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ABSTRACT

We prove rigorous scaling laws for measures of the vertical heat transport enhancement in two models of convection driven by uniform internal heating at infinite Prandtl number. In the first model, a layer of incompressible fluid is bounded by horizontal plates held at the same constant temperature and convection reduces the fraction of the total dimensionless heat input per unit volume and time escaping the layer through the bottom boundary. We prove that this fraction decreases no faster than $O(R^{-2})$, where R is a "flux" Rayleigh number quantifying the strength of the internal heating relative to diffusion. The second model, instead, has a perfectly insulating bottom boundary, so all heat must escape through the top one. In this case, we prove that the Nusselt number, defined as the ratio of the total-to-conductive vertical heat flux, grows no faster than $O(R^4)$. These power-law bounds improve on exponential results available for fluids with finite Prandtl number. The proof combines the background method with a minimum principle for the fluid's temperature and with Hardy–Rellich inequalities to exploit the link between the vertical velocity and temperature available at infinite Prandtl number.

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I. INTRODUCTION

Convective flows driven by internal sources of heat have attracted renewed interest in recent years.[1–6] Such flows are commonly encountered in geophysics, where atmospheric convection[7] and mantle convection[8,9] are typical examples. They also exhibit unique features not seen in boundary-driven Rayleigh–Bénard convection: for instance, it has recently been observed experimentally that internally heated (IH) convection can transport heat more efficiently than Rayleigh–Bénard convection.[10,11] Nevertheless, the former remains much less studied.

Even for simple configurations, such as a uniformly heated fluid layer between parallel horizontal plates, it is a largely open challenge to rigorously predict the mean (i.e., space- and time-averaged) vertical heat flux or other related quantities that measure how much convection can enhance or hinder the transport of heat through the fluid. In particular, one would like to determine how such quantities depend on the heating strength and on the fluid's Prandtl number Pr, defined as the ratio between the fluid's kinematic viscosity ν and its thermal diffusivity κ.

One source of difficulty when trying to answer these questions is that the mean thermal dissipation, the mean viscous dissipation, and the mean vertical convective heat flux cannot all be related to each other via *a priori* relationships. This is in contrast with Rayleigh–Bénard convection, where such relationships enable one to rigorously bound the convective heat transfer through variational analysis of the mean thermal dissipation.[12] Applying the same strategy to IH flows yields bounds on the mean temperature of the fluid[13–16] but not the convective heat flux.

This variational strategy was recently extended by taking into account a minimum principle for the temperature, leading to bounds on the mean convective heat flux that vary exponentially with the heating rate.[2,5] This exponential dependence does not match numerical simulations,[15,17,18] so one wonders if it is optimal. Here, we show that it is not for fluids with *infinite* Prandtl number. In particular, we derive bounds that vary as algebraic powers of the heating rate.

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Before stating our results precisely, we must introduce the mathematical model of IH convection we study. We consider a horizontally periodic layer of incompressible fluid with constant density ρ and specific heat capacity c_p, whose motion is governed by the Boussinesq equations at infinite Prandtl number.[14] The fluid is confined between two horizontal plates at a distance d from each other and is heated at a constant rate Q per unit volume. We immediately make the problem non-dimensional by taking d as the characteristic length scale, d^2/κ as the time scale, and $d^2Q/\kappa\rho c_p$ as the temperature scale. With these choices, the fluid occupies the horizontally periodic domain $\Omega = \mathbb{T}_{[0,L_x]} \times \mathbb{T}_{[0,L_y]} \times [0,1]$ and satisfies

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1a}$$

$$\nabla p = \Delta \boldsymbol{u} + R\ T\hat{\boldsymbol{z}}, \tag{1b}$$

$$\partial_t T + \boldsymbol{u} \cdot \nabla T = \Delta T + 1, \tag{1c}$$

where $\boldsymbol{u} = (u, v, w)$ is the fluid's velocity in cartesian components, p is the pressure, and T is the temperature. The unit forcing in (1c) represents the non-dimensional internal heating rate. The flow is controlled by a "flux" Rayleigh number that measures the destabilizing effect of heating compared to the stabilizing effects of diffusion,

$$R := \frac{g\alpha Q d^5}{\rho c_p \nu \kappa^2}. \tag{2}$$

Here, g is the acceleration of gravity, α is the thermal expansion coefficient of the fluid, and all other quantities have been defined above. We will use overbars to denote infinite-time averages, angled brackets to indicate volume averages, and angled brackets with a subscript h for averages over only the horizontal directions,

$$\langle f \rangle_h = \frac{1}{L_x L_y} \int_0^{L_x} \int_0^{L_y} f(x,y,z,t)\ \mathrm{d}y\ \mathrm{d}x, \tag{3a}$$

$$\langle f \rangle = \int_0^1 \langle f \rangle_h\ \mathrm{d}z, \tag{3b}$$

$$\overline{\langle f \rangle} = \limsup_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \langle f \rangle \mathrm{d}t. \tag{3c}$$

Having introduced the basic set of equations governing IH convection at infinite Pr, we now turn to identifying space- and time-averaged quantities that measure the enhancement in vertical heat transport due to convection. Contrary to the case of Rayleigh–Bénard convection, which quantity is used depends on the boundary conditions for the top ($z = 1$) and bottom ($z = 0$) of the fluid layer. In this work, we consider the two different cases sketched in Fig. 1, which we described in detail below alongside our main results. Following the nomenclature introduced in previous work,[15] we refer to the two cases as the IH1 and IH3 configurations.

In the IH1 configuration of Fig. 1(a), the velocity satisfies no-slip conditions and the temperature of both boundaries is held at a constant value, which can be taken as zero without loss of generality. Hence, we enforce

$$\text{IH1}: \boldsymbol{u}|_{z\in\{0,1\}} = \boldsymbol{0}, \quad T|_{z=0} = 0, \quad T|_{z=1} = 0. \tag{4}$$

In this case, the enhancement in vertical heat transport can be quantified through the reduction in the fraction of heat, which leaves the domain through the bottom boundary in a statistically stationary regime. Indeed, let

$$\mathcal{F}_T := -\overline{\langle \partial_z T \rangle_h}|_{z=1}, \tag{5a}$$

$$\mathcal{F}_B := \overline{\langle \partial_z T \rangle_h}|_{z=0} \tag{5b}$$

be the nondimensional mean heat fluxes through the top and bottom boundaries. Averaging z (1c) over space and time shows that $\mathcal{F}_T$ and $\mathcal{F}_B$ are related to the mean vertical convective heat flux, $\overline{\langle wT \rangle}$, via

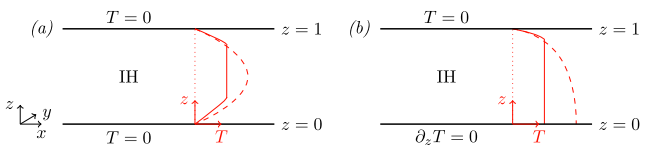


FIG. 1. IH convection with (a) isothermal boundaries (4) and (b) insulating lower boundary (8). In both panels, IH represents the uniform unit internal heat generation. Red dashed lines (—) indicate the conductive temperature profiles. Red solid lines (—) are cartoons illustrating turbulent mean temperature profiles.

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$$\mathcal{F}_T = \frac{1}{2} + \overline{\langle wT \rangle}, \tag{6a}$$

$$\mathcal{F}_B = \frac{1}{2} - \overline{\langle wT \rangle}. \tag{6b}$$

When the fluid does not move, so $\overline{\langle wT \rangle} = 0$, all heat added to the fluid is transported to the boundaries by conduction in a perfectly symmetric way, giving $\mathcal{F}_T = \mathcal{F}_B = \frac{1}{2}$. Convection breaks this symmetry, causing more heat to escape through the top boundary. In fact, one can prove that $0 \le \overline{\langle wT \rangle} \le \frac{1}{2}$ uniformly in R and Pr.[17] While the zero lower bound is saturated by the (possibly unstable) no-flow state, saturating the upper bound of $\frac{1}{2}$ would require a flow that transports heat upward so efficiently that all heat escapes the domain through the top boundary. Our first main result, however, implies that such a flow does not exist.

Theorem 1 (isothermal boundaries, $Pr = \infty$)**.** *Let $\boldsymbol{u} = (u, v, w)$ and T solve (1) subject to the no-slip isothermal boundary conditions (4). There exist constants $c, R_0 > 0$ such that*

$$\overline{\langle wT \rangle} \le \frac{1}{2} - c\, R^{-2} \quad \forall R > R_0. \tag{7}$$

Using (6), one immediately obtains Rayleigh-dependent bounds on $\mathcal{F}_T$ and $\mathcal{F}_B$.

Corollary 1.1. Under the same assumptions as Theorem 1, $\mathcal{F}_T \le 1 - c\, R^{-2}$ and $\mathcal{F}_B \ge c\, R^{-2}$ for all $R > R_0$.

The Proof of Theorem 1, presented in Sec. II D, actually provides the explicit constants $c = 216$ and $R_0 = 1892$. Our bounds therefore apply to the entire regime where convection is observed, since the conductive state is globally attractive only for $R \le 269\,26.6$.[15] The stated values of c and R_0, however, are suboptimal in the sense that one could increase the value of c at the expense of increasing R_0. In particular, one could optimize various constants in our proofs to increase c and R_0 until $R_0 = 26\,926.6$, so our bounds apply only to the convective regime.

For the IH3 configuration in Fig. 1(b), the velocity satisfies no-slip conditions, the top boundary is held at a constant zero temperature, and the bottom boundary is a perfect thermal insulator. This gives

$$\text{IH3}: \boldsymbol{u}|_{z\in\{0,1\}} = \boldsymbol{0}, \quad \partial_z T|_{z=0} = 0, \quad T|_{z=1} = 0. \tag{8}$$

Since no heat can escape through the bottom boundary, the enhancement of heat transport due to convection can no longer be quantified by looking at the boundary fluxes $\mathcal{F}_T$ and $\mathcal{F}_B$. Indeed, one has $\mathcal{F}_T = 1$ and $\mathcal{F}_B = 0$ at all values of R because no heat can leave the fluid layer through the bottom boundary. Instead, one can define a more classical Nusselt number Nu as the ratio of the space- and time-averaged total heat flux to the conductive heat flux. Since the latter is equal to the mean temperature difference $\overline{\langle \delta T \rangle}_h = \overline{\langle T|_{z=0} \rangle}_h - \overline{\langle T|_{z=1} \rangle}_h$ between the boundaries, one obtains

$$Nu := \frac{\overline{\langle \delta T \rangle}_h + \overline{\langle wT \rangle}}{\overline{\langle \delta T \rangle}_h}. \tag{9}$$

This can be rewritten in terms of $\overline{\langle wT \rangle}$ alone as

$$Nu = \frac{1}{1 - 2\overline{\langle wT \rangle}} \tag{10}$$

using the identity $\overline{\langle T|_{z=0} \rangle}_h - \overline{\langle T|_{z=1} \rangle}_h = \frac{1}{2} - \overline{\langle wT \rangle}$, which is obtained upon averaging $(1 - z)\cdot$(1c) over space and infinite time.

The uniform bounds $0 \le \overline{\langle wT \rangle} \le \frac{1}{2}$ from Ref. 19 hold also for the IH3 case and guarantee that $Nu \ge 1$. However, they do not rule out the possibility that Nu becomes infinite at a finite value of R. The second main result of this work confirms that this somewhat non-physical situation is indeed impossible because $\overline{\langle wT \rangle}$ is always smaller than $\frac{1}{2}$ by an $O(R^{-4})$ quantity.

Theorem 2 (insulating bottom and isothermal top, $Pr = \infty$)**.** *Let $\boldsymbol{u} = (u, v, w)$ and T solve (1) subject to the boundary conditions (8). There exist constants $c, R_0 > 0$ such that*

$$\overline{\langle wT \rangle} \le \frac{1}{2} - c\, R^{-4} \quad \forall R > R_0. \tag{11}$$

We immediately conclude from (10) that Nu is finite at every finite value of R and cannot increase faster than $O(R^4)$.

Corollary 2.1. Under the same assumptions as Theorem 2, $Nu \le cR^4$ for all $R > R_0$.

The Proof of Theorem 2, given in Sec. III D, provides the explicit but suboptimal constants $c \approx 0.0107$ and $R_0 = 2961$. Since a state of no flow is stable only when $R \le 1429.86$,[19] our bounds therefore apply to every value of R at which convection can occur. As in the IH1 configuration discussed previously, the values of c and R_0 could be optimized if desired.

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Let us conclude this Introduction with some general remarks about the proofs of Theorems 1 and 2. The first key ingredient in these proofs is a variational problem giving an upper bound on $\overline{\langle wT\rangle}$. This variational problem is derived by enforcing a minimum principle for the fluid's temperature within the classical "background method,"[12,20,21] which for simplicity we formulate using the language of a more general framework for bounding infinite-time averages[22–25] (the link between the two approaches has recently seen detailed discussed[23,26]). Using this minimum principle is essential to obtain bounds on $\overline{\langle wT\rangle}$ that asymptote to $\frac{1}{2}$ from below. This has already been shown for the IH1 configuration at finite Pr: for this case, without the minimum principle, one obtains only $\overline{\langle wT\rangle} \lesssim R^{1/5}$,[2] while with it one can prove that $\overline{\langle wT\rangle} \leq \frac{1}{2} - O(R^{1/5}\exp(-R^{3/5}))$ uniformly in Pr.[5] A similar (but not identical) exponentially varying bound of $\overline{\langle wT\rangle} \leq \frac{1}{2} - O(R^{-1/5}\exp(-R^{3/5}))$ uniformly in Pr was also obtained for the IH3 configuration.[5]

The second key ingredient in our proofs are estimates of Hardy–Rellich type, obtained by observing that the reduced momentum equation (1b) determines the vertical velocity field as a function of the temperature field. Specifically, taking the vertical component of the double curl of (1b) gives

$$\Delta^2 w = -R\,\Delta_h T, \tag{12}$$

where $\Delta_h := \partial_x^2 + \partial_y^2$ is the horizontal Laplacian. Using the no-slip boundary conditions with the incompressibility condition (1a), the vertical velocity w satisfies

$$w|_{z=0} = \partial_z w|_{z=0} = w|_{z=1} = \partial_z w|_{z=1} = 0. \tag{13}$$

Equation (12) was exploited in Rayleigh–Bénard convection to improve the scaling of upper bounds on Nu.[27] This was achieved by using (12) to derive inequalities of the Hardy–Rellich type (see Lemma 4) that help the construction of a background field with a logarithmically varying stable stratification in the bulk.[27,28] Here, we use the same inequalities to construct (different) background fields suited to IH convection, which will enable us to bound $\overline{\langle wT\rangle}$ in the infinite Pr limit.

The rest of this paper is structured as follows: Sec. II details the Proof of Theorem 1, while Theorem 2 is proven in Sec. III. Some technical steps of the proof are relegated to the Appendixes A and B to streamline the argument. Further discussion and perspective for possible future work are offered in Sec. IV.

II. BOUNDS FOR THE IH1 CONFIGURATION

We first consider the IH1 configuration, where the top and bottom plates are held at zero temperature. In Sec. II A, we show that $\overline{\langle wT\rangle}$ can be bounded from above by constructing suitably constrained functions of the vertical coordinate z. Section II B describes parametric *Ansätze* for such functions, while Sec. II C establishes auxiliary results that simplify the verification of the constraints and the evaluation of the bound. We then prove Theorem 1 in Sec. II D by prescribing R-dependent values of the free parameters in our *Ansätze*.

To simplify the notation, we introduce two sets of temperature fields that encode the thermal boundary conditions and the pointwise nonnegativity constraint implied by the minimum principle,

$$\mathcal{H}_n := \{T \in H^1(\Omega) : \quad T \text{ is horizontally periodic } \ \&\,(4) \trianglelefteq (8)\}, \tag{14a}$$

$$\mathcal{H}_+ := \{T \in \mathcal{H}_n : T(\boldsymbol{x}) \geq 0 \text{ a.e. } \boldsymbol{x} \in \Omega\}, \tag{14b}$$

where $n = 1$ or 3, depending on the boundary conditions of IH1 and IH3 respectively. In Sec. II, T belongs to $\mathcal{H}_1$.

A. Bounding framework

To bound $\overline{\langle wT\rangle}$, we employ the auxiliary function method.[22,26] The method relies on the observation that the time derivative of any bounded functional $\mathcal{V}\{T(t)\}$ along solutions of the Boussinesq equations (1) averages to zero over infinite time, so

$$\overline{\langle wT\rangle} = \overline{\langle wT\rangle + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}\{T(t)\}}. \tag{15}$$

If $\mathcal{V}$ is chosen such that the quantity being averaged on the right-hand side is bounded above pointwise in time, then this pointwise bound is also an upper bound on $\overline{\langle wT\rangle}$.

Following analysis at finite Prandtl number,[2] we restrict our attention to quadratic functionals taking the form

$$\mathcal{V}\{T\} = \left\langle \frac{\beta}{2}|T|^2 - [\tau(z) + z - 1]T \right\rangle, \tag{16}$$

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which are parameterized by a positive constant $\beta \in \mathbb{R}_+$ and a piecewise-differentiable function $\tau : [0,1] \to \mathbb{R}$ with the square-integrable derivative. We require τ to satisfy

$$\tau(0) = 1, \qquad \tau(1) = 0, \tag{17}$$

so the coefficient multiplying T in (16) vanishes at $z = 0$ and $z = 1$. This choice enables us to integrate by parts without picking up boundary terms when calculating $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}\{T(t)\}$. To ensure that the resulting expression $\langle wT\rangle + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}\{T\}$ can be bounded from above pointwise in time, we also require that the pair (β, τ) satisfies a condition called the *spectral constraint*.

Definition 1 (spectral constraint). The pair (β, τ) is said to satisfy the *spectral constraint* if

$$\left\langle \beta|\nabla T|^2 + \tau' wT \right\rangle \geq 0 \qquad T \in \mathcal{H}_n, \tag{18}$$

where $w = -R\Delta^{-2}\Delta_h T$ solves (12) with the boundary conditions (13).

If the spectral constraint is satisfied, then it is possible to bound $\overline{\langle wT\rangle}$ from above in terms of τ, β, and another suitably constrained function $\lambda : [0,1] \to \mathbb{R}$.

Proposition 1 (bounding framework, IH1). Suppose that the pair (β, τ) satisfies the spectral constraint and the boundary conditions in (17). Furthermore, let $\lambda \in L^2(0,1)$ be a nondecreasing function such that $\langle\lambda\rangle = -1$. Then,

$$\overline{\langle wT\rangle} \leq \frac{1}{2} + \left\langle \frac{1}{4\beta}\left|\tau' - \lambda - \beta\left(z - \frac{1}{2}\right)\right|^2 - \tau \right\rangle := U(\tau, \lambda, \beta).$$

Proof. A standard calculation using integration by parts, the incompressibility condition (1a), and the boundary conditions on $\boldsymbol{u}$ and T yields

$$\begin{aligned} \overline{\langle wT\rangle} &= \overline{\langle wT\rangle + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}\{T(t)\}} \\ &= \frac{1}{2} + \overline{\left\langle -\beta|\nabla T|^2 - \tau' wT + (\tau' - \beta z)\partial_z T - \tau \right\rangle}. \end{aligned} \tag{19}$$

The infinite-time average on the right-hand side is bounded above by the largest value of the argument over the global attractor of the infinite-Prandtl-number Boussinesq equations (1). The minimum principle for the temperature field implies that the global attractor is contained in the set $\mathcal{H}_+$ defined in (14b). Consequently, we can estimate

$$\overline{\langle wT\rangle} \leq \frac{1}{2} + \sup_{\substack{T \in \mathcal{H}_+ \\ w = -R\Delta^{-2}\Delta_h T}} \left\langle -\beta|\nabla T|^2 - \tau' wT + (\tau' - \beta z)\partial_z T - \tau \right\rangle. \tag{20}$$

This upper bound is finite if and only if unless the pair (β, τ) satisfies the spectral constraint (cf. Definition 1), in which case the supremum over T can be evaluated using a technical convex duality argument detailed in Appendix A. The result of the argument is that the bound in (20) is equivalent to

$$\overline{\langle wT\rangle} \leq \frac{1}{2} + \inf_{\substack{\lambda \in L^2(0,1) \\ \lambda \text{nondecreasing} \\ \langle\lambda\rangle = -1}} \left\langle \frac{1}{4\beta}\left|\tau' - \lambda - \beta\left(z - \frac{1}{2}\right)\right|^2 - \tau \right\rangle. \tag{21}$$

This inequality clearly implies the upper bound on $\overline{\langle wT\rangle}$ stated in the proposition, which is therefore proven. □

Remark 1. The function λ that arises when deriving (21) from (20) can be viewed as a Lagrange multiplier enforcing the pointwise non-negativity of temperature fields in the set $\mathcal{H}_+$. Further details regarding this interpretation (with slightly different notation) are given in previous work (Ref. 2, Sec. 4 4).

Remark 2. The best upper bound on $\overline{\langle wT\rangle}$ provable with our approach is found upon minimizing the expression $U(\tau, \lambda, \beta)$ over all choices of τ, λ, and β that satisfy the conditions of Proposition 1. This is hard to do analytically but can be done computationally using a variety of numerical schemes.[26] We leave such computations to future work and focus on proving Theorem 1 by constructing suboptimal τ, λ, and β analytically.

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B. Ansätze

To prove the upper bound on $\overline{\langle wT\rangle}$, we seek $\beta > 0$, $\tau(z)$, and $\lambda(z)$ that satisfy the conditions of Proposition 1 and make the quantity $U(\beta,\tau,\lambda)$ as small as possible. To simplify this task, we restrict τ to take the form

$$\tau(z) := \begin{cases} 1 - \dfrac{z}{\delta}, & 0 \le z \le \delta, \\ A \ln\left(\dfrac{z(1-\delta)}{\delta(1-z)}\right), & \delta \le z \le 1-\varepsilon, \\ A \ln\left(\dfrac{(1-\varepsilon)(1-\delta)}{\varepsilon\delta}\right)\left(\dfrac{1-z}{\varepsilon}\right), & 1-\varepsilon \le z \le 1, \end{cases} \tag{22}$$

and λ to be given by

$$\lambda(z) := \begin{cases} -\dfrac{1}{\delta}, & 0 \le z \le \delta, \\ 0, & \delta \le z \le 1. \end{cases} \tag{23}$$

These piecewise-defined functions, sketched in Fig. 2, are fully specified by the bottom boundary layer width $\delta \in (0, \frac{1}{2})$, the top boundary layer width $\varepsilon \in (0, \frac{1}{2})$, and the parameter $A > 0$ that determines the amplitude of τ in the bulk of the layer.

We also fix

$$\beta := \left\langle |\tau' - \lambda|^2 \right\rangle^{\frac{1}{2}} \left\langle |z - \frac{1}{2}|^2 \right\rangle^{-\frac{1}{2}} = 2\sqrt{3} \left\langle |\tau' - \lambda|^2 \right\rangle^{\frac{1}{2}}. \tag{24}$$

This choice is motivated by the desire to minimize the right-hand side of the inequality

$$\frac{1}{4\beta}\left\langle \left|\tau' - \lambda - \beta\left(z - \frac{1}{2}\right)\right|^2 \right\rangle \le \frac{\beta}{2}\left\langle |z - \frac{1}{2}|^2 \right\rangle + \frac{1}{2\beta}\left\langle |\tau' - \lambda|^2 \right\rangle, \tag{25}$$

which is used later in Lemma 2 by estimating from above the value of the bound $U(\beta,\tau,\lambda)$ for our choices of τ and λ.

For any choice of the parameters δ, ε, and A, the function τ satisfies the boundary conditions in (17), while λ is nondecreasing and satisfies the normalization condition $\langle\lambda\rangle = -1$. To establish Theorem 1 using Proposition 1, we only need to specify parameter values such that $U(\beta,\tau,\lambda) \le \frac{1}{2} - O(R^{-2})$ while ensuring that the pair (β,τ) satisfies the spectral constraint. For the purposes of simplifying the algebra in what follows, we shall fix

$$A(\delta) = \frac{3\sqrt{15}}{20}\delta^{\frac{3}{2}}, \tag{26}$$

from the outset. As explained in Remark 5, this choice arises when insisting that the upper estimate on $U(\beta,\tau,\lambda)$ derived in Lemma 2 be strictly less than $\frac{1}{2}$ for some values of δ and ε, at least when all other constraints on these parameters are ignored.

C. Preliminary estimates

We now derive a series of auxiliary results that make it simpler to specify the boundary layer widths δ and ε. The first result gives estimates on the value of β in (24).

Lemma 1 (estimates on β). Let $\tau(z)$, $\lambda(z)$, and β be given by (22), (23), and (24), respectively, with A specified by (26). Suppose that the boundary layer widths δ and ε satisfy

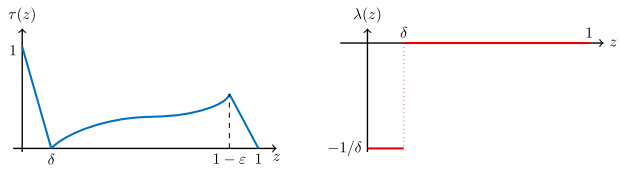


FIG. 2. Sketches of the functions $\tau(z)$ in (22) and $\lambda(z)$ in (23) used to prove Theorem 1.

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$$\delta \le \frac{1}{6}, \tag{27a}$$

$$\varepsilon \le \frac{1}{3}, \tag{27b}$$

$$\delta \ln^2\left(\frac{1}{\delta^2}\right) \le \varepsilon. \tag{27c}$$

Then,

$$\frac{9}{2\sqrt{5}}\delta \le \beta \le \frac{9}{2}\delta. \tag{28}$$

Remark 3. Condition (27c) is key to prove the auxiliary results of this section. The other two restrictions, instead, are introduced to more easily keep track of constants in our estimates, which is necessary to obtain an explicit prefactor for the $O(R^{-2})$ term. We have not attempted to optimize this prefactor.

Remark 4. Condition (27c) implies that $\delta \le \varepsilon$. We will use this fact often in the proofs of this section.

Proof of Lemma 1. It suffices to estimate $\langle|\tau' - \lambda|^2\rangle^{\frac{1}{2}}$ from above and below. For a lower bound, substitute our choices for τ and λ to estimate

$$\langle|\tau' - \lambda|^2\rangle = \int_\delta^1 |\tau'(z)|^2 \, dz \ge \int_\delta^{1-\varepsilon} |\tau'(z)|^2 \, dz = A^2 \int_\delta^{1-\varepsilon} \left|\frac{1}{z} + \frac{1}{1-z}\right|^2 dz.$$

Expanding the square, dropping the non-negative term $\frac{2}{z(1-z)}$, and recalling that $\varepsilon \le \frac{1}{3}$ and $\delta \le \frac{1}{6} < \frac{1}{3}$ by assumption, we can further estimate

$$\langle|\tau' - \lambda|^2\rangle \ge A^2\left(\int_\delta^{2/3} \frac{1}{z^2} \, dz + \int_{1/3}^{2/3} \frac{1}{(1-z)^2} \, dz\right) = \frac{A^2}{\delta}. \tag{29}$$

Taking the square root of both sides and substituting for the value of A from (26) gives $\langle|\tau' - \lambda|^2\rangle^{1/2} \ge (3\sqrt{15}/20)\ \delta$, which combined with (24) proves the lower bound on β stated in (28).

For the upper bound on β, recall that condition (27c) implies $\delta \le \varepsilon$, so $1 - \varepsilon \le 1 - \delta \le 1$. Then,

$$\begin{aligned}\frac{\langle|\tau' - \lambda|^2\rangle}{A^2} &= \int_\delta^{1-\varepsilon} \left|\frac{1}{z} + \frac{1}{1-z}\right|^2 dz + \frac{1}{\varepsilon}\ln^2\left(\frac{(1-\varepsilon)(1-\delta)}{\varepsilon\delta}\right) \\ &\le \int_\delta^{1-\delta} \left|\frac{1}{z} + \frac{1}{1-z}\right|^2 dz + \frac{1}{\varepsilon}\ln^2\left(\frac{1}{\delta^2}\right).\end{aligned}$$

Using the inequality $(a+b)^2 \le 2a^2 + 2b^2$, we can further estimate

$$\begin{aligned}\frac{\langle|\tau' - \lambda|^2\rangle}{A^2} &\le 2\left(\int_\delta^{1-\delta} \frac{1}{z^2} \, dz + \int_\delta^{1-\delta} \frac{1}{(1-z)^2} \, dz\right) + \frac{1}{\varepsilon}\ln^2\left(\frac{1}{\delta^2}\right) \\ &= \frac{4}{\delta}\left(\frac{1-2\delta}{1-\delta}\right) + \frac{1}{\varepsilon}\ln^2\left(\frac{1}{\delta^2}\right).\end{aligned}$$

Finally, we observe that $\frac{1-2\delta}{1-\delta} \le 1$ for all $\delta \ge 0$ and apply (27c) to arrive at

$$\langle|\tau' - \lambda|^2\rangle \le \frac{5A^2}{\delta}. \tag{30}$$

Substituting our choice of A from (26) and taking a square root gives $\langle|\tau' - \lambda|^2\rangle^{1/2} \le (3\sqrt{3}/4)\delta$, which combined with (24) yields the upper bound on β in (28). □

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Our second auxiliary result estimates the upper bound $U(\beta,\tau,\lambda)$ on $\overline{\langle wT\rangle}$ from Proposition 1 in terms of the bottom boundary layer width δ alone.

Lemma 2 (estimates on $U(\beta,\tau,\lambda)$). Let $\tau(z)$, $\lambda(z)$, and β be given by (22), (23), and (24), respectively, with A specified by (26). Suppose the boundary layer widths δ and ε satisfy the conditions of Lemma 1. Then,

$$U(\beta,\tau,\lambda) \le \frac{1}{2} - \frac{\delta}{8}. \tag{31}$$

Proof. Using inequality (25), our choice of β from (24), and the upper bound on $\langle|\tau'-\lambda|^2\rangle$ from (30) yields

$$U(\beta,\tau,\lambda) \le \frac{1}{2} + \frac{\sqrt{3}}{6}\left\langle|\tau'-\lambda|^2\right\rangle^{\frac{1}{2}} - \langle\tau\rangle \le \frac{1}{2} + \frac{\sqrt{15}A}{6}\,\delta^{-\frac{1}{2}} - \langle\tau\rangle.$$

Moreover, since we have chosen τ to be a non-negative function, we can estimate $\langle\tau\rangle = \int_0^1 \tau(z)\ \mathrm{d}z \ge \int_0^\delta \tau(z)\ \mathrm{d}z = \frac{1}{2}\delta$ to obtain

$$U(\beta,\tau,\lambda) \le \frac{1}{2} + \frac{\sqrt{15}A}{6}\,\delta^{-\frac{1}{2}} - \frac{1}{2}\delta. \tag{32}$$

Substituting our choice of A from (26) into this inequality gives (31). □

Remark 5. The right-hand side of (32) can be strictly smaller than $\frac{1}{2}$ only if $A \lesssim \delta^{3/2}$. It is this observation that dictates the choice of A in (26). For any fixed value of R, one should choose $A \sim \delta^{3/2}$ with a (possibly R-dependent) prefactor that optimizes the balance between the positive and negative terms, subject to constraints on A, δ, and all other parameters that ensure the spectral constraint. To simplify our proof, however, we choose to fix this prefactor *a priori* irrespective of R.

Our final auxiliary result gives sufficient conditions on δ and ε that ensure the spectral constraint (cf. Definition 1) is satisfied.

Lemma 3 (sufficient conditions for the spectral constraint). Let $\tau(z)$, $\lambda(z)$, and β be given by (22), (23), and (24), respectively, with A specified by (26). Suppose the boundary layer widths δ and ε satisfy the conditions of Lemma 1. Suppose further that

$$\delta \le (24\sqrt{3})^2\, R^{-2} \tag{33a}$$

and

$$\varepsilon^3\delta^{\frac{1}{2}}\ln^2\left(\frac{1}{\delta^2}\right) \le 8\sqrt{3}R^{-1}. \tag{33b}$$

Then, the pair (β,τ) satisfies the spectral constraint.

The proof of this result relies on Hardy–Rellich inequalities established in Ref. 14, which extract a positive term from the *a priori* indefinite term $\langle\tau' wT\rangle$.

Lemma 4 (Hardy–Rellich inequalities[14]). Let $T, w : \Omega \to \mathbb{R}$ be horizontally periodic functions such that $\Delta^2 w = -R\Delta_h T$ subject to velocity boundary conditions (13). Then,

$$\left\langle\frac{wT}{z}\right\rangle \ge \frac{4}{R}\left\langle\frac{w^2}{z^3}\right\rangle \tag{34a}$$

and

$$\left\langle\frac{wT}{1-z}\right\rangle \ge \frac{4}{R}\left\langle\frac{w^2}{(1-z)^3}\right\rangle. \tag{34b}$$

Proof of Lemma 3. Let $\mathbb{1}_{(a,b)}$ denote the indicator function of the interval (a,b). Define the functions

$$f(z) := \left[\frac{1}{\delta} + \frac{A}{z(1-z)}\right]\mathbb{1}_{(0,\delta)}(z), \tag{35a}$$

$$g(z) := \left[\frac{1}{\varepsilon}\ln\left(\frac{(1-\varepsilon)(1-\delta)}{\varepsilon\delta}\right) + \frac{1}{z(1-z)}\right]\mathbb{1}_{(1-\varepsilon,1)}(z). \tag{35b}$$

Given our choice of τ from (22), we can rewrite the spectral constraint as

$$0 \le \left\langle\beta|\nabla T|^2 + \tau' wT\right\rangle = \mathcal{F}\{T\} + \mathcal{G}\{T\},$$

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where

$$\mathcal{F}\{T\} := \frac{\beta}{2}\left\langle |\nabla T|^2 \right\rangle + A\left\langle \frac{wT}{z} \right\rangle - \langle f(z) wT \rangle, \tag{36a}$$

$$\mathcal{G}\{T\} := \frac{\beta}{2}\left\langle |\nabla T|^2 \right\rangle + A\left\langle \frac{wT}{1-z} \right\rangle - A\langle g(z) wT \rangle. \tag{36b}$$

Here, w is determined as a function of T by solving (12) subject to the boundary conditions in (13). We shall prove that $\mathcal{F}$ and $\mathcal{G}$ are individually non-negative.

First, let us consider $\mathcal{F}$. The Hardy–Rellich inequality (34a) gives

$$\mathcal{F}\{T\} \geq \frac{\beta}{2}\left\langle |\nabla T|^2 \right\rangle + \frac{4A}{R}\left\langle \frac{w^2}{z^3} \right\rangle - \langle f(z) wT \rangle. \tag{37}$$

Next, we estimate $\langle f(z) wT \rangle$. Since T vanishes at $z = 0$ by virtue of the thermal boundary conditions in (4), we can use the fundamental theorem of calculus and the Cauchy–Schwarz inequality to estimate $|T(\cdot, z)| \leq \sqrt{z}\left(\int_0^1 |\nabla T|^2 \mathrm{d}z\right)^{1/2}$. Squaring both sides and taking the horizontal average of which give $\left\langle T^2 \right\rangle_h \leq z\left\langle |\nabla T|^2 \right\rangle$. Then, the use of the Cauchy–Schwarz inequality, substitution for $\left\langle T^2 \right\rangle_h$, and Young's inequality gives

$$\begin{aligned}
\langle f(z) wT \rangle &\leq \int_0^1 |f(z)| \langle |wT| \rangle_h \, \mathrm{d}z \leq \int_0^1 |f(z)| \left\langle w^2 \right\rangle_h^{1/2} \left\langle T^2 \right\rangle_h^{1/2} \mathrm{d}z \\
&\leq \left\langle |\nabla T|^2 \right\rangle^{\frac{1}{2}} \int_0^1 f(z) z^2 \left\langle \frac{w^2}{z^3} \right\rangle_h^{\frac{1}{2}} \mathrm{d}z \\
&\leq \left\langle |\nabla T|^2 \right\rangle^{\frac{1}{2}} \left\langle f(z)^2 z^4 \right\rangle^{\frac{1}{2}} \left\langle \frac{w^2}{z^3} \right\rangle^{\frac{1}{2}} \\
&\leq \frac{\beta}{2}\left\langle |\nabla T|^2 \right\rangle + \frac{1}{2\beta}\left\langle f(z)^2 z^4 \right\rangle \left\langle \frac{w^2}{z^3} \right\rangle.
\end{aligned}$$

Upon using the lower bound on β from (28) to estimate the last term from above, we obtain

$$\langle f(z) wT \rangle \leq \frac{\beta}{2}\left\langle |\nabla T|^2 \right\rangle + \frac{\sqrt{5}}{9\delta}\left\langle f(z)^2 z^4 \right\rangle \left\langle \frac{w^2}{z^3} \right\rangle. \tag{38}$$

This can be substituted into (37) along with our choice of A from (26) to find

$$\mathcal{F}\{T\} \geq \frac{\sqrt{5}}{9}\delta^{\frac{3}{2}}\left(\frac{27\sqrt{3}}{5R} - \frac{\left\langle f(z)^2 z^4 \right\rangle}{\delta^{\frac{5}{2}}} \right) \left\langle \frac{w^2}{z^3} \right\rangle. \tag{39}$$

To conclude, we show that the term in parentheses is non-negative when δ satisfies $\delta \leq \frac{1}{6}$ and (33a). To do this, we observe that the function $f(z)$ in (35a) is non-negative function and that it is nonzero only if $z \leq \delta$. We can therefore bound it from above on the interval $(0, \delta)$ using the estimates $\frac{1}{1-z} \leq \frac{1}{1-\delta} \leq \frac{6}{5}$ and, consequently, obtain

$$\frac{\left\langle f(z)^2 z^4 \right\rangle}{\delta^{\frac{5}{2}}} \leq \frac{1}{\delta^{\frac{5}{2}}} \int_0^\delta \left(\frac{1}{\delta} + \frac{6A}{5z} \right)^2 z^4 \, \mathrm{d}z = \left(\frac{1}{5} + \frac{9\sqrt{15}}{100}\delta^{\frac{3}{2}} + \frac{81}{500}\delta^3 \right) \delta^{\frac{1}{2}}.$$

Using the assumption that $\delta \leq \frac{1}{6}$ to estimate the expression in parentheses by its value at $\delta = \frac{1}{6}$, followed by an application of assumption (33a) to estimate the remaining $\delta^{1/2}$ term in terms of R, we arrive at the desired inequality

$$\frac{\left\langle f(z)^2 z^4 \right\rangle}{\delta^{\frac{5}{2}}} \leq \frac{9\delta^{\frac{1}{2}}}{40} \leq \frac{27\sqrt{3}}{5R}.$$

Analogous arguments show that $\mathcal{G}\{T\}$ is non-negative. Using the Hardy–Rellich inequality (34b)

$$\mathcal{G}\{T\} \geq \frac{\beta}{2}\left\langle |\nabla T|^2 \right\rangle + \frac{4A}{R}\left\langle \frac{|w|^2}{(1-z)^3} \right\rangle - A\langle g(z) wT \rangle. \tag{40}$$

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To estimate the last term, we use (in order) the inequality $\langle T^2\rangle_h \le (1-z)\langle|\nabla T|^2\rangle$, the Cauchy–Schwarz inequality, and Young's inequality,

$$\begin{aligned}\langle g(z)wT\rangle &\le \int_0^1 |g(z)|\langle|wT|\rangle_h \mathrm{d}z \le \int_0^1 |g(z)|\langle w^2\rangle_h^{1/2}\langle T^2\rangle_h^{1/2}\mathrm{d}z\\ &\le \langle|\nabla T|^2\rangle^{\frac12}\int_0^1 g(z)(1-z)^2\left\langle\frac{w^2}{(1-z)^3}\right\rangle_h^{\frac12}\mathrm{d}z\\ &\le \langle|\nabla T|^2\rangle^{\frac12}\langle g(z)^2(1-z)^4\rangle^{\frac12}\left\langle\frac{w^2}{(1-z)^3}\right\rangle^{\frac12}\\ &\le \frac{\beta}{2A}\langle|\nabla T|^2\rangle + \frac{A}{2\beta}\langle g(z)^2(1-z)^4\rangle\left\langle\frac{w^2}{(1-z)^3}\right\rangle.\end{aligned}$$

Using the lower bound on β from (28) gives

$$\langle g(z)wT\rangle \le \frac{\beta}{2A}\langle|\nabla T|^2\rangle + \frac{\sqrt{5}A}{9\delta}\langle g(z)^2(1-z)^4\rangle\left\langle\frac{w^2}{(1-z)^3}\right\rangle, \tag{41}$$

which can be substituted into (40) along with the value of A from (26) to obtain

$$\mathcal{G}\{T\} \ge \frac{3\delta^{\frac32}}{16\sqrt{5}}\left(\frac{16\sqrt{3}}{R} - \delta^{\frac12}\langle g(z)^2(1-z)^4\rangle\right)\left\langle\frac{w^2}{(1-z)^3}\right\rangle. \tag{42}$$

To conclude the argument, we show that the term in parentheses is non-negative. To demonstrate this, we first estimate $g(z)$ on the interval $(1-\varepsilon,1)$ from above using the assumption that $\varepsilon \le \frac13$, so $\frac23 \le z \le 1$ and $\frac{1}{z(1-z)} \le \frac{3}{2(1-z)}$. Thus,

$$g(z) \le \frac{1}{\varepsilon}\ln\left(\frac{1}{\delta^2}\right) + \frac{3}{2(1-z)} \quad \forall z \in (1-\varepsilon,1).$$

Then, we use the assumptions $\delta \le \frac16$ and $\delta \le \varepsilon$ (cf. Remark 4) to observe that $\ln\left(\frac{1}{\varepsilon\delta}\right) \le \ln\left(\frac{1}{\delta^2}\right) \le \ln^2\left(\frac{1}{\delta^2}\right)$ and $\frac34 \le \frac34\ln^2\left(\frac{1}{\delta^2}\right)$. Combining these estimates with the upper bound on g derived above gives

$$\begin{aligned}\delta^{\frac12}\langle g(z)^2(1-z)^4\rangle &\le \delta^{\frac12}\int_{1-\varepsilon}^1\left(\frac{1}{\varepsilon}\ln\left(\frac{1}{\delta^2}\right) + \frac{3}{2(1-z)}\right)^2(1-z)^4\,\mathrm{d}z\\ &= \delta^{\frac12}\varepsilon^3\left(\frac15\ln^2\left(\frac{1}{\delta^2}\right) + \frac34\ln\left(\frac{1}{\delta^2}\right) + \frac34\right)\\ &\le 2\delta^{\frac12}\varepsilon^3\ln^2\left(\frac{1}{\delta^2}\right)\\ (\text{by (33}b\text{)}) &\le \frac{16\sqrt{3}}{R},\end{aligned}$$

as desired. This concludes the Proof of Lemma 3. □

D. Proof of Theorem 1

It is now straightforward to prove the upper bound on $\overline{\langle wT\rangle}$ by specifying boundary layer widths δ and ε that satisfy the conditions of Lemmas 1, 2, and 3.

Since the estimate for the resulting upper bound obtained in Lemma 2 is minimized when δ is as large as possible, we choose the largest value consistent with (33a),

$$\delta = (24\sqrt{3})^2\, R^{-2}. \tag{43}$$

With this choice of δ, conditions (27c) and (33b) require ε to satisfy

$$\frac{27\ 648}{R^2}\ln^2\left(\frac{R}{24\sqrt{3}}\right) \le \varepsilon \le \left(\frac{1}{48}\right)^{\frac13}\ln^{-\frac23}\left(\frac{R}{24\sqrt{3}}\right), \tag{44}$$

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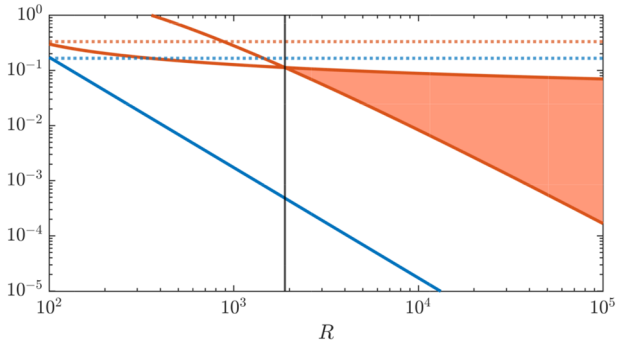


FIG. 3. Variation with R of the allowed values for the bottom boundary layer width ε (shaded region), determined by condition (27c) in Lemma 1 and condition (33b) in Lemma 3 when the bottom boundary layer width δ (blue solid lines) is chosen as in (43). Also shown are the uniform upper bounds $\delta \le \frac{1}{6}$ (blue dashed lines) and $\varepsilon \le \frac{1}{3}$ (red dashed lines) imposed on these variables. A black vertical line marks the Rayleigh number $R_0 \simeq 1891.35$ above which all constraints on δ and ε are satisfied.

which is possible for $R > R_0 \simeq 1891.35$ (cf. Fig. 3). For $R > R_0$, any choice of ε in this range is feasible. The optimal value could be determined at the expense of more complicated algebra either by optimizing the full bound $U(\beta, \tau, \lambda)$ or by deriving better ε-dependent estimates for it. However, we expect that any ε-dependent terms will contribute only higher-order corrections to our bound on $\overline{\langle wT \rangle}$.

To conclude the Proof of Theorem 1, there remains to verify that our choice of δ is no larger than $\frac{1}{6}$ and that any ε satisfying (44) is no larger than $\frac{1}{3}$. It is easily checked that both conditions hold when $R \ge R_0$ (see Fig. 3 for an illustration). For all such values of R, therefore, Lemma 2 and our choice of δ yield the upper bound $\overline{\langle wT \rangle} \le U(\beta, \tau, \lambda) \le \frac{1}{2} - cR^{-2}$ with $c = 216$.

III. BOUNDS FOR THE IH3 CONFIGURATION

We now move on to studying IH convection in the IH3 configuration, where the top boundary is maintained at constant (zero) temperature and the bottom boundary is insulating. First, in Sec. III A, we derive a bounding framework for $\overline{\langle wT \rangle}$ following steps similar to those used for the IH1 case (cf. Sec. II A). In Sec. III B, we present *Ansätze* for τ and λ, with which we obtain crucial estimates in Sec. III C, which give the bound in Sec. III D. Throughout this section, T belongs to $\mathcal{H}_3$. Observe that this changes the set of temperature fields over which the spectral constraint in Definition 1 is imposed. The notation $\mathcal{H}_+$ still denotes the subset of temperature fields in $\mathcal{H}_3$ that are non-negative pointwise almost everywhere.

A. Bounding framework

Upper bounds on $\overline{\langle wT \rangle}$ for the IH3 configuration can be derived using a quadratic auxiliary function $\mathcal{V}\{T\}$ similar to that used for the IH1 case. Precisely, we still take $\mathcal{V}$ to be defined as in (16), where the positive constant β and the piecewise-differentiable square-integrable function $\tau(z)$ are tunable parameters. However, this time we impose only the boundary condition

$$\tau(1) = 0. \tag{45}$$

These changes result in the following family of parameterized upper bounds on $\overline{\langle wT \rangle}$:

Proposition 2 (bounding framework, IH3). Suppose that the pair (β, τ) satisfies the spectral constraint (cf. Definition 1) and the boundary condition in (45). Furthermore, let $\lambda \in L^2(0,1)$ be a nondecreasing function such that $\lambda(0) = -1$. Then,

$$\overline{\langle wT \rangle} \le \frac{1}{2} + \left\langle \frac{1}{4\beta} \left| \tau' - \lambda - \beta z \right|^2 - \tau \right\rangle := U(\tau, \lambda, \beta)$$

Proof. Proceeding as in the Proof of Proposition 1 shows that

$$\overline{\langle wT \rangle} \le \frac{1}{2} + \sup_{\substack{T \in \mathcal{H}_+ \\ w = -R\Delta^{-2}\Delta_h T}} \left\langle -b|\nabla T|^2 - \tau' wT + (\tau - \beta z + 1)\partial_z T - \tau \right\rangle. \tag{46}$$

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The supremum on the right-hand side can be evaluated using the convex duality argument summarized in Appendix B, leading to the equivalent inequality

$$\overline{\langle wT\rangle} \le \frac{1}{2} + \inf_{\substack{\lambda\in L^2(0,1)\\ \lambda \text{ nondecreasing}\\ \lambda\ge -1}} \left\langle \frac{1}{4\beta}\left|\tau' - \lambda - \beta z\right|^2 - \tau \right\rangle. \tag{47}$$

This clearly implies the upper bound stated in the proposition. □

B. Ansätze

The procedure for the proof of an upper bound on $\overline{\langle wT\rangle}$ is the same as that employed for isothermal boundaries. We construct $\beta > 0$, $\tau(z)$, and $\lambda(z)$ that satisfy the conditions of Proposition 2, while trying to minimize the corresponding bound $U(\beta,\tau,\lambda)$. Due to the Neumann boundary condition on T at $z = 0$, we can no longer employ the Poincaré estimates used in Sec. II C to control the sign-indefinite term in the spectral constraint at the bottom boundary. Instead, we modify $\tau(z)$ in $(\delta, 1-\varepsilon)$ to increase slower than logarithmically in z and use results established in previous work on Rayleigh–Bénard convection.[28] The function $\tau(z)$ is hence chosen to have the form

$$\tau(z) := \begin{cases} \delta - z, & 0 \le z \le \delta, \\ \dfrac{A}{1-\alpha}\left(z^{1-\alpha} - \delta^{1-\alpha}\right) - A\,\ln\left(\dfrac{1-z}{1-\delta}\right), & \delta \le z \le 1-\varepsilon, \\ \dfrac{A\,B}{\varepsilon}(1-z), & 1-\varepsilon \le z \le 1, \end{cases} \tag{48}$$

where

$$B = B(\epsilon,\delta,\alpha) := \frac{(1-\varepsilon)^{1-\alpha} - \delta^{1-\alpha}}{1-\alpha} + \ln\left(\frac{1-\delta}{\varepsilon}\right). \tag{49}$$

On the other hand, the Lagrange multiplier $\lambda(z)$ is still chosen to be

$$\lambda(z) := \begin{cases} -1, & 0 \le z \le \delta, \\ 0, & \delta \le z \le 1. \end{cases} \tag{50}$$

These piecewise functions, sketched in Fig. 4, are fully specified by the bottom and top boundary layer widths $\delta, \varepsilon \in (0, \frac{1}{2})$, the constant $A > 0$, and the exponent $\alpha \in (0,1)$ driving the behavior of $\tau(z)$ in the bulk.

For β, we take

$$\beta := \left\langle |\tau' - \lambda|^2\right\rangle^{\frac{1}{2}} \left\langle z^2\right\rangle^{-\frac{1}{2}} = \sqrt{3}\left\langle |\tau' - \lambda|^2\right\rangle^{\frac{1}{2}}. \tag{51}$$

This choice is motivated by minimizing the right-hand side of the estimate,

$$\frac{1}{4\beta}\left\langle |\tau' - \lambda - \beta z|^2\right\rangle \le \frac{\beta}{2}\left\langle z^2\right\rangle + \frac{1}{2\beta}\left\langle |\tau' - \lambda|^2\right\rangle = \frac{1}{\sqrt{3}}\left\langle |\tau' - \lambda|^2\right\rangle^{\frac{1}{2}} = \frac{\beta}{3}, \tag{52}$$

which is used in Lemma 6 to estimate the value of the bound $U(\beta,\tau,\lambda)$ from above when τ and λ are define by (48) and (50), respectively.

For any choice of the parameters δ, ε, A, and α, the function τ satisfies the boundary conditions in (45), while λ is nondecreasing and satisfies the condition $\lambda(0) = -1$. Thus, to establish Theorem 2 using Proposition 2, we need only specific parameter values such that $U(\beta,\tau,\lambda) \le \frac{1}{2} - O(R^{-4})$ while ensuring that (β,τ) satisfy the spectral constraint. For the purposes of simplifying the algebra in what follows, we shall fix

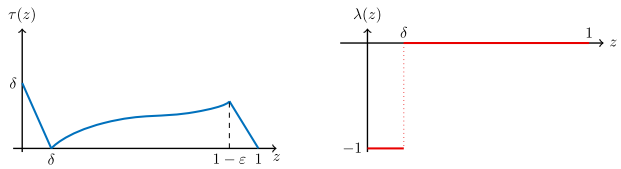


FIG. 4. Sketch of the functions $\tau(z)$ (48) and $\lambda(z)$ in (50) used to prove Theorem 2.

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$$A(\delta,\alpha) = \frac{2\sqrt{3}}{9}\sqrt{2\alpha-1}\ \delta^{\alpha+\frac{3}{2}} \tag{53}$$

from the outset. This choice arises when insisting that the upper estimate on $U(\beta,\tau,\lambda)$ derived in Lemma 6 should be strictly less than $\frac{1}{2}$ for suitable choices of δ and ε, at least when all other constraints on these parameters are ignored.

C. Preliminary estimates

We now derive auxiliary results that simplify the choice of the exponent α and of the boundary widths δ and ε. The first gives estimates on the value of β in (51).

Lemma 5 (estimates on β). Let $\tau(z)$, $\lambda(z)$, and β be given by (48), (50), and (51) with A specified in (53). Suppose that $\alpha \in (\frac{1}{2},1)$ and the boundary layer widths δ and ε satisfy

$$\delta \le \frac{1}{3}\left(\frac{1}{2}\right)^{\frac{1}{2\alpha-1}}, \tag{54a}$$

$$\varepsilon \le \frac{1}{3}, \tag{54b}$$

$$(2\alpha-1)\ \delta^{2\alpha-1}B(\epsilon,\delta,\alpha)^2 \le \varepsilon. \tag{54c}$$

Then,

$$\frac{\sqrt{2}}{3}\ \delta^2 \le \beta \le \frac{4}{3}\ \delta^2. \tag{55}$$

Remark 6. Condition (54a) and the bounds on α imposed in the Lemma imply that $0 \le \delta \le \frac{1}{6}$. These uniform bounds will be used repeatedly in the following proofs:

Proof of Lemma 5. It suffices to estimate $\langle|\tau'-\lambda|^2\rangle^{\frac{1}{2}}$ from above and below. For a lower bound, we can substitute our choices of τ and λ and then estimate

$$\langle|\tau'-\lambda|^2\rangle = \int_\delta^1 |\tau'(z)|^2 dz \ge \int_\delta^{1-\varepsilon} |\tau'(z)|^2 dz = A^2\int_\delta^{1-\varepsilon}\left|\frac{1}{z^\alpha}+\frac{1}{1-z}\right|^2 dz.$$

Dropping the positive term $1/(1-z)$ from the integrand and integrating the rest gives

$$\langle|\tau'-\lambda|^2\rangle \ge A^2\left(\frac{\delta^{1-2\alpha}-(1-\varepsilon)^{1-2\alpha}}{2\alpha-1}\right).$$

For every $\alpha \in (\frac{1}{2},1)$, the second term inside the parentheses can be estimated upon observing that constraints (54a) and (54b) imply

$$(1-\varepsilon)^{1-2\alpha} \le \left(\frac{2}{3}\right)^{1-2\alpha} \le \frac{1}{2^{2\alpha}}\delta^{1-2\alpha} \le \frac{1}{2}\delta^{1-2\alpha}. \tag{56}$$

Thus, we obtain

$$\langle|\tau'-\lambda|^2\rangle \ge A^2\frac{\frac{1}{2}\delta^{1-2\alpha}}{2\alpha-1}. \tag{57}$$

Taking the square root of (57) and using (53) gives $\langle|\tau'-\lambda|^2\rangle^{\frac{1}{2}} \ge \frac{\sqrt{6}}{9}\ \delta^2$, which combined with (51) proves the lower bound on β stated in (55).

To prove the upper bound on β, we start by using the inequality $(a+b)^2 \le 2a^2+2b^2$, evaluating exactly the integral of $\tau'-\lambda$, and dropping the negative terms to get

$$\begin{aligned}\frac{\langle|\tau'-\lambda|^2\rangle}{A^2} &= \int_\delta^{1-\varepsilon}\left|\frac{1}{z^\alpha}+\frac{1}{1-z}\right|^2 dz + \frac{B^2}{\varepsilon}\\ &\le \int_\delta^{1-\varepsilon}\frac{2}{z^{2\alpha}}+\frac{2}{(1-z)^2}dz + \frac{B^2}{\varepsilon}\\ \left(\text{since } \alpha > \frac{1}{2}\right) &\le \frac{2\delta^{1-2\alpha}}{2\alpha-1}+\frac{2}{\varepsilon}+\frac{B^2}{\varepsilon}.\end{aligned} \tag{58}$$

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Using assumption (54c), the second and final term in (58) can be estimated from above to arrive at

$$\frac{\langle|\tau'-\lambda|^2\rangle}{A^2} \le \frac{\delta^{1-2\alpha}}{2\alpha-1}\left(2+\frac{2}{B^2}+1\right). \tag{59}$$

Next, we observe that for all $\varepsilon \in (0, \frac{1}{3})$, $\delta \in (0, \frac{1}{6})$, and $\alpha \in (\frac{1}{2}, 1)$, we can estimate

$$B(\varepsilon,\delta,\alpha) \ge B\left(\frac{1}{3},\frac{1}{6},\frac{1}{2}\right) = \frac{\sqrt{6}}{3} + \ln\left(\frac{5}{2}\right) > \sqrt{2},$$

so $B^{-2} \le 1/2$. Using this estimate in (59), taking a square root, and substituting in the value of A given in (53) leads to the inequality $\langle|\tau'-\lambda|^2\rangle^{1/2} \le (4\sqrt{3}/9)\,\delta^2$. Combining this with (51) yields the upper bound on β stated in (55) and concludes the Proof of Lemma 5. □

The second auxiliary result of this section estimates the upper bound $U(\beta,\tau,\lambda)$ on $\overline{\langle wT\rangle}$ given by Proposition 2 using only the bottom boundary layer width δ.

Lemma 6 (estimates on U(β, τ, λ)). Let $\tau(z)$, $\lambda(z)$, and β be specified by (48), (50), and (51), respectively, with A given by (53). Suppose α and the boundary layer widths δ, ε satisfy the conditions of Lemma 5. Then,

$$U(\beta,\tau,\lambda) \le \frac{1}{2} - \frac{\delta^2}{18}. \tag{60}$$

Proof. Inequality (52) and the upper bound on β from (55) give

$$U(\beta,\tau,\lambda) \le \frac{1}{2} + \frac{\beta}{3} - \langle\tau\rangle \le \frac{1}{2} + \frac{4}{9}\delta^2 - \langle\tau\rangle. \tag{61}$$

The result follows upon observing that τ is non-negative, so $\langle\tau\rangle = \int_0^1 \tau(z)\mathrm{d}z \ge \int_0^\delta \tau(z)\mathrm{d}z = \frac{1}{2}\delta^2$. □

Remark 7. The right-hand side of (61) can be strictly smaller than $\frac{1}{2}$ when δ^2 is small only if $A \lesssim \delta^{\alpha+3/2}$. This observation dictates the choice of A in (53). For any fixed value of R, one should choose $A \sim \delta^{\alpha+3/2}$ with a (possibly R-dependent) prefactor that optimizes the balance between the positive and negative terms, subject to constraints on A, δ, and all other parameters that ensure the spectral constraint. To simplify our proof, however, we choose to fix this prefactor *a priori* irrespective of R.

Our final auxiliary result gives the sufficient conditions on δ and ε that ensure that the spectral constraint (cf. Definition 1) is satisfied.

Lemma 7 (sufficient conditions for spectral constraint). Let $\tau(z)$, $\lambda(z)$, and β be specified by (48), (50), and (51), respectively, with A given by (53). Suppose that α and the boundary layer widths satisfy the conditions of Lemma 5. Furthermore, let

$$h(\alpha) = 2(2\alpha-1)(1-\alpha^2) \tag{62}$$

and suppose that

$$\delta \le \varepsilon, \tag{63a}$$

$$\delta \le h(\alpha)\ R^{-2}, \tag{63b}$$

$$\frac{\sqrt{2\alpha-1}}{(1-\alpha)^2}\ \delta^{\alpha-\frac{1}{2}}\varepsilon^3 \ln^2(\delta^{-1}) \le \frac{4}{3}\sqrt{6}R^{-1}. \tag{63c}$$

Then, the pair (β,τ) satisfies the spectral constraint.

Unlike the analogous result obtained in Sec. II C, Lemma 7 cannot be proven using only the Hardy–Rellich inequalities stated in Lemma 4. The lack of a fixed boundary temperature at $z = 0$ makes it impossible to gain sufficient control on the contribution of the bottom boundary layer to the quadratic form in (18). This difficulty can be overcome using the following result, obtained as a particular case

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of a more general analysis by Whitehead and Wittenberg[28] [Eqs. (59) and (77)], which upon setting (in their notation) $v_1 = \frac{\alpha}{2} - v_2$ and $v_2 = \frac{1}{2}\left(-1 + \sqrt{2(1-\alpha^2)}\right)$.

Lemma 8 (adapted from Ref. 28). Fix $\alpha \in (\frac{1}{2}, 1)$ and $\mu, R > 0$. Suppose $\varphi(z) : [0,1] \to \mathbb{R}$ be a non-negative function, satisfying

$$\left\langle \varphi(z) z^{1+\alpha/2} \right\rangle^2 \leq \frac{\mu}{R} \frac{\sqrt{2}(3-\alpha)(2+\alpha)\sqrt{1-\alpha^2}}{3+\alpha}. \tag{64}$$

Then, for every w and T solving (12) subject to the boundary conditions (13),

$$\left\langle \frac{1}{2}|\nabla T|^2 + \mu \frac{wT}{z^\alpha} - \varphi(z) wT \right\rangle \geq 0. \tag{65}$$

We are now ready to prove Lemma 7.

Proof of Lemma 7. Let $\mathbb{1}_{(a,b)}$ denote the indicator function of the interval (a,b) and define the functions

$$f(z) := \left[1 + \frac{A(\delta,\alpha)}{z^\alpha} + \frac{A(\delta,\alpha)}{1-z}\right] \mathbb{1}_{(0,\delta)}(z), \tag{66a}$$

$$g(z) := \left[\frac{B(\varepsilon,\delta,\alpha)}{\varepsilon} + \frac{1}{z^\alpha} + \frac{1}{1-z}\right] \mathbb{1}_{(1-\varepsilon,1)}(z). \tag{66b}$$

Given our choice of τ, we can rewrite the spectral constraint as

$$0 \leq \left\langle \beta|\nabla T|^2 + \tau' wT \right\rangle = \mathcal{F}\{T\} + \mathcal{G}\{T\},$$

where

$$\mathcal{F}\{T\} := \frac{\beta}{2}\left\langle |\nabla T|^2 \right\rangle + A\left\langle \frac{wT}{z^\alpha} \right\rangle - \langle f(z)\ wT \rangle, \tag{67a}$$

$$\mathcal{G}\{T\} := \frac{\beta}{2}\left\langle |\nabla T|^2 \right\rangle + A\left\langle \frac{wT}{1-z} \right\rangle - A\langle g(z)\ wT \rangle. \tag{67b}$$

Observe that $\mathcal{F}\{T\}$ and $\mathcal{G}\{T\}$ are functionals of the temperature field only because w is determined as a function of T by solving (12) subject to the boundary conditions in (13). We shall prove that $\mathcal{F}\{T\}$ and $\mathcal{G}\{T\}$ are individually non-negative for all temperatures T from the space $\mathcal{H}_3$, which is sufficient for the spectral constraint to hold.

To prove that $\mathcal{F}\{T\} \geq 0$, we apply Lemma 8 with $\varphi(z) = f(z)/\beta$ and $\mu = A/\beta$, where f is given by (66a) and A given by (53). We therefore need to check that

$$\left\langle f(z) z^{1+\frac{\alpha}{2}} \right\rangle^2 \leq \frac{\beta A}{R} \frac{\sqrt{2}(3-\alpha)(2+\alpha)\sqrt{1-\alpha^2}}{3+\alpha}. \tag{68}$$

To verify this inequality, we first bound from above the weighted integral on the left-hand side. By assumption, we have $0 \leq z \leq \delta \leq \frac{1}{6}$ and $\alpha \in (\frac{1}{2}, 1)$, from which we obtain $\frac{1}{1-z} \leq \frac{1}{z^\alpha}$. Using this estimate and the definition of A from (53), we can therefore estimate

$$\left\langle f(z) z^{1+\frac{\alpha}{2}} \right\rangle \leq \int_0^\delta z^{1+\frac{\alpha}{2}} + 2A z^{1-\frac{\alpha}{2}} \mathrm{d}z = \left[\frac{2}{4+\alpha} + \frac{8\sqrt{3}}{9} \frac{\sqrt{2\alpha-1}}{(4-\alpha)} \delta^{\frac{3}{2}}\right] \delta^{2+\frac{\alpha}{2}}.$$

Using again that $0 \leq \delta \leq \frac{1}{6}$ and $\alpha \in (\frac{1}{2}, 1)$, the bracketed expression can be bounded from above to obtain

$$\left\langle f(z) z^{1+\frac{\alpha}{2}} \right\rangle \leq \left[\frac{4}{9} + \frac{8\sqrt{3}}{9} \cdot \frac{1}{3} \cdot \left(\frac{1}{6}\right)^{\frac{3}{2}}\right] \delta^{2+\frac{\alpha}{2}} < \frac{1}{2} \delta^{2+\frac{\alpha}{2}}. \tag{69}$$

Next, we bound from below the right-hand-side of (68). Using the lower bound on β from Lemma 5, the definition (53) of A, and the fact that $\alpha \in (\frac{1}{2}, 1)$, we have

$$\begin{aligned} \frac{\beta A \sqrt{2}(3-\alpha)(2+\alpha)}{(3+\alpha)} &\geq \frac{4\sqrt{3}}{27} \delta^{\alpha+\frac{7}{2}} \sqrt{2\alpha-1} \cdot \frac{(3-\alpha)(2+\alpha)}{(3+\alpha)} \\ &\geq \frac{5}{9\sqrt{3}} \delta^{\alpha+\frac{7}{2}} \sqrt{2\alpha-1}. \end{aligned} \tag{70}$$

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Combining (69) and (70), we conclude that (68) holds if

$$\delta^{\frac{1}{2}} \le \frac{20}{9\sqrt{3}} \frac{\sqrt{(2\alpha-1)(1-\alpha^2)}}{R} \le \sqrt{2} \frac{\sqrt{(2\alpha-1)(1-\alpha^2)}}{R},$$

which is true because δ satisfies (63a) by assumption. This proves that $\mathcal{F}\{T\} \ge 0$, as desired.

We now prove that $\mathcal{G}\{T\}$ is also non-negative. This can be done following the same steps used in Sec. II C. The Hardy–Rellich inequality (34b) gives

$$\mathcal{G}\{T\} \ge \frac{\beta}{2}\left\langle |\nabla T|^2 \right\rangle + \frac{4A}{R}\left\langle \frac{w^2}{(1-z)^3} \right\rangle - A\langle g(z) w T \rangle. \tag{71}$$

To estimate the last term, as before, we use the inequality $\left\langle T^2 \right\rangle_h \le (1-z)\left\langle |\nabla T|^2 \right\rangle$, the Cauchy–Schwarz inequality, and Young's inequality,

$$\langle g(z) w T \rangle \le \frac{\beta}{2A}\left\langle |\nabla T|^2 \right\rangle + \frac{A}{2\beta}\left\langle g(z)^2 (1-z)^4 \right\rangle \left\langle \frac{w^2}{(1-z)^3} \right\rangle.$$

Using the lower bound on β from (55) gives

$$\langle g(z) w T \rangle \le \frac{\beta}{2A}\left\langle |\nabla T|^2 \right\rangle + \frac{3\sqrt{2}\, A}{4\delta^2}\left\langle g(z)^2 (1-z)^4 \right\rangle \left\langle \frac{w^2}{(1-z)^3} \right\rangle,$$

which can be substituted into (71) along with the value of A from (53) to obtain

$$\mathcal{G}\{T\} \ge \frac{\sqrt{6(2\alpha-1)}\, A}{6}\left(\frac{4\sqrt{6}}{\sqrt{2\alpha-1}\, R} - \delta^{\alpha-\frac{1}{2}}\left\langle g(z)^2 (1-z)^4 \right\rangle \right) \left\langle \frac{|w|^2}{(1-z)^3} \right\rangle. \tag{72}$$

To conclude the argument, we need to show that the term in the parentheses is non-negative. To demonstrate this, we first estimate from above the function $g(z)$ given in (66b) on the interval $(1-\varepsilon, 1)$. Our assumption that $\delta \le \varepsilon$ implies that $1-\varepsilon \le 1-\delta \le 1$ and $\ln\left(\frac{1}{\varepsilon}\right) \le \ln\left(\frac{1}{\delta}\right)$. Thus, for all $\delta \le \frac{1}{6}$ and $\alpha \in \left(\frac{1}{2}, 1\right)$, the first term in $g(z)$ can be bounded as

$$\frac{B}{\varepsilon} \le \frac{1}{\varepsilon}\left(\frac{1}{1-\alpha} + \ln\left(\frac{1}{\delta}\right) \right) \le \frac{2\ln\left(\frac{1}{\delta}\right)}{\varepsilon(1-\alpha)}. \tag{73}$$

To estimate the other terms in $g(z)$, we observe that the assumptions $\varepsilon \le \frac{1}{3}$ and $\alpha \in \left(\frac{1}{2}, 1\right)$ imply that $\frac{1}{z^\alpha} \le \frac{1}{2(1-\alpha)(1-z)}$ and $\frac{1}{1-z} \le \frac{1}{2(1-\alpha)(1-z)}$. Consequently, we arrive at

$$g(z) \le \frac{2\ln\left(\frac{1}{\delta}\right)}{\varepsilon(1-\alpha)} + \frac{1}{(1-\alpha)(1-z)}. \tag{74}$$

Finally, using (74) and evaluating the integral in the parentheses of (72) with the fact that $\ln\left(\frac{1}{\delta}\right) \le \ln^2\left(\frac{1}{\delta}\right)$ and $\frac{1}{3} \le \frac{1}{3}\ln\left(\frac{1}{\delta}\right)$ give

$$\begin{aligned}
\delta^{\alpha-\frac{1}{2}}\left\langle g(z)^2 (1-z)^4 \right\rangle &\le \delta^{\alpha-\frac{1}{2}} \int_{1-\varepsilon}^{1} \left(\frac{2\ln\left(\frac{1}{\delta}\right)}{\varepsilon(1-\alpha)} + \frac{1}{(1-\alpha)(1-z)} \right)^2 (1-z)^4 \, \mathrm{d}z \\
&= \frac{\delta^{\alpha-\frac{1}{2}}\varepsilon^3}{(1-\alpha)^2}\left[\frac{4}{5}\ln^2\left(\frac{1}{\delta}\right) + \ln\left(\frac{1}{\delta}\right) + \frac{1}{3} \right] \\
&\le \frac{3\,\delta^{\alpha-\frac{1}{2}}\varepsilon^3 \ln^2\left(\frac{1}{\delta}\right)}{(1-\alpha)^2} \\
(\text{by (63b)}) &\le \frac{4\sqrt{6}}{\sqrt{2\alpha-1}\, R}.
\end{aligned} \tag{75}$$

This concludes the Proof of Lemma 7. □

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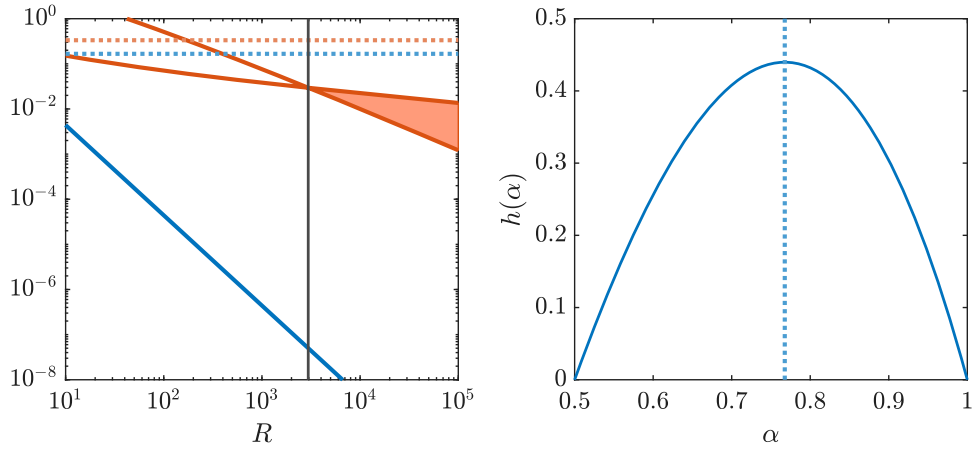


FIG. 5. (a) Variation with R of the allowed values for the bottom boundary layer width δ (76) (blue solid line) and the feasible region of ε (78) (shaded region). Also shown are uniform upper bounds of $\delta \le \frac{1}{6}$ (blue dashed line) and $\varepsilon \le \frac{1}{3}$ (red dashed line) imposed on the variables. A black vertical line marks the Rayleigh number, $R_0 \approx 2960.89$, above which all constraints are satisfied. (b) Plot of the function $h(\alpha)$ (62) (blue solid line). Shown also is the optimal $\alpha^* = (1+\sqrt{13})/6$ (blue dashed line).

D. Proof of Theorem 2

To prove Theorem 2, we only need to specify R-dependent values for α and for the boundary layer widths δ and ε that satisfy the conditions of Lemmas 5, 6, and 7.

Motivated by the desire to minimize the upper bound on $U(\beta,\tau,\lambda)$ stated in Lemma 6, we choose

$$\delta = h(\alpha^*)R^{-2}, \tag{76}$$

where $\alpha^* = (1+\sqrt{13})/6$ is the unique maximizer of $h(\alpha)$ on the interval $(\frac{1}{2},1)$ [see Fig. 5(b)]. With these choices, conditions (54c) and (63b) require ε to satisfy

$$c_0 R^{-4\alpha^*+2} B(\delta,\varepsilon,\alpha^*)^2 \le \varepsilon \le c_2 R^{-\frac{2-2\alpha^*}{3}} \ln^{-\frac{2}{3}}(\delta^{-1}), \tag{77}$$

where c_0, c_1, and c_2 are non-negative constants independent of R. Using the upper bound on B from (73), it suffices to find ε such that

$$\frac{4c_0}{(1-\alpha^*)^2} R^{-4\alpha^*+2} \ln^2\left(\frac{R^2}{h(\alpha^*)}\right) \le \varepsilon \le c_2 R^{-\frac{2-2\alpha^*}{3}} \ln^{-\frac{2}{3}}\left(\frac{R^2}{h(\alpha^*)}\right). \tag{78}$$

Figure 5 shows that suitable values of ε exist when $R \ge R_0 \approx 2960.89$. One can also check that for all such values of R and any ε in the range given by (78), one has $\delta \le \frac{1}{6}$, $\varepsilon \le \frac{1}{3}$, and $\delta \le \varepsilon$. We have therefore verified all conditions of Lemmas 5, 6, and 7.

To conclude the Proof of Theorem 2, we simply substitute our choice of δ from (76) into Lemma 6 to find the upper bound $\overline{\langle wT\rangle} \le U(\beta,\tau,\lambda) \le \frac{1}{2} - cR^{-4}$, where $c = \frac{h^2(\alpha^*)}{18} \simeq 0.0107$.

Remark 8. The top boundary layer width ε is not uniquely determined in our construction. Its optimal value could be obtained by considering more refined estimates on $U(\beta,\tau,\lambda)$ than Lemma 6, but we expect such estimates to provide only higher-order corrections to the eventual bound on $\overline{\langle wT\rangle}$.

IV. CONCLUSIONS

We have proven bounds on the enhancement of heat transport by convection for two configurations of infinite-Prandtl-number convection driven by internal heating between no-slip boundaries. In the first case, where both boundaries are held at a constant temperature, we find $\overline{\langle wT\rangle} \le \frac{1}{2} - O(R^{-2})$ for all sufficiently large R (cf. Theorem 1). This result implies that the outward heat fluxes through the top and bottom are bounded by $\mathcal{F}_T \le 1 - O(R^{-2})$ and $\mathcal{F}_B \ge O(R^{-2})$, respectively. In the second configuration, where the top boundary remains isothermal but the bottom one is insulating (no-flux condition), we find $\overline{\langle wT\rangle} \le \frac{1}{2} - O(R^{-4})$ (cf. Theorem 2). In this case, we conclude from (10) that the

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Nusselt number is bounded above by $Nu \leq O(R^4)$. Explicit suboptimal values for the prefactors in the Rayleigh-dependent terms were also obtained (cf. Secs. II D and III D).

All of these results were derived using the background method, which we formulated as a search over quadratic auxiliary functionals of the form (16) and augmented using a minimum principle for the fluid's temperature. Similar to previous works on infinite-Prandtl-number Rayleigh–Bénard convection, the background temperature fields used vary linearly in thin boundary layers and increase either logarithmically (IH1 configuration) or as a power law (IH3 configuration) in the bulk of the fluid layer. This bulk behavior enables us to use Hardy–Rellich inequalities[14] (Lemma 4) and a previously established integral estimate[28] (Lemma 8) that were originally developed in the context of Rayleigh–Bénard convection. In contrast to the latter, however, our background fields lack symmetry in the vertical direction, which reflects the lack of vertical symmetry of IH convection problems.

In our choice of background fields, allowing the bottom boundary layer width δ to be smaller than the top boundary layer width ε is essential to prove Theorems 1 and 2. For the IH1 configuration, forcing $\delta = \varepsilon$ worsens the R-dependent correction to $\frac{1}{2}$ in Theorem 1 to $O(R^{-2}\ln^{-2}(R))$. For the IH3 configuration, instead, no upper bound on $\overline{\langle wT\rangle}$ that asymptotes to $\frac{1}{2}$ from below as R increases can be obtained with our method of proof if $\delta = \varepsilon$. This boundary layer asymmetry contrasts the construction of background fields for IH convection at finite Pr,[5] where taking $\delta \neq \varepsilon$ appears to bring no qualitative improvement to the exponentially varying upper bounds on $\overline{\langle wT\rangle}$. We also stress that the *a priori* uniform limits on the allowed values of δ and ε imposed throughout Secs. II and III have been chosen with the only goal of simplifying the algebra in our proofs. Varying these limits affects the prefactors of the R-dependent terms in Theorems 1 and 2, as well as the range of R values for which they hold. Both could be optimized further if desired.

One crucial difference between our constructions for the IH1 and IH3 configurations is the leading-order behavior of the background temperature fields—or, more precisely, of the function $\tau(z)$—as the bottom boundary layer edge is approached from the bulk region. For the IH1 configuration, it suffices for τ to have the same logarithmic behavior as the background temperature fields used to study Rayleigh–Bénard convection.[27] For the IH3 configuration, however, this choice does not work due to the loss of control on the temperature of the bottom boundary, and we are instead forced to take $\tau(z) \sim z^{1-\alpha}$ with $\alpha \in (0,1)$. This modification was already used in the context of Rayleigh–Bénard convection between imperfectly conducting boundaries,[28] where the optimal exponent α depended logarithmically on the Rayleigh number. Within our proof, instead, the optimal α is a constant. Whether this difference is due to our choice of estimates or the inherent differences between Rayleigh-Bénard and IH convection remains an open question.

More generally, we do not know whether the upper bounds on $\overline{\langle wT\rangle}$ stated in Theorems 1 and 2 are qualitatively sharp. To check if the $O(R^{-2})$ and $O(R^{-4})$ corrections to the asymptotic value of $\frac{1}{2}$ are optimal within our bounding framework, one could employ a variation of the computational approach taken in previous studies by the authors[2,26] and optimize the tunable parameters τ, β, and λ in full. A more interesting but also more challenging problem is to identify which convective flows maximize $\overline{\langle wT\rangle}$ and the corresponding optimal scaling of this quantity with R. Considerable insight in this direction can be gained through direct numerical simulations, which to the best of our knowledge have only been performed at finite Pr numbers for the IH1 and IH3 models studied here.[19,29] These works report values for $\overline{\langle wT\rangle}$ or related quantities, but it is not clear that their functional dependence on R should remain the same at infinite Prandtl number. Other promising approaches could be the calculation of steady but unstable solution of the Boussinesq equations (1), which may transport heat more efficiently than turbulence as recently observed in the context of Rayleigh–Bénard convection[30,31] and the explicit design of flows that optimize heat transport.[6,32–35] Finally, it would be interesting to investigate if more sophisticated PDE analysis techniques used for Rayleigh–Bénard convection[36] can be extended to IH convection to interpolate between the algebraic bounds on $\overline{\langle wT\rangle}$ proved in this paper for infinite-Pr fluids with the finite-Pr exponential bounds.[5]

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Ali Arslan: Formal analysis (lead); Investigation (lead); Writing – original draft (equal); Writing – review & editing (equal). **Giovanni Fantuzzi**: Supervision (equal); Writing – original draft (equal); Writing – review & editing (equal). **John Craske**: Supervision (equal); Writing – original draft (equal); Writing – review & editing (equal). **Andrew Wynn**: Supervision (equal); Writing – original draft (equal); Writing – review & editing (equal).

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX A: CONVEX DUALITY FOR THE IH1 CONFIGURATION

The equivalence between (20) and (21) follow from a relatively standard convex duality argument. It will be enough to show that

$$\sup_{T\in\mathcal{H}_+} -\Phi\{T\} = \inf_{\substack{\lambda\in L^2(0,1)\\ \lambda \text{ nondecreasing}\\ \langle\lambda\rangle=-1}} \frac{1}{4\beta}\left\langle\left|\tau' - \lambda - \beta\left(z-\frac{1}{2}\right)\right|^2\right\rangle, \tag{A1}$$

where

$$\Phi\{T\} := \left\langle \beta|\nabla T|^2 - R\tau'(\Delta^{-2}\Delta_h T)T - (\tau' - \beta z)\partial_z T\right\rangle.$$

To establish this identity, we start by rewriting the maximization on the left-hand side as a minimization problem for the Legendre transform of Φ. Recall that the Legendre transform of a functional $\Lambda : \mathcal{H} \to \mathbb{R}$ is the functional

$$\Lambda^*\{\mu\} := \sup_{T\in\mathcal{H}}(\mu\{T\} - \Lambda\{T\}),$$

which acts on the dual space $\mathcal{H}^*$ of bounded linear functionals $T \mapsto \mu\{T\}$ on $\mathcal{H}$. We shall write $\mathcal{H}_+^* \subset \mathcal{H}^*$ for the subset of non-negative bounded linear functionals on $\mathcal{H}$, meaning that $\mu \in \mathcal{H}_+^*$ if and only if $\mu \in \mathcal{H}^*$ and $\mu\{T\} \geq 0$ for all $T \in \mathcal{H}_+$.

Lemma 9. If the pair (β,τ) satisfies the spectral constraint, $\sup_{T\in\mathcal{H}_+} -\Phi\{T\} = \inf_{\mu\in\mathcal{H}_+^*}\Phi^*\{\mu\}$.

Proof. Define a functional Ψ from $\mathcal{H}$ into $\mathbb{R}\cup\{+\infty\}$ via

$$\Psi\{T\} := \begin{cases} 0, & T\in\mathcal{H}_+,\\ +\infty & \text{otherwise.}\end{cases}$$

Its Legendre transform is

$$\Psi^*\{\mu\} = \begin{cases} 0 & \text{if } -\mu\in\mathcal{H}_+^*,\\ +\infty & \text{otherwise.}\end{cases}$$

We claim that

$$\begin{aligned}\sup_{T\in\mathcal{H}_+} -\Phi\{T\} &= -\inf_{T\in\mathcal{H}_+}\Phi\{T\}\\ &= -\inf_{T\in\mathcal{H}}(\Phi\{T\}+\Psi\{T\})\\ &= \inf_{\mu\in\mathcal{H}^*}\left(\Phi^*\{-\mu\}+\Psi^*\{\mu\}\right)\\ &= \inf_{\mu\in\mathcal{H}_+^*}\Phi^*\{\mu\}.\end{aligned}$$

The first, second, and fourth equalities are immediate consequences of the definitions of inf, sup, Ψ, and Ψ^*. The third one, instead, follows from the Fenchel–Rockafellar minmax theorem when $\mathcal{H}$ is viewed as a Hilbert space with the inner product $\langle\nabla T_1\cdot\nabla T_2\rangle$ and norm $\langle|\nabla T|^2\rangle$. To apply this theorem as stated in Ref. 37 (Theorem 1.12), we need to verify that the functionals Φ and Ψ are convex and that Φ is continuous (with respect to the norm on $\mathcal{H}$) at some $T_0 \in \mathrm{dom}(\Phi)\cap\mathrm{dom}(\Psi) = \mathcal{H}_+$.

The convexity of Ψ is obvious, while that of Φ follows from the assumption that the pair (β,τ) satisfies the spectral constraint (cf. Definition 1). To see this, write $\Phi\{T\} = \mathcal{Q}\{T,T\} - (\tau'-\beta z)\partial_z T$, where $\mathcal{Q}$ is the bilinear form

$$\mathcal{Q}\{T_1,T_2\} := \left\langle \beta\nabla T_1\cdot\nabla T_2 - R\tau'(\Delta^{-2}\Delta_h T_1)T_2\right\rangle,$$

and observe that the spectral constraint ensures $\mathcal{Q}\{T,T\}\geq 0$ for all T in the linear space $\mathcal{H}$. Thus, for any $\lambda\in[0,1]$, we can set $\bar\lambda = 1-\lambda$ and estimate

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$$\begin{aligned}\Phi\{\lambda T_1 + \bar{\lambda} T_2\} &= \mathcal{Q}\{\lambda T_1 + \bar{\lambda} T_2, \lambda T_1 + \bar{\lambda} T_2\} - (\tau' - \beta z)\partial_z[\lambda T_1 + \bar{\lambda} T_2] \\ &= \lambda\Phi\{T_1, T_1\} + \bar{\lambda}\Phi\{T_2, T_2\} - \lambda\bar{\lambda}\mathcal{Q}\{T_1 - T_2, T_1 - T_2\} \\ &\le \lambda\Phi\{T_1, T_1\} + \bar{\lambda}\Phi\{T_2, T_2\},\end{aligned}$$

proving that Φ is convex. The continuity of Φ at any $T_0 \in \mathcal{H}_+$ follows because Φ is continuous on the whole space $\mathcal{H}$. Indeed, the terms $\langle|\nabla T|^2\rangle$ and $\langle(\tau' - \beta z)\partial_z T\rangle$ in the expression for $\Phi\{T\}$ are clearly continuous on $\mathcal{H}$. To see that the remaining term is also continuous, it is enough to establish that $T_k \to T$ in $\mathcal{H}$ implies $\varphi_k := \Delta^{-2}\Delta_h T_k \to \Delta^{-2}\Delta_h T := \varphi$ in $\mathcal{H}$. This can be shown by combining the Poincaré inequalities $\langle\varphi^2\rangle \lesssim \langle|\Delta\varphi|^2\rangle$ and $\langle T^2\rangle \lesssim \langle|\nabla T|^2\rangle$ with the estimate

$$\langle|\Delta\varphi|^2\rangle = \left|\langle\varphi\Delta^2\varphi\rangle\right| = |\langle\varphi\Delta_h T\rangle| = |\langle\Delta_h\varphi T\rangle| \le \langle|\Delta\varphi|^2\rangle^{\frac{1}{2}}\langle|T|^2\rangle^{\frac{1}{2}},$$

which implies $\langle|\Delta\varphi|^2\rangle \le \langle T^2\rangle$. This concludes the Proof of Lemma 9. □

Next, we prove that since $\Phi\{T\}$ is invariant under horizontal translations of the temperature field T, the minimization of its Legendre transform Φ^* can be restricted to functionals $\mu \in \mathcal{H}_+^*$ that are translation invariant. Specifically, for any real numbers r, s, define the translation map $\mathcal{S}_{r,s} : \mathcal{H} \to \mathcal{H}$ and its adjoint $\mathcal{S}_{r,s}^* : \mathcal{H}^* \to \mathcal{H}^*$ via

$$\begin{aligned}\mathcal{S}_{r,s}T &:= T(x + r, y + s, z). \\ \mathcal{S}_{r,s}^*\mu &:= \mu \circ \mathcal{S}_{r,s}.\end{aligned}$$

The functional $\mu \in \mathcal{H}^*$ is translation invariant if $\mathcal{S}_{r,s}^*\mu = \mu$ for all r and s.

Lemma 10. Suppose $\Phi : \mathcal{H} \to \mathbb{R}$ satisfies $\Phi\{\mathcal{S}_{r,s}T\} = \Phi\{T\}$ for all $r, s \in \mathbb{R}$. Then,

$$\inf_{\mu\in\mathcal{H}^*}\Phi^*\{\mu\} = \inf_{\substack{\mu\in\mathcal{H}^* \\ \mu \text{ transl. inv.}}}\Phi^*\{\mu\}.$$

Proof. It suffices to show that for every $\nu \in \mathcal{H}^*$, there exists a translation-invariant $\mu \in \mathcal{H}^*$ such that $\Phi^*\{\mu\} = \Phi^*\{\nu\}$. Since every temperature field $T \in \mathcal{H}$ is horizontally periodic, such a μ can be constructed simply by averaging the functionals $\mathcal{S}_{r,s}^*\nu$ over horizontal translations r and s, i.e., by letting

$$\mu(T) := \fint_{r,s}\mathcal{S}_{r,s}^*\nu\{T\} = \fint_{r,s}\nu\{\mathcal{S}_{r,s}T\} \quad \forall T \in \mathcal{H}.$$

To show that $\Phi^*\{\mu\} = \Phi^*\{\nu\}$, we establish the complementary inequalities $\Phi^*\{\mu\} \le \Phi^*\{\nu\}$ and $\Phi^*\{\mu\} \ge \Phi^*\{\nu\}$. For the first one, use the translation invariance of Φ and the definition of the Legendre transform to estimate

$$\nu\{\mathcal{S}_{r,s}T\} - \Phi\{T\} = \nu\{\mathcal{S}_{r,s}T\} - \Phi\{\mathcal{S}_{r,s}T\} \le \Phi^*\{\nu\}$$

for all $T \in \mathcal{H}$ and all horizontal shifts $r, s \in \mathbb{R}$. Averaging over horizontal shifts shows that $\mu\{T\} - \Phi\{T\} \le \Phi^*\{\nu\}$ for all $T \in \mathcal{H}$, which implies $\Phi^*\{\mu\} \le \Phi^*\{\nu\}$.

To obtain the reverse inequality, observe that by definition of Φ^*, for any $\varepsilon > 0$, there exists $T_\varepsilon \in \mathcal{H}$ such that $\nu\{T_\varepsilon\} - \Phi\{T_\varepsilon\} \ge \Phi^*\{\nu\} - \varepsilon$. Then, since Φ is translation invariant,

$$\begin{aligned}\Phi^*\{\nu\} - \varepsilon &\le \nu\{T_\varepsilon\} - \Phi\{T_\varepsilon\} \\ &= (\mathcal{S}_{r,s}^*\nu)\{\mathcal{S}_{-r,-s}T_\varepsilon\} - \Phi\{\mathcal{S}_{-r,-s}T_\varepsilon\} \\ &\le \Phi^*\{\mathcal{S}_{r,s}^*\nu\}.\end{aligned}$$

Upon averaging this inequality over all horizontal shifts $r, s \in \mathbb{R}$ and applying Jensen's inequality to Φ^*, which is concave because it is the supremum of linear functions, we find

$$\Phi^*(\nu) - \varepsilon \le \fint_{r,s}\Phi^*\left(\mathcal{S}_{r,s}^*\nu\right) \le \Phi^*\left(\fint_{r,s}\mathcal{S}_{r,s}^*\nu\right) = \Phi^*(\mu).$$

Letting $\varepsilon \to 0$ yields $\Phi^*\{\mu\} \ge \Phi^*\{\nu\}$, as desired. Lemma 10 is therefore proved. □

To establish identity (A1), we now need to show that its right-hand side coincides with the infimum of Φ^* over translation-invariant functionals $\mu \in \mathcal{H}_+^*$. For this, we use a characterization of such μ established in (Appendix C, Arslan *et al.* 2021).[2]

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Lemma 11. Let $\mathcal{H}_+^*$ be the set of positive linear functionals on the temperature space $\mathcal{H}$ defined in (14a). If $\mu \in \mathcal{H}_+^*$ is translation invariant, there exists a nondecreasing function $\lambda \in L^2(0,1)$ with $\langle\lambda\rangle = -1$ such that $\mu\{T\} = \langle -\lambda(z)\ \partial_z T\rangle$.

Thanks to this representation, all that remains to do is to calculate

$$\Phi^*\{\mu\} = \sup_{T\in\mathcal{H}} \big\langle (\tau' - \lambda + \beta z)\partial_z T - \beta|\nabla T|^2 + R\tau'(\Delta^{-2}\Delta_h T)T\big\rangle. \tag{A2}$$

To solve this maximization problem, let $\eta(z) = \langle T\rangle_h(z)$ be the horizontal mean of T and set $\xi = T - \eta$. Since $\Delta_h\eta = 0$ and $\langle\xi\rangle_h(z) = 0$ by construction, we can therefore substitute $T = \eta + \xi$ in (A2) and solve the equivalent problem

$$\Phi^*\{\mu\} = \sup_{\substack{\eta=0 \\ \xi=0}} \sup_{\substack{\text{if } z\in\{0,1\} \\ \text{if } z\in\{0,1\} \\ \langle\xi\rangle_h=0}} \Big\{ \Big\langle (\tau' - \lambda + \beta z)\eta' - \beta|\eta'|^2 - \beta|\nabla\xi|^2 + R\tau'(\Delta^{-2}\Delta_h\xi)\xi \Big\rangle$$
$$\Big\langle (\tau' - \lambda + \beta z)\partial_z\xi - 2\beta\ \eta'\ \partial_z\xi + R\tau'\eta\ (\Delta^{-2}\Delta_h\xi)\Big\rangle \Big\}. \tag{A3}$$

The boundary conditions on η and ξ follow from those on T. The three terms on the second line vanish identically because $\langle\xi\rangle_h(z) = 0$ at all $z \in [0,1]$. To verify this claim, observe that

$$\langle f(z)\partial_z\xi\rangle = \int_0^1 f(z)\langle\xi\rangle_h'(z)\,\mathrm{d}z = 0$$

for any function $f(z)$ that depends only on the vertical direction. Similarly, one can show that

$$\big\langle R\tau'\eta\ (\Delta^{-2}\Delta_h\xi)\big\rangle = 0$$

because the function $w = -R\Delta^{-2}\Delta_h T = -R\Delta^{-2}\Delta_h\xi$ also has zero horizontal mean. Indeed, taking the horizontal average of (12) yields the ODE $\langle w\rangle_h''(z) = 0$, whose only solution satisfying the boundary conditions in (13) is $\langle w\rangle_h(z) = 0$. The minimization in (A3) therefore simplifies into

$$\Phi^*\{\mu\} = \sup_{\substack{\eta=0 \\ \xi=0}} \sup_{\substack{\text{if } z\in\{0,1\} \\ \text{if } z\in\{0,1\} \\ \langle\xi\rangle_h=0}} \Big\langle (\tau' - \lambda - \beta z)\eta' - \beta|\eta'|^2 - \beta|\nabla\xi|^2 - R\tau' w\xi \Big\rangle. \tag{A4}$$

Since the pair (β,τ) was assumed to satisfy the spectral constraint (cf. Definition 1), the choice $\xi = 0$ is optimal. The optimal η, instead, satisfies the Euler–Lagrange equation $2\beta\eta'' = (\tau' - \lambda - \beta z)'$. Solving this equation using the boundary conditions, the constraints $\tau(0) = 1$ and $\tau(1) = 0$, and the normalization $\langle\lambda\rangle = -1$ gives $\eta' = \frac{1}{2b}[\tau' - \lambda - b(z - \frac{1}{2})]$, which can be substituted back into (A4) to give

$$\Phi^*\{\mu\} = \frac{1}{4\beta}\left\langle \left|\tau' - \lambda' - \beta\left(z - \frac{1}{2}\right)\right|^2 \right\rangle. \tag{A5}$$

Minimizing the left-hand side over translation invariant μ in $\mathcal{H}_+^*$ is the same as minimizing the right-hand side over λ satisfying the conditions in Lemma 11, which is exactly the problem on the right-hand side of (A1).

APPENDIX B: CONVEX DUALITY FOR THE IH3 CONFIGURATION

The equivalence between the upper bounds (46) and (47) for the IH3 configuration follows from the identity

$$\sup_{T\in\mathcal{H}_+} -\Phi\{T\} = \inf_{\substack{\lambda\in L^2(0,1) \\ \lambda \text{ nondecreasing} \\ \lambda\geq -1}} \left\langle \frac{1}{4\beta}|\tau' - \lambda - \beta z|^2 \right\rangle, \tag{B1}$$

where

$$\Phi\{T\} := \big\langle \beta|\nabla T|^2 - R\tau'(\Delta^{-2}\Delta_h T)T - (\tau' - \beta z + 1)\partial_z T\big\rangle. \tag{B2}$$

This identity can be proven using a convex duality argument analogous to that in Appendix A. Indeed, Lemmas 9 and 10 apply to the functional Φ considered in this section with no changes to their proofs. Consequently,

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$$\sup_{T\in\mathcal{H}_+} -\Phi\{T\} = \inf_{\substack{\mu\in\mathcal{H}^*\\ \mu\ \text{transl. inv.}}} \Phi^*\{\mu\}. \tag{B3}$$

To calculate the Legendre transform Φ^*, however, we must replace Lemma 11 with a different characterization of translation-invariant linear functionals $\mu \in \mathcal{H}_+^*$. This is due to the different boundary conditions imposed on the temperature space $\mathcal{H}$.

Lemma 12. Let $\mathcal{H}_+^*$ be the set of positive linear functionals on the temperature space $\mathcal{H}$ defined in (14a). If $\mu \in \mathcal{H}_+^*$ is translation invariant, there exists a nondecreasing function $\lambda \in L^2(0,1)$ non-negative almost everywhere and such that $\mu\{T\} = \langle -\lambda(z)\ \partial_z T\rangle$.

Proof. Straightforward modifications to the Proof of Lemma 3 (Arslan *et al.*, 2021)[2] reveal that any translation-invariant $\mu \in \mathcal{H}^*$ admits the representation $\mu\{T\} = \langle -\lambda(z)\ \partial_z T\rangle$ for some function $\lambda \in L^2(0,1)$. If μ is also positive, then the argument in the proof of Ref. 2 (Lemma 2) shows that λ must be nondecreasing. To see that, we must have $\lambda(z_0) \geq 0$ at almost every $z_0 \in (0,1)$, fix $\varepsilon > 0$ sufficiently small, and consider the temperature field $T_\varepsilon \in \mathcal{H}_+$ given by

$$T_\varepsilon(\boldsymbol{x}) := \begin{cases} 2, & z \in (0, z_0-\varepsilon),\\ 2-\varepsilon^{-1}(z-z_0+\varepsilon), & z\in(z_0-\varepsilon, z_0+\varepsilon),\\ 0, & z\in(z_0+\varepsilon,1).\end{cases}$$

Then, since μ is a positive functional by assumption,

$$0 \leq \mu(T_\varepsilon) = \frac{1}{\varepsilon}\int_{z_0-\varepsilon}^{z_0+\varepsilon} \lambda(z)\mathrm{d}z.$$

By Lebesgue's differentiation theorem, the right-hand side tends to $2\lambda(z_0)$ for almost all $z_0 \in (0,1)$ as $\varepsilon \to 0$. Thus, we must have $\lambda \geq 0$ almost everywhere on (0,1). □

To conclude the argument, we need to calculate $\Phi^*\{\mu\}$ for translation-invariant $\mu \in \mathcal{H}_+^*$, which by Lemma 12 is given by

$$\Phi^*\{\mu\} = \sup_{T\in\mathcal{H}} \left\langle \left(\tau' - \lambda - \beta z + 1\right)\partial_z T - \beta\,|\nabla T|^2 + R\tau'(\Delta^{-2}\Delta_h T)T\right\rangle.$$

Since the pair (β,τ) was assumed to satisfy the spectral constraint, this maximization problem can be restricted to temperature fields that depend only on the vertical coordinate z (this can be proven by splitting T into its horizontal mean η and a perturbation ξ with zero horizontal mean, as outlined at the end of Appendix A). The optimal value can then be shown to be

$$\Phi^*\{\mu\} = \frac{1}{4\beta}\left\langle \left|\tau' - \lambda - \beta z + 1\right|^2\right\rangle$$

and can be substituted into (B3) to arrive at

$$\sup_{T\in\mathcal{H}_+} -\Phi\{T\} = \inf_{\substack{\lambda\in L^2(0,1)\\ \lambda\ \text{nondecreasing}\\ \lambda\geq 0}} \frac{1}{4\beta}\left\langle \left|\tau' - \lambda - \beta z + 1\right|^2\right\rangle.$$

Changing the optimization variable on the right-hand side to $\tilde{\lambda} = \lambda - 1$ yields (B1).

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