

Modelling of the high-frequency fundamental symmetric Lamb wave using a new boundary element formulation

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Abstract

This paper presents a new boundary element formulation for modelling the fundamental symmetric Lamb wave (S0 mode) propagating in the high frequency range. At such high frequencies, the S0 mode reveals significant dispersive character. Conventional BEM formulations governed by the generalised plane stress theory fail to accurately represent the dispersive properties of the S0 mode and to handle out-of-plane loads because the effects of thickness-stretch are not considered. Therefore, a new BEM formulation is proposed based on the dynamic fundamental solutions which are derived for the first time for a higher-order plate theory (Kane-Mindlin) taking into account coupling between extensional motion and the first mode of thickness vibration. Only plate edges are required to be discretized using simple line elements in the proposed BEM formulation. Three benchmark examples are presented where the solutions from the new BEM formulation are shown to be in excellent agreement with analytical and three-dimensional finite element results. Furthermore, the advantage of the proposed formulation is demonstrated through comparisons with the BEM results based on the generalised plane stress theory.

Keywords: Kane-Mindlin theory; Boundary element method; Lamb wave; S0 mode; high-frequency wave

1. Introduction

Ultrasonic Lamb waves are widely used for the non-destructive testing (NDT) and structural health monitoring (SHM) [1]. The simulation of these waves based on analytical or numerical approaches plays a very important role in understanding the physical mechanisms, validating damage detection algorithms and providing a guide to experiments. Existing simulation tools have been compared and reviewed by Willberg et al. [2].

Analytical methods allow to obtain dispersion curves and dynamic responses accurately, but they are limited to plates with simple and regular geometries. Some semi-analytical methods, such as a semi-analytical finite element method (SAFE method) [3], are very suitable to obtain dispersion curves of layered composite plates, but it is very difficult to use these methods to carry out dynamic response analysis for real bounded plate structures. Although the finite difference method (FDM) is able to be used for such response analysis, its main drawback is the difficulty in handling irregular geometries, such as cracks in plates.

In order to conduct the simulation for plates with complex geometries and load distributions, the finite element method (FEM) is often used with the help of existing commercial FEM codes. However, since the working frequencies are in the kHz and MHz ranges, the requirements of small element size and time increment make the FEM computationally very expensive, especially for large structures [4, 5]. The computational efficiency can be improved using higher order finite element methods, such as a spectral element method (SEM) [6], which were summarised and compared in Ref. [7].

In spite of the popularity of the above-mentioned FE-based methods, it is very attractive to apply a boundary element method (BEM) to model Lamb waves for several reasons. First, the BEM allows reducing the dimension of the problems by one, and thus reducing the number of degrees of freedom in calculations substantially. Secondly, the BEM is very suitable for modelling cracks, which are treated easily as extra boundaries in the analysis. Thus, the BEM is a powerful tool for investigating the interaction of Lamb waves with cracks. Thirdly, the efforts in meshing and remeshing are reduced using the BEM. For example, when crack parameters, such as length, are changed, only crack surfaces are required to be remodelled, but the mesh associated with ordinary boundaries and the corresponding block matrix in the global coefficient matrix of the system of equations do not need to be updated.

The above-mentioned advantages make the BEM more efficient to solve guided wave scattering problems. Furthermore, with the help of the BEM, it is much easier to obtain the scattering patterns in a large number of scenarios involving different crack lengths, orientations, locations, shapes and numbers, which is very useful for the design and validation of damage detection algorithms. For example, the BEM is a very promising numerical method for an artificial neural network-based detection algorithm [8] requiring sufficient crack scenarios for the network training.

The BEM has been used to solve some Lamb wave problems, such as the wave reflection, mode conversion and interaction with defects [9-12]. However, these research were based on the plane strain assumption and unable to consider all the two in-plane dimensions. Thus, these BEM formulations could not be used to model real 3D Lamb wave problems. In order to solve these problems, the BEM formulations based on the theory of elasticity in three dimensions or based on plate/shell theories should be used.

Zou et al. [13] have applied the three-dimensional BEM to the health monitoring of a 3D beam structure, in which the crack was modelled using a dual boundary element method [14]. Then Zou and Aliabadi [15] adopted the basic idea behind the time-domain SEM [6] and introduced the spectral elements into the BEM to form a so-called boundary spectral element method, or a spectral BEM. In their research, the spectral BEM was shown to be more time-efficient in modelling ultrasonic waves in 3D structures than the conventional BEM.

The 3D BEM formulation is general and suitable for structures with arbitrary complex geometries, but this formulation performs badly for thin plate problems due to poor accuracy in the numerical integrations. Thus, the BEM formulations based on plate theories are more suitable for the modelling of Lamb waves, especially for the cases in which the frequency is not too high. These cases are very common and of great interest in the SHM [16-20] because in order to simplify the wave signals and make the analysis easier, the damage detection processes are often conducted below the cut-off frequency of the first antisymmetric Lamb mode (A1) [21]. In these cases, only the fundamental symmetric Lamb mode (S0) and fundamental antisymmetric Lamb mode (A0) are excited.

The A0 mode is able to be modelled using the existing boundary element formulations for the Mindlin plate bending theory [22, 23]. When it comes to the S0 mode, the 2D BEM based on the generalised plane stress theory [24] can be used readily. However, this 2D formulation has two disadvantages. First, only one dilatational wave, which is nondispersive, is captured using the generalised plane stress theory. This wave can be treated as the S0 mode reasonably only at low products of frequency and plate thickness. When the plate thickness is comparable to the wavelength of the high-frequency S0 Lamb wave, accurate results cannot be obtained using such a simple theory [25]. Secondly, this 2D formulation does not consider thickness-direction expansion and contraction of the plate. Thus, it is unable to describe the displacement component of the S0 mode in the thickness direction, and to analyse the responses of the S0 mode when the plate is subjected to out-of-plane loads.

In order to overcome these two disadvantages, a new boundary element formulation can be derived based on a first-order plate theory proposed by Kane and Mindlin (K-M theory) [26], which has been used in both analytical models [27, 28] and numerical analysis [29]. In such theory, the thickness-stretch displacement [30] is treated as an independent variable, and the variation of this displacement across the plate thickness is assumed to be linear. It should be noted that although this linear assumption is not adequate for describing the wave behaviour of the S0 mode in thickness direction exactly at higher frequencies, the K-M theory is capable of describing the dispersive behaviour of the S0 mode accurately below the cut-off frequency of the A1 mode for isotropic materials [31].

The boundary element formulations for the K-M theory and the Mindlin bending theory can be combined together to accurately simulate the wave propagation of the S0 and A0 modes simultaneously below the cut-off frequency of the A1 mode. It is worth noting that conventional shell elements in commercial FEM codes are capable of modelling the A0 mode in such frequency range, but these elements only allow modelling the S0 mode at low frequencies. This is because the generalised plane stress assumption is used for these elements and hence the dispersion effect of the S0 mode cannot be considered. However, this simulation can be conducted using the time-domain SEM based on novel first-order plate elements proposed by Zak [32]. Although the above-mentioned SEM formulation is efficient in modelling the S0 and A0 modes, the application of the plate-theory-based BEM formulations to such simulation is still of great value because only plate edges need to be discretized using simple line elements and the number of degrees of freedom is significantly reduced. In addition, the efficiency of the BEM can be further improved by using spectral boundary elements [33].

All of the above-mentioned promising applications of the BEM rely on the existence of a new formulation for the K-M theory, which requires corresponding dynamic fundamental solutions. In this paper, such fundamental solutions are first derived employing a Fourier transformation method [22]. The dynamic problems are solved in the Laplace domain, which leads to a boundary-only formulation because the direct evaluation of the domain integral resulting from the inertial term can be avoided. Durbin's method [34] is used to carry out numerical inversion of the Laplace transforms. On the basis of the newly derived fundamental solutions, a boundary element formulation for the analyses of dynamic K-M plate problems is first proposed. Three benchmark examples are presented where BEM results are shown to be in excellent agreement with analytical results and those from 3D FEM simulation.

2. Laplace transform fundamental solutions

The stress state in an isotropic plate is shown in Figure 1. Roman indices (i,j,k) vary from 1 to 3 and Greek indices (α, β) vary from 1 to 2 throughout this paper. The x_1 - x_2 plane is located at the mid-surface of the plate, and the x_3 -axis is the normal to the x_1 - x_2 plane.

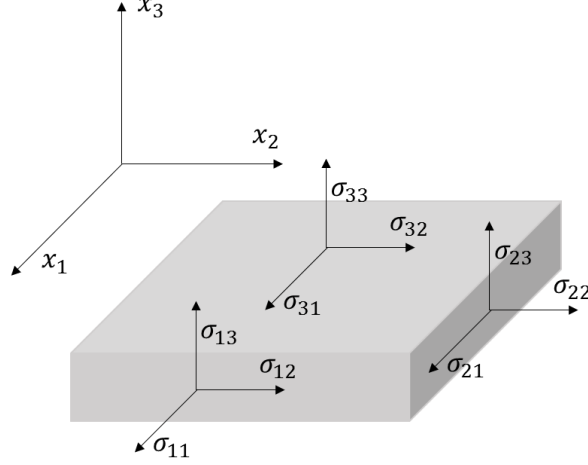


Figure 1. Stresses on the surfaces of a plate.

According to Kane and Mindlin [26], the three-dimensional displacements u_i of the plate are approximated using the following expressions:

$$\begin{aligned} u_1(x_1, x_2, x_3, t) &= v_1(x_1, x_2, t) \\ u_2(x_1, x_2, x_3, t) &= v_2(x_1, x_2, t) \\ u_3(x_1, x_2, x_3, t) &= v_3(x_1, x_2, t)x_3 \end{aligned} \quad (1)$$

where v_i are two-dimensional displacements which are independent of x_3 ; v_3 is also called “thickness-stretch displacement” [30].

Based on the above assumption, the stress equilibrium equations in terms of stress resultants can be written as:

$$\begin{aligned} N_{\alpha\beta,\beta} + F_\alpha &= 2\rho h \frac{\partial^2 v_\alpha}{\partial t^2} \\ R_{\alpha,\alpha} - N_{33} + F_3 &= \frac{2\rho h^3}{3} \frac{\partial^2 v_3}{\partial t^2} \end{aligned} \quad (2)$$

where

$$\begin{aligned}
N_{\alpha\beta} &= \int_{-h}^h \sigma_{\alpha\beta} dx_3 = 2h \left[\lambda (v_{\gamma,\gamma} + \kappa v_3) \delta_{\alpha\beta} + \mu (v_{\alpha,\beta} + v_{\beta,\alpha}) \right] \\
N_{33} &= \int_{-h}^h \sigma_{33} dx_3 = 2h \left[\kappa^2 (\lambda + 2\mu) v_3 + \kappa \lambda v_{\gamma,\gamma} \right] \\
R_\alpha &= \int_{-h}^h \sigma_{\alpha 3} x_3 dx_3 = \frac{2h^3 \mu}{3} v_{3,\alpha} \\
F_\alpha &= \sigma_{3\alpha} \big|_{x_3=h} - \sigma_{3\alpha} \big|_{x_3=-h} \\
F_3 &= h \left(\sigma_{33} \big|_{x_3=h} + \sigma_{33} \big|_{x_3=-h} \right)
\end{aligned} \tag{3}$$

$\lambda = E\nu / [(1+\nu)(1-2\nu)]$ and $\mu = E / [2(1+\nu)]$ are elastic constants; E , ν , ρ and h represent Young's modulus, Poisson's ratio, the density and half plate thickness, respectively; t denotes the time. A shear coefficient κ is used to improve results, which is similar to the shear correction factor used in the Mindlin plate bending theory [35]. $N_{\alpha\beta}$ and N_{33} are stress resultants; R_α are generalised shear forces which cause the pinching effect; F_α and F_3 represent applied surface tractions.

The displacement equations of motion are given by:

$$2h\mu\nabla^2 v_\alpha + 2h(\lambda + \mu)v_{\beta,\beta\alpha} + 2\kappa h\lambda v_{3,\alpha} + F_\alpha = 2\rho h \frac{\partial^2 v_\alpha}{\partial t^2} \tag{4}$$

$$\frac{2h^3\mu}{3}\nabla^2 v_3 - 2\kappa h\lambda v_{\beta,\beta} - 2\kappa^2 h(\lambda + 2\mu)v_3 + F_3 = \frac{2\rho h^3}{3} \frac{\partial^2 v_3}{\partial t^2} \tag{5}$$

The Laplace transform of Eqs. (4) - (5) can be written as:

$$2h\mu\nabla^2 \tilde{v}_\alpha + 2h(\lambda + \mu)\tilde{v}_{\beta,\beta\alpha} + 2\kappa h\lambda \tilde{v}_{3,\alpha} + \tilde{F}_\alpha = 2\rho h p^2 \tilde{v}_\alpha \tag{6}$$

$$\frac{2h^3\mu}{3}\nabla^2 \tilde{v}_3 - 2\kappa h\lambda \tilde{v}_{\beta,\beta} - 2\kappa^2 h(\lambda + 2\mu)\tilde{v}_3 + \tilde{F}_3 = \frac{2\rho h^3 p^2}{3} \tilde{v}_3 \tag{7}$$

where p is the Laplace transform parameter. The Laplace transform of a function $f(\mathbf{x}, t)$ is defined as:

$$\mathcal{L}[f(\mathbf{x}, t)] = \tilde{f}(\mathbf{x}, p) = \int_0^\infty f(\mathbf{x}, t) e^{-pt} dt. \tag{8}$$

Regardless of body forces $\tilde{F}_i$, Eqs. (6) - (7) can be decomposed into three Helmholtz-type differential equations:

$$(\nabla^2 - \zeta_i^2) \tilde{\varphi}_i = 0, \quad i = 1, 2, 3, \quad (9)$$

where $\tilde{\varphi}_i$ represent three displacement potentials. This decomposition means that the Kane-Mindlin theory can represent three types of elastic waves with wave numbers

$$\zeta_\beta^2 = \frac{3\kappa^2}{2\alpha_2 h^2} \left[(\alpha_1 + \alpha_2) \frac{p^2}{\varpi^2} + 1 + (-1)^\beta D \right], \quad \zeta_3^2 = \frac{p^2}{c_2^2}, \quad (10)$$

in which

$$D = \sqrt{\left[(\alpha_1 + \alpha_2) \frac{p^2}{\varpi^2} + 1 \right]^2 - 4\alpha_1\alpha_2 \frac{p^2}{\varpi^2} \left(1 + \frac{p^2}{\varpi^2} \right)}; \quad (11)$$

the cut-off frequency ϖ is equal to $\sqrt{3}\kappa c_1 / h$ and the constants α_β denote c_β^2 / c_3^2 ; the wave velocities $c_1 = \sqrt{(\lambda + 2\mu) / \rho}$ and $c_3 = \sqrt{E / [\rho(1 - \nu^2)]}$ have the same expressions as those of the longitudinal wave in the plane strain theory and generalised plane stress theory, respectively. The shear wave velocity c_2 is equal to $\sqrt{\mu / \rho}$.

The transformed displacements $\tilde{v}_i$ are expressed in terms of three independent potentials $\tilde{\varphi}_i$ as:

$$\begin{aligned} \tilde{v}_1 &= \tilde{\varphi}_{1,1} + \tilde{\varphi}_{2,1} + \tilde{\varphi}_{3,2} \\ \tilde{v}_2 &= \tilde{\varphi}_{1,2} + \tilde{\varphi}_{2,2} - \tilde{\varphi}_{3,1} \\ \tilde{v}_3 &= e_1 \tilde{\varphi}_1 + e_2 \tilde{\varphi}_2 \end{aligned} \quad (12)$$

in which

$$e_\beta = -\frac{\lambda + 2\mu}{\kappa\lambda} \left(\zeta_\beta^2 - \frac{p^2}{c_1^2} \right) \quad (13)$$

Suppose that the applied force F_2 is a unit point force applied at the origin and is represented by the Dirac delta function $\delta(t)\delta(x_1, x_2)$. It is convenient to analyse this problem using half of the plate ($x_2 \geq 0$). The displacements $\tilde{v}_i$ have the following symmetry properties:

$$\tilde{v}_1(x_1, x_2) = -\tilde{v}_1(-x_1, x_2), \quad \tilde{v}_2(x_1, x_2) = \tilde{v}_2(-x_1, x_2), \quad \tilde{v}_3(x_1, x_2) = \tilde{v}_3(-x_1, x_2) \quad (14)$$

Thus, with the help of the Fourier cosine and sine transforms, the general solutions for Eq. (9) can be obtained as:

$$\tilde{\varphi}_1 = \frac{2}{\pi} \int_0^\infty A_1 e^{-\beta_1 x_2} \cos(sx_1) ds, \quad \tilde{\varphi}_2 = \frac{2}{\pi} \int_0^\infty A_2 e^{-\beta_2 x_2} \cos(sx_1) ds, \quad \tilde{\varphi}_3 = \frac{2}{\pi} \int_0^\infty A_3 e^{-\beta_3 x_2} \sin(sx_1) ds, \quad (15)$$

in which $\beta_j^2 = s^2 + \zeta_j^2$.

Substituting Eq. (15) into Eq. (12) yields:

$$\tilde{v}_1 = -\frac{2}{\pi} \int_0^\infty [sA_1 e^{-\beta_1 x_2} + sA_2 e^{-\beta_2 x_2} + \beta_3 A_3 e^{-\beta_3 x_2}] \sin(sx_1) ds \quad (16)$$

$$\tilde{v}_2 = -\frac{2}{\pi} \int_0^\infty [\beta_1 A_1 e^{-\beta_1 x_2} + \beta_2 A_2 e^{-\beta_2 x_2} + sA_3 e^{-\beta_3 x_2}] \cos(sx_1) ds \quad (17)$$

$$\tilde{v}_3 = \frac{2}{\pi} \int_0^\infty [e_1 A_1 e^{-\beta_1 x_2} + e_2 A_2 e^{-\beta_2 x_2}] \cos(sx_1) ds \quad (18)$$

where A_i are unknown coefficients which can be determined using the boundary conditions.

Substituting Eqs. (16) - (18) into Eq. (3) gives:

$$\begin{aligned} \tilde{N}_{11} = & -\frac{4h}{\pi} \int_0^\infty \left\{ [(\lambda + 2\mu)s^2 - \lambda(\kappa e_1 + \beta_1^2)] A_1 e^{-\beta_1 x_2} \right. \\ & \left. + [(\lambda + 2\mu)s^2 - \lambda(\kappa e_2 + \beta_2^2)] A_2 e^{-\beta_2 x_2} + 2\mu s \beta_3 A_3 e^{-\beta_3 x_2} \right\} \cos(sx_1) ds \end{aligned} \quad (19)$$

$$\begin{aligned} \tilde{N}_{22} = & \frac{4h}{\pi} \int_0^\infty \left\{ [(\lambda + 2\mu)\beta_1^2 + \lambda(\kappa e_1 - s^2)] A_1 e^{-\beta_1 x_2} \right. \\ & \left. + [(\lambda + 2\mu)\beta_2^2 + \lambda(\kappa e_2 - s^2)] A_2 e^{-\beta_2 x_2} + 2\mu s \beta_3 A_3 e^{-\beta_3 x_2} \right\} \cos(sx_1) ds \end{aligned} \quad (20)$$

$$\tilde{N}_{21} = \frac{4h\mu}{\pi} \int_0^\infty [2s\beta_1 A_1 e^{-\beta_1 x_2} + 2s\beta_2 A_2 e^{-\beta_2 x_2} + (s^2 + \beta_3^2) A_3 e^{-\beta_3 x_2}] \sin(sx_1) ds \quad (21)$$

$$\tilde{R}_1 = \frac{4h^3\mu}{3\pi} \int_0^\infty [se_1 A_1 e^{-\beta_1 x_2} + se_2 A_2 e^{-\beta_2 x_2}] \sin(sx_1) ds \quad (22)$$

$$\tilde{R}_2 = -\frac{4h^3\mu}{3\pi} \int_0^\infty [e_1 \beta_1 A_1 e^{-\beta_1 x_2} + e_2 \beta_2 A_2 e^{-\beta_2 x_2}] \cos(sx_1) ds \quad (23)$$

The boundary conditions along the axis x_1 have the following forms:

$$v_1 = 0, v_3 = 0, N_{22} = -\frac{1}{2} \delta(t) \delta(x_1, 0). \quad (24)$$

Considering the expressions (16), (18) and (20), the boundary conditions lead to the following equations:

$$\begin{aligned} sA_1 + sA_2 + \beta_3 A_3 &= 0 \\ e_1 A_1 + e_2 A_2 &= 0 \\ 2h \left\{ \left[(\lambda + 2\mu) \beta_1^2 + \lambda (\kappa e_1 - s^2) \right] A_1 + \left[(\lambda + 2\mu) \beta_2^2 + \lambda (\kappa e_2 - s^2) \right] A_2 + 2\mu s \beta_3 A_3 \right\} &= -\frac{1}{4} \end{aligned} \quad (25)$$

Solving these equations, we have:

$$A_1 = -\frac{e_2}{8h\mu(e_2 - e_1)\zeta_3^2}, A_2 = \frac{e_1}{8h\mu(e_2 - e_1)\zeta_3^2}, A_3 = \frac{s}{8h\mu\beta_3\zeta_3^2} \quad (26)$$

Substituting Eq. (26) into Eqs. (16) - (18) and using the integral form of the zero-order modified Bessel function of the second kind

$$K_0(\zeta_i r) = \int_0^\infty \frac{\cos(sx_1)}{\beta_i} e^{-\beta_i x_2} ds, \quad (27)$$

in which $r = \sqrt{x_1^2 + x_2^2}$, the fundamental solutions $\tilde{U}_{2i}$ can be written in the following forms:

$$\tilde{U}_{21} = \tilde{v}_1 = \frac{1}{4\pi h\mu(e_2 - e_1)\zeta_3^2} \frac{\partial^2}{\partial x_1 \partial x_2} [e_2 K_0(z_1) - e_1 K_0(z_2) - (e_2 - e_1) K_0(z_3)] \quad (28)$$

$$\begin{aligned} \tilde{U}_{22} = \tilde{v}_2 &= \frac{1}{4\pi h\mu(e_2 - e_1)\zeta_3^2} \frac{\partial^2}{\partial x_2^2} [e_2 K_0(z_1) - e_1 K_0(z_2) - (e_2 - e_1) K_0(z_3)] \\ &+ \frac{1}{4\pi h\mu} K_0(z_3) \end{aligned} \quad (29)$$

$$\tilde{U}_{23} = \tilde{v}_3 = \frac{3}{4\pi h^3 \mu(e_2 - e_1)} \frac{\partial}{\partial x_2} [K_0(z_1) - K_0(z_2)] \quad (30)$$

in which $z_i = \zeta_i r$.

In a similar manner, the fundamental solutions $\tilde{U}_{1i}$ can be obtained easily using the above method.

When the external force $F_3 = \delta(t)\delta(x_1, x_2)$ is applied at the origin of the plate, only half of the plate ($x_2 \geq 0$) is analysed due to symmetry properties. It can be verified easily that the displacements and stress resultants have the same expressions as those in Eqs. (16) - (23).

Fundamental solutions can be obtained after considering the boundary conditions along the axis x_1 :

$$v_2 = 0, N_{21} = 0, R_2 = -\frac{1}{2}\delta(t)\delta(x_1, 0). \quad (31)$$

The detailed expressions of the fundamental solutions $\tilde{U}_{ij}$ and $\tilde{T}_{ij}$ are listed in Appendix A.

3. Boundary integral formulation in the Laplace domain

In this section, the boundary integral equation for Kane-Mindlin plate theory is derived for the first time.

In the Laplace domain, the stress equations of motion in Eq. (2) have the following forms:

$$\begin{aligned} \tilde{N}_{\alpha\beta,\beta} + \tilde{F}_\alpha &= 2\rho h p^2 \tilde{v}_\alpha \\ \tilde{R}_{\alpha,\alpha} - \tilde{N}_3 + \tilde{F}_3 &= \frac{2}{3}\rho h^3 p^2 \tilde{v}_3 \end{aligned} \quad (32)$$

Thus, an integral identity based on the above equilibrium equations can be written as:

$$\int_{\Omega} \left[\left(\tilde{N}_{\alpha\beta,\beta} + \tilde{F}_\alpha - 2\rho h p^2 \tilde{v}_\alpha \right) \tilde{v}_\alpha^* + \left(\tilde{R}_{\alpha,\alpha} - \tilde{N}_3 + \tilde{F}_3 - \frac{2}{3}\rho h^3 p^2 \tilde{v}_3 \right) \tilde{v}_3^* \right] d\Omega = 0 \quad (33)$$

where $\tilde{v}_\alpha^*$ and $\tilde{v}_3^*$ are transformed displacements in another independent self-equilibrated state; Ω denotes the plate domain.

Repeated integration by parts (i.e. using the divergence theorem), and considering the relationship between the tractions and stress resultants $t_\alpha = N_{\alpha\beta}n_\beta$, $t_3 = R_\alpha n_\alpha$, Eq. (33) becomes

$$\begin{aligned} &\int_{\Gamma} \tilde{t}_j \tilde{v}_j^* d\Gamma + \int_{\Omega} \tilde{F}_j \tilde{v}_j^* d\Omega \\ &= \int_{\Omega} \tilde{R}_\alpha \tilde{v}_{3,\alpha}^* d\Omega + \int_{\Omega} \tilde{N}_{\alpha\beta} \tilde{v}_{\alpha,\beta}^* d\Omega + \int_{\Omega} \tilde{N}_{33} \tilde{v}_3^* d\Omega + \int_{\Omega} 2\rho h p^2 \tilde{v}_\alpha \tilde{v}_\alpha^* d\Omega + \int_{\Omega} \frac{2}{3}\rho h^3 p^2 \tilde{v}_3 \tilde{v}_3^* d\Omega \end{aligned} \quad (34)$$

where Γ represents the plate edge boundaries.

Substituting Eq. (3) into Eq. (34), we have:

$$\int_{\Gamma} \tilde{t}_j \tilde{v}_j d\Gamma + \int_{\Omega} \tilde{F}_j \tilde{v}_j d\Omega = \int_{\Gamma} \tilde{t}_j \tilde{v}_j^* d\Gamma + \int_{\Omega} \tilde{F}_j \tilde{v}_j^* d\Omega \quad (35)$$

which means Betti's reciprocal work theorem is still valid for the Kane-Mindlin plates.

When the state $(.)^*$ represents the fundamental state, Eq. (35) can be written as

$$\begin{aligned}\tilde{v}_i(\mathbf{X}', p) = & \int_{\Gamma} \tilde{U}_{ij}(\mathbf{X}', \mathbf{x}, p) \tilde{t}_j(\mathbf{x}, p) d\Gamma - \int_{\Gamma} \tilde{T}_{ij}(\mathbf{X}', \mathbf{x}, p) \tilde{v}_j(\mathbf{x}, p) d\Gamma \\ & + \int_{\Omega} \tilde{U}_{ij}(\mathbf{X}', \mathbf{X}, p) \tilde{F}_j(\mathbf{X}, p) d\Omega\end{aligned}\quad (36)$$

where $\mathbf{X}'$ is an arbitrary source point in the plate domain Ω ; $\mathbf{X}$ and $\mathbf{x}$ denote a field point in the domain Ω and on the boundary Γ , respectively.

The displacement boundary integral equation can be obtained by moving the source point $\mathbf{X}'$ to the plate edge boundary $\mathbf{x}'$:

$$\begin{aligned}c_{ij}(\mathbf{x}') \tilde{v}_j(\mathbf{x}', p) = & \int_{\Gamma} \tilde{U}_{ij}(\mathbf{x}', \mathbf{x}, p) \tilde{t}_j(\mathbf{x}, p) d\Gamma - C \int_{\Gamma} \tilde{T}_{ij}(\mathbf{x}', \mathbf{x}, p) \tilde{v}_j(\mathbf{x}, p) d\Gamma \\ & + \int_{\Omega} \tilde{U}_{ij}(\mathbf{x}', \mathbf{X}, p) \tilde{F}_j(\mathbf{X}, p) d\Omega\end{aligned}\quad (37)$$

where $C \int_{\Gamma}$ denotes Cauchy principal value integral; $c_{ij}(\mathbf{x}')$ are coefficients which are determined by the geometry at the source point $\mathbf{x}'$ [36]. The corresponding boundary integral equation for static problems can be obtained easily by setting $p = 0$.

4. Treatment of singularities in the transformed fundamental solutions

Singular integrals exist in the above-derived boundary integral equation. To compute these singular integrals, the types of singularities in the transformed fundamental solutions should be understood firstly. As shown in Appendix C, the transformed fundamental solutions have the same singularities, which do not contain the Laplace transform parameter p , as those for the static problems. Thus, it is convenient to write the transformed fundamental solutions into two parts:

$$\tilde{U}_{ij}(\mathbf{x}', \mathbf{x}, p) = U_{ij}^{\text{static}}(\mathbf{x}', \mathbf{x}) + \tilde{U}_{ij}^{\text{m}}(\mathbf{x}', \mathbf{x}, p) \quad (38)$$

$$\tilde{T}_{ij}(\mathbf{x}', \mathbf{x}, p) = T_{ij}^{\text{static}}(\mathbf{x}', \mathbf{x}) + \tilde{T}_{ij}^{\text{m}}(\mathbf{x}', \mathbf{x}, p) \quad (39)$$

where U_{ij}^{static} and T_{ij}^{static} are static fundamental solutions (see Appendix B); $\tilde{U}_{ij}^{\text{m}}$ and $\tilde{T}_{ij}^{\text{m}}$ are modified terms which do not contain any singularities. Based on such separation, the results of the integrals due to the singular parts of $\tilde{U}_{ij}$ and $\tilde{T}_{ij}$ can be obtained through the calculation of those involving U_{ij}^{static} and T_{ij}^{static} .

In the static fundamental solutions, there are only two types of singularities: weak singularities of order $\ln(r)$ and strong singularities of order $1/r$. The strong singular integrals involving $T_{\alpha\beta}^{\text{static}}$

are evaluated indirectly through the rigid body motion. The plate can move arbitrarily along the direction x_1 or x_2 as a rigid body. In these cases, the plate is not subjected to any surface tractions, and the rigid body movements can be expressed as:

$$v_1 = C, v_2 = 0, v_3 = 0 \quad (40)$$

$$v_2 = C, v_1 = 0, v_3 = 0 \quad (41)$$

where C is an arbitrary constant.

It should be noted that surface tractions are required to maintain the state in which the generalised displacement v_3 is equal to a constant and the in-plane displacements v_α are zero. Thus, the rigid body motion cannot be used to compute the strong singular integrals involving T_{33}^{static} . However, such singular integrals can be estimated analytically.

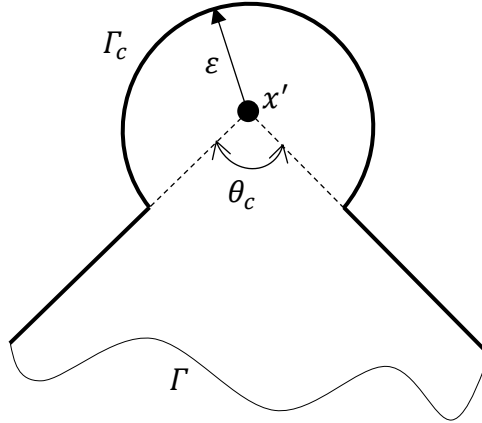


Figure 2. Treatment of the boundary around the source point $\mathbf{x}'$.

Consider a source point $\mathbf{x}'$ at a boundary corner with an inner angle θ_c . As shown in Figure 2, it is convenient to use an artificial arc Γ_c centred at the source point $\mathbf{x}'$, with an infinitesimal radius ϵ , to represent the boundary around the point $\mathbf{x}'$ [36]. The singular integral involving T_{33}^{static} is divided into two parts:

$$\begin{aligned} \int_{\Gamma} T_{33}^{\text{static}}(\mathbf{x}', \mathbf{x}) \tilde{v}_3(\mathbf{x}, p) d\Gamma &= \lim_{\epsilon \rightarrow 0} \int_{\Gamma_c} T_{33}^{\text{static}}(\mathbf{x}', \mathbf{x}) \tilde{v}_3(\mathbf{x}, p) d\Gamma \\ &+ \lim_{\epsilon \rightarrow 0} \int_{\Gamma - \Gamma_c} T_{33}^{\text{static}}(\mathbf{x}', \mathbf{x}) \tilde{v}_3(\mathbf{x}, p) d\Gamma \end{aligned} \quad (42)$$

in which the second integral on the right hand side can be treated as a Cauchy principal-value integral.

When continuous boundary elements are used for the discretization, the above-mentioned Cauchy principal-value integral exists. Also, such principal-value integral is equivalent to the finite-part integral of the divergent integral. As explained in Ref. [36], the definitions of one-sided and two sided finite-part integrals are used when the collocation is carried out at an end node and an inner node of an element, respectively. Based on these definitions, this integral can then be evaluated analytically.

Substituting Eq. (C. 7) into Eq. (42), the first integral on the right hand side of Eq. (42) is determined using the polar coordinate system as follows:

$$\lim_{\varepsilon \rightarrow 0} \int_{\Gamma_c} T_{33}^{\text{static}}(\mathbf{x}', \mathbf{x}) \tilde{v}_3(\mathbf{x}, p) d\Gamma = \tilde{v}_3(\mathbf{x}', p) \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_c} -\frac{1}{2\pi\varepsilon} d\Gamma = -\tilde{v}_3(\mathbf{x}', p) + \frac{\theta_c}{2\pi} \tilde{v}_3(\mathbf{x}', p) \quad (43)$$

Thus, the displacement boundary integral equation regarding $\tilde{T}_{3j}$ in Eq. (37) can be rewritten as follows:

$$\begin{aligned} \frac{\theta_c}{2\pi} \tilde{v}_3(\mathbf{x}', p) = & \int_{\Gamma} \tilde{U}_{3j}(\mathbf{x}', \mathbf{x}, p) \tilde{t}_j(\mathbf{x}, p) d\Gamma - C \int_{\Gamma} \tilde{T}_{3j}(\mathbf{x}', \mathbf{x}, p) \tilde{v}_j(\mathbf{x}, p) d\Gamma \\ & + \int_{\Omega} \tilde{U}_{3j}(\mathbf{x}', \mathbf{X}, p) \tilde{F}_j(\mathbf{X}, p) d\Omega \end{aligned} \quad (44)$$

5. Numerical implementation

The numerical implementation is the same as that in the BEM based on the Mindlin plate bending theory [22]. Quadratic isoparametric line elements [36] are used to discretize plate edges. Regardless of the body forces $\tilde{F}_j$, the discretized displacement boundary integral equation is given by:

$$\begin{aligned} c_{ij}^l \tilde{v}_j^l(p) = & \sum_{m=1}^M \sum_{n=1}^3 \left[\tilde{t}_j^{nm}(p) \int_{-1}^1 \tilde{U}_{ij}(\xi, p) N_n(\xi) J_m(\xi) d\xi - \right. \\ & \left. \tilde{v}_j^{nm}(p) C \int_{-1}^1 \tilde{T}_{ij}(\xi, p) N_n(\xi) J_m(\xi) d\xi \right] \end{aligned} \quad (45)$$

where $N_n(\xi)$ and $J_m(\xi)$ are the shape functions and the Jacobians; l denotes the collocation node number and M is the total number of boundary line elements.

Eq. (45) can be written in a matrix form:

$$\tilde{\mathbf{H}}(p) \tilde{\mathbf{v}}(p) = \tilde{\mathbf{G}}(p) \tilde{\mathbf{t}}(p) \quad (46)$$

where $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{G}}$ depend on the integrals involving $\tilde{T}_{ij}$ and $\tilde{U}_{ij}$, respectively. After boundary conditions in the Laplace domain are applied, Eq. (46) becomes:

$$\tilde{\mathbf{A}}(p)\tilde{\mathbf{x}}(p) = \tilde{\mathbf{f}}(p) \quad (47)$$

where $\tilde{\mathbf{f}}$ contains known coefficients, and unknown variables are stored in $\tilde{\mathbf{x}}$.

After solving the above algebraic equations, unknown displacements and tractions are obtained in the Laplace domain. These results are converted back to the time domain through the numerical inversion of the Laplace transforms given by Durbin [34]:

$$f(t) = \frac{e^{at}}{T} \left\{ \frac{1}{2} \text{Re}[\tilde{f}(a)] + \sum_{k=1}^K \text{Re}[\tilde{f}(p)] \cos k \frac{\pi}{T} t - \sum_{k=1}^K \text{Im}[\tilde{f}(p)] \sin k \frac{\pi}{T} t \right\} \quad (48)$$

$$, p = a + ik \frac{\pi}{T},$$

where p , K , and T are the Laplace transform parameter, the maximum number of Laplace term, and the total time period, respectively. The real part a of the Laplace transform parameter is determined using the relationship $aT = 3$, which leads to satisfactory results.

6. Wave Dispersion

Two dispersive dilatational waves and one nondispersive shear wave are represented in the Kane-Mindlin plate theory. After replacing the Laplace parameter p with $-i\omega$ in Eqs. (9) and (10), the governing equations for these three waves can be obtained.

In order to study dispersion relations, the potentials are assumed to have the following harmonic forms:

$$\varphi_i = \Phi_i e^{i(\mathbf{k}_i \cdot \mathbf{x} - \omega t)}, \quad i = 1, 2, 3, \quad (49)$$

where Φ_i , $\mathbf{k}_i$ and ω represent the wave amplitude, the wave vector and the frequency, respectively.

Substituting Eq. (49) into the wave equations modified from Eq. (9) yields:

$$k_i^2 = \zeta_i^2, \quad i = 1, 2, 3, \quad (50)$$

where k_i is the magnitude of the wave vector $\mathbf{k}_i$.

The group velocity c_i^g is given by:

$$c_i^g = \frac{d\omega}{dk_i}, \quad i = 1, 2, 3. \quad (51)$$

Thus, differentiating Eq. (50) with respect to k_i gives:

$$c_\beta^g = \frac{2k_\beta \alpha_2 h^2}{3\kappa^2} \frac{\varpi^2}{\omega} \frac{D}{(\alpha_1 + \alpha_2) \left\{ D - (-1)^\beta \left[(\alpha_1 + \alpha_2) \frac{\omega^2}{\varpi^2} - 1 \right] \right\} + 2\alpha_1 \alpha_2 \left(1 - \frac{2\omega^2}{\varpi^2} \right)}, \quad c_3^g = c_2, \quad (52)$$

in which

$$D = \sqrt{\left[(\alpha_1 + \alpha_2) \frac{\omega^2}{\varpi^2} - 1 \right]^2 + 4\alpha_1 \alpha_2 \frac{\omega^2}{\varpi^2} \left(1 - \frac{\omega^2}{\varpi^2} \right)} \quad (53)$$

When the frequency tends to infinity, the group velocities of the three waves become:

$$\lim_{\omega \rightarrow \infty} c_1^g = c_2, \quad \lim_{\omega \rightarrow \infty} c_2^g = c_1, \quad \lim_{\omega \rightarrow \infty} c_3^g = c_2 \quad (54)$$

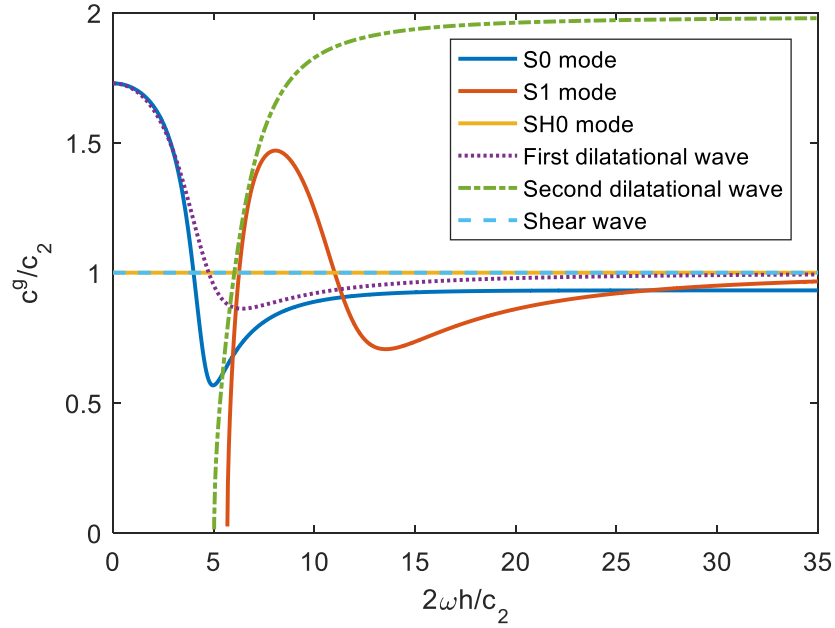


Figure 3. Comparison between the dimensionless group velocity dispersion curves obtained based on the K-M theory and those for the first two symmetric Lamb wave modes and SH0 mode ($\nu = 0.33$).

Figure 3 presents the dimensionless group velocity c^g/c_2 as a function of the dimensionless frequency $2\omega h/c_2$ for the three waves when the Poisson's ratio is set to 0.33. It should be noted that the cut-off frequency of the second dilatational wave is equal to ϖ , which has been defined

in Eq. (11). Furthermore, Figure 3 shows the relationship between these three waves and the first two symmetric Lamb wave modes (S0 and S1) and the fundamental shear horizontal guided wave mode (SH0). The first and second dilatational waves can be used to approximately represent the S0 and S1 Lamb wave modes, respectively. The K-M theory gives a much better approximation for the S0 mode than the S1 mode. The shear coefficient κ in Eqs. (6) - (7) is able to be adjusted to achieve the optimal approximation to the S0 mode at different working frequencies.

The shear coefficient used for plotting Figure 3 is equal to 0.73. This value enables the first dilatational wave to represent the S0 mode accurately (relative error <1%) below a dimensionless frequency of 3.18, which is higher than the dimensionless cut-off frequency of the A1 mode (π). When it comes to aluminium, the shear velocity c_2 is around 3100 m/s. For this material, the S0 mode can be modelled accurately using the K-M theory below a frequency-thickness product of 1.57 MHz×mm. By contrast, only below such product of 0.49 MHz×mm, the generalised plane stress theory can provide a satisfactory approximation (relative error <1%) to the S0 mode.

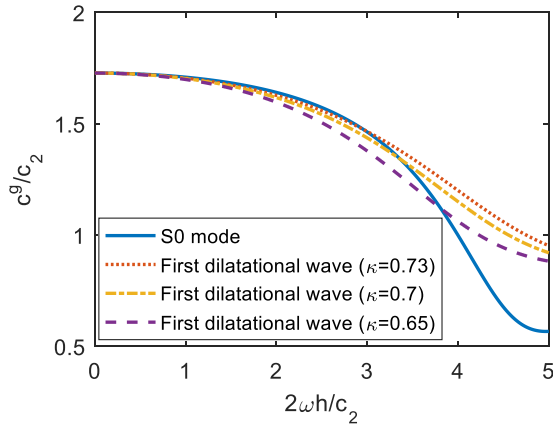


Figure 4. Comparison between the dimensionless group velocity dispersion curve for the S0 mode and such dispersion curves for the first dilatational wave obtained using different shear coefficients ($\nu = 0.33$).

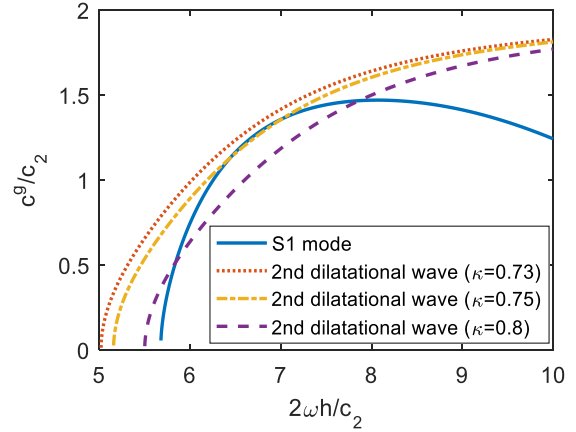


Figure 5. Comparison between the dimensionless group velocity dispersion curve for the S1 mode and such dispersion curves for the second dilatational wave obtained using different shear coefficients ($\nu = 0.33$).

The influence of the shear coefficient on the dispersion curve of the first dilatational wave is shown in Figure 4. In order to model the S0 mode at higher frequencies, the shear coefficient should be reduced. However, although the reduced coefficient enables the dispersion curve of the first dilatational wave to intersect that of the S0 mode at a higher frequency, the former curve deviates from the exact curve more seriously at neighbour frequencies of the intersection point.

Thus, the first dilatational wave, generally speaking, cannot be used to represent the S0 mode accurately at relatively much higher frequencies.

Figure 5 illustrates how the dispersion curve of the second dilatational wave varies with the shear coefficient. As such coefficient increases, the dispersion curve of the second dilatational wave is shifted toward higher frequencies and able to intersect the dispersion curve of the S1 mode. When the shear factor is equal to 0.75, the second dilatational wave can be used to approximate the S1 mode with a relative error of less than 1% over a dimensionless frequency range from 6.4 to 7.1. This range is much shorter than that regarding the first dilatational wave. It is important to note that although the second dilatational wave is able to be used to represent the S1 mode at some frequencies, the first dilatational wave fails to represent the S0 mode accurately at such frequencies. It means that the K-M theory cannot model the S0 and S1 modes accurately at the same time.

It can also be seen from Figure 5 that the fundamental shear horizontal wave (SH0) is able to be represented accurately using the shear wave. It should be noted that the SH0 wave is nondispersive and both the K-M theory and the generalised plane stress theory are able to describe it accurately over the entire frequency range.

In engineering applications, such as the guided wave based structural health monitoring, Lamb waves are often excited using the signals in the form of Hann-window-modulated sinusoidal tone burst:

$$F(t) = \frac{1}{2} A_m \sin(2\pi f_c t) \left[1 - \cos\left(\frac{2\pi f_c t}{N_w}\right) \right] H\left(\frac{N_w}{f_c} - t\right) \quad (55)$$

where f_c denotes the central frequency and N_w is the number of cycles; A_m is the amplitude and H is the Heaviside function. The optimal shear coefficient can be determined based on the frequency spectrum of this signal. Figure 6 shows the frequency spectrum of the signal obtained using the Fourier transform. It can be seen that the energy of the signal is highly concentrated in the main lobe ranging from $f_c - 2f_c / N_w$ to $f_c + 2f_c / N_w$. Within such frequency range, the least square method [29] is used to obtain the optimal shear coefficient.

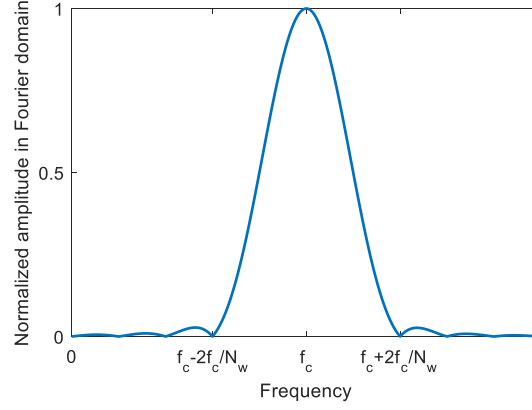


Figure 6. Frequency spectrum of the Hann-window-modulated sinusoidal tone burst.

7. Numerical examples

In the following examples, the coordinate origin is located at the centre of a plate. In order to demonstrate the wider applicability of the K-M theory, two-dimensional BEM analyses based on the generalised plane stress theory [33, 37] are conducted for comparisons. Since the shell elements in Abaqus are not suitable for modelling the high-frequency S0 mode and Abaqus code does not provide Kane-Mindlin plate elements, three-dimensional FEM simulations based on brick elements are carried out to validate BEM results.

7.1. An infinite plate under in-plane loading

An infinite plate, which is made of aluminium with the properties $E = 72.7 \text{ GPa}$, $\nu = 0.33$ and $\rho = 2700 \text{ kg/m}^3$, has a thickness ($2h$) of 10 mm. As shown in Figure 7, two point loads F_1 along the x_1 -axis are applied symmetrically on the surfaces of the plate at two points ($x_1 = 0$, $x_2 = 0$, $x_3 = \pm h$).

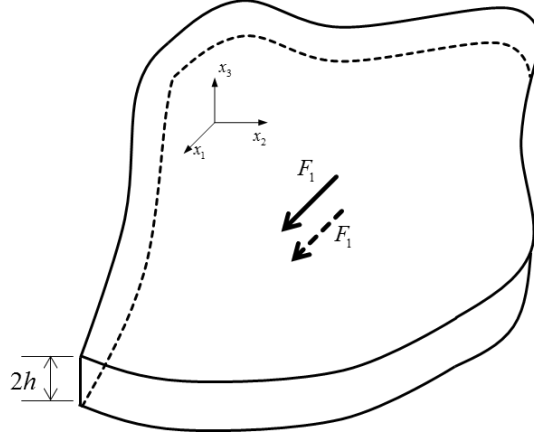


Figure 7. An infinite plate subjected to two point loads along the x_1 -axis on the top and bottom surfaces.

For this simple example, Lamb wave responses have been expressed analytically by Achenbach and Xu [38] under harmonic loads. When the Hann-window-modulated sinusoidal tone burst is adopted as the excitation signal, the responses in time domain are able to be obtained by inverse Fourier transform of the product of the harmonic responses and the Fourier transform of the excitation signal [39]. Such analytical solutions can be used to examine the applicability of the Kane-Mindlin theory in terms of modelling Lamb waves. The BEM responses in time domain are obtained by inversion of Laplace transform fundamental solutions in Eq. (A.1).

Four cases are investigated here, in which the parameters of the Hann-window-modulated sinusoidal tone burst are given in Table 1. In addition, the shear factor κ and the number of Laplace terms are also listed in this table.

Table 1. Computational parameters in four cases

	Case 1	Case 2	Case 3	Case 4
Central frequency [kHz]	50	150	190	340
Number of cycles	5	10	15	50
Shear factor	0.73	0.73	0.7	0.75
Number of Laplace terms	100	300	300	900

The displacement responses u_i were normalized with respect to the peak amplitude of the S0 mode and measured at the point A ($x_1 = 1$ m, $x_2 = 0$, $x_3 = 5$ mm) in the first three cases, and the point B ($x_1 = 2$ m, $x_2 = 0$, $x_3 = 5$ mm) was used as the measurement point in the last case to obtain the displacement response u_i normalized with respect to the peak amplitude of the S1 mode. Figs. 8-11 present the normalized displacement responses u_i at the measurement point in the four

cases, respectively. The responses were obtained using the analytical solutions given by Achenbach and Xu [38], dynamic fundamental solutions for the Kane-Mindlin theory, and dynamic fundamental solutions for the generalised plane stress theory given in Ref. [37].

At a lower frequency (50 kHz), it can be seen from Figure 8 that the S0 mode is nearly non-dispersive by comparing the shapes of the displacement signal and the excitation signal. Thus, both the K-M theory and the generalised plane stress theory are able to be used to model the S0 mode at such frequency. Figure 9 shows that the generalised plane stress theory fails to represent the S0 mode accurately as the frequency increases. This is because the dispersion of the S0 mode cannot be disregarded at higher frequencies. By contrast, the Kane-Mindlin theory still provides a satisfactory approximation for modelling such high-frequency S0 mode wave. However, as the frequency further increases, the Kane-Mindlin theory is unable to describe the dispersion behaviour of the S0 mode accurately, and thus it cannot be used to model the S0 mode, which can be seen from Figure 10. In order to model the S0 mode accurately at higher frequencies, higher-order plate theories, such as the Mindlin-Medick plate theory [40], should be used.

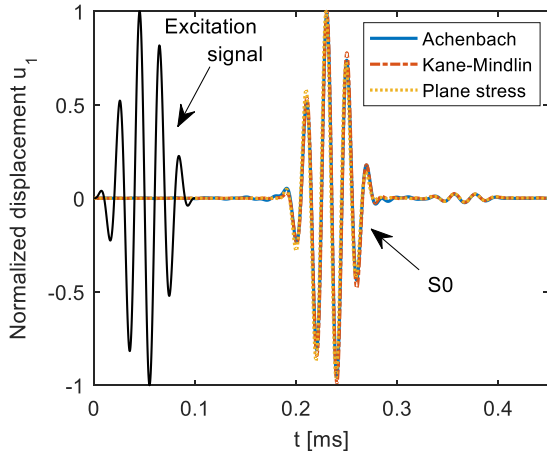


Figure 8. Comparison of normalized displacements u_1 obtained based on the analytical solutions given by Achenbach, fundamental solutions for the Kane-Mindlin theory and generalised-plane-stress theory at the point A ($x_1=1\text{m}$, $x_2=0$, $x_3=5\text{mm}$) of the plate (10 mm thickness) subjected to in-plane point loads(50 kHz).

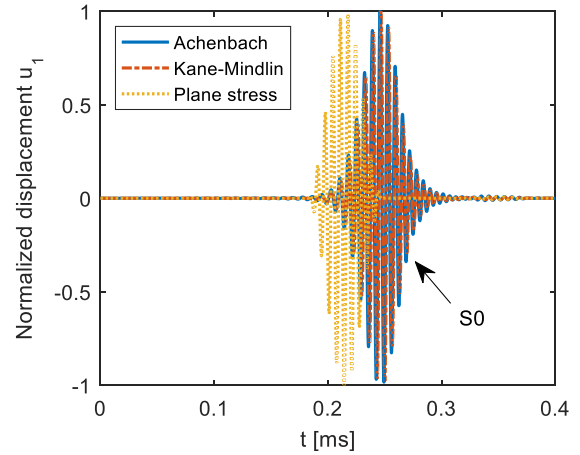


Figure 9. Comparison of normalized displacements u_1 obtained based on the analytical solutions given by Achenbach, fundamental solutions for the Kane-Mindlin theory and generalised-plane-stress theory at the point A ($x_1=1\text{m}$, $x_2=0$, $x_3=5\text{mm}$) of the plate (10 mm thickness) subjected to in-plane point loads (150 kHz).

As mentioned earlier, the Kane-Mindlin theory is also able to be used to model the S1 mode at some frequencies, which is demonstrated in Figure 11. However, at such high frequency, the Kane-Mindlin theory loses the ability to accurately describe the S0 mode.

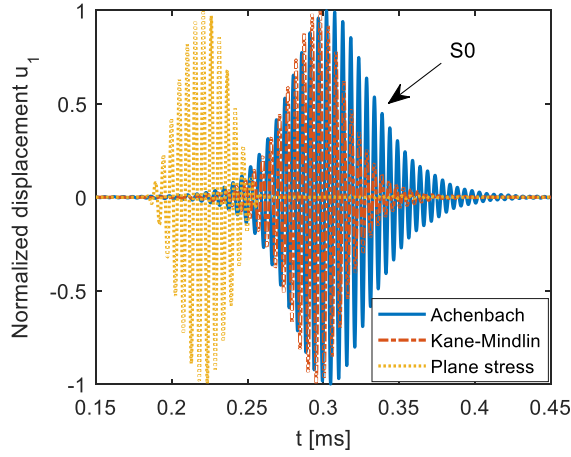


Figure 10. Comparison of normalized displacements u_1 obtained based on the analytical solutions given by Achenbach, fundamental solutions for the Kane-Mindlin theory and generalised-plane-stress theory at the point A ($x_1 = 1\text{ m}$, $x_2 = 0$, $x_3 = 5\text{ mm}$) of the plate (10 mm thickness) subjected to in-plane point loads (190 kHz).

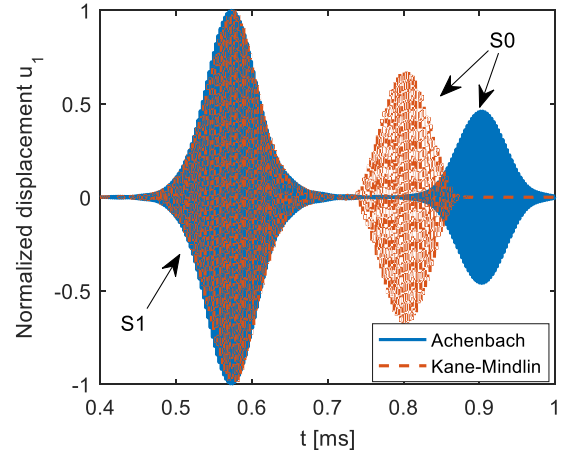


Figure 11. Comparison of normalized displacements u_1 obtained based on the analytical solutions given by Achenbach, fundamental solutions for the Kane-Mindlin theory and generalised-plane-stress theory at the point B ($x_1 = 2\text{ m}$, $x_2 = 0$, $x_3 = 5\text{ mm}$) of the plate (10 mm thickness) subjected to in-plane point loads (340 kHz).

7.2. A square plate under out-of-plane loading

A square plate with four free edges (Figure 12) has an edge length ($2l$) of 1 m and a thickness ($2h$) of 10 mm. This plate is subjected to vertical loads (F_3 and $-F_3$) which are applied at two points ($x_1 = 0$, $x_2 = 0$, $x_3 = \pm h$). The material properties of the plate are the same as those in the first example.

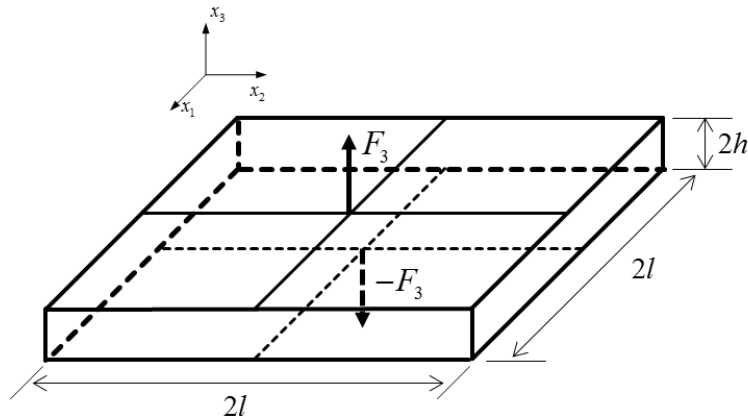


Figure 12. A square plate subjected to two out-of-plane point loads on the top and bottom surfaces.

A six-cycle 100 kHz Hann-windowed tone burst was used as the excitation load, which only generated the S0 mode. In this case, the generalised plane stress theory fails to model the generation of the S0 mode since only in-plane loads are able to be considered in this theory. However, the K-M theory is still suitable for this case. In the K-M BEM simulation, the shear factor was set to 0.78 and 200 Laplace terms were used.. Each plate edge was discretized using 50 quadratic boundary line elements, which resulted in 5.3 nodes per wavelength of the S0 mode at 100 kHz.

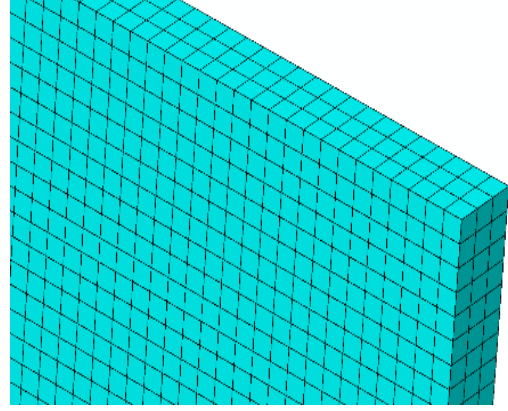


Figure 13. A detail view of a portion of the mesh for one-eighth of the square plate (half plate thickness) under out-of-plane loading in Abaqus/Explicit.

It should be noted that Achenbach and Xu [38] have also given the analytical solutions, as in the first example, for an infinite plate under such out-of-plane loads. Thus, these solutions can be used to validate the response of the direct arrival S0 mode (first S0 mode) obtained using the K-M BEM. The responses of the S0 mode reflected from the plate edges in the K-M BEM simulation were validated using Abaqus/Explicit. In the finite element analysis, due to symmetry, only one-eighth of the square plate was modelled using 333,666 linear brick elements (C3D8R), leading to 1,342,680 degrees of freedom, and there were 3 layers of elements across half-plate thickness. Figure 13 illustrates a detail view of a portion of this finite element mesh created using an approximate global size of 1.5 mm, which is equivalent to 36 nodes per wavelength of the S0 at 100 kHz.

Figure 14 shows the displacement u_1 , which was normalized with respect to the peak amplitude of the first S0 mode, obtained using the analytical solutions given by Achenbach and Xu [38], the K-M BEM and the 3D Abaqus/Explicit at the point A ($x_1 = 0.25\text{m}$, $x_2 = 0$, $x_3 = 5\text{ mm}$) of the plate. When it comes to the first wave packet representing the direct arrival S0 mode, the K-M

BEM signal agrees very well with both the analytical solution given by Achenbach and Xu [38] and the FEM result. In addition, the reflected S0 mode signal from the K-M boundary element analysis agrees very well with the FEM results. Since the analytical solutions were obtained based on the infinite plate assumption, the reflected waves from plate edges could not be captured using these solutions and thus they were not observed in Figure 14.

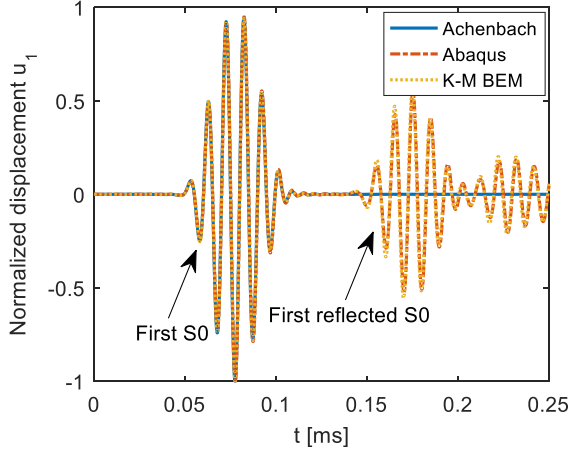


Figure 14. Comparison of normalized displacements u_1 at the point A ($x_1 = 0.25\text{m}$, $x_2 = 0$, $x_3 = 5\text{mm}$) of the square plate (10 mm thickness) subjected to out-of-plane loads obtained based on the analytical solutions given by Achenbach, using the Abaqus/Explicit and the K-M BEM.

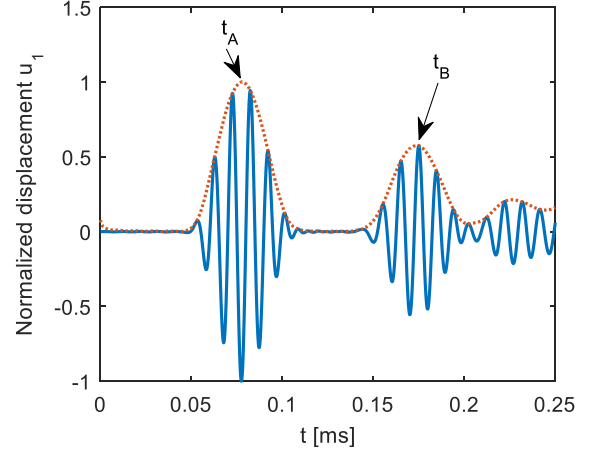


Figure 15. The normalized displacement signal u_1 at the point A ($x_1 = 0.25\text{m}$, $x_2 = 0$, $x_3 = 5\text{mm}$) of the square plate (10 mm thickness) subjected to out-of-plane loads obtained using the K-M BEM and its envelope obtained using the Hilbert transform.

The above BEM-based signals can be used to estimate the group velocity of the S0 mode with the help of the Hilbert transform [7]. After the transformation, the envelope of the signals was plotted in Figure 15. The envelope peaks of the first and the second wave packet appear at t_A and t_B , respectively. During the time period $t_B - t_A$, the S0 mode travels a distance d_{AB} of 0.5 m. Thus, the group velocity c^g of the S0 mode at 100 kHz can be estimated as 5219 m/s using the expression $c^g = d_{AB} / (t_B - t_A)$. The corresponding analytical value based on Eq. (52) is 5230 m/s. The relative error of such velocity obtained using the K-M boundary element simulation is only 0.2%.

This numerical example has also been analysed by Zak [32] using a spectral element method, in which a linear displacement variation across the thickness (first-order approximation) was assumed. Such approximation is the same as that used in the K-M theory in terms of modelling the symmetric Lamb wave modes. In Ref. [32], 36-node spectral finite plate elements were used

for the discretization, and the mesh consisted of 63,000 nodes, which resulted in 189,000 degrees of freedom and 13.3 nodes per wavelength of the S0 mode at 100 kHz. On the basis of such mesh, the relative error of the group velocity of the S0 mode was 0.3%. In the K-M BEM, only 400 nodes were used in the discretization of the plate edges and the total number of degrees of freedom was 1,200. The node density of 5.3 nodes per wavelength resulted in the relative error of 0.2%. Thus, compared with the spectral element method, modelling the S0 mode using the K-M BEM leads to a significant reduction in the number of degrees of freedom.

7.3. A free plate with one circular actuation region

In structural health monitoring, both contact-type devices, such as piezoelectric patches, and non-contact type devices, such as the laser, can be used to actuate Lamb waves [41]. When the device produces a circular actuation region on one surface of a plate, omni-directional Lamb waves are generated. In practice, since the actuation region is often very small compared with the plate, a line-force model can be used to represent the actuation effect [42, 43]. In such model, the equivalent line traction applied on the plate surface is normal to the circumference of the circular region. Thus, in the boundary integral equation, the domain integrals associated with this equivalent traction reduce to line integrals.

Figure 16 illustrates a square free plate with a circular actuation region. The plate has a thickness $2h$ of 2 mm and an edge length $2l$ of 0.6 m. The plate is made of aluminium with the properties $E = 69 \text{ GPa}$, $\nu = 0.33$ and $\rho = 2700 \text{ kg/m}^3$. The origin of the coordinate system is at the centre of the plate. The actuation region with a radius R of 5 mm is centred at the origin.

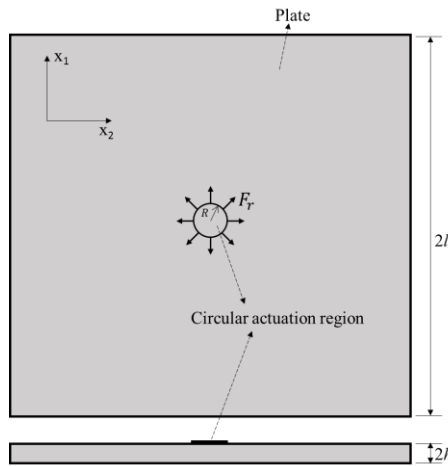


Figure 16. A square plate with a circular actuation region on the top surface.

It should be noted that this configuration can generate both symmetric and antisymmetric Lamb modes simultaneously. The line traction F_r is in the form of a five-cycle 300 kHz Hann-windowed tone burst. At such frequency, only the S0 mode and the A0 mode are generated. This traction is decomposed into symmetric tractions and anti-symmetric tractions with respect to the middle plane of the plate, as shown in Figure 17. The symmetric and anti-symmetric tractions excite the S0 mode and A0 mode, respectively. The proposed K-M boundary element formulation can be used to obtain the responses of the S0 mode, while those of the A0 mode are able to be analysed using the existing boundary element formulations based on the Mindlin plate bending theory [22, 23]. In the Mindlin plate bending theory, the A0 mode is excited by the bending moment $M_r = F_r h$ due to the anti-symmetric tractions [44].

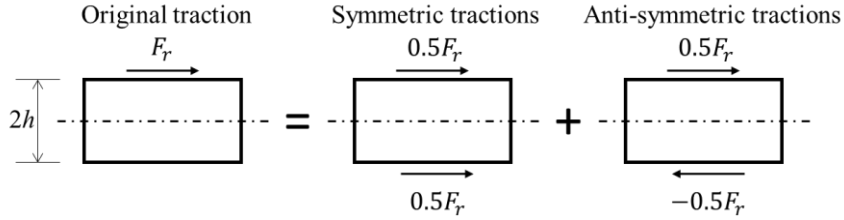


Figure 17. Decomposition of the original traction into symmetric loads and anti-symmetric loads.

The propagation and the reflection of the S0 mode and the A0 mode are decoupled for the simple single plate in this example. Thus, the dynamic responses of the S0 mode and the A0 mode can be obtained using the K-M BEM and the Mindlin-bending BEM, respectively. These two analyses were carried out separately and the dynamic response of the plate was obtained as the superposition of the responses of these two modes. However, when it comes to assembled plate structures, these two problems should be solved as a whole. The boundary element influence matrices for the K-M BEM formulation and the Mindlin-bending BEM formulation should be inserted into one global matrix, and subsequent calculations should be based on the manipulations of this global matrix. The detailed assembly method and the components of the global matrix are the same as those in the static boundary element analysis of assembled plates, which can be found in Ref. [36, 45, 46].

In the K-M BEM, 80 quadratic line elements were used to discretize each plate edge, which resulted in 1,920 degrees of freedom and 4.7 nodes per wavelength of the S0 mode at 300 kHz. The shear factor κ was set to 0.73 and 200 Laplace terms were used. In this example, since the group velocity of the S0 mode (5268.5 m/s) is much higher than that of the A0 mode (2974.8 m/s) at 300 kHz, the reflected waves of the A0 mode from the plate edges cannot be observed in

the time period of interest. Thus, when the BEM based on the Mindlin plate bending theory [22, 23] was used to model the A0 mode, the plate edges can be disregarded and do not need to be discretized. Similar to the shear factor κ in the K-M theory, the shear correction factor κ_M used in the Mindlin bending theory [35] is also adjustable and the optimal κ_M for this example was 0.94.

Raghavan and Cesnik [39] have given the analytical solutions for an infinite plate with a circular actuation region. Thus, their solutions can be used to validate the direct arrival S0 mode and A0 mode in the BEM results. When it comes to the validation of the reflected wave signals from the plate edges, the Abaqus/Explicit solver was used. In the finite element simulation, due to symmetry, a quarter of the plate was modelled and the mesh consisting 22,442,140 C3D8R brick elements was created on the basis of an approximate global size of 0.2 mm. There were 10 layers of elements across the plate thickness and the total number of degrees of freedom was 74,158,128. The mesh resulted in 90 nodes per wavelength of the S0 mode and 51 nodes per wavelength of the A0 mode at 300 kHz.

The displacement u_1 was measured at the point A, which was located on the top surface of the plate ($x_3 = 1\text{mm}$) and along the x_1 -axis at a distance 0.15 m from the origin, and normalized with respect to the peak amplitude of the first S0 mode. Figure 18 presents the normalized displacements u_1 obtained using the analytical solution given by Raghavan and Cesnik [39], the 3D Abaqus/Explicit and the BEM. Since the analytical solution is on the basis of an infinite plate, boundary-reflected waves cannot be observed in Figure 18. The first reflected S0 mode is caused due to the reflection of the S0 mode from one plate edge ($x_1 = 0.3\text{m}$), while the reflected S0 waves from two plate edges ($x_2 = \pm 0.3\text{m}$) result in the second reflected S0 mode. It should be noted that the amplitude of the A0 mode obtained using the Mindlin-bending BEM is underestimated, which has also been observed in Ref. [47]. The reason for this phenomenon is that the Mindlin bending theory is an approximate theory and it cannot represent all the features of the A0 mode accurately. Apart from this, the BEM results agree very well with both the analytical solution and the FEM results.

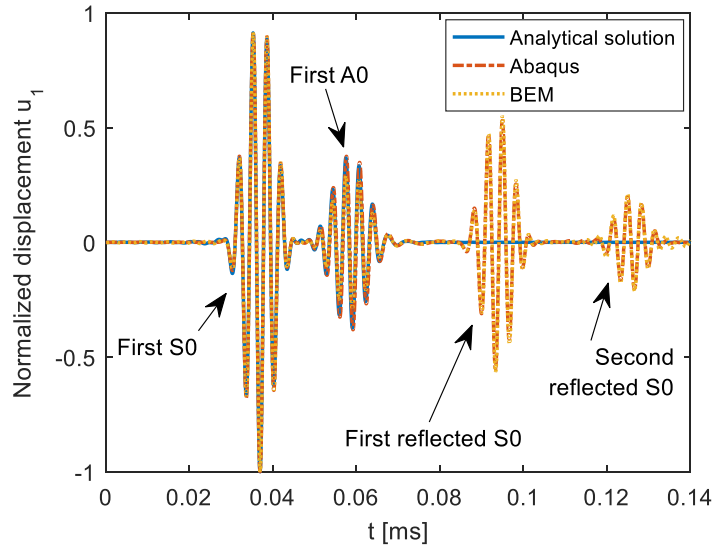


Figure 18. Comparison of normalized displacements u_1 at the point A ($x_1 = 0.15$ m, $x_2 = 0$, $x_3 = 1$ mm) of the plate (2 mm thickness) with one circular actuation region obtained using the analytical solution, the Abaqus/Explicit and the BEM.

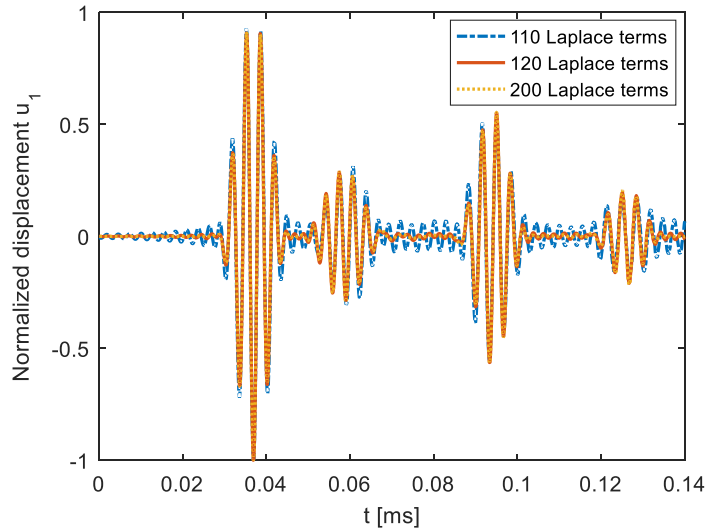


Figure 19. The influence of the number of Laplace terms on the BEM-based normalized displacement u_1 at the point A ($x_1 = 0.15$ m, $x_2 = 0$, $x_3 = 1$ mm) of the plate (2 mm thickness) with one circular actuation region.

The number of Laplace terms is an important factor affecting the computational efficiency. In the above examples, larger number of Laplace terms than strictly necessary was used to ensure the convergence. In order to increase the computational efficiency, fewer Laplace terms should be used. In the third example, the BEM results were obtained on the basis of 200 Laplace terms. However, it can be seen from Figure 19 that 120 Laplace terms are enough to achieve convergence. Furthermore, the number of Laplace terms can be halved by using a variant [23] of

the numerical inversion formula Eq.(48). However, this variant formula leads to inaccurate results toward the end of the time period. Thus, the selection between these two formulas should be based on the requirements on both accuracy and efficiency. The detailed comparisons between these two formulas have been given in Ref. [23].

8. Discussion

In this paper, the application of the BEM was extended to a new set of problems. The authors do not consider the new BEM as a replacement of other computational methods but rather providing a valuable alternative to problems that can only be solved numerically. Some comments should be made regarding the advantages and disadvantages of the proposed BEM formulation.

In contrast to the spectral plate finite element formulation [32], the main advantage of the proposed BEM formulation is that only plate edges are discretized and a significant reduction in the degrees of freedom can be achieved. In addition, a so-called dual boundary element method [23], which is more suitable for modelling cracked plates since crack surfaces can be treated simply as extra plate edges, can be formulated using the proposed dynamic fundamental solutions and BEM formulation.

The BEM formulation presented in this paper is suitable for isotropic materials; however by deriving additional fundamental solutions, it can be extended to anisotropic plate and shell structures. A well-known drawback of the BEM formulation compared to the FEM is the fully populated non-symmetric matrices leading to longer solution times for the same number of degrees of freedom. However, recent development of fast solvers [48] has significantly reduced the solution times. For material nonlinear problems, in general it is difficult to maintain the boundary-only formulation and often the non-linear region requires internal discretization.

There are also several other mesh reduction methods that have been applied to plate and shell problems [49, 50]. All of these methods avoid having to discretize the problem domain. Some of the mesh free methods even avoid having to discretize the boundary and simply distribute field points uniformly throughout the domain. The scaled boundary finite element method [51] avoids having to use the fundamental solutions through transformation of the governing partial differential equations into ordinary differential equations which allow semi-analytical integration. For complex domains, such as a domain with inner cracks or holes, this method requires the division of the problem domain into sub-domains with only their boundary discretized. Extensive review and comparisons of the above methods are beyond the scope of this

work, however, to the best of our knowledge, there is no application of the above mesh reduction methods to K-M plate problems reported in the open literature.

9. Conclusions

In this paper, dynamic fundamental solutions for the Kane-Mindlin plate theory are derived and presented in the Laplace transform domain for the first time. Based on these fundamental solutions, a new boundary element formulation is proposed for the dynamic analysis of Kane-Mindlin plates. This new formulation is capable of dealing with the thickness-stretch displacement of the S0 mode and out-of-plane loads, and accurately modelling the high-frequency S0 Lamb wave mode, which cannot be conducted using the conventional 2D BEM formulations based on the generalised plane stress theory. The results obtained using the proposed formulation agree very well with analytical results and those from three-dimensional finite element analyses. The proposed BEM formulation and the existing BEM formulations based on the Mindlin plate bending theory can be combined together for the modelling of the S0 mode and A0 mode simultaneously. This indicates that the BEM formulations on the basis of plate theories are very suitable for the numerical analyses of Lamb-wave based structural health monitoring problems. In the future work, it is planned to investigate the interaction of the S0 mode with cracks based on the newly derived formulation and fundamental solutions.

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Appendix A

The transformed fundamental solutions in the displacement boundary integral equation can be arranged as follows:

$$\tilde{U}_{\alpha\beta} = \frac{1}{4\pi h\mu(e_2 - e_1)\zeta_3^2} (\psi\delta_{\alpha\beta} - \chi r_{,\alpha} r_{,\beta}) \quad (\text{A.1})$$

where

$$\psi = \frac{(e_2 - e_1)\zeta_3[K_1(z_3) + z_3K_0(z_3)] - e_2\zeta_1K_1(z_1) + e_1\zeta_2K_1(z_2)}{r},$$

$$\chi = \frac{(e_2 - e_1)\zeta_3[2K_1(z_3) + z_3K_0(z_3)] - e_2\zeta_1[2K_1(z_1) + z_1K_0(z_1)] + e_1\zeta_2[2K_1(z_2) + z_2K_0(z_2)]}{r};$$

$K_0(z_i)$ and $K_1(z_i)$ are modified zero- and first-order Bessel functions of the second kind, respectively.

$$\tilde{U}_{\alpha 3} = -\tilde{U}_{3\alpha} = -\frac{3}{4\pi h^3 \mu (e_2 - e_1)} [\zeta_1 K_1(z_1) - \zeta_2 K_1(z_2)] r_{,\alpha} \quad (\text{A.2})$$

$$\tilde{U}_{33} = \frac{3}{4\pi h^3 \mu (e_2 - e_1)} [-e_1 K_0(z_1) + e_2 K_0(z_2)] \quad (\text{A.3})$$

$$\begin{aligned} \tilde{T}_{\alpha\beta} = \frac{1}{2\pi(e_2 - e_1)\zeta_3^2} & \left\{ \left(\psi_{,r} - \frac{\chi}{r} \right) (\delta_{\alpha\beta} r_{,n} + n_{\alpha} r_{,\beta}) - 2 \frac{\chi}{r} (n_{\beta} r_{,\alpha} - 2 r_{,\alpha} r_{,\beta} r_{,n}) - 2 \chi_{,r} r_{,\alpha} r_{,\beta} r_{,n} \right. \\ & \left. + \frac{\lambda}{\mu} \left[\psi_{,r} - \frac{\chi}{r} - \chi_{,r} - \frac{3\kappa\zeta_3^2}{h^2} (\delta_1 K_1(z_1) - \delta_2 K_1(z_2)) \right] r_{,\alpha} n_{\beta} \right\}, \end{aligned} \quad (\text{A.4})$$

where

$$\begin{aligned} \psi_{,r} &= - \left[(e_2 - e_1)\zeta_3^3 K_1(z_3) + \frac{\chi}{r} \right]; \\ \chi_{,r} &= - \left[2 \frac{\chi}{r} + (e_2 - e_1)\zeta_3^3 K_1(z_3) - e_2\zeta_1^3 K_1(z_1) + e_1\zeta_2^3 K_1(z_2) \right] \end{aligned}$$

$$\tilde{T}_{\alpha 3} = \frac{1}{2\pi(e_2 - e_1)} \left\{ [\zeta_1^2 A(z_1) - \zeta_2^2 A(z_2)] r_{,\alpha} r_{,n} - [\zeta_1 K_1(z_1) - \zeta_2 K_1(z_2)] \frac{n_{\alpha}}{r} \right\} \quad (\text{A.5})$$

$$\begin{aligned} \tilde{T}_{3\alpha} &= -\frac{3}{2\pi h^2 (e_2 - e_1)} \left\{ \frac{\lambda}{\mu} [\zeta_1^2 K_0(z_1) - \zeta_2^2 K_0(z_2) + \kappa(e_1 K_0(z_1) - e_2 K_0(z_2))] \right. \\ & \quad \left. + \frac{2\mu}{\lambda r} (\zeta_1 K_1(z_1) - \zeta_2 K_1(z_2)) \right] n_{\alpha} + 2 [\zeta_1^2 A(z_1) - \zeta_2^2 A(z_2)] r_{,\alpha} r_{,n} \Big\} \end{aligned} \quad (\text{A.6})$$

$$\tilde{T}_{33} = \frac{1}{2\pi(e_2 - e_1)} [e_1 \zeta_1 K_1(z_1) - e_2 \zeta_2 K_1(z_2)] r_{,n} \quad (\text{A.7})$$

where $A(z_i) = K_0(z_i) + \frac{2}{z_i} \left(K_1(z_i) - \frac{1}{z_i} \right)$.

Appendix B

The fundamental solutions for static Kane-Mindlin plate problems, which are obtained by modifying the fundamental solutions given in Ref. [52], are as follows:

$$U_{\alpha\beta}^{\text{static}} = \frac{1}{16\pi h\mu(1-\nu)} \left\{ \left[1-\nu^2 + \frac{4\nu^2}{z^2} - \frac{4\nu^2}{z} K_1(z) - 2\nu^2 K_0(z) \right] r_{,\alpha} r_{,\beta} - \left[(1-\nu)(3-\nu) \ln r + \frac{2\nu^2}{z^2} - \frac{2\nu^2}{z} K_1(z) \right] \delta_{\alpha\beta} \right\} \quad (\text{B. 1})$$

where $z = \eta r$ and $\eta = \frac{\kappa}{h} \sqrt{\frac{6}{(1-\nu)}}$.

$$U_{\alpha 3}^{\text{static}} = -U_{3\alpha}^{\text{static}} = \frac{\nu}{8\pi h\kappa\mu} \left[\frac{1}{r} - \eta K_1(z) \right] r_{,\alpha} \quad (\text{B. 2})$$

$$U_{33}^{\text{static}} = \frac{3}{4\pi h^3 \mu} K_0(z) \quad (\text{B. 3})$$

$$T_{\alpha\beta}^{\text{static}} = \frac{1}{4\pi(1-\nu)r} \left\{ - \left[2\nu^2 A(z) + 2\nu^2 z K_1(z) - (1-\nu)^2 \right] r_{,\alpha} n_{\beta} + 2 \left[4\nu^2 A(z) + \nu^2 z K_1(z) + \nu^2 - 1 \right] r_{,\alpha} r_{,\beta} r_{,n} - \left[2\nu^2 A(z) + (1-\nu)^2 \right] (\delta_{\alpha\beta} r_{,n} + n_{\alpha} r_{,\beta}) \right\} \quad (\text{B. 4})$$

$$T_{\alpha 3}^{\text{static}} = \frac{\nu h^2 \eta^2}{12\pi\kappa} \left[A(z) r_{,\alpha} r_{,n} - \frac{1}{z} \left(K_1(z) - \frac{1}{z} \right) n_{\alpha} \right] \quad (\text{B. 5})$$

$$T_{3\alpha}^{\text{static}} = \frac{\nu \eta^2}{2\pi\kappa} \left[B(z) n_{\alpha} - A(z) r_{,\alpha} r_{,n} \right] \quad (\text{B. 6})$$

$$T_{33}^{\text{static}} = -\frac{\eta}{2\pi} K_1(z) r_{,n} \quad (\text{B. 7})$$

where $B(z) = K_0(z) + \frac{1}{z} \left(K_1(z) - \frac{1}{z} \right)$.

Appendix C

When the distance r between the source point $\mathbf{x}'$ and the field point $\mathbf{x}$ tends to zero, the limits of the transformed fundamental solutions have the following forms:

$$\lim_{r \rightarrow 0} \tilde{U}_{\alpha\beta} = -\frac{3-4\nu}{16\pi h\mu(1-\nu)} \ln r \delta_{\alpha\beta} = \lim_{r \rightarrow 0} U_{\alpha\beta}^{\text{static}} \quad (\text{C. 1})$$

$$\lim_{r \rightarrow 0} \tilde{U}_{\alpha 3} = -\lim_{r \rightarrow 0} \tilde{U}_{3\alpha} = 0 = \lim_{r \rightarrow 0} U_{\alpha 3}^{\text{static}} = -\lim_{r \rightarrow 0} U_{3\alpha}^{\text{static}} \quad (\text{C. 2})$$

$$\lim_{r \rightarrow 0} \tilde{U}_{33} = -\frac{3}{4\pi h^3 \mu} \ln r = \lim_{r \rightarrow 0} U_{33}^{\text{static}} \quad (\text{C. 3})$$

$$\lim_{r \rightarrow 0} \tilde{T}_{\alpha\beta} = -\frac{1}{4\pi(1-\nu)r} \left[(1-2\nu)(\delta_{\alpha\beta} r_{,n} + n_{\alpha} r_{,\beta} - n_{\beta} r_{,\alpha}) + 2r_{,\alpha} r_{,\beta} r_{,n} \right] = \lim_{r \rightarrow 0} T_{\alpha\beta}^{\text{static}} \quad (\text{C. 4})$$

$$\lim_{r \rightarrow 0} \tilde{T}_{\alpha 3} = -\frac{\kappa\nu}{4\pi(1-\nu)} n_{\alpha} \ln r = \lim_{r \rightarrow 0} T_{\alpha 3}^{\text{static}} \quad (\text{C. 5})$$

$$\lim_{r \rightarrow 0} \tilde{T}_{3\alpha} = -\frac{3\kappa\nu}{2\pi h^2(1-\nu)} n_{\alpha} \ln r = \lim_{r \rightarrow 0} T_{3\alpha}^{\text{static}} \quad (\text{C. 6})$$

$$\lim_{r \rightarrow 0} \tilde{T}_{33} = -\frac{1}{2\pi r} r_{,n} = \lim_{r \rightarrow 0} T_{33}^{\text{static}} \quad (\text{C. 7})$$

where U_{ij}^{static} and T_{ij}^{static} are static fundamental solutions (see Appendix B for details).

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