

Fast evaluation of the fundamental singularities of two-dimensional doubly periodic Stokes flow

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Abstract Analytical representations of the flows associated with doubly periodic arrangements of point singularities of two-dimensional Stokes flow are derived. The analysis makes use of a conformal mapping from a concentric annulus to a rectangle. A natural mathematical object known as the Schottky-Klein prime function is used to derive the solutions avoiding the theory of elliptic functions. The new expressions have the advantage of being immediately amenable to fast evaluation without the need for Ewald or other fast summation methods or acceleration methods for lattice sums.

Keywords Stokes flow · doubly periodic · singularity theory · function theory

1 Introduction

The study of low-Reynolds-number, or Stokes, flows is an important area of fluid dynamics with a rich history. Renewed interest in the theory of such flows has emerged as new problems arise in burgeoning fields such as microfluidics, nonlinear electrokinetics, low-Reynolds-number swimming, superhydrophobic surfaces and the study of active suspensions.

In an important paper, Hasimoto [6] studied solutions for the fundamental singularities of the Stokes equations in two- and three-dimensional periodic arrays. He expressed the results in terms of Fourier series, which is natural for periodic solutions. Unfortunately, the use of such representations in practice is complicated by their poor convergence properties; Hasimoto [6] proposed the use of Ewald summation techniques to improve the convergence, and work in this direction

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continues [9]. Pozrikidis [12] has since developed these ideas involving efficient summation methods in various directions. The monograph by Pozrikidis [11] gives a useful summary of the relevant theory.

The two-dimensional problem has recently received attention, not least from Hasimoto himself [7] who revisited the problem, fifty years on, to find expressions for the doubly periodic fundamental solutions in terms of the classical elliptic functions. The results reported in [7] are a translation, and simplification, of results featured in some earlier lecture notes written in Japanese [5]. A similar rewriting of the doubly periodic solutions in terms of elliptic functions has also recently been attempted by Peterson, Chen and Ding [10]. Greengard and Kropinski [4] constructed integral equations amenable to fast multipole methods and applicable to various interfacial problems in doubly periodic domains. Other work on boundary integral methods applied to doubly periodic free boundary problems in Stokes flow has been performed by Van de Vorst [15] and Pozrikidis [13].

The present paper focusses on doubly periodic arrangements of the fundamental singularities of Stokes flow in two dimensions and presents what are believed to be new representations of the associated flow fields. The results are not just of purely theoretical interest; it turns out that the new representations lend themselves naturally to fast evaluation, thereby obviating the need for fast summation techniques or other convergence acceleration methods typically needed with other representations. Moreover, while our method can be related to the approach via standard elliptic function theory advanced by other authors [7, 10], it avoids that theory and is built around the introduction of a single special function, denoted herein by $P(\zeta, \rho)$, the properties of which can be used to systematically derive the solutions. The function $P(\zeta, \rho)$ is essentially the so-called *Schottky-Klein prime function* of a concentric annulus [3].

In contrast, Hasimoto [5] expresses the solution for doubly periodic Stokeslet arrays in terms of the Weierstrass $\wp$ -function but does not discuss any numerical issues associated with its evaluation. It is well known that the Weierstrass $\wp$ -function is difficult to evaluate efficiently and it is usually advisable to employ further classical identities relating the $\wp$ -function to the Jacobi theta functions which are amenable to fast evaluation. As it turns out, the Schottky-Klein prime function $P(\zeta, \rho)$ used here is directly related to the first Jacobi theta function (see the appendix of [2], and equation (89)) so our method can alternatively be viewed as a construction built directly on the theory of theta functions without any diversion through elliptic function theory. Peterson, Chen and Ding [10] also find expressions for the fundamental solutions in terms of the Weierstrass $\wp$ -function and, while they do concern themselves with the matter of its numerical evaluation, their methods are based on a classical infinite sum representation of $\wp$ over its doubly periodic array of poles. Greengard & Kropinski [4] similarly base their numerical algorithms on ideas for accelerating the numerical evaluations of conditionally convergent lattice sums.

2 Stokes flows in two dimensions

Consider a region of incompressible fluid of viscosity η governed by the Stokes equations

$$\nabla p = \eta \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \quad (1)$$

where $\mathbf{u} = (u, v)$ is the two-dimensional velocity field and p is the fluid pressure. It is well-known (Langlois [8]) that an incompressible solution of the Stokes equations for the velocity field (u, v) can be written in terms of a streamfunction $\psi(x, y)$ with

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \quad (2)$$

The streamfunction satisfies the biharmonic equation

$$\nabla^4 \psi = 0, \quad (3)$$

where ∇^2 is the two-dimensional Laplacian. The general solution of (3) can be written

$$\psi = \text{Im}[\bar{z}f(z) + g(z)], \quad (4)$$

where $f(z)$ and $g(z)$ are analytic functions (which can have isolated singularities) in the fluid region and are often referred to as *Goursat functions* (Langlois [8]). These analytic functions are related to physical quantities via

$$\begin{aligned} 4f'(z) &= \frac{p}{\eta} - i\omega, \\ -\overline{f(z)} + \bar{z}f'(z) + g'(z) &= u - iv, \end{aligned} \quad (5)$$

where ω is the fluid vorticity. This connection with analytic functions is important because we propose to harness the powerful mathematical results of analytic function theory to find expressions for the fundamental singularities of Stokes flows which can be evaluated efficiently. Since we will consider doubly periodic configurations with period l in the x -direction and period h in the y -direction it is natural to non-dimensionalize lengths with respect to the period width l , say.

3 Fundamental singularities of Stokes flow

In this section we express what is meant by the fundamental singularities of Stokes flow in the language of the isolated singularities of the analytic functions $f(z)$ and $g'(z)$ (the function $g(z)$, if needed, can be found by simple integration). We focus here on the Stokeslet, stresslet, force quadrupole, along with some irrotational (or “source”) singularities. Higher-order singularities can be treated similarly.

3.1 Free-space Stokeslet

A Stokeslet at a point z_0 is associated with a singularity of $f(z)$ having the form

$$f(z) = \mu \log(z - z_0), \quad (6)$$

where μ is a generally complex constant which sets the timescale of the motion; we will see later that it is related to the strength of the Stokeslet. The complex velocity is given by

$$u - iv = -\overline{f(z)} + \bar{z}f'(z) + g'(z) = -\bar{\mu} \log(\bar{z} - \bar{z}_0) + \frac{\mu \bar{z}}{z - z_0} + g'(z). \quad (7)$$

In order that the velocity is both single-valued and logarithmically singular at z_0 we must pick

$$g'(z) = -\bar{\mu} \log(z - z_0) - \frac{\mu \bar{z}_0}{z - z_0}, \quad (8)$$

so that

$$u - iv = -\bar{\mu} \log |z - z_0|^2 + \mu \left(\frac{\bar{z} - \bar{z}_0}{z - z_0} \right). \quad (9)$$

This expression is single-valued and logarithmically singular at z_0 . In summary, if $f(z)$ and $g'(z)$ locally have the form, near z_0 ,

$$\begin{aligned} f(z) &= \mu \log(z - z_0) + \text{analytic function}, \\ g'(z) &= -\bar{\mu} \log(z - z_0) - \frac{\mu \bar{z}_0}{z - z_0} + \text{analytic function}, \end{aligned} \quad (10)$$

where $\mu \in \mathbb{C}$, then we say there is a Stokeslet at z_0 .

We can introduce the stress tensor $\sigma_{ij} = -p\delta_{ij} + 2\eta e_{ij}$ where e_{ij} is the usual fluid rate-of-strain tensor. It can be shown [8] that, in complex form, the fluid stress $\sigma_{ij}n_j$ exerted across some point on a contour in the fluid with outward normal n_i is given by

$$2\eta i \frac{dH}{ds}, \quad H \equiv f(z) + z \overline{f'(z)} + \overline{g'(z)}. \quad (11)$$

The force exerted on the fluid by a Stokeslet at z_0 is therefore given by

$$\lim_{\epsilon \rightarrow 0} \oint_{|z - z_0| = \epsilon} 2\eta i \frac{dH}{ds} ds = \left[2\eta i H \right]_{|z - z_0| = \epsilon}, \quad (12)$$

where the square brackets denote the change in the enclosed quantity on a single anticlockwise traversal of the circle $|z - z_0| = \epsilon$. This equals

$$\left[2\eta i (\mu \log(z - z_0) - \mu \log(\bar{z} - \bar{z}_0)) \right]_{|z - z_0| = \epsilon} = -8\pi\eta\mu, \quad (13)$$

and gives what is usually known as the strength of the Stokeslet.

3.2 Free-space stresslet

Suppose that $f(z)$ has a simple pole at z_0 so that

$$f(z) = \frac{\mu}{z - z_0}, \quad (14)$$

where $\mu \in \mathbb{C}$. The complex velocity is given by

$$u - iv = -\frac{\bar{\mu}}{z - z_0} - \frac{\mu \bar{z}}{(z - z_0)^2} + g'(z). \quad (15)$$

In order that the velocity field is singular like $1/|z - z_0|$, we must choose

$$g'(z) = \frac{\mu \bar{z}_0}{(z - z_0)^2}, \quad (16)$$

so that

$$u - iv = -\frac{\bar{\mu}}{z - z_0} - \frac{\mu}{z - z_0} \left(\frac{\overline{z - z_0}}{z - z_0} \right). \quad (17)$$

In summary, if $f(z)$ and $g'(z)$ locally have the form, near z_0 ,

$$\begin{aligned} f(z) &= \frac{\mu}{z - z_0} + \text{analytic function}, \\ g'(z) &= \frac{\mu \bar{z}_0}{(z - z_0)^2} + \text{analytic function}, \end{aligned} \quad (18)$$

where $\mu \in \mathbb{C}$, then we say there is a stresslet at z_0 . The parameter μ determines the strength of the stresslet.

3.3 Free-space force quadrupole

Suppose that $f(z)$ has a double pole at z_0 so that

$$f(z) = \frac{\mu}{(z - z_0)^2}, \quad (19)$$

where $\mu \in \mathbb{C}$. Then the complex velocity is given by

$$u - iv = -\frac{\bar{\mu}}{(z - z_0)^2} - \frac{2\mu \bar{z}}{(z - z_0)^3} + g'(z). \quad (20)$$

In order that the velocity field is singular like $1/|z - z_0|^2$, we must choose

$$g'(z) = \frac{2\mu \bar{z}_0}{(z - z_0)^3}, \quad (21)$$

which implies that

$$u - iv = -\frac{\bar{\mu}}{(z - z_0)^2} - \frac{2\mu(\overline{z - z_0})}{(z - z_0)^3}. \quad (22)$$

In summary, if $f(z)$ and $g'(z)$ locally have the form, near z_0 ,

$$\begin{aligned} f(z) &= \frac{\mu}{(z - z_0)^2} + \text{analytic function}, \\ g'(z) &= \frac{2\mu \bar{z}_0}{(z - z_0)^3} + \text{analytic function}, \end{aligned} \quad (23)$$

where $\mu \in \mathbb{C}$, then we say there is a force quadrupole at z_0 . The parameter μ determines the strength of the quadrupole.

3.4 Source singularities

Note that function $g(z)$ can additionally have its own singularities that are independent of those imposed by the choice of $f(z)$; we shall refer to these as source singularities. Suppose $f(z)$ is analytic and $g(z)$ has a logarithmic pole at z_0 , i.e.,

$$g(z) = \mu \log(z - z_0). \quad (24)$$

The Goursat functions have, therefore, the form

$$\begin{aligned} f(z) &= \text{analytic function}, \\ g'(z) &= \frac{\mu}{z - z_0} + \text{analytic function}. \end{aligned} \quad (25)$$

If $\mu \in \mathbb{R}$ we say that there is a source/sink at z_0 (with mass flux given by $2\pi\mu$). If $\mu \in i\mathbb{R}$ we say that there is a rotlet at z_0 .

If $g(z)$ has a simple pole at z_0 , i.e.,

$$g(z) = \frac{\mu}{z - z_0}, \quad (26)$$

where $\mu \in \mathbb{C}$, then the Goursat functions

$$\begin{aligned} f(z) &= \text{analytic function}, \\ g'(z) &= -\frac{\mu}{(z - z_0)^2} + \text{analytic function}, \end{aligned} \quad (27)$$

are for a (source) dipole at z_0 . The parameter μ determines the strength of the dipole.

If $g(z)$ has now a second-order pole at z_0 , i.e.,

$$g(z) = \frac{\mu}{(z - z_0)^2}, \quad (28)$$

where $\mu \in \mathbb{C}$, then the Goursat functions

$$\begin{aligned} f(z) &= \text{analytic function}, \\ g'(z) &= -\frac{2\mu}{(z - z_0)^3} + \text{analytic function}, \end{aligned} \quad (29)$$

are for a (source) quadrupole at z_0 . The parameter μ determines the strength of the quadrupole.

4 Conformal mapping

To study doubly periodic arrays of singularities we will exploit the conformal mapping

$$z(\zeta) = -i \log \zeta. \quad (30)$$

This simple map transplants the annulus $\rho < |\zeta| < 1$ to a period rectangle in the $z = x + iy$ plane occupying the region

$$0 \leq x \leq 2\pi = l, \quad 0 \leq y \leq -\log \rho = h, \quad (31)$$

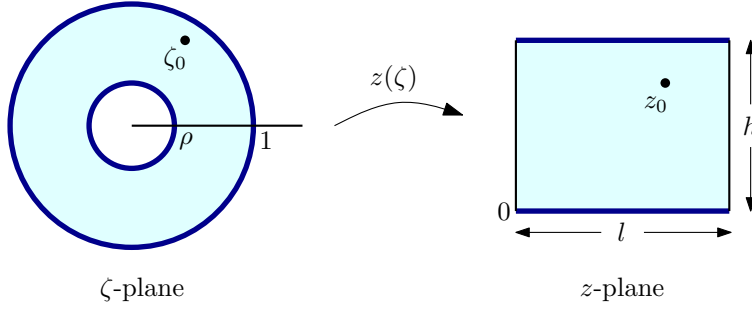


Fig. 1: Conformal mapping from the annulus $\rho < |\zeta| < 1$ in ζ -plane to the period rectangle in the physical z -plane occupying the region $0 \leq x \leq 2\pi = l$, $0 \leq y \leq -\log \rho = h$.

as illustrated in Figure 1. Note that changing the x -period simply requires multiplication of (30) by the appropriate factor. Under this conformal map, the unit circle $|\zeta| = 1$ corresponds to $0 \leq x \leq l$, $y = 0$, the inner circle $|\zeta| = \rho$ corresponds to $0 \leq x \leq l$, $y = h$ and the interval $[\rho, 1]$ (in ζ -plane) to the vertical sides of the period window $x = 0, l$, $0 \leq y \leq h$.

The key idea of our approach is to show that analytical expressions, with fast convergence properties, for the fundamental singularities of doubly periodic Stokes flow can be derived in terms of the variable ζ . It is then a simple matter, if required, to re-express the final results as functions of z using the relation, following from (30), that

$$\zeta = e^{iz}. \quad (32)$$

The representative point singularity in the principal period rectangle (Figure 1) is located at z_0 and has a preimage at ζ_0 such that

$$z_0 = z(\zeta_0) = -i \log \zeta_0. \quad (33)$$

5 Function theory

Our derivation will proceed by introducing a single special function defined by the convergent infinite product expansion

$$P(\zeta, \rho) \equiv (1 - \zeta) \prod_{k=1}^{\infty} (1 - \rho^k \zeta)(1 - \rho^k \zeta^{-1}). \quad (34)$$

This function is essentially the *Schottky-Klein prime function* [3] associated with the annulus $\rho < |\zeta| < 1$. Standard methods [1] can be used to show that this product converges uniformly for any ζ for all $0 < \rho < 1$. It can be verified directly, from the infinite product representation (34), that it satisfies the two functional relations

$$P(\rho\zeta, \rho) = -\zeta^{-1}P(\zeta, \rho), \quad P(\zeta^{-1}, \rho) = -\zeta^{-1}P(\zeta, \rho). \quad (35)$$

It is also explicit, from its definition as an infinite product (34), that $P(\zeta, \rho)$ has simple zeros at $\zeta = \rho^k$, $k \in \mathbb{Z}$.

We will also introduce various ζ -derivatives of the function $P(\zeta, \rho)$, specifically,

$$K(\zeta, \rho) \equiv \zeta \frac{\partial \log P(\zeta, \rho)}{\partial \zeta}, \quad L(\zeta, \rho) \equiv \zeta \frac{\partial K(\zeta, \rho)}{\partial \zeta}, \quad M(\zeta, \rho) \equiv \zeta \frac{\partial L(\zeta, \rho)}{\partial \zeta}. \quad (36)$$

These also satisfy functional relations derivable from (35). Indeed it is easy to establish from (35) that

$$\begin{aligned} K(\rho\zeta, \rho) &= K(\zeta, \rho) - 1, & K(\zeta^{-1}, \rho) &= 1 - K(\zeta, \rho), \\ L(\rho\zeta, \rho) &= L(\zeta, \rho), & L(\zeta^{-1}, \rho) &= L(\zeta, \rho), \\ M(\rho\zeta, \rho) &= M(\zeta, \rho), & M(\zeta^{-1}, \rho) &= M(\zeta, \rho). \end{aligned} \quad (37)$$

It can be shown, from the fact that $P(\zeta, \rho)$ has a simple zero at $\zeta = 1$, that

$$K(\zeta/\zeta_0, \rho) = -\frac{i}{z - z_0} + \text{analytic function}. \quad (38)$$

On repeated differentiation with respect to z , and use of the fact that

$$\frac{d}{dz} = i\zeta \frac{d}{d\zeta}, \quad (39)$$

we find

$$\begin{aligned} K(\zeta/\zeta_0, \rho) &= -\frac{i}{z - z_0} + \text{analytic function}, \\ L(\zeta/\zeta_0, \rho) &= \frac{1}{(z - z_0)^2} + \text{analytic function}, \\ M(\zeta/\zeta_0, \rho) &= \frac{2i}{(z - z_0)^3} + \text{analytic function}. \end{aligned} \quad (40)$$

It will also be important, for later use, to note that

$$\begin{aligned} \log \zeta \, K(\zeta/\zeta_0, \rho) &= izK(\zeta/\zeta_0, \rho) \sim i(z - z_0 + z_0) \left[-\frac{i}{z - z_0} + \mathcal{O}(1) \right] \\ &= \frac{z_0}{z - z_0} + \mathcal{O}(1), \end{aligned} \quad (41)$$

$$\begin{aligned} \log \zeta \, L(\zeta/\zeta_0, \rho) &= izL(\zeta/\zeta_0, \rho) \sim i(z - z_0 + z_0) \left[\frac{1}{(z - z_0)^2} + \mathcal{O}(1) \right] \\ &= \frac{iz_0}{(z - z_0)^2} + \frac{i}{z - z_0} + \mathcal{O}(1), \end{aligned} \quad (42)$$

and

$$\begin{aligned} \log \zeta \, M(\zeta/\zeta_0, \rho) &= izM(\zeta/\zeta_0, \rho) \sim i(z - z_0 + z_0) \left[\frac{2i}{(z - z_0)^3} + \mathcal{O}(1) \right] \\ &= -\frac{2z_0}{(z - z_0)^3} - \frac{2}{(z - z_0)^2} + \mathcal{O}(1). \end{aligned} \quad (43)$$

The parametric derivatives

$$\rho \frac{\partial \log P(\zeta, \rho)}{\partial \rho}, \quad \rho \frac{\partial K(\zeta, \rho)}{\partial \rho}, \quad \rho \frac{\partial L(\zeta, \rho)}{\partial \rho} \quad (44)$$

N	$h = 1$	$h = 2\pi = l$	$h = 10$
4	0.065539589764414	0.496736344937702	0.499920552699510
8	0.066902571082943	0.496736344937725	0.499920552699510
16	0.066928689208131	0.496736344937725	0.499920552699510
32	0.066928697974832	0.496736344937725	0.499920552699510
64	0.066928697974832	0.496736344937725	0.499920552699510

Table 1: Computation of $P(\zeta, \rho)$ using (46) for $h = 1, 2\pi, 10$ (with corresponding parameter $\rho = e^{-h}$) at point $\zeta = (1 + \rho)/2$. The sum (46) is truncated to include terms $n = -N, \dots, N$, while the sum and product in (47) to $n = 1, \dots, N$. The coefficients of the series (46) are given by powers of the parameter ρ with exponents that depend quadratically on n so, since $0 < \rho < 1$, the convergence of this sum is rapid. The table shows the extremely rapid convergence of representation (46) which implies that only few terms are required for high accuracy. The fast convergence also pertains to (48)-(51), since those expressions are directly derivable from (46).

will also be needed. On differentiation of the first of the functional relations in (37) with respect to ρ , followed by multiplication by ρ , we find that as $\zeta \mapsto \rho\zeta$ then

$$\begin{aligned}
\rho \frac{\partial \log P(\zeta, \rho)}{\partial \rho} &\mapsto \rho \frac{\partial \log P(\zeta, \rho)}{\partial \rho} - K(\zeta, \rho) + 1, \\
\rho \frac{\partial K(\zeta, \rho)}{\partial \rho} &\mapsto \rho \frac{\partial K(\zeta, \rho)}{\partial \rho} - L(\zeta, \rho), \\
\rho \frac{\partial L(\zeta, \rho)}{\partial \rho} &\mapsto \rho \frac{\partial L(\zeta, \rho)}{\partial \rho} - M(\zeta, \rho).
\end{aligned} \tag{45}$$

Rapidly convergent representations:

While the infinite product representation (34) is helpful in, for example, deriving the functional relations (35), a crucial observation is that $P(\zeta, \rho)$ also has the rapidly convergent Laurent series representation

$$P(\zeta, \rho) = A \sum_{n=-\infty}^{\infty} (-1)^n \rho^{n(n-1)/2} \zeta^n, \tag{46}$$

where $A = A(\rho)$ is given by

$$A(\rho) = \prod_{n=1}^{\infty} (1 + \rho^n)^2 / \sum_{n=1}^{\infty} \rho^{n(n-1)/2}. \tag{47}$$

The derivation of (46) is a straightforward exercise whereby a general Laurent series, with *a priori* unknown coefficients, is substituted into the first functional identity in (35) and then coefficients of different powers of ζ are compared. It is crucial to note that the coefficients of the series (46) are given by powers of the parameter ρ with exponents that depend quadratically on n so, since $0 < \rho < 1$, the convergence of this sum is rapid. Table 1 shows the extremely fast convergence of representation (46) for truncation parameter N (the sum (46) is truncated to include terms $n = -N, \dots, N$, while the sum and product in (47) to $n = 1, \dots, N$) which suggests that only a few terms are, in general, required to provide accurate computations.

Given that $P(\zeta, \rho)$ can be evaluated efficiently by means of (46), it follows immediately that all other derivative functions have expressions in terms of rapidly convergent sums, namely

$$K(\zeta, \rho) = \frac{r_1(\zeta, \rho)}{s(\zeta, \rho)}, \quad L(\zeta, \rho) = \frac{r_2(\zeta, \rho)}{[s(\zeta, \rho)]^2}, \quad M(\zeta, \rho) = \frac{r_3(\zeta, \rho)}{[s(\zeta, \rho)]^3} \quad (48)$$

and

$$\begin{aligned} \rho \frac{\partial \log P(\zeta, \rho)}{\partial \rho} &= t(\rho) + \frac{r_4(\zeta, \rho)}{s(\zeta, \rho)}, \\ \rho \frac{\partial K(\zeta, \rho)}{\partial \rho} &= \frac{r_5(\zeta, \rho)}{[s(\zeta, \rho)]^2}, \quad \rho \frac{\partial L(\zeta, \rho)}{\partial \rho} = \frac{r_6(\zeta, \rho)}{[s(\zeta, \rho)]^3}, \end{aligned} \quad (49)$$

where all functions have series representations

$$t(\rho) = \sum_{n=1}^{\infty} \frac{2n\rho^n}{1+\rho^n} - \sum_{n=1}^{\infty} \frac{n(n-1)}{2} \rho^{n(n-1)/2} \bigg/ \sum_{n=1}^{\infty} \rho^{n(n-1)/2} \quad (50)$$

and

$$\begin{aligned} s(\zeta, \rho) &= \sum_{n=-\infty}^{\infty} (-1)^n \rho^{n(n-1)/2} \zeta^n, \\ r_1(\zeta, \rho) &= \sum_{n=-\infty}^{\infty} (-1)^n \rho^{n(n-1)/2} n \zeta^n, \\ r_2(\zeta, \rho) &= \left[\sum_{n=-\infty}^{\infty} (-1)^n \rho^{n(n-1)/2} n^2 \zeta^n \right] s(\zeta, \rho) - [r_1(\zeta, \rho)]^2, \\ r_3(\zeta, \rho) &= \left[\sum_{n=-\infty}^{\infty} (-1)^n \rho^{n(n-1)/2} n^3 \zeta^n \right] [s(\zeta, \rho)]^2 - [r_1(\zeta, \rho)]^3 - 3r_1(\zeta, \rho)r_2(\zeta, \rho), \\ r_4(\zeta, \rho) &= \sum_{n=-\infty}^{\infty} (-1)^n \frac{n(n-1)}{2} \rho^{n(n-1)/2} \zeta^n, \\ r_5(\zeta, \rho) &= \left[\sum_{n=-\infty}^{\infty} (-1)^n \frac{n^2(n-1)}{2} \rho^{n(n-1)/2} \zeta^n \right] s(\zeta, \rho) - r_1(\zeta, \rho)r_4(\zeta, \rho), \\ r_6(\zeta, \rho) &= \left[\sum_{n=-\infty}^{\infty} (-1)^n \frac{n^3(n-1)}{2} \rho^{n(n-1)/2} \zeta^n \right] [s(\zeta, \rho)]^2 \\ &\quad - r_4(\zeta, \rho) \left[[r_1(\zeta, \rho)]^2 + r_2(\zeta, \rho) \right] - 2r_1(\zeta, \rho)r_5(\zeta, \rho). \end{aligned} \quad (51)$$

6 Doubly periodic array of Stokeslets

Hasimoto [5] found solutions for doubly periodic Stokeslet arrays in terms of the Weierstrass $\wp$ -function. The expressions found by Hasimoto [5] correspond to Goursat functions $f(z)$ and $g'(z)$ with local form, near z_0 , given by (10) along with

some additional source singularities in $g'(z)$. Although our method can be used to obtain solutions identical to those found in [5], we derive expressions for a doubly periodic Stokeslet with local form (10) without additional source singularities.

We now seek an expression for a doubly periodic array of Stokeslets with strength given by (13) for some $\mu \in \mathbb{C}$. The representative Stokeslet in the fundamental period rectangle shown in Figure 1 is located at z_0 and has a preimage at ζ_0 with

$$z_0 = -i \log \zeta_0. \quad (52)$$

We proceed by introducing the composed functions

$$F(\zeta) \equiv f(z(\zeta)), \quad G(\zeta) \equiv g'(z(\zeta)). \quad (53)$$

Since $f(z)$ must have a logarithmic singularity at $z = z_0$ then $F(\zeta)$ will have a logarithmic singularity at $\zeta = \zeta_0$. Let

$$F(\zeta) = \mu \log P(\zeta/\zeta_0, \rho) + a \log(\zeta/\zeta_0) + b[\log(\zeta/\zeta_0)]^2, \quad (54)$$

where $\mu \in \mathbb{C}$, $a \in \mathbb{R}$ and $b \in \mathbb{C}$. We have chosen a to be real for the following reason. If $f(z) \mapsto f(z) + cz$ for some complex constant c then $f'(z) \mapsto f'(z) + c$ and the velocity becomes

$$u - iv \mapsto u - iv - \overline{c}z + c\overline{z} \quad (55)$$

which is invariant if c is purely real. But since $z = -i \log \zeta$ then $f(z) \mapsto f(z) + cz$ with c purely real corresponds to $F(\zeta) \mapsto F(\zeta) - ic \log \zeta$. Hence, these inconsequential changes in $F(\zeta)$ are eliminated by restricting a to be real in (54).

It is clear that $F(\zeta)$ as given in (54) has logarithmic singularities at the points

$$\zeta = \zeta_0 \rho^k, \quad k \in \mathbb{Z} \quad (56)$$

implying that $f(z)$ has logarithmic singularities of the form (6) at z_0 . It follows, from the chain rule, that

$$f'(z) = \frac{F'(\zeta)}{z'(\zeta)} = i\mu K(\zeta/\zeta_0, \rho) + ia + 2ib \log(\zeta/\zeta_0). \quad (57)$$

The velocity field can be written as

$$\begin{aligned} u - iv &= -\overline{F(\zeta)} + \overline{z(\zeta)} \frac{F'(\zeta)}{z'(\zeta)} + G(\zeta) \\ &= -\overline{\mu} \log \overline{P(\zeta/\zeta_0, \rho)} - a \log(\overline{\zeta/\zeta_0}) - \overline{b} [\log(\overline{\zeta/\zeta_0})]^2 \\ &\quad + i \log \overline{\zeta} [i\mu K(\zeta/\zeta_0, \rho) + ia + 2ib \log(\zeta/\zeta_0)] + G(\zeta) \\ &= -\overline{\mu} \log \overline{P(\zeta/\zeta_0, \rho)} - 2a \log(\overline{\zeta/\zeta_0}) - a \log \overline{\zeta_0} \\ &\quad - \mu \log \overline{\zeta} K(\zeta/\zeta_0, \rho) - 2b \log \overline{\zeta} \log(\zeta/\zeta_0) - \overline{b} [\log(\overline{\zeta/\zeta_0})]^2 + G(\zeta), \end{aligned} \quad (58)$$

where $G(\zeta)$ remains to be determined. Let $b = b_r + ib_i$ and suppose

$$\begin{aligned} G(\zeta) &= -\overline{\mu} \log P(\zeta/\zeta_0, \rho) - 2a \log(\zeta/\zeta_0) + a \log \overline{\zeta_0} - \mu \log \zeta K(\zeta/\zeta_0, \rho) \\ &\quad - b_r [\log(\zeta/\zeta_0)]^2 + ib_i [\log(\zeta/\zeta_0)]^2 + 2b \log \overline{\zeta_0} \log(\zeta/\zeta_0) + \tilde{G}(\zeta), \end{aligned} \quad (59)$$

where $\tilde{G}(\zeta)$ is to be found. The velocity field becomes

$$\begin{aligned} u - iv = & -\bar{\mu} \log |P(\zeta/\zeta_0, \rho)|^2 - 2a \log |\zeta/\zeta_0|^2 - \mu \log |\zeta|^2 K(\zeta/\zeta_0, \rho) \\ & - b_r [\log |\zeta/\zeta_0|^2]^2 + ib_i \left[\log(\bar{\zeta}/\bar{\zeta}_0) - \log(\zeta/\zeta_0) \right]^2 + \tilde{G}(\zeta). \end{aligned} \quad (60)$$

In order that the velocity field is invariant as $\zeta \mapsto \zeta e^{2\pi i}$ we require

$$b_i = 0. \quad (61)$$

The term $\mu \log |\zeta|^2 K(\zeta/\zeta_0, \rho)$ is, however, too singular at ζ_0 . Therefore let

$$\tilde{G}(\zeta) = \mu \log |\zeta_0|^2 K(\zeta/\zeta_0, \rho) - \mu \log \rho^2 \left[\rho \frac{\partial \log P(\zeta/\zeta_0, \rho)}{\partial \rho} \right]. \quad (62)$$

The velocity field is then

$$\begin{aligned} u - iv = & -\bar{\mu} \log |P(\zeta/\zeta_0, \rho)|^2 - 2a \log |\zeta/\zeta_0|^2 - \mu \log |\zeta/\zeta_0|^2 K(\zeta/\zeta_0, \rho) \\ & - \mu \log \rho^2 \left[\rho \frac{\partial \log P(\zeta/\zeta_0, \rho)}{\partial \rho} \right] - b_r [\log |\zeta/\zeta_0|^2]^2. \end{aligned} \quad (63)$$

For this to be invariant as $\zeta \mapsto \rho\zeta$, we must set

$$2b_r \log \rho = \operatorname{Re}[\mu], \quad a = -\frac{\operatorname{Re}[\mu]}{2}. \quad (64)$$

A local analysis near z_0 gives

$$G(\zeta) = -\bar{\mu} \log(z - z_0) - \mu \log \zeta K(\zeta/\zeta_0, \rho) + \mu \log |\zeta_0|^2 K(\zeta/\zeta_0, \rho) + \dots \quad (65)$$

implying that

$$\begin{aligned} G(\zeta) = & -\mu \left(\frac{z_0}{z - z_0} \right) + \mu \log |\zeta_0|^2 \left(\frac{-i}{z - z_0} \right) - \bar{\mu} \log(z - z_0) + \dots \\ = & -\bar{\mu} \log(z - z_0) - \frac{\mu \bar{z}_0}{z - z_0} + \dots, \end{aligned} \quad (66)$$

as required by comparison with (10).

In summary, the Goursat functions for a doubly periodic array of Stokeslets with one at z_0 in the principal period window and having strength given by (13) are

$$\begin{aligned} F(\zeta) = & \mu \log P(\zeta/\zeta_0, \rho) + a \log(\zeta/\zeta_0) + b [\log(\zeta/\zeta_0)]^2, \\ G(\zeta) = & -\bar{\mu} \log P(\zeta/\zeta_0, \rho) - 2a \log(\zeta/\zeta_0) + a \log \bar{\zeta}_0 \\ & - \mu \log \zeta K(\zeta/\zeta_0, \rho) + \mu \log |\zeta_0|^2 K(\zeta/\zeta_0, \rho) - \mu \log \rho^2 \left[\rho \frac{\partial \log P(\zeta/\zeta_0, \rho)}{\partial \rho} \right] \\ & + 2b \log \bar{\zeta}_0 \log(\zeta/\zeta_0) - b [\log(\zeta/\zeta_0)]^2, \end{aligned}$$

with $a = -\frac{\operatorname{Re}[\mu]}{2}, \quad b = \frac{\operatorname{Re}[\mu]}{2 \log \rho}.$ (67)

Using (5), the associated pressure, vorticity and velocity fields are

$$\begin{aligned} \frac{p}{\eta} - i\omega &= 4i\mu K(\zeta/\zeta_0, \rho) + 4ia + 8ib \log(\zeta/\zeta_0), \\ u - iv &= -\bar{\mu} \log |P(\zeta/\zeta_0, \rho)|^2 - 2a \log |\zeta/\zeta_0|^2 - \mu \log |\zeta/\zeta_0|^2 K(\zeta/\zeta_0, \rho) \\ &\quad - \mu \log \rho^2 \left[\rho \frac{\partial \log P(\zeta/\zeta_0, \rho)}{\partial \rho} \right] - b [\log |\zeta/\zeta_0|^2]^2. \end{aligned} \quad (68)$$

It is readily verified that this expression for $u - iv$ is invariant under all the transformations

$$\zeta \mapsto \rho\zeta, \quad \zeta_0 \mapsto \rho\zeta_0, \quad \zeta \mapsto e^{2\pi i}\zeta, \quad \zeta_0 \mapsto e^{2\pi i}\zeta_0. \quad (69)$$

As mentioned above, the representations for a doubly periodic Stokeslet found by Hasimoto [5] correspond to Goursat functions $f(z)$ and $g'(z)$ with local form given by (10) along with some additional source singularities in $g'(z)$. Therefore, since pressure p and vorticity ω can be expressed in terms of $f(z)$ only, we have verified that the first expression in (68) is identical to that found by Hasimoto [5].

If there is a single Stokeslet in each period window then this implies the existence of a pressure gradient across the cell. Since

$$\frac{p}{\eta} - i\omega = 4i\mu K(\zeta/\zeta_0, \rho) + 4ia + 8ib \log(\zeta/\zeta_0), \quad (70)$$

it follows that as $\zeta \mapsto \zeta e^{2\pi i}$ then

$$\frac{p}{\eta} \mapsto \frac{p}{\eta} - \frac{8\pi \text{Re}[\mu]}{\log \rho}, \quad (71)$$

while as $\zeta \mapsto \rho\zeta$ then

$$\frac{p}{\eta} \mapsto \frac{p}{\eta} + 4\text{Im}[\mu]. \quad (72)$$

Similar properties have been pointed out by Greengard & Kropinski [4].

Figure 2 shows the streamline pattern for a doubly periodic array of Stokeslets of strength (13) with $\mu = 1$ located at the centre of each period window. To plot these streamlines, the rapidly convergent representations (46)-(51) which appear in (68) have been truncated to only $N = 16$. Similar plots computed using different methods are presented in Pozrikidis [14].

7 Doubly periodic array of stresslets

Following similar steps to the previous section, one can derive expressions for a doubly periodic array of stresslets. We will continue to denote the relevant Goursat functions by $f(z)$ and $g(z)$.

It turns out that the full analysis can be bypassed by noticing that the Goursat functions for the doubly periodic array of stresslets can be derived from those for a doubly periodic array of Stokeslets by computing the parametric derivatives:

$$f(z) \mapsto -\frac{\partial f(z)}{\partial z_0}, \quad g'(z) \mapsto -\frac{\partial g'(z)}{\partial z_0}. \quad (73)$$

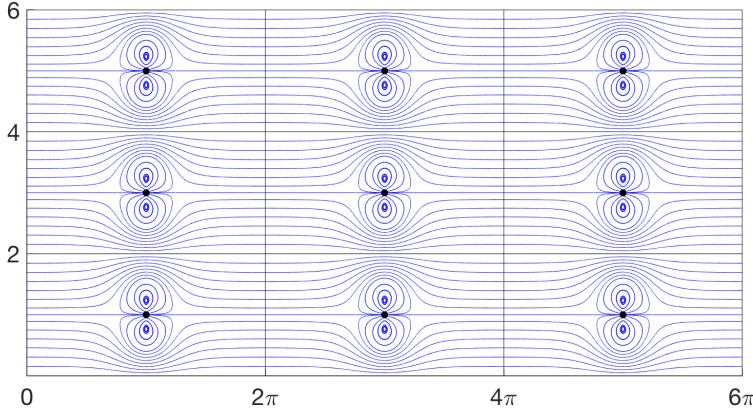


Fig. 2: Streamline pattern for a doubly periodic array of Stokeslets in a periodic lattice. The principal period window is $0 \leq x \leq l = 2\pi$, $0 \leq y \leq h = 2$. The principal Stokeslet is located at $z_0 = l/2 + ih/2 = \pi + i$ and its strength is given by (13) with $\mu = 1$.

To verify this claim, we apply the parametric derivatives (73) in (10) and obtain

$$\begin{aligned} f(z) &= \frac{\mu}{z - z_0} + \text{locally analytic function}, \\ g'(z) &= \frac{\mu \bar{z}_0}{(z - z_0)^2} - \frac{\bar{\mu}}{z - z_0} + \text{locally analytic function}. \end{aligned} \quad (74)$$

The first terms in $f(z)$ and $g'(z)$ correspond to a stresslet of strength proportional to μ at z_0 as given by (18), while the second term in $g'(z)$ corresponds to an additional source/sink and/or rotlet singularity. If we seek the Goursat functions for a doubly periodic array of stresslets without any of these additional source/sink or rotlet singularities we can compute the parametric derivatives then simply subtract terms associated to the simple pole in $g'(z)$. In general, on taking parametric derivatives in this way, care must be taken to ensure the resulting flows are as required and, as we will see, small adjustments of the resulting Goursat functions must sometimes be made.

Using the fact that

$$\frac{\partial}{\partial z_0} = \frac{\partial \zeta_0}{\partial z_0} \frac{\partial}{\partial \zeta_0} = i\zeta_0 \frac{\partial}{\partial \zeta_0}, \quad (75)$$

computation of the parametric derivatives (73) of (67) gives

$$\begin{aligned} F(\zeta) &= i\mu K(\zeta/\zeta_0, \rho) + \lambda \log(\zeta/\zeta_0) + ia, \\ G(\zeta) &= -i\mu K(\zeta/\zeta_0, \rho) - i\mu \log \zeta L(\zeta/\zeta_0, \rho) + i\mu \log |\zeta_0|^2 L(\zeta/\zeta_0, \rho) \\ &\quad - i\mu \log \rho^2 \left[\rho \frac{\partial K(\zeta/\zeta_0, \rho)}{\partial \rho} \right] - \lambda \log(\zeta/\zeta_0) + \lambda \log \bar{\zeta}_0 \\ &\quad - i\bar{\mu} K(\zeta/\zeta_0, \rho) - 2ia, \end{aligned} \quad (76)$$

where we have replaced constant $2ib$ by $\lambda \in \mathbb{R}$ (as already discussed in the previous section, we can, without loss of generality, take this to be a real constant), since

this will need to be redefined. The associated velocity field is given by

$$\begin{aligned} u - iv = & 2\text{Im}[\mu K(\zeta/\zeta_0, \rho)] - 4\lambda \log |\zeta/\zeta_0| - i\mu \log |\zeta/\zeta_0|^2 L(\zeta/\zeta_0, \rho) \\ & - i\mu \log \rho^2 \left[\rho \frac{\partial K(\zeta/\zeta_0, \rho)}{\partial \rho} \right] - i\bar{\mu} K(\zeta/\zeta_0, \rho) + \lambda \log(\zeta/\zeta_0) - ia. \end{aligned} \quad (77)$$

Note that term $-i\bar{\mu} K(\zeta/\zeta_0, \rho)$ corresponds to a simple pole at z_0 in $g'(z)$; this term together with term $\lambda \log(\zeta/\zeta_0)$ in (77), must be removed from $G(\zeta)$ in order to obtain expressions for the flow generated by a doubly periodic array of torque-free stresslets without any other induced singularities. In addition, the constant terms ia in $F(\zeta)$ and $-2ia$ in $G(\zeta)$ which correspond to a background uniform flow $u - iv = -ia$ should also be omitted. On subtraction of these terms, we find that the Goursat functions for a doubly periodic array of torque-free point stresslets with a representative at z_0 in the principal period window are

$$\begin{aligned} F(\zeta) &= i\mu K(\zeta/\zeta_0, \rho) + \lambda \log(\zeta/\zeta_0), \\ G(\zeta) &= -i\bar{\mu} K(\zeta/\zeta_0, \rho) - i\mu \log \zeta L(\zeta/\zeta_0, \rho) + i\mu \log |\zeta_0|^2 L(\zeta/\zeta_0, \rho) \\ &\quad - i\mu \log \rho^2 \left[\rho \frac{\partial K(\zeta/\zeta_0, \rho)}{\partial \rho} \right] - 2\lambda \log(\zeta/\zeta_0) + \lambda \log \bar{\zeta}_0, \end{aligned} \quad (78)$$

The choice of λ remains to be determined. This follows from the requirement that $u - iv$ is invariant as $\zeta \mapsto \rho\zeta$. The associated complex velocity field is

$$\begin{aligned} u - iv = & i\bar{\mu} K(\zeta/\zeta_0, \rho) - i\mu K(\zeta/\zeta_0, \rho) - i\mu \log |\zeta/\zeta_0|^2 L(\zeta/\zeta_0, \rho) \\ & - i\mu \log \rho^2 \left[\rho \frac{\partial K(\zeta/\zeta_0, \rho)}{\partial \rho} \right] - 4\lambda \log |\zeta/\zeta_0|. \end{aligned} \quad (79)$$

From (79), as $\zeta \mapsto \rho\zeta$, we find

$$u - iv \mapsto u - iv + i\mu - i\bar{\mu} - 4\lambda \log \rho. \quad (80)$$

We must therefore pick

$$\lambda = -\frac{\text{Im}[\mu]}{2 \log \rho}. \quad (81)$$

This choice also ensures the velocity field is invariant under the transformation $\zeta_0 \mapsto \rho\zeta_0$.

The associated pressure, vorticity and velocity fields can be written as

$$\begin{aligned} \frac{p}{\eta} - i\omega &= 4i\mu L(\zeta/\zeta_0, \rho) + 4i\lambda, \\ u - iv &= 2\text{Im}[\mu K(\zeta/\zeta_0, \rho)] - i\mu \log |\zeta/\zeta_0|^2 L(\zeta/\zeta_0, \rho) \\ &\quad - i\mu \log \rho^2 \left[\rho \frac{\partial K(\zeta/\zeta_0, \rho)}{\partial \rho} \right] - 4\lambda \log |\zeta/\zeta_0|. \end{aligned} \quad (82)$$

Figure 3 shows the streamline pattern in a typical period window generated by a doubly periodic array of stresslets with $\mu = 1$ and where each singularity is located at the centre of the period window. The representations (48)-(51) which appear in (82) have been again truncated to $N = 16$.

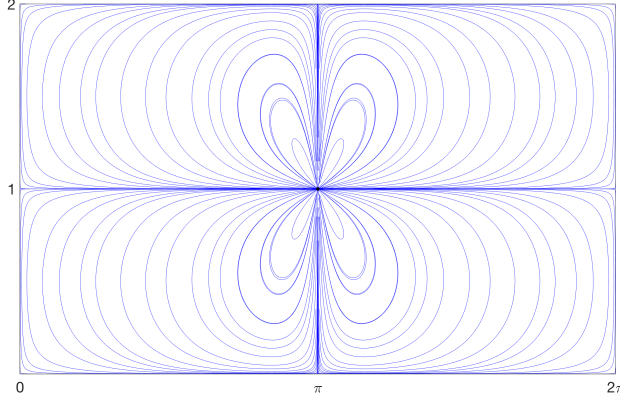


Fig. 3: Streamline pattern in a period window generated by a doubly periodic array of stresslets. The principal period window is $0 \leq x \leq l = 2\pi$, $0 \leq y \leq h = 2$. The principal stresslet is located at $z_0 = l/2 + ih/2 = \pi + i$ with $\mu = 1$.

8 Doubly periodic array of force quadrupoles

Again, the full analysis can be bypassed by noticing that the Goursat functions $f(z)$ and $g'(z)$ for a doubly periodic array of force quadrupoles can be derived from those of a similar array of torque-free stresslets by computing the parametric derivatives:

$$f(z) \mapsto \frac{\partial f(z)}{\partial z_0}, \quad g'(z) \mapsto \frac{\partial g'(z)}{\partial z_0}. \quad (83)$$

Computation of the parametric derivatives (83) of (78) gives

$$\begin{aligned} F(\zeta) &= \mu L(\zeta/\zeta_0, \rho) - i\lambda, \\ G(\zeta) &= -2\mu L(\zeta/\zeta_0, \rho) - \mu \log \zeta M(\zeta/\zeta_0, \rho) + \mu \log |\zeta_0|^2 M(\zeta/\zeta_0, \rho) \\ &\quad - \mu \log \rho^2 \left[\rho \frac{\partial L(\zeta/\zeta_0, \rho)}{\partial \rho} \right] + 2i\lambda, \end{aligned} \quad (84)$$

where λ is given by (81). Note that the constant terms $-i\lambda$ in $F(\zeta)$ and $2i\lambda$ in $G(\zeta)$ correspond to a background uniform flow $u - iv = i\lambda$ which we omit since we wish to find the Goursat functions associated with a doubly periodic array of force quadrupoles without any other induced flow.

In summary, the Goursat functions for a doubly periodic array of force quadrupoles with local behaviour (23) at z_0 in the principal period window can be written as

$$\begin{aligned} F(\zeta) &= \mu L(\zeta/\zeta_0, \rho), \\ G(\zeta) &= -2\mu L(\zeta/\zeta_0, \rho) - \mu \log \zeta M(\zeta/\zeta_0, \rho) + \mu \log |\zeta_0|^2 M(\zeta/\zeta_0, \rho) \\ &\quad - \mu \log \rho^2 \left[\rho \frac{\partial L(\zeta/\zeta_0, \rho)}{\partial \rho} \right]. \end{aligned} \quad (85)$$

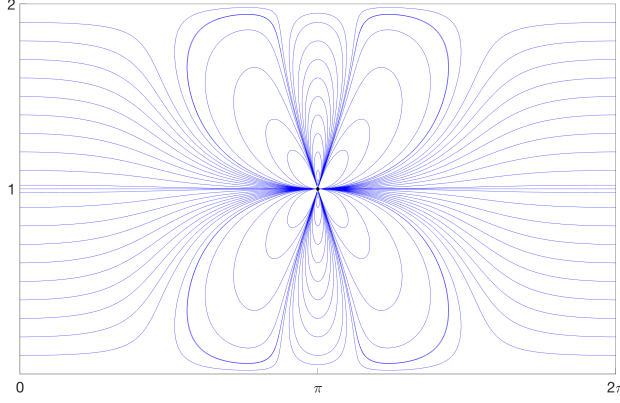


Fig. 4: Streamline pattern in a typical period window generated by a doubly periodic array of force quadrupoles. The principal period window is $0 \leq x \leq l = 2\pi$, $0 \leq y \leq h = 2$. The principal force quadrupole is located at $z_0 = l/2 + ih/2 = \pi + i$ with $\mu = 1$.

The associated pressure, vorticity and velocity fields are given by

$$\begin{aligned} \frac{p}{\eta} - i\omega &= 4i\mu M(\zeta/\zeta_0, \rho), \\ u - iv &= -\overline{\mu L(\zeta/\zeta_0, \rho)} - 2\mu L(\zeta/\zeta_0, \rho) - \mu \log |\zeta/\zeta_0|^2 M(\zeta/\zeta_0, \rho) \\ &\quad - \mu \log \rho^2 \left[\rho \frac{\partial L(\zeta/\zeta_0, \rho)}{\partial \rho} \right]. \end{aligned} \quad (86)$$

Figure 4 shows the streamline pattern in a typical period window generated by a doubly periodic array of force quadrupoles with $\mu = 1$; the singularity is located at the centre of the period window. The representations (48)–(51) which appear in (86) have been truncated to $N = 16$.

9 Doubly periodic arrays of source dipoles and quadrupoles

Using (40), the expression for a doubly periodic array of source dipoles with local behaviour (27) at z_0 in the principal period window is

$$F(\zeta) = 0, \quad G(\zeta) = -\mu L(\zeta/\zeta_0, \rho). \quad (87)$$

The expression for a doubly periodic array of source quadrupoles with local behaviour (29) at z_0 in the principal period window is

$$F(\zeta) = 0, \quad G(\zeta) = i\mu M(\zeta/\zeta_0, \rho). \quad (88)$$

Figures 5-6 show streamline patterns in a typical period window generated by doubly periodic arrays of source dipoles and source quadrupoles, respectively; each has $\mu = 1$ and the singularity is located at the centre of the period window. The representations (48)–(51) which appear in (87) and (88) have been truncated to $N = 16$.

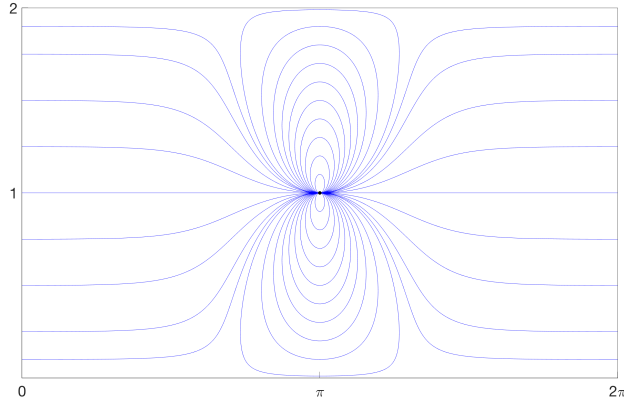


Fig. 5: Streamline pattern in a period window generated by a doubly periodic array of source dipoles. The principal period window is $0 \leq x \leq l = 2\pi$, $0 \leq y \leq h = 2$. The principal point dipole is located at $z_0 = l/2 + ih/2 = \pi + i$ with $\mu = 1$.

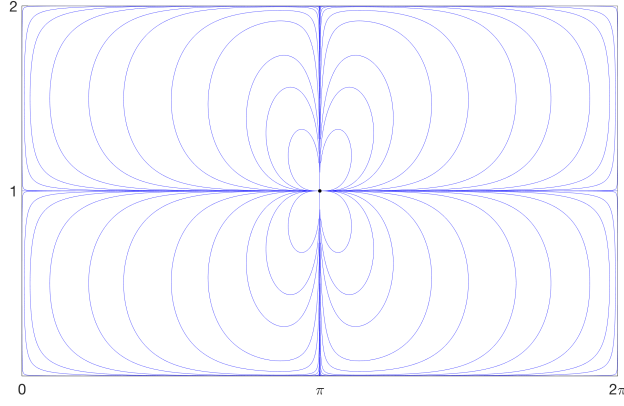


Fig. 6: Streamline pattern in a period window generated by a doubly periodic array of source quadrupoles. The principal period window is $0 \leq x \leq l = 2\pi$, $0 \leq y \leq h = 2$. The principal point quadrupole is located at $z_0 = l/2 + ih/2 = \pi + i$ with $\mu = 1$.

10 Discussion

For the doubly periodic arrays of singularities the function $P(\zeta, \rho)$ on which we have built our derivation of the solutions is related to the first Jacobi theta function Θ_1 via the identity [2]:

$$P(\zeta, \rho) = -\frac{i \exp(iz/2)}{C \rho^{1/8}} \Theta_1(z/2, \sqrt{\rho}), \quad C = \prod_{k=1}^{\infty} (1 - \rho^k). \quad (89)$$

This fact will not be surprising in light of work by other authors [4, 5, 10] who have presented alternative representations of the same doubly periodic arrays in terms of the Weierstrass $\wp$ -function and its derivatives. As mentioned in the introduction, to evaluate these $\wp$ -functions efficiently it is necessary to employ classical identities relating the $\wp$ -function to the Jacobi theta functions, including Θ_1 , for which methods of effective numerical evaluation are known. The approach advocated in this paper is, in essence, to completely avoid lattice sums, and the $\wp$ -function, and to derive the solutions *directly* in terms of functions with numerically convenient representations. This was conveniently achieved by considering a conformal mapping to a ζ -annulus and using $P(\zeta, \rho)$, the so-called *Schottky-Klein prime function* of this annulus. It is worth pointing out that this paper is a generalization, to Stokes flows, of a similar rederivation of the equations of motion for point vortex lattices presented by one of the authors [2].

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