COMPLEX LAGRANGIAN SUBMANIFOLDS IN HOLOMORPHIC SYMPLECTIC VARIETIES AND DIFFERENTIAL GRADED ALGEBRAS

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Abstract

We present various applications of deformation quantisation techniques to holomorphic symplectic geometry. Let $i : \mathrm{L} \hookrightarrow \mathrm{X}$ be a compact Kähler Lagrangian in a holomorphic symplectic variety $\mathrm{X}/\mathbf{C}$.

- We calculate the space $\mathrm{Ext}_{\mathscr{O}_\mathrm{X}}\left(i_*\mathrm{K}_\mathrm{L}^{1/2}, i_*\mathrm{K}_\mathrm{L}^{1/2}\right)$ which counts massless open strings connecting a Lagrangian D-brane[(1)] wrapped on L to itself whose gauge bundle is given by $\mathrm{K}_\mathrm{L}^{1/2}$, by proving that there is a spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{L}, \Omega_\mathrm{L}^q) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_\mathrm{X}}\left(i_*\mathrm{K}_\mathrm{L}^{1/2}, i_*\mathrm{K}_\mathrm{L}^{1/2}\right)$$

  which degenerates on the second page.

- Building on the degeneration of the spectral sequence, we show that the endomorphism differential graded algebra

$$\mathrm{RHom}_{\mathscr{O}_\mathrm{X}}\left(i_*\mathrm{K}_\mathrm{L}^{1/2}, i_*\mathrm{K}_\mathrm{L}^{1/2}\right)$$

  is formal.

- We furthermore obtain extensions of the above stated results to smoothly intersecting Lagrangian submanifolds and discuss some progress on possible applications to the Fukaya category of a hyperkähler manifold via Seiberg-Witten duality.

(1) The D-brane corresponding to $i_*\mathrm{K}_\mathrm{L}^{1/2}$ in the derived category of X.

- We define a notion of A_n-formality for A_∞-modules over graded algebras and show that the obstructions are Hochschild cohomology classes. We prove auxiliary results on the behaviour of formality in families of A_∞-modules which extend earlier work of Kaledin and Lunts on A_∞-algebras.

- Finally, examples showing the non-triviality of the spectral sequence

$$E_2^{p,q} = H^p(L, \Omega_L^q) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_X}\left(i_*\mathscr{L}, i_*\mathscr{L}\right)$$

  for general line bundles $\mathscr{L}$ supported on L are discussed.

To my family

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1 Introduction

1.1 Degeneration of the spectral sequence

Let $i : Z \subset X$ be a locally complete intersection subvariety and choose a line bundle $\mathscr{L}$ on Z. The standard isomorphism $\mathrm{Hom} \cong \Gamma \circ \mathscr{H}om$ gives $\mathrm{RHom} \cong \mathrm{R}\Gamma \circ \mathrm{R}\mathscr{H}om$, so we get a Grothendieck spectral sequence, in this case the local-to-global Ext spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(X, \mathscr{E}xt^q_{\mathscr{O}_X}(i_*\mathscr{L}, i_*\mathscr{L})) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_X}(i_*\mathscr{L}, i_*\mathscr{L}).$$

We can rewrite this spectral sequence in more familiar terms. First, we observe that the $\mathscr{E}xt$-sheaves are independent of $\mathscr{L}$.

Lemma 1.1.1. *There are natural isomorphisms*

$$\mathscr{E}xt^q_{\mathscr{O}_X}(i_*\mathscr{L}, i_*\mathscr{L}) \cong \mathscr{E}xt^q_{\mathscr{O}_X}(i_*\mathscr{O}_L, i_*\mathscr{O}_L) \otimes_{\mathscr{O}_L} \mathscr{L} \otimes_{\mathscr{O}_L} \mathscr{L}^\vee \cong \mathscr{E}xt^q_{\mathscr{O}_X}(i_*\mathscr{O}_L, i_*\mathscr{O}_L).$$

Proof. Indeed, the composition defines a natural map

$$\mathscr{H}om_{\mathscr{O}_X}(i_*\mathscr{O}_L, i_*\mathscr{O}_L) \otimes_{\mathscr{O}_L} \mathscr{L} \otimes_{\mathscr{O}_L} \mathscr{L}^\vee \to \mathscr{H}om_{\mathscr{O}_X}(i_*\mathscr{L}, i_*\mathscr{L})$$

which we see is an isomorphism working locally on Z. Now the $q > 0$ cases follow from δ-effaceability. □

Now, we are ready to calculate the $\mathscr{E}xt$-sheaves:

Proposition 1.1.2. *There are natural isomorphisms* $\mathscr{E}xt^q_{\mathscr{O}_X}(i_*\mathscr{L}, i_*\mathscr{L}) \cong i_* \wedge^q \mathscr{N}_{Z/X}$.

Proof. Since the question is local, we may assume $\mathscr{L} = \mathscr{O}_Z$ and that Z is the zero locus of a regular section $s \in H^0(X, \mathscr{O}_X^c)$ where $c = \operatorname{codim}(Z, X)$. Then we have a Koszul resolution

$$\cdots \to \mathscr{O}_X^{\binom{c}{2}} \to \mathscr{O}_X^c \xrightarrow{s^\vee} \mathscr{O}_X \to i_*\mathscr{O}_Z \to 0.$$

Taking $\mathscr{H}om_{\mathscr{O}_X}(-, i_*\mathscr{O}_Z)$ gives a complex

$$i_*\mathscr{O}_Z \to i_*\mathscr{O}_Z^c \to i_*\mathscr{O}_Z^{\binom{c}{2}} \to \cdots$$

with all arrows zero. Hence, $\mathscr{E}xt^q_{\mathscr{O}_X}(i_*\mathscr{O}_Z, i_*\mathscr{O}_Z) \cong i_*\mathscr{O}_Z^{\binom{c}{q}}$ and since s yields a regular sequence, the natural map $\mathscr{O}_X^c \to \mathscr{I}_Z$ induces an isomorphism $\mathscr{O}_Z^c \cong \mathscr{I}_Z/\mathscr{I}_Z^2$. □

So the spectral sequence takes a simpler form:

$$E_2^{p,q} = H^p(Z, \wedge^q \mathcal{N}_{Z/X}) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}_X}(i_*\mathscr{L}, i_*\mathscr{L})$$

and, as we will see later, the differentials on E_2 come from classes in

$$\mathrm{Ext}^2_{\mathcal{O}_X}(\wedge^q \mathcal{N}_{Z/X}, \wedge^{q-1} \mathcal{N}_{Z/X}).$$

Let KS be the Kodaira-Spencer class, i.e. the extension class of

$$0 \to \mathcal{T}_Z \to i^*\mathcal{T}_X \to \mathcal{N}_{Z/X} \to 0.$$

Example 1.1.3. For $q = 1$, we have an explicit description of the differential, showing it's often non-zero in general.

The differential acts via $\mathcal{N}_{Z/X} \to \mathcal{O}_Z[2]$ which is (the adjoint of) the class

$$\alpha_{\mathscr{L}} : \mathscr{L} \to \mathscr{L} \otimes \mathcal{N}^\vee_{Z/X}[2],$$

measuring the obstruction to extending $\mathscr{L}$ from Z to $2Z$[(2)].

This is purely Hodge-theoretic, i.e. $c_1(\mathscr{L})$ extends topologically and its $(0,2)$ part is the obstruction to it being of type $(1,1)$, so a holomorphic line bundle.

The differential is a map $H^p(Z, \mathcal{N}_{Z/X}) \to H^{p+2}(Z, \mathcal{O}_Z)$ which is the contraction with the

[(2)] $2Z$ is the first infinitesimal neighbourhood of Z in X given by the ideal $\mathcal{I}_Z^2$.

class $\alpha_{\mathscr{L}} \in \mathrm{H}^2(\mathrm{Z}, \mathscr{N}_{\mathrm{Z/X}}^{\vee})$ given by contracting $\mathrm{c}_1(\mathscr{L}) \in \mathrm{H}^1(\mathrm{Z}, \Omega_{\mathrm{Z}})$ with

$$\mathrm{KS} \in \mathrm{H}^1(\mathrm{Z}, \mathscr{N}_{\mathrm{Z/X}}^{\vee} \otimes \mathscr{T}_{\mathrm{Z}}).$$

- Let $i : \mathrm{L} \subset \mathrm{X}$ be a smooth complex Lagrangian in a (compact) holomorphic symplectic variety X.
- Denote the holomorphic symplectic form by σ.
- Fix a line bundle $\mathscr{L} \in \mathrm{Pic}(\mathrm{L})$.

We are interested in the self-Ext groups of $i_*\mathscr{L}$. These are naturally computed using the local-to-global spectral sequence introduced earlier

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{L}, \wedge^q \mathscr{N}_{\mathrm{L/X}}) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_{\mathrm{X}}}(i_*\mathscr{L}, i_*\mathscr{L}).$$

We see that the holomorphic symplectic form σ gives an isomorphism $\mathscr{N}_{\mathrm{L}} \simeq \Omega_{\mathrm{L}}$ by considering the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{T}_{\mathrm{L}} & \longrightarrow & i^*\mathscr{T}_{\mathrm{X}} & \longrightarrow & \mathscr{N}_{\mathrm{L/X}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle i^*\sigma} & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{N}_{\mathrm{L/X}}^{\vee} & \longrightarrow & i^*\Omega_{\mathrm{X}}^1 & \longrightarrow & \Omega_{\mathrm{L}}^1 & \longrightarrow & 0 \end{array}$$

Hence, in our symplectic setting, the spectral sequence takes the form

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{L}, \Omega_{\mathrm{L}}^q) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_{\mathrm{X}}}(i_*\mathscr{L}, i_*\mathscr{L}).$$

Remark 1.1.4.

- E_2 is de Rham cohomology $H(L/\mathbf{C})$ of L.
- The second page is independent of the line bundle.
- The differential d_2, however, isn't, e.g. the class $\alpha_{\mathscr{L}}$ mentioned earlier clearly depends on $\mathscr{L}$.

Let, as above, $i : L \hookrightarrow X$ be a smooth compact Kähler Lagrangian in a holomorphic symplectic (not necessarily proper!) variety over $\mathbf{C}$. Recall that, for any $\mathscr{L} \in \mathrm{Pic}(L)$, there exists a local-to-global Ext spectral sequence:

$$E_2^{p,q} = H^p(L, \Omega_L^q) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_X}(i_*\mathscr{L}, i_*\mathscr{L}).$$

Therefore, as explained already above, the second page gives de Rham cohomology of L, independent of $\mathscr{L}$, but the differentials depend on $\mathscr{L}$. Our first main result is on the degeneration of this spectral sequence for certain class of line bundles $\mathscr{L}$.

Theorem 1.1.5. *Let $i : L \hookrightarrow X$ be a smooth complex Lagrangian in a compact hyperkähler variety $X/\mathbf{C}$, and let $\mathscr{L}$ be a quantisable line bundle on L, such as $K_L^{1/2}$. Then the local-to-global* Ext *spectral sequence*

$$E_2^{p,q} = H^p(L, \Omega_L^q) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_X}(i_*\mathscr{L}, i_*\mathscr{L})$$

degenerates on the second page.

Remark 1.1.6.

- Recall that in the holomorphic case the existence of a line bundle

$K_L^{1/2}$ is equivalent to the existence of a spin structure on L. A choice of spin structure corresponds to a choice of a square root $K_L^{1/2}$, see [Ati71].

- In fact, as we will explain later, the quantisable line bundles are precisely the twists of $K_L^{1/2}$ by 2-torsion line bundles.

Remark 1.1.7. The proof is, unsurprisingly, via deformation quantisation. This doesn't require X compact Kähler and, in fact, the proof shows we don't even need L Kähler - it is enough it be smooth, compact with Hodge-to-de Rham spectral sequence degenerating on E_1. Examples of such manifolds are given by any manifold satisfying the $\partial\overline{\partial}$-lemma, e.g. non-projective Moishezon manifolds.

1.2 Formality results

Recall that a differential graded algebra A is formal if $A \simeq HA$ as differential graded algebras. In particular, all higher Massey products vanish. The converse is not true - there are non-formal differential graded algebras with vanishing Massey products. We now give two (algebraic) examples of non-fromal dg algebras. Afterwards, we shall also exhibit an example coming from geometry.

Example 1.2.1.

- Let $A = \mathbf{Z}[t]/t^4$ with t in degree 1 and $dt = 2$. Note that A is not graded commutative since $t^2 \neq 0$. Then HA is an exterior algebra over $\mathbf{Z}/2$ on a single generator in degree 2. There is no quasi-isomorphism $A \simeq HA$ since $[\mathbf{Z}/2, A] = 0$ as we can see by resolving the trivial dga $\mathbf{Z}/2$.

- Let A be the exterior algebra on t, u, v all in degree 1 with $dt = du = 0$ and $dv = tu$.

The Massey product $\langle t, u, v\rangle \neq 0$ and hence A is not formal. See [DGMS75] for a geometric description of this example.

We have defined what it means for a differential graded algebra to be formal and saw two examples of non-formal differential graded algebras. The next example shows that the are non-formal dg algebras arising naturally in algebraic geometry.

Example 1.2.2. Let X be a smooth scheme and $i : Z \subset X$ be a smooth subscheme. In [AC12] it is shown that the dga $i^*i_*\mathscr{O}_Z$ is formal iff the obstruction $\alpha_{\mathscr{N}^\vee_{Z/X}}$ to lifting $\mathscr{N}^\vee_{Z/X}$ to the first infinitesimal neighbourhood 2Z of Z in X vanishes.
An example where $\alpha_{\mathscr{N}^\vee_{Z/X}} \neq 0$ is given by taking a $\mathbf{P}^1 \times \mathbf{P}^1$ embedded in $\mathbf{P}^5$ via $\mathscr{O}(1)\boxtimes\mathscr{O}(2)$. In this case, the map

$$H^1(KS) : H^1(Z, \Omega_Z) \to H^2(Z, \mathscr{N}^\vee_{Z/X})$$

is non-zero since $\dim\left(H^1(Z, i^*\Omega_{\mathbf{P}^5})\right) = 1$ via pulling back the Euler sequence for projective space. Now $\alpha_{\det(\mathscr{N}^\vee_{Z/X})} \neq 0$ for $H^1(Z, \Omega_Z)$ is two dimensional with basis given by $\det(\mathscr{N}^\vee_{Z/X}) = \mathscr{O}(-4)\boxtimes\mathscr{O}(-10)$ (adjunction) and $i^*\mathscr{O}_{\mathbf{P}^5}(1)$, so, as the latter line bundle extends, we necessarily have $\alpha_{\det(\mathscr{N}^\vee_{Z/X})} \neq 0$ since otherwise $H^1(KS)$ would be identically zero, a contradiction.

Remark 1.2.3. Pushing forward $i^*i_*\mathscr{O}_Z$, one can show that $\tau^{\geq p-1}\tau^{\leq p}(i_*i^*i_*\mathscr{O}_Z)$ are formal for all p which means all $d_2 = 0$.

Let's briefly mention a conjecture of Grivaux inspired by this observation and the fact that there are no known obstructions to formality in $\mathbf{D}^b(X)$ - compare to the result of Arinkin-Caldararu mentioned in Example 1.2.2.

Conjecture 1.2.4 ([Gri14]). Let X be a smooth scheme and $i : Z \subset X$ be a smooth subscheme. Then $R\mathscr{H}om_{\mathscr{O}_X}(i_*\mathscr{O}_Z, i_*\mathscr{O}_Z)$ and $i_*\mathscr{O}_Z \otimes_{\mathscr{O}_X} i_*\mathscr{O}_Z$ split in $\mathbf{D}^b(X)$.

Remark 1.2.5. Note that while $\tau^{\geq -2}\tau^{\leq -1}(i_*i^*i_*\mathscr{O}_Z)$ is formal, $\tau^{\geq -2}\tau^{\leq -1}(i^*i_*\mathscr{O}_Z)$ is not always formal, in fact it's formal iff $i^*i_*\mathscr{O}_Z$ is. This reflects the fact that i_* is not faithful.

By Kadeishvili's theorem the cohomology of a differential graded algebra is naturally an A_∞-algebra. The A_∞-structure on HA measures failure of formality for A. Kähler formality of Deligne et al. (see [DGMS75]) says that $\Omega^*(L, \mathbf{C})$ is a formal differential graded algebra, hence $H^*(L/\mathbf{C})$ is formal as an A_∞-algebra. In light of

$$H^k(L/\mathbf{C}) = \oplus_{p,q} H^p(L, \Omega_L^q) = \mathrm{Ext}^k_{\mathscr{O}_X}(i_*\mathscr{L}, i_*\mathscr{L}),$$

it is natural to ask whether the same goes for $\mathrm{Ext}_{\mathscr{O}_X}(i_*\mathscr{L}, i_*\mathscr{L})$, i.e. if the differential graded algebra $\mathrm{RHom}_{\mathscr{O}_X}(i_*\mathscr{L}, i_*\mathscr{L})$ is formal. We confirm this in the case where $\mathscr{L}$ is a square root of the canonical bundle of L.

Theorem 1.2.6. *Let* $X/\mathbf{C}$ *be holomorphic symplectic and let* $i : L \hookrightarrow X$ *be a smooth compact Kähler Lagrangian submanifold whose canonical bundle admits a square root. Then the differential graded algebra* $\mathrm{RHom}_{\mathscr{O}_X}\left(i_*K_L^{1/2}, i_*K_L^{1/2}\right)$ *is formal, in fact, quasi-isomorphic to the de Rham algebra* $H(L/\mathbf{C})$.

1.3 Results on pairs of Lagrangians

Consider two smooth Lagrangians $i : L \hookrightarrow X$, $j : M \hookrightarrow X$ such that $L \cap M$ is smooth. For any choice of line bundles $\mathscr{L} \in \mathrm{Pic}(L)$ and $\mathscr{M} \in \mathrm{Pic}(M)$, we have a local-to-global Ext

spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{L}\cap\mathrm{M}, \Omega^{q-c}_{\mathrm{L}\cap\mathrm{M}} \otimes \mathscr{K}) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_\mathrm{X}}(i_*\mathscr{L}, j_*\mathscr{M}),$$

where $\mathscr{K} := \det \mathscr{N}_{\mathrm{L}\cap\mathrm{M}/\mathrm{M}} \otimes \mathscr{L}^\vee\big|_{\mathrm{L}\cap\mathrm{M}} \otimes \mathscr{M}|_{\mathrm{L}\cap\mathrm{M}}$.

Theorem 1.3.1. *Let* X/**C** *be a holomorphic symplectic variety and suppose that* $i : \mathrm{L} \hookrightarrow \mathrm{X}$, $j : \mathrm{M} \hookrightarrow \mathrm{X}$ *are smooth Lagrangians with a compact Kähler intersection* $\mathrm{L}\cap\mathrm{M}$ *of codimension* c *in* L. *Assume that* $\mathrm{K}_\mathrm{L}^{1/2}$ *and* $\mathrm{K}_\mathrm{M}^{1/2}$ *exist and let*

$$\mathscr{K}_{\mathrm{or}} := \left(\mathrm{K}_\mathrm{L}^{1/2}\big|_{\mathrm{L}\cap\mathrm{M}} \otimes \mathrm{K}_\mathrm{M}^{1/2}\big|_{\mathrm{L}\cap\mathrm{M}}\right)^\vee \otimes \mathrm{K}_{\mathrm{L}\cap\mathrm{M}}.$$

Then the Ext *local-to-global spectral sequence*

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{L}\cap\mathrm{M}, \Omega^{q-c}_{\mathrm{L}\cap\mathrm{M}} \otimes \mathscr{K}_{\mathrm{or}}) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_\mathrm{X}}\left(i_*\mathrm{K}_\mathrm{L}^{1/2}, j_*\mathrm{K}_\mathrm{M}^{1/2}\right)$$

degenerates on the second page. In particular,

$$\mathrm{Ext}^k_{\mathscr{O}_\mathrm{X}}\left(i_*\mathrm{K}_\mathrm{L}^{1/2}, j_*\mathrm{K}_\mathrm{M}^{1/2}\right) = \oplus_{p,q}\mathrm{H}^p(\mathrm{L}\cap\mathrm{M}, \Omega^{q-c}_{\mathrm{L}\cap\mathrm{M}} \otimes \mathscr{K}_{\mathrm{or}}) = \mathrm{H}^{k-c}(\mathrm{L}\cap\mathrm{M}, \tilde{\mathfrak{K}}_{\mathrm{or}}),$$

where $\tilde{\mathfrak{K}}_{\mathrm{or}}$ *is the local system corresponding to the* 2*-torsion line bundle* $\mathscr{K}_{\mathrm{or}}$.

Remark 1.3.2. As above, we remark that the proof shows it is enough to assume the intersection $\mathrm{L}\cap\mathrm{M}$ smooth, compact such that its Hodge-to-de Rham spectral sequence for the local system $\tilde{\mathfrak{K}}_{\mathrm{or}}$ degenerates on E_1.

Example 1.3.3. Notice that $\mathrm{Ext}_{\mathscr{O}_X}\left(i_*K_L^{1/2}, j_*K_M^{1/2}\right)$ is a graded module over the graded algebra $\mathrm{Ext}_{\mathscr{O}_X}\left(i_*K_L^{1/2}, i_*K_L^{1/2}\right)$. A theorem of Deligne (see [GM88], [Sim92]) asserts that $\Omega^*(L \cap M, \tilde{\mathfrak{K}}_{or})$ is a formal dg module over $\Omega^*(L \cap M, \mathbf{C})$. Let $\Omega^*_{\bar{\partial}}$ be the complex of $\bar{\partial}$-closed forms with differential ∂. The diagram

$$\begin{array}{ccccc} \Omega^*(L, \mathbf{C}) & \longleftarrow & \Omega^*_{\bar{\partial}}(L, \mathbf{C}) & \longrightarrow & H(L/\mathbf{C}) \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^*(L \cap M, \mathbf{C}) & \leftarrow & \Omega^*_{\bar{\partial}}(L \cap M, \mathbf{C}) & \rightarrow & H(L \cap M/\mathbf{C}) \end{array}$$

shows $\Omega^*(L \cap M, \tilde{\mathfrak{K}}_{or})$ is also formal over $\Omega^*(L, \mathbf{C})$. Hence $H(L \cap M, \tilde{\mathfrak{K}}_{or})$ is formal as an A_∞-module over $H(L/\mathbf{C})$.

We prove the same goes for the Ext modules:

Theorem 1.3.4. *Let $X/\mathbf{C}$ be holomorphic symplectic. Suppose that $i : L \hookrightarrow X$, $j : M \hookrightarrow X$ are compact Kähler Lagrangians with a smooth intersection. Assume that their canonical bundles admit square roots. Then $\mathrm{RHom}_{\mathscr{O}_X}\left(i_*K_L^{1/2}, j_*K_M^{1/2}\right)$ is a formal differential graded module over the (formal) differential graded algebra $\mathrm{RHom}_{\mathscr{O}_X}\left(i_*K_L^{1/2}, i_*K_L^{1/2}\right)$. Moreover, we have a quasi-isomorphism of pairs*

$$\left(\mathrm{RHom}_{\mathscr{O}_X}\left(i_*K_L^{1/2}, i_*K_L^{1/2}\right), \mathrm{RHom}_{\mathscr{O}_X}\left(i_*K_L^{1/2}, j_*K_M^{1/2}\right)\right) \simeq \left(H(L/\mathbf{C}), H^{*-c}(L \cap M, \tilde{\mathfrak{K}}_{or})\right),$$

where c is the codimension of $L \cap M$ in L.

Remark 1.3.5. A few observations regarding Theorem 1.3.4:

1. The proof shows that it is enough to assume L compact Kähler and M smooth such

that $L \cap M$ is smooth.

2. A variant for the dg module structure over $\mathrm{RHom}_{\mathscr{O}_X}\left(j_*K_M^{1/2}, j_*K_M^{1/2}\right)$ can be formulated, reversing the assumptions on L and M in the part 1. of the remark.

1.4 Method of proof

The proof of Theorem 1.2.6 involves two main ingredients. We first observe that the dg algebra $\mathrm{RHom}_{\mathscr{O}_X}\left(i_*K_L^{1/2}, i_*K_L^{1/2}\right)$ can be deformed over $\mathbf{C}[[\hbar]]$ to the de Rham complex - this involves results of Schapira et al. on deformation quantisation [DS07], [KS08], hence the square root of the canonical bundle. The deformation is the de Rham complex $\Omega^*(L, \mathbf{C})$ over the generic point, therefore it is generically formal by Kähler formality of [DGMS75]. Applying semicontinuity yields degeneration of the spectral sequence, proving the degeneration for $K_L^{1/2}$. The formality requires a second ingredient - it is a theorem of Kaledin that generically formal dg algebras are formal (under certain conditions which our family satisfies).

The proofs of Theorem 1.3.1 and Theorem 1.3.4 follow the same lines, however, to prove the formality, we need Proposition 2.8.5 - the analogue to the result of Kaledin for A_∞-modules.

Kaledin in [Kal07] treats formality of dg algebras as triviality of the normal cone deformation. Our approach to formality of A_∞-modules is also motivated by the deformation to the normal cone as we now explain, although we phrase our results without reference to deformation theory, opting to use the language of A_n-algebras and modules instead.

Let A be a graded algebra over R. Consider a minimal A_∞-module M over A. Assume A, M projective over R. Let $\tilde{A} = A[h]$ be the trivial deformation of A over $R[h]$. Con-

sider the graded $R[h]$-module $M[h]$. Then $(m_2^M, m_3^M h, m_4^M h^2, \cdots)$ turns $M[h]$ into an A_∞-module over $A[h]$ since the defining relations $(*'_m)$ for A_∞-modules are homogeneous. Letting $\tilde{M}$ be the so defined A_∞-module, observe that the general fibre is M. We write $M(2)$ for the minimal (formal) A_∞-module $(M, (m_2^M, 0, 0, \cdots)$, so the central fibre of $\tilde{M}$ is $\tilde{M}/h = M(2)$. In light of this, the following definition seems natural.

Definition 1.4.1. The A_∞-module $\tilde{M}$ is the deformation of M to the normal cone.

One should think of $\tilde{M}$ as an A_∞-deformation of $M(2)$ to M. Notice that the formality of the A_∞-module $\tilde{M}$ is the same as triviality of $\tilde{M}$ as a deformation, i.e. in either case we are asking for a quasi-isomorphism $\tilde{M} \simeq M(2)[h]$. Reducing modulo $h-1$, we see that this implies that M is formal. The converse is also true. Hence formality of M is equivalent to triviality of the deformation $\tilde{M}$, so we can use obstruction theory and cohomology, standard deformation theory tools, to find formality criteria for A_∞-modules.

1.5 Context

The work of Solomon-Verbitsky

In [SV19] they study the Fukaya category of I-holomorphic graded spin Lagrangians in a hyperkähler variety (X, I, J, K, g), equipped with the symplectic form $\omega_J = g(J\cdot, \cdot)$. Recall that spin is needed to set up Floer theory and is equivalent to choosing square roots of the canonical bundles in the complex case. When $L \cap M$ is smooth, they show that the Floer coboundary operator μ_1 on $CF(L, M)$ coincides with the de Rham differential, hence $HF(L, M)$ is the de Rham cohomology of $L \cap M$, up to Maslov index shifts and tensoring by Λ. Moreover, for $CF(L, L)$, μ_2 is the wedge product of differential forms

up to sign, while $\mu_k = 0$ for $k \geq 3$. Thus, in the compact case, they recover a formality result of Ivan Smith: the proof of Kähler formality as in [DGMS75] shows that the Floer A_∞-algebra $CF(L, L)$ is formal. We note that their results might be thought of as mirror to ours.

More generally, Solomon and Verbitsky consider a collection of I-holomorphic graded spin compact Lagrangians $\mathcal{L}$ and define a Fukaya A_∞-category $\widehat{\mathcal{A}_\mathcal{L}}$. In light of the formality of $CF(L, L)$, the following conjecture of Ivan Smith is natural.

Conjecture 1.5.1. $\widehat{\mathcal{A}_\mathcal{L}}$ is a formal A_∞-category.

The category $\widehat{\mathcal{A}_\mathcal{L}}$ is known to be intrinsically formal in the case of the A_k-Milnor fibre by [ST01] and [AS16] shows formality of $\widehat{\mathcal{A}_\mathcal{L}}$ in the case of a transverse nilpotent slice of the adjoint quotient $\mathfrak{sl}_2 k(\mathbf{C}) \to \mathbf{C}^{2k-1}$ and a distinguished (finite) collection of Lagrangians defined in [SS06]. We answer a similar question in the setting of virtual de Rham cohomology - see Proposition 1.5.5 below.

Kapustin's Seiberg-Witten duality and deformation quantisaion

Another source of motivation for us comes from Kapustin's conjectural Seiberg-Witten duality [Kap05] between type A and B-branes on hyperkähler manifolds which pairs well with the results of [SV19]. One way to make the conjectural duality of Kapustin precise is via deformation quantisaion modules of Kashiwara-Schapira [KS12], cf. Conjecture 1.5.2. We exploit this idea, using the deformation quantisation for complex Lagrangians [DS07], to show that the analogue of the Solomon-Verbitsky degeneration holds for the B-model. As explained, this should be regarded as a Seiberg-Witten duality between commutative

and non-commutative spaces. In particular, everything happens on the same manifold and no T-duality is involved.

In earlier work [Kap04] Kapustin conjectures that the Fukaya category of a hyperkähler variety (X, I, J, K) with symplectic form ω_J should be equivalent to a non-commutative deformation of the derived category on the holomorphic symplectic manifold $(X, I, \sigma_I = \omega_J + i\omega_K)$. We suggest a precise deformation:

Conjecture 1.5.2. The Fukaya category of a hyperkähler variety (X, I, J, K) with symplectic form ω_J is quasi-equivalent to the differential graded category of DQ modules on the holomorphic symplectic manifold $(X, I, \sigma_I = \omega_J + i\omega_K)$.

Following [SV19] and Conjecture 1.5.2, one can speculate that the category $\widehat{\mathcal{A}}_{\mathcal{L}}$ should be quasi-equivalent to the full subcategory $\mathcal{D}_{\mathcal{L}}$ of the category of DQ modules on X whose objects are a choice of a simple holonomic module $\mathscr{D}_L$ along L for each $L \in \mathcal{L}$. Our results show that $\mathcal{D}_{\mathcal{L}}$ is well-defined up to quasi-equivalence. In fact, we can strengthen Theorem 1.2.6:

Theorem 1.5.3. *Let $X/\mathbf{C}$ be holomorphic symplectic and let $i : L \hookrightarrow X$ be a smooth compact Kähler spin Lagrangian submanifold. Then, for $\mathcal{L} = \{L\}$, the differential graded category $\mathcal{D}_{\mathcal{L}}$ is formal.*

In light of the results above, [Conjecture 1.5.1] and the Seiberg-Witten duality [Conjecture 1.5.2], we expect:

Conjecture 1.5.4. Let $\mathcal{L}$ be a Solomon-Verbitsky collection of Lagrangians in X. The differential graded category $\mathcal{D}_{\mathcal{L}}$ is formal.

Thus, Theorem 1.5.3 proves Conjecture 1.5.4 in the case of one Lagrangian. Our results Theorem 1.3.1 and Theorem 1.3.4 on pairs of cleanly intersecting Lagrangians are the first step towards Conjecture 1.5.4. We are not able to show formality of the differential graded category in general but our weaker results concerning A_∞-modules show that it is well-defined up to quasi-equivalence.

Behrend and Fantechi

The construction of a differential graded category $\mathcal{C}$ of Lagrangians in a holomorphic symplectic variety X is sketched in [BF09]. This construction depends on their constructible virtual de Rham complex $\left(\mathscr{E}_{\mathrm{BF}} := \mathscr{E}xt_{\mathscr{O}_X}\left(i_*K_L^{1/2}, j_*K_M^{1/2}\right), \mathrm{d}\right)$. Locally X is the cotangent bundle of M and L is given by the graph of $\mathrm{d}f$ for some $f \in \Gamma(M, \mathscr{O}_M)$, hence $L \cap M$ is the critical locus $\mathrm{crit} f$. Then the sheaves $\mathscr{E}_{\mathrm{BF}}$ are the cohomology sheaves of $(\Omega_M, \mathrm{d}f\wedge)$. Since $\mathrm{d}f\wedge$ anticommutes with the de Rham differential d, we see that d descends to a differential on $\mathscr{E}_{\mathrm{BF}}$, still denoted d. The main result of [BF09] is that these locally defined differentials glue. The cohomology of the morphism complexes $\mathrm{Hom}_{\mathcal{C}}(L, M)$ is given by the virtual de Rham cohomology $R\Gamma(\mathscr{E}_{\mathrm{BF}}, \mathrm{d})$ of the intersection $L \cap M$. There is a spectral sequence computing $R\Gamma(\mathscr{E}_{\mathrm{BF}}, \mathrm{d})$:

$$E_1^{p,q} = H^q(X, \mathscr{E}_{\mathrm{BF}}^p) \Rightarrow R^{p+q}\Gamma(\mathscr{E}_{\mathrm{BF}}, \mathrm{d}). \tag{1}$$

When $L \cap M$ is compact Kähler, it degenerates by Hodge theory, hence our results prove a corrected version of [BF09, Conjecture 5.8]:

Proposition 1.5.5. *Let* $X/\mathbf{C}$ *be a holomorphic symplectic variety and suppose that* $i : L \hookrightarrow$

X *and* $j : M \hookrightarrow X$ *are smooth Lagrangians such that* $K_L^{1/2}$ *and* $K_M^{1/2}$ *exist and* $L \cap M$ *is smooth compact. Assuming that the spectral sequence* (1) *degenerates on* E_1*, e.g. if* $L \cap M$ *is furthermore Kähler, we have*

$$R^k\Gamma(\mathscr{E}_{BF}, d) = \mathrm{Ext}^k_{\mathscr{O}_X}\left(i_*K_L^{1/2}, j_*K_M^{1/2}\right).$$

Remark 1.5.6. This conjecture is the analogue of the formality of $\widehat{\mathcal{A}_{\mathcal{L}}}$ in the virtual setting.

D*-branes and Mirror symmetry*

Dirichlet branes are important objects introduced by Polchinski in [Pol95] and play crucial role in both the SYZ conjecture [SYZ96] and the homological mirror symmetry conjecture of Kontsevich [Kon95]. They have been related to derived categories of coherent sheaves [Sha99], [KO04], giving the latter physics meaning. More precisely, massless boundary Ramond sector states of open strings connecting Dirichlet branes wrapped on complex submanifolds in a Calabi-Yau manifold X with holomorphic gauge bundles should be parametrised by the Ext groups between the bundles.

1.6 Plan

In §2 we start by reviewing general results on A_∞-algebras and conclude with the formality theorems of Kaledin [Kal07]. In the next paragraph we define A_∞-modules and recall standard results such as Kadeishvili's theorem on minimal models. Then we have a paragraph on (bi-graded) Hochschild cohomology of modules over graded algebras. It culminates in some results on base-change for Hochschild cohomology, mirroring state-

ments in [Lun10]. After that, we develop obstruction theory for extending A_n-modules to A_{n+1}-modules as well as A_n-morphisms to A_{n+1}-morphisms between modules - this is motivated by similar ideas in [Kon97] for L_∞-algebras and [LH03] dealing with A_∞-algebras. The last paragraph contains our results on formality of A_∞-modules.

In the next section, §3, we recall results of [BBD+15] on perverse sheaves on d-critical loci - a structure that exists on the intersection of two Lagrangians. Then we have a reminder on deformation quantisation modules, following [DS07], [KS12]. In particular, we compare the compatibility of [DS07] to the results by Ginzburg et al. in [BGKP16]. We conclude by relating perverse sheaves on Lagrangian intersections and simple holonomic DQ modules.

In §4 we start by applying deformation quantisation to prove the degeneration of the local-to-global Ext spectral sequence, for square roots of the canonical bundle, in both cases of single Lagrangian and a pair of cleanly intersecting Lagrangians. We conclude with our main results on formality.

1.7 Conventions

- We work with A_∞-algebras and modules over a (unital) ring R and we shall assume that R is a commutative algebra over a field k of characteristic 0.
- A scheme over $\mathbf{C}$ or $\mathbf{C}$-scheme X means separated scheme of finite type over $\mathbf{C}$.
- A curve, resp. surface, is a separated scheme over $\mathbf{C}$ of dimension 1, resp. 2.
- A variety is an integral (reduced and irreducible) scheme over $\mathbf{C}$.

- $\mathrm{Coh}(\mathrm{X})$ is the abelian category of coherent sheaves on a $\mathbf{C}$-scheme X.
- $\mathbf{D}^b(\mathrm{X})$ is the bounded derived category of the abelian category $\mathrm{Coh}(\mathrm{X})$.
- A complex $\mathscr{F}$ of $\mathscr{O}_\mathrm{X}$-modules is called perfect if it locally quasi-isomorphic to a bounded complex of locally free sheaves. An object $\mathscr{F} \in \mathbf{D}^b(\mathrm{X})$ is perfect if it can be represented by a perfect complex.
- Ω_X is the cotangent sheaf of a scheme X/$\mathbf{C}$.
- K_X is the canonical bundle of a smooth variety X/$\mathbf{C}$.
- ω_X is the dualising complex of a scheme X.

2 A_∞-algebras

A good general reference for A_∞-algebras is [LH03]. We recall A_∞-algebras over a ring R, their bar construction, Kadeishvili's theorem on minimal models and formality results of Kaledin-Lunts. Then we review definitions and standard results on A_∞-modules. The final paragraphs of the section contain Hochschild cohomology, obstruction theory and our results on formality of A_∞-modules. We furthermore extend many of Lunts' applications of the cohomological formality criterion to A_∞-modules.

2.1 A_∞*-algebras*

Definition 2.1.1. Let $n \in \mathbf{N} \cup \{\infty\}$. An A_n-algebra is a graded R-module A equipped with a family of R-linear morphisms

$$m_i : A^{\otimes i} \to A$$

of degree $2-i$ for $1 \leq i \leq n$ such that for all $m \leq n$ we have

$$\sum_{j+k+l=m} (-1)^{jk+l} m_{j+1+l}(\mathrm{id}^{\otimes j} \otimes m_k \otimes \mathrm{id}^{\otimes l}) = 0. \qquad (*_m)$$

Remark 2.1.2.

1. If $m_i = 0$ for all $i \neq 2$, then A is a graded algebra.
2. If $m_i = 0$ for all $i \neq 1, 2$, A is a differential graded algebra.

Example 2.1.3. Let A be an A_∞-algebra. Then the first relation is

$$(*_1) \quad m_1 m_1 = 0,$$

i.e. m_1 is a differential. The second relation is

$$(*_2) \quad m_1 m_2 = m_2(m_1 \otimes \mathrm{id} + \mathrm{id} \otimes m_1),$$

meaning that m_1 is a derivation for the multiplication m_2. The third equation shows m_2 is associative up to the homotopy m_3:

$$(*_3) \quad m_2(m_2 \otimes \mathrm{id} - \mathrm{id} \otimes m_2) = m_1 m_3 + m_3(m_1 \otimes \mathrm{id}^{\otimes 2} + \mathrm{id} \otimes m_1 \otimes \mathrm{id} + \mathrm{id}^{\otimes 2} \otimes m_1).$$

In particular, any minimal A_n-algebra is associative. The cohomology of an A_n-algebra is a graded associative algebra.

Definition 2.1.4. Let $n \in \mathbf{N} \cup \{\infty\}$. Let A, B be A_n-algebras over R. An A_n-morphism $f : A \to B$ between A_n-algebras is a family of R-linear morphisms

$$f_i : A^{\otimes i} \to B$$

of degree $1 - i$ for $1 \leq i \leq n$ such that for all $m \leq n$ we have

$$\sum_{j+k+l=m} (-1)^{jk+l} f_{j+1+l}(\mathrm{id}^{\otimes j} \otimes m_k \otimes \mathrm{id}^{\otimes l}) = \sum_{i_1+\cdots+i_r=m} (-1)^s m_r(f_{i_1} \otimes \cdots \otimes f_{i_r}), \qquad (**_m)$$

where we set

$$s = \sum_{2 \le u \le r} \Big((1 - i_u) \sum_{1 \le v \le u} i_v \Big).$$

The composition of $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{B} \to \mathrm{C}$ is defined by

$$(g \circ f)_n = \sum_r \sum_{i_1 + \cdots + i_r = n} (-1)^s g_r(f_{i_1} \otimes \cdots \otimes f_{i_r}).$$

Example 2.1.5. Let $f : \mathrm{A} \to \mathrm{B}$ be an A_∞-morphism. Then

$$(**_1) \quad f_1 m_1 = m_1 f_1,$$

that is, f_1 is a morphism of complexes, i.e. A_1-morphisms are just morphisms of complexes. The second relation is

$$(**_2) \quad f_1 m_2 = m_2(f_1 \otimes f_1) + m_1 f_2 + f_2(m_1 \otimes \mathrm{id} + \mathrm{id} \otimes m_1),$$

measuring the compatibility of f_1 with the multiplications of A and B.

Remark 2.1.6. We denote the category of A_n-algebras and A_n-morphisms by Alg_n and the category of A_∞-algebras by Alg_∞. The category of differential graded algebras is a non-full subcategory of Alg_∞.

Definition 2.1.7. Let n be a positive integer or ∞.

1. A morphism

$$f = (f_1, f_2, \cdots, f_n) : \mathrm{A} \to \mathrm{B}$$

of A_n-algebras is a quasi-isomorphism if f_1 is a quasi-isomorphism of the underlying complexes.

2. A and B are said to be quasi-isomorphic if there exist A_n-algebras $C_1, \cdots C_m$ and quasi-isomorphisms $A \leftarrow C_1 \rightarrow \cdots \leftarrow C_m \rightarrow B$.

Definition 2.1.8. Let n be a positive integer or ∞.

1. An A_n-algebra A is called minimal if $m_1 = 0$.

2. A minimal model for A is a minimal A_n-algebra B equipped with an A_n quasi-isomorphism $B \rightarrow A$.

Theorem 2.1.9. *(Kadeishvili [Kad80]) Let* A *be an* A_∞*-algebra over* R *such that* HA *is a projective* R*-module. For any choice of a quasi-isomorphism* $f_1 : HA \rightarrow A$ *of complexes of* R*-modules, there exists a minimal* A_∞*-structure on* HA*, with* m_2^{HA} *being induced by* m_2*, and an* A_∞*-quasi-isomorphism* $f : HA \rightarrow A$ *lifting* f_1*.*

Definition 2.1.10. Let A be an A_∞-algebra.

1. A is called A_n-formal if it is A_n-quasi-isomorphic to the A_n-algebra (HA, m^{HA}), where m_2^{HA} induced by m_2 and $m_i^{HA} = 0$ for $i \neq 2$.

2. A is called formal if it is A_∞-quasi-isomorphic to the graded associative algebra HA, viewed as an A_∞-algebra.

We have the following two important results due to Kaledin and Lunts.

Theorem 2.1.11. *(Lunts [Lun10]) Let* A *be a minimal* A_∞*-algebra over* R *which is projective as an* R*-module. Then* A *is formal iff it is* A_n*-formal for all* n*.*

Furthermore, Kaledin, in [Kal07], shows that A_n-formality is measured by a cohomology class, called the Kaledin class, which gives the next result.

Theorem 2.1.12. *(Kaledin-Lunts [Kal07], [Lun10]) Let* R *be an integral domain with field of fractions* $k(\eta)$*. Consider a minimal* A_∞*-algebra* A *over* R *which is a finite projective* R*-module. Assume that the Hochschild cohomology group with compact supports* $\mathrm{HH}^2_c(\mathrm{A}(2))$ *is torsion-free. If* $\mathrm{A}_\eta = k(\eta)\otimes_{\mathrm{R}}\mathrm{A}$ *is formal, then* A *is formal. In particular,* $\mathrm{A}_\mathfrak{p}$ *is formal for all* $\mathfrak{p}\in \mathrm{Spec}\mathrm{R}$*.*

2.2 The bar construction

Let A be a graded R-module endowed with morphisms

$$m_i : \mathrm{A}^{\otimes i} \to \mathrm{A}.$$

For $i \geq 1$ we have a bijection

$$\begin{aligned}\mathrm{Hom}(\mathrm{A}^{\otimes i},\mathrm{A}) &\to \mathrm{Hom}((\mathrm{A}[1])^{\otimes i},\mathrm{A}[1])\\ m_i &\mapsto \mathrm{d}_i = (-1)^{i-1+\deg m_i}[1]\circ m_i\circ([1]^{-1})^{\otimes i},\end{aligned}$$

where $[1]:\mathrm{A}\to\mathrm{A}[1]$ is the canonical degree -1 morphism. Remark that in our case m_i are of degree $2-i$, so the corresponding d_i have degree 1. The morphisms d_i define a

unique morphism

$$\overline{\mathrm{T}}(\mathrm{A}[1]) \to \mathrm{A}[1],$$

which by the universal property of the reduced tensor coalgebra corresponds to a unique degree 1 coderivation

$$\mathrm{d} : \overline{\mathrm{T}}(\mathrm{A}[1]) \to \overline{\mathrm{T}}(\mathrm{A}[1]).$$

Lemma 2.2.1. *The morphisms m_i define an A_∞-algebra structure on* A *iff* d *is a differential, i.e.* $\mathrm{d}^2 = 0$.

Definition 2.2.2. The bar construction of an A_∞-algebra A is the differential graded coalgebra $\mathcal{B}(\mathrm{A}) := (\overline{\mathrm{T}}(\mathrm{A}[1]), \mathrm{d})$.

Let A, B be graded objects. For $i \geq 1$ we have a bijection

$$\begin{aligned} \mathrm{Hom}(\mathrm{A}^{\otimes i}, \mathrm{B}) &\to \mathrm{Hom}((\mathrm{A}[1])^{\otimes i}, \mathrm{B}[1]) \\ f_i &\mapsto \mathrm{F}_i = (-1)^{i-1+\deg f_i}[1]_{\mathrm{B}} \circ f_i \circ ([1]_{\mathrm{A}}^{-1})^{\otimes i}. \end{aligned}$$

If f_i are of degree $1-i$, the maps F_i define a degree 0 morphism of coalgebras

$$\mathrm{F} : \mathcal{B}(\mathrm{A}) \to \mathcal{B}(\mathrm{B}).$$

Lemma 2.2.3. *Let* A, B *be A_∞-algebras and suppose given $f_i \in \mathrm{Hom}(\mathrm{A}^{\otimes i}, \mathrm{B})$ of degree $1-i$. The morphisms f_i define an A_∞ morphism iff F is compatible with the differentials,*

i.e. we have a bijection

$$\mathrm{Hom}_{\mathrm{Alg}_\infty}(\mathrm{A},\mathrm{B}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{coalg}}(\mathcal{B}(\mathrm{A}),\mathcal{B}(\mathrm{B})).$$

2.3 A_∞*-modules*

Definition 2.3.1. Let $n \in \mathbf{N} \cup \{\infty\}$ and let A be an A_n-algebra over R. An A_n-module over A is a graded R-module M together with a family of morphisms

$$m_i^{\mathrm{M}} : \mathrm{M} \otimes \mathrm{A}^{\otimes i-1} \to \mathrm{M}$$

of degree $2-i$ for all $1 \leq i \leq n$ such that for all $1 \leq m \leq n$

$$\sum_{j+k+l=m, j\geq 1} (-1)^{jk+l} m^{\mathrm{M}}_{j+1+l}(\mathrm{id}^{\otimes j} \otimes m_k \otimes \mathrm{id}^{\otimes l}) + \sum_{k+l=m} (-1)^l m^{\mathrm{M}}_{1+l}(m_k^{\mathrm{M}} \otimes \mathrm{id}^{\otimes l}) = 0. \quad (*'_m)$$

Definition 2.3.2. Let $n \in \mathbf{N} \cup \{\infty\}$ and let A be an A_n-algebra and suppose M, N are A_n-modules over A. A morphism of A_n-modules is a family of R-linear morphisms

$$f_i : \mathrm{M} \otimes \mathrm{A}^{\otimes i-1} \to \mathrm{N}$$

of degree $1-i$ for $1 \leq i \leq n$ such that for all $1 \leq m \leq n$

$$\sum_{j+k+l=m, k\geq 1} (-1)^{jk+l} f_{j+1+l}(\mathrm{id}^{\otimes j} \otimes m_k \otimes \mathrm{id}^{\otimes l}) = \sum_{r+s=m, r\geq 1, s\geq 0} m_{s+1}(f_r \otimes \mathrm{id}^{\otimes s}). \quad (**'_m)$$

The composition of $f : \mathrm{L} \to \mathrm{M}$ and $g : \mathrm{M} \to \mathrm{N}$ is defined by

$$(g \circ f)_n = \sum_{k+l=n} g_{l+1}(f_k \otimes 1^{\otimes l}).$$

Remark 2.3.3. Let A be an A_∞-algebra and M be an A_∞-module over A. Then

1. $(\mathrm{M}, m_1^{\mathrm{M}})$ is a complex;

2. If $f : \mathrm{M} \to \mathrm{N}$ is a morphism of A_∞-modules, f_1 is a morphism of complexes

$$f_1 : (\mathrm{M}, m_1^{\mathrm{M}}) \to (\mathrm{N}, m_1^{\mathrm{N}}).$$

Example 2.3.4. If A is an A_∞-algebra, then the morphisms $m_i : \mathrm{A}^{\otimes i} \to \mathrm{A}$ define an A_∞-module structure on A over A.

Remark 2.3.5. If A is a differential graded algebra regarded as A_∞-algebra, then any differential graded module over A is canonically A_∞-modules and the category of differential graded modules over A is a non-full subcategory of the category of A_∞-modules over A.

Definition 2.3.6. Let n be a positive integer or ∞. Let A be an A_n-algebra and suppose M and N are A_n-modules over A.

1. An A_n-morphism $f = (f_1, f_2, \cdots, f_n) : \mathrm{M} \to \mathrm{N}$ is a quasi-isomorphism if f_1 is a quasi-isomorphism of complexes.

2. M and N are said to be quasi-isomorphic if there exist A_n-modules $\mathrm{M}_1, \cdots \mathrm{M}_m$ and

quasi-isomorphisms $\mathrm{M} \leftarrow \mathrm{M}_1 \rightarrow \cdots \leftarrow \mathrm{M}_m \rightarrow \mathrm{N}$.

Definition 2.3.7. Let n be a positive integer or ∞. Let A be an A_n-algebra and consider an A_n-module M over A.

1. M is called minimal if $m_1^{\mathrm{M}} = 0$.

2. A minimal model for M is a pair $(\mathrm{A}', \mathrm{M}')$, consisting of a minimal A_n-algebra A' and a minimal A_n-module M' over it, together with quasi-isomorphisms $f : \mathrm{A}' \rightarrow \mathrm{A}$ and $g : \mathrm{M}' \rightarrow f^*\mathrm{M}$, where $f^*\mathrm{M}$ is the restriction of M along f.

Remark 2.3.8. We shall say that (f, g) is a morphism of pairs $(\mathrm{A}', \mathrm{M}') \rightarrow (\mathrm{A}, \mathrm{M})$.

Theorem 2.3.9. *(Kadeishvili, [Kad80]) Let* A *be an* A_∞*-algebra and consider an* A_∞*-module* M *over* A*. Assume that* HA *and* HM *are projective* R*-modules. Then, for any choice of quasi-isomorphisms*

$$f_1 : \mathrm{HA} \rightarrow \mathrm{A}, g_1 : \mathrm{HM} \rightarrow \mathrm{M}$$

of complexes of R*-modules, inducing the identity in cohomology, there exists a minimal* A_∞*-module structure on* HM *over the* A_∞*-algebra* HA*, with* m_2^{HM} *induced by* m_2^{M}*, such that there exists a quasi-isomorphism of pairs*

$$(f, g) : (\mathrm{HA}, \mathrm{HM}) \rightarrow (\mathrm{A}, \mathrm{M}),$$

lifting (f_1, g_1)*, i.e. a minimal model for* M*. It is unique up to* A_∞*-isomorphism.*

2.4 The bar construction for A_∞-modules

Let A and M be graded R-modules. For $i \geq 1$ we have a bijection

$$\begin{aligned} \mathrm{Hom}(M \otimes A^{\otimes i-1}, M) &\to \mathrm{Hom}(M[1] \otimes (A[1])^{\otimes i-1}, M[1]) \\ m_i^M &\mapsto d_i^M = (-1)^{i-1+\deg m_i^M} s \circ m_i \circ (s^{-1})^{\otimes i}. \end{aligned}$$

Let A be an A_∞-algebra and let $(\mathcal{B}(A))^+$ be its coaugmented bar construction. Then the d_i^M define a unique comodule coderivation

$$d^M : M[1] \otimes (\mathcal{B}(A))^+ \to M[1] \otimes (\mathcal{B}(A))^+.$$

Lemma 2.4.1. *The morphisms m_i^M define an A_∞-module structure of M over A iff the coderivation d^M is a differential.*

Let A, M, N be graded R-modules. For all $i \geq 1$ we have a bijection

$$\begin{aligned} \mathrm{Hom}(M \otimes A^{\otimes i-1}, N) &\to \mathrm{Hom}(M[1] \otimes (A[1])^{\otimes i-1}, N[1]) \\ f_i &\mapsto F_i = (-1)^{i-1+\deg f_i} s_B \circ f_i \circ (s_A^{-1})^{\otimes i}. \end{aligned}$$

and, if A is an A_∞-algebra, the F_i induce a morphism of $(\mathcal{B}(A))^+$-comodules

$$F : M \otimes (\mathcal{B}(A))^+ \to N \otimes (\mathcal{B}(A))^+.$$

Lemma 2.4.2. *Let* A *be an* A_∞*-algebra and suppose given graded objects* M, N*. For all* $i \geq 1$ *there is a bijection*

$$\mathrm{Hom}^{1-i+n}(\mathrm{M} \otimes \mathrm{A}^{\otimes i-1}, \mathrm{N}) \xrightarrow{\sim} \mathrm{Hom}^{n}_{\mathcal{B}(\mathrm{A})}(\mathrm{M} \otimes (\mathcal{B}(\mathrm{A}))^{+}, \mathrm{N} \otimes (\mathcal{B}(\mathrm{A}))^{+}).$$

Furthermore, if M *and* N *are* A_∞*-modules, then we get an induced bijection between morphisms* $\mathrm{M} \to \mathrm{N}$ *of* A_∞*-modules and degree* 0 *morphisms of differential graded* $(\mathcal{B}(\mathrm{A}))^{+}$*-comodules.*

2.5 Differential graded pairs

Definition 2.5.1. Let A, B be differential graded algebras over R and let M (resp. N) be a differential graded module over A (resp. B).

1. If $f : \mathrm{A} \to \mathrm{B}$ is a morphism of differential graded algebras and $g : \mathrm{M} \to f^{*}\mathrm{N}$ is a morphism of differential graded modules, where f^{*} denotes restriction along f, we say that the pair $(f, g) : (\mathrm{A}, \mathrm{M}) \to (\mathrm{B}, \mathrm{N})$ is a differential graded morphism of pairs.

2. The pairs (A, M) and (B, N) are differential graded quasi-isomorphic which we denote $(\mathrm{A}, \mathrm{M}) \simeq (\mathrm{B}, \mathrm{N})$, if there exist pairs

   $$(\mathrm{A}_1, \mathrm{M}_1), \cdots, (\mathrm{A}_m, \mathrm{M}_m)$$

   and quasi-isomorphisms of pairs

   $$(\mathrm{A}, \mathrm{M}) \leftarrow (\mathrm{A}_1, \mathrm{M}_1) \to \cdots \leftarrow (\mathrm{A}_m, \mathrm{M}_m) \to (\mathrm{B}, \mathrm{N}).$$

3. If A is a formal differential graded algebra, we say that M is differential graded formal if the pairs (A, M) and (HA, HM) are differential graded quasi-isomorphic.

Let A, B be differential graded algebras over R such that HA and HB are projective R-modules, equpped with their minimal A_∞-algebra structures. Given two differential graded modules M and N over A and B, respectively, assume that HM and HN are projective over R, so can be given minimal A_∞-module structures over HA and HB.

Proposition 2.5.2. *The pairs* (A, M) *and* (B, N) *are dg quasi-isomorphic if and only if the pairs* (HA, HM) *and* (HB, HN) *are* A_∞*-quasi-isomorphic.*

2.6 Hochschild cohomology

Let A be a graded algebra. We are going to define Hochschild cohomology for a graded module M over A.

Let $C^{p,q}(A, M) = \mathrm{Hom}^q(M \otimes A^{\otimes p}, M)$. The module structure of M over A is a graded morphism of degree 0, denoted by $m_2^M : M \otimes A \to M$. We can endow the modules $C^{p,q}(A, M)$ with a differential, called the Hochschild differential:

$$\begin{gathered} d : C^{p,q}(A, M) \to C^{p+1,q}(A, M) \\ f \mapsto \sum (-1)^l f(\mathrm{id}^{\otimes j} \otimes m_2^A \otimes \mathrm{id}^{\otimes l}) - m_2^M(f \otimes \mathrm{id}) + (-1)^p f(m_2^M \otimes \mathrm{id}^p). \end{gathered}$$

A calculation shows that $d^2 = 0$, so we indeed have a differential, the associated complex is called the Hochschild complex.

Definition 2.6.1. The Hochschild cohomology group $HH^{p,q}(A, M)$ of a graded mod-

ule M over a graded algebra A is the p^{th} cohomology group of the Hochschild complex $(C^{*,q}(A, M), d)$.

Example 2.6.2. Suppose $f : M[\epsilon] \to M[\epsilon]$ is an $A[\epsilon]$-automorphism lifting the identity. Then, writing

$$f = f^0 + f^1\epsilon,$$

we have by assumption $f^0 = \mathrm{id}_M$, and A-linearity implies

$$f^1(ma) = f^1(m)a$$

which is to say that f^1 is a $(0,0)$-cocycle.

Definition 2.6.3. An infinitesimal A_∞-deformation of an A-module M is an A_∞-module structure on $M[\epsilon]$ over $A[\epsilon]$ extending the A-module structure on M.

Example 2.6.4. The $(1,0)$-cocycles are precisely the $A[\epsilon]$-module structures on $M[\epsilon]$, i.e. they correspond to infinitesimal deformations. Indeed, if

$$m : M[\epsilon] \otimes A[\epsilon] \to M[\epsilon]$$

is the multiplication, we decompose it as

$$m = m^0 + m^1\epsilon,$$

where m^0 is the A-module multiplication on M. As m defines a module structure, we get

$$m^1(m,a)a' + m^1(ma,a') = m^1(m,aa'),$$

i.e. m^1 is a $(1,0)$-cocycle. Notice that there is a canonical $\mathrm{A}[\epsilon]$-module structure on $\mathrm{M}[\epsilon]$.

Definition 2.6.5. We call an infinitesimal A_∞-deformation of M trivial if it is quasi-isomorphic to this canonical one.

Example 2.6.6. We note that $(1,0)$-coboundaries correspond to trivial deformations by a similar calculation, hence infinitesimal deformations of M are classified by $\mathrm{HH}^{1,0}(\mathrm{A},\mathrm{M})$. More generally, assume we are given an infinitesimal A_∞-deformation of M. Write

$$m = m^0 + m^1\epsilon,$$

where m^0 is the A-module structure on M and $m^1 = (m_2^1, m_3^1, \cdots)$. Notice the relations ($*'_m$) are homogeneous in ϵ, so as $\epsilon^2 = 0$ we see that each m_i^1 is a cocycle in the Hochschild complex of M and, conversely, any collection of cocycles satisfies ($*'_m$) for all m. Similarly, we see coboundaries correspond to trivial A_∞-deformations, hence

$$\prod_{n\geq 1} \mathrm{HH}^{n,1-n}(\mathrm{A},\mathrm{M})$$

classifies infinitesimal A_∞-deformations.

Remark 2.6.7. Remark that here we assume the deformation parameter is in degree 0 with respect to the internal grading of our objects. If we consider *derived* deformations, i.e.

allow the deformation parameter to have a non-zero degree with respect to the internal grading, then we get different cohomology group, e.g. if it is in degree 1, the cohomology group classifying derived infinitesimal deformations becomes $\mathrm{HH}^{1,-1}(\mathrm{A},\mathrm{M})$.

Remark 2.6.8. Given two graded modules M and N over A, let $\mathrm{C}^{p,q}(\mathrm{A},\mathrm{M},\mathrm{N}) = \mathrm{Hom}^q(\mathrm{M}\otimes \mathrm{A}^{\otimes p},\mathrm{N})$. It carries a Hochschild differential and we define the Hochschild cohomology of the pair (M, N), denoted $\mathrm{HH}^{p,q}(\mathrm{A},\mathrm{M},\mathrm{N})$, to be the cohomology of the resulting complex.

Proposition 2.6.9. *Let* A *be a graded algebra over* R *and let* M *be a graded module over* A. *Assume that* A *and* M *are finite projective over* R. *Suppose that* $\mathrm{R}\to\mathrm{Q}$ *is a morphism of commutative rings. Write* $\mathrm{A}_\mathrm{Q} := \mathrm{A}\otimes_\mathrm{R}\mathrm{Q}$ *and similarly for* M_Q. *Then we have*

1. $\mathrm{C}^{p,q}(\mathrm{A}_\mathrm{Q},\mathrm{M}_\mathrm{Q}) = \mathrm{C}^{p,q}(\mathrm{A},\mathrm{M})\otimes_\mathrm{R}\mathrm{Q}$.

2. *Assuming* Q *flat over* R, $\mathrm{HH}^{p,q}(\mathrm{A}_\mathrm{Q},\mathrm{M}_\mathrm{Q}) = \mathrm{HH}^{p,q}(\mathrm{A},\mathrm{M})\otimes_\mathrm{R}\mathrm{Q}$.

Proof. Clearly it is enough to prove the first assertion. Since A and M are finite projective over R, so are $\mathrm{M}\otimes\mathrm{A}^{\otimes p}$ for all p, hence we have isomorphisms of Q-modules

$$\begin{aligned}\mathrm{Hom}_\mathrm{Q}(\mathrm{M}_\mathrm{Q}\otimes_\mathrm{Q}\mathrm{A}_Q^{\otimes p},\mathrm{M}) &\cong \mathrm{Hom}_\mathrm{R}(\mathrm{M}\otimes_\mathrm{R}\mathrm{A}^{\otimes p},\mathrm{M}_\mathrm{Q})\\ &\cong \mathrm{Hom}_\mathrm{R}(\mathrm{M}\otimes_\mathrm{R}\mathrm{A}^{\otimes p},\mathrm{M})\otimes_\mathrm{R}\mathrm{Q}.\end{aligned}$$

□

Proposition 2.6.10. *Suppose that* R *is Noetherian. Consider a graded algebra* A *over* R *and a graded module* M *over* A. *Assume that* A, M *are finite projective and that for all* $p,q\in\mathbf{Z}$ *the* R*-module* $\mathrm{HH}^{p,q}(\mathrm{A},\mathrm{M})$ *is projective. Then, for any morphism of commutative*

rings $\mathrm{R} \to \mathrm{Q}$*, we have*

$$\mathrm{HH}^{p,q}(\mathrm{A}_{\mathrm{Q}}, \mathrm{M}_{\mathrm{Q}}) = \mathrm{HH}^{p,q}(\mathrm{A}, \mathrm{M}) \otimes_{\mathrm{R}} \mathrm{Q}.$$

Proof. This follows immediately from the next lemma. □

Lemma 2.6.11. *Let* R *be Noetherian and assume* (K, d) *is a bounded below complex of finite projective* R*-modules such that each* R*-module* $\mathrm{H}^p(\mathrm{K})$ *is projective. Then, for each* p*,* $\mathrm{Im}(\mathrm{d}^p)$ *is projective over* R *and hence* K *is homotopy equivalent to its cohomology* $\oplus_p \mathrm{H}^p(\mathrm{K})[-p]$*.*

2.7 Obstruction theory

In this paragraph we formulate obstruction theory for A_n-modules. Our goal is to apply it to a problem where we are extending an A_n-morphism f and we only care for keeping f_1 fixed. In particular, a theory for obstructions where the last component is allowed to vary is good enough for us and that's what we develop here. There are more general versions where one allows the last r components to vary for some $r < n$. Compared to the case where we keep all components fixed, our approach has the advantage that the corresponding obstructions are cohomology classes rather than equations in the space Hochschild cochains.

We show how the Hochschild cohomology defined in the previous section controls the obstructions to extending A_n-modules to A_{n+1}-modules by allowing the last multiplication to vary as well as A_n-morphisms to A_{n+1}-morphisms, varying the last component. We focus on modules, but analogous versions for algebras can also be formulated.

Proposition 2.7.1. *Let* A *be a minimal* A_n*-algebra. Let* M *be a graded* R*-module and suppose*

$$m_i^{\mathrm{M}} : \mathrm{M} \otimes \mathrm{A}^{i-1} \to \mathrm{M}, \quad 2 \leq i \leq n+1,$$

are graded morphisms of degree $2-i$*. Assume that for* $1 \leq i \leq n$ *the* m_i^{M} *define an* A_n*-structure on* M*. Then the subexpression of* $(*'_{n+1})$ *given by*

$$\sum_{j+1+l,k\neq 1,2} (-1)^{jk+l} m_{j+1+l}(\mathrm{id}^{\otimes j} \otimes m_k \otimes \mathrm{id}^{\otimes l})$$

defines a cocycle in $(\mathrm{C}(\mathrm{A}(2),\mathrm{M}(2));\mathrm{d})$ *which we denote by* $\mathrm{c}(m_3^{\mathrm{M}},\cdots,m_n^{\mathrm{M}})$ *and equation* $(*'_{n+1})$ *becomes*

$$\mathrm{c}(m_3^{\mathrm{M}},\cdots,m_{n-1}^{\mathrm{M}}) + \mathrm{d}(m_n^{\mathrm{M}}) = 0.$$

Proposition 2.7.2. *Let* A *be a minimal* A_∞*-algebra. Let* M *and* N *be two minimal* A_∞*-modules over* A*. Suppose given*

$$f_i : \mathrm{M} \otimes \mathrm{A}^{\otimes i-1} \to \mathrm{N}, \quad 1 \leq i \leq n+1,$$

of degree $1-i$ *such that the morphisms* f_i*, for* $1 \leq i \leq n$*, define an* A_n*-morphism. The subexpression of* $(**'_{n+1})$

$$\sum_{k\neq 1,2} (-1)^{jk+l} f_{j+1+l}(\mathrm{id}^{\otimes j} \otimes m_k \otimes \mathrm{id}^{\otimes l}) - \sum_{s\neq 0,1} m_{s+1}(f_r \otimes \mathrm{id}^{\otimes s})$$

defines a cocycle in $(\mathrm{C}(\mathrm{A}(2),\mathrm{M}(2),\mathrm{N}(2));\mathrm{d})$*, denoted by* $\mathrm{c}(f_1,\cdots,f_{n-1})$*. Then equation*

$(**'_{n+1})$ *becomes*

$$c(f_1, \cdots, f_{n-1}) + d(f_n) = 0.$$

2.8 Formality of A_∞ *modules*

Definition 2.8.1. Let A be an A_∞-algebra. Let $n \in \mathbf{N} \cup \{\infty\}$ and assume that A is A_n-formal. We say that an A_∞-module M over A is A_n-formal if there exists an A_n-quasi-isomorphism of pairs $(HA, HM) \to (A, M)$, where HA (resp. HM) is the ordinary graded associative algebra (resp. module).

Remark 2.8.2.

1. In other words M is A_n-formal if it admits a minimal A_n-model with vanishing higher multiplications.

2. Notice that our definition of A_n-formality for a module M over A assumes that the A_∞-algebra A is A_n-formal.

3. When $n = \infty$, we drop the A_∞ and simply say that M is formal.

4. By Proposition 2.5.2, under the usual projectivity assumptions, a differential graded module M over a formal differential graded algebra A is differential graded formal iff its minimal model HM is formal as an A_∞-module over the (formal) A_∞-algebra HA.

Recall that if A is a minimal A_∞-algebra, then A(2) stands for the underlying graded associative algebra, so if A is just a graded associative algebra, then A = A(2).

Similarly, let M be a minimal A_∞-module over a graded algebra A. Then M(2) denotes the A_∞-module on the graded space M with structure morphisms given by

$$m_2^{M(2)} = m_2^M \text{ and } m_i^{M(2)} = 0 \text{ for } i \neq 2,$$

i.e. M(2) is the underlying graded associative A-module, considered as an A_∞-module - this is possible precisely because M is minimal and A is just a graded associative algebra, so has no higher multiplications, and the A_∞-relations for M(2) reduce to associativity. In this notation, M is a formal A_∞-module if it is quasi-isomorphic to M(2).

Remark 2.8.3. Let M be a minimal A_∞-module over a graded algebra A. If M is formal, there exists a quasi-isomorphism $f : M \to M(2)$ lifting the identity on the underlying complexes.

We are ready to state our main results on the formality of A_∞-modules.

Proposition 2.8.4. *Let* A *be a graded algebra. An* A_∞*-module* M *over* A *is formal if and only if it is* A_n*-formal for all* $n \in \mathbf{N}$.

Proposition 2.8.5. *Let* R *be an integral domain with field of fractions* $k(\eta)$. *Let* A *be a graded* R*-algebra and let* M *be a minimal* A_∞*-module over* A *such that both* A *and* M *are finite projective as* R*-modules. Assume that* $HH^{n,1-n}(A, M(2))$ *is torsion-free for all* n. *If* $M_\eta = k(\eta) \otimes_R M$ *is formal, then* M *is formal and there exists a quasi-isomorphism* $f : M \to M(2)$ *lifting the identity on the underlying complexes. In particular, the fibres* $M_\mathfrak{p}$ *are formal for all* $\mathfrak{p} \in \mathrm{Spec}(R)$.

Proof of Propositions 2.8.4 and 2.8.5. We are going to construct the quasi-isomorphism f inductively. Since M and M(2) have the same underlying A_2-modules, we set $f_1 = \mathrm{id}$, and for any f_2, we have a quasi-isomorphism $(\mathrm{id}, f_2) : M \to M(2)$ of A_2-modules over A. Then $[c(\mathrm{id})] = [m_3]$ is the Massey product which vanishes generically since M_η is formal. Since $HH^{2,-1}(A, M(2))$ is torsion-free, it follows that $[c(\mathrm{id})]$ vanishes everywhere. Hence, we can choose f_2 such that

$$c(\mathrm{id}) + d(f_2) = 0.$$

There exists an A_∞-module structure $M^{(2)} = (M, m^{(2)})$ on the graded R-module M such that

$$m_2^{(2)} = m_2^M, m_3^{(2)} = 0 \text{ and } \tilde{f_2} := (\mathrm{id}, f_2, 0, \cdots) : M \to M^{(2)}$$

is a quasi-isomorphism of A_∞-modules. Indeed, $M^{(2)}$ can be constructed inductively as follows: we treat the multiplications $m^{(2)}$ of $M^{(2)}$ as variables and require that the map $(\mathrm{id}, f_2, 0, \cdots)$ defines a quasi-isomorphism of A_∞-modules. The corresponding equations for $m^{(2)}$ can be uniquely solved as they are all of the form

$$\sum_{j+k+l=n, k\geq 2} (-1)^{jk+l} f_{j+1+l}(\mathrm{id}^{\otimes j} \otimes m_k \otimes \mathrm{id}^{\otimes l}) = m_n^{(2)} + m_{n-1}^{(2)}(f_2 \otimes \mathrm{id}^{\otimes n-2}),$$

where $j + 1 + l = 1$ or 2. For example, $n = 2$ implies that $m_2^{(2)} = m_2$ and $n = 3$ is just the equation $c(\mathrm{id}) + d(f_2) = m_3^{(2)}$, i.e. $m_3^{(2)} = 0$.

Assume by induction that we have constructed $f_2, \cdots, f_n$ and $\mathrm{M}^{(2)}, \cdots, \mathrm{M}^{(n)}$ with

$$m_2^{(i)} = m_2^{\mathrm{M}} \text{ and } m_j^{(i)} = 0 \text{ for all } 3 \leq j \leq i+1$$

such that, for all $2 \leq i \leq n$, the maps

$$\tilde{f_i} \coloneqq (\mathrm{id}, 0, \cdots, 0, f_i, 0, \cdots) : \mathrm{M}^{(i-1)} \to \mathrm{M}^{(i)}$$

are quasi-isomorphisms of A_∞-modules.

In order to construct the pair $(f_{n+1}, \mathrm{M}^{(n+1)})$, it suffices to show that there exists f_{n+1} such that

$$(\mathrm{id}, 0, \cdots, 0, f_{n+1}, 0) : \mathrm{M}^{(n)} \to \mathrm{M}^{(n)}(2)$$

is an A_{n+2}-morphism, i.e. A_{n+2}-formality of $\mathrm{M}^{(n)}$. We know that

$$(\mathrm{id}, 0, \cdots, 0, f_{n+1}) : \mathrm{M}^{(n)} \to \mathrm{M}^{(n)}(2)$$

is an A_{n+1}-morphism for any f_{n+1}. The goal is to show f_{n+1} can be chosen so that in fact any f_{n+2} would give an A_{n+2}-morphism $\mathrm{M}^{(n)} \to \mathrm{M}^{(n)}(2)$.

By Proposition 2.7.2 it suffices that the cohomology class $[\mathrm{c}(\mathrm{id}, 0, \cdots, 0)]$ vanish. Notice that the function

$$(g_2, \cdots, g_n) \mapsto \mathrm{c}(\mathrm{id}, g_2, \cdots, g_n)$$

is constant precisely because the corresponding higher multiplications in $\mathrm{M}^{(n)}$ vanish, and the same applies for the localised version $\mathrm{M}_\eta^{(n)}$. Since M_η, thus also $\mathrm{M}_\eta^{(n)}$, is formal,

we see that

$$[c(\mathrm{id}, 0, \cdots, 0)]_\eta = 0.$$

Furthermore, $\mathrm{HH}^{n+1,-n}(A, M(2))$ is torsion-free, so we obtain that $[c(\mathrm{id}, 0, \cdots, 0)] = 0$, hence there exists f_{n+1} such that

$$(\mathrm{id}, 0, \cdots, 0, f_{n+1}, f_{n+2}) : M^{(n)} \to M^{(n)}(2)$$

is an A_{n+2}-morphism for any choice of f_{n+2}.

Then, as above, the standard construction gives an A_∞-module $M^{(n+1)}$ such that

$$m_2^{(n+1)} = m_2^M \text{ and } m_j^{(n+1)} = 0 \text{ for all } 3 \leq j \leq n+2$$

and the map

$$\tilde{f}_{n+1} \coloneqq (\mathrm{id}, 0, \cdots, 0, f_{n+1}, 0, \cdots) : M^{(n)} \to M^{(n+1)}$$

is a quasi-isomorphism of A_∞-modules.

The infinite composition

$$f \coloneqq \cdots \circ \tilde{f}_n \circ \cdots \circ \tilde{f}_2 : M \to M(2)$$

defines the required quasi-isomorphism and we note that it is well-defined because composition with $\tilde{f}_n$ leaves the components in weights $i < n$ fixed. □

We record a few useful corollaries following closely the A_∞-algebra versions of [Lun10].

Corollary 2.8.6. *Suppose that* R *is Noetherian. Let* A *be a graded algebra over* R *and let* M *be a minimal* A_∞*-module over* A*. Suppose that* A *and* M *are finite projective* R*-modules and* $\mathrm{HH}^{p,q}(\mathrm{A},\mathrm{M}(2))$ *is projective for all* $p,q \in \mathbf{Z}$*. Then the set*

$$\mathrm{F}(\mathrm{M}) = \{\mathfrak{p} \in \mathrm{Spec}(\mathrm{R}) \mid \textit{the } \mathrm{A}_\infty\textit{-module } \mathrm{M}_x \textit{ is formal}\}$$

is closed under specialisation.

Proof. Assume that $\mathrm{F}(\mathrm{M})$ is non-empty. Let $\mathfrak{p} \in \mathrm{F}(\mathrm{M})$ and consider its closure

$$\bar{\mathfrak{p}} = \mathrm{Spec}(\mathrm{Q}) \subset \mathrm{Spec}(\mathrm{R}).$$

The ring Q is an integral domain and the base-changes $\mathrm{A}_\mathrm{Q} = \mathrm{A} \otimes_\mathrm{R} \mathrm{Q}$ and $\mathrm{M}_\mathrm{Q} = \mathrm{M} \otimes_\mathrm{R} \mathrm{Q}$ are finite projective over Q. Furthermore, by Proposition 2.6.10 we have

$$\mathrm{HH}^{n,1-n}(\mathrm{A}_\mathrm{Q},\mathrm{M}(2)_\mathrm{Q}) = \mathrm{HH}^{n,1-n}(\mathrm{A},\mathrm{M}(2)) \otimes_\mathrm{R} \mathrm{Q},$$

so $\mathrm{HH}^{n,1-n}(\mathrm{A}_\mathrm{Q},\mathrm{M}(2)_\mathrm{Q})$ is projective over Q, in particular torsion-free. Hence M_Q is formal by Proposition 2.8.5. □

Proposition 2.8.7. *Suppose that* R *is Noetherian and* $\mathrm{I} \subset \mathrm{R}$ *is an ideal such that* $\cap_l \mathrm{I}^l = 0$*. Let* A *be a graded algebra over* R *and let* M *be a minimal* A_∞*-module over* A*. Suppose that* A *and* M *are finite projective* R*-modules and that* $\mathrm{HH}^{p,q}(\mathrm{A},\mathrm{M}(2))$ *is projective for all* $p,q \in \mathbf{Z}$*. If* M/I^l *is a formal* A_∞*-module over the graded algebra* A/I^l *for all* $l \in \mathbf{N}$*, then* M *is formal.*

Proof. Since M is projective over R and minimal, it is A_2-formal. Then, as in the proof of Proposition 2.8.5, to show it is A_3-formal, it is enough to prove that the Massey class $[m_3]$ vanishes in $HH^{2,-1}(A, M(2))$. Using Proposition 2.6.10 we have

$$HH^{p,q}(A/I^l, M(2)/I^l) = HH^{p,q}(A, M(2)) \otimes_R R/I^l \text{ for all } p, q \in \mathbf{Z}.$$

By assumption $[m_3] \otimes 1 = 0$ in $HH^{2,-1}(A/I^l, M(2)/I^l)$ for all $l \geq 1$. There is an exact sequence $0 \to \cap_l I^l \to R \to \prod_l R/I^l$, since $\cap_l I^l = 0$ and $HH^{p,q}(A, M(2))$ is projective over R, we conclude that $[m_3] = 0$. By induction M is A_n-formal for all $n \in \mathbf{N}$, done. □

Proposition 2.8.8. *Suppose that* R *is Noetherian with trivial Jacobson radical* $\mathfrak{J}(R)$. *Let* A *be a graded associative algebra over* R, *and let* M *be a minimal* A_∞*-module over* A. *Suppose that* A *and* M *are finite projective* R*-modules and that* $HH^{p,q}(A, M(2))$ *is projective for all* $p, q \in \mathbf{Z}$. *If* $M_\mathfrak{m}$ *is formal for all closed points* $\mathfrak{m} \in \mathrm{Spec}(R)$, *then* M *is formal. In particular,* $M_\mathfrak{p}$ *is formal for all* $\mathfrak{p} \in \mathrm{Spec}(R)$.

Proof. As in the previous proposition, we prove that M is A_n-formal for all $n \in \mathbf{N}$. The proof is exactly the same using the exact sequence $0 \to \mathfrak{J}(R) \to R \to \prod_\mathfrak{m} R/\mathfrak{m}$, Proposition 2.6.10 and projectivity of $HH^{p,q}(A, M(2))$. □

3 Perverse sheaves and DQ modules

In this section we shall briefly review some results on DQ-modules and perverse sheaves on Lagrangian intersections we need for applications. This is mainly to fix notation, in particular we refer the reader to the original papers for proofs. The material on perverse sheaves on Lagrangian intersections and, more generally, on d-critical loci is due to Joyce et al. and we refer to [BBD+15], [Joy15]. A general reference for DQ-modules, which we also closely follow, is Kashiwara and Schapira [KS12].

3.1 Perverse sheaves on d*-critical loci*

The local setting, we are interested in, is as follows: for a complex manifold X and a function f on X, we consider the intersection $\mathrm{crit} f := \mathrm{X} \cap \Gamma_{\mathrm{d}f}$, where $\Gamma_{\mathrm{d}f} \subset \Omega_{\mathrm{X}}$ is the Lagrangian, given by the graph of $\mathrm{d}f \in \Gamma(\mathrm{X}, \Omega_{\mathrm{X}})$. In particular, if X is the cotangent bundle of a complex manifold M, then for any function f on M, we can consider the Lagrangian $\mathrm{L} = \Gamma_{\mathrm{d}f}$, so $\mathrm{crit} f = \mathrm{L} \cap \mathrm{M}$, i.e. in this case the intersection of the Lagrangians can be written as the critical locus of f. It turns out that this remains true locally in the general case where X is a holomorphic symplectic and L, M are two Lagrangians.

On $\mathrm{crit} f \subset X$, we have a naturally defined perverse sheaf of vanishing cycles $\mathscr{P}_{\mathrm{X},f}$ which is the image of $\mathbf{C}_{\mathrm{X}}[\dim \mathrm{X}]$ under the vanishing cycles functor over the critical values of f.

We are going to consider the global versions of the constructions discussed above. Let $\mathrm{X}/\mathbf{C}$ be a scheme and suppose we are given an embedding of X into a smooth scheme S

with ideal sheaf $\mathscr{I}$, then we define the complex of derived 1-jets

$$\mathbf{J}^1_X := \left[\mathscr{O}_S/\mathscr{I}^2 \to \Omega_S|_X\right],$$

in degrees -1 and 0. It can be shown that it is independent of the embedding and is naturally quasi-isomorphic to the cone of the de Rham differential $\mathrm{d} : \mathscr{O}_X \to \mathbf{L}_X$, where we let

$$\mathbf{L}_X := \left[\mathscr{I}/\mathscr{I}^2 \to \Omega_S|_X\right]$$

be the truncated cotangent complex. We are interested in the sheaf $\mathscr{S}_X = \mathscr{H}^{-1}(\mathbf{J}^1_X)$.

Definition 3.1.1. (Joyce [Joy15]) A structure of a d-critical locus on X is a choice of $s \in \Gamma(X, \mathscr{S}_X)$ such that for any $x \in X$ there exists an open U, containing x, and a closed embedding of U into a smooth S together with a function f on S such that $s|_U = f$ in $\Gamma(U, \mathscr{O}_S/\mathscr{I}^2)$ and $U = \operatorname{crit} f \subset S$. The triple (U, S, f) is a chart for the d-critical locus X.

Let (X, s) be a d-critical locus. Then, given a chart (U, S, f), we can consider the canonical bundle $K_S|_{U_{red}}$ and ask whether we can glue these line bundles for a covering by critical charts. The answer is no, but we can glue their squares $K_S^{\otimes 2}|_{U_{red}}$ to get a line bundle on X_{red}. Suppose that X is of the form $\operatorname{crit} f$, then the obstruction is a ± 1-cocycle, hence we can glue the squares to get a line bundle on X_{red}.

Proposition 3.1.2. *(Joyce [Joy15]) Let (X, s) be a d-critical locus. There exists a unique line bundle $K_{(X,s)}$ on X_{red} such that for any chart (U, S, f) we have an isomorphism*

$$\lambda_{(U,S,f)} : K_{(X,s)}|_{U_{red}} \cong K_S^{\otimes 2}|_{U_{red}}$$

such that for any étale morphism $\varphi : (U, S, f) \to (V, T, g)$ *of charts, this means that the map* $\varphi : S \to T$ *is étale,* $\varphi|_U : U \hookrightarrow V$ *is the canonical inclusion and* $f = g \circ \varphi$*, we have*

$$\lambda_{(U,S,f)} = \det(d\varphi)^{\otimes 2}|_{U_{red}} \circ \lambda_{(V,T,g)}|_{U_{red}}.$$

Example 3.1.3. If X is smooth, then $K_{(X,0)} = K_X^{\otimes 2}$. We get an extra K_X factor because, as a derived scheme, the critical locus $\mathrm{crit}(0 : X \to \mathbf{C})$ is the shifted cotangent bundle $\Omega_X[1]$.

Theorem 3.1.4 (Brav et al. [BBD+15])**.** *Let* (X, s) *be a* d*-critical locus. Assume that its canonical bundle* $K_{(X,s)}$ *admits a square root, called orientation of* (X, s)*. Then there exists a perverse sheaf* $\mathscr{P}_{(X,s)}$ *on* X *such that if* (U, S, f) *is a chart, then we have a natural isomorphism*

$$\mathscr{P}_{(X,s)}|_U \simeq \mathscr{P}_{S,f} \otimes_{\mathbf{C}} \mathfrak{K}_{or},$$

where $\mathfrak{K}_{or}$ *is the local system associated to* $K_{(X,s)}^{-1/2}|_{U_{red}} \otimes K_S|_{U_{red}}$.

The next result is key in the construction of a perverse sheaf on the intersection of complex Lagrangians. It shows that we are in the context of Theorem 3.1.4 and the perverse sheaf we are after is a special case of the sheaf which appears in Theorem 3.1.4.

Proposition 3.1.5. *(Bussi [Bus14]) Let* X *be a holomorphic symplectic variety. Suppose given two Lagrangians* L *and* M *in* X*. Then then intersection* $L \cap M$ *admits a structure of a*

d-*critical locus* $(L \cap M, s)$ *with canonical bundle*

$$K_{(L\cap M,s)} = K_L|_{L\cap M_{red}} \otimes K_M|_{L\cap M_{red}}.$$

Remark 3.1.6. The d-critical charts are given by the holomorphic Lagrangian neighbourhood theorem and Bussi shows that these are compatible, hence yield the claimed d-critical structure.

Corollary 3.1.7. *Consider the* d-*critical locus* $(L \cap M, s)$ *and assume that the line bundle* $K_L|_{L\cap M_{red}} \otimes K_M|_{L\cap M_{red}}$ *admits a square root. Then there exists a perverse sheaf*

$$\mathscr{P}_{L,M} \simeq \mathscr{P}_{M,L}$$

on $L \cap M$ *with the properties described in Theorem 3.1.4.*

Lemma 3.1.8. *Let* L, M *be two Lagrangians intersecting cleanly, then there is an isomorphism*

$$K_{L\cap M} \otimes K_{L\cap M} \cong K_L|_{L\cap M} \otimes K_M|_{L\cap M}.$$

Proof. Let $\mathbf{E}_{LM} \to \mathbf{L}_{L\cap M}$ be the symmetric obstruction theory on the intersection. Recall that

$$\mathbf{E}_{LM} = \left[\Omega_X|_{L\cap M} \xrightarrow{-\text{res},\text{res}} \Omega_L|_{L\cap M} \oplus \Omega_M|_{L\cap M}\right],$$

the map $\mathbf{E}_{LM} \to \mathbf{L}_{L\cap M}$ is defined via the quasi-isomorphism

$$\mathbf{E}_{LM} \simeq \left[\mathscr{I}_{LX}/\mathscr{I}_{LX}^2|_{L\cap M} \to \Omega_M|_{L\cap M}\right],$$

and the symmetry comes from the holomorphic form on X. Then, we shall compute the determinant of $\mathbf{E}_{LM}$ in two ways. On the one hand, using that X has trivial canonical bundle, we get:

$$\det \mathbf{E}_{LM} = (\det \Omega_X|_{L\cap M})^\vee \otimes \det(\Omega_L|_{L\cap M} \oplus \Omega_M|_{L\cap M}) \cong K_L|_{L\cap M} \otimes K_M|_{L\cap M}.$$

Now, we can also calculate the determinant using the cohomology sheaves of the complex $\mathbf{E}_{LM}$ and obtain

$$\det \mathbf{E}_{LM} = \left(\det \mathscr{H}^{-1}(\mathbf{E}_{LM})\right)^\vee \otimes \det \mathscr{H}^0(\mathbf{E}_{LM}) \cong K_{L\cap M} \otimes K_{L\cap M}. \qquad \square$$

Corollary 3.1.9. *Let* $L \cap M$ *be smooth. Then* $(L \cap M, s)$ *is oriented and for any choice of* $K_{(L\cap M,s)}^{1/2}$ *we have* $\mathscr{P}_{L,M} = \mathfrak{K}_{or}[\dim(L \cap M)]$*, where* $\mathfrak{K}_{or}$ *is the local system associated to* $K_{(L\cap M,s)}^{-1/2} \otimes K_{L\cap M}$.

Proof. Indeed, our assumption means that $(L \cap M, 0)$ is the unique d-critical structure on the intersection. Hence,

$$\mathscr{P}_{L,M} \cong \mathscr{P}_{L\cap M,0} \otimes_{\mathbf{C}} \mathfrak{K}_{or} = \mathbf{C}_{L\cap M}[\dim(L \cap M)] \otimes_{\mathbf{C}} \mathfrak{K}_{or} = \mathfrak{K}_{or}[\dim(L \cap M)]. \qquad \square$$

We let X be a complex manifold. Let $\mathbf{C}[[\hbar]]$ be the ring of formal power series in $\hbar$, and $\mathbf{C}((\hbar))$ its field of fractions, i.e. the field of formal Laurent series. Define a sheaf of $\mathbf{C}[[\hbar]]$-algebras:

$$\mathscr{O}_X[[\hbar]] = \varprojlim \mathscr{O}_X \otimes_{\mathbf{C}} \mathbf{C}[[\hbar]]/\hbar^n.$$

Definition 3.2.1. A star product on $\mathscr{O}_X[[\hbar]]$ is a $\mathbf{C}[[\hbar]]$-bilinear associative multiplication $\star$ such that

$$f \star g = \sum_{i \geq 0} \mathrm{P}_i(f,g)\hbar^i, \text{ where } f, g \in \mathscr{O}_X,$$

such that P_i are holomorphic bidifferential operators with $\mathrm{P}_0(f,g) = fg$ and

$$\mathrm{P}_i(f,1) = \mathrm{P}_i(1,f) = 0$$

for all $i \geq 1$. The pair $(\mathscr{O}_X[[\hbar]], \star)$ is called a star algebra.

Definition 3.2.2. A deformation quantisation algebra (DQ-algebra) on a complex manifold X is a sheaf of $\mathbf{C}[[\hbar]]$-algebras $\mathscr{A}_X$ locally isomorphic to a star algebra as a $\mathbf{C}[[\hbar]]$-algebra.

Example 3.2.3. Let $\mathscr{A}_X$ be a DQ-algebra on X. Let $\pi : \mathscr{A}_X \to \mathscr{A}_X/\hbar\mathscr{A}_X \cong \mathscr{O}_X$. For any $f, g \in \mathscr{O}_X$, choose lifts $\tilde{f}, \tilde{g}$ such that $\pi(\tilde{f}) = f$ and $\pi(\tilde{g}) = g$. Then define a bracket

$$\{f,g\} = \pi(\hbar^{-1}(\tilde{f}\tilde{g} - \tilde{g}\tilde{f})).$$

This is independent of the choices made and defines a Poisson structure on X.

Example 3.2.4. Let X be a complex manifold. The cotangent bundle Ω_X supports a filtered sheaf of $\mathbf{C}$-algebras $\hat{\mathscr{E}}_{\Omega_X}$ of formal microdifferential operators. We start by recalling its definition. Fix $(x_1, \cdots, x_n)$ coordinates on X, and let

$$(x_1, \cdots, x_n, \xi_1, \cdots, \xi_n)$$

be the induced coordinates on Ω_X. Let $\mathscr{O}_{\Omega_X}(m)$ be the sheaf of homogeneous functions in the fibre coordinates on Ω_X of degree m, i.e.

$$\Big(\sum \xi_j \partial/\partial \xi_j - m\Big) f(x, \xi) = 0.$$

We define the sheaf of formal microdifferential operators of order $\leq m$ by

$$\hat{\mathscr{E}}_{\Omega_X}(m) = \prod_{j \in \mathbf{N}} \mathscr{O}_{\Omega_X}(m - j).$$

In order to get a sheaf globally on Ω_X, we glue these sheaves on overlaps using the transformation rule for total symbols of differential operators. Then, taking the limit over $m \in \mathbf{Z}$, we get the sheaf of formal microdifferential operators on Ω_X:

$$\hat{\mathscr{E}}_{\Omega_X} = \varinjlim_{m \in \mathbf{Z}} \hat{\mathscr{E}}_{\Omega_X}(m).$$

Note that there are products $\hat{\mathscr{E}}_{\Omega_X}(l) \otimes_{\mathbf{C}} \hat{\mathscr{E}}_{\Omega_X}(m) \to \hat{\mathscr{E}}_{\Omega_X}(l+m)$, given by

$$(f \star g)_l(x,\xi) = \sum_{\substack{l=i+j-|\alpha| \\ \alpha \in \mathbf{N}}} \frac{1}{\alpha!} (\partial/\partial\xi_1)^{\alpha_1} \cdot\cdot (\partial/\partial\xi_n)^{\alpha_n} f_i(x,\xi) (\partial/\partial x_1)^{\alpha_1} \cdot\cdot (\partial/\partial x_n)^{\alpha_n} g_j(x,\xi).$$

In particular $\hat{\mathscr{E}}_{\Omega_X}$ and $\hat{\mathscr{E}}_{\Omega_X}(0)$ are sheaves of (non-commutative) $\mathbf{C}$-algebras. Notice that the total symbol of a differential operator is a polynomial in $\xi_1, \cdots, \xi_n$, so essentially we are just allowing symbols which are general holomorphic functions rather than just polynomials.

Let t be the coordinate on $\mathbf{C}$ and $(t;\tau)$ - the symplectic coordinates on $\Omega_{\mathbf{C}}$. Let $\Omega_{X\times\mathbf{C},\tau\neq 0}$ be the open subset of $\Omega_{X\times\mathbf{C}}$ where $\tau \neq 0$. We have a map

$$\rho : \Omega_{X\times\mathbf{C},\tau\neq 0} \to \Omega_X, \quad (x,t;\xi,\tau) \mapsto (x,\tau^{-1}\xi).$$

Define the subsheaf of operators independent of t:

$$\hat{\mathscr{E}}_{\Omega_{X\times\mathbf{C}},\hat{t}}(0) = \{P \in \hat{\mathscr{E}}_{\Omega_{X\times\mathbf{C}}}(0) \text{ such that } [P, \partial_t] = 0\}.$$

Then, letting $\hbar$ act as τ^{-1}, we define the canonical DQ-algebra on Ω_X by

$$\hat{\mathscr{W}}_X(0) = \rho_* \hat{\mathscr{E}}_{\Omega_{X\times\mathbf{C}},\hat{t}}(0).$$

The $\hbar$-localisation of $\hat{\mathscr{W}}_X(0)$ is denoted by $\hat{\mathscr{W}}_X$.

Definition 3.2.5. Let X be a topological space. An R-algebroid on X is an R-linear stack

$\mathscr{A}$ which is locally non-empty and any two objects in $\mathscr{A}(U)$ are locally isomorphic for any open $U \subset X$.

Example 3.2.6. Fix a topological space X. Let $\mathscr{A}$ be a sheaf of R-algebras on X. We consider the prestack $U \to \mathscr{A}(U)^+$, where $\mathscr{A}(U)^+$ is the R-linear category with one object whose endomorphisms are given by $\mathscr{A}(U)$. The associated stack is denoted by $\mathscr{A}^+$. It's an R-algebroid.
Conversely, suppose that $\mathscr{A}$ is an algebroid. If $\mathscr{A}(X)$ is non-empty, choose any element $\tau \in \mathscr{A}(X)$. We have an equivalence $\mathscr{A} \simeq \mathscr{H}om(\tau, \tau)^+$ of R-algebroids.

Given an R-algebroid $\mathscr{A}$ over X, let $\mathscr{M}_R$ be the stack of sheaves of R-modules on X, we define the R-linear abelian category of modules over $\mathscr{A}$ by

$$\mathrm{Mod}(\mathscr{A}) = \mathrm{Fct}(\mathscr{A}, \mathscr{M}_R).$$

Definition 3.2.7. A deformation quantisation algebroid (DQ-algebroid) on X is a $\mathbf{C}[[\hbar]]$-algebroid $\mathscr{A}$ such that, for any open $U \subset X$ and $\tau \in \mathscr{A}(U)$, the $\mathbf{C}[[\hbar]]$-algebra $\mathscr{H}om(\tau, \tau)$ is a DQ-algebra on U.

Remark 3.2.8. If X is a holomorphic symplectic variety, then the holomorphic Darboux theorem implies that locally we have canonical DQ-algebras associated with X, but they won't generally glue to a global DQ-algebra. The obstruction was identified first by Kashiwara in the contact case, see [Kas96]. One has to twist by "half forms", i.e. by the twisted sheaf of half top forms and its dual. In the holomorphic setting, it is a theorem of Polesello and Schapira [PS04] that the corresponding twisted DQ-algebras glue

and we obtain a DQ-algebroid still denoted $\hat{\mathscr{W}}_X(0)$.

Any other DQ-algebroid $\mathscr{A}_X$ on X will be equivalent to $\hat{\mathscr{W}}_X(0)\otimes_{\mathbf{C}[[\hbar]]}\mathscr{L}$ for some invertible $\mathbf{C}[[\hbar]]$-algebroid $\mathscr{L}$. Hence holomorphic DQ-algebroids are classified by $H^2\left(X,\mathbf{C}[[\hbar]]^*\right)$.

Example 3.2.9. Example 3.2.3 shows that any DQ-algebroid on X induces a Poisson structure on X. Conversely, it is a theorem of Kontsevich [Kon03] that in the C^∞ setting (locally for algebraic varieties) any Poisson structure is induced by some DQ-algebroid. The global algebraic quantisation is due to Yekutieli [Yek05] and Van den Bergh [VdB07], and by Calaque et al. [CDH07] for complex manifolds.

Remark 3.2.10. If $\mathscr{A}_X$ is a DQ-algebroid, the local notions of being locally free, coherent, flat, etc. make sense for an $\mathscr{A}_X$-module $\mathscr{D}$.

Definition 3.2.11. Let $\mathscr{R}$ be a sheaf of commutative $\mathbf{C}$-algebras.

- An $\mathscr{R}$-algebroid is a $\mathbf{C}$-algebroid $\mathscr{A}$ together with a morphism of sheaves of $\mathbf{C}$-algebras $\mathscr{R}\to\mathscr{E}nd(\mathrm{id}_{\mathscr{A}})$.
- An $\mathscr{R}$-algebroid $\mathscr{A}$ is called invertible if $\mathscr{R}|_U\to\mathscr{E}nd(\tau)$ is an isomorphism for every open $U\subset X$ and any $\tau\in\mathscr{A}(U)$.

Let $\iota:\mathbf{C}\to\mathbf{C}[[\hbar]]$ be the canonical inclusion, define a $\mathbf{C}$-algebroid $\iota^*\mathscr{A}_X$ by taking the stack associated with the prestack $\mathscr{B}$ given by

$$\mathscr{B}(U)=\mathscr{A}_X(U)\text{ and }\mathrm{Hom}_{\mathscr{B}(U)}(\sigma,\tau)=\mathrm{Hom}_{\mathscr{A}_X(U)}(\sigma,\tau)/\hbar\mathrm{Hom}_{\mathscr{A}_X(U)}(\sigma,\tau).$$

This $\mathbf{C}$-algebroid is an invertible $\mathscr{O}_X$-algebroid. There are functors of $\mathbf{C}$-algebroids

$$\mathscr{A}_X \to \iota^* \mathscr{A}_X \to \mathscr{O}_X$$

and an equivalence of $\mathbf{C}$-algebroids $\iota^* \mathscr{A}_X \simeq \mathscr{A}_X/\hbar\mathscr{A}_X \simeq \mathscr{O}_X$. In particular, we get a functor preversing boundedness and coherence

$$\iota^* : \mathbf{D}(\mathscr{A}_X) \to \mathbf{D}(\iota^* \mathscr{A}_X) \quad \iota^* : \mathscr{D} \mapsto \mathbf{C} \otimes_{\mathbf{C}[[\hbar]]} \mathscr{D}.$$

The $\hbar$-localisation of a DQ-algebroid $\mathscr{A}_X$ is $\mathscr{A}_X^{\mathrm{loc}} = \mathbf{C}((\hbar)) \otimes_{\mathbf{C}[[\hbar]]} \mathscr{A}_X$. More generally, we have a functor

$$\mathrm{loc} : \mathbf{D}^{\mathrm{b}}(\mathscr{A}_X) \to \mathbf{D}^{\mathrm{b}}(\mathscr{A}_X^{\mathrm{loc}}).$$

Lemma 3.2.12. *Let $\mathscr{D} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X)$. Then we have $\mathrm{Supp}(\mathscr{D}) = \mathrm{Supp}(\iota^*\mathscr{D})$. In particular, $\mathrm{Supp}(\mathscr{D})$ is a closed analytic subset of X.*
If $\mathscr{E} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{A}_X^{\mathrm{loc}})$, then $\mathrm{Supp}(\mathscr{E})$ is a closed analytic subset of X, coisotropic for the Poisson structure defined by $\mathscr{A}_X$.

Remark 3.2.13. Note that we do not have a global equivalence $\iota^* \mathscr{A}_X \simeq \mathscr{O}_X$ of invertible $\mathscr{O}_X$-algebroids. This is generally only true locally. In the algebraic case, the vanishing of $H^2(X, \mathscr{O}_X^*)$ in the Zariski topology implies that $\iota^* \mathscr{A}_X \simeq \mathscr{O}_X$ as $\mathscr{O}_X$-algebroids globally.
The second statement in the lemma is known as Gabber's theorem and doesn't hold for coherent $\mathscr{A}_X$-modules - note that any closed analytic subset of X can be the support of such a module since any coherent $\mathscr{O}_X$-module is a coherent $\mathscr{A}_X$-module.

Theorem 3.2.14. *(Kashiwara-Schapira [KS12]) Let* X *be a complex manifold endowed with a DQ-algebroid* $\mathscr{A}_X$*. Let*

$$\mathscr{D},\mathscr{E} \in \mathbf{D}^b_{coh}(\mathscr{A}_X)$$

and suppose that $\mathrm{Supp}(\mathscr{D}) \cap \mathrm{Supp}(\mathscr{E})$ *is compact. Then* $\mathrm{RHom}_{\mathscr{A}_X}(\mathscr{D},\mathscr{E})$ *is a perfect complex of* $\mathbf{C}[[\hbar]]$*-modules.*

Definition 3.2.15. Let X be complex manifold endowed with a DQ-algebroid $\mathscr{A}_X$, and let Y be a smooth submanifold of X. A coherent $\mathscr{A}_X$-module $\mathscr{D}$ supported on Y is called simple if $\iota^*\mathscr{D}$ is concentrated in degree 0 and $H^0(\iota^*\mathscr{D})$ is an invertible $\mathscr{O}_Y \otimes_{\mathscr{O}_X} \iota^*\mathscr{A}_X$-module.

Definition 3.2.16. Let X be a holomorphic symplectic variety equipped with a DQ-algebroid $\mathscr{A}_X$.

1. An $\mathscr{A}_X^{loc}$-module is called holonomic if it is coherent and its support is a Lagrangian subvariety of X.

2. An $\mathscr{A}_X$-module $\mathscr{D}^0$ is called holonomic if it is coherent, $\hbar$-torsion free and its $\hbar$-localisation $(\mathscr{D}^0)^{loc}$ is holonomic.

3. Let L be a smooth Lagrangian. An $\mathscr{A}_X^{loc}$-module $\mathscr{D}$ is called simple holonomic if there exists locally an $\mathscr{A}_X$-module $\mathscr{D}^0$, simple along L, which generates it, i.e. $(\mathscr{D}^0)^{loc} \simeq \mathscr{D}$.

Theorem 3.2.17. *(Kashiwara-Schapira [KS08]) Let* X *be a holomorphic symplectic variety of dimension* $2n$*, equipped with a DQ-algebroid* $\mathscr{A}_X$*. Suppose that* $\mathscr{D}$ *and* $\mathscr{E}$ *are two*

holonomic $\mathscr{A}_X^{loc}$-modules. Then the complex $\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{loc}}(\mathscr{D},\mathscr{E})[n]$ is a perverse sheaf.

The next result is crucial for our applications. It says that the square roots of complex Lagrangians admit deformation quantisation by simple DQ-modules.

Theorem 3.2.18. *(D'Agnolo-Schapira [DS07]) Let X be a holomorphic symplectic variety and let $i : L \hookrightarrow X$ be a smooth compact Lagrangian. Assume that the canonical bundle K_L of L admits a square root. Then, for any choice of a square root $K_L^{1/2}$, there exists a simple $\hat{\mathscr{W}}_X(0)$-module $\mathscr{D}_L$, supported on L, which quantises $K_L^{1/2}$.*

Remark 3.2.19. Notation as in the previous theorem, consider the short exact sequence

$$1 \to \mathbf{C}^* \to \mathscr{O}_L^* \xrightarrow{\mathrm{dlog}} \mathrm{d}\mathscr{O}_L \to 0.$$

It induces a long exact sequence in cohomology, and we are interested in the folloing part of it:

$$\mathrm{H}^1(L,\mathbf{C}^*) \to \mathrm{H}^1(L,\mathscr{O}_L^*) \xrightarrow{\alpha} \mathrm{H}^1(L,\mathrm{d}\mathscr{O}_L) \xrightarrow{\delta} \mathrm{H}^2(L,\mathbf{C}^*).$$

Let $\mathbf{C}_{K_L^{1/2}}$ be the $\mathbf{C}$-algebroid associated to the class $\delta(\frac{1}{2}\alpha(c_1(K_L)))$. Then a general version of the above theorem asserts that there exists a simple $\hat{\mathscr{W}}_X(0)\otimes\mathbf{C}_{K_L^{1/2}}$-module along L, i.e. in general we only get a twisted $\hat{\mathscr{W}}_X(0)$-module.

Notice that $\mathbf{C}_{K_L^{1/2}}$ is trivial iff there exists a line bundle $\mathscr{L}$ such that $K_L \otimes \mathscr{L}^{\otimes 2}$ admits a flat connection, hence $\mathscr{L}$ can be quantised. In particular, this agrees with the results of [BGKP16] since $\hat{\mathscr{W}}_X(0) \simeq \hat{\mathscr{W}}_X(0)^{\mathrm{op}}$ implies that the Atiyah class $\mathrm{At}(\hat{\mathscr{W}}_X(0), L) = 0$.

Indeed, $\hat{\mathscr{W}}_X(0)$ is canonical in the sense of [BK04], that is, its non-commutative period in $\mathrm{H}^2(X)\mathbf{C}[[\hbar]]$ is the constant power series. Note that this is always possible to achieve by

allowing stacky deformations. If we instead require that it be a deformation as a sheaf, it may happen that no canonical deformation quantisations exist.

Remark 3.2.20. In fact, the first Rozansky-Witten class $\mathrm{RW}_1(\mathrm{X})$ of X is an obstruction for the canonical deformation quantisation algebroid to be a sheaf of algebras.

The next theorem relates the perverse sheaf of Joyce et al. on the d-critical locus $(\mathrm{L}\cap\mathrm{M}, s)$ over $\mathbf{C}(\!(\hbar)\!)$ to the deformation quantisation of the Lagrangians.

Theorem 3.2.21 ([BBD+15]). *Let* X *be a holomorphic symplectic variety of dimension* $2n$*, equipped with the canonical DQ-algebroid* $\hat{\mathscr{W}}_{\mathrm{X}}(0)$*. Suppose that* L *and* M *are smooth Lagrangians and assume that* $\mathrm{K}_{\mathrm{L}}^{1/2}$ *and* $\mathrm{K}_{\mathrm{M}}^{1/2}$ *exist. Let* $\mathscr{D}_{\mathrm{L}}^0$ *and* $\mathscr{D}_{\mathrm{M}}^0$ *be two simple holonomic* $\hat{\mathscr{W}}_{\mathrm{X}}(0)$*-modules, supported on* L *and* M*, respectively, as in Theorem 3.2.18. Then we have an isomorphism of perverse sheaves*

$$\mathrm{R}\mathscr{H}om_{\hat{\mathscr{W}}_{\mathrm{X}}}\big(\mathscr{D}_{\mathrm{L}}, \mathscr{D}_{\mathrm{M}}\big)[n] \xrightarrow{\sim} \mathbf{C}(\!(\hbar)\!) \otimes_{\mathbf{C}} \mathscr{P}_{\mathrm{L},\mathrm{M}},$$

where $\mathscr{D}_{\mathrm{L}}$ *is the* $\hbar$*-localisation* $\mathbf{C}(\!(\hbar)\!) \otimes_{\mathbf{C}[\![\hbar]\!]} \mathscr{D}_{\mathrm{L}}^0$ *and similarly for* $\mathscr{D}_{\mathrm{M}}$.

Remark 3.2.22. The theorem is much easier when $\mathrm{L}\cap\mathrm{M}$ is smooth and that's all we need. See Proposition 4.4.2 for a (stronger) statement and proof in the case of a single smooth Lagrangian $\mathrm{L} = \mathrm{M}$.

4 Applications

We begin with a few standard results on calculations of local $\mathscr{E}xt$ sheaves and their multiplicative structure on locally complete intersections. In the second and third paragraphs we prove degeneration of the spectral sequences for a single Lagrangian and a pair of cleanly intersecting Lagrangians, respectively. We conclude with the formality of the endomorphism dg algebra

$$\mathrm{RHom}_{\mathscr{O}_X}\left(i_*\mathrm{K}_{\mathrm{L}}^{1/2}, i_*\mathrm{K}_{\mathrm{L}}^{1/2}\right)$$

and the dg module

$$\mathrm{RHom}_{\mathscr{O}_X}\left(i_*\mathrm{K}_{\mathrm{L}}^{1/2}, j_*\mathrm{K}_{\mathrm{M}}^{1/2}\right)$$

over it.

4.1 Sheaves on locally complete intersections

Proposition 4.1.1. *Let $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ be a locally complete intersection. Set $c = \mathrm{codim}(\mathrm{Z}, \mathrm{X})$ and let $\mathscr{F}$ be a coherent sheaf on Z. Then*

$$\mathscr{H}^{-k}(i^*i_*\mathscr{F}) \cong \begin{cases} \mathscr{F} \otimes_{\mathscr{O}_\mathrm{Z}} \wedge^k \mathscr{N}_{\mathrm{Z/X}}^\vee, \ 0 \leq k \leq c \\ 0, \ \textit{otherwise.} \end{cases}$$

Proof. Since i is a closed embedding, it suffices to show the claimed isomorphisms after pushing forward along i, i.e.

$$i_*\mathscr{H}^{-k}(i^*i_*\mathscr{F}) \cong i_*(\mathscr{F} \otimes_{\mathscr{O}_\mathrm{Z}} \wedge^k \mathscr{N}_{\mathrm{Z/X}}^\vee) \text{ for } 0 \leq k \leq c$$

and $i_*\mathscr{H}^{-k}(i^*i_*\mathscr{F}) \cong 0$ for $k > c$. Using the projection formula,

$$i_*\mathscr{H}^{-k}(i^*i_*\mathscr{F}) \cong \mathscr{T}or_k^{\mathscr{O}_X}(i_*\mathscr{F}, i_*\mathscr{O}_Z).$$

In order to compute the $\mathscr{T}or$-sheaves, we may work locally and the proof goes exactly as in Proposition 1.1.2. □

Proposition 4.1.2. *Let $i : Z \hookrightarrow X$ be a locally complete intersection of codimension c. Let $\mathscr{F}$ and $\mathscr{G}$ be coherent sheaves on Z.*

1. *Assume $\mathscr{F}$ locally free, then we have*

$$\mathscr{E}xt^i_{\mathscr{O}_X}(i_*\mathscr{F}, i_*\mathscr{G}) \cong \begin{cases} i_*(\wedge^i \mathscr{N}_{Z/X} \otimes_{\mathscr{O}_Z} \mathscr{F}^\vee \otimes_{\mathscr{O}_Z} \mathscr{G}),\ 0 \le i \le c \\ 0,\ \textit{otherwise.} \end{cases}$$

2. *The Yoneda product coincides with the usual cup product. More precisely, let $\mathscr{F}, \mathscr{G}$ be locally free sheaves, $\mathscr{H}$ any coherent sheaf, then the Yoneda multiplication*

$$\mathscr{E}xt^i_{\mathscr{O}_X}(i_*\mathscr{G}, i_*\mathscr{H}) \otimes \mathscr{E}xt^j_{\mathscr{O}_X}(i_*\mathscr{F}, i_*\mathscr{G}) \to \mathscr{E}xt^{i+j}_{\mathscr{O}_X}(i_*\mathscr{F}, i_*\mathscr{H})$$

corresponds under the above isomorphisms to

$$i_*(\wedge^i \mathscr{N}_{Z/X} \otimes \mathscr{G}^\vee \otimes \mathscr{H}) \otimes i_*(\wedge^j \mathscr{N}_{Z/X} \otimes \mathscr{F}^\vee \otimes \mathscr{G}) \to i_*(\wedge^{i+j} \mathscr{N}_{Z/X} \otimes \mathscr{F}^\vee \otimes \mathscr{H})$$

given by exterior product and the natural map $\mathscr{G} \otimes \mathscr{G}^\vee \to \mathscr{O}_Z$.

Proof. For the first assertion, note that

$$\mathrm{R}\mathscr{H}om_{\mathscr{O}_X}(i_*\mathscr{F}, i_*\mathscr{G}) \cong i_*\mathrm{R}\mathscr{H}om_{\mathscr{O}_Z}(i^*i_*\mathscr{F}, \mathscr{G})$$

and apply Proposition 4.1.1, noting that the cohomology of $i^*i_*\mathscr{F}$ is locally free.

The second part is a local statement, so we shall assume Z is the zero locus of a regular section s of a rank c vector bundle $\mathscr{E}$ on X, and that $\mathscr{F}, \mathscr{G}$ and $\mathscr{H}$ are restrictions to Z of $\tilde{\mathscr{F}}, \tilde{\mathscr{G}}$ and $\tilde{\mathscr{H}}$ on X. Tensoring the Koszul resolution for $i_*\mathscr{O}_Z$ with $\tilde{\mathscr{F}}$ and $\tilde{\mathscr{G}}$ we get resolutions for $i_*\mathscr{F}$ and $i_*\mathscr{G}$ which allow us to compute the R$\mathscr{H}om$s. Then the R$\mathscr{H}om$ multiplication is induced by the wedge product $\wedge^k\mathscr{E} \otimes \wedge^l\mathscr{E} \to \wedge^{k+l}\mathscr{E}$ followed by the contraction $\tilde{\mathscr{G}} \otimes \tilde{\mathscr{G}} \to \mathscr{O}_X$. Taking cohomology, we get the claim on the level of $\mathscr{E}xt$s. □

4.2 Degeneration of the spectral sequence

Let (X, σ) be a holomorphic symplectic variety. Recall that a subvariety L is called Lagrangian if $\sigma|_L = 0$ and $\dim L = \frac{1}{2}\dim X$. If $i : L \hookrightarrow X$ is a smooth Lagrangian we have $\mathscr{T}_X \cong \Omega^1_X$ via the symplectic form, hence $i^*\mathscr{T}_X \cong i^*\Omega^1_X$. There is a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{T}_L & \longrightarrow & i^*\mathscr{T}_X & \longrightarrow & \mathscr{N}_{L/X} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle i^*\sigma} & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{N}^\vee_{L/X} & \longrightarrow & i^*\Omega^1_X & \longrightarrow & \Omega^1_L & \longrightarrow & 0 \end{array}$$

which shows we have isomorphisms of vector bundles

$$\Omega^q_L \cong \wedge^q \mathscr{N}_{L/X}.$$

Hence the second page of the local-to-global Ext spectral sequence, in the Lagrangian case, is $\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{L}, \Omega_{\mathrm{L}}^q)$.

Theorem 4.2.1. *Let* $\mathrm{X}/\mathbf{C}$ *be holomorphic symplectic, and consider a compact Kähler Lagrangian* $i : \mathrm{L} \hookrightarrow \mathrm{X}$ *whose canonical bundle admits a square root. Then the local-to-global* Ext *spectral sequence*

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{L}, \Omega_{\mathrm{L}}^q) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_{\mathrm{X}}}\left(i_*\mathrm{K}_{\mathrm{L}}^{1/2}, i_*\mathrm{K}_{\mathrm{L}}^{1/2}\right)$$

degenerates on the second page. Hence

$$\mathrm{H}^k(\mathrm{L}/\mathbf{C}) = \oplus_{p,q}\mathrm{H}^p(\mathrm{L}, \Omega_{\mathrm{L}}^q) = \mathrm{Ext}^k_{\mathscr{O}_{\mathrm{X}}}\left(i_*\mathrm{K}_{\mathrm{L}}^{1/2}, i_*\mathrm{K}_{\mathrm{L}}^{1/2}\right).$$

Proof. This proof was envisaged by Thomas, and Petit helped us make the initial sketch rigorous. It will be enough to show that

$$\dim_{\mathbf{C}}\left(\mathrm{Ext}^i_{\mathscr{O}_{\mathrm{X}}}\left(i_*\mathrm{K}_{\mathrm{L}}^{1/2}, i_*\mathrm{K}_{\mathrm{L}}^{1/2}\right)\right) \geq \dim_{\mathbf{C}}\left(\mathrm{H}^i(\mathrm{L}/\mathbf{C})\right).$$

Let $\mathscr{A}_{\mathrm{X}}$ be the canonical quantisation $\hat{\mathscr{W}}_{\mathrm{X}}(0)$ of X. We fix a square root $\mathrm{K}_{\mathrm{L}}^{1/2}$ of the canonical bundle. There exists a simple $\mathscr{A}_{\mathrm{X}}$-module $\mathscr{D}_{\mathrm{L}}^0$ on L quantising $\mathrm{K}_{\mathrm{L}}^{1/2}$. Let $\mathscr{A}_{\mathrm{X}}^{\mathrm{loc}}$ be the localisation $\mathbf{C}(\!(\hbar)\!) \otimes_{\mathbf{C}[\![\hbar]\!]} \mathscr{A}_{\mathrm{X}}$. Then $\mathscr{D}_{\mathrm{L}}^0$ localises to a simple holonomic DQ-module $\mathscr{D}_{\mathrm{L}}$ over $\mathscr{A}_{\mathrm{X}}^{\mathrm{loc}}$. Then Theorem 3.2.21 implies that

$$\mathrm{R}\mathscr{H}om\left(\mathscr{D}_{\mathrm{L}}, \mathscr{D}_{\mathrm{L}}\right) \simeq \mathbf{C}(\!(\hbar)\!)_{\mathrm{L}},$$

so we get $\mathrm{Ext}^i_{\mathscr{A}_X^{\mathrm{loc}}}\left(\mathscr{D}_L, \mathscr{D}_L\right) = \mathrm{H}^i\left(L, \mathbf{C}((\hbar))\right)$. Hence the universal coefficients theorem implies that

$$\dim_{\mathbf{C}((\hbar))}\left(\mathrm{Ext}^i_{\mathscr{A}_X^{\mathrm{loc}}}\left(\mathscr{D}_L, \mathscr{D}_L\right)\right) = \dim_{\mathbf{C}}\left(\mathrm{H}^i(L/\mathbf{C})\right).$$

Furthermore Theorem 3.2.14, which requires L compact, states that

$$\mathrm{RHom}_{\mathscr{A}_X}\left(\mathscr{D}_L^0, \mathscr{D}_L^0\right) \in \mathrm{Perf}\left(\mathrm{Spec}\left(\mathbf{C}[[\hbar]]\right)\right).$$

Then we can apply the semicontinuity theorem on $\mathbf{C}[[\hbar]]$ to get that

$$\dim_{\mathbf{C}}\left(\mathrm{H}^i\left(\mathbf{C}\otimes_{\mathbf{C}[[\hbar]]}\mathrm{RHom}_{\mathscr{A}_X}\left(\mathscr{D}_L^0, \mathscr{D}_L^0\right)\right)\right) \geq \dim_{\mathbf{C}((\hbar))}\left(\mathrm{H}^i\left(\mathbf{C}((\hbar))\otimes_{\mathbf{C}[[\hbar]]}\mathrm{RHom}_{\mathscr{A}_X}\left(\mathscr{D}_L^0, \mathscr{D}_L^0\right)\right)\right).$$

It's enough to observe that there is a quasi-isomorphism

$$\mathbf{C} \otimes_{\mathbf{C}[[\hbar]]} \mathrm{RHom}_{\mathscr{A}_X}\left(\mathscr{D}_L^0, \mathscr{D}_L^0\right) \simeq \mathrm{RHom}_{\mathscr{O}_X}\left(i_*\mathrm{K}_L^{1/2}, i_*\mathrm{K}_L^{1/2}\right)$$

and the projection formula implies that

$$\mathbf{C}((\hbar)) \otimes_{\mathbf{C}[[\hbar]]} \mathrm{RHom}_{\mathscr{A}_X}\left(\mathscr{D}_L^0, \mathscr{D}_L^0\right) \simeq \mathrm{RHom}_{\mathscr{A}_X^{\mathrm{loc}}}\left(\mathscr{D}_L, \mathscr{D}_L\right).$$

Thus,

$$\begin{aligned}\dim_{\mathbf{C}}\left(\mathrm{Ext}^i_{\mathscr{O}_X}\left(i_*\mathrm{K}_L^{1/2}, i_*\mathrm{K}_L^{1/2}\right)\right) &\geq \dim_{\mathbf{C}((\hbar))}\left(\mathrm{Ext}^i_{\mathscr{A}_X^{\mathrm{loc}}}\left(\mathscr{D}_L, \mathscr{D}_L\right)\right) \\ &= \dim_{\mathbf{C}}\left(\mathrm{H}^i(L/\mathbf{C})\right).\end{aligned}$$

□

4.3 Degeneration in case of pairs of Lagrangians

Having dealt with the case of one Lagrangian, we now turn to pairs of Lagrangians. Let $i : \mathrm{L} \hookrightarrow \mathrm{X}$ and $j : \mathrm{M} \hookrightarrow \mathrm{X}$ be smooth submanifolds. We need a few standard results computing $\mathscr{E}xt$-sheaves on smooth intersections. The argument in Lemma 1.1.1 extends readily to the following generalisation:

Proposition 4.3.1. *Let $\mathscr{F}$ be locally free on L and $\mathscr{G}$ be coherent on M. Then, we have a natural isomorphism for all $p \geq 0$:*

$$\mathscr{E}xt^p_{\mathscr{O}_\mathrm{X}}(i_*\mathscr{F}, j_*\mathscr{G}) \cong \mathscr{E}xt^p_{\mathscr{O}_\mathrm{X}}(i_*\mathscr{O}_\mathrm{L}, j_*\mathscr{O}_\mathrm{M}) \otimes \mathscr{G}|_{L\cap M} \otimes \mathscr{F}^\vee|_{L\cap M}.$$

Proposition 4.3.2. *Assuming $\mathrm{L} \cap \mathrm{M}$ smooth, we have*

$$\mathscr{E}xt^p_{\mathscr{O}_\mathrm{X}}(i_*\mathscr{O}_\mathrm{L}, j_*\mathscr{O}_\mathrm{M}) \cong \wedge^{p-c}\mathscr{N} \otimes \det\mathscr{N}_{\mathrm{L}\cap\mathrm{M}/\mathrm{M}},$$

where $c = \mathrm{rk}\mathscr{N}_{\mathrm{L}\cap\mathrm{M}/\mathrm{M}}$, $\mathscr{N} := \mathscr{T}_\mathrm{X}|_{\mathrm{L}\cap\mathrm{M}}/(\mathscr{T}_\mathrm{L}|_{\mathrm{L}\cap\mathrm{M}} + \mathscr{T}_\mathrm{M}|_{\mathrm{L}\cap\mathrm{M}})$ is the excess normal bundle.

We begin by assuming L is the zero locus of a regular section s of a vector bundle $\mathscr{E}$ on X. Then we have a Koszul resolution of $i_*\mathscr{O}_\mathrm{L}$.

Lemma 4.3.3. *Assume $\mathrm{M} \subseteq \mathrm{L}$. Then*

$$\mathscr{E}xt^p_{\mathscr{O}_\mathrm{X}}(i_*\mathscr{O}_\mathrm{L}, j_*\mathscr{O}_\mathrm{M}) \cong j_* \wedge^p \mathscr{N}_{\mathrm{L}/\mathrm{X}}|_\mathrm{M}.$$

Proof. This follows at once from the Koszul resolution since, after restricting to M, all differentials vanish as $M \subseteq L$. □

Recall that L and M intersect properly if

$$\dim L \cap M + \dim X = \dim L + \dim M.$$

Lemma 4.3.4. *Assume* L *and* M *intersect properly. Then*

$$\mathscr{E}xt^p_{\mathscr{O}_X}(i_*\mathscr{O}_L, j_*\mathscr{O}_M) \cong \begin{cases} \det \mathscr{N}_{L\cap M/M} \text{ if } p = c \\ 0 \text{ if } p \neq c. \end{cases}$$

Proof. Our assumption implies that $s|_M$ is regular and $L \cap M = Z(s|_M)$. Hence the complex computing the $\mathscr{E}xt$ sheaves is exact everywhere except at the right end, so we get

$$\mathscr{E}xt^p_{\mathscr{O}_X}(i_*\mathscr{O}_L, j_*\mathscr{O}_M) \cong 0 \text{ for } p \neq 0$$

and

$$\begin{aligned} \mathscr{E}xt^c_{\mathscr{O}_X}(i_*\mathscr{O}_L, j_*\mathscr{O}_M) &\cong \operatorname{coker}(\wedge^{c-1}\mathscr{E}|_M \xrightarrow{\wedge s|_M} \wedge^c \mathscr{E}|_M) \\ &\cong \wedge^c \mathscr{E}|_{L\cap M} \\ &\cong \wedge^c \mathscr{N}_{L\cap M/M}. \end{aligned}$$ □

Remark 4.3.5. We note that Lemma 4.3.3 and Lemma 4.3.4 remain true without the as-

sumption that $L = Z(s)$.

Lemma 4.3.6. *For any point $x \in L \cap M$ there exists an open neighbourhood $U \subset X$ and a smooth subvariety $W \subset U$ containing L such that*

- $(L \cap M) \cap U = W \cap M$ *scheme-theoretically and*
- W *and* M *intersect properly.*

Proof. Let $n = \dim M - \dim(L \cap M)$. If $n = 0$, then $M \subseteq L$ and we can take $W = X$. If $n > 0$, then $T_{M,}$ is not contained in $T_{L,x}$ since $L \cap M$ is smooth, so its tangent space is the intersection of the tangent spaces of L and M.
Since $T_{L,x}$ is the intersection of all hypersurfaces through x that contain L, we can find a hypersurface X' which contains L and $T_{M,x} \not\subset T_{X',x}$. Let $M' = M \cap X'$. Then,

$$L \cap M' = L \cap M$$

and hence we can apply induction since now we have

$$\dim M' - \dim(L \cap M) = n - 1.$$

This concludes the proof. □

Proof of Proposition 4.3.2. Working locally on X, using Lemma 4.3.6, we have a smooth W with $L \subset W$ and $L \cap M = W \cap M$ with W and M intersecting properly. The change of

rings spectral sequence for $\mathscr{E}xt$ sheaves

$$\mathrm{E}_2^{p,q} = \mathscr{E}xt^p_{\mathscr{O}_\mathrm{W}}(\mathscr{O}_\mathrm{L}, \mathscr{E}xt^q_{\mathscr{O}_\mathrm{X}}(\mathscr{O}_\mathrm{W}, \mathscr{O}_\mathrm{M})) \Rightarrow \mathscr{E}xt^{p+q}_{\mathscr{O}_\mathrm{X}}(\mathscr{O}_\mathrm{L}, \mathscr{O}_\mathrm{M})$$

which by Lemma 4.3.4 has non-zero entries on the second page only for

$$q = c = \operatorname{codim}(\mathrm{L} \cap \mathrm{M} \subset \mathrm{M})$$

given by

$$\mathrm{E}_2^{p,c} = \wedge^p \mathscr{N}_{\mathrm{L/W}}|_{\mathrm{L}\cap\mathrm{M}} \otimes \det \mathscr{N}_{\mathrm{L}\cap\mathrm{M/M}}.$$

In order to conclude, we note that the natural map

$$\mathscr{N}_{\mathrm{L/W}}|_{\mathrm{L}\cap\mathrm{M}} \cong \mathscr{T}_\mathrm{W}|_{\mathrm{L}\cap\mathrm{M}}/\mathscr{T}_\mathrm{L}|_{\mathrm{L}\cap\mathrm{M}} \hookrightarrow \mathscr{T}_\mathrm{X}|_{\mathrm{L}\cap\mathrm{M}}/\mathscr{T}_\mathrm{L}|_{\mathrm{L}\cap\mathrm{M}} \twoheadrightarrow \mathscr{T}_\mathrm{X}|_{\mathrm{L}\cap\mathrm{M}}/(\mathscr{T}_\mathrm{L}|_{\mathrm{L}\cap\mathrm{M}} + \mathscr{T}_\mathrm{M}|_{\mathrm{L}\cap\mathrm{M}})$$

is an isomorphism for dimension reasons - checking fibrewise suffices since $\mathrm{L} \cap \mathrm{M}$ is reduced. Hence,

$$\mathscr{E}xt^p_{\mathscr{O}_\mathrm{X}}(i_*\mathscr{O}_\mathrm{L}, j_*\mathscr{O}_\mathrm{M}) \cong \wedge^{p-c}\mathscr{N} \otimes \det \mathscr{N}_{\mathrm{L}\cap\mathrm{M/M}}.$$

This concludes the proof locally on X. Tracing through the identification, we see that these do not depend on the choice of W and so the isomorphisms exist globally. □

In the Lagrangian case, which we assume from now on, we have an exact sequence:

$$0 \to \mathscr{T}_{\mathrm{L}\cap\mathrm{M}} \to \mathscr{T}_\mathrm{L}|_{\mathrm{L}\cap\mathrm{M}} \oplus \mathscr{T}_\mathrm{M}|_{\mathrm{L}\cap\mathrm{M}} \to \mathscr{T}_\mathrm{X}|_{\mathrm{L}\cap\mathrm{M}} \to \Omega_{\mathrm{L}\cap\mathrm{M}} \to 0,$$

hence

$$\mathscr{E}xt^p_{\mathscr{O}_X}(i_*\mathscr{O}_L, j_*\mathscr{O}_M) \cong \Omega^{q-c}_{L\cap M} \otimes \det \mathscr{N}_{L\cap M/M}.$$

The adjunction formula yields an isomorphism

$$\det \mathscr{N}_{L\cap M/M} \cong K_M^\vee\big|_{L\cap M} \otimes K_{L\cap M},$$

hence by Lemma 3.1.8 we obtain

$$\det \mathscr{N}_{L\cap M/L} \otimes \det \mathscr{N}_{L\cap M/M} \cong \mathscr{O}_{L\cap M}.$$

Corollary 4.3.7. *Assume that there exist square roots* $K_L^{1/2}$ *and* $K_M^{1/2}$ *which is equivalent in the holomorphic setting to requiring that* L *and* M *carry spin structures. We define the orientation bundle*

$$\mathscr{K}_{or} := \left(K_L^{1/2}\big|_{L\cap M} \otimes K_M^{1/2}\big|_{L\cap M}\right)^\vee \otimes K_{L\cap M}$$

and set $n = \operatorname{codim}(L, X)$, *so* $c = n - \dim(L \cap M)$. *Then*

$$\mathscr{E}xt^p_{\mathscr{O}_X}\left(i_*K_L^{1/2}, j_*K_M^{1/2}\right) \cong \Omega^{q-c}_{L\cap M} \otimes \mathscr{K}_{or}.$$

Remark 4.3.8. Notice that the line bundle $\mathscr{K}_{or}$ is torsion, in fact of order 2. Hence the monodromy representation associated to the local system $\mathfrak{K}_{or}$, arising from $\mathscr{K}_{or}$, is unitary - see beginning of next proof for a brief sketch.

Theorem 4.3.9. *Let* $X/\mathbf{C}$ *be a holomorphic symplectic variety and suppose that* $i : L \hookrightarrow X$,

$j : \mathrm{M} \hookrightarrow \mathrm{X}$ *are smooth Lagrangians with a compact Kähler intersection* $\mathrm{L} \cap \mathrm{M}$. *Assume that* $\mathrm{K}_{\mathrm{L}}^{1/2}$ *and* $\mathrm{K}_{\mathrm{M}}^{1/2}$ *exist. Then the* Ext *local-to-global spectral sequence*

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{L} \cap \mathrm{M}, \Omega_{\mathrm{L}\cap\mathrm{M}}^{q-c} \otimes \mathscr{K}_{\mathrm{or}}) \Rightarrow \mathrm{Ext}_{\mathscr{O}_{\mathrm{X}}}^{p+q}\left(i_* \mathrm{K}_{\mathrm{L}}^{1/2}, j_* \mathrm{K}_{\mathrm{M}}^{1/2}\right)$$

degenerates on the second page. In particular,

$$\mathrm{Ext}_{\mathscr{O}_{\mathrm{X}}}^{k}\left(i_* \mathrm{K}_{\mathrm{L}}^{1/2}, j_* \mathrm{K}_{\mathrm{M}}^{1/2}\right) = \oplus_{p,q} \mathrm{H}^p(\mathrm{L} \cap \mathrm{M}, \Omega_{\mathrm{L}\cap\mathrm{M}}^{q-c} \otimes \mathscr{K}_{\mathrm{or}}) = \mathrm{H}^{k-c}(\tilde{\mathfrak{K}}_{\mathrm{or}}).$$

Proof. We have that $\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{L} \cap \mathrm{M}, \Omega_{\mathrm{L}\cap\mathrm{M}}^{q-c} \otimes \mathscr{K}_{\mathrm{or}})$, where c is the codimension of $\mathrm{L} \cap \mathrm{M}$ in L and M, and $\mathscr{K}_{\mathrm{or}}$ is defined in Corollary 4.3.7. The line bundle $\mathscr{K}_{\mathrm{or}}$ is 2-torsion, hence admits a flat Chern connection, so the associated representation of $\pi_1(\mathrm{L} \cap \mathrm{M}, \mathrm{pt})$ is unitary. As a consequence the Hodge-to-de Rham spectral sequence degenerates on E_1, so, analogously to the case of one Lagrangian, it will be enough to show that

$$\dim_{\mathbf{C}}\left(\mathrm{Ext}_{\mathscr{O}_{\mathrm{X}}}^{i}\left(i_* \mathrm{K}_{\mathrm{L}}^{1/2}, j_* \mathrm{K}_{\mathrm{M}}^{1/2}\right)\right) \geq \dim_{\mathbf{C}}\left(\mathrm{H}^{i-c}(\tilde{\mathfrak{K}}_{\mathrm{or}})\right).$$

Let $\mathscr{A}_{\mathrm{X}}$ be the canonical quantisation $\hat{\mathscr{W}}_{\mathrm{X}}(0)$ of X. Fix square roots $\mathrm{K}_{\mathrm{L}}^{1/2}$ and $\mathrm{K}_{\mathrm{M}}^{1/2}$ of the canonical bundles of L and M. Let $\mathscr{D}_{\mathrm{L}}^0$ and $\mathscr{D}_{\mathrm{M}}^0$ be the simple $\mathscr{A}_{\mathrm{X}}$-modules on L and M quantising $\mathrm{K}_{\mathrm{L}}^{1/2}$ and $\mathrm{K}_{\mathrm{M}}^{1/2}$, respectively. Then we let

$$\mathscr{D}_{\mathrm{L}} = \mathbf{C}((\hbar)) \otimes_{\mathbf{C}[[\hbar]]} \mathscr{D}_{\mathrm{L}}^0 \text{ and } \mathscr{D}_{\mathrm{M}} = \mathbf{C}((\hbar)) \otimes_{\mathbf{C}[[\hbar]]} \mathscr{D}_{\mathrm{M}}^0$$

be the corresponding $\hbar$-localisations.

Since the intersection $\mathrm{L} \cap \mathrm{M}$ is smooth, Theorem 3.2.21 implies that

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}_{\mathrm{X}}^{\mathrm{loc}}}\left(\mathscr{D}_{\mathrm{L}}, \mathscr{D}_{\mathrm{M}}\right) \simeq \mathbf{C}((\hbar))_{\mathrm{L}\cap\mathrm{M}} \otimes_{\mathbf{C}} \mathfrak{K}_{\mathrm{or}}[-c],$$

so we conclude that

$$\mathrm{Ext}^{i}_{\mathscr{A}_{\mathrm{X}}^{\mathrm{loc}}}\left(\mathscr{D}_{\mathrm{L}}, \mathscr{D}_{\mathrm{M}}\right) = \mathrm{H}^{i-c}(\mathrm{L} \cap \mathrm{M}, \mathbf{C}((\hbar)) \otimes_{\mathbf{C}} \mathfrak{K}_{\mathrm{or}}).$$

Hence the universal coefficients theorem implies that

$$\dim_{\mathbf{C}((\hbar))}\left(\mathrm{Ext}^{i}_{\mathscr{A}_{\mathrm{X}}^{\mathrm{loc}}}\left(\mathscr{D}_{\mathrm{L}}, \mathscr{D}_{\mathrm{M}}\right)\right) = \dim_{\mathbf{C}}\left(\mathrm{H}^{i-c}(\mathrm{L} \cap \mathrm{M}, \mathfrak{K}_{\mathrm{or}})\right).$$

Moreover Theorem 3.2.14, which requires $\mathrm{L} \cap \mathrm{M}$ compact, states that

$$\mathrm{RHom}_{\mathscr{A}_{\mathrm{X}}}\left(\mathscr{D}^{0}_{\mathrm{L}}, \mathscr{D}^{0}_{\mathrm{M}}\right) \in \mathrm{Perf}\left(\mathrm{Spec}\left(\mathbf{C}[[\hbar]]\right)\right).$$

Thus, we can apply the semicontinuity theorem on $\mathbf{C}[[\hbar]]$ to conclude that

$$\dim_{\mathbf{C}}\left(\mathrm{H}^{i}\left(\mathbf{C} \otimes_{\mathbf{C}[[\hbar]]} \mathrm{RHom}_{\mathscr{A}_{\mathrm{X}}}\left(\mathscr{D}^{0}_{\mathrm{L}}, \mathscr{D}^{0}_{\mathrm{M}}\right)\right)\right) \geq \dim_{\mathbf{C}((\hbar))}\left(\mathrm{H}^{i}\left(\mathbf{C}((\hbar)) \otimes_{\mathbf{C}[[\hbar]]} \mathrm{RHom}_{\mathscr{A}_{\mathrm{X}}}\left(\mathscr{D}^{0}_{\mathrm{L}}, \mathscr{D}^{0}_{\mathrm{M}}\right)\right)\right).$$

Now observe that there is a quasi-isomorphism

$$\mathbf{C} \otimes_{\mathbf{C}[[\hbar]]} \mathrm{RHom}_{\mathscr{A}_{\mathrm{X}}}\left(\mathscr{D}^{0}_{\mathrm{L}}, \mathscr{D}^{0}_{\mathrm{M}}\right) \simeq \mathrm{RHom}_{\mathscr{O}_{\mathrm{X}}}\left(i_{*}\mathrm{K}^{1/2}_{\mathrm{L}}, j_{*}\mathrm{K}^{1/2}_{\mathrm{M}}\right)$$

and the projection formula implies that

$$\mathbf{C}((\hbar)) \otimes_{\mathbf{C}[[\hbar]]} \mathrm{RHom}_{\mathscr{A}_X}\Big(\mathscr{D}_L^0, \mathscr{D}_M^0\Big) \simeq \mathrm{RHom}_{\mathscr{A}_X^{\mathrm{loc}}}\Big(\mathscr{D}_L, \mathscr{D}_M\Big).$$

Hence

$$\begin{aligned}\dim_{\mathbf{C}}\Big(\mathrm{Ext}^i_{\mathscr{O}_X}\Big(i_*\mathrm{K}_L^{1/2}, j_*\mathrm{K}_M^{1/2}\Big)\Big) &\geq \dim_{\mathbf{C}((\hbar))}\Big(\mathrm{Ext}^i_{\mathscr{A}_X^{\mathrm{loc}}}\Big(\mathscr{D}_L, \mathscr{D}_M\Big)\Big) \\ &= \dim_{\mathbf{C}}\Big(\mathrm{H}^{i-c}(L \cap M, \tilde{\mathfrak{K}}_{\mathrm{or}})\Big). \qquad \square\end{aligned}$$

4.4 Formality

We begin with some technical preliminary results that we use throughout. The next lemma gives sufficient condition for the cohomology of a perfect complex over $\mathbf{C}[[\hbar]]$ to be free.

Lemma 4.4.1. *Let $\iota : \mathbf{C} \hookrightarrow \mathbf{C}[[\hbar]]$ be the inclusion of the central fibre and let*

$$C \in \mathrm{Perf}\Big(\mathrm{Spec}\Big(\mathbf{C}[[\hbar]]\Big)\Big).$$

Suppose that for all $i \in \mathbf{Z}$ we have

$$\dim_{\mathbf{C}}\Big(\mathrm{H}^i(\iota^* C)\Big) = \dim_{\mathbf{C}((\hbar))}\Big(\mathrm{H}^i(\mathbf{C}((\hbar)) \otimes_{\mathbf{C}[[\hbar]]} C)\Big).$$

Then the cohomology $\mathrm{H}(C)$ is free (of finite rank) over $\mathbf{C}[[\hbar]]$.

Proof. Consider the exact triangle

$$\mathrm{C} \xrightarrow{\hbar} \mathrm{C} \to \iota^*\mathrm{C} \to \mathrm{C}[1].$$

It induces a long exact sequence in cohomology

$$\mathrm{H}^i(\mathrm{C}) \xrightarrow{\hbar} \mathrm{H}^i(\mathrm{C}) \to \mathrm{H}^i(\iota^*\mathrm{C}) \to \mathrm{H}^{i+1}(\mathrm{C}) \xrightarrow{\hbar} \mathrm{H}^{i+1}(\mathrm{C}).$$

Hence there are exact sequences

$$0 \to \mathbf{C} \otimes_{\mathbf{C}[[\hbar]]} \mathrm{H}^i(\mathrm{C}) \to \mathrm{H}^i(\iota^*\mathrm{C}) \to \mathrm{Tor}_1^{\mathbf{C}[[\hbar]]}\left(\mathbf{C}, \mathrm{H}^{i+1}(\mathrm{C})\right) \to 0.$$

In particular, we get that

$$\dim_{\mathbf{C}}\left(\mathbf{C} \otimes_{\mathbf{C}[[\hbar]]} \mathrm{H}^i(\mathrm{C})\right) \leq \dim_{\mathbf{C}}\left(\mathrm{H}^i(\iota^*\mathrm{C})\right).$$

Since $\mathrm{H}^i(\mathrm{C})$ is finitely generated, we may write it as

$$\mathrm{H}^i(\mathrm{C}) = \mathbf{C}[[\hbar]]^{d_i} \oplus \mathbf{C}[[\hbar]]/\hbar^{k_1} \oplus \cdots \oplus \cdots \oplus \mathbf{C}[[\hbar]]/\hbar^{k_{r_i}},$$

where $k_1, \cdots, k_{r_i}, r_i \in \mathbf{N}$. Notice that

$$\dim_{\mathbf{C}}\left(\mathbf{C} \otimes_{\mathbf{C}[[\hbar]]} \mathrm{H}^i(\mathrm{C})\right) = d_i + r_i \text{ and } \dim_{\mathbf{C}((\hbar))}\left(\mathbf{C}((\hbar)) \otimes_{\mathbf{C}[[\hbar]]} \mathrm{H}^i(\mathrm{C})\right) = d_i.$$

It follows by flatness of $\mathbf{C}((\hbar))$ that

$$\dim_{\mathbf{C}((\hbar))}\Big(\mathbf{C}((\hbar))\otimes_{\mathbf{C}[[\hbar]]}\mathrm{H}^i(\mathrm{C})\Big)=\dim_{\mathbf{C}((\hbar))}\Big(\mathrm{H}^i(\mathbf{C}((\hbar))\otimes_{\mathbf{C}[[\hbar]]}\mathrm{C})\Big).$$

Hence, $r_i = 0$ and $\mathrm{H}^i(\mathrm{C})$ is free. □

The next proposition calculates the derived endomorphism algebra of a simple holonomic DQ-module over $\mathbf{C}((\hbar))$.

Proposition 4.4.2. *Let* $\mathrm{X}/\mathbf{C}$ *be a holomorphic symplectic variety, endowed with its canonical DQ- algebroid* $\hat{\mathscr{W}}_{\mathrm{X}}(0)$. *Suppose* L *is a smooth Lagrangian in* X *and let* $\mathscr{D}_{\mathrm{L}}$ *be a simple holonomic* $\hat{\mathscr{W}}_{\mathrm{X}}$*-module on* L. *Then the quasi-isomorphism*

$$\mathrm{R}\mathscr{H}om_{\hat{\mathscr{W}}_{\mathrm{X}}}\Big(\mathscr{D}_{\mathrm{L}},\mathscr{D}_{\mathrm{L}}\Big)\simeq\mathbf{C}((\hbar))_{\mathrm{L}}$$

is compatible with the multiplicative structures, i.e. it is a quasi-isomorphism of dg algebras.

Proof. The question is local, we may assume $\mathrm{X}=\Omega_{\mathrm{L}}$ and L is the zero section in Ω_{L}. We shall fix coordinates $(z_1,\cdots,z_n)$ on L. Further, since any two simple holonomic $\hat{\mathscr{W}}_{\mathrm{X}}$-modules are locally isomorphic, we may let $\mathscr{D}_{\mathrm{L}}=\mathscr{O}_{\mathrm{L}}((\hbar))$. Then the Koszul complex $\mathrm{K}(\mathscr{O}_{\mathrm{L}}((\hbar)),\partial_1,\cdots,\partial_n)$, associated with the coregular sequence $\partial_1,\cdots,\partial_n$ acting (on the left) on $\mathscr{O}_{\mathrm{L}}((\hbar))$, gives a multiplicative model for $\mathrm{R}\mathscr{H}om_{\hat{\mathscr{W}}_{\mathrm{X}}}\Big(\mathscr{O}_{\mathrm{L}}((\hbar)),\mathscr{O}_{\mathrm{L}}((\hbar))\Big)$ and the natural quasi-isomorphism $\mathbf{C}((\hbar))_{\mathrm{L}}\to\mathrm{K}(\mathscr{O}_{\mathrm{L}}((\hbar)),\partial_1,\cdots,\partial_n)$ is multiplicative. □

We are now ready to prove our main formality result.

Theorem 4.4.3. *Let* $X/\mathbf{C}$ *be holomorphic symplectic and let* $i : L \hookrightarrow X$ *be a smooth compact Kähler Lagrangian submanifold whose canonical bundle admits a square root. Then the differential graded algebra* $\mathrm{RHom}_{\mathscr{O}_X}\left(i_*K_L^{1/2}, i_*K_L^{1/2}\right)$ *is formal, in fact, quasi-isomorphic to the de Rham algebra* $H(L/\mathbf{C})$.

Proof. As before, let $\mathscr{A}_X$ be the canonical quantisation $\hat{\mathscr{W}}_X(0)$ of X. We fix a square root $K_L^{1/2}$ of the canonical bundle. There exists a simple $\mathscr{A}_X$-module $\mathscr{D}_L^0$ on L quantising $K_L^{1/2}$. Let $\mathscr{A}_X^{\mathrm{loc}}$ be the localisation $\mathbf{C}((\hbar)) \otimes_{\mathbf{C}[[\hbar]]} \mathscr{A}_X$ and denote $\mathscr{D}_L = \mathbf{C}((\hbar)) \otimes_{\mathbf{C}[[\hbar]]} \mathscr{D}_L^0$ the $\hbar$-localised simple holonomic DQ-module over $\mathscr{A}_X^{\mathrm{loc}}$.
Consider the differential graded algebra $\mathrm{RHom}_{\mathscr{A}_X}\left(\mathscr{D}_L^0, \mathscr{D}_L^0\right)$ over $\mathbf{C}[[\hbar]]$. Applying Proposition 4.4.2, we have that

$$\mathrm{R}\mathscr{H}om_{\mathscr{A}_X^{\mathrm{loc}}}\left(\mathscr{D}_L, \mathscr{D}_L\right) \simeq \mathbf{C}((\hbar))_L.$$

Degeneration of the spectral sequence on $\hbar = 0$ allows us to apply Lemma 4.4.1, so the cohomology of $\mathrm{RHom}_{\mathscr{A}_X}\left(\mathscr{D}_L^0, \mathscr{D}_L^0\right)$ is free of finite rank over $\mathbf{C}[[\hbar]]$. In particular, $\mathrm{Ext}_{\mathscr{A}_X}\left(\mathscr{D}_L^0, \mathscr{D}_L^0\right)$ is a formal deformation of the graded algebra $\mathrm{Ext}_{\mathscr{O}_X}\left(i_*K_L^{1/2}, i_*K_L^{1/2}\right)$ to the de Rham algebra $H(L/\mathbf{C})((\hbar))$. It follows that

$$\dim_{\mathbf{C}}\left(\mathrm{HH}^{p,q}\left(H(L/\mathbf{C})\right)\right) \leq \dim_{\mathbf{C}}\left(\mathrm{HH}^{p,q}\left(\mathrm{Ext}_{\mathscr{O}_X}\left(i_*K_L^{1/2}, i_*K_L^{1/2}\right)\right)\right). \tag{2}$$

Furthermore, collapse of the spectral sequence on $\hbar = 0$ also implies that there exists a

filtration F on the graded algebra $\mathrm{Ext}_{\mathscr{O}_X}\left(i_*\mathrm{K}_{\mathrm{L}}^{1/2}, i_*\mathrm{K}_{\mathrm{L}}^{1/2}\right)$ such that

$$\mathrm{Gr}_{\mathrm{F}}\left(\mathrm{Ext}_{\mathscr{O}_X}\left(i_*\mathrm{K}_{\mathrm{L}}^{1/2}, i_*\mathrm{K}_{\mathrm{L}}^{1/2}\right)\right) \cong \mathrm{H}(\mathrm{L}/\mathbf{C}),$$

as graded algebras, so the associated (completed) Rees algebra is a formal deformation of the graded algebra $\mathrm{H}(\mathrm{L}/\mathbf{C})$ to $\mathrm{Ext}_{\mathscr{O}_X}\left(i_*\mathrm{K}_{\mathrm{L}}^{1/2}, i_*\mathrm{K}_{\mathrm{L}}^{1/2}\right) \otimes_{\mathbf{C}} \mathbf{C}((\hbar))$. The Rees deformation gives the opposite of (2), hence

$$\dim_{\mathbf{C}}\left(\mathrm{HH}^{p,q}\left(\mathrm{H}(\mathrm{L}/\mathbf{C})\right)\right) = \dim_{\mathbf{C}}\left(\mathrm{HH}^{p,q}\left(\mathrm{Ext}_{\mathscr{O}_X}\left(i_*\mathrm{K}_{\mathrm{L}}^{1/2}, i_*\mathrm{K}_{\mathrm{L}}^{1/2}\right)\right)\right)$$

and by Lemma 4.4.1 the Hochschild cohomology groups $\mathrm{HH}^{p,q}\left(\mathrm{Ext}_{\mathscr{A}_X}\left(\mathscr{D}_{\mathrm{L}}^0, \mathscr{D}_{\mathrm{L}}^0\right)\right)$ are free over $\mathbf{C}[[\hbar]]$ for all $p, q \in \mathbf{Z}$. As a consequence, the Hochschild cohomology groups with compact supports of Lunts, $\mathrm{HH}_c^i\left(\mathrm{Ext}_{\mathscr{A}_X}\left(\mathscr{D}_{\mathrm{L}}^0, \mathscr{D}_{\mathrm{L}}^0\right)\right)$, are free over $\mathbf{C}[[\hbar]]$. It follows, by formality of L, that the differential graded algebra $\mathrm{RHom}_{\mathscr{A}_X^{\mathrm{loc}}}\left(\mathscr{D}_{\mathrm{L}}, \mathscr{D}_{\mathrm{L}}\right)$ is formal. Hence, $\mathrm{RHom}_{\mathscr{A}_X}\left(\mathscr{D}_{\mathrm{L}}^0, \mathscr{D}_{\mathrm{L}}^0\right)$ is formal by Theorem 2.1.12.

In order to conclude the formality proof, we note that there is a quasi-isomorphism

$$\mathrm{RHom}_{\mathscr{O}_X}\left(i_*\mathrm{K}_{\mathrm{L}}^{1/2}, i_*\mathrm{K}_{\mathrm{L}}^{1/2}\right) \simeq \mathbf{C} \otimes_{\mathbf{C}[[\hbar]]} \mathrm{RHom}_{\mathscr{A}_X}\left(\mathscr{D}_{\mathrm{L}}^0, \mathscr{D}_{\mathrm{L}}^0\right).$$

The cohomology of $\mathrm{RHom}_{\mathscr{A}_X}\left(\mathscr{D}_{\mathrm{L}}^0, \mathscr{D}_{\mathrm{L}}^0\right)$ is free over $\mathbf{C}[[\hbar]]$, so it follows that the dg algebra

$$\mathrm{RHom}_{\mathscr{O}_X}\left(i_*\mathrm{K}_{\mathrm{L}}^{1/2}, i_*\mathrm{K}_{\mathrm{L}}^{1/2}\right)$$

is formal.

To prove the last statement of the theorem, note that $\mathrm{Ext}_{\mathscr{A}_X}\left(\mathscr{D}_{\mathrm{L}}^0, \mathscr{D}_{\mathrm{L}}^0\right)$ is a generically

constant deformation which leaves the dimension of $\mathrm{HH}^{2,0}$ unchanged. The classical results on deformations of Gerstenhaber in [Ger74] extend to the graded case, thus the family $\mathrm{Ext}_{\mathscr{A}_X}\left(\mathscr{D}_L^0, \mathscr{D}_L^0\right)$ must be the trivial deformation - we give a summary of the proof, referring the reader to [Ger74, Section 3] for details (in the non-graded case).

Let $\mathrm{HH} := \mathrm{HH}\left(\mathrm{Ext}_{\mathscr{O}_X}\left(i_*\mathrm{K}_L^{1/2}, i_*\mathrm{K}_L^{1/2}\right)\right)$, $\mathrm{HH}_\hbar := \mathrm{HH}\left(\mathrm{Ext}_{\mathscr{A}_X}\left(\mathscr{D}_L^0, \mathscr{D}_L^0\right)\right)$ and $\mathrm{HH}_{(\hbar)}$ is the localisation of $\mathrm{HH}_\hbar$ at $\hbar$. The reduction mod $\hbar$ gives a map $\mathrm{HH}_\hbar \to \mathrm{HH}$ and since $\mathrm{HH}_\hbar$ is free over $\mathbf{C}[[\hbar]]$, taking a section of the reduction map, we get $\mathrm{HH}_\hbar \cong \mathrm{HH}[[\hbar]]$.

We have an isomorphism of graded algebras

$$\varphi_\hbar : \mathrm{Ext}_{\mathscr{A}_X}\left(\mathscr{D}_L^0, \mathscr{D}_L^0\right) \otimes_{\mathbf{C}[[\hbar]]} \mathbf{C}((\hbar)) \cong \mathrm{H}(L/\mathbf{C}) \otimes_{\mathbf{C}} \mathbf{C}((\hbar)).$$

Write $m_\hbar$ and m_{dR} for the multiplications of $\mathrm{Ext}_{\mathscr{A}_X}\left(\mathscr{D}_L^0, \mathscr{D}_L^0\right)$ and $\mathrm{H}(L/\mathbf{C})$, respectively. Then by definition

$$m_\hbar = \varphi_\hbar^{-1} m_{\mathrm{dR}}(\varphi_\hbar, \varphi_\hbar). \tag{3}$$

Differentiating (3) with respect to $\hbar$ gives

$$m_\hbar' = -\mathrm{d}_\hbar(\varphi_\hbar^{-1}\varphi_\hbar'), \tag{4}$$

where $\mathrm{d}_\hbar$ is the Hochschild differential of the graded algebra $\mathrm{Ext}_{\mathscr{A}_X}\left(\mathscr{D}_L^0, \mathscr{D}_L^0\right)$, i.e.

$$[m_\hbar'] = 0 \text{ in } \mathrm{HH}^{2,0}_{(\hbar)}.$$

Writing

$$m_\hbar = m + m_r\hbar^r + \cdots, \, r \geq 1,$$

we see that the left side of (4) is

$$r m_r \hbar^{r-1} + (r+1) m_{r+1} \hbar^r + \cdots .$$

Hence we deduce that

$$m_r + (r+1)/r \cdot m_{r+1} \hbar + (r+2)/r \cdot m_{r+2} \hbar^2 + \cdots$$

is a $\hbar$-torsion class in $\mathrm{HH}^{2,0}_{\hbar}$, lifting the class $[m_r] \in \mathrm{HH}^{2,0}$, so it must vanish. Thus $[m_r] = 0$ too. If $m_r = \mathrm{d}(\psi^r)$, then we can kill m_r using $\mathrm{id} - \psi^r \hbar^r$. By induction we get ψ^i for all $i \geq r$, the infinite composition

$$\psi_\hbar = \left(\left(\mathrm{id} - \psi^r \hbar^r \right) \circ \left(\mathrm{id} - \psi^{r+1} \hbar^{r+1} \right) \circ \cdots \right)$$

makes sense and we have $\psi_\hbar^{-1} m_\hbar(\psi_\hbar, \psi_\hbar) = m$, showing the triviality of $m_\hbar$.
Since $\mathrm{Ext}_{\mathscr{O}_X}\left(i_* \mathrm{K}_L^{1/2}, i_* \mathrm{K}_L^{1/2} \right)$ and $\mathrm{H}(L/\mathbf{C})$ are finite dimensional over $\mathbf{C}$ and become isomorphic after extending scalars to $\mathbf{C}((\hbar))$, they are already isomorphic over $\mathbf{C}$. □

Corollary 4.4.4. *Let* $X/\mathbf{C}$ *be holomorphic symplectic, and consider a compact Kähler Lagrangian* $i : L \hookrightarrow X$ *such that* $\mathrm{K}_L^{1/2}$ *exists. Then*

$$\mathrm{Ext}_{\mathscr{O}_X}\left(i_* \mathrm{K}_L^{1/2}, i_* \mathrm{K}_L^{1/2} \right) \cong \mathrm{H}(L/\mathbf{C})$$

as graded algebras.

Theorem 4.5.1. *Let* $\mathrm{X}/\mathbf{C}$ *be holomorphic symplectic and suppose that* $i : \mathrm{L} \hookrightarrow \mathrm{X}$, $j : \mathrm{M} \hookrightarrow \mathrm{X}$ *are compact Kähler Lagrangians with a smooth intersection. Asuume their canonical bundles admit square roots. Then*

$$\mathrm{RHom}_{\mathscr{O}_\mathrm{X}}\left(i_*\mathrm{K}_\mathrm{L}^{1/2}, j_*\mathrm{K}_\mathrm{M}^{1/2}\right)$$

is a formal differential graded module over the (formal) differential graded algebra

$$\mathrm{RHom}_{\mathscr{O}_\mathrm{X}}\left(i_*\mathrm{K}_\mathrm{L}^{1/2}, i_*\mathrm{K}_\mathrm{L}^{1/2}\right).$$

Moreover, we have a quasi-isomorphism of pairs

$$\left(\mathrm{RHom}_{\mathscr{O}_\mathrm{X}}\left(i_*\mathrm{K}_\mathrm{L}^{1/2}, i_*\mathrm{K}_\mathrm{L}^{1/2}\right), \mathrm{RHom}_{\mathscr{O}_\mathrm{X}}\left(i_*\mathrm{K}_\mathrm{L}^{1/2}, j_*\mathrm{K}_\mathrm{M}^{1/2}\right)\right) \simeq \left(\mathrm{H}(\mathrm{L}/\mathbf{C}), \mathrm{H}^{*-c}(\mathrm{L}\cap\mathrm{M}, \tilde{\mathfrak{K}}_{\mathrm{or}})\right),$$

where c *is the codimension of* $\mathrm{L}\cap\mathrm{M}$ *in* L.

Proof. Let $\mathscr{A}_\mathrm{X}$ be the canonical quantisation $\hat{\mathscr{W}}_\mathrm{X}(0)$ of X. We shall fix square roots $\mathrm{K}_\mathrm{L}^{1/2}$ and $\mathrm{K}_\mathrm{M}^{1/2}$ of the canonical bundles of L and M. Let $\mathscr{D}_\mathrm{L}^0$ and $\mathscr{D}_\mathrm{M}^0$ be simple $\mathscr{A}_\mathrm{X}$-modules on L and M quantising $\mathrm{K}_\mathrm{L}^{1/2}$ and $\mathrm{K}_\mathrm{M}^{1/2}$, respectively. Then we let $\mathscr{D}_\mathrm{L} = \mathbf{C}((\hbar)) \otimes_{\mathbf{C}[[\hbar]]} \mathscr{D}_\mathrm{L}^0$ and $\mathscr{D}_\mathrm{M} = \mathbf{C}((\hbar)) \otimes_{\mathbf{C}[[\hbar]]} \mathscr{D}_\mathrm{M}^0$ be the corresponding $\hbar$-localisations.

The complex $\mathrm{RHom}_{\mathscr{A}_\mathrm{X}}\left(\mathscr{D}_\mathrm{L}^0, \mathscr{D}_\mathrm{M}^0\right)$ is a dg module over $\mathrm{RHom}_{\mathscr{A}_\mathrm{X}}\left(\mathscr{D}_\mathrm{L}^0, \mathscr{D}_\mathrm{L}^0\right)$. Since the

intersection $L \cap M$ is smooth, Theorem 3.2.21 implies that

$$R\mathscr{H}om_{\mathscr{A}_X^{loc}}\left(\mathscr{D}_L, \mathscr{D}_M\right) \simeq \mathbf{C}((\hbar))_{L\cap M} \otimes_{\mathbf{C}} \tilde{\mathfrak{K}}_{or}[-c].$$

It follows by Theorem 4.3.9 and Lemma 4.4.1 that the cohomology of

$$\mathrm{RHom}_{\mathscr{A}_X}\left(\mathscr{D}_L^0, \mathscr{D}_M^0\right)$$

is free over $\mathbf{C}[[\hbar]]$. In particular, we see that $\mathrm{Ext}_{\mathscr{A}_X}\left(\mathscr{D}_L^0, \mathscr{D}_M^0\right)$ is a formal deformation of the graded module $\mathrm{Ext}_{\mathscr{O}_X}\left(i_*K_L^{1/2}, j_*K_M^{1/2}\right)$ to $H^{*-c}(L \cap M, \tilde{\mathfrak{K}}_{or})((\hbar))$. It follows that

$$\dim_{\mathbf{C}}\left(HH^{p,q}\left(H(L/\mathbf{C}), H^{*-c}(L\cap M, \tilde{\mathfrak{K}}_{or})\right)\right) \leq \dim_{\mathbf{C}}\left(HH^{p,q}\left(H(L/\mathbf{C}), \mathrm{Ext}_{\mathscr{O}_X}\left(i_*K_L^{1/2}, j_*K_M^{1/2}\right)\right)\right).$$

As in the previous theorem, degeneration of the spectral sequence on $\hbar = 0$ gives a filtration F on $\mathrm{Ext}_{\mathscr{O}_X}\left(i_*K_L^{1/2}, j_*K_M^{1/2}\right)$ such that

$$\mathrm{Gr}_F\left(\mathrm{Ext}_{\mathscr{O}_X}\left(i_*K_L^{1/2}, j_*K_M^{1/2}\right)\right) \cong H^{*-c}(L \cap M, \tilde{\mathfrak{K}}_{or})$$

as graded modules over $H(L/\mathbf{C})$. The Rees deformation argument then implies the opposite of the above inequality, so we conclude

$$\dim_{\mathbf{C}}\left(HH^{p,q}\left(H(L/\mathbf{C}), H^{*-c}(L\cap M, \tilde{\mathfrak{K}}_{or})\right)\right) = \dim_{\mathbf{C}}\left(HH^{p,q}\left(H(L/\mathbf{C}), \mathrm{Ext}_{\mathscr{O}_X}\left(i_*K_L^{1/2}, j_*K_M^{1/2}\right)\right)\right).$$

Thus, by Lemma 4.4.1, the Hochschild cohomology groups

$$\mathrm{HH}^{p,q}\Big(\mathrm{Ext}_{\mathscr{A}_X}\big(\mathscr{D}_L^0,\mathscr{D}_L^0\big),\mathrm{Ext}_{\mathscr{A}_X}\big(\mathscr{D}_L^0,\mathscr{D}_M^0\big)\Big)$$

are free over $\mathbf{C}[[\hbar]]$ for all $p,q\in\mathbf{Z}$.

We observed in the beginning of the proof of Theorem 4.3.9 that the local system $\tilde{\mathfrak{K}}_{or}$ corresponds to a unitary representation of the fundamental group of $L\cap M$. Then, as explained on page Example 1.3.3, we conclude that $\mathrm{RHom}_{\mathscr{A}_X^{loc}}\big(\mathscr{D}_L,\mathscr{D}_M\big)$ is a formal dg module over the dg algebra $\mathrm{RHom}_{\mathscr{A}_X^{loc}}\big(\mathscr{D}_L,\mathscr{D}_L\big)$. Hence $\mathrm{RHom}_{\mathscr{A}_X}\big(\mathscr{D}_L^0,\mathscr{D}_M^0\big)$ is a formal dg module over the dg algebra $\mathrm{RHom}_{\mathscr{A}_X}\big(\mathscr{D}_L^0,\mathscr{D}_L^0\big)$ by Proposition 2.8.5.

Next we note that there is a natural quasi-isomorphism of dg modules

$$\mathrm{RHom}_{\mathscr{O}_X}\big(i_*K_L^{1/2},j_*K_M^{1/2}\big)\simeq\mathbf{C}\otimes_{\mathbf{C}[[\hbar]]}\mathrm{RHom}_{\mathscr{A}_X}\big(\mathscr{D}_L^0,\mathscr{D}_M^0\big),$$

associated with the quasi-isomorphism of dg algebras

$$\mathrm{RHom}_{\mathscr{O}_X}\big(i_*K_L^{1/2},i_*K_L^{1/2}\big)\simeq\mathbf{C}\otimes_{\mathbf{C}[[\hbar]]}\mathrm{RHom}_{\mathscr{A}_X}\big(\mathscr{D}_L^0,\mathscr{D}_L^0\big).$$

It is now enough to recall that the cohomology of $\mathrm{RHom}_{\mathscr{A}_X}\big(\mathscr{D}_L^0,\mathscr{D}_M^0\big)$ is free over $\mathbf{C}[[\hbar]]$, hence $\mathrm{RHom}_{\mathscr{O}_X}\big(i_*K_L^{1/2},j_*K_M^{1/2}\big)$ is formal over $\mathrm{RHom}_{\mathscr{O}_X}\big(i_*K_L^{1/2},i_*K_L^{1/2}\big)$.

For the last assertion, we observe that $\mathrm{Ext}_{\mathscr{A}_X}\big(\mathscr{D}_L^0,\mathscr{D}_M^0\big)$ is a generically constant deformation that leaves $\mathrm{HH}^{1,0}$ constant. Such a deformation must be trivial, the proof being similar to the one for deformations of algebras discussed in the proof of Theorem 4.4.3. Thus, $\mathrm{Ext}_{\mathscr{O}_X}\big(i_*K_L^{1/2},j_*K_M^{1/2}\big)$ and $\mathrm{H}^{*-c}(L\cap M,\tilde{\mathfrak{K}}_{or})$ are isomorphic since they are finite

dimensional over $\mathbf{C}$ and become isomorphic upon extending scalars to $\mathbf{C}((\hbar))$. $\square$

5 Algebraic structures on the spectral sequence and non-triviality

5.1 Homological algebra

We collect some general results on spectral sequences. For a scheme X, let FCoh(X) be the quasi-abelian category of finitely filtered objects of Coh(X). We denote $\mathbf{D}^{\pm}\mathrm{F}(\mathrm{X})$ the filtered derived categories of Deligne which are localisations with respect to the class of filtered quasi-isomorphisms. Naturally we have filtered derived functors and for a left exact $\mathrm{T} : \mathrm{FCoh}(\mathrm{X}) \to \mathrm{Vect}$ and any $(\mathrm{K}, \mathrm{F}) \in \mathbf{D}^{\pm}\mathrm{F}(\mathrm{X})$, we get a spectral sequence

$$\mathrm{E}_1^{p,q} = \mathrm{R}^{p+q}\mathrm{T}(\mathrm{Gr}_{\mathrm{F}}^{p}(\mathrm{K})) \Rightarrow \mathrm{R}^{p+q}\mathrm{T}(\mathrm{K}).$$

This construction is a functor from the filtered derived category to the category of cohomological spectral sequences of vector spaces.

Definition 5.1.1. Let $\mathrm{E}_2^{'p,q} \Rightarrow \mathrm{H}^{'p+q}, \mathrm{E}_2^{''p,q} \Rightarrow \mathrm{H}^{''p+q}, \mathrm{E}_2^{p,q} \Rightarrow \mathrm{H}^{p+q}$ be spectral sequences. A pairing of $\mathrm{E}_2^{'p,q} \Rightarrow \mathrm{H}^{'p+q}$ and $\mathrm{E}_2^{''p,q} \Rightarrow \mathrm{H}^{''p+q}$ to $\mathrm{E}_2^{p,q} \Rightarrow \mathrm{H}^{p+q}$ is a family of maps

$$\cup_{\mathrm{r}} : \mathrm{E}_r^{'p,q} \otimes \mathrm{E}_r^{''s,t} \to \mathrm{E}_r^{p+s,q+t} \text{ and } \cup : \mathrm{H}^{'p} \otimes \mathrm{H}^{''q} \to \mathrm{H}^{p+q}$$

such that $\cup_r$ are compatible with the differentials, $\cup_{r+1}$ is induced by $\cup_r$,

$$\mathrm{colim}\cup_r = \cup_\infty \text{ and } \cup_\infty = \mathrm{gr}(\cup).$$

Definition 5.1.2. Given a Grothendieck spectral sequence

$$E_2^{p,q}(S) = R^pTR^qG(S) \Rightarrow H^{p+q}(S)$$

such that the functors RT, RG and H have cup products, we shall say it has cup products if, given a pairing $S_1 \otimes S_2 \to S$, we have a natural pairing of the corresponding spectral sequences and, in addition, the products on E_2 and H are the ones induced by the functors.

Proposition 5.1.3. *Let $\mathscr{F}$ a be coherent sheaf on* X. *Then the local-to-global* Ext *spectral sequence*

$$E_2^{p,q} = H^p(X, \mathscr{E}xt^q_{\mathscr{O}_X}(\mathscr{F}, \mathscr{F})) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{F})$$

has cup products.

Remark 5.1.4. More generally, there are also pairings of the local-to-global Ext in the case of different sheaves arising from the compostion in local $R\mathscr{H}om$.

As already mentioned in the introduction, the local-to-global Ext spectral sequence arises from the natural isomorphism of derived functors

$$\mathrm{RHom} \cong R\Gamma \circ R\mathscr{H}om.$$

We shall give an alternative way of constructing it via filtered complexes. This has the advantage of giving a geometric interpretation of the differentials as obstructions to for-

mality of certain objects. For $(\mathscr{F}, \mathrm{d}) \in \mathbf{D}^{\mathrm{b}}(\mathrm{X})$, define the canonical truncations:

$$\tau^{\leq p}\mathscr{F} = \begin{cases} \mathscr{F}^i, & \text{for } i < p \\ \ker(\mathrm{d}^p), & \text{for } i = p \\ 0, & \text{for } i > p \end{cases}$$

$$\tau^{\geq p}\mathscr{F} = \begin{cases} 0, & \text{for } i < p \\ \mathscr{F}^p/\mathrm{im}(\mathrm{d}^{p-1}), & \text{for } i = p \\ \mathscr{F}^i, & \text{for } i > p. \end{cases}$$

The inclusion $\tau^{\leq p}\mathscr{F} \hookrightarrow \mathscr{F}$ induces isomorphisms on $\mathscr{H}^i$ for $i \leq p$ and the canonical projection $\mathscr{F} \to \tau^{\geq p}\mathscr{F}$ induces isomorphisms on $\mathscr{H}^i$ for $i \geq p$. There are canonical exact triangles

$$\tau^{\leq p-1}\mathscr{F} \to \tau^{\leq p}\mathscr{F} \to \mathscr{H}^p(\mathscr{F})[-p] \to \tau^{\leq p-1}\mathscr{F}[1],$$

$$\mathscr{H}^{p-1}(\mathscr{F})[-p+1] \to \tau^{\geq p-1}\mathscr{F} \to \tau^{\geq p}\mathscr{F} \to \mathscr{H}^{p-1}(\mathscr{F})[-p+2].$$

Let's set $\tau^{[p-1,p]}\mathscr{F} = \tau^{\geq p-1}\tau^{\leq p}\mathscr{F}$, so that there's an exact triangle:

$$\mathscr{H}^{p-1}(\mathscr{F})[-p+1] \to \tau^{[p-1,p]}\mathscr{F} \to \mathscr{H}^p(\mathscr{F})[-p] \xrightarrow{\delta_{-p}(\mathscr{F})[-p]} \mathscr{H}^{p-1}(\mathscr{F})[-p+2],$$

and the elements

$$\delta_{-p}(\mathscr{F}) \in \mathrm{Ext}^2_{\mathscr{O}_X}(\mathscr{H}^p(\mathscr{F}), \mathscr{H}^{p-1}(\mathscr{F}))$$

realise universally the second differential d_2 of the spectral sequence of a filtered derived functor applied to $\mathscr{F}$.

Remark 5.1.5. In particular, if $i : Z \hookrightarrow X$ is a locally complete intersection, and $\mathscr{L}$ is a line bundle on Z, the spectral sequence

$$E_2^{p,q} = H^p(X, \mathscr{E}xt^q_{\mathscr{O}_X}(i_*\mathscr{L}, i_*\mathscr{L})) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_X}(i_*\mathscr{L}, i_*\mathscr{L})$$

can be realised from $i^!i_*\mathscr{L}$ with its canonical filtration. Indeed, Grothendieck-Verdier duality gives a filtered quasi-isomorphism between canonically filtered complexes

$$R\mathscr{H}om_{\mathscr{O}_X}(i_*\mathscr{L}, i_*\mathscr{L}) \simeq i_*R\mathscr{H}om_{\mathscr{O}_L}(\mathscr{L}, i^!i_*\mathscr{L})$$

and applying filtered derived global sections functor, we get a spectral sequence using (5.1) which, by Deligne's décalage theorem (see [Del71]), is the local-to-global Ext spectral sequence after renumbering $E_r^{p,q} \mapsto E_{r+1}^{2p+q,-p}$.

Let $i : Z \hookrightarrow X$ be a locally complete intersection. The normal bundle of Z in X is denoted by $\mathscr{N}_{Z/X}$. For any coherent $\mathscr{F}$ on Z, we have a canonical exact triangle

$$\mathscr{F} \otimes_{\mathscr{O}_Z} \mathscr{N}^\vee_{Z/X}[1] \to \tau^{\geq -1} i^*i_*\mathscr{F} \to \mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_Z} \mathscr{N}^\vee_{Z/X}[2].$$

Definition 5.1.6. The extension class of the above triangle

$$\alpha_{\mathscr{F}} \in \mathrm{Ext}^2_{\mathscr{O}_Z}(\mathscr{F}, \mathscr{F} \otimes_{\mathscr{O}_Z} \mathscr{N}^\vee_{Z/X})$$

is called the deformation-obstruction class of $\mathscr{F}$.

Remark 5.1.7. In the special case of $\mathscr{N}_{Z/X}^{\vee}$, the obstruction class $\alpha_{\mathscr{N}_{Z/X}^{\vee}}$ is sometimes called the HKR class because of its close relationship to Hochschild-Kostant-Rosenberg-like theorems.

Given a locally complete intersection $i : Z \hookrightarrow X$, we have the conormal exact sequence

$$0 \to \mathscr{N}_{Z/X}^{\vee} \to i^*\Omega_X \to \Omega_Z \to 0,$$

whose class is called the Kodaira-Spencer class $KS \in \mathrm{Ext}^1_{\mathscr{O}_Z}(\Omega_Z, \mathscr{N}_{Z/X}^{\vee})$.
We can extend the definitions of the Atiyah class and the obstruction class $\alpha_{\mathscr{F}}$ to the derived category. Consider the diagonal embedding

$$\Delta : X \hookrightarrow X \times X$$

and let 2Δ be the first infinitesimal neighbourhood of Δ in $X \times X$, i.e. it is defined by the square of the ideal sheaf of the diagonal. The canonical exact sequence of a closed embedding becomes

$$0 \to \Delta_*\Omega_X \to \mathscr{O}_{2\Delta} \to \Delta_*\mathscr{O}_X \to 0$$

Its extension class is called the universal Atiyah class $At \in \mathrm{Ext}^1_{\mathscr{O}_{X\times X}}(\Delta_*\mathscr{O}_X, \Delta_*\Omega_X)$. Now we use Fourier-Mukai functors to evaluate these universal classes at particular objects in the derived category: the Atiyah class of an object $\mathscr{F} \in \mathbf{D}^b(X)$ is then

$$\mathrm{At}(\mathscr{F}) = \Phi_{\mathrm{At}}(\mathscr{F}),$$

where

$$\Phi_{\mathrm{At}}(\mathscr{F}) : \Phi_{\Delta_*\mathscr{O}_X}(\mathscr{F}) \to \Phi_{\Delta_*\Omega_X}(\mathscr{F}).$$

To define the universal obstruction class, consider the closed embedding

$$\tilde{i} = \mathrm{id} \times i : Z \times Z \hookrightarrow Z \times X,$$

so that the complex $\tilde{i}^*\tilde{i}_*\Delta_*\mathscr{O}_Z$ gives an Ext^2 class

$$\alpha_{\Delta_*\mathscr{O}_Z} : \Delta_*\mathscr{O}_Z \to \Delta_*\mathscr{N}^\vee_{Z/X}[2].$$

As above we can evaluate it at any object in the derived category $\mathbf{D}^b(X)$.

Proposition 5.1.8 ([AC12], [HT10]). *Suppose $i : Z \hookrightarrow X$ is a locally complete intersection and consider $\mathscr{F} \in \mathbf{D}^b(X)$. The class $\alpha_{\mathscr{F}}$ is the product of $\mathrm{At}(\mathscr{F})$ and KS, i.e.*

$$\alpha_{\mathscr{F}} = (\mathrm{id}_{\mathscr{F}} \otimes \mathrm{KS}) \circ \mathrm{At}(\mathscr{F}).$$

The next theorem gives a geometric interpretation of the obstruction class defined above (and justifies the terminology!).

Theorem 5.1.9. *(Huybrechts-Thomas [HT10], Grivaux [Gri15]) Let $j : X \hookrightarrow 2X$ be a first order thickeing of a Noetherian separated scheme. Suppose $\mathscr{F} \in \mathbf{D}^b(X)$ is a perfect complex. Then $\mathscr{F}$ extends to a perfect complex $\mathscr{F}'$ on $2X$ iff $\alpha_{\mathscr{F}} = 0$.*

Theorem 5.1.10. *(Arinkin-Căldăraru [AC12]) Let $i : Z \hookrightarrow X$ be a locally complete in-*

tersection. Suppose $\mathscr{F}$ is locally free, $\mathscr{F}_0$ - quasi-coherent on Z*. Consider a morphism* $\mathrm{m} : \mathscr{F} \otimes \mathscr{N}_{\mathrm{Z/X}}^{\vee} \to \mathscr{F}_0$*. Then the obstruction to existence of an exact sequence*

$$0 \to i_*\mathscr{F}_0 \to \mathscr{G} \to i_*\mathscr{F} \to 0,$$

such that the ideal sheaf of Z *acts on $\mathscr{G}$ via* m *is* $\mathrm{m} \circ \alpha_{\mathscr{F}}$*.*

We are ready explain the relationship between these obstruction classes and the differentials on the second page of the spectral sequence. Thinking of $\alpha_{\mathscr{L}}$ as an extension class, it is clear that $\delta_0(i^*i_*\mathscr{L}) = \alpha_{\mathscr{L}}$ as $i^*i_*\mathscr{L}$ is concentrated in non-positive degrees. For the other Ext classes, we have to work harder. Let us factor $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ as follows:

$$\mathrm{Z} \overset{j}{\hookrightarrow} 2\mathrm{Z} \overset{2i}{\hookrightarrow} \mathrm{X},$$

where as usual 2Z is the first infinitesimal neighbourhood of Z in X. In the case of a structure sheaf $\mathscr{O}_{\mathrm{Z}}$, one can calculate $\delta_1(i^*i_*\mathscr{O}_{\mathrm{Z}})$ in terms of the deformation-obstruction classes:

Proposition 5.1.11. *(Arinkin-Căldăraru) Let* $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ *be a locally complete intersection. Then we have*

$$\delta_1(i^*i_*\mathscr{O}_{\mathrm{Z}}) = \alpha_{\mathscr{N}_{\mathrm{Z/X}}^{\vee}}.$$

Proof. Consider the exact sequence

$$0 \to j_*\mathscr{N}_{\mathrm{Z/X}}^{\vee} \to \mathscr{O}_{2\mathrm{Z}} \to j_*\mathscr{O}_{\mathrm{Z}} \to 0$$

apply j^* to get an exact triangle

$$j^*j_*\mathscr{N}_{Z/X}^\vee \to \mathscr{O}_Z \to j^*j_*\mathscr{O}_Z \to j^*j_*\mathscr{N}_{Z/X}^\vee[1],$$

hence an isomorphism

$$\tau^{<0}j^*j_*\mathscr{O}_Z \cong j^*j_*\mathscr{N}_{Z/X}^\vee[1]$$

in $\mathbf{D}^b(X)$. This implies that

$$\delta_1(j^*j_*\mathscr{O}_Z) = \delta_1(j^*j_*\mathscr{N}_{Z/X}^\vee[1]) = \delta_0(j^*j_*\mathscr{N}_{Z/X}^\vee) = \alpha_{\mathscr{N}_{Z/X}^\vee}.$$

We let $\mathrm{Alt} : \mathscr{F}^{\otimes p} \to \bigwedge^p \mathscr{F}$ be the natural quotient. Consider an exact triangle

$$\mathrm{Sym}^2\mathscr{N}_{Z/X}^\vee[2] \to \tau^{\geq -2}j^*j_*\mathscr{O}_Z \to \mathscr{C} \to \mathrm{Sym}^2\mathscr{N}_{Z/X}^\vee[3],$$

where the first map is the inclusion

$$\mathrm{Sym}^2\mathscr{N}_{Z/X}^\vee \to \mathscr{N}_{Z/X}^\vee \otimes_{\mathscr{O}_Z} \mathscr{N}_{Z/X}^\vee$$

followed by the canonical morphisms

$$\mathscr{N}_{Z/X}^\vee \otimes_{\mathscr{O}_Z} \mathscr{N}_{Z/X}^\vee \to \mathscr{H}^{-2}(j^*j_*\mathscr{O}_Z) \text{ and } \mathscr{H}^{-2}(j^*j_*\mathscr{O}_Z) \to \tau^{\geq -2}j^*j_*\mathscr{O}_Z.$$

By construction we have $\delta_1(\mathscr{C}) = \mathrm{Alt} \circ \delta_1(j^*j_*\mathscr{O}_Z)$, and since

$$\tau^{\geq -2}i^*i_*\mathscr{O}_Z \to \tau^{\geq -2}j^*j_*\mathscr{O}_Z \to \mathscr{C}$$

is an isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathrm{X})$, we conclude

$$\delta_1(i^*i_*\mathscr{O}_\mathrm{Z}) = \delta_1(\mathscr{C}) = \mathrm{Alt} \circ \delta_1(j^*j_*\mathscr{O}_\mathrm{Z}) = \mathrm{Alt} \circ \alpha_{\mathscr{N}_{\mathrm{Z/X}}^\vee}.$$

Theorem 5.1.10, and the exact sequence

$$0 \to i_*\mathrm{Sym}^2\mathscr{N}_{\mathrm{Z/X}}^\vee = \mathscr{I}_\mathrm{Z}^2/\mathscr{I}_\mathrm{Z}^3 \to \mathscr{I}_\mathrm{Z}/\mathscr{I}_\mathrm{Z}^3 \to i_*\mathscr{N}_{\mathrm{Z/X}}^\vee \to 0$$

imply that

$$\mathrm{Alt} \circ \alpha_{\mathscr{N}_{\mathrm{Z/X}}^\vee} = \alpha_{\mathscr{N}_{\mathrm{Z/X}}^\vee},$$

since $\mathrm{Sym} \circ \alpha_{\mathscr{N}_{\mathrm{Z/X}}^\vee} = 0$, that is, the obstruction class of the conormal bundle is skew-symmetric and we are done. □

Remark 5.1.12. It would be nice to have similar geometric interpretation for the higher δ's in the general case.

Remark 5.1.13. In general, if $\mathscr{F} \in \mathrm{Perf}(\mathrm{Z})$ lifts to the first infinitesimal neighbourhood, we have $\delta_1(i^*i_*\mathscr{F}) = \alpha_{\mathscr{F}\otimes\mathscr{N}_{\mathrm{Z/X}}^\vee}$. We note that Arinkin and Căldăraru in [AC12] show that $i^*i_*\mathscr{F}$ is formal iff $\alpha_\mathscr{F}$ and $\alpha_{\mathscr{F}\otimes\mathscr{N}_{\mathrm{Z/X}}^\vee}$ vanish. In the case of structure sheaf, formality is understood in the (stronger) sense of differential graded algebras.

Proposition 5.1.14. *Let* $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ *be a locally complete intersection of codimension* c *and consider* $\mathscr{L} \in \mathrm{Pic}(\mathrm{Z})$. *Define*

$$\tilde{\delta}_q(\mathscr{L}) := \delta_{c-q}(i^*i_*\mathscr{L}) \otimes \mathrm{id}_{\mathscr{L}^\vee} \otimes \mathrm{id}_{\det\mathscr{N}_{\mathrm{Z/X}}}$$

The differential d_2 *of*

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{X}, \mathscr{E}xt^q_{\mathscr{O}_\mathrm{X}}(i_*\mathscr{L}, i_*\mathscr{L})) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_\mathrm{X}}(i_*\mathscr{L}, i_*\mathscr{L})$$

can be described as

$$\mathrm{d}_2^{p,q} = \mathrm{R}^p\Gamma\tilde{\delta}_q(\mathscr{L}) : \mathrm{H}^p(\mathrm{Z}, \wedge^q \mathscr{N}_{\mathrm{Z/X}}) \to \mathrm{H}^{p+2}(\mathrm{Z}, \wedge^{q-1} \mathscr{N}_{\mathrm{Z/X}}).$$

Proof. As already explained Grothendieck-Verdier duality gives an isomorphism in the derived category $\mathbf{D}^\mathrm{b}\mathrm{F}(\mathrm{X})$ between canonically filtered complexes

$$\mathrm{R}\mathscr{H}om_{\mathscr{O}_\mathrm{X}}(i_*\mathscr{L}, i_*\mathscr{L}) \simeq i_*\mathrm{R}\mathscr{H}om_{\mathscr{O}_\mathrm{Z}}(\mathscr{L}, i^!i_*\mathscr{L}).$$

By definition (see Remark 5.1.5),

$$\mathrm{d}_2^{p,q} = \mathrm{R}^p\Gamma\delta_{-q}(\mathrm{R}\mathscr{H}om_{\mathscr{O}_\mathrm{X}}(i_*\mathscr{L}, i_*\mathscr{L})),$$

hence we are done since there is a canonical isomorphism

$$i_*\mathrm{R}\mathscr{H}om_{\mathscr{O}_\mathrm{Z}}(\mathscr{L}, i^!i_*\mathscr{L}) \simeq i_*(\mathscr{L}^\vee \otimes_{\mathscr{O}_\mathrm{Z}} i^!i_*\mathscr{L})$$

in $\mathbf{D}^\mathrm{b}\mathrm{F}(\mathrm{X})$. □

Proposition 5.1.15. *Let* $i : \mathrm{Z} \hookrightarrow \mathrm{X}$ *be a locally complete intersection and consider any*

$\mathscr{F} \in \mathrm{Perf}(Z)$. *Then* $i_*\alpha_{\mathscr{F}} = 0$. *Furthermore, the morphism*

$$\mathrm{Ext}^2_{\mathscr{O}_Z}(\mathscr{F}, \mathscr{F} \otimes_{\mathscr{O}_Z} \mathscr{N}^\vee_{Z/X}) \to \mathrm{Ext}^2_{\mathscr{O}_X}(i_*\mathscr{F}, i_*(\mathscr{F} \otimes_{\mathscr{O}_Z} \mathscr{N}^\vee_{Z/X}))$$

is injective iff $\alpha_{\mathscr{F}} = 0$.

Remark 5.1.16. We observe that there are two obvious cases

$$\mathscr{L} = \mathscr{O}_Z \text{ and } \mathscr{L} = \det \mathscr{N}_{Z/X}$$

in which the differential $d_2^{p,c} : H^p(Z, \wedge^c \mathscr{N}_{Z/X}) \to H^{p+2}(Z, \wedge^{c-1} \mathscr{N}_{Z/X})$ vanishes. For the structure sheaf, it's enough to note that $\delta_0(i^* i_* \mathscr{O}_Z) = 0$, while for $\det \mathscr{N}_{Z/X}$, we use Proposition 5.1.15.

Recall that Serre duality asserts that the pairing

$$H^p(L, \Omega^q_L) \otimes H^{n-p}(L, \Omega^{n-q}_L) \to H^n(L, K_L)$$

is perfect, where we set $n = \dim(L) = \mathrm{codim}(L, X)$ and K_L denotes the canonical bundle of L. Since d_2 kills $H^n(L, K_L)$, the higher differentials kill the appropriate subquotients, and d_2 is multiplicative, we see that the diagram

$$\begin{array}{ccc} H^p(L, \Omega^q_L) \otimes H^{n-p}(L, \Omega^{n-q}_L) & \underset{\mathrm{id}\otimes d_2}{\overset{d_2\otimes \mathrm{id}}{\rightleftarrows}} & H^{p+2}(L, \Omega^{q-1}_L) \otimes H^{n-p-2}(L, \Omega^{n-q+1}_L) \\ & \searrow \quad \mathbf{C} \quad \swarrow & \end{array}$$

commutes up to sign, hence the following diagram, and its variants for the higher differentials, commutes up to sign, i.e. the differentials are compatible with Serre duality:

$$\begin{array}{ccc} \mathrm{H}^p(\mathrm{L}, \Omega_{\mathrm{L}}^q) & \xrightarrow{\mathrm{d}_2} & \mathrm{H}^{p+2}(\mathrm{L}, \Omega_{\mathrm{L}}^{q-1}) \\ \downarrow & & \downarrow \\ \mathrm{H}^{n-p}(\mathrm{L}, \Omega_{\mathrm{L}}^{n-q})^\vee & \xrightarrow{\mathrm{d}_2^\vee} & \mathrm{H}^{n-p-2}(\mathrm{L}, \Omega_{\mathrm{L}}^{n-q+1})^\vee. \end{array}$$

Definition 5.1.17. A graded Lefschetz structure, in an abelian category $\mathscr{A}$, is a pair (H, L), where $\mathrm{H} \in \mathscr{A}$ is a graded object and L is a nilpotent endomorphism of H, such that $\mathrm{L}(\mathrm{H}^i) \subset \mathrm{H}^{i+2}$ and $\mathrm{L}^i : \mathrm{H}^{-i} \xrightarrow{\sim} \mathrm{H}^i$.

Example 5.1.18. Let X be a smooth projective variety of dimension n with an ample line bundle $\mathscr{L}$. Then, if ω is the first Chern class of $\mathscr{L}$ and

$$\mathrm{L}_\omega \coloneqq \omega \cup : \mathrm{H}^{n-*}(X/k) \to \mathrm{H}^{n-*}(X/k),$$

the pair $(\mathrm{H}^{n-*}(X/k), \mathrm{L}_\omega)$ is a graded Lefschetz structure (in fact, it is a $(-)$Hodge-Lefschetz structure - an example of a mixed Hodge structure).

5.2 Traces

We briefly review traces, mainly to fix notation. Suppose $\mathscr{F} \in \mathrm{Perf}(\mathrm{X})$ and let $\mathscr{V}$ be a vector bundle on X. Then we define the sheaf trace

$$\mathrm{Tr} : \mathrm{R}\mathscr{H}om_{\mathscr{O}_\mathrm{X}}(\mathscr{F}, \mathscr{F} \otimes_{\mathscr{O}_\mathrm{X}} \mathscr{V}) \to \mathscr{V}$$

as the compostion of the natural isomorphism

$$\mathrm{R}\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{V})\simeq\mathscr{F}^\vee\otimes_{\mathscr{O}_X}\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{V}$$

and the natural map $\mathscr{F}^\vee\otimes_{\mathscr{O}_X}\mathscr{F}\to\mathscr{O}_X$. Applying RΓ, and taking cohomology, we get the cohomological trace map of Illusie (see [Ill71]):

$$\mathrm{Tr}:\mathrm{Ext}^k_{\mathscr{O}_X}(\mathscr{F},\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{V})\to\mathrm{H}^k(X,\mathscr{V}).$$

If $\mathscr{F},\mathscr{G}\in\mathrm{Perf}(X)$, we also have a partial trace:

$$\mathrm{Tr}_{\mathscr{F}}:\mathrm{Ext}^k_{\mathscr{O}_X}(\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{G},\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{G}\otimes_{\mathscr{O}_X}\mathscr{V})\to\mathrm{Ext}^k_{\mathscr{O}_X}(\mathscr{G},\mathscr{G}\otimes_{\mathscr{O}_X}\mathscr{V}).$$

The trace is additive in the following sense: let $\mathscr{F}\to\mathscr{F}_2\to\mathscr{F}_3\to\mathscr{F}_1[1]$ be an exact triangle, and suppose $\alpha_i\in\mathrm{Ext}^k_{\mathscr{O}_X}(\mathscr{F}_i,\mathscr{F}_i\otimes_{\mathscr{O}_X}\mathscr{V})$ induce a morphism of exact triangles (notice $\mathscr{V}$ is flat so tensoring with $\mathscr{V}$ is exact), then $\mathrm{Tr}\alpha_1-\mathrm{Tr}\alpha_2+\mathrm{Tr}\alpha_3=0$. It is also multiplicative: if $\alpha\in\mathrm{Ext}^k_{\mathscr{O}_X}(\mathscr{F},\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{V})$ and $\beta\in\mathrm{Ext}^j_{\mathscr{O}_X}(\mathscr{V},\mathscr{E})$, for a vector bundle $\mathscr{E}$, then $\mathrm{Tr}((\mathrm{id}\otimes\beta)\circ\alpha)=\beta\circ\mathrm{Tr}\alpha$ in $\mathrm{H}^{i+j}(X,\mathscr{E})$. Applying this to the case of an obstruction class $\alpha_{\mathscr{V}}$, we get

$$\mathrm{Tr}\alpha_{\mathscr{V}}=\mathrm{Tr}((\mathrm{id}\otimes\mathrm{KS})\circ\mathrm{At}\mathscr{V})=\mathrm{KS}\circ\mathrm{Tr}(\mathrm{At}\mathscr{V})=\alpha_{\det\mathscr{V}}.$$

We give two examples for which the spectral sequence is non-degenerate, in fact, in both cases $\mathrm{d}_2^{0,1} \neq 0$.

Example 5.3.1. Suppose X is a K3-fibred Calabi-Yau and $i : \mathrm{S} \hookrightarrow \mathrm{X}$ is a smooth fibre. Assuming that S contains a $\mathbf{P}^1$, with normal bundle $\mathscr{O}_{\mathbf{P}^1}(-1) \oplus \mathscr{O}_{\mathbf{P}^1}(-1)$, which doesn't deform in X. We may suppose further that $\mathscr{O}_{\mathrm{S}}(\mathbf{P}^1)$ does not deform to first order as our surface moves in the fibration. Then the sheaf $i_*\mathscr{O}_{\mathrm{S}}(\mathbf{P}^1)$ does not deform in X, even to first order, hence,

$$\mathrm{Ext}^1_{\mathscr{O}_{\mathrm{X}}}(i_*\mathscr{O}_{\mathrm{S}}(\mathbf{P}^1), i_*\mathscr{O}_{\mathrm{S}}(\mathbf{P}^1)) = 0.$$

The spectral sequence has non-trivial terms on the second page

$$\mathrm{E}_2^{0,1} = \mathrm{H}^0(\mathrm{S}, \mathscr{O}_{\mathrm{S}}) = \mathbf{C},$$

$$\mathrm{E}_2^{2,0} = \mathrm{H}^2(\mathrm{S}, \mathscr{O}_{\mathrm{S}}) = \mathbf{C}.$$

For dimension reasons, $\mathrm{d}^{0,1} : \mathrm{E}_2^{0,1} \to \mathrm{E}_2^{2,0}$ is the only possibly non-zero differential. If it were to vanish, then we would get

$$\mathrm{Ext}^1_{\mathscr{O}_{\mathrm{X}}}(i_*\mathscr{O}_{\mathrm{S}}(\mathbf{P}^1), i_*\mathscr{O}_{\mathrm{S}}(\mathbf{P}^1)) \neq 0,$$

a contradiction. Thus,

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{S}, \wedge^q \mathscr{N}_{\mathrm{S/X}}) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_{\mathrm{X}}}(i_*\mathscr{O}_{\mathrm{S}}(\mathbf{P}^1), i_*\mathscr{O}_{\mathrm{S}}(\mathbf{P}^1))$$

does not degenerate on the second page and, in particular, there are normal directions which do not correspond to massless states of open strings.

Example 5.3.2. Let $S \to \mathbf{P}^1$ be an elliptic K3 surface and fix a smooth fibre E_1. Let E_2 be another smooth fibre which is isomorphic to E_1. Consider the Lagrangian surface

$$i : Z = E_1 \times E_2 \hookrightarrow S^{[2]}$$

in the Hilbert square of S. Then the line bundle $i_*\mathscr{O}(\Delta)$, where Δ is the diagonal, does not deform in a normal direction $\eta \in H^0(Z, \mathscr{N}_{Z/S^{[2]}})$ which moves E_1 and fixes E_2. By our description of the differentials on the second page, we see that

$$d_2^{0,1}(\eta) \in H^2(Z, \mathscr{O}_Z)$$

is the contraction of η with the class $\alpha_{\mathscr{O}(\Delta)} \in H^2(Z, \mathscr{N}^\vee_{Z/S^{[2]}})$ given by the natural map

$$H^2(Z, \mathscr{N}^\vee_{Z/S^{[2]}}) \otimes H^0(Z, \mathscr{N}_{Z/S^{[2]}}) \to H^2(Z, \mathscr{O}_Z).$$

This measures precisely the obstruction to deforming $i_*\mathscr{O}(\Delta)$ along η. Hence,

$$d_2^{0,1}(\eta) \neq 0$$

which shows that

$$E_2^{p,q} = H^p(Z, \wedge^q \mathscr{N}_{Z/X}) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_{S^{[2]}}}(i_*\mathscr{O}(\Delta), i_*\mathscr{O}(\Delta))$$

does not degenerate on the second page. Of course, it degenerates on E_3 for dimension reasons. If fact, since we have a Serre duality structure on the spectral sequence in the Lagrangian case, we see in dimension 2, that is, for Lagrangian surfaces, $d_2^{0,1}$ is the only possible obstruction to non-degeneracy, i.e. the spectral sequence degenerates on E_2 if and only if $d_2^{0,1} = 0$.

We record the observation at the end of Example 5.3.2 as a separate proposition.

Proposition 5.3.3. *Let $i : L \hookrightarrow X$ be a smooth complex Lagrangian surface in a compact hyperkähler fourfold $X/\mathbf{C}$, and let $\mathscr{L}$ be a line bundle on L. Then the local-to-global* Ext *spectral sequence*

$$E_2^{p,q} = H^p(L, \Omega_L^q) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_X}(i_*\mathscr{L}, i_*\mathscr{L})$$

degenerates on the second page if and only if $d_2^{0,1} = 0$.

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