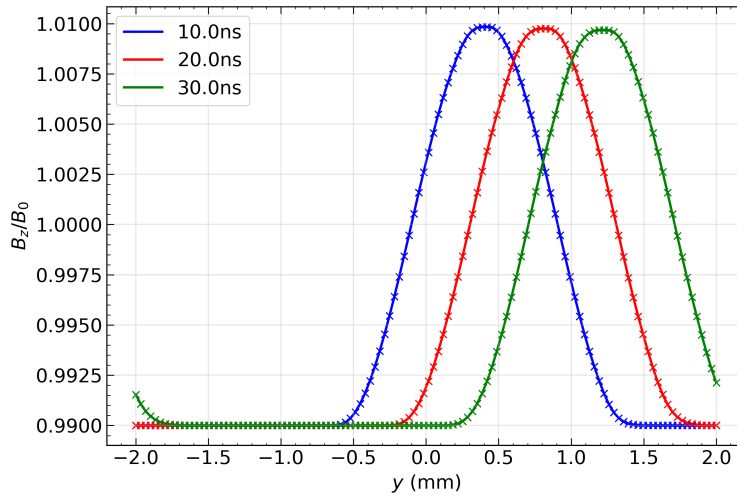


Figure 6.4: Profiles of the magnetic field over time in the advection test problem, taken along the y axis. The solid lines show the result calculated using the original Gorgon solver and the crosses show the profile calculated using the inertia solver.

Making these assumptions, we can write the evolution of the magnetic field as

$$\begin{aligned}\frac{\partial B_z}{\partial t} &= -\frac{\partial}{\partial y}(u_y B_z) \\ &= -u_0 \frac{\partial B_z}{\partial y},\end{aligned}\tag{6.21}$$

which is an advection equation for B_z with constant velocity u_0 in direction $\hat{\mathbf{y}}$. Figure 6.4 shows the magnetic field profile over time for the inertia solver and for the existing Gorgon solver, taken on a line along the y axis. The result from the original Gorgon solver, shown in solid lines, matches with the solution from the inertia solver, shown with crosses. Both solutions show the magnetic field being advected along the y axis at the expected velocity, with minimal numerical dissipation. We find that the van Leer flux correction, which raises the spatial accuracy of the advection solver to second order, is important for minimising this dissipation. The result from this test problem indicates that the ideal MHD term is implemented correctly within Gorgon.

6.5.3 whistler wave test

As described in section 6.3.4, the Hall term introduces new wave modes that an explicit scheme needs to resolve for numerical stability. In this section, we discuss the whistler mode and show a test problem that validates the electron inertia solver for this wave.

6.5.3.1 Dispersion Relation

We compare the results from a numerical solution to the dispersion relation for the wave. This dispersion relation depends on the model used and here we summarise them. In Appendix B.3 we show the full derivation for the separate models. We consider a plasma where there is no density gradient ($\nabla n = 0$) or hydrodynamic response, so n , T and $\mathbf{u}$ remain constant. Initially, we have a constant and uniform guide magnetic field $\mathbf{B} = B_0 \hat{\mathbf{z}}$. We assume that the x and y components of the electromagnetic fields are perturbed by a transverse wave travelling along the z direction, and so they have the form:

$$\begin{aligned}\mathbf{B} &= \delta B_x \hat{\mathbf{x}} + \delta B_y \hat{\mathbf{y}} + B_0 \hat{\mathbf{z}} \\ \mathbf{E} &= \delta E_x \hat{\mathbf{x}} + \delta E_y \hat{\mathbf{y}} \\ \mathbf{j} &= \delta j_x \hat{\mathbf{x}} + \delta j_y \hat{\mathbf{y}},\end{aligned}\tag{6.22}$$

where all perturbations have the plane wave form $\psi(z, t) \propto \exp(ikz - i\omega t)$ for some wavenumber k and frequency ω . We then substitute these perturbations into the governing equations and linearise. For Hall MHD, which neglects the displacement current and electron inertia, the dispersion relation is given by

$$\omega - k^2 v_A \lambda_i = 0,\tag{6.23}$$

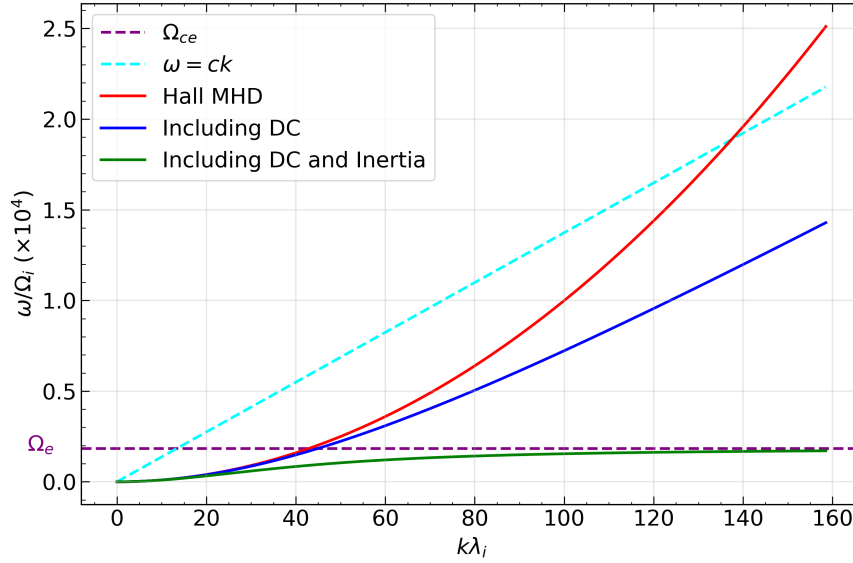


Figure 6.5: Plot showing the frequency ω of the whistler mode using different models. The cyan blue line shows the dispersion relation for electromagnetic waves and the red line shows the relation for Hall MHD, which neglects displacement current and electron inertia. The blue line shows the dispersion relation if we allow a finite displacement current, and the green line shows the result when finite electron inertia is accounted for.

where v_A is the Alfvén velocity and λ_i is the ion inertial length. For the case where there is a finite displacement current but the electron inertia is neglected, the dispersion relation is modified to

$$\omega^4 - \omega^2 \left(2c^2 k^2 + \frac{c^4}{(v_A \lambda_i)^2} \right) + c^4 k^4 = 0 \quad (6.24)$$

Finally, when the electron inertia is accounted for as well, we again obtain a different dispersion relation:

$$\omega^2 \left(1 - \frac{\omega_{pe}^2}{(\omega^2 - c^2 k^2)} \right)^2 - \Omega_e^2 = 0, \quad (6.25)$$

where $\omega_{pe} = \sqrt{\frac{n_e e^2}{m_e \epsilon_0}}$ is the electron plasma frequency and $\Omega_e = \frac{eB}{m_e}$ is the electron cyclotron frequency. In figure 6.5, we plot these dispersion relations for a particular value of n and B . It should first be noted that the whistler frequency only becomes significant for $k\lambda_i \gg 1$, or alternatively for length scales smaller than the ion inertial length. It is clear that the different models, shown by the solid lines, all have the same asymptotic behaviour as $k\lambda_i$ approaches 0 but very different behaviour for $k \rightarrow \infty$. For Hall MHD, there is nothing to limit the whistler frequency for high k (or alternatively short length scales). At some point, the whistler waves travel faster than light waves (shown by the cyan line). Therefore, short wavelength noise will cause the explicit

timestep to fall drastically. It is for this reason that Hall MHD simulations are often challenging. Assuming a finite speed of light has the effect of causing the whistler mode to asymptote to the speed of light for $k \rightarrow \infty$. Physically, this represents electric fields ‘shorting out’ the whistler waves above a certain frequency. It should also be noted that using an artificially reduced value of c , which is commonly done in pulsed power modelling as discussed in section 6.3.4, causes the whistler waves to asymptote at that reduced value instead. This offers an alternative approach for damping the high speed whistler waves. Including the effect of inertia limits the whistler frequency to the electron cyclotron frequency Ω_e . High wave modes have no group velocity, causing them not to propagate. This is usually already resolved if we are resolving the light waves as well. This limiting behaviour on the whistler waves is a physical way to control the whistler timestep and significantly reduce the computational burden of solving the Hall effect within an MHD code, demonstrating why including electron inertia is a useful tool.

6.5.3.2 Numerical test problem

In this section, we discuss the setup of the whistler wave test problem within Gorgon. We take the initial conditions from Huba [107]. The domain is one dimensional along $\hat{\mathbf{z}}$, with $L_z = 0.2\text{m}$. The simulations in this chapter use 480 cells to ensure that short wavelength modes are still well resolved. We consider a fully ionised deuterium plasma with constant density $n_e = n_i = 10^{18}\text{m}^{-3}$ and no fluid velocity. We also assume that the plasma is highly conductive, with $\eta = 10^{-8}\Omega\text{m}$. We set the temperature to be constant so that the plasma $\beta_0 = 10^{-4}$. This gives the value:

$$T = \frac{\beta B_0^2}{4\mu_0 n}, \quad (6.26)$$

where B_0 is the magnitude of the constant guide field. The magnetic field is initialised to

$$\mathbf{B} = \delta B_x \hat{\mathbf{x}} + \delta B_y \hat{\mathbf{y}} + B_0 \hat{\mathbf{z}}, \quad (6.27)$$

where $B_0 = 0.1\text{T}$, $\delta B_x = 0.01B_0 \sin(kz)$ and $\delta B_y = 0.01B_0 \cos(kz)$. It is important that $|\delta B_{x,y}| \ll B_0$ to ensure a linear response. We define the wavenumber $k = \frac{2\pi m}{L_z}$ for mode number m . The current density is set to $\mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{B}$ and the electric field to $\mathbf{E} = \frac{\mathbf{j} \times \mathbf{B}}{en_e}$. The boundary conditions are periodic in the z direction.

Figure 6.6 shows the magnetic field perturbations for mode $m = 2$ along the z axis at a few time snapshots. The perturbations are travelling waves, with δB_x and δB_y out of phase with each other. There is a slight variation in the amplitude of the perturbations over time, on the order of 5%. The reason for this is unknown, but it is believed to be related to the finite displacement current as artificially increasing the value of c reduces

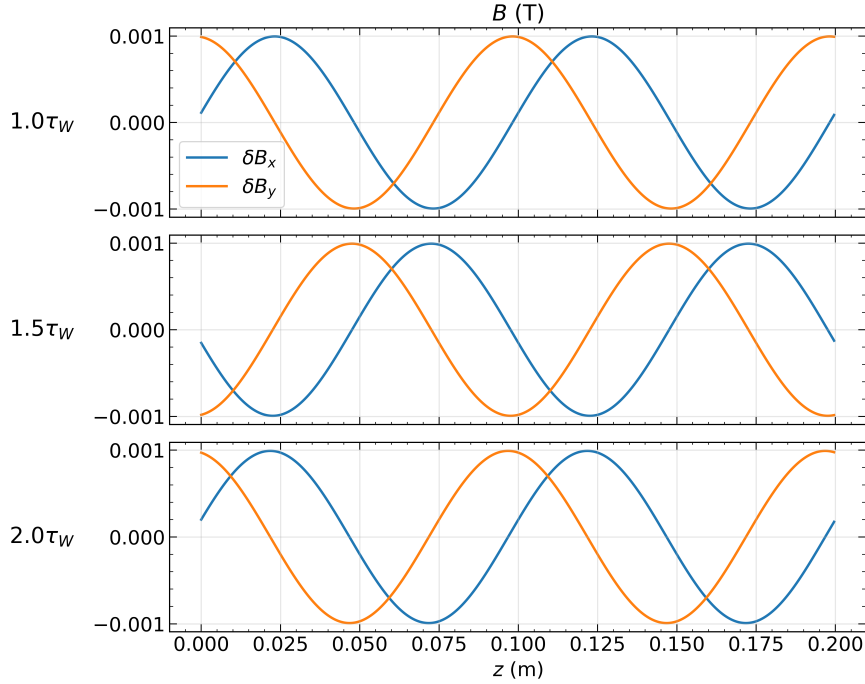


Figure 6.6: Plots showing the evolution of the perturbations in the magnetic field over time, for a whistler wave with mode number $m = 2$. The waves are shown at intervals of the whistler period τ_W .

these amplitude oscillations. We also find that if the Hall effect is not included within the update the waves do not propagate at all, as expected since the whistler wave is a Hall driven mode. These results indicate that the electron inertia model can correctly recreate the whistler mode.

We vary the mode number m and calculate the frequency of the whistler waves produced by Gorgon. We plot these frequencies as crosses over the theoretical dispersion relation in figure 6.7. The original Gorgon MHD solver, which treats the Hall as an advection velocity, agrees well with the red line for Hall MHD [205]. However as we need to resolve the cell crossing time at this advection velocity for stability, this causes the simulations to be highly computationally expensive compared to the other models. We find that the displacement current with no electron inertia limit asymptotes to the speed of light, as expected. The black and orange lines show that the whistler branch asymptotes to an artificially reduced speed of light, as asserted above. With enough of a reduction in c , this can be an effective way to control timesteps for the Hall MHD model although the behaviour may not be entirely physical since we are effectively capping the whistler speed numerically. Finally, we find that the calculated frequencies using the inertia model agree well with the theory for $k \lesssim 300$. At higher values of k , it becomes difficult to meaningfully calculate a frequency for the wave as the waves barely propagate. This makes the uncertainty in calculating the frequency high so it is not shown on the graph. However, we find that the phase velocity does asymptote to zero for the higher mode waves.

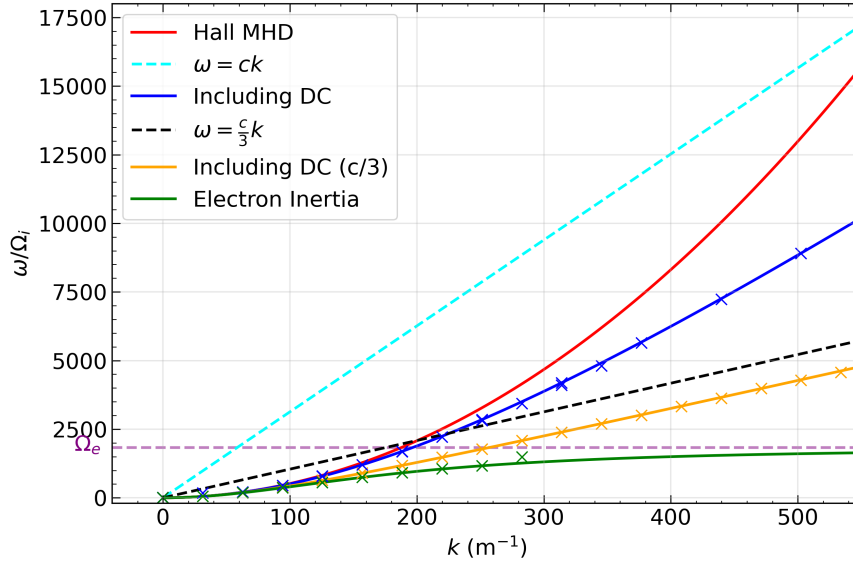


Figure 6.7: Dispersion relation of whistler waves calculated using various models. The crosses mark calculated frequencies from Gorgon. The red line shows the dispersion relation for Hall MHD and the green line for the electron inertia model. The cyan line and black lines show the dispersion relations for light waves with speed c and $c/3$ respectively. The blue and orange lines show the dispersion relation when accounting for the finite displacement current (but with no electron inertia) with speeds of light c and $c/3$ respectively.

6.5.4 Hall drift wave test

In this section, we discuss the other wave mode that is introduced by the Hall effect: the Hall drift wave.

6.5.4.1 Dispersion Relation

The Hall drift wave is brought about by the fact that the electron drift velocity $\mathbf{v}_e = -\frac{\mathbf{j}}{en_e}$ depends on the electron density. Considering the Hall MHD system of equations:

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} \\ \mathbf{E} &= -\mathbf{u} \times \mathbf{B} + \frac{\mathbf{j} \times \mathbf{B}}{en_e},\end{aligned}\tag{6.28}$$

we can rewrite the evolution of the magnetic field as:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{u} \times \mathbf{B} - \frac{\mathbf{j}}{en_e} \times \mathbf{B} \right).\tag{6.29}$$

The first term in the bracket on the right hand side leads to the frozen-in-flow behaviour of ideal MHD, so by analogy the second term leads to an advection of the magnetic field with the electron drift velocity $\mathbf{v}_e$. In an inhomogeneous plasma where n_e is a function of space, this will lead to a shearing of the magnetic field due to the variation of $\mathbf{v}_e$ over

space. The wave will travel in the $\mathbf{B} \times \nabla n$ direction. This wave is known as a Hall drift wave. In this section, we derive the dispersion relation for this wave in the Hall MHD limit. This wave was first described by J. D. Huba [107, 186, 206].

We consider a fully ionised deuterium plasma in the x - y plane with an initially constant magnetic field B_0 out of plane in the $\hat{z}$ direction. We assume that the resistivity is negligible, the temperature is constant and there is no fluid velocity or hydrodynamic response. The density profile is assumed to be static in time with a profile in the $\hat{x}$ direction $n_e = n_i = n(x)$. The magnetic field is then perturbed by $\delta B_z = B_1 \exp(iky - i\omega t)$ with $B_1 \ll B_0$. Linearising Ampère's Law and the Hall MHD Ohm's Law given in equation (6.28), we obtain:

$$\begin{aligned}\delta j_x(y, t) &= \frac{ik}{\mu_0} \delta B_z \\ \delta E_y(x, y, t) &= -\frac{B_0 \delta j_x}{en_e}.\end{aligned}\tag{6.30}$$

Substituting these expressions into the linearised Faraday's Law for $\mathbf{B}$, we find:

$$\begin{aligned}i\omega \delta B_z &= \frac{\delta j_x B_0}{e} \frac{1}{n_e^2} \frac{dn_e}{dx} \\ &= \frac{ik B_0}{\mu_0 e} \frac{1}{n_e L_n} \delta B_z,\end{aligned}\tag{6.31}$$

where we have used the definition of the density gradient length scale $L_n^{-1} = \frac{1}{n_e} \frac{dn_e}{dx}$. It is possible to rewrite this in terms of the Alfvén speed and ion inertial length as:

$$\omega = \frac{v_A}{L_n} k \lambda_i.\tag{6.32}$$

As $k\lambda_i \rightarrow 0$, then ω also tends to 0. This is because if $k\lambda_i \ll 1$, then the wavelength of the drift wave is much larger than the ion inertial length and so the Hall effect is expected to have little impact. It should also be noted that sharp density gradients drive rapid Hall drift waves as $v = \omega/k \rightarrow \infty$ as $L_n \rightarrow 0$. Within Hall MHD, there is no mechanism to prevent these waves going superluminal at large density gradients, such as at a vacuum-plasma interface. In the literature, there is little discussion of ways of mitigating the dispersion behaviour of this wave. The past work has almost exclusively focused on ways to control the whistler wave. However, we have found that it is the Hall drift wave that makes Hall simulations involving a vacuum region intractable with an explicit code.

We have not been able to derive a linear dispersion relation in the case with finite displacement current or electron inertia. We discuss a possible reason for this in the next section.

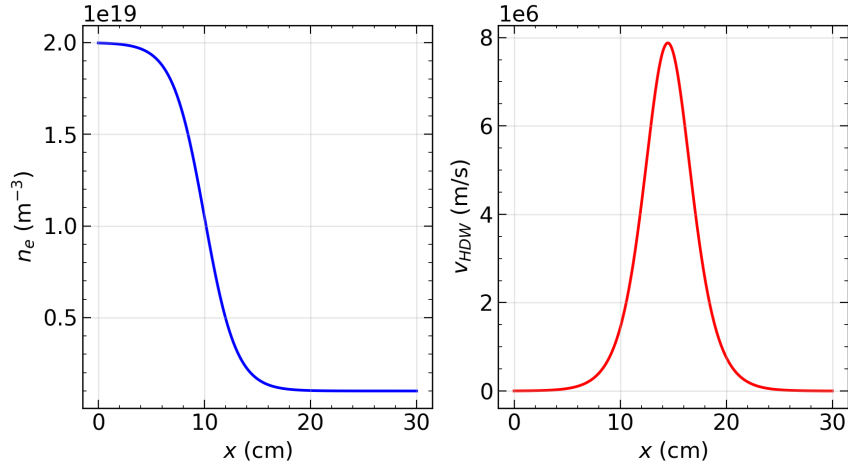


Figure 6.8: Plot showing the initial condition used for the electron density $n_e(x)$ for the Hall drift wave (in blue) and the resulting phase velocity for the wave v_{HDW} (in red).

6.5.4.2 Numerical test problem

In this section, we present the numerical setup for the Hall drift wave and show the dispersion relation obtained from Gorgon. We take the initial conditions from Huba [107], for comparison to past work.

The problem is 2D with $L_x = 30\text{cm}$ and $L_y = 20\text{cm}$. We have found convergence with $(N_x, N_y) = (300, 200)$ cells, though finer resolution is needed to resolve the higher modes for the test problem. The magnetic field is initialised with only a $\hat{z}$ component, given by:

$$B_z = B_0 + \delta B \cos\left(\frac{2\pi m y}{L_y}\right), \quad (6.33)$$

where $B_0 = 0.1\text{T}$, $\delta B/B_0 = 0.01$ and m is the mode number. The density profile is given by:

$$n_e(x) = \frac{n_0}{2} \left[(1 + A) + (1 - A) \tanh\left(\frac{x - x_0}{\Delta}\right) \right], \quad (6.34)$$

where $n_0 = 10^{18}\text{m}^{-3}$, $A = 20$, $x_0 = L_x/2$ and $\Delta = 0.3\text{cm}$. The plasma is isothermal, with temperature chosen so that $\beta = \frac{2n_0 T}{B_0^2/2\mu_0} = 10^{-4}$. The boundary conditions are periodic in the $\hat{y}$ direction and reflective in the $\hat{x}$ direction. We have found that using transmissive boundaries in $\hat{x}$ does not change the answer. Figure 6.8 shows the initial profiles for $n_e(x)$ and the corresponding phase velocity v_{HDW} calculated from the dispersion relation. It can be seen that the phase velocity peaks around $x \approx 14$ cm. This is the velocity with which the magnetic field travels in the negative y direction throughout the box. Due to the variation of the phase velocity, the magnetic field shears and travels faster in the centre of the box. This can be seen in figure 6.9 which shows the

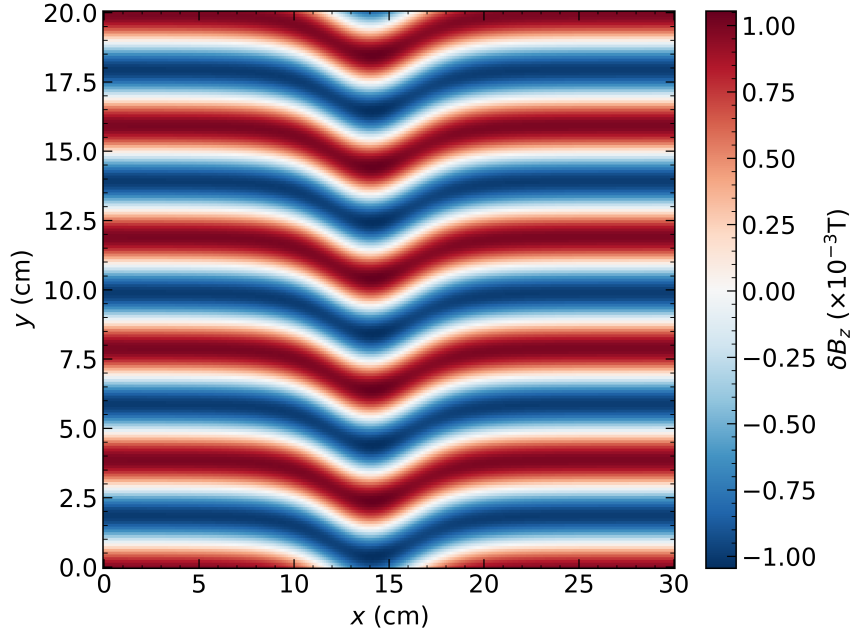


Figure 6.9: Plot showing the perturbation in the B_z field in the x - y plane for the Hall drift wave problem for mode number $m = 5$ at time $t = 2.6\text{ns}$.

perturbation in the B_z field for $m = 5$ at $t = 2.6\text{ns}$. There is no x or y component generated, as expected. The magnetic field is moving in the negative $\hat{y}$ direction, which is the $\mathbf{B} \times \nabla n_e$ direction. We find that if there is no density gradient, the magnetic field does not propagate at all. Initially, the magnetic field has no variation in the x direction. After some time, it has generated a profile in this direction due to the shearing. The peak velocity is where the magnetic field has perturbed the most from its initial condition. This is located at $x \approx 14\text{cm}$, as expected. This profile is consistent with results from the literature [107]. This indicates that the electron inertia model correctly reproduces the Hall drift mode.

We obtain a dispersion relation by varying the mode number and measuring the speed v of the resulting drift waves. We then calculate a frequency ω from this using $\omega = vk$. We measure these quantities along the line $x = 15\text{cm}$, to ensure we are sampling the point of maximum velocity. Figure 6.10 shows this dispersion relation. The solid blue line shows the frequency predicted by equation (6.32); it is linear as predicted by Hall MHD theory excluding both the displacement current and electron inertia. The existing Hall MHD model in Gorgon, shown by orange diamonds, agrees well with theory, although it begins to disagree for large k . This is due to worsening resolution for the higher wave numbers. The red crosses show the results calculated using a model that includes displacement current but not electron inertia. The dispersion relation in this case agrees well with theory until $k \approx 250\text{m}^{-1}$, after which the frequencies of the drift waves fall below the dispersion relation. The displacement current has a damping effect on the drift waves, which is consistent with its behaviour on the high wavenumber whistler waves. The green circles show the dispersion relation calculated when both a

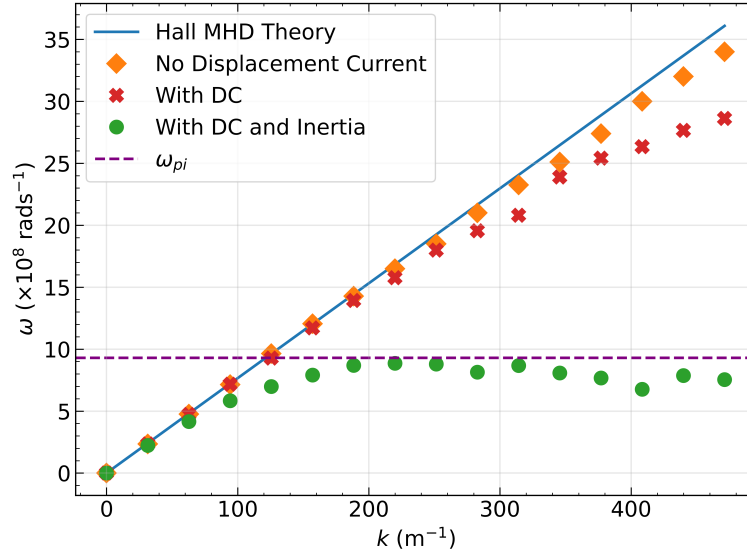


Figure 6.10: The dispersion relation of Hall drift waves calculated using different models. The solid line shows the theoretical relation for Hall MHD, which is well matched by the orange diamonds showing the frequency calculated using the Hall MHD model in Gorgon. Different dispersion characteristics are obtained for the displacement current (red crosses) and displacement current with electron inertia (green circles) models. The value of the ion plasma frequency for the characteristic density n_0 is shown by the purple line.

displacement current and a finite electron inertia are retained. The frequencies calculated from the inertia model agree with theory for small k but they begin to differ around $k \approx 80 \text{ m}^{-1}$. The frequency of the modes saturates at the ion plasma frequency for high k . Similarly to the whistler waves, the phase velocity tends to zero for increasing k for this wave branch which makes determining the frequency inaccurate and so the data is not smooth. This trending behaviour demonstrates that, as with the whistler waves, the electron inertia can significantly reduce the computational cost of simulations by increasing the maximum stable timestep. As with the displacement current, the electron inertia has a damping effect on the drift modes. We have found that reducing the value of c by a factor of 10 and including electron inertia can reduce the phase speed of the drift waves to almost zero for high k . Whilst this is useful numerically, further work is needed to understand whether these high mode drift waves have some observable impact on experiments that is being unphysically damped by the numerics. Although we have no theoretical dispersion relation to compare the electron inertia model to, the new solver produces consistent answers in the $k \rightarrow 0$ limit and reasonable qualitative behaviour compared to the other MHD models used. Therefore, we are confident that it is correctly modelling the Hall effect in inhomogeneous plasmas.

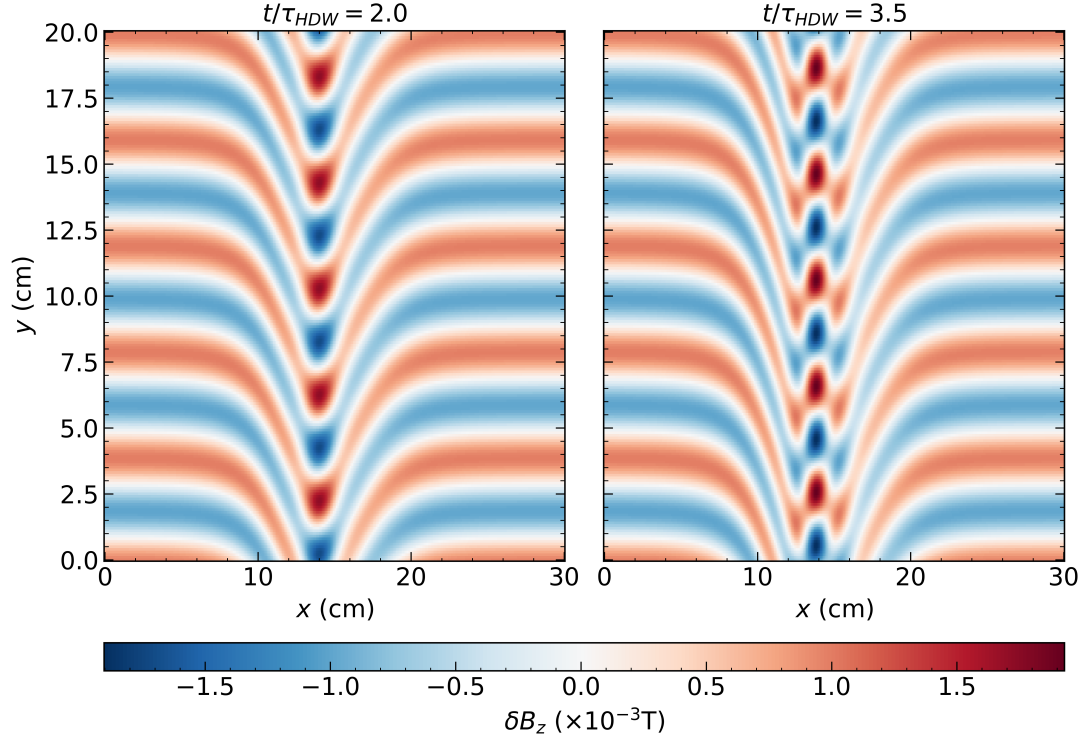


Figure 6.11: Plots showing how the B_z perturbation for a Hall drift wave varies over time, relative to the wave period τ_{HDW} . This period is calculated using the Hall MHD dispersion relation in equation (6.32).

6.5.4.3 Nonlinear behaviour of the Hall drift wave

After a couple of wave periods, the drift wave goes nonlinear. In figure 6.11, we show the magnetic field at two times for mode $m = 5$ using the inertia model. This is the same simulation as shown in figure 6.9. On the left, magnetic field is being advected away from the shearing region to the region of highest phase velocity at $x \approx 15\text{cm}$. The size of the perturbation in the shearing region is decreased while the perturbation amplitude grows along $L_x/2$. Initially, there is only a j_x component to the current. As the magnetic field shears, it gains a gradient in x , which generates a j_y component as well. j_y grows as the field shears more. The nonlinearity becomes observable when $|j_y| \sim |j_x|$, which occurs after only a couple of wave periods for the inertia model. At later times, shown in the right hand figure, a secondary shearing region develops. At this point, the magnetic field at $x \approx L_x/2$ no longer travels at the predicted Hall drift wave speed and higher modes are beginning to be coupled into the motion. The mechanism for this is not known and requires further investigation. We find that at even later times, this region itself shears into smaller regions that propagate at different speeds. When we extract a frequency from these drift waves, we are careful to do it before higher modes become coupled by the nonlinearity and the linear dispersion is not meaningful. We note that whilst the drift waves ultimately go nonlinear in all models, including without displacement current, the electron inertia model generates

nonlinearity significantly earlier than the other two models. It is also important to note however that the electron inertia model is not entirely physically consistent, as we neglect charge separation on scales smaller than the electron inertial length c/ω_{pe} . Therefore, we would expect that the nonlinear evolution of these drift waves is affected by physics that we are not including in our model. One effect of the Hall term is to drive current structures (and therefore also magnetic field structures) to increasingly small spatial scales by generating vorticity in the magnetic field [202, 207, 208]. This small scale structure creates increasingly large gradients which subsequently reduces the stable timestep for simulations. A mechanism to saturate the effect of the Hall on small spatial scales, such as electron inertia, is needed to maintain numerical stability. We discuss this more in the context of the equilibrium z-pinch that we model in section 6.6. Whilst in this test problem both our geometry and our resolution prevent the generation of vortices, qualitatively figure 6.11 shows this propensity of the Hall effect to generate smaller scale structure.

6.5.5 Ohmic heating test problem

Ohmic heating plays an important role in pulsed power experiments. For example, the resistive heating of wires produces the plasma flows that are studied in many MAGPIE experiments [83, 209]. A common problem in resistive MHD simulations of pulsed power experiments is the observation of extremely high surface temperatures due to Ohmic heating at the vacuum-plasma interface [210]. It has been proposed that the Hall effect helps to resolve this problem [63], although this is not directly tested by this problem. Therefore, it is important that the current densities from the inertia model cause physical amounts of Ohmic heating. We validate this using a test problem generated from our self-similar code, described in detail in chapter 4.7.3.

We use the unity β test problem described in section 4.7.3. We consider the case without the Nernst effect, since this has not yet been implemented within the inertia solver, and without the Ettingshausen effect. We use the single temperature version of Gorgon, where we do not allow the electron and ion temperatures to separate. The simulation is 1D along x with $L_x = 3\text{mm}$ and $N_x = 3000$, with transmissive boundary conditions to let pressure waves escape the domain. We have found that the temperature profiles are converged with a smaller number of cells, but it is better to have more cells to compare to the self-similar solution. We use the Braginskii resistivity model. The magnetic field, temperature and density profiles are initialised as described in section 4.7.3. The current density and electric field are initialised as $\mathbf{j} = \nabla \times \mathbf{B}/\mu_0$ and $\mathbf{E} = \eta\mathbf{j}$ respectively. The Hall effect has no effect due to the geometry of the problem. Other than during an early relaxation period, the initial electric field and current density are not relevant since the profiles converge to the theoretical profiles within a few timesteps anyway.

Figure 6.12 shows the temperature, density, magnetic field and velocity profiles obtained when using the old Gorgon solver (in red) and the electron inertia solver (in

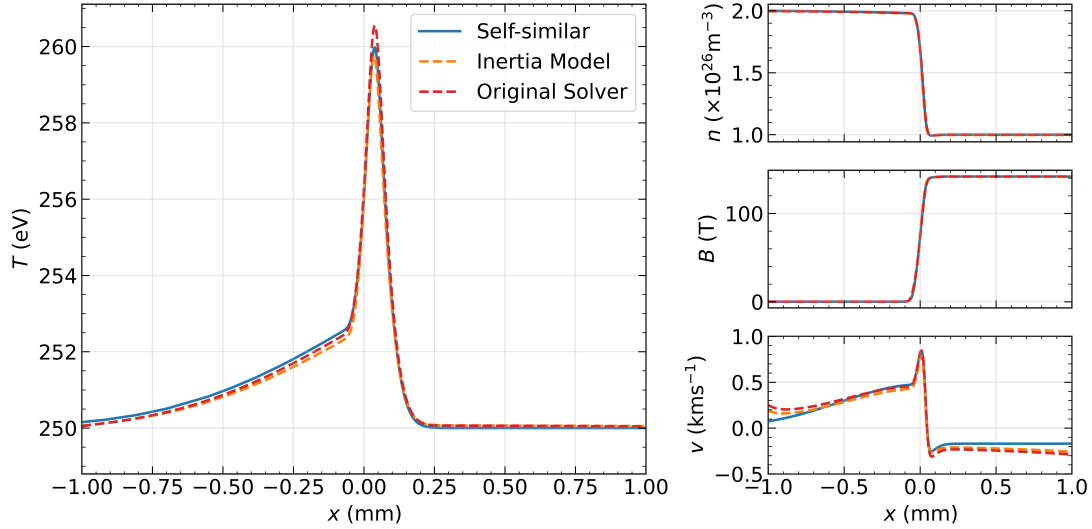


Figure 6.12: Profiles of temperature (left), density (right, top), magnetic field (right, middle) and velocity (right, bottom) for the Ohmic heating test problem described in section 6.5.5. The results from the self-similar code are shown in blue, those from the inertia model developed in this chapter in orange and those from the original Gorgon solver (without displacement current or electron inertia) in red.

orange). The comparison is made at 3.5ns to allow the initially discontinuous profiles to settle to pressure balance. The old Gorgon solver used in this comparison has no displacement current or electron inertia. The self-similar solution is shown in solid blue. Resistive diffusion causes the initially step function magnetic field to relax into the denser plasma on the left hand side of the domain. As it does so, the finite resistivity of this plasma causes magnetic energy to be converted into the thermal energy of the plasma, producing the visible peak in temperature. The plasma then expands to maintain pressure balance. The asymmetry in the temperature profile is due to the magnetisation of thermal conduction on the right hand side of the profile. The inertia solver produces nearly identical profiles to the original Gorgon solver, indicating that our new method has not affected the calculation of Ohmic heating. This is to be expected, because there is little effect due to the displacement current or electron inertia for these parameters and therefore we should reproduce the same current densities as the original solver.

6.5.6 Magnetic cavity power flow test

It has been suggested in past literature that the Hall effect can play an important role in power flow in pulsed power experiments [116–118]. Studying the impact of the Hall effect in this situation would be a thesis chapter in itself, but in this section we describe a test problem inspired by this situation that can stress test the Hall effect in a challenging parameter regime. This test problem is taken from ref [118].

We consider a magnetic field applied near an axisymmetric electrode. This is intended

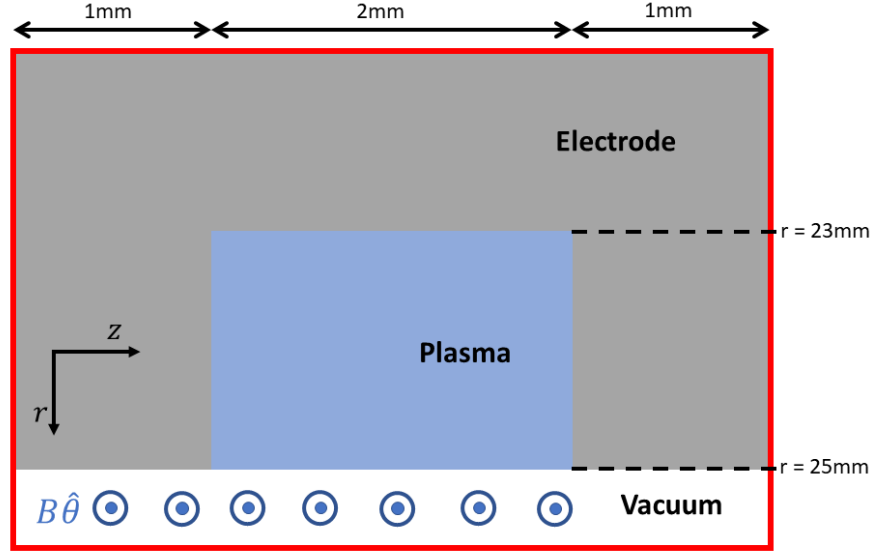


Figure 6.13: Diagram showing the setup for the power flow problem described in section 6.5.6. There is an electrode (shown in grey), a low density plasma (shown in blue) and a vacuum region (shown in white). The densities and resistivities for these regions are given in table 6.2. An axial current is applied in the electrode, producing an azimuthal magnetic field B_θ .

to be somewhat like the load of a pulsed power device such as a wire array. Figure 6.13 is a diagram of this electrode. The electrode (shown in grey) is assumed to be made of partially ionised aluminium with solid density and a very low resistivity. There is a region of low density plasma inside this electrode (shown in blue). This may be surface plasma ablated from the surface of the electrode, and consequently we assume a much lower density and a much higher resistivity for this plasma. We assume that there is a vacuum region that separates this electrode from the edge of the domain. An axial current is applied in the electrode, which produces an azimuthal magnetic field which ultimately moves across the vacuum and diffuses into the plasma. We are interested in the structure of the magnetic field as it diffuses.

The inertia solver is not yet implemented in cylindrical geometry, so when implementing this test problem within Gorgon we approximate it as Cartesian. This is a reasonable approximation as we are only considering the edge 2mm of the electrode and thus are not expecting convergent geometry effects. In figure 6.13, we treat the (r, θ, z) directions as (x, y, z) . Table 6.2 shows the values of resistivity, density and effective atomic number $\bar{Z}$ used for the different regions of the problem. We assume that the density and resistivity within each material are uniform. A current profile is applied at the centre of the electrode with the waveform:

$$I(t) = 20 \times 10^6 \sin^2 \left(\frac{2\pi t}{100\text{ns}} \right), \quad (6.35)$$

which is intended to resemble a rising current pulse on the Z machine. This is actually initialised in the simulation by applying a B_y magnetic field in the bottom row of cells

	Electrode	Low Density Plasma	Vacuum
η (Ωm)	1.46×10^{-8}	2.94×10^{-4}	10^{12}
ρ (kgm^{-3})	2.7×10^3	10^{-3}	10^{-6}
$\bar{Z}$	2.44	1.8065	N/A

Table 6.2: Table showing the values of resistivity η , mass density ρ and effective atomic number $\bar{Z}$ for the different regions of the power flow problem discussed in section 6.5.6.

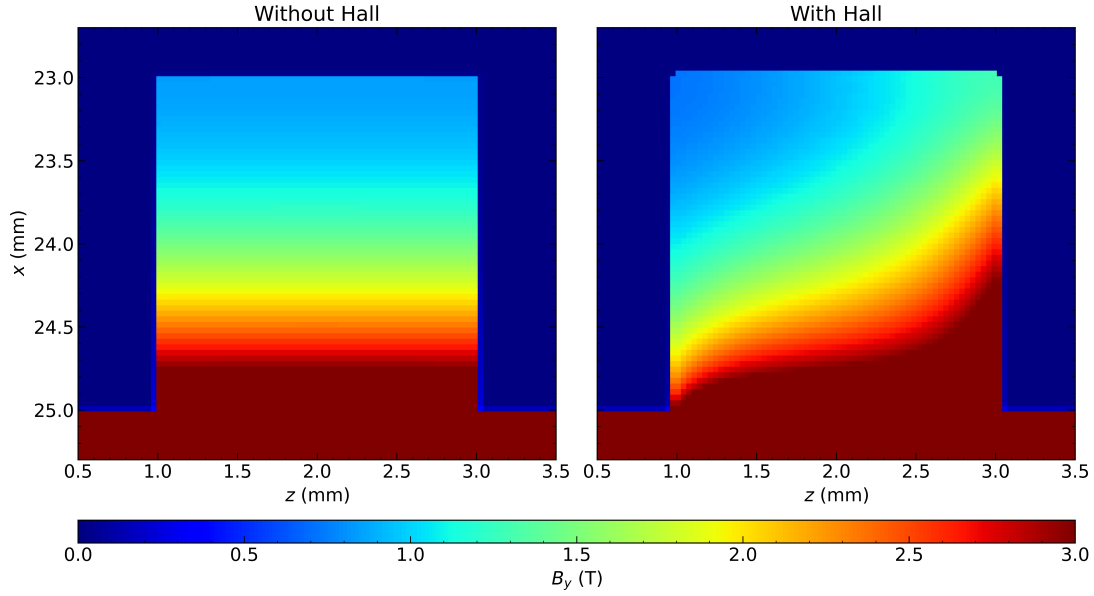


Figure 6.14: Plot showing the $\hat{y}$ component of magnetic field for the power flow test problem described in section 6.5.6. The left plot shows the magnetic field in the no Hall effect case, and the right plot shows B_y when the Hall is included in the calculation.

in the domain with value:

$$B(t) = \frac{\mu_0 I(t)}{2\pi R}, \quad (6.36)$$

where $R = 25\text{mm}$. Note that we still apply the magnetic field as if it were acting at a radial distance of 25mm from the centre of the electrode. We do not set the currents or fields within the electrode to zero, allowing them to evolve self-consistently. Outflow boundary conditions are used for fields on the top, left and right of the box. Finally, we assume that the plasma is static with no hydrodynamic motion and that the electrode is not hydrodynamically affected by the large fields moving into it.

Figure 6.14 shows the magnetic field profile calculated using the inertia solver across the domain at a time of 10ns for two separate cases. In the left plot, the Hall effect is set to zero in the calculation. On the right hand side, it is included within the solution. The inclusion of the Hall effect changes the structure of the magnetic field, making it asymmetric and curved compared to the resistive MHD case. To see why this is, consider Faraday's Law. To simplify for this qualitative description, we assume that there is no

movement of the plasma or electrodes and that the displacement current is negligible. This means that the current only has j_x and j_z components and the magnetic field only has a B_z component. Under these assumptions, we can write Faraday's Law as:

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} \\ &= -\nabla \times \left(\eta \mathbf{j} + \frac{\mathbf{j} \times \mathbf{B}}{en_e} \right).\end{aligned}\quad (6.37)$$

Using the identity $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$ and the fact that $\mathbf{j} = \nabla \times \mathbf{B}/\mu_0$, we can rewrite the former equation as:

$$\frac{\partial B_y}{\partial t} = \frac{\eta}{\mu_0} \nabla^2 B_y - \frac{B_y}{en_e^2} \left(\frac{\partial n_e}{\partial z} j_z B_y + \frac{\partial n_e}{\partial x} j_x B_y \right). \quad (6.38)$$

The first term on the right hand side corresponds to resistive diffusion and the second corresponds to the Hall effect. In the absence of the Hall effect, this just causes diffusion of magnetic field through the resistive low density plasma inside the electrode. As shown on the left of figure 6.14, this means that the magnetic field front is flat. The Hall effect introduces an asymmetry. Considering the right hand side of the domain: $\partial n_e / \partial z$ is positive and $j_z = \partial B_y / \partial x$ is negative. Therefore, this leads to $\partial B_y / \partial t$ being larger than the only resistive diffusion case and so the magnetic field moves forward faster on the right hand side. A similar argument can be made to show that the magnetic field moves less quickly on the left hand side. This generates the curved structure seen on the right hand side of the figure. This is the same mechanism that gave rise to the Hall drift wave earlier in this chapter. An alternative way of viewing this effect is that the inclusion of the Hall effect leads to an additional advection of the magnetic field in the $\mathbf{B} \times \nabla n_e$ direction. This is the mechanism of the Hall drift wave that was discussed earlier. This is in the $-\hat{x}$ direction on the right hand side of the domain, $-\hat{z}$ on the top and $+\hat{x}$ on the left, leading to the observed field structure. It should be noted that due to the interpolation required due to the staggered grid, there is an additional ‘‘corner’’ cell of zero magnetic field in the right hand figure. It is important to correctly characterise the power flow within transmission lines for a successful pulsed power experiment and so the fact that the Hall effect has an observable impact on the field structure indicates that it is crucial in fluid based power flow modelling.

This test problem has also been carried out for the PERSEUS code and a quantitatively very similar magnetic field profile is obtained. This can be seen in figure 6.15. This is a challenging test problem, and the fact that the new inertia solver has produced physically meaningful results that are consistent with the past work validates it. In particular, there are no artifacts in the magnetic field across the vacuum-plasma boundary and no fields or currents within the electrode region. As discussed in section 6.3.2, the lack of numerical artifacts on the vacuum-plasma boundary is the best achievable outcome for a code that does not account for charge separation effects. We do not manually set the fields or currents within the electrode region to zero, this is

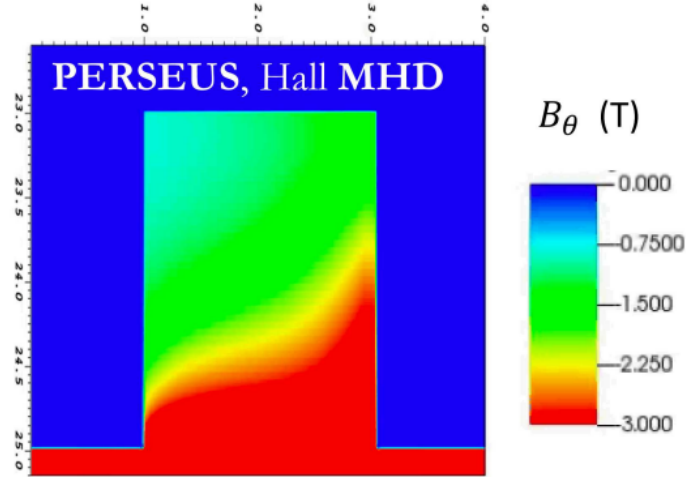


Figure 6.15: The magnetic field profile produced by the PERSEUS extended MHD code for the cavity power flow problem described in section 6.5.6. Reproduced with permission from ref [118].

self-consistently achieved, indicating that the new solver produces expected behaviour within highly conductive regions. Additionally, despite a difference of six orders of magnitude in density between the electrode and plasma region (and nine between the electrode and vacuum), we do not produce any noticeable artifacts in the magnetic field in these regions. This suggests that the new solver is robust to these extreme density gradients, which is encouraging for a code attempting to model vacuum-plasma boundaries. This situation is quite indicative of the sort of experimental platform that Gorgon is used to model: with a vacuum region, large density gradients and mixed regions of resistive and conductive plasma. This is promising for use of this solver for laboratory astrophysics experimental modelling.

6.5.7 Farmer Hall drift wave test

6.5.7.1 Background

When we considered the Hall drift wave in section 6.5.4, we assumed that the plasma did not respond hydrodynamically to the density gradient applied or the travelling magnetic field perturbations. In Farmer *et al.* [115], this approximation is relaxed and the hydrodynamic behaviour of the plasma was included in the linear dispersion relation for Hall MHD. The density gradient drives a drift wave in the magnetic field due to the Hall effect. This altered magnetic field then exerts a force on the plasma via the $\mathbf{j} \times \mathbf{B}$ term, which causes the density profile and fluid velocity to change. This alters the dispersion characteristics of the drift wave.

Similarly to the drift wave, we can obtain a dispersion relation for this “Farmer drift wave”. By applying this as a test problem to our inertia solver, we can assess whether the Hall term is correctly coupled to the hydrodynamics and ensures that it does not

drive new numerical instabilities. We can write the governing equations for this system (in the limit of no displacement current or electron inertia) as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (6.39)$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = \mathbf{j} \times \mathbf{B} \quad (6.40)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[\left(\mathbf{u} - \frac{\mathbf{j}}{en_e} \right) \times \mathbf{B} \right] - \nabla \times (\eta \mathbf{j}), \quad (6.41)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \quad (6.42)$$

where we have assumed that $\beta \ll 1$ and thus that the thermal pressure can be ignored. Similarly to the Hall drift wave, the magnetic field is assumed to be in the $\hat{z}$ direction whilst all quantities only vary in the x - y plane. This suppresses the whistler wave which travels along the magnetic field. The resistivity is assumed to be negligible and there are no Biermann battery or thermoelectric effects included. No dispersion relation has been obtained yet in the displacement current or inertia cases.

The linear dispersion relation is complex and derived in full in ref. [115]. Here, we just summarise that derivation. For there to be an effect due to the Hall term in this geometry, there needs to be an initial density gradient which is assumed to be in the $\hat{x}$ direction: $\rho_0 = \rho(x)$. Otherwise there is an initial uniform magnetic field $\mathbf{B}_0 = B_0 \hat{z}$ and no initial fluid velocity. To obtain a linear dispersion relation, the variables are perturbed by a small quantity:

$$\rho = \rho_0 + \rho_1(x, y, t) \quad (6.43)$$

$$B_z = B_0 + B_1(x, y, t) \quad (6.44)$$

$$u_x = u_{x,1}(x, y, t) \quad (6.45)$$

$$u_y = u_{y,1}(x, y, t), \quad (6.46)$$

where the perturbations have form $f_1(x, y, t) \sim \psi(x) \exp(iky - i\omega t)$; that is they are travelling waves in the $\hat{y}$ direction with some spatial profile in $\hat{x}$. Performing the normal linearisation process on equations (6.39) to (6.42) produces a Sturm-Liouville type equation that can be solved to obtain the $\psi(x)$ profiles of the perturbations. As it is a Sturm-Liouville problem, these $\psi(x)$ profiles are a family of eigenfunctions with associated eigenfrequencies. For this test problem, we only consider the second of these eigenfrequencies (if we were to order them in magnitude). We also only consider a single mode with wavenumber $k_1 = 10^3 \text{m}^{-1}$. This is for comparison to the literature [115]. We assume that if our solver is able to solve the problem for this case, it will also work for the other eigenfunctions and wavenumbers.

An important dimensionless parameter that appears when the equations are linearised is $h = \frac{c}{\omega_{pi} L_x} = \frac{\lambda_i}{L_x}$. This term arises due to the density gradient initially imposed in the problem. A large value of h indicates that $L_x \ll \lambda_i$ and therefore that the Hall term

has a larger effect.

6.5.7.2 Numerical test problem

As discussed in the previous section, we obtain the x profile of the perturbations by solving the linearised system of governing equations. We then initialise the simulation with these profiles, to give:

$$B_z(x, y) = B_0 + \alpha B_1(x) \sin(k_1 y) \quad (6.47)$$

$$\rho(x, y) = \rho_0(x) + \alpha \rho_1(x) \cos(k_1 y) \quad (6.48)$$

$$u_x(x, y) = \alpha u_{x,1}(x) \sin(k_1 y) \quad (6.49)$$

$$u_y(x, y) = \alpha u_{y,1}(x) \cos(k_1 y), \quad (6.50)$$

where $\alpha = 10^{-2}$ is a small constant applied to ensure a linear response. The temperature is held constant at $T = 0.25\text{eV}$ and it is assumed that the plasma is composed of fully ionised aluminium with charge to mass ratio of $13/27$. The initial density profile in x is chosen to be:

$$\rho_0(x) = \rho_i \left(\frac{x}{L_x} \right)^2, \quad (6.51)$$

where ρ_i is varied in different runs to change the value of h and the impact of the Hall term. Table 6.3 shows the different values of h that were used in this work and their corresponding ρ_i . In the original work, $B_0 = 400\text{T}$. However, we found that this value of magnetic field caused the Alfvén speed for the lowest density cases to approach a non-negligible fraction of the speed of light ($v_A \sim 0.3c$ for $h = 5$). This in turn interacted with the Boris correction used in Gorgon to control nearly relativistic velocities and the test problem did not propagate linearly. Therefore, in this work we instead chose to use $B_0 = 4\text{T}$. The domain is 2D in x and y , with $L_x = 0.1\text{cm}$ and $L_y = 2\pi L_x$. We found good convergence with $(N_x, N_y) = (98, 612)$. The boundaries are periodic in y , reflective for momentum in x and we set B_z to have zero gradient across the x boundary.

Figure 6.16 shows ρ , u_x , u_y and B_z after half a wave period for $h = 1.0$. These profiles are calculated using the inertia solver in Gorgon. Each variable has a distinct profile in x , which then shifts upwards in the y direction without alteration, since these profiles are eigenfunctions of the linearised problem. These plots are consistent with the profiles produced in the literature [115]. The inertia solver evolves an equation for $\partial \mathbf{j} / \partial t$, rather than just assuming $\mathbf{j} = \nabla \times \mathbf{B} / \mu_0$ as is done when deriving the test problem. The fact that the inertia solver still correctly solves for the wave motion is important therefore as it indicates that it produces physical currents that are being coupled correctly to the hydrodynamic motion by the $\mathbf{j} \times \mathbf{B}$ term. This shows that we are relaxing to the correct Hall MHD limit despite both temporally and spatially underresolving the electron inertial scales [114]. Lineouts of the magnetic field are taken in the x and y directions, with the locations of these lineouts indicated by the yellow and

$\rho_i (\times 10^{-5} \text{kg/m}^3)$	$h = c/\omega_{pi} L_x$
149	0.5
37.2	1.0
16.6	1.5
9.31	2.0
5.96	2.5
4.14	3.0
3.04	3.5
2.33	4.0
1.84	4.5
1.49	5.0

Table 6.3: Table showing the values of density ρ_i used in the Farmer drift wave problem. A larger value of h indicates the Hall term has a larger impact. These values are the same as in ref [115].

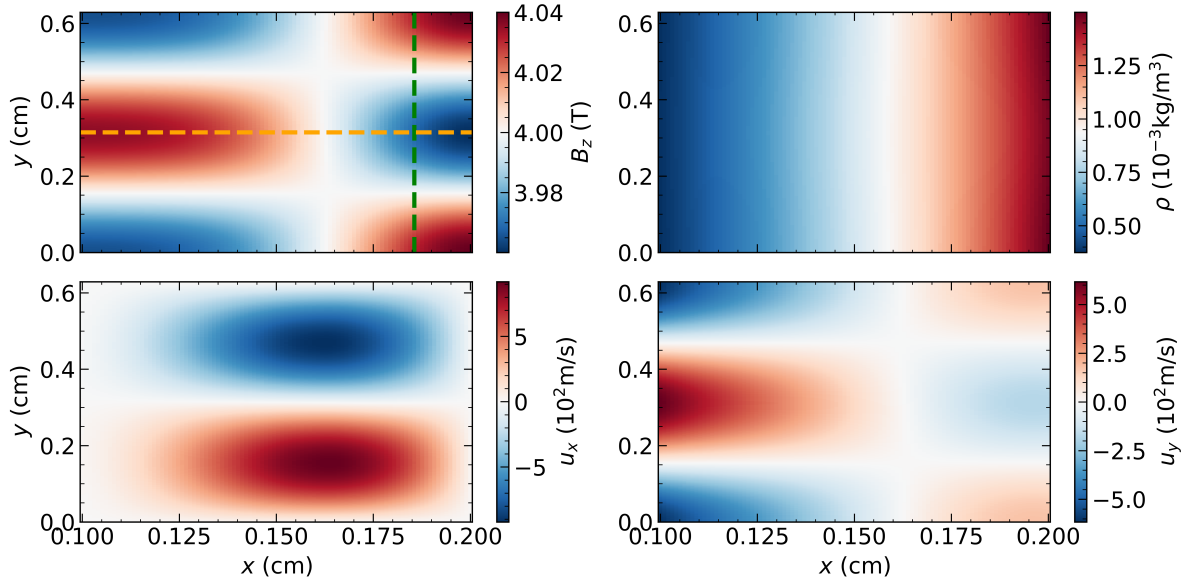


Figure 6.16: Plots showing the magnetic field B_z , the mass density ρ , the x component of the velocity u_x and the y component of velocity u_y for the Farmer test problem after half a wave period. The yellow and green dashed lines on the top left plot show the directions of lineouts taken, shown in figure 6.17.

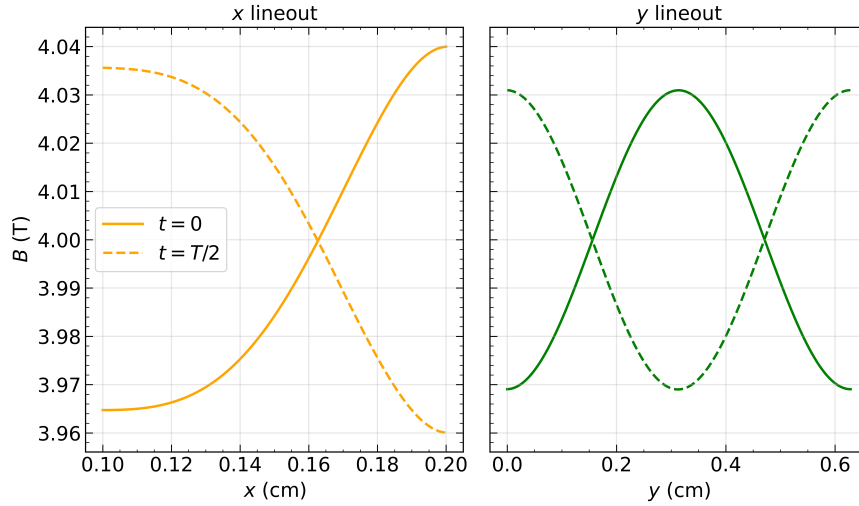


Figure 6.17: Lineouts of the magnetic field from the Farmer test problem, indicated by the lines in figure 6.16. The left plot shows the lineout along x and the right the lineout along y . The solid lines should the initial condition and the dashed lines show the profiles after half a wave period.

green dashed lines respectively in figure 6.16. These lineouts are plotted in figure 6.17. The wave propagates in the positive y direction, with the x profile only changing due to this motion. As with the advection problem, we find the van Leer flux correction when calculating the $\mathbf{u} \times \mathbf{B}$ fluxes is necessary to minimise numerical diffusivity. As with the Hall drift wave, we find that the problem goes nonlinear after a few wave periods. The plasma profiles begin to shear and the wave no longer purely travels without alteration in the y direction.

The phase velocity of the Farmer drift wave was calculated for a range of different h , this is shown in figure 6.18. The phase velocity of the drift wave increases with h , as expected since the Hall velocity j/en driving this effect is inversely proportional to density. There is good agreement between all models used and the theoretical phase velocity, indicating that we are correctly reproducing the drift wave from the literature. The data calculated using the inertia model, shown using the green points, begins to lie below the theoretical curve for large values of h . This suggests that the electron inertia acts as a limiting factor in the drift wave speed for low densities, which would be consistent with the results from the whistler and non-hydrodynamic drift waves. This illustrates why some mechanism is needed to make the Hall effect practical to implement for vacuum simulations. We do note however that these densities are nearing the minimum values that can practicably be used in a pulsed power simulation and charge separation physics is likely to have an effect once the length scale becomes comparable to the electron inertial length. A direction of future work could be to explore how the evolution of these drift wave modes changes in very low density plasmas where charge separation cannot be ignored. The points at $h = 0$ use the same density as indicated in table 6.3 for $h = 0.5$, but the Hall effect is turned off in the calculation. It should

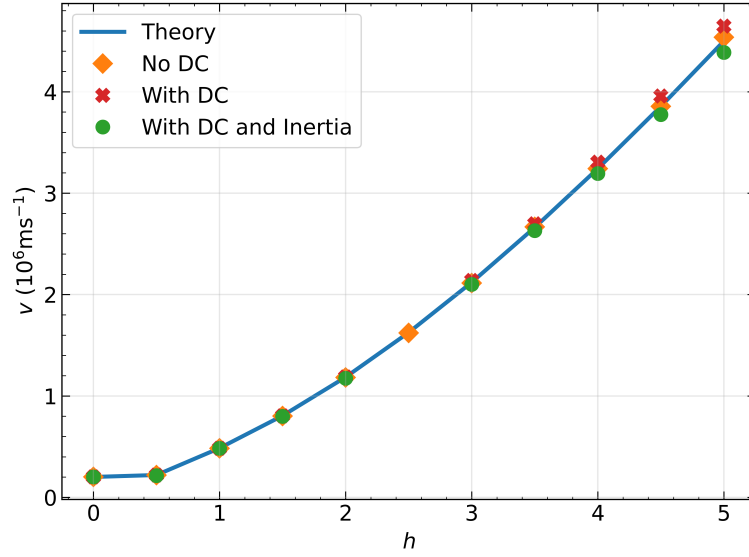


Figure 6.18: Plot showing how the phase velocity of the Farmer drift wave (discussed in section 6.5.7) varies as a function of the dimensionless parameter $h = \frac{c}{\omega_{pi} L_x}$, for various different models. The blue solid line shows the theoretical phase velocities, the orange diamonds the results with Hall MHD (not including the displacement current) and the red crosses indicate the velocities calculated using a model with displacement current. The green points show the results for the electron inertia model.

be noted that the velocity is not zero, as it was in the original drift wave. Due to the inclusion of the hydrodynamic motion, in the limit of a negligible Hall effect, the Farmer drift wave becomes an Alfvén wave and is well described by ideal MHD. The new field solver also recreates this limit. In this work and in the literature, the Farmer drift wave is only evolved for a single wavenumber, with $k_1 = 10^3 \text{m}^{-1}$. For the non-hydrodynamic drift wave, we found that higher modes went nonlinear earlier and that the effect of electron inertia was to cause the frequency of higher mode waves to asymptote at a lower value than the Hall MHD case. We would expect similar qualitative behaviour for the Farmer wave.

6.6 Hall MHD simulations of an $m=0$ MHD instability mode

In the previous section, we carried out extensive validation of the Hall solver and confirmed that it is consistent with theoretical results and past work. In this section, we apply the Hall solver to a more physical situation to determine the impact of the Hall effect in a more realistic scenario. As highlighted in the introduction, there is a plethora of applications where it is believed that Hall physics is having an important impact and we only study a single case in this work. The further refinement of this Hall solver and use in problems such as power flow or studying laboratory astrophysics experiments like

those carried out on MAGPIE is a significant avenue of potential future work.

We study the evolution an $m=0$ mode in a z-pinch as this problem for our Hall model. There are a few reasons for this. The edge of a z-pinch is an example of a vacuum-plasma interface, which is what we have primarily been interested in for this work. In addition, the problem is a good stress test for the newly developed Hall solver. The density gradients are extreme, requiring robust treatment of the drift waves discussed in this chapter as well as management of the Alfvén speed in the low density vacuum region. Resistive MHD simulations of z-pinchs are plagued by high temperature at the vacuum-plasma interface due to runaway Ohmic heating; therefore this problem is also important to indicate whether our model can produce sensible values for temperature at the interface. Finally, the z-pinch is arguably the most basic situation that still has some experimental interest. Wire array targets and x-pinchs are still regularly fired on MAGPIE, both of which depend on accurate modelling of z-pinch physics to make physical predictions.

6.6.1 Problem Set-up

We model the problem in 3D Cartesian geometry; implementation of the Hall solver in cylindrical geometry is still ongoing. This substantially increases the computational overhead of simulations, but is necessary to capture the convergent effects of the geometry on the problem. The z-pinch is assumed to be an axisymmetric cylinder of fully ionised deuterium plasma at the start of the simulation. In reality, the pinch would begin as a wire or a gas puff through which a large current is passed, ionising the gas and producing a plasma. There is important physics in this process such as the electrothermal instability that can seed perturbations that lead to instabilities [61, 211]. Modelling of this so-called “cold-start” process is beyond the scope of this thesis, but is usually done in Gorgon so is a regime in which our Hall solver should be tested in the future. We use transmissive boundary conditions in the x and y directions although in reality there is a substantial vacuum region around the outside of the z-pinch that isolates the plasma from these boundaries. Periodic boundaries are used in the axial direction.

We begin our simulations in ideal MHD equilibrium, using the parabolic z-pinch profiles used elsewhere in this thesis:

$$n(r) = \begin{cases} n_0 \left[1 - \left(\frac{r}{a} \right)^2 \right], & \text{if } r \leq a \\ 0, & \text{otherwise} \end{cases} \quad (6.52)$$

$$B_\theta(r) = \begin{cases} \frac{\mu_0 I}{2\pi a^2} r, & \text{if } r \leq a \\ \frac{\mu_0 I}{2\pi r}, & \text{otherwise} \end{cases}, \quad (6.53)$$

where I is the applied current, n_0 is the number density on axis and a is the pinch

radius. For the work shown here, $a = 240\mu\text{m}$ and the line density is $N_l = 10^{20}\text{m}^{-3}$. The peak density on axis is related to the line density by:

$$n_0 = \frac{2N_l}{\pi a^2}. \quad (6.54)$$

A minimum mass density of $\rho = 10^{-2}\text{kg/m}^3$ is chosen to maintain reasonable Alfvén speeds in the low density vacuum. Any densities that fall below this are set to vacuum. The magnetic field profile chosen implies an axial current density of $j_z = \frac{I}{\pi a^2}$ in the positive z direction. The numerical speed of light $c_{MHD} = 3 \times 10^8 \text{ m/s}$ here and a resolution of $8 \mu\text{m}$ is used. An external current is applied using the method described in section 3.3.5. We use a current that varies in time with the profile:

$$I(t) = 10^8 t^{\frac{1}{3}} A. \quad (6.55)$$

A rising current is necessary to increase the magnetic field over time to compensate for the Ohmic heating that occurs as magnetic field diffuses into the pinch, as otherwise the initial equilibrium is not stable. This specific $I \propto t^{\frac{1}{3}}$ relationship is known as the Haines-Hammel curve [36] and has been verified to lead to a stationary z-pinch by past simulations and theory [127]. We begin our simulations at $t = 10\text{ns}$ as dI/dt is not defined at $t = 0$. In effect, we are assuming that the z-pinch has formed and reached an MHD equilibrium that we then perturb by the beginning of our simulation. From the Bennett relation, we have an initial temperature of $T_e = T_i = 72\text{eV}$. It is worth remarking that we have only chosen a particular set of pinch parameters as a test of the Hall solver. Our conclusions are not generally true for all pinches. Comprehensive theoretical work on z-pinch stability in the regime of Hall MHD has been carried out (for a review see ref. [36]) and other sets of pinch parameters should be further investigated.

We have verified that if no initial perturbation is applied, the equilibrium remains stationary. An axial perturbation in the radial velocity is applied to drive the pinch to go m=0 unstable:

$$v_r(r, z) = \begin{cases} -v_0 \left[1 + \cos\left(\frac{2\pi m z}{L_z}\right) \right] \frac{r}{a}, & \text{if } r \leq a \\ 0, & \text{otherwise} \end{cases}, \quad (6.56)$$

where m is the mode number, L_z is the length of the pinch in the axial direction and v_0 is the initial size of the perturbation. For this work, we set $v_0 = 10^3\text{m/s}$ to drive a reasonable instability growth in practical computation time. This is still subsonic however. Figure 6.19 shows the form of this velocity perturbation.

The parabolic profile we are using here is among the more challenging initial conditions for z-pinch simulations and has not yet been studied computationally with the Hall effect. Past work on Hall MHD simulations of z-pinch edges has used the far shallower Bennett profiles and therefore not had the issues with a numerical vacuum that need to

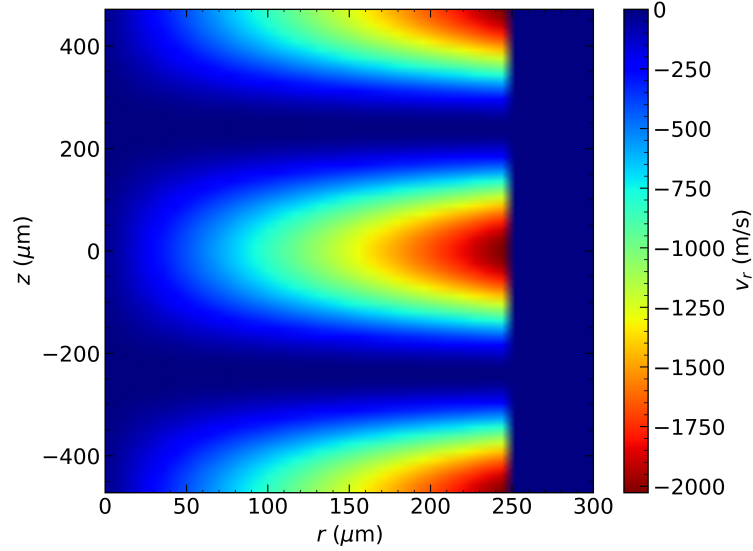


Figure 6.19: Figure showing how the radial velocity of the z-pinch varies over the simulation domain for mode number $m = 2$. There are periodic boundary conditions at the top and bottom of the domain. The velocity perturbation seeds an $m=0$ instability.

be handled in this work [202, 204].

6.6.2 Evolution of the pinch

In this section, we compare the evolution of the pinch in the case where the Hall effect is included to the result from resistive MHD. We consider just a single mode perturbation, with $ka = 3.8$. Figure 6.20 shows how the mass density evolves for the pinch over time in both cases. The time is normalised to the characteristic growth time of the instability, which is the time taken for an Alfvén wave to cross the pinch radius. We calculate this Alfvén time using the magnetic field at $r = a$ and the average density over a circular cross-section $\hat{\rho}$, to give:

$$\tau_A = \frac{a}{v_A} = \frac{a\sqrt{\mu_0\hat{\rho}}}{B_\theta(r=a)}. \quad (6.57)$$

For our parameters, this has a value $\tau_A = 1.7\text{ns}$. We show a lineout of the $y-z$ plane for a particular value of x . We have found that the pinch remains azimuthally symmetric and thus the value of x does not matter. This highlights that we could carry out this simulation in 2D $r-z$ cylindrical geometry and also attain better resolution, but the implementation of the solver in this geometry is still ongoing.

Initially, we only have a perturbation in the radial velocity. It takes some time for this information to be transmitted from the edge to smaller radii inside the pinch, so at first there is not much evolution in either case as can be seen from the first two panels. At $t = 0.58\tau_A$, there are high frequency perturbations in the Hall case that do not occur in the no Hall case. As highlighted before when discussing the drift wave,

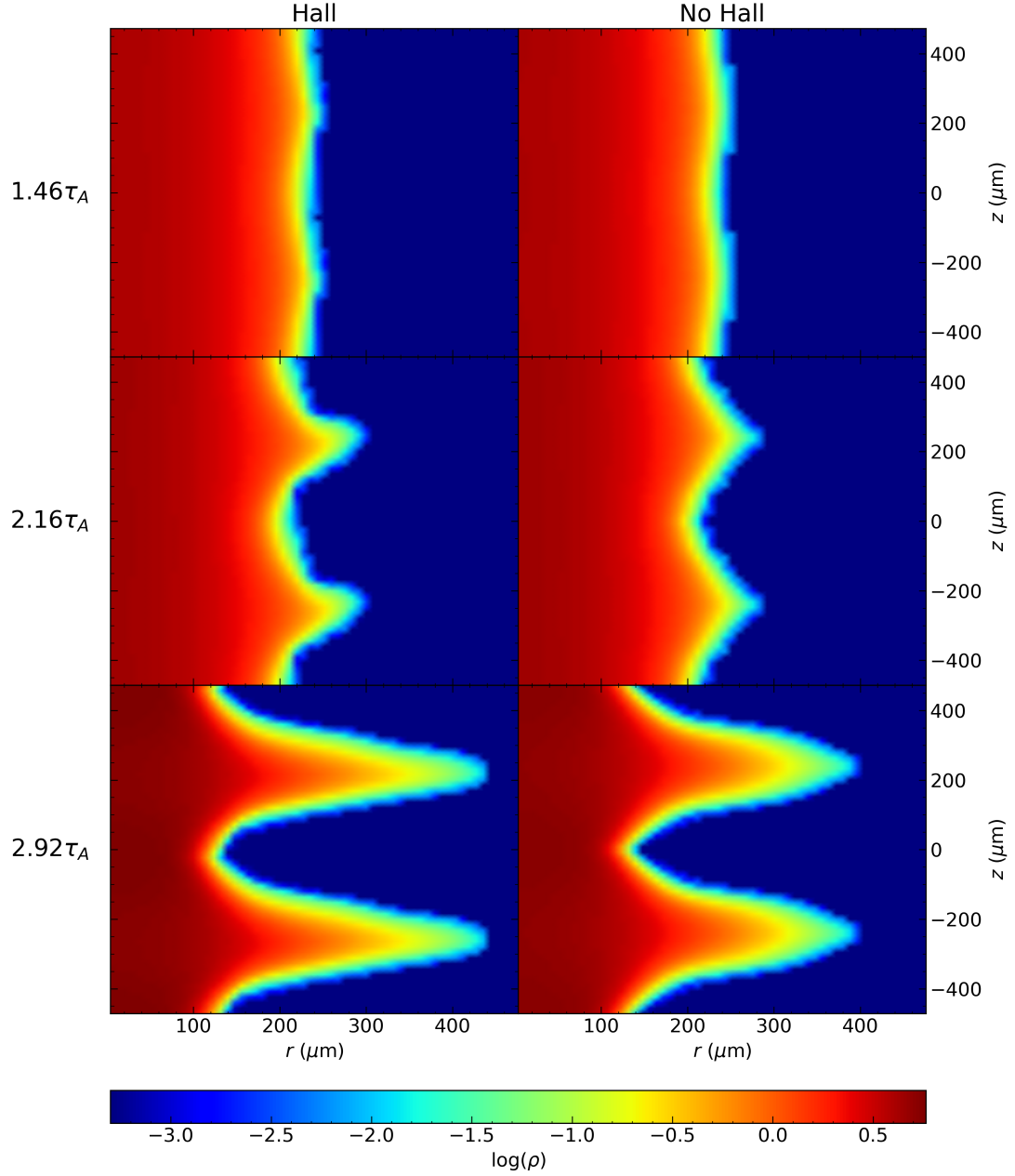


Figure 6.20: Plots showing how the mass density of the z -pinch evolves over time in the Hall and no Hall cases. The time is normalised to the characteristic growth time of the instability, the time for an Alfvén wave to cross the pinch radius, denoted τ_A . The plots are shown on a logarithmic scale. The current flow is in the positive z direction in these plots.

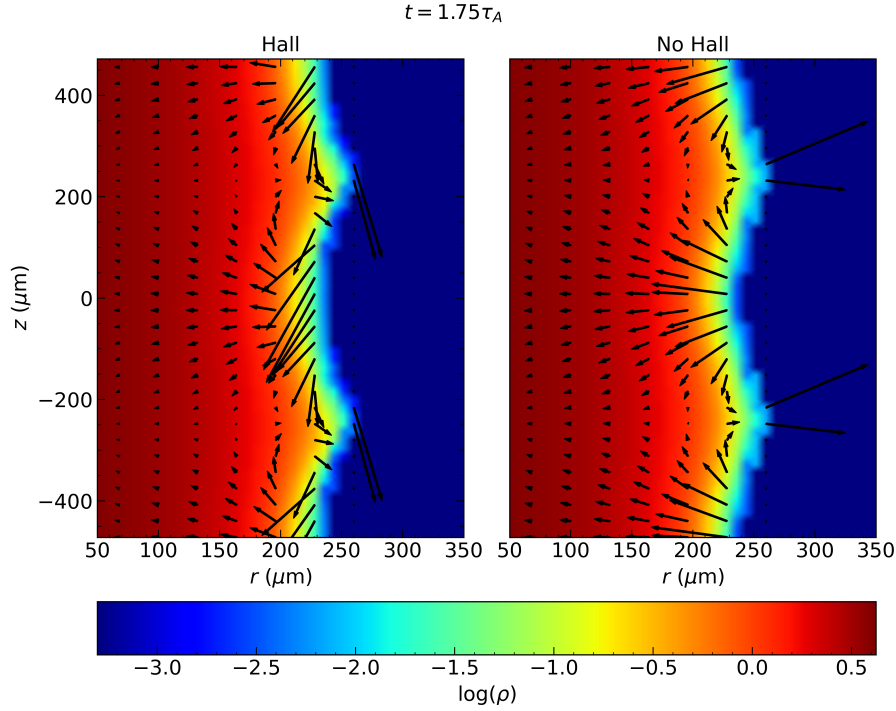


Figure 6.21: Plot showing how the macroscopic fluid velocity vector, shown by the black arrows, varies across the domain. The velocity vectors are plotted over the density profile. The angle of the arrow represents the direction of the vector in the $r - z$ plane and the size represents the magnitude.

this is a characteristic behaviour of the Hall effect. It drives small scale structure in the magnetic field, and thus also the density. Electron inertia should saturate this short wavelength structure at the electron inertial length c/ω_{pe} , but it is not practical to resolve that length scale with our grid. This is discussed later in this section. We find that a higher resolution grid still produces this high frequency noise, as expected.

After approximately an Alfvén crossing time, the instability does settle to the dominant mode seeded by the velocity perturbation. An axial perturbation in the radius of the pinch emerges due to the compression by the magnetic field and perturbed radial velocity. Where the radius is minimum, the magnetic field is larger than if the radius were unperturbed since $B_\theta \propto \frac{1}{r}$. This therefore leads to a higher magnetic pressure here that exceeds the thermal pressure, pushing plasma away from this neck region into the bulging regions. This leads to a lower thermal pressure in the neck, causing it to be compressed more, increasing B_θ and causing the cycle to repeat. The reverse argument applies in the bulging regions, leading to formation of the characteristic sausage shape of the instability. This can be seen in the second set of panels in the figure.

It is clear after a couple of Alfvén crossing times that the “bulge” of the instability shears upwards in the Hall case, where it remains symmetric in the no Hall case. The

electric field associated with the Hall effect is given by:

$$\mathbf{E}_{Hall} = \frac{\mathbf{j} \times \mathbf{B}}{en_e}. \quad (6.58)$$

In our case, $\mathbf{B}$ is in the $\hat{\theta}$ direction and $\mathbf{j}$ is in the $\hat{z}$ direction. This means that the Hall electric field is in the inward radial (i.e. $-\hat{r}$) direction. The ions and electrons in the plasma therefore undergo an $\mathbf{E} \times \mathbf{B}$ drift in the $-\hat{r} \times \hat{\theta} = -\hat{z}$ direction. This leads to a fluid velocity in the negative axial direction. It is the combination of this axial flow and the radial inward flow that leads to the observed shearing. Figure 6.21 shows the formation of the lobe at $t = 1.75\tau_A$ to highlight this. The velocity vector is plotted over the density profile. The direction of the arrow represents the direction of the velocity in the $r - z$ plane and the size represents the magnitude. Towards the centre of the z-pinch, the Hall and no Hall cases are nearly indistinguishable. There is a fluid flow away from the “neck” regions of the pinch towards the bulge regions, which is what develops the instability as discussed above. The significant difference between the Hall and no Hall cases is clear around $r = 225\mu\text{m}$. In the no Hall case, the fluid flow is still away from the neck region to the bulge albeit with a large magnitude. In the Hall case, the velocity vectors have been deflected downwards by the Hall-driven $\mathbf{E} \times \mathbf{B}$ drift leading to the shearing of the density that we observe. It can be seen that the drift at the bottom of the bulge regions causes the velocity vector to actually divert radially outwards; the lobe regions are caused to curl upwards by the Hall effect. It has previously been observed in both kinetic [208] and Hall MHD [203] simulations of Magneto Rayleigh Taylor instabilities that the Hall effect can cause a vorticity to develop, leading to a curling of the MRT spikes in a way that is qualitatively similar to what we observe in this case. As remarked in chapter 1, past laboratory astrophysics experiments have observed curling of plasma jets in a preferential direction that depends on the polarity of the current flow [83]. We believe the effect described in this paragraph is responsible for these observations.

It is worth noting that grid imprinting is apparent in both figure 6.20 and figure 6.21. This is due to attempting to resolve a cylindrically symmetric process on a relatively coarse Cartesian grid. It can be mitigated through higher grid resolution, but since we are limited to 3D Cartesian simulation at present, the computational demands are high. Figure 6.22 shows the density in the Hall/no Hall case for a resolution of $4\mu\text{m}$ (double that shown before). It can be seen that the profiles are qualitatively very similar, with an asymmetry in the Hall case that does not occur in the no Hall case. We do find that the high frequency noise that occurs at the start of the simulation seeds a secondary mode of the instability in the higher resolution case. This is significantly smaller in amplitude than the main mode seeded by the velocity perturbation however. Further investigation of this is needed.

At later times, as shown by the third set of panels of figure 6.20, the growth of the instability accelerates. By this point, the instability has become nonlinear. Here, the difference between the Hall and no Hall cases is less pronounced, though there is still

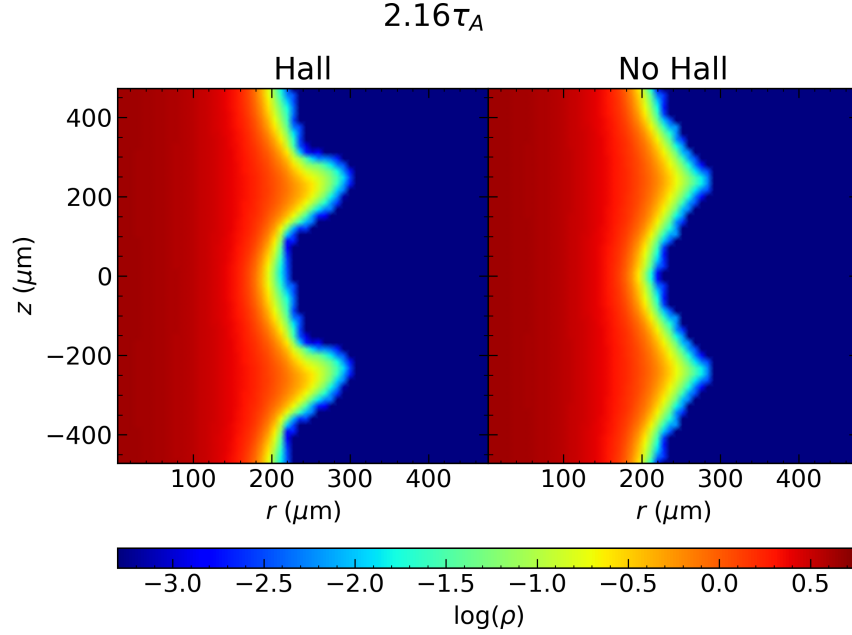


Figure 6.22: Plot showing how the density compares in the Hall/no Hall case for a resolution double that shown in figure 6.20.

some slight shearing in the density profile. This is because as the radius decreases, the density increases and therefore the ion inertial length decreases. Towards the middle of the pinch, there is not much difference between the Hall and no Hall cases as the density is too high for the Hall term to have a significant effect. The main difference between the two cases is that the Hall effect leads to an extended low-density “tail” in the bulge regions of the instability. This may be due to the increased $m = 0$ growth on the surface in the Hall case, but the reason for it is still not understood.

6.6.3 Magnetic field profile

The density evolves in the way that we discussed in the last section due to the Hall effect changing the magnetic field profile. In this section, we discuss the evolution of the magnetic field to elucidate further the impact of the Hall term. Figure 6.23 shows the evolution of the magnetic field over time. The red line in the figure is the contour of $B_\theta = 170\text{T}$, chosen to highlight the differences between the Hall and no Hall cases. Initially, the magnetic field is axially uniform. As the pinch implodes, the magnetic field is higher where the pinch radius is smaller since $B_\theta \propto \frac{1}{r}$. As can be seen from the contours at $t = 1.46\tau_A$, there is little difference between the Hall and no Hall cases at this point. At $2.16\tau_A$, it is clear that the Hall effect leads to an increased magnetic field perturbation; with a reduction in magnetic field in the regions between the areas of higher field. In other words, the Hall effect increases the size of the magnetic field perturbation. It can be seen from the contours that the lobes of high magnetic field are sheared upwards by the Hall term where they remain symmetric in the no Hall case. As in the density, in the final set of panels in the figure, the differences between

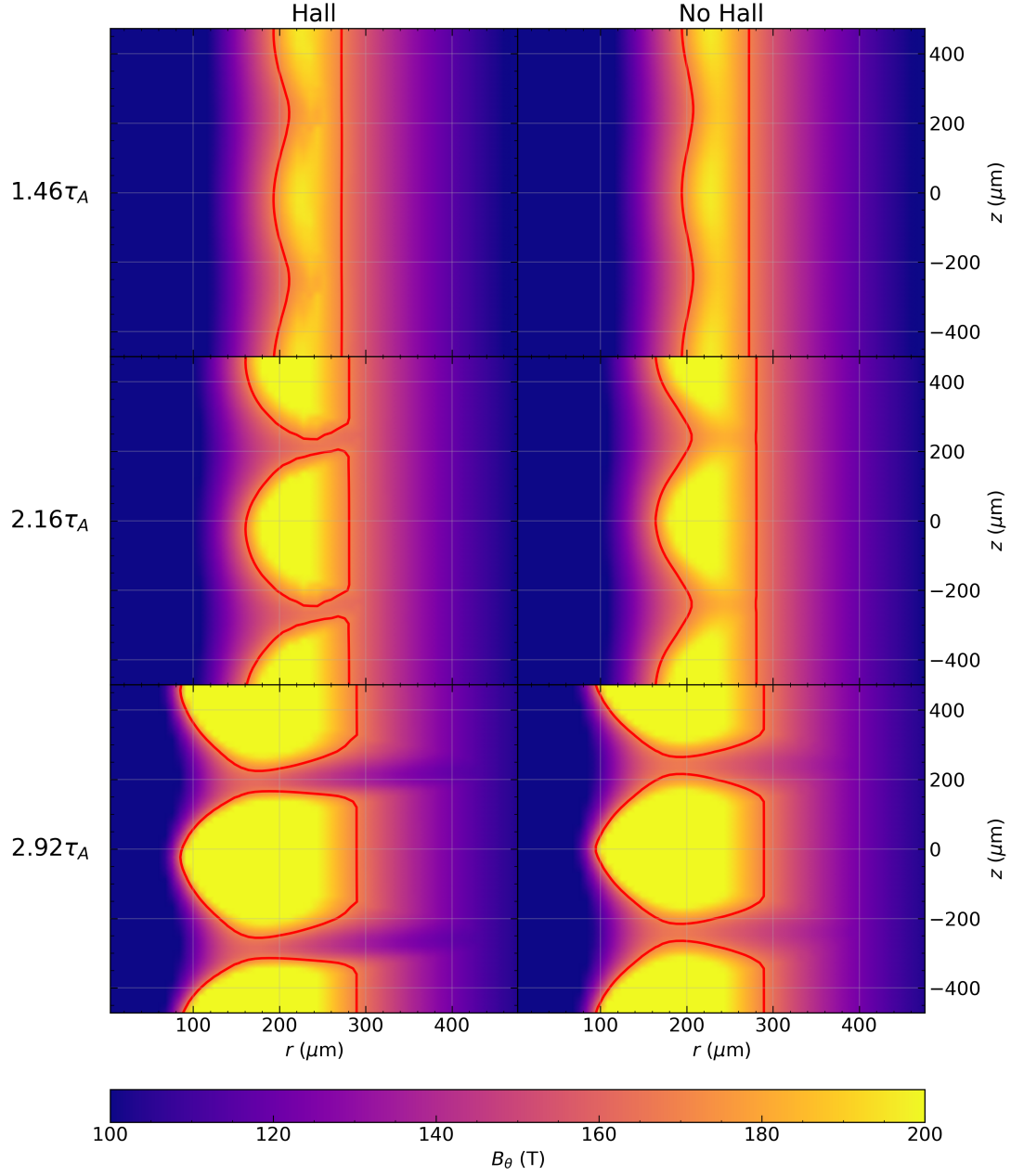


Figure 6.23: Plots showing the evolution of the azimuthal B_θ magnetic field in the $m=0$ instability problem over time. The case with the Hall term included in the calculation and with the Hall term excluded are compared. The red line is the contour of $B_\theta = 170\text{T}$, added to highlight the differences.

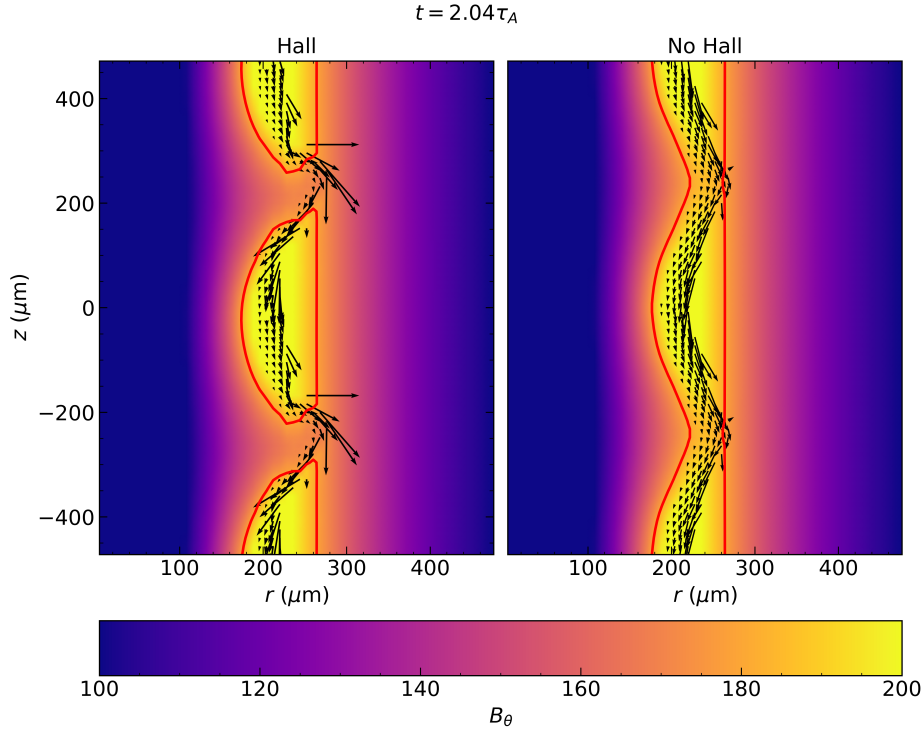


Figure 6.24: Figure showing the electron drift velocity vector $\mathbf{v}_e = -\frac{\mathbf{j}}{en_e}$ plotted over the magnetic field profile. The angle of the arrows is the direction of the vector and the size is the magnitude. The red line is the contour of $B_\theta = 180\text{T}$.

the Hall and no Hall case are less stark. However, there is a shearing in the Hall case that does not exist in the no Hall case. The overall effect is that the Hall term exacerbates the magnetic field perturbation and causes shearing in the lobes of magnetic field, corresponding to the same shear that was seen in the density.

In figure 6.24, we plot the electron drift velocity vector $\mathbf{v}_e = -\frac{\mathbf{j}}{en_e}$ over the magnetic field profile. This is shown by the black arrows. The Hall effect can be thought of as the advection of magnetic field with the electron drift velocity (compared to frozen-in-flow which is advection with the bulk fluid velocity). It is clear that the regions of decreased magnetic field that we highlighted in the previous figure correspond to regions of increased electron drift velocity. There is a large drift velocity round the edge of the $m=0$ “lobes”, which advects magnetic field away from these regions to exacerbate the perturbation. It is clear as well that the electron flow is driving a vorticity in the magnetic field profile, consistent with past theoretical work [207] and other results in this chapter.

6.6.4 Ohmic heating

As the magnetic field compresses the plasma, it diffuses into the plasma due to the finite resistivity. The Ohmic heating this causes is important to include as the increased

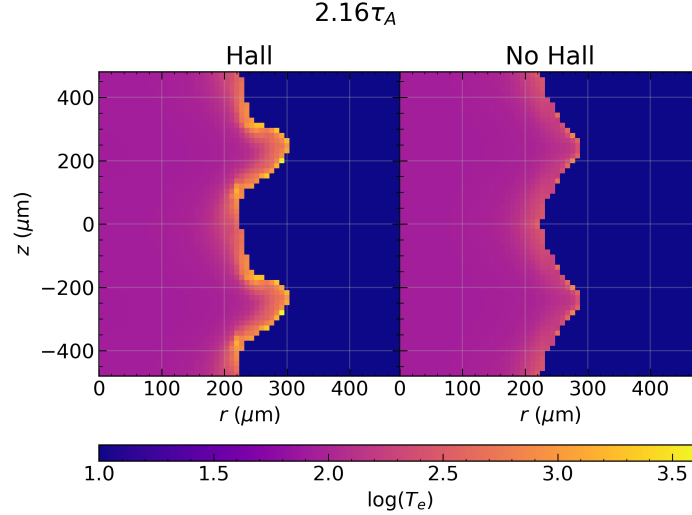


Figure 6.25: Plot showing the electron temperature (in eV) across the domain at a time of $2.16\tau_A$, for both the Hall on and Hall off cases. The temperature of the vacuum is set to a floor value of 10^{-2} eV.

plasma temperature also increases the thermal pressure and changes the growth of the instability. Figure 6.25 shows the electron temperature of the plasma at the point where the asymmetric lobes have formed. The vacuum region is set to a floor temperature of $T_e = 10^{-2}$ eV, though the value is not relevant because it does not take apart in the hydrodynamics. The temperature is higher at the surface of the plasma, where there is a spike in the current density. The Hall effect leads to a larger spike in current density, which is why the surface temperatures are higher in the case with the Hall effect turned on. It can be seen that the regions of highest temperature in the Hall case correspond to where the current density is higher at the top and the bottom of the plasma bulges, as can also be seen in figure 6.24. The temperature is then diffused inwards by thermal conduction, though the high temperatures are mostly localised to the surface of the pinch. The temperatures obtained by the plasma are quite extreme, particularly in the Hall on case (reaching as high as 10 keV in some cells). Though the edge of the pinch is observed to be hot in experiments, it is unlikely that the temperature would reach values as high as this. This indicates that some additional mechanism is needed to limit the current densities at the vacuum-plasma interface, particularly when the Hall effect is included. It has been suggested in prior studies that the Hall effect itself limits this current density [63], but we do not find this to be the case in our work. The Hall effect in fact worsens the problem through the production of larger current densities on the surface.

Figure 6.26 shows the ion temperature across the domain. The ions are not directly heated by Ohmic heating, but we find that they also display significantly higher temperatures at the surface of the pinch. This is due to energy gained through collisions with the electrons. We do not include the Ettingshausen effect in these simulations, which would likely mitigate the temperature separation as was seen in chapter 4. It should be

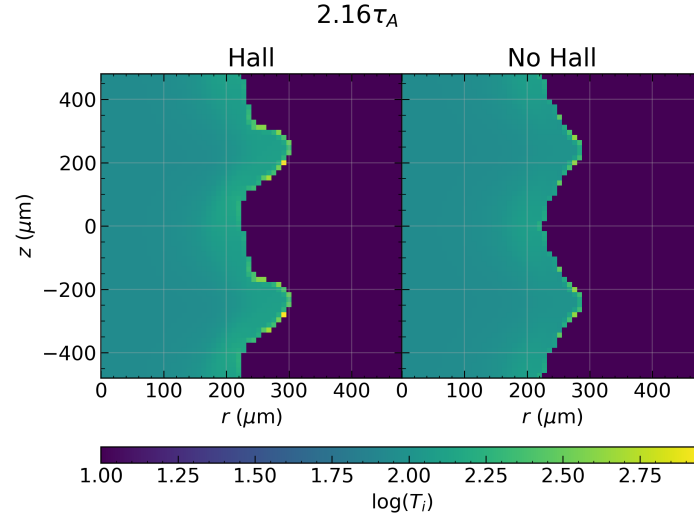


Figure 6.26: Plot showing the ion temperature (in eV) across the domain at a time of $2.16\tau_A$, for both the Hall on and Hall off cases. The temperature of the vacuum is set to a floor value of 10^{-2}eV .

noted that the ion temperatures are significantly lower than the electron temperatures at the surface in both cases.

6.6.5 Impact on growth rate

It is not possible to obtain a single value for the growth rate of an $m=0$ instability when resistive diffusion and Ohmic heating are included, because the growth rate varies over time [212, 213]. However, we can still study how the amplitude of the instability varies with time. We define the quantities $\psi(z, t)$ and $\xi(z, t)$:

$$\psi(z, t) = \int_0^a 2\pi r \rho(r, z, t) dr \quad (6.59)$$

$$\xi(z, t) = \int_0^a 2\pi r B_\theta(r, z, t) dr. \quad (6.60)$$

These are the radially and azimuthally integrated mass density and magnetic field respectively. They are metrics of the size of the perturbation. As we are studying just a single mode perturbation, we can take the amplitudes of these sinusoidally-varying quantities over time to see the growth rate of the instability. The variation of this amplitude is shown in figure 6.27 for both the Hall on and Hall off cases. It can be seen that both ψ and ξ increase exponentially, as they should for an instability. For both ψ and ξ , at early times the instability grows faster with the Hall term turned on. This is the point where the Hall is exacerbating the magnetic field perturbation as we discussed in section 6.6.3. At later times however, the Hall and no Hall cases grow at the same rate (shown by the solid and dashed lines being parallel at later times). This is as we discussed in figure 6.20, where the Hall effect has less impact at late times due to the higher densities in the neck regions of the pinch.

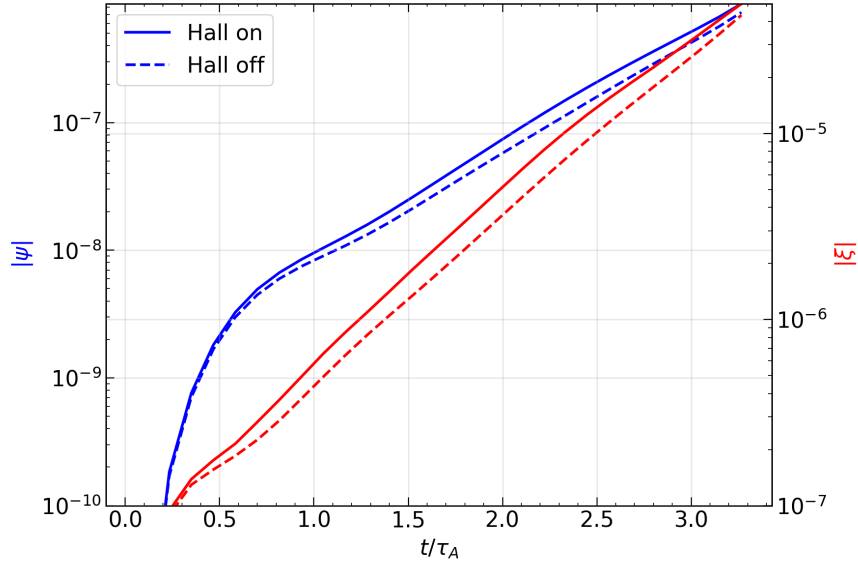


Figure 6.27: Plot showing how the amplitude of the perturbation in radially integrated density (ψ) and the perturbation in radially integrated magnetic field (ξ) vary over time. The solid lines show the Hall on case and the dashed lines show the Hall off case.

We remark that the difference in the growth rate between the Hall and no Hall cases is small for the parameters we have used. Extensive theoretical work has found parameter regimes where the Hall effect can either stabilise the $m=0$ mode, or even significantly enhance the instability growth [36]. Different plasma parameters, such as the line density or pinch current, will produce a different regime of Hall effect. Furthermore, we have only studied a single mode number. Different mode numbers will also grow at different rates. Analysis of the linear equations has found that the stability of the pinch can depend on the equilibrium profiles chosen as well [214]. A study of how this instability changes for different parameters is an intended avenue of future work. For the purpose of this chapter, we have shown that the new Hall solver in Gorgon can produce physically meaningful answers in an extremely challenging test case.

6.6.6 The effect of electron inertia

Throughout these $m=0$ simulations, we tested the effect of the electron inertia terms on the evolution. We found that the inertia had very little effect on the results, not quantitatively changing the growth rate or the qualitative structure of the vacuum-plasma interface. This was surprising, since past work has suggested that electron inertia is key for a robust Hall solution [63, 114]. Also in theory, the inclusion of electron inertia should cause the short-wavelength structure in the $\mathbf{B}$ field produced by the Hall effect to saturate at the electron inertial scale $\lambda_e = \frac{c}{\omega_{pe}}$. The test problems that we discussed also showed that the dispersion relations should be changed significantly by the inclusion of the inertial terms.

However, for pulsed power simulations, there is almost no situation that can be prac-

tically simulated where the electron inertia will have a significant effect. This is just due to the high densities involved. For the $m=0$ simulations, the electron inertial length varies from approximately $1\mu\text{m}$ at the edge of the pinch to $0.1\mu\text{m}$ at the core. Both of these values are well below anything that could be sensibly resolved. We also significantly underresolve the electron plasma frequency in time (allowed by our implicit scheme). Therefore, the electron inertia has no effect in our case. This suggests methods other than that of electron inertia would be needed to stabilise a Hall solver, as discussed in the next section.

In these simulations, for computational tractability, we used a relatively high cutoff density of $\rho = 10^{-2} \text{ kg/m}^3$. Using a lower cutoff density, there would be more low density plasma in the simulation which could lead to the Hall effect having a stronger effect. This should be investigated further. However, even with the current cutoff density, we still find electron drift velocities on the surface of the pinch with magnitude $\approx 10^7 \text{ m/s}$. Given that the numerical speed of light always needs to be larger than the electron drift velocity, this severely limits the size of the timesteps that can be taken. This will be worsened with lower cutoff densities.

6.6.7 Implementation challenges

As intended, the $m=0$ problem exposed several implementation challenges in the Hall solver that should be discussed.

It is common to use a reduced speed of light within pulsed power simulations to keep the timescales tractable. However, when using a reduced speed of light we found that the inertia solver was unstable if we did not ensure that the chosen numerical speed of light was greater than the electron drift velocity everywhere. This proved to be problematic because the current density, and hence the electron drift velocity, on the surface of the pinch increases during the course of the implosion. This means that it is difficult to a priori decide on a numerical speed of light. As a result, we often used numerical speeds of light that were significantly higher than they needed to be for stability, increasing simulation runtime. For the parameters we present here, we found stability problems if we chose $c < 10^7 \text{ m/s}$. If the electron inertia was not included in the update then we found that a reduced speed of light did not make the solver unstable but it did qualitatively change the answers. In this case, there was no observable difference between the Hall and no Hall cases and the growth rate of the instability was reduced. This is likely because a reduced speed of light is artificially damping the Hall effect in this case.

As we have highlighted multiple times in this chapter, the Hall effect drives short wavelength structure in the magnetic field and hence also in the density. This is an issue for simulations because it requires an increasingly small timestep to resolve due to the Hall drift and whistler wave timesteps. A robust Hall solver needs some strategy to mitigate this. One potential method would be introducing hyper-resistivity into Ohm's Law, as discussed in section 6.1.2. This is an additional term of the form $\frac{\nu}{en_e} \nabla^2 \mathbf{j}$.

This is often used in conventional Hall MHD solvers and has the effect of dissipating small-scale structure in the EM fields, as desired [112, 113, 198, 200, 201]. However it would require an adaptation of the method that we have developed in this chapter as it involves a derivative of the current density so the same source-hyperbolic term split would not be possible. Alternatively, it has been speculated that the electron inertia will saturate the current density and length scale at reasonable values [114]. However, as we discussed in the previous section, these values are still not practically resolvable for most cases that we are interested in. In principle, we could increase the effective electron mass since this would increase the electron inertial length and make it easier to resolve. However, for the simulations presented in this chapter, we would need to use an electron-ion mass ratio of $\frac{m_e}{m_i} \approx \frac{1}{25}$ to make resolving the inertial length practical. These are quite extreme electron masses. A final approach is to use anomalous resistivity, which would introduce an additional resistivity in the lowest density parts of the plasma. This reduces the current density at the vacuum-plasma interface, which we observed was likely too high. It also has the effect of dissipating the smallest scale structures, which tend to occur at the edge where the density is lowest and the Hall effect is strongest. Preliminary simulations using anomalous resistivity do not show a significantly changed growth rate, but the surface current density (and thus the temperature) is reduced. This is promising but further investigation is needed.

There are several places of “user choice” in the update, most notably in the averaging scheme used to obtain the necessary components at electric field locations on the Yee grid. We have used the same averaging scheme throughout all our results presented in this thesis, but have found that a different averaging scheme (for example interpolating everything to cell centres and performing the update there instead) can produce qualitatively different answers. These different implementations still pass the test problems we laid out in section 6.5, but vary for more complex problems such as the evolution of the $m=0$. This also explains why we find quite different conclusions to some of the suggested results from the PERSEUS code, as a result of our different implementations [114]. A comprehensive comparison of different Hall implementations (e.g. staggered grid vs cell centered grid with divergence cleaning), but within the same core code with the same hydrodynamics, would be a valuable and important piece of work to try to isolate the issues in different algorithms and produce a solver that gives consistent answers irrespective of small implementation changes.

7 Conclusions

7.1 Summary of Results

In this thesis, we have used several different models to better understand the physics of vacuum-plasma interfaces, particularly at the edge of a z-pinch. These models include both semi-analytic, extended MHD and kinetic models. Here, we briefly summarise the physics results and highlight potential avenues of future work.

7.1.1 Self-similar model

In chapter 4, the impact of various extended MHD terms was explored using a semi-analytic self-similar equilibrium model.

We showed in section 4.5 that the equations of extended MHD in 1D planar geometry admit self-similar solutions in the subsonic limit. We derived the form of the self-similar equations in terms of a variable $\eta \propto \frac{x}{\sqrt{t}}$. In section 4.6, a numerical method to solve these equations as a boundary value problem was laid out. The generality of our approach is an extension of past work that only studied a particular regime of plasma β , or that had restrictive boundary conditions that meant it was only suitable for certain problems.

In section 4.5.3, it was shown that the plasma β is a key parameter determining which magnetised transport term dominates in equilibrium problems. Therefore, in section 4.7, we performed a parameter scan over β to demonstrate the different regimes. In high β plasmas, which are most characteristic of magneto-inertial fusion experiments, the thermal transport is dominated by thermal conduction. The only role of magnetic field in the evolution of the equilibrium is via the magnetisation of the conductivity. The magnetic field is advected by the Nernst and frozen-in-flow effects, with little impact from resistive diffusion. This is consistent with past work. In contrast, in the low β regime, the resistive diffusion dominates over the Nernst effect in magnetic field transport. Thermal transport is complicated in this regime as all terms in the energy equation can play a role depending on the initial temperature profile. Notably however, the Ohmic heating becomes significant in low β plasma and can be the dominant effect in cases with a strong magnetic field gradient, such as at a plasma-vacuum interface. It was found that as the β reduced, the length scale of the magnetic field gradient also decreased. Magnetised transport is gradient driven and thus this had a key effect in determining the balance of the different transport terms. We considered several order unity β cases, concluding that all terms can play a role and that the balance of them is

highly dependent on the initial profiles for temperature and magnetic field. We found that it was not possible to characterise the dominant effect solely by a dimensionless parameter, underlining the need for a quick model like that developed in this chapter to establish which magnetised transport effects may play a role in a particular situation.

Through comparison of the self-similar model to the two-temperature MHD model in the Gorgon code, it was shown in section 4.8 that the Ettingshausen effect plays a key role at a vacuum-plasma interface. Excessively high electron temperatures in resistive MHD simulations can be mitigated by including the Ettingshausen term which advects electron temperature down the magnetic field gradient towards the bulk of the plasma.

Finally, in section 4.9, the design of a potential experiment to measure the Nernst effect in a pulsed power setting was considered. It has been proposed that one could observe the Nernst in plasma outflow from a wire array colliding with a cold wall. Whilst this is not an equilibrium situation, an estimate using the self-similar code indicated that this plasma is not sufficiently hot to observe a significant Nernst effect. A hotter plasma with temperature $T_e \sim 100\text{eV}$ would produce a more substantial difference in the magnetic field due to the Nernst advection of magnetic field into the cold wall. However, this would be an extremely difficult measurement to make, indicating that laser-plasma experiments may be more suitable to measure the Nernst effect due to their higher plasma temperatures.

7.1.2 Kinetic simulations of a vacuum-plasma interface

In chapter 5, we carried out fully kinetic particle-in-cell simulations of the edge region of a z-pinch in order to investigate how the structure differed from that predicted by resistive MHD.

In section 5.3, we carried out 1D simulations using a Cartesian approximation to the cylindrical geometry of the z-pinch. Despite simulations starting in MHD pressure balance, electrons accelerate into the initially vacuum region where they are confined by the magnetic field on a longer timescale. At the same time, ions are accelerated into the vacuum by the resulting charge separation electric field set up by the electron motion. The ions are deflected on a much longer time and spatial scale than the electrons leading to a charge separation layer, with an electric field dominated by the Hall term. This indicates that the Hall effect is the most important extended MHD term to include to model the vacuum-plasma interface more physically. It was found that whilst the distribution function is non-Maxwellian during the formation of the charge separation layer, this is a transient effect and on longer timescales the distribution is not anisotropic. However, further work using more resolution or a continuum code is needed due to difficulties in resolving the distribution function well using PIC.

In section 5.4, 2D simulations motivated by an $m=0$ instability in z-pinch were carried out. It was found that lobes formed in the magnetic field due to advection of field away from high density regions. This is the same behaviour as expected in resistive MHD. However, unlike in resistive MHD, the lobes in magnetic field then began to shear

downwards in the direction of the current flow. At later times, this sheared magnetic field also caused the plasma density to shear. We compared this result to Hall MHD simulations of the same problem carried out with Gorgon and found close qualitative agreement. This emphasises that the Hall effect is crucial for more accurately modelling a z-pinch edge.

The steep density gradient at the z-pinch edge required large numbers of particles to resolve, making the PIC simulations extremely computationally expensive and limiting runtime. In section 5.5.1, the features of an ideal kinetic code for simulating vacuum-plasma interfaces were highlighted. Whilst PIC is a powerful tool, a continuum method would likely be better for this problem due to the density gradient. The computational cost of such a code could be improved by using a hybrid model as the electrons are fairly collisional except in very low densities.

7.1.3 Hall physics model with electron inertia

In chapter 6, we developed a new MHD solver for the Gorgon code that included the effect of the Hall term semi-implicitly. This is essential as the wave modes produced by the Hall term require impractically small timesteps if an explicit implementation is used.

The algorithm for this solver is described in section 6.3. The problem is complicated by the staggered grid and numerical vacuum used within Gorgon. Quantities need to be interpolated to the correct position on the staggered grid, distinguishing whether those quantities are calculated from a plasma or a vacuum cell. This introduces a degree of freedom of choice in implementation; both in where quantities are interpolated to and which quantities should be averaged from vacuum cells. We make a particular choice in this work which is described in the section. However, experience has shown that different choices can produce qualitatively different results for realistic 3D situations whilst still passing test problems. This is discussed more in the future work section of this chapter.

In section 6.5, a testing framework for the new MHD solver was laid out. Test problems for the resistive diffusion, frozen-in-flow field and Ohmic heating were carried out. The new MHD solver does not use any of the machinery from the previous solver so it was important to test that it was still able to recreate resistive MHD results. We found that the new solver produced virtually identical answers to the previous Gorgon solver. The dispersion relation of the wave modes driven by the Hall effect, the Hall drift and Whistler waves, was then tested. It was found that the new solver produced values consistent with theory, indicating that the Hall effect is implemented correctly. In doing this, it was found that reducing the numerical speed of light and introducing electron inertia had a damping effect on these wave modes, making them more tractable to simulate than in a normal Hall MHD model. By use of the Farmer Hall drift wave [115], we found that the Lorentz force produced by the new solver was also consistent with theoretical predictions. Finally, we tested that the new code correctly handled numerical

vacuum regions using a test motivated by power flow problems [118] involving a magnetic field diffusing across a vacuum into a low density plasma surrounded by an electrode. The Hall effect causes the rate of diffusion to be dependent on the direction of current flow, leading to anisotropy. Physical behaviour at the vacuum-plasma interface was enforced by ensuring that electromagnetic fields were continuous across the boundary.

In section 6.6, we used the newly developed Hall solver to simulate $m=0$ modes in an equilibrium z-pinch. This was intended as a highly challenging test problem to indicate whether the solver would be appropriate for the pulsed power regimes that we intend to use it in. The Hall effect caused the evolution of the instability to differ qualitatively from resistive MHD. There was an asymmetry in the shape of the high density bulges that form as the pinch implodes, with them curving upwards in the direction of current flow. This is consistent with the results from the PIC simulations in chapter 5. The Hall effect produces a radially inward electric field that becomes larger as the density decreases. In combination with the radial implosion of the z-pinch, the $\mathbf{E} \times \mathbf{B}$ drift driven by the electric field causes the shearing seen in the density. The Hall effect also makes the magnetic field axially asymmetric and exacerbates the size of the perturbation. This is due to driving vorticity in the electron drift velocity which moves magnetic field away from the edge of the high density bulges of the plasma. This effect causes the instability to grow faster when the Hall effect is included, compared to resistive MHD, although past theoretical work [36] indicates that this conclusion is strongly dependent on the mode number, density of the pinch and current applied to it.

7.2 Future Work

7.2.1 Improvements to PIC simulations

The PIC simulations we carried out in chapter 5 were limited by the geometry in which we were carrying them out, as well as the computational demands of having to model the density gradient using finite particles.

In that chapter, we discussed the ideal computational tools for modelling the vacuum-plasma interface problem. We argued that to resolve the lowest densities, a continuum kinetic code is needed. There are many questions that could be answered with such an approach to the problem. Anomalous resistivity is a kinetic process that is heuristically added to many hydrocodes [122]. Simulations have self-consistently generated it in test problems using PIC codes [123], but whether it naturally emerges in the simulation of a z-pinch edge is an unanswered question. Another question is how kinetic physics alters the generalised Ohm's law in a z-pinch plasma. Terms such as the Nernst or Ettingshausen effects are derived in classical transport assuming a reasonably collisional plasma [86]. The role of the Nernst effect has been explored in the non-local limit [215], but other terms have not been studied.

Both of these studies would require kinetic electrons however. With a hybrid code with kinetic ions and fluid electrons, it would be interesting to see whether the Hall

effect in particular actually emerges on fluid timescales. There has been much exploration of Hall MHD and extended MHD simulations of pulsed power plasmas [148, 150], particularly with the PERSEUS code [114]. It would be valuable to see whether these effects attributed to the Hall term actually emerge from a more detailed model, or if some other physics is ultimately more relevant.

7.2.2 Improvements to Hall MHD solver

The electron pressure term has the same scaling as the Hall effect and both theory [186] and past simulations [148, 196] have indicated that it is important in the plasma-vacuum interface regime. This is also shown by the results of our PIC simulations in chapter 5. This term would need to be added to the solver. As the electron pressure does not differ whilst the EM fields are being updated, it could be included in the semi-implicit source term update with the Hall and resistive diffusion terms. The electron pressure leads to the Biermann effect, generating magnetic fields where there are non-parallel density and temperature gradients. Testing that including this term within the source terms does not produce erroneously large self-generated fields at the edge of the pinch (where the pressure gradient is large) would be important.

We currently assume that the resistivity is an isotropic scalar η . In reality, the effect of the magnetic field means that the resistivity is an anisotropic tensor [86]. These anisotropic $\eta_{\wedge}$ components to the resistivity could in principle be added to the source term update in the solver. Past implementations of Hall have described the term as a correction to this anisotropic resistivity [63], so including these off-diagonal terms may have important consequences. However, there is little discussion of these terms in the literature and so their potential impact has not yet been characterised.

Due to difficulties in their implementation, the hyperbolic terms in the electron inertia equation have not been included. Whilst in principle they could be added and other work has done this (e.g. [114, 202]), in most practical situations that would be simulated by Gorgon the electron inertial terms have little effect as we discussed in chapter 6. In a situation where they are dominant, it is likely that charge separation or other two fluid effects could no longer be neglected.

A consistent problem with using resistive MHD to model pulsed power experiments is that it tends to overestimate the current density on the vacuum-plasma interface [68], sometimes predicting superluminal electron drift velocities. This in turn leads to unphysical Ohmic heating and high surface temperatures [63]. It was found that just including electron inertia does little to improve this because the inertial length and timescales are often unresolved. Addition of relativistic effects to the generalised Ohm's Law has been shown to limit the electron drift velocity to the numerical speed of light [68]. This could be done to make simulations more tractable, particularly if the speed of light is reduced.

Our analysis in section 2.4 indicated that the electron drift velocity regularly exceeds the ion sound speed at the vacuum-plasma interface. Theoretical work has indicated

this could lead to anomalous resistivity effects [36], though there is little experimental evidence for this phenomenon. As anomalous resistivity is just a correction to the existing η , it would be straightforward to include within the new solver. In some later simulations, we implemented this correction and found that it did lead to lower vacuum-plasma interface currents and less high frequency structure in the magnetic field of the pinch at early times. However, a recent review paper has produced a better model for η_{anom} , validated by PIC simulations [123]. Implementing this model and more comprehensive testing to see the effect of anomalous resistivity would be interesting future work. A debate in the field is whether anomalous resistivity, electron inertia or the Hall term is the dominant effect in maintaining physical vacuum-plasma interface currents. This question ultimately could be answered by the new Hall solver.

7.2.3 A comparison of Hall implementations

It was found when implementing the Hall effect that there were several places of “free choice” in the algorithm, mostly to do with Gorgon’s staggered grid meaning the choice of where to do the source term update is arbitrary. Slightly different implementations still pass the test problems that we presented, but produce qualitatively different answers for more complex simulations such as the $m = 0$ instability. A comparison of different Hall implementations to each other and to experiment would be a useful exercise. Implementing a cell centered version of MHD, using divergence cleaning, in Gorgon would mean that it would be possible to test different Hall implementations whilst using the same hydrodynamics. This would make it possible to assess which implementation produces the most physical answer to experimental-like problems, which is ultimately the aim for this solver.

In a similar vein, comparison of the results of our Hall implementation to a true two-fluid code would be interesting. It is unclear whether the usage of electron inertia within a single-fluid code is valid and a two-fluid code could answer whether the predictions are inconsistent with a more detailed code.

7.2.4 Characterisation of the $m=0$ problem

When modelling the $m = 0$ instability in chapter 6, we only considered a single mode number, line density and applied current. Theory indicates that the growth rate of the instability depends on these parameters [36]. A comprehensive study of this problem with parabolic profiles and a Hall MHD solver has not been done and could be done using the solver as it currently stands. The simulations would best be carried out in 2D r - z cylindrical geometry since 3D Cartesian is expensive.

7.2.5 Application of the Hall solver to laboratory experiments

Part of the motivation for this thesis was results from MagLIF and laboratory astrophysics experiments on MAGPIE that suggested resistive MHD modelling of pulsed power experiments was insufficient to explain certain observations. Theoretical analysis

indicating that these experiments are in a regime where the ion inertial length is relevant implies that the Hall effect may be important. However, this is at best speculative. Application of the new Hall solver to experimental situations where we believe the Hall effect may be relevant could yield some greater understanding of the behaviour of these experiments. There are very few extended MHD codes with the capability to study the Hall effect in a pulsed power environment so there is a rich field of situations that could be modelled using the new solver.

Appendices

A Numerics Appendices

A.1 Detailed Discretisation of the New Gorgon MHD Solver

In chapter 6, we described a new MHD solver for the Gorgon code that would solve the equations of extended MHD, including Faraday’s Law:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \quad (\text{A.1})$$

Ampère’s Law:

$$\frac{\partial \mathbf{E}}{\partial t} = c^2 \nabla \times \mathbf{B} - c^2 \mu_0 \mathbf{j}, \quad (\text{A.2})$$

and a reduced form of the generalised Ohm’s Law, including electron inertia:

$$\frac{\partial \mathbf{j}}{\partial t} = \frac{e^2 n_e}{m_e} \left[\mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{1}{en_e} (\mathbf{j} \times \mathbf{B}) - \eta \mathbf{j} \right]. \quad (\text{A.3})$$

In this appendix, we show the detailed discretisation of this field solver.

For the first step in the process, we update the electric field by half a timestep using the hyperbolic part of the update:

$$\mathbf{E}' = \mathbf{E}^n + c^2 \frac{\Delta t}{2} \nabla \times \mathbf{B}^n, \quad (\text{A.4})$$

where we calculate $\mathbf{B}^n = \nabla \times \mathbf{A}^n$. This is an explicit update and requires us to resolve:

$$\Delta t = \frac{\min(\Delta x, \Delta y, \Delta z)}{c}, \quad (\text{A.5})$$

for numerical stability.

The $\mathbf{u} \times \mathbf{B}$ electric field is important as it leads to ideal MHD’s frozen-in-flow condition. Therefore, handling it carefully is crucial. We calculate $\mathbf{u} \times \mathbf{B}$ using an upwinding method with van Leer flux correction, described in Ref. [136]. We then perform a frame transformation, defining the electric field in the rest frame of the plasma:

$$\mathbf{E}'' = \mathbf{E}' + \mathbf{u} \times \mathbf{B}^n. \quad (\text{A.6})$$

We note that:

$$\begin{aligned}\frac{\partial \mathbf{E}''}{\partial t} &= \frac{\partial \mathbf{E}'}{\partial t} + \frac{\partial}{\partial t}(\mathbf{u} \times \mathbf{B}^n) \\ &= \frac{\partial \mathbf{E}'}{\partial t},\end{aligned}\tag{A.7}$$

since during the source term update we assume the magnetic field is fixed.

We use a backwards Euler method for the source term update, so can write:

$$\begin{aligned}\frac{\mathbf{E}''^{n+1/2} - \mathbf{E}''}{\Delta t/2} &= -c^2 \mu_0 \mathbf{j}^{n+1/2} \\ \frac{\mathbf{j}^{n+1/2} - \mathbf{j}^n}{\Delta t/2} &= \frac{e^2 n_e}{m_e} \left[\mathbf{E}''^{n+1/2} - \frac{1}{en_e} (\mathbf{j}^{n+1/2} \times \mathbf{B}^n) - \eta \mathbf{j}^{n+1/2} \right].\end{aligned}\tag{A.8}$$

We can rewrite this update as:

$$\begin{pmatrix} E''_x \\ E''_y \\ E''_z \\ j_x \\ j_y \\ j_z \end{pmatrix}^n = \underline{\underline{\mathbf{M}}} \cdot \begin{pmatrix} E''_x \\ E''_y \\ E''_z \\ j_x \\ j_y \\ j_z \end{pmatrix}^{n+1/2},\tag{A.9}$$

where

$$\underline{\underline{\mathbf{M}}} = \begin{pmatrix} 1 & 0 & 0 & c^2 \mu_0 \Delta t/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & c^2 \mu_0 \Delta t/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & c^2 \mu_0 \Delta t/2 \\ -\frac{e^2 n_e \Delta t/2}{m_e} & 0 & 0 & 1 + \eta \frac{e^2 n_e \Delta t}{m_e} & \frac{e B_z^n \Delta t/2}{m_e} & -\frac{e B_y^n \Delta t/2}{m_e} \\ 0 & -\frac{e^2 n_e \Delta t/2}{m_e} & 0 & -\frac{e B_z^n \Delta t/2}{m_e} & 1 + \eta \frac{e^2 n_e \Delta t/2}{m_e} & \frac{e B_x^n \Delta t/2}{m_e} \\ 0 & 0 & -\frac{e^2 n_e \Delta t/2}{m_e} & \frac{e B_y^n \Delta t/2}{m_e} & -\frac{e B_x^n \Delta t/2}{m_e} & 1 + \eta \frac{e^2 n_e \Delta t/2}{m_e} \end{pmatrix}.\tag{A.10}$$

Therefore, we calculate new values using

$$\begin{pmatrix} E''_x \\ E''_y \\ E''_z \\ j_x \\ j_y \\ j_z \end{pmatrix}^{n+1/2} = \underline{\underline{\mathbf{M}}}^{-1} \cdot \begin{pmatrix} E''_x \\ E''_y \\ E''_z \\ j_x \\ j_y \\ j_z \end{pmatrix}^n,\tag{A.11}$$

Due to the Yee grid, we need to perform this update 3 times for every cell, in each

electric field location (i.e. each edge of the grid). As described in section 6.3.1, we average quantities to each location to ensure that we have all information in the correct location for the update.

To solve this, we need to invert matrix $\underline{\underline{\mathbf{M}}}$. This matrix can be written in block matrix form as:

$$\underline{\underline{\mathbf{M}}} = \begin{pmatrix} \underline{\underline{\mathbf{A}}} & \underline{\underline{\mathbf{B}}} \\ \underline{\underline{\mathbf{C}}} & \underline{\underline{\mathbf{D}}} \end{pmatrix}, \quad (\text{A.12})$$

where $\underline{\underline{\mathbf{A}}}$ etc. are all 3×3 matrices. If we assume that $\underline{\underline{\mathbf{A}}}$ and $\underline{\underline{\mathbf{D}}}$ are invertible, then it has been shown [216] that the inverse of this matrix can be written as:

$$\begin{pmatrix} \underline{\underline{\mathbf{A}}} & \underline{\underline{\mathbf{B}}} \\ \underline{\underline{\mathbf{C}}} & \underline{\underline{\mathbf{D}}} \end{pmatrix}^{-1} = \begin{pmatrix} (\underline{\underline{\mathbf{A}}} - \underline{\underline{\mathbf{B}}}\underline{\underline{\mathbf{D}}}^{-1}\underline{\underline{\mathbf{C}}})^{-1} & \underline{\underline{\mathbf{0}}} \\ \underline{\underline{\mathbf{0}}} & (\underline{\underline{\mathbf{D}}} - \underline{\underline{\mathbf{C}}}\underline{\underline{\mathbf{A}}}^{-1}\underline{\underline{\mathbf{B}}})^{-1} \end{pmatrix} \begin{pmatrix} \underline{\underline{\mathbf{I}}} & -\underline{\underline{\mathbf{B}}}\underline{\underline{\mathbf{D}}}^{-1} \\ -\underline{\underline{\mathbf{C}}}\underline{\underline{\mathbf{A}}}^{-1} & \underline{\underline{\mathbf{I}}} \end{pmatrix}, \quad (\text{A.13})$$

where $\underline{\underline{\mathbf{I}}}$ is the identity matrix and $\underline{\underline{\mathbf{0}}}$ is the zero matrix. We know that $\underline{\underline{\mathbf{A}}}$ and $\underline{\underline{\mathbf{B}}}$ are always invertible since they are proportional to the identity matrix. $\underline{\underline{\mathbf{C}}}$ is invertible as long as the density is not 0, which requires the careful treatment of the vacuum that we discuss in section 6.3.1. We check during the solve that the inverse of $\underline{\underline{\mathbf{D}}}$ is defined. We use $\underline{\underline{\mathbf{M}}}^{-1}$ to obtain $\mathbf{E}^{n+1/2}$ and $\mathbf{j}^{n+1/2}$. We then convert back to the lab frame to obtain a final value for $\mathbf{E}^{n+1/2}$:

$$\mathbf{E}^{n+1/2} = \mathbf{E}^{n+1/2} - \mathbf{u} \times \mathbf{B}^n, \quad (\text{A.14})$$

using this to update $\mathbf{A}$:

$$\mathbf{A}^{n+1} = \mathbf{A}^n - \Delta t \mathbf{E}^{n+1/2}. \quad (\text{A.15})$$

To complete the MHD solution, we now calculate $\mathbf{B}^{n+1} = \nabla \times \mathbf{A}^{n+1}$ and update $\mathbf{E}^n$ and $\mathbf{j}^n$ to timestep $n + 1$ using equations A.4 to A.14, but taking a full timestep Δt rather than a half timestep. In effect, we carry out the same hyperbolic and then source update of $\mathbf{E}$ and $\mathbf{j}$, but with updated values for $\mathbf{B}$ and over a full timestep. This is similar to a predictor-corrector scheme and is done to obtain second order accuracy in time [131].

A.2 Implicit Hall solver without electron inertia

For comparison to the existing Gorgon solver, we also developed an implicit Hall solver with no electron inertia. We did not use this solver for the results in this thesis, but we summarise it here.

The governing equations in the case without electron inertia are now:

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} \\ \frac{\partial \mathbf{E}}{\partial t} &= c^2 \nabla \times \mathbf{B} - c^2 \mu_0 \mathbf{j} \\ \mathbf{E} &= -\mathbf{u} \times \mathbf{B} + \frac{1}{en_e} \mathbf{j} \times \mathbf{B} + \eta \mathbf{j},\end{aligned}\tag{A.16}$$

where now the current is calculated directly from the generalised Ohm's law, rather than being evolved explicitly. We still perform the same operator splitting of the hyperbolic and source terms in the update, carrying out the hyperbolic update explicitly and the source update implicitly. We still use the same predictor-corrector method within the update.

However, our source term update is now different. We note that we can rewrite the generalised Ohm's Law as:

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \underline{\underline{\mathbf{M}}} \cdot \begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix},\tag{A.17}$$

where we have absorbed $\mathbf{u} \times \mathbf{B}$ into $\mathbf{E}$ as above and where

$$\underline{\underline{\mathbf{M}}} = \begin{pmatrix} \eta & \frac{B_z}{en_e} & -\frac{B_y}{en_e} \\ -\frac{B_z}{en_e} & \eta & \frac{B_x}{en_e} \\ \frac{B_y}{en_e} & -\frac{B_x}{en_e} & \eta \end{pmatrix}\tag{A.18}$$

Note that as in the electron inertia update, interpolation is necessary to ensure that all components are located at the same point in the grid. We use the same process as the other scheme. Therefore, we can calculate $\mathbf{j}$ given $\mathbf{E}$ by inverting this 3×3 matrix. The source term update is then given by

$$\frac{\mathbf{E}^{n+1} - \mathbf{E}}{\Delta t} = -c^2 \mu_0 \mathbf{j}^{n+1},\tag{A.19}$$

where $\mathbf{j}^{n+1} = \underline{\underline{\mathbf{M}}}^{-1} \cdot \mathbf{E}^{n+1}$. With some algebra, this can be rearranged to give

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}^n = \underline{\underline{\mathbf{U}}} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}^{n+1},\tag{A.20}$$

from which it is straightforward to get the advanced electric field values via another inversion of the 3×3 matrix $\underline{\underline{\mathbf{U}}}$. The remainder of the MHD update is the same. We still use the same predictor-corrector method described in the previous section.

We have checked and this method gives identical answers to the electron inertia solver if we apply an artificial multiplier to the electron mass in equation 6.9 to reduce m_e to arbitrarily small values.

B Theory Appendices

B.1 Moments of the Kinetic Equation

In section 2.2, we discussed how fluid equations can be obtained by taking moments of the kinetic equation. In this section, we show the specifics of this. The kinetic equation is given by:

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla_r f_\alpha + \frac{q_\alpha}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f_\alpha = \left(\frac{\partial f_\alpha}{\partial t} \right)_{coll}, \quad (\text{B.1})$$

for a distribution function $f(\mathbf{r}, \mathbf{v}, t)$. We define $\mathbf{a}_\alpha = \frac{q_\alpha}{m_\alpha} (\mathbf{E} + \mathbf{v} \times \mathbf{B})$ as the acceleration.

For an arbitrary function of velocity $\psi(\mathbf{v})$, the moment is given by:

$$\begin{aligned} \int_v \psi \frac{\partial f}{\partial t} d^3v + \int_v \psi \mathbf{v} \cdot \nabla_r f d^3v + \int_v \psi \mathbf{a} \cdot \nabla_v f d^3v \\ = \int_v \psi \left(\frac{\partial f}{\partial t} \right)_{coll} d^3v, \end{aligned} \quad (\text{B.2})$$

where each of the integrals is over the whole velocity space and we have dropped the species index α for ease. Using the definition of the velocity space average of ψ :

$$\langle \psi \rangle = \frac{\int_v \psi f d^3v}{\int_v f d^3v}, \quad (\text{B.3})$$

we can rewrite the first term as:

$$\int_v \psi \frac{\partial f}{\partial t} d^3v = \frac{\partial}{\partial t} n \langle \psi \rangle. \quad (\text{B.4})$$

The second term can be simplified by using the product rule: $\nabla_r \cdot (\psi f \mathbf{v}) = \psi f \nabla_r \cdot \mathbf{v} + \psi \mathbf{v} \cdot \nabla_r f + f \mathbf{v} \cdot \nabla_r \psi$ and noticing that $\mathbf{v}$ is independent of $\mathbf{r}$:

$$\begin{aligned} \int_v \psi \mathbf{v} \cdot \nabla_r f d^3v &= \nabla_r \cdot \int_v \mathbf{v} \psi f d^3v - \int_v f \mathbf{v} \cdot \nabla_r \psi d^3v \\ &= \nabla_r \cdot (n \langle \mathbf{v} \psi \rangle), \end{aligned} \quad (\text{B.5})$$

since $\psi(\mathbf{v})$ is not a function of position. For the third term, we again use the product rule:

$$\int_v \psi \mathbf{a} \cdot \nabla_v f d^3v = \int_v \nabla_v \cdot (\psi f \mathbf{a}) d^3v - \int_v f \mathbf{a} \cdot (\nabla_v \psi) d^3v - \int_v f \psi (\nabla_v \cdot \mathbf{a}) d^3v. \quad (\text{B.6})$$

The final term on the right hand side is zero is $\nabla_v \cdot \mathbf{a} = 0$; that is that the acceleration component a_i is independent of the corresponding velocity component v_i [85]. This is obviously true for the electric field force. Since the magnetic force component is proportional to $\mathbf{v} \times \mathbf{B}$, it can be shown that $\nabla_v \cdot (\mathbf{v} \times \mathbf{B}) = 0$. Therefore, the final term on the right hand side vanishes. We handle the first term by the divergence theorem:

$$\int_v \nabla_v \cdot (\psi f \mathbf{v}) d^3v = \oint_S (\psi f \mathbf{v}) \cdot d\mathbf{S}, \quad (\text{B.7})$$

where the surface integral is over a closed, spherically symmetric surface of radius v in velocity space. We enforce that the distribution function drops to zero faster than $1/v^2$ as $v \rightarrow \infty$ and so $f d\mathbf{S} \rightarrow 0$ [85]. Therefore, this integral gives zero. Finally, we can rewrite this term as:

$$\int_v \psi \mathbf{a} \cdot \nabla_v f d^3v = - \int_v f \mathbf{a} \cdot (\nabla_v \psi) d^3v = -n \langle \mathbf{a} \cdot \nabla_v \psi \rangle. \quad (\text{B.8})$$

Rewriting the moment equation therefore, we have for the evolution of $\langle \psi \rangle$ [85]:

$$\frac{\partial}{\partial t} (n \langle \psi \rangle) + \nabla_r \cdot (n \langle \mathbf{v} \psi \rangle) - n \langle \mathbf{a} \cdot \nabla_v \psi \rangle = \int_v \psi \left(\frac{\partial f}{\partial t} \right)_{coll} d^3v. \quad (\text{B.9})$$

The zeroth moment results from setting $\psi = 1$. This yields $\langle \psi \rangle = 1$ and $\langle \nabla_v \psi \rangle = 0$ and thus that:

$$\frac{\partial n}{\partial t} + \nabla_r \cdot (n \mathbf{u}) = 0, \quad (\text{B.10})$$

which is the continuity equation. We have assumed that the collision integral in this case gives zero, which means that collisions do not change the number of particles, only their velocity distribution. This is true for the applications that we are interested in, where we do not care about recombination or ionisation effects from collisions.

The first moment sets $\psi = m\mathbf{v}$, which gives:

$$\frac{\partial(mn\mathbf{u})}{\partial t} + \nabla_r \cdot \underline{\underline{P}} - nq(\mathbf{E} + \mathbf{u} \times \mathbf{B}) = \mathbf{R}, \quad (\text{B.11})$$

where:

$$\mathbf{R} = m \int_v \mathbf{v} \left(\frac{\partial f}{\partial t} \right)_{coll} d^3v, \quad (\text{B.12})$$

is the change in momentum $mn\mathbf{u}$ due to collisions. This is known as the momentum equation, since we can rewrite $mn = \rho$. It is convenient to split the velocity $\mathbf{v}$ into the mean component $\mathbf{u}$ and a component due to the random thermal motion of particles $\mathbf{w}$:

$$\mathbf{v} = \mathbf{u} + \mathbf{w} \quad (\text{B.13})$$

As $\mathbf{w}$ is random in velocity space, $\langle \mathbf{w} \rangle = 0$. Note that therefore $\langle \mathbf{v} \rangle = \mathbf{u}$ as previously

defined. The vector $\mathbf{w}$ is the velocity of particles in the reference frame in which the fluid as a whole is at rest. Therefore, we can use this to rewrite the total pressure tensor:

$$\begin{aligned}
 \underline{\underline{P}} &= \int_v m \mathbf{v} \mathbf{v} f d^3v \\
 &= \int_v m (\mathbf{u} + \mathbf{w})(\mathbf{u} + \mathbf{w}) f d^3w \\
 &= mn \mathbf{u} \mathbf{u} + mn (\mathbf{u} \langle \mathbf{w} \rangle + \langle \mathbf{w} \rangle \mathbf{u}) + mn \langle \mathbf{w} \mathbf{w} \rangle \\
 &= mn \mathbf{u} \mathbf{u} + mn \langle \mathbf{w} \mathbf{w} \rangle.
 \end{aligned} \tag{B.14}$$

The first term on the right hand side is the ram pressure and the second is the thermal pressure due to particle fluctuations in the rest frame of the fluid [86]. We can write this fluctuation part as:

$$\begin{aligned}
 \pi_{ij} &= mn \langle w_i w_j \rangle \\
 &= p \delta_{ij} + \sigma_{ij},
 \end{aligned} \tag{B.15}$$

where δ_{ij} is the Kronecker delta. The first term is the scalar pressure and σ_{ij} is the anisotropic part of the stress tensor. Note that we can write each of these parts from the line above as:

$$p = \frac{nm}{3} \langle w^2 \rangle \tag{B.16}$$

$$\sigma_{ij} = nm \langle w_i w_j - \frac{w^2}{3} \delta_{ij} \rangle, \tag{B.17}$$

without loss of information. Putting all of this together in the momentum equation, we have that:

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \nabla_r \cdot (\rho \mathbf{u} \mathbf{u}) + \nabla_r p + \nabla_r \cdot \underline{\underline{\sigma}} = nq(\mathbf{E} + \mathbf{u} \times \mathbf{B}) + \mathbf{R}. \tag{B.18}$$

The energy equation is the second moment of the kinetic equation, setting $\psi = \frac{1}{2}mv^2$, to yield

$$\frac{\partial}{\partial t} \left(\frac{mn}{2} \langle v^2 \rangle \right) + \nabla_r \cdot \left(\frac{mn}{2} \langle v^2 \mathbf{v} \rangle \right) - \frac{nm}{2} \langle \mathbf{a} \cdot \nabla_v v^2 \rangle = \int_v \frac{mv^2}{2} \left(\frac{\partial f}{\partial t} \right)_{coll} d^3v. \tag{B.19}$$

We can write that $\nabla_v v^2 = 2\mathbf{v}$ and so therefore:

$$\begin{aligned}
 \frac{nm}{2} \left\langle \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) 2\mathbf{v} \right\rangle &= qn \mathbf{E} \cdot \langle \mathbf{v} \rangle \\
 &= qn \mathbf{E} \cdot \mathbf{u}.
 \end{aligned} \tag{B.20}$$

We use the same method of splitting $\mathbf{v} = \mathbf{u} + \mathbf{w}$ to transform the first term:

$$\begin{aligned} \left\langle \frac{\rho}{2} v^2 \right\rangle &= \langle (\mathbf{w} + \mathbf{u}) \cdot (\mathbf{w} + \mathbf{u}) \rangle \\ &= \frac{1}{2} \rho \langle w^2 \rangle + \frac{1}{2} \rho u^2. \end{aligned} \quad (\text{B.21})$$

The first term here is the thermal energy of the plasma and the second is the kinetic energy. We can perform a similar manipulation for the term in the spatial derivative:

$$\begin{aligned} \left\langle \frac{\rho v^2}{2} \mathbf{v} \right\rangle &= \frac{\rho}{2} \left\langle [(\mathbf{u} + \mathbf{w}) \cdot (\mathbf{u} + \mathbf{w})] (\mathbf{u} + \mathbf{w}) \right\rangle \\ &= \frac{\rho}{2} u^2 \mathbf{u} + \rho \mathbf{u} \langle \mathbf{w} \mathbf{w} \rangle + \frac{\rho}{2} \langle w^2 \mathbf{w} \rangle + \frac{\rho}{2} \mathbf{u} \langle w^2 \rangle. \end{aligned} \quad (\text{B.22})$$

We use the definitions of the scalar pressure and stress tensors:

$$\langle w_i w_j \rangle = \frac{3p}{\rho} \delta_{ij} + \frac{1}{\rho} \sigma_{ij}, \quad (\text{B.23})$$

and define the heat flux:

$$\mathbf{q} = \frac{\rho}{2} \langle w^2 \mathbf{w} \rangle. \quad (\text{B.24})$$

$\mathbf{q}$ represents the transport of energy due to the random motion of particles in the reference frame where the fluid as a whole is at rest [86]. This lets us rewrite:

$$\left\langle \frac{\rho v^2}{2} \mathbf{v} \right\rangle = \frac{\rho}{2} u^2 \mathbf{u} + \mathbf{q} + \frac{5}{2} p \mathbf{u} + \underline{\underline{\sigma}} : \mathbf{u}, \quad (\text{B.25})$$

where the last term represents the contraction of the stress tensor with the fluid velocity vector. We define the collision term:

$$Q = \int_v \frac{mv^2}{2} \left(\frac{\partial f}{\partial t} \right)_{coll} d^3v. \quad (\text{B.26})$$

We can substitute these results into equation (B.19) to finally obtain:

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho u^2 + \frac{3}{2} p \right) + \nabla_r \cdot \left(\left[\frac{1}{2} u^2 + \frac{5}{2} p \right] \mathbf{u} + \underline{\underline{\sigma}} : \mathbf{u} + \mathbf{q} \right) = qn\mathbf{E} \cdot \mathbf{u} + Q, \quad (\text{B.27})$$

This is a statement of the conservation of total energy, made of thermal and kinetic energy contributions. However, the equations of continuity and momentum conservation already ensure that kinetic energy is correctly conserved, so it is useful to remove this contribution from the total energy balance to just give an equation for the internal energy [86]. By using the continuity and momentum equations, we can eliminate the kinetic energy terms to give:

$$\frac{3}{2} \frac{\partial p}{\partial t} + \nabla \cdot \left(\frac{3}{2} p \mathbf{u} \right) + p \nabla \cdot \mathbf{u} + \underline{\underline{\sigma}} : \nabla \mathbf{u} + \nabla \cdot \mathbf{q} = Q', \quad (\text{B.28})$$

where $\underline{\underline{\sigma}} : \nabla \mathbf{u} = \sigma_{ij} \frac{\partial u_i}{\partial x_j}$ is the contraction of the stress tensor with the gradient of the flow velocity. The collision term is now:

$$\begin{aligned} Q' &= \int_v \frac{mv^2}{2} \left(\frac{\partial f}{\partial t} \right)_{coll} d^3v - \mathbf{R} \cdot \mathbf{u} \\ &= \int_w \frac{mw^2}{2} \left(\frac{\partial f}{\partial t} \right)_{coll} d^3w \end{aligned} \quad (\text{B.29})$$

This is the form in which this equation is solved in Gorgon and that we discuss in chapter 2. The second term on the left hand side is the advection of energy density by the fluid flow and the third and fourth terms are the work done by pdV heating. As before, we have the divergence of the heat flux and the effect of collisions as well. Note that in chapter 2, we drop the apostrophe on Q' in the final equation.

B.2 Derivation of the Generalised Ohm's Law

In order to model a plasma using MHD, an expression for the current density $\mathbf{j}$ is needed to evolve Maxwell's equations for $\mathbf{E}$ and $\mathbf{B}$. This is obtained in this work using the Generalised Ohm's Law, which we rely on throughout this thesis. In this section, we show how it is derived.

We start with the equations of motion for the electron and ion populations, derived in section 2.2.6:

$$\frac{\partial \rho_e \mathbf{u}_e}{\partial t} + \nabla \cdot (\rho_e \mathbf{u}_e \mathbf{u}_e) = -\nabla \cdot \underline{\underline{\mathbf{P}_e}} - en_e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) + \mathbf{R}_e \quad (\text{B.30})$$

$$\frac{\partial \rho_i \mathbf{u}_i}{\partial t} + \nabla \cdot (\rho_i \mathbf{u}_i \mathbf{u}_i) = -\nabla \cdot \underline{\underline{\mathbf{P}_i}} + Zen_i(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) + \mathbf{R}_i, \quad (\text{B.31})$$

where $\underline{\underline{\mathbf{P}_e}}$ and $\underline{\underline{\mathbf{P}_i}}$ are the electron and ion pressure tensors respectively. We multiply the ion momentum equation by $\frac{m_e}{m_i}$ and subtract the electron momentum equation, to obtain:

$$\begin{aligned} &\frac{\partial}{\partial t} \left(\frac{m_e}{m_i} m_i n_i \mathbf{u}_i - m_e n_e \mathbf{u}_e \right) + \nabla \cdot \left(\frac{m_e}{m_i} m_i n_i \mathbf{u}_i \mathbf{u}_i - m_e n_e \mathbf{u}_e \mathbf{u}_e \right) = \quad (\text{B.32}) \\ &\nabla \cdot (\underline{\underline{\mathbf{P}_e}} - \frac{m_e}{m_i} \underline{\underline{\mathbf{P}_i}}) + e \left[(n_e + Z \frac{m_e}{m_i} n_i) \mathbf{E} + (n_e \mathbf{u}_e + Z \frac{m_e}{m_i} n_i \mathbf{u}_i) \times \mathbf{B} \right] - \left(\mathbf{R}_e - \frac{m_e}{m_i} \mathbf{R}_i \right). \end{aligned}$$

Due to conservation of momentum in collisions, $\mathbf{R}_e = -\mathbf{R}_i$. In this work, we assume that we have quasineutrality and thus that $n_e \approx Zn_i \approx n$. Finally, we drop all the $\mathcal{O}\left(\frac{m_e}{m_i}\right)$ terms as $m_e \ll m_i$. By using the definition of the current that $\mathbf{j} = en(\mathbf{u}_i - \mathbf{u}_e)$, this then lets us simplify the above equation to:

$$\frac{m_e}{e} \frac{\partial \mathbf{j}}{\partial t} + m_e \nabla \cdot [n(\mathbf{u}_i \mathbf{u}_i - \mathbf{u}_e \mathbf{u}_e)] = \nabla \cdot \underline{\underline{\mathbf{P}_e}} + en(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) - \mathbf{R}_e. \quad (\text{B.33})$$

It is useful to rearrange to find an expression for the electron velocity in terms of the

current and the centre of mass velocity $\mathbf{u} \approx \mathbf{u}_i$:

$$\begin{aligned}\mathbf{u}_e &= \mathbf{u}_i - \frac{\mathbf{j}}{en} \\ &\approx \mathbf{u} - \frac{\mathbf{j}}{en}.\end{aligned}\tag{B.34}$$

Substituting this into the above equation, we get:

$$\frac{m_e}{ne^2} \frac{\partial \mathbf{j}}{\partial t} + \frac{m_e}{ne^2} \nabla \cdot \left(\mathbf{u}\mathbf{j} + \mathbf{j}\mathbf{u} - \frac{\mathbf{j}\mathbf{j}}{en} \right) = \frac{1}{en} \nabla \cdot \underline{\underline{\mathbf{P}_e}} + \mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{1}{en} \mathbf{j} \times \mathbf{B} - \frac{\mathbf{R}_e}{en}.\tag{B.35}$$

A full expression for the change in electron momentum due to collisions with ions can be found in Braginskii [86], but in the simplest case, it can be written as:

$$\frac{\mathbf{R}_e}{en} = \underline{\underline{\eta}} \cdot \mathbf{j},\tag{B.36}$$

where in general $\underline{\underline{\eta}}$ is an anisotropic tensor resistivity. More complete models that account for e.g. thermoelectric effects can be included. For this work, we assume $\eta_{ij} = \eta_{\perp} \delta_{ij}$ and that $\nabla \cdot \underline{\underline{\mathbf{P}_e}} = \nabla p_e$ (i.e. that we have a scalar resistivity and electron pressure).

Putting all of this together, the generalised Ohm's Law can be written as:

$$\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot \left(\mathbf{u}\mathbf{j} + \mathbf{j}\mathbf{u} - \frac{\mathbf{j}\mathbf{j}}{en} - \frac{e}{m_e} p_e \mathbf{I} \right) = \frac{ne^2}{m_e} \left(\mathbf{E} + \mathbf{u} \times \mathbf{B} - \frac{1}{en_e} \mathbf{j} \times \mathbf{B} - \eta \mathbf{j} \right)\tag{B.37}$$

B.3 Derivation of the Whistler wave dispersion relations

In section 6.5.3, we discussed the Whistler wave mode. Understanding the dispersion characteristics of this wave is important because it dictates the timestep in an explicit code, and should be resolved for accuracy in an implicit code. In this appendix, we derive the dispersion relations for the various different models considered in chapter 6.

B.3.1 Hall MHD

In Hall MHD, commonly used in space physics [107, 193], we assume massless electrons and an infinite speed of light. Therefore, we can ignore displacement current and electron inertia and write the evolution of the magnetic field as:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E},\tag{B.38}$$

where $\mathbf{E} = \frac{1}{en_e} \mathbf{j} \times \mathbf{B}$. We have assumed that $\mathbf{u} = 0$ and $\eta = 0$ for the purpose of this derivation, and that there is no hydrodynamic response so n and $\mathbf{u}$ do not change. We assume that the plasma has constant density $n_e = n_i = n$ and that initially $\mathbf{B} = B_0 \hat{\mathbf{z}}$. We then assume plane wave type perturbations along the z direction, so we can assume

that the magnetic field and the current have a form similar to 6.22:

$$\begin{aligned}\mathbf{B}(z, t) &= \delta B_x(z, t)\hat{\mathbf{x}} + \delta B_y(z, t)\hat{\mathbf{y}} + B_0\hat{\mathbf{z}} \\ \mathbf{j}(z, t) &= \delta j_x(z, t)\hat{\mathbf{x}} + \delta j_y(z, t)\hat{\mathbf{y}},\end{aligned}\tag{B.39}$$

with all perturbed quantities proportional to $\exp(ikz - i\omega t)$. For example, we can write $\delta B_x = \delta \tilde{B}_x \exp(ikz - i\omega t)$ where $\delta \tilde{B}_x$ is a constant, complex amplitude. We can use this relationship to linearise equation B.38, noting that

$$\begin{aligned}\frac{1}{\mu_0} \nabla \times \mathbf{B} &= -\frac{1}{\mu_0} \frac{\partial \delta B_y}{\partial z} \hat{\mathbf{x}} + \frac{1}{\mu_0} \frac{\partial \delta B_x}{\partial z} \hat{\mathbf{y}} \\ &= \delta j_x \hat{\mathbf{x}} + \delta j_y \hat{\mathbf{y}} \\ &= -\frac{ik}{\mu_0} \delta B_y \hat{\mathbf{x}} + \frac{ik}{\mu_0} \delta B_x \hat{\mathbf{y}},\end{aligned}\tag{B.40}$$

and that

$$\mathbf{j} \times \mathbf{B} = B_0 \delta j_y \hat{\mathbf{x}} - B_0 \delta j_x \hat{\mathbf{y}},\tag{B.41}$$

since the product of two perturbed quantities is zero to first order. Combining these we can see that

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t} &= -i\omega \delta B_x \hat{\mathbf{x}} - i\omega \delta B_y \hat{\mathbf{y}} \\ &= -\frac{1}{en} \left(-\frac{k^2 B_0}{\mu_0} (-\delta B_y \hat{\mathbf{x}} + \delta B_x \hat{\mathbf{y}}) \right).\end{aligned}\tag{B.42}$$

We then use the fact that this must be true for both components for all ω, k to find a set of simultaneous equations which we solve to obtain:

$$\omega^2 = \frac{k^2 B_0}{en\mu_0}.\tag{B.43}$$

We then use the definitions of $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$, $\omega_{pi} = \sqrt{\frac{n e^2}{m_i \epsilon_0}}$, $\lambda_i = \frac{c}{\omega_{pi}}$ and $v_A = \frac{B_0}{\sqrt{\mu_0 m_i n}}$ to write the dispersion relation in the same form as equation 6.23:

$$\omega = k v_A^2 \lambda_i\tag{B.44}$$

B.3.2 Hall MHD with displacement current

In this section, we consider the case where we now allow a finite displacement current. The governing equations become:

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} \\ \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} &= \nabla \times \mathbf{B} - \mu_0 \mathbf{j} \\ \mathbf{E} &= \frac{\mathbf{j} \times \mathbf{B}}{en_e}.\end{aligned}\tag{B.45}$$

We make the same assumptions as previously: that there is no ∇n and that there is no hydrodynamic response. Therefore, the unperturbed state is given by:

$$\begin{aligned}\mathbf{E} &= 0 \\ \mathbf{j} &= 0 \\ \mathbf{B} &= B_0 \hat{\mathbf{z}},\end{aligned}\tag{B.46}$$

and we can write the quantities after perturbation in the same form as equation 6.22:

$$\begin{aligned}\mathbf{B} &= \delta B_x \hat{\mathbf{x}} + \delta B_y \hat{\mathbf{y}} + B_0 \hat{\mathbf{z}} \\ \mathbf{E} &= \delta E_x \hat{\mathbf{x}} + \delta E_y \hat{\mathbf{y}} \\ \mathbf{j} &= \delta j_x \hat{\mathbf{x}} + \delta j_y \hat{\mathbf{y}}.\end{aligned}\tag{B.47}$$

We next substitute these relationships into the governing equations and neglect all terms second order or higher in perturbed quantities. This yields:

$$\begin{aligned}-i\omega(\delta B_x \hat{\mathbf{x}} + \delta B_y \hat{\mathbf{y}}) &= ik(\delta E_y \hat{\mathbf{x}} - \delta E_x \hat{\mathbf{y}}) \\ -i\omega\mu_0\epsilon_0(\delta E_x \hat{\mathbf{x}} + \delta E_y \hat{\mathbf{y}}) &= -ik(\delta B_y \hat{\mathbf{x}} - \delta B_x \hat{\mathbf{y}}) \\ &\quad - \mu_0(\delta j_x \hat{\mathbf{x}} + \delta j_y \hat{\mathbf{y}}) \\ \frac{en}{B_0}(\delta E_x \hat{\mathbf{x}} + \delta E_y \hat{\mathbf{y}}) &= \delta j_y \hat{\mathbf{x}} - \delta j_x \hat{\mathbf{y}}.\end{aligned}\tag{B.48}$$

Substituting the linearised Ohm's Law into the linearised Ampère's Law, we obtain:

$$\begin{aligned}\omega\mu_0\epsilon_0\delta E_x &= k\delta B_y + i\frac{en\mu_0}{B_0}\delta E_y \\ \omega\mu_0\epsilon_0\delta E_y &= -k\delta B_x - i\frac{en\mu_0}{B_0}\delta E_x.\end{aligned}\tag{B.49}$$

We then use the linearised Faraday's Law to obtain expressions for $\delta E_{x,y}$ in terms of $\delta B_{x,y}$, which we substitute into the prior result to obtain:

$$\begin{aligned} \left(k - \frac{\omega^2 \mu_0 \epsilon_0}{k}\right) \delta B_x &= -i \frac{\mu_0 e n \omega}{k B_0} \delta B_y \\ \left(k - \frac{\omega^2 \mu_0 \epsilon_0}{k}\right) \delta B_y &= -i \frac{\mu_0 e n \omega}{k B_0} \delta B_x. \end{aligned} \quad (\text{B.50})$$

We can combine these two equations to finally obtain:

$$\left(\frac{\omega^2 \mu_0 \epsilon_0}{k} - k\right)^2 \left(\frac{k B_0}{\mu_0 e n \omega}\right)^2 = 1 \quad (\text{B.51})$$

Using the definitions of c , v_A and λ_i from the previous section yields:

$$\left(\frac{\omega^2}{k c^2} - k\right)^2 \left(\frac{k}{\omega} v_A \lambda_i\right)^2 = 1, \quad (\text{B.52})$$

which can be expanded to result in

$$\omega^4 - \omega^2 \left(2k^2 c^2 + \frac{c^4}{(v_A \lambda_i)^2}\right) + k^4 c^4 = 0. \quad (\text{B.53})$$

This is a polynomial in ω that can be solved numerically for a given value of k . There are four solutions, but the frequency of the Whistler mode is given by the smallest positive real part of these solutions.

B.3.3 Electron Inertia

The final model that we explore is when we account for a finite electron mass. This changes the governing equations to:

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} \\ \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} &= \nabla \times \mathbf{B} - \mu_0 \mathbf{j} \\ m_e \frac{\partial \mathbf{j}}{\partial t} &= e^2 n_e \left[\mathbf{E} - \frac{\mathbf{j} \times \mathbf{B}}{e n_e} \right]. \end{aligned} \quad (\text{B.54})$$

We follow exactly the same process as in the previous 2 subsections, using the same form of the perturbation as from the Hall MHD with displacement current case. Linearising

and substituting in the form of the perturbation, we obtain:

$$\begin{aligned}
-i\omega(\delta B_x \hat{\mathbf{x}} + \delta B_y \hat{\mathbf{y}}) &= ik(\delta E_y \hat{\mathbf{x}} - \delta E_x \hat{\mathbf{y}}) \\
-\frac{i\omega}{c^2}(\delta E_x \hat{\mathbf{x}} + \delta E_y \hat{\mathbf{y}}) &= -ik(\delta B_y \hat{\mathbf{x}} - \delta B_x \hat{\mathbf{y}}) \\
&\quad -\mu_0(\delta j_x \hat{\mathbf{x}} - \delta j_y \hat{\mathbf{y}}) \\
-i\omega(\delta j_x \hat{\mathbf{x}} + \delta j_y \hat{\mathbf{y}}) &= \frac{e^2 n}{m_e}(\delta E_x \hat{\mathbf{x}} + \delta E_y \hat{\mathbf{y}}) \\
&\quad + \frac{e}{m_e} B_0(\delta j_y \hat{\mathbf{x}} - \delta j_x \hat{\mathbf{y}}).
\end{aligned} \tag{B.55}$$

Matching components, this is a set of 6 simultaneous equations for 6 unknown variables. First, we use the linearised Faraday's Law and the linearised Ampère's Law to find an expression relating $\delta E_{x,y}$ to $\delta j_{x,y}$. This yields:

$$\delta E_{x,y} = \frac{c^2 \mu_0 \omega}{i(\omega^2 - c^2 k^2)} \delta j_{x,y}. \tag{B.56}$$

We then take this result and substitute it into the linearised electron inertia equation, noting that $\Omega_e = \frac{eB_0}{m_e}$ and using the definition of ω_{pe} :

$$\begin{aligned}
\delta j_x &= \frac{\omega_{pe}^2}{\omega^2 - c^2 k^2} \delta j_x + i \frac{\Omega_e}{\omega} \delta j_y \\
\delta j_y &= \frac{\omega_{pe}^2}{\omega^2 - c^2 k^2} \delta j_y - i \frac{\Omega_e}{\omega} \delta j_x.
\end{aligned} \tag{B.57}$$

It is helpful to let $A = \frac{\omega_{pe}^2}{\omega^2 - c^2 k^2}$, since then we can write:

$$\begin{aligned}
(1 - A)\delta j_x &= i \frac{\Omega_e}{\omega} \delta j_y \\
(1 - A)\delta j_y &= -i \frac{\Omega_e}{\omega} \delta j_x,
\end{aligned} \tag{B.58}$$

which can be combined to give:

$$(1 - A)^2 = -i^2 \left(\frac{\Omega_e}{\omega} \right)^2. \tag{B.59}$$

Rearranging and substituting in the value of A , we obtain:

$$\omega^2 \left(1 - \frac{\omega_{pe}^2}{(\omega^2 - c^2 k^2)} \right)^2 - \Omega_e^2 = 0, \tag{B.60}$$

where $\omega_{pe} = \sqrt{\frac{e^2 n}{m_e \epsilon_0}}$ is the plasma frequency and $\Omega_e = \frac{eB_0}{m_e}$. Similarly to above, this dispersion relation can be expanded and the resulting sixth order polynomial solved numerically to obtain the Whistler frequency for a given k .

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4.8	[3]	AIP Publishing	
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Permission 1

10/07/2023, 17:39

Email - Farrow, Griffin M M - Outlook

Re: Figure permissions for "Increased Ion Temperature and Neutron Yield Observed in Magnetized Indirectly Driven D 2 -Filled Capsule Implosions on the National Ignition Facility"

Moody, John D. <moody4@llnl.gov>

Mon 10/07/2023 16:53

To: Farrow, Griffin M M <griffin.farrow15@imperial.ac.uk>

Cc: Sio, Hong <sio1@llnl.gov>

This email from moody4@llnl.gov originates from outside Imperial. Do not click on links and attachments unless you recognise the sender. If you trust the sender, add them to your [safe senders list](#) to disable email stamping for this address.

Hi Griffin,

Yes, you have my permission to use figures 1 and 2 from the PRL. Hong Sio just had his PoP accepted which covers the material he presented at his invited talk last APS on the magnetized implosions. He has some figures of the hot spot temperature and yield vs B-field which may of interest to you.

I've copied him on this email in case you'd like to contact him.

Let us know if you need any additional information. Thanks, John

From: "Farrow, Griffin M M" <griffin.farrow15@imperial.ac.uk>

Date: Monday, July 10, 2023 at 8:16 AM

To: John Moody <moody4@llnl.gov>

Subject: Figure permissions for "Increased Ion Temperature and Neutron Yield Observed in Magnetized Indirectly Driven D 2 -Filled Capsule Implosions on the National Ignition Facility"

Dear John,

I'm a PhD student at Imperial College, working with Jerry Chittenden on magneto-inertial fusion. I think we met last summer at Livermore for a few days, I also work with Sam O'Neill. I'm writing up my thesis on different ways of modelling magneto-inertial fusion and I wanted to discuss the magnetized NIF platform that you developed. I was wondering whether I could use a figure from your PRL publication "Increased Ion Temperature and Neutron Yield Observed in Magnetized Indirectly Driven D 2 -Filled Capsule Implosions on the National Ignition Facility". The figures are figures 1 and 2 from that publication. I've already ask for permission from the journal (attached).

The journal requires I also seek permission from you, I just wanted to check that it was okay if I use these figures?

Yours sincerely,

Griffin Farrow



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Author:	J. D. Moody et al.
Publication:	Physical Review Letters
Publisher:	American Physical Society
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Author(s):	Griffin Farrow
Expected completion date:	Jul. 2023

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Name:	Griffin Farrow
Affiliation:	Individual
Email Id:	mgmf46@gmail.com
Country:	United Kingdom

Permission 2

10/07/2023, 17:50

Email - Farrow, Griffin M M - Outlook

Re: [EXTERNAL] Permission to use a figure

Hamlin, Nathaniel Donald <ndhamli@sandia.gov>

Mon 10/07/2023 17:48

To: Farrow, Griffin M M <griffin.farrow15@imperial.ac.uk>

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Hi Griffin,

Sure, feel free to use that figure or any others that would be useful.

Nat

From: Farrow, Griffin M M <griffin.farrow15@imperial.ac.uk>

Date: Monday, July 10, 2023 at 10:39 AM

To: Hamlin, Nathaniel Donald <ndhamli@sandia.gov>

Subject: [EXTERNAL] Permission to use a figure

Hi Nat,

Hope you're doing alright and stuff at Sandia is going along well!

I'm pretty sure I've already asked this, but I've lost the email and so I'm sending it again - apologies!

I was wondering: I am writing my thesis at the moment, and want to talk about the power flow test problem that you talk about in this talk: "Power-Flow Modeling using Fluid and PIC Simulation Codes", url here: <https://www.osti.gov/servlets/purl/1641597>. I was wondering if it would be possible to reuse the figure showing the PERSEUS magnetic field profile on slide 12 in this talk within my thesis, so I can compare results using our code. I would obviously cite you properly, but the university requires us to have permission in writing from the original authors of figures for theses...

Many thanks,

Griffin

Permission 3

12/07/2023, 09:25

Email - Farrow, Griffin M M - Outlook

RE: [EXTERNAL] Permissions for a figure from one of your publications

Gomez, Matthew R <mrgomez@sandia.gov>

Tue 11/07/2023 23:49

To:Farrow, Griffin M M <griffin.farrow15@imperial.ac.uk>

Cc:Jennings, Christopher <cjennin@sandia.gov>

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Hi Griffin,

I am ok with you using that figure. Chris was the one that ran those simulations, so I wanted to include him here as well.

Matt

From: Farrow, Griffin M M <griffin.farrow15@imperial.ac.uk>

Sent: Tuesday, July 11, 2023 3:35 AM

To: Gomez, Matthew R <mrgomez@sandia.gov>

Subject: [EXTERNAL] Permissions for a figure from one of your publications

You don't often get email from griffin.farrow15@imperial.ac.uk. [Learn why this is important](#)

Dear Matt,

I'm a PhD student at Imperial College, working with Jerry Chittenden on magneto-inertial fusion. I'm writing up my thesis on different ways of modelling magneto-inertial fusion and I wanted to discuss the magnetized NIF platform that you developed. I was wondering whether I could use a figure from your PRL publication "Experimental Demonstration of Fusion-Relevant Conditions in Magnetized Liner Inertial Fusion". The figure is figure 1 from that publication. I've already ask for permission from the journal (attached).

The journal requires I also seek permission from you, I just wanted to check that it was okay if I use this figure?

Yours sincerely,

Griffin Farrow



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Title:	Experimental Demonstration of Fusion-Relevant Conditions in Magnetized Liner Inertial Fusion
Author:	M. R. Gomez et al.
Publication:	Physical Review Letters
Publisher:	American Physical Society
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Affiliation:	Individual
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Permission 4

11/07/2023, 16:39

Email - Farrow, Griffin M M - Outlook

RE: [EXTERNAL] Permission to use a figure from one of your publications

Awe, Thomas J <tjawe@sandia.gov>

Tue 11/07/2023 16:38

To:Farrow, Griffin M M <griffin.farrow15@imperial.ac.uk>

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Hi Griffin,

No problem. You have my permission. Let me know if you need something more official.

Thanks.

Tom

From: Farrow, Griffin M M <griffin.farrow15@imperial.ac.uk>

Sent: Tuesday, July 11, 2023 11:32 AM

To: Awe, Thomas J <tjawe@sandia.gov>

Subject: [EXTERNAL] Permission to use a figure from one of your publications

You don't often get email from griffin.farrow15@imperial.ac.uk. [Learn why this is important](#)

Dear Dr Awe,

I'm a PhD student at Imperial College, working with Jerry Chittenden on magneto-inertial fusion. I'm writing up my thesis on different ways of modelling magneto-inertial fusion and I wanted to discuss the helical instability saw in some MagLIF experiments. I was wondering whether I could use a figure from your PRL publication "Observations of modified three-dimensional instability structure for imploding z-pinch liners that are premagnetized with an axial field". The figures are figures 2 and 5 in that publication.

The journal requires I also seek permission from you, I just wanted to check that it was okay if I use those figures?

Yours sincerely,

Griffin Farrow



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Title:	Observations of Modified Three-Dimensional Instability Structure for Imploding Z-Pinch Liners that are Premagnetized with an Axial Field
Author:	T. J. Awe et al.
Publication:	Physical Review Letters
Publisher:	American Physical Society
Cost:	USD \$ 0.00

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Name:	Griffin Farrow
Affiliation:	Individual
Email Id:	mgmf46@gmail.com
Country:	United Kingdom

Permission 5

13/07/2023, 14:43

Email - Farrow, Griffin M M - Outlook

RE: Using some figures from your PRL

Russell, Danny R <Daniel.Russell@frm2.tum.de>

Thu 13/07/2023 14:39

To: Farrow, Griffin M M <griffin.farrow15@imperial.ac.uk>

This email from Daniel.Russell@frm2.tum.de originates from outside Imperial. Do not click on links and attachments unless you recognise the sender. If you trust the sender, add them to your [safe senders list](#) to disable email stamping for this address.

Hi Griffin,

Of course! Glad to hear they are being used 😊

Good luck with the writing,

Danny

Dr Danny R Russell (he/him)
Postdoctoral Researcher
FRM II - NEPOMUC
Technische Universität München

From: [Farrow, Griffin M M](#)
Sent: 13 July 2023 12:06
To: daniel.russell@frm2.tum.de
Subject: Using some figures from your PRL

Hey Danny,

Hope Germany is great and you're having a wonderful time in your new job!
I'm in the process of thesis writing and I'm trying to write the motivation section at the start of my thesis. I'm trying to talk about some Magpie experiments that show there is potentially some beyond MHD physics and I want to talk about your shock experiments. I wanted to use some figures from your PRL in my thesis. I've also got to request permissions from APS, but I was wondering whether it would be alright if I used Figures 1 and 2.

Many thanks!!

Griffin



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Title:	Perpendicular Subcritical Shock Structure in a Collisional Plasma Experiment
Author:	D. R. Russell et al.
Publication:	Physical Review Letters
Publisher:	American Physical Society
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