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Choking of liquid flows

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It is well-known that laminar flow of a liquid in a duct is predicted to choke if the viscosity of the liquid increases exponentially with increasing pressure. In other words, the pressure drop in the duct is predicted to become unbounded when the volumetric flow rate reaches a critical finite value. Choking is not observed in practice, however: the reason why is investigated here. It is shown that choking is always predicted to occur if the viscosity is independent of temperature or heat generation by viscous dissipation is neglected. If the viscosity decreases exponentially with increasing temperature and heat generation is not neglected, however, and if the temperature field is fully developed or if the flow is adiabatic, it is shown that choking is predicted not to occur.

1. Introduction

The viscosity μ of a liquid increases with increasing pressure p and decreasing temperature T. For many liquids, it may be assumed that the dependence of μ on p and T is as follows:

$$\mu = \mu_0 \exp[\zeta_p\{p-p_r\}] \exp[-\zeta_T\{T-T_r\}]. \tag{1}$$

Here μ_0, ζ_p and ζ_T are non-negative material constants; p_r and T_r are a reference pressure and a reference temperature, respectively. For a typical liquid, $\zeta_p \sim 0.001$ bar^{-1} and $\zeta_T \sim 0.01$ K^{-1}. Thus significant variations in viscosity typically arise if the pressure varies by of order 1000 bar or if the temperature varies by of order 100 K.

It is well-known (see for example Denn 1981) that, if the variation of μ with T is neglected (i.e. $\zeta_T = 0$), laminar flow in a duct is predicted to choke and the pressure drop $P \to \infty$ when the volumetric flow rate Q reaches some critical finite value. Thus, for flow in a pipe of length L and radius R ($\ll L$), the flow chokes when $Q \to \pi R^4/8\,\mu_0 L\zeta_p$. The pressure drop $P^\dagger$ when $\zeta_p = 0$ and $\zeta_T = 0$ is given by the Hagen–Poiseuille law:

$$P^\dagger = 8\,\mu_0 LQ/\pi R^4. \tag{2}$$

Thus choking occurs when $P^\dagger \to 1/\zeta_p$, i.e. when $P^\dagger \sim 1000$ bar. Values of this order occur in many practical cases, such as in polymer-melt processing machinery, lubrication systems, hydraulic actuators and oil wells. Choking is never observed, however. The reason why is addressed here.

2. Flow equations

Consider axisymmetric laminar flow in a pipe. Let z and r denote the axial and radial coordinates, respectively, and let u_z and u_r denote the corresponding velocity components. Using the argument given for example in Penwell, Porter & Middleman (1971), the liquid may be assumed to be incompressible, so that its density ρ is

constant. (Note that variations in p and T cause variations in ρ. Variations in μ can be ascribed to variations in ρ, instead of in p and T, by a free-volume argument. This is not done here.) Then the mass conservation equation is

$$\frac{\partial u_z}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(ru_r)=0 \tag{3}$$

or, in integrated form,

$$Q=\int_0^R 2\pi r u_z \,\mathrm{d}r. \tag{4}$$

An order-of-magnitude analysis of the momentum conservation equations yields

$$\left|\frac{\partial p}{\partial r}\right| \Big/ \left|\frac{\partial p}{\partial z}\right| \sim \frac{R}{L} \ll 1,$$

so that radial pressure variations may be neglected. If the lubrication approximation is valid, the axial momentum conservation equation is

$$-\frac{\mathrm{d}p}{\mathrm{d}z}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu\frac{\partial u_z}{\partial r}\right)=0, \tag{5}$$

which may be partially integrated to yield

$$\frac{\partial u_z}{\partial r}=\frac{r}{2\mu_0}\exp\left[\zeta_T\{T-T_\mathrm{r}\}\right]\frac{\mathrm{d}p}{\mathrm{d}z}\exp\left[-\zeta_p\{p-p_\mathrm{r}\}\right]. \tag{6}$$

If axial conduction is negligible, the energy conservation equation is

$$\rho\gamma\left(u_z\frac{\partial T}{\partial z}+u_r\frac{\partial T}{\partial r}\right)=\frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\mu\left(-\frac{\partial u_z}{\partial r}\right)^2, \tag{7}$$

where γ and α are the specific heat and thermal conductivity of the liquid, respectively. Incorporating (6) into (7) yields

$$\rho\gamma\left(u_z\frac{\partial T}{\partial z}+u_r\frac{\partial T}{\partial r}\right)=\frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{r^2}{4\mu_0}\exp\left[\zeta_T\{T-T_\mathrm{r}\}\right]\left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)^2\exp\left[-\zeta_p\{p-p_\mathrm{r}\}\right]. \tag{8}$$

The boundary conditions are

$$\left.\begin{aligned}
&\frac{\partial u_z}{\partial r}=0, \quad u_r=0, \quad \frac{\partial T}{\partial r}=0 && \text{at } r=0,\\
&u_z=0, \quad u_r=0, \quad T=T_\mathrm{w} && \text{at } r=R,\\
&T=T_0, \quad p=p_0 && \text{at } z=0,\\
&p=p_L && \text{at } z=L.
\end{aligned}\right\} \tag{9}$$

It is convenient to choose the reference pressure $p_\mathrm{r}=p_L$. Choice of a convenient reference temperature T_r depends on whether the temperature field is fully developed or developing. The Graetz number Gz and Reynolds number Re are defined thus:

$$Gz=\frac{4\rho\gamma Q}{\pi\alpha L}, \quad Re=\frac{2\rho Q}{\pi\mu_0 R}. \tag{10}$$

The Graetz number may be interpreted as the ratio of axial convection to radial conduction. Thus a flow is fully developed and it is convenient to choose $T_r = T_w$ if Gz is small, which in practice means $Gz \ll 1$; a flow is developing otherwise and it is convenient to choose $T_r = T_0$. Axial conduction is negligible compared with axial convection if $Gz \gg R^2/L^2$. The flow is laminar if $Re < \sim 2000$ and the lubrication approximation is valid if $Re \ll L/R$. Note that, by implication, the viscosity used to determine Re is that at p_r and T_r.

3. Temperature-independent viscosity

When the viscosity is independent of temperature, i.e. $\zeta_T = 0$, the velocity field can be determined independently of the temperature field. Thus, as shown in Denn (1981) and Penwell *et al.* (1971), (6) may be integrated to yield

$$P = -\frac{1}{\zeta_p} \ln (1 - \zeta_p P^\dagger), \tag{11}$$

where the pressure drop $P = (p_0 - p_L)$ and $P^\dagger$ is given by the Hagen–Poiseuille law (2). It follows that $P \to \infty$ when $P^\dagger \to 1/\zeta_p$: the flow is then choked.

The occurrence of choking is unaffected by the precise functional dependence of viscosity μ on pressure p. Thus, if μ increases with p not as $\exp[\zeta_p\{p - p_r\}]$ but as $[1 + \zeta_p\{p - p_r\}]^n$, were $n > 0$, it is easy to show that choking occurs only if $n > 1$ and $P^\dagger \to 1/(n-1)\zeta_p$. More generally, integration of (6) yields

$$Q = \frac{\pi R^4}{8L} \int_{p_L}^{p_0} \frac{1}{\mu} \mathrm{d}p, \tag{12}$$

so that choking occurs for any functional dependence of μ on p such that

$$\int_{p_L}^{p_0} \frac{1}{\mu} \mathrm{d}p < \infty \quad \text{for} \quad (p_0 - p_L) \to \infty.$$

4. No heat generation

When heat generation by viscous dissipation is neglected, the term involving pressure in (8) vanishes. The only term involving pressure is thus in (6), which may be rewritten

$$\frac{\partial u_z}{\partial r} = \frac{r}{2\mu_0} \exp[\zeta_T\{T - T_r\}] \frac{\mathrm{d}}{\mathrm{d}z}\left(-\frac{1}{\zeta_p} \exp[-\zeta_p\{p - p_r\}]\right). \tag{13}$$

When the viscosity is independent of pressure, i.e. $\zeta_p = 0$, the only effect on (13) is that the term

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(-\frac{1}{\zeta_p} \exp[-\zeta_p\{p - p_r\}]\right)$$

is replaced by $\mathrm{d}p/\mathrm{d}z$. On integration, this means that the pressure drop P when $\zeta_p \neq 0$ is related to the pressure drop $P^\ddagger$ when $\zeta_p = 0$ (to be distinguished from $P^\dagger$ given by (2) which is the pressure drop when $\zeta_p = 0$ and $\zeta_T = 0$) thus:

$$P = -\frac{1}{\zeta_p} \ln (1 - \zeta_p P^\ddagger). \tag{14}$$

It follows that $P \to \infty$ as $P^{\ddagger} \to 1/\zeta_p$: the flow is then choked. (Note that this assumes that it is possible that $P^{\ddagger} \to 1/\zeta_p$. Because heat generation is neglected, the temperature is bounded above by its maximum value on the boundaries of the pipe, i.e. by the greater of T_0 and T_w. Hence the viscosity is bounded below and the pressure drop is unbounded for an unbounded volumetric flowrate. Thus it is possible that $P^{\ddagger} \to 1/\zeta_p$.)

5. Temperature-dependent viscosity and heat generation

When the viscosity is independent of temperature or heat generation is neglected, choking is predicted to occur. When, on the other hand, the viscosity depends on temperature and heat generation is not neglected, no general prediction of the occurrence of choking appears to be possible. Choking is, however, predicted not to occur in two asymptotic cases: when the Graetz number is unbounded or when it is small, i.e. when $Gz \to \infty$ or when $Gz < \sim 1$. The physical relevance of these cases may be seen by examining a particular polymer-melt processing operation. It is typically the case in injection moulding that $Gz \sim 1000$, but Gz can be as small as 0.1 for large mouldings and, at the start of injection, is unbounded; also $Re \sim 0.1$ and $L/R \sim 100$. (Of course, polymer melts are non-Newtonian: the viscosity μ of a melt depends not just on temperature T and pressure p but also on shear rate $\dot{\gamma}$. It is straightforward but tedious to show that the occurrence of choking is unaffected by the dependence of μ on $\dot{\gamma}$ (see also Denn 1981).)

If the flow is adiabatic, i.e. $Gz \to \infty$, the flow-average temperature $\bar{T}$ is given by

$$\bar{T} = T_0 + \frac{1}{\rho\gamma}(p_0 - p). \tag{15}$$

Then, as shown in Denn (1981), if radial variations in viscosity are neglected and μ is evaluated at $\bar{T}$ (which is, admittedly, unlikely to be a good approximation since it is not a good one when $\zeta_p = 0$), it follows that

$$\exp(\epsilon\zeta_p P) - \exp(-\zeta_p P) = (1+\epsilon)\zeta_p P^{\dagger}, \tag{16}$$

where $\epsilon = \zeta_T/\rho\gamma\zeta_p$. It follows that, provided $\epsilon \neq 0$, $P < \infty$ for $P^{\dagger} < \infty$: the flow cannot choke.

If $Gz < \sim 1$, the convection terms on the left-hand side of (8) may be neglected: the temperature field is fully developed. Define the dimensionless power χ and the Nahme number Na:

$$\chi = \frac{\zeta_T Q P}{8\pi\alpha L}, \quad Na = \frac{\zeta_T R^4 P^2}{64\alpha\mu_0 L^2}. \tag{17}$$

The Nahme number may be interpreted as the ratio of the temperature rise because of heat generation by viscous dissipation to the temperature difference required significantly to alter the viscosity. It is typically the case in injection moulding that $Na \sim 10$. (Note that this definition of Na is based on the pressure drop P and not, as is more common, on the volumetric flow rate Q.) Then it follows that

$$u_z = \frac{4Q}{\pi R^2}\int_{\sigma}^{1} \sigma \left[\frac{\left(\frac{2}{\chi}\Big/\left(-\frac{L}{P}\frac{\mathrm{d}p}{\mathrm{d}z}\right)\right)}{\left(\frac{2}{\chi}\Big/\left(-\frac{L}{P}\frac{\mathrm{d}p}{\mathrm{d}z}\right)\right) - 1 + \sigma^4}\right]^2 \exp[-\zeta_p\{p - p_r\}]\left(-\frac{L}{P}\frac{\mathrm{d}p}{\mathrm{d}z}\right)\frac{Na}{\chi}\,\mathrm{d}\sigma \tag{18}$$

and
$$T = T_w + \frac{2}{\zeta_T} \ln \left[\frac{\left(\frac{2}{\chi} \Big/ \left(-\frac{L}{P}\frac{dp}{dz}\right)\right)}{\left(\frac{2}{\chi} \Big/ \left(-\frac{L}{P}\frac{dp}{dz}\right)\right) - 1 + \sigma^4} \right] \tag{19}$$

(see equations (14) and (15) of Richardson 1986) where $\sigma = r/R$. Note that $\partial T/\partial z \neq 0$: the temperature field is thus fully developed in the sense that convection is negligible but not in the sense that the temperature field is independent of axial position. Also

$$-\frac{L}{P}\frac{dp}{dz} = \frac{\chi}{Na \exp\left(-\zeta_p(p-p_r)\right) + \frac{1}{2}\chi^2}, \tag{20}$$

which may be integrated to yield

$$\frac{1}{\zeta_p P}\left(1 - \exp\left(-\zeta_p P\right)\right) Na = \chi - \tfrac{1}{2}\chi^2 \tag{21}$$

(which is the equivalent of equation (17) in Richardson 1986). By an argument analogous to that given in Richardson (1986), it can be shown that $dP/dQ > 0$ for small Q and $dP/dQ < 0$ for large Q. Thus there is a maximum pressure drop P_{max} which is given by

$$P_{max} = -\frac{1}{\zeta_p} \ln\left(1 - \tfrac{1}{2}\zeta_p P^\dagger\right) \tag{22}$$

and is such that

a specified value of $P < P_{max}$ determines two values of Q;
a specified value of $P = P_{max}$ determines one value of Q;
a specified value of $P > P_{max}$ determines no value of Q.

By contrast

a specified value of Q always determines one value of P.

The occurrence of a maximum pressure drop would seem to preclude the possibility that the flow can choke. This is confirmed by the fact that (21) implies that $0 \leqslant \chi \leqslant 2$ so that the power which must be supplied to the flow $QP < \infty$. Clearly, a finite flow rate Q implies a bounded pressure drop P: choking cannot occur.

6. Conclusion

The occurrence of choking of liquid flows appears to be related to whether heat generation by viscous dissipation affects the viscosity. In particular, choking occurs when the Nahme number Na (defined by (17) when the viscosity is given by (1)) vanishes, which implies no heat generation or a temperature-independent viscosity or both. The physical reason for this is that, if $Na = 0$, a high pressure causes a high viscosity which in turn causes a high pressure gradient; this in its turn causes a high pressure and so on, leading to choking. When the Nahme number is non-zero, in contrast, choking does not occur, at least for $Gz \to \infty$ or $Gz < \sim 1$. The reason is presumably that, if $Na \neq 0$, a high pressure causes a temperature rise because there is heat generation: although the high pressure tends to cause a high viscosity, the temperature rise tends to cause a low viscosity: the net effect is an intermediate viscosity and no choking. Because $Na \neq 0$ for any real flow, choking presumably cannot occur in practice.

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REFERENCES

DENN, M. M. 1981 *Polymer Engng Sci.* **21**, 65–68.
PENWELL, R. C., PORTER, R. S. & MIDDLEMAN, S. 1971 *J. Polymer Sci.* A-2, **9**, 731–745.
RICHARDSON, S. M. 1986 *Rheol. Acta* **25**, 372–379.