

Metric Collapsing on Calabi-Yau 3-folds

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Abstract

The aim of this thesis is to describe the collapsing Calabi-Yau (CY) metrics on a CY 3-fold with a Lefschetz K3 fibration, from both the a priori estimate perspective and the gluing perspective. Collapsing CY metric is a well studied subject, but most of the previous works concentrate on the behaviour away from the singular fibres, and the full description of the metric was only available in a very small number of cases, mostly relying on very favourable gluing ansatz.

From the nonlinear perspective, the essential realisation is that by restricting the type of singularities, and under some conjecture in pluripotential theory, then a small neighbourhood of the singular fibre has a local noncollapsing bound, which enables us to understand the pointed Gromov-Hausdorff limit of the singular fibre in the scale where the fibre volume is 1.

This gives strong heuristic evidence that there is a much finer scale near the nodal points in the fibration, where the scaled limit is a CY metric on $\mathbb{C}^3$ with maximal volume growth and singular tangent cone at infinity. This model CY metric is rigorously constructed using a noncompact version of Yau's solution to the Calabi conjecture.

The model metric enables a gluing description of the collapsing CY metrics. The difficulty of the gluing lies in the coarse nature of the gluing ansatz, and the fact that the metric has many types of characteristic behaviours at different scales. We overcome this by developing a sharp linear theory, using some earlier ideas of Gabor Székelyhidi.

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Chapter 1

Introduction

1.1 Motivating questions

The general context of this thesis lies in the crossroad of **Riemannian geometry**, concerned with analytic understanding of Einstein's equation, and **special holonomy**, concerned with solutions of Einstein's equation with special linear algebraic structures parallel under the Levi-Civita connection. Our main focus is on **Calabi-Yau metrics**, namely metrics with holonomy contained in $SU(n)$, or equivalently a Kähler metric (X, g, J, ω) with a nowhere vanishing holomorphic volume form Ω , satisfying the complex Monge-Ampère equation

$$\omega^n = \text{const} \cdot \Omega \wedge \overline{\Omega}.$$

A fundamental question in Riemannian geometry is to describe limiting behaviour of sequences of complete metrics (M_i, g_i) under control on the Ricci tensor [2]. A basic fact is **Bishop-Gromov monotonicity**: if (M, g) has non-negative Ricci curvature, then the renormalised volume of metric balls is a decreasing function of the radius:

$$\frac{\text{Vol}(B_g(r))}{r^{\dim M}} \searrow, \quad \text{as } r \nearrow.$$

This leads to a dichotomy: a sequence of $\dim M$ -dimensional complete Riemannian manifolds (M_i, g_i) with $\text{Ric} \geq 0$ is called **non-collapsing** if there is a uniform bound from below

$$\frac{\text{Vol}(B_{g_i}(r))}{r^{\dim M}} \geq \kappa > 0, \quad \forall r \leq \text{diam}(M_i, g_i),$$

and called **collapsing** if otherwise. A large part of Cheeger-Colding theory is founded on the principle that a non-collapsing bound imposes a strong sense of *coercivity* on the Riemannian metrics. For example, we can extract a subsequential limit space (M_∞, g_∞) in the Gromov-Hausdorff sense, and under the assumption $\text{Ric} = 0$ and a uniform lower bound on the diameters, the limit space is smooth away from a closed subsets of Hausdorff codimension at least 4. From the analytic perspective, the importance of the non-collapsing condition is that

it guarantees many fundamental inequalities when coupled with a Ricci bound, such as the Sobolev inequality and the Harnack inequality. Consequently, the general theory in the non-collapsing case is far better understood than the collapsing case.

A heuristic explanation of this dichotomy is that collapsing generally means *information loss* in the limit. To illustrate this difference, by a result of Donaldson and Sun [13], given a noncollapsing sequence of compact polarised Calabi-Yau n -folds $(X_i, L_i, g_i, \omega_i, \Omega_i)$ with unit volume, then they all embed into a common fixed projective space $\mathbb{CP}^N$, and after subsequence and projective linear transformations they converge both metrically and algebraically into a normal variety inside $\mathbb{CP}^N$. That is, one can recover the complex geometry from the limiting variety by flat deformations. In contrast, in a general collapsing situation one does not expect to recover the sequence solely from the metric limit. To do so requires more information on *scaling limits* ('bubbles') and *adiabatic structures*. Depending on the severity of the degeneration mechanism, this additional information may have *algebraic* nature, as will be the case for this thesis, or *transcendental* nature, as in the case of large complex structure limit.

The second major thread of this thesis comes from **Kähler geometry**, and in particular a consequence of Yau's solution of the **Calabi conjecture**:

Theorem 1.1.1. [49] *A compact Kähler manifold X with a nowhere vanishing holomorphic volume form Ω admits a unique Calabi-Yau metric (g, ω, J, Ω) within any given Kähler class.*

Yau's proof, which we shall sketch in Section 1.3.1, involves a priori estimates to solve a highly nonlinear PDE, but it gives little information on what the Calabi-Yau metric actually looks like. In particular, the Calabi-Yau structure depends on natural **moduli** parameters involving the period data of Ω and the Kähler class of ω , and the central question is to relate this moduli information to Riemannian geometry.

Question 1.1.2. How does the Calabi-Yau metric behave when the complex structure or the Kähler class is allowed to degenerate?

Notable progress has been made by Tosatti and collaborators [42][43][44][25], who consider Calabi-Yau metrics on a fixed compact complex manifold with a **holomorphic fibration**, and the collapsing happens by changing the Kähler class to shrink the fibres, as shall be outlined in Section 1.3.3. Their work however gives no information on the metric behaviour near the singular fibres.

In this thesis the attention is focused on the simplest nontrivial special case of Calabi-Yau 3-folds with a **holomorphic K3 Lefschetz fibration**, and the main results are a complete description of the collapsing metric up to small fluctuations, including the neighbourhood of the singular fibres. A fuller summary will be given in Section 1.4. There are good grounds to hope for large classes of generalisations, although some restriction on the nature of the singularity of the fibration is necessary, and the author thinks the key condition should be that the singular fibres have *klt* singularities.

Aside from the intrinsic theoretic interest of metric collapsing, it has also emerged in recent years as a **construction technique** for special holonomy

metrics, particularly in the context of G_2 and $Spin(7)$ -holonomy, where a general existence theorem like Yau's solution of the Calabi conjecture is still lacking. A prototype is the construction by Haskins et al. of S^1 -invariant G_2 -metrics with small S^1 fibres collapsing down to a Calabi-Yau 3-fold [21]. A recent proposal of Donaldson, which is an initial motivation for this thesis, suggests another collapsing mechanism where the G_2 -manifolds admit a coassociative K3 fibration over a 3-dimensional base, and the K3 fibres shrink down to zero size in the limit (*cf.* Section 1.2.4).

1.2 Philosophy of collapsing problems

A very general strategy to approach a collapsing problem involves two steps: classify all the **scaling limits** ('bubbles') which control microscopic behaviour, and describe the **adiabatic structure** which controls how the bubbles vary over a large scale; this arises as soon as the bubbles have some moduli.

An upshot is that any collapsing problem is intrinsically **multi-scaled**: over the largest length scale the manifold resembles a lower dimensional object, while over the characteristic length scale of the deepest bubbles the manifold resembles a higher dimensional object. The large scale behaviour is *not* controlled by the same PDE governing the microscopic features, and identifying that adiabatic structure usually requires some additional geometric insight.

A metric (connection, function, etc) involves an infinite amount of data, but only a small amount of microscopic information effectively contributes to large scale effects. A general principle is that if we perform some version of **Fourier analysis** on the fibres, then only the zeroth Fourier mode has long range effect, while the higher Fourier modes have only localised effects.

Example 1.2.1. (Laplacian on a squashed torus) As a toy situation for analysis on collapsing manifolds, consider a torus T^2 with fundamental domain $[0, 1] \times [0, \epsilon] \subset \mathbb{R}_{x,y}^2$ with the standard Euclidean structure, where $0 < \epsilon \ll 1$, and the task is to describe the solution to the Poisson equation $\Delta u = f$ for a given source f with $\int_{T^2} f = 0$.

Define the fibrewise average function $\bar{f}(x) = \epsilon^{-1} \int_0^\epsilon f dy$, so $f_2 = f - \bar{f}$ has fibrewise average zero. Then $u = u_1 + u_2$, where $u_1(x)$ is independent of y and solves the Poisson equation on the large circle S_x^1 with source $\bar{f}$, while u_2 has fibrewise average zero and solves $\Delta u_2 = f_2$. The key point is that the Green operator acting on f_2 has an exponentially decaying Schwartz kernel, meaning that on points at distance d to the support of f_2 , the function u_2 is suppressed by an exponential factor $O(e^{-Cd/\epsilon})$.

In short, at large scale the torus behaves like a circle and only the zeroth Fourier mode has large distance effects. In physicists' language, massless particles are responsible for long range forces.

Example 1.2.2. (Holomorphic sections on a fibred space) As another toy example, let $f : X \rightarrow Y$ be a holomorphic fibration map between complex algebraic varieties, and L be a line bundle on X . Finding global holomorphic sections of $L \rightarrow X$ amounts to solving a Cauchy-Riemann equation on X . A common strategy is to first solve the equation over a local base, which means

taking the pushforward f_*L , and then find the global section of f_*L , which amounts to solving a Cauchy-Riemann equation on Y . We have

$$H^0(X, L) = H^0(Y, f_*L).$$

This example suggests that *long range effects are related to cohomology*.

Example 1.2.3. (Atiyah-Floer) Take two Riemann surfaces Σ and S with Kähler metrics g_Σ and g_S . Here Σ is compact and S may be not. The product has a family of metrics $g_\epsilon = \epsilon g_\Sigma + g_S$ as $\epsilon \rightarrow 0$. A connection A on a principal bundle over $\Sigma \times S$ is called an ASD instanton if its curvature $F_A = - * F_A$. Assuming A_ϵ is an ASD connection with respect to the metric g_ϵ , and assume naïvely that $A_\epsilon \rightarrow A$ smoothly as $\epsilon \rightarrow 0$, then taking the limit of the ASD condition,

$$\begin{cases} F_A|_{\Sigma \times \{s\}} = 0, \forall s \in S, \\ F_A \text{ is of Dolbeault type } (1,1). \end{cases}$$

Thus A defines fibrewise a *flat connection* on Σ , inducing a *holomorphic map* from S to the *moduli space of flat connection* on Σ . This holomorphic map emerges as the *adiabatic structure* which controls large distance behaviour of the ASD connections. This example is the tip of an iceberg called the Atiyah-Floer conjecture, which relates instanton Floer homology to Lagrangian Floer homology of the moduli space of flat connections [6].

Example 1.2.4. (Donaldson's adiabatic K3 fibration) Donaldson [12] recently proposed that G_2 -manifolds M with coassociative K3 fibrations can collapse down to a 3-dimensional base B by shrinking fibres. Let us first work over a local base where the fibration is a submersion. The pointwise linear algebraic model of coassociative submanifold inside a G_2 -manifold is $\mathbb{R}^4 \subset \mathbb{R}_{x_i}^4 \oplus \mathbb{R}_{t_j}^3 = \mathbb{R}^7$, with a 3-form ϕ , an induced metric g and the Hodge star 4-form $*\phi$,

$$\phi = \sum_1^3 \omega_j dt_j - dt_1 dt_2 dt_3, \quad g = \sum_1^4 dx_i^2 + \sum_1^3 dt_j^2, \quad *\phi = \frac{1}{2} \omega_1^2 - \sum_{cyc} \omega_i dt_j dt_k,$$

where $\omega_1, \omega_2, \omega_3$ are the standard basis of $\Lambda_+^2 \mathbb{R}^4$. The pointwise model for scaling down the fibres is

$$\phi_\epsilon = \epsilon \sum_1^3 \omega_j dt_j - dt_1 dt_2 dt_3, \quad g_\epsilon = \epsilon \sum_1^4 dx_i^2 + \sum_1^3 dt_j^2, \quad *_\epsilon \phi_\epsilon = \frac{\epsilon^2}{2} \omega_1^2 - \epsilon \sum_{cyc} \omega_i dt_j dt_k.$$

The G_2 -condition is

$$d\phi_\epsilon = 0, \quad d *_\epsilon \phi_\epsilon = 0.$$

Assuming nice convergence, Donaldson identified the formal limit.

First, the 2-forms $\omega_i = \iota_{\frac{\partial}{\partial t_i}} \phi$ induce a hypersymplectic structure on the K3 fibres, which in the scaled limit becomes a *hyperkähler structure*. By the Torelli theorem, the *moduli space* of hyperkähler structures is identified by the hyperkähler periods, namely the *cohomology classes* $[\omega_i]$. The closed condition $d\phi = 0$ implies some integrability on these classes, so that there exists a map $h : B \rightarrow H^2(K3)$ with $[\omega_i] = \frac{\partial h}{\partial t_i}$. Donaldson shows how to conversely obtain a *closed G_2 -structure* from the period data h . The map h immerses B as a

spacelike 3-dimensional submanifold of $H^2(K3)$, which has mixed signature (3,19). Taking the limit of the coclosed condition leads to the requirement that this submanifold has zero mean curvature. Such a submanifold is called *maximal*, encoding the *adiabatic structure* of collapsing coassociative fibrations. The author solved the Dirichlet problem for graphical maximal submanifolds in [30].

To get compact examples one has to work over a global base and allow for singular fibres. Donaldson's proposal is to allow *Kovalev-Lefschetz singularities*, meaning essentially that the critical locus of the fibration map consists of circles, transverse to which the fibration is modelled on a Lefschetz fibration for Calabi-Yau 3-folds.

The main focus of this thesis is the dimensionally reduced case of a Calabi-Yau 3-fold with a Lefschetz K3 fibration over $\mathbb{CP}^1$. Here the adiabatic structure is the *period integrals* of the K3 fibres.

Example 1.2.5. (SYZ conjecture) This roughly says that generic Calabi-Yau metrics on polarised 3-folds near the large complex structure limit admit special Lagrangian T^3 fibrations, with approximately flat fibres, over a real 3-dimensional base B which is topologically S^3 . Moreover, the base admits an affine structure with singularity, and a compatible *real Monge-Ampère metric*, which agrees with the Gromov-Hausdorff limit of the degenerating CY metrics.

This fits into our philosophy: the bubbles in question are flat $T^3 \times \mathbb{R}^3$. Flat Euclidean metrics on T^3 have some *moduli*, and the large scale variation in this moduli is governed by the *real Monge-Ampère equation*.

1.3 Calabi conjecture and ramifications

We highlight some key points of Yau's solution to the Calabi conjecture [49], and discuss its proliferation in the contexts of weaker regularity assumption [7], noncompact complete setting [23], and collapsing families [42][43][44].

1.3.1 Summary of Yau's proof

We now give a very sketchy account of (the modern view on) Yau's proof of Theorem 1.1.1. Fix a background Kähler metric ω in the given class, and the task is to find a Kähler potential ϕ solving the **complex Monge-Ampère equation** for any given density function e^f satisfying the cohomological constraint $\int_X e^f \omega^n = \int_X \omega^n$,

$$\omega_\phi^n = (\omega + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^f \omega^n, \quad \int_X \phi \omega^n = 0, \quad (1.1)$$

In particular for $e^f \omega^n = \text{const} \cdot \Omega \wedge \bar{\Omega}$ one obtains the Calabi-Yau metric. The uniqueness follows from a simple integration by parts argument.

Remark 1.3.1. *A good feature is that (1.1) is a second order scalar PDE, and the space of Kähler potentials has convexity properties. In contrast many other problems in special holonomy involve tensorial PDEs posed on topologically nontrivial configuration spaces, which are much less tractable.*

The existence proof follows the continuity method, namely to deform through the space of Kähler metrics as f deforms from zero to the desired function. Viewing (1.1) as defining a map from the potential to the density function, we need to show the map is **submersive** and **proper**. The submersion property is a consequence of standard Hodge theory on the Laplacian. To show properness one needs **a priori estimates** on ϕ depending only on X and bounds on f , and for that matter we are free to impose any order of regularity on f .

The first step is to bound $\|\phi\|_{C^0}$ (known as ‘ C^0 -estimate’), using a technique called **Moser iteration**. We rewrite the equation as

$$(1 - e^f)\omega^n = -\sqrt{-1}\partial\bar{\partial}\phi \wedge (\omega^{n-1} + \omega^{n-2} \wedge \omega_\phi + \dots + \omega_\phi^{n-1}).$$

For any $p > 1$, multiply the equation by $|\phi|^{p-2}$ and integrate by parts,

$$\int_X |\nabla |\phi|^{p/2}|_\omega^2 \omega^n \leq \frac{C(n, \|f\|_{L^\infty})p^2}{p-1} \int_X |\phi|^{p-1} \omega^n. \quad (1.2)$$

The key is that a weaker norm on RHS controls a stronger norm on LHS. This is reverse to the direction of the Poincaré inequality

$$\int_X u^2 \omega^n \leq C \int_X |du|_\omega^2 \omega^n, \quad \int_X u = 0, \quad (1.3)$$

and the **Sobolev inequality**

$$\int_X |u|^{\frac{2n}{n-1}} \omega^n \leq C \left(\int_X |du|_\omega^2 \omega^n + \int_X |u|^2 \omega^n \right). \quad (1.4)$$

The combined force is that $\|\phi\|_{L_1^2} \leq C$, and one can inductively bound $\|\phi\|_{L^p}$ for large p , which turns out to be uniform in p . Taking the limit $p \rightarrow \infty$, this gives a bound $\|\phi\|_{L^\infty} \leq C(X, \|f\|_{L^\infty})$.

The second step is to bound $\|\partial\bar{\partial}\phi\|_{C^0}$ (known as $C^{1,\bar{1}}$ -estimate), *i.e.* to prove a uniform equivalence between ω and $\omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi$. The starting point is a differential inequality by local calculations (in the analyst’s convention of Laplacian)

$$\Delta_{\omega_\phi} \log \text{Tr}_\omega \omega_\phi \geq -C \text{Tr}_{\omega_\phi} \omega - C.$$

Subtracting a large enough multiple of the identity

$$\Delta_{\omega_\phi} \phi = n - \text{Tr}_{\omega_\phi} \omega,$$

we get a differential inequality

$$\Delta_{\omega_\phi} (\log \text{Tr}_\omega \omega_\phi - C\phi) \geq \text{Tr}_{\omega_\phi} \omega - C'.$$

Using the a priori bound on ϕ , an application of the **maximum principle** then shows $\text{Tr}_\omega \omega_\phi \leq C$.

Remark 1.3.2. This $C^{1,\bar{1}}$ -estimate argument has **global nature**: if we only know (1.1) on a standard unit ball in $\mathbb{C}^n$ with a given C^0 -bound on ϕ , there are counterexamples for the $C^{1,\bar{1}}$ -bound. The solution can develop singularities.

The third step is to get higher order estimates. Standard elliptic theory implies it is sufficient to have a $C^{2,\alpha}$ bound on ϕ . Evans-Krylov theory bridged the gap between $C^{1,1}$ -bound and $C^{2,\alpha}$ -bound (cf. [36] Chapter 2, Section 4 for details). This argument is of **local nature**, and crucially uses that $\log \det(\partial_i \partial_{\bar{j}} \phi)$ is a *concave* function of the matrix $(\partial_i \partial_{\bar{j}} \phi)$, and the main tool is a Harnack inequality. The geometric insight is that higher order regularity is a manifestation of the local Euclidean nature of Kähler manifolds.

While Yau's method is firmly rooted in Riemannian geometry (aka elliptic analysis), some significant subsequent developments focus more on the potential function, and draw tools from complex analysis (aka **pluripotential theory**). The exemplary result originally due to Kolodziej is

Theorem 1.3.3. (cf. *Theorem A* in [7]*) *Fix $p > 1$. On a fixed Kähler manifold (X, ω) , given a density function e^f satisfying $\int_X e^f \omega^n = \int_X \omega^n$, then the solution ϕ to (1.1) satisfies a Hölder bound in terms of an upper bound on $\|e^f\|_{L^p}$:*

$$\|\phi\|_{C^\alpha} \leq C(X, \|e^f\|_{L^p}).$$

The Hölder exponent α can be taken to be any positive number smaller than $\frac{2}{np^+1}$, where p^* is the conjugate exponent of p .*

This circle of ideas enable one to solve complex Monge-Ampère equation with very **weak regularity**, which is useful for studying singularity formation. The drawback is that the regularity from pluripotential theory is too weak to control the metric directly, and Riemannian geometry of singular solutions is an active research area.

1.3.2 Complete noncompact Calabi-Yau metrics

The thesis of H.-J. Hein (cf. Chapter 3,4 in [23]) sets out a framework for solving the complex Monge-Ampère equation (1.1) and its linear cousin the Poisson equation on complete noncompact manifolds, building on the seminal paper by Tian and Yau [40]. We explain Hein's results in a manner which highlights its logical structure. The ambient complete manifold M is *assumed* to satisfy the following analytic properties:

- There is a $C^{k,\alpha}$ quasi-atlas with $k \geq 3$, meaning a collection of charts on which the complex structure and the metric have $C^{k,\alpha}$ bounds, and the injectivity radius/regularity scale in these charts are bounded below. This assumption allows one to speak of (unweighted) Hölder spaces, and is typically easy to satisfy.
- There is a function $\rho(x)$ uniformly equivalent to the distance function $\text{dist}(0, x)$ outside the unit ball, which satisfies $|\nabla \rho| + \rho|\sqrt{-1}\partial\bar{\partial}\rho| \leq C$. This assumption is useful in integration by part arguments.
- We need the **weighted Sobolev inequality** on functions: assume the power law volume growth $\text{Vol}(B(r)) \sim r^\beta$ with rate $\beta > 2$. For $1 \leq p \leq \frac{\dim_{\mathbb{R}} M}{\dim_{\mathbb{R}} M - 2}$ and functions u with L^2 -gradient,

$$\left(\int |u|^{2p} (1 + \rho)^{p(\beta-2)-\beta} d\text{Vol} \right)^{1/p} \leq C \int |\nabla u|^2.$$

These inequalities differ from the standard Sobolev inequalities in the sense that they do not require the manifold to have maximal volume growth, which makes them remarkably flexible.

The output of this package is:

- **(Poisson equation case)** Let $f \in C^{0,\alpha}$ satisfy $|f| \leq C\rho^{-q}$ for given $\beta > q > 2$. Then there is a unique $C^{2,\alpha}$ solution to $\Delta u = f$ with decay estimate $|u| \leq C\rho^{2-q+\epsilon}$, where $\epsilon \ll 1$ is any fixed small number satisfying $2-q+\epsilon < 0$.
- **(Complex Monge-Ampère equation case, cf. [23, Proposition 4.1])** Denote ω as the ambient Kähler form. Let $f \in C^{2,\alpha}$ satisfy $|f| \leq C\rho^{-q}$ for $2 < q < \beta$. Then there is some $0 < \alpha' \leq \alpha$ and $\phi \in C^{4,\alpha'}$ solving (1.1), with decay estimate $|\phi| \leq C\rho^{2-q+\epsilon}$, where $\epsilon \ll 1$ is any fixed small number. If in addition $f \in C^{k,\bar{\alpha}}(M)$ for some $k \geq 3$, then ϕ belongs to $C_{\text{loc}}^{k+2,\bar{\alpha}}(M)$.

The key idea in Hein's package is to construct a priori L^∞ estimates and power law decay estimates on potentials via the method of **weighted Moser iteration**, which hinges on the weighted Sobolev inequalities. A major problem is to ensure the boundary terms drop out in integration by part calculations. In the Poisson equation case, this is done by solving Dirichlet problem on larger and larger domains. In the complex Monge-Ampère equation case Hein considers a family of *viscosity equations*

$$(\omega + \sqrt{-1}\partial\bar{\partial}u_\epsilon)^n = e^{f+\epsilon u_\epsilon}\omega^n,$$

and derives uniform estimates for all small $\epsilon > 0$. The desired solution follows by taking a subsequential limit, which is *not a priori unique*.

Here we have separated the assumptions on the ambient manifolds from the decay assumptions to emphasize their distinct nature. It is essential to assume **faster than quadratic decay** on the source function f , because the method needs the potential $u = O(\rho^{2-q})$ to be bounded. Hein's method respects compact group actions.

Hein also gave sufficient conditions for the weighted Sobolev inequality:

Definition 1.3.4. A complete noncompact Riemannian manifold (M, g) is called $SOB(\beta)$ if there is a point $x_0 \in M$, a fixed small number $\delta > 0$, and $C \geq 1$, such that any two points $p, q \in M$ with $d(p, x_0) = d(q, x_0) = s$ are connected within the annulus $A(x_0, (1-\delta)s, (1+\delta)s)$ whenever $s > C$. Moreover $\text{Vol}(B(x_0, s)) \leq Cs^\beta$ for all $s \geq C$, and $\text{Vol}(B(x, (1-\frac{1}{C})r(x))) \geq \frac{1}{C}r(x)^\beta$, and $\text{Ricci} \geq -Cr(x)^{-2}$ if $r(x) = d(x_0, x) \geq C$.

Proposition 1.3.5. (cf. [23] Prop. 3.2) Suppose the complete manifold (M, g) satisfies $SOB(\beta)$ for $\beta > 2$, then the weighted Sobolev inequality holds.

Remark 1.3.6. The definition of $SOB(\beta)$ used in [23] requires also the connectedness of the annulus $A(s, t)$ for any $t > s > C$, but this stronger condition is not needed for his weighted Sobolev inequality. The condition $\beta > 2$ reflects a basic distinction between parabolic ($\beta \leq 2$) and non-parabolic ($\beta > 2$) manifolds. For instance the Green's function on $\mathbb{R}^n$ is positive for $n \geq 3$, but changes sign for $n = 2$.

1.3.3 Collapsing Calabi-Yau metrics à la Tosatti

We consider collapsing Calabi-Yau metrics on a fixed complex manifold under the degeneration of the Kähler class. More precisely, let (X, ω_X) be a compact n -dimensional Kähler manifold with a nowhere vanishing holomorphic volume form Ω . Let $\pi : X \rightarrow Y$ be a holomorphic fibration to an m -dimensional base manifold Y where $m < n$, with connected fibres denoted by X_y for $y \in Y$. By the adjunction formula, the smooth fibres have trivial canonical bundle in their own right. We normalise the fibre volume by $\int_{X_y} \omega_X^{n-m} = 1$. Away from the discriminant locus $S \subset Y$, the fibration is a smooth submersion. Let ω_Y be a Kähler metric on Y , and consider for each $0 < t \leq 1$ the unique Calabi-Yau metric $\tilde{\omega}_t$ in the Kähler class defined by $\omega_t = \pi^* \omega_Y + t \omega_X$. We are interested in uniform estimates for $\tilde{\omega}_t$ in terms of the geometry of the fibration, and the asymptotic behaviour as $t \rightarrow 0$.

This has been investigated by Tosatti in [42] and subsequently improved by Tosatti et al [43][44]. Their basic picture is that the collapsing metric *away from the singular fibres* $\pi^{-1}(S)$ involves two characteristic scales:

- **(base scale)** Over any compact subset in $Y \setminus S$, the Calabi-Yau metric $\tilde{\omega}_t$ is uniformly equivalent to ω_t with constants independent of t . As $t \rightarrow 0$, with suitable normalisation the Kähler potential of $\tilde{\omega}_t$ relative to ω_t converges in $C_{loc}^{1,1}(X \setminus f^{-1}(S), \omega_X)$, and the Kähler form converges in C^0 to the pullback of the **generalised Kähler Einstein metric** $\tilde{\omega}_Y$ on the base which satisfies

$$\text{Ric}(\tilde{\omega}_Y) = \text{Weil Petersson metric},$$

or its integrated form

$$\tilde{\omega}_Y^m = \frac{\int_Y \omega_Y^m}{\int_X \sqrt{-1}^{n^2} \Omega \wedge \bar{\Omega}} \pi_*(\sqrt{-1}^{n^2} \Omega \wedge \bar{\Omega}), \quad (1.5)$$

where π_* denotes the pushforward of the volume measure.

- **(fibre scale)** For $y \in Y \setminus S$, the fibrewise metric $\frac{1}{t} \tilde{\omega}_t|_{X_y}$ converges smoothly to the Calabi-Yau metric $\omega_{SRF,y}$ on the fibre X_y in the class $[\omega_X|_{X_y}]$.

Tosatti's picture may be called the **semi-Ricci-flat** description, as the fibre metrics are approximately Ricci flat. In terms of our general philosophy on collapsing (cf. Section 1.2), the *complete bubbles* are $\omega_{SRF,y} \times \mathbb{C}^m$, and the *adiabatic structure* is the generalised KE metric, which carries the information of the period data of the fibres. An insight of this thesis is that the complete bubbles become more nuanced near the singular fibre.

Remark 1.3.7. We may rescale $\tilde{\omega}_t$ for convenience. In the Tosatti convention above $\int_{X_y} \tilde{\omega}_t^{n-m} = t^{n-m}$, where the base has diameter $O(1)$. A rescaled convention where the fibre volume $\int_{X_y} \tilde{\omega}_t^{n-m} = 1$ is more useful for gluing purposes.

Remark 1.3.8. The generalised KE metric on Y is in general a singular Kähler metric defined as the unique solution to (1.5) in the class $[\omega_Y]$, with singularity along S . Since the RHS measure can be highly degenerate along S , solving this

equation is in general very difficult, relying on deep pluripotential theory [14]. However for $m = 1$, on the 1-dimensional base a volume measure is the same as a singular Kähler form, and the generalised KE metric amounts to **period integrals**.

This enables Tosatti et al [18] to prove that in the special case $m = 1$, the Gromov-Hausdorff limit of $(X, \tilde{\omega}_t)$ exists as $t \rightarrow 0$, is homeomorphic to Y , and is the metric completion of $(Y \setminus S, \tilde{\omega}_Y)$. This rather difficult result requires more control near S , and uses some advanced Hodge theory and collapsed Cheeger-Colding theory.

The general ideas of Tosatti's work are variations on Yau's arguments, involving Moser iteration and maximum principles. Two additional inputs from pluripotential theory involve a uniform bound on the Kähler potential of $\tilde{\omega}_t$ relative to ω_t (cf. Proposition 2.1.1), and the solution to the generalised KE metric.

Example 1.3.9. (Gross-Wilson metric [19]) The prototype is a collapsing elliptic K3 surface fibred over $\mathbb{P}^1$ with 24 nodal fibres, where the fibres volume shrinks down to small size. Away from the nodal fibres, the fibrewise metrics $t^{-1}\tilde{\omega}_t$ converge to the Euclidean metrics on elliptic fibres, but the convergence rate is *not uniform* in all fibres. Around the nodal fibres there is an explicit S^1 -invariant metric model called the *Ooguri-Vafa metric*, which describes the local metric up to exponentially small errors. Although $t^{-1}\tilde{\omega}_t$ has unit fibre volume, the fibre diameters are *not uniformly bounded*, because the Euclidean metrics on the I_1 family of elliptic curves become very long and thin near the nodal fibre.

We remark to interested readers that the author [29] has generalised the Ooguri-Vafa metric to Calabi-Yau 3-folds.

1.4 Lefschetz K3 fibred 3-folds: overview

The basic setting of this thesis is a special case of Section 1.3.3. Let X be a compact **Calabi-Yau 3-fold with a Lefschetz K3 fibration** $\pi : X \rightarrow Y = \mathbb{P}^1$. Given reference Kähler metrics ω_X on X and ω_Y on Y with volume normalisation convention

$$\int_{X_y} \omega_X^2 = 1, \quad \int_Y \omega_Y = 1, \quad \int_X \sqrt{-1} \Omega \wedge \bar{\Omega} = 1, \quad (1.6)$$

we aim to describe the collapsing family of Calabi-Yau metrics $\tilde{\omega}_t$ in the Kähler class of $\omega_t = t\omega_X + \pi^*\omega_Y$ for $0 < t \ll 1$. Denote S as the finite set of critical values of π . For simplicity, we assume each singular fibre contains only one nodal point.

This problem can be approached from both the nonlinear perspective and the gluing perspective. We will largely follow the historical order of discovery, and the 3 main Chapters of this thesis are respectively based on the author's papers [27][26][28], whose contents shall be summarized in the 3 Sections below.

1.4.1 Nonlinear perspective

A central message of Chapter 2 is that under more restrictive hypotheses about the nature of the singularity for the fibration, some key features of the collapsing metrics $\tilde{\omega}_t$ near the singular fibres can be derived from a priori estimates by improving Tosatti's methods. We shall focus on Lefschetz K3 fibred 3-folds, although many arguments are of quite general character. The geometric picture stands *in contrast* to the Gross-Wilson metric Example 1.3.9.

A key feature of Lefschetz K3 fibrations is that the singular fibre is *normal* and has **klt singularity**. Over a local base with coordinate y , using the adjunction formula, a section of the relative canonical bundle Ω_y is defined by $\Omega = dy \wedge \Omega_y$. An analytic manifestation of the klt condition is a uniform upper bound in all fibres (*cf.* Lemma 1.4.10)

$$A_y = \int_{X_y} \Omega_y \wedge \bar{\Omega}_y \leq C,$$

or equivalently a pointwise upper bound for the pushforward volume measure

$$\pi_*(\sqrt{-1}\Omega \wedge \bar{\Omega}) \leq C\omega_Y. \quad (1.7)$$

Another manifestation is that each fibre X_y admits a Calabi-Yau metric $\omega_{SRF,y}$ in the Kähler class $[\omega_X|_{X_y}]$, with uniformly bounded diameters

$$\text{diam}(X_y, \omega_{SRF,y}) \leq C, \quad \forall y \in Y. \quad (1.8)$$

This fact is easy in our K3 Lefschetz fibred situation because Yau's proof also applies to orbifolds, and smoothings of the nodal K3 are obtained by gluing Eguchi-Hanson metrics to the K3 orbifold. Heuristically, the existence of a Calabi-Yau metric on every fibre asserts a strong sense of polystability.

Remark 1.4.1. *For more general fibrations, the existence of singular Calabi-Yau metrics on normal Calabi-Yau varieties with klt singularities follows from difficult pluripotential theory [14]. The uniform diameter bound is given in [35, Theorem 2.1], for the Calabi-Yau metrics on a smoothing family of a normal Calabi-Yau variety with klt singularities such that the relative canonical bundle is trivial. In particular this is true for Lefschetz fibrations with $n \geq 3$. In contrast, the above properties are not true for an I_1 -type elliptic fibration.*

As a consequence of the **volume condition** (1.7), we show

Proposition 1.4.2. (*cf.* Section 2.1) There is a uniform lower bound $\tilde{\omega}_t \geq C\omega_t$ which holds independent of t on X .

Remark 1.4.3. *The uniform upper bound $\tilde{\omega}_t \leq C\omega_t$ is not true near the nodes, because the Calabi-Yau volume form $\tilde{\omega}_t^3$ is much larger than ω_t^3 at the nodes.*

The next key condition is the **uniform fibre diameter bound** for all $0 < t \leq 1$ and all $y \in Y$,

$$\text{diam}(X_y, t^{-1}\tilde{\omega}_t) \leq C. \quad (1.9)$$

We attempt to prove this a priori by developing general tools to bound the distance function of Kähler metrics from knowledge on the volume density:

Theorem 1.4.4. (cf. Section 2.2.1) Fix $p > 1$. On a fixed compact Kähler manifold (M, ω) of dimension n , assume another Kähler metric $\omega_\psi = \omega + \sqrt{-1}\partial\bar{\partial}\psi$ satisfies $\omega_\psi^n = h\omega^n$, then the distance function in the new metric $d_{\omega_\psi}(x, y)$ has a Hölder bound with respect to the original metric, with constants and Hölder exponent only depending on (M, ω) and an upper bound on $\|h\|_{L^p}$.

We want to apply this idea to control the rescaled fibre metrics $t^{-1}\tilde{\omega}_t|_{X_y}$, up to the singular fibre. We shall make use of a very special case of the following conjecture, which is an extension of the Kolodziej estimate Theorem 1.3.3, allowing the Kähler class to degenerate to a *semiample* class without strict positivity.

Conjecture 1.4.5. (See [7], discussion after Theorem A*) Given a deformation family of smooth compact n -dimensional Kähler manifolds $(\mathcal{X}_y, \omega_y)$ over a small disc, consider a family of semiample classes $[\theta_y]$ on the fibres, which have smooth semi-positive representing forms θ_y , with uniformly C^∞ bounded local potentials. Given solutions to the Monge-Ampère equation in the class $[\theta_y]$,

$$\theta'_y = \theta_y + \sqrt{-1}\partial\bar{\partial}\psi_y, \quad \theta_y'^n = h_y\theta_y^n, \quad \sup_{\mathcal{X}_y}\psi_y = 0, \quad (1.10)$$

then for $p > 1$, a uniform L^p bound on the volume density h_y implies a uniform Hölder estimate on the Kähler potential ψ_y . The Hölder exponent α can be taken as $0 < \alpha < \frac{2}{1+np^*}$, where p^* is the conjugate exponent of p .

Using this and some special algebro-geometric facts in dimension 3, we show

Theorem 1.4.6. (cf. Section 2.2.1) Assume Conjecture 1.4.5, the fibre diameter bound (1.9) holds uniformly for all t and all fibres.

Remark 1.4.7. The author expects that (1.9) holds in the higher dimensional Lefschetz fibration case, and more generally for fibrations over Riemann surfaces with klt singular fibres. As a piece of evidence, recall (cf. Section 1.3.3) Tosatti et al showed that on all smooth fibres $t^{-1}\tilde{\omega}_t|_{X_y}$ converge to the fibrewise CY metrics $\omega_{SRF,y}$ as $t \rightarrow 0$, with convergence rate dependent on the fibre, and we mentioned in Remark 1.4.1 that $\omega_{SRF,y}$ have uniform diameter bounds in the klt case. The main difficulty of the general case lies in the pluripotential theoretic problem of achieving uniform Hölder type bounds on the Kähler potential near the singular fibre.

This uniform fibre diameter bound implies that the neighbourhood of any fibre with the metric $t^{-1}\tilde{\omega}_t$ satisfies a **uniform local non-collapsing** condition, which links the work to Gromov-Hausdorff convergence theory, and enables the application of many standard geometric inequalities. In particular, we can take the **non-collapsed pointed Gromov-Hausdorff limit** of $t^{-1}\tilde{\omega}_t$ around the nodes as $t \rightarrow 0$.

Assuming Conjecture 1.4.5, then aside from reproving all of Tosatti's results in Section 1.3.3 for Lefschetz K3 fibrations, we also obtain strong control near the nodal fibres:

Theorem 1.4.8. (cf. Section 2.3.4) Assume Conjecture 1.4.5. As $t \rightarrow 0$, the family of CY metrics $t^{-1}\tilde{\omega}_t$ based at the nodal point converges in the Gromov-Hausdorff sense to the metric product $X_0 \times \mathbb{C}$, where the nodal K3 fibre X_0

is equipped with the orbifold CY metric $\omega_{SRF}|_{X_0}$, and the $\mathbb{C}$ factor has the Euclidean metric.

1.4.2 Exotic metric on $\mathbb{C}^3$

A number of nonlinear heuristic insights from Theorem 1.4.8 motivate the construction of a new complete CY metric $\omega_{\mathbb{C}^3}$ on $\mathbb{C}^3$:

- The Gromov-Hausdorff limit of $t^{-1}\tilde{\omega}_t$ based at the nodal point is *singular* along a nodal line in $X_0 \times \mathbb{C}$, which suggests the *Riemannian curvature blows up near the vanishing cycles* as $t \rightarrow 0$ in the fibre scale. In the Gromov-Hausdorff limit all vanishing cycles collapse down to points, and the local structure is not captured by $X_0 \times \mathbb{C}$.
- If we scale up the CY metrics $\tilde{\omega}_t$ around the point with maximal Riemannian curvature, until $\max |\text{Rm}| = 1$, and consider the subsequential limit as $t \rightarrow 0$, then the metrics should *converge smoothly to a nontrivial complete CY metric bubble*. This geometry happens at a much *finer length scale* compared to the fibre scale, which the author calls the **quantisation scale**. The reason for the terminology is that the semi-Ricci-flat description of Tosatti is formally analogous to the semiclassical limit in quantum mechanics, and the quantisation scale is where it breaks down as a C^0 -approximation to the metric.
- In the quantisation scale, the nontrivial topology of the K3 fibres disappears into spatial infinity. This suggests the metric bubble should be $\mathbb{C}^3$. The local information of the *Lefschetz fibration* should be captured by the bubble.
- Since the metric at the fibre scale is locally non-collapsed already, Bishop-Gromov monotonicity (cf. 1.1) implies that the metric at the quantisation scale is also locally noncollapsed. Passing to the limit, the metric bubble $\omega_{\mathbb{C}^3}$ should satisfy the **maximal volume growth** condition

$$\text{Vol}(B_{\omega_{\mathbb{C}^3}}(r)) \geq \kappa r^6, \quad \text{for some } \kappa > 0. \quad (1.11)$$

- The **tangent cone at infinity** for the metric bubble should match up with the local tangent cone of $X_0 \times \mathbb{C}$, which is $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$.

The main result of Chapter 3 is:

Theorem 1.4.9. *There is a complete Calabi-Yau metric $\omega_{\mathbb{C}^3}$ of maximal volume growth on $\mathbb{C}^3$ with the standard Euclidean holomorphic volume form, whose tangent cone at infinity is the singular cone $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$.*

We briefly summarize the method. It is natural to write down a so called semi-Ricci-flat metric as a first approximation, which restricts to the Eguchi-Hanson metrics on the quadric fibres of $f = z_1^2 + z_2^2 + z_3^2 : \mathbb{C}^3 \rightarrow \mathbb{C}$. In terms of Kähler potential this naïve ansatz is

$$\phi_{\text{naive}} = \frac{1}{2}|f|^2 + \sqrt{H + |f|}, \quad H = \sum_i |z_i|^2, \quad (1.12)$$

where $\sqrt{H + |f|}$ is the Kähler potential of the Eguchi-Hanson metrics on the fibres, and $\frac{1}{2}|f|^2$ corresponds to a Euclidean metric on the base $\mathbb{C}$. This is a singular metric admitting $SO(3, \mathbb{R}) \times U(1)$ -**symmetry**, suggesting that the construction of Calabi-Yau metric is a cohomogeneity two problem. The singularity issue arises because $\sqrt{H + |f|}$ is not smooth along $f = 0$. We then choose a **regularisation** of the metric ansatz (*cf.* Main ansatz 3.2.3), whose volume form is asymptotic to a multiple of the standard Euclidean volume form on $\mathbb{C}^3$. We carefully examine the error terms involved in the approximate metric, and then apply Hein's thesis (*cf.* Section 1.3.2) to perturb this into a genuine Calabi-Yau metric.

This Calabi-Yau metric is interesting from several other viewpoints. One context is Gromov-Hausdorff convergence theory in Riemannian geometry. Our metric is a nontrivial Ricci-flat metric on $\mathbb{C}^3$ with maximal volume growth, whose tangent cone at infinity is the singular variety $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$; this falsifies a conjecture [41, Remark 5.3] in the field. The neighbourhood of vanishing cycles in the Lefschetz fibration are asymptotically collapsed to the nodal line, similar to the picture in the unpublished work of Hein and Naber on A_k -type singularities. From the viewpoints of Euclidean geometry this is counterintuitive as the collapsed region apparently carries a large amount of Lebesgue measure. The behaviour of the Laplacian is likewise non-standard in this region; in fact the cohomogeneity two problem of inverting the Laplacian effectively reduces to solving ODEs, in contrast to the Euclidean intuition, which suggests a genuine dependence of the Laplacian on both variables.

This work has since been generalised to higher dimensions by the recent independent work of R. Conlon and F. Rochon [3], and G. Székelyhidi [39], where the fibration f is replaced by more complicated fibrations $\mathbb{C}^n \rightarrow \mathbb{C}$ defined using a large class of weighted homogenous polynomials. Their difficulty comes from the computational complexity of estimating the main error terms in the ansatz, and developing the linear theory to correct these errors which tend to have very slow decay rate near infinity. Some ideas of Székelyhidi's linear theory are sketched in Section 3.5. Section 3.7 contains some informal discussions on the moduli question of classifying complete Calabi-Yau metrics on $\mathbb{C}^n$ with maximal volume growth.

1.4.3 Gluing perspective

The a priori estimate method is not well-suited to detect the geometry at extremely small length scales. On the other hand, the gluing method has been used to some effect in collapsing problems, such as J. Fine's construction of cscK metrics on fibred complex surfaces [9], and Gross and Wilson's construction of CY metrics on elliptic K3 surfaces [19].

The main goal of Chapter 4 is to give a gluing description of the rescaled collapsing metrics $t^{-1}\tilde{\omega}_t$ (*cf.* Theorem 4.4.1). The main consequence is that Theorem 1.4.8 holds unconditionally, and that $\omega_{\mathbb{C}^3}$ arises as a **blow up limit** of the collapsing metrics $\tilde{\omega}_t$ near the node (*cf.* Theorem 4.0.4).

We now describe the gluing models in various regions.

The generalised Kähler Einstein metric $\tilde{\omega}_Y$

The generalised Kähler Einstein metric $\tilde{\omega}_Y$ on the base Y is the collapsing limit as $t \rightarrow 0$. Since the base Y is complex one-dimensional, we can write down $\tilde{\omega}_Y$ rather more explicitly (cf. Remark 1.3.8). Let Ω be the holomorphic volume form on X , normalised to $\int_X \sqrt{-1}\Omega \wedge \bar{\Omega} = 1$. Under our normalisation convention $\int_Y \omega_Y = \int_Y \tilde{\omega}_Y = 1$, then $\tilde{\omega}_Y$ is just the pushforward of the volume form

$$\tilde{\omega}_Y = \pi_*(\sqrt{-1}\Omega \wedge \bar{\Omega}). \quad (1.13)$$

Picking a holomorphic local coordinate y on Y , we can write $\Omega = dy \wedge \Omega_y$, where by adjunction Ω_y defines a holomorphic section of the relative canonical bundle. The formula (1.13) boils down to

$$\tilde{\omega}_Y = \sqrt{-1}dy \wedge d\bar{y} \int_{X_y} \Omega_y \wedge \bar{\Omega}_y = A_y \sqrt{-1}dy \wedge d\bar{y}. \quad (1.14)$$

For our Lefschetz fibration,

Lemma 1.4.10. *The function $A_y = \int_{X_y} \Omega_y \wedge \bar{\Omega}_y$ is Lipschitz in y .*

Proof. We focus on the fibration $\pi : \mathcal{X} \rightarrow D_y$ over a small disc around the nodal fibre X_0 , and take the square root fibration $\mathcal{X}' \rightarrow D_{\sqrt{y}}$. After taking a small resolution $\tilde{\mathcal{X}}' \rightarrow \mathcal{X}'$, the fibration $\tilde{\mathcal{X}}' \rightarrow D_{\sqrt{y}}$ becomes a submersion. Since Ω_y still defines a holomorphic section of the relative canonical bundle for the new fibration, the submersion property shows that A_y is a smooth function over the square root base, namely A_y is a smooth function of $\sqrt{y}$. But A_y is an even function of $\sqrt{y}$, hence Lipschitz in y .

We take a closer examination at the singularity of A_y , which is not needed for the proof. Notice that Ω_y is a closed 2-form on X_y , and so is $\bar{\Omega}_y$. This means the fibrewise integral A_y only depends on the cohomology class of Ω_y , which is the same data as the period integrals. Let $\Sigma \in H_2(X_y)$ be the class of a vanishing cycle. If a 2-cycle $\alpha \in H_2(X_y)$ is monodromy invariant, or equivalently it is orthogonal to Σ under the intersection product, then the period integral $\int_\alpha \Omega_y$ is smooth in y , because we can make the representing cycles avoid the nodal point. To understand $\int_\Sigma \Omega_y$, we again pass to the family $\tilde{\mathcal{X}}' \rightarrow D_{\sqrt{y}}$. This vanishing cycle class Σ becomes the class of the exceptional $\mathbb{P}^1$ when we take the small resolution. Again $\int_\Sigma \Omega_y$ is smooth in $\sqrt{y}$. Furthermore, the Picard-Lefschetz formula implies that $\int_\Sigma \Omega_y$ is an odd function of $\sqrt{y}$, and the nature of period integrals implies this function is holomorphic in $\sqrt{y}$, so

$$\int_\Sigma \Omega_y = g(y)\sqrt{y},$$

where g is a holomorphic function in y . The Lefschetz fibration imposes a further nondegeneracy condition on the deformation of the nodal fibre, which being translated into period integrals means $g(0) \neq 0$. Combining these discussions, the class $[\Omega_y] \in H^2(X_y)$ is the sum of a smooth monodromy invariant part and an orthogonal part $(g(y)\sqrt{y})\Sigma$. This implies

$$A_y = \int_{X_y} [\Omega_y] \wedge [\bar{\Omega}_y] = \text{smooth term} - 2|g(y)|^2|y|.$$

The factor -2 comes from $\Sigma \cdot \Sigma = -2$. Notice $|g(y)|^2$ is smooth in y , but the modulus function $|y|$ is not C^1 in y , despite being smooth in $\sqrt{y}$. From the nondegeneracy condition $g(0) \neq 0$, we see the Lipschitz regularity is the sharp statement. $\square$

Clearly $A_y \geq C > 0$, so $\tilde{\omega}_Y$ is uniformly equivalent to ω_Y on Y . The failure of smoothness of A_y constrains the regularity of the metric ansatz we can produce.

Semi-Ricci-flat metric

We solve the Monge-Ampère equation on the K3 fibres X_y to find a function ψ_y , such that $\omega_X|_{X_y} + \sqrt{-1}\partial\bar{\partial}\psi_y$ is the Calabi-Yau metric on X_y . Set $\psi = \psi_y$ as a global function on X , then the semi-Ricci-flat metric ω_{SRF} is defined by

$$\omega_{SRF} = \omega_X + \sqrt{-1}\partial\bar{\partial}\psi + \frac{1}{t}\pi^*\tilde{\omega}_Y. \quad (1.15)$$

Its defining property is that the fibrewise restriction is Ricci-flat, and as such captures key features of the collapsing metrics $\tilde{\omega}_t$ suitably away from the nodes. However, its definition involves the ambiguity of a form pulled back from the base, is not necessarily positive definite, and can be quite singular near $\pi^{-1}(S)$. Because of these difficulties we need to **regularize** ω_{SRF} before using it as a gluing ansatz away from the nodes, which involve some technical complexity.

Embedding of model metric on $\mathbb{C}^3$

We sketch why it is possible to glue in the model Calabi-Yau metric $\omega_{\mathbb{C}^3}$ on $\mathbb{C}^3$ to the semi-Ricci-flat metric, modulo more technical issues involving regularization and gluing errors.

First, the local behaviour of ω_{SRF} amounts to understanding the CY metrics on the smoothing fibres of the K3 orbifold. A relatively standard description is that these CY metrics are obtained by gluing (suitably framed) Eguchi-Hanson metrics on affine quadrics to the orbifold K3 metric. In suitable local coordinates $\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3$ on X , the potential for the fibrewise metric is approximated locally by the Eguchi-Hanson potential $\sqrt{r^4 + |y|}$, where

$$r = \left(\sum_i |\mathfrak{z}_i|^2\right)^{1/4}, \quad y = \sum_i \mathfrak{z}_i^2.$$

The base term $t^{-1}\tilde{\omega}_Y$ can be locally replaced by a Euclidean metric $\frac{A_0}{t}\sqrt{-1}dy \wedge d\bar{y}$. Thus

$$\omega_{SRF} \sim \sqrt{-1}\partial\bar{\partial}(\sqrt{r^4 + |y|} + \frac{A_0}{t}|y|^2).$$

Now we perform the coordinate change

$$\mathfrak{z}_i = \left(\frac{t}{2A_0}\right)^{1/3} z_i, \quad r = \left(\frac{t}{2A_0}\right)^{1/6} H^{1/4}, \quad y = \left(\frac{t}{2A_0}\right)^{2/3} f, \quad (1.16)$$

so that

$$\sqrt{r^4 + |y|} + \frac{A_0}{t}|y|^2 = \left(\frac{t}{2A_0}\right)^{1/3}(\sqrt{H + |f|} + \frac{1}{2}|f|^2),$$

which matches up with the asymptote at infinity of the scaled model metric $(\frac{t}{2A_0})^{1/3}\omega_{\mathbb{C}^3}$. The formula (1.16) defines an embedding map of a large complex ball in $\mathbb{C}^3$ into X :

$$F_t : \{|z_1|^2 + |z_2|^2 + |z_3|^2 \lesssim t^{-2/3}\} \subset \mathbb{C}^3 \rightarrow \{\mathfrak{z}_1^2 + \mathfrak{z}_2^2 + \mathfrak{z}_3^2 \lesssim 1\} \subset X. \quad (1.17)$$

The semi-Ricci-flat approximation to the collapsing CY metric breaks down completely inside the small ball $\{|z| \lesssim 1\}$, which is the **quantisation scale** mentioned in Section 1.4.2.

Gluing construction: general ideas

The metric ansatz is constructed by gluing a rescaled version of the model metric $\omega_{\mathbb{C}^3}$ to the regularized semi-Ricci-flat metric. This ansatz suffers from rather large gluing errors, and one needs to work with rather coarse function spaces to perturb this into the actual CY metric $\tilde{\omega}_t$.

The key issue is to understand the harmonic analysis of the Laplace operator for the metric ansatz. The difficulty is the simultaneous presence of several scales with very different characteristic behaviours, an issue inherent in any collapsing problem and made more acute by the presence of singular fibres. The technique is largely drawn from the work of G. Székelyhidi [39]. It involves analysing mapping properties of weighted Hölder spaces for every model geometry at each scale, decomposing the functions into pieces each sensitive only to one particular scale, inverting the Laplacian approximately on individual pieces using the various model Green operators, and patching the pieces to an approximate global solution. The main advantage of this method, aside from giving a fairly explicit description of the Green operator, is that it allows us to derive a t -independent bound on a suitable operator norm, and in this sense this linear theory is optimal.

It is worth pointing out that following the recent works [3][39], many other new examples of complete CY metrics on $\mathbb{C}^n$ are now known, which are strong candidates for modelling collapsing fibrations with higher dimensional fibres. Such examples are likely to provide a vast generalisation of our gluing construction.

Remark 1.4.11. *Since the model metric on $\mathbb{C}^3$ is constructed by a non-compact version of Yau's theorem, and the collapsing Calabi-Yau 3-fold metric also exists by Yau's theorem, what have we gained from this gluing construction? Answer:*

- **Local universality:** *even though there are plenty of Lefschetz K3 fibrations, the local model near the nodal point is unique up to scale.*
- *A nontrivial power law between the **regularity scale** with the **degeneration parameter** t in the Kähler class.*
- *A similar gluing construction might be carried out for the G_2 -analogue, where existence is not guaranteed by general theorems.*

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Chapter 2

Nonlinear perspective

We work in the setting of Section 1.4.1. Assuming a conjecture in pluripotential theory, we prove a uniform fibre diameter bound for a Lefschetz K3 fibred Calabi-Yau 3-fold, which reduces the study of the collapsing metric $\tilde{\omega}_t$ to a locally non-collapsed situation, and we identify the Gromov-Hausdorff limit of the rescaled neighbourhood of the singular fibre.

2.1 The uniform lower bound on the metric

This Section is a variation on Tosatti's arguments [42] in the presence of the volume condition (1.7). We write $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}\phi$ for some Kähler potential ϕ depending on t . The Calabi-Yau condition for $\tilde{\omega}_t$ is

$$\tilde{\omega}_t^3 = a_t t^2 \sqrt{-1} \Omega \wedge \bar{\Omega}, \quad (2.1)$$

where a_t is a cohomological constant. Under the normalisation (1.6),

$$a_t = t^{-2} \int_X \omega_t^3 = 3 + t \int_X \omega_X^3. \quad (2.2)$$

In the limit a_t converges to $a_0 = 3$.

The following result is taken from pluripotential theory:

Proposition 2.1.1. (*cf.* main theorem in [15]) There is a uniform constant such that $\|\phi\|_{L^\infty} \leq C$.

This leads to a lower bound on the horizontal component of $\tilde{\omega}_t$:

Proposition 2.1.2. There is a uniform bound $\mathrm{Tr}_{\tilde{\omega}_t} \pi^* \omega_Y \leq C$.

Proof. (*cf.* [42, Lemma 3.1]) Since $\tilde{\omega}_t$ is Ricci-flat and ω_Y has bounded Riemannian curvature, the Chern-Lu formula implies

$$\Delta_{\tilde{\omega}_t} \log \mathrm{Tr}_{\tilde{\omega}_t} \omega_Y \geq -C \mathrm{Tr}_{\tilde{\omega}_t} \omega_Y.$$

Here we abuse notation to write $\pi^*\omega_Y$ as ω_Y . To cancel the RHS, we notice

$$\Delta_{\tilde{\omega}_t}\phi = n - \text{Tr}_{\tilde{\omega}_t}\omega_t \leq n - \text{Tr}_{\tilde{\omega}_t}\omega_Y.$$

Subtracting a large multiple of the second inequality from the first,

$$\Delta_{\tilde{\omega}_t}(\log \text{Tr}_{\tilde{\omega}_t}\omega_Y - C\phi) \geq \text{Tr}_{\tilde{\omega}_t}\omega_Y - C'.$$

The maximum principle together with Proposition 2.1.1 imply the result. $\square$

Define the oscillation to be $\text{osc} = \sup - \inf$.

Lemma 2.1.3. *The fibrewise oscillation of the Kähler potential ϕ satisfies the uniform bound*

$$\text{osc}_{X_y}\phi \leq Ct. \quad (2.3)$$

Proof. Notice first that on each smooth fibre X_y , the forms ω_X , $\omega_{SRF,y}$ and $t^{-1}\tilde{\omega}_t$ all belong to the same cohomology class. We can write on X_y

$$\omega_X = \omega_{SRF,y} + \sqrt{-1}\partial\bar{\partial}(v - \frac{\phi}{t}), \quad t^{-1}\tilde{\omega}_t = \omega_{SRF,y} + \sqrt{-1}\partial\bar{\partial}v.$$

Their volume densities have upper bounds. We compute

$$\frac{\tilde{\omega}_t^2|_{X_y}}{\omega_{SRF,y}^2} = \frac{\tilde{\omega}_t^2 \wedge \omega_Y}{\omega_{SRF,y}^2 \wedge \omega_Y} = \frac{\tilde{\omega}_t^2 \wedge \omega_Y}{A_y^{-1}\Omega_y \wedge \overline{\Omega}_y \wedge \omega_Y} \leq C \frac{\tilde{\omega}_t^3(\text{Tr}_{\tilde{\omega}_t}\omega_Y)}{\Omega_y \wedge \overline{\Omega}_y \wedge \omega_Y} \leq Ct^2, \quad (2.4)$$

using the volume condition (1.7) to conclude $A_y \leq C$, and using Proposition 2.1.2. Observe also that

$$\frac{\omega_t^2|_{X_y}}{\omega_{SRF,y}^2} \leq Ct^2.$$

We view these as L^∞ -bounds on the RHS of the Monge-Ampère equation satisfied by ω_X and $t^{-1}\tilde{\omega}_t$ on X_y , with the background metric $\omega_{SRF,y}$. Since these $\omega_{SRF,y}$ are Ricci flat, with unit volume and uniformly bounded diameters by (1.8), they have uniformly bounded Sobolev and Poincaré constants using the isoperimetric inequality of Croke [4], so Yau's C^0 -estimate yields

$$\text{osc}_{X_y}v \leq C, \text{ and } \text{osc}_{X_y}(v - \frac{\phi}{t}) \leq C,$$

hence (2.3) holds on all smooth fibres, and therefore extends by continuity to all fibres. This concludes the proof. $\square$

Now we follow [42] to define the fibrewise average function of ϕ :

$$\underline{\phi} = \int_{X_y} \phi \omega_X^2. \quad (2.5)$$

This $\underline{\phi}$ is a continuous function on Y , but needs not be smooth at the nodes of the fibration, so we need a little extra care.

There is a measure inequality

$$\sqrt{-1}\partial\bar{\partial}\underline{\phi} = \pi_*(\sqrt{-1}\partial\bar{\partial}\phi \wedge \omega_X^2) \geq -\pi_*(\omega_t \wedge \omega_X^2) \geq -C(\omega_Y + t\pi_*(\omega_X^3)) \geq -C\omega_Y.$$

Plurisubharmonicity is preserved under pull back to X . We identify $\underline{\phi}$ with its pullback to X . Then taking the trace with respect to $\tilde{\omega}_t$, we get

$$\Delta_{\tilde{\omega}_t} \underline{\phi} \geq -C \operatorname{Tr}_{\tilde{\omega}_t} \omega_Y \geq -C.$$

Since ϕ is bounded in L^∞ , so is $\underline{\phi}$, hence $\underline{\phi}$ must be in L^2_1 on X .

Now we can prove the lower bound in Theorem 1.4.2:

Proof. (Theorem 1.4.2)

Apply the Chern-Lu formula to get

$$\Delta_{\tilde{\omega}_t} \log \operatorname{Tr}_{\tilde{\omega}_t} \omega_X \geq -C \operatorname{Tr}_{\tilde{\omega}_t} \omega_X.$$

Essentially the same computation as in Proposition 2.1.2 gives

$$\Delta_{\tilde{\omega}_t} (\log \operatorname{Tr}_{\tilde{\omega}_t} \omega_X - \frac{C}{t} (\phi - \underline{\phi})) \geq \operatorname{Tr}_{\tilde{\omega}_t} \omega_X - \frac{\operatorname{Const}}{t}.$$

The rest of the proof is the classical maximum principle for quasilinear elliptic equations in the weak formulation. Let $u = \log \operatorname{Tr}_{\tilde{\omega}_t} \omega_X - \frac{C}{t} (\phi - \underline{\phi})$, then by the bound on the fibrewise oscillation of ϕ , we have

$$\Delta_{\tilde{\omega}_t} u \geq C(e^u - e^{C' - \log t}), \quad (2.6)$$

for some uniform positive constant C, C' , where everything is interpreted in the distributional sense, and we know a priori that u is continuous and $u \in L^2_1$. Now multiply the above inequality by the test function $(u - C' + \log t)_+$, and integrate by parts. We get

$$\int_{u > C' - \log t} |\nabla u|_{\tilde{\omega}_t}^2 \tilde{\omega}_t^n + \int_{u > C' - \log t} C(e^u - e^{C' - \log t})(u - C' + \log t) \tilde{\omega}_t^n \leq 0,$$

whence the set $\{u > C' - \log t\}$ has zero measure. So in fact $u \leq C' - \log t$, hence $\operatorname{Tr}_{\tilde{\omega}_t} \omega_X \leq Ct^{-1}$. This combines with the estimate $\operatorname{Tr}_{\tilde{\omega}_t} \omega_Y \leq C$ to give $\operatorname{Tr}_{\tilde{\omega}_t} \omega_t \leq C$, as desired. $\square$

Define $H = \frac{\omega_X^2 \wedge \omega_Y}{\omega_X^3}$ as in [42]. On our Lefschetz fibration $H \gtrsim 1$ away from the nodes, and up to uniform equivalence $H \sim \sum |z_i|^2$ in the local picture $y = z_1^2 + z_2^2 + z_3^2$ near the nodes.

Corollary 2.1.4. There is a uniform upper bound $\tilde{\omega}_t \leq \frac{C}{H} \omega_t$.

Proof. By simultaneous diagonalisation of ω_t and $\tilde{\omega}_t$ at a point, one shows

$$\operatorname{Tr}_{\omega_t} \tilde{\omega}_t \leq (\operatorname{Tr}_{\tilde{\omega}_t} \omega_t)^2 \frac{\tilde{\omega}_t^3}{\omega_t^3}.$$

The volume ratio $\frac{\tilde{\omega}_t^3}{\omega_t^3} \leq \frac{C}{H}$, so $\operatorname{Tr}_{\tilde{\omega}_t} \omega_t \leq C$ implies $\operatorname{Tr}_{\omega_t} \tilde{\omega}_t \leq \frac{C}{H}$. $\square$

Remark 2.1.5. Near the node, this implies a logarithmic diameter bound for small $y = z_1^2 + z_2^2 + z_3^2$:

$$\operatorname{diam}(X_y, t^{-1} \tilde{\omega}_t) \leq -C \log |y|,$$

which falls short of a uniform bound.

In particular, in the subset $\{H \geq C\} \subset X$, namely the region away from the nodal points but not necessarily away from the singular fibres, there is a uniform equivalence

$$C^{-1}\omega_t \leq \tilde{\omega}_t \leq C\omega_t.$$

Around any given point $P \in \{H \geq C\}$, we can apply Evans-Krylov theory mentioned in Section 1.3.1 to a $t^{-1}\omega_t$ -ball of diameter $O(1)$, to deduce that $t^{-1}\tilde{\omega}_t$ has uniform C^∞ bound with respect to the background metric $t^{-1}\omega_t$.

Corollary 2.1.6. Inside $\{H \geq C\}$,

$$\left\| \nabla_{\omega_X}^{(k)} \frac{1}{t} \tilde{\omega}_t|_{X_y} \right\|_{L^\infty} \leq C(k), \quad \left\| \nabla_{\omega_X}^{(k)} (\text{Tr}_{\tilde{\omega}_t} \omega_Y)|_{X_y} \right\|_{L^\infty} \leq C(k).$$

2.2 Diameter estimates

2.2.1 General bounds on distance functions

We develop some general tools for estimating diameters and distance functions on a compact Kähler manifold directly from the Kähler potential, which are of independent interest.

Theorem 2.2.1. *Let (M, ω) be a fixed n -dimensional compact Kähler manifold, and consider a new metric $\omega_\psi = \omega + \sqrt{-1}\partial\bar{\partial}\psi$.*

- *If $\|\psi\|_{C^\alpha} \leq C$, then the distance function in the new metric $d_{\omega_\psi}(x, y)$ has a $C^{\alpha/2}$ Hölder bound in terms of the original metric, with constants only depending on (X, ω) and $\|\psi\|_{C^\alpha}$.*
- *Without any control on ψ , then*

$$\int_{M \times M} d\mu_x d\mu_y \exp\left(\frac{d_{\omega_\psi}(x, y)}{C}\right) \leq C', \quad (2.7)$$

where $d\mu = \omega^n$ is the reference measure, and $C, C' > 0$ denote constants independent of ψ .

Proof. Work in a coordinate chart $B(x, R)$ on M , where ω is uniformly equivalent to the Euclidean metric. Consider the distance function on $y \in B(x, R)$ defined by $\rho_x(y) = d_{\omega_\psi}(x, y)$, which makes sense because ω_ψ is smooth. Then $|\nabla_{\omega_\psi} \rho_x| \leq 1$. Thus

$$|\nabla_{\omega} \rho_x|^2 \leq \text{Tr}_{\omega} \omega_\psi.$$

The next idea comes from the Chern-Levine inequality. For $r < R/3$, we can choose a cutoff function χ supported on $B(2r)$ and equal to one on $B(r)$, such that $\sqrt{-1}\partial\bar{\partial}\chi \leq \frac{C}{r^2}\omega$. We first consider the case with some Hölder bound on potential. This gives

$$\begin{aligned} \int_{B(r)} (\text{Tr}_{\omega} \omega_\psi) \omega^n &= n \int_{B(r)} \omega_\psi \omega^{n-1} \leq n \int \chi \omega_\psi \omega^{n-1} \leq Cr^{2n} + n \int \chi \sqrt{-1}\partial\bar{\partial}\psi \omega^{n-1} \\ &= Cr^{2n} + n \int (\psi - \inf_{B(2r)} \psi) \sqrt{-1}\partial\bar{\partial}\chi \omega^{n-1} \leq Cr^{2n} + C(\text{osc}_{B(2r)} \psi) r^{2n-2} \leq Cr^{2n-2+\alpha}. \end{aligned}$$

Hence $\int_{B(r)} |\nabla_\omega \rho_x|^2 \omega^n \leq C r^{2n-2+\alpha}$. This means the function ρ_x is bounded in the Morrey norm. In particular by Morrey embedding theorem [16, Theorem 7.19], and the trivial observation $\rho_x(x) = 0$,

$$\|\rho_x\|_{C^{\alpha/2}(B(x, R/3))} \leq C,$$

where C depends only on the (M, ω) and bound on ψ . This implies

$$d_{\omega_\psi}(x, y) \leq C d_\omega(x, y)^{\alpha/2} \quad (2.8)$$

on coordinate charts. The general Hölder bound for any $x, y \in M$ follows from the triangle inequality.

For the case with no assumption on the potential, we will use some ideas in the standard proof of Skoda integrability theorem [50, Theorem 3.1]. For convenience, we assume the normalisation $\int_M \omega^n = 1$. We notice that by the monotonicity formula in the theory of Lelong numbers, the estimate

$$\int_{B(r)} \omega_\psi \omega^{n-1} \leq C r^{2n-2} \int_M \omega_\psi \omega^{n-1} \leq C r^{2n-2}$$

holds always. This combined with the John-Nirenberg inequality [16, Theorem 7.21] gives

$$\int_M \exp\left(\frac{\rho_x - \bar{\rho}_x}{C}\right) \omega^n \leq C', \quad (2.9)$$

where $\bar{\rho}_x$ is the average of ρ_x , and the constants are independent of x or ω_ψ . Write the measure $d\mu = \omega^n$, and define $\bar{\rho} = \int_M \int_M d_{\omega_\psi}(x, y) d\mu_x d\mu_y$. Then by Fubini theorem and Jensen convexity inequality,

$$\begin{aligned} \int_M d\mu_y \exp\left(\frac{\bar{\rho}_y - \bar{\rho}}{C}\right) &= \int_M d\mu_y \exp\left(\int_M \frac{d_{\omega_\psi}(x, y) - \bar{\rho}_x}{C} d\mu_x\right) \\ &\leq \int_M \int_M \exp\left(\frac{d_{\omega_\psi}(x, y) - \bar{\rho}_x}{C}\right) d\mu_x d\mu_y \leq C'. \end{aligned}$$

By an application of Fubini theorem (cf. [35, the proof of lemma 2.2]),

$$\int_M \int_M d_{\omega_\psi}(x, y)^2 d\mu_x d\mu_y \leq C \int_M \text{Tr}_\omega \omega_\psi d\mu \leq C,$$

so the number $\bar{\rho}$ is controlled by a constant, using Cauchy-Schwarz. Thus

$$\int_M d\mu_y \exp\left(\frac{\bar{\rho}_y}{C}\right) \leq C', \quad (2.10)$$

and therefore

$$\begin{aligned} \int_{M \times M} d\mu_x d\mu_y \exp\left(\frac{d_{\omega_\psi}(x, y)}{C}\right) &\leq \left(\int_{M \times M} d\mu_x d\mu_y \exp\left(\frac{d_{\omega_\psi}(x, y) - \bar{\rho}_x}{C/2}\right)\right)^{1/2} \\ &\quad \left(\int_{M \times M} d\mu_x d\mu_y \exp\left(\frac{\bar{\rho}_x}{C/2}\right)\right)^{1/2} \leq C', \end{aligned} \quad (2.11)$$

implying the claim. $\square$

Remark 2.2.2. *Intuitively, one may think the potential has the same order as the distance squared, which explains the way their Hölder exponents are related.*

Combined with the deep Kolodziej estimate Theorem 1.3.3 in pluripotential theory, this implies Theorem 1.4.4.

Remark 2.2.3. *Notice there is no requirement on the Ricci curvature of ω_ψ . Theorem 1.4.4 can be contrasted with the general Riemannian situation, where even the L^∞ bound on the volume form is very far from controlling the diameter. The moral is that Kähler geometry is far more rigid than Riemannian geometry.*

2.2.2 The uniform diameter bound

Now we return to the collapsing metric problem.

Corollary 2.2.4. On the nodal K3 fibre X_0 , the diameter bound holds uniformly in t :

$$\text{diam}(t^{-1}\tilde{\omega}_t|_{X_0}) \leq C.$$

Proof. On the nodal K3 fibre, the volume density of $t^{-1}\tilde{\omega}_t|_{X_0}$ has L^∞ bound with respect to the orbifold K3 metric (cf. (2.4)). The Kolodziej estimate Theorem 1.3.3 and the discussions in Section 2.2.1 apply equally well to orbifolds. $\square$

Assuming Conjecture 1.4.5 which is a generalisation of the Kolodziej estimate to mildly degenerate situations, we prove Theorem 1.4.6.

Proof. We first make a base change to the square root fibration. The local complex geometry near the node is

$$\begin{array}{ccc} \{z_1^2 + z_2^2 + z_3^2 = w^2\} & \longrightarrow & \mathbb{C}_{z_1, z_2, z_3}^3 \\ \tilde{\pi} \downarrow & & \downarrow y = z_1^2 + z_2^2 + z_3^2 \\ \mathbb{C}_w & \xrightarrow{y=w^2} & \mathbb{C}_y \end{array}$$

The fibred product has a three-fold ODP singularity at the origin, and admits a small resolution $\tilde{\mathcal{X}}$. Locally $\tilde{\mathcal{X}}$ can be described explicitly as

$$\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) = \{[U : V] \in \mathbb{P}^1, (u_1, v_1) \otimes (u_2, v_2) | Uv_1 = V u_1\}.$$

Here u_1, u_2, v_1, v_2 are not well defined by themselves, but $u_1 u_2, u_1 v_2, u_2 v_1, u_2 v_2$ are. The resolution map is

$$z_1 + \sqrt{-1}z_2 = u_1 u_2, \quad w + z_3 = v_1 u_2, \quad w - z_3 = v_2 u_1, \quad z_1 - \sqrt{-1}z_2 = v_1 v_2.$$

The point is that the family $\tilde{\mathcal{X}} \rightarrow \mathbb{C}_w$ has the same fibres as the square root fibration $\tilde{\pi}$ for $w \neq 0$, and makes the fibration a *submersion*. This means for $|y| \lesssim 1$ the smooth fibres of π can be endowed with Kähler metrics $\omega_{\tilde{\mathcal{X}}_w}$ which have *uniformly bounded geometry*.

Using (2.4), the volume density has upper bound on $\tilde{\mathcal{X}}_w$:

$$(t^{-1}\tilde{\omega}_t|_{X_y})^2 \leq C\omega_{SRF,y}^2 \leq \frac{C}{H}\omega_X|_{X_y}^2 \leq C\omega_{\tilde{\mathcal{X}}_w}^2.$$

The subtlety is that the classes $[t^{-1}\tilde{\omega}_t] = [\omega_X|_{X_y}]$ degenerate to the boundary of the Kähler cone as $y \rightarrow 0$, so the Kolodziej estimate on the potential no longer applies, and one is forced to invoke a generalised conjectural statement 1.4.5.

Once we have the Hölder bound on the potential, the argument for a Hölder bound on the distance function proceeds as before, and in particular proves the uniform diameter bound (1.9). $\square$

This uniform fibre diameter bound (1.9) is a very important condition. In particular it leads to a **local non-collapsing bound** in the neighbourhood of any given fibre (including the singular fibres) as $t \rightarrow 0$, so we can apply the well developed machinery of noncollapsing Cheeger-Colding theory. More delicate consequences will be treated later.

Corollary 2.2.5. Assuming the uniform fibre diameter bound (1.9), $t^{-1}\tilde{\omega}_t$ satisfies the **local volume non-collapsing** estimate: there are constants, such that for any central point P , and for any $1 \lesssim R \lesssim t^{-1/2}$,

$$\text{Vol}(B_{t^{-1}\tilde{\omega}_t}(P, R)) \geq CR^2. \quad (2.12)$$

Moreover for any $R \lesssim 1$,

$$\text{Vol}(B_{t^{-1}\tilde{\omega}_t}(P, R)) \geq CR^6. \quad (2.13)$$

Proof. Assume first that $R \gtrsim 1$. Any fibre contains a subregion $\{H \gtrsim 1\}$ where $\tilde{\omega}_t$ is uniformly equivalent to ω_t . Thus if $d_{\omega_Y}(y, y') \lesssim Rt^{1/2}/C$, then the $t^{-1}\tilde{\omega}_t$ -distance between the two fibres X_y and $X_{y'}$ is $O(R)$. Now (1.9) says each fibre has $t^{-1}\tilde{\omega}_t$ -diameter $O(R)$. Hence we can reach any point on a nearby fibre within $O(R)$ distance, so the ball $B_{t^{-1}\tilde{\omega}_t}(P, CR)$ contains the preimage of $B_{\omega_Y}(\pi(P), Rt^{1/2})$. Since the volume form of $t^{-1}\tilde{\omega}_t$ is $a_t t^{-1} \sqrt{-1} \Omega \wedge \bar{\Omega}$, we obtain the estimate (2.12). The other estimate (2.13) follows from Bishop-Gromov monotonicity using the Ricci flatness of $\tilde{\omega}_t$. $\square$

2.3 Convergence results

2.3.1 Convergence of fibrewise metrics

Let $t \ll 1$ throughout the discussion. In the presence of smooth bounds in Cor 2.1.6, a convergence bound on the fibrewise potential implies a convergence result for fibre metrics. We will crucially use the 1-dimensional base condition.

We fix a central fibre X_0 , which can be singular. Consider a coordinate ball $\{|y| \leq R\} \subset Y$. Let $\omega_{Y,0} = A_0 \sqrt{-1} dy \wedge d\bar{y}$ be a Euclidean metric on $\{|y| \leq R\}$, scaled so that $\omega_{Y,0}$ is equal to $\tilde{\omega}_Y$ at the origin.

Proposition 2.3.1. Assume the uniform fibre diameter bound (1.9), then we have a **concentration estimate** for $\text{Tr}_{\tilde{\omega}_t} \tilde{\omega}_Y$ uniform for all choices of X_0 :

$$\max_{|y| \leq t^{1/2}} \text{Tr}_{\tilde{\omega}_t} \tilde{\omega}_Y \leq 1 + \frac{C}{|\log t|}, \quad (2.14)$$

$$t^{-3} \|\text{Tr}_{\tilde{\omega}_t} \tilde{\omega}_Y - 1\|_{L^1_{\tilde{\omega}_t}(|y| \leq t^{1/2})} \leq \frac{C}{|\log t|}. \quad (2.15)$$

Proof. By the Chern-Lu formula, using the fact that $\omega_{Y,0}$ has zero curvature, we have

$$\Delta_{\tilde{\omega}_t}(\log \operatorname{Tr}_{\tilde{\omega}_t} \omega_{Y,0}) \geq 0.$$

The rest of the argument is essentially an effective version of the fact that $\mathbb{C}$ cannot support bounded subharmonic functions.

Notice that $\log |y|$ is a pluriharmonic function on $X \setminus X_0$, so in particular is harmonic with respect to the Laplacian $\Delta_{\tilde{\omega}_t}$. We can compare the function $\log \operatorname{Tr}_{\tilde{\omega}_t} \omega_{Y,0}$ with $C \log |y| + C'$ using the maximum principle to achieve the following three circle type inequality:

$$\begin{aligned} & \max_{|y|=Dt^{1/2}} \log \operatorname{Tr}_{\tilde{\omega}_t} \omega_{Y,0} - \max_{|y|=t^{1/2}} \log \operatorname{Tr}_{\tilde{\omega}_t} \omega_{Y,0} \\ & \leq \frac{\log D}{\log Rt^{-1/2}} (\max_{|y|=R} \log \operatorname{Tr}_{\tilde{\omega}_t} \omega_{Y,0} - \max_{|y|=t^{1/2}} \log \operatorname{Tr}_{\tilde{\omega}_t} \omega_{Y,0}) \end{aligned}$$

where D is a large constant to be chosen later. By proposition 2.1.2, we have $\log \operatorname{Tr}_{\tilde{\omega}_t} \omega_{Y,0} \leq C$. Moreover $\max_{|y|=t^{1/2}} \log \operatorname{Tr}_{\tilde{\omega}_t} \omega_{Y,0} \geq C$ because any fibre contains a nonempty region $\{H \gtrsim 1\}$ where ω_t is uniformly equivalent to $\tilde{\omega}_t$. Therefore

$$\max_{|y| \leq Dt^{1/2}} \log \operatorname{Tr}_{\tilde{\omega}_t} \omega_{Y,0} - \max_{|y| \leq t^{1/2}} \log \operatorname{Tr}_{\tilde{\omega}_t} \omega_{Y,0} \leq \frac{C}{|\log t|}. \quad (2.16)$$

This says on the region $\{|y| \leq Dt^{1/2}\}$ the maximum is almost achieved in the interior for the subharmonic function $\log \operatorname{Tr}_{\tilde{\omega}_t} \omega_{Y,0}$. We will use this to conclude the function $\log \operatorname{Tr}_{\tilde{\omega}_t} \omega_{Y,0}$ is close to a constant.

Now the Ricci-flat metric $\tilde{\omega}_t$ satisfies the local volume noncollapsing estimate (2.13) on the region $\{|y| \leq Dt^{1/2}\}$, so it satisfies the scale invariant Sobolev inequality and the Poincaré inequality on $\{|y| \leq Dt^{1/2}\}$ (cf. [22]). Moreover, using the fibre diameter bound we can arrange the constant D to be large enough, such that there is a point $P \in X_0$ and a radius r with

$$\{|y| \leq t^{1/2}\} \subset B_{\tilde{\omega}_t}(P, r) \subset B_{\tilde{\omega}_t}(P, 4r) \subset \{|y| \leq Dt^{1/2}\}$$

We have the following Harnack inequality (see [20], Page 83-89 for a proof which only relies on Sobolev and Poincaré inequalities):

Lemma 2.3.2. *If u is a nonnegative superharmonic function in an n -dimensional ball $B(P, 4r)$, then for $1 \leq p < 2n/(2n-2)$, there is a constant independent of r, u , such that*

$$r^{-2n/p} \|u\|_{L^p(B_{\tilde{\omega}_t}(P, 2r))} \leq C \inf_{B(P, r)} u. \quad (2.17)$$

Applying this to the function $u = -\log \operatorname{Tr}_{\tilde{\omega}_t} \omega_{Y,0} + \max_{|y| \leq Dt^{1/2}} \log \operatorname{Tr}_{\tilde{\omega}_t} \omega_{Y,0}$,

$$t^{-3} \|u\|_{L^1_{\tilde{\omega}_t}(|y| \leq t^{1/2})} \leq \frac{C}{|\log t|}.$$

This is an effective version of the strong maximum principle.

Since $u \geq 0$ we have $u \geq C(1 - e^{-u})$. Using also Proposition 2.1.2, we deduce a normalised L^1 bound

$$t^{-3} \left\| -\operatorname{Tr}_{\tilde{\omega}_t} \omega_{Y,0} + \max_{|y| \leq t^{1/2}} \operatorname{Tr}_{\tilde{\omega}_t} \omega_{Y,0} \right\|_{L^1_{\tilde{\omega}_t}(|y| \leq t^{1/2})} \leq \frac{C}{|\log t|}.$$

This says the deviation of the maximum from the average is small of order $O(\frac{1}{|\log t|})$. But the average value of $\text{Tr}_{\tilde{\omega}_t} \omega_{Y,0}$ is known:

$$\pi_*((\text{Tr}_{\tilde{\omega}_t} \omega_{Y,0}) \tilde{\omega}_t^3) = 3\pi_*(\tilde{\omega}_t^2 \wedge \omega_{Y,0}) = 3t^2 \pi_*(\omega_X^2) \wedge \omega_{Y,0} = 3t^2 \omega_{Y,0},$$

So

$$\int_{|y| \leq t^{1/2}} (\text{Tr}_{\tilde{\omega}_t} \omega_{Y,0}) \tilde{\omega}_t^3 = 3t^2 \text{Vol}_{\omega_{Y,0}}(|y| \leq t^{1/2}).$$

Observe also

$$\int_{|y| \leq t^{1/2}} \tilde{\omega}_t^3 = a_t t^2 \int_{|y| \leq t^{1/2}} \sqrt{-1} \Omega \wedge \bar{\Omega} = a_t t^2 \text{Vol}_{\tilde{\omega}_Y}(|y| \leq t^{1/2}),$$

where the a_t is the volume normalising constant in (2.1). In our choice of normalisation $\omega_{Y,0}$ is locally C^0 -close to $\tilde{\omega}_Y$, so the average value of $\text{Tr}_{\tilde{\omega}_t} \omega_{Y,0}$ is approximately $\frac{3}{a_t} \sim 1$, and the estimate follows. $\square$

The fibrewise volume form can be expressed in terms of $\text{Tr}_{\tilde{\omega}_t} \tilde{\omega}_Y$:

$$\begin{aligned} \frac{\tilde{\omega}_t|_{X_y}^2}{\omega_{SRF,y}^2} &= \frac{\tilde{\omega}_t^2 \tilde{\omega}_Y}{\omega_{SRF,y}^2 \tilde{\omega}_Y} = \frac{(\text{Tr}_{\tilde{\omega}_t} \tilde{\omega}_Y) \tilde{\omega}_t^3}{3A_y^{-1} \Omega_y \wedge \bar{\Omega}_y \wedge \tilde{\omega}_Y} \\ &= \frac{a_t t^2 \text{Tr}_{\tilde{\omega}_t} \tilde{\omega}_Y \sqrt{-1} \Omega \wedge \bar{\Omega}}{3\sqrt{-1} \Omega_y \wedge \bar{\Omega}_y \wedge dy \wedge d\bar{y}} = \frac{a_t t^2 \text{Tr}_{\tilde{\omega}_t} \tilde{\omega}_Y}{3}, \end{aligned}$$

Comparing this with the concentration estimate Proposition 2.3.1, the volume density of the normalised fibre metric $t^{-1} \tilde{\omega}_t|_{X_y}$ with respect to the background metric $\omega_{SRF,y}$ is almost bounded above by one, and satisfies an L^1 concentration estimate. Now recall we introduced a function v in our proof of the fibrewise oscillation estimate (2.3), defined by $t^{-1} \tilde{\omega}_t|_{X_y} = \omega_{SRF,y} + \sqrt{-1} \partial \bar{\partial} v$. Standard integration by parts argument for the Monge-Ampère equation yields

$$\int_{X_y} v(\omega_{SRF,y}^2 - (t^{-1} \tilde{\omega}_t|_{X_y})^2) \geq C \int_{X_y} |\nabla v|_{\omega_{SRF,y}}^2 \omega_{SRF,y}^2.$$

But the oscillation of v is bounded in L^∞ , hence

Proposition 2.3.3. Assume the uniform fibre diameter bound (1.9). The fibrewise potential has the convergence estimate

$$\int_{X_y} |\nabla v|_{\omega_{SRF,y}}^2 \omega_{SRF,y}^2 \leq \frac{C}{|\log t|}.$$

Theorem 2.3.4. In the setup above, in the region $\{H \geq C\}$, namely away from the critical points, the normalised fibrewise metric $t^{-1} \tilde{\omega}_t|_{X_y}$ converges to $\omega_{SRF,y}$ with constructive estimates to all orders: for any small $\epsilon > 0$,

$$\left\| \nabla_{\omega_X}^{(k)} \left(\omega_{SRF,y} - \frac{1}{t} \tilde{\omega}_t|_{X_y} \right) \right\|_{L^\infty(X_y \cap \{H \geq C\})} \leq \frac{C(k, \epsilon)}{|\log t|^{1/2-\epsilon}}. \quad (2.18)$$

Proof. Away from the critical points $\omega_{SRF,y}$ is smoothly equivalent to ω_X . Given the convergence estimate $\|\nabla v\|_{L^2}^2 = O(\frac{1}{|\log t|})$, and the smooth bound in Corollary 2.1.6, it is a relatively standard fact that the higher order derivatives of v are small to any given order. The precise effective version is (2.18). $\square$

2.3.2 Convergence in the base scale

As mentioned in Section 1.3.3, when the base dimension $m = 1$, Tosatti et al [18] isometrically identified the Gromov-Hausdorff limit of $(X, \tilde{\omega}_t)$ as $t \rightarrow 0$ in general. The aim of this Section is to give a much easier proof in our special situation.

Theorem 2.3.5. *Assume the uniform fibre diameter bound (1.9), then the sequence of Ricci flat spaces $(X, \tilde{\omega}_t)$ converges in GH sense to Y with the generalised KE metric $\tilde{\omega}_Y$ defined by (1.13).*

There is a natural comparison map $\pi : (X, \tilde{\omega}_t) \rightarrow (Y, \tilde{\omega}_Y)$, and the task is to show this is an approximate isometry.

Using $|\nabla \pi|^2 = \text{Tr}_{\tilde{\omega}_t} \tilde{\omega}_Y$, the upper bound in Proposition 2.3.1 reads $|\nabla \pi|_{L^\infty} \leq 1 + \frac{C}{|\log t|}$, so the distance is almost decreasing:

$$d_{\tilde{\omega}_Y}(\pi(x_1), \pi(x_2)) \leq d_{\tilde{\omega}_t}(x_1, x_2) \left(1 + \frac{C}{|\log t|}\right), \quad \forall x_1, x_2.$$

For a bound in the reverse direction, fix a small $\epsilon > 0$.

Lemma 2.3.6. *If γ is a path of length $l(\gamma)$ in $(Y, \tilde{\omega}_Y)$ which stays distance ϵ bounded away from the singular locus S , then γ admits a lift to the total space $(X, \tilde{\omega}_t)$ which has length at most $l(\gamma)(1 + \frac{C(\epsilon)}{|\log t|^{1/2-\epsilon}})$.*

Proof. Since we are away from the singular fibres, the concentration estimate in Proposition 2.3.1 coupled with smooth bounds from Evans-Krylov theory, imply (cf. also (2.18))

$$|\text{Tr}_{\tilde{\omega}_t} \tilde{\omega}_Y - 1| \leq \frac{C}{|\log t|^{1/2-\epsilon}}.$$

The interpretation of this is that π is almost a Riemannian submersion. A Riemannian metric on the total space splits the tangent bundle on X into the vertical and the horizontal parts, which defines an Ehresmann connection. The above inequality means the horizontal lift of any unit tangent vector on Y is also a unit tangent vector up to $O(\frac{1}{|\log t|^{1/2-\epsilon}})$ error. From this the claim is clear. $\square$

Corollary 2.3.7. *If x, x' are two points in X , then*

$$d_{\tilde{\omega}_t}(x, x') \leq \left(1 + \frac{C}{|\log t|^{1/2-\epsilon}}\right) d_{\tilde{\omega}_Y}(\pi(x), \pi(x')) + C\epsilon + Ct^{1/2} \quad (2.19)$$

Proof. Take a curve γ joining $\pi(x)$, $\pi(x')$, which almost minimises the length: $l(\gamma) \leq d_{\tilde{\omega}_0}(f(x), f(x')) + \epsilon$. Then γ can spend at most $C\epsilon$ length in the ϵ neighbourhood of any point in S . Observe also that S is a finite set. Now the diameter of the preimage of the ϵ neighbourhood of any singular point in S is bounded by $C\epsilon + Ct^{1/2}$, using the fibre diameter bound. For the part of γ with distance at least ϵ away from S , we can use the previous lemma to construct a lift, and the claim follows. $\square$

Notice the diameter of $(Y, \tilde{\omega}_Y)$ is bounded. Thus for fixed ϵ , we can choose t sufficiently small, such that

$$d_{\tilde{\omega}_t}(x_1, x_2) - C\epsilon \leq d_{\tilde{\omega}_Y}(\pi(x_1), \pi(x_2)) \leq d_{\tilde{\omega}_t}(x_1, x_2) + \epsilon, \quad \forall x_1, x_2,$$

which is equivalent to the GH convergence claim.

2.3.3 Gradient estimate

This is a continuation of Section 2.3.1. The aim is to improve certain aspects of the concentration estimate (Proposition 2.3.1), and show that $d\pi$ is in some sense approximately parallel.

Proposition 2.3.8. In the setup of last Section, we have the gradient estimate

$$\int_{|y| \leq t^{1/2}} \left(\frac{|\nabla d\pi|^2}{\text{Tr}_{\tilde{\omega}_t} \omega_{Y,0}} - |\partial \log \text{Tr}_{\tilde{\omega}_t} \omega_{Y,0}|^2 \right) \tilde{\omega}_t^3 \leq \frac{Ct^2}{|\log t|}. \quad (2.20)$$

Proof. The starting point is that the Chern-Lu formula has a more refined version (cf. [34, Proposition 7.1])

$$\Delta_{\tilde{\omega}_t} \log \text{Tr}_{\tilde{\omega}_t} \omega_{Y,0} = \frac{|\nabla d\pi|^2}{|d\pi|^2} - |\partial \log |d\pi|^2|^2.$$

Here the RHS is pointwise non-negative by Cauchy-Schwarz. The norm square of the differential $d\pi$ is just the trace: $|d\pi|^2 = \text{Tr}_{\tilde{\omega}_t} \omega_{Y,0}$. This is consistent with the interpretation of $d\pi$ as a section of $\text{Hom}(TX, TY)$, and the natural norms come from the metric $\tilde{\omega}_t$ and $\omega_{Y,0}$.

Now let χ be a cutoff function on the base Y supported in $\{|y| \leq t^{1/2}\}$, and equal to one on $|y| \leq \frac{1}{2}t^{1/2}$. We require

$$-\frac{C}{t} \omega_{Y,0} \leq \sqrt{-1} \partial \bar{\partial} \chi \leq \frac{C}{t} \omega_{Y,0},$$

so the Laplacian $|\Delta_{\tilde{\omega}_t} \chi| \leq \frac{C}{t}$. Multiplying the test function χ to both sides of the refined Chern-Lu formula, we get

$$\int_{|y| \leq t^{1/2}} \chi \left(\frac{|\nabla d\pi|^2}{|d\pi|^2} - |\partial \log |d\pi|^2|^2 \right) = \int \Delta_{\tilde{\omega}_t} \chi (\log \text{Tr}_{\tilde{\omega}_t} \omega_{Y,0} - \max_{|y| \leq t^{1/2}} \log \text{Tr}_{\tilde{\omega}_t} \omega_{Y,0}).$$

Using the proof of the concentration estimate, the L^1 of the log term is controlled, so the RHS is estimated by $\frac{Ct^2}{|\log t|}$. This implies

$$\int_{|y| \leq \frac{1}{2}t^{1/2}} \left(\frac{|\nabla d\pi|^2}{|d\pi|^2} - |\partial \log |d\pi|^2|^2 \right) \tilde{\omega}_t^3 \leq \frac{Ct^2}{|\log t|}.$$

This is the claim up to cosmetic changes of constants. $\square$

2.3.4 Gromov Hausdorff limit around the singular fibre

Consider the convergence of the metric $t^{-1}\tilde{\omega}_t$ restricted to the singular fibre X_0 . The expected limit is the nodal CY metric $\omega_{SRF,0}$. Notice we certainly cannot expect $t^{-1}\tilde{\omega}_t|_{X_0}$ to have a uniform two sided bound by $\omega_{SRF,0}$, because this is not true even for $t = 1$. Instead we should study a weaker notion of convergence, such as the Gromov-Hausdorff limit, which is sensible in the light of the local volume noncollapsing estimate (2.12) (2.13).

Let P be the nodal point, and look at the pointed sequence of Ricci flat spaces $Z_t = (\pi^{-1}B_{\omega_Y}(0, R) \subset X, t^{-1}\tilde{\omega}_t)$. Local noncollapsing implies that after passing to a subsequence, there is some complex 3-dimensional Gromov-Hausdorff limit space (Z, ω_∞) , containing an open dense regular locus Z^{reg} where the limiting metric is smooth, whose complement has Hausdorff codimension at least 4. Moreover Z^{reg} has a natural limiting complex structure, such that the limiting metric is Kähler. (See [13] for a nice summary.) We shall suppress below mentions of subsequence to avoid overloading notation, and tacitly understand a Gromov-Hausdorff metric is fixed on the disjoint union $Z_t \sqcup Z$, which displays the GH convergence. We will also assume implicitly that $t \ll 1$.

We wish to identify the complex structure. Some heuristic first: since everything away from the fibre X_0 is pushed to infinity by scaling, the limit as a complex variety should be the normal neighbourhood of X_0 , which is just the trivial product $X_0 \times \mathbb{C}$ in the case of a smooth fibre, and the guess is that the same is true for the singular fibre.

More formally, we build comparison maps as inspired by [13]. Let u denote the standard coordinate on $\mathbb{C}$, and $\omega_{\mathbb{C}}$ refers to the standard Euclidean metric on $\mathbb{C}$. Define the holomorphic maps

$$f_t : (Z_t, t^{-1}\tilde{\omega}_t) \rightarrow (X \times \mathbb{C}, \omega_X + \omega_{\mathbb{C}}), \quad x \mapsto (x, u = t^{-1/2}\pi(x)).$$

By the uniform bound $\text{Tr}_{\tilde{\omega}_t} \omega_t \leq C$, there is a Lipschitz bound on f_t independent of t , so the Gromov-Hausdorff limit inherits a Lipschitz map f_∞ into $X \times \mathbb{C}$. By the interior regularity of holomorphic functions, the limiting map f_∞ is holomorphic. We identify the image:

Lemma 2.3.9. *The image of f_∞ is $X_0 \times \mathbb{C}$.*

Proof. First we show the image is contained in $X_0 \times \mathbb{C}$. For any fixed radius D ,

$$\text{Image}(f_t) \cap \text{pr}_{\mathbb{C}}^{-1}(B_{\omega_{\mathbb{C}}}(0, D)) \subset \{(x, u) \in X \times \mathbb{C} : \text{dist}_{\omega_X}(x, X_0) \leq CDt^{1/2}, |u| \leq D\}.$$

By continuity,

$$\text{Image}(f_\infty) \cap \text{pr}_{\mathbb{C}}^{-1}(B(0, D)) \subset \{(x, u) \in X \times \mathbb{C} : \text{dist}_{\omega_X}(x, X_0) = 0, |u| \leq D\},$$

so the first claim is proved.

Next we show the image contains $X_0 \times \mathbb{C}$. Using the fibre diameter bound, for any $(x, u) \in X_0 \times \mathbb{C}$ with $|u| \leq D$, and t sufficiently small depending on D , there is always a point $Q_t \in Z_t$, whose $\frac{1}{t}\tilde{\omega}_t$ -distance to P is bounded by $C(1 + D)$, and $d(f_t(Q), (x, u)) \leq CDt^{1/2}$ in the metric on the target. Then using the compactness of the ball $B(P, C(1 + D)) \subset Z$, we can extract a subsequential limit Q , which has to map to (x, u) by continuity. $\square$

Recall the function H measures the severity of singular effect. Now H is a continuous function on X_0 , so defines a function on Z by pulling back via f_∞ .

Proposition 2.3.10. The map f_∞ is a biholomorphism $\{H > 0\} \subset Z \rightarrow X_0^{\text{reg}} \times \mathbb{C}$.

Proof. Using $C^{-1}\omega_t \leq \tilde{\omega}_t \leq \frac{C}{H}\omega_t$, the map f_t is bi-Lipschitz when restricted to $\{H \geq \epsilon\}$ for any $\epsilon > 0$, with smooth bounds on $\tilde{\omega}_t$ depending on ϵ . Passing to the limit shows that f_∞ is injective and is a local complex isomorphism on $\{H > 0\} \subset Z$. The rest essentially follows the above Lemma. $\square$

The sequence of Calabi-Yau metrics carry holomorphic volume forms, inducing canonically defined volume measures. On $\{H \geq \epsilon > 0\}$ the metrics $\frac{1}{t}\tilde{\omega}_t$ subconverge to ω_∞ in the C^∞ -topology. There the Monge-Ampère equations

$$(t^{-1}\tilde{\omega}_t)^3 = a_t t^{-1} \sqrt{-1} \Omega \wedge \bar{\Omega} = a_t t^{-1} \sqrt{-1} \Omega_y \wedge \bar{\Omega}_y \wedge dy \wedge d\bar{y} = a_t \sqrt{-1} \Omega_y \wedge \bar{\Omega}_y \wedge du \wedge d\bar{u}$$

converge to

$$\omega_\infty^3 = 3\sqrt{-1} \Omega_0 \wedge \bar{\Omega}_0 \wedge du \wedge d\bar{u}.$$

Integrating over the region $\{H > 0\} \cap \{|u| \leq D\} \subset Z$, for D any fixed number,

$$\begin{aligned} \int_{\{H>0\} \cap \{|u|\leq D\}} \omega_\infty^3 &= 3 \int_{X_0} \Omega_0 \wedge \bar{\Omega}_0 \int_{\{|u|\leq D\} \subset \mathbb{C}} \sqrt{-1} du d\bar{u} \\ &= \lim \int_{\{|u|\leq D\} \subset Z_t} a_t \sqrt{-1} t^{-1} \Omega \wedge \bar{\Omega} = \lim \int_{\{|u|\leq D\} \subset Z_t} (t^{-1}\tilde{\omega}_t)^3. \end{aligned}$$

Here the intermediate equality uses the continuity of $A_y = \int_{X_y} \Omega_y \wedge \bar{\Omega}_y$ (cf. Lemma 1.4.10).

But in a non-collapsing situation with non-negative Ricci curvature, we have the measured Gromov-Hausdorff convergence (cf. Cheeger's book [2], Page 66), which implies

$$\liminf \text{Vol}_{\frac{1}{t}\tilde{\omega}_t}(\{|u| \leq D\} \subset Z_t) \geq \text{Vol}_{\omega_\infty}(\{|u| \leq D\} \subset Z).$$

Comparing the above,

Proposition 2.3.11. (Full measure property) The set $H > 0$ inside Z must have full measure on each cylinder $\{|u| \leq D\} \subset Z$, so the set $H = 0$ has measure zero in Z . In particular, every ball in Z contains a point with $H > 0$.

We now study the metric ω_∞ over the smooth region $X_0^{\text{reg}} \times \mathbb{C}$.

Proposition 2.3.12. Over the smooth part of $X_0 \times \mathbb{C}$, the limiting metric restricts fibrewise to the Calabi-Yau metric $\omega_{SRF,0}$ on X_0 .

Proof. On $\{H \geq \epsilon > 0\}$, we have the convergence estimate

$$t^{-1}\tilde{\omega}_t|_{X_y} = \omega_{SRF,y} + O\left(\frac{1}{|\log t|^{1/2-\epsilon}}\right).$$

But $\omega_{SRF,y}$ is continuous in y at $y = 0$, and $t \rightarrow 0$ implies $y \rightarrow 0$ in a bounded region of the Gromov-Hausdorff metric, so $t^{-1}\tilde{\omega}_t|_{X_y} \rightarrow \omega_{SRF,0}$. $\square$

We also need information about the horizontal component of the metric. We comment that the coordinate u on $\mathbb{C}$ requires an implicit choice of a local chart on the base Y . We use the convention that the Euclidean metric $\omega_{\mathbb{C}}$ agrees with $t^{-1}\omega_{Y,0}$ (cf. Section 2.3.1) under the identification $u = t^{-1/2}y = t^{-1/2}\pi(x)$.

Proposition 2.3.13. The metric ω_{∞} over the smooth part of $X_0 \times \mathbb{C}$ satisfies the Riemannian submersion property $\text{Tr}_{\omega_{\infty}} \omega_{\mathbb{C}} = 1$.

Proof. The concentration estimate 2.3.1 can be interpreted as

$$\left\| \text{Tr}_{\frac{1}{t}\tilde{\omega}_t} \omega_{\mathbb{C}} - 1 \right\|_{L^1(\{|u| \leq 1\}, \frac{1}{t}\tilde{\omega}_t)} \leq \frac{C}{|\log t|}.$$

In particular, upon passing to the L^1 limit, the trace $\text{Tr}_{\omega_{\infty}} \omega_{\mathbb{C}} = 1$. It is clear that $u \leq 1$ is not an essential restriction. $\square$

Proposition 2.3.14. Over the smooth part of $X_0 \times \mathbb{C}$, the differential du is parallel with respect to ω_{∞} .

Proof. Recall the gradient estimate (2.20), which can be written in a scaled version

$$\int_{|u| \leq 1} \left(\frac{|\nabla du|_{\frac{1}{t}\tilde{\omega}_t}^2}{\text{Tr}_{\frac{1}{t}\tilde{\omega}_t} \omega_{\mathbb{C}}} - |\partial \log \text{Tr}_{\frac{1}{t}\tilde{\omega}_t} \omega_{\mathbb{C}}|_{\frac{1}{t}\tilde{\omega}_t}^2 \right) (t^{-1}\tilde{\omega}_t)^3 \leq \frac{C}{|\log t|}.$$

Restricted to any compact set in the smooth locus, the trace term converges smoothly to the constant one, so in the limit

$$\int |\nabla du|_{\omega_{\infty}}^2 \omega_{\infty}^3 = 0.$$

It is clear that $u \leq 1$ is not an essential restriction. We get the qualitative statement that $\nabla du = 0$ pointwise on the smooth locus. $\square$

Proposition 2.3.15. The limiting metric $\omega_{\infty} = \omega_{SRF,0} + \omega_{\mathbb{C}}$ over the smooth locus of $X_0 \times \mathbb{C}$.

Proof. The parallel differential du induces a parallel $(1,0)$ type vector field by the complexified Hamiltonian construction: $\iota_V \omega_{\infty} = d\bar{u}$. In particular V is a holomorphic vector field. But on the nodal K3 surface, there is no holomorphic vector field on X_0^{reg} , so V must lie in the subbundle $T\mathbb{C} \subset T(X_0^{\text{reg}} \times \mathbb{C})$. Moreover, on each fibre X_0 , V must be a constant multiple of $\frac{\partial}{\partial u}$. We can then write $V = \lambda(u) \frac{\partial}{\partial u}$, where λ is a holomorphic function in u . Since du and V are both parallel, the quantity $\lambda = du(V)$ must be a constant.

We know ω_{∞} restricted to the fibres is just $\omega_{SRF,0}$. By construction, the vector field V defines the Hermitian orthogonal complement of the holomorphic tangent space of the fibres. Now $V = \lambda \frac{\partial}{\partial u}$ where the constant is specified by the Riemannian submersion property. The claim follows. $\square$

Lemma 2.3.16. The metric space $(X_0 \times \mathbb{C}, \omega_{SRF,0} + \omega_{\mathbb{C}})$ is the metric completion of its smooth locus.

Proof. Clear from the orbifold nature of X_0 . $\square$

Corollary 2.3.17. The holomorphic inverse map f_∞^{-1} defined on the smooth locus $(X_0^{\text{reg}} \times \mathbb{C}, \omega_{SRF,0} + \omega_{\mathbb{C}})$ extends to a continuous map

$$f_\infty^{-1} : (X_0 \times \mathbb{C}, \omega_{SRF,0} + \omega_{\mathbb{C}}) \rightarrow Z,$$

which is a right inverse of f_∞ , and isometric to its image.

Theorem 2.3.18. *Assume the uniform fibre diameter bound. Then the map $f_\infty : (Z, \omega_\infty) \rightarrow (X_0 \times \mathbb{C}, \omega_{SRF,0} + \omega_{\mathbb{C}})$ is an isometry, and naturally an isomorphism of normal varieties.*

Proof. The right inverse f_∞^{-1} must be surjective using the full measure property of the set $\{H > 0\}$ on Z . This combined with the above corollary proves the isometry statement, so in particular implies homeomorphism. We know f_∞ is a biholomorphism

$$\{H > 0\} \subset Z \simeq X_0^{\text{reg}} \times \mathbb{C}.$$

Since a point in Z with $H = 0$ is a metric singularity, it cannot be a smooth point in the complex geometric sense, so $\{H > 0\} \subset Z$ coincides with the regular locus Z^{reg} , and f_∞ is an isomorphism between regular loci. This allows us to identify the structure sheaf of Z with that of the normal variety $X_0 \times \mathbb{C}$. $\square$

Chapter 3

A new complete Calabi-Yau metric on $\mathbb{C}^3$

Motivated by the study of collapsing Calabi-Yau 3-folds with a Lefschetz K3 fibration, we construct a complete Calabi-Yau metric on $\mathbb{C}^3$ with maximal volume growth, which in the appropriate scale models the collapsing metric near the nodal point. This new Calabi-Yau metric has singular tangent cone at infinity $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$, and its Riemannian geometry has certain non-standard features near the singularity of the tangent cone, which are more typical of adiabatic limit problems. The proof uses an existence result in H-J. Hein's PhD thesis to perturb an asymptotic approximate solution into an actual solution, and the main difficulty lies in correcting the slowly decaying error terms.

3.1 The Stenzel metric

We review the Stenzel construction of Ricci-flat metrics on the smoothings of ordinary double point ('ODP') singularities [38], to set up some notations and to derive some useful formulae. Let $n \geq 3$. We start with the standard quadratic function

$$f = \sum_1^n z_i^2 : \mathbb{C}^n \rightarrow \mathbb{C},$$

and consider the fibres $X_y = \{\sum z_i^2 = y\}$. For later reference, let us establish a few conventions:

- The standard Euclidean metric on $\mathbb{C}^n$ is $\omega_E = \frac{1}{2}\sqrt{-1}\sum dz_i \wedge d\bar{z}_i$; we will sometimes also denote by ω_E its restriction to fibres.
- The volume form is $\text{vol}_E = \frac{1}{n!}\omega_E^n = \frac{1}{2^n}\widehat{\text{vol}_E}$, where the normalised volume form is $\widehat{\text{vol}_E} = \prod_i(\sqrt{-1}dz_i \wedge d\bar{z}_i)$.
- The pullback of the standard metric from the base $\mathbb{C}$ is $\frac{1}{2}\sqrt{-1}df \wedge d\bar{f}$.
- The norm-squared function on $\mathbb{C}^n$ is $H = \sum |z_i|^2$.

- The holomorphic n -form on $\mathbb{C}^n$ is $\Omega = dz_1 \dots dz_n$.

We consider the ansatz $\phi = F(H)$ for the Kähler potential on X_y , where F is locally smooth; this amounts to imposing $SO(n, \mathbb{R})$ symmetry. We compute the metric

$$\begin{aligned}\bar{\partial}\phi &= F'(H)\bar{\partial}H, \\ \sqrt{-1}\partial\bar{\partial}\phi &= F'\sqrt{-1}\partial\bar{\partial}H + F''\sqrt{-1}\partial H \wedge \bar{\partial}H.\end{aligned}$$

Using the binomial theorem, we have

$$(\sqrt{-1}\partial\bar{\partial}\phi)^{n-1} = F'^{n-1}(\sqrt{-1}\partial\bar{\partial}H)^{n-1} + (n-1)F'^{n-2}F''(\sqrt{-1}\partial\bar{\partial}H)^{n-2} \wedge (\sqrt{-1}\partial H \bar{\partial}H).$$

To understand the terms, we compute the forms over $\mathbb{C}^n$:

$$\begin{aligned}(\sqrt{-1}\partial\bar{\partial}H)^{n-1} &= (n-1)! \sum_{i=1}^n \prod_{j \neq i} (\sqrt{-1}dz_j \wedge d\bar{z}_j), \\ \sqrt{-1}df \wedge d\bar{f} &= 4 \sum_{i,j} \sqrt{-1}z_i \bar{z}_j dz_i \wedge d\bar{z}_j, \quad \sqrt{-1}\partial H \wedge \bar{\partial}H = \sqrt{-1} \sum_{i,j} z_j \bar{z}_i dz_i \wedge d\bar{z}_j, \\ \frac{1}{4}(\sqrt{-1}\partial\bar{\partial}H)^{n-1} \wedge (\sqrt{-1}df \wedge d\bar{f}) &= (n-1)! \widehat{H vol_E}, \\ \frac{1}{4}(\sqrt{-1}\partial\bar{\partial}H)^{n-2} \wedge (\sqrt{-1}\partial H \wedge \bar{\partial}H) \wedge (\sqrt{-1}df \wedge d\bar{f}) &= (n-2)!(H^2 - |y|^2) \widehat{vol_E}.\end{aligned}$$

Thus

$$\frac{1}{4}(\sqrt{-1}\partial\bar{\partial}\phi)^{n-1} \wedge (\sqrt{-1}df \wedge d\bar{f}) = (n-1)! \widehat{vol_E} \{HF'^{n-1} + (H^2 - |y|^2)F'^{n-2}F''\}. \quad (3.1)$$

By the adjunction formula the quadric fibres also have trivial canonical bundles, and we can write down the holomorphic volume form Ω_y on X_y by $\Omega = df \wedge \Omega_y$. This gives

$$\widehat{vol_E} = \sqrt{-1}^{n^2} \Omega \wedge \bar{\Omega} = \sqrt{-1}df \wedge d\bar{f} \wedge (\sqrt{-1}^{(n-1)^2} \Omega_y \wedge \bar{\Omega}_y). \quad (3.2)$$

The defining condition for the Calabi-Yau metric on X_y is

$$(\sqrt{-1}\partial\bar{\partial}\phi)^{n-1} = \sqrt{-1}^{(n-1)^2} \Omega_y \wedge \bar{\Omega}_y. \quad (3.3)$$

We can wedge both sides by $\sqrt{-1}df \wedge d\bar{f}$, and use (3.1) (3.2) to see the following identity holds in $\Omega_{\mathbb{C}^n}^{n,n}|_{X_y}$:

$$4(n-1)!\{HF'^{n-1} + (H^2 - |y|^2)F'^{n-2}F''\} = 1.$$

We can simplify further by a scaling observation: let $F = |y|^{\frac{n-2}{n-1}} \tilde{F}(\frac{H}{|y|})$, then $\tilde{F}$ satisfies the ODE

$$x\left(\frac{d\tilde{F}}{dx}\right)^{n-1} + (x^2 - 1)\left(\frac{d\tilde{F}}{dx}\right)^{n-2} \frac{d^2\tilde{F}}{dx^2} = \frac{1}{4(n-1)!}.$$

Now make the change of variable $x = \frac{H}{|y|} = \cosh \tau$. Then the equation becomes

$$\frac{d}{d\tau} \left[\left(\frac{d\tilde{F}}{d\tau} \right)^{n-1} \right] = \frac{1}{4(n-2)!} \sinh^{n-2} \tau.$$

If we impose the initial condition $\frac{d\tilde{F}}{d\tau}(0) = 0$ to ensure smooth extension of the potential across the vanishing cycle $\{H = |y|\}$, then the solution defines up to a numerical factor the Stenzel metric.

Example 3.1.1. For $n = 3$, the above equation implies $\frac{d\tilde{F}}{d\tau} = \frac{1}{\sqrt{2}} \sinh(\frac{\tau}{2})$, $\tilde{F} = \sqrt{2} \cosh(\frac{\tau}{2}) = \sqrt{\frac{H}{|y|} + 1}$. Thus the Kähler potential is $\phi = F(H) = \sqrt{H + |y|}$. The Stenzel metric in this special case is also known as the **Eguchi-Hanson metric**.

3.2 A general symmetric ansatz

We consider Kähler potentials on $\mathbb{C}^n$ of the form $\phi = F(H, \eta)$ where $\eta = |f|^2$. This amounts to imposing the **symmetry** under $SO(n, \mathbb{R}) \times U(1)$. Notice that this is not a cohomogeneity-one problem, so it may be hard to write down explicit solutions to interesting equations. We shall nevertheless exhibit solutions whose asymptotic behaviour resembles a Calabi-Yau metric.

We imitate the computations about the Stenzel metric, but now we need to keep track of all partial derivatives, because we are now working on the entire $\mathbb{C}^n$ instead of quadric fibres. Using that f is holomorphic,

$$\bar{\partial}\phi = F_H \bar{\partial}H + F_\eta f d\bar{f},$$

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial}\phi &= F_H \sqrt{-1}\partial\bar{\partial}H + F_{HH} \sqrt{-1}\partial H \wedge \bar{\partial}H + \sqrt{-1}F_{H\eta}(\bar{f}df \wedge \bar{\partial}H + f\partial H \wedge d\bar{f}) \\ &\quad + (\eta F_{\eta\eta} + F_\eta) \sqrt{-1}df \wedge d\bar{f}. \end{aligned}$$

Now using the binomial theorem, one can expand

$$\begin{aligned} (\sqrt{-1}\partial\bar{\partial}\phi)^n &= (F_H \sqrt{-1}\partial\bar{\partial}H + F_{HH} \sqrt{-1}\partial H \wedge \bar{\partial}H)^n \\ &\quad + n(F_H \sqrt{-1}\partial\bar{\partial}H + F_{HH} \sqrt{-1}\partial H \wedge \bar{\partial}H)^{n-1} \wedge \\ &\quad \{(\eta F_{\eta\eta} + F_\eta) \sqrt{-1}df \wedge d\bar{f} + \sqrt{-1}F_{H\eta}(\bar{f}df \wedge \bar{\partial}H + f\partial H \wedge d\bar{f})\} \\ &\quad + \binom{n}{2} (F_H \sqrt{-1}\partial\bar{\partial}H + F_{HH} \sqrt{-1}\partial H \wedge \bar{\partial}H)^{n-2} \wedge (2\eta F_{H\eta}^2 df d\bar{f} \wedge \partial H \bar{\partial}H). \end{aligned}$$

The individual terms can be computed using formulae in Section 3.1. We arrive at a formula for the volume form:

Proposition 3.2.1. For a symmetric Kähler potential $\phi = F(H, \eta)$,

$$\begin{aligned} (\sqrt{-1}\partial\bar{\partial}\phi)^n &= n! \widehat{\text{vol}}_E \{ \{ F_H^n + F_H^{n-1} F_{HH} H \} \\ &\quad + 4(\eta F_{\eta\eta} + F_\eta) \{ H F_H^{n-1} + (H^2 - \eta) F_H^{n-2} F_{HH} \} \\ &\quad + 4F_{H\eta} \eta F_H^{n-1} - 4F_{H\eta}^2 \eta F_H^{n-2} (H^2 - \eta) \} \}. \end{aligned} \quad (3.4)$$

Example 3.2.2. If the potential does not depend on η , then

$$(\sqrt{-1}\partial\bar{\partial}\phi)^n = n!\widehat{\text{vol}}_E\{F_H^n + F_H^{n-1}F_{HH}H\}.$$

For $F = \frac{1}{2}H$, one recovers the Euclidean metric and its volume form.

Main Ansatz 3.2.3. Let $n = 3$. We define a regularized version of the semi-Ricci-flat metric (1.12) by

$$\phi_\infty = F(H, \eta) = \frac{1}{2}\eta + \sqrt{H + a(\eta, H)}, \quad a(\eta, H) = \sqrt{\eta + \sqrt{H + 1}}. \quad (3.5)$$

This is the main ansatz in this Chapter. It is a Kähler metric, using the fact that if ψ, ψ' are plurisubharmonic then so is $\log(e^\psi + e^{\psi'})$. Our notation suggests ϕ_∞ models the metric asymptote near infinity.

Lemma 3.2.4. *The volume form of the metric ansatz (3.5) is*

$$\begin{aligned} \frac{(\sqrt{-1}\partial\bar{\partial}\phi_\infty)^3}{\widehat{\text{vol}}_E} &= \frac{3}{2} \left\{ 1 + \frac{1}{4a(H+a)^{1/2}} + \frac{H^{1/2}}{4a(H+a)} - \frac{H^{1/2}}{(H+a)^2} + \frac{3}{4(H+a)\sqrt{H}} \right. \\ &\quad \left. - \frac{H-a}{4aH^{3/2}} + \frac{2H^3 - 5aH^2 - 6a^2H - 3a^3}{8(H+a)^2H^{5/2}a} \right\} + O\left(\frac{1}{a^3}\right). \end{aligned} \quad (3.6)$$

Proof. We give some intermediate steps:

$$\begin{aligned} a_\eta &= \frac{1}{2a}, \quad a_{\eta\eta} = -\frac{1}{4a^3}, \quad a_{\eta H} = -\frac{1}{8a^3\sqrt{H+1}}, \\ a_H &= \frac{1}{4a\sqrt{H+1}}, \quad a_{HH} = -\frac{1}{16a^3(H+1)} - \frac{1}{8a(H+1)^{3/2}}, \\ F_H &= \frac{1}{2\sqrt{H+1}}(1 + a_H), \quad F_{HH} = -\frac{1}{4(H+a)^{3/2}}(1 + a_H)^2 + \frac{a_{HH}}{2\sqrt{H+a}}, \\ F_\eta &= \frac{1}{2} + \frac{1}{2\sqrt{H+a}}a_\eta, \quad F_{\eta\eta} = \frac{1}{2\sqrt{H+a}}a_{\eta\eta} - \frac{1}{4(H+a)^{3/2}}a_\eta^2, \\ F_{H\eta} &= \frac{1}{2\sqrt{H+a}}a_{H\eta} - \frac{1}{4(H+a)^{3/2}}a_\eta(1 + a_H). \end{aligned}$$

To aid computation, let us introduce the quantities

$$\begin{aligned} Q_1 &= F_H^3 + F_H^2 H F_{HH}, \quad Q_2 = \eta F_{\eta\eta} + F_\eta, \\ Q_3 &= H F_H^2 + (H^2 - \eta) F_H F_{HH}, \quad Q_4 = F_{H\eta} F_H^2 - F_{H\eta}^2 F_H (H^2 - \eta). \end{aligned}$$

Our general formula (3.4) for the volume form reads

$$\frac{(\sqrt{-1}\partial\bar{\partial}\phi_\infty)^3}{6\widehat{\text{vol}}_E} = Q_1 + 4Q_2Q_3 + 4Q_4.$$

We are interested in the asymptote as $H \rightarrow \infty$. It is elementary but useful to notice the basic inequalities for a, H, η , such as $H \geq \eta^{1/2}$, $a \geq \eta^{1/2}$, $a \geq H^{1/4}$, and

$a = O(H)$ for $H > 1$. After substituting in the expressions for derivatives, and extracting leading order terms,

$$\begin{aligned}
Q_1 &= -\frac{H}{16(H+a)^{5/2}} + \frac{1}{8(H+a)^{3/2}} + O\left(\frac{1}{H^2a}\right), \\
Q_2 &= \frac{1}{2} + \frac{1}{8a\sqrt{H+a}} - \frac{1}{16(H+a)^{3/2}} + O\left(\frac{1}{a^3}\right), \\
Q_3 &= \frac{1}{8} + \frac{H^{1/2}}{32a(H+a)} - \frac{H^{1/2}}{8(H+a)^2} + \frac{3}{32(H+a)\sqrt{H}} - \frac{H-a}{32aH^{3/2}} \\
&\quad + \frac{2H^3 - 5aH^2 - 6a^2H - 3a^3}{64(H+a)^2H^{5/2}a} + O\left(\frac{1}{a^3}\right), \\
Q_4 &= -\frac{1}{32} \frac{a}{(H+a)^{5/2}} - \frac{H-a}{128(H+a)^{5/2}} + O\left(\frac{1}{a^3}\right).
\end{aligned}$$

Combining these computations gives the formula. $\square$

It is worth pointing out for later purposes that the terms we are discarding also have higher order bounds, which can be obtained inductively using the following principle: write all contributions in terms of products of power law expressions in $H+a$, H , a etc; each H -derivative picks up a multiplicative factor $O(\frac{1}{H})$ (the factor coming from a_H are further suppressed); each η -derivative involves differentiating an expression of a , so the derivative picks up a multiplicative factor $O(\frac{1}{a^2})$. The result is that the error term $E = O(\frac{1}{a^3})$ in formula (3.6) has **higher order bounds** $|\frac{\partial^{i+j}}{\partial H^i \partial \eta^j} E| = O(\frac{1}{H^i a^{3+2j}})$.

A cruder version of the asymptotic formula as $H \rightarrow \infty$ is:

$$(\sqrt{-1}\partial\bar{\partial}\phi_\infty)^3 = \frac{3}{2} \widetilde{vol_E} \left\{ 1 + \frac{1}{4H^{1/2}a} + O\left(\frac{1}{H^{3/2}}\right) + O\left(\frac{1}{a^3}\right) \right\}. \quad (3.7)$$

To take two extreme cases, in the region where $|f| \sim H^{1/4} \gg 1$ then $a \sim H^{1/4}$, and $O(\frac{1}{H^{3/2}}) = O(\frac{1}{a^6})$ is insignificantly small. In the region where $|f| \sim H \gg 1$, then $a \sim H$, and $O(\frac{1}{H^{3/2}}) = O(\frac{1}{a^{3/2}})$ is the dominating error term.

3.3 Riemannian geometry of the main ansatz

We study the asymptotic properties of the metric ansatz (3.5). As a preliminary remark, the volume form of the metric (3.5) is asymptotic to the volume form of the Euclidean metric. At one stage we will make use of a non-holomorphic parametrisation, so we shall distinguish the Kähler form ω and the associated Riemannian metric tensor g .

3.3.1 Asymptotic metric

Now we study the properties of the metric for $H \gg 1$, only to the leading order. The asymptotic expression is

$$\begin{aligned} \omega = \sqrt{-1}\partial\bar{\partial}\phi_\infty &\sim \frac{1}{2}\sqrt{-1}df \wedge d\bar{f} + \frac{\sqrt{-1}}{2\sqrt{H+a}}\partial\bar{\partial}H - \frac{\sqrt{-1}}{4(\sqrt{H+a})^3}\partial H \wedge \bar{\partial}H \\ &\sim \frac{1}{2}\sqrt{-1}df \wedge d\bar{f} + \frac{\sqrt{-1}}{2\sqrt{H+|f|}}\partial\bar{\partial}H - \frac{\sqrt{-1}}{4(\sqrt{H+|f|})^3}\partial H \wedge \bar{\partial}H \end{aligned} \quad (3.8)$$

This follows readily from differentiating (3.5), and discarding the terms coming from differentiating a since these do not affect the leading asymptote.

The **distance function** has large distortion from Euclidean behaviour:

Lemma 3.3.1. *For $P \in \{H > 1\}$, there is a uniform equivalence*

$$C(|f| + H^{1/4}) \leq d(0, P) \leq C'(|f| + H^{1/4}). \quad (3.9)$$

In other words, $d(0, \cdot)$ is uniformly equivalent to the function a on $\{H > 1\}$.

Proof. Using the asymptotic formula, one sees $C\omega \geq \sqrt{-1}df \wedge d\bar{f}$, so $|f| \leq Cd(0, P)$. Similarly one sees for $H > 1$, $C\omega \geq \frac{\sqrt{-1}}{\sqrt{H}}\partial\bar{\partial}H$, so along any path parametrised by the arclength,

$$\frac{d|z|}{ds} \leq C|z|^{1/2},$$

which integrates to give $H^{1/4} \leq Cd(0, P)$.

Now for the reverse direction, we take the radial path joining the origin and the point P , and compute the length. This gives $d(0, P) \leq C(|f| + H^{1/4})$. $\square$

Lemma 3.3.2. *Let $r \gg 1$. We have the **maximal volume growth** condition:*

$$Cr^6 \leq \text{Vol}(B_g(0, r)) \leq C'r^6. \quad (3.10)$$

Proof. The description of the distance function implies that

$$\{H^{1/4} \leq Cr\} \cap \{|f| \leq Cr\} \subset B_g(0, r) \subset \{H^{1/4} \leq C'r\} \cap \{|f| \leq C'r\}$$

for some appropriate constants C and C' . Thus we can consider instead the set $\{H \leq r^4\} \cap \{\eta \leq r^2\}$ and estimate its volume. This is an exercise in Lebesgue integration theory.

Since the asymptotic volume measure is comparable to the Lebesgue measure,

$$\text{Vol}_\omega(\{H \leq r^4\} \cap \{\eta \leq r^2\}) \sim \text{Vol}_{\text{Lebesgue}}(\{H \leq r^4\} \cap \{\eta \leq r^2\}).$$

By the homogeneity of the Lebesgue measure, the RHS is

$$r^{12} \text{Vol}_{\text{Lebesgue}}(\{H \leq 1\} \cap \{\eta \leq r^{-6}\}).$$

Now for very large r , the set $\{H \leq 1\} \cap \{\eta \leq r^{-6}\}$ is approximately the product of $\{H \leq 1\} \cap X_0$ with the disc $\{\eta \leq r^{-6}\} \subset \mathbb{C}$ where X_0 is the nodal central fibre, so the above

$$\sim r^{12} \text{Area}(\{H \leq 1\} \cap X_0) \text{Area}(\{\eta \leq r^{-6}\} \subset \mathbb{C}) \sim r^6.$$

Here the area of $\{H \leq 1\} \cap X_0$ is induced from the restriction of the Euclidean metric. The claimed result follows. $\square$

We define a family of cones in $\mathbb{C}^3$ by $E_\epsilon = \{|f| \geq \epsilon H\}$, for $0 < \epsilon < 1$. In practice ϵ is a fixed small number. Our next result says these cones are highly squashed in the metric g .

Lemma 3.3.3. *Take $r > 1$. Let $P \in B_g(r) \cap E_\epsilon$, then the distance of P to the complement of E_ϵ has an upper bound*

$$d_g(P, \mathbb{C}^3 \setminus E_\epsilon) \leq \frac{C}{\epsilon^{1/4}} r^{1/4}. \quad (3.11)$$

Proof. We start by observing that on the affine quadric $\{f = 1\}$ with the Eguchi-Hanson metric (cf. Section 3.1), there is a uniform diameter bound $\text{diam}(\{H \leq \frac{1}{\epsilon}\}) \leq \frac{C}{\epsilon^{1/4}}$ for $0 < \epsilon < 1$, because at large distance the Eguchi-Hanson metric is asymptotic to the orbifold metric on X_0 , and the distance to the origin on the flat orbifold X_0 is $\sqrt{2}H^{1/4}$. By scaling, on the fibre X_f with the Eguchi-Hanson metric, the diameter bound reads

$$\text{diam}(\{H \leq \frac{|f|}{\epsilon}\} \subset X_f) \leq \frac{C}{\epsilon^{1/4}} |f|^{1/4}.$$

Now if $\eta \geq 1$, then the metric g restricted to the fibre X_f is uniformly equivalent to the Eguchi-Hanson metric, because g is by construction essentially the semi-Ricci flat approximation, for large values of η . Thus if $P \in X_f \cap E_\epsilon$, we can move the point inside X_f and exit the region E_ϵ within distance $\frac{C}{\epsilon^{1/4}}(\eta^{1/8} + 1) \leq \frac{C}{\epsilon^{1/4}}(r^{1/4} + 1)$. Here we add in the constant 1 to make the argument work also for $\eta \leq 1$. This means $d(P, \mathbb{C}^3 \setminus E_\epsilon) \leq \frac{C}{\epsilon^{1/4}}(r^{1/4} + 1) \leq \frac{C}{\epsilon^{1/4}} r^{1/4}$. $\square$

Remark 3.3.4. *If we normalise the metric by a factor $\frac{1}{r^2}$, then $P \in B_{\frac{1}{r^2}g}(1)$, and this estimate says $d_{\frac{1}{r^2}g}(P, \mathbb{C}^3 \setminus E_\epsilon) \leq \frac{C}{\epsilon^{1/4}r^{3/4}}$. Intuitively, for large r and fixed ϵ , this means the region E_ϵ has very small solid angle.*

Now we concentrate on the region $(\mathbb{C}^3 \setminus E_\epsilon) \cap \{H \gg 1\}$, with ϵ a fixed small number. In this region, up to $O(\epsilon)$ error, the asymptote of ω as $H \rightarrow \infty$ simplifies further to

$$\omega \sim \frac{1}{2} \sqrt{-1} df \wedge d\bar{f} + \frac{\sqrt{-1}}{2\sqrt{H}} \partial \bar{\partial} H - \frac{\sqrt{-1}}{4H^{3/2}} \partial H \wedge \bar{\partial} H.$$

Ehresmann's theorem suggests the fibration f restricted to the region $(\mathbb{C}^3 \setminus E_\epsilon) \cap \{H \gg 1\}$ is a smooth fibration over $\mathbb{C}$. We find a simple parametrisation over the base $\mathbb{C}$:

$$\Psi : X_0 \times \mathbb{C} \rightarrow \mathbb{C}^3, \quad z = w + \frac{f\bar{w}}{2|w|^2}, \quad (w, f) \in X_0 \times \mathbb{C}. \quad (3.12)$$

The region $(\mathbb{C}^3 \setminus E_\epsilon) \cap \{H \gg 1\}$ corresponds roughly to $\{|f| \leq \epsilon|w|^2, |w|^2 \gg 1\}$. The parametrisation Ψ is not exactly holomorphic, but only approximately so up to an error of order $\frac{\partial z}{\partial \bar{w}} = O(\epsilon)$. The function H pulls back to $|w|^2(1 + O(\epsilon))$. The pullback Riemannian metric Ψ^*g can be expanded from the above metric asymptotic formula. After writing out $dz_i = dw_i + df \frac{\bar{w}_i}{2|w|^2} + \frac{f}{2|w|^2} d\bar{w}_i - \frac{f\bar{w}_i}{2|w|^4} d|w|^2$, and discarding small terms of order $O(\epsilon)$,

Lemma 3.3.5. *The Riemannian metric g associated to ω admits the approximate formula in the region $\{|f| \leq \epsilon H, H \gg 1\}$:*

$$\Psi^*g = \operatorname{Re}(df \otimes d\bar{f} + |w|^{-1} \sum dw_i \otimes d\bar{w}_i - \frac{1}{2}|w|^{-3} \sum \bar{w}_i dw_i \otimes w_j d\bar{w}_j)(1 + O(\epsilon)) \quad (3.13)$$

i.e. up to relative error of order $O(\epsilon)$, the metric g is asymptotic to the product metric g_{product} on $X_0 \times \mathbb{C}$, where the X_0 factor is equipped with the Stenzel metric, which agrees with the flat orbifold metric on $X_0 \simeq \mathbb{C}^2/\mathbb{Z}_2$.

The information at our disposal now allows us to identify the **tangent cone at infinity** for the metric g .

Proposition 3.3.6. The tangent cone of $(\mathbb{C}^3, g)$ at infinity is isometrically $X_0 \times \mathbb{C}$ with the product metric.

Proof. Clearly only asymptotic properties of the metric are relevant. Fix a small number ϵ . Consider large values of r , and apply the scaling factor $\frac{1}{r^2}$ to the metric g . In the ball $B_{\frac{1}{r^2}g}(0, D)$ of fixed radius D , if there is a point P in the region E_ϵ , then by the lemma 3.3.3, its distance to $\mathbb{C}^3 \setminus E_\epsilon$ is controlled above by $\frac{CD^{1/4}}{\epsilon^{1/4}r^{3/4}} < C\epsilon$ for large r . Thus the only important contribution comes from the region $(\mathbb{C}^3 \setminus E_\epsilon) \cap \{H \geq C\}$. Up to $O(\epsilon)$ error, $(\mathbb{C}^3 \setminus E_\epsilon) \cap B_{\frac{1}{r^2}g}(0, D)$ is Gromov-Hausdorff close to the metric space

$$(B_{r^{-2}g_{\text{product}}}(0, D) \cap \{|f| \leq \epsilon|w|^2\}, \frac{1}{r^2}g_{\text{product}}) \subset X_0 \times \mathbb{C}.$$

Now we observe that $X_0 \times \mathbb{C}$ with the product metric is conical, and scaling features imply that for large r the region $\{|f| \geq \epsilon|w|^2\}$ is unimportant. Thus the metric space is Gromov-Hausdorff close to $(B_{r^{-2}g_{\text{product}}}(0, D), \frac{1}{r^2}g_{\text{product}})$, which is the same as $(B_{g_{\text{product}}}(0, rD), g_{\text{product}})$. From this we see that as $r \rightarrow \infty$, $B_{\frac{1}{r^2}g}(0, D)$ Gromov-Hausdorff converges to $(B_{g_{\text{product}}}(0, D), g_{\text{product}})$, for any fixed D , so the tangent cone is indeed $X_0 \times \mathbb{C}$ with the product metric. $\square$

Remark 3.3.7. (Jumping of complex structure) *The complex structure on the tangent cone at infinity differs from the original manifold $\mathbb{C}^3$. This can be explained by the flat family of subvarieties of $\mathbb{C}_y \times \mathbb{C}_{z_1, z_2, z_3}^3$ parametrised by $t \in \mathbb{C}$:*

$$\{(y, z_1, z_2, z_3) | ty = z_1^2 + z_2^2 + z_3^2\},$$

whose generic fibre is just a graph over $\mathbb{C}^3$, and the central fibre is $\mathbb{C} \times X_0$; the computation of Gromov Hausdorff limit involves a scaling procedure, which is reflected here by the parameter t .

Remark 3.3.8. (Algebraicity) The distance $d(0, \cdot) \sim a$ is bi-Hölder but not bi-Lipschitz to the Euclidean distance. Thus the ring of holomorphic functions with polynomial growth agrees with the standard polynomial ring on affine $\mathbb{C}^3$, but the filtration structure coming from the growth rate is different. In particular $\deg(z_i) = 2$ for $i = 1, 2, 3$ and $\deg(f) = 1$. The Spec of the associated graded ring is the tangent cone at infinity:

$$\{(y, z_1, z_2, z_3) | z_1^2 + z_2^2 + z_3^2 = 0\} \subset \mathbb{C}_y \times \mathbb{C}_{z_1, z_2, z_3}^3.$$

Remark 3.3.9. (Measure theoretic paradox) The link of the tangent cone at infinity is singular. The singularity corresponds to the node in X_0 , and the cone E_ϵ collapses to the nodal line in $X_0 \times \mathbb{C}$. This behaviour is counterintuitive as E_ϵ seems to carry a large amount of Lebesgue measure, which disappears in the limit, but general theory of noncollapsing Gromov-Hausdorff convergence implies measure theoretic convergence. But this is not a contradiction because in a large ball with respect to the metric g , the set E_ϵ only occupies a very small portion of the measure.

H-J. Hein pointed out an analogy with the Taub-NUT metric which may help to understand this measure theoretic picture. By a well known observation of LeBrun [33], the Taub-NUT space is biholomorphic to $\mathbb{C}^2$ with the standard holomorphic volume form. In fact the holomorphic symplectic moment map makes $\mathbb{C}^2$ into a fibration of affine quadrics, exactly analogous to our situation. The counterintuitive fact then is that the Taub-NUT metric has cubic volume growth. The explanation again lies in the large distortion of distance. The distinctive feature of $n = 2$ is that the quadric fibres are isomorphic to $\mathbb{C}^*$, whose natural cylindrical metric has linear growth rate, which is not maximal. The Taub-NUT metric is asymptotically described by the semi-Ricci flat metric coming from the quadric fibration; the growth rate is quadratic in the base direction and linear in the fibre direction, thus the overall growth is cubic.

Next we consider the geometry around a central point $P \in E_\epsilon \cap \{H \gg 1\}$, so $f(P)$ is large. We can define a local diffeomorphism around P ,

$$\Psi_P : X_{f(P)} \times \mathbb{C}_\zeta \rightarrow \mathbb{C}^3, \quad z = w(1 + \frac{\zeta}{2f(P)}), \quad (w, \zeta) \in X_{f(P)} \times \mathbb{C}.$$

The region $\{|\zeta| \ll |f(P)|, |w|^{1/2} \ll |f(P)|\}$ roughly corresponds to $\{|f - f(P)| \ll |f(P)|, H^{1/4} \ll |f(P)|\}$, where the quantity $f = f(z) \sim f(P) + \zeta$, $H \sim |w|^2$, and $\sqrt{H + |f|} \sim \sqrt{|w|^2 + |f(P)|}$. The asymptotic metric formula (3.8) gives

$$\omega \sim \frac{1}{2} \sqrt{-1} d\zeta \wedge d\bar{\zeta} + \sqrt{-1} \partial \bar{\partial} \sqrt{|w|^2 + |f(P)|},$$

namely the product of the Euclidean metric on $\mathbb{C}_\zeta$ with the Eguchi-Hanson metric on $X_{f(P)}$. The diameter of this local region is $o(|f(P)|)$, and the curvature scale around the vanishing cycle is $O(|f(P)|^{1/4})$.

Proposition 3.3.10. (Blow up limits around vanishing cycles) Let $P_i \in E_\epsilon$ with $f(P_i) \rightarrow \infty$. Then the pointed Gromov-Hausdorff limit of the rescaled metrics $(\mathbb{C}^3, |f(P_i)|^{-1/2} g, P_i)$ is $EH_1 \times \mathbb{C}$, where EH_1 denotes the Eguchi-Hanson metric on X_1 .

Remark 3.3.11. *This indicates that the tangent cone at infinity, which is the scaled metric limit around a fixed base point, does not capture all asymptotic information; this phenomenon happens in general if the tangent cone at infinity has singular link. The presence of the flat $\mathbb{C}$ factor is compatible with the splitting theorem in Riemannian geometry applied to radial directions. The appearance of many different scaling factors $|f(P_i)|^{-1/2}$ means this metric is multi-scaled.*

3.3.2 Topological discussions

The topological situation is related to the general story of Milnor fibrations associated to hypersurface singularities. Let us dispense with the metric geometry for a moment. In the ODP context, the sphere $S^5 = \{H = 1\}$ intersects $\mathbb{C}^3 \setminus X_0$ in an open set, which is just the complement of the link $X_0 \cap S^5$ of X_0 . We have a fibration over the circle

$$S^5 \cap (\mathbb{C}^3 \setminus X_0) \rightarrow S^1, \quad z \mapsto \frac{f}{|f|}.$$

The fibre is topologically identified with the Milnor fibre X_1 . This is because via the radial projection map $z \mapsto \frac{z}{|f|^{1/2}}$, this fibration is identified with

$$f : f^{-1}(S^1) \subset \mathbb{C}^3 \rightarrow S^1.$$

The monodromy around the circle is described by the standard Picard-Lefschetz formula (cf. Theorem 3.16 [47]).

The analogous picture on the tangent cone $X_0 \times \mathbb{C}$ is the following. We have a fibration

$$\{|w| + |f|^2 = 1\} \cap (X_0 \times \mathbb{C} \setminus X_0 \times \{0\}) \rightarrow S^1, \quad (w, f) \mapsto \frac{f}{|f|}.$$

(We write the sphere $\{|w| + |f|^2 = 1\}$ here because the radial distance on $X_0 \times \mathbb{C}$ is uniformly equivalent to $\sqrt{|w| + |f|^2}$.) The fibre is homeomorphic to $X_0 \simeq \mathbb{C}^2/\mathbb{Z}_2$. The effect of the metric geometry is to collapse the vanishing cycle to a point, thereby changing X_1 to X_0 .

More explicitly, in our metric g , the vanishing cycle $\{H = |f|\}$ on X_f for large $|f|$ has diameter proportional to $|f|^{1/4}$, but the distance of the vanishing cycle to the origin is of order $|f|$. The tangent cone only sees the regions whose diameter has at least linear growth rate, so the vanishing cycle disappears in the limit. This phenomenon also appears in the announced work of Hein and Naber on A_k type singularities.

The link $S(\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C})$ of the tangent cone $X_0 \times \mathbb{C} \simeq \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ is singular along a circle S^1 . The appearance of the circle is explained by the $U(1)$ symmetry of the whole setup. Metrically, the points on the circle correspond to the Gromov Hausdorff limit of the vanishing cycles.

It is also interesting to make some comparisons of our situation with Joyce's QALE spaces (cf. [31], and the subsequent generalisation into QAC spaces [5]). The main common feature is that the tangent cone at infinity is a finite

group quotient $\mathbb{C}^n/G = \mathbb{C}^3/\mathbb{Z}_2$, where the main complication comes from the non-free action of $\mathbb{Z}_2$. However, our setup does not seem to fit naturally into Joyce's framework, not even at the topological level. Joyce requires the ambient manifold to be a resolution of the quotient $\mathbb{C}^n/G$, which is not a natural structure on $\mathbb{C}^3$. Joyce understands $\mathbb{C}^n/G$ in terms of a stratification by fixed point sets of subgroups of G , and describes the structure of his resolution in a stratified manner in terms of what he calls the local product resolution; the simplest local picture looks like

$$Y \times \mathbb{C}^m \rightarrow \mathbb{C}^{n-m}/H \times \mathbb{C}^m,$$

where Y is a resolution of $\mathbb{C}^{n-m}/H$ and H is a subgroup of G . Here Y plays a similar role as X_1 . His framework does not typically incorporate the Picard-Lefschetz monodromy behaviour, as his product structures are usually globally trivial over stratified pieces. At the metric level, on the local piece $Y \times \mathbb{C}$, Joyce requires asymptotic convergence of the metric to the product metric (this type of phenomenon also happens for QAC spaces [5]). In our case the vanishing cycles do not have constant size, but instead grow slowly as $O(a^{1/4})$ with the distance to the origin, so the metric in E_ϵ cannot converge to a product metric.

3.3.3 The Ricci curvature

Having examined the leading order behaviour of the metric, let us compute the Ricci tensor for $H \gg 1$. The formula for the Ricci form is

$$\text{Ric} = -\sqrt{-1}\partial\bar{\partial}\log\det(g_{i\bar{j}}).$$

Suppose we write $v = \det(g_{i\bar{j}})$ for the volume density function. Then

$$\text{Ric} = \frac{\sqrt{-1}\partial v \wedge \bar{\partial} v}{v^2} - \frac{\sqrt{-1}\partial\bar{\partial} v}{v}.$$

The nature of our ansatz means v depends only on H, η . We have

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial} v &= v_H \sqrt{-1}\partial\bar{\partial} H + v_{HH} \sqrt{-1}\partial H \wedge \bar{\partial} H + \sqrt{-1}v_{H\eta}(\bar{f}df \wedge \bar{\partial} H + f\partial H \wedge d\bar{f}) \\ &\quad + (\eta v_{\eta\eta} + v_\eta)\sqrt{-1}df d\bar{f}. \end{aligned}$$

$$\sqrt{-1}\partial v \wedge \bar{\partial} v = v_H^2 \sqrt{-1}\partial H \wedge \bar{\partial} H + \sqrt{-1}v_H v_\eta(\partial H \wedge f d\bar{f} + \bar{f} df \wedge \bar{\partial} H) + v_\eta^2 \sqrt{-1}df \wedge d\bar{f}.$$

The asymptotic expression for v is given by (3.7). To leading order, v is a constant. The worst term in (3.7) is $\frac{1}{H^{1/2}a}$. It is clear by computation that the terms showing up in the quadratic expression $\sqrt{-1}\partial v \wedge \bar{\partial} v$ are much smaller than the terms in the second derivative $\sqrt{-1}\partial\bar{\partial} v$, reflecting the fact that the approximation is good enough for nonlinear effect to be insignificant in the asymptotic regime. Comparing the computation with the asymptotic metric (3.8), we see $-\frac{C}{aH}\omega \leq \text{Ric} \leq \frac{C}{aH}\omega$. Thus

$$|\text{Ricci tensor}| \leq \frac{C}{aH}g.$$

Recall the distance to the origin is comparable to a . In the metric g , a typical point has $H \sim a^4$, so the Ricci tensor decays as $O(\frac{1}{a^5})$, which is quite fast. But

in the worst region E_ϵ , H is comparable to a , so this ansatz only gives *quadratic decay of the Ricci curvature*.

The last observation relates to the work of Hein (*cf.* Section 1.3.2).

Proposition 3.3.12. The metric ansatz (3.5) satisfies the condition $SOB(6)$ (*cf.* Definition 1.3.4).

Proof. The main point is that we know the maximal volume growth and the at worst quadratic Ricci curvature decay. The completeness property is clear from the asymptotic formula.

The connectedness property in our definition of $SOB(6)$ is implied by the connectedness of the link of the tangent cone at infinity. $\square$

3.3.4 Regularity scale and weighted Hölder spaces

Hein's package (*cf.* Section 1.3.2) also requires the existence of $C^{k,\alpha}$ -quasi atlas.

Lemma 3.3.13. *Our metric ansatz (3.5) has a $C^{k,\alpha}$ quasi-atlas for any k .*

Proof. It is enough to consider $H \gg 1$. For any such point $x = (x_1, x_2, x_3) \in \mathbb{C}^3$, without loss of generality $|x_1| = \max |x_i|$. Then we take the local coordinates $z'_1 = f - f(x)$, $z'_2 = \frac{1}{H(x)^{1/4}}(z_2 - x_2)$, $z'_3 = \frac{1}{H(x)^{1/4}}(z_3 - x_3)$. By the asymptotic metric formula (3.8), in a coordinate ball of radius $O(H^{1/4})$ the metric g is uniformly equivalent to the standard Euclidean metric, and in fact with higher order regularity. $\square$

The weighted Hölder spaces are in general designed to capture two key features, *locality* and *regularity scale*. The weight function w is specified up to uniform equivalence by

$$w \sim \begin{cases} 1 & H \lesssim 1, \\ \frac{H^{1/4}}{a} & H \gtrsim 1, \end{cases}$$

We define the double weighted Hölder space $C_{\delta,\tau}^{k,\alpha}(\mathbb{C}^3)$ with ambient metric g . The Hölder seminorm of a tensor T is given by

$$[T]_{0,\alpha} = \sup a^\alpha w^\alpha \sup_{d(z,z') \lesssim H^{1/4}} \frac{|T(z) - T(z')|}{d(z,z')^\alpha}.$$

Here $d(z, z') \lesssim H^{1/4}$ requires the local region to have bounded geometry, so we can compare $T(z)$ with $T(z')$ using parallel transport along a geodesic. The weighted norm of tensors are then defined by

$$\|T\|_{C_{\delta,\tau}^{k,\alpha}(\mathbb{C}^3)} = \sum_{j=0}^k a^{-\delta+j} w^{-\tau+j} |\nabla^j T|_{L^\infty} + [a^{-\delta+k} w^{-\tau+k} \nabla^k T]_{0,\alpha}.$$

For $H \gtrsim 1$, an estimate in this norm is the Hölder version of $|\nabla^k T| = O(a^\delta w^\tau H^{-k/4})$, compatible with the regularity scale $\sim H^{1/4}$.

3.4 Correcting asymptotic errors: ODE approach

3.4.1 The asymptotic Laplacian

It is important in the study of the Monge-Ampère equation to have a well controlled linear theory of the Laplace operator. We want an asymptotic formula for the Laplacian of locally smooth functions $b(H, \eta)$ depending only on H, η .

Lemma 3.4.1. *For the metric (3.5), the formula for the Laplacian is*

$$\begin{aligned} (\sqrt{-1}\partial\bar{\partial}b)(\sqrt{-1}\partial\bar{\partial}\phi_\infty)^2 &= \widehat{\text{vol}}_E \left\{ \left(\frac{3H+a}{\sqrt{H+a}} + O\left(\frac{1}{a}\right) \right) b_H + \left(\frac{2(H^2-a^2)}{\sqrt{H+a}} + O\left(\frac{H}{a}\right) \right) b_{HH} \right. \\ &\quad \left. + \left(O\left(\frac{\eta}{a}\right) \right) b_{H\eta} + \left(1 + O\left(\frac{1}{aH^{1/2}}\right) \right) (\eta b_{\eta\eta} + b_\eta) \right\}. \end{aligned} \quad (3.14)$$

Proof. We take the first variation of our formula for the volume form (3.4), namely to change ϕ to $\phi + tb$, and differentiate with respect to t at $t = 0$. The LHS becomes $3(\sqrt{-1}\partial\bar{\partial}\phi_\infty)^2 \wedge (\sqrt{-1}\partial\bar{\partial}b)$. The RHS becomes a linear combination of $b_H, b_{HH}, b_{H\eta}, b_{\eta\eta}, b_\eta$. For example, the coefficient of b_{HH} is

$$6\widehat{\text{vol}}_E \{ F_H^2 H + 4(\eta F_{\eta\eta} + F_\eta)(H^2 - \eta)F_H \} = 6\widehat{\text{vol}}_E \left(\frac{(H^2 - a^2)}{\sqrt{H+a}} + O\left(\frac{H}{a}\right) \right),$$

which after cancelling the factor 3 gives a term in (3.14). The other contributions are similar. $\square$

The interesting feature of this expression is its inhomogeneity. Think about the region E_ϵ , where $|f|$ and H are comparable. If we take a homogeneous expression b , such as $(|f| + H)^\alpha$, and recall that H and η are homogeneous in z_i of degree 2 and 4 respectively, then each H differentiation brings down the degree by 2, and each η differentiation brings down the degree by 4. Then purely on the ground of dimensional analysis, only the terms with b_H and b_{HH} dominate. Then the insight from this formula is that if we just want to invert the Laplacian approximately in the region $E_\epsilon \cap \{H \gg 1\}$, then we only need to solve a **second order ODE** rather than a PDE. The suppression of the η derivative can be interpreted as saying the fibres behave as if they were independent in the region E_ϵ . This phenomenon is typical in adiabatic limit problems, where the metric ansatz ultimately comes from.

Lemma 3.4.2. *The function a is comparable to the distance to the origin for $H > 1$, and globally satisfies the estimate*

$$|da| + a|\sqrt{-1}\partial\bar{\partial}a| \leq C.$$

Proof. The gradient bound is easy. To see the bound on $\partial\bar{\partial}a$, we notice that a is plurisubharmonic, so it suffices to prove $a|\Delta a| \leq C$. This follows from applying formula (3.14) and the partial derivative computations leading to (3.6). $\square$

3.4.2 Approximately inverting the Laplacian

The problem with applying Hein's result to construct a genuine Calabi-Yau metric is that the error terms to the volume form (cf. (3.6)) have slower than quadratic decay. More precisely, in the region where H is comparable to a^4 , all error terms are of order at most $O(\frac{1}{a^3})$, so the approximation is already good enough. But in the region E_ϵ , the worst error term has order $O(\frac{1}{aH^{1/2}})$. Notice also its square is of order $O(\frac{1}{a^2H}) = O(\frac{1}{a^3})$ and in particular decays faster than quadratically, which indicates that nonlinear effects are asymptotically unimportant.

We shall attempt to remove the leading errors by explicitly inverting the Laplacian up to an admissible amount of error. The result is

Lemma 3.4.3. *There exists a function $b = \frac{1}{a}\tilde{b}(\frac{H}{a})$ defined for $H \gg 1$, where $\tilde{b}$ is smooth with growth control*

$$\tilde{b}(x) = O(\log x), \quad \frac{d^k \tilde{b}}{dx^k} = O(x^{-k}), \forall k \geq 1,$$

such that

$$\begin{aligned} (\sqrt{-1}\partial\bar{\partial}b) \wedge (\sqrt{-1}\partial\bar{\partial}\phi_\infty)^2 &= \frac{1}{2}\widetilde{vol_E} \left\{ \frac{1}{4a(H+a)^{1/2}} + \frac{H^{1/2}}{4a(H+a)} - \frac{H^{1/2}}{(H+a)^2} \right. \\ &\quad \left. + \frac{3}{4(H+a)\sqrt{H}} - \frac{H-a}{4aH^{3/2}} + O\left(\frac{\log(2H/a)}{a^3}\right) \right\}. \end{aligned}$$

Lemma 3.4.4. *There exists a function $\mathfrak{b} = \frac{1}{a^2}\tilde{\mathfrak{b}}(\frac{H}{a})$ defined for $H \gg 1$, where $\tilde{\mathfrak{b}}$ is smooth with growth control*

$$\tilde{\mathfrak{b}}(x) = O(x^{-1/2}), \quad \frac{d^k \tilde{\mathfrak{b}}}{dx^k} = O(x^{-k}), \forall k \geq 1,$$

such that

$$(\sqrt{-1}\partial\bar{\partial}\mathfrak{b}) \wedge (\sqrt{-1}\partial\bar{\partial}\phi_\infty)^2 = \frac{1}{2}\widetilde{vol_E} \left\{ \frac{2H^3 - 5aH^2 - 6a^2H - 3a^3}{8(H+a)^2H^{5/2}a} + O\left(\frac{1}{a^4}\right) \right\}.$$

We begin with examples of treating individual terms before describing the general algorithm.

Example 3.4.5. We treat the error term $\frac{1}{4a(H+a)^{1/2}}$ in detail. The leading order terms for the Laplacian of a function $b(H, \eta)$ is

$$(\sqrt{-1}\partial\bar{\partial}b)(\sqrt{-1}\partial\bar{\partial}\phi_\infty)^2 \sim \widetilde{vol_E} \left\{ \frac{3H+a}{\sqrt{H+a}}b_H + \frac{2(H^2-a^2)}{\sqrt{H+a}}b_{HH} \right\}$$

As an intuitive guideline, although a and H are not independent variables, in the region E_ϵ they behave as if they are, because a is $\eta^{1/2}$ up to lower order corrections. Now we solve a related ODE with variable x :

$$\frac{3x+y}{\sqrt{x+y}}h' + \frac{2(x^2-y^2)}{\sqrt{x+y}}h'' = \frac{1}{4y(x+y)^{1/2}}.$$

We can rewrite the ODE as

$$\frac{d}{dx}\{2(x-y)\sqrt{x+y}\frac{dh}{dx}\} = \frac{1}{4y(x+y)^{1/2}}.$$

Thus the ODE is reducible to elementary integration. The first integration gives

$$\frac{dh}{dx} = \frac{1}{4y(x-y)\sqrt{x+y}}(\sqrt{x+y} - \sqrt{2y}),$$

where we choose the normalisation that for $x = y$ the derivative $\frac{dh}{dx}$ remains bounded. This is to ensure the smooth dependence of h on x . Integrate further to get

$$h(x) = \frac{1}{2y} \log\left(\frac{\sqrt{x+y} + \sqrt{2y}}{\sqrt{y}}\right).$$

Here the normalisation is chosen to display homogeneity behaviour.

We then substitute H for x and a for y . This gives an approximate solution

$$b^{(1)}(H, \eta) = \frac{1}{2a} \log\left(\frac{\sqrt{H+a} + \sqrt{2a}}{\sqrt{a}}\right).$$

By construction, the expression

$$\frac{3H+a}{\sqrt{H+a}}b_H^{(1)} + \frac{2(H^2-a^2)}{\sqrt{H+a}}b_{HH}^{(1)} = \frac{1}{4a(H+a)^{1/2}} + O\left(\frac{\log(\frac{2H}{a})}{a^3}\right)$$

The error term $O(\frac{\log(\frac{2H}{a})}{a^3})$ appears because a also depends on H , but the relative error is quite small, so the situation is improved. Now we use the formula (3.14) to see that in fact

$$(\sqrt{-1}\partial\bar{\partial}b^{(1)})(\sqrt{-1}\partial\bar{\partial}\phi_\infty)^2 = \widetilde{vol}_E\left\{\frac{1}{4a(H+a)^{1/2}} + O\left(\frac{\log(2H/a)}{a^3}\right)\right\}.$$

This means we can invert the Laplacian on this error term approximately, as desired.

Remark 3.4.6. *In Euclidean potential theory, if the forcing term decays slower than quadratically near infinity, then in general the solution to the Poisson equation will grow at infinity. One explanation that we obtain a decaying potential in our context, is that our forcing term is $O(\frac{1}{a^3})$ in the region where $H \sim \eta^2$, and the slow decay occurs only in a very small solid angle.*

Example 3.4.7. As a second example, we treat the term $\frac{1}{4H^{3/2}}$. We first write down the associated ODE:

$$\frac{d}{dx}\{2(x-y)\sqrt{x+y}\frac{dh}{dx}\} = \frac{3x+y}{\sqrt{x+y}}h' + \frac{2(x^2-y^2)}{\sqrt{x+y}}h'' = \frac{1}{4x^{3/2}}.$$

The first integration gives

$$\frac{dh}{dx} = \frac{1}{4(x-y)\sqrt{x+y}}\left(\frac{1}{y^{1/2}} - \frac{1}{x^{1/2}}\right),$$

where the normalisation is chosen to make $\frac{dh}{dx}$ bounded at $y = x$. The RHS clearly exhibits homogeneity features, so we can find an integral in the form

$$h(x, y) = \frac{1}{y} \tilde{h}\left(\frac{x}{y}\right),$$

where

$$\frac{d\tilde{h}}{dx} = \frac{1}{4(x-1)\sqrt{x+1}} \left(1 - \frac{1}{x^{1/2}}\right).$$

The detailed expression of the RHS is less important than its qualitative features, namely that it is a smooth expression in x , and its asymptotic growth at large x is $O(\frac{1}{x^{3/2}})$. This implies $\tilde{h}(x) = O(\frac{1}{x^{1/2}})$. We then define

$$b^{(2)}(H, \eta) = \frac{1}{a} \tilde{h}\left(\frac{H}{a}\right),$$

so by construction

$$\frac{3H+a}{\sqrt{H+a}} b_H^{(2)} + \frac{2(H^2-a^2)}{\sqrt{H+a}} b_{HH}^{(2)} = \frac{1}{4H^{3/2}} + O\left(\frac{1}{a^{5/2}H^{1/2}}\right).$$

Using the Laplacian formula (3.14), one can see further that

$$(\sqrt{-1}\partial\bar{\partial}b^{(2)})(\sqrt{-1}\partial\bar{\partial}\phi_\infty)^2 = \widehat{vol_E}\left\{\frac{1}{4H^{3/2}} + O\left(\frac{1}{a^{5/2}H^{1/2}}\right)\right\}.$$

This means the singular term can be approximately cancelled in this example.

The features of these examples mostly carry over to the other error terms in (3.6) with minor changes. We can write down with obvious modifications an ODE for each error term. Explicit integrals in most other cases are more difficult to compute. However, by the homogeneity of the ODE, in the case of Lemma 3.4.3 we can always write

$$h(x, y) = \frac{1}{y} \tilde{h}\left(\frac{x}{y}\right),$$

where $\tilde{h}$ solves the ODE with parameter $y = 1$. The algorithm is that we perform the first integral with the requirement that $\frac{d\tilde{h}}{dx}$ stays bounded near $x = 1$, integrate to find h , substitute H for x and a for y . It is clear that in each case the result is a smooth function $b(H, \eta)$ for $H \gg 1$.

It remains to control the behaviour of the function $\tilde{h}(x)$ for large x , in order to estimate all the errors. For the terms

$$\frac{1}{4a(H+a)^{1/2}}, \quad \frac{H^{1/2}}{4a(H+a)}, \quad -\frac{1}{4aH^{1/2}},$$

the relevant $\tilde{h}$ has leading order growth (up to constant factors) $\tilde{h} \sim \log x$, $\frac{d\tilde{h}}{dx} \sim x^{-1}$. For the terms

$$-\frac{H^{1/2}}{(H+a)^2}, \quad \frac{3}{4(H+a)\sqrt{H}}, \quad \frac{1}{4H^{3/2}},$$

$\tilde{h}$ has leading order growth (up to constant factors) $\tilde{h} \sim x^{-1/2}$, $\frac{d\tilde{h}}{dx} \sim x^{-3/2}$.

It follows that in the first three cases, we can cancel the error terms up to $O(\frac{\log(H/a)}{a^3})$, and in the last three cases, we can cancel the error terms up to $O(\frac{1}{H^{1/2}a^{5/2}})$. In summary, we have the lemma 3.4.3.

The situation of Lemma 3.4.4 is analogous. The main modification is that the homogeneity behaviour is slightly different, so we should instead write

$$h(x, y) = \frac{1}{y^2} \tilde{h}\left(\frac{x}{y}\right).$$

Remark 3.4.8. *It should be pointed out that this ODE method improves the error only in the bad region with $\eta \gg H^{1/2}$, i.e. the region collapsing to the singularity of the tangent cone. It is typical in adiabatic problems involving a local fibration structure, that one can improve an approximate solution by solving a related equation along the fibres. This is indeed what happens here.*

In the region where η is comparable to $H^{1/2}$, the Laplacian genuinely depends on both H and η derivatives, so ODE reduction is not possible in general, but the error there is already acceptably small.

Remark 3.4.9. *The form $\sqrt{-1}\partial\bar{\partial}(b + \mathbf{b})$ compared to $\sqrt{-1}\partial\bar{\partial}\phi_\infty$ is of order $O(\frac{1}{aH^{1/2}}) + O(\frac{\log(2H/a)}{a^3})$, so its square has order $O(\frac{1}{a^3})$. This again confirms the fact that the nonlinear effect is asymptotically weak.*

3.5 Alternative approach to invert Laplacian

We now digress to sketch another approach to invert the Laplacian, which is a variant of Székelyhidi's work [39] generalising our construction. The general gist is that if a metric is built out of a possibly infinite collection of local models each having a well controlled Green operator, and if these local models do not overlap too much, then one can produce a **parametrix** for the global Green operator by **decomposition and patching**. The method is very flexible, well adapted to multiscaled problems, and often yields sharp mapping properties. This will be useful in Chapter 4.

Proposition 3.5.1. Let $-2 < \tau < 0$ and let $-4 < \delta < 0$. There exists an operator $P_{\mathbb{C}^3} : C_{\delta-2, \tau-2}^{k, \alpha}(\mathbb{C}^3) \rightarrow C_{\delta, \tau}^{k+2, \alpha}(\mathbb{C}^3)$ with $\|P_{\mathbb{C}^3}\mathbf{f}\|_{C_{\delta, \tau}^{k+2, \alpha}} \leq C \|\mathbf{f}\|_{C_{\delta-2, \tau-2}^{k, \alpha}}$, such that $\|\Delta P_{\mathbb{C}^3}\mathbf{f} - \mathbf{f}\|_{C_{\delta-2, \tau-2}^{k, \alpha}} \ll \|\mathbf{f}\|$.

Remark 3.5.2. *The Proposition applies to any metric with the same asymptotic behaviour at infinity, and a Banach iteration argument actually allows us to produce a bounded right inverse to Δ_g .*

The rest of this Section is a sketch proof of Proposition 3.5.1.

The geometry on $(\mathbb{C}^3, g)$ is made up of 3 types of basic building blocks: flat Euclidean $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C} = X_0 \times \mathbb{C}_f$ in the generic asymptotic region, scaled copies of $EH_1 \times \mathbb{C} = X_1 \times \mathbb{C}$ near the vanishing cycles, and a compact region. We need mapping properties of the Laplacian on the building blocks. The weighted Hölder norms $\|T\|_{C_{\delta, \tau}^{k, \alpha}}$ on $X_0 \times \mathbb{C}_f$ is designed to agree up to uniform equivalence

with the norms on $(\mathbb{C}^3, g)$ (cf. Section 3.3.4) over the region $\{H \gtrsim 1\} \cap (\mathbb{C}^3 \setminus E_\epsilon)$, and an estimate in this norm $\|T\| \lesssim 1$ is the Hölder version of

$$|\nabla^k T| = O((H^{1/4} + |f|)^{\delta-\tau} H^{(\tau-k)/4}),$$

where $H^{1/4}$ is uniformly equivalent to the distance to the nodal line, and $H^{1/4} + |f|$ is uniformly equivalent to the distance to the origin in $X_0 \times \mathbb{C}$. Similarly the weighted Hölder norms $\|T\|_{C_\tau^{k,\alpha}}$ on $X_1 \times \mathbb{C}$ are designed such that $\|T\| \lesssim 1$ is the Hölder version of

$$|\nabla^k T| = O(w^{\tau-k}),$$

where the Eguchi-Hanson factor X_1 inherits the functions w , which is a smoothed version of the distance to the vanishing cycle.

Lemma 3.5.3. *Given $-2 < \tau < 0$ and $-4 < \delta < 0$, then $\Delta_{X_0 \times \mathbb{C}} : C_{\delta, \tau}^{k+2, \alpha} \rightarrow C_{\delta-2, \tau-2}^{k, \alpha}$ has a bounded inverse.*

One can start from the weighted Schauder estimate

$$\|u\|_{C_{\delta, \tau}^{k+2, \alpha}} \leq C \|\Delta u\|_{C_{\delta-2, \tau-2}^{k, \alpha}} + C \sup (H^{1/4} + |f|)^{-\delta+\tau} H^{-\tau/4} |u|,$$

and then show by a blow up argument

$$\sup (H^{1/4} + |f|)^{-\delta+\tau} H^{-\tau/4} |u| \leq C \|\Delta u\|_{C_{\delta-2, \tau-2}^{k, \alpha}}.$$

The choice of weight ensures local L^1 integrability on $C_{\delta-2, \tau-2}^{k, \alpha}$ functions, and rules out harmonic functions in $C_{\delta, \tau}^{k+2, \alpha}$. Alternative methods include using the Green representation, or viewing $X_0 \times \mathbb{C}$ as a cone with singular link and perform spectral decomposition to functions on the link. A more general statement is Proposition 8 in [39].

Lemma 3.5.4. *Given $-2 < \tau < 0$, then $\Delta_{X_1 \times \mathbb{C}} : C_\tau^{k+2, \alpha} \rightarrow C_{\tau-2}^{k, \alpha}$ has a bounded inverse.*

This can be proved by Fourier transform in the $\mathbb{C}$ direction. A more general statement is Proposition 9 in [39]. We comment that Hein's package on the Poisson equation (cf. Section 1.3.2) also applies to this setup. In particular, if the source function $\mathfrak{f}$ is supported in a unit ball inside $X_1 \times \mathbb{C}$, then $\Delta_{X_1 \times \mathbb{C}}^{-1} u$ has improved decay $O(\text{dist}(\cdot, \text{supp}(\mathfrak{f}))^{-4+\epsilon})$, and elliptic bootstrap will give decay estimates on higher derivatives.

Lemma 3.5.5. *Given $-4 < \delta < 0$ and $\tau < 0$, and a function $\mathfrak{f}$ compactly supported in $\{H \lesssim 1\} \subset (\mathbb{C}^3, g)$ with bound $\|\mathfrak{f}\|_{C_{\delta-2, \tau-2}^{k, \alpha}} \leq 1$, then there is a function u on $\mathbb{C}^3$ with $\Delta_g u = \mathfrak{f}$ and $\|u\|_{C_{\delta, \tau}^{k+2, \alpha}} \leq C$.*

Here we apply Hein's package on the Poisson equation (cf. Section 1.3.2) to find a solution $\Delta_g u = \mathfrak{f}$ with decay estimate $|u| = O(a^{-4+\epsilon})$ for any fixed $0 < \epsilon \ll 1$, so in particular $|u| \leq C a^\delta w^\tau$. The higher order estimates come from elliptic bootstrap.

Now we describe an algorithm for producing the parametrix $P_{\mathbb{C}^3}$. Let $\mathfrak{f}$ be a given function on $(\mathbb{C}^3, g)$ with $\|\mathfrak{f}\|_{C_{\delta-2, \tau-2}^{k, \alpha}} = 1$.

- We find an infinite collection of points $P_i \in E_\epsilon \cap \{H \gtrsim 1\}$, such that the balls $B_g(P_i, |f(P_i)|^{1/4})$ cover $E_\epsilon \cap \{H \gtrsim 1\}$, but at any point at most C balls intersect. Using a partition of unity, we can **decompose**

$$f = f_{X_0 \times \mathbb{C}} + f_{H \lesssim 1} + \sum_i f_i,$$

where $f_{X_0 \times \mathbb{C}}$, $f_{H \lesssim 1}$ and f_i are supported respectively in $(\mathbb{C}^3 \setminus E_\epsilon) \cap \{H \gtrsim 1\}$, $\{H \lesssim 1\}$ and $B_g(P_i, |f(P_i)|^{1/4})$. We can demand that any summand has $C_{\delta, \tau}^{k, \alpha}(\mathbb{C}^3)$ -norm $\lesssim 1$.

- Via the local diffeomorphism Ψ we may regard $f_{X_0 \times \mathbb{C}}$ as a function on $X_0 \times \mathbb{C}$, supported away from the nodal line. Applying Lemma 3.5.3 we can produce a function $u'_{X_0 \times \mathbb{C}} = \Delta_{X_0 \times \mathbb{C}}^{-1} f_{X_0 \times \mathbb{C}}$. After cutting off $u'_{X_0 \times \mathbb{C}}$ at the scale $|f| \sim H$, we obtain a function $u_{X_0 \times \mathbb{C}}$, which can be regarded as a function on $\mathbb{C}^3$ supported away from the vanishing cycles. The compatibility of norms imply $\|u\|_{C_{\delta, \tau}^{k+2, \alpha}(\mathbb{C}^3)} \leq C$. Moreover, because the metric deviation between g and $g_{X_0 \times \mathbb{C}}$ is of order $O(\epsilon)$ on $(\mathbb{C}^3 \cap E_\epsilon) \cap \{H \gg 1\}$,

$$\|\Delta_g u_{X_0 \times \mathbb{C}} - f_{X_0 \times \mathbb{C}}\|_{C_{\delta-2, \tau-2}^{k, \alpha}((\mathbb{C}^3 \cap E_\epsilon) \cap \{H \gg 1\})} \lesssim \epsilon.$$

Outside this region the cutoff procedure introduces some new errors, which can be absorbed into f_i and $f_{H \lesssim 1}$. The net effect is that we may assume $f_{X_0 \times \mathbb{C}} = 0$.

- Via the diffeomorphism Ψ_{P_i} we may regard f_i as a function on $X_{f(P_i)} \times \mathbb{C}$ supported near the vanishing cycle. After metric scaling this geometry is modelled on $EH_1 \times \mathbb{C}$. After applying Lemma 3.5.4 we find a function u'_i solving the model Poisson equation. After cutting off u'_i outside distance $\epsilon|f(P_i)|$ from P_i , we obtain a function u_i , which can be regarded as a compactly function on $\mathbb{C}^3$ compactly supported in $B_g(P_i, \epsilon|f(P_i)|)$. Using Lemma 3.5.4 and ensuing remarks, one can extract *decay estimates* on u_i inside $B_g(P_i, \epsilon|f(P_i)|)$:

$$|\nabla^j u_i| \leq C a^{\delta-3\tau/4} H^{-j/4} (d_g(\cdot, P_i) a^{-1/4} + 1)^{-4+\epsilon'}, \quad j \leq k+2,$$

and there is a Hölder version. This essentially means that within the ball $B_g(P_i, |f(P_i)|^{1/4})$, then $|\nabla^2 u_i| \lesssim C a^{\delta-2} w^{\tau-2}$, and outside the ball there is almost quartic decay. Another consequence is that on the region of cutoff

$$|\nabla^2 u_i| \leq C a^{\delta-3\tau/4} H^{-1/2} (\epsilon a^{3/4})^{-4+\epsilon'},$$

so $|\Delta_g u_i| \leq C a^{\delta-3\tau/4} H^{-1/2} (\epsilon a^{3/4})^{-4+\epsilon'}$, which has a fast decay factor.

- We need to prevent *cummulative error*, and the key is *local finiteness* and *decay estimates*. Let $u_{EH} = \sum_i u_i$ be the locally finite sum of all local solutions to the Poisson equations coming from these Eguchi-Hanson pieces. We wish to bound u_{EH} . Notice $u_{EH}(Q)$ at a point Q only receives contributions from pieces with $|f(P_i)| \sim a(Q)$. Thus

$$\begin{aligned} |u_{EH}(Q)| &\leq C a^{\delta-3\tau/4} \sum_{|f(P_i)| \sim a} (d_g(\cdot, P_i) a^{-1/4} + 1)^{-4+\epsilon'} \\ &\leq C a^{\delta-3\tau/4} (w a^{3/4})^{\epsilon'-2} \leq C a^\delta w^\tau, \end{aligned}$$

where we used $\tau > -2 + \epsilon'$ in the last inequality. The higher order version says $\|u_{EH}\|_{C_{\delta,\tau}^{k+2,\alpha}(\mathbb{C}^3)} \leq C$.

- Next we examine the error $\Delta_g u_{EH} - \sum_i \mathfrak{f}_i$. This arises from two reasons: the *metric deviation* of g from the model $X_{f(P_i)} \times \mathbb{C}$, and the *cutoff error*. The first is controlled by $\epsilon \|\nabla_g^2 u_{EH}\|_{C_{\delta-2,\tau-2}^{k,\alpha}} \ll 1$ on $\{H \geq A\}$ for some $A \gg 1$ depending on ϵ . The cutoff error coming from one u_i is bounded by $Ca^{\delta-3\tau/4} H^{-1/2} (\epsilon a^{3/4})^{-4+\epsilon'}$, and at any point Q there are $O(a^{3/2})$ copies of u_i which contribute, so the cutoff error is bounded on $\{H \geq A\}$ by

$$Ca^{\delta-3\tau/4} H^{-1/2} (\epsilon a^{3/4})^{-2+\epsilon'} \ll Ca^\delta H^{-1/2} \leq Ca^\delta w^\tau H^{-1/2} \sim Ca^{\delta-2} w^{\tau-2}.$$

To summarize, $|\Delta_g u_{EH} - \sum_i \mathfrak{f}_i| \ll a^\delta w^\tau H^{-1/2}$ in $\{H \geq A \gg 1\}$. The higher order version is $\|\Delta_g u_{EH} - \sum_i \mathfrak{f}_i\|_{C_{\delta-2,\tau-2}^{k,\alpha}(H \geq A)} \ll 1$. Inside the compact region $\{H \lesssim A\}$ some new source function is generated by the algorithm, and we absorb such terms into $\mathfrak{f}_{H \lesssim 1}$.

- At this stage we can effectively assume the source function is supported in the compact region $\{H \lesssim A\}$. Then one appeals to Lemma 3.5.5 to find $u_{H \lesssim 1}$ solving the Poisson equation $\Delta_g u_{H \lesssim 1} = \mathfrak{f}_{H \lesssim 1}$ with weighted Hölder bounds.
- We now **patch** $u = u_{X_0 \times \mathbb{C}} + u_{EH} + u_{H \lesssim 1}$, which satisfies $\|u\|_{C_{\delta,\tau}^{k+2,\alpha}(\mathbb{C}^3)} \leq C$ and $\|\Delta_g u - \mathfrak{f}\|_{C_{\delta-2,\tau-2}^{k,\alpha}(\mathbb{C}^3)} \ll 1$ as required.

3.6 Perturbation into the Calabi-Yau metric

We improve the metric ansatz to allow for the application of Hein's existence result reviewed in Section 1.3.2. For this we define a metric

$$\omega' = \sqrt{-1} \partial \bar{\partial} (\phi_\infty - (b + \mathfrak{b})\chi). \quad (3.15)$$

where b is defined in Lemma 3.4.3 and $\mathfrak{b}$ in Lemma 3.4.4, $0 \leq \chi \leq 1$ is a cutoff function, vanishes in the ball $\{H \leq R^2\}$, and equals to 1 outside $\{H \leq 4R^2\}$. If the parameter R is chosen large enough, we can ensure ω' is positive definite.

For $H \gg 1$, the volume form of this modified ansatz is

$$\begin{aligned} \omega'^3 &= (\sqrt{-1} \partial \bar{\partial} \phi_\infty)^3 - 3(\sqrt{-1} \partial \bar{\partial} \phi_\infty)^2 \wedge \sqrt{-1} \partial \bar{\partial} (b + \mathfrak{b}) + O\left(\frac{1}{a^3}\right) \\ &= \frac{3}{2} \widetilde{\text{vol}_E} \left\{ 1 + O\left(\frac{\log(2H/a)}{a^3}\right) \right\}. \end{aligned}$$

Crudely speaking, the error is now at most $O(\frac{1}{a^{3-\epsilon}})$ for any small $\epsilon > 0$.

Theorem 3.6.1. *There is a smooth $SO(3, \mathbb{R}) \times U(1)$ -invariant relative Kähler potential $\phi'_{\mathbb{C}^3}$ on $\mathbb{C}^3$ solving the Monge-Ampère equation*

$$\omega_{\mathbb{C}^3} = \omega + \sqrt{-1} \partial \bar{\partial} \phi'_{\mathbb{C}^3}, \quad \omega_{\mathbb{C}^3}^3 = \frac{3}{2} \widetilde{\text{vol}_E}, \quad \|\phi'_{\mathbb{C}^3}\|_{C_{-1+\epsilon,0}^{k,\alpha}} \leq C(k, \alpha, \epsilon). \quad (3.16)$$

In particular the metric $\omega_{\mathbb{C}^3}$ is Calabi-Yau, and is uniformly equivalent to ω .

Proof. Since ω' deviates from $\sqrt{-1}\partial\bar{\partial}\phi_\infty$ by only a small smooth perturbation of order $O(\frac{1}{aH^{1/2}}) + O(\frac{\log(2H/a)}{a^3})$, the condition $SOB(6)$ and the existence of $C^{k,\alpha}$ quasi-atlas still hold for ω' (cf. Definition 1.3.4, Section 3.3.3). So Hein's theorem gives a smooth Kähler potential ψ solving

$$\omega_{\mathbb{C}^3}^3 = (\omega' + \sqrt{-1}\partial\bar{\partial}\psi)^3 = \frac{3}{2}\overline{\text{vol}_E}, \quad |\psi| \leq Ca^{-1+\epsilon}.$$

Since our setup is symmetric under the action of $SO(3, \mathbb{R}) \times U(1)$, and Hein's method works equivariantly with respect to the group action, we can assume the solution is symmetric.

We may write $\phi'_{\mathbb{C}^3} = \psi - (b + \mathfrak{b})\chi$, so

$$\omega_{\mathbb{C}^3} = \omega' + \sqrt{-1}\partial\bar{\partial}\psi = \omega + \sqrt{-1}\partial\bar{\partial}\phi'_{\mathbb{C}^3}, \quad |\phi'_{\mathbb{C}^3}| \leq Ca^{-1+\epsilon}.$$

By elliptic bootstrap

$$|\nabla_g^k \phi'_{\mathbb{C}^3}| \leq C(k)a^{-1+\epsilon}(H+1)^{-k/4} \sim C(k)a^{-1+\epsilon-k}w^{-k},$$

hence the decay estimate $\|\phi'_{\mathbb{C}^3}\|_{C_{-1+\epsilon,0}^{k,\alpha}} \leq C$. $\square$

Remark 3.6.2. *We think the decay rate estimate is not far from optimal because our correction term to the potential is $b = O(\frac{\log(2H/a)}{a})$.*

As a consequence of the decay estimate, the leading order asymptote of ω agrees with that of $\omega_{\mathbb{C}^3}$, hence

Corollary 3.6.3. The asymptotic properties of ω , including maximal volume growth and the tangent cone at infinity, are also true for $\omega_{\mathbb{C}^3}$.

Remark 3.6.4. *This CY metric is a counterexample to a previous conjecture, which says any complete CY metric on $\mathbb{C}^n$ with maximal volume growth is flat (cf. [41], Remark 5.3).*

Proposition 3.6.5. (Uniqueness in the asymptotic class) If a relative Kähler potential u satisfies $(\omega_{\mathbb{C}^3} + \sqrt{-1}\partial\bar{\partial}u)^3 = \omega_{\mathbb{C}^3}^3$, and $\|u\|_{C_{-1+\epsilon,0}^{2,\alpha}} \leq C$, then $u = 0$.

Proof. Integration by parts argument as in Yau's proof gives

$$\int_{\mathbb{C}^3} |\nabla|u|^p|^2 \omega_{\mathbb{C}^3}^3 = 0, \quad p \gg 1.$$

where decay properties of u at infinity enables us to drop boundary terms. $\square$

3.7 Remarks on moduli questions

3.7.1 Moduli questions and possible generalisations

Our Calabi-Yau metric $\omega_{\mathbb{C}^3}$ is a nontrivial Ricci flat metric on $\mathbb{C}^3$. This is interesting in relation to the following fundamental question: on a given noncompact

complex manifold with a global holomorphic volume form, can we classify the complete Calabi-Yau metrics? What if we impose maximal volume growth condition? How about the case of $\mathbb{C}^n$?

Székeleyhidi [39] and Conlon-Rochon [3] generalised our construction to an infinite class of examples using more complicated fibrations on $\mathbb{C}^n$, starting from some ansatz which desingularises a semi-Ricci flat metric. There the Ricci-flat metrics on singular central fibres come from more abstract existence results in Sasaki geometry, and one needs to invest more in the linear theory.

This question is also very relevant for understanding what happens near the (isolated) critical points of a very collapsed Calabi-Yau manifold fibred over some Riemann surface, with klt singular fibres. The expectation is that there should be a complete Calabi-Yau metric on a model space fibred over $\mathbb{C}$, possibly biholomorphic to $\mathbb{C}^n$, whose asymptotic behaviour is a semi-Ricci flat metric, such that after appropriate scaling it describes the collapsing metric near the critical points on the total space of the fibration. (The klt condition is imposed for the fibrewise Calabi-Yau metric to make sense; if we drop this condition, a prototype example is the Ooguri-Vafa metric, which is an incomplete metric modelling the collapsing elliptically fibred K3 surface near a singular fibre [19], and whose local behaviour near the node is modelled on the Taub-NUT metric which does not have maximal volume growth.) A description of the tangent cone at infinity is likely to involve stability issues, as understood for A_k -type singularities by Hein and Naber.

On the negative side, any meaningful classification program must deal with singular tangent cones, which can be a major source of difficulty.

3.7.2 Deformation and uniqueness questions

We can also ask the weaker question of classifying Calabi-Yau metrics arising as deformations of a given one, which I believe is more tractable. This depends on the class of deformations, which in turn depends on the function spaces in which one sets up the Monge-Ampère equation. Let us formulate a number of more concrete questions:

Question 3.7.1. Can we classify all Calabi-Yau metrics with the given holomorphic volume form which are uniformly equivalent to $\omega_{\mathbb{C}^3}$?

Instead of giving a complete answer, we remark that $SL(3, \mathbb{C})$ gives automorphisms of $\mathbb{C}^3$ as a complex manifold. But the uniform equivalence forces a compatibility condition with the fibration structure, so reduces this symmetry to $SO(3, \mathbb{C}) \times \langle e^{2\pi i/3} \rangle$. Clearly $SO(3, \mathbb{R}) \times \langle e^{2\pi i/3} \rangle$ is a subgroup preserving also the metric structure. So the moduli space contains at least the homogeneous space $SO(3, \mathbb{C})/SO(3, \mathbb{R}) \simeq SL(2, \mathbb{C})/SU(2)$. These examples are different as Calabi-Yau metrics on the fixed complex manifold $\mathbb{C}^3$, but they are isometric as Riemannian manifolds by a linear change of coordinates.

A more subtle deformation of our metric can be obtained by a simultaneous scaling of our metric and the coordinates, such that the volume form is

preserved. Viewed alternatively, if we start from the ansatz

$$\phi = \frac{1}{2\lambda^{2/3}}\eta + \lambda^{1/3}\sqrt{H + (\eta + \lambda\sqrt{H + \lambda^{2/3}})^{1/2}},$$

where $\lambda > 0$, and solve the Monge-Ampère equation as before, we would get a new Calabi-Yau metric with the same volume form.

Question 3.7.2. Given the leading order asymptote for the Kähler form, then is our Calabi-Yau metric unique?

The uniqueness holds assuming sufficient decay of the perturbation term (*cf.* Prop 3.6.5), and the question is about the optimal decay condition we need.

Question 3.7.3. What are the formal deformations of our example?

This involves linearising the Calabi-Yau equation. We obtain formally an equation on a closed $(1,1)$ form β :

$$\omega_{\mathbb{C}^3}^2 \wedge \beta = 0. \tag{3.17}$$

This says β is primitive. On the space $\mathbb{C}^3$, closedness and exactness are the same. It is clear from the Hodge identities that β is harmonic. For example, the nontrivial deformations discussed above provide particular solutions. The answer in general depends on the function space. In particular, we may ask what are the bounded solutions, and how they relate to moduli questions.

Chapter 4

A gluing construction of collapsing metrics

We use the gluing method to give a refined description of the collapsing CY metrics $t^{-1}\tilde{\omega}_t$ on Calabi-Yau 3-folds with a Lefschetz K3 fibration as $t \ll 1$, which has the scaling convention that $\int_{X_y} (t^{-1}\tilde{\omega}_t)^2 = 1$. A particular consequence is

Theorem 4.0.4. (cf. Section 4.4.1) *There exists a 1-parameter family of holomorphic embedding maps F_t from large Euclidean balls in $\mathbb{C}^3$ to a neighbourhood of the node inside X , and a fixed number A_0 depending on the geometry of $\pi : X \rightarrow Y$, such that as $t \rightarrow 0$, the scaled CY metrics $(\frac{2A_0}{t})^{1/3} F_t^*(t^{-1}\tilde{\omega}_t)$ converge in $C_{loc}^0(\mathbb{C}^3)$ to the model CY metric $\omega_{\mathbb{C}^3}$.*

Here the function A_y amounts to the fibrewise period integrals (cf. Section 1.4), and A_0 is its value at $y = 0$. All constants are uniform for sufficiently small t unless stated otherwise.

4.1 CY metrics on smoothings of nodal K3

We now describe the CY metrics on the K3 fibres which are small deformations of any chosen nodal fibre. The basic picture is that these are obtained by gluing scaled versions of the Eguchi-Hanson metric to the CY metric on the nodal K3 fibre. This Section will be brief since there are many gluing constructions of very similar nature in the literature, e.g. [11][37], but we want to give enough details to keep track of the key estimates for later use.

On the nodal central fibre X_0 , we write the orbifold CY metric as $\omega_{SRF}|_{X_0} = \omega_X|_{X_0} + \sqrt{-1}\partial\bar{\partial}\psi_0$, solving the Monge-Ampère equation

$$(\omega_X|_{X_0} + \sqrt{-1}\partial\bar{\partial}\psi_0)^2 = \frac{1}{A_0}\Omega_0 \wedge \bar{\Omega}_0, \quad \int_{X_0} \psi_0 \omega_X^2 = 0, \quad A_0 = A_y|_{y=0}, \Omega_0 = \Omega_y|_{y=0}.$$

At the nodal point P , the fibration π induces (up to scale) a complex symmetric bilinear form on the tangent space $T_P X$, or equivalently an $SO(3, \mathbb{C}) \simeq$

$SL(2, \mathbb{C})/\mathbb{Z}_2$ structure. The orbifold CY metric singles out a Hermitian metric on $\mathbb{C}^2/\mathbb{Z}_2$, or equivalently an $SU(2)/\mathbb{Z}_2 = SO(3, \mathbb{R})$ structure, so we have a preferred Hermitian structure $|\cdot|$ on $T_P X$. We can then choose local coordinates $\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3$ on an open neighbourhood $U_1 \subset X$ and a local coordinate y on Y , where the fibration π is represented by $y = \mathfrak{z}_1^2 + \mathfrak{z}_2^2 + \mathfrak{z}_3^2$, and $\psi_0 = c_0 + r^2 + O(r^4)$ with $r = (|\mathfrak{z}_1|^2 + |\mathfrak{z}_2|^2 + |\mathfrak{z}_3|^2)^{1/4}$. Here the innocuous constant c_0 is a matter of normalisation, which appears because we impose $\int_{X_0} \psi_0 \omega_X^2 = 0$. Using the compatibility condition $dy \wedge \Omega_y = \Omega$ at $y = 0$, one calculates the normalisation on the holomorphic volume form at the nodal point to be

$$\begin{aligned} \sqrt{-1}\Omega \wedge \bar{\Omega}|_{\mathfrak{z}=0} &= \Omega_0 \wedge \bar{\Omega}_0 \wedge \sqrt{-1}dy \wedge d\bar{y}|_{\mathfrak{z}=0} = \sqrt{-1}A_0(\sqrt{-1}\partial\bar{\partial}r^2)^2 \wedge dy \wedge d\bar{y}|_{\mathfrak{z}=0} \\ &= A_0 \prod_i \sqrt{-1}d\mathfrak{z}_i d\bar{\mathfrak{z}}_i, \end{aligned}$$

so up to a constant in $U(1)$, the holomorphic volume form is locally given in U_1 by $\Omega = \sqrt{A_0}d\mathfrak{z}_1 d\mathfrak{z}_2 d\mathfrak{z}_3(1 + O(\mathfrak{z}))$.

We focus on the fibres over the small local base $\{|y| < \epsilon_1\}$ with $\epsilon_1 \ll 1$. For convenience, we extend the function r on $X_y \cap U_1$ smoothly to the whole X_y , such that outside the coordinate neighbourhood r is of order 1. In $X_y \cap U_1$, one has the scaled Eguchi-Hanson metric EH_y given by $\sqrt{-1}\partial\bar{\partial}\sqrt{r^4 + |y|}$. Then the function r can be thought as a smoothed out version of the distance to the vanishing cycle. This allows us to define the weighted Hölder spaces $C_\beta^{k,\alpha}(X_y)$ on X_y . The weighted Hölder norm of a function f on X_y can be defined by

$$\begin{aligned} \|f\|_{C_\beta^{k,\alpha}} &= \|f\|_{C^{k,\alpha}(X_y \setminus \{r < c\}), \omega_X} + \sum_{j \leq k} \sup_{X_y \cap U_1} r^{-\beta+j} |\nabla_{EH_y}^j f| \\ &\quad + \sup_{d_{EH_y}(x,x') \lesssim r(x), x, x' \in X_y \cap U_1} r(x)^{-\beta+\alpha+k} \frac{|\nabla_{EH_y}^k f(x) - \nabla_{EH_y}^k f(x')|}{d_{EH_y}(x, x')^\alpha}, \end{aligned}$$

where the difference of two tensors at nearby points are compared by parallel transport along the unique minimal geodesic joining them. The constant c is meant to be small enough to make $\{r < c\}$ contained in the coordinate neighbourhood U_1 . Similarly, one can define the $C_\beta^k(X_y)$ norm by setting α to zero, and it is easy to extend these definition to tensors. A useful feature of Hölder norms, which will be used repeatedly later, is that they are *local*, namely if the manifold is covered by several regions with some overlap, then it suffices to estimate the Hölder norms on each individual region.

Let $1 \ll \Lambda_1 \ll \epsilon_1^{-1/4}$ be a large number. Then for any y with $|y| < \epsilon_1$, we can find a diffeomorphism $G_{0,y}$ between $X_y \setminus \{r < \Lambda_1|y|^{1/4}\}$ (namely the complement of a neighbourhood of the vanishing cycle) and an open subset of $X_0 \setminus \{r < \frac{1}{2}\Lambda_1|y|^{1/4}\}$. We can demand these diffeomorphisms to depend smoothly on y , namely they fit into a fibration preserving diffeomorphism

$$G_0 : \{x \in X : |y| < \epsilon\} \setminus \{r < \Lambda_1|y|^{1/4}\} \rightarrow U' \subset X_0 \times \{|y| < \epsilon_1\}. \quad (4.1)$$

(This can be defined, for instance, by flowing along the vector fields orthogonal to the fibres under the ω_X metric. Or one can prescribe the diffeomorphism explicitly in the coordinate neighbourhood U_1 and try to extend it outside U_1 , similar to [37]. Many reasonable constructions will satisfy the desired estimates.)

The diffeomorphism $G_{0,y}$ is approximately holomorphic: we can arrange so that on $X_y \setminus \{r < \Lambda_1|y|^{1/4}\}$ where $G_{0,y}$ is defined,

$$|\Omega_y - G_{0,y}^* \Omega_0|_{\omega_X} \leq \frac{C|y|}{|\mathfrak{z}|^2} |\Omega_y|_{\omega_X} = \frac{C|y|}{r^4} |\Omega_y|_{\omega_X}.$$

That is, the variation of complex structure causes an error of order $O(\frac{|y|}{r^4})$. Furthermore, we can compare potentials, holomorphic volume forms and the background metric ω_X on X_0 and X_y to higher order. For instance,

$$\begin{aligned} \|G_{0,y}^* r^2 - r^2\|_{C_{-2}^k((U_1 \cap X_y) \setminus \{r < \Lambda_1|y|^{1/4}\})} &\leq C|y|, \\ \|G_{0,y}^* \Omega_0 - \Omega_y\|_{C_{-4}^k(X_y \setminus \{r < \Lambda_1|y|^{1/4}\})} &\leq C|y|, \\ \|G_{0,y}^*(\omega_X|_{X_0}) - \omega_X|_{X_y}\|_{C_{-2}^k(X_y \setminus \{r < \Lambda_1|y|^{1/4}\})} &\leq C|y|. \end{aligned} \quad (4.2)$$

The various power law behaviours can be seen quite easily from dimensional analysis. It is enough to examine what happens inside the coordinate neighbourhood $U_1 \subset \mathbb{C}^3$. The point is that to the leading order, expressions like r^2 , Ω_y and ω_X have some homogeneity behaviour under the scaling $\mathfrak{z} \rightarrow \lambda \mathfrak{z}$, and the diffeomorphism $G_{0,y}$ would approximately respect this homogeneity, so the problem reduces by scaling to the case with $|\mathfrak{z}| \sim 1, |y| \ll 1$, where estimates of the above type are clear. This type of argument will be tacitly used many times later when we assert good properties about diffeomorphisms.

One can now construct an approximate CY metric ω'_y , essentially by gluing the Eguchi-Hanson metric EH_y to $G_{0,y}^*(\omega_{SRF}|_{X_0})$ at scale $r \sim |y|^{1/6}$. The gluing region is then contained in the coordinate neighbourhood U_1 by $\epsilon_1 \ll 1$, and avoids the vicinity of the vanishing sphere $\{r < \Lambda_1|y|^{1/4}\}$.

Let $\gamma_1(s)$ be a cutoff function,

$$\gamma_1(s) = \begin{cases} 1 & \text{if } s > 2, \\ 0 & \text{if } s < 1. \end{cases}$$

and let $\gamma_2 = 1 - \gamma_1$. We define

$$\omega'_y = \omega_X|_{X_y} + \sqrt{-1} \partial \bar{\partial} \left\{ \gamma_1 \left(\frac{r}{|y|^{1/6}} \right) G_{0,y}^*(\psi_0 - c_0) + \gamma_2 \left(\frac{r}{|y|^{1/6}} \right) \sqrt{r^4 + |y|} \right\}. \quad (4.3)$$

The diffeomorphism is well defined on the support of γ_1 , so this expression makes sense. We remark that there can be many minor variants to the gluing ansatz.

Lemma 4.1.1. *If $\epsilon_1 \ll 1$, then ω'_y is positive definite, namely a Kähler metric, and for $-2 \leq \beta < 0$ satisfies*

$$\omega_y'^2 = \frac{1}{A_y} (1 + f'_y) \Omega_y \wedge \bar{\Omega}_y, \quad \|f'_y\|_{C_{\beta-2}^{k,\alpha}(X_y)} \leq C(k, \alpha) |y|^{-\frac{1}{6}\beta + \frac{2}{3}}.$$

Here the constants are independent of y as long as $|y| < \epsilon_1$.

Remark 4.1.2. *In particular $|f'_y|_{L^\infty} = O(|y|^{\frac{1}{12}(\beta+2)}) \ll 1$, meaning that the nonlinear effect is weak.*

Proof. When $r > 2|y|^{1/6}$, this ω'_y is just $\omega_X|_{X_y} + \sqrt{-1}\partial\bar{\partial}G_{0,y}^*\psi_0$, which we would like to compare to $G_{0,y}^*(\omega_X|_{X_0} + \sqrt{-1}\partial\bar{\partial}\psi_0)$. To control their difference, we examine

$$r^2|\partial\bar{\partial}G_{0,y}^*(\psi_0 - r^2 - c_0) - G_{0,y}^*\partial\bar{\partial}(\psi_0 - r^2 - c_0)|_{EH_y} \leq \frac{C|y|}{r^4} \sum_{j=1}^2 |r^j \nabla^j(\psi_0 - r^2 - c_0)| \leq C|y|.$$

The first inequality uses the general observation that the relative error caused by variation of complex structure is of order $O(\frac{|y|}{r^4})$, and the second uses that $|\nabla^k(\psi - c_0 - r^2)| = O(r^{4-k})$ on the orbifold X_0 . This can be contrasted with

$$r^2|\partial\bar{\partial}G_{0,y}^*r^2 - G_{0,y}^*\partial\bar{\partial}r^2|_{EH_y} \leq \frac{C|y|}{r^2},$$

which is the dominant error term for small r . The higher order estimates proceed in the same fashion, and one eventually gets

$$\left\| \omega_X|_{X_y} + \sqrt{-1}\partial\bar{\partial}G_{0,y}^*\psi_0 - G_{0,y}^*(\omega_X|_{X_0} + \sqrt{-1}\partial\bar{\partial}\psi_0) \right\|_{C_{-4}^k(X_y \cap \{r > 2|y|^{1/6}\})} \leq C|y|.$$

This easily implies the positive definiteness of ω'_y in this region. Moreover,

$$\left\| \omega_y'^2 - G_{0,y}^*\omega_{SRF}|_{X_0}^2 \right\|_{C_{-4}^k(X_y \cap \{r > 2|y|^{1/6}\})} \leq C|y|.$$

But we know

$$\omega_{SRF}|_{X_0}^2 = A_0^{-1}\Omega_0 \wedge \bar{\Omega}_0, \quad |A_0 - A_y| \leq C|y|, \quad \left\| \Omega_y - G_{0,y}^*\Omega_0 \right\|_{C_{-4}^k(X_y \cap \{r > 2|y|^{1/6}\})} \leq C|y|,$$

so we can assemble the facts to see

$$\left\| \omega_y'^2 - A_y^{-1}\Omega_y \wedge \bar{\Omega}_y \right\|_{C_{-4}^k(X_y \cap \{r > 2|y|^{1/6}\})} \leq C|y|. \quad (4.4)$$

Now we analyse the region $\{|y|^{1/6} < r < 2|y|^{1/6}\}$, where the cutoff error is supported. The term $\omega_X|_{X_y}$ is of order $O(r^2)$ small compared to the Eguchi-Hanson metric EH_y . To understand the deviation of ω'_y from EH_y , it suffices to examine $\sqrt{-1}\partial\bar{\partial}\{\gamma_1(\frac{r}{|y|^{1/6}})(G_{0,y}^*(\psi_0 - c_0) - \sqrt{r^4 + |y|})\}$. We have

$$\begin{aligned} \left\| r^2 - \sqrt{r^4 + |y|} \right\|_{C_{-2}^k(X_y \cap U_1)} &\leq C|y|, \\ \left\| G_{0,y}^*r^2 - r^2 \right\|_{C_{-2}^k((U_1 \cap X_y) \setminus \{r < \Lambda_1|y|^{1/4}\})} &\leq C|y|, \\ \left\| G_{0,y}^*(\psi_0 - c_0 - r^2) \right\|_{C_{-2}^k((U_1 \cap X_y) \setminus \{r < \Lambda_1|y|^{1/4}\})} &\leq Cr^2r^4 \leq C|y|, \end{aligned}$$

so $\left\| G_{0,y}^*(\psi_0 - c_0) - \sqrt{r^4 + |y|} \right\|_{C_{-2}^k(\{|y|^{1/6} < r < 2|y|^{1/6}\})} \leq C|y|$. Using also $\left\| \gamma_1(\frac{r}{|y|^{1/6}}) \right\|_{C_0^k(X_y)} \leq C$, we see

$$\left\| \partial\bar{\partial}\{\gamma_1(\frac{r}{|y|^{1/6}})(G_{0,y}^*(\psi_0 - c_0) - \sqrt{r^4 + |y|})\} \right\|_{C_{-4}^k(\{|y|^{1/6} < r < 2|y|^{1/6}\})} \leq C|y|.$$

In particular, there is a pointwise estimate

$$|\partial\bar{\partial}\{\gamma_1(\frac{r}{|y|^{1/6}})(G_{0,y}^*(\psi_0 - c_0) - \sqrt{r^4 + |y|})\}|_{EH_y} \leq C|y||y|^{-4/6} = O(|y|^{1/3}) = O(r^2).$$

We observe $r \sim |y|^{1/6}$ is precisely the scale where various error sources are of comparable strength. We can now easily see the positive definiteness of ω_y in this region.

There is yet another source of error coming from the holomorphic volume form. Since $\Omega = \sqrt{A_0}(1 + O(\mathfrak{z}))d\mathfrak{z}_1d\mathfrak{z}_2d\mathfrak{z}_3$ where $O(\mathfrak{z})$ is a holomorphic function, and $\Omega = \Omega_y \wedge dy$, we can check from the explicit volume form of the Eguchi-Hanson metric, that

$$|\nabla_{EH_y}^k \{(\sqrt{-1}\partial\bar{\partial}\sqrt{r^4 + |y|})^2 - A_y^{-1}\Omega_y \wedge \bar{\Omega}_y\}|_{EH_y} = O(r^{2-k})$$

Combining these discussions,

$$\|\omega_y'^2 - A_y^{-1}\Omega_y \wedge \bar{\Omega}_y\|_{C_{-4}^k(X_y \cap \{|y|^{1/6} < r < 2|y|^{1/6}\})} \leq C|y|. \quad (4.5)$$

Finally, when $r < |y|^{1/6}$, the error to the volume form is of order $O(r^2)$ with good higher order estimates. These combined with (4.4), (4.5) imply the claim. $\square$

To perturb the approximate metric ω_y' into the actual CY metric $\omega_{SRF}|_{X_y}$ on X_y , we need the crucial mapping property of the Laplacian on the weighted Hölder spaces.

Lemma 4.1.3. *(Compare [37] Proposition 3.2) If $-2 < \beta < 0$ and $|y| < \epsilon_1$, then the Laplacian $\Delta_{\omega_y'} : C_{\beta}^{k+2,\alpha}(X_y) \rightarrow C_{\beta-2}^{k,\alpha}(X_y)$ restricted to the subspaces of functions with $\int_{X_y} f\omega_y'^2 = 0$ is an isomorphism, and the inverse satisfies a uniform estimate in f and y*

$$\|\Delta_{\omega_y'}^{-1}f\|_{C_{\beta}^{k+2,\alpha}(X_y)} \leq C(k, \alpha, \beta) \|f\|_{C_{\beta-2}^{k,\alpha}(X_y)}. \quad (4.6)$$

Remark 4.1.4. *This can be proved using the weighted Schauder estimates*

$$\|u\|_{C_{\beta}^{k+2,\alpha}(X_y)} \leq C \|\Delta u\|_{C_{\beta-2}^{k,\alpha}(X_y)} + C \|r^{-\beta}u\|_{L^\infty}$$

and a standard blow up argument.

The implicit function theorem then implies in a standard fashion that

Proposition 4.1.5. *(CY metrics on the **smoothing** of the nodal K3 fibre) Let $-2 < \beta < 0$, and $|y| < \epsilon_1 \ll 1$. There is a unique potential function ψ_y' with $\int_{X_y} \psi_y'\omega_y'^2 = 0$, such that*

$$\omega_{SRF}|_{X_y} = \omega_y' + \sqrt{-1}\partial\bar{\partial}\psi_y', \quad (\omega_{SRF}|_{X_y})^2 = A_y^{-1}\Omega_y \wedge \bar{\Omega}_y,$$

with the uniform estimate in y ,

$$\|\psi_y'\|_{C_{\beta}^{k+2,\alpha}} \leq C(k, \alpha, \beta)|y|^{-\frac{1}{6}\beta + \frac{2}{3}}. \quad (4.7)$$

Remark 4.1.6. *In particular $|\psi_y'| \leq C|y|^{\frac{2}{3} - \frac{1}{6}\beta}r^\beta$, so $|\int_{X_y} \psi_y'\omega_X^2| = O(|y|^{\frac{2}{3} - \frac{1}{6}\beta})$. There is a different normalisation convention for the potential,*

$$\omega_{SRF}|_{X_y} = \omega_X + \sqrt{-1}\partial\bar{\partial}\psi_y, \quad \int_{X_y} \psi_y\omega_X^2 = 0. \quad (4.8)$$

The advantage of (4.8) is that it makes sense also for fibres outside $\{|y| < \epsilon_1\}$, so is more useful for the global construction of the semi-Ricci-flat metric. We have $\psi_y = \psi'_y + \gamma_1(\frac{r}{|y|^{1/6}})G_{0,y}^*(\psi_0 - c_0) + \gamma_2(\frac{r}{|y|^{1/6}})\sqrt{r^4 + |y|} + c_0 + c'_0(y)$, where $c'_0(y)$ is a constant on X_y with $|c'_0(y)| \leq C(\beta)|y|^{\frac{2}{3} - \frac{1}{6}\beta}$, for any $-2 < \beta < 0$ and $|y| < \epsilon_1$.

We take the opportunity to consider **deformation** of the CY metrics $\omega_{SRF}|_{X_y}$ as the complex structure varies with y . When $|y| \geq \epsilon_1$ so X_y is bounded away from the singular fibre, then it is a standard fact that the potential ψ_y solving (4.8) deforms smoothly with y . In particular we can take a trivialisation $G_{y'}$ for the fibration around a given fibre $X_{y'}$, which induce diffeomorphisms $G_{y',y}$ identifying sufficiently nearby fibres X_y with $X_{y'}$, and then the potentials are compared as $|\psi_y - \psi_{y'}| \leq C|y - y'|$. We would like to extend this kind of Lipschitz bound to fibres with $|y'| < \epsilon_1$.

Given a fibre $X_{y'}$ with $|y'| < \epsilon_1$, we can take a fibration preserving trivialisation $G_{y'}$ over the disc $\{y : |y - y'| \leq \epsilon_2|y'|\}$,

$$G_{y'} : \{x \in X : |y' - \pi(x)| \leq \epsilon_2|y'|\} \rightarrow X_{y'} \times \{|y - y'| \leq \epsilon_2|y'|\} \subset X_{y'} \times \mathbb{C}. \quad (4.9)$$

This can be defined, for example, by flowing along the vector field obtained by the orthogonal horizontal lift of tangent vector fields on Y , using an ambient metric ω_X . For $|y - y'| \leq \epsilon_2|y'|$, this induces the diffeomorphisms $G_{y',y}$ from X_y to $X_{y'}$, depending smoothly on y . We can demand

$$|\Omega_y - G_{y',y}^* \Omega_{y'}|_{\omega_X} \leq \frac{C|y - y'|}{|y|^2} |\Omega_y|_{\omega_X} = \frac{C|y - y'|}{r^4} |\Omega_y|_{\omega_X},$$

namely the variation of complex structure causes errors of order $O(\frac{|y - y'|}{r^4})$. The analogue of (4.2) is

$$\begin{aligned} \|G_{y',y}^* \Omega_{y'} - \Omega_y\|_{C_{-4}^k(X_y)} &\leq C|y - y'|. \\ \|G_{y',y}^*(\omega_X|_{X_{y'}}) - \omega_X|_{X_y}\|_{C_{-2}^k(X_y)} &\leq C|y - y'|. \end{aligned}$$

We put an approximate CY metric on X_y as

$$\omega_y'' = \omega_X|_{X_y} + \sqrt{-1} \partial \bar{\partial} G_{y',y}^* \psi_{y'}.$$

This can be compared to $G_{y',y}^*(\omega_{SRF}|_{X_{y'}}) = G_{y',y}^*(\omega_X|_{X_{y'}} + \sqrt{-1} \partial \bar{\partial} \psi_{y'})$. To estimate their difference, the main issue is to control the norm of $G_{y',y}^*(\partial \bar{\partial} \psi_{y'}) - \partial \bar{\partial} G_{y',y}^* \psi_{y'}$. We first examine the pointwise bound measured against $\omega_{SRF}|_{X_y}$.

$$r^2 |G_{y',y}^*(\partial \bar{\partial} \psi_{y'}) - \partial \bar{\partial} G_{y',y}^* \psi_{y'}| \leq \frac{C|y - y'|}{r^4} \sum_{j=1}^2 |r^j \nabla^j \psi_{y'}| \leq \frac{C|y - y'|}{r^2}.$$

The first inequality uses that the relative error caused by the variation of complex structure is of order $O(\frac{|y - y'|}{r^4})$, and the second inequality makes use of Proposition 4.1.5 and its ensuing Remark to control $\psi_{y'}$. This estimate can easily be improved to higher orders, to give

$$\|\omega_y'' - G_{y',y}^*(\omega_{SRF}|_{X_{y'}})\|_{C_{-4}^k(X_y)} \leq C|y - y'|.$$

One can assemble the facts to show for $-2 \leq \beta < 0$,

$$\|(\omega_y'')^2 - A_y^{-1} \Omega_y \wedge \bar{\Omega}_y\|_{C_{\beta-2}^{k,\alpha}(X_y)} \leq C|y - y'| |y'|^{-\frac{1}{4}(\beta+2)}.$$

In particular the volume error is $O(\frac{|y-y'|}{r^4})$ small in L^∞ norm, which for $|y - y'| \leq \epsilon_2 |y'| \leq C\epsilon_2 r^4 \ll r^4$ is small in absolute norm. This signifies that nonlinear effect is weak. Then one can use Lemma 4.1.3 and the implicit function theorem to solve

$$(\omega_y'' + \sqrt{-1} \partial \bar{\partial} \psi_y'')^2 = A_y^{-1} \Omega_y \wedge \bar{\Omega}_y,$$

with estimate $\|\psi_y''\|_{C_{\beta}^{k+2,\alpha}(X_y)} \leq C(k, \alpha, \beta) |y - y'| |y'|^{-\frac{1}{4}(\beta+2)}$ for $-2 < \beta < 0$. Comparing this with (4.8), and installing the suitable integral normalisation condition, we get

Lemma 4.1.7. *The function ψ_y defined by (4.8) satisfies the Lipschitz type estimate: for $-2 < \beta < 0$,*

$$\|G_{y',y}^* \psi_{y'} - \psi_y\|_{C_{\beta}^{k+2,\alpha}(X_y)} \leq C(k, \alpha, \beta) |y - y'| |y'|^{-\frac{1}{4}(\beta+2)}$$

uniformly for $|y'| < \epsilon_1, |y - y'| \leq \epsilon_2 |y'|$.

Remark 4.1.8. *As mentioned before, for $|y'| \geq \epsilon_1, |y - y'| \leq \epsilon_2 |y'|$, we have the easier analogue: for $-2 < \beta < 0$,*

$$\|G_{y',y}^* \psi_{y'} - \psi_y\|_{C^{k+2,\alpha}(X_y)} \leq C(k, \alpha, \beta) |y - y'|.$$

Here we can use the usual Hölder norm, and it is understood that $y, y' \in Y$ do not come close to other critical values in S . When y, y' go beyond the coordinate neighbourhood, then $|y - y'|$ should be replaced by the qualitatively similar expression $d_{\omega_Y}(y, y')$.

Remark 4.1.9. *Comparing Proposition 4.1.5 and Lemma 4.1.7, if we consider $|y - y'| \sim t^{1/2}$, then ψ_y' and $G_{y',y}^* \psi_{y'} - \psi_y$ have comparable norm estimates when $|y| \sim t^{\frac{6}{14+\beta}}$.*

4.2 Construction of metric ansatz

4.2.1 Embedding the model metric $\omega_{\mathbb{C}^3}$

Recall from Chapter 3 the CY metric $\omega_{\mathbb{C}^3} = \sqrt{-1} \partial \bar{\partial} \phi_{\mathbb{C}^3}$ on $\mathbb{C}^3$ with standard coordinates z_1, z_2, z_3 . The asymptotic formula with remainder estimate is

$$\phi_{\mathbb{C}^3} = \phi_{\infty} + \phi'_{\mathbb{C}^3}, \quad \phi_{\infty} = \frac{1}{2} |\tilde{y}|^2 + \sqrt{H + a}, \quad \|\phi'_{\mathbb{C}^3}\|_{C_{-1+\epsilon,0}^{k,\alpha}(\mathbb{C}^3)} \leq C, \quad (4.10)$$

where

$$H = |z_1|^2 + |z_2|^2 + |z_3|^2, \quad \tilde{y} = z_1^2 + z_2^2 + z_3^2, \quad a = \sqrt{|\tilde{y}|^2 + \sqrt{H + 1}}.$$

Here we introduced the notation $\tilde{y}$ as f will be reserved later. The volume form is $\omega_{\mathbb{C}^3}^3 = \frac{3}{2} \prod_{i=1}^3 \sqrt{-1} dz_i \wedge d\bar{z}_i$. According to the motivations explained in Section 1.4.3, we perform the change of coordinates (1.16)

$$\mathfrak{z}_i = \left(\frac{t}{2A_0}\right)^{1/3} z_i, \quad r = \left(\frac{t}{2A_0}\right)^{1/6} H^{1/4}, \quad y = \left(\frac{t}{2A_0}\right)^{2/3} \tilde{y},$$

which defines an embedding map F_t of a large complex ball in $\mathbb{C}^3$ into X :

$$F_t : \{|z_1|^2 + |z_2|^2 + |z_3|^2 \lesssim t^{-2/3}\} \subset \mathbb{C}^3 \rightarrow \{\mathfrak{z}_1^2 + \mathfrak{z}_2^2 + \mathfrak{z}_3^2 \lesssim 1\} \subset X.$$

4.2.2 Weighted Hölder spaces on X

In section 4.2.3 we shall construct an ansatz metric θ_t on X , and estimate the error of its volume form. Since the actual construction is rather complicated, it is helpful to keep in mind the following cruder picture:

- In the region $U_1 \cap \{|y| < \epsilon_1\} \simeq F_t^{-1}(U_1 \cap \{|y| < \epsilon_1\})$, the metric θ_t is approximately $(\frac{t}{2A_0})^{1/3} \omega_{\mathbb{C}^3}$.
- The region with $|y| < \epsilon_1$ and suitably away from the vanishing cycles, can be identified via the diffeomorphism G_0 with a subset of $X_0 \times \{|y| < \epsilon_1\}$ (cf. (4.1)), and θ_t is approximately the product metric $\omega_{SRF}|_{X_0 + \frac{A_0}{t} \sqrt{-1} dy \wedge d\bar{y}}$.
- For $|y| \gtrsim \epsilon_1$, the metric θ_t is essentially the semi-Ricci-flat metric ω_{SRF} . Since we are staying away from singular fibres, ω_{SRF} is uniformly equivalent to $\omega_X + t^{-1} \tilde{\omega}_Y$.

These descriptions are compatible on overlaps.

The purpose of this Section is to introduce the **weighted Hölder spaces** on X , adapted to these local geometries.

We first set up the weighted Hölder spaces $C_{\delta, \tau}^{k, \alpha}(X_0 \times \mathbb{C})$ on $X_0 \times \mathbb{C}$ equipped with the product metric $\omega_{SRF}|_{X_0 + \frac{1}{t} A_0 \sqrt{-1} dy \wedge d\bar{y}}$. It is convenient to substitute the variable $\zeta = (\frac{t}{2A_0})^{-1/2} y$, so the metric becomes $\omega_{SRF}|_{X_0 + \frac{1}{2} \sqrt{-1} d\zeta \wedge d\bar{\zeta}}$. Recall on X_0 we have a function r , uniformly equivalent to the distance to the node. Let $\rho = \sqrt{r^2 + |\zeta|^2}$ and $w' \sim \frac{r}{\rho}$. The weighted Hölder norms on $X_0 \times \mathbb{C}$ are

$$\|f\|_{C_{\delta, \tau}^{k, \alpha}(X_0 \times \mathbb{C})} = \sum_{j=0}^k \sup \rho^{-\delta+j} w'^{-\tau+j} |\nabla^j f| + [\rho^{-\delta+k} w'^{-\tau+k} \nabla^k f]_{0, \alpha}. \quad (4.11)$$

where for any tensor T ,

$$[T]_{0, \alpha} = \sup_{d(x, x') \leq r(x)/10} \rho(x)^\alpha w'(x)^\alpha \frac{|T(x) - T(x')|}{d(x, x')^\alpha}$$

These weighted norms are adapted to viewing $X_0 \times \mathbb{C}$ as having a local conical singularity at the origin with singular link, and are designed to resemble the weighted Hölder spaces for $(\mathbb{C}^3, \omega_{\mathbb{C}^3})$.

Define the set $U_2 = \{|y| < \epsilon_1, r > \Lambda_1|y|^{1/4}, r > t^{1/6}\} \subset X$, which can be identified via the diffeomorphism G_0 with an open subset $G_0(U_2) \subset X_0 \times \mathbb{C}$. This allows one to compute the weighted Hölder norm on $U_2 \simeq G_0(U_2)$. Similarly one can compute the weighted Hölder norm on U_1 by viewing it as $F_t^{-1}(U_1) \subset \mathbb{C}^3$, using the metric $\omega_{\mathbb{C}^3}$. Let $U_3 = \{x \in X : |y| > \frac{\epsilon_1}{2}\} \subset X$ be the subset of X staying away from all singular fibres. On U_3 it makes sense to compute the usual $C^{k,\alpha}$ norm using the metric $\omega_X + \frac{1}{t}\omega_Y$.

Now we can define the weighted Hölder spaces $C_{\delta,\tau,t}^{k,\alpha}(X)$. The weighted norm is

$$\|f\|_{C_{\delta,\tau,t}^{k,\alpha}(X)} = t^{-\frac{\delta}{6}} \|f\|_{C_{\delta,\tau}^{k,\alpha}(U_1, \omega_{\mathbb{C}^3})} + \|f\|_{C_{\delta,\tau}^{k,\alpha}(U_2)} + t^{\frac{1}{2}(\delta-\tau)} \|f\|_{C^{k,\alpha}(U_3)} \quad (4.12)$$

Similarly, one can define the $C_{\delta,\tau,t}^k$ norm, namely by setting α to zero. The definitions also extend to tensors, with a suitable twist to the powers of t to maintain compatibility with differentiation. For instance, for a 2-form θ

$$\|\theta\|_{C_{\delta-2,\tau-2,t}^{k,\alpha}(X)} = t^{-\frac{\delta}{6}} \|\theta\|_{C_{\delta-2,\tau-2}^{k,\alpha}(U_1, \omega_{\mathbb{C}^3})} + \|\theta\|_{C_{\delta-2,\tau-2}^{k,\alpha}(U_2)} + t^{\frac{1}{2}(\delta-\tau)} \|\theta\|_{C^{k,\alpha}(U_3)}.$$

To see (4.12) is a reasonable definition, we can check that on the mutual overlap of U_1 , U_2 and U_3 , the different definitions of norms are equivalent up to a bounded factor independent of t . On $U_1 \cap U_2$, the metric $t^{1/3}\omega_{\mathbb{C}^3} \sim t^{1/3}\sqrt{-1}\partial\bar{\partial}\phi_\infty$ is uniformly equivalent to the metric $G_0^*(\omega_{SRF}|_{X_0} + \frac{1}{t}A_0\sqrt{-1}dy\wedge d\bar{y})$. The weight functions are related on $U_1 \cap U_2$ up to bounded factors by

$$\rho \sim t^{1/6}a, \quad r \sim t^{1/6}H^{1/4}, \quad w' \sim w.$$

This is enough to conclude the equivalence of $t^{-\delta/6} \|\cdot\|_{C_{\delta,\tau}^1(U_1, \omega_{\mathbb{C}^3})}$ with $\|\cdot\|_{C_{\delta,\tau}^1(U_2)}$ on $U_1 \cap U_2$. The higher order equivalence is similar. Likewise with $U_1 \cap U_3$ and $U_2 \cap U_3$.

4.2.3 Regularising the semi-Ricci-flat metric

The aim of this Section is to produce an approximate CY metric θ_t on X . The heuristic idea, as explained in Section 1.4.3, is to glue a scaled copy of $\omega_{\mathbb{C}^3}$ to the semi-Ricci-flat metric ω_{SRF} . This is complicated by the need to regularise ω_{SRF} . The rough idea of this regularisation is to replace ω_{SRF} by local product metrics when we are far from the vanishing cycles, and utilise the construction of the model metric on $\mathbb{C}^3$ when we are close to the vanishing cycles.

To save writing, we will pretend there is only one nodal fibre for π , although the presence of many nodal fibres causes no extra difficulty. We define a partition of unity $\{\chi_i\}_{i=0}^N$ on the base Y , such that $\chi_0 = 1$ on $\{|y| \leq t^{\frac{6}{14+\tau}}\}$ and $\text{supp}(\chi_0) \subset \{|y| \leq 2t^{\frac{6}{14+\tau}}\}$. For $1 \leq i \leq N$, the supports of χ_i are contained in $\{|y| \geq t^{\frac{6}{14+\tau}}\}$ for some fixed number $-2 < \tau < 0$, each centred around a point y_i with ω_Y -diameter scale $t^{1/2}$. We can demand that $0 \leq \chi_i \leq 1$, and all these χ_i have uniform C^k bounds with respect to the smooth metric $t^{-1}\omega_Y$ for any given positive integer k . Moreover, at each point in Y the number of non-vanishing χ_i is bounded independent of t , even though the total number $N \sim O(\frac{1}{t})$.

We can now write down the metric ansatz θ_t as

$$\begin{aligned} \theta_t = \omega_X + \frac{1}{t}\tilde{\omega}_Y + \sqrt{-1}\partial\bar{\partial}\left\{\sum_{i=1}^N \chi_i G_{y_i}^* \psi_{y_i} + \chi_0(c_0 + \gamma_1\left(\frac{r}{t^{1/10} + t^{1/12}\rho^{1/6}}\right)G_0^*(\psi_0 - c_0) \right. \\ \left. + \gamma_2\left(\frac{r}{t^{1/10} + t^{1/12}\rho^{1/6}}\right)\left(\frac{t}{2A_0}\right)^{1/3}(\phi'_{\mathbb{C}^3} + \sqrt{H+a})\right\}. \end{aligned} \quad (4.13)$$

Remark 4.2.1. We explain the meaning of this construction, in the order of decreasing length scales, before carrying out the error estimates. The positivity of θ_t will be clear in the course of these estimates. As a caveat θ_t is not smooth, due to the non-smoothness of $\tilde{\omega}_Y$ (cf. Section 1.4.3).

- When $|y| \geq 2t^{\frac{6}{14+\tau}}$, including in particular $|y| \geq \epsilon_1$, we are far from the singular fibre, and the construction is $\theta_t = \omega_X + \frac{1}{t}\tilde{\omega}_Y + \sqrt{-1}\partial\bar{\partial}\sum_{i=1}^N \chi_i G_{y_i}^* \psi_{y_i}$. Recall from (4.8) that ψ_{y_i} is the potential of the CY metric on X_{y_i} , which we can graft to its nearby fibres using the diffeomorphism G_{y_i} (here G_{y_i} is well defined over the support of χ_i , and only a small number of χ_i actually contribute around a given fibre X_y). The resulting θ_t is close to the semi-Ricci-flat metric. Remark 4.1.9 explains the choice of power $t^{\frac{6}{14+\tau}}$.
- When $t^{\frac{6}{14+\tau}} \leq |y| \leq 2t^{\frac{6}{14+\tau}}$, the metric θ_t starts to receive contribution from the nodal fibre X_0 (here the diffeomorphism G_0 is well defined on the support of the cutoff functions and is used to graft the potential on X_0 to X_y), but the fluctuation effect of $\omega_{\mathbb{C}^3}$ is not yet significant. The expression inside χ_0 plays the same role as the potential of the approximate metric ω_y on X_y as in (4.3).
- When $|y| < t^{\frac{6}{14+\tau}}$ but $r > (t^{1/10} + t^{1/12}\rho^{1/6})$, the metric θ_t is essentially

$$\theta_t \sim \omega_X + \frac{1}{t}\tilde{\omega}_Y + \sqrt{-1}\partial\bar{\partial}G_0^*\psi_0,$$

which is approximately the product metric on $U_2 \subset X_0 \times \mathbb{C}$. We now summarize the basic numerical properties of the cutoff scales. For $|y| \gtrsim t^{3/5}$, namely $|\tilde{y}| \gtrsim t^{-1/15}$, the term $t^{1/12}\rho^{1/6} \geq |y|^{1/6}$ dominates the term $t^{1/10}$, so the cutoff scale of $\gamma_1\left(\frac{r}{t^{1/10} + t^{1/12}\rho^{1/6}}\right)$ is comparable to the cutoff scale of $\gamma_1\left(\frac{r}{|y|^{1/6}}\right)$ in agreement with the gluing scale for ω_y in (4.3), explaining our choices of exponents in the cutoff functions. For $|y| < t^{3/5}$, the cutoff scale of $\gamma_1\left(\frac{r}{t^{1/10} + t^{1/12}\rho^{1/6}}\right)$ is comparable to the cutoff scale of $\gamma_1\left(\frac{r}{t^{1/10}}\right)$, deviating from the gluing scale of ω_y . The transition between these two behaviours happens at $|y| \sim t^{3/5}$, for which the cutoff scale is $r \sim t^{1/10}$, $\rho \sim t^{1/10}$, and in terms of the coordinates on $\mathbb{C}^3$ this means a $\sim H^{1/4} \sim t^{-1/15}$, $w \sim 1$, and the effect of regularisation on the semi-Ricci-flat metric becomes appreciable.

- When $|y| < t^{\frac{6}{14+\tau}}$, and $r < t^{1/10} + t^{1/12}\rho^{1/6}$, the metric is

$$\theta_t = \omega_X + \frac{1}{t}\tilde{\omega}_Y + \sqrt{-1}\partial\bar{\partial}\left(\frac{t}{2A_0}\right)^{1/3}(\phi'_{\mathbb{C}^3} + \sqrt{H+a}).$$

This region is contained in U_1 , so we may use the coordinates on $F_t^{-1}(U_1) \subset \mathbb{C}^3$. If we replace $\frac{1}{t}\tilde{\omega}_Y$ by its leading term

$$\frac{1}{t}A_0\sqrt{-1}dy \wedge d\bar{y} = \left(\frac{t}{2A_0}\right)^{1/3}\frac{\sqrt{-1}}{2}d\tilde{y} \wedge d\bar{\tilde{y}} = \left(\frac{t}{2A_0}\right)^{1/3}\sqrt{-1}\partial\bar{\partial}\left(\frac{1}{2}|\tilde{y}|^2\right),$$

then we recognise that

$$\theta_t \sim \omega_X + \left(\frac{t}{2A_0}\right)^{1/3}\sqrt{-1}\partial\bar{\partial}\left\{\sqrt{H+a} + \frac{1}{2}|\tilde{y}|^2 + \phi'_{\mathbb{C}^3}\right\} = \omega_X + \left(\frac{t}{2A_0}\right)^{1/3}\omega_{\mathbb{C}^3}.$$

But ω_X is in fact far smaller than $\left(\frac{t}{2A_0}\right)^{1/3}\omega_{\mathbb{C}^3}$, so we are left with $\theta_t \sim \left(\frac{t}{2A_0}\right)^{1/3}\omega_{\mathbb{C}^3}$. The subset $\{H \lesssim 1\}$ sits at the ‘quantisation scale’, where the semi-Ricci-flat approximation breaks down completely.

We now turn to the error estimates, beginning with the regions where the semi-Ricci-flat behaviour is dominant. We first calculate how much the metric θ_t restricted in the fibre direction deviates from the CY metric on the fibres. This is a familiar problem given the work in Section 4.1.5, so we will only indicate main modifications.

Lemma 4.2.2. *Fix $-2 < \tau < 0$. When $|y| \geq t^{\frac{6}{14+\tau}}$, the deviation of $\theta_t|_{X_y}$ from the CY metric $\omega_{SRF}|_{X_y}$ is estimated by*

$$\|\theta_t|_{X_y} - \omega_{SRF}|_{X_y}\|_{C_{\tau-2}^{k,\alpha}(X_y)} \leq C(k, \alpha, \tau)t^{1/2}|y|^{-\frac{1}{4}(\tau+2)}. \quad (4.14)$$

When $|y| < t^{\frac{6}{14+\tau}}$, but $r \gtrsim t^{1/10} + t^{1/12}\rho^{1/6}$, the deviation of $\theta_t|_{X_y}$ from $G_{0,y}^*(\omega_{SRF}|_{X_0})$ is estimated by

$$\|\theta_t|_{X_y} - G_{0,y}^*(\omega_{SRF}|_{X_0})\|_{C_{-4}^{k,\alpha}(X_y \cap \{r \gtrsim t^{1/10} + t^{1/12}\rho^{1/6}\})} \leq C(k, \alpha, \epsilon, \tau) \max(|y|, t^{3/5-\epsilon}) \quad (4.15)$$

where $\epsilon > 0$ can be made arbitrarily small.

Proof. When $|y| \geq 2t^{\frac{6}{14+\tau}}$, since the support of χ_i has ω_Y -length scale $\sim t^{1/2}$, we use Lemma 4.1.7 and the ensuing Remark to see that $\|\psi_y - G_{y_i,y}^*\psi_{y_i}\|_{C_{\tau}^{k+2,\alpha}(X_y)} \leq Ct^{1/2}|y|^{-\frac{1}{4}(\tau+2)}$ whenever $\chi_i \neq 0$ at y . Since at any y the number of non-vanishing χ_i is bounded independent of t , these errors cannot accumulate, so adding up $\chi_i(\psi_y - G_{y_i,y}^*\psi_{y_i})$ and applying $\sqrt{-1}\partial\bar{\partial}$ in the fibre direction, we see (4.14).

When $|y| \sim t^{\frac{6}{14+\tau}}$, we can make a few simplifications to (4.13) with negligible effects. The cutoff function $\gamma_1(\frac{r}{t^{1/10} + t^{1/12}\rho^{1/6}})$ is practically replaceable by $\gamma_1(\frac{r}{|y|^{1/6}})$, and likewise with γ_2 . We can also replace $\sqrt{H+a}$ with $\sqrt{H+|\tilde{y}|}$, and use the estimate (3.16) to drop the $\phi'_{\mathbb{C}^3}$ term in (4.13). Then the potential term in (4.13) proportional to χ_0 is reduced to

$$c_0 + \gamma_1\left(\frac{r}{|y|^{1/6}}\right)G_{0,y}^*(\psi_0 - c_0) + \gamma_2\left(\frac{r}{|y|^{1/6}}\right)\sqrt{r^4 + |y|},$$

which by Proposition 4.1.5 and its ensuing Remark, deviates from ψ_y by $\psi'_y + c'_0(y)$, with estimate

$$\|\psi'_y + c'_0(y)\|_{C_{\tau}^{k,\alpha}(X_y)} \leq C|y|^{-\frac{1}{6}\tau + \frac{2}{3}} \sim Ct^{1/2}|y|^{-\frac{1}{4}(\tau+2)}.$$

This contribution is comparable in strength to $\|\psi_y - G_{y_i, y}^* \psi_{y_i}\|_{C_\tau^{k+2, \alpha}(X_y)}$, so we have (4.14) as in the previous case.

When $|y| < t^{\frac{6}{14+\tau}}$, but $r > 2(t^{1/10} + t^{1/12}\rho^{1/6})$, we have $\theta_t = \omega_X + \frac{1}{t}\tilde{\omega}_Y + \sqrt{-1}\partial\bar{\partial}G_0^*\psi_0$. Restricted to the fibres, this situation is identical with what we saw in Lemma 4.1.1, and

$$\|\theta_t|_{X_y} - G_{0, y}^*(\omega_{SRF}|_{X_0})\|_{C_{-4}^{k, \alpha}(X_y \cap \{r > 2(t^{1/10} + t^{1/12}\rho^{1/6})\})} \leq C|y|.$$

When $|y| < t^{\frac{6}{14+\tau}}$, and $r \sim t^{1/10} + t^{1/12}\rho^{1/6}$, we have contributions from the cutoff region. As mentioned in Remark 4.2.1 there are two subcases. When $|y| \gtrsim t^{3/5}$, the cutoff function $\gamma_1(\frac{r}{t^{1/10} + t^{1/12}\rho^{1/6}})$ can be practically replaced by $\gamma_1(\frac{r}{|y|^{1/6}})$, and likewise with γ_2 . We are in a situation similar to Lemma 4.1.1, and the main correction term is $t^{1/3}\phi'_{\mathbb{C}^3}$, which by (3.16) is of order $O(t^{1/3}a^{-1+\epsilon})$. (In fact there is another error term caused by the deviation of $t^{1/3}\sqrt{H+a}$ from $t^{1/3}\sqrt{H+|\tilde{y}|}$, which has to do with regularisation. This error is of order $O(t^{1/3}a^{-1})$, which is a little less significant than $t^{1/3}\phi'_{\mathbb{C}^3}$.) The correction effect of $t^{1/3}\phi'_{\mathbb{C}^3}$ to the metric is of order $O(t^{1/3}a^{-1+\epsilon}r^{-2})$. The relative strength of this new error source compared to the error already present in the previous case, is of order

$$O\left(\frac{t^{1/3}a^{-1+\epsilon}r^{-2}}{|y|r^{-4}}\right) = O\left(\frac{t^{\frac{1}{2}-\frac{1}{6}\epsilon}r^2}{\rho^{1-\epsilon}|y|}\right) = O\left(\frac{t^{\frac{1}{2}-\frac{1}{6}\epsilon}}{\rho^{1-\epsilon}|y|^{2/3}}\right) = O\left(\frac{t^{1-\frac{2}{3}\epsilon}}{|y|^{5/3-\epsilon}}\right).$$

When we come near $|y| \sim t^{3/5}$, this new error source $t^{1/3}\phi'_{\mathbb{C}^3}$ overwhelms by a relative factor $O(t^{-\frac{1}{15}\epsilon}) = O(t^{-\epsilon})$, while for $|y| \gg t^{3/5}$, this new error is not significant. Thus in this new region $\{t^{3/5} \lesssim |y| < t^{\frac{6}{14+\tau}}, r \sim t^{1/10} + t^{1/12}\rho^{1/6}\}$, the previous estimate is changed to

$$\|\theta_t|_{X_y} - G_{0, y}^*(\omega_{SRF}|_{X_0})\|_{C_{-4}^{k, \alpha}(X_y \cap \{r \sim t^{1/12}\rho^{1/6}\})} \leq C \max(|y|, t^{3/5-\epsilon}).$$

On the other hand, if $|y| < t^{3/5}$, then $\gamma_1(\frac{r}{t^{1/10} + t^{1/12}\rho^{1/6}})$ can be practically replaced by $\gamma_1(\frac{r}{t^{1/10}})$, and likewise with γ_2 . The main errors are caused by the variation of complex structures, the deviation of the nodal K3 metric from the flat orbifold metric on $\mathbb{C}^2/\mathbb{Z}_2$, and the presence of $t^{1/3}\phi'_{\mathbb{C}^3}$. Since we are working at the scale $r \sim t^{1/10}$, the various sources of error for the potential are of order

$$O\left(\frac{|y|}{r^2}\right) = O(t^{2/5}), \quad O(r^4) = O(t^{2/5}), \quad O(t^{1/3}a^{-1+\epsilon}) = O(t^{\frac{2}{5}-\epsilon}).$$

The error for the metric is of order $O(\frac{t^{\frac{2}{5}-\epsilon}}{r^2}) = O(t^{1/5-\epsilon})$. The higher order derivative estimate involves no extra difficulty. Hence for $|y| < t^{3/5}$ and $r \sim t^{1/10}$,

$$\|\theta_t|_{X_y} - G_{0, y}^*(\omega_{SRF}|_{X_0})\|_{C_{-4}^{k, \alpha}(X_y \cap \{r \sim t^{1/10}\})} \leq Ct^{\frac{3}{5}-\epsilon}.$$

A more uniform way to present these estimate is that for $|y| < t^{\frac{6}{14+\tau}}$, and $r \gtrsim t^{1/10} + t^{1/12}\rho^{1/6}$, there is the estimate (4.15) where the exponent $\epsilon > 0$ can be made arbitrarily small. $\square$

Staying still in this region, we wish to estimate how much the volume form of θ_t fails to be Calabi-Yau. The defining condition of the CY metric $\tilde{\omega}_t$ is

$$(t^{-1}\tilde{\omega}_t)^3 = a_t t^{-1} \sqrt{-1} \Omega \wedge \bar{\Omega}, \quad a_t = t \int_X (t^{-1}[\omega_Y] + [\omega_X])^3 = 3 + t \int_X \omega_X^3,$$

under the normalisation $\int \sqrt{-1} \Omega \wedge \bar{\Omega} = 1$, $\int_Y [\omega_Y] = 1$, $\int_{X_y} \omega_X^2 = 1$. Writing

$$\theta_t^3 = a_t t^{-1} (1 + f_t) \sqrt{-1} \Omega \wedge \bar{\Omega}, \quad (4.16)$$

the task is to estimate the error f_t in the weighted Hölder norm introduced in Section 4.2.2.

Lemma 4.2.3. *Let $-2 < \tau < 0$ and $\delta > \frac{3}{4}\tau - \frac{1}{2}$, $\delta < \frac{2}{3} + \frac{5\tau}{6}$, then in the region $\{|y| \gtrsim t^{\frac{6}{14+\tau}}\}$ and the region $\{|y| < t^{\frac{6}{14+\tau}}, r \gtrsim t^{1/10} + t^{1/12} \rho^{1/6}\}$, the volume error is estimated as $\|f_t\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}} \leq C(\alpha, \delta, \tau) t^{\delta'}$, where $\delta' = -\frac{1}{2}\tau + \frac{1}{2}\delta + \frac{6}{14+\tau}(\frac{2}{3} + \frac{5\tau}{6} - \delta)$.*

Proof. The dominant term of θ_t^3 is $\frac{3}{t} \omega_{SRF}|_{X_y}^2 \tilde{\omega}_Y = \frac{3}{t} \sqrt{-1} \Omega \wedge \bar{\Omega}$. The deviation comes from two sources: the fibrewise deviation of $\theta_t^2|_{X_y}$ from $\omega_{SRF}|_{X_y}^2$, and also $(\theta_t - \frac{1}{t} \tilde{\omega}_Y)^3$, which involves understanding the horizontal component of $\theta_t - \frac{1}{t} \tilde{\omega}_Y$ and can be thought as fluctuation of the generalised KE metric $\tilde{\omega}_Y$.

Consider first the region with $|y| \geq 2t^{\frac{6}{14+\tau}}$. Fibrewise deviation from Calabi-Yau metric causes an error $f'_t = \frac{\theta_t|_{X_y}^2 \wedge \tilde{\omega}_Y}{\sqrt{-1} \Omega \wedge \bar{\Omega}} - 1 = \frac{\theta_t|_{X_y}^2}{\omega_{SRF}|_{X_y}^2} - 1$, whose pointwise magnitude is controlled by

$$\begin{aligned} |f'_t| &\leq C \|\theta_t|_{X_y} - \omega_{SRF}|_{X_y}\|_{C_{\tau-2}^0(X_y)} r^{\tau-2} \leq C t^{1/2} |y|^{-\frac{1}{4}(\tau+2)} r^{\tau-2} \\ &\leq C t^{1/2} (t^{1/2} \rho)^{-\frac{1}{4}(\tau+2)} \rho^{\tau-\delta} r^{\tau-2} \rho^{\delta-\tau} \\ &\leq C t^{\frac{6}{14+\tau}(\frac{3}{4}\tau-\delta-\frac{1}{2})} t^{\frac{1}{2}(1+\delta-\tau)} r^{\tau-2} \rho^{\delta-\tau} \\ &\leq C t^{\frac{6}{14+\tau}(\frac{3}{4}\tau-\delta-\frac{1}{2})+\frac{1}{2}(1+\delta-\tau)} w^{\tau-2} \rho^{\delta-2} = C t^{\delta'} w^{\tau-2} \rho^{\delta-2}. \end{aligned}$$

where we used (4.14) and $\delta > \frac{3}{4}\tau - \frac{1}{2}$. This is the first step towards estimating f'_t in the $C_{\delta-2, \tau-2, t}^{0, \alpha}(X \cap \{|y| > 2t^{\frac{6}{14+\tau}}\})$ norm in this region. Estimating the vertical derivatives of f'_t poses no further difficulty.

We now make some general comments about horizontal differentiation. Near a given fibre $X_{y'}$, there is a trivialisation around a small normal neighbourhood, for example induced by the diffeomorphism G_{y_i} where $|y_i - y'| \leq \epsilon_2 |y'|$. This will induce some horizontal distribution, which allows us to lift the vector fields on the base Y to X . In the coordinate neighbourhood U_1 with coordinates $\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3$, a particular lift of $\frac{\partial}{\partial y}$ is given by $\sum \frac{\bar{\mathfrak{z}}_i}{2|\mathfrak{z}|^2} \frac{\partial}{\partial \mathfrak{z}_i}$, which is orthogonal to the fibres with respect to the standard Euclidean metric in these coordinates. Now if the trivialisation is chosen well, its induced horizontal lift of $\frac{\partial}{\partial y}$ will differ from $\sum \frac{\bar{\mathfrak{z}}_i}{2|\mathfrak{z}|^2} \frac{\partial}{\partial \mathfrak{z}_i}$ by some vertical vector field whose ω_X -magnitude is $O(\frac{1}{|\mathfrak{z}|}) = O(r^{-2})$, or equivalently its magnitude with respect to the Eguchi-Hanson metric EH_y is $O(r^{-3})$. This measures the deviation between horizontal lifts for any two different good choices of trivialisations, such as G_{y_i} and G_{y_j} where $|y_i - y_j| \leq \epsilon_2 |y_i|$.

In particular, given a function $f \in C_{\tau}^{k, \alpha}(X_{y_i})$ on a very nearby fibre X_{y_i} , then $G_{y_i}^* f$ defines a function near $X_{y'}$. To estimate the magnitude of its gradient

in the horizontal direction, we can fix a good auxiliary trivialisation around $X_{y'}$, equip the normal neighbourhood with an ambient metric comparable to the product metric $\omega_{SRF}|_{X_{y'}} + \frac{1}{t}\sqrt{-1}A_{y'}dy \wedge d\bar{y}$, find the horizontal lift v of $\sqrt{t}\frac{\partial}{\partial y}$ under the good trivialisation, make v act on $G_{y_i}^*f$, and then compute the magnitude of the derivative. (The normalisation on v ensures it is roughly of unit length in our ambient metric.) But G_{y_i} also provides a good trivialisation, hence another lift v' of $\sqrt{t}\frac{\partial}{\partial y}$, with $|v - v'|_{EH_{y_i}} = O(t^{1/2}r^{-3})$. Tautologically $v'(G_{y_i}^*f) = 0$, so

$$|v(G_{y_i}^*f)| = |(v - v')(G_{y_i}^*f)| \leq Ct^{1/2}r^{-3}|\nabla_{X_{y_i}}f|_{EH_{y_i}} \leq Ct^{1/2}r^{-3}r^{\tau-1}\|f\|_{C_\tau^{1,\alpha}(X_{y_i})}.$$

We may think of $v(G_{y_i}^*f)$ suggestively as the horizontal derivative of $G_{y_i}^*f$, and write it schematically as $\sqrt{t}\frac{\partial G_{y_i}^*f}{\partial y}$. Similarly we make sense of $\sqrt{t}\frac{\partial G_{y_i}^*f}{\partial \bar{y}}$. Continuing in a similar fashion, if we differentiate $G_{y_i}^*f$ by k times and measure it using the ambient metric, then as long as our choices of trivialisations are well behaved (meaning $v - v'$ have good higher order weighted Hölder estimates), we will get

$$t^{k/2}|\frac{\partial^k G_{y_i}^*f}{\partial^k y \partial^{k-j} \bar{y}}| \leq Ct^{k/2}r^{-3k}r^{\tau-k}\|f\|_{C_\tau^{k,\alpha}(X_{y_i})}.$$

The main effect of horizontal differentiation along a unit vector, compared to vertical differentiation, is that it brings about an extra factor of $O(t^{1/2}r^{-3})$ for each derivative. This principle also works for tensors. The ultimate reason is an approximate homogeneity under $\mathfrak{z} \rightarrow \lambda\mathfrak{z}$, which reduces the problem to the case where $r \sim 1, |y'| \ll 1$.

As a special observation, as long as $r \gg t^{1/6}$, horizontal differentiation is suppressed by vertical differentiation. Using these principles, we see in particular

$$\|f'_t\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(X \cap \{|y| > 2t^{\frac{6}{14+\tau}}\})} \leq Ct^{\delta'}.$$

But the metric $\tilde{\omega}_Y$ is only Lipschitz, so the best improvement is the $C_{\delta-2,\tau-2,t}^1$ bound.

Staying in the region $\{|y| > 2t^{\frac{6}{14+\tau}}\}$, we also need to estimate the error $f_t'' = \frac{(\theta_t - t^{-1}\tilde{\omega}_Y)^3}{3t^{-1}\sqrt{-1}\Omega \wedge \bar{\Omega}}$. Since θ_t is approximately $\omega_{SRF}|_{X_y}$ on the fibre, the size of $(\theta_t - t^{-1}\tilde{\omega}_Y)^3$ depends on knowing the horizontal part of $\theta_t - t^{-1}\tilde{\omega}_Y$ (the horizontal-vertical mixed terms also play a role, whose contributions can be treated similarly). This in turn requires understanding the horizontal second derivative of $G_{y_i}^*\psi_{y_i}$, and the horizontal component of $\sqrt{-1}\partial\bar{\partial}\{\chi_i(G_{y_i}^*\psi_{y_i} - G_{y_j}^*\psi_{y_j})\}$ when the support of χ_i and χ_j overlap. The former is estimated by $\frac{Ct}{r^6}$ using Section 4.1 and the above principles concerning horizontal differentiation. Notice in our region this error is insignificant compared to f'_t :

$$\frac{Ct}{r^6} \ll Ct^{1/2}|y|^{-\frac{1}{4}(\tau+2)}r^{\tau-2}.$$

The new feature of the latter term $\sqrt{-1}\partial\bar{\partial}\{\chi_i(G_{y_i}^*\psi_{y_i} - G_{y_j}^*\psi_{y_j})\}$ comes from differentiating χ_i , which by Lemma 4.1.7 can be controlled. For instance,

$$|(G_{y_i}^*\psi_{y_i} - G_{y_j}^*\psi_{y_j})\sqrt{-1}\partial\bar{\partial}\chi_i| \leq C|G_{y_i}^*\psi_{y_i} - G_{y_j}^*\psi_{y_j}| \leq Ct^{1/2}|y|^{-\frac{1}{4}(\tau+2)}r^\tau,$$

which is again dominated by $Ct^{1/2}|y|^{-\frac{1}{4}(\tau+2)}r^{\tau-2}$; the same happens for all terms involving differentiating χ_i . At each given point only a bounded number of y_i contribute, so the errors do not accumulate, and the horizontal part of $\theta_t - \frac{1}{t}\tilde{\omega}_Y$ is dominated by $Ct^{1/2}|y|^{-\frac{1}{4}(\tau+2)}r^{\tau-2}$, whence the same holds for $|f_t''|$. All these indicate that f_t'' is less significant compared to f_t' . Proceeding further,

$$\|f_t''\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(X \cap \{|y| > 2t^{\frac{6}{14+\tau}}\})} \leq Ct^{\delta'}.$$

The total error f_t can be expressed as

$$f_t = -1 + \frac{3}{a_t}(1 + f_t' + f_t'').$$

Here the normalising constant is $\frac{3}{a_t} = 1 + O(t)$, so combining the above discussions, in the region $\{|y| > 2t^{\frac{6}{14+\tau}}\}$,

$$\|f_t\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(X \cap \{|y| > 2t^{\frac{6}{14+\tau}}\})} \leq Ct^{\delta'}.$$

In the region $\{|y| \sim t^{\frac{6}{14+\tau}}\}$, there are new contributions from the terms in (4.13) inside χ_0 . The arguments are very similar, the new ingredient being that $\phi'_{\mathbb{C}^3}$ is negligible by (3.16). The result is

$$\|f_t\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(X \cap \{|y| \sim t^{\frac{6}{14+\tau}}\})} \leq Ct^{\delta'}.$$

Next, we focus on the region with $|y| < t^{\frac{6}{14+\tau}}$, but $r \gtrsim (t^{1/10} + t^{1/12}\rho^{1/6})$. As before we start with the contribution f_t' measuring the failure of fibrewise Calabi-Yau condition, making use of (4.15) and the fact that the fibrewise holomorphic volume form Ω_y is close to Ω_0 . In the subcase of $|y| > t^{3/5}$, since $-2 < \tau < 0$, $\delta < \frac{2}{3} + \frac{5\tau}{6}$, $0 < \epsilon \ll 1$ and $r \gtrsim |y|^{1/6}$,

$$\begin{aligned} |f_t'| &\leq C \max(t^{3/5-\epsilon}, |y|)r^{-4} \leq C(\max(t^{3/5-\epsilon}, |y|)\rho^{\tau-\delta}r^{-2-\tau})\rho^{\delta-2}w^{\tau-2} \\ &\leq Ct^{-\frac{1}{2}(\tau-\delta)}(\max(t^{3/5-\epsilon}, |y|)|y|^{\tau-\delta}|y|^{\frac{1}{6}(-2-\tau)})\rho^{\delta-2}w^{\tau-2} \\ &\leq Ct^{-\frac{1}{2}\tau+\frac{1}{2}\delta}t^{\frac{6}{14+\tau}}(2/3+5\tau/6-\delta)\rho^{\delta-2}w^{\tau-2} \\ &= Ct^{\delta'}\rho^{\delta-2}w^{\tau-2}. \end{aligned}$$

On the other hand, when $r > 2(t^{1/10} + t^{1/12}\rho^{1/6})$, the fluctuation error f_t'' is

$$|f_t''| \leq C \frac{t}{r^6} \ll |y|r^{-4},$$

so is insignificant compared to f_t' . At the cutoff scale, there is an extra term coming from $t^{1/3}\phi'_{\mathbb{C}^3}$, which gives a contribution to f_t'' of order $O(t^{1/3}a^{-1+\epsilon}r^{-2})$, which is again dominated by $C \max(t^{3/5-\epsilon}, |y|)r^{-4}$. Proceeding further,

$$\|f_t\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(\{t^{3/5} \lesssim |y| < t^{\frac{6}{14+\tau}}, r \gtrsim t^{1/10} + t^{1/12}\rho^{1/6}\})} \leq Ct^{\delta'}.$$

In the region $\{|y| < t^{3/5}, r \gtrsim t^{1/10}\}$, the error due to failure of fibrewise Calabi-Yau condition can be estimated by (4.15)

$$|f_t'| \leq Ct^{3/5-\epsilon}r^{-4} \leq Ct^{2/5-\delta/10-\epsilon}\rho^{\delta-2}w^{\tau-2},$$

and the fluctuation error is

$$|f_t''| \leq \begin{cases} C \frac{t}{r^6} \ll Ct^{3/5} r^{-4}, & r > 2(t^{1/10} + t^{1/12} \rho^{1/6}) \\ Ct^{1/3} a^{-1+\epsilon} r^{-2} \leq Ct^{3/5-\epsilon} r^{-4}, & r \sim t^{1/10}, \end{cases}$$

so $|f_t'|$ is the dominant error. Proceeding as usual,

$$\|f_t\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(\{|y| < t^{3/5}, r \geq t^{1/10}\})} \leq Ct^{2/5-\delta/10-\epsilon} \ll t^{\delta'}.$$

Combining all the discussions above gives the claim. $\square$

We now turn attention to the region $\{|y| < t^{\frac{6}{14+\tau}}, r < t^{1/10} + t^{1/12} \rho^{1/6}\} \subset U_1$, where the metric θ_t is predominantly $(\frac{t}{2A_0})^{1/3} \omega_{\mathbb{C}^3}$.

Lemma 4.2.4. *Let $-2 < \tau < 0$ and $\delta < \frac{2}{3} + \frac{5\tau}{6}$, then in the region $\{|y| < t^{\frac{6}{14+\tau}}, r < t^{1/10} + t^{1/12} \rho^{1/6}\}$, the volume form error $\|f_t\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}} \leq C(\alpha, \delta, \tau) t^{\delta'}$.*

Proof. As explained in Remark 4.2.1, in this region the deviation of θ_t from $(\frac{t}{2A_0})^{1/3} \omega_{\mathbb{C}^3}$ arises from $\frac{1}{t}(A_y - A_0)\tilde{\omega}_Y$ and ω_X .

On $F_t^{-1}(U_1) \subset \mathbb{C}^3$, the metric $F_t^* \omega_X$ is comparable to the Euclidean metric $t^{2/3} \sqrt{-1} \partial \bar{\partial} H$, whence $\|F_t^* \omega_X\|_{C_{2,2}^{k, \alpha}(U_1, \omega_{\mathbb{C}^3})} \leq Ct^{2/3}$. In particular in the region $\{|y| < t^{\frac{6}{14+\tau}}, r < t^{1/10} + t^{1/12} \rho^{1/6}\}$, using that $\delta < \frac{2}{3} + \frac{5\tau}{6}$ and $-2 < \tau < 0$,

$$|\omega_X|_{\omega_{\mathbb{C}^3}} \leq Ct^{2/3} (H+1)^{1/2} \leq Ct^{\frac{6}{14+\tau}(\frac{2}{3} + \frac{5\tau}{6} - \delta) + \frac{2}{3}\delta - \frac{1}{2}\tau} a^{\delta-2} w^{\tau-2} = Ct^{\delta'+\delta/6} a^{\delta-2} w^{\tau-2},$$

Since A_y is Lipschitz in y , the term $\frac{1}{t}(A_y - A_0)\tilde{\omega}_Y$ is controlled by

$$\left| \frac{1}{t}(A_y - A_0)\tilde{\omega}_Y \right|_{\omega_{\mathbb{C}^3}} \leq C|y| |t^{-1}\tilde{\omega}_Y|_{\omega_{\mathbb{C}^3}} \leq Ct^{1/3}|y| \ll t^{2/3}(H+1)^{1/2},$$

so this contribution is insignificant compared to ω_X . Hence $f_t''' = \frac{\theta_t^3}{(\frac{t}{2A_0})\omega_{\mathbb{C}^3}^3} - 1$ satisfies the estimate $|f_t'''| \leq Ct^{\delta'+\delta/6-1/3} a^{\delta-2} w^{\tau-2}$. Proceeding further,

$$\|f_t'''\|_{C_{\delta-2, \tau-2}^{0, \alpha}(\{|y| < t^{\frac{6}{14+\tau}}, r < t^{1/10} + t^{1/12} \rho^{1/6}\}, \omega_{\mathbb{C}^3})} \leq Ct^{\delta'+\delta/6-1/3}.$$

This cannot be improved to higher orders because we do not have higher order control for A_y .

$$\text{Using } \omega_{\mathbb{C}^3}^3 = \frac{3}{2} \prod_1^3 \sqrt{-1} dz_i d\bar{z}_i,$$

$$\left(\frac{t}{2A_0}\right) \omega_{\mathbb{C}^3}^3 = \frac{3}{2} \left(\frac{t}{2A_0}\right) \prod_1^3 \sqrt{-1} dz_i d\bar{z}_i = \frac{3A_0}{t} \prod \sqrt{-1} d\mathfrak{z}_i d\bar{\mathfrak{z}}_i,$$

which we compare to

$$\Omega = \sqrt{A_0} d\mathfrak{z}_1 d\mathfrak{z}_2 d\mathfrak{z}_3 (1 + O(\mathfrak{z})), \quad \sqrt{-1} \Omega \wedge \bar{\Omega} = A_0 \prod \sqrt{-1} d\mathfrak{z}_i d\bar{\mathfrak{z}}_i (1 + O(r^2)),$$

where $O(\mathfrak{z})$ denotes a fixed holomorphic function. We see that

$$\left(\frac{t}{2A_0}\right) \omega_{\mathbb{C}^3}^3 = a_t t^{-1} \sqrt{-1} \Omega \wedge \bar{\Omega} (1 + O(t) + O(r^2)).$$

Both this error and f_t''' contribute to $f_t = \frac{\theta_t^3}{a_t t^{-1} \sqrt{-1} \Omega \wedge \bar{\Omega}} - 1$. In total

$$\|f_t\|_{C_{\delta-2, \tau-2}^{0, \alpha}(\{|y| < t^{\frac{6}{14+\tau}}, r < t^{1/10} + t^{1/12} \rho^{1/6}\}, \omega_{\mathbb{C}^3})} \leq C t^{\delta' + \delta/6 - 1/3}.$$

Finally, to convert this into the desired weighted Hölder norm $C_{\delta-2, \tau-2, t}^{0, \alpha}$ on X , we need to multiply by an extra factor $t^{-\frac{\delta-2}{6}}$ (cf. Section 4.2.2). $\square$

Combining the above lemmas,

Proposition 4.2.5. (Global volume form error) Let $-2 < \tau < 0$ and $\frac{3}{4}\tau - \frac{1}{2} < \delta < \frac{2}{3} + \frac{5\tau}{6}$. Then

$$\|f_t\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(X)} \leq C t^{\delta'}. \quad (4.17)$$

Remark 4.2.6. An important test for the strength of an estimate in $C_{\delta-2, \tau-2, t}^{0, \alpha}$ -norm, is that it should imply that f is small in L^∞ norm, which is the chief indication that nonlinear effects of the Monge-Ampère equation are weak. Notably, in the region $U_1 \simeq F_t^{-1}(U_1) \subset \mathbb{C}^3$, for $\delta \leq \frac{1}{2} + \frac{3}{4}\tau$ and $-2 < \tau < 0$,

$$\|f\|_{L^\infty(U_1)} \leq C \|f\|_{C_{\delta-2, \tau-2}^{k, \alpha}(U_1, \omega_{\mathbb{C}^3})} \leq C \|f\|_{C_{\delta-2, \tau-2, t}^{k, \alpha}(X)} t^{\frac{\delta-2}{6}}.$$

The constraint on the weight is to ensure $a^{\delta-2} w^{\tau-2} \leq C$. Thus (4.17) is only useful for gluing purposes when

$$-2 < \tau < 0, \quad \frac{3}{4}\tau - \frac{1}{2} < \delta \leq \frac{1}{2} + \frac{3}{4}\tau, \quad \delta' + \frac{\delta-2}{6} > 0.$$

The constraint $\delta < \frac{2}{3} + \frac{5}{6}\tau$ is implied by the others. These constraints have solutions, for instance, if we specialize to $\tau = -\frac{2}{3}$, then we need $-\frac{3}{13} < \delta \leq 0$.

4.2.4 Metric deviation

In the course of estimating the volume form error we have essentially showed the closeness of θ_t to various local models. We now wish to state a coarser version of these estimates, valid on somewhat larger regions. This can be viewed as a quantified statement for the intuition discussed at the beginning of Section 4.2.2, and will be useful in Section 4.3.3.

Let $\Lambda_2 \gg 1, \Lambda_3 \gg 1$ be two large numbers, and $0 < \epsilon_3 \leq \epsilon_1$ be a small number, all to be fixed independent of t . We demand that $\epsilon_3 \ll (\frac{1}{\Lambda_1 \Lambda_2^2})^{12}$ is so small that the set $\{r > \Lambda_2^{-2}(t^{1/10} + t^{1/12} \rho^{1/6}), |y| < \epsilon_3\}$ is contained in U_2 . For technical convenience, we impose further that $\epsilon_3^{1/3} \Lambda_2^8 \ll \epsilon_3^{1/7}$.

Proposition 4.2.7. Given $\epsilon_3, \Lambda_2, \Lambda_3$ as above, then as long as t is sufficiently small, the following estimates hold.

- In the region $\{r \lesssim t^{1/10} + t^{1/12} \rho^{1/6}\} \cap \{|y| < \epsilon_3\} \subset U_1$, the metric θ_t deviates from the scaled $\mathbb{C}^3$ model metric by

$$\left\| \theta_t - \left(\frac{t}{2A_0} \right)^{1/3} \omega_{\mathbb{C}^3} \right\|_{C_{0,0,t}^{0, \alpha}} \leq C(\alpha) \epsilon_3^{1/7}.$$

- In the region $\{r > \Lambda_2^{-2}(t^{1/10} + t^{1/12}\rho^{1/6}), |y| < \epsilon_3\} \subset U_2$, which can be identified as a subset of $X_0 \times \mathbb{C}$ via the trivialisation G_0 , the metric θ_t deviates from the product metric by

$$\left\| \theta_t - G_0^*(\omega_{SRF}|_{X_0} + \frac{1}{t} A_0 \sqrt{-1} dy \wedge d\bar{y}) \right\|_{C_{0,0,t}^{0,\alpha}} \leq C(\alpha) \epsilon_3^{1/7}.$$

- Let $X_{y'}$ be any fibre with $|y'| > \frac{\epsilon_3}{2}$. For $|y - y'| \leq \Lambda_3 t^{1/2} \ll \epsilon_2 |y'|$, the trivialisation $G_{y'}$ is well defined. The metric θ_t deviates from the product metric in the region $\{|y - y'| \leq \Lambda_3 t^{1/2}\}$ by

$$\left\| \theta_t - G_{y'}^*(\omega_{SRF}|_{X_{y'}} + \frac{1}{t} A_{y'} \sqrt{-1} dy \wedge d\bar{y}) \right\|_{C_{0,0,t}^{0,\alpha}} \leq C(\alpha, \epsilon_3) \Lambda_3 t^{1/2}.$$

Here if y' goes beyond the coordinate neighbourhood, then $A_{y'} \sqrt{-1} dy \wedge d\bar{y}$ should be replaced by the local Euclidean metric on the base which best approximates $\tilde{\omega}_Y$.

Proof. (Sketch) For brevity, we will only indicate how to estimate the metric deviation in the L^∞ norm. The weighted Hölder improvement is no more difficult, given the methods in Lemmas 4.2.3 and 4.2.4.

When $|y| < t^{\frac{6}{14+\tau}}$ the first two estimates are essentially extractable from the calculations in Lemma 4.2.3 and Lemma 4.2.4, the point being that the metric deviation is bounded by some positive power of t , so when t is sufficiently small all these terms are negligible compared to any bound independent of t .

Let $|y| \gtrsim t^{\frac{6}{14+\tau}}$. We consider first the deviation of θ_t from $(\frac{t}{2A_0})^{1/3} \omega_{\mathbb{C}^3}$. In the region $\{r \lesssim t^{1/10} + t^{1/12}\rho^{1/6}\} \cap \{t^{\frac{6}{14+\tau}} \leq |y| < \epsilon_3\}$, around a given fibre $X_{y''}$ the metric deviation is primarily the deviation between $(\frac{t}{2A_0})^{1/3} \omega_{\mathbb{C}^3}$ and the local product metric $G_{y''}^*(\omega_{SRF}|_{X_{y''}} + \frac{1}{t} \sqrt{-1} A_{y''} dy \wedge d\bar{y})$. In the base direction, the only deviation which is not suppressed by a power of t comes from $\frac{1}{t} (A_{y''} - A_0) \sqrt{-1} dy \wedge d\bar{y}$, which is of order $O(|y|) = O(\epsilon_3)$. In the fibre direction, we can use Proposition 4.1.5 to estimate the deviation of $\omega_{SRF}|_{X_{y''}}$ from the Eguchi-Hanson metric on fibres, which is of order $O(|y|^{\frac{2}{3} - \frac{1}{6}\beta + \frac{1}{4}(\beta-2)}) = O(\epsilon_3^{\frac{1}{12}(\beta+2)})$ for any $-2 < \beta < 0$. The fibrewise deviation of $(\frac{t}{2A_0})^{1/3} \omega_{\mathbb{C}^3}$ from the Eguchi-Hanson metric is negligible in this region. In particular, these errors are all controlled by $C\epsilon_3^{1/7}$. The exponent is not optimal; anything less than $\frac{1}{6}$ will do.

Next we consider the deviation of θ_t from the product metric $G_0^*(\omega_{SRF}|_{X_0} + \frac{1}{t} A_0 \sqrt{-1} dy \wedge d\bar{y})$, in the region $\{r > \Lambda_2^{-2}(t^{1/10} + t^{1/12}\rho^{1/6}), t^{\frac{6}{14+\tau}} \lesssim |y| < \epsilon_3\}$. Apart from the $O(\epsilon^{1/7})$ error in the previous case, there is an additional contribution, caused by the deviation of Eguchi-Hanson metric from $G_0^*(\omega_{SRF}|_{X_0})$, which is of order $O(\frac{|y|}{r^4}) = O(\frac{|y|}{|y|^{2/3}\Lambda_2^{-8}}) = O(\epsilon_3^{1/3}\Lambda_2^8)$. By our imposed assumptions, this term is also dominated by $O(\epsilon_3^{1/7})$.

The last claim of this Proposition deals with fibres bounded away from the singular fibre, and is therefore easy. $\square$

4.3 Inverting the Laplacian

We first describe the harmonic analysis on various model spaces of θ_t at different scales, and then produce a parametrix of the Green operator via decomposition and patching, by ideas similar to Section 3.5.

Let $\mathcal{A}_{\delta-2, \tau-2, t}^{0, \alpha}(X)$ be the space of $\partial\bar{\partial}$ -exact (1,1) type forms completed under the $C_{\delta-2, \tau-2, t}^{0, \alpha}$ norm on 2-forms. The trace over θ_t defines a bounded map

$$\mathrm{Tr}_{\theta_t} : \mathcal{A}_{\delta-2, \tau-2, t}^{0, \alpha}(X) \rightarrow C_{\delta-2, \tau-2, t}^{0, \alpha}(X).$$

The image lies in the subspace $\{\int_X f \theta_t^3 = 0\}$ by construction. Our main result for the linear analysis is

Proposition 4.3.1. Let $-2 + \alpha < \delta < 0$ and $-2 + \alpha < \tau < 0$. Then there exists a right inverse $\mathcal{R}$ to Tr_{θ_t} on the subspace of average zero functions,

$$\mathcal{R} : \{f \in C_{\delta-2, \tau-2, t}^{0, \alpha}(X) : \int_X f \theta_t^3 = 0\} \rightarrow \mathcal{A}_{\delta-2, \tau-2, t}^{0, \alpha},$$

with norm bound $\|\mathcal{R}\| \leq C(\delta, \tau, \alpha)$ independent of t .

We shall use the Riemannian Laplacian $\Delta_{\theta_t} = 2 \mathrm{Tr}_{\theta_t} \sqrt{-1} \partial \bar{\partial}$. The right inverse $\mathcal{R}$ can be thought schematically as $\mathcal{R} = 2 \sqrt{-1} \partial \bar{\partial} \Delta_{\theta_t}^{-1}$. It maps a real valued function to a real (1,1)-form. We emphasize that the t -independent bound is optimal, which is the main strength of the method.

4.3.1 Harmonic analysis for $\omega_{\mathbb{C}^3}$

The mapping property on $C_{\delta, \tau}^{k, \alpha}(\mathbb{C}^3, \omega_{\mathbb{C}^3})$ follows from Section 3.5.

Proposition 4.3.2. Let $-2 < \tau < 0$, and $-4 < \delta < 0$. Then there exists a bounded right inverse $P_{\mathbb{C}^3} : C_{\delta-2, \tau-2}^{0, \alpha}(\mathbb{C}^3) \rightarrow C_{\delta, \tau}^{2, \alpha}(\mathbb{C}^3)$ to the Laplacian $\Delta_{\omega_{\mathbb{C}^3}}$.

4.3.2 Harmonic analysis for $K3 \times \mathbb{C}$

We sketch the harmonic analysis on $X_y \times \mathbb{C}_\zeta$ with the product metric $\omega_{SRF}|_{X_y} + \frac{\sqrt{-1}}{2} d\zeta \wedge d\bar{\zeta}$, following established methods of G. Székelyhidi [39], T. Walpuski [48] and S. Brendle [1]. Here we consider either X_y is one of the nodal fibres, or y is bounded away from the critical value set S , so that X_y has bounded geometry. Of particular importance to us is an exponential decay property when the forcing term has fibrewise average zero and compact support, which will be exploited later to localise the Green operators.

We begin by working with the usual Hölder space $C^{k, \alpha}(X_y \times \mathbb{C})$. Denote $C^{k, \alpha, ave}(X_y \times \mathbb{C})$ as the subspace of functions with average zero on X_y fibres.

Lemma 4.3.3. *The Laplacian $\Delta : C^{2, \alpha, ave}(X_y \times \mathbb{C}) \rightarrow C^{0, \alpha, ave}(X_y \times \mathbb{C})$ is an isomorphism, with bounded inverse $P_y = \Delta^{-1}$, $\|P_y\| \leq C(y)$. Moreover, if the forcing term $f \in C^{0, \alpha, ave}$ is supported in $X_y \times \{|\zeta| < B\}$ for some $B > 1$, then outside of $X_y \times \{|\zeta| < B + 1\}$ the function $P_y f$ has exponential decay:*

$$|P_y f| \leq C(y, \alpha) e^{-m(|\zeta| - B)} \|f\|_{C^{0, \alpha}}, \quad |\zeta| \geq B + 1. \quad (4.18)$$

Similarly for the higher derivatives. The constants $C(y)$ are independent of B , and are uniform for y bounded away from S , or $y \in S$.

Proof. (sketch) By standard Schauder estimate

$$\|u\|_{C^{2,\alpha}} \leq C \|\Delta u\|_{C^{0,\alpha}} + C \|u\|_{L^\infty}.$$

Applying Lemma 7.5 in [48], the kernel of $\Delta : C^{2,\alpha}(X_y \times \mathbb{C}) \rightarrow C^{0,\alpha}(X_y \times \mathbb{C})$ must be constant on the $\mathbb{C}$ factor, so must be a global constant. It is clear from Fourier decomposition in the fibre direction that Δ restricts to a map between the subspaces of functions with fibrewise average zero, where the kernel of Δ is removed. Then a standard blow up argument shows the coercivity estimate $\|u\|_{C^{2,\alpha}} \leq C \|\Delta u\|_{C^{0,\alpha}}$.

For the surjectivity claim, we can use Fourier analysis to invert Δ for smooth functions with fibrewise average zero, whose Fourier transform in the $\mathbb{C}$ direction has compact support. One can then remove the compact support assumption by an approximation argument in the weak topology, using the above coercivity estimate (cf. Page 23, [39] for a very similar argument).

For the exponential decay estimate, one considers the fibrewise L^2 integral $h(\zeta) = \int_{X_y} u^2(\cdot, \zeta)$, which is a function on $\mathbb{C}$. Since $\Delta u = 0$ for $|\zeta| > B$, we have

$$\Delta_{\mathbb{C}} h = \int_{X_y} 2|\nabla_{\mathbb{C}} u|^2 + 2u \Delta_{\mathbb{C}} u = \int_{X_y} 2|\nabla_{\mathbb{C}} u|^2 - 2u \Delta_{X_y} u = \int_{X_y} 2|\nabla u|^2.$$

Now since the fibrewise average is imposed to be zero, by the Poincaré inequality $\int_{X_y} |\nabla u|^2 \geq m'^2 h$ for some $m' > 0$, so we have

$$\Delta_{\mathbb{C}} h \geq m'^2 h, \quad |\zeta| > B.$$

The L^∞ bound on h is already bounded in terms of $\|\Delta u\|_{C^\alpha}$. We now compare h on $\{|\zeta| \geq B\}$ with a positive supersolution

$$\tilde{h}_\epsilon = C \|\Delta u\|_{C^\alpha} \log(2|\zeta|/B) e^{-m'(|\zeta|-B)} + \epsilon e^{\epsilon|\zeta|}, \quad 0 < \epsilon \ll 1$$

satisfying $\Delta_{\mathbb{C}} \tilde{h}_\epsilon \leq m'^2 \tilde{h}_\epsilon$, to deduce $h \leq \tilde{h}_\epsilon$ in the region $\{|\zeta| \geq B\}$. Sending $\epsilon \rightarrow 0$ gives an exponential decay estimate on h with any decay rate $0 < m < m'$, and since u is harmonic outside $\{|\zeta| \leq B\}$, by elliptic regularity this implies exponential decay on u and all the higher derivatives. $\square$

Now let X_0 be a nodal K3 fibre. Recall the double weighted Hölder space $C_{\delta,\tau}^{k,\alpha}(X_0 \times \mathbb{C})$ from Section 4.2.2, which is adapted to viewing $X_0 \times \mathbb{C}$ as a space with local conical singularity at the origin with singular link. Recall the vertical distance to the nodal line is comparable to r , the distance in the base direction is $|\zeta|$, and $\rho = \sqrt{|\zeta|^2 + r^2}$ essentially measures the distance to the origin.

Lemma 4.3.4. *Let $-2 < \tau < 0$, and $\delta > -2$, then functions f in $C_{\delta-2,\tau-2}^{0,\alpha}(X_0 \times \mathbb{C})$ can be integrated along X_0 fibres,*

$$\pi_* f(\zeta) = \int_{X_0 \times \{\zeta\}} f, \quad |\pi_* f(\zeta)| \leq C(1 + |\zeta|)^{\delta-\tau} \|f\|_{C_{\delta-2,\tau-2}^{0,\alpha}(X_0 \times \mathbb{C})}.$$

If moreover $\min(\tau, \delta) > -2 + \alpha$, then for $|\zeta - \zeta'| \leq 1$,

$$|\pi_* f(\zeta) - \pi_* f(\zeta')| \leq C(1 + |\zeta|)^{\delta-\tau} |\zeta - \zeta'|^\alpha \|f\|_{C_{\delta-2,\tau-2}^{0,\alpha}(X_0 \times \mathbb{C})}.$$

Proof. For the first claim, notice

$$|f| \leq C \rho^{\delta-2} w'^{\tau-2} \|f\| = C \rho^{\delta-\tau} r^{\tau-2} \|f\| \leq C \|f\| \max(r^{\delta-2}, r^{\tau-2}),$$

so if $r^{\delta-2}, r^{\tau-2}$ are both integrable along fibres, then $|\pi_* f(\zeta)| \leq C \|f\|$ for $|\zeta| \lesssim 1$. When $|\zeta| > 1$, then ρ is uniformly equivalent to $|\zeta|$, so $|f| \leq C |\zeta|^{\delta-\tau} r^{\tau-2}$, hence $|\pi_* f(\zeta)| \leq C |\zeta|^{\delta-\tau} \|f\|$.

The weighted Hölder statement uses the integrability of $r^{\delta-2-\alpha}, r^{\tau-2-\alpha}$ instead. $\square$

For such weights we can make sense of the subspace $C_{\delta-2, \tau-2}^{0, \alpha, ave}(X_0 \times \mathbb{C}) \subset C_{\delta-2, \tau-2}^{0, \alpha}(X_0 \times \mathbb{C})$ of functions with zero average on fibres. Likewise with the subspace $C_{\delta, \tau}^{2, \alpha, ave}(X_0 \times \mathbb{C}) \subset C_{\delta, \tau}^{2, \alpha}(X_0 \times \mathbb{C})$. Observe also that L^2 integrals on fibres make sense for functions in $C_{\delta, \tau}^{2, \alpha}(X_0 \times \mathbb{C})$.

Proposition 4.3.5. Let $-2 + \alpha < \tau < 0$, and $-2 + \alpha < \delta < 0$. The Laplacian $\Delta : C_{\delta, \tau}^{2, \alpha, ave}(X_0 \times \mathbb{C}) \rightarrow C_{\delta-2, \tau-2}^{0, \alpha, ave}(X_0 \times \mathbb{C})$ is an isomorphism, with bounded inverse $P_{y=0} = \Delta^{-1}$. Moreover, if the forcing term $f \in C_{\delta-2, \tau-2}^{0, \alpha, ave}$ is supported in $X_0 \times \{|\zeta| < B\}$, then outside of $X_0 \times \{|\zeta| < B+1\}$ the function $P_{y=0}f$ has exponential decay:

$$|P_{y=0}f| \leq C(\alpha, \tau, \delta) e^{-m(|\zeta|-B)} \|f\|_{C_{\delta-2, \tau-2}^{0, \alpha}}, \quad |\zeta| \geq B+1. \quad (4.19)$$

Proof. From elliptic estimates

$$\|u\|_{C_{\delta, \tau}^{2, \alpha}} \leq C \|\Delta u\|_{C_{\delta-2, \tau-2}^{0, \alpha}} + C \|\rho^{-\delta} w'^{-\tau} u\|_{L^\infty}.$$

We claim the coercivity estimate $\|u\|_{C_{\delta, \tau}^{2, \alpha}} \leq C \|\Delta u\|_{C_{\delta-2, \tau-2}^{0, \alpha}}$. If this fails, then we consider a blow up sequence with $\|\rho^{-\delta} w'^{-\tau} u_i\|_{L^\infty} = 1$, $\|\Delta u_i\|_{C_{\delta-2, \tau-2}^{0, \alpha}} \rightarrow 0$, and $|\rho^{-\delta} w'^{-\tau} u_i(x_i)| > \frac{1}{2}$, where x_i does not lie on the nodal line. The elliptic estimates provide a uniform $C_{\delta, \tau}^{2, \alpha}$ bound, and we will use Arzela-Ascoli to extract a subsequence to reach a contradiction.

- If x_i tends to the origin, and $w'(x_i)$ is uniformly bounded positively from below, then we perform a metric scaling so that the distance from x_i to the origin is normalised to 1, and scale the function u_i to have value 1 at x_i . Passing to the scaled limit, we get a nontrivial harmonic function on the flat $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ away from the nodal line, with a double power law bound of the type $O(\rho^\delta w'^\tau)$. (Here we abuse notation to denote by ρ and w' the corresponding quantities on $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$.) For our range of weights the harmonic function is in L_{loc}^2 , so extends over the real codimension 4 nodal line by standard elliptic regularity. But our weights also force decay in the $\mathbb{C}^2/\mathbb{Z}_2$ direction, so the harmonic function is trivial, contradiction.
- If $r(x_i) \rightarrow 0$, and $w'(x_i) \rightarrow 0$, then we perform a metric scaling so that the distance from x_i to the nodal line is normalised to be 1, and scale the function u_i to have value 1 at x_i . Passing to a scaled limit, we get a nontrivial harmonic function on $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$, with a power law bound of the type $O(r^\tau)$, which implies a contradiction similar to the previous case.

- If x_i stays in a bounded region of $X_0 \times \mathbb{C}$, and $r(x_i)$ is uniformly bounded positively from below, then after passing to the limit, we obtain a nontrivial harmonic function u in $C_{\delta, \tau}^{2, \alpha, ave}(X_0 \times \mathbb{C})$, which must extend smoothly across the nodal line because it is locally integrable. We note that the fibrewise average of u is zero. Then we can consider the fibrewise L^2 integral $h(\zeta) = \int_{X_0 \times \{\zeta\}} |u|^2$ satisfying $\Delta_{\mathbb{C}} h \geq m'^2 h$ for $m' > 0$ as in Lemma 4.3.3. As $\zeta \rightarrow \infty$ the harmonic function $u = O(|\zeta|^{\delta - \tau})$, so comparison principle implies $h \leq C + \epsilon e^{\epsilon|\zeta|}$. Taking the limit $\epsilon \rightarrow 0$ shows that $h \leq C$, hence u is bounded, so we can apply Lemma 4.3.3 to conclude $u = 0$, contradiction.
- If $r(x_i)$ stays bounded below, but $\rho(x_i) \rightarrow \infty$, then we normalise u_i to have value 1 at x_i . Passing to the scaled limit we get a nontrivial harmonic function u on $X_0 \times \mathbb{C}$ with fibrewise average zero, and there is a bound of type $u = O(r^\tau)$. Then u must extend smoothly across the nodal line, and a similar argument as before shows $u = 0$, contradiction.

In particular $\Delta : C_{\delta, \tau}^{2, \alpha, ave}(X_0 \times \mathbb{C}) \rightarrow C_{\delta-2, \tau-2}^{0, \alpha, ave}(X_0 \times \mathbb{C})$ is injective. For the surjectivity claim, we need to solve $\Delta u = f$ for a given $f \in C_{\delta-2, \tau-2}^{0, \alpha, ave}(X_0 \times \mathbb{C})$. For this we can use a sequence of functions $f_i \in C^{0, \alpha, ave}(X_0 \times \mathbb{C})$ which weakly converge to f on compact subsets of the complement of the nodal line, and are uniformly bounded in $C_{\delta-2, \tau-2}^{0, \alpha, ave}$; such a sequence can be produced using cutoff functions and integration on fibres. Using Lemma 4.3.3 we find $u_i \in C^{2, \alpha}$ solving $\Delta u_i = f_i$. Notice for our range of weights $C^{2, \alpha} \subset C_{\delta, \tau}^{2, \alpha}$. By the coercivity estimate u_i is a bounded sequence in $C_{\delta, \tau}^{2, \alpha, ave}$, from which we can extract a weak limit u , giving the desired solution to $\Delta u = f$.

The exponential decay argument is as in Lemma 4.3.3. $\square$

Remark 4.3.6. *If $\delta > 0$, then the surjectivity statement must fail. Given a smooth function $u \in C_{\delta, \tau}^{2, \alpha}(X_0 \times \mathbb{C})$, then the positive weight forces u to vanish at the origin. But for a smooth forcing function f , in general we cannot solve $\Delta u = f$ with fibrewise average zero and the vanishing condition at the origin.*

Remark 4.3.7. *We did not consider the function spaces $C_{\delta, \tau}^{k, \alpha}(X_0 \times \mathbb{C})$ for larger k , because we do not have higher regularity estimates for the metric ansatz θ_t (cf. Section 4.2.3), and because the higher order derivatives $\nabla^k u = O(\rho^{\delta-k} w^{\tau-k})$ can fail to be L^1 integrable along fibres when k is large.*

4.3.3 Decomposition, patching and parametrix

Let f be a function with $\|f\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(X)} = 1$. We aim to prove Proposition 4.3.1 by finding an approximate solution to $\Delta_{\theta_t} u = f$ subject to the condition $\int_X f \theta_t^3 = 0$. For simplicity of presentation, we pretend there is only one nodal fibre in X . We shall assume throughout this Section that $-2 + \alpha < \tau < 0$ and $-2 + \alpha < \delta < 0$.

Remark 4.3.8. *In this Section, we regard the parameters $\delta, \tau, \alpha, \epsilon_1, \epsilon_2, \Lambda_1$ as fixed once for all, and all constants are allowed to depend on them; we will introduce some additional parameters $\epsilon_3, \Lambda_2, \Lambda_3$ to be fixed in due course, and the dependence of various estimates on these parameters will be explicitly tracked*

down. It is also our standing assumption that t is sufficiently small with respect to the choices of all other parameters.

We will **decompose** f both **spatially** and **spectrally**, so that on each constituent of f the harmonic analysis becomes simpler.

Recall we have a standard cutoff function

$$\gamma_1(s) = \begin{cases} 1 & \text{if } s > 2, \\ 0 & \text{if } s < 1. \end{cases}$$

and $\gamma_2 = 1 - \gamma_1$. Let $\Lambda_2 \gg 1$ be a large number to be fixed independent of t . On the region $\{|y| < \epsilon_1\}$, we build cutoff functions

$$\eta_1 = \gamma_1\left(\frac{r\Lambda_2}{t^{1/10} + t^{1/12}\rho^{1/6}}\right), \quad \eta_2 = 1 - \eta_1.$$

These will be used to decompose the function near the singular fibre into regions modelled by $\mathbb{C}^3$ and $X_0 \times \mathbb{C}$. We also need a partition of unity $\{\chi'_i\}_{i=0}^{N'}$ on the base Y . Here χ'_0 is equal to 1 for $|y| \leq \epsilon_3/2$ and vanishes for $|y| > 2\epsilon_3/3$, where $0 < \epsilon_3 \leq \epsilon_1$ is some small number to be fixed independent of t , satisfying the constraints listed in Section 4.2.4. For $i = 1, 2, \dots, N'$, the functions χ'_i have supports contained in $\{|y| \geq \epsilon_3/2\}$, each having length scale $\sim t^{1/2}$ in the ω_Y metric, and containing a point y'_i which we think of as the centre of that support. We can demand $0 \leq \chi'_i \leq 1$, and all these χ'_i have uniform C^k bounds with respect to the metric $t^{-1}\omega_Y$ for any given positive k . Moreover, at each point in Y there are only a bounded number of nonvanishing χ'_i , even though $N' \sim O(\frac{1}{t})$.

We now decompose f into pieces.

- For $i = 1, \dots, N'$, we use the trivialisation $G_{y'_i}$ to identify $\chi'_i f$ as a compactly supported function $(G_{y'_i}^{-1})^*(\chi'_i f)$ on $X_{y'_i} \times \mathbb{C}$, and use integration on fibre to decompose it into the sum of a function f_i with fibrewise average zero in $X_{y'_i} \times \mathbb{C}$, and a function f'_i on the base:

$$\chi'_i f = f_i + f'_i.$$

Clearly the supports of f_i and f'_i are contained in $\text{supp}(\chi'_i)$. The norms $\|f_i\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}}, \|f'_i\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}} \lesssim 1$ because integration on fibre is a bounded operator for $-2 + \alpha < \delta < 0$ and $-2 + \alpha < \tau < 0$. The reader is encouraged to think of functions on the base as the *zeroth Fourier mode* and fibrewise average zero functions as the *higher Fourier modes*.

- For $i = 0$, we use the trivialisation G_0 to identify $\chi'_0 \eta_1 f$ as a compactly supported function $(G_0^{-1})^*(\chi'_0 \eta_1 f)$ on $X_0 \times \mathbb{C}$. We integrate on the fibres of $X_0 \times \mathbb{C}$ to obtain a locally defined function on $\mathbb{C}$,

$$f'_0(y) = \frac{\int_{X_0 \times \{y\}} (G_0^{-1})^*(\eta_1 f) \omega_{SRF}|_{X_0}^2}{\int_{X_0 \times \{y\}} \eta_1 \omega_{SRF}|_{X_0}^2}.$$

Notice that $\eta_1 = 1$ on most of the measure of the fibres, so the denominator is approximately 1. The function $\chi'_0 f'_0$ is well defined on the support of

χ'_0 , and has a weighted Hölder bound $\|\chi'_0 f'_0\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}} \leq C$ by Lemma 4.3.4. Then we write

$$f_{0,1} = \chi'_0 \eta_1(f - f'_0(y)), \quad f_{0,2} = \chi'_0 \eta_2(f - f'_0(y)).$$

By construction $f_{0,1}$ and $f_{0,2}$ are supported where the cutoff functions are supported, their $C_{\delta-2, \tau-2, t}^{0, \alpha}$ norms $\lesssim 1$, and $f_{0,1}$ has fibrewise average zero. This gives a decomposition $\chi'_0 f = f_{0,1} + f_{0,2} + \chi'_0 f'_0$.

To summarize, we obtain $f_{0,1}, f_1, f_2, \dots, f_{N'}$ which have some fibrewise average zero property, a function $f_{0,2}$ supported near the nodal point, and a function on the base $f_Y = \chi'_0 f'_0 + f'_1 + \dots + f'_{N'}$, which has a weighted Hölder bound $\|f_Y\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(X)} \leq C$ with constant independent of t , because even though we are summing over many terms, near any given fibre only a small number of them contribute. These give a decomposition of f :

$$f = f_{0,1} + f_{0,2} + f_1 + f_2 + \dots + f_{N'} + f_Y. \quad (4.20)$$

The strategy is then to use the harmonic analysis we developed on various model geometries to *divide and conquer* all these pieces, at least approximately.

We first deal with $f_1, f_2, \dots, f_{N'}$, which are supported far away from the singular fibre. Via the trivialisations $G_{y'_i}$, we regard these as functions on $X_{y'_i} \times \mathbb{C}$ with average zero, so we can apply the Green operator $P_{y'_i}$ provided by Lemma 4.3.3 to solve the Poisson equation on $X_{y'_i} \times \mathbb{C}$. To put these local solutions back on X , we use the cutoff function $\tilde{\chi}'_i = \gamma_2(\frac{|y-y'_i|}{\Lambda_3 t^{1/2}})$, where Λ_3 is a large number to be fixed independent of t . We write

$$P_1 f = \sum_{i=1}^{N'} \tilde{\chi}'_i P_{y'_i} f_i. \quad (4.21)$$

Notice the cutoff procedure makes the function well defined on X .

Lemma 4.3.9. *Given ϵ_3 , then we can choose Λ_3 sufficiently large, so that for sufficiently small t with respect to all previous choices,*

$$\left\| \Delta_{\theta_t} P_1 f - \sum_{i=1}^{N'} f_i \right\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(X)} \leq \frac{1}{100}, \quad \|P_1 f\|_{C_{\delta, \tau, t}^{2, \alpha}(X)} \leq C(\epsilon_3, \alpha, \delta, \tau).$$

Proof. In this proof we shall not keep track of dependence on $\delta, \tau, \alpha, \epsilon_3$. By Lemma 4.3.3 there is a uniform estimate,

$$\|P_{y'_i} f_i\|_{C^{2, \alpha}(X_{y'_i} \times \mathbb{C})} \leq C \|f_i\|_{C^{0, \alpha}(X_{y'_i} \times \mathbb{C})},$$

and a uniform exponential decay estimate with rate m on $|P_{y'_i} f_i|$ and the higher derivatives, for $|y - y'_i| \gtrsim t^{1/2}$, namely outside the support of χ'_i .

We examine the norm of $\Delta_{\theta_t} P_{y'_i} f_i - f_i$. The error comes from two sources: the cutoff at scale $|y - y'_i| \sim \Lambda_3 t^{1/2}$, and the deviation of θ_t from the product metric on $X_{y'_i} \times \mathbb{C}$. Because of the exponential decay, the cutoff error

is of order $O(\exp(-m\Lambda_3) \|f_i\|_{C^{0,\alpha}})$. The metric deviation error is of order $O(t^{1/2}\Lambda_3 \|P_{y'_i} f_i\|_{C^{2,\alpha}})$ using Proposition 4.2.7. For t sufficiently small with respect to all previous choices, the errors suppressed by a power of t can be ignored, so

$$\|\Delta_{\theta_t} P_{y'_i} f_i - f_i\|_{C^{0,\alpha}(X_{y'_i} \times \mathbb{C})} \leq C \exp(-m\Lambda_3) \|f_i\|_{C^{0,\alpha}(X_{y'_i} \times \mathbb{C})}.$$

The norm on $C^{0,\alpha}(X_{y'_i} \times \mathbb{C})$ differs from the $C_{\delta-2,\tau-2,t}^{0,\alpha}(X)$ norm by a factor of order $t^{\frac{1}{2}(\delta-\tau)}$ (cf. Section 4.2.2). These weight factors cancel out to give

$$\|\Delta_{\theta_t} P_{y'_i} f_i - f_i\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(X)} \leq C e^{-m\Lambda_3} \|f_i\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(X)} \leq C.$$

Now we want to sum the contributions over i . Our cutoff procedure ensures that each forcing term f_i can only influence the points with $|y - y'_i| \lesssim \Lambda_3 t^{1/2}$, so each point y only receives contributions from $O(\Lambda_3^2)$ terms. Thus

$$\left\| P_1 f - \sum_{i=1}^{N'} f_i \right\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(X)} \leq C \Lambda_3^2 e^{-m\Lambda_3} \|f\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(X)}.$$

When Λ_3 is chosen to be sufficiently large, we can make this coefficient arbitrarily small. Once we fix this choice, the claims of the Lemma are clear. $\square$

Remark 4.3.10. *The intuition is that by the exponential decay property, the forcing functions with fibrewise average zero have localised effect for the Poisson equation. They almost do not interact with the rest of the manifold, so their effects do not accumulate.*

Next we deal with $f_{0,1}$, which is supported on the support of $\chi'_0 \eta_1$ and thus can be regarded as a function on $X_0 \times \mathbb{C}$ with some fibrewise average zero property. We prepare a cutoff function $\tilde{\chi}'_0$ on Y , supported in $\{|y| < \epsilon_3\}$, is equal to 1 on $\{|y| \leq \frac{3}{4}\epsilon_3\}$, and $|\nabla_{t^{-1}\omega_Y}^k \tilde{\chi}'_0| \leq C(k)(t^{1/2}\epsilon_3^{-1})^k$. We also need a logarithmic cutoff function

$$\tilde{\eta}_1 = \gamma_1 \left(\frac{\log\left(\frac{r\Lambda_2^3}{t^{1/10}+t^{1/12}\rho^{1/6}}\right)}{\log \Lambda_2} \right),$$

which equals 1 on the support of $f_{0,1}$, and whose gradient is supported in the range $\frac{1}{\Lambda_2} \leq \frac{r}{t^{1/10}+t^{1/12}\rho^{1/6}} \leq \frac{1}{\Lambda_2}$, involving $O(\log \Lambda_2)$ dyadic scales. Its key property is that $\|d\tilde{\eta}_1\|_{C_{-1,-1,t}^{k,\alpha}} \leq \frac{C}{\log \Lambda_2}$, namely that we can gain an extra factor of order $O(\frac{1}{\log \Lambda_2})$ compared to ordinary cutoff functions. Then we set

$$P_{0,1}f = \tilde{\chi}'_0 \tilde{\eta}_1 P_{y=0} f_{0,1}, \quad (4.22)$$

where we recall from Lemma 4.3.5 that $P_{y=0}$ is the Green operator on $X_0 \times \mathbb{C}$ for the product metric.

Similarly we deal with $f_{0,2}$, which is supported on the support of $\chi'_0 \eta_2$, so can be regarded as a function on $\mathbb{C}^3$. We need an extra logarithmic cutoff function

$$\tilde{\eta}_2 = \gamma_2 \left(\frac{2 \log\left(\frac{r\Lambda_2^{3/2}}{2(t^{1/10}+t^{1/12}\rho^{1/6})}\right)}{\log \Lambda_2} \right),$$

which equals 1 on the support of $f_{0,2}$, and whose gradient is supported in the range $\frac{2}{\Lambda_2} \leq \frac{r}{t^{1/10} + t^{1/12} \rho^{1/6}} \leq \frac{2}{\Lambda_2^{1/2}} \ll 1$. Its key property is that $\|d\tilde{\eta}_2\|_{C_{-1,-1,t}^{k,\alpha}} \leq \frac{C}{\log \Lambda_2}$. Then we set

$$P_{0,2}f = \left(\frac{t}{2A_0}\right)^{1/3} \tilde{\chi}'_0 \tilde{\eta}_2 P_{\mathbb{C}^3} f_{0,2}, \quad (4.23)$$

where we recall from Proposition 4.3.2 that $P_{\mathbb{C}^3}$ is the Green operator on $(\mathbb{C}^3, \omega_{\mathbb{C}^3})$. The scaling factor is inserted to account for the relation between θ_t and $\omega_{\mathbb{C}^3}$, so that $\Delta_{\theta_t} P_{0,2}f$ is approximately $f_{0,2}$.

Lemma 4.3.11. *We can choose $\Lambda_2 \gg 1$, $\epsilon_3 \ll 1$ subject to the constraints in Section 4.2.4, such that for t sufficiently small with respect to all previous choices,*

$$\|\Delta_{\theta_t} P_{0,1}f - f_{0,1}\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(X)} \leq \frac{1}{100}, \quad \|P_{0,1}f\|_{C_{\delta,\tau,t}^{2,\alpha}(X)} \leq C(\delta, \tau, \alpha),$$

and moreover

$$\|\Delta_{\theta_t} P_{0,2}f - f_{0,2}\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(X)} \leq \frac{1}{100}, \quad \|P_{0,2}f\|_{C_{\delta,\tau,t}^{2,\alpha}(X)} \leq C(\delta, \tau, \alpha).$$

Proof. The norm $\|\Delta_{\theta_t} P_{0,1}f - f_{0,1}\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(X_0 \times \mathbb{C})}$ is caused by the cutoff error and the deviation of θ_t from the product metric on $X_0 \times \mathbb{C}$. The cutoff error caused by $\tilde{\chi}'_0$ is suppressed by a power of t , hence negligible. The cutoff error caused by $\tilde{\eta}_1$ is of order $O(\frac{1}{\log \Lambda_2} \|P_{y=0}f_{0,1}\|_{C_{\delta,\tau}^{2,\alpha}(X_0 \times \mathbb{C})})$. The metric deviation error is of order $O(\epsilon_3^{1/7} \|P_{y=0}f_{0,1}\|_{C_{\delta,\tau}^{2,\alpha}(X_0 \times \mathbb{C})})$ using Proposition 4.2.7, provided the constraints in Section 4.2.4 are satisfied. On the support of $P_{0,1}f$, the $C_{\delta-2,\tau-2,t}^{0,\alpha}(X_0 \times \mathbb{C})$ norm and the $C_{\delta-2,\tau-2,t}^{0,\alpha}(X)$ norm are equivalent. Combining these discussions,

$$\begin{aligned} \|\Delta_{\theta_t} P_{0,1}f - f_{0,1}\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(X)} &\leq C\left(\frac{1}{\log \Lambda_2} + \epsilon_3^{1/7}\right) \|P_{y=0}f_{0,1}\|_{C_{\delta,\tau}^{2,\alpha}(X_0 \times \mathbb{C})} \\ &\leq C\left(\frac{1}{\log \Lambda_2} + \epsilon_3^{1/7}\right) \|f_{0,1}\|_{C_{\delta-2,\tau-2}^{0,\alpha}(X_0 \times \mathbb{C})} \\ &\leq C\left(\frac{1}{\log \Lambda_2} + \epsilon_3^{1/7}\right) \|f_{0,1}\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(X)} \\ &\leq C\left(\frac{1}{\log \Lambda_2} + \epsilon_3^{1/7}\right). \end{aligned}$$

Now setting $\Lambda_2 \gg 1$ and $\epsilon_3 \ll 1$ subject to the constraints in Section 4.2.4, we can make the coefficient arbitrarily small. The first couple of claims follow.

The claims about $P_{0,2}f$ are almost completely analogous, except that we need to insert some scaling factors such as $t^{\frac{1}{6}(\delta-2)}$ which account for the difference between the $C_{\delta-2,\tau-2,t}^{0,\alpha}(X)$ norm and the $C_{\delta-2,\tau-2}^{0,\alpha}(\mathbb{C}^3, \omega_{\mathbb{C}^3})$ norm. $\square$

Now consider the function f_Y on the base Y . We will eventually reduce the question of inverting the Laplacian on f_Y to a question essentially on the base. We first summarize the information about f_Y .

Lemma 4.3.12. *The function f_Y satisfies the bound*

$$|f_Y(y)| \leq C \begin{cases} (1 + t^{-1/2}|y|)^{\delta-\tau}, & |y| < \epsilon_1, \\ t^{\frac{\tau-\delta}{2}}, & |y| \geq \epsilon_1, \end{cases}$$

and for $|y - y'| \leq t^{1/2}$, there is a weighted Hölder bound

$$|f_Y(y) - f_Y(y')| \leq C(t^{-1/2}|y - y'|)^\alpha \begin{cases} (1 + t^{-1/2}|y|)^{\delta-\tau}, & |y| < \epsilon_1, \\ t^{\frac{\tau-\delta}{2}}, & |y| \geq \epsilon_1. \end{cases}$$

Moreover, the average values

$$\bar{f} = \frac{\int_X f \theta_t^3}{\int_X \theta_t^3}, \quad \overline{f_Y} = \frac{\int_Y f \tilde{\omega}_Y}{\int_Y \tilde{\omega}_Y} = \int_Y f \tilde{\omega}_Y$$

differ by only a ‘small amount’: $|\bar{f} - \overline{f_Y}| \leq Ct^{\frac{1}{2}(\tau-\delta)}\epsilon_3^{\delta-\tau+2}$. In particular if $\bar{f} = 0$ then $|\overline{f_Y}| \leq Ct^{\frac{1}{2}(\tau-\delta)}\epsilon_3^{\delta-\tau+2}$. Here constants only depend on δ, τ, α , and t is assumed to be sufficiently small as usual.

Proof. The weighted Hölder control on f_Y is immediate from its construction and Lemma 4.3.4.

We explain why the average values are close to each other. On the set $\{|y| > \epsilon_3\}$ which has most of the θ_t^3 -measures, $\theta_t^3 \sim \frac{3}{t}\omega_{SRF}|_{X_y}^2 \tilde{\omega}_Y$ up to some relative error suppressed by a power of t , and f_Y is essentially just the integration along fibres using the $\omega_{SRF}|_{X_y}^2$ measure, so

$$\int f \theta_t^3 \sim \int \frac{3}{t} \tilde{\omega}_Y \int_{X_y} f \omega_{SRF}|_{X_y}^2 \sim \int \frac{3}{t} f_Y \tilde{\omega}_Y, \quad \int \theta_t^3 \sim \int \frac{3}{t} \tilde{\omega}_Y,$$

and cancelling factors gives $\bar{f} - \overline{f_Y} \sim 0$. After neglecting the small error caused by the set $\{|y| > \epsilon_3\}$, and making a very crude estimate for the error caused by the set $\{|y| \leq \epsilon_3\}$, we arrive at

$$|\bar{f} - \overline{f_Y}| \leq C \frac{\int_{|y| \leq \epsilon_3} |f| \theta_t^3}{\int_X \theta_t^3} \leq C \frac{\int_{|y| \leq \epsilon_3} (1 + t^{-1/2}|y|)^{\delta-\tau} \tilde{\omega}_Y}{\int_Y \tilde{\omega}_Y} \leq Ct^{\frac{1}{2}(\tau-\delta)}\epsilon_3^{\delta-\tau+2}.$$

□

A requirement for solving the Poisson equation is that the forcing term has zero average. Thus we would like to decompose f_Y into an average zero function and a small error function. Let χ_0'' be a cutoff function on Y supported in $\{|y| > \epsilon_1\}$, and equals 1 on a subset of Y with at least half of the ω_Y -measure, and has bounded higher derivatives with respect to $t^{-1}\omega_Y$. Let

$$f_Y' = f_Y - \overline{f_Y} \frac{\chi_0''}{\int_Y \chi_0'' \tilde{\omega}_Y}, \quad f_Y'' = \overline{f_Y} \frac{\chi_0''}{\int_Y \chi_0'' \tilde{\omega}_Y}.$$

By construction $\int_Y f_Y' \tilde{\omega}_Y = 0$, the function f_Y' satisfies the same kind of weighted Hölder estimate as f_Y described in Lemma 4.3.12, and if $\bar{f} = 0$, then

$$\|f_Y''\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(X)} \leq C|\overline{f_Y}| t^{\frac{1}{2}(\delta-\tau)} \leq C\epsilon_3^{\delta-\tau+2}.$$

We now define $P_Y f$ to be the unique solution to the Poisson equation on Y :

$$\sqrt{-1}\partial\bar{\partial}(P_Y f) = \frac{1}{2t}f'_Y\tilde{\omega}_Y, \quad \int_Y (P_Y f)\tilde{\omega}_Y = 0. \quad (4.24)$$

Lemma 4.3.13. *Assume that the average value $\bar{f} = 0$. If ϵ_3 is chosen to be sufficiently small, then for sufficiently small t with respect to all previous choices,*

$$\|\Delta_{\theta_t} P_Y f - f_Y\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(X)} \leq \frac{1}{100}, \quad \|\partial\bar{\partial}(P_Y f)\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(X)} \leq C(\delta, \tau, \alpha).$$

Proof. The estimate on $\|\partial\bar{\partial}(P_Y f)\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(X)}$ is immediate from the definition of $P_Y f$ and the fact that $\|\frac{1}{t}\tilde{\omega}_Y\|_{C_{0,0,t}^{0, \alpha}(X)} \leq C$.

We estimate $\|\Delta_{\theta_t} P_Y f - f_Y\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(X)}$. Since $\|f_Y''\| = O(\epsilon_3^{\delta-\tau+2})$ can be made arbitrarily small by choosing $\epsilon_3 \ll 1$, we focus on $\|\Delta_{\theta_t} P_Y f - f'_Y\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(X)}$. By a standard formula of the Laplacian $\sqrt{-1}\partial\bar{\partial}(P_Y f) \wedge \theta_t^2 = \frac{1}{6}(\Delta_{\theta_t} P_Y f)\theta_t^3$, so

$$\Delta_{\theta_t} P_Y f = f'_Y \frac{3\tilde{\omega}_Y \wedge \theta_t^2}{t\theta_t^3}.$$

The point is that to control the Laplacian, we do not actually need to estimate $P_Y f$; all we need is the tautological properties of $\sqrt{-1}\partial\bar{\partial}P_Y f$.

In the region $\{|y| \gtrsim t^{\frac{6}{14+\tau}}\} \cup \{|y| \leq t^{\frac{6}{14+\tau}}, r \gtrsim t^{1/10} + t^{1/12}\rho^{1/6}\}$ where the semi-Ricci-flat behaviour is dominant, the quantity $\frac{3\tilde{\omega}_Y \wedge \theta_t^2}{t\theta_t^3}$ is very close to 1. Following almost the same calculations in Lemma 4.2.3, one can extract an estimate in this region

$$\left\| \frac{\tilde{\omega}_Y \wedge \theta_t^2}{\sqrt{-1}\Omega \wedge \bar{\Omega}} - 1 \right\|_{C_{0,0,t}^{0, \alpha}} \leq Ct^{\frac{\tau+2}{2(14+\tau)}}, \quad \left\| \frac{t\theta_t^3}{3\sqrt{-1}\Omega \wedge \bar{\Omega}} - 1 \right\|_{C_{0,0,t}^{0, \alpha}} \leq Ct^{\frac{\tau+2}{2(14+\tau)}},$$

so $\left\| \frac{3\tilde{\omega}_Y \wedge \theta_t^2}{t\theta_t^3} - 1 \right\|_{C_{0,0,t}^{0, \alpha}} \leq Ct^{\frac{\tau+2}{2(14+\tau)}}$, hence in this region

$$\|\Delta_{\theta_t} P_Y f - f'_Y\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}} \leq Ct^{\frac{\tau+2}{2(14+\tau)}} \|f'_Y\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}} \leq Ct^{\frac{\tau+2}{2(14+\tau)}}.$$

This term is suppressed by a power of t , hence negligible.

In the region $\{|y| < t^{\frac{6}{14+\tau}}, r < t^{1/10} + t^{1/12}\rho^{1/6}\}$ there is a very coarse estimate

$$\left\| \frac{3\tilde{\omega}_Y \wedge \theta_t^2}{t\theta_t^3} - 1 \right\|_{C_{0,0,t}^{0, \alpha}} \leq C,$$

hence

$$\|\Delta_{\theta_t} P_Y f - f'_Y\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}} \leq C \|f'_Y\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}} = C \|f_Y\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}}.$$

The last equality is because $f'_Y = f_Y$ in this region. But the bounds on f_Y in Lemma 4.3.12 imply that

$$\|f_Y\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(\{|y| < t^{\frac{6}{14+\tau}}, r < t^{1/10} + t^{1/12}\rho^{1/6}\})} \leq C \sup_r \{r^{2-\tau}, r^{2-\delta}\} \leq Ct^{\frac{2}{14+\tau}}.$$

Conceptually, this extra gain of $t^{\frac{2}{14+\tau}}$ factor is because integration on fibre has a regularising effect for a certain range of weights. This suppression factor makes $\|\Delta_{\theta_t} P_Y f - f'_Y\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}}$ negligible. $\square$

Remark 4.3.14. *The subtlety is that we estimated $\sqrt{-1}\partial\bar{\partial}P_Y f$ without trying to control $P_Y f$. The first reason is that geometrically speaking, metric quantities are primary, and the potential is only auxiliary. The second reason is that the weighted Hölder space for $P_Y f$ is quite awkward to work with, due to the fact that $\delta - \tau + 2 > 0$.*

We now sum over the pieces to define an approximate Green operator

$$Pf = P_1 f + P_{0,1} f + P_{0,2} f + P_Y f \quad (4.25)$$

Combining Lemma 4.3.9, 4.3.11, 4.3.13 and fixing $\epsilon_3 \ll 1, \Lambda_2 \gg 1, \Lambda_3 \gg 1$ to satisfy all the constraints, we see

Corollary 4.3.15. Let $-2 + \alpha < \delta < 0$ and $-2 + \alpha < \tau < 0$. Let f be a function with $\|f\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(X)} = 1$ and $\int_X f \theta_t^3 = 0$. Then one can choose $\epsilon_3, \Lambda_2, \Lambda_3$ such that

$$\|\Delta_{\theta_t} Pf - f\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(X)} \leq \frac{1}{25}, \quad \|\partial\bar{\partial}Pf\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(X)} \leq C(\delta, \tau, \alpha).$$

Finally, we prove Proposition 4.3.1.

Proof. (Proposition 4.3.1) Let f be a function in $C_{\delta-2, \tau-2, t}^{0, \alpha}(X)$ with integral zero. We can define

$$u = P \sum_{j=0}^{\infty} (1 - \Delta_{\theta_t} P)^j f,$$

which converges because $\|\Delta_{\theta_t} P - 1\| \leq \frac{1}{25}$. Although we do not have good control on u directly, we do know

$$\|\partial\bar{\partial}u\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}} \leq C \|\partial\bar{\partial}P\| \|f\| \leq C \|f\|.$$

Using our convention for the Laplacian, the $\partial\bar{\partial}$ exact $(1,1)$ -form $\theta = 2\sqrt{-1}\partial\bar{\partial}u$ solves $\text{Tr}_{\theta_t} \theta = \Delta_{\theta_t} u = f$, so setting

$$\mathcal{R}(f) = \theta = 2\sqrt{-1}\partial\bar{\partial}P \sum_{j=0}^{\infty} (1 - \Delta_{\theta_t} P)^j f \quad (4.26)$$

gives the desired right inverse $\mathcal{R}$ with bounds. We remark that $\mathcal{R}$ maps a real valued function to a real $(1,1)$ -form. $\square$

4.3.4 Exponential localising property

An interesting consequence of the parametrix construction in Section 4.3.3 is that away from the singular fibres, the effect of the forcing term is very localised.

Proposition 4.3.16. In the setup of Proposition 4.3.1, if the forcing function f is supported away from the neighbourhood of a smooth fibre $\{x \in X : |\pi(x) - y'| \leq d\} \subset X \setminus \{|y| \leq \epsilon_1\}$, then

$$\|\mathcal{R}f\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(\{x \in X : |\pi(x) - y'| \leq \frac{d}{2}\})} \leq C(\delta, \tau, \alpha) \exp\left(\frac{-m(\delta, \tau, \alpha)d}{t^{1/2}}\right) \|f\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(X)},$$

where the exponential decay constants do not depend on f, d, y', t .

Proof. Without loss of generality the set $\{|y - y'| \leq d\} \subset Y$ has less than $\frac{1}{3}$ of the total ω_Y -measure. Using the flexibility of the definition of the cutoff function χ_0'' , we may arrange that χ_0'' is supported away from $\{|y - y'| \leq d\} \subset Y$. Then in the construction of the operator $1 - \Delta_{\theta_t}P$, only the part $1 - \Delta_{\theta_t}P_1$ can propagate a forcing term supported outside $\{x \in X : |y - y'| \leq d\} \subset X$ into this set.

The key point now is that the construction of $\mathcal{R}$ in (4.26) involves an iteration of the operator $1 - \Delta_{\theta_t}P$, and each iteration can propagate the support towards $X_{y'}$ only by a very small amount of ω_Y -distance, of order $\sim O(\Lambda_3 t^{1/2})$. Thus it requires about $O(\frac{d}{\Lambda_3 t^{1/2}})$ iterations to propagate the forcing term into the set $\{|y - y'| \leq \frac{d}{2}\} \subset X$. Each iteration results in a damping factor $\frac{1}{25}$ on the norm of the forcing term, so

$$\|\mathcal{R}f\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(\{x \in X : |\pi(x) - y'| \leq \frac{d}{2}\})} \lesssim \left(\frac{1}{25}\right)^{\frac{-Cd}{t^{1/2}}} \|f\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(X)}$$

as required. $\square$

4.4 Collapsing Calabi-Yau metric

4.4.1 Perturbation to the Calabi-Yau metric

We carry out the main gluing construction. To avoid complication, we choose the weights $\tau = -\frac{2}{3}$ and $-\frac{3}{13} < \delta < 0$ (cf. Remark 4.2.6). The elementary numerical properties of these weights are summarised as

$$\begin{cases} \|f\|_{C_{0,0,t}^{0,\alpha}(X)} \leq C t^{\frac{1}{6}\delta - \frac{1}{3}} \|f\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(X)}, \\ \|fg\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(X)} \leq C \|g\|_{C_{0,0,t}^{0,\alpha}(X)} \|f\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(X)}, \end{cases}$$

where f, g are arbitrary elements of the function spaces. The same statement holds for forms.

Theorem 4.4.1. Let $\tau = -\frac{2}{3}$ and $-\frac{3}{13} < \delta < 0$. Then for sufficiently small t , the Calabi-Yau metric $t^{-1}\tilde{\omega}_t$ in the class $[\theta_t] = [\omega_X] + t^{-1}[\omega_Y]$ is close to θ_t ,

$$\|t^{-1}\tilde{\omega}_t - \theta_t\|_{C_{\delta-2,\tau-2,t}^{0,\alpha}(X)} \leq C(\delta, \alpha) t^{\frac{23}{60} + \frac{1}{20}\delta}. \quad (4.27)$$

Proof. We wish to find the CY metric $t^{-1}\tilde{\omega}_t$ as a small perturbation of the metric ansatz θ_t . Using the defining conditions (cf. Section 4.2.3)

$$(t^{-1}\tilde{\omega}_t)^3 = a_t t^{-1} \sqrt{-1} \Omega \wedge \bar{\Omega}, \quad \theta_t^3 = a_t t^{-1} (1 + f_t) \sqrt{-1} \Omega \wedge \bar{\Omega},$$

it suffices to find a function $f \in C_{\delta-2, \tau-2, t}^{0, \alpha}(X)$ with $\int_X f \theta_t^3 = 0$, such that

$$(\theta_t + \mathcal{R}f)^3 = \frac{1}{1 + f_t} \theta_t^3, \quad \|f\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(X)} \leq C t^{\frac{23}{60} + \frac{1}{20} \delta}. \quad (4.28)$$

Once we found f , then setting $\tilde{\omega}_t = \theta_t + \mathcal{R}f$ gives the CY metric with estimate. Although θ_t and f can fail to be smooth, the equation $(t^{-1} \tilde{\omega}_t)^3 = a_t t^{-1} \sqrt{-1} \Omega \wedge \bar{\Omega}$ would imply the smoothness of $t^{-1} \tilde{\omega}_t$, so $t^{-1} \tilde{\omega}_t$ is an honest CY metric, and hence the unique CY metric in this class.

We now focus on solving (4.28). Define the nonlinear operator $\mathcal{F}$ acting on the subspace $\{f : \int_X f \theta_t^3 = 0\} \subset C_{\delta-2, \tau-2, t}^{0, \alpha}(X)$,

$$\mathcal{F}(f) = \frac{(\theta_t + \mathcal{R}f)^3}{\theta_t^3} - 1.$$

Notice $\mathcal{F}(f)$ automatically has zero integral. We restrict attention to the open subset

$$\mathcal{U} = \{f \in C_{\delta-2, \tau-2, t}^{0, \alpha}(X) : \int_X f \theta_t^3 = 0, \|f\|_{C_{0,0,t}^{0, \alpha}(X)} < \epsilon_4\} \subset C_{\delta-2, \tau-2, t}^{0, \alpha}(X)$$

where $\epsilon_4 \ll 1$ is a small constant independent of t , and we separate $\mathcal{F}$ into the linearisation and the nonlinearity,

$$\mathcal{F}(f) = \text{Tr}_{\theta_t} \mathcal{R}f + Q(f) = f + Q(f).$$

Here $Q(0) = 0$. The equation (4.28) can be cast in the form of a fixed point equation

$$f = \frac{-f_t}{1 + f_t} - Q(f).$$

Using the basic numerical properties of the weights, for $u, v \in \mathcal{U}$, the non-linearity Q satisfies

$$\begin{aligned} \|Q(u) - Q(v)\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(X)} &\leq C(\|u\|_{C_{0,0,t}^{0, \alpha}(X)} + \|v\|_{C_{0,0,t}^{0, \alpha}(X)}) \|u - v\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(X)} \\ &\leq C \epsilon_4 \|u - v\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(X)} \\ &\leq \frac{1}{2} \|u - v\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(X)}. \end{aligned}$$

The contraction property in the last step can be ensured by choosing ϵ_4 sufficiently small. On the other hand, our volume error estimate (4.17) reads

$$\|f_t\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(X)} \leq C t^{\frac{23}{60} + \frac{1}{20} \delta},$$

which implies by the numerical properties of the weights that

$$\left\| \frac{-f_t}{1 + f_t} \right\|_{C_{\delta-2, \tau-2, t}^{0, \alpha}(X)} \leq C t^{\frac{23}{60} + \frac{1}{20} \delta}, \quad \left\| \frac{-f_t}{1 + f_t} \right\|_{C_{0,0,t}^{0, \alpha}(X)} \leq C t^{\frac{13}{60} \delta + \frac{1}{20}} \ll \epsilon_4.$$

The Banach fixed point theorem then yields a solution $f \in \mathcal{U}$ to equation (4.28) with estimates, as required. $\square$

Remark 4.4.2. *A particular consequence of our bound is*

$$|t^{-1}\tilde{\omega}_t - \theta_t|_{\theta_t} \leq C(\delta)t^{\frac{1}{20} + \frac{13}{60}\delta}, \quad -\frac{3}{13} < \delta < 0.$$

Notice that the exponent $0 < \frac{1}{20} + \frac{13}{60}\delta < \frac{1}{20}$ is rather small, which indicates that our metric ansatz θ_t is a rather coarse approximation. Our gluing construction is only made possible because of the optimal nature of the linear theory.

We now sketch the proof of Theorem 1.4.8 and 4.0.4, which are immediate consequences of the main gluing theorem 4.4.1.

Proof. By construction θ_t is C^0 close to the product metric $G_0^*(\omega_{SRF}|_{X_0} + \frac{A_0}{t}\sqrt{-1}dy \wedge d\bar{y})$ in the region $\{r > t^{1/10} + t^{1/12}\rho^{1/6}, |y| < t^{\frac{6}{14+\tau}}\}$, with error suppressed by a power of t . The region $\{r < t^{1/10} + t^{1/12}\rho^{1/6}, |y| < t^{\frac{6}{14+\tau}}\}$ is negligible in the GH convergence because every point is within $\tilde{\omega}_t$ distance $t^{\frac{1}{14+\tau}}$ to the previous region. The region $\{|y| > t^{\frac{6}{14+\tau}}\}$ is invisible to the pointed GH limit because its $t^{-1}\tilde{\omega}_t$ -distance to the nodal point $\gtrsim t^{\frac{-(2+\tau)}{2(14+\tau)}}$, which diverges to infinity. Thus the pointed GH limit of $t^{-1}\tilde{\omega}_t$ is $X_0 \times \mathbb{C}$ with the product metric, proving Theorem 1.4.8.

On any fixed Euclidean ball inside $F_t^{-1}(U_1) \subset \mathbb{C}^3$ centred at the origin, the metric ansatz has the asymptotic formula $\theta_t \sim (\frac{t}{2A_0})^{1/3}\omega_{\mathbb{C}^3}$ as $t \rightarrow 0$, so Theorem 4.4.1 implies Theorem 4.0.4 as well. $\square$

Remark 4.4.3. *As a digression, the exponential localising property discussed in Section 4.3.4 implies that the CY metric near a smooth fibre is locally determined up to exponentially small corrections from the rest of the manifold. Thus contrary to the low global regularity of our metric ansatz θ_t , the actual CY metric $t^{-1}\tilde{\omega}_t$ may have much better regularity near a given smooth fibre, such as admitting a formal power series expansion in t as in J. Fine's work [9].*

4.4.2 Open directions

In this Section we speculate how this work may be generalised to describe collapsing CY metrics $\tilde{\omega}_t$ on more complicated fibrations $\pi : X \rightarrow Y$ over a 1-dimensional base Y , where $[\tilde{\omega}_t]$ lies in the Kähler class $[t\omega_X + \omega_Y]$, and $0 < t \ll 1$. To begin with, we point out that our gluing construction depends essentially on the fact that all fibres admit (possibly singular) CY metrics, and on the existence of the model CY metric $\omega_{\mathbb{C}^3}$ on $\mathbb{C}^3$, which fits well with the singularity in the fibration π . The 1-dimensional base assumption is also essential because much less is known about the generalised KE metrics on Y for higher dimensions.

Following [3][39] a large class of examples of complete CY metrics on $\mathbb{C}^n$ are now known, which generalise the model metric $\omega_{\mathbb{C}^3}$. In particular, on the total space of the standard higher dimensional Lefschetz fibration $f : \mathbb{C}^n \rightarrow \mathbb{C}$ where $f = \sum z_i^2$, there is a complete CY metric $\omega_{\mathbb{C}^n}$, whose asymptotic behaviour at infinity approximates the semi-Ricci-flat metric on $\mathbb{C}^n$, namely that in the fibre direction it approximates the Stenzel metrics on the fibres, and in the horizontal direction it is predominantly the pullback of the Euclidean metric on the base.

Now suppose a projective CY manifold X admits a Lefschetz fibration $\pi : X \rightarrow Y$, where the fibres have complex dimension at least 3. By adjunction, the fibres are Calabi-Yau varieties in their own right, and admit (possibly singular) Calabi-Yau metrics. The result of Hein and Sun [24] says in particular that the CY metrics on the singular fibres are modelled on the Stenzel metric near the nodal point, with polynomial rate of convergence. One expects that using standard gluing arguments, the CY metrics on the smoothing fibres are modelled on the Stenzel metric in the region close to the vanishing cycles. Thus it seems very plausible that the collapsing metric $\tilde{\omega}_t$ is obtained by gluing a suitably scaled copy of $\omega_{\mathbb{C}^n}$ to a suitably regularised version of the semi-Ricci-flat metric on X . It is also conceivable to extend this picture to more complicated fibrations with isolated critical points, using the model metrics provided by [3][39].

In a slightly different vein, the strategy of producing complete CY metrics in [3][39] does not depend in an essential way on the ambient manifold being $\mathbb{C}^n$. To give a special interesting example to indicate the possible generalisation, it is a folklore speculation that there may be a non-standard complete CY metric ω_{Q_ϵ} on the affine quadric $Q_\epsilon = \{\zeta_1^2 + \zeta_2^2 + \zeta_3^2 + \zeta_4^2 = \epsilon^2\}$, which admits a Lefschetz fibration by projecting to the first coordinate

$$f : Q_\epsilon \rightarrow \mathbb{C}, \quad (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \mapsto \zeta_1.$$

There are exactly two singular fibres, corresponding to $\zeta_1 = \pm\epsilon$. The expected behaviour is that asymptotically near infinity ω_{Q_ϵ} looks like the semi-Ricci-flat metric on Q_ϵ , namely that restricted to the fibres it approximates the Eguchi-Hanson metrics on the fibres, and in the horizontal direction it is dominated by the pullback of the Euclidean metric $\frac{\sqrt{-1}}{2}d\zeta_1 \wedge d\bar{\zeta}_1$; in particular ω_{Q_ϵ} has maximal volume growth rate and tangent cone $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ at spatial infinity. As $\epsilon \rightarrow 0$, it is expected that ω_{Q_ϵ} converges to a CY metric ω_{Q_0} on the conifold Q_0 , with local tangent cone at the origin isometric to the Stenzel cone, and tangent cone at infinity isometric to $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$. This would be one of the simplest examples of such conjectural transition behaviours between different Calabi-Yau cones. Alternatively, these ω_{Q_ϵ} can be regarded as a family of metrics on the fixed complex manifold Q_1 , by changing the relative size of the fibre compared to the base, much like our setup in the compact case.

Now the relevance of ω_{Q_ϵ} to the collapsing metrics comes in when we ‘collide two singular fibres’. More formally, let X_ϵ be a polarised family of projective CY 3-folds admitting Lefschetz K3 fibrations over $Y = \mathbb{P}^1$, such that in a neighbourhood containing two critical points, the fibration is modelled by $f : Q_\epsilon \rightarrow \mathbb{C}$. The collapsing CY metric $\tilde{\omega}_{t,\epsilon}$ depends on both the small collapsing parameter t and the small degeneration parameter ϵ in the picture.

- When $\epsilon > 0$ is fixed and $t \rightarrow 0$, the situation is a collapsing Lefschetz K3 fibration, covered by the gluing construction in this paper. The $t^{-1}\tilde{\omega}_{t,\epsilon}$ distance between the two nearby singular fibres is of order $O(\epsilon t^{-1/2})$. The quantisation scale, namely the $t^{-1}\tilde{\omega}_{t,\epsilon}$ -length scale of the $\mathbb{C}^3$ bubble embedded in X_ϵ , is of order $O((\frac{t}{A_0})^{1/6})$. To understand how A_0 depends on ϵ , we notice that for ζ_1 very close to ϵ , the Lefschetz fibration is approximately

$$y = -2\epsilon(\zeta_1 - \epsilon) = \zeta_2^2 + \zeta_3^2 + \zeta_4^2,$$

which means

$$A_0 \sim \int_{X_{y=0}} \Omega_0 \wedge \bar{\Omega}_0 \sim \int_{X_{y=0}} \frac{\Omega \wedge \bar{\Omega}}{dy \wedge d\bar{y}} \sim \frac{1}{\epsilon^2} \int_{X_{y=0}} \frac{\Omega \wedge \bar{\Omega}}{d\zeta_1 \wedge d\bar{\zeta}_1} \sim O\left(\frac{1}{\epsilon^2}\right).$$

Hence the quantisation scale is $O(t^{1/6}\epsilon^{1/3})$.

- When we decrease ϵ until $\epsilon \sim t$, then $O(\epsilon t^{-1/2}) = O(t^{1/6}\epsilon^{1/3}) = O(t^{1/2})$, namely the quantisation scale is comparable to the $t^{-1}\tilde{\omega}_{t,\epsilon}$ -distance between the two critical points, so the interaction between the two $\mathbb{C}^3$ bubbles become significant. On the other hand, if we substitute $\zeta_i = \epsilon\zeta'_i$, and scale the metric by a factor t^{-1} so that the distance scale becomes of order 1, then as $t \sim \epsilon \rightarrow 0$ we expect to see a blow up limit complex analytically isomorphic to Q_1 , which is up to some scaling factor isometric to some member of the family of model metrics on Q_1 .
- When $\epsilon = 0$, the CY manifolds develop a local conical singularity, and by Hein and Sun's result the CY metric $\tilde{\omega}_{t,0}$ is locally modelled on the Stenzel cone, at least on some extremely small scale. When $\epsilon \ll t$, the effect of collapsing is insignificant in a very small region near the conical point, and one sees the usual behaviour of the smoothing of the conical singularity.

These discussions are meant to suggest that there is a numerous supply of non-compact complete CY metrics associated to a fibration structure, and these examples are intimately tied to the local behaviour of collapsing metrics on compact CY manifolds admitting fibration structures.

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