Stochastic Optimal Control for Piecewise Deterministic Markov Processes

by

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To Marta

ABSTRACT

Piecewise Deterministic Markov Processes form a general class of non-diffusion processes. A path of a process in this class consists of deterministic continuous motion interrupted by random jumps or by boundary jumps. The class provides a wide framework for the study of optimization in models arising in operational research and investment planning such as queues, storage processes, capacity expansion models, hydropower management and others.

In this work, the control problem of Piecewise Deterministic Markov Processes is studied and the characterization provided justifies local optimality criteria under broad conditions.

As a preliminary stage, an elementary control problem involving only one jump is analysed. It is then recast as deterministic control problem, defined jointly over the class of controls at a subset of the boundary domain and the class of continuously acting controls in the interior domain. The first yields a pointwise extremization problem that can be solved independently. The remaining problem is a standard optimal control problem with Lagrange cost and end point constraints, the novelty here is that the value function is shown to be globally Lipschitzian. For this purpose, a geometrical extension for vector fields and other data beyond the original bounded domain is proposed, and an additional deterministic control problem is introduced for which final end point constraints are absent. Each possible solution for the latter is shown to comply with constraints

for the original problem, thus a global exact penalty method for this class of deterministic problems is established.

The simpler stochastic control problem of Piecewise Deterministic Process is related to the general stochastic control problem in a further stage. They are bridged together by the identification of a certain lower envelope operation arising in connection with the simpler problem and then, by an embbeding of solutions that relies upon a fixed point argument.

A Maximum Principle for nonsmooth data and a Verification Theorem which gives necessary and sufficient conditions of optimality in terms of a modified Hamilton-Jacobi-Bellman equation are established. A sequential criterion of optimality is also provided with a direct appeal for numerical recursive procedures for solutions.

CONTENTS

Chapter 4: An Embedding for P_1 and Algorithms for the Stochastic PDP problem

CHAPTER 1

MAIN RESULTS AND EXAMPLES

1.1 Introduction

Piecewise Deterministic Processes (PDP) form a general class of continuous-time homogeneous Markov processes. The paths are right-continuous with left-hand limits and discontinuities are nonaccumulative. Between jumps, the path is an integral curve defined on the state domain by a time-invariant vector field. In the interior domain, jumps take place in accordance with an underlying distribution, whereas if the boundary set is attained, a jump occurs instantaneously. In both cases, the process can either restart afresh in the interior domain, or be killed by jumping into an absorbing state with some probability.

Let $E \subset \mathbb{R}^n$ be the state domain and E_∂ be a subset of the boundary set for E. A point in $\mathbb{R}^n$ outside E is denoted by Δ. The process can be characterized locally by the following elements:

(i) Vector field $f : E \to \mathbb{R}^n$.

(ii) Jump rate $\lambda : E \to \mathbb{R}_+$.

(iii) Transition measures:

$$Q : \mathrm{int}\{E\} \to \underline{\underline{P}}(E \bigcup \{\Delta\}) \qquad P: E_\partial \to \underline{\underline{P}}(E \bigcup \{\Delta\})$$

where $\underline{\underline{P}}(E \bigcup \{\Delta\})$ is the set of probability measures on $E \bigcup \{\Delta\}$.

Definition (i) provides the integral curve $t \to x(t,z)$. For each $z \in E$, it is uniquely defined by

$$x(t,z) = z + \int_0^t f(x(s,z))ds \quad t \in [0,t^*(z)]$$

where

$$t^*(z) = \inf\{t \geq 0: x(t,z) \in E_\partial\}$$

with the convention $\inf \phi = +\infty$.

A sample path (x_t), $t \geq 0$ starting from $z \in E$ is constructed as follows. Let $T_1(\omega)$ be the jumping time:

$$T_1(\omega) = \inf\{t \geq 0: x_{t-}(\omega) \neq x_t(\omega)\}$$

We define

$$x_t(\omega) = x(t,z) \qquad \text{for } t \in [0,T_1(\omega)) \quad .$$

Applying (ii), the first jump time $T_1(\omega)$ has the distribution:

$$P_z(T_1 > t) = \exp(-\int_0^t \lambda(x(s,z))ds) I_{\{t<t^*(z)\}}.$$

The conditional distribution of x_{T_1} is specified by the transition measures in (iii)

$$P_z(x_{T_1} \in A | x_t \neq x_{t^-})$$

$$= I_{\{t<t^*(z)\}} Q(A,x_{t^-}) + I_{\{t=t^*(z)\}} P(A,x_{t^-})$$

we take $x_{0-} = z$, so that $x_{t-} = x(t,z)$ on $[0,t^*(z) \wedge T_1(\omega)]$. The markovian process restarts at x_{T_1} or remains at Δ indefinitely. The construction of a PDP based on the quadruple (f,λ,Q,P) was established by Davis in [D1].

The class of PDP's comprises a wide variety of Markov process in which deterministic motion and random discontinuities underlie the general behaviour. The class provides a conceptual model that is appropriate to describe dynamical processes and physical phenomena in presence of uncertainty or disturbances which are discontinuous Markov processes. Examples in queues and storage processes are to be found in Davis [D1] and Soner [S2]. In section 1.2 the planning model studied by Davis *et al.* [D2] is presented, in section 1.3 a model in hydroelectricity management is introduced.

Our main concern is to study controlled PDP's. It signifies that each of the three elements (f,λ,Q) can be parameterized by an extra variable in the set $U \subset \mathbb{R}^p$, whereas the transition measure P can be parameterized by an extra variable in the set $U_\partial \subset \mathbb{R}^q$. The set U_∂ may or may not coincide with set U, it is introduced to allow independent parameterization at the boundary set E_∂. Sets U and U_∂ are understood to be compact sets. To avoid unnecessary remarks, we suppose that for each $u \in U$ or $u \in U_\partial$ as appropriate, $Q(\cdot,u)$ and $P(\cdot,u)$ are defined or can be suitably extended to the entire closed set E. Let (T_k), $k = 1,2,...$ be the sequence of jump times of a sample path, and let us denote $z_k = x_{T_k}$, $k = 1,2,...$. Admissible controls are mesurable functions $u = u_0/u_\partial$ or $u = u_0/\tilde{u}_\partial$ with $u_0 : \mathbb{R}_+ \times E \to U$, $u_\partial : \mathbb{R}_+ \times E \to U_\partial$ and $\tilde{u}_\partial : E \to U_\partial$ such that a control process is:

$$(1.1)\quad u_0(t,z)/u_\partial(t,z)I_{\{t\le T_1\}} + \sum_{k=1} u_0(t-T_k,z_k)/u_\partial(t-T_k,z_k)I_{\{T_k<t\le T_{k+1}\}} \quad , \quad t \ge 0$$

or

$$(1.2)\quad u_0(t,z)/\tilde{u}_\partial(x_{t-})I_{\{t\le T_1\}} + \sum_{k=1} u_0(t-T_k,z_k)/\tilde{u}_\partial(x_{t-})I_{\{T_k<t\le T_{k+1}\}} \quad , \quad t \ge 0$$

The admissible class of controls $\underline{\underline{U}}$ is formed by the product $\underline{\underline{U}}_0 \times \underline{\underline{U}}_\partial$ which contains controls in the interior set, u_0 and in the boundary subset E_∂, u_∂ in the form specified by (1.1). $\underline{\underline{U}}$ is alternatively formed by $\underline{\underline{U}}_0 \times \tilde{\underline{\underline{U}}}_\partial$, which contains all controls $u_0/\tilde{u}_\partial$ specified by (1.2). At some stage, we wish to assert the uniqueness of solutions for the differential equation

$$\frac{d}{dt}\, x(t) = f\,(x(t),u(t)), \qquad x(0) = z\ .$$

In general that can not be achieved if the feedback control $u\,(x(\cdot))$ is employed. For the study of the single jump problem in chapter 3, a mixture of feedback controls for the boundary subset and controls of form $u(\cdot,z)$ for the interior domain turns out to be appropriate. In the characterization of chapter 4 both classes have the form indicated by (1.1).

The aim is to solve the stochastic PDP control problem, thereby minimizing the associated cost. We seek the optimal control u in

$\underline{\underline{U}}$, in the sense that the functional

$$E_z^u\left[\int_0^\tau \ell(x_{t^-},u_t)d\mu_t\right]$$

is minimized for each $z \in E$. E_z^u denotes the expectation with respect to the measure P_z^u, τ is the jumping time to the state Δ, $\ell(\cdot,\cdot)$ in the local cost (non-negative) and $(\mu_t),t\geq 0$ is a an increasing process, to be specified below. For each $t\in[0,\tau(\omega)]$ and control $u = u_0/u_\partial$ in the admissible class, the instantaneous cost is allowed to assume the following form

$$\ell(x_{t^-},u_t)d\mu_t = \ell_0(x_{t^-},u_{0,t})dt + \ell_\partial(x_{t^-},u_{\partial,t})I_{\{x_{t^-}\in E_\partial\}}$$

$\ell_\partial(\cdot,\cdot)$ arises in connection with costs of jumping from the boundary subset and $\ell_0(\cdot,\cdot)$ is related to running costs. Costs associated with jumps from the interior set can also be taken into account by a redefinition of $\ell_0(\cdot,\cdot)$.

The framework of non-smooth analysis is employed in this work, and we consider the data of the problem (ℓ_0,ℓ_∂) and of the PDP (f,λ) to be bounded measurable functions which are Lipschitz continuous on E for each fixed $u \in U$. Similar requirements are needed for the measures Q and P. For the precise details, see conditions (DC1) to (DC6) in chapter 2, sections 2.3 and 2.4.

The main results are stated in the sequel. Let $\Sigma^+(K)$ be the class of nonegative bounded Lipschitzian functions of rank K defined on E. We define functions:

$$(1.3) \qquad H : \mathbb{R} \times \mathbb{R}^n \times E \times U \times \bigcup_{K<\infty} \Sigma^+(K) \to \mathbb{R}$$

by

$$H(q,p,z,a,[\psi])$$

$$= \langle p,f(z,u)\rangle - \ell_0(z,u) - \lambda(z,u)\int_E [\psi(y)+q]dQ(y;z,u)$$

and

$$(1.4) \qquad F : E \times U_\partial \times \bigcup_{K<\infty} \Sigma^+(K) \to \mathbb{R}$$

by

$$F(z,u,[\psi]) = \ell_\partial(z,u) + \int_E f(y)dP(y;z,u).$$

Let us denote by $V : E \to \mathbb{R}$ the value function for the stochastic PDP problem:

$$V(z) = \inf_{u \in \underline{\underline{U}}} E_z^u \left[\int_0^\tau \ell(x_{t^-},u_t)d\mu_t \right].$$

Theorem 1.1: Let V^1 be an arbitrary function of class $\Sigma^+(K)$, for some K. Let a sequence $\{V^i\}$ be defined in $\bigcup_{K<\infty} \Sigma^+(K)$ by the modified Hamilton-Jacobi-Bellman equation

$$(1.5) \qquad \min_{\xi \in \partial V^i(z)} -\max_{u \in U} H(-V^i(z), -\xi,z,u,[V^{i-1}]) = 0$$

for each $z \in E$,

$$(1.6) \qquad V^i(z) + \max_{u \in U_\partial} - F(z,u,[V^{i-1}]) = 0$$

for each $z \in E_\partial$.

It follows that $V^i(z) \to V(z)$ uniformly on E.

Here $\partial V(\cdot)$ indicates the generalized gradient in the sense of Clarke. Expression (1.6) is the boundary condition of the "p.d.e." (1.5) for Lipschitzian function V^i, whereby $\{V^i\}$ can be sequentially constructed. The result is valid under conditions (A1) to (A5) introduced in chapter 3, section 3.1 and condition (A6) in chapter 4, section 4.1. For details and comments see remarks and lemmas in those sections. Provided that conditions (A8) and (A9) of chapter 4, section 4.6 are satisfied, the problem of determining the sequence $\{V^i\}$ can be handled numerically by the procedure proposed by Gonzalez and Rofmann [G1]. Given a sequence $\{\bar{V}^i\}$ of approximations to $\{V^i\}$, proposition 4.1 of chapter 4, section 4.6 supplies the number I of approximations that suffices to attain the value V within a pre-specified precision.

In addition to the sequential characterization in theorem 1.1, direct criteria of optimality can be established. For that purpose, a reinforced version of condition (DC1) in chapter 2, section 2.3 is needed. This is condition (A7), chapter 4, section 4.5

Verification Theorem:

Theorem 1.2: Suppose that (A1)-(A7) hold. A necessary and sufficient condition for a control u in $\underline{\underline{U}}$ with $u = u_0/u_\partial$ and $u_0 \in \underline{\underline{U}}_0$, $u_\partial \in \underline{\underline{U}}_\partial$, to be optimal is that

$$\min_{\xi \in \partial V^u(x(t,z))} - H(-V^u(x(t,z)),-\xi,x(t,z),u_0(t,z),[V^u]) = 0$$

for each $z \in E$ and $t \in [0,t^*_{u_0}(z)]$. In addition

$$V^u(x(t^*_{u_0},z)) - F(x(t^*_{u_0},z),u_\partial(t^*_{u_0},z),[V^u]) = 0$$

for each $z \in E$. The notations $x(\cdot,z) = x_{u_0}(\cdot,z)$ and $t^*_{u_0} = t^*_{u_0}(z) = \inf\{t \geq 0: x_{u_0}(t,z) \in E_\partial\}$ are used.

Maximum Principle:

Theorem 1.3: Let $u = u_0/u_\partial$ be a solution for the stochastic PDP problem, where u_0 is a control of class $\underline{\underline{U}}_0$ and u_∂ is a control of class $\underline{\underline{U}}_\partial$. Let $t \to x_{u_0}(t,z)$ be a trajectory on $[0,t^*_{u_0}(z)]$ with $x_{u_0}(0,z) = z$, which coincides with the left continuous version of (x_t) on the interval $(T_i(\omega), T_{i+1}(\omega)]$ with $x_{T_i}(\omega) = z$ for some i, $i = 0,1,\ldots$. Suppose that conditions (A1) - (A7) hold. Then there exists

(a) $F_\partial : E \to \mathbb{R}$ a Lipschitzian function.

(b) $p : [0,t^*_{u_0}(z)] \to \mathbb{R}^n$, $q : [0,t^*_{u_0}(z)] \to \mathbb{R}$

absolutely continuous function for each i, such that the following relations are verified

(i) $F_\partial(x(t^*_{u_0},z)) = F(x(t^*_{u_0},z),\ u_\partial(t^*_{u_0},z))$

$= \inf_{u \in U_\partial} F(x(t^*_{u_0},z),u)$

(ii) $-\dot{p}(t) + \lambda(x(t,z),u_0(t,z))p(t)$

$\in \partial_x H(q(t),p(t),x(t,z),u_0(t,z),[V^u])$ a.e.

$-\dot{q}(t) = H(q(t),p(t),x(t,z),u_0(t,z),[V^u])$

$-\langle p(t),f(x(t,z),u_0(t,z))\rangle$ a.e.

(iii) $\max_{v \in U} H(q(t),p(t),x(t,z),v,[V^u])$

$= H(q(t),p(t),x(t,z),u_0(t,z),[V^u])$

$= 0$ a.e.

Whenever $t^*_{u_0}(z) < \infty$:

(iv) $[q(t^*_{u_0})\ p(t^*_{u_0})] + r\xi \in [0 \quad -c\partial d_E(x(t^*_{u_0},z)]$

for some nonegative scalar c and a n+1-dimensional vector ξ so that

$\xi \in [F_\partial(x(t^*_{u_0},z))\ \partial F_\partial(x(t^*_{u_0},z)].$

Whenever $t^*_{u_0}(z) = +\infty$:

(v) $|q| + |p| < \infty.$

The notations $x(\cdot,z) = x_{u_0}(\cdot,z)$ and $t^*_{u_0} = t^*_{u_0}(z)$ are used and in (ii) $\partial_x H$ indicates the partial generalized gradient with respect to the third variable, see [C1].

The above assertions are drawn from the study of a smaller PDP control problem, presented in chapter 3. It possesses a completely deterministic counterpart which is thoroughly employed. The analysis is based on deterministic control theory and the appropriate regularity follows by introducing a further deterministic equivalence. In chapter 4, the solution for this sub-problem is related to the original PDP control problem by virtue of an embedding, justified by Dynamic Programming.

Earlier results for control of PDP's under the present formulation can be found in Vermes [V1]. The works of Pliska [P1], Rishel [R1], Vermes [V2] and Soner [S2] deal with problems which can be reformulated as particular cases of controlled PDP's. In the work of Vermes [V1] the assumption (A4') of chapter 3, section 3.1 was employed. It imposes conditions in the family of vector fields that the example of section 4.2 does not comply with. It was possible to introduce an alternative assumption called calmness ((A5), chapter 3, section 3.1) which is a version of the local calmness condition, of deterministic control theory (see [C2]). It is easy to check that (A5) is automatically satisfied when (A4') holds true. The result of Vermes is based on existence and convex theory and it provides an upper envelope characterization for the value function. Namely, $V = \sup \Psi$, where Ψ is a $C^1(E)$ sub-class, which comprises all functions ψ satisfying the Hamilton-Jacobi-Bellman inequalities :

$$\max_{u \in U} H(-\psi(z), -\nabla\psi(z), z, u, [\psi]) \leq 0$$

for $z \in E$,

$$\psi(z) + \max_{u \in U_\partial} - F(z,u) \leq 0$$

for $z \in E_\partial$.

A necessary and sufficient condition in a limiting form is also provided in that work. In the present context, theorem 1.1. gives the value function as the uniform limit of a sequence of Lipschitz continuous functions, hence it is a continuous function. The sequence is explicitly constructed by the Hamiltonian (1.3) and boundary condition (1.4), so that this form of solution has an appeal for numerical recursive methods. Additionally, if Q and P are Lipschitz continuous in the sense established in condition (A7), chapter 4, section 4.5, the value itself is Lipschitz continuous and direct criteria of optima lity can be provided in the form of a Verification Theorem and a Maximum Principle.

The work is arranged as follows: In chapter 2 the PDP is presented and a isomorphism between the natural filtration and a discrete process is established. The controlled PDP and the stochastic PDP control problem are stated precisely. In chapter 3 a study of regularity is developed for problem P_1, a PDP control problem with a single jump. Two standard formulations of deterministic optimal control are introduced denoted $P_{d,z}$ and $\tilde{P}_{d,z}$ and it is shown they have solutions which are equivalent to solutions to P_1 in an appropriate sense. In chapter 4 the original problem is regained and solutions are cast in the form of solutions for P_1. Theorem 1.1 is proven in section 4.4 and theorems 1.2 and 1.3 in section 4.5.

Numerical methods are suggested in sections 4.6. In the remainder of this chapter we present two applications for which the main features are captured by the controlled PDP model and the stochastic PDP control problem.

1.2 Capacity Expansion Under Uncertainty [D2]

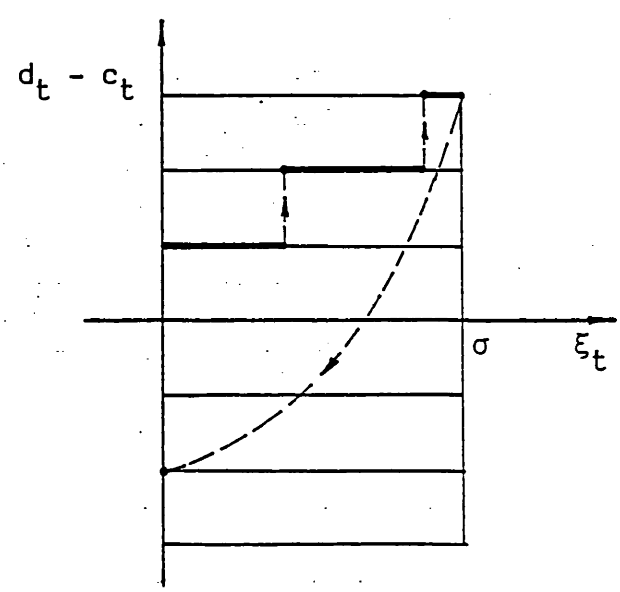
Capacity expansion is the process of providing new facilities over time to meet uncertain demand. Typical examples are the planning of expansion of electrical power systems, steel industry, major computer and communication systems, etc. These are large scale projects for which management decisions to enlarge should be made well ahead in time to avoid undercapacity and carefully enough to avoid surplus and unnecessary waste of resources. The time span between the decision in favour of an increase in the capacity and the ultimate completion and provision of a new unit should be considered in the planning of these projects, due to significant periods of installation or construction. Suppose that at some instant the decision to start the construction of a new unity is taken, supported by forecasts of demand, costs, and completion time, so that investment is channelled into the project with some specified rate. It may happen that during the period of installation, the actual demand deviates from the forecast value. In that case corrections in the rate of investment should be introduced accordingly, acting as a control variable. The problem described can be posed in the framework of controlled PDP's. For clarity, we give a simplified version which underlines the features of uncertainty of future demand and no instantaneous provision of capacity. It illustrates simultaneously the need of such a general formulation.

The demand (d_t) is modelled as a random point process so that it increases by discrete amounts at random times. The installed capacity (c_t) is known and it increases by K units whenever a project is completed. The total investment required in a single project is a constant σ. Let (u_t) be the rate of investment, then for each t

$$\xi_t = \int_{\tau}^{t} u_t \, dt \leq \sigma$$

where τ is the last time before t at which a project was completed. The new project becomes operational at time the integral equals σ. Suppose that the demand is exponentially distributed with rate λ. One can define a controlled PDP as follows:

(1) (x_t) is a two dimensional process taking values on $E = Z \times [0, \sigma]$ and the boundary subset E_∂ is $Z \times \{\sigma\}$ (Z is the integer lattice), such that $(x_t) = (x_t^1\ x_t^2) = (d_t - c_t,\ \xi_t)$.



(2) The quadruple of local characteristics is given by

$$f(x,u) = (0 \quad u) \quad ; \quad \lambda(x,u) = \lambda$$

$$Q(\cdot;x,u) = \delta_{(x^1+1,x^2)} \quad ; \quad P(\cdot;x,u) = \delta_{(x^1-K,0)}$$

($\delta_{(x^1,x^2)}$ denotes the Dirac measure at $x = (x^1\ x^2)$). By reaching E_∂ a unit is completed and K units of capacity are made available. The objective is to choose an investment strategy so as to satisfy demand while avoiding wasteful over-capacity. A function $(z,u) \to \ell_0(z,u)$ provides the local cost and the control problem consists on minimizing the discounted infinite horizon cost

$$E_z^u \left[\int_0^\infty e^{-\delta t}\ \ell_0(x_t,u_t)dt \right]$$

over the class of admissible policies. Here admissible policies are interior controls (no control at E_∂), defined by

$$u(t,(d_0 - c_0,\ \xi_0))I_{\{t \leq T_1\}}$$

$$+ \sum_{k=1} u(t-T_k,(d_{T_k} - c_{T_k},\ \xi_{T_k}))I_{\{T_k < t \leq T_{k+1}\}} \in [0,\rho],\ t \geq 0$$

where ρ is the maximum rate of investment. The infinite horizon problem can be transformed into the standard problem by introducing $\tau(\omega)$, the jumping time into state Δ, as a random variable with exponential distribution of rate δ.

1.3 Management of Hydropower Production

The hydropower system we consider here is composed of I_0 independent water reservoirs which should jointly supply a demand for electricity, denoted by $(D(t)), t \geq 0$. Each of these reservoirs can represent a single dam or an aggregate of various dams. It may happen that a number of dams share the same supply of water or are located in a same region, thereby the water inlets are replenished by the same rain pattern; for the purpose here, they behave like a single unit. These considerations are not directly relevant in the PDP modelling, but they may be essential to reduce the problem to an acceptable size. The model in the form presented here is perhaps oversimplified, but the reader will be able to see how to elaborate it without leaving the general framework of controlled PDP's.

We proceed with the definition of the model for the i-th dam. The following variables are introduced

$v_i(\cdot)$ = input flow of water

$h_i(\cdot)$ = disposable stock of water

$u_i(\cdot)$ = water flow employed to generate power

$s_i(\cdot)$ = water flow released through spillway.

They can take the values as indicated below

$$v_i(\cdot) \geq 0 \quad , \qquad h_i(\cdot) \in [\underline{h}_i, \bar{h}_i] \tag{1.9}$$

$$u_i(\cdot) \in [0, \bar{u}_i] \quad , \quad s_i(\cdot) \geq 0.$$

In normal operation the dynamical behaviour of a dam is given by

$$(1.10) \qquad \frac{d}{dt} h_i(t) = v_i(t) - u_i(t) - s_i(t) .$$

The input flow $(v_i(t)), t \geq 0$ is a stochastic process with jumps. When a rainfall occurs upstream the water flow increases instantaneously by a random amount w. The rainfall depends only of seasonal variations, so that an annual cyclical rate can be attributed by some non-negative function

$$t \to \lambda_i(t)$$

thus, at some time t_0, the expected number of rainshowers in the interval $(t_0, t_0 + \Delta t]$ is given by

$$\lambda_i(t_0)\Delta t + O(\Delta t)$$

The total amount w of precipitation water at each rainfall varies along the year, so that w has a seasonal distribution

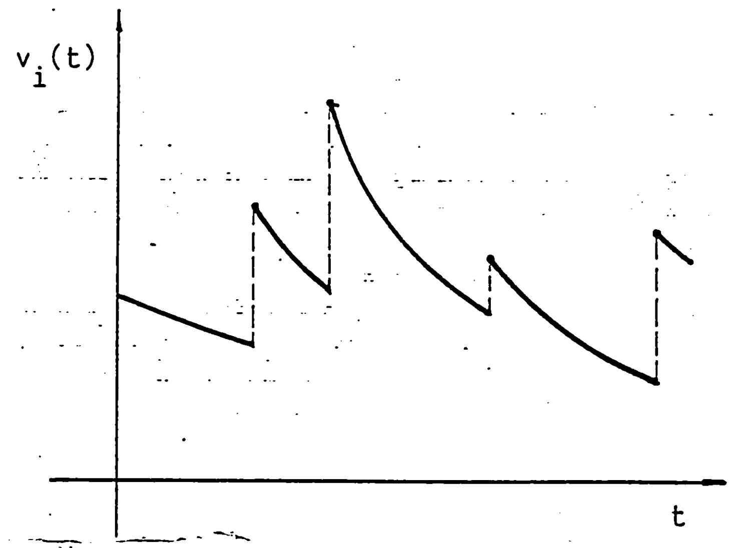
$$t \to P(w\epsilon \cdot \ |T=t)$$

where T is the time a rainfall is observed.

Between any two consecutive rainfalls the input flow of water is deterministic, and the variation $\frac{d}{dt} v_i(t)$ depends only upon the present flow $v_i(t)$ itself and known variables like evaporation at each time of the year, etc. Hence, it can be described analytically by an o.d.e:

$$\frac{d}{dt} v_i(t) = g(t, v_i(t)).$$

The expression (1.10) provides the dynamical relation of the variables when the operation of a dam is normal, namely



$h_i(\cdot) \in (\underline{h}_i, \bar{h}_i)$ and the amount of water released $u_i(\cdot) + s_i(\cdot)$ is free. We associate the status "0" (=normal) to this type of operation.

If the water level reaches the value $\bar{h}_i$, the variables of the problem are related by

$$(1.11) \qquad \begin{aligned} dh_i(t) &= 0 \\ v_i(t) &= u_i(t) + s_i(t) \end{aligned}$$

the operation assumes the abnormal status +1 (= maximum level). The normal operation status is re-established when the water level drops to a certain value $\bar{h}_i - \Delta\bar{h}$ and the input flow reaches a lower bound, say when $v_i(t)$ equals $\bar{u}_i$

Suppose that the water level reaches the minimum operational level $\underline{h}_i$. The dynamical equation is

$$(1.12) \qquad \frac{d}{dt} h_i(t) = v_i(t).$$

The output of the reservoir is sealed so that $u_i(t) = s_i(t) = 0$ and the stock of water can only increase. The abnormal status -1 (= no power) is attributed. The operation returns to the normal status when the water reaches a prespecified level above $\underline{h}_i$, say $\Delta\underline{h}$.

A controlled PDP model for the dam is now established. The state process $(x_i(t))$, $t \geq 0$ is given by

$$(n_i(t),\ v_i(t),\ h_i(t)),\ t \geq 0$$

$x_i(\cdot)$ takes its values in the state domain E with $E = E_{-1} \cup E_0 \cup E_{+1}$

and we define the boundary subset $E_\partial = E_{-1,\partial} \cup E_{0,\partial} \cup E_{+1,\partial}$ such that

$$E_{-1} = \{-1\} \times \mathbb{R}_+ \times [\underline{h}_i,\ \underline{h}_i + \Delta\underline{h}]$$

$$E_0 = \{0\} \times \mathbb{R}_+ \times [\underline{h}_i, \bar{h}_i]$$

$$E_{+1} = \{+1\} \times [\bar{u}_i, \infty) \times \{\bar{h}_i\}$$

and

$$E_{-1,\partial} = \{-1\} \times \mathbb{R}_+ \times \{\underline{h}_i + \Delta\underline{h}\}$$

$$E_{0,\partial} = \{0\} \times \mathbb{R}_+ \times (\{\underline{h}_i\} \cup \{\bar{h}_i\})$$

$$E_{+1,\partial} = \{+1\} \times \{\bar{u}_i\} \times \{\bar{h}_i\}.$$

Let A be a set of the measurable space $(E, \underline{\underline{E}})$ of form $A = N \times [v', v''] \times H$. On $n_i = -1$ no power is generated and the PDP data are

$$f(x_i; u_i, s_i) = (0 \quad g(t, v_i) \quad v_i)$$

$$\lambda(x_i; u_i, s_i) = \lambda(t)$$

$$Q(A; x_i; u_i, s_i) = \delta_{-1}(N)\delta_{h_i}(H) \int_{[v'-v_i,\, v''-v_i]} dP(w) \mid T=t)$$

$$P(A; x_i; u_i, s_i) = \delta_0(N)\delta_{h_i}(H)\ \delta_{v_i}\ [v', v'']$$

($\delta_y(B)$ denotes the Dirac measure at y of set B). On $n_i = 0$ the operation is normal and the PDP data are

$$f(x_i;u_i,s_i) = (0 \quad g(t,v_i) \quad v_i-u_i-s_i)$$

$$\lambda(x_i;u_i,s_i) = \lambda(t)$$

$$Q(A;x_i;u_i,s_i) = \delta_0(N)\delta_{h_i}(H)\int_{[v'-v_i,v''-v_i]} dP(w|T=t)$$

if $x_i \in E_{0,\partial}$ with $h_i = \underline{h}_i$

$$P(A;x_i;u_i,s_i) = \delta_{-1}(N)\ \delta_{h_i}(H)\delta_{v_i}[v',v'']$$

if $x_i \in E_{0,\partial}$ with $h_i = \bar{h}_i$

$$P(A;x_i;u_i,s_i) = \delta_{+1}(N)\delta_{h_i}(H)\ \delta_{v_i}[v',v'']$$

Finally, on $n_i = +1$ the water stock is at maximum level and the PDP data are

$$f(x_i;u_i,s_i) = (0 \quad g(t,v_i) \quad 0)$$

$$\lambda(x_i;u_is_i) = \lambda(t)$$

$$Q(A;x_i;u_i,s_i) = \delta_{+1}(N)\delta_{h_i}(H)\int_{[v'-v_i,v''-v_i]} dP(w|T=t)$$

$$P(A;x_i;u_i,s_i) = \delta_0(N)\delta_{\bar{h}_i-\Delta\bar{h}}(H)\delta_{v_i}\ [v',v'']$$

In order to transform this process in a time-homogeneous PDP, it is only necessary to include the time variable t in the state vector; the resulting augmented process is clearly a PDP and does not depend explicitly upon time.

Each dam i has a function $d_i(x_i, u_i)$ associated, so that

$$d_i(+1, h_i, u_i) = P_i(\bar{h}_i, u_i)$$

$$d_i(0, h_i, u_i) = P_i(h_i, u_i)$$

$$d_i(-1, h_i, u_i) = 0$$

and $P_i(h_i, u_i)$ is the power generated when the water level is h_i and the flow of water to activate the generator is u_i. Suppose that the demand D(t) is deterministic and it is always supplied, either by the system of I_0 dams or by some import of energy, so that

$$D(t) = \sum_{i=1}^{I_0} p_i(x_i(t), u_t(t)) + P(t)$$

where P(t) is the amount of electricity imported at time t.

For each unit i let $\ell_{0,i}(x_i; u_i, s_i)$ be a function in which the running costs of operation are expressed, such as operation with low efficiency, wear-out, etc. and let $\ell_{\partial,i}(x_i; u_i, s_i)$ be a function in which stoppages ($n_i = -1$), water spilled during dry seasons ($n_i = +1$), etc., are penalized. The control problem seeks the best strategy by minimizing the expected cost of operation, while avoiding imports or over production of electricity in the long run:

$$E\left\{\int_0^\infty e^{-\alpha t}\left\{C\left[D(t) - \sum_{i=1}^{I_0} d_i(x_i(t);u_i(t),s_i(t))\right]\right.\right.$$

$$+ \sum_{i=1}^{I_0} \ell_{o,i}\ (x_i(t);\ u_i(t),\ s_i(t))$$

$$\left.\left. + \left[\sum_{i=1}^{I_0} \ell_{\partial,i}(x_i(t);u_i(t),s_i(t))\right] I_{\{x_i(t-)\epsilon E_\partial\}}\right\}dt\right\}$$

($x_i(t-) = \lim_{s\uparrow t} x_i(s)$). Function $C(\cdot)$ is the price of imports or a penalization for overproduction, depending on the sign of the argument. A strategy defines $u_i(t)$ and $s_i(t)$ for all i as a function of all observations $x_i(T)$, $i = 1,2,..$ and time interval $t-T$, where T is the last preceding time with respect to t that a rain-shower was observed.

The model seems to fit better than equivalent models for management of electricity production which uses the diffusion model to characterize the rainfall pattern, see Delebecque and Quadrat [D3], Bensoussan and Lions [B2]. The idea of an instantaneous precipitation of rain water is better conveyed by a jump than by a continuous path of a noisy process. In addition, the PDP enables us to introduce different dynamics when the water stock reaches the lower and upper level of saturation and to instantaneously penalize the operation in any of these levels.

CHAPTER 2

2.1 Definitions

Let E be a closed subset of $\mathbb{R}^n$ called state domain set. E is endowed with the Borel σ-algebra $\underline{\underline{E}}$; a distinguished point Δ outside E is added to form the measurable space

$$(E \bigcup \{\Delta\},\ \underline{\underline{E}} * \sigma\{\Delta\})$$

A subset of $E \setminus \text{int}\{E\}$ is denoted by E_∂; this is the active boundary for E and it is defined in (2.2) below.

Consider a probability space $(\Omega, \underline{\underline{F}}, P)$. The Piecewise Deterministic Process (PDP) is postulated as follows:

(P1) A PDP $(x_t(\omega))$, $t \in \mathbb{R}_+$ is a family of random functions defined on $(\Omega, \underline{\underline{F}}, P)$ with values in $(E \bigcup \{\Delta\},\ \underline{\underline{E}} * \sigma\{\Delta\})$

(P2) The paths $t \to x_t(\omega)$ are right-continuous with left hand limits with isolated discontinuities. If $x_t(\omega) = \Delta$ then $x_s(\omega) = \Delta$ for all $s \geq t$. Let $\tau(\omega)$ be defined by

$$\tau(\omega) = \min\{t \in \mathbb{R}_+ : x_t(\omega) = \Delta\}$$

An ordered collection $(T_k(\omega))$, $k = 0,1,\dots$ can be defined in the following way:

$$T_0(\omega) = 0$$

For $k \geq 1$

$$T_k(\omega) = \inf\{t \in \mathbb{R}_+ \setminus [0, T_{k-1}(\omega)] : x_t(\omega) \neq x_{t-}(\omega)\}$$

if $T_{k-1}(\omega) < +\infty$;

$$T_k(\omega) = +\infty$$

if $T_{k-1}(\omega) = +\infty$.

For each $z \in E \bigcup \{\Delta\}$, $t \to x(t,z)$ is an integral curve on $[0,t^*(z)]$ uniquely determined by a stationary deterministic vector field defined on E and extended trivially to $E \bigcup \{\Delta\}$. Whenever $z \in E$

(2.1) $\quad x(\cdot,z) \in \text{int}\{E\} \qquad$ on $[0, t^*(z))$

and

$$x(0,z) = z, \qquad x(t^*(z),z) \in E_\partial$$

where E_∂ is the set

(2.2) $\quad E_\partial = \{z \in E : x(-t,z) \in \text{int}\{E\}, \text{ for some } t > 0\}$

and $t^* : E \to \mathbb{R}_+$ is the function

(2.3) $$t^*(z) = \begin{cases} +\infty & \text{if } x(t,z) \notin E_\partial \quad , \text{ for all } t \in \mathbb{R}_+ \\ \inf\{t \in \mathbb{R}_+ : x(t,z) \in E_\partial\}, & \text{otherwise} \end{cases}$$

Whenever $z = \Delta$, $x(\cdot,z) = z$.

(P3) For each $z \in E \bigcup \{\Delta\}$ and $k = 0,1,..$

$$\sup_{\omega \in \Omega} \{T_{k+1}(\omega) - T_k(\omega) : x_{T_k}(\omega) = z\} \leq t^*(z)$$

Denote $z_k(\omega) = x_{T_k}(\omega)$, $k = 0,1,..$

(P4) The path of a PDP is characterized by

$$x_t(\omega) = \sum_k x(t-T_k(\omega), z_k(\omega)) I_{\{\omega: T_k \le t < T_{k+1}\}}, \qquad t \in \mathbb{R}_+$$

Let $(\overset{\circ}{\underline{\underline{F}}})$, $t \in \mathbb{R}_+$ be the natural increasing filtration:

$$\overset{\circ}{\underline{\underline{F}}}_t = \sigma\{x_s, s \le t\}, \qquad t \in \mathbb{R}_+$$

As a consequence of (P2), the filtration defined above is right-continuous. Denote $(\underline{\underline{F}}_t)$, $t \in \mathbb{R}_+$, the augmented version of $(\overset{\circ}{\underline{\underline{F}}}_t)$, $t \in \mathbb{R}_+$, which contains all subsets of P-null sets of $\underline{\underline{F}}$.

(P5) Let $A \in \underline{\underline{E}} * \sigma\{\Delta\}$. For all $s \le t$

$$P(x_t \in A | \underline{\underline{F}}_s) = P_{x_s}(x_t \in A)$$

Conditions (P1) to (P5) characterizes a right-continuous Markov process.

Let $(Y^k, \underline{\underline{Y}}^k)$ for $k = 0,1,\ldots$ be a copy of the measurable space

$$(\mathbb{R}_+ \times E \bigcup \{\Delta\}, \quad B(\mathbb{R}_+) * \underline{\underline{E}} * \sigma\{\Delta\}).$$

We define

(2.4) $$\Sigma = \prod_{k=0}^{\infty} Y^k$$

and

$$\sigma = \sigma\{ \prod_{k=0}^{\infty} \underline{\underline{Y}}^k \}$$

A collection of mappings (Γ_k), $k = 0,1,\ldots$ from $(\Omega,\underline{\underline{F}},P)$ into (Σ,σ) is defined as follows. From each k, Γ_k is given by

$$\gamma_0 \times \gamma_1 \times \ldots \times \gamma_k : \Omega \to \Sigma$$

such that

$$\gamma_i : \Omega \to Y^i \qquad i = 0,1,\ldots,k$$

with

$$\gamma_0(\omega) = (T_0(\omega), z_0(\omega)) = (0,z)$$

and

$$\gamma_i(\omega) = (T_i(\omega),\ z_i(\omega)) = (T_i(\omega),\ x_{T_i}(\omega))$$

for $i = 1,2,\ldots,k$.

Lemma 2.1 For each $t \in \mathbb{R}_+$ and k with $k = 0,1,\ldots$

$$\overset{\circ}{\underline{\underline{F}}}_t \bigcap \{T_k \leq t < T_{k+1}\} = \Gamma_k^{-1}(\sigma) \bigcap \{T_k \leq t < T_{k+1}\}$$

Proof: (P2) states the right-continuity of a PDP hence it readily follows that a PDP is a separable process. Let A be a set expressed by

$$A = \bigcap_{i \geq 0} A_i \tag{2.5}$$

where

$$A_i = \{\omega : x_{t_i}(\omega) \in C_i\}$$

for some $C_i \in \underline{\underline{E}}$ and $t_i(t)$ a non random time such that $t_i(t) \leq t$ for $i = 0,1,\ldots$ Applying the definition of paths for PDP given by (P4), one obtains

$$A_i \bigcap \{T_k \le t_i(t) < T_{k+1}\}$$

$$= \{x(t_i(t) - T_k, z_k) \in C_i\} \bigcap \{T_k \le t_i(t) < T_{k+1}\} = \Gamma_k^{-1}(B_{i,k})$$

where $B_{i,k}$ is the set:

$$\{(T_k(\omega), z_k(\omega)) : (t_i(t) - T_k, z_k) \in x^{-1}(C_i), T_k \le t_i(t) < T_{k+1}\} .$$

Thus $\Gamma_k^{-1}(B_{i,k}) \in \underline{\underline{F}}_{t_i(t)}$. Now, we write

$$A_i \bigcap \{T_k \le t < T_{k+1}\}$$

$$= \left(\bigcup_{n=0}^{k} A_i \bigcap \{T_n \le t_i(t) < T_{n+1}\}\right) \bigcap \{T_k \le t < T_{k+1}\}$$

$$= \left(\bigcup_{n=0}^{k} \Gamma_n^{-1}(B_{i,n})\right) \bigcap \{T_k \le t < T_{k+1}\})$$

one can denote $B = \bigcap_{i \ge 0} \bigcup_{n=0}^{k} B_{i,n}$. Clearly

$$A \bigcap \{T_k \le t < T_{k+1}\} = \Gamma_k^{-1}(B) \bigcap \{T_k \le t < T_{k+1}\}$$

since A defined by (2.5) is a generating class for $\overset{\circ}{\underline{\underline{F}}}_t$, one has

$$\overset{\circ}{\underline{\underline{F}}}_t \bigcap \{T_k \le t < T_{k+1}\} \subset \Gamma_k^{-1}(\sigma) \bigcap \{T_k \le t < T_{k+1}\}$$

In order to obtain the opposite inclusion, one uses the equivalence indicated by (P2)

$$\sum_{i \geq 0} I_{\{T_i \leq t\}} = \sum I_{\{x_s \neq x_{s-}, s \leq t\}}$$

and the expression in the right-hand side is $\overset{\circ}{\underline{\underline{F}}}_t$-measurable so that

$$(2.6) \qquad \sigma\{T_0, T_1, \ldots, T_k\} \bigcap \{T_k \leq t < T_{k+1}\} \bigcap \{T_k < \infty\}$$

$$\subset \overset{\circ}{\underline{\underline{F}}}_t \bigcap \{T_k \leq t < T_{k+1}\} \bigcap \{T_k < \infty\}$$

The inclusion above also holds on $\{T_k = +\infty\}$ because $(\overset{\circ}{\underline{\underline{F}}}_t)$, $t \geq 0$ is right-continuous. This same property of $(\overset{\circ}{\underline{\underline{F}}}_t)$, $t \geq 0$ also yields

$$\{x_{T_k}(\omega) \in A\} \in \overset{\circ}{\underline{\underline{F}}}_{T_k} \qquad k = 0, 1, \ldots$$

since $z_k = x_{T_k}$, $k = 0, 1, \ldots$, it follows that

$$(2.7) \qquad \sigma\{z_0, z_1, \ldots, z_k\} \bigcap \{T_k \leq t < T_{k+1}\} \subset \overset{\circ}{\underline{\underline{F}}}_t \bigcap \{T_k \leq t < T_{k+1}\}$$

In view of (2.6) and (2.7) we have

$$\Gamma_k^{-1}(\sigma) \bigcap \{T_k \leq t < T_{k+1}\} \subset \overset{\circ}{\underline{\underline{F}}}_t \bigcap \{T_k \leq t < T_{k+1}\}$$

and the result is proven.

Lemma 2.2: Let $T(\omega)$ be a $\overset{\circ}{\underline{\underline{F}}}_t$-stopping time. The equivalence in lemma 2.1 holds if t and $\overset{\circ}{\underline{\underline{F}}}_t$ are replaced by T and $\overset{\circ}{\underline{\underline{F}}}_T$

Proof: The proof proceeds by steps. Initially, let $S(\omega)$ be a $\overset{\circ}{\underline{\underline{F}}}_t$-stopping time, taking values in a countable subset of $\mathbb{R}_+$ denoted by

$\{s_n\}$, n = 0,1... We want to show

$$(2.8)\qquad \underline{\underline{F}}_S \cap \{T_k \le S < T_{k+1}\} = \Gamma_k^{-1}(\sigma) \cap \{T_k \le S < T_{k+1}\}$$

for each k = 0,1,.. In order to verify the above equation one simply writes

$$\underline{\underline{F}}_S \cap \{S = s_n\} = \underline{\underline{F}}_{s_n} \cap \{S = s_n\}$$

so that, from lemma 2.1 one has

$$\underline{\underline{F}}_S \cap \{T_k \le S < T_{k+1}\}$$

$$= \bigcup_n (\underline{\underline{F}}_{s_n} \cap \{S = s_n\} \cap \{T_k \le s_n < T_{k+1}\})$$

$$= \bigcup_n (\Gamma_k^{-1}(\sigma) \cap \{S = s_n\} \cap \{T_k \le s_n < T_{k+1}\})$$

$$= \Gamma_k^{-1}(\sigma) \cap \{T_k \le S < T_{k+1}\}$$

thus the result is asserted for countable-valued $\overset{\circ}{\underline{\underline{F}}}_t$-stopping times. Next, we consider an arbitrary $\overset{\circ}{\underline{\underline{F}}}_t$-stopping time $T(\omega)$ and define $S_n(\omega)$ by

$$S_n(\omega) = \min \left\{ \frac{i}{2^n} : T(\omega) < \frac{i}{2^n} \right\}$$

$S_n(\omega)$ is also a $\overset{\circ}{\underline{\underline{F}}}_t$-stopping time and it is countable-valued; hence (2.8) holds for such a random time. These considerations yield

$$(2.10)\qquad \overset{\circ}{\underline{\underline{F}}}_T \cap \{T_k \le S_n < T_{k+1}\} \subset \Gamma_k^{-1}(\sigma) \cap \{T_k \le S_n < T_{k+1}\}$$

since, in view of (2.9) $\overset{\circ}{\underline{\underline{F}}}_T \subset \overset{\circ}{\underline{\underline{F}}}_{S_n}$

Now, let us denote by G_n the following ω-set

$\{x_t(\omega) = x(t-T(\omega), x_T(\omega)),\ t \in [T, T + 2^{-n})\}$

One concludes that

$$\{T_k \leq S_n < T_{k+1}\} \cap G_n = \{T_k \leq T < T_{k+1}\} \cap G_n$$

and by virtue of (2.10)

$$\overset{\circ}{\underline{F}}_T \cap \{T_k \leq T < T_{k+1}\} \cap G_n \subset \Gamma_k^{-1}(\sigma) \cap \{T_k \leq T < T_{k+1}\} \cap G_n$$

or, equivalently one can maintain that for each $A \in \overset{\circ}{\underline{F}}_T$, there exists a set $B_n \in \Gamma_k^{-1}(\sigma)$ such that

$$A \cap \{T_k \leq T < T_{k+1}\} \cap G_n = B_n \cap \{T_k \leq T < T_{k+1}\} \cap G_n$$

$n = 0, 1, \ldots$

The sequence (G_n) is monotone increasing therefore (B_n) also is. Moreover, because $(x_t(\omega))$ is right-continuous process for each $\omega \in \Omega$, it follows that $G_n \uparrow \Omega$, so that

$$A \cap \{T_k \leq T < T_{k+1}\} = \Big(\bigcup_{i \geq 0} \bigcap_{n \geq i} B_n\Big) \cap \{T_k \leq T < T_{k+1}\}$$

Denote B the set $\bigcup_{i \geq 0} \bigcap_{n \geq i} B_n$; clearly $B \in \Gamma_k^{-1}(\sigma)$ on $\{T_k \leq T < T_{k+1}\}$, which implies the following inclusion

$$(2.11) \quad \overset{\circ}{\underline{\underline{F}}}_T \bigcap \{T_k \leq T < T_{k+1}\} \subset \Gamma_k^{-1}(\sigma) \bigcap \{T_k \leq T < T_{k+1}\}$$

Next, let $\{S_n\}$ be a sequence of countable-valued $\overset{\circ}{\underline{\underline{F}}}_t$-stopping times such that $S_n \downarrow T$. The right-continuity of $\overset{\circ}{\underline{\underline{F}}}_T$ readily provides the inclusion opposite to (2.11) because

$$\Gamma_k^{-1}(\sigma) \bigcap \{T_k \leq T < T_{k+1}\}$$
$$= \lim_n \Gamma_k^{-1}(\sigma) \bigcap \{T_k \leq S_n < T_{k+1}\} \subset \lim_n \underline{\underline{F}}_{S_n} \bigcap \{T_k \leq S_n < T_{k+1}\}$$
$$= \underline{\underline{F}}_T \bigcap \{T_k \leq T < T_{k+1}\}$$

upon applying (2.8) with $S(\omega) = S_n(\omega)$ for each n; the lemma is thereby proven.

Lemmas 2.1 and 2.2 establish a correspondence between $(\overset{\circ}{\underline{\underline{F}}}_t)$-measurable sets of $(\Omega, \underline{\underline{F}})$ and the measurable space defined in (2.4)(Σ, σ). This correspondence can be extended to all $(\underline{\underline{F}}_t)$ - measurable sets of $(\Omega, \underline{\underline{F}})$ by means of mappings (Γ_k), $k = 0, 1, \ldots$ For each $A \in \underline{\underline{F}}$ with $P(A) = 0$ let $N \in \sigma$ be set

$$N = \bigcap_i \{\Gamma_i(\omega) : \omega \in A\}$$

Denote $\underline{\underline{N}}$ the σ-algebra generated by all sets $N \in \sigma$ as above and denote

$$\hat{\Gamma}_k^{-1}(\sigma) = \Gamma_k^{-1}(\sigma) \bigcup \{\bigcup_i \Gamma_i^{-1}(\underline{\underline{N}})\}$$

This construction yields the following generalization of lemma 2. 2

Lemma 2.3: Let $T(\omega)$ be a $\underline{F}_t$-stopping time

$$\underline{F}_T \bigcap \{T_k \leq T < T_{k+1}\} = \hat{\Gamma}_k^{-1}(\sigma) \bigcap \{T_k \leq T < T_{k+1}\}$$

for each $k = 0,1,\ldots$

The lemma below provides a representation for random variables

Lemma 2.4: Let $T(\omega)$ be a $\underline{F}_t$-stopping time. For each r.v. $y(\omega)$ which is $\underline{F}_T$-measurable there exists Borel-measurable functions $\{Y_k\}$, $k = 0,1,\ldots$, such that for each k

$$y(\omega) I_{\{\omega: T_k \leq T < T_{k+1}\}} = Y_k(T_0, z_0, \ldots, T_k, z_k) I_{\{\omega: T_k \leq T < T_{k+1}\}}$$

Proof: By applying lemma 2.3, one obtains that for each $A \in \underline{F}_T$ there exists a $B \in \sigma$ such that

$$I_A(\omega) I_{\{\omega: T_k \leq T < T_{k+1}\}} = I_B(T_0, z_0, \ldots, T_k, z_k) I_{\{\omega: T_k \leq T < T_{k+1}\}}$$

A direct consequence of the relation above is that the representation in the lemma is valid for any simple r.v. $y(\omega)$ which is $\underline{F}_T$-measurable. The lemma is verified for each $y(\omega)$ which is $\underline{F}_T$-measurable by expressing it as a limit of a family of simple r.v.'s.

2.2 Markov Piecewise Deterministic Processes

At this stage the markovian feature of a PDP will be directly employed. The structure of σ-fields and the representation for r.v.'s of the last section yield some valuable properties.

Lemma 2.5: A PDP is a strong Markov process.

Proof: Let $T(\omega)$ be a $\underline{\underline{F}}_t$-stopping time. The process is adapted to $\underline{\underline{F}}_t$ and is right-continuous. The progressive measurability of (x_t) follows from that, hence x_t in $\underline{\underline{F}}_T$-measurable r.v. In order to verify the strong Markov property, it suffices to show that for each $z \in E$

$$E_z[f(x_{t+T})] = E_z[E_{x_T}[f(x_{t+T})]]$$

see [B4, proposition I.8.2]. Using the definition of paths for a PDP in (P4), we can write:

$$E_{x_T}[f(x_{t+T})]I_{\{\omega:T_k\leq T<T_{k+1}\}}$$

$$= \quad E_{x(T-T_k,z_k)}[f(x_{t+T})]I_{\{\omega:T_k\leq T<T_{k+1}\}}$$

Now one applies the representation of lemma 2.4 to obtain

$$T(\omega)I_{\{\omega:T_k\leq T<T_{k+1}\}} = t_k(T_0,z_0,\ldots,T_k,z_k)I_{\{\omega:T_k\leq T<T_{k+1}\}}$$

Thus

$$E_{x_T}[f(x_{t+T})]I_{\{\omega:T_k\leq T<T_{k+1}\}} = E_{x(t_k-T_k,z_k)}[f(t+T)]I_{\{\omega:T_k\leq T<T_{k+1}\}}$$

the r.v. above is measurable with respect to $\hat{\Gamma}_k^{-1}(\sigma)$, therefore

$$= E[f(x_{t+T}) | \hat{\Gamma}_k^{-1}(\sigma)] I_{\{\omega : T_k \leq T < T_{k+1}\}}$$

using the equivalence of σ-fields in lemma 2.3, one can write

$$= E[f(x_{t+T}) | \underline{\underline{F}}_T] I_{\{\omega : T_k \leq T < T_{k+1}\}}$$

To complete the proof one expresses

$$E[f(x_{t+T})] = E[\sum_k E[f(x_{t+T}) | \underline{\underline{F}}_T] I_{\{T_k \leq T < T_{k+1}\}}]$$

$$= E[\sum_k E_{x_T}[f(x_{t+T})] I_{\{T_k \leq T < T_{k+1}\}}] = E[E_{x_T}[f(x_{t+T})]].$$

The Markov PDP will be made more explicit. The construction in the remark proceeding lemma 2.3 indicates that a PDP can be equivalently defined by the stochastic kernel of the discrete process $(T_k(\omega), z_k(\omega))$, $k = 0,1,\ldots$ Furthermore the strong Markov property and the time-homogeneous character of a PDP make evident that the discrete process is characterized by the distribution

(2.14) $\quad P_z\{(T_1 > t) \bigcap (z_1 \in A)\} \qquad t \in \mathbb{R}_+,\ A \in \underline{\underline{E}} * \sigma\{\Delta\}$

for each $z \in E \bigcup \{\Delta\}$

Let us denote for each $z \in E \bigcup \{\Delta\}$

$$F_t^A(z) = P_z\{(T_1 > t) \bigcap (z_1 \in A)\} \quad t \in \mathbb{R}_+,\ A \in \underline{\underline{E}} * \sigma\{\Delta\}$$

and let $F_t(z) = F_t^{E\{\Delta\}}(z)$. It is evident that the measure $F_t^A(z)$ on $(\mathbb{R}_+, B(\mathbb{R}_+))$ is absolutely continuous with respect to that defined by $F_t(z)$. Hence a positive real-valued measurable function $t_1 \to \mathring{Q}_t^A(z)$ exists such that

$$F_t^A(z) = -\int_{[t,\infty)} \mathring{Q}_t^A(z)dF_t(z)$$

Since $(E\bigcup\{A\}, \underline{\underline{E}} * \sigma\{\Delta\})$ is metrizable, we invoke the result of [B1, proposition 7.17] to assert the existence of regular conditional probabilities, so that one can always identify

$$P_z(z_1 \in A|T_1 = t) = \mathring{Q}_t^A(z) \qquad \text{a.s.}$$

In most applications the PDP is constructed from local distributions and parameters defined as follows

(D1) Transition measures:

$$Q : E \to \underline{\underline{P}}(E\bigcup\{\Delta\}) \qquad P : E \to \underline{\underline{P}}(E\bigcup\{\Delta\})$$

($\underline{\underline{P}}(A)$ is the set of probability measures on A). One relates

$$\mathring{Q}_t^A(z) = Q(A,x(t,z))$$

on $0 \leq t < t^*(z)$.

$$\mathring{Q}_t^A(z) = P(A, x(t,z))$$

on $t = t^*(z)$

Q and P satisfy $Q(E_\partial,\cdot) = P(E_\partial,\cdot) = 0$

(D2) Jump rate:

$\lambda : E \to \mathbb{R}_+$, bounded and Lipschitz continuous.

Because the process is Markov, one possible characterization for $t \to F_t(z)$ is by means of an exponential type of distribution with rate function $\lambda(\cdot)$ as follows:

$$dF_t(z) = -\lambda(x(t,z))\, F_t(z)dt$$

on $0 \leq t < t^*(z)$,

$$dF_t(z) = -F_{t^-}(z)$$

on $t = t^*(z)$.

Alternatively, $t \to F_t(z)$ can be defined by means of a density function in the interior domain:

(D2') $g : E \to \mathbb{R}$, bounded and Lipschitz continuous such that

$$dF_t(z) = -g(x(t,z))dt$$

on $0 \leq t < t^*(z)$,

$$dF_t(z) = -F_{t^-}(z)$$

on $t = t^*(z)$.

For consistency one has the condition

$$\int_0^{t^*(z)} g(x(t,z))dt \leq 1, \quad \text{for all } z \in E$$

Clearly, definitions (D2) and (D2') are equivalent, since we always can take

$$(2.15)\quad g(x(t,z)) = \lambda(x(t,z)) \exp(-\int_0^t \lambda(x(s,z))ds)$$

or

$$(2.16)\quad \lambda(x(t,z) = g(x(t,z))(1 - \int_0^t g(x(s,z))ds)^{-1} .$$

The motivation to provide two different definitions for $t \to F_t(z)$ relies on the fact that the conditions imposed in (D2) and in (D2') are non-equivalent.

The defining function of the vector field is now identified:

(D3) $f : E \to \mathbb{R}^n$, bounded and Lipschitz continuous.

Thus the integral curve $t \to x(t,z)$ is uniquely defined by the o.d.e.

$$\frac{d}{dt} x(t,z) = f(x(t,z)), \qquad 0 \leq t \leq t^*(z)$$

$$x(0,z) = z$$

for each $z \in E$.

The Markov kernel mentioned in (2.14) can be defined by the local characteristics in (D1), (D2) and (D3) as follows. Let $A \in \underline{\underline{E}} * \sigma\{\Delta\}$

(2.17):

$$P_z\{(T_1 > t)\bigcap(z_1 \in A)\} =$$

$$I_{\{t<t^*(z)\}}\{\int_t^{t^*(z)} \int_A dQ(y;\ x(s,z))\ \lambda(x(s,z))\ \exp(-\int_0^s \lambda(x(r,z))dr)ds$$

$$+ \int_A dP(y;\ x(t^*(z),z))\exp(-\int_0^{t^*(z)} \lambda(x(r,z))dr)\}$$

or equivalently, by the local characteristics given in (D1), (D2') and (D3)

$$(2.18)\quad I_{\{t<t^*(z)\}}\Big\{\int_0^{t^*(z)}\int_A dQ(y;\ x(s,z))g(x(s,z))ds$$

$$+\int_A dP(y;x(t^*(z),z))(1-\int_0^{t^*(z)} g(x(s,z))ds)\Big\}$$

Proposition 2.1: A process constructed from the Markov Kernel (2.17) or (2.18) is a PDP in accordance with the axiomatic definitions (P1) - (P5).

2.3 Controlled Piecewise Deterministic Processes

Let U and U_∂ be compact sets of $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively, called control sets. We are interested in PDP's that are parametrized by functions on the mentioned sets, called control functions. For the purpose here, a control function u is a pair u_0/u_∂ such that

$u_0 \in \underline{\underline{U}}_0$ and $u_\partial \in \underline{\underline{U}}_\partial$ or $u_\partial \in \tilde{\underline{\underline{U}}}_\partial$ where $\underline{\underline{U}}_0$, $\underline{\underline{U}}_\partial$ and $\tilde{\underline{\underline{U}}}_\partial$

are the classes

$$\underline{\underline{U}}_0 = \{u_0\colon \mathbb{R}_+ \times E \to U, \text{ measurable}\},$$

$$\underline{\underline{U}}_\partial = \{u_\partial\colon \mathbb{R}_+ \times E \to U_\partial, \text{ measurable}\},$$

and

$$\tilde{\underline{\underline{U}}}_\partial = \{u_\partial\colon \qquad E \to U_\partial, \text{ measurable}\}.$$

The class $\underline{\underline{U}}_0$ contains the continuous time controls acting in the interior of E whereas $\underline{\underline{U}}_\partial$ or $\tilde{\underline{\underline{U}}}_\partial$ contains controls acting at the subset E_∂ of the boundary for E. The domain of definition is extended to E for technical reasons in chapter 3.

The admissible class of controls is denoted by $\underline{\underline{U}}$ and is formed by one of the product class $\underline{\underline{U}}_0 \times \underline{\underline{U}}_\partial$ or $\underline{\underline{U}}_0 \times \tilde{\underline{\underline{U}}}_\partial$. The control process on $\underline{\underline{U}}_0 \times \underline{\underline{U}}_\partial$ is defined by means of the discrete process $(T_k(\omega), z_k(\omega))$, $k = 0,1,..$:

$$\sum_k u_0(t - T_k(\omega), z_k(\omega)) I_{\{\omega: T_k < t \leq T_{k+1}\}} \quad , \qquad t \geq 0$$

$$\sum_k u_\partial(t - T_k(\omega), z_k(\omega)) I_{\{\omega: T_k < t \leq T_{k+1}\}} \quad , \qquad t \geq 0$$

whereas if $\tilde{\underline{\underline{U}}}_\partial$ is employed, the controls are direct feedback controls, defined in terms of the original process

$$u_\partial(x_{t-}(\omega)) \quad , \qquad t \geq 0$$

Since $t \to x_{t-}(\omega)$ is continuous on $\{\omega: T_k < t \leq T_{k+1}\}$ the conditions on measurability for $\underline{\underline{U}}_\partial$ and $\tilde{\underline{\underline{U}}}_\partial$ are equivalent.

The reasons for using the classes $\underline{\underline{U}}_0$ and $\underline{\underline{U}}_\partial$ are threefold. Firstly, the correspondence between (x_t), $t \geq 0$ and the discrete process (T_k, z_k), $k = 0,1,...$ enable us to employ them, since noting "stochastically relevant" happens between jumps, although the controls provided are not strictly equivalents to feedback controls. Moreover, the resulting continuous time controlled process is Markov provided that the state process to be considered is the augmented process $(\tau_t(\omega),\ z_t(\omega),\ x_t(\omega))$, $t \geq 0$ where

$$\tau_t(\omega) = \min\{t - T_k(\omega) : T_k \in (T_k),\ T_k \leq t\}$$

$$z_t(\omega) = \{z_k(\omega) : k = \max\{i : T_i \in (T_k),\ T_i \leq t\}\}$$

Secondly, in order to define the process the trajectory $t \to x_{u_0}(t,z)$ should be uniquely defined for a given control u_0 and point z in E. If u_0 is feedback control of form $u_0(x_{u_0}(\cdot,z))$, the uniqueness of trajectories for each u_0 and z is lost in general. Thirdly, in order to define some operators associated with a control problem, the control functions between two consecutive jumps are seen as a mapping from E into the space of time functions on U and on U_∂. These are functions of initial conditions $x_{T_k}(\omega)$, $k = 0,1,..,$ and time since last jump $t - T_k(\omega)$, here $\underline{\underline{U}}_0 \times \underline{\underline{U}}_\partial$ provides the right setting.

We proceed with the characterization of the family of controlled PDP's; the local descriptions (Q, P, λ, f) or (Q, P, g, f) will be employed.

(DC1) Controlled transition measures:

$$Q : E \times U \to \underline{\underline{P}}(E \cup \{\Delta\}) \qquad P : E \times U_\partial \to \underline{\underline{P}}(E \cup \{\Delta\})$$

such that

(i) $Q(\cdot,\cdot)$ and $P(\cdot,\cdot)$ are measurable functions

(ii) For any function $h : E \to \mathbb{R}$ which is Lipschitz continuous

$$z \to \int_E h \, dQ(z,u)$$

and

$$z \to \int_E h \, dP(z,u)$$

are Lipschitz continuous functions, for each u in U or in U_∂ respectively

(DC2) Controlled jump rate:

$\lambda : E \times U \to \mathbb{R}_+$ is bounded measurable function such that $z \to \lambda(z,u)$ is Lipschitz continuous for each u in U

(DC2') Controlled jump density:

$g : E \times U \to \mathbb{R}$ is bounded measurable function such that $z \to g(z,u)$ is Lipschitz continuous for each u in U.

(DC3) Controlled vector field:

$f : E \times U \to \mathbb{R}^n$ is bounded measurable function such that $z \to f(z,u)$ is Lipschitz continuous for each u in U

(DC4) Admissible class of controls:

$$\underline{\underline{U}} = \underline{\underline{U}}_\circ \times \underline{\underline{U}}_\partial \text{ or } \underline{\underline{U}}_\circ \times \tilde{\underline{\underline{U}}}_\partial .$$

The subset E_∂ of $E \setminus \text{int}\{E\}$ in the controlled case is given by

(2.19) $$E_\partial = \{z \in E: x_{u_0}(-t,z) \in \text{int}\{E\}, \text{ some } t > 0\}$$

Following these definitions we can characterize the family of controlled process, denoted by $\{(x_t), \underline{\underline{U}}\}$. Suppose that $\underline{\underline{U}} = \underline{\underline{U}}_0 \times \underline{\underline{U}}_\partial$ and (Q, P, λ, f) is the local description which satisfies the requirements on boundness and Lipschitz continuity. For each $u = u_0/u_\partial$ with $u_0 \in \underline{\underline{U}}_0$ and $u_\partial \in \underline{\underline{U}}_\partial$, the transition kernel for the associate discrete process is

(2.20) $$P_z^u\{(T_1 > t) \cap (z_1 \in A)\}$$

$$= I_{\{t<t^*(z))\}} \cdot \{ \int_t^{t^*_{u_0}} \int_A dQ(y; x_{u_0}(s,z), u_0(s,z)) \cdot$$

$$\lambda(x_{u_0}(s,z), u_0(s,z)) \exp(-\int_t^s \lambda(x_{u_0}(r,z)), u_0(r,z))dr)ds$$

$$+ \int_A dP(y; x_{u_0}(t^*_{u_0}(z), z), u_\partial(t^*_{u_0}(z), z)) \cdot$$

$$\exp(-\int_t^{t^*_{u_0}(z)} \lambda(x_{u_0}(r,z), u_0(r,z))dr)\}$$

where

$$\frac{d}{dt} x_{u_0}(t,z) = f(x_{u_0}(t,z), u_0(t,z))$$

on $0 \leq t \leq t^*_{u_0}(z)$ and

$$t^*_{u_0}(z) = \inf\{t \geq 0 : x_{u_0}(t,z) \in E_\partial\}$$

for each $z \in E$, $t \in \mathbb{R}_+$ and $A \in \underline{\underline{E}} * \sigma\{\Delta\}$.

The transition (2.20) characterizes each element of the controlled PDP family $\{(x_t), \underline{\underline{U}}\}$ by the same reasons the transition (2.17) does in the non-controlled case.

2.4 The Stochastic PDP Control Problem

The control problem concerns $\{(x_t), \underline{\underline{U}}\}$. The aim is to minimize the expected value of a functional over $\underline{\underline{U}}$. Initially, let us introduce a counting process (p^*_t), $t \geq 0$ where

$$(2.21) \quad p^*_t(\omega) = \sum_i I_{\{\omega:\, T_i \leq t,\, x_{T_i^-} \in E_\partial\}}$$

A time rate (μ_t), $t \geq 0$ is given by the relation

$$\mu_t(\omega) = (t \quad p^*_t(\omega))'$$

((') indicates the transpose of a vector).

(DC5) In addition let

$\ell_0 : E \times U \to \mathbb{R}_+$ and $\ell_\partial : E \times U_\partial \to \mathbb{R}_+$ be bounded measurable functions such that $z \to \ell_0(z,u)$ and $z \to \ell_\partial(z,u)$ are Lipschitz continuous for each u in U or u in U_∂ as appropriate.

Denote $\ell(\cdot,\cdot) = (\ell_0(\cdot,\cdot) \quad \ell_\partial(\cdot,\cdot))$. The stochastic PDP control problem is defined as follows

$$\text{minimize } E_z^u\left[\int_0^\tau \ell(x_{t-},u_t)d\mu_t \right]$$

over $\underline{\underline{U}}$, for each z in E.

The final time $\tau(\omega)$ is the jumping time into the state Δ, in conformity with (P2) in section 2.1. It is assumed throughout that

$$E_z^u[\tau] < \infty \tag{2.22}$$

for each z in E and u in $\underline{\underline{U}}$.

CHAPTER 3

THE SINGLE JUMP PROBLEM

This chapter can be described in brief words as a temporary retreat to a situation simpler than the one proposed originally by the PDP stochastic control problem. We return to the general case in chapter 4. At this point the main concern is to study the single jump stochastic problem P_1, presented in the sequel. It will serve two important purposes. Firstly, P_1 leads to a precise formulation to the deterministic problem within a PDP stochastic problem which appear "pathwisely" between any two jumps. If one is armed with P_1, proposition 3.1 establishes the equivalence of a stochastic solution to solutions of two distinct deterministic problems: one is a simple pointwise optimization problem on the boundary subset E_∂, the other is the deterministic optimal control problem of Langrange type, denoted $P_{d,z}$. Most of the deterministic counterpart for P_1 is shown to comply with the theory of deterministic optimal control in a almost straightforward manner, for the purpose of providing necessary conditions for optimality. Although a non-standard situation arises in $P_{d,z}$ when the optimal solution spans an unbounded internal, an appropriate collection of results to handle that problem is contained in the theory of non-smooth optimization. By means of an independent argument, relying on the general assumptions for this chapter (A1) to (A5) it is possible to identify the solution in an unbounded time interval with the behaviour of a sequence of solutions for standard problems. From that instance, a Maximum Principle for problem P_1 is derived in theorem 3.3.

The second important issue in this approach refers to the relationship between P_1 and the general PDP stochastic control problem. This is best understood if one associates to P_1 an operator acting on a certain class of functions defined on the state domain. If one can justify the equivalence between the PDP stochastic problem and a succession of single jump problems, the former problem is characterized when a class of functions is shown to be stable with respect to the operator associated with P_1. In order to maintain the problem within the framework of local methods of optimization, we will require more than the minimal characterization. It turns out that the class of continuous Lipschitz functions on the domain is the appropriate class, since it is the largest class for which local methods are developed in the theory of optimal control. Theorem 3.1 asserts this fundamental property for problem P_1 whenever assumptions (A1) to (A5) are verified. For this purpose, the equivalence of P_1 and the optimal control problem Pd,z is recognized. The $P_{d,z}$ formulation is employed only as an intermediary stage between P_1 and a relaxed version $\tilde{P}_{d,z}$ since not much can be deduced from $P_{d,z}$ insofar as regularity is concerned. $\tilde{P}_{d,z}$ is an artifically extended problem with unconstrained domain set; the geometry of the boundary set E_∂ and the data originally defined on E supply the elements in the definition of the family of vector fields and other data outside the former domain. The value for problem $\tilde{P}_{d,z}$ is shown to posses the required regularity and the conclusion of the theorem ultimately follows from that verification, since the solutions for $\tilde{P}_{d,z}$ are shown to agree with solutions for $P_{d,z}$, for each $z \in E$.

A verification theorem is provided in terms of a modified Hamilton-Jacobi-Bellman equation, in which generalized gradients are involved to express partial derivatives in the proper sense. Theorem 3.2 supplies necessary and sufficient conditions for optimality as it is customary in smooth control problems.

3.1 Definitions, Assumptions and Main Results for P_1

Throughout this chapter we study a PDP control problem denoted by P_1, a stochastic problem for $\{(x_t),\underline{U}\}$ with a single jump: the horizon of control is either the killing time for the original problem or the first jump time, whichever occurs first. P_1 is formally defined as

P_1:

$$\text{minimize } E_z^u\Big[\int_0^{T_1} \ell(x_{t^-},u_t)d\mu_t + I_{\{T_1<\tau\}}\,\psi(x_{T_1})\Big]$$

for each $z \in E$, over admissible class of controls $\underline{U}$.

Here $\underline{U}$ is understood to be the product class $\underline{U}_0 \times \tilde{\underline{U}}_\partial$ ($\underline{U}_0$ = class of controls which are initial point and time dependent $\tilde{\underline{U}}_\partial$ = class of feedback controls on the boundary). The integral part of the cost for P_1 aggregates the data of the original control problem, whereas the extra term included in the cost is defined as follows

(3.1) $\psi : E \to \mathbb{R}$ is Lipschitz continuous function, nonegative and bounded.

When a choice in the control class $\underline{\underline{U}}$ is made, the expected cost to incour as a function of starting point is expressed by the function $L^u : E \to \mathbb{R}$ such that

$$(3.2) \qquad L^u(z) = E_z^u\Big[\int_0^{T_1 \wedge \tau} \ell(x_{t^-}, u_t)\,d\mu_t + I_{\{T_1<\tau\}}\psi(x_{T_1})\Big].$$

The appellation "cost function for u" will be used for each function $L^u(\cdot)$. The family $\{L^u(\cdot) : u \in U\}$ is certainly bounded since $\ell(\cdot,\cdot)$ and $\psi(\cdot)$ are nonegative and bounded above, so that

$$0 \le L^u(z) \le E_z^u[\,|\ell_0|T_1 + (|\ell_\partial| + |\psi|)I_{\{T_1<\tau\}}]$$

$$\le E_z^u[\,|\ell_0|\tau] + |\ell_\partial| + |\psi|$$

In view of (2.22) in the last exprression indicated is bounded.

Problem P_1 is characterized by function $L : E \to \mathbb{R}$ defined by

$$(3.3) \qquad L(z) = \inf_{u \in \underline{\underline{U}}} L^u(z)$$

for each $z \in E$.

The appellation "value function for P_1" or "value function" for short, will be used throughout.

The following assumptions on the problem will be in force:

(A1) Let E_∂ be the subset of the boundary of E identified in (2.19). Denote $d_A : \mathbb{R}^n \to \mathbb{R}_+$ the distance function to a set $A \subset \mathbb{R}^n$. The domain set E is an arbitrary closed set which satisfies conditions (i) and (ii):

(i) $\sup_{z \in E} d_{E_\partial}(z) < \infty$

(ii) E is convex/concave-smooth set at E_∂. More precisely, let $\partial d_E(\cdot)$ be the generalized gradient of the distance function to E, $d_E(\cdot)$. If z is a point in E_∂ and E is concave in a neighbourhood of z, then

$$\partial d_E(z) = \left\{\alpha \frac{v}{|v|} : 0 \leq \alpha \leq 1\right\} \qquad \text{for some } v \in \mathbb{R}^n,\ V \neq 0$$

(A2) For each point z in the domain E

$$\sup_{u \in \underline{\underline{U}}} \int_{\mathbb{R}_+} P_z^u\,(T_1 > t)e^{Kt}dt < \infty$$

(K is the Lipschitz rank of $f(\cdot,u)$ for each $u \in U$).

(A3) For each point z in the domain E a control $t \to u(t,z)$ should exist such that for some $K^* > 0$ independent of choice of z, it yields

$$t_u^*(z) = \inf\{t \geq 0 : x_u(t,z) \in E_\partial\} \leq K^* d_{E_\partial}(z) .$$

(A4) For each point z in the boundary set E_∂

$$\inf_{u \in U} \lambda(z,u) > K.$$

(A5) (Calmness) Let $E_{+\delta}$ be the set

$$\{z \in \mathbb{R}^n : d_E(z) \leq \delta\}.$$

Suppose that each data of problem P_1 is continuously extended to the set $E_{+\delta}$ and let $L_{+\delta} : E_{+\delta} \to \mathbb{R}$ be the value for P_1 with $E_{+\delta}$ as the new state domain set. P_1 is calm problem, in the sense that a monotone increasing right-continuous function ρ with $\rho(0) = 0$ should exist for each proper extension so that

$$-L_{+\delta}(z) + L(z) \leq \rho(\delta)$$

for all $z \in E$ is verified.

Assumptions (A4) and the calmness condition (A5) can be replaced by the stronger hypothesis:

(A4') For each point z in the boundary set E_∂

$$\inf\{\langle v, f(z,u)\rangle : u \in U,\ v \in \partial d_E(z),\ |v| = 1\} > 0.$$

Before proceeding to the main results, we pause to provide context and some interpretations to the assumptions (A1) to (A5).

(R1) The state domain set is not necessarily compact, although (A1) part(i) may ultimately lead to that implication. If E_∂ is itself a compact set, that should also be the case of set E.

(R2) If E_∂ is defined by some function $g:\mathbb{R}^n \to \mathbb{R}$ such that $E_\partial = \{z \in \mathbb{R}^n : g(z) = c_0\}$ for some constant c_0, (A1) part(ii) is equivalent to the requirement that $g(\cdot)$ be continuously differentiable near any point $z \in E_\partial$, whenever E is concave near z. That is direct consequence of a characterization for $\partial d_E(\cdot)$ given by Clarke in proposition 2.5.6 of [C1].

(R3) Suppose that the PDP model satisfies for each $z \in E$

$$\inf_{u \in U} \lambda(z,u) Q(E;z,u) > 0$$

and for each $z \in E_\partial$

$$\inf_{u \in U_\partial} P(E;z,u) > 0.$$

An equivalent statement for (A2) is

$$\sup_{z \in E, u \in U} E_z^u[e^{KT_1} I_{\{T_1 < \tau\}}] < \infty$$

(R4) Assumption (A3) entails a controllability condition that conforms with the boundness of velocity set $\{f(\cdot,u): u \in U\}$. As for the condition required by (A2), it can be satisfied trivially if a finite number t_0 exists, such that

$$P_z^u(\tau>t) \to 0 \quad \text{as } t \uparrow t_0.$$

for all z and u. In the situation indicated, (A4) need not be satisfied, although this alternative setting will not be treated here in detail. Throughout τ can be r.v. with a distribution of exponential type for which the rate is given by

$$\lambda(z,u)Q(\Delta,z,u)$$

a bounded function on $E \times U$ (i.e $P(\Delta;\, z,\, u)$ may be identically null on $E_\partial \times U_\partial$). This situation is more complex analytically and we can tell in advance that unbounded time interval of control will arise in the equivalent deterministic problem $P_{d,z}$ in this connection.

(R5) Since $\lambda(\cdot,u)$ and $f(\cdot,u)$ are continuous on E for each u, (A4) or (A4') imposes nonvoid condition on λ and f.

(R6) Assumption (A5) bears similarity with a condition known in deterministic optimal control theory as local calmness (see [C2]). It can be expressed as follows.

Let $P_{d,z}$ be a deterministic optimal control problem and let $L(z)$ indicate the correspondent value. $P_{d,z}$ has a solution given by the pair trajectory/control $(x(\cdot),u(\cdot))$. Any trajectory $x'(\cdot)$ should satisfy

$$x'(0) = z \quad ; \quad x'(t^*) \in B \quad , \quad t^* < \infty$$

for $B \subset \mathbb{R}^n$, in order to be admissible for $P_{d,z}$. In addition, the optimal trajectory $x(\cdot)$ is supposed to be in the interior domain. An extra control problem is defined locally, in which all pairs $(x'(\cdot), u'(\cdot))$ with $x'(\cdot)$ contained in a ε-tube about $x(\cdot)$ are involved. In this auxiliary problem, the end point set constraint B is altered while all other data are kept as they were previously defined for $P_{d,z}$. The set B is replaced by

$$B + \{v\} = \{z \in \mathbb{R}^n : z + v \in B\}$$

for some $v \in \mathbb{R}^n$. Let us denote by $L_{v,\varepsilon}(z)$, the value for the problem just described. If no admissible trajectory exists, then $L_{v,\varepsilon}(z)$ is taken to be $+\infty$. Herewith the local calmness property is defined:

The problem $P_{d,z}$ is said to be locally calm at the solution $x(\cdot)$ if one can find numbers ε and K so that

$$L_{v,\varepsilon}(z) - L(z) \geq -K\ |v|$$

holds for each $v \in \mathbb{R}^n$.

It was shown in [C2] that local calmness is the weakest condition whereby one can expect the value function of a deterministic problem to be locally Lipschitz function. The local sense refers to the ε-tube about the optimal trajectory $x(\cdot)$.

In this chapter we seek regularity for the value of P_1 strong enough to validate the Hamilton-Jacobi-Bellman equation presented in theorems 3.1 and 3.2. Therefore one is only content if the value for P_1 is uniformly Lipschitz function in the state domain E as a whole. Although for many reasons, the result mentioned can not be directly adapted to handle the deterministic counterpart of P_1, a requirement along the lines of local calmness can be expected to have much significance in this context.

(R7) A direct consequence of calmness condition (A5) is that P_1 cannot be improved at any rate if the time

$$t_u^*(z) = \inf\{t \geq 0 : x_u(t,z) \in E_\partial\}$$

which has the stochastic interpretation

$$P_z^u(x_{T_1^-} \in E_\partial) = P_z^u(T_1 = t_u^*(z))$$

were replaced by a more relaxed final time $\tilde{t}_u(z)$ for the deterministic trajectory $x_u(\cdot,z)$, with

$$\tilde{t}_u(z) \in \{t \geq 0 : x_u(t,z) \in E_\partial\}$$

for all $z \in E$. If the above fact is not verified, function ρ as specified in (A5) is clearly non-existent.

Let $H_1 : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times E \times U \to \mathbb{R}$ be defined by

$$(3.4) \quad H_1(r,q,p,z,u) = \langle p,f(z,u)\rangle - r\{\ell_0(z,u)+\lambda(z,u)\int_E [\psi(y)+q]dQ(y;z,u)\}.$$

Let also $F : E \times U_\partial \to \mathbb{R}$ be the function

$$(3.5)\qquad F(z,u) = \ell_\partial(z,u) + \int_E \psi(y)dP(y;z,u).$$

Function H_1 is the Hamiltonian for P_1 while F is used to disclose conditions at the boundary set E_∂. The main results in this chapter are the following theorems.

Theorem 3.1: Suppose (A1) to (A5) hold. Then the value for P_1, $L(\cdot)$, is a uniformly Lipschitzian function on E. It satisfies

$$(3.6)\qquad \min_{\xi \in \partial L(z)} - \max_{u \in U} \quad H_1(1,-L(z),-\xi,z,u) = 0$$

for each $z \in E$.

$$(3.7)\qquad L(z) + \max_{u \in U_\partial} -F(z,u) = 0$$

for each $z \in E_\partial$.

The equivalence between P_1 and $P_{d,z}$ is established by proposition 3.2 whereas $P_{d,z}$ is shown to agree with the related problem $\tilde{P}_{d,z}$ in the domain set E, in proposition 3.4. Proposition 3.3 in other hand, asserts that the value for $\tilde{P}_{d,z}$ is uniformly Lipschitz function, thereby problem P_1 inherits the Lipschitzian behaviour. The differential expression and boundary condition in theorem 3.1 are readily deduced from similar equations in theorem 3.2 by taking $L(\cdot) = L^u(\cdot)$ for optimal u and $t = 0$ for each each z.

Theorem 3.2: (Verification theorem) Suppose (A1) to (A5) hold. A necessary and sufficient condition for a control u in the class $\underline{\underline{U}}$

with $u = u_0/u_\partial$ to be a solution for P_1 is that the cost function for u satisfies for each $z \in E$

$$(3.8) \quad \min_{\xi \in \partial L^u(x(t,z))} - \max_{u \in U} \quad H_1(1,-L^u(x(t,z)),-\xi,x(t,z),u)$$

$$= \min_{\xi \in \partial L^u(x(t,z))} - H_1(1,-L^u(x(t,z)),-\xi,x(t,z),u_0(t,z))$$

$$= 0$$

$$\text{for } u_0 \in \underline{\underline{U}}_0 \qquad \text{and} \qquad t \in [0,t^*(z)]_{u_0}$$

and also

$$(3.9) \quad L^u(x(t^*_{u_0},z)) + \max_{u \in U_\partial} - F(x(t^*_{u_0},z),u)$$

$$= L^u(x(t^*_{u_0},z)) - F(x(t^*_{u_0},z),\ u_\partial(x(t^*_{u_0},z)))$$

$$= 0$$

$$\text{for } u_\partial \in \underline{\underline{\tilde{U}}}_\partial \qquad \text{and} \qquad t^*_{u_0} = t^*_{u_0}(z) < \infty.$$

The Lipschitzian characterization for the value of P_1 supplies the essential argument for theorem 3.2. The modified version of the Hamilton-Jacobi-Bellman equation (3.8) and (3.9) which involves generalized gradients for L is a global version of similar local result presented in Clarke [C1] and Clarke and Vinter [C2]. In particular, the characterization is also employed to derive the Maximum Principle for P_1 in the case optimal solutions span an infinite time interval (i.e. $t^*_u(z) = \infty$).

Theorem 3.3: (Maximum Principle for P_1). Let $u = u_0/u_\partial$ be a solution for P_1, where u_0 is a control of class $\underline{\underline{U}}_0$ and u_∂ a control of class $\underline{\underline{\tilde{U}}}_\partial$. Let $t \to x(t,z)$ be a trajectory on $[0,t^*_{u_0}(z)]$ with $x(0,z) = z$, which coincides with the left-continuous version of (x_t) on the internal $[0,T_1(\omega)]$ with $x_0 = z$, for each $z \in E$. Suppose that (A1) to (A5) hold. Then there exists:

(a) $F_\partial : E \to \mathbb{R}$ a Lipschitz function;

(b) $p:[0,t^*_{u_0}(z)] \to \mathbb{R}$ absolutely continuous functions

for each $z \in E$;

(c) a scalar r equal to 0 or 1 for each $z \in E$.

The following relations are verified:

(i) $$F_\partial(x(t^*_{u_0},z)) = F(x(t^*_{u_0},z),u_\partial(x(t^*_{u_0},z))) = \inf_{u\in U_\partial} F(x(t^*_{u_0},z),u)$$

(ii) $$-(\dot{q}(t)\ \dot{p}(t)) + (0\ \lambda(x(t,z),u_0(t,z))p(t)) \in \partial_{(r,x)}H_1(r,q(t),p(t),x(t,z),u_0(t,z)) \quad \text{a.e. } t \in [0,t^*_{u_0}(z)]$$

(iii) $$\max_{u\in U} H_1(r,q(t),p(t),x(t,z),u) = H_1(r,q(t),p(t),x(t,z),u_0(t,z) = 0 \quad \text{a.e. } t \in [0,t^*_{u_0}(z)].$$

Whenever $t^*_{u_0}(z) < \infty$:

(iv) $(q(t^*_{u_0})\ p(t^*_{u_0})) + r\ \xi \in (0\quad -c\partial d_E(x(t^*_{u_0},z)))$

for some nonegative scalar c and a n+1-dimensional vector ξ so that

$$\xi \in (F_\partial(x(t^*_{u_0},z)),\ \partial F_\partial(x(t^*_{u_0},z))) .$$

Whenever $t^*_{u_0}(z) = \infty$

(v) $|q| + |p| < \infty.$

It is clear from the definition of a PDP that the path of the process is continuous and deterministic in nature of the interval $[0,T_1(\omega)]$. That fact is acknowledged in the initial stage of verification of results presented in the theorems above, in which P_1 is equally posed by a completely deterministic problem. The latter will possess solutions that when restricted to the interval $[0,T_1(\omega)]$, they will agree with solution to the former. The main considerations in the next section is this equivalence, and it turns out that problem $P_{d,z}$ for each z in E epitomizes the desired relation.

Problem $P_{d,z}$ involves trajectories for $f(\cdot,\cdot)$, $t \to x(t,z)$, which satisfies the initial end point constraint $x(0,z) = z$ and $(t^*_{u_0},x(t^*_{u_0}, z))$ should comply with a final end point constraint in the set $[0,\infty) \times E_\partial \bigcup \{\infty\} \times E$. This dual behaviour arises as a direct consequence of the definition of a PDP process. The domain set E can be bounded subset of $\mathbb{R}^n$, therefore a path may reach E_∂, the boundary subset for E in a finite time interval with positive probability; in that case $t^*_u(z) < \infty$ and the first jump time $T_1(\omega)$ has a finite upper bound. In another conceivable situation, the set E_∂ will not be visited with probability one, here $t_u^*(z) = \infty$ and any jump occurs only in an exponential fashion. Although this complementary behaviour arises naturally each of these cases brings about non-related type of solutions, while one has to comply with explicit boundary constraint, the other has no upper bound for the horizon of control. That is an adverse situation insofar regularity is concerned. The fact that end point constraints generally induce irregular values is well known in the deterministic theory and infinite time problems naturally have divergent values. In order to endow P_1 with the desired regularity, this "double identity" ill-behaviour is overcome by the introduction of a second deterministic equivalence, $\tilde{P}_{d,z}$, for which solutions present an unified quality.

Whenever end point constraints are explicitly prescribed in a deterministic control problem, the concept of normality is relevant to necessary conditions for optimality (see [C1], section 3.4). A deterministic control problem is said to be normal if the multipliers $(r\ q(\cdot)\ p(\cdot))$ for an optimal pair $(x(\cdot),u(\cdot))$ can be chosen with r equal to 1. It refers therefore, to possible non-trivial necessary conditions yielded by the maximum principle. It is shown in [C2] that the local calmness of a problem is implied by a strong version of normality. As regards to problem $P_{d,z}$, it is possible to show the normality at each point z in E, so that the necessary conditions given by theorem 3.3 are never trivial.

Corollary: In theorem 3.3 the number r can be chosen equal to 1 for each z on E.

The unconstrained problem $\tilde{P}_{d,z}$ offers the basic argument for the proof of the corollary. However this result is already hinted by the strong similarity presented by the Hamiltonian-Jacobi-Bellman equation in theorem 3.2 and the Maximum Principle in theorem 3.3. If one takes for each $t \in [0, t_u^*(z)]$

$$(3.10) \qquad r = 1$$

$$q(t) = -L^u(x(t,z))$$

$$p(t) = \arg\min_{\xi \in \partial L^u(x(t,z))} \{-\max_{u \in U} H(1, -L^u(x(t,z)), -\xi, x(t,z), u)\}$$

equations (3.8) and (3.9) in theorem 3.2 and conditions (iii) and (i) in theorem 3.3 are rendered equal. In this situation, conditions (iv) and (v) are easily verified, the only non established connection is condition (ii), since without further arguments one can not assert the absolutely continuity of function $p(\cdot)$ defined by (3.10).

The main difficulty to establish a maximum principle for solutions in an unbounded time interval resides on the fact that variational arguments do not apply, consequently the analysis should rely on some indirect argument. In a unbounded time interval small variations on controls may produce trajectories that violates any neighbourhood of a pre-specified trajectory. Moreover, in a bounded state domain, any trajectory produced by variation on controls may reach the boundary set and the essential sense of continuity for small perturbations is lost. The maximum principle conditions are alternatively obtained here by employing a sequence of finite time interval problems in a neighbourhood of solutions for $P_{d,z}$ with unbounded time interval and showing convergence of multipliers and solutions.

3.2 A Deterministic Counterpart for P_1

In this section P_1 is reformulated as a deterministic problem. Proposition 3.1 unveils a deterministic equivalence involving class of controls $\underline{\underline{U}}_0$ and $\tilde{\underline{\underline{U}}}_\partial$. The minimization over $\tilde{\underline{\underline{U}}}_\partial$ is done independently leaving the minimization over $\underline{\underline{U}}_0$, namely problem $P_{d,z}$ for all z in E. That version is accounted for in the deterministic control theory and proposition 3.2 establishes the required equivalence with P_1.

For each control u in $\underline{\underline{U}}$, let $x_u(\cdot,z)$ with $x_u(0,z) = z$ be a trajectory on $[0,t_u^*(z)]$ which coincides with the left-continuous version of (x_t) on $[0,T_1(\omega)]$. Recall that $t_u^*(z) \geq T_1(\omega)$and whenever it is clear from the context we will use abbreviated notation:

$$(3.11)\qquad x(t) = x(t,z) = x_u(t,z)$$

$$t^* = t_u^* = t_u^*(z) = \inf\{t \geq 0 : x_u(t,z) \in E_\partial\}.$$

Let $\Lambda : [0,t^*] \to \mathbb{R}$ be defined by

$$(3.12)\qquad \Lambda(t) = \int_0^t \lambda(x(s,z),\ u_0(s,z))ds.$$

let also $f_0 : E \times U \to \mathbb{R}$ be defined by

$$(3.13) \qquad f_0(z,u) = \ell_0(z,u) + \lambda(z,u)\int_E \psi(y)dQ(y;z,u) \quad .$$

and recall $F: E \times U_\partial \to \mathbb{R}$ defined in (3.5).

Proposition 3.1 $z' \to L^u(z')$ the cost function for $u = u_0/u_\partial$ has the following equivalent representation

$$(3.14) \qquad L^u(z) = \int_0^{t^*} e^{-\Lambda(t)} f_0(x(t,z),u_0(t,z))dt$$
$$+ \; e^{-\Lambda(t^*)} F(x(t^*,z),u_\partial(x(t^*,z)))$$

for each z in E.

Proof: The proposition follows by applying the distribution $P^u_z(z_1 \in A)$ defined in (2.20). Firstly, use the fact that $t \to x(t,z)$ is the left-continuous version of (x_t) on $[0,T_1(\omega)]$. Secondly, apply the explicit form of controls in classes $\underline{\underline{U}}_0$ and $\underline{\underline{\tilde{U}}}_\partial$. In conformity with data in the cost for P_1 we write the integral part as

$$\int_0^{T_1(\omega)} \ell(x_{t^-}(\omega),u_t(\omega))d\mu_t(\omega) =$$

$$\int_0^{T_1(\omega)} \ell_0(x(t,z),u_0(t,z))dt + \ell_\partial(x(t^*,z),u_\partial(x(t^*,z)))I_{\{\omega:T_1<\tau,T_1=t^*\}}$$

so that the only r.v.'s involved is $T_1(\omega)$, thus

$$E_z^u[\int_0^{T_1} \ell(x_{t^-},u_t)d\mu_t] =$$

$$\int_0^{t^*} \{\lambda(x(t,z),u_0(t,z))\exp(-\int_0^t \lambda(x(s,z),u_0(s,z))ds)\cdot$$

$$\int_0^t \ell_0(x(s,z),u_0(s,z))ds\}dt + \exp(-\int_0^{t^*} \lambda(x(t,z),u_0(t,z))dt)\cdot$$

$$\{\int_0^{t^*} \ell_0(x(t,z),u_0(t,z))dt + \ell_\partial(x(t^*,z),u_\partial(x(t^*,z))\}$$

by an interchange in the order of integration and introducing the notation in (3.12), one obtains

$$(3.15) \qquad E_z^u[\int_0^{T_1\wedge\tau} \ell(x_{t^-},u_t)d\mu_t] =$$

$$\int_0^{t^*} e^{-\Lambda(t)} \ell_0(x(t,z),u_0(t,z))dt + e^{-\Lambda(t^*)}\ell_\partial(x(t^*,z),u_\partial(x(t^*,z))).$$

The remaining term in the cost for P_1 can be calculated similarly, by direct application of $P^u_z(z_1 \epsilon A)$ using notation in (3.12):

$$(3.16) \qquad E^u_z[I_{\{T_1<\tau\}}\psi(x_{T_1})] = E^u_z[I_{\{T_1<\tau\}}E[\psi(x_{T_1})|T_1]] =$$

$$\int_0^{t^*} e^{-\Lambda(t)} \lambda(x(t,z),\ u_0(t,z)) \int_E \psi(y)dQ(y;x(t,z),u_0(t,z))dt$$

$$e^{-\Lambda(t^*)}\int_E \psi(y)dP(y;x(t^*,z),u_\partial(x(t^*,z))).$$

The representation in the proposition is readily obtained from (3.15) and (3.16), using notation in (3.13) and function F defined in (3.5).

Note in passing that another equivalent representation to the cost for P_1 can be provided if the density function for jumps were used. If g: $E \times U \to \mathbb{R}$ is the function defined in chapter 2, (DC2') we denote

$$G(t) = 1 - \int_0^t g(x(s,z),u_0(s,z))ds$$

and obtain from proposition 3.1 and the equivalence indicated by (2.15) and (2.16) the following representation:

$$(3.17) \qquad L^u(z) = \int_0^{t^*} \{G(t)\ell_0(x(t,z),u_0(t,z)) + g(x(t,z),u_0(t,z)) \int_E \psi(y)dQ(y;x(t,z),u_0(t,z))\}dt + G(t)F(x(t^*,z),u_\partial(x(t^*,z)))$$

for each z in E, $u = u_0/u_\partial$, $u_0 \in \underline{\underline{U}}_0$, $u_\partial \in \tilde{\underline{\underline{U}}}_\partial$.

We leave the problem with jumps characterized by a "local density" g at this stage, hereupon it is supposed that the characterization of jumps by rate is the setting which supplies bounded Lipschitz continuous data. Any result in the sequel can be easily interpreted in the context of jump density by a mere replacement of (3.14) by (3.17).

The solution for P_1 prescribes equivalently the minimization of the functional in the right-hand side of (3.14) for each point z in E. The expression does not involve any r.v. hence the problem posed is a completely deterministic one. However, because class of controls $\underline{\underline{U}}_0$ and $\tilde{\underline{\underline{U}}}_\partial$ are distinct and cannot be reduced to a single class, the problem at this stage is non-standard. One consistent approach is to solve the problem within one class and give in succession a conditional characterization within the second class. Note in (3.14) that only an ordinary pointwise minimum problem within class $\tilde{\underline{\underline{U}}}_\partial$ is required, the one which involves the final cost function F on the set E_∂. For its simplicity, this problem is handled first; lemma 3.1 provides the complete answer.

Let $F_\partial : E \to \mathbb{R}$ be the function

$$(3.18) \qquad F_\partial(z) = \inf_{u \in \tilde{\underline{\underline{U}}}_\partial} F(z,u).$$

Lemma 3.1 $z' \to F_\partial(z')$ is uniformly Lipschitzian function on E.

Proof: Recall the definition of F given by (3.5). It follows from the general assumption on data that $(z',u') \to F(z',u')$ is measurable function on $E \times U_\partial$ and the family $\{F(\cdot,u) : u \in U_\partial\}$ is uniformly Lipschitzian on E. To prove the lemma we express F_∂ equivalently by

$$F_\partial(z) = \inf_{u \in U_\partial} F(z,u) \qquad \text{for each } z \text{ in } E.$$

In view of (3.18) and the lemma above, proposition 3.1 can be alternatively stated with function F_∂ replacing F and the remaining problem is the restriction of P_1 with respect to class of controls $\underline{\underline{U}}_0$. A standard deterministic formulation is provided in the sequel for the problem within $\underline{\underline{U}}_0$ and proposition 3.2 formally establishes the connection with P_1.

For each z in E the deterministic optimal control problem $P_{d,z}$ is defined by:

$P_{d,z}$:

$$\text{minimize} \int_0^{t^*} e^{-\Lambda(t)} f_0(x(t),u(t))dt + e^{-\Lambda(t^*)} F_\partial(x(t^*))$$

on Ω, the class formed by all pairs $(x(\cdot),u(\cdot))$ with

$\dot{x}(\cdot) = f(x(\cdot),u(\cdot))$

$u(\cdot) \in U$, measurable function

$x(\cdot) \in E$

on $[0,t^*]$. Also

$x(0) = z$

$(t^*,x(t^*)) \in \inf\{t \geq 0: x(t) \in E_\partial\} \times E$

with the convention that $\inf \phi = +\infty$.

Let $L^u_{P_d}(z)$ be the cost of $P_{d,z}$ for some pair in Ω and let $L_{P_d}(z)$ be the value for $P_{d,z}$, so that

(3.19) $\quad L_{P_d}(z) = \inf_\Omega L^u_{P_d}(z)$.

Recall that L is the value function for P_1. The following proposition is a direct consequence of the equivalences expressed in (3.14), (3.18) and definition of $P_{d,z}$.

Proposition 3.2: $\quad L(z) = L_{P_d}(z)$ for each z in E.

$P_{d,z}$ is a standard Lagrange problem of deterministic optimal control with end point and state constraints. Although state constraints performed by condition $x(\cdot) \in E$ are present in $P_{d,z}$, they are not active constraints for any pair in Ω, except possibly at $t = t^*$. $P_{d,z}$ is the first formal deterministic equivalent to P_1 with respect to class of controls $\underline{\underline{U}}_0$. Whenever u_0 is a solution to P_1, with $u_0 \in \underline{\underline{U}}_0$, for each $z \in E$ a pair $(x(\cdot),u(\cdot))$ which solves $P_{d,z}$ exists such that

$$u_0(\cdot,z) = u(\cdot) \qquad \text{on } [0,T_1(\omega)]$$

for all $z \in E$. The converse assertion is not always true since a solution for P_1 should satisfy stricter measurability conditions (the class $\underline{\underline{U}}_0$ contains functions which are measurable with respect to $B(R_+)$ and $\underline{\underline{E}}$, jointly).

It turns out that the equivalent problem $P_{d,z}$ falls within the scope of optimal control theory whenever a solution complies with $t^* < \infty$. In this situation, a modified version of Pontryagin Principle for non-smooth data applies and the main argument for the Maximum Principle for P_1 is thereby supplied. However, the $P_{d,z}$ formulation does not provide much insight over the issue of regularity, mainly because explicit final end point constraints are present. With that respect, we turn our attention to a further deterministic equivalent, problem $\tilde{P}_{d,z}$.

3.3 A Problem with Unbounded Domain

A second formal equivalence to problem P_1 is introduced here. The set E is no longer seen as the domain restriction and an extension of vector fields and other data of P_1 to the entire Euclidean space will be considered. Denote $\tilde{f}: \mathbb{R}^n \times U \to \mathbb{R}^n$,

$\tilde{f}_0 \colon \mathbb{R}^n \times U \to \mathbb{R}$, $\tilde{\lambda} \colon \mathbb{R} \times U \to \mathbb{R}$ and $\tilde{F}_\partial \colon \mathbb{R}^n \to \mathbb{R}$ functions to be defined precisely later; in addition denote

$$\tilde{\Lambda}(t) = \int_0^t \tilde{\lambda}(x(s),u(s))ds \tag{3.20}$$

$$\Delta(t) = \int_0^t d_E(x(s))ds \tag{3.21}$$

for each $t \in \mathbb{R}_+^*$ and some arbitrary control $u(\cdot)$. ($\mathbb{R}_+^*$ = extended nonnegative real numbers).

For each z in $\mathbb{R}^n$ and numbers K_1, K_2 and K_3, let $\tilde{P}_{d,z}$ be the deterministic optimal control problem defined as follows:

$\tilde{P}_{d,z}$:

$$\text{minimize} \int_0^{t_f} e^{-(\tilde{\Lambda}(t)+K_1\Delta(t))}[\tilde{f}_0(x(t),u(t))+K_2 d_E(x(t))]dt$$
$$+ \quad e^{-(\Lambda(t_f)+K_1\Delta(t_f))}[\tilde{F}_\partial(x(t_f))+K_3 d_{E_\partial}(x(t_f))]$$

on $\tilde{\Omega}$, the class containing all pairs $(x(\cdot),u(\cdot))$ which satisfy

$$\dot{x}(\cdot) = \tilde{f}(x(\cdot),u(\cdot))$$
$$u(\cdot) \in U, \text{ measurable function}$$

on $[0,t_f]$. In addition

$$x(0) = z \qquad (t_f,x(t_f)) \in \mathbb{R}_+^* \times \mathbb{R}^n$$

In order to make $\tilde{P}_{d,z}$ meaningful we take

$$\tilde{f}(\cdot,\cdot) = f(\cdot,\cdot) \ ; \quad \tilde{f}_0(\cdot,\cdot) = f_0(\cdot,\cdot) \ ;$$

$$\tilde{\lambda}(\cdot,\cdot) = \lambda(\cdot,\cdot) \ ; \qquad \text{on set } E \times U, \text{ and}$$

$$\tilde{F}_\partial(\cdot) = F_\partial(\cdot) \qquad \text{on set } E.$$

These will be extended to $\mathbb{R}^n \times U$ and $\mathbb{R}^n$ in a geometrical manner. The underlying concepts are furnished by the lemmas in the next section.

As usual, let us define $L^u_{\tilde{P}_d}(z)$ as the cost of $\tilde{P}_{d,z}$ at some z in $\mathbb{R}^n$ for some pair in $\tilde{\Omega}$. We also define $L_{\tilde{P}_d}(z)$ the value for $\tilde{P}_{d,z}$ in the obvious way.

$$L_{\tilde{P}_d}(z) = \inf_{\tilde{\Omega}} L^u_{\tilde{P}_d}(z).$$

The most important feature of $\tilde{P}_{d,z}$ is the following.

Proposition 3.3: For large enough K_1, $z' \to L_{\tilde{P}_d}(z')$ is uniformly Lipschitzian function on a neighbourhood of set E, provided that (A1), (A2) and (A4) hold.

The next assertion establishes the correspondence between problems P_1 and $\tilde{P}_{d,z}$; propositions 3.3 and 3.4 jointly supply the main result on regularity for P_1.

Proposition 3.4: For K_1, K_2 and K_3 large enough

$$L(z) = L_{\tilde{P}_d}(z) \qquad \text{for each } z \in E,$$

provided that (A3) and (A5) hold.

3.4 An extension for the Data

We proceed by making functions $\tilde{f}$, $\tilde{f}_0$, $\tilde{\lambda}$ and $\tilde{F}_\partial$ available for use in $\tilde{P}_{d,z}$ by a constructive approach. The tools are geometrical arguments, developed analytically by lemmas 3.2, 3.3 and 3.4 in the sequel. We first quote a characterization for generalized gradients of Lipschitzian function and in particular for ∂d_E at points where d_E is differentiable.

Lemma 3.2

(i) [C1, 2.5.1] Let f be a Lipschitz function near z and suppose that S is any set of Lebesgue null measure in $\mathbb{R}^n$ and Ω_f is the set at which f is not differentiable. Then

$$(3.23) \qquad \partial f(z) = \mathrm{co}\{\lim \nabla f(x_i);\ x_i \to z,\ x_i \notin S \cup \Omega_f\}$$

(ii) [C1, 2.5.4] Let $\nabla d_E(z)$ exist and be different from zero. Then $z \notin E_\partial$ and there is a unique point z' in E_∂ satisfying

$$(3.24) \qquad d_E(z) = |z-z'|.$$

Moreover

$$(3.25) \qquad \nabla d_E(z) = \frac{z-z'}{|z-z'|} \ .$$

Let us denote $N(E;\delta)$ the open set containing all points which belong to the δ-neighbourhood of E but not to E, namely

(3.26) $$N(E;\delta) = \{z \in \mathbb{R}^n : d_E(z) < \delta\} \setminus E.$$

The convex/concave-differentiable characterization for E given in (A1) part (ii) is used extensively in the following lemma.

Lemma 3.3

(i) A number $\bar{\delta} > 0$ can be found such that if a point z belongs to the set $N(E,\bar{\delta})$ there exists a unique point z' in E_∂ satisfying

$$d_E(z) = |z-z'|$$

(ii) $d_E(\cdot)$ is continuously differentiable on set $N(E,\bar{\delta})$.

Proof: (i) We want to show equivalently that for each point z' in E_∂ there exists a number δ, strictly positive such that the set

$$\{z \in N(E,\delta) : \exists \text{ many points } z' \text{ in } E \text{ s.t. } d_E(z) = |z-z'|\}$$

is empty. Suppose initially that E is convex near z' . It is clear that a number δ as indicated exists. Otherwise if $d_E(z) = |z-z'| = |z-z''| < \delta$ holds for $z'' \in E_\partial$ and $z' \neq z''$ it follows that for each $\alpha \in (0,1)$

$$|z - [\alpha(z''-z') + z']| = |\alpha(z-z'') + (1-\alpha)z - (1-\alpha)z'|$$
$$\alpha|z-z''| + (1-\alpha)|z-z'| = d_E(z)$$

thus

$\alpha(z''-z') + z' \notin E$ for each $\alpha \in (0,1)$

which denies the fact that E is convex near z'.

Suppose now that E is concave near z'. We first calculate $\partial d_E(z')$. From the characterization given by lemma 3.2 part (i) follows that $0 \in \partial d_E(z')$ since one can take a sequence $\{z_i\}$ with $z_i \to z'$ while avoiding the set $S \bigcup \Omega_f$ and satisfying $z_i \in \text{int}\,\{E\}$ for each i. Hence by (A1) part (ii) and lemma 3.2

$$\nabla d_E(z_i) \to \frac{v}{|v|}\ , \qquad \text{some } v \in \mathbb{R}^n \tag{3.27}$$

for each sequence $\{z_i\}$ with

$z_i \notin S \bigcup \Omega_f$

$z_i \in N(E,\delta)$ for some $\delta > 0$.

Now contrary to the statement of the lemma, suppose that

$$\{z \in N(E,\delta) \ :\ \exists\ z', z'' \in E \text{ s.t. } d_E(z) = |z-z'| = |z-z''|\}$$

is non empty for each $\delta > 0$. Consequently, there exist sequences of numbers $\{\delta_i\}$ and of points in $\mathbb{R}^n$ $\{z_i\}$, such that $\delta_i \to 0$, $z_i \to z'$ and for each i

$$z_i \in N(E,\delta_i)$$

$$d_E(z_i) = |z_i - z_i'| = |z_i - z_i''|$$

for z_i' and z_i'' points of E_∂ with $z_i' \neq z_i''$. Since the set where $d_E(\cdot)$ is differentiable is a Lebesgue full measure in $\mathbb{R}^n$ it is also a dense set in $\mathbb{R}^n$. Consequently, there exists sequences in $\mathbb{R}^n$,

$\{y_i\}$ and $\{w_i\}$, such that for each i $d_E(\cdot)$ is differentiable near y_i and w_i, moreover y_i and w_i are arbitrarily near x_i so that

$$\nabla d_E(y_i) = \frac{z_i - z_i'}{|z_i - z_i'|} + \varepsilon$$

$$\nabla d_E(w_i) = \frac{z_i - z_i''}{|z_i - z_i''|} + \varepsilon$$

for $\varepsilon > 0$ arbitrarily small holds, upon applying lemma 3.2 part (ii). It readily follows that $y_i \to z'$, $w_i \to z'$ as $i \to \infty$ and from lemma 3.2 part (i)

$$\lim \nabla d_E(y_i) \in \partial d_E(z')$$

$$\lim \nabla d_E(w_i) \in \partial d_E(z').$$

But the fact that

$$\lim \nabla d_E(y_i) \neq \lim \nabla d_E(w_i)$$

contradicts (3.27). Therefore a nonegative number δ should exists for each point z' in E_∂ such that the set

$$\{z \in N(E,\delta): \ d_E(z) = |z-z'|\}$$

comprises only points with unique nearest point in E. One then takes $\bar{\delta}$ as the infimum number δ as above, over all points z' in E_∂.

(ii) We use the representation for the generalized gradients and $\nabla d_E(\cdot)$ of lemma 3.2 to write for any point $z \in N(E,\bar{\delta})$

$$\partial d_E(z) = \text{co}\{\lim \frac{z_i - z_i'}{|z_i - z_i'|} : z_i \to z,\ z_i \notin S\bigcup \Omega_{d_E}\}$$

$$= \{ \frac{z-z'}{|z-z'|} \}$$

upon using part (i) of the lemma. Since $\partial d_E(z)$ is a singleton for each z in $N(E,\bar{\delta})$ it follows from the corollary for [C1, 2.2.4] in the fact that $d_E(\cdot)$ is continuously differentiable on $N(E,\bar{\delta})$.

Lemma 3.4: Let y and z be points in $N(E,\bar{\delta})$ with $d_E(y) = d_E(z)$. Then

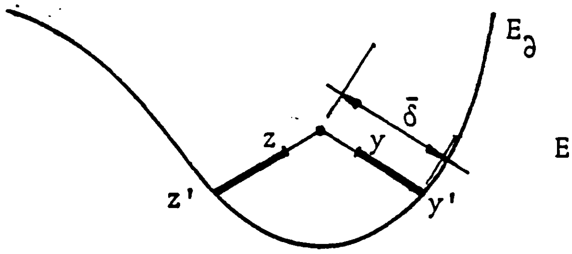
$$(1-\bar{\delta}^{-1} d_E(z))|z'-y'| \leq |z-y|$$

where z' and y' are the closest points in E to z and y respectively.

Proof: Whenever $|z'-y'| \geq 2\bar{\delta}$ one can write

$$\begin{aligned} |z'-y'| &= |z'-z+z-y+y-y'| \\ &\leq |z'-z| + |y'-y| + |z-y| \\ &= 2\, d_E(z) + |z-y| \\ &\leq |z'-y'|\bar{\delta}^{-1} d_E(z) + |z-y| \end{aligned}$$

and the expression in the lemma is verified. Now, suppose that $|z'-y'| < 2\bar{\delta}$. Clearly, the more E is concave near z' and y', the stricter estimate it will be.



We proceed by constructing two triangles:

(i) $\bar{z}$, z',y', such that

$$|\bar{z}-z'| = |\bar{z}-y'| = \bar{\delta}$$

$\bar{z}$,z,z',y', are points in the same plane

(ii) $\bar{y}$,z',y', such that

$$|\bar{y}-y'| = |\bar{y}-z'| = \bar{\delta}$$

$\bar{y}$,y,z',y' are points in the same plane.

Since $\bar{\delta}$ is the number mentioned by lemma 3.3 part (i) either $\bar{z}$ or y is the point nearest possible to E that can have z' and y' simultaneously as nearest points in E. Hence the edge $\bar{z}-\bar{y}'$ of triangle (i) yields the direction in which the normal vector to E at y', y-y', can best approach the normal vector to E at z', z-z'. A similar conclusion can be drawn from triangle (ii), in that the edge $\bar{y}$-z' yields the best direction the normal vector to E at z', z-z' could have, in order to yield the minimum possible value of |z-y|. Hence, the minimum distance |z-y| is attained when y-y' is colinear to $\bar{z}$-z' or equivalently z-z' is colinear to $\bar{y}$-z'. In that situation, triangles (i) and (ii) coincide; applying the representation for $\nabla d_E(.)$ given by lemma 3.2 part (ii) and in view of lemma 3.3 part (ii) one obtains:

$$|z-y| = |d_E(z)\nabla d_E(z) + z' - d_E(y)\nabla d_E(y) - y'|$$

$$\geq \left| d_E(z) \frac{\bar{z}-z'}{|\bar{z}-z'|} + z' - d_E(y) \frac{\bar{z}-y'}{|\bar{z}-y'|} - y' \right|$$

Using the fact that $d_E(z) = d_E(y)$ and the construction given to the triangle (i) it follows that

$$|z-y| \geq (1-\bar{\delta}^{-1}d_E(z))|z'-y'|$$

hence the lemma is proven.

We proceed with the constructive definition of data for $\tilde{P}_{d,\cdot}$ based on data for $P_{d;\cdot}$. Each function will have the general features claimed in the definition of $\tilde{P}_d$ and it will be shown in lemma 3.5 that the extensions are locally Lipschitz in a most useful sense.

Definition: The data for $\tilde{P}_{d,\cdot}$.

Let $\varepsilon > 0$ be an arbitrary given number.

($\tilde{D}1$) Let δ_1 and δ_2 be positive number such that

$$\delta_2 < \bar{\delta}$$

$$\delta_1 < \min\left[\delta_2,\ (|f_0|+|\lambda F_\partial|+\tilde{K}_\partial|f|)^{-1}\varepsilon,\ \rho^{-1}(\varepsilon)\right]$$

where $\bar{\delta}$ is the number appearing in lemma 3.3 part (i) : $|f_0|$, $|\lambda F_\partial|$ and $|f|$ are the supremum norm of functions indicated and $\tilde{K}_\partial$ will be specified by (3.45). Recall that constant K^* and function $\rho(\cdot)$ are mentioned in (A3) and (A5) respectively.

$(\tilde{D}2)$ For each point z in $N(E,\delta_2)$, denote z' the point in E_∂ which satisfies

$$d_E(z) = |z-z'|.$$

$(\tilde{D}3)$

Let $\xi_0 > 0$ be arbitrarily small. The family of vector fields in $\mathbb{R}^n$ is defined by $\tilde{f} : \mathbb{R}^n \times U \to \mathbb{R}^n$ as follows. For each $u \in U$ and $z' \in E_\partial$, a point with the meaning attributed in $(\tilde{D}2)$, let $\tilde{f}(z,u)$ be

$$\tilde{f}(z,u) = \begin{cases} f(z,u) \quad , & \text{if } z \in E; \\ \xi_0 \ \delta_1^{-2} \nabla d_E(z) d_E^2(z) + f(z',u)(1-\delta_1^{-2} d_E^2(z)) & \\ & \text{if } z \in N(E,\delta_1); \\ \xi_0 \ \nabla d_E(z)[1-(\delta_2-\delta_1)^{-2}(d_E(z)-\delta_1)^2] & \\ & \text{if } z \in N(E,\delta_2)\setminus N(E,\delta_1) \\ 0 \quad , & \text{otherwise.} \end{cases}$$

$(\tilde{D}4)$ For each of the remaining data of $\tilde{P}_d$,. the extension to $\mathbb{R}^n$ is represented as follows. For each $u \in U$ and $z' \in E_\partial$, a point with the meaning attributed in $(\tilde{D}2)$ let $\tilde{\Gamma}(z,u)$ be

$$\tilde{\Gamma}(z,u) = \begin{cases} \Gamma(z,u), & \text{if } z \in E \\ \Gamma(z',u)(1-\delta_1^{-2} d_E^2(z)) & \text{if } z \in N(E,\delta_1) \\ 0 & \text{otherwise} \end{cases}$$

with $\Gamma = f_0$, λ and F_∂.

Since $\delta_1 < \delta_2 < \bar{\delta}$ the extension proposed is nonambiguous by lemma 3.3. In view of ($\tilde{D}3$) and ($\tilde{D}4$) we observe that

$$(3.28) \qquad |\tilde{f}|_{\mathbb{R}^n} = |f|_E + \xi_0$$

for ξ_0 arbitrarily small and also

$$|\tilde{f}_0|_{\mathbb{R}^n} = |f_0|_E \quad , \quad |\tilde{\lambda}|_{\mathbb{R}^n} = |\lambda|_E \quad , \quad |\tilde{F}_\partial|_{\mathbb{R}^n} = |F_\partial|_E .$$

Moreover, ($\tilde{D}3$) establishes that

$$(3.29) \qquad \tilde{f}(z,u) = \xi_0 \cdot \nabla d_E(z)$$

for any point z in $N(E,\delta_2)$ which satisfies $d_E(z) = \delta_1$ and all $u \in U$. Further important features of the extension proposed is given in the next lemma:

Lemma 3.5: For each $\delta \geq 0$ and each u in U $\tilde{f}(\cdot,u)$, $\tilde{f}_0(\cdot,u)$ $\tilde{\lambda}(\cdot,u)$ and $\tilde{F}_\partial(\cdot)$ are locally Lipschitzian functions on $\text{clo}\{N(E,\delta) \bigcup E\}$ with rank given by

$$K_\Gamma(1+c_1\delta) + |\Gamma| c_2\delta + c_3\delta$$

for some positive constants c_1, c_2 and c_3, and for $\Gamma = \tilde{f}$ or $\tilde{f}_0$ or $\tilde{\lambda}$ or $\tilde{F}_\partial$ and K_Γ the correspondent Lipschitz rank of f or f_0 or λ or F_∂ on the set E.

Proof: The fact that $f(\cdot,u)$, $\tilde{f}(\cdot,u)$, $\tilde{f}_0(\cdot,u)$ $\tilde{\lambda}(\cdot,u)$ and $F_\partial(\cdot)$ are continuous function on $E \bigcup N(E,\delta_2)$ for each u is a direct conse-

quence of ($\tilde{D}3$) and ($\tilde{D}4$). Hence it suffices to verify that each function is Lipschitzan with the rank specified for any two arbitrary points in E, in $N(E,\delta_1)$ and in $N(E,\delta_2)\setminus N(E,\delta_1)$. In a first stage, let us take $\delta \le \delta_2$ and two
arbitrary points z and y in the set $N(E,\delta)$ such that

$$(3.30) \qquad d_E(y) \le d_E(z).$$

We label w a third point in $N(E,\delta)$ which satisfies

$$(3.31) \qquad \begin{aligned} d_E(w) &= d_E(y) \\ d_E(w) &= |w-z'| \qquad \text{if } |z-y| \le |z'-v'| \end{aligned}$$

$$(3.32) \qquad \begin{aligned} d_E(w) &= d_E(z) \\ d_E(w) &= |w-y'| \qquad \text{if } |z-y| > |z'-y'|. \end{aligned}$$

That is, when $|z-y| \le |z'-y'|$ E is "concave near z' and y'" and w is a point which possesses the distance to E identical to the value attained by y and the normal vectors to E: w-w' and z-z' are colinear, with z'=w'. When $|z-y| > |z'-y'|$, E is "convex near z' and y'" and the role of $d_E(w)$ and w-w' are interchanged. The point w specified above can always be found, moreover

$$(3.33) \qquad |z-y| \ge |z-w|, \qquad |z-y| \ge |y-w|.$$

At this point we can derive some estimates. Since $d_E(\cdot)$ is Lipschitz of rank one and z and y are in $N(E,\delta)$, we have that

$$(3.34) \qquad |d_E^2(z) - d_E^2(y)| \le 2\delta|z-y|.$$

Also, in view of the representation for $\nabla d_E(\cdot)$ given by lemma 3.2 part (ii) valid on set $N(E,\delta)$ and (3.31) one obtains

$$(3.35) \qquad \begin{aligned} & d_E(y)|\nabla d_E(z) - \nabla d_E(y)| \\ & = |w-w' - (y-y')| \\ & \le |w-y| + |w'-y'| \\ & \le [1+(1-\bar{\delta}^{-1}d_E(y))^{-1}]|w-y| \\ & \le [1+(1-\bar{\delta}^{-1}d_E(z))^{-1}]|z-y| \end{aligned}$$

upon applying lemma 3.4 and expression (3.33). Likewise, we obtain in the case (3.32)

$$(3.36) \qquad d_E(z)|\nabla d_E(z) - \nabla d_E(y)| \le [1+(1-\bar{\delta}^{-1}d_E(z))^{-1}]|z-y|$$

We proceed by evaluating the Lipschitz rank for $\tilde{f}(\cdot,u)$. On set E $\tilde{f}$ agrees with f, hence the rank is K as indicated by the lemma with $\delta=0$. For $0 < \delta \le \delta_1$ we take arbitrary points z and y in $N(E,\delta)$ satisfying $d_E(y) \le d_E(z)$ and evaluate

$$\begin{aligned} & |\nabla d_E(z)d_E^2(z) - \nabla d_E(y)d_E^2(y)| \\ = \; & |\nabla d_E(z)(d_E^2(z)-d_E^2(y)) + d_E^2(y)(\nabla d_E(z) - \nabla d_E(y))| \\ = \; & |\nabla d_E(y)(d_E^2(z)-d_E^2(y)) + d_E^2(z)(\nabla d_E(z) - \nabla d_E(y))|. \end{aligned}$$

We employ the second expression above when (3.31) holds and the estimate (3.35) applies. Otherwise we employ the third expression above since (3.32) is verified and the estimate (3.36) applies. In any of these cases one obtains

$$(3.37)\qquad |\nabla d_E(z)d_E^2(z) - \nabla d_E(y)d_E^2(y)|$$
$$\leq \delta\{3 + (1-\bar{\delta}^{-1}\delta_1)^{-1}\}|z-y|$$

upon also applying (3.34). Now, we evaluate for each $u \in U$

$$(3.38)\qquad |f(z',u)(1-\delta_1^{-2}d_E^2(z)) - f(y',u)(1-\delta_1^{-2}d_E^2(y))|$$
$$\leq (1-\delta_1^{-2}d_E^2(z))\,|f(z',u) - f(y',u)| + |f|\delta_1^{-2}\,|d_E^2(y) - d_E^2(z)|$$
$$\leq K(1-\delta_1^{-2}\,d_E^2(z))|z'-y'| + 2\delta|f|\delta_1^{-2}\,|z-y|.$$

The choice of point w in (3.31) or (3.32), together with lemma 3.4 imply that

$$|z'-y'| = |w'-y'| \leq [1-\bar{\delta}^{-1}d_E(y)]^{-1}|w-y|$$

holds if $|z-y| \leq |z'-y'|$, or

$$|z'-y'| = |z'-w'| \leq [1-\bar{\delta}^{-1}d_E(z)]^{-1}|w-z|$$

holds if $|z-y| > |z'-y'|$.

Now, in view of (3.33) and the fact that $d_E(z) \geq d_E(y)$ we unify both expression as follows:

$$|z'-y'| \leq [1-\bar{\delta}^{-1}d_E(z)]^{-1}|z-y|.$$

Hence we can attribute the following estimate to (3.38)

$$(3.39)\quad |f(z',u)(1-\delta_1^{-2}d_E^2(z)) - f(y',u)(1-\delta_1^{-2}d_E^2(y))|$$

$$< \left\{ K \frac{1-\delta_1^{-2}\, d_E^2(z)}{1-\bar{\delta}^{-1}d_E(z)} + 2\delta_1^{-2}\, |f|\delta \right\} \quad |z-y|.$$

notice that

$$t \to \frac{1-\delta_1^{-2}t^2}{1-\bar{\delta}^{-1}\, t}$$

is continuous on the closed interval $[0,\delta]$ and it is equal to one when $t=0$. We now choose constant c_1 indicated in the lemma so that

$$(3.40)\qquad \frac{1-\delta_1^{-2}t^2}{1-\bar{\delta}^{-1}t} \leq 1 + c_1 t \qquad \text{on the interval } [0,\delta_1]$$

is verified; constant c_2 is simply taken as

$$(3.41)\qquad c_2 = 2\delta_1^{-2}\,.$$

Finally in view of (3.37), (3.39), (3.40) and (3.41) we can deduce from definition ($\tilde{D}3$) that

$$(3.42)\qquad |\tilde{f}(z,u) - \tilde{f}(y,u)| \leq \{K(1+c_1)\delta + c_2|f|\ \delta + d_3\delta\}|z-y|$$

for each u in U and z and y in $\operatorname{clos}\{E\bigcup N(E,\delta)\}$ with $\delta \leq \delta_1$

and $d_3 = [3+(1-\bar{\delta}^{-1}\delta_1)^{-1}]\xi_0\delta_1^{-1}$.

It remains to verify the Lipschitzian property of $\tilde{f}$ on $N(E,\delta_2)\setminus N(E,\delta_1)$. We take arbitrary points z and y in that set with $d_E(y) \leq d_E(z) < \delta$ and definition ($\tilde{D}3$) yields for each u in U the following relation

$$\begin{aligned} &|\tilde{f}(z,u) - \tilde{f}(y,u)| \\ &\leq \xi_0(\delta_2-\delta_1)^{-2} \ |\nabla d_E(z)[(d_E(y)-\delta_1)^2 - (d_E(z)-\delta_1)^2]| \\ &+ \xi_0[1-(\delta_2-\delta_1)^{-2}(d_E(y)-\delta_1)^2] \cdot |\nabla d_E(z) - \nabla d_E(y)| \end{aligned}$$

since $\delta_1 \leq d_E(\cdot) < \delta_2$ on $N(E,\delta_2)\setminus N(E,\delta_1)$ we apply (3.35) or (3.36) to the second term to obtain

$$\begin{aligned} (3.43) \quad &|\tilde{f}(z,u) - \tilde{f}(y,u)| \\ &\leq 2\xi_0(\delta_2-\delta_1)^{-2}(d_E(z) - \delta_1)|z-y| \\ &+ \xi_0\delta_1^{-1} \ [1+(1-\bar{\delta}_1^{-1}\delta_2)^{-1}] \cdot [1-(\delta_2-\delta_1^{-2})^{-2} \ (d_E(y)-\delta_1)^2]\cdot|z-y|. \end{aligned}$$

Now function $t \to 1-(\delta_2-\delta_1)^{-2}(t-\delta_1)^2$ is equally zero when $t=\delta_1$, hence we adopt a number h_2 so that

$$1-(\delta_2-\delta_1)^{-2}(t-\delta_1)^2 \leq h_2(t-\delta_1)$$

is verified on the interval $[\delta_1,\delta_2]$. This, together with the fact that z and y are points in the set $N(E,\delta)$ imply the following estimate for (3.43):

$$(3.44)\qquad |\tilde{f}(z,u) - \tilde{f}(y,u)| \le h_3(\delta-\delta_1)|z-y| \le h_3\,\delta\,|z-y|$$

for each u in U and z and y in $N(E,\delta)\setminus N(E,\delta_1)$, $\delta_1 < \delta \le \delta_2$. Constant h_3 above is

$$h_3 = 2\xi_0(\delta_2-\delta_1)^{-2} + \xi_0\,\delta_1^{-1}[1+(-\bar{\delta}^{-1}\delta_2)^{-1}].$$

Outside set $E\bigcup N(E,\delta_2)$ the expression of lemma is trivially satisfied for each extension. In view of (3.42) and (3.44) the lemma is proven for $\tilde{f}$ with

$$c_3 = \max[d_3,h_3].$$

For each of the remaining data $\tilde{f}_0$, $\tilde{\lambda}$ and $\tilde{F}_\partial$ we can derive the result from an expression similar to (3.39). For arbitrary points z and y in the set $N(E,\delta)$ with $\delta \le \delta_1$ it follows from ($\tilde{D}4$) that

$$|\Gamma(z,u) - \Gamma(y,u)|$$

$$= |\Gamma(z',u)(1-\delta_1^{-2}\,d_E^2(z)) - \Gamma(y',u)(1-\bar{\delta}_1^{-2}\,d_E^2(y))|$$

$$\le K_\Gamma\{\frac{1-\delta_1^{-2}\,d_E^2(z)}{1-\bar{\delta}^{-1}\,d_E(z)} + 2\delta_1^{-2}|\Gamma|\delta\}\cdot|z-y|$$

$$\le \{K_\Gamma(1+c_1\delta) + c_2|\Gamma|\delta\}\cdot|z-y|$$

for each u in U, upon applying (3.39), (3.40) and (3.41). Because each function is trivial outside set $E \bigcup N(E,\delta_1)$ the lemma is thereby proven.

For further use we denote

$$(3.45) \qquad \begin{aligned} \tilde{K}_0 &= K_0(1+c_1\delta_2) + |f_0|c_2\delta_2 + c_3\delta_2 \\ \tilde{K}_\lambda &= K_\lambda(1+c_1\delta_2) + |\lambda|c_2\delta_2 + c_3\delta_2. \\ \tilde{K}_\partial &= K_\partial(1+c_1\delta_2) + |F_\partial|c_2\delta_2 + c_3\delta_2 \end{aligned}$$

It is a direct consequence of the proposed extension that any trajectory initiated on the set $E \bigcup N(E,\delta_2)$ will remain indefinitely on the set $\mathrm{clos}\{E \bigcup N(E,\delta_2)\}$ hence the constants in (3.45) are the uniform Lipschitz rank for function $\tilde{f}_0, \tilde{\lambda}$ and $\tilde{F}_\partial$ respectively.

3.5 Proof of Proposition 3.3

As a preliminary for the proof, we establish three inequalities:

(I1) For any two bounded, non-negative and measurable functions $\phi(\cdot)$ and $\phi'(\cdot)$

$$\int_0^t \max[\phi(s),\phi'(s)]ds \le 2\max\left[\int_0^t \phi(s)ds, \int_0^t \phi'(s)ds\right]$$

for each $t \ge 0$.

That follows directly from the relations

$$\int_0^t \max[\phi(s),\phi'(s)]ds - \max[\int_0^t \phi(s)ds, \int_0^t \phi'(s)ds]$$

$$\leq \int_0^t \max[\phi(s),\phi'(s)]ds - \int_0^t \phi(s)ds$$

$$= \int_0^t \max[0,\phi'(s)-\phi(s)]ds \leq \int_0^t \phi'(s)ds$$

$$\leq \max[\int_0^t \phi(s)ds, \int_0^t \phi'(s)ds].$$

(I2) (Gronwall-Bellman estimate) Let $(x(\cdot),u(\cdot))$ and $(x'(\cdot),u'(\cdot))$ be pairs in the admissible class $\tilde{\Omega}$ for $\tilde{P}_{d,\cdot}$, such that $u(\cdot) = u'(\cdot)$ and

$$x(0) = z, \qquad x'(0) = z'$$

for some z and z' in $E \bigcup N(E,\delta_2)$. Then

$$|x(t) - x'(t)|$$

$$\leq \quad |z-z'| \cdot \exp\{K[t + c_0 \int_0^t \max[d_E(x(s)),d_E(x'(s))]ds]\}$$

for each $t \geq 0$ with $c_0 = c_1 + K^{-1}(c_2|f| + c_3)$. Constants c_1, c_2 and c_3 are given by lemma 3.5.

First notice that lemma 3.5 yields

$$
\begin{aligned}
&|\tilde{f}(x(t),u(t)) - \tilde{f}(x'(t),u'(t))| \\
&\leq \{K(1 + c_1 \max[d_E(x(t)),d_E(x'(t))]) \\
&\qquad + (c_2|f| + c_3) \max[d_E(x(t)),d_E(x'(t))]\}|x(t)-x'(t)| \\
&= K(1+c_0 \max[d_E(x(t),d_E(x'(t))]) \cdot | x(t) - x'(t)|
\end{aligned}
$$

for each $t \geq 0$, provided that $u(t) = u'(t)$. That is the situation proposed in (I2), therefore one obtains

$$
\begin{aligned}
&|x(t) = x'(t)| \\
&\leq |z-z'| + \int_0^t |\tilde{f}(x(s),u(s))- \tilde{f}(x'(s),u'(s))|ds \\
&\leq |z-z'| + K \int_0^t (1+c_0\max[d_E(x(s)),d_E(x'(s))])|x(s)-x'(s)|ds.
\end{aligned}
$$

The Gronwall-Bellman inequality can be applied to the above expression thereby the final expression in (I2) is established.

(I3) Let $(x(\cdot), u(\cdot))$ and $(x'(\cdot),u'(\cdot))$ be pairs in the admissible control class $\tilde{\Omega}$ for $\tilde{P}_d$,. such that $u(\cdot) = u(\cdot)$ and

$$x(0) = z, \quad x'(0) = z'$$

for some z and z' in $E \bigcup N(E,\delta_2)$. Let $\gamma : \mathbb{E}^n \times U \to \mathbb{E}_+$ be a nonegative Lipschitzian function of rank K_γ on the set $E \bigcup N(E,\delta_2)$ for each u in U. Denote for each t

$$\Gamma_1(t) = \int_0^t \gamma(x(s),u(s))ds$$

$$\Gamma_2(t) = \max\left[\int_0^t \gamma(x(s),u(s))ds,\ \int_0^t \gamma(x'(s),u'(s))ds\right].$$

Then for each t

$$\begin{aligned}&|e^{-\Gamma_1(t)} - e^{-\Gamma_2(t)}|\\ &\leq K_\gamma \min\Big[e^{-\Gamma_1(t)} \int_0^t e^{-\Gamma_2(s)+\Gamma_1(s)}|x(s) - x'(s)|ds,\\ &\qquad e^{-\Gamma_2(t)}\int_0^t e^{-\Gamma_1(s)+\Gamma_2(s)}|x(s) - x'(s)|ds\Big].\end{aligned}$$

Similarly, the above estimate holds for $\Gamma_1(\cdot)$ and $\Gamma_2(\cdot)$ defined by

$$\Gamma_1(t) = \int_0^t \gamma(x(s),u(s))ds$$

$$\Gamma_2(t) = \int_0^t \gamma(x'(s),u'(s))ds.$$

For that one solves the linear differential equations

$$\dot{m}_1(t) = -\gamma^+(t)m_1(t) + e^{-\Gamma_1(t)}(\gamma(x(t),u(t))-\gamma^+(t))$$

and

$$\dot{m}_2(t) = -\gamma(x(t),u(t))m_2(t) + e^{-\Gamma_2(t)}(\gamma^+(t)-\Gamma(x(t),u(t)))$$

where

$$\gamma^+(t)=\begin{cases}\gamma(x(t),u(t)), & \text{if } \int_0^t \gamma(x(s),u(s)ds \geq \int_0^t \gamma(x'(s),u'(s))ds \\ \gamma(x'(t),u(t)), & \text{otherwise.}\end{cases}$$

It can be easily verified that when $m_1(0) = m_2(0) = 0$, the solution for the d.e.'s above are respectively

$$m_1(t) = e^{-\Gamma_1(t)} - e^{-\Gamma_2(t)}$$

$$m_2(t) = e^{-\Gamma_2(t)} - e^{-\Gamma_1(t)} \quad .$$

Moreover the solution for the first d.e. yields

$$|m_1(t)| \leq e^{-\Gamma_1(t)} \int_0^t e^{-\Gamma_2(s)+\Gamma_1(s)} |\gamma(x(s),u(s))-\gamma(x'(s),u'(s))|ds$$

and equivalently for the second

$$|m_2(t)| \leq e^{-\Gamma_2(t)} \int_0^t e^{-\Gamma_1(s)+\Gamma_2(s)} |\gamma(x'(s),u'(s))-\gamma(x(s),u(s))|ds$$

thus (I3) is verified. For the second case mentioned in I3 one simply replaces $\gamma^+(\cdot)$ by $\gamma(x'(\cdot),u'(\cdot))$ to obtain the same estimate for $|m(\cdot)|$ and $|m_2(\cdot)|$.

At this point the actual proof for proposition 3.3 will be given. The constant K_1 appearing in the cost for $\tilde{P}_{d,\cdot}$ is chosen to be

(3.46) $\quad K_1 > K \max[4\, c_0, 2\delta_1^{-1}]$

where c_0 appears in (I2) and δ_1 in ($\tilde{D}1$).

The Lipschitzian continuity property of $L_{\tilde{P}_d}(\cdot)$ is verified if one can show that

$$|L^u_{\tilde{P}_d}(z) - L^u_{\tilde{P}_d}(z')| \le K_L |z - z'|$$

holds for some finite number K_L and for all admissible controls in $\tilde{\Omega}$ and points z and z' in the set $E \bigcup N(E, \delta_2)$. In the sequel we employ the local Lipschitz rank for $\tilde{f}$ as defined by lemma 3.5 and apply(I2), whereas for other data we use the uniform Lipschitz constants supplied in (3.45). For ease of notation, hereafter every composite function involving $(x(\cdot),u(\cdot))$ will be denoted in this proof as a function of time only; also, we drop the symbol (~) for the extended functions (e.g. $\tilde{f}_0(x(t),u(t)) = f_0(t)$; $\tilde{f}_0(x'(t),u(t)) = f_0'(t)$). In addition, let $t \to \Delta^+(t)$ and $t \to \Delta^*(t)$ be defined on $[0,t_f]$ respectively by

(3.47) $$\Delta^+(t) = \max\left[\int_0^t d_E(x(s))ds, \int_0^t d_E(x'(s))ds\right]$$

(3.48) $$\Delta^*(t) = \int_0^t \max[d_E(x(s)), d_E(x'(s))]ds.$$

For any control $u:[0,t_f] \to U$ the absolute value of the difference

$$L^u_{\tilde{P}_d}(z) - L^u_{\tilde{P}_d}(z')$$

is clearly dominated by the sum of the following expressions:

$$(3.49)\quad \left| \int_0^{t_f} e^{-\Lambda(t)-K_1\Delta^+(t)}[f_0(t) + K_2 d_E(t) - f_0'(t) - K_2 d_E'(t)]dt \right|$$

$$(3.50)\quad \left| \int_0^{t_f} e^{-\Lambda(t)}[e^{-K_1\Delta(t)} - e^{-K_1\Delta^+(t)}](f_0(t) + K_2 d_E(t))dt \right|$$

$$(3.51)\quad \left| \int_0^{t_f} [e^{-\Lambda(t)-K_1\Delta^+(t)} - e^{-\Lambda'(t)-K_1\Delta(t)}](f_0'(t) + K_2 d_E'(t))dt \right|$$

$$(3.52)\quad \left| e^{-\Lambda(t_f)-K_1\Delta^+(t_f)}[F_\partial(t_f)+K_3 d_{E_\partial}(t_f)-F_\partial'(t_f)-K_3 d_{E_\partial}'(t_f)] \right|$$

$$(3.53)\quad \left| e^{-\Lambda(t_f)}[e^{-K_1\Delta(t)} - e^{-K_1\Delta^+(t)}](F_\partial(t_f) + K_3 d_{E_\partial}(t_f)) \right|$$

$$(3.54)\quad \left| [e^{-\Lambda(t_f)-K_1\Delta^+(t_f)} - e^{-\Lambda'(t_f)-K_1\Delta(t_f)}](F_\partial'(t_f) + K_3 d_{E_\partial}'(t_f)) \right|.$$

A first estimate for (3.49) is

$$\left\{(\tilde{K}_0+K_2) \int_0^{t_f} \exp[-\Lambda(t)-K_1\Delta^+(t) + K(t+c_0\Delta^*(t)]dt\right\}|z-z'|$$

upon applying the fact that $d_E(\cdot)$ and $\tilde{f}_0(\cdot,u)$ are Lipschitz function of rank one and $\tilde{K}_\partial$ respectively. We then apply the Gronwall-Bellman estimate (I2) with notation in (3.48). Recall the choice of K_1 indicated in (3.46) to obtain ·

$$-K_1 \ \Delta^+(t) + K \ c_0 \ \Delta^*(t)$$

$$\leq (2Kc_0 - K_1)\Delta^+(t)$$

$$\leq - \frac{-K_1}{2} \ \Delta^+(t)$$

as a direct consequence of (I1), upon identifying $\Delta^+(\cdot)$ and $\Delta^*(\cdot)$ in (3.47) and (3.48) respectively. Therefore

$$(3.55) \qquad \exp(-K_1\Delta^+(t) + K \ c_1\Delta^*(t)) \leq \exp(- \frac{K_1}{2} \ \Delta^+(t))$$

for all $t \geq 0$ and a less strict estimate for (3.49) is

$$(3.56) \quad \{(\tilde{K}_0+K_2)\int_0^{t_f} \exp(-\Lambda(t) - \frac{K_I}{2} \ \Delta^+(t) + Kt)dt\}|z-z'|.$$

Using the same arguments as above, we obtain an upper estimate for (3.52):

$$(3.57) \quad \{(\tilde{K}_\partial + K_3)\exp(-\Lambda(t_f) - \frac{K_I}{2} \ \Delta^+(t_f) + Kt_f)\}|z-z'|.$$

We will be given similar estimates for the remaining expressions, after an appropriate use of inequality (I3). First, let $\Upsilon(z,u) = K_1 d_E(z)$ so that we have for (I3)

$$\Gamma_1(t) = K_1\Delta(t)$$

$$\Gamma_2(t) = K_1\Delta^+(t).$$

We apply (I3), (I2) and (3.55) in that order to assert for each t

$$(3.58)\quad |e^{-K_1\Delta(t)} - e^{-K_1\Delta^+(t)}|$$

$$\leq K_1\ e^{-K_1\Delta(t)} \int_0^t e^{-K_1\Delta^+(s)+K_1\Delta(s)} |x(s) - x'(s)| ds$$

$$\leq \{K_1\ e^{-K_1\Delta(t)} \int_0^t e^{[-K_1\Delta^+(s)+K_1\Delta(s)+K(s+c_0\Delta^*(s))]} ds\} |z-z'|$$

$$\leq \{K_1\ e^{-K_1\Delta(t)} \int_0^t e^{[(-K_1/2)\Delta^+(s)+K_1\Delta(s)+Ks]} ds\} |z-z'|.$$

Expression (3.58) and an interchange of order of integration yield the following upper estimate for (3.50)

$$(3.59)\quad \{K_1 L^u_{\tilde{P}_d}(z) \int_0^{t_f} \exp(-\Lambda(t) -(K_1/2)\Delta^+(t) + Kt)dt\} |z-z'|$$

upon applying the inequality

$$\int_t^{t_f} e^{-\Lambda(s)-K_1\Delta(s)} [f_0(s)+K_2 d_E(s)]ds \leq e^{-\Lambda(t)-K_1\Delta(t)} L^u_{\tilde{P}_d}(z).$$

Equivalently, expression (3.58) also yields an estimate for (3.53):

$$(3.60)\quad \{K_1(|F_\partial|+K_3|d_{E_\partial}|) \int_0^{t_f} \exp(-\Lambda(t) - (K_1/2)\Delta^+(t)+Kt)dt\} |z-z'|$$

upon using the fact that

$$e^{-\Lambda(t_f)-K_1\Delta(t_f)} \int_0^{t_f} e^{((-K_1/2)\Delta^+(t) + K_1\Delta(t)+Kt)} dt$$

$$\leq \int_0^{t_f} e^{(-\Lambda(t)-(K_1/2)\Delta^+(t) + Kt)} dt$$

Notice that $|d_{E_\partial}|$ is meaniful since it is finite on $\mathrm{clos}\{E\bigcup N(E,\delta_2)\}$ following assumption (A1) part (i).

For the remaining expressions we denote $\Upsilon(z,u) = \tilde{\lambda}(z,u)$ so that $\Gamma_1(\cdot)$ and $\Gamma_2(\cdot)$ in (I3) become

$$\Gamma_1(\cdot) = \Lambda(\cdot), \quad \Gamma_2(\cdot) = \Lambda'(\cdot).$$

We apply (I3) and (I2) successively to obtain for each t

(3.61) $|e^{-\Lambda(t)} - e^{-\Lambda'(t)}|$

$$\leq \tilde{K}_\lambda e^{-\Lambda'(t)} \int_0^t e^{-\Lambda(s)+\Lambda'(s)}|x(s) - x'(s)|ds$$

$$\leq \{\tilde{K}_\lambda e^{-\Lambda'(t)} \int_0^t e^{(-\Lambda(s)+\Lambda'(s)+K(s+c_0\Delta^*(s)))} ds\}|z-z'|.$$

In order to give estimates for (3.51) and (3.54) wee first write

$$|e^{-\Lambda(\cdot)-K_1\Delta^*(\cdot)} - e^{-\Lambda'(\cdot)-K_1\Delta(\cdot)}|$$

$$\leq e^{-\Lambda'(\cdot)}|e^{-K_1\Delta^+(\cdot)} - e^{-K_1\Delta(\cdot)}| + e^{-K_1\Delta^+(\cdot)}|e^{-\Lambda(\cdot)} - e^{-\Lambda'(\cdot)}|$$

and employ the inequalities in (3.58) and (3.61). Whence, after a interchange of order in the integration the following estimate is supplied for (3.51)

$$(3.62)\quad L^{u}_{\tilde{P}d}(z)\Big\{\tilde{K}_\lambda \int_0^{t_f} \exp(-\Lambda(t) - \frac{K_1}{2}\Delta^+(t) + Kt)dt$$

$$+ K_1 \int_0^{t_f} \exp(-\Lambda'(t) - \frac{K_1}{2}\Delta^+(t) + Kt)dt\Big\}|z-z'|.$$

Finally upon applying (3.58) and (3.61) and using a similar justification to the one given to estimate (3.60) it follows that

$$(3.63)\quad (|F_\partial|+K_3|d_{E_\partial}|)\Big\{\tilde{K}_\lambda \int_0^{t_f} \exp(-\Lambda(t) - \frac{K_1}{2}\Delta^+(t) + Kt)dt$$

$$+ K_1\int_0^{t_f} \exp(-\Lambda'(t) - \frac{K_1}{2}\Delta^+(t) + Kt)dt\Big\}|z-z'|.$$

Therefore, in view of (3.56), (3.57), (3.59), (3.60), (3.62) and (3.63) we can assert that

$$(3.64)\quad |L^{u}_{\tilde{P}d}(z) - L^{u}_{\tilde{P}d}(z')|$$

$$\leq \Big\{r_1 \int_0^{t_f} \exp(-\Lambda(t) - \frac{K_1}{2}\Delta^+(t) + Kt)dt$$

$$+ r_2 \exp(-\Lambda(t_f) - \frac{K_I}{2} \Delta^+(t_f) + Kt_f)$$

$$+ r_3 \int_0^{t_f} \exp(-\Lambda'(t) - \frac{K_I}{2} \Delta^+(t) + Kt)dt\}|z-z'|$$

for any z and z' in $\mathrm{clos}\{E \bigcup N(E,\delta_2)\}$, where number r_1, r_2 and r_3 are

$$r_1 = \tilde{K}_0 + K_2 + (K_1+\tilde{K}_\lambda)(|L^u_{\tilde{P}d}| + |F_\partial|+K_3|d_{E_\partial}|)$$

$$r_2 = \tilde{K}_\partial + K_3$$

$$r_3 = K_1(|L^u_{\tilde{P}d}| + |F_\partial| + K_3 \ |d_{E_\partial}|).$$

The finiteness of the expression within brackets in the right-hand side of (3.64) will now be shown for any points z and z' in $\mathrm{clos}\{E \bigcup N(E,\delta_2)\}$. Initially suppose that

$$t_f \leq t^*(z), \quad t_f \leq t^*(z).$$

In that case we use the probabilistic interpretation for data so that the expression within brackets is bounded by

$$(r_1+r_2) \int_0^\infty P^u_z(T_1 > t)e^{Kt}dt + r_3 \int_0^\infty P^u_{z'}(T_1>t) \ e^{Kt}dt$$

and assumption (A2) provides the argument.

Now suppose that

$$t_f > t^*(z) \quad \text{and/or} \quad t_f > t^*(z').$$

In view of the result for trajectories which remain in E, to show the finiteness of the expression within brackets in (3.64) it suffices to show for some $d_1 \geq 0$ and $d_2 > 0$ that

$$-\Lambda(t) - \frac{K_1}{2}\,\Delta^+(t) + Kt < d_1 - d_2 t$$

for any point $z \in \operatorname{clos}\{N(E,\delta_2)\}$ with $x(0) = z$ and t large enough. Suppose that assumption (A4) is in force. Hence, using the definition for the extension $\tilde{\lambda}(\cdot,\cdot)$ in ($\tilde{D}4$) and the choice of parameter K_1 indicated by (3.46) we have for each u in U and z in $\operatorname{clos}\{N(E,\delta_2)\}$ that

$$\tilde{\lambda}(z,u) + \frac{K_I}{2}\,d_E(z)$$

$$= \lambda(z',u)(1-\delta_1^{-2}d_E^2(z)) + \frac{K_I}{2}\,d_E(z)$$

$$> K(1-\delta_1^{-2}d_E^2(z)) + 2K\delta_1^{-1}\,d_E(z) \geq K.$$

Therefore we can take $d_1 = 0$ and

$$d_2 = \inf_{z\in E_\partial,\ u\in U} \lambda(z,u) - K.$$

Let us suppose that (A4') is in force, and let ξ_1 be the positive number such that

$$(3.65) \quad \xi_1 \leq \inf_{z'\in E_\partial, u\in U} \{\langle v,f(z',u)\rangle : v \in \partial d_E(z), |v| = 1\}.$$

Lemma 3.3 part (ii) asserts that $d_E(\cdot)$ in continuously differentiable in $N(E,\delta_2)$ since $\delta_2<\bar{\delta}$. Initially, suppose that $x(t)$ belongs to $N(E,\delta_1)$ for some t. One can justify the following relation by employing lemma 3.2 part (i) and part (ii)

$$\frac{d}{dt}(d_E(x(t)) = \langle \nabla d_E(x(t)) \ , \ \tilde{f}(x(t),u(t))\rangle$$

$$= \langle v \ , \ \tilde{f}(x(t),u(t))\rangle$$

for some $v \in \partial d_E(x'(t))$ with $x'(t)$ the nearest point in E to $x(t)$. In view of definition ($\tilde{D}3$) for $\tilde{f}(\cdot,u)$ on $N(E,\delta_1)$ it follows that

$$\frac{d}{dt}(d_E(x(t)) \geq \xi_0\delta_1^{-2}d_E^2(x(t)) + \xi_1(1-\delta_1^{-2}d_E^2(x(t)))$$

since ξ_0 enters ($\tilde{D}3$) as an arbitrary parameter, we can choose $\xi_0 = \xi_1$ to obtain

$$d_E(x(t)) - d_E(x(0)) \geq \xi_1 t$$

for any interval $[0,t]$, provided that $x(0)$ and $x(t)$ are in $\text{clos}\{N(E,\delta_1)\}$. In addition, because $d_E(x(\cdot))$ is strictly monotone it follows that

$$d_E(x(\cdot)) \geq \delta_1 \text{ on } \text{clos}\{N(E,\delta_2)\backslash N(E,\delta_1)\}.$$

One finally obtains

$$-\Lambda(t) - \frac{K_1}{2}\Delta^+(t) + Kt$$

$$\leq -\frac{K_1}{2}\Delta(t) + Kt \quad \leq -\frac{K_1}{2}\left(-\frac{\delta_1^2}{2} + \delta_1 t\right) + Kt$$

for $t \geq \delta_1\xi_1^{-1}$. With the choice indicated in (3.46) for K_1 one can take $d_1 = \frac{K_1\delta_1^2}{4}$ and $d_2 = \frac{K_1\delta_1}{2} - K$

and the proposition is demonstrated.

3.6 Proof of Proposition 3.4

The proof evolves in two phases. Initially we suppose that the data of the problem satisfies the stronger condition (A4'). Thereupon we claim that any potential solution for $\tilde{P}_{d,\cdot}$ should satisfy the end point constraints, in that case $L^u_{P_d}(\cdot) = L^u_{\tilde{P}_d}(\cdot)$ holds and u also is a potential solution for $P_{d,\cdot}$. In a further stage, assumption (A4') is replaced by the weaker condition in (A5) (calmness). Presently, constants K_2 and K_3 which appear in the cost of $P_{d,\cdot}$ are chosen as indicated:

(3.66) $K_2 > K_3(|\lambda| + K_1\delta_2)$

(3.67) $K_3 > \max\{[|f_o|+|\lambda F_\partial|+\tilde{K}_\partial|f|]K^*,\ \xi_o^{-1}[\tilde{K}_\partial|f|+(|\lambda|+K_1\delta_2)|F_\partial|]\}$.

Let $t \to u(t)$ be a control defined on $[0,t_f]$ and $(x(\cdot),u(\cdot))$ a pair in $\tilde{\Omega}$. Suppose that $(x(\cdot),u(\cdot))$ is a potentially optimal pair for $\tilde{P}_{d,z}$, some $z \in E$ such that $t_f \neq t^*(z) = \inf\{t \geq 0 : x(t) \in E_\partial\}$. We show that a better candidate $(\bar{x}(\cdot),\bar{u}(\cdot))$ in $\tilde{\Omega}$ exists such that $t \to \bar{u}(t)$ is defined on $[0,t^*(z)]$ and $\bar{u}(\cdot) = u(\cdot)$ on the set $[0,\ t_f \wedge t^*(z)]$.

Initially suppose that $t_f < \infty$, $t_f < t^*(z)$ holds. In this situation (A3) asserts that a control $\bar{u}(\cdot)$ should exist on $[0,t^*_{\bar{u}}(z)]$ such that

$$t^*_{\bar{u}}(z) - t_f \leq K^* d_{E_\partial}(x(t_f)).$$

We evaluate the difference of costs, using the fact that $t \to F_\partial(x(t))$ is Lipschitzian function:

$$(3.68)\quad L^u_{\tilde{P}_d}(z) - L^{\bar{u}}_{\tilde{P}_d}(z)$$

$$= e^{-\Lambda(t_f)}\,[F_\partial(\bar{x}(t_f)) + K_3 d_{E_\partial}(\bar{x}(t_f))] -$$

$$\int_{t_f}^{t^*_{\bar{u}}(z)} e^{-\Lambda(t)} f_0(\bar{x}(t),\bar{u}(t))dt \quad - \quad e^{-\Lambda(t^*_{\bar{u}}(z))} F_\partial(\bar{x}(t^*_{\bar{u}}(z))$$

$$\geq \qquad e^{-\Lambda(t_f)} K_3\, d_{E_\partial}(\bar{x}(t_f)) -$$

$$\int_{t_f}^{t^*_{\bar{u}}(z)} e^{-\Lambda(t)}[f_0(\bar{x}(t),\bar{u}(t)) + \lambda(\bar{x}(t),\bar{u}(t)F_\partial(\bar{x}(t))$$

$$+ \sup\{\langle v,f(\bar{x}(t),\bar{u}(t))\rangle : v \in \partial F_\partial(x(t))\}]dt$$

applying the choice for K_3 indicated in (3.67), we have

$$> e^{-\Lambda(t_f)}(|f_0| + |\lambda F_\partial| + \tilde{K}_\partial |f|)\,(t^*_{\bar{u}}(z) - t_f)$$

$$- \int_{t_f}^{t^*_{\bar{u}}(z)} e^{-\Lambda(t)}[f_0(\bar{x}(t),\bar{u}(t)) + \lambda(\bar{x}(t),\bar{u}(t))F_\partial(\bar{x}(t))$$

$$+ \sup\{\langle v, f(\bar{x}(t),\bar{u}(t))\rangle : \quad v \in \partial F_\partial(x(t))\}]dt$$

thus $L^{u}_{\tilde{P}_d}(z) > L^{\bar{u}}_{\tilde{P}_d}(z)$.

Now suppose that $t^*(z) < t_f \leq \infty$, $t^*(z) < \infty$ holds; since condition (A4') is in force $x(t), t > t^*(z)$ can never re-enter set E so that $d_E(x(t_f)) > 0$. One takes $\bar{u}(\cdot)$ as the restriction for $u(\cdot)$ on $[0,t^*(z)]$ and evaluates

(3.69) $$L^{u}_{\tilde{P}_d}(z) - L^{\bar{u}}_{\tilde{P}_d}(z)$$

$$= \int_{t^*(z)}^{t_f} e^{-\tilde{\Lambda}(t)-K_1\Delta(t)}\,[\tilde{f}_0(x(t),u(t)) + K_2 d_E(x(t))]dt$$

$$+ \quad e^{-\tilde{\Lambda}(t_f)-K_1\Delta(t_f)}[\tilde{F}_\partial(x(t)) + K_3 d_{E_\partial}(x(t_f))] \; - e^{-\tilde{\Lambda}(t^*(z))}F_\partial(x(t^*(z))$$

$$\geq \int_0^{t_f} e^{-\tilde{\Lambda}(t)-K_1\Delta(t)}\{\langle v(t),\tilde{f}(x(t),u(t))\rangle + K_3\ \langle \Delta d_E(x(t)), f(x(t),u(t))\rangle$$

$$- \quad (\lambda(x(t),u(t)) + K_1 d_E(x(t))[\tilde{F}_\partial(x(t)) + K_3 d_{E_\partial}(x(t))] + K_2\ d_E(x(t))\}dt$$

for $v(t) \in \partial\tilde{F}_\partial(x(t))$ a.e. Here we apply the fact that $t \to \tilde{F}_\partial(x(t))$ is Lipschitzan function hence absolutely continuous. Moreover $d_E(x(\cdot))$ is monotone and $x(\cdot) \in N(E,\delta_2)$ on the interval $(t^*(z),t_f]$ hence $\nabla d_E(x(\cdot))$ exists and coincides with $\nabla d_{E_\partial}(x(\cdot))$ on the specified interval. Lemma 3.2 part (i) and part (ii) assert that

$$\nabla d_E(x(\cdot)) \in \partial d_E(x'(\cdot))$$

for $x'(t)$ the nearest point in E to $x(t)$ for each t. In view of definition ($\tilde{D}3$) for $\tilde{f}(\cdot,\cdot)$ we have that

$$\begin{aligned}
&\langle \nabla d_E \langle x(\cdot)), \tilde{f}(x(\cdot),u(\cdot))\rangle \\
&\geq \xi_0 \delta_1^{-2} d_E^2(x(\cdot)) + \xi_1(1-\delta_1^{-2} d_E^2(x(\cdot)), \quad \text{if } x(\cdot) \in N(E,\delta_1) \\
&\geq \xi_0(1 - (\delta_2-\delta_1)^2(d_E(x(\cdot)) - \delta_1)^2), \quad \text{if } x(\cdot) \in N(E,\delta_2) \setminus N(E,\delta_1)
\end{aligned}$$

where ξ_1 is the positive number which satisfies

$$\xi_1 \leq \inf_{z' \in E_\partial, u \in U} \{\langle v,f(z',u)\rangle : v \in \partial d_E(z'), |v| = 1\}.$$

As in the proof of proposition 3.3 in the part following (3.65), we choose ξ_0 to be equal to ξ_1. Applying the relations above and the fact that $|v(\cdot)| \leq \tilde{K}_\partial$ to (3.69), it yields

$$L^{u}_{\tilde{P}_d}(z) - L^{\bar{u}}_{\tilde{P}_d}(z)$$

$$\geq \int_0^{t_f} e^{-\tilde{\Lambda}(t)-K_1\Delta(t)} \{I_{\{x(t)\epsilon N(E,\delta_1)\}}[-K_\partial|f|(1 - \delta_1^{-2}d_E^2(x(t)))$$

$$+ K_3\xi_0 - (\lambda+K_1\delta_2)(|F_\partial|(1-\delta_1^{-2}d_E^2(x(t))) + K_3 d_{E_\partial}(x(t))) + K_2 d_E x(t))]$$

$$+ I_{\{x(t)\epsilon N(E,\delta_2)\setminus N(E,\delta_1)\}}[K_2 d_E(x(t))$$

$$+ K_3\xi_0(1 - (\delta_2 - \delta_1)^2(d_E(x(t)) - \delta_1)^2]\}dt.$$

This expression is strictly positive, upon employing the choice for parameters K_2 and K_3 estipulated by (3.66) and (3.67) respectively. We have shown that a solution for $\tilde{P}_{d,z}$ should comply with the end point constraint for $P_{d,z}$ for any pair $(x(\cdot), u(\cdot))$ in Ω with $t^*(z) < \infty$, $x(t^*) \in E_\partial$. It is clear that any pair $(x(\cdot),u(\cdot))$ in Ω with $t^*(z) = +\infty$ and $x(\cdot) \in E$ is also contained in the class $\tilde{\Omega}$. In addition because $F_\partial(\cdot)$ and $d_{E_\partial}(\cdot)$ are bounded on $E \bigcup N(E,\delta_2)$ (the second as result of (A1) part (i)) it follows that

$$e^{-\Lambda(t)}[F_\partial(x(t)) + K_3 d_{E_\partial}(x(t))] \to 0, \qquad \text{as } t \to \infty$$

as a consequence of (A2), since in particular when $t_u^*(z) = +\infty$

$$e^{-\Lambda(t)} = P_z^u(T_1>t) \to 0 , \qquad \text{as } t \to \infty$$

is implied. The result of the proposition is thereby established when (A4') is in force.

To complete the proof we verify the result with condition (A5) replacing (A4') (recall that earlier (A4) served as an alternative condition to (A4') in proposition (3.3). For sake of argument, suppose that problem $P_{d,\cdot}$ is defined on the domain set

$$E_{+\delta_1} = \operatorname{clos}\{E \bigcup N(E,\delta_1)\}$$

hence the boundary set is now $E_{\partial+\delta_1}$ defined in the obvious way

$$E_{\partial+\delta_1} = E_{+\delta_1} \setminus \operatorname{int}\{E_{+\delta_1}\}.$$

Now, from lemma 3.2 part (ii) and lemma 3.3 part (ii) we verify that for each point $z \in E_{\partial+\delta_1}$ there exists a unique normal to the set $E_{+\delta_1}$ at z expressed by $\frac{z-z'}{|z-z'|}$ where z' is the nearest point to z in E. Then $E_{+\delta_1}$ is smooth set and it follows from lemma 3.2 part (i) that

$$\partial d_{E_{+\delta_1}}(z) = \{\alpha \frac{v}{|v|} : 0 \leq \alpha \leq 1\}$$

for each $z \in E_{\partial+\delta_1}$ with $v = z-z'$. In view of definition ($\tilde{D}3$) for $\tilde{f}(\cdot,\cdot)$ and the remark (3.29) afterwards, it follows that for each $z \in E_{\partial+\delta_1}$

$$\inf\{\langle v,\tilde{f}(z,u)\rangle : u \in U,\ v \in \partial d_{E_{+\delta_1}}(z),\ |v| = 1\} \geq \xi_0.$$

Thus problem $P_{d,}$. with extended domain $E_{+\delta_1}$ satisfies condition (A4'), thereupon the results of the present proposition applies. If $L_{+\delta_1} : E_{+\delta_1} \to \mathbb{R}$ is the value for the above mentioned problem, propositions 3.3 and 3.4 state that $L_{+\delta_1}(\cdot)$ is Lipschitzian function on the entire domain. Applying the choice for δ_1 indicated in definition ($\tilde{D}1$) and using the calmness condition (A5) one obtains

$$-L_{+\delta_1}(z) + L(z) \leq \varepsilon$$

for each $z \in E$ and for an arbitrary $\varepsilon > 0$.

At this point we want to show that the difference above is $\geq -\varepsilon$, thereby showing the result of the proposition. For this purpose, the following lemma is given.

Lemma 3.6 There exists controls in $\underline{\underline{U}}_0$, admissible for P_1 with augmented domain $E_{+\delta_1}$ such that

$$t^*_{\delta_1}(z') = \inf\{t \geq 0 : x(t,z') \in E_{\partial+\delta_1}\} \leq \delta_1$$

for each $z' \in E_\partial$.

At any point z' in E_∂ we apply the control indicated by the lemma and evaluate

$$L_{+\delta_1}(z')$$

$$\leq \int_0^{t^*_{\delta_1}(z')} e^{-\tilde{\Lambda}(t)} \tilde{f}_0(x(t),u(t))dt + e^{-\Lambda(t^*_{\delta_1})} \tilde{F}_\partial(x(t^*_{\delta_1}))$$

$$\leq \int_0^{t^*_{\delta_1}(z')} e^{-\tilde{\Lambda}(t)}[\tilde{f}_0(x(t),u(t)) - \tilde{\lambda}(x(t),u(t))\tilde{F}_\partial(x(t))$$

$$+ \langle v(t),\tilde{f}(x(t),u(t))\rangle]dt + \tilde{F}_\partial(z')$$

where $v(t) \in \partial\tilde{F}_\partial(x(t))$. Since $\tilde{F}_\partial(\cdot) = F_\partial(\cdot)$ on E_∂ we can conclude that

$$L_{+\delta_1}(z') \leq F_\partial(z') + (|f_0| + |\lambda F_\partial| + \tilde{K}_\partial|f|)\delta_1 \leq F_\partial(z') + \varepsilon$$

the last expression is obtained from the choice of parameter δ_1 made in $(\tilde{D}1)$. This clearly yields

$$L_{+\delta_1}(z) \leq L(z) + \varepsilon$$

for each $z \in E$, which is the relation sought. The proposition is thereby proved.

Proof of lemma 3.6:

Initially, it is shown that for each point z' in E_∂, a control value $\bar{u}$ and a vector $v \in \partial d_E(z')$ exist so that

$$(3.70) \qquad \langle v, f(z', \bar{u})\rangle \geq (K^*)^{-1}$$

where K^* is the constant appearing in (A3). Let $t \to u(t,z)$ be the control in $\underline{U}_0$ for P_1 with the original domain E that has the property claimed in (A3), and let $t \to x(t,z)$ be the corresponding trajectory for each $z \in E$. One then can write

$$(K^*)^{-1}\, t_u^*(z) \leq d_{E_\partial}(z)$$

and

$$d_{E_\partial}(z) = \int_0^{t_u^*(z)} \langle -v(t), f(x(t,z), u(t,z))\rangle dt$$

for some $t \to v(t)$ such that $v(t) \in \partial d_{E^c}(x(t,z))$ (E^c = complementary set for E). Now suppose that $z' \in E_\partial$ and E is smooth near z'. It is clear that any point $z \in \mathrm{int}\{E\}$ near enough z' should possess a unique nearest point in E^c, hence $\nabla d_{E^c}(\cdot)$ exists for any such points by the reasons given in lemma 3.3 part (ii). We can evaluate

$$(3.71) \qquad (K^*)^{-1} \leq \liminf\left\{ \frac{d_{E_\partial}(z)}{t_u^*(z)} : z \to z',\ z \in \mathrm{int}\{E\}\right\}$$

$$\leq \liminf\left\{ \frac{1}{t_u^*(z)} \int_0^{t_u^*(z)} \langle -\nabla d_{E^c}(x(t,z)), f(\dot{x}(t,z), u(t,z))\rangle dt \right.$$

$$\left. : z \to z',\ z \in \mathrm{int}\{E\}\right\}$$

$$\leq \quad \liminf\left\{ \frac{1}{t_u^*(z)} \int_0^{t_u^*(z)} \max_{u\in U} \langle \nabla d_{E^c}(x(t,z)), f(x(t,z),u)\rangle dt \right.$$

$$\left. : z \to z',\ z \in \mathrm{int}\{E\} \right\}$$

since $t \to \max_{u\in U} \langle -\nabla d_{E^c}(x(t,z)), f(x(t,z),u)\rangle$ is continuous we have

$$\leq \quad \liminf\left\{ \frac{1}{t_u^*(z)} \left[\{\max_{u\in U} \langle -\nabla d_{E^c}(z), f(z,u)\rangle \cdot t_u^*(z) + 0(t_u^*(z)) \right] \right.$$

$$\left. : z \to z',\ z \in \mathrm{int}\{E\} \right\}$$

$$= \max_{u\in U} \langle v, f(z',u)\rangle$$

for some $v \in -\partial d_{E^c}(z') = \partial d_E(z)$ with $|v| = 1$. Because $f(\cdot,u)$ is continuous and points where E is smooth form a dense set in E_∂, we have shown (3.70).

We wish to supply the bound

$$\text{(3.72)} \qquad \frac{d}{dt}\,(d_E(x_u(t,z'))) \geq 1$$

for a.a. $t \in [0, t_{\delta_1}^*(z')]$, each $z' \in E_\partial$ and some control u, from which the result follows. We construct such a control u in the following way

$$\text{(3.73)} \quad u(t, z') = \begin{cases} \bar{u} & \text{on } t \in [0,\varepsilon_0] \\ \text{arbitrary, admissible} & \text{on } t > \varepsilon_0 \end{cases}$$

here $\bar{u}$ is the control value satisfying (3.70) for each z' and ε_0 is an arbitrary number such that $0 < \varepsilon_0 < K^* \delta_1$. Recall that by (A1) part (ii) E can only be non-smooth near z' if E is convex near z'. It follows from [C1, 2.4.4] that $d_E(\cdot)$ is convex near z' and [C1, 2.2.7] implies that the generalized gradient set $\partial d_E(z')$ coincides with the sub-differential of $d_E(\cdot)$ at z' in the sense of convex analysis. Employing the latter characterization, we write for any such a point z' and any $v \in \partial d_E(z')$:

$$(3.74) \qquad d_E(z'+rt) \geq \langle v,r\rangle t \ , \quad \text{for all } r \in \mathbb{R}^n.$$

Using the choice of control u(•,z') indicated by (3.73) and (3.74) with $r = f(z',\bar{u})$, we calculate

$$(3.75) \qquad \liminf_{t\to 0} t^{-1} d_E(x_u(t,z'))$$

$$= \liminf_{t\to 0} t^{-1}\{d_E(z' + f(z',\bar{u})t) + O(t)\}$$

$$\geq \langle v, f(z'\bar{u})\rangle \qquad \text{for each } v \in \partial d_E(z')$$

upon applying the continuity of $f(\cdot,\bar{u})$. When E is non-convex near z', the representation given by (A1) part (ii) is

$$\partial d_E(z') = \{\alpha \frac{r}{|r|} : 0 \leq \alpha \leq 1\} \text{ for some } r \in \mathbb{R}^n,\ r \neq 0.$$

In this case, (3.75) clearly holds and equality is attained when $v = \dfrac{r}{|r|}$. As a result of (3.75) the choice of controls in (3.73) and relation (3.70), we have that

$$d_E(x_u(t,z')) \geq (K^*)^{-1}\varepsilon_0$$

at $t = \varepsilon_0$ for all $z' \in E_\partial$. Consequently, $x(t,z') \in N(E,\delta_1)$ for $t \geq \varepsilon_0$ and we can evaluate $\frac{d}{dt}(d_E(x_u(t,z')))$ having in mind definition $(\tilde{D}3)$ and lemma 3.3 part (ii)

$$\frac{d}{dt}(d_E(x_u(t,z'))) = \langle \nabla d_E(x_u(t,z')), \tilde{f}(x_u(t,z'),u(t,z'))\rangle$$

$$= \xi_0\ \delta_1^{-1} d_E^2(x_u(t,z'))$$

$$+ \langle \nabla d_E(x_u(t,z')), f(x_u(t,z'),u(t,z'))\rangle(1 - \delta_1^{-2} d_E^2(x_u(t,z')))$$

$$\geq (\xi_0 + |f|)\delta_1^{-2} d_E^2(x_u(t,z')) - |f|$$

ξ_0 is the constant appearing in $(\tilde{D}3)$, the definition of $\tilde{f}$. Since ξ_0 is arbitrary, we can choose

$$\xi_0 > [1 + |f|(1 - \delta_1^{-2}(K^*\varepsilon_0)^2)]\delta_1^2(K^*\varepsilon_0)^{-2}]$$

to obtain (3.72) for $t \geq \varepsilon_0$. Hence, the maximum time spent by any such a trajectory in the set $N(E,\delta_1)$ is $\delta_1 + \varepsilon_0$, since ε_0 is arbitrary, the estimate of the lemma is valid.

3.7 Proof of Theorem 3.2

We will use throughout the equivalence of problems P_1 and $P_{d,z}$ together with the Lipschitz characterization for the value function. The lemma in the sequel will prove useful.

Lemma 3.7 For each $z \in \text{int}\{E\}$

$$\min_{\xi \in \partial L(x), u \in U} \{\langle \xi, f(z,u)\rangle - \lambda(z,u)L(z) + f_0(z,u)\} \geq 0$$

Proof: Let z be a point in $\text{int}\{E\}$. A control value $\bar{u} \in U$ is given and let z' be a point in some neighbourhood of z such that

$$(3.76) \qquad z = x(\bar{t}) = z' + \int_0^{\bar{t}} f(x(t),\bar{u})dt \quad , \text{ some } \bar{t} > 0$$

with $z' = x(0)$.

It follows from the optimality principle that

$$(3.77) \quad e^{-\Lambda(t)} L(x(t)) \leq \int_t^{\bar{t}} e^{-\Lambda(s)} f_0(x(s),\bar{u})ds + e^{-\Lambda(\bar{t})}L(x(\bar{t}))$$

for each $t \in [0,\bar{t}]$. In addition the Lipschitz continuity of $f(\cdot,\bar{u})$ and (3.76) yield the following estimate

$$(3.78) \quad \left| \int_t^{\bar{t}} [f(x(s),u) - \bar{f}(z,u)]ds \right| \leq \frac{1}{2} K|f|(\bar{t}-t)^2$$

for each $t \in [0,\bar{t}]$.

Initially suppose that $L(\cdot)$ is differentiable at point z. Thus we can employ (3.77) and evaluate the limit

$$(3.79) \quad 0 \leq \lim_{t \to \bar{t}} (\bar{t}-t)^{-1} \{ \int_t^{\bar{t}} e^{-\Lambda(s)} f_0(x(\bar{s}),u)ds$$

$$+ e^{-\Lambda(\bar{t})}L(x(\bar{t})) - e^{-\Lambda(t)}L(x(t))\}$$

$$= e^{-\Lambda(\bar{t})} f_0(z,\bar{u}) + \lim_{t\to\bar{t}} (\bar{t}-t)^{-1} \{ e^{-\Lambda(\bar{t})} L(z) - e^{-\Lambda(t)} L(z - \int_t^{\bar{t}} f(x(s)),\bar{u}) ds \}$$

$$\leq e^{-\Lambda(\bar{t})} f_0(z,\bar{u}) + \lim_{t\to\bar{t}} (\bar{t}-t)^{-1} \{ e^{-\Lambda(\bar{t})} L(z) - e^{-\Lambda(t)} L(z-(\bar{t}-t) f(z,\bar{u}))$$

$$+ e^{-\Lambda(t)} K_L \cdot \left| \int_t^{\bar{t}} [f(z,\bar{u}) - f(x(s),\bar{u})] ds \right| \}$$

$$\leq f_0(z,\bar{u}) - \lambda(z,\bar{u}) L(z) + \langle \nabla L(z), f(z,\bar{u}) \rangle$$

upon denoting K_L the Lipschitz rank of $L(\cdot)$, applying estimate (3.78) and the fact that $0 < e^{-\Lambda(\bar{t})} < 1$.

Now we can deduce an appropriate version of (3.79) for arbitrary point z in E. Firstly recall the characterization for generalized gradients given in lemma 3.2 part (i). Secondly note that $f_0(\cdot,\bar{u})$ $\lambda(\cdot,\bar{u}), f(\cdot,\bar{u})$ and $L(\cdot)$ are continuous, so that for each $\xi \in \partial L(z)$ we have

$$f_0(z,\bar{u}) - \lambda(z,\bar{u}) L(z) + \langle \xi, f(z,\bar{u}) \rangle$$

$$\in \text{co}\{ \lim [f_0(z_i,\bar{u}) - \lambda(z_i,\bar{u}) L(z_i) + \langle \nabla L(z_i), f(z_i,\bar{u}) \rangle]$$

$$: z_i \to z, \quad z_i \notin S \bigcup \Omega_L \} \geq 0$$

ξ and $\bar{u}$ are arbitrary elements of compact sets, thereby the lemma is verified.

Necessity: Let $u \in \underline{\underline{U}}$ be a solution for P_1 so that $u = u_0/u_\partial$ for u_0 of class $\underline{\underline{U}}_0$ and u_∂ of class $\tilde{\underline{\underline{U}}}_\partial$; by definition

$$L(\cdot) = L^u(\cdot)$$

on E, therefore theorem 3.1 asserts that $L^u(\cdot)$ is Lipschitz continuous function on E. In particular, at any point $z \in E_\partial$

$$L^u(z) = F(z,u_\partial(z)) = \min_{u \in \tilde{\underline{\underline{U}}}_\partial} F(z,u).$$

The above relation yields the boundary condition (3.9) in view of definition of $t^*_{u_0}(\cdot)$.

Let $(x(\cdot),u(\cdot))$ be the corresponding solution for $P_{d,z}$ at some $z \in \mathrm{int}\{E\}$, namely

$$(x(\cdot),u(\cdot)) = (x(\cdot,z),u_0(\cdot,z)) \quad \text{on } [0,T_1(\omega)]$$

with $u_0 \in \underline{\underline{U}}_0$ optimal for P_1. Let $t \in (0,t^*_{u_0}(z))$ be a Lebesgue point for each of the following functions

$$(3.80) \quad t \to x(t), \quad t \to e^{-\Lambda(t)}, \; t \to \int_0^t e^{-\Lambda(s)} f_0(x(s),u(s))ds.$$

Since $(x(\cdot),u(\cdot))$ is optimal for $P_{d,z}$, we invoke the optimality principle to obtain

$$e^{-\Lambda(t)}L(x(t)) = \int_t^{\bar{t}} e^{-\Lambda(s)}f_0(x(s),u(s))ds + e^{-\Lambda(\bar{t})}L(x(\bar{t}))$$

for each $t \in [0,\bar{t}]$. We now evaluate

$$(3.81) \qquad 0 \geq \liminf_{t\to\bar{t}} (\bar{t}-t)^{-1}\{\int_t^{\bar{t}} e^{-\Lambda(s)}f_0(x(s),u(s))ds$$

$$+ e^{-\Lambda(\bar{t})}L(x(\bar{t})) - e^{-\Lambda(t)}L(x(t))\}$$

$$= e^{-\Lambda(\bar{t})}\{f_0(x(\bar{t}),u(\bar{t})) - \lambda(x(\bar{t}),u(\bar{t}))L(x(\bar{t}))$$

$$+ \liminf_{t\to\bar{t}} (\bar{t}-t)^{-1}(L(x(\bar{t})) - L(x(t)))\}$$

and the limit above can be expressed as

$$(3.82) \quad \liminf_{t\to\bar{t}} (\bar{t}-t)^{-1}(L(x(\bar{t})) - L(x(\bar{t}) - \int_t^{\bar{t}} f(x(s),u(s))ds)]$$

$$\geq \liminf_{t\to\bar{t}} (\bar{t}-t)^{-1}[L(x(\bar{t}) - L(x(\bar{t}) - (\bar{t}-t)f(x(\bar{t}),u(\bar{t}))]$$

$$+ \liminf_{t\to\bar{t}} - (\bar{t}-t)^{-1} K_L\, O(\bar{t}-t)$$

$$\geq - \limsup_{\substack{z\to x(\bar{t})\\ t\to\bar{t}}} (\bar{t}-t)^{-1}[L(z - (\bar{t}-t)f(x(\bar{t}),u(\bar{t}))) - L(z)]$$

by definition the last term coincides with

$$-(L)^{\circ}(x(\bar{t}), -f(x(\bar{t}),u(\bar{t})))$$

where $(L)^{\circ}(z,v)$ indicates the generalized directional derivative for L at a point z in the direction v. Also by definition follows that

$$(L)^{\circ}(z,v) = \max_{\xi e \partial L(z)} \langle \xi, v \rangle.$$

When (3.81) and (3.82) are brought together it follows that

$$(3.83) \quad 0 \geq \min_{\xi e \partial L(x(\bar{t}))} \langle \xi, f(x(\bar{t}), u(\bar{t})) \rangle$$

$$- \lambda(x(\bar{t}), u(\bar{t}))L(x(\bar{t})) + f_0(x(\bar{t}), u(t))$$

for each $\bar{t}$ with the property required in (3.81), provided that $(x(\cdot), u(\cdot))$ is a solution for $P_{d,z}$ with $z \in \operatorname{int}\{E\}$. The set formed by all Lebesgue time $\bar{t}$ is known to be of full measure in the interval $[0, t_u^*(z)]$, as consequence the set mentioned is also a dense set in $[0, t_u^*(z)]$. Let t be arbitrary in $[0, t_u^*(z))]$ and $\{\bar{t}_i\}$ a sequence such that (3.83) is satisfied for each i and $\bar{t}_i \to t$ as $i \to \infty$. The following relations can be justified

$$0 \leq \liminf_{i \to \infty} \min\{\langle \xi, f(x(t_i), u) \rangle - \lambda(x(\bar{t}_i), u)L(x(\bar{t}_i))$$

$$+ f_0(\bar{x}(t_i), u) : \xi \in \partial L(\bar{x}(t_i)), \ u \in U\}$$

$$\geq \min\{\langle \xi, f(x(t), u) \rangle - \lambda(x(t), u)L(x(t)) + f_0(x(t), u)$$

$$: \xi \in \bigcap_{i=0}^{\infty} \bigcup_{j \geq i}^{\infty} \partial L(x(\bar{t}_j)), \ u \in U\}$$

$$= \min\{\langle \xi, f(x(t), u) \rangle - \lambda(x(t), u)L(x(t)) + f_0(x(t), u)$$

$$: \xi \in \partial L(x(t)), \ u \in U\}$$

$$= \limsup_{i \to \infty} \min\{\langle \xi, f(x(\bar{t}_i), u) \rangle - \lambda(x(\bar{t}_i), u)L(x(\bar{t}_i))$$

$$+ \ f_0(x(\bar{t}_i), u) \ : \xi \in \partial L(x(\bar{t}_i)), \ u \in U\}$$

the above relies upon continuity of $f(\cdot,u)$, $f_0(\cdot,u)$, $\lambda(\cdot,u)$ and $L(\cdot)$ for each u and upper semicontinuity of set function $\partial L(\cdot)$ [C1,2.1.5(d)]. In view of lemma 3.7 we now obtain

$$\min_{\xi \in \partial L(x(t))} \langle \xi, f(x(t),u(t))\rangle$$

$$-\lambda(x(t),u(t))L(x(t)) + f_0(x(t),u(t))$$

$$= \min_{\xi \in \partial L(x(t))} - \max_{u \in U} \{ \langle -\xi, f(x(t),u)\rangle$$

$$-\lambda(x(t),u)(-L(x(t))) - f_0(x(t),u)\} = 0.$$

In order to deduce relation (3.8) from the expressions above one replaces $f_0(\cdot,\cdot)$ by its defining function given by (3.13), then recognizes the Hamiltonian function given by (3.4). Now replace the optimal pair $(x(\cdot),u(\cdot))$ for $P_{d,z}$ by the correspondent optimal control u_0 for P_1 at z.

<u>Sufficiency</u>: Let $u = u_0/u_\partial$ be a control satisfying (3.8) and (3.9). Let z be a point in the boundary set E_∂. It follows from (3.9) that

$$L^u(x(t^*_{u_0}(z),z)) = L^u(z) \tag{3.84}$$

$$= -\max_{u \in U_\partial} - F(z,u)$$

hence u_∂ is optimal within $\underline{\underline{\tilde{U}}}_\partial$.

Let z be a point in $\mathrm{int}\{E\}$ and let $(\tilde{x}(\cdot),\tilde{u}(\cdot))$ be an admissible pair for $P_{d,z}$. The optimality of $u = u_0/u_\partial$ is proved if

$$(3.85)\quad L^u(z) \le \int_0^{t^*_{\tilde u}(z)} e^{-\Lambda(t)} f_0(\tilde x(t),\tilde u(t))dt + e^{-\Lambda(t_{\tilde u}(z^*)} F_\partial(\tilde x(t^*_{\tilde u})$$

holds for any pair $(\tilde x(\cdot),\tilde u(\cdot))$ in Ω and arbitrary initial point z.

In order to verify (3.85), first notice that $t \to L^u(\tilde x(t))$ is Lipschitz continuous and hence absolutely continuous function. Its derivative $\frac{d}{dt} L^u(\tilde x(t))$ is defined a.e. and by definition of generalized gradients

$$(3.86)\quad \frac{dt}{dt} L^u(\tilde x(t)) \in \partial_t L^u(\tilde x(t)) \qquad \text{a.e.}$$

(see [C1,2.2.2]). In addition, [C1,2.6.6] specifies a chain rule for generalized gradients, which applied here yields

$$(3.87)\quad \partial_t L^u(\tilde x(t)) \in \mathrm{co}\{\langle\xi,\eta\rangle : \xi \in \partial L^u(x(t)), \eta \in \partial x(t)\}$$

$$\overset{\text{a.e.}}{=} \quad \mathrm{co}\{\langle\xi,f(\tilde x(t),\tilde u(t))\rangle : \xi \in \partial L^u(x(t))\}.$$

Now we are entitled to write the following

$$e^{-\Lambda(t^*_{\tilde u})} L^u(x(t^*_{\tilde u})) - L^u(z)$$

$$= \int_0^{t^*_{\tilde u}} \bar e^{\Lambda(t)}[\frac{d}{dt} L^u(\tilde x(t)) - \lambda(\tilde x(t),\tilde u(t))L^u(\tilde x(t))]dt$$

$$\geq \int_0^{t^*_{\tilde{u}}} e^{-\Lambda(t)}[\min_{\xi\in\partial L^u(\tilde{x}(t))} \langle\xi, f(\tilde{x}(t),\tilde{u}(t))\rangle$$

$$-\lambda(\tilde{x}(t),\tilde{u}(t))L^u(\tilde{x}(t))]dt$$

$$= \int_0^{t^*_u} e^{-\Lambda(t)}[\min_{\xi\in\partial L^u(\tilde{x}(t))} -H(1,-L^u(\tilde{x}(t)),-\xi,\tilde{x}(t),\tilde{u}(t))$$

$$-f_0(\tilde{x}(t),\tilde{u}(t)]dt$$

$$\geq \int_0^{t^*_u} e^{-\Lambda(t)}[\min_{\xi\in\partial L^u(\tilde{x}(t))} -\max_{u\in U} H(1,-L^u(\tilde{x}(t)),-\xi,\tilde{x}(t),u)$$

$$- f_0(\tilde{x}(t),\tilde{u}(t))]dt$$

$$\geq -\int_0^{t^*_u} e^{-\Lambda(t)} f_0(\tilde{x}(t),\tilde{u}(t))dt$$

upon applying (3.86) and (3.87) for the first inequality, using the definition of Hamiltonian function in (3.4) and of $f_0(\cdot,\cdot)$ in (3.13) together with the expression (3.8) in the theorem. The above inequalities yield the desired expression (3.85) if one identifies $F_\partial(\cdot)$ with $L^u(\cdot)$ on the boundary set E_∂. That follows from (3.18) and (3.84), thereby showing the optimality of u for problem P_1.

3.8 Proof of theorem 3.3

The optimization problem determined by P_1 within class of controls $\tilde{\underline{U}}_\partial$ is expressed by (3.18), in view of proposition 3.1. It follows that if $u_\partial(\cdot)$ is optimal for P_1 on E the statement (i) is necessarily true. The fact that $F_\partial(\cdot)$ is a Lipschitzian function is asserted by lemma 3.1.

We use the deterministic formulation $P_{d,z}$ with the restriction of $F_\partial(\cdot)$ on E_∂ as terminal cost function for the control problem. The equivalence shown in proposition 3.2 enables us to solve the optimization problem P_1 within class of controls $\underline{U}_0$ employing that deterministic representation.

It is convenient in the context of the proof to replace the exponential term in the cost by an extra differential relation

$$\dot{x}_0(t) = -x_0(t)\lambda(x(t),u(t))$$

with

$$x_0(0) = 1$$

and we denote $(x_0(\cdot)\ x(\cdot))$ by $\bar{x}(\cdot)$. Problem $P_{d,z}$ can be equivalently posed as follows

$$\text{(3.88)}\quad \text{minimize} \int_0^{t^*} y_0(t)f_0(y(t),u(t))dt + y_0(t^*)F_\partial(y(t^*))$$

on $\bar{\Omega}$, the class formed by all pairs $(\bar{y}(\cdot),u(\cdot))$ with $\bar{y}(\cdot) = (y_0(\cdot)\ y(\cdot))$, such that for each $t \in [0,t_f]$, $t_f \geq t^*$

$$\frac{d}{dt}\,\bar{y}(t) = \mathrm{col}[-y_0(t)\lambda(y(t),u(t)) \quad f(y(t),u(t))]$$

$$y(t) \in E$$

$$u(t) \in U, \text{ measurable function.}$$

Also

$$\bar{y}(0) = (1 \quad z)$$

$$(t^*, y(t^*)) \in \inf\{t \geq 0 : y(t) \in E_\partial\} \times E.$$

For an optimal control u_0 in $\underline{\underline{U}}_0$ we denote $(\bar{x}(\cdot),\bar{u}(\cdot))$ the correspondent solution for $P_{d,z}$ at some $z \in E$, namely

$$(3.89) \qquad (\bar{x}(\cdot),\bar{u}(\cdot)) = [(e^{-\Lambda(\cdot)}x(\cdot,z)),\ u_0(\cdot,z)] \text{ on } [0,T_1(\omega)].$$

In order to apply the necessary conditions for an optimal control we summarize the functional properties of data for $P_{d,z}$:

$(x_0 \ x) \to x_0\, F_\partial(x)$ is a uniformly Lipschitz function on $[0,1] \times E$;

$$(x_0 \ x) \to x_0\, f_0(x,u)\ ,\ (x_0 \ x) \to x_0\, \lambda(x,u)\ ,\ (x_0 x) \to f(x_1,u)$$

are uniformly Lipschitz functions on $[0,1] \times E$ for each $u \in U$;

$$u \to f_0(x,u),\ u \to \lambda(x,u),\ \ u \to f(x,u)$$

are measurable functions on U for each $x \in E$.

Note that the end point constraint for $P_{d,z}$ implies two distinct forms of behaviour for admissible solutions due to the following set inclusion

$$\inf\{t \geq 0 : x(t) \in E_\partial\} \times E \subset [0, \infty) \times E_\partial \bigcup \{+\infty\} \times E.$$

We divide the analysis in two cases:

(a) $(t_u^*(z), x(t_u^*(z))) \in [0,\infty) \times E_\partial$

(b) $(t_u^*(z), x(t_u^*(z))) \in \{+\infty\} \times E$

for $(x(\cdot),u(\cdot))$, a solution to $P_{d,z}$.

Case (a): Here the time interval $[0,t_f]$ can be taken to be finite and the data satisfy the properties summarized above. In this situation the deterministic maximum principle in the non-smooth version is applicable. We refer to theorem 5.2.1 and theorem 5.2.3 of Clarke [C1] and identify the data for $P_{d,z}$ with the correspondent data in the theorems. Note in passing that admissible trajectories for $P_{d,z}$ activates states constraint $d_E(x(\cdot)) \leq 0$ at most in a Lebesgue null measure set of $[0,t_u^*]$, namely in $\{t_u^*\}$. In this situation the end point constraint $x(t_u^*) \in E_\partial$ fully accounts for state constraints.

The Hamiltonian function for problem $P_{d,z}$ with notation (3.88) is defined as follows

(3.90) $H(\bar{r},\bar{q},\bar{p}, \bar{x},u) = \langle \bar{p},f(x,u)\rangle - \bar{q}\, x_0\, \lambda(x,u) - \bar{r}\, x_0\, f_0(x,u)$

for some $\bar{r}$ and $\bar{q}$ in $\mathbb{R}$, $\bar{p}$ in $\mathbb{R}^n$, $\bar{x} = (x_0\ x)$ in $\mathbb{R}^{n+1}$ and u in U.

The statements in the sequel are drawn from the theorems mentioned and they are valid with $\bar{r}$ equal to 0 or 1. Statement (i) of theorem 5.2.1 asserts that if $\bar{x}(\cdot) = (x_0(\cdot)\ \ x(\cdot))$ such that $(\bar{x}(\cdot),\bar{u}(\cdot))$ is an optimal pair for $P_{d,z}$, the adjoint equation for such a pair defines absolutely continuous functions $\bar{q}(\cdot)$ and $\bar{p}(\cdot)$ as follows

$$(3.91)\quad -\frac{d}{dt}\,(\bar{q}(t)\ \bar{p}(t)) \in \partial_{\bar{x}}\ H(\bar{r},\bar{q}(t),\bar{p}(t),\bar{x}(t),\bar{u}(t))$$

a.e. on $[0,t_u^*]$. Assertion (ii) of theorem 5.2.3 provides the transversality condition

$$(3.92)\qquad \bar{q}(t_u^*) = -\,\bar{r}\ F_\partial(x(t_u^*))$$

$$(3.93)\quad \bar{p}(t_u^*) \in -\bar{r}\ x_0(t_u^*)\cdot \partial F_\partial\,(x(t_u^*)) - \partial d_E(x(t_u^*)).$$

Problem $P_{d,z}$ is time-homogeneous with free end time t_u^*.

Assertions (i) and (ii) of theorem 5.2.3 implies that the optimal pair and the adjoint vectors are related by the Hamiltonian in the following way. For each $v \in U$

$$(3.94)\qquad H(\bar{r},\bar{q}(\cdot),\bar{p}(\cdot),\ \bar{x}(\cdot),v)$$

$$\leq\ H(\bar{r},\bar{q}(\cdot),\bar{p}(\cdot),\bar{x}(\cdot),\bar{u}(\cdot)) = 0$$

a.e. on $[0,t_u^*]$. We now can rearrange the expressions if we redefine the adjoint vector by means of the relation

$$(\bar{r}\ \bar{q}(\cdot)\ \bar{p}(\cdot)) = (r\quad q(\cdot)\ x_0(\cdot)\cdot p(\cdot))$$

since $x_0(\cdot) \neq 0$, $(r\quad q(\cdot)\ p(\cdot))$ above is well defined. In addition, we introduce the Hamiltonian $H_1 : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times U \to \mathbb{R}$ defined by

$$(3.95)\quad H_1(r,q,p,x,u) = \langle p,f(x,u)\rangle - r\left[f_0(x,u) + \lambda(x,u)q\right] .$$

In view of these definitions expressions (3.91), (3.92), (3.93) and (3.94) can be equivalently expressed by expressions (3.96), (3.97), (3.98) and (3.99) below

$$(3.96)\quad -\frac{d}{dt}\,(q(t)\quad p(t)) \in \partial_{(r,x)}H_1(r,q(t),p(t),x(t),\bar{u}(t)) + (0\quad -\lambda(x(t),\bar{u}(t))p(t))$$

a.e. on $[0,t_u^*]$;

$$(3.97)\qquad q(t_u^*) = -rF_\partial(x(t_u^*))$$

$$(3.98)\qquad p(t_u^*) \in -\, r\partial F_\partial(x(t_u^*)) - \partial d_E(x(t_u^*))$$

$$(3.99)\qquad H_1(r,q(\cdot),p(\cdot),x(\cdot),v) \leq H_1(r,q(\cdot),p(\cdot),x(\cdot),\bar{u}(\cdot)) \text{ a.e. on } [0,t^*],$$

for each $v \in U$.

We now recognize in (3.95) the Hamiltonian defined in (3.4) upon applying the expression of $f_0(\cdot,\cdot)$ in (3.13). Statements (ii), (iii) and (iv) are the stochastic equivalents to (3.96), (3.99) and (3.97)/(3.98) above, respectively.

Case (b) The situation here is not covered directly by the results employed in the previous case due to the fact that $[0,t_u^*]$ is an unbounded interval. Let $(\bar{x}(\cdot),\bar{u}(\cdot))$ with $\bar{x}(\cdot) = (x_0(\cdot)\ x(\cdot))$ be a solution for $P_{d,z}$ in conformity with the notation in (3.88). We denote the ε-tube about $x(\cdot)$ by $T(x,\varepsilon)$, namely

$$T(x,\varepsilon) = \{v \in \mathbb{R}^n : |x(t) - v| < \varepsilon,\ t \geq 0\}.$$

At this point we introduce a function $\psi : T(x,\varepsilon) \to \mathbb{R}$ and sequence of numbers $\{\underline{t}_i\}$ and $\{\bar{t}_i\}$ where for each i $\underline{t}_i < \bar{t}_i$ and $\underline{t}_i \to \infty$ as $i \to \infty$. With these elements, a collection of deterministic problems $\{P_i\}$ can be defined on $T(x,\varepsilon)$ as follows. For each i, P_i is the problem

$$\text{minimize} \int_0^{t_i} y_0(t) f_0(y(t),u(t))dt + y_0(t_i)\psi(y(t_i))$$

on the class Ω_i. This class comprises all pairs $(\bar{y}(\cdot),\ u(\cdot))$ which satisfies

$$\frac{d}{dt}\,\bar{y}(t) = \frac{d}{dt}\,(y_0(t)\ y(t)) = \mathrm{col}\big[(-y_0(t)\lambda(y(t),u(t))\quad f(y(t),u(t))\big]$$

$$y(t) \in T(x,\varepsilon)$$

$$y(0) = (1,z)$$

$$y(t_i) \text{ free}, \quad t_i \in [\underline{t}_i, \bar{t}_i].$$

For the moment we hypothesize two properties for function ψ that are convenient; the existence of such a function will be demonstrated in a further passage.

(a_1) For each i

$$\inf_{\Omega_i} \left\{ \int_0^{t_i} y_0(t) f_0(y(t),u(t))dt + y_0(t_i)\psi(y(t_i)) \right\}$$

$$= \int_0^{\infty} x_0(t)\, f_0(x(t),\bar{u}(t))dt$$

(a_2) $\psi(\cdot)$ is bounded, uniformly Lipschitzian on $T(x,\varepsilon)$.

Under (a_2) the necessary conditions in the non-smooth version is applicable for P_i. In fact, it is a simpler problem than the finite time problem treated in this proof as case (a). In P_i there are no end point constraints for trajectories; moreover in view of (a_1) follows that there exists a solution which is in the interior of $T(x;\varepsilon)$, namely the restriction of $(\bar{x}(\cdot),\bar{u}(\cdot))$ to the interval $[0,t_i]$. As a consequence, the restriction of $(\bar{x}(\cdot),\bar{u}(\cdot))$ should satisfy the necessary conditions of optimality for each P_i since it is always a true solution for P_i. Proceeding in the way indicated above, one identifies function H in (3.90) as the Hamiltonian for P_i and obtains the adjoint equation and the correspondent transversality condition deduced from the same assertions employed to derive (3.91) and (3.92)/(3.93):

$$(3.100)\quad -\frac{d}{dt}\,(\bar{q}_i(t)\;\bar{p}_i(t)) \in \partial_{\bar{x}}\quad H(\bar{r},\bar{q}_i(t),\bar{p}_i(t),\bar{x}(t),\bar{u}(t))$$

a.e. $t \in [0,t_i]$;

$$(3.101)\qquad \bar{q}_i(t_i) = -\bar{r}\;\psi(x(t_i))$$

$$\bar{p}_i(t_i) \in -\;\bar{r}\;x_0(t_i)\;\partial\psi(x(t_i)).$$

The transversality condition for $t \to H(\bar{r},\bar{q}(t),\bar{p}(t),\bar{x}(t),\bar{u}(t))$ and the time-homogeneous character of data yield for each $u \in U$

$$(3.102)\qquad H(\bar{r},\bar{q}_i(\cdot),\bar{p}_i(\cdot),\bar{x}(\cdot),u)$$

$$\leq\; H(\bar{r},\bar{q}_i(\cdot),\bar{p}_i(\cdot),\bar{x}(\cdot),\bar{u}(\cdot)) = h$$

a.e. $t \in [0,t_r]$, where

$$h \in \partial d_{[\underline{t}_i,\bar{t}_i]}(t_i)$$

Since sequences $\{\underline{t}_i\}$ and $\{\bar{t}_i\}$ are arbitrary and (a_1) hold, the value for P_i does not depend on the particular choice of final time t_i. Thus t_i can be taken in the set $\mathrm{int}[\underline{t}_i,\bar{t}_i]$ for each i so that $h \in \{0\}$ for all i.

We now turn our attention to the limiting conditions provided by the family $\{P_i\}$. Note in passing that $\bar{q}_i(\cdot)$ and $\bar{p}_i(\cdot)$ are non-trivial functions for each problem P_i if only $\bar{r}$ is equal to 1, hence we can take $r=1$ and write (3.101) accordingly. In addition, we can write (3.102) for any i and j as

$$\max_{u \in U} H(1,\bar{q}_i(\cdot),\ \bar{q}_i(\cdot),\bar{x}(\cdot),u)$$

$$= \max_{u \in U} H(1,\bar{q}_j(\cdot),\ \bar{q}_j(\cdot),\bar{x}(\cdot),u) = 0$$

a.e. on $[0,t_i] \bigcap [0,t_j]$

Since $(\bar{q}_i(\cdot)\ \ \bar{p}_i(\cdot))$ and $(\bar{q}_j(\cdot)\ \ \bar{p}_j(\cdot))$ are continuous and satisfy the differential relation (3.100) we can take $\bar{q}_i(\cdot) = \bar{q}_j(\cdot)$, $\bar{p}_i(\cdot) = \bar{p}_j(\cdot)$ on $[0,t_i]\ \ [0,t_j]$ for each i and j, and that enable us to define

$$\bar{q}(\cdot) = \bar{q}_i(\cdot) \quad ; \quad \bar{p}(\cdot) = \bar{p}_i(\cdot)$$

on $[0,t_i]$ for each i. As a consequence, $\bar{q}(\cdot)$ and $\bar{p}(\cdot)$ satisfy

$$-\frac{d}{dt}\,(\bar{q}(t)\ \bar{p}(t)) \in \partial_{\bar{x}}\ H(\bar{r},\bar{q}(t),\bar{p}(t),\bar{x}(t),\bar{u}(t))$$

a.e. t on $[0,\infty]$;

$$\bar{q}(t) = -\psi(x(t))$$

$$\bar{p}(t) \in -x_0(t)\ \partial\psi(x(t))$$

for each $t \in [0,\infty]$ since $\{t_i\}$ is arbitrary. Also

$$\max_{u \in U} H(\bar{r},\bar{q}(\cdot),\bar{p}(\cdot),\bar{x}(\cdot),u) = 0$$

a.e. on $[0,\infty]$.

Since $x_0(\cdot) > 0$ on $[0,\infty)$, we now can redefine the adjoint vectors by means of the relation

$$(\bar{r} \quad \bar{q}(\cdot) \quad \bar{p}(\cdot)) = (r \quad q(\cdot) \quad x_0(\cdot).p(\cdot))$$

In addition, if we introduce the Hamiltonian function H_1 in (3.95) the necessary conditions can be rewritten as follows. There exists absolutely continuous function $q(\cdot)$ and $p(\cdot)$ on $[0,\infty)$ and number r equal to 1 such that

$$(3.103) \qquad -\frac{d}{dt}(q(t) \quad p(t)) \in \partial_{(r,x)}H_1(r,q(t),p(t),x(t),\bar{u}(t)) + (0 \quad -\lambda(x(t),\bar{u}(t))p(t))$$

a.e. $t \in [0,\infty)$.

$$(3.104) \qquad H_1(r,q(t),p(t),x(t),v)$$

$$\leq H_1(r,q(t),p(t),x(t),\bar{u}(t)) = 0 \quad \text{a.e. } t \in [0,\infty),$$

for each $v \in U$

$$(3.105) \qquad q(t) = -\psi(x(t))$$
$$p(t) \in \partial\psi(x(t))$$

for each $t \in [0,\infty)$.

Statements (ii) and (iii) of theorem 3.3 are the stochastic equivalents of (3.103) and (3.104) whereas one acknowledges (a_2) to obtain statement (v) from (3.105).

In order to complete the proof, a function $\psi(\cdot)$ with properties specified by (a_1) and (a_2) should be provided. Applying the optimality principle one can write for deterministic problems $P_{d,z}$ and $P_{d,x(t_i)}$

$$L(z) = L^{u}_{P_d}(z) = \int_0^\infty x_0(t) f_0(x(t),\bar{u}(t))dt$$

$$= \inf_{\bar{\Omega}} \int_0^\infty y_0(t)\, f_0(y(t),u(t))dt$$

$$= \inf_{\Omega_i} \Big\{ \int_0^{t_i} y_0(t)\, f_0(y(t),u(t))dt$$

$$+ \inf_{\bar{\Omega}} \int_{t_i}^\infty y_0(t)\, f_0(y(t),u(t))dt \Big\}$$

$$= \inf_{\Omega_i} \Big\{ \int_0^{t_i} y_0(t)\, f_0(y(t),u(t))dt + y_0(t_i) L(y(t_i)) \Big\}.$$

Therefore $L(\cdot)$ satisfies (a_1) and is bounded function. In addition, proposition 3.4 states that $L(\cdot) = L_{\tilde{P}_d}(\cdot)$ on E which is a consequence of the fact that any solution to problem $\tilde{P}_{d,z}$ coincides with a solution for $P_{d,z}$. Finally proposition 3.3 asserts that $L_{\tilde{P}_d}(\cdot)$ is uniformly Lipschitzian function in a neighbourhood of E. It follows that $L_{\tilde{P}_d}(\cdot)$ satisfies (a_1) and (a_2) for ε small enough, thereby the theorem is proven.

Proof of Corollary: For the case (b) in the theorem (i.e. $t_u^* = +\infty$) the normality was already asserted and it was stated before necessary conditions (3.103) - (3.105). To verify the normality when the end point constraint satisfies

$$(3.106) \qquad (t_u^*,\ x(t_u^*)) \in [0,\infty) \times E_\partial$$

we invoke problem $\tilde{P}_{d,z}$. In this formulation, state and end point constraints are immaterial, but solutions for $\tilde{P}_{d,z}$ are also solutions for $P_{d,z}$ and conversely. The normality of $\tilde{P}_{d,z}$ for solutions which satisfies (3.106) is seen from the transversality condition provided in this situation:

$$q(t_u^*) = -r[F_\partial(x(t_u^*)) + K_3 d_{E_\partial}(x(t_u^*))] = -r\ F_\partial(x(t_u^*))$$

$$p(t_u^*) \in -r[\partial F_\partial(x(t_u^*)) + K_3 \partial d_{E_\partial}(x(t_u^*))]$$

$q(\cdot)$ and $p(\cdot)$ are as usual, absolutely continuous functions and they satisfy an autonomous linear set of differential equations, namely the adjoint equation. If r is taken equal 0 it is clear from the transversality condition above that $q(\cdot)$ and $p(\cdot)$ are identically null vector functions, thus the normality of the problem $P_{d,z}$ readily follows from the equivalence shown in proposition 3.4.

CHAPTER 4

AN EMBEDDING FOR P_1 AND ALGORITHMS FOR THE STOCHASTIC PDP PROBLEM

The stochastic PDP control problem proposed in chapter 2 will be approached, now that one is armed with the results for the single jump problem P_1 developed in the previous chapter. By virtue of condition (A6) in section 4.1 or (A6') in section 4.2 and an operator associated with the single jump problem P_1 , the characterization for the general problem is provided by theorem 4.1. Dynamic programming furnishes the main argument and an embedding for solutions to the smaller problem is ultimately justified. Although the operators involved are not contractive in the usual sense, it is shown that they possess unique fixed point in some specified class. The analysis yields the optimality criterion in limiting form stated in theorem 1.1 and the criteria in closed form established by theorems 1.2 and 1.3, these two with the provision of condition (A7). Recursive methods arise directly from these dual criteria: in the first phase an explicit sequence of functions is constructed, inspired by the assertions in theorem 1.1; the second stage consists entirely of solving a deterministic optimal control problem. It is shown in section 4.6 that numerical methods for the first phase are available under conditions (A8) and (A9). A hypothesis on uniform contraction stated in proposition 4.1 is employed to calculate in advance the number of iterations I for a sequence $\{\bar{V}^i\}_{i=1,2,\ldots I}$ of approximations for the value, to reach some arbitrary given precision.

4.1 The basic assumption

Let us introduce some notation for the stochastic PDP problem. Let $V^u : E \to \mathbb{R}$ be the cost function for some u in $\underline{\underline{U}}$:

$$(4.1) \qquad V^u(z) = E_z^u\left[\int_0^\tau \ell(x_{t-}, u_t)d\mu_t\right]$$

for each z in E, and let $V : E \to \mathbb{R}$ be the value for the problem:

$$(4.2) \qquad V(z) = \inf_{u \in \underline{\underline{U}}} V^u(z)$$

for each z in E. The value function $V(\cdot)$ is clearly bounded below by zero and we assume that the problem is well posed in that there exist bounded elements in the family $\{V^u(\cdot): u \in \underline{\underline{U}}\}$. In the definition of a PDP process it was originally required that jump times should be distinguishable from each other. We now reinforce that assumption, at least for a subclass in $\underline{\underline{U}}$.

(A6) For each number $\varepsilon > 0$, a control u in $\underline{\underline{U}}$ satisfies

$$(4.3) \qquad V^u(\cdot) < V(\cdot) + \varepsilon$$

if ε is small and u satisfies (4.3) then

$$(4.4) \qquad E_z^u\left[\sum_k I_{\{T_k<\tau\}}\right] < \infty$$

is verified at each z in E.

An equivalent statement for (A6), (A6') in which ε-optimal controls are not assumed a priori, and some sufficient conditions will be established in the sequel to illustrate the significance of the condition above. Note in passing that in particular

$$(4.5) \qquad \lim T_k > \tau \qquad P^u\text{-} \quad \text{a.s.}$$

holds for controls mentioned in (A6).

Lemma 4.1 In view of (A6)

$$(4.6) \qquad E_z^u\Big[\int_0^\tau (\lambda(x_{t-},u_t)dt + dp_t^*)\Big] < \infty$$

holds for each z in E and control u satisfying (4.3) for ε–small enough. Here (p_t^*), $t \geq 0$ is counting process defined in (2.21)

Proof: Let (n_t) be the counting process defined by

$$n_t(\omega) = \sum_k I_{\{T_k(\omega)\leq t\}} \qquad , \quad t \geq 0$$

upon applying the definition of (p_t^*), we have

$$n_t(\omega) = \sum_k I_{\{T_k\leq t:\ x_{T_k^-} \in \operatorname{int}\{E\}\}} + p_t^* , \quad t \geq 0$$

For a control with the property indicated in the lemma we write the survivor function $(F_t(z))$ as defined by (D2) in chapter 2, so that (2.20) yields

$$P_z^u(T_1>t) = -\int_{[t,\infty)} dF_t(z)$$

$$= \left[\int_t^{t^*_{u_0}(z)} e^{-\Lambda(s)}\lambda(x(s,z),u_0(s,z))ds + e^{-\Lambda(t^*_{u_0}(z))}\right] I_{\{t<t^*_{u_0}(z)\}}$$

for $u = u_0/u_\partial$.

A process denoted (A_t), called compensator for (n_t), is defined recursively by

$$A_t = \int_0^{T_1\wedge t} -\frac{dF_s(z)}{F_{s^-}(z)} \qquad \text{for } t \in [0,T_1]$$

$$A_t = A_{T_i} + \int_0^{T_{i+1}\wedge t} -\frac{dF_s(z)}{F_{s-}(z)} \qquad \text{for } t \in (T_i, T_{i+1}]$$

$i = 1,2..$

Hence the compensator for (n_t) in a controlled PDP has the explicit form

$$A_t = \int_0^t [\lambda(x_{s-},u_s)ds + dp^*_s]$$

If u is a control which satisfies (4.3), in view of (A6) it should also satisfy (4.4). In this circumstances (n_t) has the unique stochastically equivalent decomposition:

$$n_{t\wedge\tau} = m_{t\wedge\tau} + A_{t\wedge\tau}$$

where $(m_{t\wedge\tau})$ is a $\underline{\underline{F}}_t$-martingale. We refer to [L1, theorem 18.1] for the conclusion above and to [L1, theorem 18.2] for the construction of the compensator process. The assertion of lemma therefore follows from the equivalence between (4.4) and (4.6).

Conditions for which (A6) is automatically satisfied can be given in terms of the data of the problem, in the interest of possible applications. Recall that the data are bounded functions.

Lemma 4.2: Condition (A6) (4.4) is implied by each of the following conditions:

(i) There exists a set $S \subset E$, satisfying

$$\inf\{d_{E_\partial}(z) : z \in S\} > 0$$

and the family of measures

$$\{P(\cdot;z,u) : z \in E_\partial,\ u \in U_\partial\}$$

is supported on S

(ii) The boundary cost $\ell_\partial(\cdot,\cdot)$ is such that

$$\inf\{\ell_\partial(z,u) : z \in E_\partial,\ u \in U_\partial\} > 0$$

Proof:

(i) Since $f(\cdot,\cdot)$ is bounded function

$$\int_0^t dp_t^*(\omega) \leq s_o^{-1}|f|t$$

where

$$s_o = \inf\{d_{E_\partial}(z): z \in S\}$$

holds, as a consequence of

$$t^*(z_i(\omega)) > s_o|f|^{-1} \qquad P^u - \text{a.s.}, \quad i = 1,2,\ldots$$

Now in view of the hypothesis on $\tau(\omega)$ in (2.22), we calculate

$$E_z^u[\int_0^\tau (\lambda(x_{t-},u_t)dt + dp_t^*)] \leq (|\lambda| + s_o^{-1}|f|)E_z^u[\tau] < \infty$$

(ii) Suppose that u is a control yielding $V^u(\cdot) \leq V(\cdot) + c_o$ for some small $c_o > 0$ and concurrently:

$$E_z^u[\sum_k^n I_{\{T_k<\tau\}}] \to \infty \qquad \text{as } n \to \infty \tag{4.7}$$

The first assertion above implies that sequence of numbers $\{c_n\}$ can be found with $c_n \to c_o$ such that for each u

$$E_z^u[\int_0^{T_n\wedge\tau} \ell(x_{t-},u_t)d\mu_t] \tag{4.8}$$

$$\leq \inf_{u\in \underline{\underline{U}}} E_z^u[\int_0^{T_n\wedge\tau} \ell(x_{t-},u_t)d\mu_t] + c_n$$

since the integrand is monotone increasing and V(·) is finite. But (4.7) and (4.8) cannot be simultaneously true since

$$E_z^u\left[\int_0^{T_n \wedge \tau} \lambda(x_{t-},u_t)d\mu_t\right]$$

$$\geq s_1 \; E_z^u\left[\int_0^{T_n \wedge \tau} dp_t^*\right]$$

where

$$s_1 = \inf\{\ell_\partial(z,u) : z \in E_\partial,\ u \in U_\partial\}$$

and in lemma 4.1 it was shown that

$$E_z^u\left[\sum_k^n I_{\{T_k<\tau\}}\right]$$

$$\leq E_z^u[\,|\lambda|\tau] + E_z^u\left[\int_0^{T_n \wedge \tau} dp_t^*\right]$$

the first term is finite from the assumptions (2.22) on $\tau(\omega)$, so that

$$E_z^u\left[\int_0^{T_n \wedge \tau} dp_t^*\right] \to \infty \qquad \text{as} \quad n \to \infty$$

thus (4.8) is denied and the assertion (ii) of lemma is proven.

4.2 Operator associated with P_1

The results obtained in chapter 3 for problem P_1 are applied to the general stochastic PDP problem in this chapter. The relationship between solutions for the two problems is made clear by the introduction of an operator representation for the cost of P_1 associated with a control. A characterization for the lower envelope operator of this family is provided by the study of P_1.

Let M_0 be the partially ordered class defined by

$$M_0 = \{\psi\colon E \to \mathbb{R}, \text{ measurable and bounded}\}$$

with

$$\psi \geq \psi' \iff \psi(z) \geq \psi'(z), \quad \text{each } z \in E,\ \psi \text{ and } \psi' \text{ in } M_0$$

Let $\Sigma^+(K)$ be the sub-class of M_0 as follows

$\Sigma^+(K) = \{\psi\colon E \to \mathbb{R}$, bounded, nonegative, Lipschitzian function of rank $K\}$

For each u in the class $\underline{\underline{U}}$, let us introduce an operator associated with problem P_1. It is defined on M_0 and is denoted by L^u such that, for each $\psi \in M_0$, $L^u[\psi]$ is defined by

$$(4.9) \qquad L^u[\psi](z) = E_z^u\Big[\int_0^{T_1} \ell(x_{t-},u_t)d\mu_t + I_{\{T_1<\tau\}}\, \psi(x_{T_1})\Big]$$

for each z in E. Note that if $\psi \geq \psi'$ then $L^u[\psi] \geq L^u[\psi']$. The operator L^u is therefore monotone.

For the purpose of expressing L^u in a standard form, we consider that the class $\underline{\underline{U}}$ of admissible controls is composed by the product $\underline{\underline{U}}_0 \times \underline{\underline{U}}_\partial$. Herewith an element of $\underline{\underline{U}}$ is pair u_0/u_∂ of the form

$$(4.10) \qquad u_0 : \mathbb{R}_+ \times E \to U \qquad u_\partial\colon \mathbb{R}_+ \times E \to U_\partial$$

such that $u_0(\cdot,z)$ and $u_\partial(\cdot,z)$ are Borel measurable functions for each z in E; $u_0(t,\cdot)$ and $u_\partial(t,\cdot)$ are universally measurable functions for each $t \in \mathbb{R}_+$.

The operator L^u applied to some ψ in the class $\Sigma^+(K)$ is now completely identified with problem P_1 with final cost ψ, if the class $\tilde{\underline{\underline{U}}}_\partial$ of feedback controls employed for P_1 and class $\underline{\underline{U}}_\partial$ can be related. For this purpose let us take a point z in E and a control $t \to u_0(t,z)$ in $\underline{\underline{U}}_0$, so that $t \to x_{u_0}(t,z)$ is the resulting trajectory. For any $\tilde{u}_\partial \in \tilde{\underline{\underline{U}}}_\partial$ one simply define $t \to u(t,z)$ in $\underline{\underline{U}}$ by

$$u(t,z) = u_0(t,z)/\tilde{u}_\partial(x_{u_0}(t,z))$$

for each $t \in [0,t^*_{u_0}(z)]$ and $z \in E$.

We introduce operator L on $\underline{\underline{\bigcup_{K\geq 0}}}\ \Sigma^+(K)$

with the following definition:

$$(4.11) \qquad L[\psi](z) = \inf_{u \in \underline{\underline{U}}} L^u[\psi](z)$$

for each z in E and ψ in $\Sigma^+(K)$ for some K. Operator L is therefore the lower envelope for the family $[L^u: u \in \underline{\underline{U}}]$ and the monotonicity of L is inherited from the family. The following lemma refers to that property and gives an operator version for some of the results in chapter 3. (theorem 3.1 and assertion preceding (3.3))

Lemma 4.3: $L : \Sigma^+(K_1) \to \Sigma^+(K_2)$, $K_2 \geq K_1$. If ψ and ψ' belongs to a class $\Sigma^+(K)$ and $\psi \geq \psi'$, then $L[\psi] \geq L[\psi']$.

The special choice of measurable class made in (4.10) for admissible controls $\underline{\underline{U}}$ is now employed.

Lemma 4.4 For each given number $\varepsilon > 0$, there exists a control $u \in \underline{\underline{U}}$ such that

$$L^u[\psi] \leq L[\psi] + \varepsilon$$

holds for any ψ in the class M_0

Proof: Initially, $\frac{\varepsilon}{2}$ - optimal controls are shown to exist in the class $\underline{\underline{U}}_\partial$. Using the construction $u_\partial(\cdot,\cdot) = \tilde{u}_\partial(x_{u_0}(\cdot,\cdot))$ for a in $\underline{\underline{U}}_\partial$, from a given control u_0 in $\tilde{\underline{\underline{U}}}_0$, and $\tilde{u}_\partial$ in $\tilde{\underline{\underline{U}}}_\partial$ one concludes that it is enough to to assert the existence of $\frac{\varepsilon}{2}$-optimal controls in $\tilde{\underline{\underline{U}}}_\partial$ in order to verify the equivalent result for controls in $\underline{\underline{U}}_\partial$. Using the pointwise minima which is involved (see (3.18)) together with lemma 3.1 we can affirm that a control value u in U_∂ which, for some $\alpha > 1$ satisfies

$$F(z,u) \leq F_\partial(z) + \frac{\varepsilon}{2\alpha}$$

at some point $z \in E$, yields the $\frac{\varepsilon}{2}$ -optimal control $u(z') = u$ for $z' \in E$ such that

$$|z-z'| < \frac{\alpha-1}{4\alpha K}\,\varepsilon$$

where $K = \sup\{\text{Lipschitz rank of } F(\cdot,u) : u \in U\}$. Using this procedure for at most a countable number of points z, an $\frac{\varepsilon}{2}$ - optimal control in $\tilde{\underline{\underline{U}}}_\partial$ is defined; this problem is solved indepedently of the minimization in $\underline{\underline{U}}_0$ and we can proceed by showing the existence of $\frac{\varepsilon}{2}$ -optimal controls in the class $\underline{\underline{U}}_0$.

Note that for a fixed point z in E by definition one can find Borel measurable function $u_0 : [0,t^*_{u_0}] \to U$ and from the first part of the proof a control $\tilde{u}_\partial \in \tilde{\underline{\underline{U}}}_\partial$ such that

$$[L^{u_0/\tilde{u}_\partial}[\psi](z) \leq \inf_{\tilde{u}_\partial \in \tilde{\underline{\underline{U}}}_\partial} L^{u_0/\tilde{u}_\partial}[\psi](z) + \frac{\varepsilon}{2}$$
$$\leq L[\psi](z) + \varepsilon$$

The functions $u_0(\cdot)$ are now embedded in the following class:

$$\Pi = \{u_0(\cdot) : u_0(\cdot) \in U \quad \lambda - \text{a.e. on } \mathbb{R}_+, \quad \text{Borel measurable}\}$$

here λ is the Lebesgue measure. Thus, for each $z \in E$ one obtains the equivalence

$$(4.12) \quad L[\psi](z) = \inf_{u_0 \in \Pi} \inf_{\tilde{u}_\partial \in \tilde{\underline{\underline{U}}}_\partial} L^{u_0/\tilde{u}_\partial}[\psi](z)$$

Now, we introduce the set of equivalence classes of Borel measurable functions $L_p[\mathbb{R}_+, \mu; U]$ for $1 \leq p < \infty$ and some finite measure μ which is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$ (e.g. $\frac{d\mu(t)}{d\lambda(t)} = e^{-\alpha t}$, $\alpha > 0$). The norm of $L_p[\mathbb{R}_+, \mu; U]$ is defined by

$$\|u\|_p = \{ \int_{\mathbb{R}_+} |u_0(t)|^p \, d\mu(t)\}^{1/p}.$$

Since U is closed and compact set, Π is clearly a close subset of the L_p-space defined above. In other hand, it is a complete and separable metric space with the metric supplied by the norm. Hence, the space Π endowed with J_p, the relative metric topology is homeomorphic to the subset Π of $L_p[\mathbb{R}_+, \mu; U]$; it follows therefore that (Π, J_p) forms a Borel space. Using the above characterization $(z, u_0) \to L^{u_0/\tilde{u}_\partial}[\psi](z)$ for each $\tilde{u}_\partial$ is a Borel function in E X Π and we can apply [B1, proposition 7.50] to conclude that for each $\varepsilon > 0$ there exists an analytically masurable (hence an universally measurable) function

$$u_0 : E \to \Pi$$

such that graph $\{u_0\} \subset E \times \Pi$ and

$$L^{\tilde{u}_0(z)/u_\partial}[\psi](z) \leq \inf_{u_0 \in \Pi} L^{\tilde{u}_0/u_\partial}[\psi](z) + \frac{\varepsilon}{2}$$

for all $z \in E$. By identifying $u_0(\cdot)$ as above with a control in the class $\underline{\underline{U}}_0$ and applying (4.12), the proof is completed.

The assertion of lemma 4.4 provides the main argument to verify the existence of ε-optimal controls for the stochastic PDP control problem. Lemma 4.6 in the next section establishes the existence of such controls based on assumption (A6') as follows

(A6') For each u in $\underline{\underline{U}}$

$$E_z^u[\sum_k I_{\{T_{k<\tau}\}}] < \infty$$

holds for all z in E

Lemma 4.6 shows that (A6) and (A6') are equivalent conditions for the purpose of the characterization given by theorem 4.1

4.3 Operator representation and criteria of optimality

Having defined the necessary elements, the connection between solutions for the single jump process and solutions for the general problem can now be established.

Lemma 4.5: SUppose that (A6) is valid. If u is a control which satisfies $V^u \leq V + \varepsilon_0$ for some number ε_0 then

$$V^u(z) = \lim_{n\to\infty} (L^u)^n[\psi](z)$$

for any ψ in the class M_0. Furthermore, V^u is unique strong such a limit. The assertion is valid for all u in $\underline{\underline{U}}$, whenever (A6') is satisfied

Proof: One evaluates

$$(L^u)^n[\psi](z) = E_z^u[\int_0^{T_n \wedge \tau} \ell(x_{t^-}, u_t) d\mu_t + I_{\{T_n < \tau\}} \psi(x_{T_n})]$$

as $n \to \infty$. The first integrand is monotone increasing and the second has an upper bound which is decreasing. The hypothesis in the lemma for control u implies that u satisfies (4.4) in (A6) or the equivalent expression in (A6'). It follows in particular that (4.5) is satisfied, thereby the limit stated is verified.

Note that the strong Markov property for PDP lemma 2.5, enable us to express V^u as follows.

$$V^u(z) = E_z^u\{ \int_0^{T_n \wedge \tau} \ell(x_{t-},u_t)d\mu_t$$

$$+ I_{\{T_n<\tau\}} E^u[\int_{T_n}^{\tau} \ell(x_{t-},u_t)d\mu_t | \underline{\underline{F}}_{T_n}]$$

$$= E_z^u[\int_0^{T_n \wedge \tau} \ell(x_{t-},u_t)d\mu_t + I_{\{T_n<\tau\}} V^u(x_{T_n})]$$

for $n = 1,2,3,\ldots$ and each z in E. Take $n = 1$ to obtain

$$V^u(\cdot) = L^u[V^u][(\cdot)$$

Hence V^u is a fixed point of $\underline{\underline{L}}^u$ for each u. Also, in view of lemma 4.5 it is the unique element of M_0 with this property for any control u mentioned in (A6) or for all controls in $\underline{\underline{U}}$ if (A6') holds.

Lemma 4.6 For each given number $\varepsilon > 0$, there exists a control $u \in \underline{\underline{U}}$

such that

$$V^u \leq V + \varepsilon$$

holds provided that (A6') is satisfied.

Proof: For any given ε, one chooses ε' such that

$$\varepsilon' = \{1 + \sup_{z \in E, u \in \underline{\underline{U}}} E_z^u[\sum_k I_{\{T_k < \tau\}}]\}^{-1} \varepsilon$$

It follows from lemma 4.5 and definitions that for each z in E and

$\psi = 0$ together with the monotonicity

$$(4.13) \quad \infty > V(z) = \inf_{u \in \underline{\underline{U}}} \lim_n (L^u)^n[\psi](z)$$

$$\geq \lim_n (\inf_{u \in \underline{\underline{U}}} L^u)^n[\psi](z)$$

$$= \lim_n (L)^n[\psi](z) = W(z) \geq 0$$

Function W defined pointwisely on E by the above limit is a fixed point for L. Now lemma 4.4 asserts that $u \in \underline{\underline{U}}$ can be found which

satisfies

$$(4.14) \qquad L^u[W] \leq L[W] + \varepsilon'$$

The relation above, the monotonicity property of L^u and definition (4.9) yield

$$(L^u)^2[W](z) \leq L^u[W + \varepsilon'](z)$$

$$= L[W](z) + E_z^u[I_{\{T_1 < \tau\}}]\varepsilon'$$

$$\leq W(z) + E_z^u[I_{\{T_1 < \tau\}} + 1]\varepsilon'$$

for each z in E, upon applying (4.14) twice. Applying the operator L^u sequentially, one obtains

$$(L^u)^n[W](z) \leq W(z) + E_z^u[\sum_k^{n-1} I_{\{T_k<\tau\}} + 1]\varepsilon'$$

Taking the limit as $n \to \infty$, lemma 4.5 yields

$$V^u(z) \leq W(z) + E_z^u[\sum_k I_{\{T_k<\tau\}} + 1]\varepsilon'$$

in view of (4.13) and the choice of ε', it follows that

$$V^u(z) \leq V(z) + \varepsilon$$

for each z in E, thereby proving the lemma.

These lemmas provide the background for the first characterization of the problem.

Theorem 4.1: $V(\cdot)$ is the unique fixed point in $\bigcup_K \Sigma^+(K)$ for the lower envelope operator.

Proof: Let ψ be arbitrary function on $\bigcup_{K<\infty} \Sigma^+(K)$. It follows from lemma 4.5 and definitions that

$$\text{(4.15)} \qquad \lim_n (L)^n[\psi](z) = \lim_n (\inf_{u \in \underline{\underline{U}}} L^u)^n[\psi](z)$$

$$\leq \inf_{u \in \underline{\underline{U}}} \lim_n (L^u)^n[\psi](z) = \inf_{u \in \underline{\underline{U}}} V^u(z) = V(z)$$

for each z in E. Denote $W(z)$ the limit in (4.15) for each z, hence

(4.16) $$0 \leq W \leq V$$

and clearly W is a fixed point for L. Moreover, it is bounded function and from lemma 4.3 one has that

(4.17) $$W \in \bigcup_{K \geq 0} \Sigma^{+}(K)$$

In order to verify the result, it suffices to show that $W \geq V$. Let $\varepsilon > 0$ be an arbitrary given number. In view of (A6) or (A6') and lemma 4.6, one can choose a control u in $\underline{\underline{U}}$ such that

(4.18) $$V^{u} < V + \varepsilon$$

Hence from (4.12) a similar relation can be stated:

(4.19) $$V^{u} < W + \varepsilon'$$

and necessarily $\varepsilon' \geq \varepsilon$. We show that ε and ε' can be chosen with $\varepsilon' \leq \varepsilon$. In view of the monotonicity property of L^{u} and the fact that V^{u} is fixed point for L^{u}, we can write

$$L^{u}[W] \leq L^{u}[V] \leq L^{u}[V^{u}] \leq W + \varepsilon'$$

upon applying (4.16) and (4.19). The relation above, together with definition (4.9) and monotonicity for L^{u} yield

$$(L^{u})^{2}[W](z) \leq L^{u}[W+\varepsilon'](z) = L^{u}[W](z) + E_{z}^{u}[I_{\{T_{1}<\tau\}}]\varepsilon'$$

$$\leq W(z) + E_{z}^{u}[I_{\{T_{1}<\tau\}}+1]\varepsilon'$$

for each z in E. When the relation is applied sequentially, one obtains

$$(L^u)^n[W](z) \leq W(z) + E_z^u[\sum_k^{n-1} I_{\{T_k<\tau\}} + 1]\varepsilon'$$

Now, in view of (4.18) lemma 4.5 is applicable so that the above expression yields

$$V^u(z) \leq W(z) + E_z^u[\sum_k 1_{\{T_k<\tau\}} + 1]\varepsilon'$$

for each z in E. Since u should satisfy the condition in (A6) or in (A6') we now are able to choose

$$\varepsilon'' = \sup_{z\in E} E_z^u[\sum_k I_{\{T_k<\tau\}} + 1]\varepsilon'$$

and obtain from (4.19)

$$V(\cdot) < W(\cdot) + \varepsilon''$$

because ε is arbitrary so is ε' and consequently ε''. In view of (4.16) we have shown that

$$V(\cdot) = W(\cdot).$$

An abstract necessary and sufficient can now be given, using the characterization for the problem in theorem 4.1.

Corollary 4.1: A control u in $\underline{\underline{U}}$ is a solution for the stochastic PDP problem if and only if

$$L^u[V^u] = L[V^u] \tag{4.20}$$

where

$$V^u(z) = \lim_n (L^u)^n[\psi](z)$$

for some $\psi \in M_0$ and each $z \in E$

4.4 Limiting criterion of optimality

The fact that the value of the problem is bounded and continuous function is disclosed by theorem 4.1. Hence, a sequence of bounded Lipschitzian functions can approximate the value function $V(\cdot)$ within arbitrary precision. One such sequence is provided by the lower envelope operator. Most importantly, this sequence can be made explicit by a local optimization method and a recursive procedure deduced from the single jump problem analysis.

Let $H : \mathbb{R} \times \mathbb{R}^n \times E \times U \times \bigcup_{K<\infty} \Sigma^+(K) \to \mathbb{R}$

be defined by

$$(4.21) \qquad H(q,p,z,u,[\psi]) = \langle p,f(z,u)\rangle - \ell_0(z,u) - \lambda(z,u) \int_E [\psi(y)-q]dQ(y;z,u)$$

and also let

$$F: E \times U_\partial \times \bigcup_{K<\infty} \Sigma^+(K) \to \mathbb{R}$$

be defined by

$$(4.22) \quad F(z,u,[\psi]) = \ell_\partial(z,u) + \int_E \psi(y)dP(y;z,u)$$

We are in position to prove the sequential criteria in theorem 1.1.

Theorem 1.1 Let V^1 be an arbitrary function of class $\Sigma^+(K)$ for some K. A sequence $\{V^i\}$ is defined in $\bigcup_{K<\infty} \Sigma^+(K)$, by the modified Hamilton-Jacobi-Bellman equation

$$\min_{\varepsilon\partial V^i(z)} -\max_{u\varepsilon U} H(-V^i(z),-\xi,z,u,[V^{i-1}]) = 0$$

for each $z\varepsilon E$,

$$V^i(z) + \max_{u\varepsilon U_\partial} - F(z,u,[V^{i-1}]) = 0$$

for each $z \varepsilon E_\partial$.

It follows that $V^i(z) \to V(z)$ uniformly on E.

Proof: The fixed point characterization for V in theorem 4.1 implies that

$$V(z) = \lim_n (L)^n[\psi](z)$$

uniformly on E, for each ψ in $\bigcup_{K<\infty} \Sigma^+(K)$. In this situation, lemma 4.3 implies that $L[\psi]$ is also a function of class $\Sigma^+(K)$ for some K. One identifies $L[\psi](\cdot)$ with the value for the single jump problem P_1, denoted by $L(\cdot)$ in chapter 3, whenever $\psi(\cdot)$ is the final cost function. In view of definition of Hamiltonian functions H_1 in (3.4) and H in (4.21), the result readily follows by denoting

$V^i = (L)^{i-1}[V^1]$, $i = 1,2,\ldots$ Theorem 1.1 above and theorem 3.2 jointly provide a characterization for minimizing sequences in $\underline{\underline{U}}$.

Corollary 4.3: Let $\{V^i\}$ be a sequence defined in theorem 1.1 and suppose that for each i a control $u^i = u^i_o/u^i_\partial$ in $\underline{\underline{U}}$ satisfies.

$$\min_{\xi\varepsilon\partial V^i(x(t,z))} -H(-V^i(x(t,z)),-\xi,x(t,z),u^i_o(t,z),[V^{i-1}]) = 0$$

for each $z \varepsilon E$ and $t \varepsilon [0,t^*_{u^i_o}(z)]$.

In addition

$$V^i(x(t^*_{u^i_o},z)) - F(x(t^*_{u^i_o},z),u^i_\partial(t^*_{u^i_o},z),[V^{i-1}]) = 0$$

for each $z \in E$.

Then $\{u^i\}$ is minimizing sequence for the stochastic PDP problem.

4.5 Direct criteria of optimality

The conditions imposed initially on measure Q and P are now strengthened to the point indicated by condition (A7) in the sequel. As a result, direct necessary and sufficient conditions for solutions are analytically justified.

(A7) The mesures Q and P are such that

$$Q : E \times U \to \underline{P}(E), \qquad P : \underline{E} \times U_\partial \to P(E)$$

are measurable functions. In addition, for each u in U or in U_∂ as appropriate, they are Lipschitz continuous with respect to a metric which is consistent with the topology of weak convergence.

Proof of Theorem 1.2 (Verification Theorem):

Theorem 1.2 is an explicit version of corollary 4.1 and the statement of sufficiency is therefore clear from the single jump problem and theorem 3.2. With the added hypothesis on measures Q and P, $V^u(\cdot)$ for a solution u to the stochastic PDP control problem must belong to the class $\Sigma^+(K)$, for some K. The statement of necessity is derived again from theorem 3.2 for the single jump problem with $V^u(\cdot)$ as final cost.

Proof of Theorem 1.3 (Maximum Principle):

Theorem 1.3 benefits from the optimality criteria in corollary 4.1. With the added hypothesis on Q and P, $V^u(\cdot)$ is a Lipschitzian function of class $\Sigma^+(K)$ for some K. In this situation, we apply the Maximum Principle for P_1, theorem 3.3 with $V^u(\cdot)$ as final cost.

4.6 Recursive Methods

Theorem 1.1 provides a local characterization for solutions in a sequential form whereas theorems 1.2 and 1.3 supply methods for calculus of solutions in local and closed form. Although the last two assert optimality properties in a very neat and economic way, they suffer from the fact that the numerical methods they entail are implicit. If they were used directly the value for the problem namely, $V^i(\cdot)$ for optimal u should be known or at least a good approximation should be available when ε-optimal control suffices. Algorithms for the stochastic PDP problem therefore should proceed in two steps: determination of (1) value, (2) control. The result of theorem 1.1 suggests an appropriate method for solving step (1) in that a recursive approximation for the value is established. Herewith step (2) can be handled by methods in association with theorems 1.2 or 1.3 after step (1) is accomplished, the optimality conditions in these theorems form the usual basis for numerical methods in Deterministic Optimal Control. Note in passing that if (A7) does not hold, the procedure is conceptually limited to the provision of ε-optimal controls in general, since step (1) should be replaced by the determination of a Lipschitzian approximation for the value, in order to validate the methods in step (2). In this situation clearly theorems 1.1, 1.2 and 1.3 still apply, and due to natural numerical imprecisions condition (A7) plays a minor role.

Some numerical methods will now be proposed. For this, we invoke the relaxed problem $\tilde{P}_{d,z}$ defined in chapter 3, section 3.3. It is a deterministic problem defined in a neighbourhood of the domain set E, no final end point constraint is present and the final time t_f is a deterministic optimal stopping for the problem. It was shown in proposition 3.3 and 3.4 that this relaxed problem and the single jump problem are equivalent provided that constants K_1, K_2 nd K_3 in $\tilde{P}_{d,z}$ are sufficiently large. At this point, we reinforce assumption (A2) of chapter 3 and the dependence of data on U and U_∂ in the following sense:

(A8) $\qquad \lambda_0 > K$

holds, with

$$\lambda_0 = \inf_{u \in U, z \in E} \lambda(z,u)$$

(A9) All data are continuous

When $\tilde{P}_{d,z}$ fulfills these extra conditions the numerical procedure proposed by Gonzalez and Rofman [G1] can be applied to obtain an approximation for the value. Their scheme seems particularly attractive for solving step (1) since the approximation for the value is calculated directly in terms of a family linear finite elements functions which provides a suitable formula for the generalized gradient, the algorithm is monotone and it appears to be computationally inexpensive. In the sense of theorem 1.1, the method can provide an approximation for the sequence $\{V^i\}$, say $\{\bar{V}^i\}$. It is not entirely clear at this point whether the approximate sequence $\{\bar{V}^i\}$ will converge to the true value V. If some error bounds can be obtained, the following proposition give an affirmative answer.

Proposition 4.1: Suppose that scalars α_1 and α_2 can be found in the interval $[0,1)$ such that for all scalars $r \neq 0$ and functions ψ in the class $\bigcup_{K<\infty} \Sigma^+(K)$

$$(4.23) \quad \alpha_1 r \leq (L)^n[\psi+r](\cdot) - (L)^n[\psi](\cdot\} \leq \alpha_2 r$$

is verified for some n. For a given ψ in $\Sigma^+(K)$ and $\varepsilon > 0$ consider the sequence $\{\bar{V}^i\}$ satisfying

$$\bar{V}^1(\cdot) = \psi(\cdot)$$

and

$$L[\bar{V}^i](\cdot) \leq \bar{V}^{i+1}(\cdot) \leq \bar{L}[V^i](\cdot) + \varepsilon \qquad i = 1,2,...$$

Then

$$|(L)^{nk}[\psi](\cdot) - \bar{V}^{nk}(\cdot)| < \varepsilon$$

where

$$\bar{\varepsilon} = \varepsilon(1-\alpha_2)^{-1} \sup_{z \in E, u \in \underline{U}} E_z^u \left[\sum_i^n I_{\{T_i<\tau\}} + 1\right]$$

In addition for all $k = 1,2,..$

$$\bar{V}^{nk}(\cdot) + \beta_k \leq \bar{V}(\cdot) \leq \bar{V}^{nk}(\cdot) + \beta_k$$

is satisfied, where

$$\beta_K = \min\left[\frac{\alpha_1}{1-\alpha_1}, \frac{\alpha_2}{1-\alpha_2}\right] \cdot \delta_K - \bar{\varepsilon}$$

$$\bar{\beta}_K = \max\left[\frac{\alpha_1}{1-\alpha_1}, \frac{\alpha}{1-\alpha_2}\right] \cdot \bar{\delta}_K + \bar{\varepsilon}$$

and

$$\delta_K = \inf_{z \in E} [\bar{V}^{nk}(z) - \bar{V}^{n(k-1)}(z)] - 2\varepsilon$$

$$\bar{\delta}_K = \sup_{z \in E} [\bar{V}^{nk}(z) - \bar{V}^{n(k-1)}(z)] + 2\varepsilon$$

Proof: From propositions 4.6 and 4.7 of Bertsekas and Shreve [B1]. It is clear from the condition in (4.23) that if ψ and ψ' belong to the class $\bigcup_{K<\infty} \Sigma^+(K)$ with $\psi \geq \psi'$ and $\sup_{z \in E} |\psi(z) - \psi'(z)| = r$ it follows that

$$0 \leq (L)^n[\psi](\cdot) - (L)^n[\psi'](\cdot)$$

$$\leq (L))^n[\psi'+r](\cdot) - (L)^n[\psi'](\cdot)$$

$$\leq \alpha_2 \sup_{z \in E} |\psi(z) - \psi'(z)|$$

If ψ and ψ' are arbitrary elements of the above class (4.23) yields the conclusion that the operator $(L)^n$ satisfies the uniform contraction assumption with ratio $\alpha_1 \vee \alpha_2$. This is exactly the condition under which the mentioned results were proven.

It follows that $V(\cdot)$ can be obtained within an arbitrary degree of precision in a finite number of iterations; moreover this number is especified as a function of the sequence of approximations $\{\bar{V}^i]$ and not as a function of the exact solutions $\{V^i\}$. For more details we refer to Bertsekas and Shreve [B1], section 4.3 and Bertsekas [B3], section 6.2.

The proposition and remarks above suggest a numerical solution for step (1), namely the determination of the value function. For step (2) an standard optimal control algorithm for non-smooth problems can give an appropriate solution. Indeed, one can use the relaxed formulation $\tilde{P}_{d,z}$ again to solve an deterministic optimal control problem with Lipschitzian bounded data without state and final end point constraints which are present in the exact formulation $P_{d,z}$. These constraints are accounted by a penalization in the cost, and proposition 3.4 asserts the equivalence of both problems

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