

Imperial College London
Department of Mathematics

A hyperbolic perspective on the Dehn surgery characterisation problem

Laura Wakelin

Submitted in partial fulfilment of the requirements
for the degree of Doctor of Philosophy at
Imperial College London (September 2023)



I hereby declare that the work presented in this thesis is entirely my own, except for the contents of the final chapter, which originates from collaborative work with Patricia Sorya. Information derived from the published and unpublished work of others has been acknowledged in the text and references are given in the bibliography.

Laura Wakelin (September 2023)

The copyright of this thesis rests with the author. Unless otherwise indicated, its contents are licensed under a Creative Commons Attribution Non-Commercial 4.0 International Licence (CC BY-NC). Under this licence, you may copy and redistribute the material in any medium or format. You may also create and distribute modified versions of the work. This is on the condition that: you credit the author and do not use it, or any derivative works, for a commercial purpose. When reusing or sharing this work, ensure you make the licence terms clear to others by naming the licence and linking to the licence text. Where a work has been adapted, you should indicate that the work has been changed and describe those changes. Please seek permission from the copyright holder for uses of this work that are not included in this licence or permitted under UK Copyright Law.

Abstract

As is the case for many questions in low-dimensional topology, the Dehn surgery characterisation problem is simple to state but difficult to solve. Given a non-trivial slope $p/q \in \mathbb{Q}$, when does the oriented homeomorphism type of the 3-manifold $\mathbb{S}_K^3(p/q)$ obtained by Dehn surgery of slope p/q on a knot $K \subset \mathbb{S}^3$ uniquely characterise the knot K ? A wide variety of techniques, ranging from Floer homology to more geometric approaches, have been successfully employed to find characterising slopes for certain knots; the construction of non-characterising slopes has also been studied extensively. However, there is currently no universal criterion to determine whether a slope is characterising or non-characterising for any given knot.

This thesis investigates the Dehn surgery characterisation problem in the case where the knot complement $\mathbb{S}_K^3$ has a hyperbolic outermost JSJ piece. The fundamental strategy involves comparing JSJ decompositions before and after Dehn filling: this allows us to reduce the problem at hand to one about hyperbolic geometry. Subsequently, two novel ideas - minimal geodesics (in the more general case) and minimal volumes (in the special case of Whitehead doubles) - can be used to determine explicit conditions which ensure that a slope is characterising. The limitations of this method are also illustrated via a construction of infinite families of pairs of multiclasp Whitehead doubles sharing a non-characterising slope.

Acknowledgements

I would first and foremost like to thank Steven Sivek for an endless supply of patience and encouragement whilst guiding me through my PhD. I have learnt an immeasurable amount from him and I truly could not have asked for a better supervisor.

I also wish to express my gratitude to the many other mentors who have shaped my mathematical journey. Thank you to my external examiner, Marc Lackenby, for all of the time invested in reading my work and helping me to find and fix the inevitable mistakes; thank you also to my internal examiner, Yankı Lekili, for taking a genuine interest in my research throughout my time at Imperial. I'm deeply indebted to Lisa Piccirillo for hosting me at MIT and being a wonderful role model for young women in low-dimensional topology. I'd also like to acknowledge Jeff Meier, Dave Gabai, Duncan McCoy, Tom Mrowka and every other mathematician who has shared their insights with me, mathematical or otherwise. Many thanks to all of my past project supervisors, particularly Jonny Evans, who originally inspired me to pursue low-dimensional topology. A special thanks goes to my high school teacher, Stella Mackenzie, for introducing me to the beauty of mathematics and going above and beyond to open those first doors for me.

I doubt I would enjoy my work so much without my brilliant collaborators. Thank you to Patricia Sorya, for sharing my fascination with the Dehn surgery characterisation problem and working with me across ICERM, Zoom, UQÀM and Oxford. Thank you to my fellow multisectors, Shintaro Fushida-Hardy and Devashi Gulati, for creating such a friendly and supportive environment in which to discover new mathematics together. Thank you to Charles Stine for persevering through the highs and lows of our ongoing project with such enthusiasm.

I'd like to thank the MPIM for giving me the opportunity to be part of such a thriving research community of low-dimensional topologists for 6 months. Thank you to Aru Ray for being exceptionally welcoming and inviting me to speak at MFO and GROW. Thank you also to Sarah, Isaac, Hiroki, Roberto and everyone else who was part of my time in Bonn - I look forward to returning next year.

This thesis would not have been possible without the continued support of the LSGNT throughout my studies. Many thanks to Nicky Townsend for working hard to run everything so smoothly. I'm grateful to my female colleagues for creating a space for each other to shine brightly. I'd also like to thank everyone in my cohort - Lambert, Jess, Abi, Riccardo, Alessio, Soham, Teresa, Ashvni, Jesse, Ivan, Hannah (and honorary members Josh and Fra) - for all of the good times that we shared; I will always cherish the memories that we made together during our time as PhD students.

Thank you to all of the people who kept me sane during my Part III year at Cambridge, particularly my friends from 67 Chesterton Road and Clare MCR: Pedro, Lotte, Lucy, Lily and Sam. Thank you to all of my other friends who have distracted me from maths every now and again, especially Hannah, who is my rock even when we are oceans apart, and Vatsal, who frequently reminds me to take a break and enjoy playing music together.

I am grateful to my flatmates for making 20 Brook House and 31 Nash House feel like home for the last 4 years, just by being their unique and authentic selves. Skye - it can't have been easy living with mathematicians, but it's been a privilege to experience your unlimited kindness every day. Infinite thanks to my soul sister, Jess, for providing many perfectly brewed cups of tea, always being there for me and giving me the chance to be a best friend.

Thank you to Soham, for filling my life with light and laughter, believing in my boundless dreams and walking alongside me every step of the way.

Finally, my deepest thanks go to my family for all of their love and support: to the best siblings in the world, Holly and Joe, for being my lifelong friends, and to my parents, Richard and Shinder, for raising me with every opportunity to fulfil my potential.

To my parents and grandparents.

*Seek the wisdom that will untie your knot.
Seek the path that demands your whole being.*

- Rumi

Contents

Abstract	3
Acknowledgements	4
List of tables	10
List of figures	11
1 Introduction	12
1.1 Literature review	13
1.2 Main results	17
1.3 Overview	22
2 Preliminaries	23
2.1 Satellite knots	23
2.2 Geometry of 3-manifolds	26
2.3 Dehn surgery	29
3 JSJ decompositions	33
3.1 Creation and compression	33
3.2 Hyperbolic knots	36
3.3 Hyperbolic patterns	37
3.4 Pattern properties	40

4	Shortest geodesics	42
4.1	Hyperbolic geometry	43
4.2	Quantitative comparison	45
4.3	Core curves correspond	46
4.4	Proof of Theorem 1.6	49
4.5	Proof of Theorem 1.8	50
5	Whitehead doubles	52
5.1	Homeomorphic link complements	53
5.2	Upper volume bounds	55
5.3	Comparing Dehn fillings	57
5.4	Proof of Theorem 1.10	59
6	Non-characterising slopes	63
6.1	Construction	64
6.2	Gluing maps	66
6.3	Proof of Theorem 1.11	67
7	The multicusped case	69
7.1	The cyclic surgery theorem	71
7.2	Lower volume bounds	74
7.3	Proof of Theorem 1.12	78
7.4	Proof of Theorem 1.13	82
7.5	Proof of Theorem 1.14	86
	Bibliography	91
A	Minimal geodesics	96
B	Minimal volumes	99

List of Tables

1.1	The values of V_k , a_k and q_k for each stage k	19
5.1	2-cusped orientable hyperbolic 3-manifolds which can be ruled out by one of the simpler elimination techniques (as described in the proof of Theorem 1.10).	62
7.1	Values of $C(K)$ when $K = W_t^n(C)$	81
7.2	Values of $C(K)$ when $K = B^n(C_0, C_1)$	86
7.3	Number of links L in the HTLinkExteriors census with $\text{sys}(L) > 0.14$	90
A.1	The length of a minimal geodesic in $\mathbb{S}_{T_t^\pm}^3$ and $\mathfrak{q}(\mathbb{S}_{T_t^\pm}^3)$ for $ t \leq 34$	97
A.2	The length of a minimal geodesic in $\mathbb{S}_{L^n}^3$ and $\mathfrak{q}(\mathbb{S}_{L^n}^3)$ for $ n \leq 34$	98
B.1	2-cusped orientable hyperbolic 3-manifolds with low volume.	100
B.2	Hyperbolic knots C with $\text{vol}(\mathbb{S}_C^3) < \text{vol}(\mathbb{S}_W^3)$ (whose complements $\mathbb{S}_C^3$ can be triangulated with up to 8 tetrahedra) and the corresponding values of $\mathcal{Q}(\mathbb{S}_C^3)$	101

List of Figures

1.1	The twist knots for which all non-integral slopes are characterising.	14
1.2	The n -clasped t -twisted Whitehead link, $W_t^n = U \cup V_t^n$	18
1.3	A pair of knots sharing a $1/q$ -surgery.	20
2.1	Seifert fibred spaces.	26
4.1	Rolfsen $-t$ -twist giving $\mathbb{S}_{W^\pm}^3(1/t) \cong \mathbb{S}_{T_t^\pm}^3$	49
4.2	Rolfsen $-n$ -twist giving $\mathbb{S}_B^3(1/n) \cong \mathbb{S}_{W^n}^3$	51
5.1	Rolfsen ∓ 1 -twist trick.	54
6.1	Rolfsen $-q$ -twist giving $\mathbb{S}_{W^n}^3(1/q) \cong \mathbb{S}_{T_q^n}^3$	63
6.2	A homeomorphism switching two knot complements.	64
6.3	Pairs of knots, $K_\pm$ and $K'_\pm$, with $\mathbb{S}_{K_\pm}^3(\pm 1) \cong \mathbb{S}_{K'_\pm}^3(\pm 1)$	65
7.1	The satellite splittings induced by $Y(p/q) \cong X'$	72
7.2	The satellite splittings induced by $\tilde{Y}(\tilde{p}/\tilde{q}) \cong X$	74
7.3	Obstructing the swapping of JSJ pieces in the $ \partial Y = 3$ case.	82
7.4	The Borromean rings $B = L6a4$ and a satellite $B(C_0, C_1)$	84
7.5	Rolfsen $-n$ -twist giving $\mathbb{S}_L^3(1/n) \cong \mathbb{S}_{B^n}^3$ for $L = L10n100$	85
7.6	The link $L = L12a1996$ and a satellite $L(C_0, C_1, C_2)$	89

Chapter 1

Introduction

Given a knot K in the 3-sphere $\mathbb{S}^3$, we can build a new 3-manifold $\mathbb{S}_K^3(p/q)$ by performing *Dehn surgery* of slope $p/q \in \mathbb{Q} \cup \{\infty\}$ on K . This involves removing a tubular neighbourhood $\nu(K) \cong \mathbb{S}^1 \times \mathbb{D}^2$ of K , leaving behind the knot complement $\mathbb{S}_K^3 := \mathbb{S}^3 \setminus \nu(K)$, and gluing it back in by a map determined by the slope p/q . This thesis investigates the difficult question of determining precisely when we can recover K from the oriented homeomorphism type of $\mathbb{S}_K^3(p/q)$.

Dehn surgery is a bridge between knot theory and 3-manifolds. It is well-known that every closed orientable 3-manifold can be obtained by surgery on a link in $\mathbb{S}^3$ [Lic62; Wal60]. This is an existence result for which uniqueness fails: all such surgery descriptions can be related by some sequence of moves [Kir78]. However, if we restrict to the case of surgery on knots, then there are several different angles from which one can investigate the question of uniqueness. The cosmetic surgery conjecture postulates that there cannot exist an orientation-preserving homeomorphism between surgeries of different slopes on the same (non-trivial) knot unless there is a self-homeomorphism of the knot complement taking one slope to the other. The Dehn surgery characterisation problem, on the other hand, asks whether it is possible to have an orientation-preserving homeomorphism between surgeries of the same (non-trivial) slope on different knots.

Definition 1.1. *A slope $p/q \in \mathbb{Q}$ is characterising for a knot $K \subset \mathbb{S}^3$ if for any knot $K' \subset \mathbb{S}^3$, the existence of an orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q)$ implies that $K = K'$.*

The purpose of this thesis is to find explicit conditions under which a slope p/q is characterising for a given knot K . We will study the case where the knot complement $\mathbb{S}_K^3$ can be assigned a special type of structure called *hyperbolic geometry* - if not on its entirety, then at least on some well-defined piece containing the boundary.

1.1 Literature review

The Dehn surgery characterisation problem, which is the complete classification of a knot's characterising and non-characterising slopes, is still an open question. The usual strategy is to focus on one particular knot or family of knots at a time.

1.1.1 Characterising slopes

Naturally, the first case to be considered was the unknot U : the following result was originally conjectured by Gordon in [Gor78].

Theorem 1.2 ([KMOS07]). *Every slope $p/q \in \mathbb{Q}$ is characterising for the unknot U .*

The $p/q = 0$ case (Property R) was proved by Gabai using taut foliations [Gab87]. The case for non-integral slopes $p/q \in \mathbb{Q} \setminus \mathbb{Z}$ was a consequence of the cyclic surgery theorem [CGLS87], whilst the $p/q = \pm 1$ case came separately [GL89]. The general case was proved by Kronheimer, Mrowka, Ozsváth and Szabó [KMOS07] using monopole Floer homology, a novel 3-manifold invariant arising from solutions to the Seiberg-Witten equations of an associated 4-manifold. The key ingredients in their proof were a surgery exact sequence relating the monopole Floer homology groups of $\mathbb{S}_K^3(n-1)$, $\mathbb{S}_K^3(n)$ and $\mathbb{S}^3$ for $p/q = n \in \mathbb{Z}$ and a non-vanishing theorem which helps to detect knot genus.

Heegaard Floer homology, first introduced by Ozsváth and Szabó [OS04] and now known to be equivalent to monopole Floer homology, is another technique which has been employed to address the Dehn surgery characterisation problem. As shown by Ozsváth and Szabó [OS02] and independently by Rasmussen [Ras03], a knot in a 3-manifold induces a filtration on Heegaard Floer homology, giving rise to a powerful knot invariant called knot Floer homology. This tool is able to detect the unknot U , as well as the trefoils $\pm T = \pm 3_1$ and the figure eight knot $S = 4_1$ as the only genus 1 fibred knots [Ghi08]. In light of this, Ozsváth and Szabó were able to prove the following result [OS19].

Theorem 1.3 ([OS19]). *Every slope $p/q \in \mathbb{Q}$ is characterising for the trefoils $\pm T$ and the figure eight knot S .*

Recent work of Baldwin and Sivek [BS22] yields a similar result regarding the next twist knots, $\pm R = \pm 5_2$, though we observe that some integral slopes are omitted.

Theorem 1.4 ([BS22]). *Every slope $p/q \in \mathbb{Q} \setminus \mathbb{Z}_{\pm}$ is characterising for the knots $\pm R$ (where $\mathbb{Z}_{+} := \mathbb{Z}_{>0}$ and $\mathbb{Z}_{-} := \mathbb{Z}_{<0}$).*

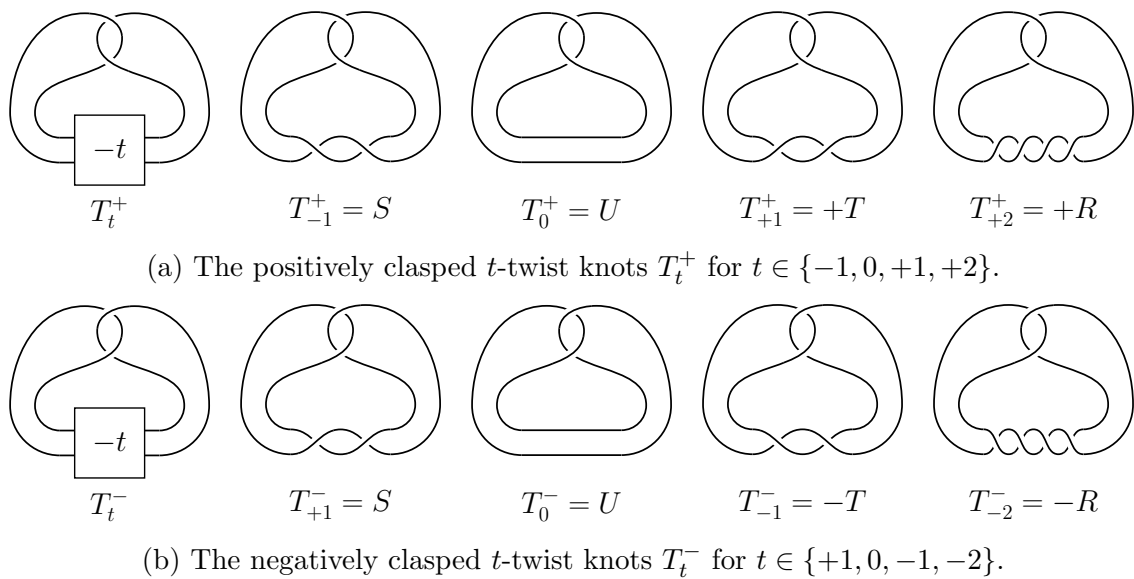


Figure 1.1: The twist knots for which all non-integral slopes are characterising.

Aside from twist knots, another family of knots whose characterising slopes have been investigated in depth is that of torus knots. Building on work of Ni and Zhang on the torus knot $T_{5,2}$ [NZ14], McCoy showed that all but finitely many of the slopes $p/q \in \mathbb{Q} \setminus \mathbb{Z}_-$ are characterising for positive torus knots [McC20]. He did this by combining genus bounds from Heegaard Floer homology with distance bounds on exceptional surgeries resulting from the 6-theorem [Lac00; Ago00]. However, it is not true that we will find that all slopes are characterising for every knot.

1.1.2 Non-characterising slopes

Non-characterising slopes are certainly known to exist. In particular, integral slopes are often exempt from strategies used to show that slopes are characterising, meaning that they provide promising candidates. For example, infinite families of knots with the same 0-surgeries were found by Osoinach in [Oso06]. Baker and Motegi discovered knots which each have infinitely many non-characterising slopes: the simplest of their examples is the hyperbolic knot 8_6 , for which all integral slopes are non-characterising [BM18]. Subsequent work of Abe and Tagami applies similar methods to a group of small knots, including the twist knot 8_1 : they use annulus twisting to show that these knots satisfy a simple condition of Baker and Motegi which ensures the existence of infinitely many non-characterising slopes [AT21]. Another application of annulus twisting is implicit in Piccirillo's proof that the Conway knot is not slice, under the guise of the dualisable patterns construction: her key idea is to construct a knot sharing the same 4-dimensional 0-trace as the Conway knot, but of course an immediate consequence is that they also have the same 3-dimensional 0-surgery [Pic20]. RBG links can also be used to find other integral non-characterising slopes.

It is not only integral slopes which can be non-characterising. For each torus knot, McCoy gave a precise description of when we can find a cable of a torus knot sharing a unique non-integral non-characterising slope [McC20, Proposition 1.5]. He also showed that every hyperbolic knot can only have finitely many non-characterising

slopes with $|q| \geq 3$ [McC19, Theorem 2]. In a different approach, Brakes introduced a method of constructing non-characterising slopes of the form $n \pm 1/q$: this relies on finding pairs of knots which can be trivialised through a specific twisting operation, thus ensuring that both admit surgeries to a 3-manifold which can be decomposed as a pair of knot complements glued together along their boundaries in a certain way [Bra80]. This strategy does not always produce distinct knots; however, in one of the results in this thesis, we will rediscover an example given by Brakes, along with an infinite family of pairs of satellite knots sharing a non-characterising slope with arbitrarily large denominator.

1.1.3 Explicit bounds

In order to classify all slopes as either characterising or non-characterising for a given knot, we are tasked with finding explicit conditions which separate the two categories. McCoy developed such a result for torus knots [McC20]. Incorporating hyperbolic knots and satellite knots, Lackenby proved via a novel geometric approach that every knot has infinitely many characterising slopes in the interval $[-1, 1]$ [Lac19]. This result has recently been extended by Sorya to the following [Sor23].

Theorem 1.5 ([Lac19; McC20; Sor23]). *Every slope p/q with $|q| \geq C(K)$ is characterising for the knot K , where $C(K)$ is some constant that depends on K .*

We can write down an explicit formula for $C(K)$ when K is a torus knot [NZ14; McC20], and Sorya showed that setting $C(K) = 2$ works when K is a composite knot [Sor23]. However, the proof of the existence of $C(K)$ in the general case is non-constructive, which motivates the primary purpose of this article: to find concrete values for $C(K)$ when K is either a hyperbolic knot or a satellite knot with certain hyperbolic features.

1.2 Main results

We will now state each of the theorems that we aim to prove in this thesis, beginning with a value for $C(K)$ where K is a hyperbolic knot and building up to the case where K is a satellite knot for which a piece $Y \subset \mathbb{S}_K^3$ that contains the boundary torus can be equipped with a hyperbolic geometry.

1.2.1 Hyperbolic knots

Given any compact orientable hyperbolic 3-manifold Y with toroidal boundary, let $\text{sys}(Y)$ denote its *systole* (the length of a shortest geodesic in Y). We first introduce the quantity

$$\mathfrak{q}(Y) := \left\lceil \sqrt{6\sqrt{3} \left(2\pi \frac{1.9793}{\text{sys}(Y)} + 28.78 \right)} \right\rceil \in \mathbb{Z}$$

which will arise in our search for $C(K)$. Note that this exceeds 35 when $\text{sys}(Y) < 0.14$.

Our main result is that for any hyperbolic knot K , we have a formula to compute an explicit (though perhaps not optimal) value for $C(K)$. Let $\mathbb{S}_K^3$ denote the complement of K .

Theorem 1.6. *Let K be a hyperbolic knot.*

Then every slope p/q with $|q| \geq C(K) = \max\{35, \mathfrak{q}(\mathbb{S}_K^3)\}$ is characterising for K .

As an example, consider the hyperbolic twist knots, such as the Stevedore knot, 6_1 .

Corollary 1.7. *Let $K = T_t^\pm$ be a hyperbolic twist knot with $|t|$ full twists.*

- *If $|t| \leq 3$, then every slope p/q with $|q| \geq 35$ is characterising for $T_t^\pm$.*
- *If $|t| \geq 4$, then every slope p/q with $|q| \geq \mathfrak{q}(\mathbb{S}_{T_t^\pm}^3)$ is characterising for $T_t^\pm$.*

1.2.2 Hyperbolic patterns

The same idea can be used to address any satellite knot by a hyperbolic pattern $P = U \cup Q$ (where $Q \subset \mathbb{S}^1 \times \mathbb{D}^2$ is a hyperbolic knot in the solid torus which we will use to build the satellite knot). The *winding number* w of the pattern P is taken to mean that of $Q \subset \mathbb{S}^1 \times \mathbb{D}^2$.

Theorem 1.8. *Let $K = P(C)$ be a satellite of a companion C by a hyperbolic pattern $P = U \cup Q$ with winding number w .*

Then every slope p/q with $\gcd(p, w) \neq 1$ and $|q| \geq \max\{35, \mathfrak{q}(\mathbb{S}_P^3)\}$ is characterising for K .

In particular, the result applies to the special case of Whitehead doubles (which have $w = 0$).

Corollary 1.9. *Let $K = W_t^n(C)$ be an n -clasped t -twisted Whitehead double.*

- *If $|n| \leq 4$, then every slope p/q with $|p| \neq 1$ and $|q| \geq 35$ is characterising for K .*
- *If $|n| \geq 5$, then every slope p/q with $|p| \neq 1$ and $|q| \geq \mathfrak{q}(\mathbb{S}_{W_t^n}^3)$ is characterising for K .*

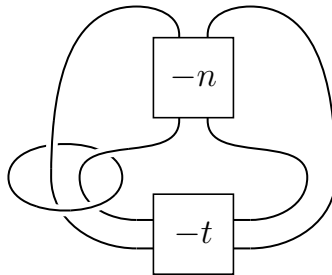


Figure 1.2: The n -clasped t -twisted Whitehead link, $W_t^n = U \cup V_t^n$.

Our strategy uses JSJ decompositions to reduce the problem to one about the pattern P , which can be addressed using lengths of minimal geodesics.

Hyperbolic volume

When the pattern is given by the Whitehead link, we can instead employ an alternative argument which relies on hyperbolic volume: crucially, the Whitehead link complement and its sister are known to be the smallest amongst all 2-cusped orientable hyperbolic 3-manifolds [Ago10]. This leads to a better bound, but relies on the conjectural completeness of the SnapPea census of such manifolds up to a given point.

Theorem 1.10. *Let $K = W_t^\pm(C)$ be a ± 1 -clasped t -twisted Whitehead double for $\pm t \in \{-1, 0, 1, 2\}$.*

Then every slope p/q with $|p| \neq 1$ and $|q| \geq q_{\min}$ is characterising for K , where $q_{\min}$ is some constant determined by the Whitehead link and independent of C .

Furthermore, if the SnapPea census of 2-cusped orientable hyperbolic 3-manifolds is complete up to stage k of Table 1.1 (namely, up to the volume bound V_k , or up to the first a_k items), then we can take $q_{\min}$ to be the corresponding value of q_k .

k	V_k	a_k	q_k
1	4.0464	4	43
2	4.4057	6	32
3	4.5275	8	30
4	4.6842	10	28
5	4.7801	16	27
6	4.8913	18	26
7	5.0212	24	25
8	5.1749	57	24

Table 1.1: The values of V_k , a_k and q_k for each stage k .

Note that we can also work in the opposite direction: starting from any volume bound $V_{\max}$, we can recover the corresponding lower bound on the denominator via the formula

$$q_{\min} := \left\lceil 2\pi \sqrt{\frac{3}{1 - (V_{\max}^{-1} \cdot \text{vol}(\mathbb{S}_W^3))^{2/3}}} \right\rceil \in \mathbb{Z}.$$

Non-characterising slopes

In order to demonstrate the reason for imposing a condition on the numerator of the slope, for each $1/q$ we construct an infinite family of pairs of distinct multiclapsed Whitehead doubles of double twist knots sharing a surgery of slope $1/q$. These are depicted in Figure 1.3 (where the extra twisting compensates for the writhe of the companion knot).

Theorem 1.11. *Let $K = W^n(T_q^m)$ and $K' = W^m(T_q^n)$ for any $q, m, n \in \mathbb{Z} \setminus \{0\}$.*

Then there exists an orientation-preserving homeomorphism $\mathbb{S}_K^3(1/q) \cong \mathbb{S}_{K'}^3(1/q)$.

In particular, when $m \neq n$, $1/q$ is a non-characterising slope for both K and K' .

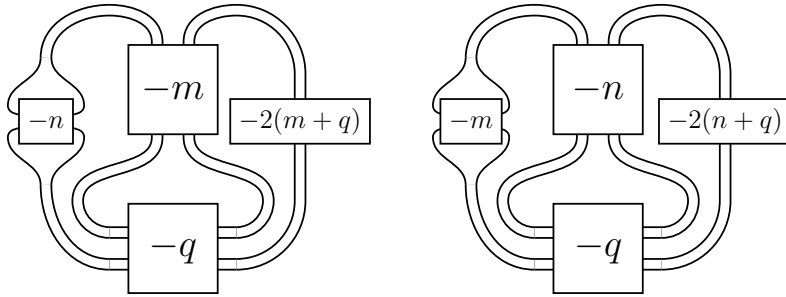


Figure 1.3: A pair of knots sharing a $1/q$ -surgery.

1.2.3 Hyperbolic multicusped outermost JSJ pieces

The minimal geodesics approach can also be used to find characterising slopes for any knot whose complement has a hyperbolic outermost JSJ piece Y with any number $m = |\partial Y|$ of cusps. Theorem 1.6 covers the $m = 1$ case (where K is just a hyperbolic knot) and Theorem 1.8 addresses the $m = 2$ case (where K is a satellite knot by a purely hyperbolic pattern $P = U \cup Q$) under the extra assumption that the winding number w of P satisfies $\gcd(p, w) \neq 1$. If we can find a way to ensure that an orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q)$ restricts to a slope-preserving homeomorphism between the JSJ pieces containing the surgery

curves using only a condition on the denominator, then the same minimal geodesics technique can be used to construct a value for $C(K)$ regardless of m .

In order to do this, we can use hyperbolic volume to obstruct the swapping of JSJ pieces. Suppose that X is a $(k - 1)$ -cusped orientable hyperbolic 3-manifold for which $\text{vol}(X) < \mathcal{V}_k$, where $\mathcal{V}_k$ denotes the least possible volume of all k -cusped orientable hyperbolic 3-manifolds. Then there is a well-defined quantity

$$\mathcal{Q}(X) := \left\lceil 2\pi \sqrt{\frac{3}{1 - (\mathcal{V}_k^{-1} \cdot \text{vol}(X))^{2/3}}} \right\rceil \in \mathbb{Z}$$

(here, the ceiling function is defined as $\lceil x \rceil := \min\{n \in \mathbb{Z} \mid n > x\}$) for which we can show that $|q| \geq \mathcal{Q}(X)$ prevents X from arising as a filling of any k -cusped hyperbolic JSJ piece. We say that a compact orientable 3-manifold M satisfies *Property H* if every hyperbolic JSJ piece $X \subset M$ satisfies this condition, and define $\mathcal{Q}(M)$ to be the maximum of all of the $\mathcal{Q}(X)$. Enforcing a $|q| \geq \mathcal{Q}(M)$ criterion when necessary and drawing on the cyclic surgery theorem [CGLS87] and the minimal geodesics technique, this leads to the following three results.

In the $m = 2$ case, we have a value for $C(K)$ in several different scenarios.

Theorem 1.12. *Let K be a satellite knot whose complement has a 2-cusped hyperbolic outermost JSJ piece Y . Write $K = P(C)$ and $Y = \mathbb{S}_P^3$ for $P = U \cup Q$.*

- *If $P(U) = Q$ is a torus knot, then $C(K) = \max\{35, \mathfrak{q}(\mathbb{S}_P^3)\}$.*
- *If $P(U) = Q$ is an unknot and $\mathbb{S}_C^3$ contains only Seifert fibred JSJ pieces, then $C(K) = \max\{35, \mathfrak{q}(\mathbb{S}_P^3)\}$.*
- *If $\mathbb{S}_C^3$ contains no torus knot complements and satisfies Property H, then $C(K) = \max\{35, \mathfrak{q}(\mathbb{S}_P^3), \mathcal{Q}(\mathbb{S}_C^3)\}$.*

In the $m = 3$ case, we must be careful when there are torus knot complements attached to Y .

Theorem 1.13. *Let K be a satellite knot whose complement has a 3-cusped hyperbolic outermost JSJ piece Y . Let $\mathbb{S}_{C_0}^3$ and $\mathbb{S}_{C_1}^3$ be the knot complements adjacent to Y .*

- *If $\mathbb{S}_{C_0}^3$ and $\mathbb{S}_{C_1}^3$ are not torus knot complements, then $C(K) = \max\{35, \mathfrak{q}(Y)\}$.*
- *If $\mathbb{S}_{C_0}^3$ is a torus knot complement and $\mathbb{S}_{C_1}^3$ is a hyperbolic knot complement satisfying $\text{vol}(\mathbb{S}_{C_1}^3) < \text{vol}(\mathbb{S}_W^3)$, then $C(K) = \max\{35, \mathfrak{q}(Y), \mathcal{Q}(\mathbb{S}_{C_1}^3)\}$.*

Finally, in the $m \geq 4$ case, we have a very strong result.

Theorem 1.14. *Let K be a satellite knot whose complement has an m -cusped hyperbolic outermost JSJ piece Y with $m \geq 4$.*

Then $C(K) = \max\{35, \mathfrak{q}(Y)\}$.

In future work, we hope to address the remaining $m = 2$ and $m = 3$ cases and obtain equally comprehensive results.

1.3 Overview

The structure of this thesis is as follows. In Chapter 2, we review some key concepts and relevant results. In Chapter 3, we use a JSJ decomposition argument to show that any knot sharing the same surgery as our fixed knot must be of a very similar type. In Chapter 4, we employ a minimal geodesic argument to show that in fact the only possibility is for both knots to be exactly the same, thus completing the proofs of Theorem 1.6 and Theorem 1.8. In Chapter 5, we discuss hyperbolic volume and describe our alternative strategy for Whitehead doubles, which proves Theorem 1.10. In Chapter 6, we construct explicit examples of pairs of multiclasp Whitehead doubles sharing a surgery of slope $1/q$ as described in Theorem 1.11. Finally, in Chapter 7, we look at satellite knots whose complements contain a hyperbolic outermost JSJ piece with any number of cusps, pioneering new methods to complete the proofs of Theorem 1.12, Theorem 1.13 and Theorem 1.14 in joint work with Patricia Sorya. All chapters except for the final one also appear in [Wak23].

Chapter 2

Preliminaries

We begin by outlining some of the basic ideas that we will draw on in this article. In particular, we review the different geometries of 3-manifolds and how these manifest for knot complements before and after Dehn filling. Throughout, we use the symbol $\cong$ to denote an orientation-preserving homeomorphism between 3-manifolds and $=$ to denote an ambient isotopy between knots or links. The notation $U \cup Q = U' \cup Q'$ corresponds to an isotopy of links which sends U to U' and Q to Q' . Our 3-manifolds will always be connected and orientable and may have empty or toroidal boundary.

2.1 Satellite knots

Recall that a *knot* K in a 3-manifold M is the image of an embedding $\mathbb{S}^1 \hookrightarrow M$ considered up to ambient isotopy (whilst a *link* L is allowed to have multiple components). We denote mirror pairs of knots by $\pm K$. Drilling out a tubular neighbourhood $\nu(K) \cong \mathbb{S}^1 \times \mathbb{D}^2$ of K from M produces the knot complement $M_K := M \setminus \nu(K)$ (and similarly for links). In this discussion, we usually take our ambient manifold M to be either $\mathbb{S}^3$ or $\mathbb{S}^1 \times \mathbb{D}^2$, occasionally generalising to the complement $\mathbb{S}_{U_m}^3$ of an unlink with $m \geq 0$ components - the complement of the empty unlink is $\mathbb{S}^3$ itself. When K is nullhomologous, we denote its *meridian* by

μ and (Seifert) *longitude* by λ . This will be our canonical choice of generators for $H_1(\partial\nu(K))$; for links, we consider each component in isolation. We say that a simple closed curve $\gamma = p\mu + q\lambda$ on $\partial\nu(K) \cong \mathbb{T}^2$ (considered up to isotopy) has *slope* $p/q \in \mathbb{Q} \cup \{\infty\}$, where $p, q \in \mathbb{Z}$ are a pair of coprime integers and $1/0 := \infty$. We may also refer to γ itself as a slope.

2.1.1 Classification

Knots in $\mathbb{S}^3$ beyond the unknot $U = U_1$ can be broadly classified into three main groups. A *torus knot* $T_{a,b}$ is one which can be drawn on the surface of an unknotted torus, in the form of a simple closed curve winding a times around the meridian and b times around the longitude for some pair of coprime integers $a, b \in \mathbb{Z}$ with $|a| > 1$ and $|b| > 1$; relaxing the coprimeness condition extends this definition to also include *torus links*. A *hyperbolic knot* K by definition is precisely one whose complement $\mathbb{S}_K^3$ admits a complete hyperbolic metric; similarly, we can also define *hyperbolic links*. Together, torus knots and hyperbolic knots form a class of knots called *simple knots*. All other knots are called *satellite knots* and can be constructed iteratively via the following process.

Definition 2.1. *The satellite knot $K = P(C)$ with pattern $P = U \cup Q$ (for a knot $Q \subset \mathbb{S}_U^3 \cong \mathbb{S}^1 \times \mathbb{D}^2$ which is neither contained in any $\mathbb{B}^3 \subset \mathbb{S}_U^3$ nor isotopic to the core of $\mathbb{S}_U^3$) and non-trivial companion knot $C \subset \mathbb{S}^3$ is the image of Q in the oriented homeomorphism $f : \mathbb{S}_U^3 \rightarrow \nu(C)$ which sends the longitude of the solid torus $\mathbb{S}_U^3 \cong \mathbb{S}^1 \times \mathbb{D}^2$ to the longitude of $\nu(C)$.*

Our work investigates both hyperbolic knots and satellite knots which can be expressed in the form $K = P(C)$ for some pattern $P = U \cup Q$ whose complement $\mathbb{S}_P^3$ is hyperbolic or contains a hyperbolic piece.

2.1.2 Whitehead doubles

As a special case, we focus on the *n-clasped t-twisted Whitehead doubles*, $K = W_t^n(C)$, which can be constructed from any companion knot C by applying the *n-clasped t-twisted Whitehead link* pattern, $W_t^n = U \cup V_t^n$, for some integers $n \neq 0$ and $t \in \mathbb{Z}$ (as depicted in Figure 1.2). One key property of this pattern is that it has linking number zero. Note that the component $V_t^n \subset \mathbb{S}_U^3$ is in fact isotopic to the (n, t) -double twist knot, $T_t^n \subset \mathbb{S}^3$. Although n and t are interchangeable for $T_t^n \subset \mathbb{S}^3$, they are uniquely determined for $V_t^n \subset \mathbb{S}_U^3$. When $t = 0$ (which we take unless otherwise stated), T_0^n is unknotted; moreover, the unknotted components of $W^n = U \cup V^n$ can then be exchanged by an isotopy, which will be an important step in our construction of the non-characterising slopes in Theorem 1.11.

Different values of n give non-homeomorphic link complements, which can be distinguished for each $|n|$ by their hyperbolic volumes (t -twisting has no effect). When $n = \pm 1$ (for which we often use the shorthand $\pm$), the complement of $W_t^\pm$ has the joint smallest volume amongst all 2-cusped orientable hyperbolic 3-manifolds [Ago10]: this is essential for our proof of Theorem 1.10. Furthermore, the component $V_t^\pm \subset \mathbb{S}_U^3$ is isotopic to a t -twist knot, $T_t^\pm \subset \mathbb{S}^3$. The values of t in the statement of the theorem come from those $T_t^\pm$, shown in Figure 1.1, for which all non-integral slopes are known to be characterising.

2.1.3 Cables and composites

The (r, s) -cable $K = J_{r,s}(C)$ of a knot C is the satellite knot with companion knot C and pattern $J_{r,s} = U \cup T_{r,s}$, where the torus knot $T_{r,s} \subset \mathbb{S}_U^3$ can be isotoped to lie on $\partial \mathbb{S}_U^3 \cong \mathbb{T}^2$ in the standard way. Note that our convention is to take the winding number of the torus knot $T_{r,s}$ inside $\mathbb{S}_U^3$ to be $|s|$ rather than r . We include patterns with $|r| = 1$ (despite $T_{1,s}$ being unknotted), but we always require $|s| > 1$ (otherwise the pattern is trivial).

Observe that there are often multiple ways to express a satellite knot in the form $P(C)$: for instance, a composite knot $C_0 \# C_1$ can be written either as a satellite $P_1(C_0)$ of C_0 by a pattern P_1 with $P_1(U) = C_1$, or as a satellite $P_0(C_1)$ of C_1 by a pattern P_0 with $P_0(U) = C_0$. Meanwhile, the satellite knots $J_{r,s}(W(C))$ can alternatively be thought of as $(J_{r,s}(W))(C)$.

2.2 Geometry of 3-manifolds

In three dimensions, topology and geometry are closely entwined. There are two important classes of 3-manifolds which form the building blocks for this relationship.

2.2.1 Seifert fibred spaces

Firstly, we define *Seifert fibred spaces*, which admit an $\mathbb{S}^1$ -fibration over an orbifold base.

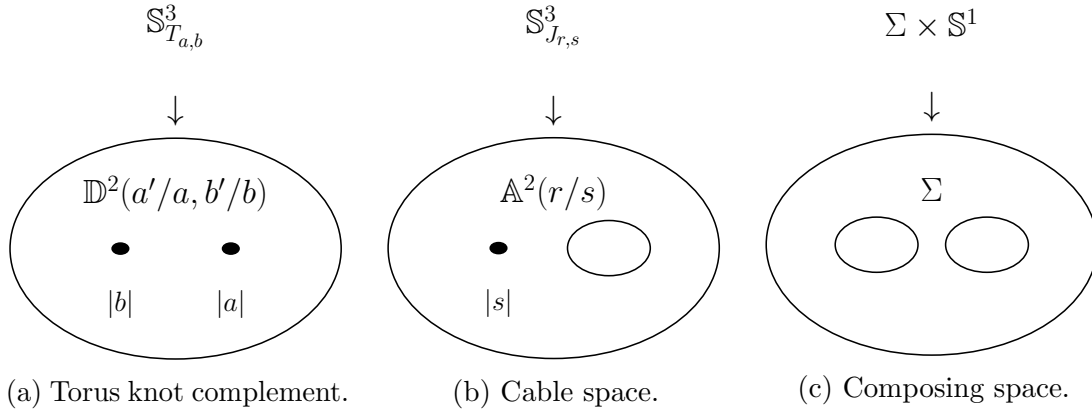


Figure 2.1: Seifert fibred spaces.

Definition 2.2. A Seifert fibred space is an $\mathbb{S}^1$ -bundle over an orbifold $\Sigma(\alpha_1/\beta_1, \dots, \alpha_n/\beta_n)$ for which each $\mathbb{S}^1$ -fibre admits a tubular neighbourhood that is homeomorphic to the $\mathbb{S}^1$ -fibred mapping torus of a $2\pi t$ -twist automorphism of $\mathbb{D}^2$ for some $t \in \mathbb{Q}$. The central $\mathbb{S}^1$ -fibre is called regular when $t \in \mathbb{Z}$ and exceptional of order $|\beta_i| > 1$ when $t = \alpha_i/\beta_i \notin \mathbb{Z}$ for each pair of coprime integers $\alpha_i, \beta_i \in \mathbb{Z}$.

For example, the complement $\mathbb{S}_{T_{a,b}}^3$ of the torus knot $T_{a,b}$ is a Seifert fibred space over the disc orbifold $\mathbb{D}^2(a'/a, b'/b)$, which has two exceptional fibres of orders $|a|$ and $|b|$; the Seifert invariants a' and b' satisfy the relationship $|ab' + ba'| = 1$. Similarly, the complement $\mathbb{S}_{J_{r,s}}^3$ of the (r, s) -cable pattern $J_{r,s}$ is a Seifert fibred space over the annulus orbifold $\mathbb{A}^2(r/s)$, which has one exceptional fibre of order $|s|$ and Seifert invariant r representing the behaviour of the fibre in the other direction - this manifold is called a *cable space*. As a final key example, a Seifert fibred space over a planar surface $\Sigma = \mathbb{S}^2 \setminus \sqcup_{i=1}^{m \geq 3} \mathbb{D}^2$ with no exceptional fibres is called a *composing space*.

2.2.2 Hyperbolic 3-manifolds

Secondly, we define *hyperbolic 3-manifolds*, which we will later revisit in more detail.

Definition 2.3. A hyperbolic 3-manifold $M \cong \mathbb{H}^3/\Gamma$ is the quotient of hyperbolic 3-space $\mathbb{H}^3$ by a freely-acting discrete subgroup Γ of the orientation-preserving isometry group $\text{Isom}^+(\mathbb{H}^3) \cong \text{PSL}(2, \mathbb{C})$.

For instance, the figure eight knot complement can be built from gluing two ideal tetrahedra in $\mathbb{H}^3$ [Thu80, Chapter 1], whilst the Whitehead link complement is constructed from a single ideal octahedron in $\mathbb{H}^3$ [Thu80, Chapter 3].

Theorem 2.4 (Hyperbolisation theorem [Thu82, Theorem 2.3]). *Let M be a compact orientable 3-manifold with toroidal boundary. Then M admits a hyperbolic structure if and only if it is irreducible and atoroidal with infinite $\pi_1(M)$.*

This hyperbolic structure has finite volume provided M is neither $\mathbb{S}^1 \times \mathbb{D}^2$ nor $\mathbb{T}^2 \times \mathbb{I}$. By Mostow rigidity [Mos73], the geometry of a finite volume hyperbolic 3-manifold is an invariant of its homeomorphism type, and thus gives us a tool for identification.

2.2.3 The JSJ decomposition theorem

Note that a finite volume hyperbolic 3-manifold cannot simultaneously be a Seifert fibred space. However, a compact orientable 3-manifold M can always be built from a combination of these types of pieces by first invoking the prime decomposition theorem [Kne29; Mil62] and then applying the following result to each irreducible constituent.

Theorem 2.5 (JSJ decomposition theorem [JS79; Joh79]). *Let M be a compact orientable irreducible 3-manifold with empty or toroidal boundary. Then there exists a minimal finite collection of essential tori in M (unique up to isotopy) called the JSJ tori such that each of the JSJ pieces formed by cutting M along these tori is either atoroidal or a Seifert fibred space.*

When M is a satellite knot complement, we can determine exactly what the JSJ pieces look like. Crucially, the JSJ tori correspond to those appearing naturally in the satellite construction.

Theorem 2.6 ([Bud06]). *Let $K \subset \mathbb{S}^3$ be a satellite knot. Then the JSJ pieces of $\mathbb{S}_K^3$ can be classified into the following types.*

1. *A Seifert fibred space over the disc orbifold $\mathbb{D}^2(a'/a, b'/b)$: namely, a torus knot complement, $\mathbb{S}_{T_{a,b}}^3$.*
 - *This JSJ piece cannot contain $\partial\nu(K)$ because K is a satellite knot.*
 - *Regular fibres have slope ab on the boundary torus.*
2. *A Seifert fibred space over the annulus orbifold $\mathbb{A}^2(r/s)$: namely, a cable space, $\mathbb{S}_{J_{r,s}}^3$.*
 - *If this JSJ piece contains $\partial\nu(K)$, then K is a cable knot.*
 - *Regular fibres have slope rs on $\partial\nu(K)$ and slope r/s on the other boundary torus.*

3. A Seifert fibred space over the planar surface $\Sigma = \mathbb{S}^2 \setminus \sqcup_{i=1}^{m \geq 3} \mathbb{D}^2$: namely, a composing space.

- If this JSJ piece contains $\partial\nu(K)$, then K is a composite knot.
- Regular fibres have meridional slope on all boundary tori including $\partial\nu(K)$.

4. A hyperbolic 3-manifold that is homeomorphic to a hyperbolic link complement, $\mathbb{S}_L^3$, such that removing one specific component of the link $L \subset \mathbb{S}^3$ leaves an unlink (possibly the empty link).

- If this JSJ piece contains $\partial\nu(K)$, then K corresponds to this specific component of L .

2.3 Dehn surgery

For the purpose of introducing our notation, we briefly recall the definition of *Dehn surgery*.

2.3.1 Definition and notation

If M' is a compact orientable 3-manifold, then *Dehn filling* describes the process of attaching a solid torus $\mathbb{S}^1 \times \mathbb{D}^2$ to a toroidal boundary component $\mathbb{T}^2 \subset \partial M'$ via a homeomorphism

$$\phi : \partial(\mathbb{S}^1 \times \mathbb{D}^2) \rightarrow \mathbb{T}^2 \subset \partial M', \quad \phi(\{pt\} \times \partial\mathbb{D}^2) = \gamma,$$

where $\gamma \subset \mathbb{T}^2$ is some simple closed curve. We may write $M'(\gamma)$ for the resulting manifold. In particular, when K is a nullhomologous knot inside a compact orientable 3-manifold M , we can glue the solid torus $\nu(K) \cong \mathbb{S}^1 \times \mathbb{D}^2$ back into the knot

complement $M' = M_K$ by a homeomorphism

$$\phi : \partial\nu(K) \rightarrow \mathbb{T}^2 \subset \partial M_K, \quad \phi(\{pt\} \times \partial\mathbb{D}^2) = p\mu + q\lambda,$$

where μ and λ denote the meridian and longitude of $\mathbb{T}^2$ induced by K and $p, q \in \mathbb{Z}$ are some pair of coprime integers. The resulting manifold is determined by the slope p/q of $\gamma = p\mu + q\lambda$, so we denote it by $M_K(p/q)$.

This process can be extended to links $L \subset M$ in the obvious way: we first choose a meridian and longitude for each component of L by considering them in isolation as knots in M , and then we can write

$$M_{L_0, \dots, L_{m-1}}(p_0/q_0, \dots, p_{m-1}/q_{m-1})$$

for Dehn surgery on an m -component link $L = L_0 \cup \dots \cup L_{m-1}$. For an unfilled i^{th} component, we may write $*$ instead of p_i/q_i , or omit the coefficient altogether when the meaning is clear from the context. In this article, whenever $L = U_{m-1} \cup J$, an unspecified filling component will always be taken to mean J . We will be particularly interested in performing Dehn surgery on 2-component links $P = U \cup Q$; the same convention then allows us to write

$$\mathbb{S}_P^3(p/q) = \mathbb{S}_{U,Q}^3(*, p/q)$$

without specifying the filling component.

2.3.2 Background results

There are several results concerning Dehn surgery on knots in $\mathbb{S}^3$ and $\mathbb{S}^1 \times \mathbb{D}^2$ which we will use in this article. An important quantity to consider when comparing Dehn fillings is the *distance* Δ between a pair of slopes, which is precisely their minimal intersection number when drawn as simple closed curves on $\mathbb{T}^2$. In particular, note

that $\Delta(p/q, 1/0) = |q|$: this is the reason behind the conditions on the denominator that appear in the following results.

First, we consider surgeries on a hyperbolic knot in $\mathbb{S}^3$.

Theorem 2.7. *Let $K \subset \mathbb{S}^3$ be hyperbolic. If $|q| \geq 2$, then $\mathbb{S}_K^3(p/q)$ is irreducible. If $|q| \geq 3$, then $\mathbb{S}_K^3(p/q)$ is atoroidal. If $|q| \geq 9$, then $\mathbb{S}_K^3(p/q)$ is hyperbolic.*

Proof. This follows from the maximal distances between exceptional fillings [GL87; GL95; LM13]. $\square$

A similar result holds when the ambient manifold is the complement of an $(m-1)$ -component unlink U_{m-1} (where $m \geq 2$).

Theorem 2.8 ([Lac19, Theorem 2.8]). *Let $J \subset \mathbb{S}_{U_{m-1}}^3$ be hyperbolic and write $L = U_{m-1} \cup J$. If $|q| \geq 2$, then $\mathbb{S}_L^3(p/q)$ is irreducible with incompressible boundary. If $|q| \geq 3$, then $\mathbb{S}_L^3(p/q)$ is hyperbolic.*

Proof. This follows from Theorem 2.4 by obstructing essential surfaces inside $\mathbb{S}_L^3(p/q)$. $\square$

Moving on to torus knots, we summarise the complete classification of surgeries.

Theorem 2.9 ([Mos71]). *Let $K = T_{r,s}$ be the (r, s) -torus knot in $\mathbb{S}^3$. Then*

$$\mathbb{S}_{T_{r,s}}^3(p/q) \cong \begin{cases} L(r, s) \# L(s, r) & p = rs, q = 1 \\ L(|p|, qs^2) & p = qrs \pm 1, \\ M & \text{otherwise,} \end{cases}$$

where M is a Seifert fibred space over $\mathbb{S}^2$ with three exceptional fibres of orders $|r|$, $|s|$ and $|p - qrs|$.

Viewing the torus knot $T_{r,s}$ as the (r, s) -cable of the unknot, the following result can be considered a generalisation of this.

Theorem 2.10 ([Gor83]). *Let $K = J_{r,s}(C)$ be the (r, s) -cable of a knot C in $\mathbb{S}^3$.*

Then

$$\mathbb{S}_{J_{r,s}(C)}^3(p/q) \cong \begin{cases} \mathbb{S}_C^3(r/s) \# L(s, r) & p = rs, q = 1 \\ \mathbb{S}_C^3(p/qs^2) & p = qrs \pm 1, \\ \mathbb{S}_C^3 \cup_{\mathbb{T}^2} M' & \text{otherwise,} \end{cases}$$

where M' is a Seifert fibred space over $\mathbb{D}^2$ with two exceptional fibres of orders $|s|$ and $|p - qrs|$.

2.3.3 Cosmetic surgeries

We also note the following advancement towards the cosmetic surgery conjecture.

Theorem 2.11 ([NW15]). *Let $K \subset \mathbb{S}^3$ and suppose that $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_K^3(p'/q')$.*

Then $p/q = \pm p'/q'$.

Combining the previous two results gives the following corollary, which will prove to be very useful when considering cable knots.

Corollary 2.12. *Let $K' = J_{r,s}(K)$ be an (r, s) -cable of K and let p/q be a slope with $|p - qrs| = 1$.*

Then there is no orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q)$.

Proof. Suppose otherwise, so that $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q) \cong \mathbb{S}_K^3(p/qs^2)$ (as $|p - qrs| = 1$). Then the slopes p/q and p/qs^2 form a pair of cosmetic surgeries on K . By Theorem 2.11, this is impossible because $|s| > 1 \implies p/q \neq \pm p/qs^2$. $\square$

We will next see how we can combine these results with Theorem 2.6 in order to investigate and compare the manifolds produced by performing Dehn surgery on knots.

Chapter 3

JSJ decompositions

In this chapter, we will use JSJ decompositions to gather crucial information about when we can have an orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p'/q')$ between manifolds obtained by Dehn surgery on knots K and K' . In particular, we will fix K to be either a hyperbolic knot or a satellite knot $P(C)$ by a hyperbolic pattern $P = U \cup Q$ and show that this forces the other knot to be of a very similar form. For the time being, we allow the surgery slopes p/q and p'/q' to differ, yielding more general results. Notation of the form $\mathbb{S}_P^3(p/q)$ always corresponds to filling the component corresponding to Q rather than U .

3.1 Creation and compression

Given a knot K , it is very easy to describe the JSJ decomposition of the complement $\mathbb{S}_K^3$; however, after Dehn filling to produce $\mathbb{S}_K^3(p/q)$, it is not always the case that the JSJ tori are formed precisely from those of $\mathbb{S}_K^3$. There are two ways in which the JSJ decomposition can change: creation of new JSJ tori and compression of existing JSJ tori. We will see that the possibility of creation of new JSJ tori can be immediately ruled out by implementing the assumption that $|q| \geq 3$, and that compression of existing JSJ tori only occurs in one very specific situation. Note that, in order for a

JSJ torus to compress during Dehn filling, the process must cause it to either bound some solid torus $\mathbb{S}^1 \times \mathbb{D}^2$ or cobound (with another JSJ torus) some thickened torus $\mathbb{T}^2 \times \mathbb{I}$.

3.1.1 The surgered piece

The following result gives a precise description of the JSJ decomposition of $\mathbb{S}_K^3(p/q)$, demonstrating that at most one JSJ torus can compress during filling and that it must come from a cable space (see also [Lac19] and [Sor23]).

Proposition 3.1. *Let K be a knot with outermost JSJ piece Y and let p/q be a slope with $|q| \geq 3$.*

Then the JSJ piece of $\mathbb{S}_K^3(p/q)$ which contains the surgery curve takes one of the following forms:

(i) $Y(p/q)$;

(ii) $\hat{Y}(p/qs^2) \iff Y$ is an (r, s) -cable space next to a JSJ piece $\hat{Y}$ and $|p - qrs| = 1$.

Proof. Let Y be the outermost JSJ piece of $\mathbb{S}_K^3$. We will use the classification of JSJ pieces in Theorem 2.6 to consider each possibility in turn.

Case (1) arises when $Y = \mathbb{S}_{T_{a,b}}^3$ for a torus knot $K = T_{a,b}$. Since $|q| \geq 2$, we can ignore the $|p - qrs| = 0$ case of Theorem 2.9 and observe that $Y(p/q) = \mathbb{S}_K^3(p/q)$ is irreducible and atoroidal. We deduce that $Y(p/q)$ is still a single Seifert fibred JSJ piece.

Case (4) happens when Y is the complement of a certain type of hyperbolic link. Since $|q| \geq 3$, $Y(p/q)$ is irreducible and atoroidal with incompressible boundary, so it remains a single JSJ piece and none of its boundary tori compress. Note that in the case where K is a hyperbolic knot, a stronger assumption of $|q| \geq 9$ is required to ensure that $Y(p/q) = \mathbb{S}_K^3(p/q)$ remains hyperbolic; the $|q| \geq 3$ assumption allows the possibility that it is Seifert fibred, but this does not affect the conclusion.

Case (3) takes place when Y is a composing space. Regular fibres have meridional slope on the boundary of Y , so $|q| \geq 1$ implies that the Seifert fibration extends to $Y(p/q)$. Since this adds an exceptional fibre of order $|q| \geq 2$, the boundary tori do not compress; moreover, the Seifert fibred structure is otherwise unchanged and unique, so it cannot extend over any adjacent JSJ pieces for precisely the reason that it did not before filling. Hence $Y(p/q)$ is a single Seifert fibred JSJ piece.

Case (2) occurs when $Y = \mathbb{S}_{J_{r,s}}^3$ is a cable space and $K = J_{r,s}(C)$ is a cable knot. Since $|q| \geq 2$, we can ignore the $|p - qrs| = 0$ case of Theorem 2.10. When $|p - qrs| > 1$, the Seifert fibred structure on Y extends to $Y(p/q)$ over the new solid torus, introducing a new exceptional fibre of order $|p - qrs| > 1$ (which prevents compression of the boundary tori). As the Seifert fibred structure is unique (unless $|s| = |p - qrs| = 2$: see [Sor23, Proposition 3.6] for a more detailed discussion of this), there is no way for it to extend over any other attached JSJ pieces, and so $Y(p/q)$ is a single Seifert fibred JSJ piece.

By Theorem 2.10, compression happens precisely when $|p - qrs| = 1$, in which case we have $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_C^3(p/qs^2)$. However, letting $\hat{Y}$ be the outermost JSJ piece of $\mathbb{S}_C^3$, we will show that the same thing cannot happen again. If $C = J_{r',s'}(C')$ and $|p - qr's's^2| = 1$, then

$$0 \leq 4|r - r's's| \leq |q||s||r - r's's| = |qrs - qr's's^2| \leq |p - qrs| + |p - qr's's^2| = 2$$

and we obtain $r/s = r's' \in \mathbb{Z}$, a contradiction. Hence the cable cannot be iterated and there is no further compression of tori. Since $|qs^2| > |q| \geq 3$ and the JSJ piece $\hat{Y}$ is not a compressing cable space, we can simply apply the same arguments as we did for each of the other possibilities for $\hat{Y}$ to deduce that the JSJ piece of $\mathbb{S}_K^3(p/q)$ containing the surgery curve in this case is precisely $\hat{Y}(p/qs^2)$. $\square$

3.1.2 The JSJ piece $\tilde{Y}$

We will often use the notation $\tilde{Y}(\tilde{p}/\tilde{q})$ to represent the JSJ piece containing the surgery curve. This encompasses both cases and reflects the fact that the JSJ piece $\tilde{Y}$ will be the one which is most important in our discussion. Moreover, the slope $\tilde{p}/\tilde{q}$ satisfies $|\tilde{q}| \geq |q|$, so it is convenient to note that lower bounds on the denominator automatically apply.

3.2 Hyperbolic knots

Now we will suppose that there is an orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p'/q')$ and use what we know about JSJ decompositions to compare K and K' .

3.2.1 Restrictions on the unknown knot

Let's begin by considering at the scenario in which K is a fixed hyperbolic knot. Proposition 3.1 provides enough information to determine the basic type of the unknown knot K' .

Proposition 3.2. *Let K be a hyperbolic knot. Suppose there is an orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p'/q')$ for some knot K' and some slopes $p/q, p'/q'$ with $|q| \geq 9$ and $|q'| \geq 3$.*

Then one of the following holds:

(i) K' is hyperbolic;

(ii) $K' = J_{r,s}(\hat{K})$ is the (r, s) -cable of a hyperbolic knot $\hat{K}$, $|p' - q'rs| = 1$ and $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{\hat{K}}^3(p'/q's^2)$.

Proof. Since K is a hyperbolic knot and $|q| \geq 9$, the filling $\mathbb{S}_K^3(p/q)$ remains hyperbolic by Theorem 2.7 and contains no JSJ tori. We have an orientation-preserving

homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p'/q')$ for some other knot K' whose complement $\mathbb{S}_{K'}^3$ has outermost JSJ piece Y' . By Proposition 3.1, the JSJ piece of $\mathbb{S}_{K'}^3(p'/q')$ containing the surgery curve - in this case, the only JSJ piece - is of the form $\tilde{Y}(\tilde{p}/\tilde{q})$; since this is hyperbolic, $\tilde{Y}$ must also be hyperbolic. Since $|\partial\tilde{Y}| = 1$, we must have $\tilde{Y} \cong \mathbb{S}_{\tilde{K}}^3$ for some hyperbolic knot $\tilde{K}$. Reconciling $\tilde{Y}$ with Y' and $\hat{Y}$ in Proposition 3.1 gives the two cases. $\square$

3.2.2 The hyperbolic knot $\tilde{K}$

We will write $(\tilde{K}, \tilde{p}/\tilde{q})$ to mean in case (i) $(K', p'/q')$ and in case (ii) $(\hat{K}, p'/q's^2)$. In both situations, $\tilde{K}$ is hyperbolic and there is an orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{\tilde{K}}^3(\tilde{p}/\tilde{q})$.

3.3 Hyperbolic patterns

We would like an analogous result for satellite knots which can be expressed as $K = P(C)$ for a hyperbolic pattern $P = U \cup Q$. However, as $\mathbb{S}_K^3(p/q)$ will now comprise more than one JSJ piece, there is an extra obstacle of checking that the orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p'/q')$ restricts to one between the JSJ pieces containing the surgery curves.

Recall the following result of Gordon.

Lemma 3.3 ([Gor83]). *Let $P = U \cup Q$ be any pattern with winding number w .*

Then $H_1(\mathbb{S}_P^3(p/q)) \cong \mathbb{Z} \oplus \mathbb{Z}/\gcd(p, w)\mathbb{Z}$ and the nullhomologous curve on $\partial\mathbb{S}_P^3(p/q)$ has slope

$$\begin{cases} p/qw^2 & \text{if } w \neq 0, \\ 1/0 & \text{if } w = 0. \end{cases}$$

Proof. Observe that $H_1(\mathbb{S}_P^3) \cong \mathbb{Z} \oplus \mathbb{Z}$ is freely generated by the meridians $[\mu_U]$ and $[\mu_Q]$. Moreover, note that $[\lambda_U] = w[\mu_Q]$ and $[\lambda_Q] = w[\mu_U]$ in $H_1(\mathbb{S}_P^3)$. After

Dehn filling the component corresponding to Q along slope p/q , we see that $p[\mu_Q] + q[\lambda_Q] = 0$ in $H_1(\mathbb{S}_P^3(p/q))$. When $w = 0$, $p[\mu_Q]$ vanishes; since $\gcd(p, 0) = |p|$, we obtain $H_1(\mathbb{S}_P^3(p/q)) \cong \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$, and we still have $[\lambda_U] = 0$ so the nullhomologous curve has slope $1/0$. When $w \neq 0$, we have $p[\mu_Q] + qw[\mu_U] = 0$, so $H_1(\mathbb{S}_P^3(p/q)) \cong \mathbb{Z} \oplus \mathbb{Z}/\gcd(p, w)\mathbb{Z}$; since $p[\lambda_U] + qw^2[\mu_U] = 0$, the nullhomologous curve has slope p/qw^2 . $\square$

Consequently, $\mathbb{S}_P^3(p/q)$ can only be homeomorphic to a knot complement when $\gcd(p, w) = 1$. This is significant when it is a single JSJ piece because by Theorem 2.6, the only possible JSJ pieces of a satellite knot complement with a single boundary component take the form of simple knot complements.

3.3.1 Restrictions on the unknown pattern

We will now see that the homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p'/q')$ restricts to a *slope-preserving* homeomorphism $\mathbb{S}_P^3(p/q) \cong \mathbb{S}_{P'}^3(p'/q')$ between the filled pattern pieces (meaning that the meridian and longitude - and hence all other slopes - on the boundary torus are identified).

Proposition 3.4. *Let $K = P(C)$ be a satellite of a knot C by a hyperbolic pattern $P = U \cup Q$ with winding number w . Suppose there is an orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p'/q')$ for some knot K' and some slopes $p/q, p'/q'$ with $\gcd(p, w) \neq 1$ and $|q|, |q'| \geq 3$.*

Then $K' = P'(C)$ is a satellite of the same knot C by a pattern P' and there is a slope-preserving homeomorphism $\mathbb{S}_P^3(p/q) \cong \mathbb{S}_{P'}^3(p'/q')$, where one of the following holds:

- (i) $P' = U' \cup Q'$ is hyperbolic;
- (ii) $P' = J_{r,s}(\hat{P})$ is the (r, s) -cable of a hyperbolic pattern $\hat{P} = \hat{U} \cup \hat{Q}$, $|p' - q'rs| = 1$ and there is a slope-preserving homeomorphism $\mathbb{S}_P^3(p/q) \cong \mathbb{S}_{\hat{P}}^3(p'/q's^2)$.

Proof. Since the outermost JSJ piece $Y = \mathbb{S}_P^3$ of $\mathbb{S}_K^3$ is hyperbolic and $|q| \geq 3$, Proposition 3.1 tells us that the JSJ piece containing the surgery curve is $Y(p/q) = \mathbb{S}_P^3(p/q)$. Consider the JSJ piece of $\mathbb{S}_{K'}^3$ containing the surgery curve, which is of the form $\tilde{Y}(\tilde{p}/\tilde{q})$ by Proposition 3.1. The homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p'/q')$ preserves the JSJ decompositions, so there must be at least one JSJ torus present in $\mathbb{S}_{K'}^3(p/q)$. If K' were a simple knot, so that $\tilde{Y} = \mathbb{S}_{K'}^3$, then $\tilde{Y}(\tilde{p}/\tilde{q}) = \mathbb{S}_{K'}^3(p'/q')$ and there are no JSJ tori. We deduce that K' is not a simple knot and therefore must be a satellite knot.

Since $\gcd(p, w) \neq 1$, Lemma 3.3 implies that the JSJ piece $\mathbb{S}_P^3(p/q)$ cannot be homeomorphic to a knot complement. By Theorem 2.6, the only possible JSJ pieces of $\mathbb{S}_{K'}^3$ with a single boundary component are simple knot complements, so the homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p'/q')$ must take $\mathbb{S}_P^3(p/q)$ to the JSJ piece $\tilde{Y}(\tilde{p}/\tilde{q})$ that was created during Dehn filling. Therefore we also have an orientation-preserving homeomorphism from the companion knot complement $\mathbb{S}_C^3$ to some other knot complement $\mathbb{S}_{C'}^3 \subset \mathbb{S}_{K'}^3$, which is not affected by the filling. Since knots are determined by their complements [GL89], it follows that $C = C'$ and additionally that the homeomorphism $\mathbb{S}_C^3 \cong \mathbb{S}_{C'}^3$ is slope-preserving. We can now express the satellite knot as $K' = P'(C)$ for a pattern P' such that $\tilde{Y}(\tilde{p}/\tilde{q}) = \mathbb{S}_{P'}^3(p'/q')$ and there is a slope-preserving homeomorphism $\mathbb{S}_P^3(p/q) \cong \mathbb{S}_{P'}^3(p'/q')$.

By Proposition 3.1, $\mathbb{S}_{P'}^3(p'/q') = \tilde{Y}(\tilde{p}/\tilde{q})$ for some JSJ piece $\tilde{Y} \subset \mathbb{S}_{K'}^3$; since this is hyperbolic by Theorem 2.8, $\tilde{Y}$ must also be hyperbolic. Since $|\partial\tilde{Y}| = 2$, we must have $\tilde{Y} \cong \mathbb{S}_{\tilde{P}}^3$ for some pattern $\tilde{P} = \tilde{U} \cup \tilde{Q}$. The two options in Proposition 3.1 yield each of the possible pattern types. $\square$

3.3.2 The hyperbolic pattern $\tilde{P}$

We will henceforth use $(\tilde{P}, \tilde{p}/\tilde{q})$ to denote in case (i) $(P', p'/q')$ and in case (ii) $(\hat{P}, p'/q's^2)$. Thus $\tilde{P} = \tilde{U} \cup \tilde{Q}$ is a hyperbolic pattern and there is a slope-preserving homeomorphism $\mathbb{S}_P^3(p/q) \cong \mathbb{S}_{\tilde{P}}^3(\tilde{p}/\tilde{q})$.

3.4 Pattern properties

Before moving onto the core of the proofs of our main theorems, we take a closer look at what a slope-preserving homeomorphism $\mathbb{S}_P^3(p/q) \cong \mathbb{S}_{P'}^3(p'/q')$ means for the patterns.

3.4.1 Winding number

We begin by looking at the special case of patterns with winding number zero.

Lemma 3.5. *Let $P = U \cup Q$ be any pattern with winding number $w = 0$.*

If there is a slope-preserving homeomorphism $\mathbb{S}_P^3(p/q) \cong \mathbb{S}_{P'}^3(p'/q')$, for some pattern $P' = U' \cup Q'$ and some slopes $p/q, p'/q'$, then P' has winding number $w' = 0$.

If $P' = J_{r,s}(\hat{P})$ for some pattern $\hat{P} = \hat{U} \cup \hat{Q}$ and $|p' - q'rs| = 1$, then $\hat{P}$ has winding number $\hat{w} = 0$.

Proof. Since our homeomorphism is slope-preserving, it should take nullhomologous curves to nullhomologous curves. When $w = 0$, the nullhomologous curve is precisely the meridian by Lemma 3.3. This forces $w' = 0$.

If $P' = J_{r,s}(\hat{P})$ and $|p' - q'rs| = 1$, then the cable space compresses and we have a slope-preserving homeomorphism $\mathbb{S}_P^3(p/q) \cong \mathbb{S}_{\hat{P}}^3(p'/q's^2)$. We conclude by the same argument that $\hat{w} = 0$. $\square$

3.4.2 Application to the unknot

We now set $p'/q' = p/q$ and use the Dehn surgery characterisation of the unknot U , trefoils $\pm T$, figure eight knot S and 5_2 knots $\pm R$ to gain information about any pattern which produces one of these knots when applied to the unknot.

Lemma 3.6. *Let $P = U \cup Q$ be a pattern with $Q \in \{U, \pm T, S, \pm R\}$ in $\mathbb{S}^3$.*

If there is a slope-preserving homeomorphism $\mathbb{S}_P^3(p/q) \cong \mathbb{S}_{P'}^3(p/q)$, for some pattern $P' = U' \cup Q'$ and some non-integral slope p/q , then $Q = Q'$ in $\mathbb{S}^3$.

If $P' = J_{r,s}(\hat{P})$ for some pattern $\hat{P} = \hat{U} \cup \hat{Q}$ and $|p - qrs| = 1$, then $Q \in \{U, \pm T\}$ and $\hat{Q} = U$ in $\mathbb{S}^3$.

Proof. Apply both patterns P and P' to the unknot by gluing a solid torus to the exterior of each filled pattern piece in the standard way. This yields a homeomorphism $\mathbb{S}_Q^3(p/q) \cong \mathbb{S}_{Q'}^3(p/q)$. However, we know that every slope p/q is characterising for Q - by [KMOS07] for U , by [OS19] for $\pm T$ and S and by [BS22] for $\pm R$ (provided $|q| \geq 2$). Therefore Q' must be isotopic to the same knot in $\mathbb{S}^3$.

If $P' = J_{r,s}(\hat{P})$, then $Q = Q' = J_{r,s}(\hat{Q})$ in $\mathbb{S}^3$. The only way for this to hold when $Q = U$ is if $\hat{Q} = U$ and $|r| = 1$. When $Q = \pm T$, we can have $\hat{Q} = U$ and $(r, s) \in \{(\pm 2, 3), (\pm 3, 2)\}$. We cannot have $Q = S$, nor $Q = \pm R$, because there is no way to express these hyperbolic knots as non-trivial cables. $\square$

Chapter 4

Shortest geodesics

The aim of this chapter is to show that, if $L = U_{m-1} \cup J$ and $\tilde{L} = \tilde{U}_{m-1} \cup \tilde{J}$ for $m-1 \geq 0$, then the existence of a slope-preserving homeomorphism $\mathbb{S}_L^3(p/q) \cong \mathbb{S}_{\tilde{L}}^3(\tilde{p}/\tilde{q})$ implies that $J = \tilde{J}$. We can do this by analysing shortest geodesics before and after Dehn filling. Setting $(J, \tilde{J})$ to be $(K, \tilde{K})$ when $m = 1$ and $(Q, \tilde{Q})$ when $m = 2$, this will enable us to complete the proofs of Theorems 1.6 and 1.8.

Theorem 1.6. *Let K be a hyperbolic knot.*

Then every slope p/q with $|q| \geq C(K) = \max\{35, \mathfrak{q}(\mathbb{S}_K^3)\}$ is characterising for K .

Theorem 1.8. *Let $K = P(C)$ be a satellite of a companion C by a hyperbolic pattern $P = U \cup Q$ with winding number w .*

Then every slope p/q with $\gcd(p, w) \neq 1$ and $|q| \geq \max\{35, \mathfrak{q}(\mathbb{S}_P^3)\}$ is characterising for K .

The argument goes as follows. We know that the slope-preserving homeomorphism $\mathbb{S}_L^3(p/q) \cong \mathbb{S}_{\tilde{L}}^3(\tilde{p}/\tilde{q})$ is homotopic to an isometry by Mostow rigidity, and therefore preserves shortest geodesics. Furthermore, if $|q| \geq \max\{35, \mathfrak{q}(\mathbb{S}_L^3)\}$, then the shortest geodesic in $\mathbb{S}_L^3(p/q)$ is the core curve c_J of the filling. This is because the geometry of $\mathbb{S}_L^3(p/q)$ away from the core curve looks very similar to that of $\mathbb{S}_L^3$, in which all geodesics are much longer by our hypothesis. By considering the corresponding

geodesics in $\mathbb{S}_L^3(\tilde{p}/\tilde{q})$, it once again transpires that the shortest geodesic can only be the core curve $c_{\tilde{J}}$ of the filling. We now have a slope-preserving homeomorphism which takes J to $\tilde{J}$; it follows that $L = \tilde{L}$.

4.1 Hyperbolic geometry

Recall that a hyperbolic 3-manifold M is a Riemannian manifold locally modelled on $\mathbb{H}^3$, which can be viewed as the upper half-space $\{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$ equipped with the metric ds^2/z^2 , where ds^2 is the usual Euclidean metric on $\mathbb{R}^3$. Here, geodesics are semicircles perpendicular to the boundary, which become vertical lines when one end lies at infinity. We can express M as the quotient of $\mathbb{H}^3$ by a freely-acting discrete subgroup Γ of $\text{Isom}^+(\mathbb{H}^3) \cong \text{PSL}(2, \mathbb{C})$.

A *horoball* is a subset of $\mathbb{H}^3$ isometric to $H_c := \{(x, y, z) \in \mathbb{H}^3 \mid z \geq c\}$; these arise either in this form or as balls tangent to the boundary plane. A *cuspidal neighbourhood* of a component of ∂M which is of the form $N \cong \mathbb{T}^2 \times [1, \infty)$ - this description can be obtained by taking the quotient of a horoball H_c by Γ . With the infimal $c < 1$ such that $\partial(H_c/\Gamma) \cong \mathbb{T}^2$ is embedded rather than immersed, N is a *maximal cusp*, whose boundary ∂N is the standard place to measure lengths of slopes on ∂M .

4.1.1 Length and area

Definition 4.1. *Let M be a compact orientable hyperbolic 3-manifold with toroidal boundary ∂M . Let γ be a slope on a component of ∂M and let N be the corresponding maximal cusp.*

- *The area $A(\partial N)$ of $\partial N \cong \mathbb{T}^2$ is the usual Euclidean area.*
- *The length $l(\gamma)$ of γ is the Euclidean length of a geodesic representative on ∂N .*
- *The normalised length $\hat{l}(\gamma)$ of γ is $\hat{l}(\gamma) := l(\gamma)/\sqrt{A(\partial N)}$.*

Lemma 4.2. *Let M be a compact orientable hyperbolic 3-manifold with toroidal boundary ∂M . Let γ be a slope on a component of ∂M and let N be the corresponding maximal cusp.*

- *The area $A(\partial N)$ of $\partial N \cong \mathbb{T}^2$ satisfies $A(\partial N) \geq 2\sqrt{3}$.*
- *The length $l(\gamma)$ of γ satisfies $l(\gamma)l(\mu) \geq \Delta(\gamma, \mu) \cdot A(\partial N)$.*
- *The normalised length $\hat{l}(\gamma)$ of γ satisfies $\hat{l}(\gamma)\hat{l}(\mu) \geq \Delta(\gamma, \mu)$.*

Proof. The first inequality follows from the lower bound of $\sqrt{3}$ on cusp volume [GHM+21, Theorem 1.2], which is half of cusp area; the others follow from a simple geometric argument (see for instance [Lac20, Lemma 8.1]). $\square$

When M is a link complement, we know that the canonical choice of meridian μ and longitude λ on each component of ∂M allows us to identify $\gamma = p\mu + q\lambda$ with $p/q \in \mathbb{Q} \cup \{\infty\}$. Recall that we have a notion of *distance* between a pair of slopes p/q and r/s , which corresponds to their geometric intersection number: $\Delta(p/q, r/s) := |ps - qr|$. In particular, substituting $\Delta(p/q, 1/0) = |q|$ into Lemma 4.2 yields an explicit relationship between the length and denominator of the slope p/q .

4.1.2 The 6-theorem

When a slope is sufficiently long, we can fill along it and maintain the hyperbolic structure on M .

Theorem 4.3 (6-theorem [Ago00; Lac00]). *Let M be a compact orientable hyperbolic 3-manifold with toroidal boundary ∂M . Let γ be a slope on a component of ∂M with length $l(\gamma) > 6$.*

Then $M(\gamma)$ is also hyperbolic.

4.2 Quantitative comparison

In order to formulate a quantitative argument, we will employ some results from [FPS22] regarding lengths of geodesics.

4.2.1 A geodesic and its meridian

The first result allows us to compare the length of a sufficiently short geodesic $c \subset M$ with the normalised length of its meridian μ viewed in the complement M_c .

Theorem 4.4 ([FPS22, Corollary 6.13]). *Let M be a complete hyperbolic 3-manifold with finite volume and let $c \subset M$ be a closed geodesic with complement M_c and meridian μ . Suppose that one or both of the following hypotheses is satisfied:*

- $\hat{l}(\mu) \geq 7.823$ with respect to the complete hyperbolic structure on M_c ;
- $l(c) \leq 0.0996$ with respect to the complete hyperbolic structure on M .

Then these quantities satisfy the inequality

$$\frac{2\pi}{\hat{l}(\mu)^2 + 16.17} < l(c) < \frac{2\pi}{\hat{l}(\mu)^2 - 28.78}.$$

In particular, let $L = U_{m-1} \cup J$ and set $M = \mathbb{S}_L^3(p/q)$ to be the result of Dehn surgery along the knot $J \subset \mathbb{S}_{U_{m-1}}^3$. If we take our geodesic to be the core curve $c_J \subset \mathbb{S}_L^3(p/q)$ of the filling, then the meridian μ of c_J viewed inside the manifold $M_{c_J} = \mathbb{S}_L^3(p/q) \setminus \nu(c_J) = \mathbb{S}_L^3$ corresponds to the surgery slope p/q for J which bounds a disc in $\mathbb{S}_L^3(p/q)$. This gives us a way to sandwich the length $l(c_J)$ of the core curve between upper and lower bounds which depend only on $\hat{l}(\mu) = \hat{l}(p/q)$.

4.2.2 A pair of geodesics

The next result allows us to analyse multiple short geodesics appearing simultaneously.

Theorem 4.5 ([FPS22, Corollary 7.20]). *Let M be a complete hyperbolic 3-manifold with finite volume and let $c \cup \gamma \subset M$ be a geodesic link in M . Suppose that $\max\{l(c), l(\gamma)\} \leq 0.0735$.*

Then γ is isotopic to a geodesic in the complete metric on the complement M_c with length at most $1.9793 \cdot l(\gamma)$.

Intuitively, this means that if both c_J and γ are sufficiently short geodesics in $M = \mathbb{S}_L^3(p/q)$, then γ must also have been very short in $M_{c_J} = \mathbb{S}_L^3$. Our strategy relies heavily on this result: we will show that if all geodesics in $\mathbb{S}_L^3$ were in fact very long, then it is impossible for any of them to have produced such a short geodesic γ inside $\mathbb{S}_L^3$. This allows us to deduce that the core curve c_J must be the unique shortest geodesic in $\mathbb{S}_L^3(p/q)$.

4.3 Core curves correspond

Given our fixed hyperbolic link $L = U_{m-1} \cup J$, we seek to demonstrate that setting $|q| \geq \max\{35, q(\mathbb{S}_L^3)\}$ forces the shortest geodesic in $\mathbb{S}_L^3(p/q)$ to be the core curve c_J .

4.3.1 Comparing lengths

We will do this by first ensuring that the following condition on the normalised length $\hat{l}(p/q)$ of the slope is satisfied. This will then provide useful information about shortest geodesics.

Lemma 4.6. *Let $L = U_{m-1} \cup J$ be a hyperbolic link and let p/q be a slope on the boundary component of $\mathbb{S}_L^3$ corresponding to J such that $|q| \geq q_0$.*

Then the normalised length of the slope p/q satisfies $\sqrt{6\sqrt{3}} \cdot \hat{l}(p/q) \geq q_0$.

Proof. By Lemma 4.2, the normalised length $\hat{l}(p/q)$ of the slope p/q satisfies the relationship

$$\hat{l}(p/q)\hat{l}(1/0) \geq \Delta(p/q, 1/0) = |q| \geq q_0.$$

Since the trivial filling of $J \subset \mathbb{S}_{U_{m-1}}^3$ produces something non-hyperbolic, Theorem 4.3 tells us that $l(1/0) \leq 6$. We also have a universal lower bound $A(\partial N) \geq 2\sqrt{3}$ on cusp area from Lemma 4.2, which implies that $\sqrt{2\sqrt{3}} \cdot \hat{l}(1/0) \leq 6$. Combining these inequalities gives

$$\hat{l}(p/q) \geq \frac{|q|}{\hat{l}(1/0)} \geq \frac{\sqrt{2\sqrt{3}}}{6} \cdot |q| \geq \frac{q_0}{\sqrt{6\sqrt{3}}}$$

as required. $\square$

We can thus bound the length of the core curve c_J in the filling $\mathbb{S}_L^3(p/q)$ in the following way. Note that L_0 is just some hyperbolic link (possibly, but not necessarily, L itself): for now, the quantity $\mathfrak{q}(\mathbb{S}_{L_0}^3)$ is used in a purely numerical manner.

Lemma 4.7. *Let $L = U_{m-1} \cup J$ be a hyperbolic link and let p/q be a slope on the boundary component of $\mathbb{S}_L^3$ corresponding to J such that $|q| \geq \max\{35, \mathfrak{q}(\mathbb{S}_{L_0}^3)\}$ for some hyperbolic link L_0 .*

Then the core curve c_J of $\mathbb{S}_L^3(p/q)$ is a geodesic with length $l(c_J)$ strictly less than $\min\{0.0706, 1.9793^{-1} \cdot \text{sys}(\mathbb{S}_{L_0}^3)\}$.

Moreover, any other geodesic γ in $\mathbb{S}_L^3(p/q)$ with length $l(\gamma) \leq l(c_J)$ must also be isotopic to a geodesic in the hyperbolic link complement $\mathbb{S}_L^3$ with length strictly less than $\min\{0.14, \text{sys}(\mathbb{S}_{L_0}^3)\}$.

Proof. Since $|q| \geq 26$, Lemma 4.6 tells us that $\hat{l}(p/q) \geq 7.823$, so we can apply Theorem 4.4 and Lemma 4.6, respectively, to obtain

$$l(c_Q) < \frac{2\pi}{\hat{l}(p/q)^2 - 28.78} \leq \frac{2\pi}{\frac{|q|^2}{6\sqrt{3}} - 28.78}$$

and then use the hypothesis $|q| \geq \max\{35, \mathfrak{q}(\mathbb{S}_{L_0}^3)\}$ to reach the desired result.

Now suppose that γ is another geodesic in $\mathbb{S}_L^3(p/q)$ with length $l(\gamma) \leq l(c_J)$. By Theorem 4.5, we can deduce that γ must have been a geodesic with respect to the complete metric on the complement $\mathbb{S}_L^3(p/q) \setminus \nu(c_J) \cong \mathbb{S}_L^3$ with length at most $1.9793 \cdot l(\gamma) < \min\{0.14, \text{sys}(\mathbb{S}_{L_0}^3)\}$, as required. $\square$

4.3.2 The slope-preserving homeomorphism

The following result ensures that when $|q|, |\tilde{q}| \geq \max\{35, \mathfrak{q}(\mathbb{S}_L^3)\}$ (note that $L_0 = L$ now) and there is a slope-preserving homeomorphism $\mathbb{S}_L^3(p/q) \cong \mathbb{S}_{\tilde{L}}^3(\tilde{p}/\tilde{q})$, we can deduce that the shortest geodesics are precisely the core curves. Note that when $m = 1$ and $\mathbb{S}_L^3(p/q) \cong \mathbb{S}_{\tilde{L}}^3(p/q)$ is a homeomorphism of closed manifolds, the term “slope-preserving” can be interpreted simply as “orientation-preserving”.

Proposition 4.8. *Let $L = U_{m-1} \cup J$, $\tilde{L} = \tilde{U}_{m-1} \cup \tilde{J}$ be hyperbolic links and let $p/q, \tilde{p}/\tilde{q}$ be slopes on the boundary components of $\mathbb{S}_L^3, \mathbb{S}_{\tilde{L}}^3$ corresponding to $J, \tilde{J}$, respectively, such that $|q|, |\tilde{q}| \geq \max\{35, \mathfrak{q}(\mathbb{S}_L^3)\}$.*

If there is a slope-preserving homeomorphism $\mathbb{S}_L^3(p/q) \cong \mathbb{S}_{\tilde{L}}^3(\tilde{p}/\tilde{q})$, then $J = \tilde{J}$.

Proof. Since $|q| \geq \max\{35, \mathfrak{q}(\mathbb{S}_L^3)\}$, Lemma 4.7 with $L_0 = L$ tells us that the core curve c_J of $\mathbb{S}_L^3(p/q)$ is a geodesic of length $l(c_J) < \min\{0.0706, 1.9793^{-1} \cdot \text{sys}(\mathbb{S}_L^3)\}$. Moreover, any other geodesic γ in $\mathbb{S}_L^3(p/q)$ which is shorter than c_J , or even just another geodesic with length $l(\gamma) < \min\{0.0706, 1.9793^{-1} \cdot \text{sys}(\mathbb{S}_L^3)\}$, must also be isotopic to a geodesic in $\mathbb{S}_L^3(p/q) \setminus \nu(c_J) \cong \mathbb{S}_L^3$, where its length is strictly less than $\min\{0.14, \text{sys}(\mathbb{S}_L^3)\}$. However, all geodesics in $\mathbb{S}_L^3$ clearly have length at least $\text{sys}(\mathbb{S}_L^3)$. Therefore the shortest geodesic in $\mathbb{S}_L^3(p/q)$ is precisely the core curve c_J .

Since $|\tilde{q}| \geq \max\{35, \mathfrak{q}(\mathbb{S}_{\tilde{L}}^3)\}$, Lemma 4.7 applied to $\tilde{L}$ with $L_0 = L$ tells us that the core curve $c_{\tilde{J}}$ of $\mathbb{S}_{\tilde{L}}^3(\tilde{p}/\tilde{q})$ is a geodesic of length $l(c_{\tilde{J}}) < \min\{0.0706, 1.9793^{-1} \cdot \text{sys}(\mathbb{S}_{\tilde{L}}^3)\}$. By Mostow rigidity, the slope-preserving homeomorphism $\mathbb{S}_L^3(p/q) \cong \mathbb{S}_{\tilde{L}}^3(\tilde{p}/\tilde{q})$ is homotopic to an isometry, which preserves geodesics and their lengths. This means that the shortest geodesic in $\mathbb{S}_{\tilde{L}}^3(\tilde{p}/\tilde{q})$ must in fact be the core curve $c_{\tilde{J}}$, as any shorter geodesic $\tilde{\gamma}$ would have to correspond to another geodesic γ in $\mathbb{S}_L^3(p/q)$ with $l(\gamma) < \min\{0.0706, 1.9793^{-1} \cdot \text{sys}(\mathbb{S}_L^3)\}$. Since the isometry preserves shortest geodesics, the core curves c_J in $\mathbb{S}_L^3(p/q)$ and $c_{\tilde{J}}$ in $\mathbb{S}_{\tilde{L}}^3(\tilde{p}/\tilde{q})$ are identified. This gives the desired slope-preserving homeomorphism taking J to $\tilde{J}$; it follows that $L = \tilde{L}$. $\square$

4.4 Proof of Theorem 1.6

Set $J = K$ to be our fixed hyperbolic knot and $\tilde{J} = \tilde{K}$ to be another hyperbolic knot for which there is an orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{\tilde{K}}^3(\tilde{p}/\tilde{q})$. We will see that the assumption $|q|, |\tilde{q}| \geq \max\{35, \mathbf{q}(\mathbb{S}_K^3)\}$ ensures that the shortest geodesics in both fillings are exactly the core curves, c_K and $c_{\tilde{K}}$. This enables us to complete the proof of Theorem 1.6.

Proof of Theorem 1.6. Suppose that $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q)$ for a hyperbolic knot K and slope p/q satisfying the hypotheses in the statement of the theorem. Recall from Proposition 3.2 that K' is either (i) another hyperbolic knot or (ii) a cable $K' = J_{r,s}(\hat{K})$ of one. In case (ii), taking $(\tilde{K}, \tilde{p}/\tilde{q}) = (\hat{K}, p/qs^2)$ in Proposition 4.8 tells us that $K' = J_{r,s}(K)$ and $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q) \cong \mathbb{S}_K^3(p/qs^2)$, which can be ruled out by Corollary 2.12. This leaves us with case (i) as the only possibility; taking $(\tilde{K}, \tilde{p}/\tilde{q}) = (K', p/q)$ in Proposition 4.8 then gives $K = K'$, as required. $\square$

4.4.1 Applications of Theorem 1.6

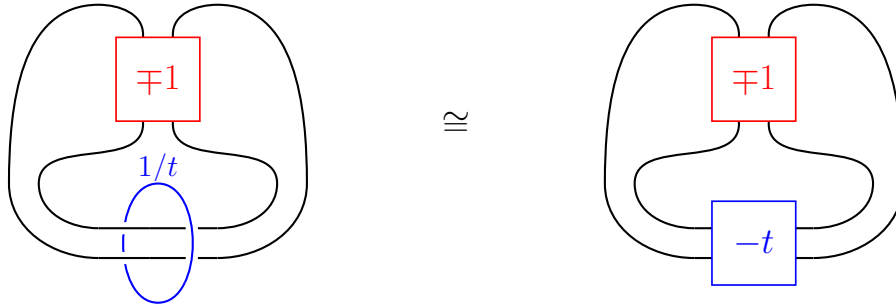


Figure 4.1: Rolfsen $-t$ -twist giving $\mathbb{S}_{W^\pm}^3(1/t) \cong \mathbb{S}_{T_t^\pm}^3$.

Proof of Corollary 1.7. For the t -twist knots $K = T_t^\pm$, the SnapPea length spectrum function returns the length $\Lambda_t^\pm$ of a shortest geodesic in the complement. These are listed in Table A.1 for $|t| \leq 34$. These values are clearly all greater than 0.14 for $|t| \leq 3$ (provided $T_t^\pm$ is hyperbolic) but less than 0.14 for $4 \leq |t| \leq 34$ by inspection.

For $|t| \geq 35$, we claim that the length of the shortest geodesic always satisfies $\Lambda_t^\pm < 0.14$ (and thus $\mathfrak{q}(\mathbb{S}_{T_t^\pm}^3) \geq 35$). Observe that $\mathbb{S}_{W^\pm}^3(1/t) \cong \mathbb{S}_{T_t^\pm}^3$ and note that the Whitehead link complement satisfies

$$\text{sys}(\mathbb{S}_{W^\pm}^3) \approx 1.061275061905 > 0.14.$$

By Lemma 4.7, if $|t| \geq \max\{35, \mathfrak{q}(\mathbb{S}_{W^\pm}^3)\} = 35$, then the core curve $c_{V^\pm}$ of $\mathbb{S}_{W^\pm}^3(1/t)$ is a geodesic whose length satisfies

$$l(c_{V^\pm}) < \min\{0.0706, 1.9793^{-1} \cdot \text{sys}(\mathbb{S}_{W^\pm}^3)\} = 0.0706 < 0.14$$

and thus $\Lambda_t^\pm < 0.14$. □

4.5 Proof of Theorem 1.8

Now we will set $L = P$ and $\tilde{L} = \tilde{P}$ to be a pair of hyperbolic patterns, $P = U \cup Q$ and $\tilde{P} = \tilde{U} \cup \tilde{Q}$, for which there is an slope-preserving homeomorphism $\mathbb{S}_P^3(p/q) \cong \mathbb{S}_{\tilde{P}}^3(p/q)$. The assumption $|q|, |\tilde{q}| \geq \max\{35, \mathfrak{q}(\mathbb{S}_P^3)\}$ will imply that the shortest geodesics in both fillings are the core curves, c_Q and $c_{\tilde{Q}}$. Putting everything together, we can complete the proof of Theorem 1.8.

Proof of Theorem 1.8. Suppose that $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q)$ for a satellite knot $K = P(C)$ and slope p/q satisfying the hypotheses in the statement of the theorem. Recall from Proposition 3.4 that, since we have $\gcd(p, w) \neq 1$ and $|q| \geq 3$, we know that $K' = P'(C)$ is also a satellite knot, by either (i) a hyperbolic pattern P' or (ii) a cable $P' = J_{r,s}(\hat{P})$ of one. In case (ii), taking $(\tilde{P}, \tilde{p}/\tilde{q}) = (\hat{P}, p/qs^2)$ in Proposition 4.8 tells us that $K' = J_{r,s}(P(C)) = J_{r,s}(K)$ and $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q) \cong \mathbb{S}_K^3(p/qs^2)$, which can be ruled out by Corollary 2.12. This leaves us with case (i) as the only possibility; taking $(\tilde{P}, \tilde{p}/\tilde{q}) = (P', p/q)$ in Proposition 4.8 then gives $K = K'$. □

4.5.1 Applications of Theorem 1.8

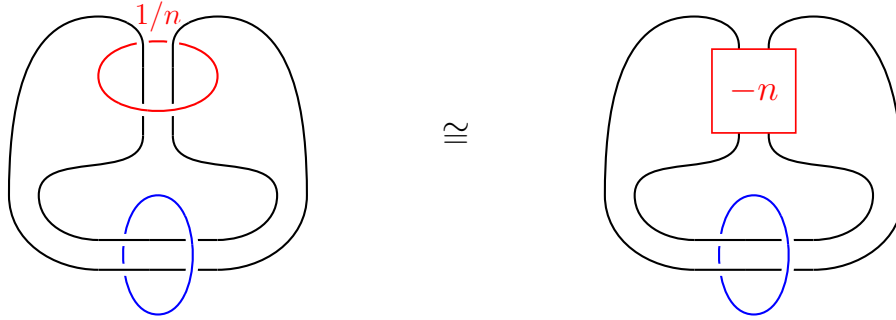


Figure 4.2: Rolfsen $-n$ -twist giving $\mathbb{S}_B^3(1/n) \cong \mathbb{S}_{W^n}^3$.

Proof of Corollary 1.9. For the n -clapsed t -twisted Whitehead links $P = W_t^n$, the SnapPea length spectrum function returns the length $\Lambda^{(n)}$ of a shortest geodesic in $\mathbb{S}_{W_t^n}^3$ (this is independent of t). These are listed for $|n| \leq 34$ in Table A.2 and are clearly all greater than 0.14 for $|n| \leq 4$ but less than 0.14 for $5 \leq |n| \leq 34$ by inspection.

For $|n| \geq 35$, we need to show that the length of the shortest geodesic always satisfies $\Lambda^{(n)} < 0.14$. Letting B denote the Borromean rings, observe that we have $\mathbb{S}_B^3(1/n) \cong \mathbb{S}_{W^n}^3$ and note that the Borromean rings complement satisfies

$$\text{sys}(\mathbb{S}_B^3) \approx 2.122550123810 > 0.14.$$

By Lemma 4.7, if $|n| \geq \max\{35, \mathfrak{q}(\mathbb{S}_B^3)\} = 35$, then the core curve c of $\mathbb{S}_B^3(1/n)$ is a geodesic whose length satisfies

$$l(c) < \min\{0.0706, 1.9793^{-1} \cdot \text{sys}(\mathbb{S}_B^3)\} = 0.0706 < 0.14$$

and thus $\Lambda^{(n)} < 0.14$. □

Chapter 5

Whitehead doubles

We will now consider the special case where our fixed hyperbolic pattern is the ± 1 -clasped t -twisted Whitehead link, $W_t^\pm = U \cup V_t^\pm$, for $\pm t \in \{-1, 0, 1, 2\}$ as in Theorem 1.10. To clarify this notation: we take $t \in \{-1, 0, 1, 2\}$ in the positively clasped case and $t \in \{1, 0, -1, -2\}$ in the negatively clasped case (just as we did for the selection of twist knots $T_t^\pm$ depicted in Figure 1.1).

The Whitehead link has many significant properties. We know that $W_t^\pm$ has linking number $w = 0$. The untwisted version, $W^\pm = U \cup V^\pm$, comprises two unknotted components which can be exchanged by an isotopy. Most importantly, the Whitehead link complement (jointly with its sister manifold, the $P(-2, 3, 8)$ pretzel link complement) achieves the least hyperbolic volume over all 2-cusped orientable hyperbolic 3-manifolds [Ago10]. This leads to an alternative method of deducing that the existence of a slope-preserving homeomorphism $\mathbb{S}_{W_t^\pm}^3(p/q) \cong \mathbb{S}_{\tilde{P}}^3(\tilde{p}/\tilde{q})$ (where $\tilde{P}$ is a hyperbolic pattern) implies that $\tilde{P} = W_t^\pm$. This will form a crucial step in our upcoming proof of Theorem 1.10.

Theorem 1.10. *Let $K = W_t^\pm(C)$ be a ± 1 -clasped t -twisted Whitehead double for $\pm t \in \{-1, 0, 1, 2\}$.*

Then every slope p/q with $|p| \neq 1$ and $|q| \geq q_{\min}$ is characterising for K , where $q_{\min}$ is some constant determined by the Whitehead link and independent of C .

Furthermore, if the SnapPea census of 2-cusped orientable hyperbolic 3-manifolds is complete up to stage k of Table 1.1 (namely, up to the volume bound V_k , or up to the first a_k items), then we can take $q_{\min}$ to be the corresponding value of q_k .

Our key ingredient will be an inequality that restricts the extent to which Dehn filling can decrease hyperbolic volume. This depends on a scaling factor which is determined by the length (and in turn by the denominator) of the filling slope. That is, for a given lower bound $q_{\min}$ on $|q|$, we can construct an upper bound $V_{\max}$ on the volume of $\mathbb{S}_{\tilde{P}}^3$ for which it is possible to have a slope-preserving homeomorphism $\mathbb{S}_{W_t^\pm}^3(p/q) \cong \mathbb{S}_{\tilde{P}}^3(\tilde{p}/\tilde{q})$. Under the additional requirement of completeness of the SnapPea census of 2-cusped orientable hyperbolic 3-manifolds up to the given $V_{\max}$, we can use the fact that $\tilde{P} = \tilde{U} \cup \tilde{Q}$ must have winding number $\tilde{w} = 0$ to systematically rule out everything except $\mathbb{S}_{\tilde{P}}^3 \cong \mathbb{S}_{W_t^\pm}^3$. Finally, since $\pm t \in \{-1, 0, 1, 2\}$, we can deduce that $\tilde{P} = W_t^\pm$.

5.1 Homeomorphic link complements

Unlike knots, links are not necessarily determined by their complements. However, in the case of the Whitehead link, there are a limited number of possibilities.

5.1.1 Twisted Whitehead links

Lemma 5.1. *Let $W_t^\pm = U \cup V_t^\pm$ be a ± 1 -clasped t -twisted Whitehead link for some $t \in \mathbb{Z}$ and let $\tilde{P} = U \cup \tilde{Q}$ be a link such that there exists an orientation-preserving homeomorphism $\mathbb{S}_{W_t^\pm}^3 \cong \mathbb{S}_{\tilde{P}}^3$.*

Then $\tilde{P}$ is a t' -twisted Whitehead link with the same clasp, $\tilde{P} = W_{t'}^\pm$, for some integer $t' \in \mathbb{Z}$.

Proof. Since $W_t^\pm$ is Brunnian with linking number zero, $\tilde{P}$ must be obtained from $W_t^\pm$ by an integer number of full twists around an unknotted component [MS01].

Each clasp sign generates a distinct set of twisted Whitehead links with orientation-preserving homeomorphic complements (where objects in opposing sets are related by orientation-reversing homeomorphisms). $\square$

We will later use the following result to analyse cables of twisted Whitehead doubles.

Lemma 5.2. *Let $W_t^\pm = U \cup V_t^\pm$ be a ± 1 -clasped t -twisted Whitehead link for $\pm t \in \{-1, 0, 1, 2\}$. If there is a slope-preserving homeomorphism $\mathbb{S}_{W_t^\pm}^3(p/q) \cong \mathbb{S}_{P'}^3(p/q)$, where $P' = J_{r,s}(W_{t'}^\pm)$ is a cable of some $W_{t'}^\pm = U' \cup V_{t'}^\pm$ and p/q is a non-integral slope with $|p - qrs| = 1$, then $t = t' = 0$ and $|r| = 1$.*

Proof. Since $V_t^\pm \in \{U, \pm T, S, \pm R\}$, we can apply Lemma 3.6 to rule out S and $\pm R$. In the case where $V_t^\pm = U$, we have $t = t' = 0$ and $|r| = 1$, as required. Alternatively, if $V_t^\pm = \pm T$, then we have $t = \pm 1$, $t' = 0$ and $(r, s) \in \{(\pm 2, 3), (\pm 3, 2)\}$. However, by the Rolfsen ∓ 1 -twist trick shown in Figure 5.1, this gives

$$L(p, q) \cong \mathbb{S}_{T_0^\pm}^3(p/q) \cong \mathbb{S}_{U, V_{\pm 1}^\pm}^3(\mp 1, p/q) \cong \mathbb{S}_{U, V_0^\pm}^3(\mp 1, p/qs^2) \cong \mathbb{S}_{T_{\mp 1}^\pm}^3(p/qs^2),$$

yet the figure eight knot $T_{\mp 1}^\pm = S$ has no lens space surgeries. $\square$

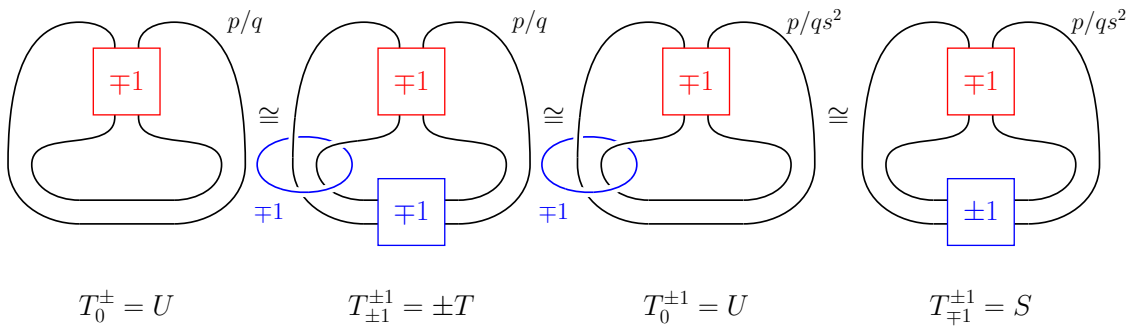


Figure 5.1: Rolfsen ∓ 1 -twist trick.

5.1.2 Linking number

In general, we can use linking number as an invariant to distinguish between link complements.

Lemma 5.3. *Let $P = U \cup Q$, $\tilde{P} = \tilde{U} \cup \tilde{Q}$ be links whose linking numbers $w, \tilde{w}$ satisfy $|w| \neq |\tilde{w}|$.*

Then there exists no orientation-preserving homeomorphism $\mathbb{S}_P^3 \cong \mathbb{S}_{\tilde{P}}^3$ between their complements.

Proof. Consider the map $H_1(\mathbb{T}^2) \rightarrow H_1(\mathbb{S}_P^3)$ induced by the inclusion $\mathbb{T}^2 \hookrightarrow \mathbb{S}_P^3$ of either toroidal boundary component. The index of the image of this map is

$$\begin{cases} |w| & \text{if } w \neq 0, \\ \infty & \text{if } w = 0. \end{cases}$$

Under any homeomorphism $\mathbb{S}_P^3 \cong \mathbb{S}_{\tilde{P}}^3$, this index would be preserved, giving $|w| = |\tilde{w}|$. □

Recall that the Whitehead link $W_t^\pm$ has linking number $w = 0$. We already know from Lemma 3.5 that, in order to have our slope-preserving homeomorphism $\mathbb{S}_{W_t^\pm}^3(p/q) \cong \mathbb{S}_{\tilde{P}}^3(\tilde{p}/\tilde{q})$, $\tilde{P}$ should also have linking number $\tilde{w} = 0$. As a consequence, we can obstruct candidates for $\tilde{P}$ by showing that for each 2-cusped orientable hyperbolic 3-manifold M with $\text{vol}(M) < V_{\max}$, if there exists an orientation-preserving homeomorphism $M \cong \mathbb{S}_{\tilde{P}}^3$ to the complement of any hyperbolic link $\tilde{P} = \tilde{U} \cup \tilde{Q}$, then $\tilde{P}$ must have linking number $\tilde{w} \neq 0$. This rules out M as a potential pattern piece.

5.2 Upper volume bounds

Our main technique in addressing the case of Whitehead doubles will be the use of hyperbolic volume.

5.2.1 Key inequalities

In particular, we rely on the following inequalities to construct our upper bound $V_{\max}$ on $\text{vol}(\mathbb{S}_{\tilde{P}}^3)$ for any link complement $\mathbb{S}_{\tilde{P}}^3$ such that $\mathbb{S}_{W_t^\pm}^3(p/q) \cong \mathbb{S}_{\tilde{P}}^3(\tilde{p}/\tilde{q})$.

Theorem 5.4 ([FKP08; Thu80]). *Let M be a compact orientable hyperbolic 3-manifold with toroidal boundary $\partial M = \sqcup_{i=1}^m \mathbb{T}_i^2$. Take a subset $\{\mathbb{T}_{i_j}^2\}_{j=1}^n$ of boundary tori together with slopes $\{\gamma_{i_j} \subset \mathbb{T}_{i_j}^2\}_{j=1}^n$. Choose horoball neighbourhoods N_{i_j} of each $\mathbb{T}_{i_j}^2$ such that each $l(\gamma_{i_j}) \geq l_{\min} > 2\pi$.*

Then $M(\gamma_{i_1}, \dots, \gamma_{i_n})$ is a hyperbolic 3-manifold such that

$$\left(1 - \left(\frac{2\pi}{l_{\min}}\right)^2\right)^{3/2} \cdot \text{vol}(M) \leq \text{vol}(M(\gamma_{i_1}, \dots, \gamma_{i_n})) < \text{vol}(M).$$

In our case, M will be the complement of a 2-component hyperbolic link (so $m = 2$) and we will fill one boundary component (so $n = 1$) along slope $\tilde{p}/\tilde{q}$.

5.2.2 Construction of $V_{\max}$

Lemma 5.5. *Let $W_t^\pm = U \cup V_t^\pm$ be a ± 1 -clasped t -twisted Whitehead link for some integer $t \in \mathbb{Z}$ and let $\tilde{P} = \tilde{U} \cup \tilde{Q}$ be a hyperbolic pattern. Suppose that $\mathbb{S}_{W_t^\pm}^3(p/q) \cong \mathbb{S}_{\tilde{P}}^3(\tilde{p}/\tilde{q})$ for some slopes $p/q, \tilde{p}/\tilde{q}$ on the boundary components of $\mathbb{S}_{W_t^\pm}^3, \mathbb{S}_{\tilde{P}}^3$ corresponding to $V_t^\pm, \tilde{Q}$, respectively, with $|\tilde{q}| \geq q_{\min}$ for some fixed $q_{\min} \geq 11$.*

Then $q_{\min}$ determines a constant

$$V_{\max} := \left(1 - 3\left(\frac{2\pi}{q_{\min}}\right)^2\right)^{-3/2} \cdot \text{vol}(\mathbb{S}_{W_t^\pm}^3)$$

such that $\text{vol}(\mathbb{S}_{\tilde{P}}^3) < V_{\max}$.

Proof. Since $|\tilde{q}| \geq q_{\min} \geq 11$, the length of the slope $\tilde{p}/\tilde{q}$ satisfies

$$l(\tilde{p}/\tilde{q}) \geq \frac{2\sqrt{3}}{6} \cdot \Delta(\tilde{p}/\tilde{q}, 1/0) = \frac{|\tilde{q}|}{\sqrt{3}} \geq \frac{q_{\min}}{\sqrt{3}} =: l_{\min} > 2\pi$$

by Lemma 4.2 and Theorem 4.3 (due to the existence of the exceptional filling

$\mathbb{S}_P^3(1/0) \cong \mathbb{S}^1 \times \mathbb{D}^2$). Therefore we can apply Theorem 5.4 to obtain the inequality

$$\left(1 - 3\left(\frac{2\pi}{q_{\min}}\right)^2\right)^{3/2} \cdot \text{vol}(\mathbb{S}_P^3) \leq \text{vol}(\mathbb{S}_P^3(\tilde{p}/\tilde{q})) = \text{vol}(\mathbb{S}_{W_t^\pm}^3(p/q)) < \text{vol}(\mathbb{S}_{W_t^\pm}^3)$$

and rearrange to reach the desired result. $\square$

Note that we require $q_{\min} \geq 11$ in order for the hypothesis $l_{\min} > 2\pi$ to be satisfied. We also need to assume that $q_{\min} \geq 24$ in order to ensure that we obtain a finite list from SnapPea; otherwise, the volume bound $V_{\max}$ becomes larger than the volume of the 3-chain link complement, which is conjecturally the smallest limit point for the set of volumes of 2-cusped orientable hyperbolic 3-manifolds.

We know that $\mathbb{S}_P^3$ is the complement of a 2-component hyperbolic link $\tilde{P} = \tilde{U} \cup \tilde{Q}$ with linking number $\tilde{w} = 0$. These constraints will allow us to eliminate most of the 2-cusped orientable hyperbolic 3-manifolds with volume at most $V_{\max}$ from the list by showing that none of them are of this form. As we will see, one way to do this is by comparing different fillings of these manifolds: for homological reasons, the existence of certain types of fillings obstructs the $\tilde{w} = 0$ condition.

5.3 Comparing Dehn fillings

When considering pairs of fillings of a 2-cusped orientable hyperbolic 3-manifold M , there are several situations which can arise that prevent M from being of the required form.

5.3.1 Multiple solid torus fillings

The simplest case occurs when there are multiple solid torus fillings.

Lemma 5.6. *Suppose that M is a 2-cusped orientable hyperbolic 3-manifold which admits distinct solid torus fillings, $M((a_1, b_1), *)$, $M((\hat{a}_1, \hat{b}_1), *)$, on the same boundary component.*

Then there exists no hyperbolic link $\tilde{P} = \tilde{U} \cup \tilde{Q}$ with winding number $\tilde{w} = 0$ such that $M \cong \mathbb{S}_{\tilde{P}}^3$.

Proof. We can think of $M((\hat{a}_1, \hat{b}_1), *)$ as a solid torus surgery on the core of the filling in the solid torus $M((a_1, b_1), *)$. Since M is hyperbolic, and thus irreducible, the core cannot be contained in any $\mathbb{B}^3 \subset \mathbb{S}^1 \times \mathbb{D}^2$. We can now apply the classification of such solid torus fillings given in [Gab90] to deduce that the core must be a Berge-Gabai knot: a 0-bridge or 1-bridge braid in $\mathbb{S}^1 \times \mathbb{D}^2$, which clearly has non-zero winding number in both cases. Moreover, if $M \cong \mathbb{S}_{\tilde{P}}^3$, then this core corresponds to one of the link components sitting inside the complement of the other. It follows that $\tilde{w} \neq 0$. $\square$

5.3.2 Homological obstruction

If we can only find a single solid torus filling on a component of M , then we check for the following situation instead.

Lemma 5.7. *Suppose that M is a 2-cusped orientable hyperbolic 3-manifold which admits a solid torus filling $M((a_1, b_1), *) \cong \mathbb{S}^1 \times \mathbb{D}^2$. Let $M((a_1, b_1), (a_2, b_2))$ be the unique $\mathbb{S}^1 \times \mathbb{S}^2$ filling of this $\mathbb{S}^1 \times \mathbb{D}^2$ and suppose that there also exists a filling $M((a'_1, b'_1), (a_2, b_2))$ whose first homology is finite.*

Then there exists no hyperbolic link $\tilde{P} = \tilde{U} \cup \tilde{Q}$ with winding number $\tilde{w} = 0$ such that $M \cong \mathbb{S}_{\tilde{P}}^3$.

Proof. Suppose for a contradiction that $M \cong \mathbb{S}_{\tilde{P}}^3$ for such a link $\tilde{P}$. Then we can express the filling data in the form of slopes $(a, b) = \alpha/\beta$ with respect to the meridian and longitude of each link component. Since $\tilde{P}$ has winding number $\tilde{w} = 0$, the first homology of a filling along both components can easily be written down. In

particular, consider the following two lens space fillings of our $\mathbb{S}^1 \times \mathbb{D}^2$:

$$\begin{aligned}\mathbb{Z} &\cong H_1(\mathbb{S}^1 \times \mathbb{S}^2) \cong H_1(M(\alpha_1/\beta_1, \alpha_2/\beta_2)) \cong \mathbb{Z}/\alpha_1\mathbb{Z} \oplus \mathbb{Z}/\alpha_2\mathbb{Z}, \\ \mathbb{Z}/2\mathbb{Z} &\cong H_1(\mathbb{RP}^3) \cong H_1(M(\alpha_1/\beta_1, \alpha'_2/\beta'_2)) \cong \mathbb{Z}/\alpha_1\mathbb{Z} \oplus \mathbb{Z}/\alpha'_2\mathbb{Z}.\end{aligned}$$

It follows that $|\alpha_1| = 1$, $|\alpha_2| = 0$ and $|\alpha'_2| = 2$. But then $H_1(M(\alpha'_1/\beta'_1, \alpha_2/\beta_2)) \cong \mathbb{Z}/\alpha'_1\mathbb{Z} \oplus \mathbb{Z}$. $\square$

5.4 Proof of Theorem 1.10

Given a slope-preserving homeomorphism $\mathbb{S}^3_{W_t^\pm}(p/q) \cong \mathbb{S}^3_{\tilde{P}}(\tilde{p}/\tilde{q})$, we need to show that $\tilde{P} = W_t^\pm$. We first claim that $\tilde{P} = W_{t'}^\pm$ for some $t' \in \mathbb{Z}$.

Proposition 5.8. *Let $W_t^\pm = U \cup V_t^\pm$ be a ± 1 -clasped t -twisted Whitehead link for some integer $t \in \mathbb{Z}$ and let $\tilde{P} = \tilde{U} \cup \tilde{Q}$ be a hyperbolic pattern. Let $p/q, \tilde{p}/\tilde{q}$ be slopes on the boundary components of $\mathbb{S}^3_{W_t^\pm}, \mathbb{S}^3_{\tilde{P}}$ corresponding to $V_t^\pm, \tilde{Q}$, respectively. Assume that the SnapPea census of 2-cusped orientable hyperbolic 3-manifolds is complete up to stage k in Table 1.1, and suppose that $|q|, |\tilde{q}| \geq q_k$.*

If there is a slope-preserving homeomorphism $\mathbb{S}^3_{W_t^\pm}(p/q) \cong \mathbb{S}^3_{\tilde{P}}(\tilde{p}/\tilde{q})$, then $\tilde{P} = W_{t'}^\pm$ for some $t' \in \mathbb{Z}$.

Proof. Suppose that $\mathbb{S}^3_{W_t^\pm}(p/q) \cong \mathbb{S}^3_{\tilde{P}}(\tilde{p}/\tilde{q})$. Since $|\tilde{q}| \geq q_k \geq 11$, Lemma 5.5 produces a constant

$$V_k := \left(1 - 3\left(\frac{2\pi}{q_k}\right)^2\right)^{-3/2} \cdot \text{vol}(\mathbb{S}^3_W)$$

such that $\text{vol}(\mathbb{S}^3_{\tilde{P}}) < V_k$. We have made sure to choose $24 \leq q_k \leq 43$: if $q_k < 24$, then V_k becomes greater than the volume of the 3-chain link complement, so there are likely to be infinitely many M with $\text{vol}(M) < V_k$ to consider; if $q_k > 43$, then V_k becomes so close to $\text{vol}(\mathbb{S}^3_W)$ that there appear to be no other M with $\text{vol}(M) < V_k$ to rule out (though we cannot say for certain).

We proceed to work through the SnapPea census of 2-cusped orientable hyperbolic 3-manifolds in the stages described in Table 1.1. Namely, for each stage k , if the list is complete up to the first a_k items, then q_k is selected to be precisely the least integer to produce a value of V_k which lies just below the highest volume in this list, leaving finitely many manifolds of smaller volume to be systematically eliminated via one of our techniques.

First, we locate the manifolds which are obviously not link complements for homological reasons. The homology $H_1(\mathbb{S}_P^3) \cong \mathbb{Z} \oplus \mathbb{Z}$ of a 2-component link complement $\mathbb{S}_P^3$ is freely generated by the homology classes of the two meridians, so anything with torsion can be immediately ruled out. It turns out that the only two to consider are $m412$ and $s596$, which both share a volume of 5.0747080321 and first homology $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. We discount these from all further discussion.

We would like to use Lemma 3.5 to show that none of the remaining manifolds can be homeomorphic to the complement of a link with winding number $\tilde{w} = 0$. We can immediately rule out the candidates which are identified by SnapPea to be the complements of links with non-zero linking number. Moreover, those possessing a component with multiple distinct $\mathbb{S}^1 \times \mathbb{D}^2$ fillings can be eliminated by Lemma 5.6: in fact, with respect to the homology basis used by SnapPea, the pair of fillings given by $(a_1, b_1) = (1, 0)$ and $(\hat{a}_1, \hat{b}_1) = (0, 1)$ works in each of these cases. This is all summarised in Table 5.1.

In general (just excluding $m412$ and $s596$), we can skip straight to the method of eliminating a manifold by Lemma 5.7. Each of the manifolds M on the list satisfies $M((1, 0), *) \cong \mathbb{S}^1 \times \mathbb{D}^2$, which in turn has a unique filling slope (a, b) that produces $M((1, 0), (a, b)) = \mathbb{S}^1 \times \mathbb{S}^2$. It is then easy to check that $H_1(M((0, 1), (a, b)))$ is finite. Table B.1 in the appendix contains (a, b) for each M on the list, which is ordered by volume and split into portions L_k with volume between V_{k-1} and V_k for $1 \leq k \leq 8$, where $V_0 := \text{vol}(\mathbb{S}_W^3)$ itself and L_9 represents the smallest pair of manifolds after the volume V_8 .

We are left with the Whitehead link complement, $m129$, as the only remaining option for $M \cong \mathbb{S}_P^3$. By Lemma 5.1, $\tilde{P} = W_{t'}^\pm$ for some $t' \in \mathbb{Z}$. $\square$

Remark 5.9. Note that we can also work in the opposite direction: starting from any volume bound $V_{\max}$, we can recover the corresponding lower bound on the denominator via the formula

$$q_{\min} := \left\lceil 2\pi \sqrt{\frac{3}{1 - (V_{\max}^{-1} \cdot \text{vol}(\mathbb{S}_W^3))^{2/3}}} \right\rceil \in \mathbb{Z}.$$

Finally, having successfully narrowed down our hyperbolic pattern to a twisted Whitehead link, $\tilde{P} = W_{t'}^\pm$, we proceed to complete the proof of Theorem 1.10.

Proof of Theorem 1.10. By Proposition 3.4, we know that $K' = P'(C)$ is a satellite knot and that there is a slope-preserving homeomorphism $\mathbb{S}_{W_t^\pm}^3(p/q) \cong \mathbb{S}_{P'}^3(p/q)$, where P' is either (i) a hyperbolic pattern itself or (ii) a cable $J_{r,s}(\hat{P})$ of a hyperbolic pattern $\hat{P}$. Let $\tilde{P}$ denote the hyperbolic pattern in each case and consider the slope-preserving homeomorphism $\mathbb{S}_{W_t^\pm}^3(p/q) \cong \mathbb{S}_{\tilde{P}}^3(\tilde{p}/\tilde{q})$. We have just shown in Proposition 5.8 that $\tilde{P} = W_{t'}^\pm$ for some $t' \in \mathbb{Z}$. In case (ii), where $(\tilde{P}, \tilde{p}/\tilde{q}) = (\hat{P}, p/qs^2)$, we have $P' = J_{\pm 1, s}(W_t^\pm)$ by Lemma 5.2; however, we can rule this out by Corollary 2.12. In case (i), where $(\tilde{P}, \tilde{p}/\tilde{q}) = (P', p/q)$, Lemma 3.6 tells us that $V_t^\pm = V_{t'}^\pm$, so $t = t'$ and $K = K'$, as required. $\square$

5.4.1 Applications of Theorem 1.10

The appeal of Theorem 1.10 is that it is the only result in this thesis which can potentially generate a value of $C(K) < 35$, as it is not restricted by the same limiting factors as results which rely on the minimal geodesics technique. By replacing the $|p| \neq 1$ condition on the numerator with a more sophisticated method of obstructing the swapping of JSJ pieces (for instance, as described later in the proof of Theorem 1.12), we could potentially reduce our bound as low as $C(K) = 24$.

Of course, this technique is entirely dependent on the completeness of the SnapPea census of 2-cusped orientable hyperbolic 3-manifolds up to a given point; however, given that the completeness of the SnapPea census of 1-cusped orientable hyperbolic 3-manifolds has been verified up to the first 14 [GHM+21], this is a very tangible problem.

M	$H_1(M)$	$ \text{Tor}(H_1(M)) $
$m412$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	2
$s596$	$\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	2

M	$\tilde{P}$ with $M \cong \mathbb{S}^3_{\tilde{P}}$	$ \tilde{w} \neq 0$
$m125$	$L13n5885$	5
$m203$	$L6a2$	3
$m295$	$L9n14$	1
$m367$	$L7a6$	1
$m391$	$L8a12$	4
$s443$	$L11n208$	2
$s578$	$L9a364$	2
$s602$	$L10a114$	5
$s647$	$L12n1027$	2
$v1060$	$L13n5895$	3
$v1263$	$L11a360$	3

M	$\mathbb{S}^1 \times \mathbb{D}^2$ fillings
$m202$	$(1,0), (0, 1), (-1, 1)$
$m329$	$(1,0), (0, 1)$
$m357$	$(1,0), (0, 1)$
$m366$	$(1,0), (0, 1)$
$m388$	$(1,0), (0, 1)$
$s503$	$(1,0), (0, 1)$
$s548$	$(1,0), (0, 1)$
$s579$	$(1,0), (0, 1)$
$s601$	$(1,0), (0, 1)$
$v1180$	$(1,0), (0, 1)$
$v1203$	$(1,0), (0, 1)$
$v1264$	$(1,0), (0, 1)$
$t02728$	$(1,0), (0, 1)$
$t02750$	$(1,0), (0, 1)$

Table 5.1: 2-cusped orientable hyperbolic 3-manifolds which can be ruled out by one of the simpler elimination techniques (as described in the proof of Theorem 1.10).

Chapter 6

Non-characterising slopes

In this chapter, we construct the non-characterising slopes that are described in Theorem 1.11 and depicted in Figure 1.3.

Theorem 1.11. *Let $K = W^n(T_q^m)$ and $K' = W^m(T_q^n)$ for any $q, m, n \in \mathbb{Z} \setminus \{0\}$.*

Then there exists an orientation-preserving homeomorphism $\mathbb{S}_K^3(1/q) \cong \mathbb{S}_{K'}^3(1/q)$.

In particular, when $m \neq n$, $1/q$ is a non-characterising slope for both K and K' .

We employ Brakes' strategy of gluing together a pair of knot complements to construct pairs of knots which share a common surgery [Bra80]. This demonstrates precisely why we require the $|p| \neq 1$ condition in Theorem 1.10.

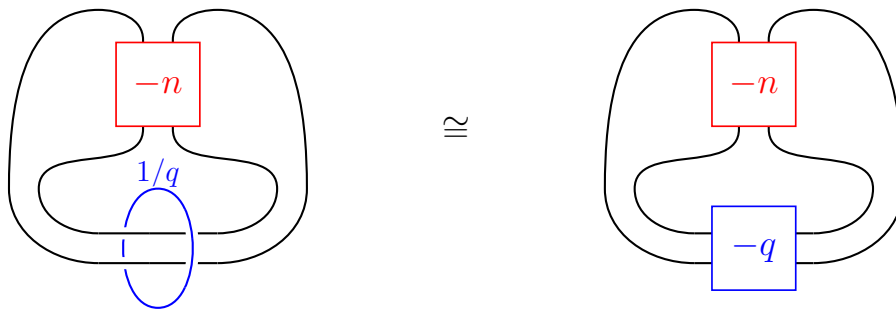


Figure 6.1: Rolfen $-q$ -twist giving $\mathbb{S}_{W^n}^3(1/q) \cong \mathbb{S}_{T_q^n}^3$.

6.1 Construction

Let $W^n = U \cup V^n$ be the n -clasped untwisted Whitehead link. We prohibit any extra twisting of V^n inside the solid torus $\mathbb{S}_U^3$ because we require our pattern to be unknotted in order to carry out the upcoming construction. Recall that W^n has winding number $w = 0$ and that the two unknotted components can be exchanged by an isotopy. We clearly require $n \neq 0$ in order for this pattern to be non-trivial.

Let T_q^m denote the (m, q) -double twist knot. We take $m \neq 0$ and $q \neq 0$ in order to ensure that T_q^m is not the unknot. Note that $T_q^m = T_m^q$. Also, observe that by the Rolfsen $-q$ -twist depicted in Figure 6.1, T_q^m can be obtained by performing $1/q$ -surgery on one component of the m -clasped Whitehead link; it does not matter which component, since they are both unknotted and can be exchanged by an isotopy.

Now, by applying an n -clasped Whitehead link pattern W^n to an (m, q) -double twist knot T_q^m , we can construct an infinite family of pairs of satellite knots sharing a $1/q$ -surgery. The idea is that, since filling the pattern piece $\mathbb{S}_{W^n}^3$ with slope $1/q$ produces an (n, q) -double twist knot complement, $\mathbb{S}_{W^n}^3(1/q) \cong \mathbb{S}_{T_q^n}^3$, it is possible for a homeomorphism between surgeries to switch the JSJ pieces.

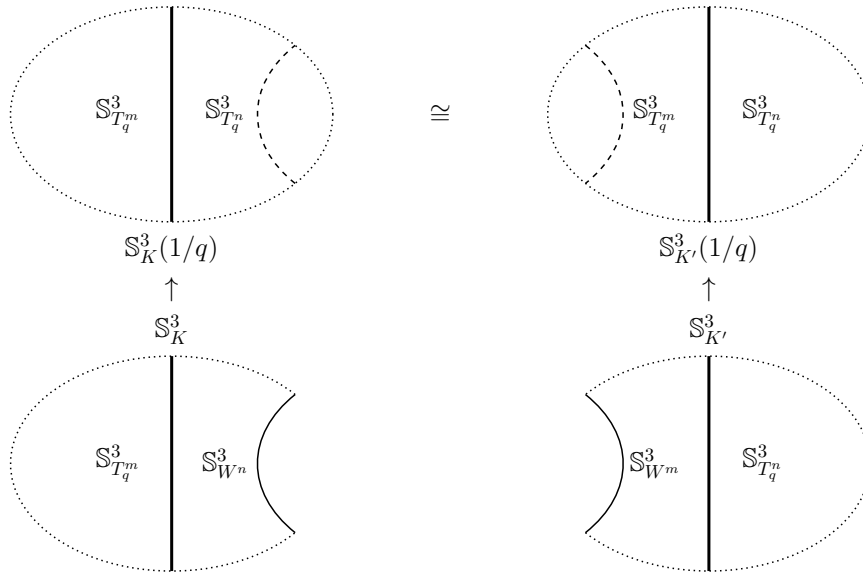


Figure 6.2: A homeomorphism switching two knot complements.

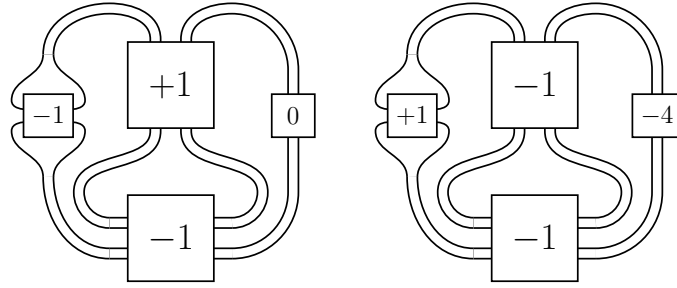
6.1.1 Special case

As a special case, we generate the following infinite family of pairs of true Whitehead doubles of true twist knots which each share a non-characterising slope of the form $1/q$.

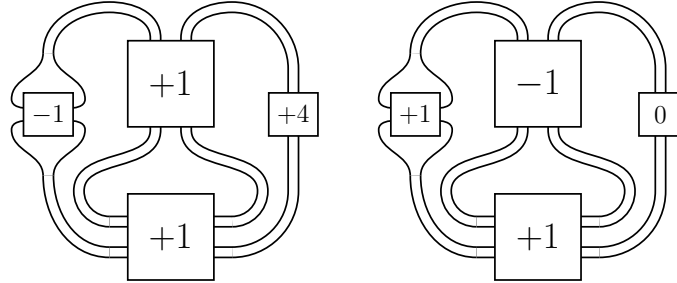
Corollary 6.1. *Let $K_q = W^+(T_q^-)$ and $K'_q = W^-(T_q^+)$ for any choice of $q \in \mathbb{Z} \setminus \{0\}$.*

Then there exists an orientation-preserving homeomorphism $\mathbb{S}_{K_q}^3(1/q) \cong \mathbb{S}_{K'_q}^3(1/q)$.

In particular, $1/q$ is a non-characterising slope for both K_q and K'_q .



(a) A pair of knots, $K_+ = W^+(S)$ and $K'_+ = W^-(+T)$, with $\mathbb{S}_{K_+}^3(+1) \cong \mathbb{S}_{K'_+}^3(+1)$.



(b) A pair of knots, $K_- = W^+(-T)$ and $K'_- = W^-(S)$, with $\mathbb{S}_{K_-}^3(-1) \cong \mathbb{S}_{K'_-}^3(-1)$.

Figure 6.3: Pairs of knots, $K_{\pm}$ and $K'_{\pm}$, with $\mathbb{S}_{K_{\pm}}^3(\pm 1) \cong \mathbb{S}_{K'_{\pm}}^3(\pm 1)$.

Furthermore, when $q = \pm 1$, we obtain the integer non-characterising slopes ± 1 and rediscover an example given in [Bra80]. The following pairs consist of orientation-preserving homeomorphic fillings which both admit JSJ decompositions into a trefoil knot complement and a figure eight knot complement, glued together along the central JSJ torus by an orientation-reversing map which exchanges meridians and longitudes. Note that these examples are mirrors of one another.

Example 6.2. Let $K_{\pm} = W^+(T_{\pm 1}^-)$ and $K'_{\pm} = W^-(T_{\pm 1}^+)$.

Then there exists an orientation-preserving homeomorphism $\mathbb{S}_{K_{\pm}}^3(\pm 1) \cong \mathbb{S}_{K'_{\pm}}^3(\pm 1)$.

In particular, ± 1 is a non-characterising slope for both $K_{\pm}$ and $K'_{\pm}$.

Figure 6.3 shows the knots $K_{\pm}$ and $K'_{\pm}$.

6.2 Gluing maps

In order to compare JSJ decompositions, it is crucial to consider gluing maps as well as JSJ pieces.

Let $K = W^n(T_q^m)$. We will carefully analyse exactly what the JSJ decomposition of $\mathbb{S}_K^3(1/q)$ looks like. We already know that there are precisely two JSJ pieces, which each take the form of a double twist knot complement: the companion piece $\mathbb{S}_{T_q^m}^3$ and the filled pattern piece $\mathbb{S}_{W^n}^3(1/q) \cong \mathbb{S}_{T_q^m}^3$. Note that if we take $|q| \geq 2$, then these are both hyperbolic.

6.2.1 Satellite construction

To describe the gluing map, we use the satellite construction to compare how the meridian and longitude of both of the double twist knots interact on the JSJ torus along which their complements meet. In particular, we show that the gluing map identifies these in pairs, and that each pair represents a nullhomologous curve on either side of the JSJ torus.

Consider the n -clasped untwisted Whitehead link pattern, $W^n = U \cup V^n$. We can view the component $V^n \subset \mathbb{S}_U^3 \cong \mathbb{S}^1 \times \mathbb{D}^2$ as sitting inside a solid torus with meridian μ and longitude λ . When we form the n -clasped Whitehead double of the companion (m, q) -double twist knot T_q^m , we make the following identifications:

$$\mu = \lambda_U = \mu_{T_q^m},$$

$$\lambda = \mu_U = \lambda_{T_q^m}.$$

The complement of K can now be expressed as $\mathbb{S}_K^3 \cong \mathbb{S}_{T_q^m}^3 \cup_{\mathbb{T}^2} \mathbb{S}_{W^n}^3$, where the union along the central JSJ torus $\mathbb{T}^2$ is specified by these identifications.

6.2.2 Effect of Dehn surgery

Now let's perform $1/q$ -surgery on K . Filling the pattern piece gives $\mathbb{S}_{W^n}^3(1/q) \cong \mathbb{S}_{T_q^n}^3$: we can see this by performing an isotopy so that the component labelled U is now sitting inside the solid torus $\mathbb{S}_{V^n}^3 \cong \mathbb{S}^1 \times \mathbb{D}^2$, and then viewing $1/q$ -surgery along V^n as applying a Rolfsen $-q$ -twist to U inside $\mathbb{S}_{V^n}^3$, as shown in Figure 6.1. This transforms U into the (n, q) -double twist knot T_q^n . In particular, observe that we now have the following identifications:

$$\begin{aligned}\mu_{T_q^m} &= \lambda_U = \lambda_{T_q^n}, \\ \lambda_{T_q^m} &= \mu_U = \mu_{T_q^n}.\end{aligned}$$

In other words, our gluing map exchanges meridians and longitudes. Note that each of these is nullhomologous on one side of the JSJ torus: the curve $\mu_{T_q^m} = \lambda_{T_q^n}$ on the pattern side by the $w = 0$ case of Lemma 3.3, and the curve $\lambda_{T_q^m} = \mu_{T_q^n}$ on the companion side since it bounds a Seifert surface.

6.3 Proof of Theorem 1.11

Proof of Theorem 1.11. Let $K = W^n(T_q^m)$ and $K' = W^m(T_q^n)$. Clearly, both $\mathbb{S}_K^3(1/q)$ and $\mathbb{S}_{K'}^3(1/q)$ admit a JSJ decomposition into two double twist knot complements, $\mathbb{S}_{T_q^m}^3$ and $\mathbb{S}_{T_q^n}^3$. Moreover, by the reasoning above, the gluing map in both cases is given by the identifications $\mu_{T_q^m} = \lambda_{T_q^n}$ and $\lambda_{T_q^m} = \mu_{T_q^n}$. Therefore $\mathbb{S}_K^3(1/q) \cong \mathbb{S}_{K'}^3(1/q)$, and $K \neq K'$ by the geometry of their complements. $\square$

6.3.1 Applications of Theorem 1.11

Proof of Corollary 6.1. Simply set $n = +1$ and $m = -1$. The result holds for all $q \in \mathbb{Z} \setminus \{0\}$. □

Chapter 7

The multicusped case

The goal of this chapter is to explore new methods for finding the constant $C(K)$ for which all slopes p/q with $|q| \geq C(K)$ are characterising for K . In particular, we will focus on the case where the outermost JSJ piece $Y \subset \mathbb{S}_K^3$ is hyperbolic with $m \geq 2$ cusps: we have already solved the $m = 1$ case in Theorem 1.6, but we have not yet eliminated the requirement for an additional constraint on the numerator in the $m = 2$ case, nor addressed the $m \geq 3$ case at all. Under certain circumstances, we will see how novel methods can be used to obstruct the swapping of JSJ pieces in an orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q)$ using only a condition on the denominator.

One of our techniques involves invoking the cyclic surgery theorem [CGLS87] in a similar way to the strategy used in [Sor23]. We will also sometimes need to assume Property H for some knot complement $\mathbb{S}_C^3 \subset \mathbb{S}_K^3$ and impose the condition $|q| \geq \mathcal{Q}(\mathbb{S}_C^3)$ discussed in the introduction. Combining this with the assumption $|q| \geq \max\{35, \mathfrak{q}(Y)\}$ originating from the minimal geodesics technique described earlier will then generate an explicit value for $C(K)$. This chapter originates from joint work with Patricia Sorya.

When $m = 2$, we use distances between exceptional fillings, and restrictions on hyperbolic volume when necessary, to prove Theorem 1.12.

Theorem 1.12. *Let K be a satellite knot whose complement has a 2-cusped hyperbolic outermost JSJ piece Y . Write $K = P(C)$ and $Y = \mathbb{S}_P^3$ for $P = U \cup Q$.*

- *If $P(U) = Q$ is a torus knot, then $C(K) = \max\{35, \mathfrak{q}(\mathbb{S}_P^3)\}$.*
- *If $P(U) = Q$ is an unknot and $\mathbb{S}_C^3$ contains only Seifert fibred JSJ pieces, then $C(K) = \max\{35, \mathfrak{q}(\mathbb{S}_P^3)\}$.*
- *If $\mathbb{S}_C^3$ contains no torus knot complements and satisfies Property H, then $C(K) = \max\{35, \mathfrak{q}(\mathbb{S}_P^3), \mathcal{Q}(\mathbb{S}_C^3)\}$.*

When $m = 3$, we can use the cyclic surgery theorem [CGLS87] as in [Sor23], and hyperbolic volume when necessary, to prove Theorem 1.13.

Theorem 1.13. *Let K be a satellite knot whose complement has a 3-cusped hyperbolic outermost JSJ piece Y . Let $\mathbb{S}_{C_0}^3$ and $\mathbb{S}_{C_1}^3$ be the knot complements adjacent to Y .*

- *If $\mathbb{S}_{C_0}^3$ and $\mathbb{S}_{C_1}^3$ are not torus knot complements, then $C(K) = \max\{35, \mathfrak{q}(Y)\}$.*
- *If $\mathbb{S}_{C_0}^3$ is a torus knot complement and $\mathbb{S}_{C_1}^3$ is a hyperbolic knot complement satisfying $\text{vol}(\mathbb{S}_{C_1}^3) < \text{vol}(\mathbb{S}_W^3)$, then $C(K) = \max\{35, \mathfrak{q}(Y), \mathcal{Q}(\mathbb{S}_{C_1}^3)\}$.*

When $m \geq 4$, the cyclic surgery theorem and a homological lemma are sufficient to prove Theorem 1.14, with no need for hyperbolic volume bounds.

Theorem 1.14. *Let K be a satellite knot whose complement has an m -cusped hyperbolic outermost JSJ piece Y with $m \geq 4$.*

Then $C(K) = \max\{35, \mathfrak{q}(Y)\}$.

The structure of this chapter is as follows. To begin with, there will be a short review of Sorya's cyclic surgery theorem strategy and an introduction to our new hyperbolic volume lower bound techniques. Combining these ideas leads to the proofs of Theorem 1.12, Theorem 1.13 and Theorem 1.14. We will intersperse our discussion with a series of applications and examples demonstrating the utility of these results.

7.1 The cyclic surgery theorem

Suppose that we have an orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q)$, where K is our fixed satellite knot with hyperbolic outermost JSJ piece Y and K' is some knot with outermost JSJ piece Y' . By Proposition 3.1, if $|q| \geq 3$, then $Y(p/q)$ is the JSJ piece of $\mathbb{S}_K^3(p/q)$ containing the surgery curve, whilst the JSJ piece of $\mathbb{S}_{K'}^3(p/q)$ containing the surgery curve is of the form $\tilde{Y}(\tilde{p}/\tilde{q})$: either (i) $Y'(p/q)$ or (ii) $\hat{Y}(p/qs^2)$ precisely when Y' is a compressing (r, s) -cable space adjacent to a JSJ piece $\hat{Y}$.

Sorya's strategy for proving the existence of the constant $C(K)$ in the general case [Sor23] uses the celebrated cyclic surgery theorem [CGLS87] to obstruct the swapping of JSJ pieces.

Theorem 7.1 (The cyclic surgery theorem [CGLS87]). *Let M be a compact orientable irreducible 3-manifold with $\partial M \cong \mathbb{T}^2$ and suppose that M is not a Seifert fibred space.*

If $\pi_1(M(\alpha))$ and $\pi_1(M(\beta))$ are both cyclic, then $\Delta(\alpha, \beta) \leq 1$.

We will see how this approach can be reinterpreted as one of the main ingredients in the construction of the constant $C(K)$ in the proofs of Theorem 1.13 and Theorem 1.14.

7.1.1 Induced satellite splittings

Suppose for a contradiction that the orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q)$ restricts to $Y(p/q) \cong X'$ for some JSJ piece $X' \subset \mathbb{S}_{K'}^3$. Observe that X' is the outermost JSJ piece of a uniquely determined companion knot complement $\mathbb{S}_C^3 \subset \mathbb{S}_{K'}^3$. This induces a satellite splitting $\mathbb{S}_{K'}^3 \cong \mathbb{S}_C^3 \cup \mathbb{S}_{P'}^3$, which enables us to write $K' = P'(C')$. Now we select the companion knot complement $\mathbb{S}_C^3 \subset \mathbb{S}_K^3$ for which $\mathbb{S}_{P'}^3(p/q) \cong \mathbb{S}_C^3$. This induces a satellite splitting $\mathbb{S}_K^3 \cong \mathbb{S}_P^3 \cup \mathbb{S}_C^3$; thus we can write $K = P(C)$.

Note that this process is not symmetric in general: starting with an orientation-preserving homeomorphism $\tilde{Y}(\tilde{p}/\tilde{q}) \cong X$ (as we will soon do when comparing hyperbolic volumes) could lead to different choices of patterns and companions. However, this section will exclusively use the satellite splittings stemming from an orientation-preserving homeomorphism $Y(p/q) \cong X'$ as described above.

Let $T_C := \mathbb{S}_C^3 \cap \mathbb{S}_P^3$ and $T_{C'} := \mathbb{S}_{C'}^3 \cap \mathbb{S}_{P'}^3$ be the JSJ tori that separate the pattern and companion in each of these splittings. In particular, observe that the orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q)$ sends T_C to $T_{C'}$, whilst swapping the pattern and companion sides by assumption: $Y(p/q) \cong X' \subset \mathbb{S}_C^3 \cong \mathbb{S}_P^3(p/q)$ and $\mathbb{S}_C^3 \cong \mathbb{S}_{P'}^3(p/q)$.

The idea is to fill $\mathbb{S}_P^3$ along the slope on T_C corresponding to $\mu_{C'}$ and then apply the cyclic surgery theorem to a pair of lens space fillings of the resulting manifold $M := \mathbb{S}_P^3(T_C; \mu_{C'})$. In this chapter, the boundary torus along which a filling takes place will usually be specified in the notation; when omitted, the filling is along the outermost boundary torus, which corresponds to the satellite knot itself.

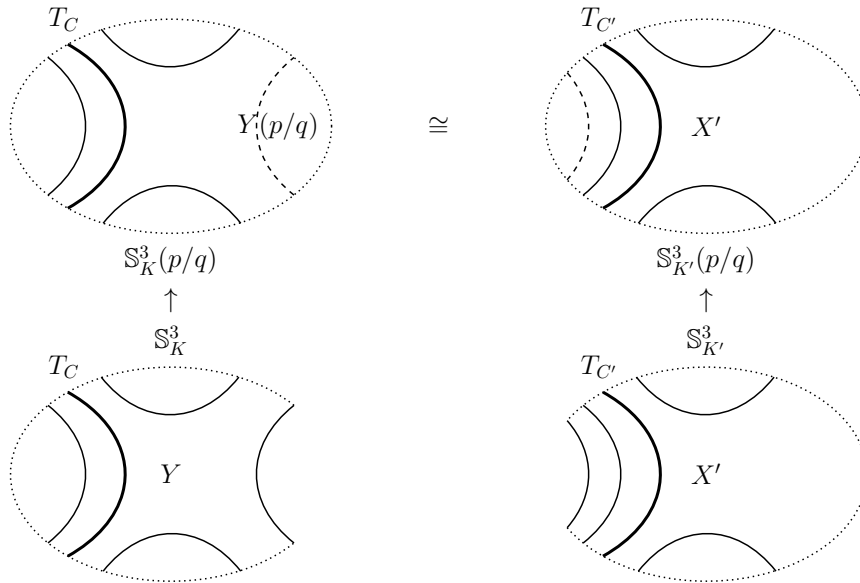


Figure 7.1: The satellite splittings induced by $Y(p/q) \cong X'$.

7.1.2 Lens space fillings

The following lemma uses the cyclic surgery theorem to show that the filling $\mathbb{S}_P^3(T_C; \mu_{C'})$ of $\mathbb{S}_P^3$ along T_C with slope $\mu_{C'}$ should produce a Seifert fibred space. This is a special case of [Sor23, Section 5.3].

Lemma 7.2. *Let K be a satellite knot whose outermost JSJ piece Y is hyperbolic and suppose there is an orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q)$ for some knot K' and some slope p/q with $|q| \geq 3$. Suppose that this restricts to $Y(p/q) \cong X' \subset \mathbb{S}_{K'}^3$ and let $K = P(C)$ and $K' = P'(C')$ be the induced satellite splittings along tori T_C and $T_{C'}$, respectively.*

Then $M := \mathbb{S}_P^3(T_C; \mu_{C'})$ is a Seifert fibred space.

Proof. Write $P = U \cup Q$ and consider the following two (not necessarily distinct) lens space fillings of M :

$$\begin{aligned} M(p/q) &\cong \mathbb{S}_{U,Q}^3(\mu_{C'}, p/q) \cong \mathbb{S}_{C'}^3(\mu_{C'}) \cong \mathbb{S}^3; \\ M(1/0) &\cong \mathbb{S}_{U,Q}^3(\mu_{C'}, 1/0) \cong \mathbb{S}_U^3(\mu_{C'}) \cong L(a, b). \end{aligned}$$

Since $\Delta(p/q, 1/0) = |q| \geq 2$, M must be a Seifert fibred space by Theorem 7.1. $\square$

It follows that $Y(T_C; \mu_{C'}) \subset M$ is also a Seifert fibred space. Since $Y \subset \mathbb{S}_P^3$ is a hyperbolic JSJ piece, this means that it must have a non-trivial exceptional filling. Note that we cannot directly use an assumption of the form $|q| \geq 3$ to prevent $Y(T_C; \mu_{C'})$ from being a Seifert fibred space (as we did for the filling $Y(p/q)$ of Y along the boundary component corresponding to K) because this filling takes place along the inner boundary torus $T_C \subset \partial Y$, which does not necessarily sit in the complement of an unlink. Instead, our strategy (except in a few special cases) will be to show that no Seifert fibred structure on $Y(T_C; \mu_{C'})$ would be able to extend over all of the adjacent JSJ pieces that form M . It will follow that M cannot actually be a Seifert fibred space, contradicting the assumption that $Y(p/q) \cong X' \subset \mathbb{S}_{K'}^3$.

7.2 Lower volume bounds

In this section, we will introduce a new technique that can be used to show that the orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q)$ must restrict to one $Y(p/q) \cong \tilde{Y}(\tilde{p}/\tilde{q})$ between the JSJ pieces containing the surgery curves. This idea will be used in the proofs of Theorem 1.12 and Theorem 1.13.

7.2.1 Hyperbolic JSJ pieces

Suppose as in the previous section that the orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q)$ does not restrict to one $Y(p/q) \cong \tilde{Y}(\tilde{p}/\tilde{q})$ between the JSJ pieces containing the surgery curves. Rather than contradicting the assumption that $Y(p/q) \cong X' \subset \mathbb{S}_{K'}^3$, our perspective will now be focused on trying to obstruct $\tilde{Y}(\tilde{p}/\tilde{q}) \cong X \subset \mathbb{S}_K^3$. In particular, we will look at the case where X is a hyperbolic JSJ piece of low volume. The idea is that, if Property H is satisfied for $\mathbb{S}_K^3 \setminus Y$, then no k -cusped hyperbolic JSJ piece $\tilde{Y} = \mathbb{S}_L^3$ can be filled to produce a $(k-1)$ -cusped hyperbolic JSJ piece $\tilde{Y}(\tilde{p}/\tilde{q}) \cong X \subset \mathbb{S}_K^3 \setminus Y$.

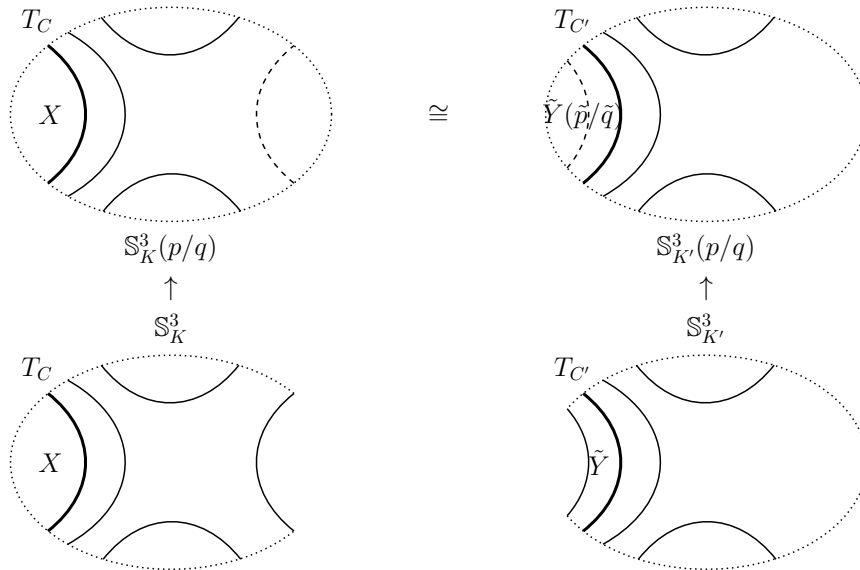


Figure 7.2: The satellite splittings induced by $\tilde{Y}(\tilde{p}/\tilde{q}) \cong X$.

7.2.2 Existence and construction

Let us recall the definition of Property H.

Definition 7.3. *A $(k - 1)$ -cusped orientable hyperbolic 3-manifold X is said to satisfy Property H if $\text{vol}(X) < \mathcal{V}_k$, where $\mathcal{V}_k$ denotes the least possible volume of all k -cusped orientable hyperbolic 3-manifolds.*

A compact orientable irreducible 3-manifold M with empty or toroidal boundary satisfies Property H if every hyperbolic JSJ piece of M satisfies Property H.

For example, $\mathcal{V}_2$ is achieved by the Whitehead link complement; hence all hyperbolic twist knot complements satisfy Property H. Since the presence of extra Seifert fibred spaces in the JSJ decomposition has no effect on this, the same goes for their cables and composites with one another.

If Property H is satisfied, then the following quantity is well-defined. (Here, the ceiling function is defined as $\lceil x \rceil := \min\{n \in \mathbb{Z} \mid n > x\}$, so that $\lceil n \rceil := n + 1$.)

Definition 7.4. *Let X be a $(k - 1)$ -cusped orientable hyperbolic 3-manifold satisfying Property H: $\text{vol}(X) < \mathcal{V}_k$. Then*

$$\mathcal{Q}(X) := \left\lceil 2\pi \sqrt{\frac{3}{1 - (\mathcal{V}_k^{-1} \cdot \text{vol}(X))^{2/3}}} \right\rceil \in \mathbb{Z}.$$

Let M be a compact orientable irreducible 3-manifold with empty or toroidal boundary satisfying Property H and let $\mathcal{X}$ denote its set of hyperbolic JSJ pieces. Then

$$\mathcal{Q}(M) := \begin{cases} \max\{\mathcal{Q}(X) \mid X \in \mathcal{X}\} & \mathcal{X} \neq \emptyset; \\ 0 & \mathcal{X} = \emptyset. \end{cases}$$

Let X be a $(k - 1)$ -cusped orientable hyperbolic 3-manifold satisfying Property H. The following lemma demonstrates how the condition $|\tilde{q}| \geq \mathcal{Q}(X)$ prevents the existence of an orientation-preserving homeomorphism $\tilde{Y}(\tilde{p}/\tilde{q}) \cong X$ for any arbitrary

k -cusped hyperbolic JSJ piece $\tilde{Y} = \mathbb{S}_{\tilde{L}}^3$. The proof will make use of the following quantity as a lower bound on slope length: $\mathcal{L}(X) := \mathcal{Q}(X)/\sqrt{3}$.

Lemma 7.5. *Let X be a $(k-1)$ -cusped orientable hyperbolic 3-manifold satisfying $\text{vol}(X) < \mathcal{V}_k$. Let $\tilde{Y} = \mathbb{S}_{\tilde{L}}^3$ be a k -cusped hyperbolic JSJ piece, where $\tilde{L} = \tilde{U}_{k-1} \cup \tilde{J}$ is a hyperbolic link, and let $\tilde{p}/\tilde{q}$ be a slope on the component of $\partial\tilde{Y}$ corresponding to $\tilde{J}$.*

If $|\tilde{q}| \geq \mathcal{Q}(X)$, then there can be no orientation-preserving homeomorphism $\tilde{Y}(\tilde{p}/\tilde{q}) \cong X$.

Proof. Since $\text{vol}(X) < \mathcal{V}_k$, the quantity $\mathcal{Q}(X)$ is well-defined. Let $\tilde{Y} = \mathbb{S}_{\tilde{L}}^3$ be a k -cusped hyperbolic JSJ piece as above and suppose that there exists a homeomorphism $\tilde{Y}(\tilde{p}/\tilde{q}) \cong X$. Then taking

$$|\tilde{q}| \geq \mathcal{Q}(X) \geq 11$$

and applying the 6-theorem ensures that the length of the slope satisfies

$$l(\tilde{p}/\tilde{q}) \geq \mathcal{L}(X) > 2\pi.$$

By the minimality of $\mathcal{V}_k$ and [FKP08], respectively, this gives the inequalities

$$\left(1 - \left(\frac{2\pi}{\mathcal{L}(X)}\right)^2\right)^{3/2} \mathcal{V}_k \leq \left(1 - \left(\frac{2\pi}{\mathcal{L}(X)}\right)^2\right)^{3/2} \cdot \text{vol}(\tilde{Y}) \leq \text{vol}(\tilde{Y}(\tilde{p}/\tilde{q})) = \text{vol}(X).$$

However, by the definition of $\mathcal{Q}(X)$, the following series of inequalities also holds:

$$\begin{aligned} \mathcal{Q}(X) &> 2\pi \sqrt{\frac{3}{1 - (\mathcal{V}_k^{-1} \cdot \text{vol}(X))^{2/3}}} \\ \iff \mathcal{L}(X) &> \frac{2\pi}{\sqrt{1 - (\mathcal{V}_k^{-1} \cdot \text{vol}(X))^{2/3}}} \\ \iff \text{vol}(X) &< \left(1 - \left(\frac{2\pi}{\mathcal{L}(X)}\right)^2\right)^{3/2} \mathcal{V}_k. \end{aligned}$$

This contradicts the assumption that $X \cong \tilde{Y}(\tilde{p}/\tilde{q})$. □

7.2.3 Examples of $\mathcal{Q}(X)$

Here are some examples of $\mathcal{Q}(X)$ for compact orientable hyperbolic 3-manifolds X satisfying Property H.

Example 7.6. Let $X = \mathbb{S}_S^3$ be the figure eight knot complement, which has a volume of approximately

$$\text{vol}(\mathbb{S}_S^3) \approx 2.0298832128.$$

This is less than the volume $\mathcal{V}_2$ of the smallest 2-cusped orientable hyperbolic 3-manifold, which is achieved by the Whitehead link complement, at approximately

$$\text{vol}(\mathbb{S}_W^3) \approx 3.6638623767.$$

This gives

$$\mathcal{Q}(\mathbb{S}_S^3) = \left\lceil 2\pi \sqrt{\frac{3}{1 - (\mathcal{V}_2^{-1} \cdot \text{vol}(\mathbb{S}_S^3))^{2/3}}} \right\rceil = 20.$$

Table B.2 contains a list of some more hyperbolic knot complements $\mathbb{S}_C^3$ with $\text{vol}(\mathbb{S}_C^3) < \mathcal{V}_2$ and the corresponding values of $\mathcal{Q}(\mathbb{S}_C^3)$.

Remark 7.7. It appears that, for a given number of tetrahedra in the triangulation, each knot C in Table B.2 is constructed by performing a Rolfsen twist along an unknotted component of either the Whitehead link $\pm W = W^\pm$ (with either clasp) or its sister the $(-2, 3, 8)$ -pretzel link $\pm P(-2, 3, 8)$ (possibly its mirror). This is unsurprising, as these are known to be precisely the links whose complements achieve the minimal 2-cusped volume $\mathcal{V}_2$ [Ago10], and Dehn filling one component must decrease hyperbolic volume [Thu80].

Unlike the minimal 2-cusped hyperbolic volume $\mathcal{V}_2$, the minimal 3-cusped hyperbolic volume $\mathcal{V}_3 = \text{vol}(\mathbb{S}_{C_3}^3)$ (where C_3 denotes the minimally twisted 3-chain link) appearing in the next example is conjectural, both in terms of its value and the link complement which realises it.

Example 7.8. Let $X = \mathbb{S}_W^3$ be the Whitehead link complement, whose volume $\mathcal{V}_2$ is stated above. This is less than the volume $\mathcal{V}_3$ of the smallest 3-cusped orientable hyperbolic 3-manifold, which is (conjecturally) the complement of the 3-chain link C_3 , at approximately

$$\text{vol}(\mathbb{S}_{C_3}^3) \approx 5.3334895669.$$

Modulo this conjecture, this gives

$$\mathcal{Q}(\mathbb{S}_W^3) = \left\lceil 2\pi \sqrt{\frac{3}{1 - (\mathcal{V}_3^{-1} \cdot \text{vol}(\mathbb{S}_W^3))^{2/3}}} \right\rceil = 24.$$

The minimal 4-cusped hyperbolic volume $\mathcal{V}_4 = \text{vol}(\mathbb{S}_{C_4}^3)$ is known explicitly and achieved by the complement of the minimally twisted 4-chain link C_4 [Yos13]. There are certainly 3-cusped orientable hyperbolic 3-manifolds X with volume less than $\mathcal{V}_4$; however, imposing the extra condition that $X \cong \mathbb{S}_L^3$ for a hyperbolic link L such that removing one component leaves the unlink seems to be very restrictive. Despite running extensive searches on SnapPea, we have been unable to find any examples of this whatsoever. The same goes when we increase the number of cusps.

7.3 Proof of Theorem 1.12

The goal of this section is to prove Theorem 1.12 (the case where $|\partial Y| = 2$).

In earlier work on this situation, the swapping of JSJ pieces is prevented by imposing an explicit condition on the numerator: assuming that $\gcd(p, w) \neq 1$, where w denotes the winding number of P , the outermost JSJ piece $Y = \mathbb{S}_P^3$ cannot (for homological reasons) become a knot complement $Y(p/q) \cong X' = \mathbb{S}_{C'}^3 \subset \mathbb{S}_{K'}^3$ after filling, and therefore must be sent to the newly created JSJ piece $\tilde{Y}(\tilde{p}/\tilde{q})$ in the other filling. However, we can actually use a more direct obstruction which arises from comparing distances between exceptional filling slopes.

The following lemma relies on $P(U) = Q$ being an unknot or a torus knot in $\mathbb{S}^3$.

Lemma 7.9. *Let $K = P(C)$ be the satellite of a companion C by a hyperbolic pattern $P = U \cup Q$, where $P(U) = Q$ is an unknot or a torus knot, and suppose there is an orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q)$ for some knot K' and some slope p/q with $|q| \geq 9$. Suppose that this restricts to $\mathbb{S}_P^3(p/q) \cong \mathbb{S}_{C'}^3 \subset \mathbb{S}_{K'}^3$, and let $K' = P'(C')$ be the induced satellite splitting.*

Then $P(U) = Q$ is an unknot, $|p| = 1$ and P' has winding number $w' = 0$.

Proof. Observe that the $|q| \geq 3$ condition ensures that $\mathbb{S}_P^3(p/q)$ is still hyperbolic after the filling; thus $\mathbb{S}_{C'}^3$ is a hyperbolic knot complement. Consider the filling

$$\mathbb{S}_{C'}^3(\mu_U) \cong \mathbb{S}_{U,Q}^3(1/0, p/q) \cong \mathbb{S}_Q^3(p/q).$$

Since $P(U) = Q$ is an unknot or a torus knot, $\mathbb{S}_Q^3(p/q)$ cannot be hyperbolic.

By [Sor23, Lemma 4.3], the slope of the meridian $\mu_{C'}$ on the boundary of $\mathbb{S}_P^3(p/q)$ with respect to the usual coordinates is (for some y)

$$\begin{cases} y/qw'^2 & w' \neq 0; \\ 1/0 & w' = 0. \end{cases}$$

The rest of the argument depends on the winding number w' of P' .

When $w' \neq 0$, the slopes $\mu_{C'} \neq \mu_U$ give two distinct exceptional fillings of the hyperbolic knot complement $\mathbb{S}_P^3(p/q) \cong \mathbb{S}_{C'}^3$ with distance

$$\Delta(\mu_{C'}, \mu_U) = \Delta(y/qw'^2, 1/0) = |q|w'^2 \geq |q| \geq 9$$

apart, exceeding the maximum possible value given in Theorem 2.7 ($|q| \leq 8$).

When $w' = 0$, the slopes $\mu_{C'} = \mu_U$ coincide and we have

$$\mathbb{S}_Q^3(p/q) \cong \mathbb{S}_{C'}^3(\mu_U) \cong \mathbb{S}_{C'}^3(\mu_{C'}) \cong \mathbb{S}^3.$$

If $P(U) = Q$ is a torus knot, then p/q is the trivial slope, which is a contradiction. If $P(U) = Q$ is unknotted, then the slope p/q gives an $\mathbb{S}^3$ filling if and only if $|p| = 1$. $\square$

We are now ready to prove Theorem 1.12.

Proof of Theorem 1.12. Let $K = P(C)$ be a satellite of any knot C by a hyperbolic pattern $P = U \cup Q$. Suppose that there is an orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q)$ for a knot K' with outermost JSJ piece Y' . Since $|q| \geq 9$, Proposition 3.1 tells us that $\mathbb{S}_P^3(p/q)$ remains hyperbolic and that the JSJ piece of $\mathbb{S}_{K'}^3(p/q)$ containing the surgery curve is of the form $\tilde{Y}(\tilde{p}/\tilde{q})$: either (i) $Y'(p/q)$ or (ii) $\hat{Y}(p/qs^2)$ precisely when Y' is a compressing (r, s) -cable space adjacent to a JSJ piece $\hat{Y}$. We claim that $\mathbb{S}_P^3(p/q) \cong \tilde{Y}(\tilde{p}/\tilde{q})$.

Suppose for a contradiction that $\mathbb{S}_P^3(p/q) \cong \mathbb{S}_{C'_0}^3 \subset \mathbb{S}_{K'}^3$, and let $K' = P'_0(C'_0)$ be the induced satellite splitting. If $P(U) = Q$ is a torus knot, then this is already impossible by Lemma 7.9. If $P(U) = Q$ is an unknot, write $\tilde{P}_0$ to be either (i) P'_0 itself or (ii) the pattern $\hat{P}_0$ for which $P'_0 = J_{r,s}(\hat{P}_0)$ precisely when there is a compressing (r, s) -cable space. By Lemma 7.9, the pattern P'_0 has winding number $w'_0 = 0$ (and thus $\tilde{P}_0$ also has winding number $\tilde{w}_0 = 0$). It follows that $\mathbb{S}_{\tilde{P}_0}^3$ must contain a hyperbolic JSJ piece: an (r', s') -cable space contributes a factor of $s' \neq 0$ and a capped-off composing space contributes a factor of 1. However, if $\mathbb{S}_{P'_0}^3(p/q) \cong \mathbb{S}_{\tilde{P}_0}^3(\tilde{p}/\tilde{q}) \cong \mathbb{S}_C^3$ contains only Seifert fibred JSJ pieces, then this is clearly impossible.

Now let's suppose instead that $\mathbb{S}_C^3$ contains no torus knot complements and satisfies Property H. Rather than obstructing $\mathbb{S}_P^3(p/q) \cong \mathbb{S}_{C'_0}^3 \subset \mathbb{S}_{K'}^3$, we will try to contradict the assumption that $\tilde{Y}(\tilde{p}/\tilde{q}) \cong X \subset \mathbb{S}_C^3$ by considering each possibility for $\tilde{Y}$ in turn. If $\tilde{Y}$ is a cable space, then $\tilde{Y}(\tilde{p}/\tilde{q})$ would be a torus knot complement, but our hypothesis ensures that there are none present in $\mathbb{S}_C^3$. If $\tilde{Y}$ is a composing space, then [Sor23, Lemma 5.4] automatically obstructs the JSJ pieces from switching. Finally, if $\tilde{Y}$ is hyperbolic, then we take $|q| \geq \mathcal{Q}(\mathbb{S}_C^3)$ and apply Lemma 7.5.

We deduce that the orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q)$ restricts to one between the JSJ pieces containing the surgery curves. Consider the hyperbolic pattern $\tilde{P}$ for which $\tilde{Y} = \mathbb{S}_{\tilde{P}}^3$. Then we in fact have a slope-preserving homeomorphism $\mathbb{S}_P^3(p/q) \cong \mathbb{S}_{\tilde{P}}^3(\tilde{p}/\tilde{q})$, because the orientation-preserving homeomorphism $\mathbb{S}_C^3 \cong \mathbb{S}_{C'}^3$ sends meridians to meridians by [GL89]. The condition $|q| \geq \max\{35, \mathfrak{q}(\mathbb{S}_P^3)\}$ is sufficient to ensure that the minimal geodesics are precisely the core curves by Proposition 4.8. This gives either $K' = J_{r,s}(K)$, which can be ruled out by Corollary 2.12, or $K = K'$, as required. $\square$

7.3.1 Applications of Theorem 1.12

Theorem 1.12 provides values of $C(K)$ when $K = P(C)$ is a satellite knot by some hyperbolic pattern $P = U \cup Q$.

Example 7.10. Let $K = W_t^n(C)$ be an n -clasped t -twisted Whitehead double of a knot C . In the special case where $n = t = \pm 1$, applying the pattern to the unknot produces a trefoil, $W_{\pm 1}^\pm(U) = \pm T$, and thus $C(K) = \max\{35, \mathfrak{q}(\mathbb{S}_W^3)\} = 35$ for any companion knot C . Note that $W_0^n(U) = U$ for all values of n . If $|n| \leq 4$, then $\text{sys}(\mathbb{S}_{W_t^n}^3) > 0.14$ (regardless of t), so $\mathfrak{q}(\mathbb{S}_{W_t^n}^3) < 35$; if $|n| \geq 5$, then $\text{sys}(\mathbb{S}_{W_t^n}^3) < 0.14$ (by the same argument as in the proof of Corollary 1.9), so $\mathfrak{q}(\mathbb{S}_{W_t^n}^3) > 35$. Taking this into account, Table 7.1 shows the other values of $C(K)$ coming from Theorem 1.12.

$C(K)$	$ n \leq 4$	$ n \geq 5$
$t = 0$ and $\mathbb{S}_C^3$ contains only Seifert fibred JSJ pieces	35	$\mathfrak{q}(\mathbb{S}_{W^n}^3)$
$\mathbb{S}_C^3$ contains no torus knot complements and satisfies Property H	$\max\{35, \mathcal{Q}(\mathbb{S}_C^3)\}$	$\max\{\mathfrak{q}(\mathbb{S}_{W^n}^3), \mathcal{Q}(\mathbb{S}_C^3)\}$

Table 7.1: Values of $C(K)$ when $K = W_t^n(C)$.

If the SnapPea census of 2-cusped orientable hyperbolic 3-manifolds is complete up to stage k of Table 1.1, then we can actually reduce 35 to q_k in the $|n| = 1$ case.

7.4 Proof of Theorem 1.13

In this section, we will prove Theorem 1.13 (the case where $|\partial Y| = 3$).

The idea is to use Lemma 7.2 to ensure that the orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q)$ must restrict to one between the pieces containing the surgery curves. Write $K = P_0(C_0)$ and $K = P_1(C_1)$ for the satellite splittings induced by the knot complements attached to Y , so that $\mathbb{S}_{P_0}^3 = \mathbb{S}_{C_0}^3 \cup Y$ and $\mathbb{S}_{P_1}^3 = \mathbb{S}_{C_1}^3 \cup Y$. Suppose that $Y(p/q) \cong X'$ for some X' which is the outermost JSJ piece of a uniquely determined companion knot complement $\mathbb{S}_{C'}^3 \subset \mathbb{S}_{K'}^3 \setminus Y$ and write $K' = P'(C')$. This splitting occurs along a torus $T_{C'} \subset \partial X'$, which corresponds to one of the unfilled $T_i \subset \partial Y(p/q)$, so $\mathbb{S}_{P_i}^3(p/q) \cong \mathbb{S}_{C'}^3$ and $\mathbb{S}_{C_i}^3 \cong \mathbb{S}_{P'}^3(p/q)$ for either $i = 0$ or $i = 1$. This is illustrated in Figure 7.3.

Lemma 7.2 says that $Y(T_i; \mu_{C'})$ is a Seifert fibred space, but we claim that a Seifert fibration of $Y(T_i; \mu_{C'})$ cannot extend to the whole of $\mathbb{S}_{P_i}^3(T_i; \mu_{C'}) = \mathbb{S}_{C_j}^3 \cup Y(T_i; \mu_{C'})$ (where $i \neq j$). The issue arises when $Y(T_i; \mu_{C'})$ admits a Seifert fibred structure that could potentially extend over an attached torus knot complement.

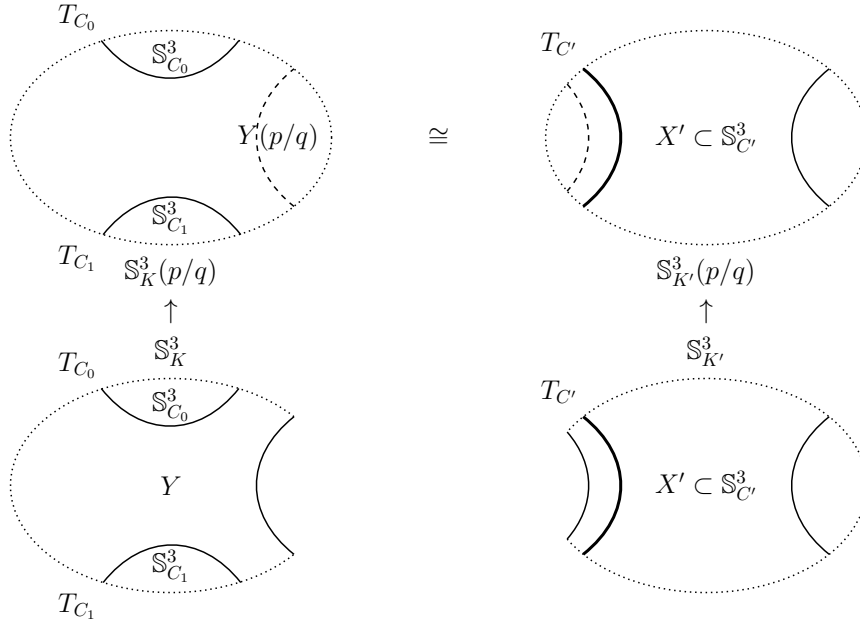


Figure 7.3: Obstructing the swapping of JSJ pieces in the $|\partial Y| = 3$ case.

The following lemma will be used in our proof to contradict Lemma 7.2.

Lemma 7.11. *Let K be a satellite knot whose outermost JSJ piece Y is hyperbolic with $m = 3$ cusps. Let $\mathbb{S}_{C_0}^3$ and $\mathbb{S}_{C_1}^3$ be the knot complements attached to Y and suppose that $\mathbb{S}_{C_1}^3$ is not a torus knot complement. Let γ_0 be a slope on T_0 .*

Then $\mathbb{S}_{P_0}^3(T_0; \gamma_0)$ is not a Seifert fibred space.

Proof. In order for the whole of $\mathbb{S}_{P_0}^3(T_0; \gamma_0) = \mathbb{S}_{C_1}^3 \cup Y(T_0; \gamma_0)$ to be a Seifert fibred space, the outermost JSJ piece of the attached knot complement $\mathbb{S}_{C_1}^3$ must admit a Seifert fibration which can extend across the whole of $\mathbb{S}_{P_0}^3(T_0; \gamma_0)$. Since C_1 is not a torus knot, it must be either a cable knot or a composite knot. In both cases, the Seifert fibration on the outermost JSJ piece of $\mathbb{S}_{C_1}^3$ is unique and does not extend over the other JSJ pieces of $\mathbb{S}_{C_1}^3$. Hence $\mathbb{S}_{P_0}^3(T_0; \gamma_0)$ is not a Seifert fibred space. $\square$

Combining this with lower volume bounds, we can now prove Theorem 1.13.

Proof of Theorem 1.13. Let K be a satellite knot whose complement has a hyperbolic outermost JSJ piece Y with $m = 3$ cusps. Suppose that there is an orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q)$ for some knot K' with outermost JSJ piece Y' . Since $|q| \geq 9$, Proposition 3.1 implies that $Y(p/q)$ is still hyperbolic and that the JSJ piece of $\mathbb{S}_{K'}^3(p/q)$ containing the surgery curve can be expressed as $\tilde{Y}(\tilde{p}/\tilde{q})$: either (i) $Y'(p/q)$ or (ii) $\hat{Y}(p/qs^2)$ precisely when Y' is a compressing (r, s) -cable space adjacent to a JSJ piece $\hat{Y}$. We first need to show that $Y(p/q) \cong \tilde{Y}(\tilde{p}/\tilde{q})$.

Suppose instead that $Y(p/q) \cong X'$ for some $X' \subset \mathbb{S}_{K'}^3$ with $|\partial X'| = 2$ which is the outermost JSJ piece of a companion knot complement $\mathbb{S}_{C'}^3 \subset \mathbb{S}_{K'}^3$, and let $K' = P'(C')$ be the induced satellite splitting. Then $\mathbb{S}_{P_i}^3(p/q) \cong \mathbb{S}_{C'}^3$ and $\mathbb{S}_{C_i}^3 \cong \mathbb{S}_{P'}^3(p/q)$ for either $i = 0$ or $i = 1$ (see Figure 7.3). We will need different methods for each of these cases, but the key hypothesis is that $\mathbb{S}_{C_1}^3$ is never a torus knot complement.

If $i = 0$, then we will use the cyclic surgery theorem. By Lemma 7.2, we know that $\mathbb{S}_{P_0}^3(T_0; \mu_{C'})$ is a Seifert fibred space. However, $\mathbb{S}_{C_1}^3$ is not a torus knot complement by assumption, so Lemma 7.11 gives an immediate contradiction.

If $i = 1$, then there is an orientation-preserving homeomorphism $\mathbb{S}_{C_1}^3 \cong \mathbb{S}_{\tilde{P}}^3(\tilde{p}/\tilde{q})$, where the hyperbolic pattern $\tilde{P}$ is either (i) P' itself or (ii) the pattern $\hat{P}$ for which $\tilde{P} = J_{r,s}(\hat{P})$ precisely when there is a compressing (r, s) -cable space. However, if $|q| \geq \mathcal{Q}(\mathbb{S}_{C_1}^3)$, then this is obstructed by Lemma 7.5.

Now that the orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q)$ is known to restrict to a slope-preserving homeomorphism between the JSJ pieces containing the surgery curves, the $|q| \geq \max\{35, \mathfrak{q}(Y)\}$ assumption ensures that core curves are sent to core curves by Proposition 4.8. Therefore either $K' = J_{r,s}(K)$, which can be ruled out by Corollary 2.12 as usual, or $K = K'$, as required. $\square$

7.4.1 Applications of Theorem 1.13

Let $L = L_K \cup L_0 \cup L_1$ be a 3-component link such that $L \setminus L_K = L_0 \cup L_1$ is a 2-component unlink. Write $K = L(C_0, C_1)$ for the satellite knot obtained by gluing the knot complements $\mathbb{S}_{C_0}^3$ and $\mathbb{S}_{C_1}^3$ to the boundary components of $Y = \mathbb{S}_L^3$ corresponding to L_0 and L_1 in the standard way. Notice that K could also be written in the usual satellite form as $K = P_0(C_0)$ or $K = P_1(C_1)$ for some suitable patterns P_0 and P_1 , but our new notation conveys more directly the JSJ decomposition of $\mathbb{S}_K^3$.

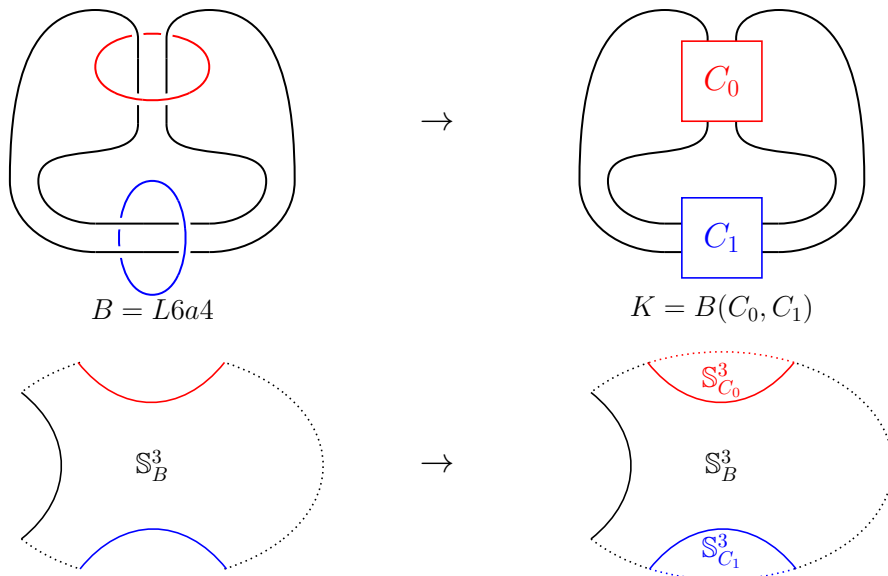


Figure 7.4: The Borromean rings $B = L6a4$ and a satellite $B(C_0, C_1)$.

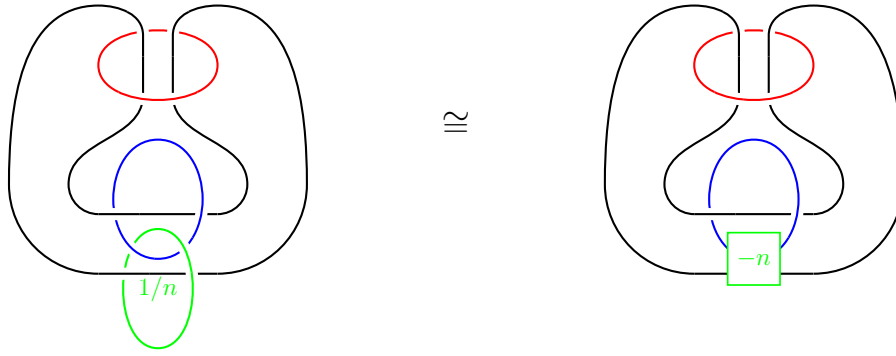


Figure 7.5: Rolfsen $-n$ -twist giving $\mathbb{S}_L^3(1/n) \cong \mathbb{S}_{B^n}^3$ for $L = L10n100$.

Example 7.12. Let $B^n(C_0, C_1)$ be a satellite of C_0 and C_1 by the generalised Borromean rings pattern B^n (this is obtained from the 4-component link $L = L10n100$ by performing a Rolfsen $-n$ -twist along a specific component, as depicted in Figure 7.5). Table A.2 lists the values of $\mathfrak{q}(\mathbb{S}_{B^n}^3)$ for each value of n and Table 7.2 shows the corresponding constant $C(K)$ arising from Theorem 1.13.

The usual Borromean rings $B = B^1$ is listed in SnapPea as $L6a4$ with complement $t12067$ and $\mathfrak{q}(\mathbb{S}_{B^1}^3) = 19$. The link B^{-1} is listed in SnapPea as $L8n5$ with complement $t12066$ and $\mathfrak{q}(\mathbb{S}_{B^{-1}}^3) = 20$. In fact, we have $\mathfrak{q}(\mathbb{S}_{B^n}^3) < 35$ for all $|n| \leq 4$.

The generalised Borromean rings $B^5 = L14a24647$ is the first for which this fails: namely, it has $\mathfrak{q}(\mathbb{S}_{B^5}^3) = 37$. The next link B^{-5} has more than 14 crossings, so it does not appear in SnapPea's HTLinkExteriors census, but it also satisfies $\mathfrak{q}(\mathbb{S}_{B^{-5}}^3) = 37$. We can check by hand that $\mathfrak{q}(\mathbb{S}_{B^n}^3) > 35$ for all $5 \leq |n| \leq 34$ (see Table A.2).

Noting that the complement $\mathbb{S}_L^3$ of the 4-component link $L = L10n100$ has the same systole as the usual Borromean rings B , we can prove (as we did earlier for the t -twist knots and n -clasped Whitehead links) that $\mathfrak{q}(\mathbb{S}_{B^n}^3) > 35$ for all $|n| \geq 35$ as well. This gives the values of $C(K)$ in Table 7.2.

The $|n| = 1$ case is rather unusual in that $\text{sys}(\mathbb{S}_{B^1}^3) \neq \text{sys}(\mathbb{S}_{B^{-1}}^3)$. It is interesting to note that $B^{-n} \neq -B^n$ in general, yet it appears that $\text{sys}(\mathbb{S}_{B^n}^3) = \text{sys}(\mathbb{S}_{B^{-n}}^3)$ and thus $\mathfrak{q}(\mathbb{S}_{B^n}^3) = \mathfrak{q}(\mathbb{S}_{B^{-n}}^3)$ for all $|n| \geq 2$. This suggests that taking $|n| \geq 2$ is sufficient to ensure that the core curve of the filling $\mathbb{S}_L^3(1/n) \cong \mathbb{S}_{B^n}^3$ is the shortest geodesic.

$C(K)$	$ n \leq 4$	$ n \geq 5$
$\mathbb{S}_{C_0}^3$ and $\mathbb{S}_{C_1}^3$ are not torus knot complements	35	$\mathfrak{q}(\mathbb{S}_{B^n}^3)$
$\mathbb{S}_{C_0}^3$ is a torus knot complement and $\mathbb{S}_{C_1}^3$ is a hyperbolic knot complement with $\text{vol}(\mathbb{S}_{C_1}^3) < \text{vol}(\mathbb{S}_W^3)$	$\max\{35, \mathcal{Q}(\mathbb{S}_C^3)\}$	$\max\{\mathfrak{q}(\mathbb{S}_{B^n}^3), \mathcal{Q}(\mathbb{S}_{C_1}^3)\}$

 Table 7.2: Values of $C(K)$ when $K = B^n(C_0, C_1)$.

There are 34590 3-component links with up to 14 crossings in the HTLinkExteriors census, which is complete up to this point. We can check that precisely 14121 of these candidates are hyperbolic and contain a 2-component unlink. Checking the systoles of their complements, we obtain the following result.

Theorem 7.13. *Let Y be the complement of a 3-component hyperbolic link $L = L_K \cup L_0 \cup L_1$ such that $L_0 \cup L_1$ is an unlink.*

If L has up to 14 crossings, then either $L = B^5$ or $\mathfrak{q}(\mathbb{S}_L^3) \leq 35$.

7.5 Proof of Theorem 1.14

Finally, we turn to the proof of Theorem 1.14 (the case where $|\partial Y| \geq 4$).

This can be addressed using Lemma 7.2. We'll suppose that $Y(p/q) \cong X'$ for some X' which is the outermost JSJ piece of a companion knot complement $\mathbb{S}_{C'}^3 \subset \mathbb{S}_{K'}^3$, and choose the satellite splittings $K = P_0(C_0)$ and $K' = P'(C')$ such that $\mathbb{S}_{P_0}^3(p/q) \cong \mathbb{S}_{C'}^3$ and $\mathbb{S}_{C_0}^3 \cong \mathbb{S}_{P'}^3(p/q)$. Given that $m \geq 4$, the following lemma can be used to show that no Seifert fibration on $Y(T_0; \mu_{C'})$ can be extended to the whole of $\mathbb{S}_{P_0}^3(T_0; \mu_{C'})$, contradicting Lemma 7.2. By convention, the meridian and longitude of each link component are taken to be those induced by viewing it as a knot in $\mathbb{S}^3$.

Lemma 7.14. *Let $L = L_K \cup L_0 \cup L_1 \cup \dots \cup L_{m-2}$ be an m -component link for $m \geq 4$ such that $L \setminus L_K$ is an unlink. Let $\gamma_0, \gamma_1, \gamma_2$ be slopes on the boundary components T_0, T_1, T_2 of the link complement $\mathbb{S}_L^3$ corresponding to L_0, L_1, L_2 , respectively.*

If γ_1 and γ_2 are homologous in the filling $\mathbb{S}_L^3(T_0; \gamma_0)$, then they must be longitudinal.

Proof. Viewing each link component as a knot in $\mathbb{S}^3$, let μ_K and λ_K denote the meridian and longitude of the L_K component and let μ_k and λ_k be the meridian and longitude of each L_k component. Since $L \setminus L_K$ is an unlink, each λ_k can be expressed as a multiple of μ_K ; in particular, if l_k denotes the linking number between L_K and L_k , then $\lambda_k = l_k \mu_K$. The slopes in question can thus be written in $H_1(\mathbb{S}_L^3; \mathbb{Z})$ as

$$\begin{aligned}\gamma_0 &= a_0 \mu_0 + b_0 \lambda_0 = a_0 \mu_0 + b_0 l_0 \mu_K; \\ \gamma_1 &= a_1 \mu_1 + b_1 \lambda_1 = a_1 \mu_1 + b_1 l_1 \mu_K; \\ \gamma_2 &= a_2 \mu_2 + b_2 \lambda_2 = a_2 \mu_2 + b_2 l_2 \mu_K.\end{aligned}$$

Now, if γ_1 and γ_2 are homologous in the filling $\mathbb{S}_L^3(T_0; \gamma_0)$, then there is some $c \in \mathbb{Z}$ for which we have $\gamma_1 - \gamma_2 = c \gamma_0$; equivalently, the following equality in $H_1(\mathbb{S}_L^3; \mathbb{Z})$ holds:

$$(a_1 \mu_1 + b_1 l_1 \mu_K) - (a_2 \mu_2 + b_2 l_2 \mu_K) = c(a_0 \mu_0 + b_0 l_0 \mu_K).$$

As $\mu_K, \mu_0, \mu_1, \mu_2$ are all linearly independent in $H_1(\mathbb{S}_L^3; \mathbb{Z})$, it follows that we have $a_1 = a_2 = 0$. Thus $\gamma_1 = \lambda_1$ and $\gamma_2 = \lambda_2$. $\square$

It remains to prove Theorem 1.14.

Proof of Theorem 1.14. Let K be a satellite knot whose outermost JSJ piece Y is hyperbolic with $m \geq 4$ cusps. Suppose that there is an orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q)$ for some knot K' with outermost JSJ piece Y' . Since $|q| \geq 9$, we know from Proposition 3.1 that $Y(p/q)$ is a hyperbolic JSJ piece and that the JSJ piece of $\mathbb{S}_{K'}^3(p/q)$ containing the surgery curve is of the form $\tilde{Y}(\tilde{p}/\tilde{q})$: either (i) $Y'(p/q)$ or (ii) $\hat{Y}(p/qs^2)$ precisely when Y' is a compressing (r, s) -cable space adjacent to a JSJ piece $\hat{Y}$. As usual, the bulk of the argument relies on obstructing the swapping of JSJ pieces in the orientation-preserving homeomorphism order to deduce that there is a slope-preserving homeomorphism $Y(p/q) \cong \tilde{Y}(\tilde{p}/\tilde{q})$. We will do this by combining Lemma 7.2 and Lemma 7.14.

Suppose once again that $Y(p/q) \cong X'$ for some $X' \subset \mathbb{S}_{K'}^3$, which is the outermost JSJ piece of a companion knot complement $\mathbb{S}_{C'}^3 \subset \mathbb{S}_{K'}^3$. Choose the satellite splittings $K = P_0(C_0)$ and $K' = P'(C')$ such that $\mathbb{S}_{P_0}^3(p/q) \cong \mathbb{S}_{C'}^3$ and $\mathbb{S}_{C_0}^3 \cong \mathbb{S}_{P'}^3(p/q)$ (as illustrated earlier in Figure 7.1). Consider the filling $\mathbb{S}_{P_0}^3(T_0; \mu_{C'})$ of $\mathbb{S}_{P_0}^3$ along the torus T_0 which induces the satellite splitting. By Lemma 7.2, we know that this must be a Seifert fibred space. We will use Lemma 7.14 to contradict this.

Since $m \geq 4$, Lemma 7.14 implies that any Seifert fibred structure on $Y(T_0; \mu_{C'})$ must be such that homologous slopes on the remaining inner boundary components are longitudinal with respect to the coordinates induced by viewing each link component as a knot in $\mathbb{S}^3$. Hence they are meridional with respect to the coordinates induced by the knot complements $\mathbb{S}_{C_1}^3, \dots, \mathbb{S}_{C_{m-2}}^3$ glued to the other sides in the satellite construction. However, recall from Theorem 2.6 that the regular fibres on the boundary of a torus knot complement or the relevant boundary component of a cable space are integral, and thus cannot coincide with the meridional slope of regular fibres on the boundary torus. Therefore the only possible JSJ pieces that could be attached to $Y(T_0; \mu_{C'})$ in a way which is compatible with the Seifert fibred structure are composing spaces, on which regular fibres do have meridional slope by Theorem 2.6. Still, composing spaces cannot make up the entire JSJ decomposition of $\mathbb{S}_{P_0}^3(p/q)$: at some point, it becomes necessary to cap off the boundary components with simple knot complements, over which the Seifert fibration cannot extend. Therefore $\mathbb{S}_{P_0}^3(p/q)$ cannot in fact be a Seifert fibred space, contradicting Lemma 7.2.

It follows that the orientation-preserving homeomorphism $\mathbb{S}_K^3(p/q) \cong \mathbb{S}_{K'}^3(p/q)$ restricts to a slope-preserving homeomorphism $Y(p/q) \cong \tilde{Y}(\tilde{p}/\tilde{q})$ as claimed. Since $|q| \geq \max\{35, q(Y)\}$, Proposition 4.8 then ensures that the core curves correspond. Applying Corollary 2.12 to rule out the $K' = J_{r,s}(K)$ case, it remains to conclude that $K = K'$. $\square$

7.5.1 Applications of Theorem 1.14

In the $m \geq 4$ case, the condition $\text{sys}(Y) > 0.14$ is very often satisfied and there is never any need for additional constraints such as Property H. As in the $m = 3$ case, we use the notation $K = L(C_0, \dots, C_{m-2})$ to describe the satellite knot obtained by attaching non-trivial knot complements $\mathbb{S}_{C_0}^3, \dots, \mathbb{S}_{C_{m-2}}^3$ to $m - 1$ unlinked boundary components of the m -cusped hyperbolic outermost JSJ piece $Y = \mathbb{S}_L^3$.

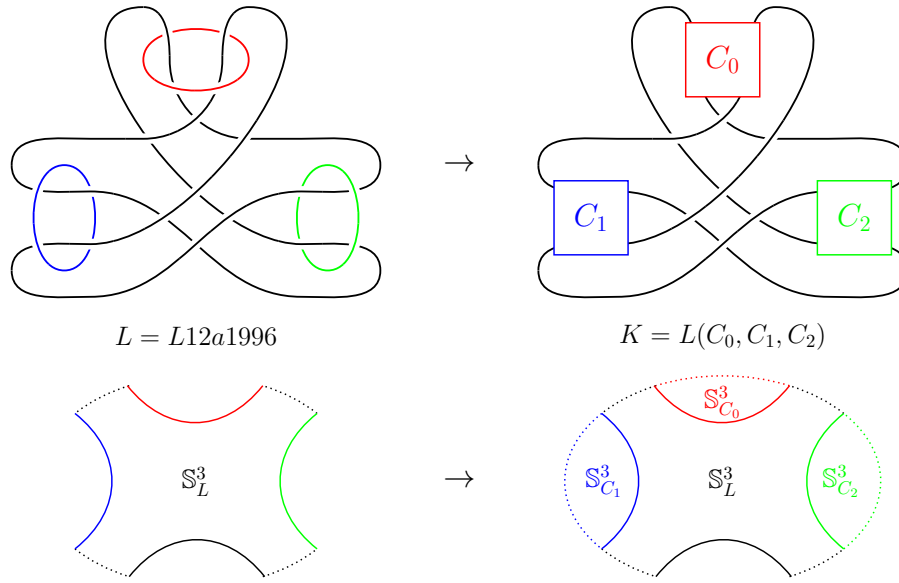


Figure 7.6: The link $L = L12a1996$ and a satellite $L(C_0, C_1, C_2)$.

Example 7.15. Let L denote the 4-component hyperbolic link listed in SnapPea as $L12a1996$. Figure 7.6 shows how a satellite knot $L(C_0, C_1, C_2)$ can be constructed from $\mathbb{S}_L^3$ by attaching three non-trivial knot complements to three unlinked boundary components. SnapPea computes

$$\text{sys}(\mathbb{S}_L^3) \approx 1.740021645305 > 0.14$$

and hence any slope with $|q| \geq 35$ is characterising for such a satellite $L(C_0, C_1, C_2)$.

The HTLinkExteriors census in SnapPea is complete for links with up to 14 crossings. Table 7.3 shows how many are m -component hyperbolic links with $L \setminus L_K = U_{m-1}$. For $m \geq 4$, all of these satisfy $\text{sys}(\mathbb{S}_L^3) > 0.14$, giving $\mathfrak{q}(\mathbb{S}_L^3) \leq 35$.

m	$L = L_K \cup L_0 \cup \dots \cup L_{m-2}$	L hyperbolic, $L \setminus L_K = U_{m-1}$	$\text{sys}(L) > 0.14$
2	77330	73659	73641
3	34590	14121	14120
4	7463	553	553
5	1101	4	4
6	85	0	0
7	4	0	0

Table 7.3: Number of links L in the HTLinkExteriors census with $\text{sys}(L) > 0.14$.

Theorem 7.16. *Let $Y = \mathbb{S}_L^3$ be the complement of a m -component hyperbolic link $L = L_K \cup L_0 \cup \dots \cup L_{m-2}$ with $m \geq 4$ such that $L \setminus L_K$ is an unlink.*

If L has up to 14 crossings, then $\mathfrak{q}(\mathbb{S}_L^3) \leq 35$.

Bibliography

- [AT21] Tetsuya Abe and Keiji Tagami. *Knots with infinitely many non-characterizing slopes*. In: *Kodai Math. J.* 44.3 (2021), pp. 395–421. ISSN: 0386-5991, 1881-5472.
- [Ago00] Ian Agol. *Bounds on exceptional Dehn filling*. In: *Geom. Topol.* 4 (2000), pp. 431–449. ISSN: 1465-3060.
- [Ago10] Ian Agol. *The minimal volume orientable hyperbolic 2-cusped 3-manifolds*. In: *Proc. Amer. Math. Soc.* 138.10 (2010), pp. 3723–3732. ISSN: 0002-9939.
- [BM18] Kenneth L. Baker and Kimihiko Motegi. *Non-characterizing slopes for hyperbolic knots*. In: *Algebr. Geom. Topol.* 18.3 (2018), pp. 1461–1480. ISSN: 1472-2747.
- [BS22] John A. Baldwin and Steven Sivek. *Characterizing slopes for 5_2* . 2022. arXiv: 2209.09805 [math.GT].
- [Bra80] W. R. Brakes. *Manifolds with multiple knot-surgery descriptions*. In: *Math. Proc. Cambridge Philos. Soc.* 87.3 (1980), pp. 443–448. ISSN: 0305-0041.
- [Bud06] Ryan Budney. *JSJ-decompositions of knot and link complements in $\mathbb{S}^3$* . In: *Enseign. Math. (2)* 52.3-4 (2006), pp. 319–359. ISSN: 0013-8584.
- [CGLS87] Marc Culler, C. McA. Gordon, J. Luecke and Peter B. Shalen. *Dehn surgery on knots*. In: *Ann. of Math. (2)* 125.2 (1987), pp. 237–300. ISSN: 0003-486X, 1939-8980.

- [FKP08] David Futer, Efstratia Kalfagianni and Jessica S. Purcell. *Dehn filling, volume, and the Jones polynomial*. In: *J. Differential Geom.* 78.3 (2008), pp. 429–464. ISSN: 0022-040X.
- [FPS22] David Futer, Jessica S. Purcell and Saul Schleimer. *Effective bilipschitz bounds on drilling and filling*. In: *Geom. Topol.* 26.3 (2022), pp. 1077–1188. ISSN: 1465-3060.
- [Gab87] David Gabai. *Foliations and the topology of 3-manifolds III*. English (US). In: *Journal of Differential Geometry* 26.3 (1987), pp. 479–536. ISSN: 0022-040X.
- [Gab90] David Gabai. *1-bridge braids in solid tori*. In: *Topology Appl.* 37.3 (1990), pp. 221–235. ISSN: 0166-8641.
- [GHM+21] David Gabai, Robert Haraway, Robert Meyerhoff, Nathaniel Thurston and Andrew Yarmola. *Hyperbolic 3-manifolds of low cusp volume*. 2021. arXiv: 2109.14570 [math.GT].
- [Ghi08] Paolo Ghiggini. *Knot Floer homology detects genus-one fibred knots*. In: *American Journal of Mathematics* 130.5 (2008), pp. 1151–1169. ISSN: 00029327, 10806377.
- [Gor78] C. McA. Gordon. *Some aspects of classical knot theory*. In: *Knot theory (Proc. Sem., Plans-sur-Bex, 1977)*, Lecture Notes in Math., 685. , 1978, pp. 1–60.
- [Gor83] C. McA. Gordon. *Dehn surgery and satellite knots*. In: *Trans. Amer. Math. Soc.* 275.2 (1983), pp. 687–708. ISSN: 0002-9947.
- [GL87] C. McA. Gordon and J. Luecke. *Only integral Dehn surgeries can yield reducible manifolds*. In: *Math. Proc. Cambridge Philos. Soc.* 102.1 (1987), pp. 97–101. ISSN: 0305-0041.
- [GL89] C. McA. Gordon and J. Luecke. *Knots are determined by their complements*. In: *J. Amer. Math. Soc.* 2.2 (1989), pp. 371–415. ISSN: 0894-0347.

- [GL95] C. McA. Gordon and J. Luecke. *Dehn surgeries on knots creating essential tori, I*. In: *Comm. Anal. Geom.* 3.3-4 (1995), pp. 597–644. ISSN: 1019-8385.
- [JS79] William H. Jaco and Peter B. Shalen. *Seifert fibered spaces in 3-manifolds*. In: *Mem. Amer. Math. Soc.* 21.220 (1979), pp. viii+192. ISSN: 0065-9266, 1947-6221.
- [Joh79] Klaus Johannson. *Homotopy equivalences of 3-manifolds with boundaries*. Vol. 761. Lecture Notes in Mathematics. Springer, Berlin, 1979, pp. ii+303. ISBN: 3-540-09714-7.
- [Kir78] Robion Kirby. *A calculus for framed links in $\mathbb{S}^3$* . In: *Invent. Math.* 45.1 (1978), pp. 35–56. ISSN: 0020-9910.
- [Kne29] H. Kneser. *Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten*. In: *Jahresbericht der Deutschen Mathematiker-Vereinigung* 38 (1929), pp. 248–259.
- [KMOS07] P. Kronheimer, T. Mrowka, P. Ozsváth and Z. Szabó. *Monopoles and lens space surgeries*. In: *Ann. of Math. (2)* 165.2 (2007), pp. 457–546. ISSN: 0003-486X.
- [Lac00] Marc Lackenby. *Word hyperbolic Dehn surgery*. In: *Invent. Math.* 140.2 (2000), pp. 243–282. ISSN: 0020-9910.
- [Lac19] Marc Lackenby. *Every knot has characterising slopes*. In: *Math. Ann.* 374.1-2 (2019), pp. 429–446. ISSN: 0025-5831.
- [Lac20] Marc Lackenby. *Dehn surgery from a hyperbolic perspective*. 2020. URL: <https://people.maths.ox.ac.uk/lackenby/dehn-surgery-icerm-28dec2020.pdf>.
- [LM13] Marc Lackenby and Robert Meyerhoff. *The maximal number of exceptional Dehn surgeries*. In: *Invent. Math.* 191.2 (2013), pp. 341–382. ISSN: 0020-9910.

- [Lic62] W. B. R. Lickorish. *A representation of orientable combinatorial 3-manifolds*. In: *Ann. of Math. (2)* 76 (1962), pp. 531–540. ISSN: 0003-486X.
- [MS01] Brian Mangum and Theodore Stanford. *Brunnian links are determined by their complements*. In: *Algebr. Geom. Topol.* 1 (2001), pp. 143–152. ISSN: 1472-2747.
- [McC19] Duncan McCoy. *On the characterising slopes of hyperbolic knots*. In: *Math. Res. Lett.* 26.5 (2019), pp. 1517–1526. ISSN: 1073-2780, 1945-001X.
- [McC20] Duncan McCoy. *Non-integer characterizing slopes for torus knots*. In: *Comm. Anal. Geom.* 28.7 (2020), pp. 1647–1682. ISSN: 1019-8385, 1944-9992.
- [Mil62] J. Milnor. *A unique decomposition theorem for 3-manifolds*. In: *Amer. J. Math.* 84 (1962), pp. 1–7. ISSN: 0002-9327.
- [Mos71] Louise Moser. *Elementary surgery along a torus knot*. In: *Pacific J. Math.* 38 (1971), pp. 737–745. ISSN: 0030-8730.
- [Mos73] G. D. Mostow. *Strong rigidity of locally symmetric spaces*. Annals of Mathematics Studies, No. 78. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1973, pp. v+195.
- [NW15] Yi Ni and Zhongtao Wu. *Cosmetic surgeries on knots in $\mathbb{S}^3$* . In: *J. Reine Angew. Math.* 706 (2015), pp. 1–17. ISSN: 0075-4102.
- [NZ14] Yi Ni and Xingru Zhang. *Characterizing slopes for torus knots*. In: *Algebraic — & Geometric Topology* 14.3 (2014), pp. 1249–1274. ISSN: 1472-2747.
- [Oso06] John K. Osoinach. *Manifolds obtained by surgery on an infinite number of knots in $\mathbb{S}^3$* . In: *Topology* 45.4 (2006), pp. 725–733. ISSN: 0040-9383.
- [OS02] P. Ozsv’ath and Z. Szabó. *Holomorphic disks and knot invariants*. In: *Advances in Mathematics* 186 (2002), pp. 58–116.

- [OS04] Peter Ozsváth and Zoltán Szabó. *Holomorphic disks and topological invariants for closed three-manifolds*. In: *Annals of Mathematics* 159.3 (2004), pp. 1027–1158. ISSN: 0003486X.
- [OS19] Peter Ozsváth and Zoltán Szabó. *The Dehn surgery characterization of the trefoil and the figure eight knot*. In: *J. Symplectic Geom.* 17.1 (2019), pp. 251–265. ISSN: 1527-5256.
- [Pic20] Lisa Piccirillo. *The Conway knot is not slice*. In: *Ann. of Math. (2)* 191.2 (2020), pp. 581–591. ISSN: 0003-486X, 1939-8980.
- [Ras03] Jacob Andrew Rasmussen. *Floer homology and knot complements*. ProQuest LLC, Ann Arbor, MI, 2003, p. 126. ISBN: 978-0496-39374-9.
- [Sor23] Patricia Sorya. *Characterizing slopes for satellite knots*. 2023. arXiv: 2307.00739 [math.GT].
- [Thu80] William P. Thurston. *The geometry and topology of 3-manifolds*. 1980.
- [Thu82] William P. Thurston. *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*. In: *Bull. Amer. Math. Soc. (N.S.)* 6.3 (1982), pp. 357–381. ISSN: 0273-0979.
- [Wak23] Laura Wakelin. *The Dehn surgery characterisation of Whitehead doubles*. 2023. arXiv: 2304.04349 [math.GT].
- [Wal60] Andrew H. Wallace. *Modifications and cobounding manifolds*. In: *Canadian J. Math.* 12 (1960), pp. 503–528. ISSN: 0008-414X.
- [Yos13] Ken’ichi Yoshida. *The minimal volume orientable hyperbolic 3-manifold with 4 cusps*. In: *Pacific J. Math.* 266.2 (2013), pp. 457–476. ISSN: 0030-8730, 1945-5844.

Appendix A

Minimal geodesics

First, we give the values of $\mathfrak{q}(\mathbb{S}_K^3)$ obtained using the minimal geodesics method when $K = T_t^\pm$ is a ± 1 -clasped t -twist knot for $|t| \leq 34$. Since we have shown that $\text{sys}(\mathbb{S}_{T_t^\pm}^3) < 0.14$ for $|t| \geq 35$, this completes the proof of Corollary 1.7.

Next, we list the values of $\mathfrak{q}(\mathbb{S}_{W^n}^3)$ contributing to $C(K)$ for all n -clasped t -twisted Whitehead doubles $K = W_t^n(C)$ with $|n| \leq 34$ (note: this is independent of t and C). Since we have shown that $\text{sys}(\mathbb{S}_{W^n}^3) < 0.14$ for $|n| \geq 35$, this completes the proof of Corollary 1.9 and yields the values of $C(K)$ given in Table 7.1.

Finally, we list $\mathfrak{q}(\mathbb{S}_{B^n}^3)$ for all satellite knots $K = B^n(C_0, C_1)$ by the generalised Borromean rings pattern B^n (which is obtained by performing $1/n$ -surgery on one component of the 4-component hyperbolic link $L10n100$) for $|n| \leq 34$. Since we have shown that $\text{sys}(\mathbb{S}_{B^n}^3) < 0.14$ for $|n| \geq 35$, this yields the values of $C(K)$ given in Table 7.2.

t	$\Lambda_t^\pm = \text{sys}(\mathbb{S}_{T_t^\pm}^3)$	$\mathfrak{q}(\mathbb{S}_{T_t^\pm}^3)$
∓ 1	1.087070144996	21
∓ 2	0.330635521631	27
∓ 3	0.153578692788	34
∓ 4	0.088638567733	42
∓ 5	0.057710538310	51
∓ 6	0.040571176947	60
∓ 7	0.030082564226	68
∓ 8	0.023196961993	77
∓ 9	0.018433283838	86
∓ 10	0.015000689183	95
∓ 11	0.012445426919	104
∓ 12	0.010491939188	113
∓ 13	0.008964992283	122
∓ 14	0.007748815892	131
∓ 15	0.006764425705	140
∓ 16	0.005956433725	149
∓ 17	0.005285065287	158
∓ 18	0.004721159018	167
∓ 19	0.004242940361	176
∓ 20	0.003833884329	185
∓ 21	0.003481266543	194
∓ 22	0.003175160372	203
∓ 23	0.002907729811	212
∓ 24	0.002672722444	221
∓ 25	0.002465100316	230
∓ 26	0.002280767426	239
∓ 27	0.002116366001	248
∓ 28	0.001969122376	257
∓ 29	0.001836729164	266
∓ 30	0.001717254246	275
∓ 31	0.001609069838	284
∓ 32	0.001510796722	293
∓ 33	0.001421260048	303
∓ 34	0.001339454037	312

t	$\Lambda_t^\pm = \text{sys}(\mathbb{S}_{T_t^\pm}^3)$	$\mathfrak{q}(\mathbb{S}_{T_t^\pm}^3)$
± 1	—	—
± 2	0.562399148646	23
± 3	0.217101464988	30
± 4	0.114457182528	38
± 5	0.070693861571	47
± 6	0.048010614991	55
± 7	0.034739404196	64
± 8	0.026304508293	73
± 9	0.020609961033	82
± 10	0.016584439142	90
± 11	0.013633629114	99
± 12	0.011406166569	108
± 13	0.009683450721	117
± 14	0.008323669024	126
± 15	0.007231550219	135
± 16	0.006341162326	144
± 17	0.005605698492	153
± 18	0.004991184252	162
± 19	0.004472474961	171
± 20	0.004030637877	180
± 21	0.003651197189	189
± 22	0.003322931515	198
± 23	0.003037033650	208
± 24	0.002786512978	217
± 25	0.002565763633	226
± 26	0.002370247869	235
± 27	0.002196260793	244
± 28	0.002040753425	253
± 29	0.001901198105	262
± 30	0.001775485070	271
± 31	0.001661842188	280
± 32	0.001558772135	289
± 33	0.001465002792	298
± 34	0.001379447782	307

Table A.1: The length of a minimal geodesic in $\mathbb{S}_{T_t^\pm}^3$ and $\mathfrak{q}(\mathbb{S}_{T_t^\pm}^3)$ for $|t| \leq 34$.

n	$\text{sys}(\mathbb{S}_{W^n}^3)$	$\mathbf{q}(\mathbb{S}_{W^n}^3)$
± 1	1.061275061905	21
± 2	0.773114038508	22
± 3	0.351756919644	26
± 4	0.197729369929	31
± 5	0.126321972231	37
± 6	0.087607243698	43
± 7	0.064305470069	49
± 8	0.049202303775	55
± 9	0.038858005544	61
± 10	0.031464306931	67
± 11	0.025996893021	73
± 12	0.021840285802	79
± 13	0.018606572877	86
± 14	0.016041419853	92
± 15	0.013972441787	98
± 16	0.012279441187	105
± 17	0.010876534341	111
± 18	0.009701032735	117
± 19	0.008706311759	124
± 20	0.007857112359	130
± 21	0.007126371214	136
± 22	0.006493036413	143
± 23	0.005940533385	149
± 24	0.005455668899	155
± 25	0.005027835778	162
± 26	0.004648427592	168
± 27	0.004310402283	175
± 28	0.004007952923	181
± 29	0.003736256576	187
± 30	0.003491280776	194
± 31	0.003269633001	200
± 32	0.003068442554	206
± 33	0.002885267124	213
± 34	0.002718018298	219

 (a) $L^n = W^n$.

n	$\text{sys}(\mathbb{S}_{B^n}^3)$	$\mathbf{q}(\mathbb{S}_{B^n}^3)$
+1	2.122550123810	19
-1	1.762747174039	20
± 2	0.773114038508	22
± 3	0.351756919644	26
± 4	0.197729369929	31
± 5	0.126321972231	37
± 6	0.087607243698	43
± 7	0.064305470069	49
± 8	0.049202303775	55
± 9	0.038858005544	61
± 10	0.031464306931	67
± 11	0.025996893021	73
± 12	0.021840285802	79
± 13	0.018606572877	86
± 14	0.016041419853	92
± 15	0.013972441787	98
± 16	0.012279441187	105
± 17	0.010876534341	111
± 18	0.009701032735	117
± 19	0.008706311759	124
± 20	0.007857112359	130
± 21	0.007126371214	136
± 22	0.006493036413	143
± 23	0.005940533385	149
± 24	0.005455668899	155
± 25	0.005027835778	162
± 26	0.004648427592	168
± 27	0.004310402283	175
± 28	0.004007952923	181
± 29	0.003736256576	187
± 30	0.003491280776	194
± 31	0.003269633001	200
± 32	0.003068442554	206
± 33	0.002885267124	213
± 34	0.002718018298	219

 (b) $L^n = B^n$.

 Table A.2: The length of a minimal geodesic in $\mathbb{S}_{L^n}^3$ and $\mathbf{q}(\mathbb{S}_{L^n}^3)$ for $|n| \leq 34$.

Appendix B

Minimal volumes

The following table contains the smallest 2-cusped orientable hyperbolic 3-manifolds from the SnapPea census, grouped by triangulation complexity and then ordered by increasing volume. For each M , we also include (a, b) for which $H_1(M((0, 1), (a, b)))$ is finite. We further divide our list into portions, denoted L_k , corresponding to those items with volume between V_{k-1} and V_k (as in Table 1.1). This illuminates the points at which the volume bounds occur and up to which we require completeness.

APPENDIX B. MINIMAL VOLUMES

	M	$\text{vol}(M)$	(a, b)
L_1	$m129$	3.6638623767	
	$m125$	3.6638623767	$(4, 3)$
L_2	$m202$	4.0597664256	$(5, 3)$
	$m203$	4.0597664256	$(0, 1)$
L_3	$m292$	4.4153324775	$(-4, 5)$
	$m295$	4.4153324775	$(3, 2)$
L_4	$m328$	4.5559188899	$(5, 4)$
	$m329$	4.5559188899	$(7, 5)$
L_5	$m357$	4.7254015851	$(11, 7)$
	$m359$	4.7254015851	$(1, 2)$
	$m366$	4.7494999819	$(-5, 7)$
	$m367$	4.7494999819	$(3, 1)$
	$s441$	4.7517019655	$(5, 6)$
	$s443$	4.7517019655	$(-4, 3)$
L_6	$m388$	4.8511707573	$(5, 8)$
	$m391$	4.8511707573	$(1, 1)$
L_7	$s503$	4.8937641326	$(-10, 7)$
	$s506$	4.8937641326	$(-6, 5)$
	$v1060$	4.9327140585	$(5, 4)$
	$v1061$	4.9327140585	$(6, 7)$
	$s548$	4.9767702943	$(-14, 9)$
	$s549$	4.9767702943	$(2, 3)$
	M	$\text{vol}(M)$	(a, b)
L_8	$s568$	5.0294944813	$(5, 3)$
	$s569$	5.0294944813	$(-7, 9)$
	$t02501$	5.0411812564	$(-6, 5)$
	$t02502$	5.0411812564	$(7, 8)$
	$s576$	5.0425492156	$(-11, 8)$
	$s577$	5.0425492156	$(9, 7)$
	$s578$	5.0448991629	$(5, 1)$
	$s579$	5.0448991629	$(-7, 10)$
	$v1178$	5.0533214945	$(-7, 6)$
	$v1180$	5.0533214945	$(13, 9)$
	$m412$	5.0747080321	—
	$s596$	5.0747080321	—
	$s601$	5.0826538415	$(-7, 11)$
	$s602$	5.0826538415	$(-2, 1)$
	$v1203$	5.0990348432	$(-17, 11)$
	$v1204$	5.0990348432	$(3, 4)$
	$s621$	5.1062718035	$(-5, 2)$
	$s622$	5.1062718035	$(8, 11)$
	$o9_05655$	5.1111665875	$(8, 9)$
	$o9_05656$	5.1111665875	$(7, 6)$
	$s637$	5.1273136230	$(19, 12)$
	$s638$	5.1273136230	$(-1, 3)$
	$s647$	5.1379412019	$(-5, 3)$
	$t02727$	5.1401504513	$(-8, 7)$
	$t02728$	5.1401504513	$(16, 11)$
	$v1252$	5.1503497145	$(9, 11)$
	$v1253$	5.1503497145	$(-7, 5)$
	$s660$	5.1549263093	$(-8, 13)$
	$s661$	5.1549263093	$(-1, 2)$
	$v1263$	5.1621342201	$(-6, 1)$
	$v1264$	5.1621342201	$(-9, 13)$
	$t02749$	5.1676956678	$(4, 5)$
	$t02750$	5.1676956678	$(-20, 13)$
L_9	$v1284$	5.1799776154	
	$v1285$	5.1799776154	

Table B.1: 2-cusped orientable hyperbolic 3-manifolds with low volume.

We also list some hyperbolic knots whose complements have volume less than that of the Whitehead link complement. These provide examples of JSJ pieces satisfying Property H and thus lead to applications of Theorem 1.12 and Theorem 1.13.

Tetrahedra	CDW	C	$\text{vol}(\mathbb{S}_C^3)$	$\mathcal{Q}(\mathbb{S}_C^3)$
2	$K2_1$	$T_{\mp 1}^{\pm} = 4_1$	2.0298832128	20
3	$K3_1$	$P(-2, 3, 7)$	2.8281220883	29
	$K3_2$	$T_{\pm 2}^{\pm} = 5_2$	2.8281220883	29
4	$K4_1$	$T_{\mp 2}^{\pm} = 6_1$	3.1639632289	37
	$K4_2$	$T_{\pm 3}^{\pm} = 7_2$	3.3317442316	46
	$K4_3$	$T_{3,8;2,1}$	3.4742477613	61
	$K4_4$	$T_{4,7;2,1}$	3.6086890618	113
5	$K5_1$	$T_{6,6;2,1}$	3.4179148372	53
	$K5_2$	$T_{\mp 3}^{\pm} = 8_1$	3.4272052463	54
	$K5_3$	$T_{\pm 4}^{\pm} = 9_2$	3.4866601463	63
	$K5_4$	$T_{5,6;3,1}$	3.5285094967	72
6	$K6_1$	$T_{\mp 4}^{\pm} = 10_1$	3.5261959907	71
	$K6_2$	$T_{\pm 5}^{\pm} = 11_{552}$	3.5538199185	80
	$K6_3$	$T_{5,11;2,1}$	3.5873867858	96
	$K6_4$	$T_{5,13;2,1}$	3.6102068815	114
7	$K7_1$	$K12a803$	3.5738825406	88
	$K7_2$	$K13a3143$	3.5889139178	97
	$K7_3$	$T_{5,16;2,1}$	3.6270400794	138
	$K7_4$	$T_{5,-17;3,-1}$	3.6352511867	156
8	$K8_1$	$K14a12741$	3.6004672628	105
	$K8_2$	$K15a54894$	3.6095391745	114
	$K8_3$	$T_{5,21;2,2}$	3.6422669304	180
	$K8_4$	$T_{5,22;3,3}$	3.6461101124	198

Table B.2: Hyperbolic knots C with $\text{vol}(\mathbb{S}_C^3) < \text{vol}(\mathbb{S}_W^3)$ (whose complements $\mathbb{S}_C^3$ can be triangulated with up to 8 tetrahedra) and the corresponding values of $\mathcal{Q}(\mathbb{S}_C^3)$.