

TURBULENT SLENDER FLOW CALCULATIONS

by

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To my parents,

"

Mr Goh Kow Ehoon &

Mdm Beh Ah Bee

"In the beginning God created the heavens and the earth."

Genesis 1:1

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Abstract

A numerical algorithm is developed for steady, three-dimensional, incompressible flow using the slender flow approximation. The mean flow governing equations are the three momentum equations and the continuity equation. The turbulence is modelled by a Reynolds-stress model involving six $\overline{u_i u_j}$ -equations and the length scale given by the ϵ -equation. The equations are discretized on a newly proposed grid structure known as the pressure staggered grid, or PSTAG. In PSTAG, the mean velocities are stored at the nodes whereas the pressures and turbulence quantities are staggered in the crossflow plane as well as between adjacent streamwise stations. The momentum equations are linearized and, in principle, are non-iterative. The elliptic pressure field in the crossflow plane is obtained from a Poisson pressure equation derived from the discretized form of the continuity and momentum equations.

Four laminar test cases were used, primarily to study the numerical performance of PSTAG. The x-momentum solution was evaluated using fully-developed duct flow while the crossflow plane solution (i.e. the coupled y- and z- momentum equations and the pressure equation) were examined using the Taylor-Green vortex. Comparisons were made between PSTAG and the more widely employed staggered grid for a flat plate laminar boundary layer and a boundary layer with a sinusoidal spanwise perturbation. Besides being faster than a staggered grid formulation, PSTAG is also stable as well as accurate.

The range of turbulent flows calculated includes the zero and adverse pressure gradient boundary layer on a flat plate, the "infinite" swept wing with zero and non-zero spanwise gradients, and finally the vortex-boundary layer interaction flow. Except for the boundary layer at zero pressure gradient, these are complex flows (i.e. flows where extra strain rates are important) and, therefore, serve as useful yardsticks to study the behaviour of turbulence models. In general, some re-modelling is required for the pressure-strain and diffusion models. A length-scale-limiting model is proposed and has proven quite successful.

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Nomenclature

a	Ratio of $ \overline{uv} $ to k , or Width of rectangular duct.
a_{ij}	Anisotropic tensor.
A_m, B_m	Coefficients in the analytic solution for the fully-developed duct.
$\tilde{A}, \tilde{B}, \tilde{C}$	Matrices in linearized momentum equations.
b	Ratio of $\overline{vv}$ to u_τ^2 , or Height of rectangular duct.
C	Constant of proportionality.
C_D, C_μ, σ_k	Empirical constants in modelled k -equation.
C_f	Skin friction coefficient.
C_k	PSTAG specification for k/u_τ^2 .
C_{uu}	PSTAG specification for $\overline{uu}/u_\tau^2$.
C_{vv}	PSTAG specification for $\overline{vv}/u_\tau^2$.
C_{vw}	PSTAG specification for $\overline{vw}/u_\tau^2$.
C_y, C_z	Courant numbers.
$C_\varepsilon, C_{\varepsilon 1}, C_{\varepsilon 2}$	Empirical constants in modelled ε -equation.
C_1, C_2, C_s	Empirical constants in modelled $\overline{u_i u_j}$ -equations.
C_{1v}, C_{2v}	Empirical constants for wall damping function.
C'_2	Empirical constant for the simplified "rapid" pressure-strain model.
d	Width of a Taylor-Green vortex.
d_y, d_z	Diffusion restriction criterion.
D_{ij}	$= -(\overline{u_i u_k} \partial U_k / \partial x_j + \overline{u_j u_k} \partial U_k / \partial x_i)$
D_{ijk}	Dissipation term in $\overline{u_i u_j u_k}$ -equation.
$\tilde{D}$	Crossflow multi-dimensional vector operator.
$\tilde{D}_{cr}$	yz-mom component of $\tilde{D}$.

$\tilde{D}_x$	x-mom component of $\tilde{D}$.
e_{ij}	Rate of strain tensor.
F_i	Body force vector in Navier-Stokes equations.
F_1, F_2, F_4	Coefficients in discretized pressure equation.
$F_{\phi 1}, F_{\phi 2}$	Coefficients in discretized ϕ -equation.
G_1, G_2, G_4	Coefficients in discretized pressure equation.
$G_{\phi 1}, G_{\phi 2}, G_{\phi 4}$	Coefficients in discretized ϕ -equation.
I	y-direction coordinate of computational grid.
J	z-direction coordinate of computational grid.
k	Grid spacing geometric progression factor.
K	Taylor-Green vortex constant defined as $(\pi/d)^2$.
l_m	Mixing-length.
l_s	Length scale.
l_{sn}	Turbulence length scale in non-constant shear stress profile.
L_{ref}	Normalising/reference length.
$\tilde{L}$	Streamwise multi-dimensional vector operator.
$\tilde{L}_{CR}$	yz-mom component of $\tilde{L}$.
$\tilde{L}_x$	x-mom component of $\tilde{L}$.
m	Integer.
n	Empirical constant in homogeneous grid turbulence.
n_i	Normal to boundary vector.
p	Fluctuating pressure.
P	Mean pressure.
P_∞	Free stream mean pressure.
P_{ij}	Generation of Reynolds-stress.
P_{ijk1}	Generation of $\overline{u_i u_j u_k}$.
P_{ijk2}	Second-order correlations in $\overline{u_i u_j u_k}$ -equation.

P_k	Production of turbulence kinetic energy.
$\vec{P}$	Source terms matrix in momentum equations.
Q	Total mean velocity in x-z plane.
Q_e	Free stream total mean velocity in x-z plane.
$\vec{Q}$	Total mean velocity vector.
R	Radius of curvature.
Re	Reynolds number.
Re_{CELL}	Cell Reynolds number.
S	Source term in the ϕ -equation.
S_{ij}	Surface integral in the pressure-strain correlation.
S_P	The part of S that is independent of ϕ .
S_ϕ	The part of S that is dependent on ϕ .
$\vec{S}$	Source term matrix in discretized momentum equations.
$\vec{S}_{CR}$	yz-mom component of $\vec{S}$.
$\vec{S}_P$	Source term in discretized pressure equation.
$\vec{S}_x$	x-mom component of $\vec{S}$.
t	Time.
t_s	Turbulence time scale.
T_{ijk}	Diffusion of $\overline{u_i u_j u_k}$.
u	Fluctuating velocity.
$\overline{u_i u_j}$	Reynolds-stress tensor.
u_τ	Surface friction velocity.
U	Mean velocity component in streamwise direction.
U_e	Free stream mean velocity in x-direction.
U_{ref}	Normalising/reference velocity.
vol	Volume

V	Mean velocity component in normal direction.
V_s	Velocity scale.
W	Mean velocity component in spanwise direction.
W_e	Free stream mean velocity in spanwise direction.
x	Streamwise direction.
y	Normal direction.
y^+	Normalised distance from the wall ($= u_\tau y/\nu$).
z	Spanwise direction.

Greek Symbols

β	Bulk viscosity.
δ	Boundary layer thickness at $U = 0.995 U_{ref}$.
δ^*	Boundary layer displacement thickness.
δ_{ij}	Kronecker delta.
δ'	Initial plane δ in the 2-D region for the protruded flat plate boundary layer.
Δ	Grid spacing.
$\Delta \vec{Q}$	Mean velocity increment vector.
$\Delta x, \Delta y, \Delta z$	Grid spacing in x-, y-, z- direction.
Δx_{LIMIT}	Maximum permissible Δx from the diffusion restriction criterion.
ε	Rate of turbulence energy dissipation.
ε_{ijk}	Alternating tensor.
ϕ	General notation for variables.
ϕ_{ijk}	Pressure-strain correlation in $\overline{u_i u_j u_k}$ -equation.
$\phi_{ij,1}$	"Slow" pressure-strain term.
$\phi_{ij,2}$	"Rapid" pressure-strain term.

$\phi_{ij,v}, \phi_v$	Pressure-strain wall "echoing" term.
ϕ_{v1}	Wall damping function in Model 1.
ϕ_{v2}	Wall damping function in Model 2.
$\phi_{1,v}$	$(\phi_{ij} + \phi_{ji})_1$ wall damping function.
$\phi_{2,v}$	$(\phi_{ij} + \phi_{ji})_2$ wall damping function.
Γ	Diffusivity.
κ	von Karman constant.
λ	Intermittency factor.
μ	Dynamic viscosity.
ν	Kinematic viscosity.
ν_t	Eddy-viscosity.
ω	Relaxation factor.
ω_i	Vorticity tensor.
ρ	Density.
σ_{ij}	Stress tensor.
τ	Total shear stress in x-z plane.
τ_v	Wall shear stress.
ϑ	Momentum thickness, or Angle in radians.
ϑ_τ	$= \tan^{-1} [W/U]$

Other Symbols

I_ε	Convection term in ε -equation.
II_ε	Generation term in ε -equation.
III_ε	Diffusion term in ε -equation.
IV_ε	Vortex stretching term in ε -equation.

V_ε	Viscous destruction term in ε -equation.
k	Turbulence kinetic energy.
∇_{CR}	Crossflow plane Laplace operator.
$\angle (Q)$	Direction of Q velocity vector.
$\angle (\overline{uv}, \overline{vw})$	Direction of shear stress vector.
$\angle (\frac{\partial U}{\partial y}, \frac{\partial W}{\partial y})$	Direction of mean velocity gradient vector.

Superscripts

n	Upstream x-station.
κ	Iteration index.
$'$	Swept wing coordinates.

Subscripts

I	y-coordinate in computational domain.
IR	Inner region of boundary layer.
J	z-coordinate in computational domain.
o	Initial plane.
$1, 2, 3$	x-, y-, z- direction.
x, y, z	x-, y-, z- direction.

Overbars

$\sim$	Instantaneous value.
$-$	Time-averaged.
$\rightarrow$	Vector.

Abbreviations

ADI	Alternating-direction-implicit.
BC	Boundary condition.
CDS	Central differencing scheme.
CFD	Computational fluid dynamics.
CPU	Central processing unit.
CV	Control volume.
HDS	Hybrid differencing scheme.
l.h.s.	Left-hand-side.
LES	Large eddy simulation.
N-S	Navier-Stokes.
PLDS	Power-law differencing scheme.
QUICK	Quadratic upstream interpolation for convective kinematics scheme.
r.h.s.	Right-hand-side.
RSTM	Reynolds-stress turbulence model.
SLOR	Successive-line-over-relaxation.
SUDS	Skew-upwind differencing scheme.
UDS	Upwind differencing scheme.
x-mom	x-component momentum equation.
y-mom	y-component momentum equation.
yz-mom	coupled y- & z- component momentum equations.
z-mom	z-component momentum equation.
1-,2-,3-D	One-, two-, three- dimensional.

1 INTRODUCTION

1.1 Computational Fluid Dynamics

The dynamics of a continuum Newtonian fluid, one whose shear stress is linearly proportional to the rate of strain, are governed by the laws of conservation of momentum, mass and energy. Applying Newton's second law of motion (force = mass $\times$ acceleration) for momentum conservation, one obtains the Navier-Stokes (N-S) equations. The continuity equation is derived from the mass conservation law. These four fundamental equations may be supplemented by a fifth equation, the energy equation, representing energy conservation. The ultimate goal of computational fluid dynamics (CFD) is to numerically solve these instantaneous equations of motion, since their non-linear mathematical character precludes any analytical solutions except for a very limited number of laminar flows in simple geometries.

Most flows encountered in nature and in engineering applications are turbulent, that is, flows that are unsteady, three-dimensional (3-D) and quasi-random. The spectrum of length scales in turbulent flows stretches from the largest eddies (typically of the order of the flow domain) to the Kolomogrov microscales which are several orders of magnitude smaller. In order to obtain an adequate resolution, the computation grid structure has to be so fine that the memory storage capacity in a supercomputer would be severely strained, let alone the feasibility of computing times. In spite

of today's advanced computer architecture (enabling fast memory access and arithmetic operations), efficient programming techniques and numerical methods, it is impracticable to solve the time-dependent N-S equations for flows of over engineering structures. Such computations, known as direct numerical simulations, are largely confined to flows in/around simple geometry at low Reynolds number (Re) for purely academic purposes.

To overcome the limitations mentioned above, one has to derive, simplify or approximate from the instantaneous N-S equations other forms of equations which are both plausible as well as amenable to solution by a digital computer. One can start by statistically decomposing the instantaneous velocity and pressure (assuming constant density and negligible body forces) in the N-S equations into a mean and a fluctuating value, then perform time- (or ensemble-) averaging to yield what are commonly known as the Reynolds equations. Although the Reynolds equations are exact, since no assumption has been introduced, it is of utmost importance to bear in mind that the averaging process will undoubtedly discard some valuable information pertinent to the flow motion.

The Reynolds equations introduce additional unknown terms of fluctuating velocities correlations which are called Reynolds-stresses and closure requires some methods of determining these "apparent" stresses. In fact, the predominant difficulty in turbulence modelling lies in the accurate prediction of the Reynolds-stresses. As the word "modelling" implies, further

approximations are necessary, hence, one can see another departure from the instantaneous N-S equations. Any assumptions made in arriving at a model have to be supported by physical reasoning and in this aspect the study of the physics of turbulence has contributed immensely. In general, one hopes that the turbulence model would incorporate as much of the flow details as possible and also, it should be applicable to a wide range of flows.

Turbulence models can be broadly categorised into two main classes - one based on the Boussinesq's eddy-viscosity, ν_t , concept and the other which solves transport equation/s for the turbulence correlations. In the ν_t -models, one assumes the Reynolds-stresses to be proportional to the mean rate of strain in much the same way as the Stokes' Law for viscous stresses in laminar flows. Following this assumption, ν_t is defined, often a function of space and flow situations. This analogy, however, has been strongly disputed on the fact that lumps of turbulent fluid are not rigid bodies and the mean free paths are not small as compared to molecular mean free paths. Despite its conceptual shortcomings, this method has been found to work well for two-dimensional (2-D) flows particularly where the streamwise shear stress is of prime importance. It can be said that ν_t -models are perhaps the most tested and popular method until recent years.

Even in some relatively simple 2-D flows (e.g. wall jets & asymmetric wall shear layers), negative values of ν_t emerge. This is conceptually inconceivable because ν_t is normally characterised by positive (turbulence) quantities of velocity

scale V_τ and length scale l_τ . Needless to say, in complex flows with more than one important Reynolds-stress ν_t -models are not suitable. This brings one to the next category or higher level of turbulence models which promises greater potential for more accurate flow description by solving the modelled form of the transport equations (derived from the Reynolds equations) for the Reynolds-stress and l_τ .

The common practice in building a turbulence model is to consider well documented simple flows (e.g. grid turbulence, homogeneous flow etc) and empirically model each process with an appropriate term and coefficient. When applying the model to a complex flow, one implicitly accepts that turbulence motions can be superposed - this cannot be true, otherwise, there will be a universal model with a universal set of constants. Although much work has gone into establishing suitable models for the Reynolds-stresses, relatively little has been done on the length scale. This is due, in part, to the lack of available experimental evidence which might otherwise permit a more detailed understanding in l_τ . Thus it is at times difficult to establish whether the weakness in a turbulence model lies solely in the model for the length scale.

A prediction method for industrial applications must primarily satisfy three basic requirements:- (a) stability, (b) accuracy, and (c) economy. The order being a guideline of the order of priority. These three aspects are interlinked and an improvement in one area often brings about problems in the other/s. Stability in a solution procedure is of paramount importance and a scheme

should be stable over a wide range of flows. Non-convergence in a scheme can, in general, be attributed to unboundness in the discretized equations, thereby, leading to a growth in the numerical errors as the solution progresses.

Accuracy consists of both numerical and physical accuracies. While physical accuracy depends strongly upon the set of equations of motion being solved (in the case of turbulent flow, the choice of turbulence model is crucial), numerical accuracy is governed by the truncated part of the discretized equations. Numerical errors can often be reduced, for example through grid refinement or using higher order discretization schemes, the former at an increase in computing costs and the latter often with associated numerical instability. It is extremely difficult, if not impossible, to quantify numerical errors in a numerical solution because these errors manifest themselves with the computer truncation errors through the millions of arithmetic operations.

An algorithm must also be economical, that is to say efficient. Program efficiency depends upon the solution procedure (i.e. iteration technique, matrix inversion, etc) as well as programming techniques and hardware. Recipes for enhancing stability and accuracy are often more costly and in some instances trade-offs must be made. As a general rule, a prediction method should never use mathematical models that are more complex than necessary.

Finally, it should be fair to say that until one can solve the

instantaneous equations of motion, a task which is highly unlikely to be accomplished in the near future, CFD should be viewed as a design tool to complement experimental research, and vice versa, rather than a supersession.

1.2 Literature Survey

1.2.1 Numerical Methods

Prediction methods for streamwise (or x-direction) parabolic 3-D flows are, in general, step-marching and utilise an extended form of the boundary layer approximations to eliminate upstream influences. Although the elliptic nature of the pressure field is retained in the crossflow plane (where y and z are the vertical and spanwise axes respectively), it is necessary to set the streamwise pressure gradient $\frac{\partial P}{\partial x}$ a priori before a marching procedure can be developed. (NOTE: The velocities corresponding to the x, y and z directions are U, V and W respectively. The suffix notation is $(x_1, x_2, x_3) \equiv (x, y, z)$ and $(U_1, U_2, U_3) \equiv (U, V, W)$.)

Patankar & Spalding (1972) presented an algorithm known by the acronym SIMPLE (Semi-Implicit Pressure Linked Equations) for this category of flow. It shares many similarities with an earlier 2-D version in Patankar & Spalding (1970). SIMPLE solves the momentum and pressure correction equations for the primitive variables (i.e. velocities and pressure are the dependent variables). The governing equations are discretized on a staggered grid arrangement using fully-implicit finite volume method and the

non-linearities are overcome by using known upstream values.

For each streamwise step SIMPLE starts with a guessed pressure field and proceeds to solve the streamwise momentum equation for a preliminary streamwise velocity distribution. It then computes an equation for streamwise pressure gradient correction to ensure mass conservation and subsequently updates both the primary velocity field and streamwise pressure gradient. Likewise, this order of calculations is also adopted in the crossflow plane. Patankar & Spalding (1972) were able to dispense with iterations at each streamwise step, which was not grossly in error, because the flow being predicted was laminar and also the streamwise step used was small. Actually, SIMPLE has to handle non-linearities in an iterative manner at each streamwise step.

SIMPLE relies on heavy under-relaxation for convergence and this undesirable aspect is due to the solution of the pressure field. The Poisson pressure correction equation, derived from the discretized form of the continuity and momentum equations, truncates terms pertaining to the errors in the secondary velocities. For a converged solution these truncated terms are zero but for an unconverged solution, Raithby & Schneider (1979) showed that the pressure correction values are overestimated and hence under-relaxation is needed.

Patankar (1981) introduced SIMPLER (SIMPLE-revised) which uses the pressure correction equation to obtain only the secondary velocities but not the pressure field. Instead, the pressure field

is calculated from a pressure equation which includes the truncated terms. An inherent advantage of SIMPLER is that since the pressure field is calculated before solving the momentum and pressure correction equations, there is no need to guess a pressure field. The calculated pressure field, being a better estimate, will also enhance speedier convergence when solving the secondary flow momentum equations.

A different approach for evaluating the secondary flow was given by Briley (1974) who assumed the secondary velocities corrections to be irrotational, therefore, solving a Poisson velocity potential equation instead of a pressure correction equation. This scheme also calculates the pressure from a Poisson equation. Unlike SIMPLE/R, the solutions of the Poisson equations require source term corrections in order to satisfy the constraint imposed by Green's first integral theorem. The mean flow equations are differenced in the streamwise direction using the unconditionally stable Crank-Nicholson method and non-linearity handled by known upstream values, thereby decoupling the momentum equations.

To discard an iterative solution, Briley & McDonald (1977) proposed a Taylor series form of linearizing the momentum and continuity equations. This compressible flow algorithm evaluates density directly from the continuity equation and through an equation of state obtains the pressure. The linearized difference equations are implicit and, due to the Taylor series expansion, are coupled. Coupling of the equations means that one cannot associate a particular variable with a particular equation,

therefore, the system of linear equations must be solved simultaneously. To reduce errors introduced by linearizations and a non-iterative solution per time step, small time steps were used, a constraint also imposed on the non-iterative Patankar & Spalding (1972) algorithm.

The Los Alamos group, since the Harlow & Welch (1965) time-dependent 2-D incompressible solver, has developed many useful programs; amongst them is the compressible and incompressible flow solver of Harlow & Amsden (1971). In the momentum equations, pressure is replaced by the density variable via a state-equation, hence, from the momentum and continuity equations a density equation can be formulated. The density equation is then converted, using a state-equation, to a Poisson pressure equation which is solved. (Note: The compressible algorithm of Briley & McDonald, 1977, solves for density and obtains the pressure using the calculated density values.) Using the calculated pressure field, the density values are obtained from the state-equation which are in turn needed in the momentum equations to determine the velocity field. This algorithm is known as the Implicit Continuous-fluid Eulerian (ICE) method.

The ease with which one can use the continuity equation to solve for density and hence pressure is lost in incompressible flows because the linkage between density and pressure via the equation of state breaks down due to the constant density property. This leads to the need to formulate an equation for pressure itself, which can be derived from the continuity equation as shown by Issa

(1983). As such, there exists a linkage between pressure and velocity. A correct pressure field would yield velocities that satisfy both the momentum and continuity equations simultaneously. This pressure-velocity linkage is often very vital to the success of attaining convergence and the differences in the various schemes available today usually lie in the different approaches taken to handle this linkage.

Issa (1986) also proposed a non-iterative scheme which handles the pressure-velocity linkage using a single-predictor and double-corrector method. This scheme, known as Pressure-Implicit with Splitting of Operators (PISO), predicts the velocity field with a guessed pressure field and subsequently corrects the velocities and pressure in two corrector steps. In each corrector step, the pressure is computed using the latest computed velocities which is in turn used to re-calculate (or update) the velocities. When computing for the velocities and pressures in the predictor and corrector stages, the momentum and pressure equations are used. This is quite different from algorithms employing correction equations. The sufficiency of two corrector steps was proven from an error analysis, showing that by the second corrector step, the errors are already one order smaller than the discretization errors in the original differenced equations.

In the method of Caretto et al. (1972), known as SIVA or Simultaneous Variable Addjustments, once the pressure at a node has been determined from the pressure equation the secondary velocities in the four surrounding nodes (of a five-node

computational stencil) are immediately updated. The solution of the momentum equations for the secondary flow was made efficient by reducing these equations to algebraic expressions by using the latest calculated values of the variables.

Upstream influence by the longitudinal pressure gradient, if not unduly strong, can be incorporated into a streamwise parabolic solution procedure by global iterations. Each sweep consists of a complete marching solution from the upstream end to the downstream end with an update on the streamwise pressure gradient at the end of each global sweep. This is often referred to as a "partially parabolic" solver and, of course, requires an increase in computer memory over boundary layer calculations since pressures must be stored for the whole domain. Cousteix et al. (1986) experimented with four different partially parabolic solvers for 3-D flows. In the first method the momentum and continuity equations (but not the pressure equation) formed a system of coupled equations giving 4x4 block-tridiagonal matrices. The second method treats the momentum and pressure equations individually while the third and fourth methods are SIMPLE and SIMPLER respectively. The main focal point in Cousteix et al. (1986) is the ease whereby one can adapt parabolic algorithms for partially parabolic situations.

In the various schemes mentioned this far, all solve for the primitive variables (i.e U, V, W & P) with special attention needed in the pressure-velocity linkage. An alternative way is solve the vorticity equations in replacement of the momentum equations. This is highly attractive because (the problematic)

pressure does not appear in the vorticity equations. However, pressure is very often an important required quantity (e.g. in the determination of drag forces acting on an aircraft) and the extra effort needed to extract it from a known velocity field is itself not a cheap process. It must be added that although pressure does not appear in the vorticity equations, the wall boundary conditions (BCs) for vorticity are related to pressure.

Caretto et al. (1972) used the vorticity-velocity formulation where crossflow momentum and continuity equations were combined to yield an equation for the streamwise vorticity. This can be done by differentiating the y-momentum equation w.r.t. z and subtracting the differential of z -momentum w.r.t. y and use of continuity in the algebraic manipulation. Secondary velocities are determined from two Poisson equations using streamwise vorticity values. In comparison with a primitive variables formulation, there is no gain with regard to the number of equations solved - one momentum equation for the primary flow, one vorticity equation for the streamwise vorticity and two Poisson equations for the crossflow.

In general, a solution algorithm involves solving discretized differential transport equations which are identifiable by terms pertaining to convection, diffusion and sources. Although these equations can be discretized with any differencing method, the solutions of the ensuing equations may not yield the correct answers, even when converged. It is known that for processes where convection is dominant, the central differencing scheme (CDS) may

produce unboundedness (i.e. lack of diagonal dominance in the matrix) leading to wiggles and possibly non-convergence when the cell Reynolds number Re_{CELL} is greater than 2. In the CFD literature, Re_{CELL} or the Peclet number (given by $\frac{\Delta U}{\Gamma}$ where Δ is the cell size, U is the velocity and Γ the diffusivity) is a measure of the ratio of convection to diffusion. Strictly speaking, the stability criterion of $Re_{\text{CELL}} \leq 2$ applies only to explicit schemes and is less tight for implicit schemes (see Chapter 6).

The numerical instabilities in CDS are attributed to the first-order derivatives (but not the second-order derivatives) and an immediate remedy is to resort to first-order accurate upwind differencing scheme (UDS) for the convection terms. The UDS is accurate so long as the velocity vectors are aligned with the grid and the flow is strongly convective, otherwise it suffers from smearing due to pseudo-diffusivities. It must be added that the UDS is the only scheme that is unconditionally bounded. In order to retain the accuracy of CDS whenever possible, Spalding (1972) devised a hybrid differencing scheme (HDS) which switches between CDS or UDS depending on whether the Peclet number is less than or greater than 2. Runchal (1972) demonstrated that although HDS has the accuracy of UDS at high Peclet numbers, it is considerably more accurate than either the CDS or UDS individually at Re_{CELL} around 2. It must be pointed out that HDS cannot eliminate the undesirable effects of pseudo-diffusivity inherent in the UDS.

To improve on the accuracy of UDS when the velocity vector is not aligned with the grid line, Raithby (1976) proposed the skew-upwind differencing scheme (SUDS) which takes into account the direction of the velocity vector. A disadvantage of this rather attractive method lies in the increased computational effort required for the larger computational stencil (i.e. a nine-node stencil in 2-D) and the need for interpolations to account for the velocity direction. Gosman & Lai (1982) introduced flux-blending to ensure unconditional boundedness for SUDS. The flux-blending coefficient, though difficult to optimise, is chosen such that it is as close to unity as possible, in other words, as near to being SUDS as possible.

Another higher order scheme for handling the problematic convection dominant transport equation is the quadratic upstream interpolation for convective kinematics scheme (QUICK) by Leonard (1979). QUICK is third-order accurate and any value is approximated by two upstream nodes and one downstream node, giving a nine-node computational stencil in 2-D. Another scheme based on the exact solution of a one-dimensional (1-D) linear pure convection-diffusion problem is the power-law differencing scheme (PLDS), derived by Patankar (1980). Its extension to multi-dimensional non-linear problems suffers a similar fate as UDS when the velocity vectors are not aligned with the grid lines, as investigated by Huang et al. (1985). In the same report, both SUDS and QUICK were found to be more superior than other schemes (an opinion also shared by Leschziner, 1980) although the authors seem to have a preference for QUICK. In another comparison of the

performance of various schemes, Syed et al. (1985) concluded that flux-blending schemes fare better than either SUDS or QUICK. (In Smith & Hutton, 1982, is a comprehensive review-cum-evaluation of work on advection discretization.)

Although the criteria for a differencing scheme are straight forward (i.e. stability, accuracy and economy), the means for evaluation are not. There are many free parameters like the overall solution algorithm, programming techniques, computer architecture etc which can affect a comparative study. It is not surprising to find different workers drawing different conclusions, and to date a consensus on the most suitable differencing scheme has yet to be reached. In a recent survey of CFD techniques, Patankar (1988) concluded that the convection-diffusion discretization still remains an unsolved problem. It is, therefore, fair to say that as long as there is no single standardised approach available to assess the performance of various schemes, care must be exercised when drawing any conclusions.

Until recently, equations have usually been discretized on a staggered grid structure because it offers a convenient way of eliminating a "checker-board" pressure equation whose solution may be oscillatory. However, a non-orthogonal staggered grid introduces many extra terms and therefore makes the equations complicated. To overcome this difficulty, Shih & Ren (1984) (see also Perić, 1985) introduced a collocated grid arrangement, albeit needing special treatment to eliminate decoupling between adjacent

pressure nodes. On closer scrutiny, the formulation of discretized equations on both the collocated and staggered grids are in principle the same and it is not surprising that Perić et al. (1988) found their performance almost identical in cartesian coordinates. Details of the collocated and staggered grids, including a new grid proposal, will be covered in chapter 4.

1.2.2 Turbulence Modelling

As mentioned in §1.1, the closure problem associated with the Reynolds equations has to be overcome by the use of a turbulence model. It has been recognized that the Reynolds-stress turbulence model (RSTM) offers an avenue towards predicting complex flows and at the same time providing more flow details than could be obtained by lower order turbulence models. Suffice to say, the origins of the RSTM could be traced to earlier works by Chou (1945) and Rotta (1951), the latter can be said to have introduced the first RSTM. Subsequently, the works of Daly & Harlow (1970), Shir (1973), Hanjalić & Launder (1972b) and Launder et al. (1975) established the basic ingredients necessary for a RSTM. There is also a low-Re version of the RSTM developed by Hanjalić & Launder (1976) (a 3-equation model) with the pressure-strain model later improved by Shima (1988).

The high rate of success in predicting the turbulent boundary layer on a flat plate at zero pressure gradient can be attributed to the fact that most turbulence models are being "set up" to calculate this flow. This flow is one of the most commonly

encountered flow in engineering, so if a turbulence model is unable to predict it satisfactorily, its usefulness is very limited. As such, useful yardsticks for assessing turbulence models are not 2-D boundary layers at zero pressure gradient. Hanjalic & Launder (1972b), Launder et al. (1975) and Younis (1982), just to cite a few, have achieved good results for the flat plate boundary layer with the RSTM. In Rodi (1984, p.300), it can be seen that the predictions by RSTM are as good as those obtained by the best ν_t -models. Even the low-Re RSTMs which fully resolve the inner layer to the wall have achieved good predictions (see Nakao, 1984).

The first motivation towards a RSTM lies in the need to predict complex flows (i.e. flows with strain rates in addition to $\frac{\partial U}{\partial y}$) where one is aware of the shortcomings exhibited by ν_t -models. Even with the RSTM, the difficulties associated with computing complex flows are not totally removed (see Bradshaw, 1975; Lakshminarayana, 1986). In the case of a relatively "mild" complex flow like a 2-D turbulent boundary layer with an adverse streamwise pressure gradient, both Hanjalic & Launder (1980) and Younis (1982) overpredicted the skin friction coefficient C_f at the last measured station of Bradshaw (1969) by about 30% (remembering that this is only a mild adverse pressure gradient situation). The problem was due to the equation for the rate of energy dissipation (ϵ -equation) returning values of l_g near the wall that were too large. The difference between this flow and one at zero pressure gradient is that the longitudinal pressure gradient causes the rate of growth of the boundary layer to be

more complicated, as well as rapid, which consequently confuses the length scale equation.

As stated in Hanjalic & Launder (1980), their previous experience in calculating flow through a nozzle indicated that the source (which contains mainly the strain rates) in the ϵ -equation has to be increased to cater for the effect of irrotational strain rates. Therefore, an invariant term proportional to $\left[k \frac{\partial U_i}{\partial x_j} \frac{\partial U_l}{\partial x_k} \epsilon_{ijm} \epsilon_{lkm} \right]$ (where k is the turbulence kinetic energy, ϵ_{ijm} & ϵ_{lkm} are the alternating tensors with $i, j, k, l, m = 1, 2$ or 3) was added to the ϵ -equation. With this term, the generation of ϵ was divided into a part containing rotational strain rates and another part containing irrotational strain rates. While the rotational strain rate coefficient was kept at the same value as in the original ϵ -equation, the irrotational strain rates coefficient was increased by threefold. The prediction of C_f with the modified ϵ -equation was in almost total agreement with the experimental data. However, when used for flows with streamline curvature the effect is opposite to what is desired, hence, the originators have since discarded this model (see Launder, 1984, p.75).

For adverse pressure gradient flows, it is l_g rather than the absolute value of ϵ calculated by the ϵ -equation that is causing the problem. This was shown in Rodi & Scheuerer (1986) where the near-wall l_g (i.e in the inner region) of a mixing-length model was found to be smaller than the l_g from a k - ϵ model, although the k - ϵ model has larger ϵ values. In the same paper, an expression was derived for the coefficient of the generation of ϵ as a

function of $\left[y, \frac{dp}{dx}, \frac{(-\overline{uv})^{0.5}}{u_\tau} \right]$, where u_τ is the surface friction velocity, by assuming local equilibrium and validity of the log-law, but not a constant shear stress profile. The equation indicates that the overall effect of an adverse pressure gradient is simply an increase in the generation of ε .

The special topic in this thesis is the prediction of 3-D flows, which are obviously more complicated and, as mentioned earlier, are good test cases for evaluation of turbulence models. The "infinite" swept wing is typical of a pressure-driven 3-D turbulent boundary layer and has been experimentally investigated by van den Berg et al. (1975), Elsenaar & Boelsma (1974) and Pontikos (1982). Generally speaking, one can rule out isotropic ν_t -models because such models are not able to reproduce the lag in shear-stress vector behind the mean velocity gradient vector. The non-zero W velocity component introduces an extra strain rate $\frac{\partial W}{\partial y}$, this is an inviscid secondary flow mechanism with skewing in the mean vorticity vector away from its original direction. $\frac{\partial W}{\partial y}$ is capable of significantly altering the turbulence structure, for instance, by reducing the ratio of the shear stress to the turbulent energy. Murthy & Lakshminarayana (1985) calculated this flow with a $k-\varepsilon$ turbulence model, and the predicted W velocity showed an increasing departure from experimental data as the flow proceeds downstream. k is quite well reproduced but over 80% of the boundary layer $\overline{uv}$ is overpredicted.

The flow on a spinning cylinder is in some respect similar to the "infinite" swept wing configuration in having non-zero W and $\frac{\partial W}{\partial y}$.

The basic distinct difference being the latter is pressure-driven while the former is stress-driven. Gibson & Younis (1986) predicted this flow with a RSTM but with constants very different from those in Launder et al. (1975). They argued that by changing the coefficients in the "slow" and "rapid" parts of the pressure-strain model from 1.5 to 3.0 and 0.6 to 0.3 respectively, the effects of extra strain rates in this flow can be accounted for. This work demonstrates that to describe 3-D flows adequately, some re-modelling of the pressure-strain correlation may be needed. The results showed good mean velocities predictions. The Reynolds-stresses are reasonably predicted except for the underprediction of $\overline{uw}$.

Liandrat et al. (1987) calculated the embedded vortex in a turbulent boundary layer with ν_t -models as well as the RSTM. It must be mentioned that this flow was also calculated by Dedoussis (1988) using a mixing-length model and the agreement with the experimental data of Bradshaw et al. (1982) is only qualitative. Liandrat et al. (1987) employed the solution algorithm of Cousteix et al. (1986) and the difficulties in computing such a complex flow is clearly evident in the number of numerical recipes deployed. To enhance numerical stability in the momentum equations, they found it necessary to convert each $\overline{u_i u_j}$, obtained from the RSTM into an equivalent ν_t and a residue. Other treatments include setting the streamwise step taken by the turbulence model to one-tenth of that taken in the mean flow solution and also special care in establishing initial BCs. The turbulence model can be described as a two-layer

model. A mixing-length model was used in the sublayer and matched with a RSTM in the log-law region. The mean velocity contours at Bradshaw et al. (1982) second measured station were adequately reproduced. No results were shown for $\overline{uu}$, $\overline{vv}$, $\overline{ww}$ and $\overline{uv}$ but the authors mentioned that these contours were less distorted than indicated by experiment. $\overline{uw}$ and $\overline{vw}$ were underpredicted, and so were the spanwise C_f values. The performance of a mixing-length and $k-\epsilon$ models were also evaluated with the conclusion that these models are not suitable for embedded-vortex flows.

In view of the relatively scarce computational work on embedded vortices in turbulent boundary layers, it is appropriate to review other work concerned with swirling secondary flow. Demuren & Rodi (1984) calculated the stress-induced secondary flow (also known as Prandtl's second kind) in a square duct with uniform inlet boundary conditions. The RSTM was simplified into algebraic equations for each $\overline{u_i u_j}$ by neglecting convective and diffusive transport terms. The streamwise velocity was well captured but the crossflow velocities were underpredicted. Demuren & Rodi (1987) presented results on a square duct with non-uniform inlet conditions, and except for regions near the corners and walls, the agreement with the experimental data of Gessner (1982) is reasonably good.

Jones & Pascau (1989) (see also Hogg & Leschziner, 1989) calculated a confined annular swirling jet with some success in the mean velocity field. The Reynolds-Stresses are not well reproduced with the normal stresses being overpredicted by about

100% in some regions. The overprediction of k contradicts the proposed modification to the ε -equation by Bardina et al. (1985), based on the argument that rotational motion (or swirl in this case) inhibits the transfer of energy in the cascade process and therefore reduces ε . This flow clearly demonstrates a weakness in the pressure-strain formulation because the exact $\overline{u_i u_j}$ generation terms, being dominant, are unable to respond adequately to the swirl-induced strain rates.

To summarise §1.2.1 and §1.2.2, the governing equations in primitive variables are the most commonly adopted equations and perhaps this is where a consensus in CFD ends. In discretizing a transport equation one can choose from a range of lower-order (e.g. UDS - more stable but less accurate) to higher-order (eg. QUICK - more accurate but less stable) differencing methods. Likewise, there are as many solution procedures being developed (e.g. SIMPLE/R, PISO, SMAC, ICE). In turbulence models, the RSTM promises potential for calculating complex flows, though the present model is by no means universal and work on it is still in progress. In spite of the vast amount of research in CFD, one can still safely say that there is at present no single numerical scheme or turbulence model capable of predicting all kinds of fluid flow.

1.3 Scope Of Present Work

The present dissertation deals with steady, incompressible 3-D flows (both laminar and turbulent) which have a predominant flow

direction and is characterised as "slender", that is, flows with length scale in the streamwise direction that is much, much larger than the crossflow plane length scale. The aims of the present work are as follows:-

- (a) Development of a fast and efficient numerical scheme based on a grid arrangement that has velocities stored at the nodes while the pressures and turbulence quantities are staggered.
- (b) Evaluation of the scheme's performance in terms of stability, accuracy and economy with several different laminar flows. A comparison between the present grid structure with the collocated and staggered grids.
- (c) Choosing a suitable form of the RSTM and incorporating it into the numerical scheme developed in (a) above.
- (d) Calculations of several complex flows using the RSTM with the intention to assess the ability of the RSTM to respond to extra strain rates.
- (e) Propose a modification to the ε -equation equation to account for the effects of a non-constant near-wall shear stress profile due to an adverse longitudinal pressure gradient.

The present numerical scheme, for laminar flows only, shares many common features with Dedoussis (1988). The different grid structure adopted in the two schemes naturally resulted in different forms of discretized equations and, hence, some differences in the solution of the discretized equations. In general, the basic solution structure for the momentum and pressure equations in both schemes are similar.

Chapter 1 presents introductory remarks pertaining to the field of CFD and reviews previous related work on numerical schemes and the use of RSTM in flow calculations. The objectives of the present work are also outlined, followed by a brief description of the thesis.

Chapter 2 discusses the governing equations of motion and the slender flow approximations. This is followed by a discussion, from an engineering viewpoint, of types of practical flows which can be treated as "slender". There is a section on the physics of turbulence followed by a brief review on current turbulence models (not including the RSTM).

Chapter 3 presents in detail the modelling of the "slow" and "rapid" parts of the pressure-strain term and the turbulent diffusion term in the Reynolds-stress transport equation. Two RSTMs were distinguished, based on the form of the near-wall damping function employed. The modelling of the ϵ -equation is also examined and a proposal to modify the length scale for adverse pressure gradient flow is presented.

Chapter 4 gives details of the numerical scheme covering the choice of grid arrangement, linearization techniques, discretization methods, alternating-direction-implicit (ADI) and successive-line-over-relaxation (SLOR) solution methods, boundary conditions (BCs) and the overall solution algorithm. The advantages and disadvantages of the present proposed grid

structure are discussed.

Chapter 5 is basically a documentation of the computer program, outlining the functions of each subroutine.

Chapter 6 presents results and discussion on four laminar flow computations, these are

- i. fully-developed flow in rectangular duct (3-D),
- ii. Taylor-Green vortex (3-D).
- iii. flat plate boundary layer (2-D),
- iv. protruded flat plate boundary layer (3-D) (where the protrusion represents a sinusoidal spanwise perturbation).

Chapter 7 presents results and discussion on five turbulent flow computations, these are

- i. flat plate boundary layer (2-D),
- ii. flat plate boundary layer with adverse longitudinal pressure gradient (2-D),
- iii. "infinite" swept wing (3-D) with zero spanwise gradient,
- iv. "infinite" swept wing (3-D) with non-zero spanwise gradient,
- v. longitudinal embedded vortex in a boundary layer (3-D).

Chapter 8 contains the conclusions and proposals for further work.

2 TURBULENT SLENDER FLOW

2.1 Equations Of Motion

2.1.1 Navier - Stokes & Continuity Equation

It is generally accepted that all kinds of fluid motion, however complicated, can be described by the N-S equations. Using an Eulerian frame of reference, the N-S equations in compact tensor notation is

$$\frac{\partial \tilde{U}}{\partial t} + \tilde{U}_j \frac{\partial}{\partial x_j} \tilde{U}_i = - \frac{1}{\tilde{\rho}} \frac{\partial \tilde{P}}{\partial x_i} + \frac{1}{\tilde{\rho}} \frac{\partial \sigma_{ij}}{\partial x_j} + F_i$$

Eq.2.1

$\tilde{U}_i$, $\tilde{P}$ and $\tilde{\rho}$ are the instantaneous velocity, pressure and density respectively, F_i the body force and the stress tensor σ_{ij} . For a Newtonian fluid,

$$\sigma_{ij} = \mu e_{ij} + \delta_{ij} \left(\beta - \frac{2}{3}\mu \right) \frac{\partial \tilde{U}_k}{\partial x_k}$$

Eq.2.2

δ_{ij} is the Kronecker delta and $\delta_{ij}=1$ if $i=j$ & $\delta_{ij}=0$ if $i \neq j$. μ is the dynamic viscosity, β the bulk viscosity and e_{ij} is the rate of strain (deformation) given by

$$e_{ij} = \left(\frac{\partial \tilde{U}_i}{\partial x_j} + \frac{\partial \tilde{U}_j}{\partial x_i} \right)$$

Using control-volume analysis, one can derive the continuity equation based on the principle of conservation of mass.

$$\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial}{\partial x_i} (\tilde{\rho} \tilde{U}_i) = 0 \quad \text{Eq.2.3}$$

For constant density, Eq.2.2 becomes

$$\sigma_{ij} = \mu e_{ij}$$

and Eq.2.3

$$\frac{\partial}{\partial x_i} \tilde{U}_i = 0 \quad \text{Eq.2.4}$$

Density and viscosity variations are affected by temperature fluctuations which can be assumed negligible at low Mach No. flows (excluding, of course, flows in combustion chambers etc) or the so called incompressible flows. When the fluid medium is liquid, "incompressibility" does not necessarily imply constant viscosity because temperature variation can be large. Although β is essentially zero for simple monatomic gases, it is of the same order of magnitude as μ or higher for diatomic or polyatomic gases and thus makes σ_{ij} rather complicated. The contribution from β is significant only with processes involving compression or rarefaction and in boundary layer flows without shocks or strong expansions, involving mainly shearing, it can be neglected.

For constant property flow with no body forces, the N-S equations Eq.2.1 become

$$\frac{\partial \tilde{U}_i}{\partial t} + \tilde{U}_j \frac{\partial}{\partial x_j} \tilde{U}_i = - \frac{1}{\rho} \frac{\partial \tilde{P}}{\partial x_i} + \nu \frac{\partial^2 \tilde{U}_i}{\partial x_j^2} \quad \text{Eq.2.5}$$

where ν is the kinematic viscosity, defined as $\nu = \frac{\mu}{\rho}$.

Although the set of equations comprising Eq.2.4 and Eq.2.5 contains four unknowns $\tilde{U}$, $\tilde{V}$, $\tilde{W}$ and $\tilde{P}$ in four equations, it is not possible to solve them analytically due to the non-linearity. The time taken for time-dependent numerical solutions for turbulent flows by computers is still immensely forbidding, if not impractical, besides an enormous amount of data needs to be stored. From an engineer's point of view, the complete behaviour of the three velocity components and pressure as functions of time and 3-D space is seldom of interest. A statistical representation of the flow variables as ensemble- or time- averaged mean and fluctuations is more realistic but of course at the expense of some loss of information.

$$\tilde{U} = U + u \quad \& \quad \tilde{P} = P + p$$

Upper case denotes the mean value and lower case denotes the fluctuation. The instantaneous continuity equation can be cast as

$$\frac{\partial}{\partial x_i} U_i = 0 \quad \text{Eq.2.6}$$

which implicitly implies the condition of $\frac{\partial}{\partial x_i} u_i = 0$.

To derive a mean quantity equation for a steady constant-property flow, replace the instantaneous quantities in Eq.2.5 with the above statistical decomposition, perform time-averaging and use the continuity equation to obtain

$$U_j \frac{\partial}{\partial x_j} U_i = - \frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j^2} - \frac{\partial \overline{u_i u_j}}{\partial x_j} \quad \text{Eq.2.7}$$

This set of equations is also known as the Reynolds equations and

together with the continuity equation Eq.2.6 does not form a closed set of equations due to the six additional unknown Reynolds-stresses $\rho \overline{u_i u_j}$. These are apparent stresses which act along the x_i -direction on a plane perpendicular to the x_j -direction. To effect closure, the Reynolds-stresses must be determined by adopting a suitable turbulence model (see §2.3.2 for a review on turbulence models). As will be shown in §2.1.3, it is possible to drive transport equations for the second order products $\overline{u_i u_j}$ which in turn contains the next higher order unknown correlations. If one proceeds in this manner, closure cannot be obtained since unknown correlations of one higher order will keep appearing.

2.1.2 Poisson Pressure Equation

In many engineering applications, it is necessary to determine the pressure distribution over a body in order to establish the pressure drag forces. Therefore, it is justifiable to treat pressure, like the velocity components, as one of the primitive variables. But unlike the velocity components, the elliptic nature of pressure makes it a rather obscure quantity to determine both experimentally and numerically. Pressure fluctuations within the flow are difficult to measure to within acceptable accuracies because the probe errors are often of the same order as the fluctuations. From a computational viewpoint, difficulties arise from the solution of the pressure equation itself and also in specifying the BCs. Although there are vorticity-based calculation methods that avoid solving for a pressure field directly, the need

(which amounts to considerable effort) to extract pressure from the velocity field remains. Basing on this argument, it is perhaps more advisable to solve a pressure equation from the onset.

For variable (compressible) density flow, the continuity equation can serve as a pressure equation via the state-equation link with density. This link is lost for constant density flow, as such, an equation for pressure has to be formulated. Though not immediately obvious, a Poisson equation for pressure can be derived by taking the divergence of the N-S equations (i.e. differentiate the x_i -component of the momentum equation w.r.t. x_i) and using the continuity equation and then perform time-averaging.

$$\frac{1}{\rho} \frac{\partial^2 p}{\partial x^2} = - \frac{\partial^2}{\partial x_i \partial x_j} (U_i U_j) - \frac{\partial^2}{\partial x_i \partial x_j} \overline{u_i u_j} \quad \text{Eq.2.8}$$

In numerical flow calculations, one solves the momentum equations and a pressure equation (instead of the continuity equation) since a correctly predicted pressure field will yield velocities that satisfy continuity. It is appropriate to briefly add that in flow computations one very rarely solves a Poisson equation of the form shown above due to numerical difficulties, but in essence a pressure equation derived from the discretized form of the momentum and continuity equations. A discussion of the numerical solution technique for the Poisson pressure equation is deferred to §4.3.

2.1.3 Reynolds - Stress Equation

The $\overline{u_i u_j}$ -equation, from which second moment closure models are approximated, can be obtained by adding u_j times the x_i -component momentum equation to u_i times the x_j -component momentum equation and then time-averaged.

$$\begin{aligned}
 \overline{u_k \frac{\partial}{\partial x_k} \overline{u_i u_j}} &= - \left[\overline{u_i u_k} \frac{\partial \overline{u_j}}{\partial x_k} + \overline{u_j u_k} \frac{\partial \overline{u_i}}{\partial x_k} \right] - 2\nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}} \\
 \text{"convection"} & \qquad \qquad \text{"generation"} \qquad \qquad \text{"destruction"} \\
 &+ \frac{\partial}{\partial x_k} \left[-\overline{u_i u_j u_k} - \frac{p}{\rho} (\delta_{jk} \overline{u_i} + \delta_{ik} \overline{u_j}) + \nu \overline{\frac{\partial u_i u_j}{\partial x_k}} \right] \\
 & \qquad \qquad \qquad \text{"diffusion"} \\
 &+ \frac{p}{\rho} \left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i} \right) \qquad \qquad \text{Eq.2.9} \\
 & \qquad \qquad \qquad \text{"pressure strain"}
 \end{aligned}$$

The above equation represents six transport equations for $\overline{u_i u_j}$ and since no approximations have been made during the derivation it is exact. By putting $i=j$ and dividing by 2, one obtains a conservation equation for the turbulence energy k where $k = \frac{1}{2} \overline{u_i u_i}$.

The term on the left-hand-side (l.h.s.) of Eq.2.9 represents transport of $\overline{u_i u_j}$ by the mean velocities and since it can be expressed (using the continuity equation) as a pure spatial gradient it is a transport term. In the k -equation, the l.h.s. is normally referred to as advection. The first term on the right-hand-side (r.h.s.) of the $\overline{u_i u_j}$ -equation is the generation of $\overline{u_i u_j}$ from its interaction with the mean rates of strain (it is not always a positive quantity). The viscous destruction term is

usually a sink for energy dissipation by viscosity. Since the small scale motion is assumed to be isotropic, this term therefore affects only k , that is, only the normal stresses.

The correlation $\overline{u_i u_j u_k}$ is the transport of $\overline{u_i u_j}$ by the velocity fluctuations in the k -direction, this is similar to the transport by U_k . The second term in the turbulent diffusion group is transport by pressure fluctuations, a term which is both difficult to measure experimentally as well as interpret physically. The last term in this group is transport by the viscous stress fluctuations. This group of diffusion terms do not affect the average levels of $\overline{u_i u_j}$ in any cross-section but merely act to promote spatial redistribution.

The pressure-strain or redistribution term denotes the interaction between the fluctuating pressures and fluctuating strain rates. It has zero trace (i.e. it is zero in the k -equation) and tends to make the turbulence more isotropic by transferring energy from the larger normal stress to those with lower intensities. It also acts to reduce the magnitude of the shear stresses. Recently, data from large eddy simulations (LES) have been used to estimate this correlation with the aim of aiding turbulence modelling.

2.2 Calculation Of Slender Flows

2.2.1 Slender Flow Approximations

The Reynolds and continuity equations, though simpler than the

time-dependent form, are elliptic in space and any solution based upon them would still require an enormous amount of computing time and memory storage. To overcome this problem, one uses physical reasoning, of course backed by experimental results, to further simplify these governing equations so that a set of equations more amenable to solution appears. It must be stressed that any simplifications made are flow dependent. However, it is not uncommon for different flow phenomena to exhibit certain similar flow characteristics (see §2.2.2) which enable some classifications to be made.

A shear layer/flow is normally identified by the presence of a large streamwise velocity gradient in the cross stream direction with strong shearing confined to a "thin" layer. The present work focusses on the type of shear flows categorised as "slender" flows. Slender flows are 3-D with a pre-dominant flow direction (i.e. the streamwise or x-wise direction) and the length scale in the streamwise direction is much, much larger than the length scale on a plane normal to it (i.e. the "crossflow" plane).

$$\frac{\partial}{\partial x} \ll \frac{\partial}{\partial y} \sim \frac{\partial}{\partial z}$$

In other words, rapid changes of flow quantities such as mean velocities and Reynolds-stresses occur in the crossflow plane. As such, streamwise derivatives can be assumed to be negligible when compared to crossflow plane derivatives and can be omitted. This, however, does not imply that the rates of strain in the x-direction can be neglected, in fact, these should not be.

The set of Reynolds equations Eq.2.7, using the slender flow approximations, can be simplified into the following conservative, normalised form.

$$\begin{aligned} \text{x-mom: } \frac{\partial UU}{\partial x} + \frac{\partial UV}{\partial y} + \frac{\partial UW}{\partial z} = & - \frac{dP}{dx} + \frac{1}{Re} \frac{\partial^2 U}{\partial y^2} + \frac{1}{Re} \frac{\partial^2 U}{\partial z^2} \\ & - \frac{\partial \overline{uv}}{\partial y} - \frac{\partial \overline{uw}}{\partial z} \end{aligned} \quad \text{Eq.2.10a}$$

$$\begin{aligned} \text{y-mom: } \frac{\partial UV}{\partial x} + \frac{\partial VV}{\partial y} + \frac{\partial VW}{\partial z} = & - \frac{\partial P}{\partial y} + \frac{1}{Re} \frac{\partial^2 V}{\partial y^2} + \frac{1}{Re} \frac{\partial^2 V}{\partial z^2} \\ & - \frac{\partial \overline{vv}}{\partial y} - \frac{\partial \overline{vw}}{\partial z} \end{aligned} \quad \text{Eq.2.10b}$$

$$\begin{aligned} \text{z-mom: } \frac{\partial UW}{\partial x} + \frac{\partial VW}{\partial y} + \frac{\partial WW}{\partial z} = & - \frac{\partial P}{\partial z} + \frac{1}{Re} \frac{\partial^2 W}{\partial y^2} + \frac{1}{Re} \frac{\partial^2 W}{\partial z^2} \\ & - \frac{\partial \overline{vw}}{\partial y} - \frac{\partial \overline{ww}}{\partial z} \end{aligned} \quad \text{Eq.2.10c}$$

The normalising parameters are length scale, L_{ref} , velocity scale, U_{ref} , and pressure P_{ref} . In Eq.2.10, the normalised form of the primitive variables are

$$x_i = \frac{x_i}{L_{ref}}; \quad U_i = \frac{U_i}{U_{ref}}; \quad u'_i = \frac{u'_i}{U_{ref}}; \quad P = \frac{P - P_{ref}}{\rho U_{ref}^2}$$

$$\text{and Reynolds Number, } Re = \frac{U_{ref} L_{ref}}{\nu}$$

In Eq.2.10, the terms omitted from the Reynolds equations are the streamwise viscous diffusion and Reynolds-stress gradients. The

omission of these terms removes their upstream influence. It is important to note that these equations are not direction invariant, therefore, the predominant flow direction must be known beforehand. This set of governing momentum equations will form the basis of the present numerical solution for the mean velocity field.

A simplification more powerful than the neglect of streamwise viscous and Reynolds-stress gradients in the slender flow approximation is the assumption that $\frac{\partial P}{\partial x}$ can be totally dissociated from the crossflow plane pressure field. The crossflow plane pressure variation is assumed to be sufficiently small such that it can be represented by a mean bulk pressure. This mean bulk pressure is to be specified by $\frac{dP}{dx}$ with "fine tuning" from the pressure field calculated from the pressure equation. The assumption $\frac{\partial P}{\partial y} \sim \frac{\partial P}{\partial z} \approx 0$, though bold is not preposterous and can be considered an extension of the well known thin shear layer approximation $\frac{\partial P}{\partial y} = 0$ (a rigorous and thorough proof can be found in Cebeci & Bradshaw, 1977, pp.39-45).

When $\frac{\partial P}{\partial x}$ is set *a priori*, the set of elliptic Reynolds equations in 3-D space becomes parabolic in the x-direction (appropriately, $\frac{\partial P}{\partial x}$ is replaced by $\frac{dP}{dx}$ in Eq.2.10a) and is amenable to a more economical space marching solution in the x-direction. These equations are commonly referred to as the parabolized N-S equations. The elliptic pressure Poisson equation is also reduced from 3-D to 2-D. In situations where upstream influence only allows $\frac{dP}{dx}$ to be assumed to a close estimate, the parabolized N-S

equations has to be iterated globally. At the end of each global streamwise sweep $\frac{dP}{dx}$ has to be updated. In such a scheme, the set of equations is partially parabolized or semi-elliptic.

The slender flow approximations, though widely applicable (see §2.2.2), must be used with caution. In some instances, streamwise curvature can be quite large such that $\frac{\partial P}{\partial y} \approx 0$ does not hold, still it may be possible to use the slender flow equations but with $\frac{\partial P}{\partial y}$ approximated by, say, the centrifugal term $\frac{\rho U^2}{R}$ where R is the radius of curvature. Extra rates of strain augmented by longitudinal or lateral curvatures or three-dimensionality, can also significantly affect the turbulence structure and consequently cause the slender flow approximations to break down. When the pressure field is significantly affected by interaction between the shear layer and the external inviscid flow, it may become difficult to specify $\frac{\partial P}{\partial x}$ a priori thus reducing the usefulness of the slender flow approximation. One can say that as long as the neglected terms are small or can be approximated (sometimes crudely), the slender flow equations can be used.

Since the slender flow equations are capable of predicting the secondary flows, it is worthwhile to briefly mention about the two types of secondary flow. Skew-induced secondary flow or Prandtl's first kind is generated by an inviscid mechanism when the spanwise vorticity elements are deflected to the streamwise direction by the mean strain rate/s. Once the streamwise vorticity is created it will be diffused by viscous stresses and Reynolds-stresses. The attenuation of skew-induced secondary flow by Reynolds-stresses

should not be confused with the stress-induced secondary flow (or Prandtl's second kind) due to Reynolds-stress gradients in the crossflow plane. Stress-induced secondary flow is a true turbulence phenomenon that has no counterpart in laminar flow. In the experimental work of Kornilov & Kharitonov (1984), the sufficiently long wing-body junction rig not only allowed the Reynolds-stress gradients to diffuse the horse-shoe vortex but also enabled stress-induced secondary flow to appear.

Stress-induced secondary flows are normally one order of magnitude smaller than skew-induced ones, and when both mechanisms are present the latter is predominant. Although the skew-induced secondary flow mechanism is inviscid, the important role played by the Reynolds-stresses in diffusing the vorticity created means a proper turbulence model is required for good prediction of the flow. A good turbulence model is also necessary for stress-induced secondary flows in order to obtain accurate crossflow velocities. Though the crossflow velocity vectors are small in comparison with U (say 2%-3%), they can have large influences on the streamwise velocity contours.

2.2.2 Practical Slender Flows

As outlined in §1.3, since one of the primary objectives in this work is to develop an efficient algorithm for turbulent slender flows, it is most appropriate to look at some practical occurrences of slender flows. Here, slender flows are taken to include "thin shear flows" because the former can be treated as an

extension of the latter from 2-D to 3-D. In brief, the thin shear layer or boundary layer approximation applies when

$$\frac{\partial}{\partial y} \gg \frac{\partial}{\partial x} \sim \frac{\partial}{\partial z}$$

Therefore, any calculation algorithm based on the slender flow equations should, in principle, be able to predict thin shear flows without any difficulties, albeit some "redundant" computations will be executed.

Turbulent shear flows, whether slender or not, can be categorised into the following three groups according to the number of fixed boundaries present:-

- (a) Free shear layers, that is, those without any boundaries. Examples include 2-D flows like the mixing layers, jets, wakes, plumes and the 3-D trailing vortices.
- (b) Turbulent shear flows bounded by one free and one fixed boundary. Examples are 2-D or 3-D boundary layers and wall jets with walls that can either be straight or curved, permeable or impermeable.
- (c) Turbulent shear flows bounded by two or more boundaries. Examples are flows in pipes, ducts, channels and along streamwise corners.

Boundary layers are strong shear flows and are by far the commonest type of shear flow experienced. Boundary layers can be 2-D or 3-D, and a simple illustration is shown in Fig.2.1. The flow over most part of the wing is a boundary sheet whereas near

the wing-body junction and wing tip the flow is slender (i.e. boundary region). In Fig.2.2 is shown the flow through blade cascades in turbomachinery (or compressor), the various flow phenomena are clearly indicated in the diagram. The oncoming flow to the blades are unsteady and very complex, a mixture of the blade wakes and vortices from the preceding row of blades. In the passage between the blades, there are the tip and hub corner vortices, and in some situations there may be shocks.

Although different basic slender flow phenomena can be quite easily distinguished, in practice they seldom exist alone. The strategy adopted in calculation methods for complicated flows (like the above examples) is to segregate a combined flow phenomena into its basic flows and perform predictions for the individual basic flows. (Even when one can identify the basic flow features, in many instances, it may not be possible to "decouple" them due to their strong interactions.) The solution procedure will require a boundary "matching" algorithm to allow for interactions between the different flow domains. If the interactions between the different flow fields are weak or negligible, a more economical method would be to overlap the computational domains and predict the more independent flow first and then use the results as BCs for the other more "dependent" flow/s. For instance, in the aircraft wing-body configuration in Fig.2.1, the boundary region flow at the wing-body junction can supply BCs for the boundary sheet flow calculation over the wing.

In many practical situations, the engineer is usually interested

in the transfer of heat (e.g. turbomachinery, heat exchangers) or species concentration (e.g. combustion chambers, chemical mixing plant) or even pollutants (e.g. plumes, river flows, planetary boundary layers, jet streams in the oceans, distribution of dust particles or foreign objects when an aircraft is close to the ground). Whatever the requirements, one still needs to accurately determine the momentum transfer first since all other transported quantities depend on the velocity field.

2.3 Turbulence Modelling

2.3.1 Physics Of Turbulence

Turbulence, being the most complicated kind of fluid motion, occurs inevitably in most flows of practical interest and its occurrence is often accepted without question. The best way to describe turbulence is to consider vortex elements with vorticity ω_i defined as twice the angular velocity.

$$\omega_i = \text{curl } U_i = \epsilon_{ijk} \left[\frac{\partial U_k}{\partial x_j} - \frac{\partial U_j}{\partial x_k} \right]$$

If a given vortex element parallel to the z-axis (with vorticity ω_z) is stretched by a rate of strain $\frac{\partial W}{\partial z}$, the cross-sectional area on the x-y plane will decrease. In order to conserve angular momentum (equal to the product of vorticity and square of the radius) both U and V must increase, this in turn stretches the other vortex elements in other directions. In other words, stretching in one direction causes a decrease in length scale and an increase in velocity scale in the other two directions. The

stretching process, also known as cascading, will produce a tangle of vortex elements with the unsteady vorticity vectors aligned in all directions. It is shown in Bradshaw (1972, pp.14-15) that continual stretching will produce smaller and smaller eddies that are more and more isotropic in character.

The complexity of turbulence can be seen by the wide spectrum of eddy sizes and frequencies and with increase in Re the spectrum also widens. The large eddies extract energy from the mean flow to feed the turbulence and through the cascade process this energy is passed down to smaller and smaller eddies until it is dissipated as heat by viscosity. Since most flows have a predominant flow direction, the large scale motion and hence the stretching of the tangle of vortex elements by the mean flow will also have a preferred direction. This implies anisotropy in the large scale motions. The large eddies are strongly dependent upon the BCs, size of the flow domain, mean strain rates as well as history effects and the structure can be said to be different for different types of flow. Since the empirical constants in a turbulence model are structural parameters, one expects their values to change whenever the flow structure changes.

In most parts of a turbulent flow, the effects of viscosity, other than its role in energy dissipation, can be neglected. Viscous stresses are relatively much smaller than Reynolds-stresses (with the exception of the transition region and the viscous sub-layer) and viscous diffusion is also negligible. Although viscosity plays no part in the vortex stretching process, a fluid without any

initial vorticity can acquire it through viscous diffusion and hence initiate the stretching process. The small eddies which are responsible for energy dissipation have been postulated as being isotropic and therefore have a universal character and are independent of the large eddies. This assumption is not strictly true because some information from the large eddies does get passed down to the dissipating eddies.

The large scale motions, which contain most of the Reynolds-stresses, are mainly responsible for the transport of momentum and also control the amount of energy to be dissipated by viscosity. Therefore, in turbulence modelling, one is concerned with modelling the behaviour of the large eddies through suitably chosen velocity scale/s and length scale/s.

2.3.2 Overview Of Turbulence Models

Turbulence models can broadly be divided into two main categories, one based on the ν_t concept and the other solving differential transport equation/s for turbulence quantities. Still sub-divisions are necessary to distinguish between the wide range of models and a common way is to identify the model according to the number of transport equations solved. The complexity of a model increases with the number of equations solved, with the hope that with increase in the number of equations, more details of the flow can be captured so that ultimately a universal model may be obtained. Since the number of available turbulence models are enormous and far too exhaustive to be covered in detail, a brief

general review will be attempted here (for a recent review see Nallasamy, 1987).

The oldest concept in turbulence modelling is the Boussinesq ν_t concept which assumes an analogy between viscous stresses in laminar flow and turbulence apparent stresses in turbulent flow, that is, the Reynolds-stresses are proportional to mean rates of strain. The definition of ν_t is

$$-\overline{u_i u_j} = \nu_t \left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right] - \frac{2}{3} \delta_{ij} k \quad \text{Eq.2.11}$$

and the turbulence or eddy viscosity is usually modelled as

$$\nu_t \propto V_s l_s \quad \text{Eq.2.12}$$

ν_t depends on the state of turbulence rather than being a fluid property. The term involving the Kronecker delta is necessary to ensure that upon contraction of indices the r.h.s. does not vanish completely due to continuity. Eq.2.11 itself does not constitute a turbulence model but it is in the determination of ν_t that a turbulence model becomes necessary. ν_t is a hybrid quantity and is defined in Eq.2.11 as a mixture of mean and turbulence scales. There is, therefore, no reason to believe that ν_t can be modelled entirely by either the mean flow scale or the turbulence scale, it should be a mixture of both and to date nobody actually knows how to deal with it.

In zero-equation models, no transport equation for turbulence quantities is involved and ν_t is algebraically specified. The simplest possible model (if it may be considered a model at all)

is one of constant ν_t obtained either empirically from experiments or more often through trial and error calculations. The assumption that ν_t is constant across the width of the shear layer and varies simply proportional to the shear layer thickness can work quite well. The first proper turbulence model was developed by Prandtl (see Schlichting, 1979, pp.579-583) and is therefore appropriately called the Prandtl mixing-length hypothesis. In a 2-D boundary layer,

$$\nu_t = l_m \left| \frac{\partial U}{\partial y} \right|$$

where the mixing-length l_m is defined as the distance travelled by a lump of fluid so that the velocity difference it experiences is equal to the transverse velocity fluctuation.

With the constant of proportionality in Eq.2.12 taken as 1.0, l_m remains the only unknown needed to specify ν_t . For simple 2-D flows, l_m can be accurately specified by empirical formulae.

$$\text{Inner layer} \quad \left\{ \begin{array}{l} l_m = \kappa y f(y^+) \\ f(y^+) = 1 - \exp\left[-\frac{y^+}{A^+}\right] \end{array} \right.$$

In the above van Driest (1956) near-wall formula for the inner layer, κ is the von Karman constant, $y^+ (= \frac{y u_\tau}{\nu})$ the dimensionless distance from the wall and the constant $A^+ = 26.0$.

$$\text{Outer layer} \quad \left\{ \begin{array}{l} \nu_t = 0.0168 \rho U_\infty \delta^* \lambda \\ \lambda = \left\{ 1 + 5.5 \left[\frac{y}{\delta} \right]^6 \right\}^{-1} \\ \text{OR,} \\ l_m = 0.09 \delta \quad (\text{ramp function}) \end{array} \right.$$

where U_∞ is the free stream mean velocity, δ the boundary layer thickness, δ^* the boundary layer displacement thickness and λ the intermittency factor. The more elaborate formula of Cebeci & Smith (1974, p.171) using the intermittency factor (which improves the calculated shear stress profile) works out to be roughly the same as the simple ramp function.

The main drawback of the l_m -model is the dependence on local flow conditions and it is therefore unable to simulate history effects. It is unsatisfactory when applied to rapidly developing or recirculating flows (i.e. when both convection and diffusion are important) and needless to say complex flows because of the difficulty in specifying l_m satisfactorily. Despite all these shortcomings, l_m -models are still widely in use in the industries today (including incompressible flow predictions) mainly because of simplicity and economy. In Rodi (1980, pp.18-20) one can find other l_m -models such as the Prandtl model for free shear flows, the Buleev model for duct flows and the von Karman model which eliminates prescribing l_m algebraically.

The next higher level of turbulence models are the one-equation models. This class endeavour to solve a transport equation for a turbulence quantity, often the turbulence velocity scale. The Bradshaw et al. (1967) model solves a shear stress differential equation derived from the k -equation by assuming $\frac{\overline{uv}}{k} = a =$ constant.

$$U \frac{\partial}{\partial x} \frac{\overline{uv}}{a} + V \frac{\partial}{\partial y} \frac{\overline{uv}}{a} = - \overline{uv} \frac{\partial U}{\partial y} - \frac{|\overline{uv}|^3}{l_s^2} - |\overline{uv}_{\max}|^{1/2} \frac{\partial}{\partial y} (G \overline{uv})$$

Turbulent diffusion is not modelled as gradient diffusion but instead depends on a bulk velocity $|\overline{uv}_{\max}|^{1/2}$. The function G is given by

$$G = \left[\frac{|\overline{uv}_{\max}|}{U_\infty^2} \right]^{1/2} f_1 \left(\frac{y}{\delta} \right)$$

The length scale l_s is given by

$$\frac{l_s}{\delta} = f_2 \left(\frac{y}{\delta} \right)$$

Both f_1 and f_2 are empirically specified. Although this model is successful in wall boundary layers it will fail in situations where $\overline{uv}$ changes sign because k is always positive. What is more important is that it is the first true transport model that does not employ the ν_t -concept.

In most of the one-equation ν_t -models, k is the single characteristic velocity scale and its choice is perhaps difficult to dispute for several reasons. k is representative of the turbulence intensities in three spatial directions and it is contained mostly in the large scale motions which are the motions a model is trying to mimic. It is also helpful that k is a scalar quantity as ϵ is. As mentioned in §2.1.3, the exact form of the k -equation can be obtained from the $\overline{u_i u_j}$ -equation, Eq.2.9, by putting $i=j$ and dividing throughout by 2.

$$U_i \frac{\partial k}{\partial x_i} = \underbrace{- \overline{u_i u_j} \frac{\partial U_i}{\partial x_j}}_{\text{"advection"}} - \underbrace{\nu \frac{\partial \overline{u_i}}{\partial x_j} \frac{\partial \overline{u_i}}{\partial x_j}}_{\text{"production"}} - \underbrace{\frac{\partial}{\partial x_i} \left(\overline{u_i \frac{u_j u_j}{2}} + \frac{u_i p}{\rho} \right)}_{\text{"destruction"}} \quad \underbrace{\quad}_{\text{"diffusion"}}$$

Eq.2.13

Each term has its usual meaning as given in §2.1.3, note that the pressure-strain term vanishes by continuity.

In the modelled k -equation, the production term is approximated by replacing $\overline{u_i u_j}$ with Eq.2.11.

$$- \overline{u_i u_j} \frac{\partial U_i}{\partial x_j} = \nu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \frac{\partial U_i}{\partial x_j}$$

ϵ is modelled as

$$\epsilon = \nu \frac{\partial \overline{u_i}}{\partial x_j} \frac{\partial \overline{u_i}}{\partial x_j} = C_D \frac{k^{3/2}}{l_s}$$

and the turbulent diffusion is assumed proportional to the gradient of k .

$$\overline{u_i \frac{u_j u_j}{2}} + \frac{u_i p}{\rho} = - \frac{\nu_t}{\sigma_k} \frac{\partial k}{\partial x_i}$$

where $\nu_t = C_\mu \frac{k^2}{\epsilon}$ and σ_k, C_μ, C_D are empirical constants.

The length scale is determined from simple empirical relations in a similar manner to the mixing-length, likewise, difficulties arise in complex flow situations. It is in general more difficult to specify l_s in a one-equation turbulence model than l_m in a l_m -model because l_s is more strongly flow dependent. Rodi (1980, pp.84-85) discussed the more sophisticated l_s model by Bernard and

his co-workers which uses an interaction formula to combine the flow dependent length scales, each determined individually. The model works well for wall boundary layers as well as free shear flows, however, predictions are poor for complex flows such as recirculating flows.

An interesting point to note is that under local equilibrium conditions, the modelled k -equation reduces to

$$\nu_t \left(\frac{\partial U}{\partial y} \right)^2 = C_D \frac{k^{3/2}}{l_s}$$

One can eliminate k from the above equation by using the relationship $\nu_t = C_\mu k^{1/2} l_s$ to arrive at

$$\nu_t = l_s^2 \left(\frac{C_\mu}{C_D} \right)^{1/2} \left(\frac{\partial U}{\partial y} \right)$$

This is analogous to the zero-equation l_m formulation and partly explains the success of l_m -models in equilibrium shear layers.

The two-equation turbulence models attempt to eliminate the difficulties associated with specifying l_s by solving a transport equation for l_s itself. The additional computational effort needed for the l_s -equation is justifiably not expensive in view of the successes enjoyed by this class of model. Physically, the length scale characterises the large scale motions and being "transportable" is, of course, affected by the flow history (an effect difficult to simulate algebraically). l_s also depends on ε , which destroys the small eddies or effectively increases the eddy size, and vortex stretching mechanism which reduces the eddy size.

It is (still) hoped that the form of the l_g -equation can duplicate all these effects.

An equation for l_g can be cast as frequency ($k^{1/2}/l_g$), vorticity (k/l_g^2) or ε ($= k^{3/2}/l_g$). Although the various forms are physically different, in essence, they are solving for the same length scale and, in principle, regardless of the equation being used the results should be the same. The choice of ε is more satisfying because it is measurable (albeit with difficulty) and Launder & Spalding (1974) showed that unlike the other forms no wall correction is required for ε .

Since the RSTM in this work also entails solving the ε -equation, it is appropriate to defer detailed discussion to §3.5. The differences between the modelled ε -equation for ν_t -models and RSTM will also be pointed out. In all classes of turbulence models which involve the ε -equation, poor predictions are often attributed (perhaps conveniently) to this equation. Such arguments are rather subjective since little is understood of the exact equation, not to mention the heavily truncated modelled form.

The constants in k - ε models are widely accepted to be non-universal and often special functions for these constants are devised to suit particular flow situations. A review of the more popular functions in various two-equation models is given by Patel et al. (1985). Two-equation models find wide applications in 2-D flows where $\overline{uv}$ is dominant and in recirculating flows where the Reynolds-stresses are not as important as the convection or the

pressure gradient terms. In other words, the ν_t concept limits satisfactory predictions of complex flows.

In view of all the shortcomings of the ν_t -models, one moves on to solve differential equations for each of the Reynolds-stresses and a length scale (normally ε) equation. This is known as second-moment closure or RSTM and, briefly, it is a multi-velocity scale model (detailed description can be found in Chapter 3). The inherent advantage of a RSTM is that they unhook the Reynolds-stresses from the mean velocity field. If a mean velocity field is strongly altered by sudden pressure gradients it can considerably change the mean strain rates and thus change the Reynolds-stresses as predicted by a ν_t -model even if the genuine Reynolds-stresses have not changed at all or have not changed much. The mean pressure gradient does not appear in the $\overline{u_i u_j}$ -equations and therefore any effects must come from the generation terms, which take some time to affect events. This is the proper time history effect of the turbulence which ν_t -models cannot reproduce.

Many workers view the RSTM as one capable of predicting complex flow fields while at the same time it has the potential to attain some degree of universality. Although second-moment closure is more reliable than ν_t -models, the approximations for the pressure redistribution term and the ε -equation (again) are highly uncertain and therefore lack universality. As compared to the other turbulence models, the computing time and storage are substantially greater and relative to the two-equation k - ε models Cebeci et al. (1986) give a factor of ≈ 7 for computing time and

≈ 2.5 for memory storage.

Algebraic-stress models were developed in order to tap the comprehensiveness of RSTM but minimising computing time by converting the differential equations for $\overline{u_i u_j}$ to algebraic expressions. This can be done by approximating the derivatives of $\overline{u_i u_j}$, found only in the transport terms in the $\overline{u_i u_j}$ -equation, to the transport of k (see Rodi, 1976).

$$\begin{aligned} \frac{D}{Dt} \overline{u_i u_j} - \text{Diff } \overline{u_i u_j} &= \frac{\overline{u_i u_j}}{k} \left[\frac{D}{Dt} k - \text{Diff } k \right] \\ &= \frac{\overline{u_i u_j}}{k} (P_k - \varepsilon) \end{aligned}$$

where P_k is the production of k , that is, $P_k = -\overline{u_i u_j} (\partial u_i / \partial x_j)$. In addition to the algebraic $\overline{u_i u_j}$ -equations, one also needs to solve the k - and ε - equations. In 2-D flows, the algebraic expressions reduce to a form analogous to the ν_t formulation where C_μ becomes a function of the ratio $\frac{P_k}{\varepsilon}$. In fact, this variation of C_μ with $\frac{P_k}{\varepsilon}$ was pointed out in Rodi (1972). Though not widely tested, algebraic-stress models have been used by Demuren & Rodi (1984 & 1987) and Eppich (1987) to predict stress-induced secondary flows in duct with rather encouraging results.

Some recent work has concentrated on solving the 3-D N-S equations for the large eddies, and approximating eddies that are smaller than the numerical grid resolution with a semi-empirical turbulence model (see Schumann & Friedrich, 1986). This is known in CFD as large eddy simulation (LES). The ability to directly simulate the large eddies is very attractive because these

turbulence carrying eddies are, in general, highly flow dependent. The modelled small eddies are much less problem-dependent and will not strongly influence the overall solution of the large scale motion. Hence, the sub-grid model can be quite universal.

At present, the difference in computing times between LES and a RSTM is very large though, optimistically, it is possible LES may one day become standard engineering design tool. This optimism is enhanced by some remarkable results obtained from using rather coarse grids. In the near future, the role of LES would be in the development of conventional turbulence models (especially the RSTM) by providing data for correlations that are difficult to measure (e.g. the pressure-strain term) and determining the values of the empirical constants.

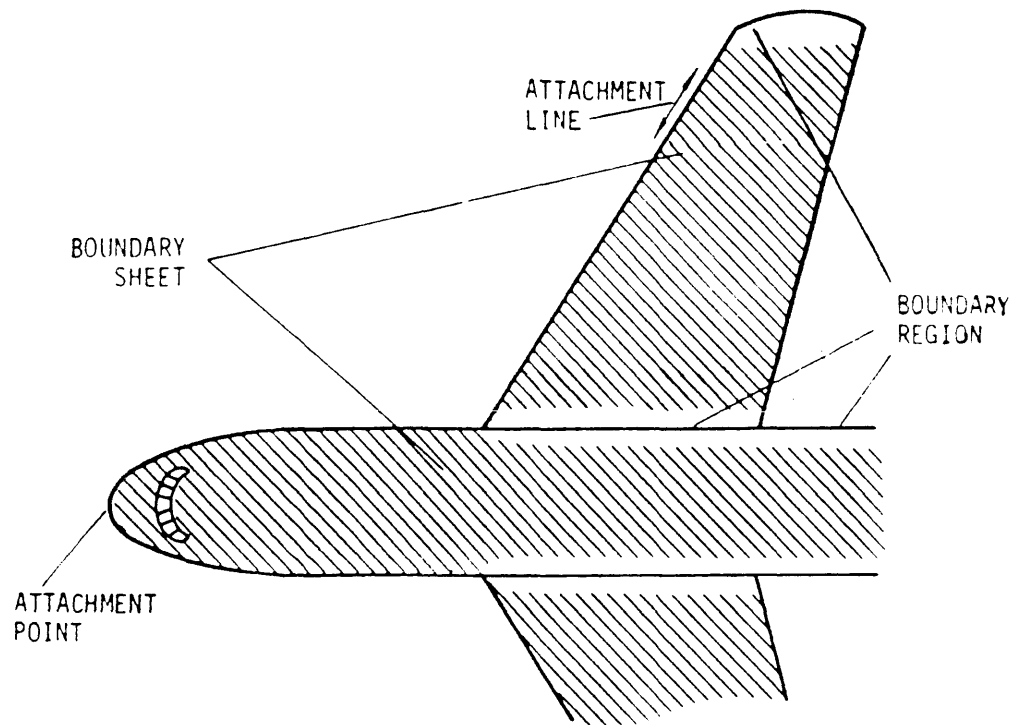


Fig.2.1 Boundary Sheets And Boundary Regions (from Cebeci & Bradshaw, 1977, p.316).

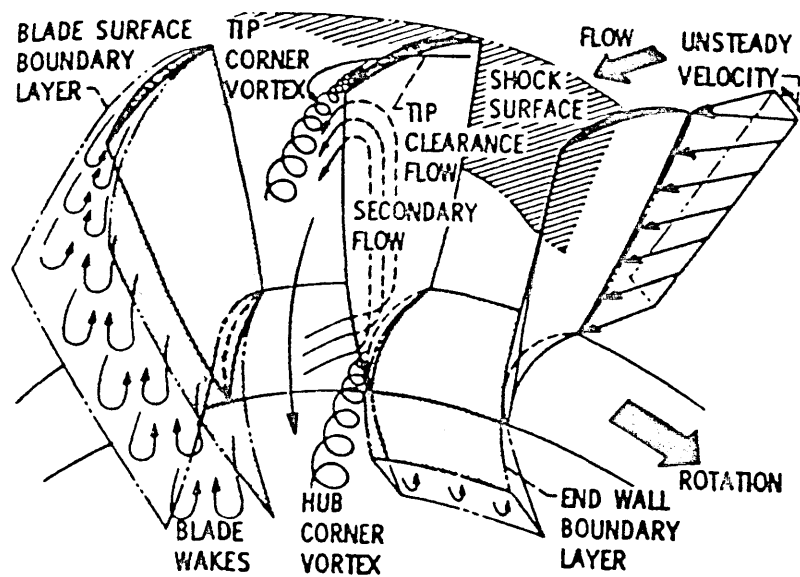


Fig.2.2 Turbomachinery Blade Flow Features (from McNally & Sockol, 1985).

3 REYNOLDS - STRESS TURBULENCE MODEL

3.1 Second - Moment Closure Model

The time-averaged $\overline{u_i u_j}$ -equations, although a "reduced" form of the time-dependent N-S equations, contain more unknowns than equations. It is therefore not possible to solve the $\overline{u_i u_j}$ transport equations without resorting to some form of "closure" approximation. The higher-order unknown quantities can be empirically related to the Reynolds-stresses etc, but the empirical relations are likely to be valid only for restricted ranges of flows. The RSTM is also known as second-moment closure model because no equation in the model involves any correlation higher than second order. Second-moment closure is perhaps the simplest turbulence model possible that can hope to achieve any generality for predicting complex strain rate fields. Thus the $\overline{u_i u_j}$ -equations, with the continuity and momentum equations, form a closed system of equations for the mean velocities, the Reynolds-stresses and, in general, the mean pressure.

The course followed in modelling the $\overline{u_i u_j}$ -equations is to express higher-order unknown terms as empirical functions of the mean velocities and Reynolds-stresses, their derivatives and a characteristic turbulence time scale t_g . Before discussing the modelling process, it is appropriate, for completeness, to present the $\overline{u_i u_j}$ -equation, Eq.2.9, again; the meaning of each term in the equation can be found in §2.1.3.

$$\begin{aligned}
\underline{u}_k \frac{\partial}{\partial x_k} \overline{u_i u_j} &= - \left[\overline{u_i u_k} \frac{\partial u_j}{\partial x_k} + \overline{u_j u_k} \frac{\partial u_i}{\partial x_k} \right] - 2\nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}} \\
\text{"convection"} & \qquad \qquad \text{"generation"} & \qquad \qquad \text{"destruction"} \\
& + \frac{\partial}{\partial x_k} \left[-\overline{u_i u_j u_k} - \frac{p}{\rho} (\delta_{jk} u_i + \delta_{ik} u_j) + \nu \overline{\frac{\partial u_i}{\partial x_k} u_j} \right] \\
& \qquad \qquad \qquad \text{"diffusion"} \\
& + \frac{p}{\rho} \left[\overline{\frac{\partial u_i}{\partial x_j}} + \overline{\frac{\partial u_j}{\partial x_i}} \right] \qquad \qquad \text{Eq.3.1} \\
& \qquad \qquad \qquad \text{"pressure strain"}
\end{aligned}$$

On the r.h.s. of Eq.3.1, the generation term can be kept in exact form, but the remaining three terms have to be approximated. It is obviously an advantage to be able to keep the generation term in its exact form since the stress-strain interactions govern the broad characteristics of the turbulence.

As mentioned in §1.3, the turbulence model to be used in this work is one which is suitable for high local Re flows, therefore, unsuitable for use in the region very close to a wall (i.e. the viscous sub-layer and buffer region) where viscous stresses are important. In deriving a model for high Re, one assumes that:-

- (a) Large scale eddies are responsible for momentum transfer.
- (b) The viscous-dependent (small scale) eddies do not carry much of the Reynolds-stress, so the equations can be modelled independent of viscosity.

Although some doubts have been thrown on assumption (b), it remains an important and useful basis for turbulence modelling.

3.2 Modelling The Reynolds - Stress Equation

3.2.1 Viscous Dissipation

Viscous destruction, unlike viscous diffusion, cannot be ignored. Its significance can easily be shown by examining the k -equation where one will find that it balances or limits the growth of k from the production terms. This is the mechanism in the cascade process whereby turbulence energy is destroyed. In approximating the rate of viscous dissipation, one assumes it to be equally partitioned among the three velocity components (this is not quite the same as assuming local isotropy).

$$2\nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}} \approx \frac{2}{3} \delta_{ij} \epsilon \quad \text{Eq.3.2}$$

It must be added that although velocity fluctuations may be small in comparison with the mean velocity, in fine scale motions the derivative of u_i is usually large.

3.2.2 Turbulent Diffusion

The turbulent diffusion process can be attributed to velocity fluctuations, pressure fluctuations and viscosity. Basing on the argument of local isotropy in high local Re , the viscous diffusion term can be omitted (it is retained in low local Re models), or at least grouped with the triple product model. The pressure diffusion term is also dropped because experimental work by (possibly the only available source) Hanjalic & Launder (1972a) showed that the pressure diffusion term in the k -equation, is

relatively unimportant. The triple product correlations are therefore the only terms in this group which require modelling.

A transport equation for $\overline{u_i u_j u_k}$ can be derived and has the form

$$\frac{D \overline{u_i u_j u_k}}{Dt} = P_{ijk1} + P_{ijk2} + T_{ijk} + \phi_{ijk} - D_{ijk}$$

where P_{ijk1} is generation due to mean strain rates, P_{ijk2} is product of second order correlations and their gradients, T_{ijk} is diffusion of quadruple products, ϕ_{ijk} is pressure-strain correlations and D_{ijk} is dissipation.

P_{ijk1} is neglected on the basis that $\overline{u_i u_j u_k}$ in the $\overline{u_i u_j}$ -equation is important only when the rate of strain is small. Although this term as well as the convective transport of $\overline{u_i u_j u_k}$ (which is also neglected) can be represented exactly, a model of this sort will be classified as third-moment closure and therefore not in line with the scope of this work. The term D_{ijk} , being zero in homogeneous flow and of constant magnitude in inhomogeneous flow, is omitted.

According to Hanjalic & Launder (1972b),

$$P_{ijk2} + T_{ijk} = C_s \frac{k}{\varepsilon} \left(\overline{u_i u_l} \frac{\partial \overline{u_j u_k}}{\partial x_l} + \overline{u_j u_l} \frac{\partial \overline{u_k u_i}}{\partial x_l} + \overline{u_k u_l} \frac{\partial \overline{u_i u_j}}{\partial x_l} \right)$$

and

$$\phi_{ijk} = - \frac{3C_1}{t_s} \overline{u_i u_j u_k}$$

The above proposal for ϕ_{ijk} can be considered as the first term in a non-linear series as obtained by Lumley (1978). The greatly

simplified or truncated form of the triple-product correlation approximation is

$$-\overline{u_i u_j u_k} = C_g \frac{k}{\varepsilon} \left(\overline{u_i u_l} \frac{\partial \overline{u_j u_k}}{\partial x_l} + \overline{u_j u_l} \frac{\partial \overline{u_k u_i}}{\partial x_l} + \overline{u_k u_l} \frac{\partial \overline{u_i u_j}}{\partial x_l} \right)$$

Eq.3.3

The ratio $\frac{k}{\varepsilon}$ can be thought of as a time scale for the diffusing motion or turbulence decay time. C_g is given a value of 0.11 for most flows but remains a parameter often optimised for specific situations. According to Lumley (1978) C_g , strictly speaking, is equal to $(3C_1)^{-1}$, therefore with $C_1=1.5$ one gets $C_g=0.22$, which is not the optimised value obtained by many different workers. (Note: C_1 is the empirical coefficient in the "rapid" part of the pressure-strain term to be described in §3.2.3).

The form derived in Eq.3.3 being greatly simplified gave cause to an even simpler version of the turbulent diffusion transport model from Daly & Harlow (1970).

$$-\overline{u_i u_j u_k} = C_g \frac{k}{\varepsilon} \overline{u_k u_l} \frac{\partial \overline{u_i u_j}}{\partial x_l}$$

Here, the value of C_g (=0.22) differs from that in Eq.3.3. Besides its simplicity, this model has a non-symmetric tensor just like the original diffusion group in Eq.3.1 and may be able to cater for the neglected pressure and viscosity diffusion terms when formulating Eq.3.3. Computationally, the differences between the two models are very slight as to render them "identical".

3.2.3 Pressure Redistribution

The final term to be modelled in the $\overline{u_i u_j}$ -equation is the pressure redistribution or pressure-strain term. As the former name appropriately implies, this term has zero trace, that is, for $i=j$ in Eq.3.1 it vanishes. The pressure-strain term acts to redistribute energy among the normal stresses, usually transferring energy from $\overline{uu}$ to $\overline{vv}$ and $\overline{ww}$. Due to the absence of ϵ in the shear stress equations, the pressure-strain term acts as a sink to limit the generation of shear stress from mean strain rates.

Modelling the pressure redistribution process is difficult because the correlation between fluctuating pressure and fluctuating rate of strain has no direct measurement to serve as a guide although recent LES data are providing some valuable data for this correlation.

$$\frac{p}{\rho} \frac{\partial u_i}{\partial x_j} = \frac{1}{4\pi} \int \left[\underbrace{\frac{\partial^3 \overline{u_k u_l u_i}}{\partial x_k \partial x_l \partial x_j}}_{\phi_{ij,1}} + 2 \underbrace{\frac{\partial \overline{u_k}}{\partial x_i} \frac{\partial^2 \overline{u_l u_i}}{\partial x_k \partial x_j}}_{\phi_{ij,2}} \right] \frac{d \text{ vol}}{r} + S_{ij} \quad \phi_{ij,v}$$

Eq.3.4

From Eq.3.4, two physical processes can be identified: $(\phi_{ij} + \phi_{ji})_1$ arising from the interactions between the turbulence components, and $(\phi_{ij} + \phi_{ji})_2$ from the mean rates of strain and turbulence. The surface integral S_{ij} is important only in close proximity to a wall and its treatment will be discussed in §3.2.4.

$(\phi_{ij} + \phi_{ji})_1$ is non-linear and usually acts in such a way as to

equalize the normal Reynolds-stresses and diminish the shear stresses to zero, in short, it is a return to isotropy term. For anisotropic homogeneous flows, the $\overline{u_i u_j}$ -equation can be written as

$$\frac{D\overline{u_i u_j}}{Dt} = (\phi_{ij} + \phi_{ji})_1 - \varepsilon_{ij}$$

By introducing the anisotropy tensor

$$a_{ij} = \left[\overline{u_i u_j} - \frac{2}{3} \delta_{ij} k \right] \frac{1}{k}$$

and putting $\varepsilon_{ij} = \frac{2}{3} \delta_{ij} \varepsilon$, after some manipulations one arrives at

$$\frac{Da_{ij}}{Dt} = (\phi_{ij} + \phi_{ji})_1 + \varepsilon a_{ij} \quad \text{Eq.3.5}$$

The proposed model for $(\phi_{ij} + \phi_{ji})_1$, first advocated by Rotta (1951) is

$$(\phi_{ij} + \phi_{ji})_1 = -C_1 \varepsilon a_{ij} \quad \text{Eq.3.6}$$

By substituting Eq.3.6 into Eq.3.5, one can deduce that a value of $C_1 > 1.0$ is necessary to ensure that decaying anisotropic homogeneous turbulence finally reaches isotropy. This does not, however, provide any firm indication of the value of C_1 and different workers have used different values ranging from 1.5-3.0. In the present work the more generally accepted value of $C_1 = 1.5$ is adopted.

It is worthwhile to note that Eq.3.6 assumes a linear return to isotropy when, in fact $(\phi_{ij} + \phi_{ji})_1$ is itself non-linear. However, from another viewpoint, Eq.3.6 can be treated as the first term in

a series involving other non-linear terms. In Lumley & Nouri (1974), Lumley (1978) and Chung (1978) extended versions of the $(\phi_{ij} + \phi_{ji})_1$ approximation have been proposed, but none has actually absorbed the non-linear terms into their model proper. Introducing non-linear terms inevitably requires optimisation of an additional constant (i.e. more empiricism) and subsequent difficulties in "tuning" it.

The second part of the redistribution process or the "rapid" part involving mean rates of strain is modelled as

$$\begin{aligned} (\phi_{ij} + \phi_{ji})_2 = & - \frac{(C_2 + 8)}{11} \left[P_{ij} - \frac{2}{3} \delta_{ij} P_k \right] \\ & - \frac{(30C_2 - 2)}{55} k \left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right] \\ & - \frac{(8C_2 - 2)}{11} \left[D_{ij} - \frac{2}{3} \delta_{ij} P_k \right] \end{aligned}$$

Eq.3.7

where P_{ij} is the generation term in Eq.3.1, that is, $P_{ij} = -[\overline{u_i u_k} (\partial U_j / \partial x_k) + \overline{u_j u_k} (\partial U_i / \partial x_k)]$, and

$$D_{ij} = - \left[\overline{u_i u_k} \frac{\partial U_k}{\partial x_j} + \overline{u_j u_k} \frac{\partial U_k}{\partial x_i} \right]$$

The above form was derived by Launder et al. (1975) as well as Naot et al. (1970) (but based on different physical arguments) by requiring the r.h.s. of Eq.3.7 to be of zero trace and also exhibits the symmetry of the integral in Eq.3.4.

For isotropic turbulence, with $(\phi_{ij} + \phi_{ji})_1 = 0$, it can be shown that Eq.3.7 reduces to

$$(\phi_{ij} + \phi_{ji})_2 = 0.4 k \left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right] \quad \text{Eq.3.8}$$

regardless of the value of C_2 . Therefore agreeing with the result obtained by Crow (1968). This, however, does not shed any light on a suitable value for C_2 . Here a value of $C_2=0.4$ is adopted. $C_2=0.4$ is now a standard value and has been arrived at by many workers through simple shear flow calculations.

Although all three terms on the r.h.s of Eq.3.7 have zero trace, the first term is generally larger than the other two. This led to a proposal to retain only the first term.

$$(\phi_{ij} + \phi_{ji})_2 = - C'_2 \left(P_{ij} - \frac{2}{3} \delta_{ij} P_k \right)$$

C'_2 adopts a value different from C_2 and in order to compensate for the neglected terms in Eq.3.7, one would expect C'_2 to have a higher value than C_2 . To satisfy Eq.3.8 for isotropic turbulence, the value of C'_2 is found to be 0.6, however, in Launder et al. (1975) better predictions were obtained with $C'_2=0.55$. The above truncated form, to be referred to as the simple pressure-strain model, was first developed by Naot et al. (1970) as a replacement for the return to isotropy term $(\phi_{ij} + \phi_{ji})_1$.

The initial appeal of the simple pressure-strain model lies in its simplicity and many workers (for example Gibson & Launder, 1976; Gibson & Younis, 1986, just to name a few) have employed it successfully in many inhomogeneous shear flows. One setback is that in homogeneous sheared turbulence it gives equal levels of $\overline{vw}$ and $\overline{ww}$, even though experiments indicate that $\overline{ww}$ is some 20%

higher than $\overline{v v}$. Since the present work has found the more comprehensive pressure-strain model Eq.3.7 unable to respond adequately in some flows with extra strain rates (see Chapter 7), one should therefore not expect the simpler model to be able to perform any better.

3.2.4 Influence Of Wall

Flows along solid surfaces are found in many practical situations, as such its importance cannot be dismissed. The presence of a wall reflects pressure perturbations (or "echoing" effects) and consequently complicates modelling of the pressure redistribution term. The method of process by process analysis of free shear flow for modelling turbulence cannot be used for near-wall effects. Thus, near-wall turbulence is difficult to model and most models simply try to mimic experimental data.

The closure of the $\overline{u_i u_j}$ -equations in the previous sections has yet to take into account the presence of wall effects (if any) denoted by ϕ_w (for ease of representation $(\phi_{ij} + \phi_{ji})_w$ is written as ϕ_w). Bradshaw (1973) has pointed out that the contribution by the surface integral in Eq.3.4 is significant as long as the distance from the wall is of the same order as the energy containing eddies - which is likely to be the case through most of the shear layer thickness. Experimental data also show that in close proximity to a wall, the energy transfer from the streamwise normal stress component to the transverse normal stress component is impeded, resulting in an appreciably larger lateral normal

stress than in homogeneous flow. Also in near-wall flow, the ratio of the shear stress $-\overline{uv}$ to k is lower than in homogeneous shear flow. Therefore, a wall damping model must be capable of reproducing, besides the reduction of the shear stress, the increase in anisotropy as the wall is approached.

In near-wall flows, both convection and diffusion of turbulence quantities are small and this has led to an inference that the wall effects must come from the pressure redistribution term. The form of the modelled wall damping functions has also to be of zero trace in order to be consistent with the fact that the pressure-strain term acts only to distribute energy among the normal stresses but not to alter the total kinetic energy level. In Launder et al. (1975), ϕ_v has been modelled in a form analogous to the "rapid" and "slow" parts found in the pressure-strain term.

$$\phi_{v1} = \left[0.125 \frac{\epsilon}{k} (\overline{u_i u_j} - \frac{2}{3} \delta_{ij} k) + 0.015 (P_{ij} - D_{ij}) \right] \frac{k^{3/2}}{\epsilon y}$$

Eq.3.9

The suffix "1" in ϕ_{v1} is used to distinguish it from a second wall damping model to be discussed below. It must be mentioned that the above form was arrived at by neglecting a term corresponding to $k (\partial U_i / \partial x_j + \partial U_j / \partial x_i)$ from the original formulation. On closer examination, one will find that Eq.3.9 predicates energy transfer from $\overline{uu}$ to $\overline{vv}$ only and does not affect the level of $\overline{ww}$. It has been found in this work, and also by Younis (1984), that the adoption of Eq.3.9 can be inordinately sensitive to the initial near-wall BCs imposed during the calculations.

The second wall correction formulation was first proposed by Shir (1973) as a wall correction for $(\phi_{ij} + \phi_{ji})_1$. Later, Gibson & Launder (1978) extended it to simulate the "echoing" effects of $(\phi_{ij} + \phi_{ji})_2$.

$$\phi_{v2} = (\phi_{1,v} + \phi_{2,v}) F\left[\frac{l_s}{y}\right] \quad \text{Eq.3.10}$$

F is a function of the ratio between a characteristic eddy length scale and the distance from the wall; $\phi_{1,v}$ and $\phi_{2,v}$ are the wall corrections for $(\phi_{ij} + \phi_{ji})_1$ and $(\phi_{ij} + \phi_{ji})_2$ respectively.

$$\phi_{1,v} = C_{1v} \frac{\varepsilon}{k} \left[\delta_{ij} \overline{u_k u_m} n_k n_m - \frac{2}{3} \overline{u_k u_i} n_k n_j - \frac{2}{3} \overline{u_k u_j} n_k n_i \right]$$

$$\phi_{2,v} = C_{2v} \left[\delta_{ij} \phi_{kmz} n_k n_m - \frac{2}{3} \phi_{ikz} n_k n_j - \frac{2}{3} \phi_{jkz} n_k n_i \right]$$

n_i is a unit vector normal to a surface, C_{1v} and C_{2v} are empirical constants taking the values 0.3 and 0.1 respectively. Although the above approximation is generally applicable to situations with more than one wall, it is most conveniently used in flows with only one wall.

Despite all the uncertainties in modelling wall effects, one can be certain that ϕ_v must be a function of (l_s/y) . Very often l_s is taken as $(k^{3/2}/\varepsilon)$ and since it is difficult to determine ε with any degree of certainty, doubts are again cast at the wall formulations. It is required that the decay function $F(l_s/y)$ acts to diminish the influence of ϕ_v as the distance from the wall increases and it is chosen such that a value of unity is achieved at close proximity to a wall. With

$$F\left(\frac{1}{y}\right) \propto \frac{k^{3/2}}{\varepsilon y}$$

and using $\varepsilon = \frac{u_\tau^3}{\kappa y}$ for near-wall calibration, one obtains

$$F\left(\frac{1}{y}\right) = \frac{a^{3/2}}{\kappa} \frac{k^{3/2}}{\varepsilon y}$$

In this work, two versions of RSTM will be used - differing only in the form of wall damping formula. In both models (convection and generation terms are used in exact form), the approximations for the viscous dissipation is Eq.3.2, turbulent diffusion Eq.3.3 and pressure-strain Eq.3.6 and Eq.3.7. Model 1 employs Eq.3.9 as the near-wall model whereas Model 2 uses Eq.3.10. Although simpler truncated models for turbulent diffusion and pressure redistribution have been discussed, it is the purpose of this work to start with the more complete model.

In a 2-D boundary layer, near-wall algebraic expressions for each of the $\overline{u_i u_j}$ -equations can be derived by assuming convective and diffusive transport to be negligible, production of k is equal to ε (i.e. local equilibrium, $\varepsilon = -\overline{uv} \frac{\partial U}{\partial y}$). Accordingly the following near-wall proportions are obtained.

	Model 1	Model 2
$\frac{\overline{uu}}{k}$	1.1	1.0
$\frac{\overline{vv}}{k}$	0.3	0.3
$\frac{\overline{ww}}{k}$	0.6	0.7
$\frac{\overline{uv}}{k}$	-0.28	-0.3

The Model 1 proportions can be seen to be in closer accord with measured data. Unlike, Model 1, the near-wall algebraic expression for each of the Reynolds-stresses in Model 2 also contains at least one other Reynolds-stress. Hence, the near-wall proportions in Model 2 are more strongly coupled and have to be determined in a rather laborious trial and error manner. The wall function constants in Model 2 (i.e. $C_{1v}=0.3$ & $C_{2v}=0.1$) are values established by the present study which are found to be consistent with the above proportions.

3.3 Modelling The Rate Of Turbulence Energy Dissipation Equation

3.3.1 The Rate Of Turbulence Energy Dissipation Equation

The appearance of ε in the normal stress equations makes ε a natural and convenient (and widely accepted) choice for determining the turbulence length scale or time scale. ε also has a precise physical meaning and can be quantified from the energy balance in the k -equation. ε can be related to the turbulence length scale by $l_s = \frac{k^{3/2}}{\varepsilon}$ and turbulence time scale by $t_s = \frac{k}{\varepsilon}$. Other length scale determining quantities such as kl_s or $k^{1/2}/l_s$ require some form of wall corrections, this may not be convenient in the prediction of a flow which combines wall bounded and free flows, for example the aerofoil boundary layer developing into the wake region.

The ε -equation, first obtained by Davydov (cited in Launder, 1984, p.16) can be derived by differentiating the x_1 -momentum equation

w.r.t. x_j , then multiplying throughout by $2\nu \frac{\partial u_i}{\partial x_j}$ and averaging. It is adequate simply to say that the mathematically complicated ϵ -equation contains terms pertaining to convection (I_ϵ), generation (II_ϵ), diffusion (III_ϵ), vortex stretching (IV_ϵ) and viscous destruction (V_ϵ). The measurement of these terms is at present hampered by available experimental techniques. Therefore, any approximations can only be verified indirectly, for example, ensuring the predicted values of ϵ is consistent with say, the l_s .

When modelling the $\overline{u_i u_j}$ -equations, one is actually attempting to model the large energy containing eddies. Appropriately, the chosen turbulence length scale should also be based on the large scale motions which, strictly speaking, is related to the rate of energy transfer (whose equation is even more complicated) rather than the dissipation rate. In fact, the terms in the fine scale ϵ -equation have usually been modelled according to the large scale motions (and hence independent of viscosity). Pope (1977) pointed out that the large scale motions, whose time scale is much smaller than the cascading time scale, should not have an immediate effect on the viscous dissipation. Also, the ϵ -equation dependence on local rather than volumetric mean flow quantities cannot be accurate.

The use of a single ϵ -equation implicitly assumes that the rate of energy transfer from the large eddies is equal to the rate of energy dissipated by viscosity at fine scales. This assumption is physically plausible but is correct only when the energy spectrum is in equilibrium. Perhaps, one should recognize the

overstretched role in using a single ϵ -equation. Hanjalić et al. (1979), amongst others, suggested a multi-time-scale model which takes into account the different ϵ associated with different range of wave numbers or eddy sizes. The model, using two interrelated time scales together with an additional mean vorticity term to enhance the effects of irrotational strain rates, showed improvements over the single-scale model in calculating the axisymmetric turbulent jet and 2-D boundary layer in adverse pressure gradient.

3.3.2 Vortex Stretching & Viscous Destruction Of ϵ

The vortex stretching process by the turbulence is responsible for energy cascading or energy transfer from large to small eddies. In the dimensional analysis of Launder (1984, p.61-63), the vortex stretching and viscous destruction, IV_ϵ and V_ϵ , terms were found to be the most dominant terms in the ϵ -equation. Rodi (1971) analysed the energy spectrum equation for homogeneous isotropic turbulence with the experimental data of Uberoi (1963) and concluded that the vortex stretching spectrum roughly balances the viscous destruction spectrum. Hence, many workers agree that both these terms are meaningfully modelled only when both terms are considered together.

In high Re flows and the existence of an inertial subrange, Davydov proposed the approximation

$$IV_\epsilon + V_\epsilon = -C_{\epsilon 2} \frac{\epsilon^2}{k} \quad \text{Eq.3.11}$$

where $C_{\epsilon 2}$ is an empirical constant to be determined. However, most practical flows do not have a sufficiently high Re to ensure that an inertial subrange exists. But since the other energy spectra do not considerably overlap the dissipation spectrum, Eq.3.11 is still fairly plausible.

To determine a suitable value for $C_{\epsilon 2}$, one turns to homogeneous grid turbulence where k can be assumed to vary inversely proportional to distance downstream of the grid in a power-law manner.

$$k = C x^{-n}$$

where C is the constant of proportionality and n is an empirical constant. One can then derive an equation for $C_{\epsilon 2}$ (see Appendix A1).

$$C_{\epsilon 2} = 1 + \frac{1}{n}$$

In the experimental data of Comte-Bellot & Corrsin (1966), n varies from 1.2 immediately downstream of the grid to 2.5 in the final period of decay (at very low Re) much further downstream. Using a value of $n=1.2$, Hanjalic & Launder (1976) chose $C_{\epsilon 2}=1.8$ (although the exact value should be 1.83). In an earlier work, Hanjalic & Launder (1972b) set $C_{\epsilon 2}=2.0$ when they deduced $n=1.0$ from the experimental data of Batchelor & Townsend (1948): subsequently, Launder et al. (1975) obtained $C_{\epsilon 2}=1.9$ using $n=1.1$. It can be seen that the range of $n=1.0-2.5$ implies a legitimate range of values for $C_{\epsilon 2}=1.4-2.0$ (the commonly used range is $C_{\epsilon 2}=1.8-2.0$). In the present computations $C_{\epsilon 2}$ is set to 1.8, a value similar to Hanjalic & Launder (1976) and Gibson & Younis

(1986).

3.3.3 Production Of ε

The creation or generation of ε is brought about by interactions between the mean and fluctuating strain rates. This is a source term in the ε -equation and has been modelled since Davydov as

$$II_{\varepsilon} = - C_{\varepsilon 1} \frac{\varepsilon}{k} \overline{u_i u_k} \frac{\partial u_i}{\partial x_k} \quad \text{Eq.3.12}$$

A similar form is also often adopted in the k - ε model but with the $\overline{u_i u_j}$ tensor replaced by the Boussinesq expression Eq.2.11. The role of the production term is very important in determining the correct level of ε . In fact, we (as well as other workers) have found it necessary to adjust the constant $C_{\varepsilon 1}$ for adverse pressure gradient flows. Perhaps this indicates that II_{ε} may not be adequately represented by the production of k .

3.3.4 Diffusion Of ε

The diffusion transport of ε is contributed by the fluctuating velocity and pressure. This term, though generally small except in separating flows, is not insignificant. According to Launder et al. (1975), the diffusion transport by pressure fluctuations is neglected, and only the velocity fluctuation part is approximated.

$$III_{\varepsilon} = C_{\varepsilon} \frac{\partial}{\partial x_k} \left(\frac{k}{\varepsilon} \overline{u_k u_l} \frac{\partial \varepsilon}{\partial x_l} \right) \quad \text{Eq.3.13}$$

This approximation, unlike the form used in a two-equation k - ε

turbulence model, is modelled in line with the turbulent diffusion term (see Eq.3.3) in the $\overline{u_i u_j}$ -equation.

In order to establish suitable values for the constants C_ϵ and $C_{\epsilon 1}$, one has to consider the near-wall flow of a 2-D boundary layer. Assuming the convection term can be neglected, that the law of the wall is valid and that the flow is in local equilibrium, the following expression can be obtained (see Eq.A1.8 in Appendix A1).

$$C_{\epsilon 1} = C_{\epsilon 2} - \kappa^2 \frac{b}{a^2} C_\epsilon \quad \text{Eq.3.14}$$

where $b = \overline{vv}/u_\tau^2$. It is obvious from Eq.3.14 that either C_ϵ or $C_{\epsilon 1}$ has to be known (or preset) before the other value can be determined. The value of C_ϵ is often obtained through "fine tuning" (or computer optimisation) during the actual computations and $C_\epsilon=0.15$ has now become a "standard" value.

Hanjalic & Launder (1972b) used values of $\kappa=0.42$, $a=0.285$ and $b=1.6$ in Eq.3.14 to arrive at $C_{\epsilon 1} = C_{\epsilon 2} - 3.45C_\epsilon$. The value of $b=1.6$ is far too large. The more appropriate values of $\kappa=0.41$, $a=0.28$ and $b=1.07$, which are consistent with the values of Model 1 in §3.2.4, would yield

$$C_{\epsilon 1} = C_{\epsilon 2} - 2.294C_\epsilon$$

In this work, using the empirical constants $C_\epsilon=0.15$ and $C_{\epsilon 2}=1.8$, the above relation gives $C_{\epsilon 1}=1.45$ (or more precisely $C_{\epsilon 1}=1.455$).

3.3.5 A Length - Scale - Limiting Model

The form of the ϵ -equation discussed so far, in general, performs quite satisfactorily in simple shear flows. However, it is widely acknowledged that in adverse pressure gradient flows the predicted level of near-wall ϵ is lower than desired due to extra strain rates, $\frac{\partial U}{\partial x}$ and/or $\frac{\partial V}{\partial y}$. In other words, the length scale is overpredicted. Several remedies have been recommended (see §1.2.2) but none has so far achieved universality outside the range of adverse pressure gradient flows. One might have to accept zonal turbulence modelling as the inevitable, whether it is the changing of empirical constants or the inclusion of new terms, when one stops short of solving the time-dependent N-S equations.

Since the near-wall flow is not greatly influenced by adverse pressure, the log-law is still valid although the shear stress values near the wall are greater than u_τ^2 . Under local equilibrium conditions, the production of k is $\left[-\overline{uv} \frac{u_\tau}{\kappa y} \right]$ and taking the velocity scale V_g as $|\overline{uv}|$, one obtains

$$l_{sn} = \frac{|\overline{uv}|^{1/2}}{u_\tau} \kappa y \quad \text{Eq.3.15}$$

where l_{sn} is the near-wall turbulence length scale for a non-constant shear stress profile. Clearly, this turbulence length scale contradicts the widely accepted fact that the near-wall length scale remains equal to κy and is largely unaffected by the presence of a pressure gradient. Although Galbraith et al. (1977) also pointed out that Eq.3.15 is more appropriate, most workers are content with using $l_g = \kappa y$ because it has proven successful on many occasions.

From the modelled ε -equation, one can also derive an algebraic expression for the near-wall length scale (see Appendix A1). For constant shear stress profile (i.e. Eq.A1.9),

$$l_s^2 = \frac{C_{\varepsilon 2} - C_{\varepsilon 1}}{C_{\varepsilon}} \frac{a^2}{b} y^2 \quad \text{Eq.3.16}$$

Eq.3.16 can be written in a form analogous to $l_s = \kappa y$ by requiring

$$\kappa = \left[\frac{C_{\varepsilon 2} - C_{\varepsilon 1}}{C_{\varepsilon}} \frac{a^2}{b} \right]^{\frac{1}{2}} \quad \text{Eq.3.17}$$

In the present work, the set of constants $C_{\varepsilon}=0.15$, $C_{\varepsilon 1}=1.45$, $C_{\varepsilon 2}=1.8$, $a=0.28$ and $b=1.07$ when substituted into Eq.3.17 yields $\kappa=0.413$, this clearly demonstrates the consistency of the chosen values. The near-wall length scale for a non-constant shear stress profile (i.e. Eq.A1.11) is

$$l_{sn}^2 = \frac{C_{\varepsilon 2} - C_{\varepsilon 1}}{C_{\varepsilon}} \frac{a^2}{b} \left[\frac{\overline{uv}^2}{u_{\tau}^4} \right] y^2$$

This equation can be re-cast, using Eq.3.17, as

$$l_{sn} = \frac{|\overline{uv}|}{u_{\tau}^2} \kappa y \quad \text{Eq.3.18}$$

In Eq.3.18, the contribution to the increase in length scale, which is proportional to $|\overline{uv}|/u_{\tau}^2$ can be split into two parts. The first part of $|\overline{uv}|^{1/2}/u_{\tau}$ has already been explained when deriving Eq.3.15, the second part of $|\overline{uv}|^{1/2}/u_{\tau}$ is due to the ε -equation model itself. It is the latter part contribution which accounts for the overprediction of the length scale by the

ε -equation even if the first part contribution, according to Eq.3.15, is correct.

When one accepts $l_s = \kappa y$, then in Eq.3.18, the length scale has to be reduced by $u_\tau^2/|\overline{uv}|$. This can be accomplished through the set of constants that makes up κ in Eq.3.17. The values of a and b can be assumed to remain unchanged, and to further simplify the formulation, C_ε can also be set as a constant. Thus the difference of $(C_{\varepsilon 2} - C_{\varepsilon 1}) = 0.35$ can be made a function of $u_\tau^4/|\overline{uv}|^2$.

$$C_{\varepsilon 1} = C_{\varepsilon 2} - 0.35 \frac{u_\tau^2}{|\overline{uv}|^2} \quad \text{Eq.3.19}$$

From Eq.3.19, the ε production model empirical constant $C_{\varepsilon 1}$ has become a variable. It must be mentioned that in the PSTAG algorithm, when $|\overline{uv}|$ falls below u_τ^2 no $C_{\varepsilon 1}$ correction is made, that is,

$$\begin{aligned} |\overline{uv}| \leq u_\tau^2 & \quad C_{\varepsilon 1} = 1.45 \\ |\overline{uv}| > u_\tau^2 & \quad 1.45 < C_{\varepsilon 1} < 1.8 \end{aligned}$$

Note that the arrangement of Eq.3.19 always ensures $C_{\varepsilon 1} < C_{\varepsilon 2}$ or more importantly the variation of $C_{\varepsilon 1}$ will not result in a negative length scale.

The above length-scale-limiting model, based on plausible argument, is both simple and cheap to implement when compared with the model of Hanjalic and Launder (1980) and Pope (1978). The latter model, developed for axisymmetric jets, consists of an additional term having the form

$$\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \left(\frac{\partial U_j}{\partial x_k} - \frac{\partial U_k}{\partial x_j} \right) \left(\frac{\partial U_k}{\partial x_i} - \frac{\partial U_i}{\partial x_k} \right)$$

The above term, being zero in 2-D is not applicable to any 2-D flows. In the prediction of 2-D adverse pressure gradient boundary layer, the results from the present length-scale-limiting model (see §7.2) are comparable to those obtained by Hanjalic' and Launder (1980).

4 NUMERICAL SCHEME

4.1 Grid Arrangement

4.1.1 Pressure Staggered Grid

In this numerical scheme a 3-D cartesian grid structure is used. All three velocity components are stored at the nodes, the pressures and turbulence quantities are staggered both in the crossflow plane and also in the streamwise direction as shown in Fig.4.1. Therefore each pressure value is situated at the centre of a control volume (CV) whose eight corners are the velocity nodes. This scheme is thus given the name PSTAG, in short for pressure staggered grid, and from here on it will be referred to by this acronym. It should be pointed out that the variable storage method in PSTAG is similar to that of Hirt et al. (1974) but this is where the similarity ends. In the latter numerical scheme, which is more suited for variable density flows, the pressure is obtained from the density and internal energy via an equation of state. This formulation does not involve a Poisson pressure equation, and is therefore not prone to a decoupled pressure equation; in other words, the variable storage arrangement is more of convenience than necessity.

In fact, PSTAG is quite similar to the collocated grid system of Peric (1985), except for the staggered locations for pressures and turbulence quantities. As such, PSTAG may be thought of as a

hybrid scheme and it will be seen in later sections that the hybrid PSTAG does in fact enjoy the advantages associated with both the collocated and staggered grid arrangements while at the same time eliminating some of their undesirable features. A comparison between PSTAG, collocated and staggered grids is given in §4.6.

In the cartesian grid shown in Fig.4.1, the node coordinates on the y-z plane are specified by (I,J) with $I=1,2,\dots,IMAX+1$ along the y-axis and $J=1,2,\dots,JMAX+1$ along the z-axis. The distances between adjacent nodes in the y-direction are defined as

$$\Delta Y_I = Y_{I+1} - Y_I$$

and the ΔY 's are related through a geometric progression,

$$\Delta Y_{I+1} = k \Delta Y_I$$

where $k = 1$ for a uniform grid and $k > 1$ for a non-uniform grid. With $k > 1$, the horizontal grid lines can be closely clustered at points near the wall (that is, near to $I = 1$) and made more sparse further away from it. This is a desirable feature, since flow quantities often change rapidly near the wall and a fine mesh may be necessary to capture the flow field more realistically. The above method of positioning and spacing grid lines in the y-direction can also be applied to grid lines in the spanwise direction. This type of grid expansion is especially useful when calculations are to be performed for flows along streamwise corners because it enables nodes to be clustered near both walls.

To advance the solution in the x-direction, the "known" x-station is denoted as x^n and calculations are marched to the x^{n+1} station (this corresponds to the usual notation in which a superscript denotes a "time-like" direction). This superscript notation is also adopted for the primitive variables so the velocity components are U^n, V^n, W^n at x^n , and $U^{n+1}, V^{n+1}, W^{n+1}$ at x^{n+1} , and the midpoint pressure values at $x^{n+1/2}$ ($= \frac{1}{2} [x^n + x^{n+1}]$) are denoted by $P^{n+1/2}$ (likewise for the turbulence quantities). Subscripts are used to denote spatial coordinates in the crossflow plane as usual: for example, $P_{I+1/2, J+1/2}^{n+1/2}$ denotes the pressure at $(I+\frac{1}{2}, J+\frac{1}{2})$ on the crossflow plane at the $x^{n+1/2}$ station.

4.1.2 Collocated Grid

In a "true" collocated grid, which is the more natural way to store the variables, all the three velocity components and the pressures are stored at the nodes as shown in Fig.4.2. It is well documented that the decoupling between adjacent pressures in the discretized pressure equation causes the solution to be unstable and may yield unrealistic results. However, this decoupling in the pressure equation can be eliminated by staggering the velocities during the formulation of the pressure equation (see Shih & Ren, 1984; Perić, 1985). Hence, the term "true" collocated grid is reserved for grids which do not employ this special treatment to overcome pressure decoupling.

4.1.3 Staggered Grid

The idea behind the staggered grid arrangement was to overcome checker-boarding of the pressure equation in the "true" collocated grid. Indeed, many available solvers for the N-S equations (or approximated forms of them) employ a staggered grid system, where the secondary flow velocities and the pressures are evaluated at points staggered with respect to the nominal node as shown in Fig.4.3. It is inappropriate here to go into any great depth on the methods in discretization of the governing equations of motion based on this grid structure. However, a close scrutiny into the coupled y-mom and z-mom (yz-mom) computational stencil will be of great interest (see §4.6.1) as it will then reveal how the flow is depicted in a staggered grid structure.

4.2 Calculation Of Velocity Field

4.2.1 Streamwise Discretization Of Momentum Equations

The set of normalised conservative parabolic Reynolds equations with slender flow approximations, Eq.2.10, can be written in the following compact matrix form,

$$\frac{\partial}{\partial x} (U \vec{Q}) = -\vec{P} - \frac{\partial}{\partial y} (V \vec{Q}) - \frac{\partial}{\partial z} (W \vec{Q}) + \frac{1}{Re} \nabla_{CR}^2 \vec{Q}$$

Eq.4.1

The $\vec{}$ overbar denotes a vector and the velocity vectors $\vec{Q}$ and source terms $\vec{P}$ are given as

$$\vec{Q} = \begin{bmatrix} U \\ V \\ W \end{bmatrix} ; \quad \vec{P} = \begin{bmatrix} \frac{dP}{dx} + \frac{\partial \overline{uv}}{\partial y} + \frac{\partial \overline{uw}}{\partial z} \\ \frac{\partial P}{\partial y} + \frac{\partial \overline{vv}}{\partial y} + \frac{\partial \overline{vw}}{\partial z} \\ \frac{\partial P}{\partial z} + \frac{\partial \overline{vw}}{\partial y} + \frac{\partial \overline{ww}}{\partial z} \end{bmatrix}$$

and the Laplace operator in the crossflow plane is denoted by

$$\nabla_{CR}^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

For laminar flows there are no Reynolds-stresses, so $\vec{P}$ contains only the pressure derivatives. It must be said that although the set of equations Eq.4.1 are in conservative form, it is not a prerequisite for a successful numerical scheme. In some flow computations, non-conservative schemes were found to be more accurate than conservative schemes (see Roache, 1982, pp.32-33). Conservative schemes are generally preferred especially in internal flows where conservation of, say, mass must be maintained.

Discretizing Eq.4.1 in the streamwise direction using the second-order accurate Crank-Nicolson method at crossflow plane location (I,J) between the x^n and x^{n+1} stations, one obtains

$$\begin{aligned} \frac{(U\vec{Q})^{n+1} - (U\vec{Q})^n}{\Delta x} = & -\vec{P}^{n+\frac{1}{2}} - \frac{1}{2} \left[\frac{\partial}{\partial y}(V\vec{Q})^{n+1} + \frac{\partial}{\partial y}(V\vec{Q})^n \right] \\ & - \frac{1}{2} \left[\frac{\partial}{\partial z}(W\vec{Q})^{n+1} + \frac{\partial}{\partial z}(W\vec{Q})^n \right] \\ & + \frac{1}{2} \frac{1}{Re} \left[\nabla_{CR}^2 \vec{Q}^{n+1} + \nabla_{CR}^2 \vec{Q}^n \right] + O(\Delta x^2) \end{aligned}$$

Eq.4.2

From the above equation, it becomes obvious that location of $\vec{P}$ at $x^{n+1/2}$ conveniently eliminates any approximation of $\vec{P}$ at x^n and x^{n+1} . On the other hand, when a fully-implicit scheme is used $\vec{P}$ should be located at x^{n+1} . This implies that the positioning of $\vec{P}$ is flexible but depends upon the type of streamwise differencing scheme used.

The Crank-Nicolson difference scheme is implicit and hence unconditionally stable, implying that the errors decay as calculations progress. Although implicit schemes allow large arbitrary x-steps to be taken (so long as these are physically plausible), the Crank-Nicolson scheme is more restrictive and too large an x-step may give rise to oscillations (see Smith, 1986, pp.122-124; Patankar & Baliga, 1978).

4.2.2 Linearization Of Momentum Equations

To handle the non-linear convective terms in Eq.4.2, some form of linearization is required. These terms are linearized, using Taylor series, by expanding about the x^n station.

$$(U_i U_j)^{n+1} = (U_i U_j)^n + U_i^n \Delta U_j + U_j^n \Delta U_i + O(\Delta x^2)$$

Eq.4.3

This still leaves the three momentum equations coupled, and Dedoussis (1988) reported that their simultaneous solution is numerically unstable for x-mom. To overcome this problem, the convective terms $(UV)^{n+1}$ and $(UW)^{n+1}$ in x-mom are linearized as

$$(UV)^{n+1} = (UV)^n + V^n \Delta U + O(\Delta x)$$

$$(UW)^{n+1} = (UW)^n + W^n \Delta U + O(\Delta x)$$

This essentially decouples x-mom from the crossflow momentum equations. Such linearizations are adequately accurate for slender flow, in which streamwise gradients are - by definition - small compared to gradients in the crossflow plane.

In fact, the form of linearization of $(UV)^{n+1}$ and $(UW)^{n+1}$ in x-mom is exactly the same as linearization using only known upstream values of V^n and W^n . If the convective terms in y-mom and z-mom are linearized in a similar way (i.e. $(U_i U_j)^{n+1} = U_i^n U_j^{n+1}$) then all three momentum equations will be decoupled. In view of the crossflow ellipticity, a strong coupling of y-mom and z-mom, achieved by using Eq.4.3, is more favourable. Cousteix (1986) found that simultaneous solution in general converges faster than sequential solution.

Applying the appropriate linearizations to the corresponding convective terms of the momentum equations, Eq.4.2 becomes

$$\begin{aligned} \frac{1}{\Delta x} \tilde{A} \Delta \tilde{Q} = & - \tilde{P}^{n+\frac{1}{2}} - \frac{1}{2} \left[- \frac{\partial}{\partial y} \tilde{B} - \frac{\partial}{\partial z} \tilde{C} + \frac{1}{Re} \nabla_{CR}^2 \right] \Delta \tilde{Q} \\ & + \left\{ \left[- \frac{\partial}{\partial y} V^n - \frac{\partial}{\partial z} W^n + \frac{1}{Re} \nabla_{CR}^2 \right] \tilde{Q}^n \right\} + O(\Delta x) \end{aligned}$$

Eq.4.4

The $\sim$ overbar denotes matrices.

$$\tilde{A} = \begin{bmatrix} 2U^n & 0 & 0 \\ V^n & U^n & 0 \\ W^n & 0 & U^n \end{bmatrix} ; \quad \tilde{B} = \begin{bmatrix} V^n & 0 & 0 \\ 0 & 2V^n & 0 \\ 0 & W^n & V^n \end{bmatrix} ; \quad \tilde{C} = \begin{bmatrix} W^n & 0 & 0 \\ 0 & W^n & V^n \\ 0 & 0 & 2W^n \end{bmatrix}$$

The velocity increment vector $\Delta \vec{Q}$ is the change from x^n to x^{n+1} .

$$\Delta \vec{Q} = \vec{Q}^{n+1} - \vec{Q}^n$$

The appearance of $\Delta \vec{Q}$ is a consequence of the Taylor series linearization technique. The solution of Eq.4.4, therefore, yields $\Delta \vec{Q}$ instead of $\vec{Q}^{n+1}$ and as pointed out by Briley and McDonald (1977) it has better error round-off characteristics.

In Eq.4.4, the terms within the curly brackets { } are also source terms, in addition to $\vec{P}^{n+1/2}$ because only known values at x^n are involved. This enables them to be lumped into a single source term denoted by $\vec{S}$.

$$\vec{S} = - \vec{P}^{n+1/2} + \left\{ \left(\frac{\partial}{\partial y} V^n - \frac{\partial}{\partial z} W^n + \frac{1}{Re} \nabla_{CR}^2 \right) \vec{Q}^n \right\}$$

Although the Crank-Nicolson x-wise discretization method is $O(\Delta x^2)$ accurate, the overall accuracy of Eq.4.2 has been reduced by one order to $O(\Delta x)$ in Eq.4.4. This is due to the form of linearizations (Eq.4.3) employed which when substituted into the numerator on the l.h.s. of Eq.4.2 has a denominator containing Δx .

In the crossflow plane all terms are discretized using CDS. For a first-order derivative of a quantity ϕ centred at (I,J), where ϕ can represent any of the primitive velocity variables or their product, for example, $\phi \equiv U$ or $\phi \equiv U_i U_j$, the differenced formula

is

$$\frac{\partial \phi}{\partial Y_{I,J}} = \frac{\phi_{I+1} - \phi_{I-1}}{\Delta Y_I + \Delta Y_{I-1}} + O(\Delta Y_I - \Delta Y_{I-1}) + O(\Delta Y^2)$$

This is $O(\Delta Y)$ accurate for a non-uniform grid and $O(\Delta Y^2)$ accurate for a uniform grid. A similar result is obtained for first-order spanwise derivatives.

For a second-order derivative centred at (I,J) , one obtains

$$\begin{aligned} \frac{\partial^2 \phi}{\partial Y^2_{I,J}} &= \frac{2}{\Delta Y_I + \Delta Y_{I-1}} \left[\frac{\phi_{I+1} - \phi_I}{\Delta Y_I} - \frac{\phi_I - \phi_{I-1}}{\Delta Y_{I-1}} \right] \\ &+ O(\Delta Y_I - \Delta Y_{I-1}) + O(\Delta Y^2) \end{aligned}$$

The order of accuracy is $O(\Delta x)$ for a non-uniform grid and $O(\Delta x^2)$ for a uniform grid which is the same as that for a first-order derivative. This form of discretization is also used for the second-order spanwise derivatives with similar order of accuracy.

The source terms in $\tilde{p}^{n+1/2}$ are first-order derivatives and when centred at (I,J) the y -derivative differenced formula is

$$\begin{aligned} \frac{\partial \phi}{\partial Y_{I,J}} &= \frac{\left[\phi_{I+\frac{1}{2}, J+\frac{1}{2}} + \phi_{I+\frac{1}{2}, J-\frac{1}{2}} \right] - \left[\phi_{I-\frac{1}{2}, J+\frac{1}{2}} + \phi_{I-\frac{1}{2}, J-\frac{1}{2}} \right]}{\Delta Y_I + \Delta Y_{I-1}} \\ &+ O(\Delta Y_I - \Delta Y_{I-1}) + O(\Delta Y^2) + O(\Delta Z)^2 \end{aligned}$$

Note that the superscript $n+1/2$ for ϕ has been omitted for clarity.

Employing the appropriate differencing formula established above for the first- and second- order derivatives in Eq.4.4, the final discretized form of the set of momentum equations is

$$\begin{aligned} \tilde{L} \frac{1}{\Delta x} \Delta \vec{Q} &= \tilde{D} \Delta \vec{Q} + \tilde{S}^{n+\frac{1}{2}} + O(\Delta x) + O(\Delta Y_{I+1} - \Delta Y_I) \\ &+ O(\Delta Y^2) + O(\Delta Z_{J+1} - \Delta Z_J) + O(\Delta Z^2) \end{aligned}$$

Eq.4.5

where $\tilde{L}$ and $\tilde{D}$ are discrete multi-dimensional vector operators in the streamwise direction and in the y-z plane respectively. The computational stencil of $\tilde{L}$ and $\tilde{D}$ (see Appendix A2) are five-point clusters: as such it becomes possible to adopt the fast and efficient alternating-direction-implicit (ADI) solution technique.

4.2.3 Solution Of Streamwise Momentum Equation

In the set of discretized slender flow equations, Eq.4.5, one can observe that x-mom decouples from the secondary flow momentum equations (which are themselves coupled) because the only secondary flow velocities involved are those at the upstream station. Thus x-mom can be solved separately first and since it is streamwise parabolic, under most circumstances no iteration is needed for each streamwise step, keeping in mind that $\frac{dP}{dx}$ must be set a priori. A non-iterative x-mom solution is permissible provided the source terms are not strongly linked to the solution of U. In laminar flows a non-iterative x-mom solution can be amply justified because the source term contains only the known $\frac{dP}{dx}$ term.

In turbulent flows, the mean velocities interact with the Reynolds-stresses and in order to ensure a strong coupling between them PSTAG uses an iterative x-mom solution. This will also minimize errors introduced by linearizations. On the other hand, a non-iterative x-mom solution is not grossly in error in view of the flow slenderness or small streamwise changes. In the latter option, the momentum equations are evaluated using Reynolds-stress gradients on a crossflow plane that lags by one streamwise step (see Dedoussis, 1988).

The x-mom component in Eq.4.5 can be written as

$$\tilde{L}_x \frac{1}{\Delta x} [\Delta U]_{I,J} = \tilde{D}_x [\Delta U]_{I,J} + \tilde{S}_x^{n+\frac{1}{2}} \quad \text{Eq.4.6}$$

where $\tilde{L}_x$, $\tilde{D}_x$ and $\tilde{S}_x^{n+\frac{1}{2}}$ represent the x-mom components of $\tilde{L}$, $\tilde{D}$ and $\tilde{S}^{n+\frac{1}{2}}$ respectively. Subscripts are used on the streamwise velocity increment ΔU (enclosed in square brackets) to denote the location wherein x-mom is being centred; this is also the centre of the computational stencil.

With x-mom discretized at (I,J), the computational stencil is centred at (I,J) and the four surrounding nodes are (I+1,J), (I-1,J), (I,J+1) and (I,J-1). This five-point cluster can be decomposed into two three-point clusters of (I+1,J), (I,J), (I-1,J) along the vertical grid lines and (I,J+1), (I,J), (I,J-1) along the horizontal grid lines. Advantage can be taken of this ability to distinguish computational clusters along axis directions in order to utilise the ADI solution method (see

Ferziger, 1981, pp.158-167; Peyret & Taylor, 1983, pp.66-68) because the three-point clusters form simple tridiagonal matrices which are easy to invert. The operator $\tilde{D}_x$ can be written as

$$\tilde{D}_x = \tilde{D}_{x,y} + \tilde{D}_{x,z}$$

to denote the use of the ADI solution technique, $\tilde{D}_{x,y}$ and $\tilde{D}_{x,z}$ being discrete operators along vertical lines and horizontal grid lines respectively. The solution procedure of Eq.4.6 using the ADI technique is carried out in two sweeps. The first sweep is the y-sweep, using $\tilde{D}_{x,y}$ to perform calculations along all the J=constant lines across the entire computational domain: once completed, the z-sweep operates using $\tilde{D}_{x,z}$, also throughout the whole computational domain along the I=constant grid lines. This procedure can be summed up as follows,

$$\left[\tilde{L}_x \frac{1}{\Delta x} - \tilde{D}_{x,y} \right] [\Delta U]_{I,J}^* = \dot{S}_x^{n+\frac{1}{2}}$$

$$\left[\tilde{L}_x \frac{1}{\Delta x} - \tilde{D}_{x,z} \right] [\Delta U]_{I,J} = \tilde{L}_x \frac{1}{\Delta x} [\Delta U]_{I,J}^*$$

where $[\Delta U]^*$ is the intermediate solution. The ADI technique may be thought of either as spatial stepping giving the solution $[\Delta U]^*$ at $x^{n+1/2}$ or as an iterative predictor-corrector method.

4.2.4 Solution Of Cross-Stream Momentum Equations

Calculation of the secondary velocities has to be performed iteratively through two loops. Within the outer loop (referred to

as the outer pressure loop), yz-mom is solved only once (i.e the solution of yz-mom is itself non-iterative) for each estimated pressure field and the computed values of the secondary flow are then used in the inner iterative loop for pressure equation calculation. Thus, at least in principle, yz-mom can be treated as streamwise-parabolic which would permit employing the efficient ADI method. The outer pressure loop establishes the pressure-velocity linkage to satisfy the continuity equation.

The coupled crossflow momentum equations of Eq.4.5 can be written as

$$\tilde{L}_{CR} \frac{1}{\Delta X} \begin{bmatrix} \Delta V \\ \Delta W \end{bmatrix}_{I,J} = \tilde{D}_{CR} \begin{bmatrix} \Delta V \\ \Delta W \end{bmatrix}_{I,J} + \tilde{S}_{CR}^{n+\frac{1}{2}} \quad \text{Eq.4.7}$$

where $\tilde{L}_{CR}$ and $\tilde{D}_{CR}$ denote the y-z plane components of $\tilde{L}$ and $\tilde{D}$ respectively. The source term, $\tilde{S}_{CR}^{n+\frac{1}{2}}$, is the sum of the crossflow component of $\tilde{S}^{n+\frac{1}{2}}$ and terms involving ΔU which have become "known" source terms. The velocity increments in square brackets are subscripted to denote the location where the crossflow momentum equations are centred when discretized.

A similar type of five-point memory cluster to that used in x-mom exists for the multi-dimensional operators in the discretized crossflow momentum equations, except that now the nodes contain two velocities instead of one. The computational stencil centred at (I,J) with neighbouring nodes (I+1,J), (I-1,J), (I,J+1) and (I,J-1) can be split into two three-point clusters of (I+1,J), (I,J), (I-1,J) along J=constant grid lines and (I,J+1), (I,J),

(I,J-1) along I=constant grid lines. These three-point clusters make possible the utilization of the ADI method, which now has to solve block 2x2 tridiagonal matrices instead of a simple tridiagonal matrix. Block tridiagonal matrices are solved in a manner similar to tridiagonal matrices except that additional matrix inversions are required for the 2x2 blocks. The block elimination technique of Isaacson & Keller (1966, pp.58-61) is adopted (see also Cebeci & Bradshaw, 1977, pp.223-231). It is worthwhile to note that this form of solution becomes non-trivial once the block matrices become large.

To use the ADI solver, the operator $\tilde{D}_{CR}$ is written as

$$\tilde{D}_{CR} = \tilde{D}_{CR,y} + \tilde{D}_{CR,z}$$

with $\tilde{D}_{CR,y}$ and $\tilde{D}_{CR,z}$ operating along grid lines in the y- and z-direction respectively. The solution of Eq.4.7 can be performed in two sweeps, the y-sweep preceding the z-sweep.

$$\left[\tilde{L}_{CR} \frac{1}{\Delta X} - \tilde{D}_{CR,y} \right] \begin{bmatrix} \Delta V \\ \Delta W \end{bmatrix}_{I,J}^* = \tilde{S}_{CR}^{n+\frac{1}{2}}$$

$$\left[\tilde{L}_{CR} \frac{1}{\Delta X} - \tilde{D}_{CR,z} \right] \begin{bmatrix} \Delta V \\ \Delta W \end{bmatrix}_{I,J} = \left[\tilde{L}_{CR} \frac{1}{\Delta X} \right] \begin{bmatrix} \Delta V \\ \Delta W \end{bmatrix}_{I,J}^*$$

where the $*$ ("starred") velocity increments are the intermediate solution.

It should now become obvious that PSTAG requires only two sweeps, in the y- and z- direction, in both the solution of x-mom and

yz-mom. It is in the calculation of yz-mom that PSTAG differs from the staggered grid formulation which is shown in Dedoussis (1988) to require an additional as well as undesirable diagonal sweep through the computational domain. The diagonal sweep is undesirable from the point of view of specifying BCs (see §4.6.2) and the increase in computational time of typically about 50% (see §6.3.3 & §6.4.3).

4.2.5 Velocity Boundary Conditions

With reference to the 3-D computational domain shown in Fig.4.4, velocity BCs are needed on five out of the six boundary planes. Due to the x-wise parabolic character of the set of governing equations, the exit plane 1'-2'-3'-4' does not need any BCs. The velocity BCs at the inlet plane 1-2-3-4 must be specified before the start of any calculations. For laminar flows it is often possible to use an established theoretical solution to generate the initial profiles. As for turbulent flows initial BCs, experimental data are used. Whenever experimental data are used as initial BCs, it is not unusual to encounter a "settling-down" period before momentum and mass continuity are satisfied.

At each crossflow plane, BCs are required along boundaries 1-2, 2-3, 3-4 and 1-4, the present scheme assumes these BCs to remain unchanged as one proceeds downstream. Implementation of streamwise dependent BCs in the yz-plane, if neccessary, can be quite easily accomplished. The yz-plane BCs are basically of two types, namely Neumann or Dirichlet. Neumann BCs are those in which the gradient

normal to a boundary is specified whereas Dirichlet BCs are those whose values on the boundary are known.

For a solid wall, the obvious choice is to adopt the Dirichlet no-slip BC.

$$U = V = W = 0$$

For free stream or lateral boundaries, one can decide between the Neumann or Dirichlet BCs. The decision is, of course, determined by the physics of the flow, hence, it is essential to have a basic understanding of the flow to be computed. In some instances, a flow may have its own unique BCs. For the present work, when lateral boundaries are encountered the zero Neumann BC is used, therefore, along 2-3 and 1-4 one has

$$\frac{\partial U}{\partial z} = \frac{\partial V}{\partial z} = \frac{\partial W}{\partial z} = 0$$

and along the free stream boundary 3-4,

$$\frac{\partial U}{\partial y} = \frac{\partial V}{\partial y} = \frac{\partial W}{\partial y} = 0$$

It is essential in 3-D flows to situate the lateral boundaries sufficiently far away such that a 2-D condition exists or at least can be assumed. Otherwise, considerable difficulties may arise due to non-convergence in the solution.

In the case of wall boundary layers at zero pressure gradient, the Dirichlet BC for U along 3-4 can be used directly.

$$U = U_e = 1.0$$

It is worth noting that in PSTAG the boundaries in the yz-plane computational domain for all three momentum equations are, unlike the staggered grid, coincident. This advantage will be discussed in §6.4.3.

4.3 Calculation Of Pressure Field

4.3.1 Formulation Of Pressure Equation

As mentioned earlier, the appearance of the fluid density as a variable in the continuity equation coupled with the fact that density can be related to pressure by an equation of state makes it rather attractive to use density as one of the main working variables. Such an approach is often adopted in compressible flow calculation methods. However, in low Mach number regimes the linkage between density and pressure weakens until eventually, in the limit of incompressibility, it no longer exists. Therefore, it is more meaningful to use pressure as a working variable in constant density flow calculation. Besides, prediction methods that solve for pressure are equally applicable to compressible flows.

Direct discretization of the Poisson pressure equation Eq.2.8 derived from substituting the continuity equation (i.e assuming zero velocity divergence) into the divergence of the N-S equations has three pitfalls: firstly, the initial velocity field may not satisfy the continuity equation; secondly, the terms in the equation may not be discretized in the same manner as those in the

momentum equations; and thirdly, zero velocity divergence may not be satisfied before convergence is achieved. To ensure consistency, the discretized Poisson equation is derived from the discretized form of the momentum and continuity equation (see Issa, 1983). Centring the continuity equation at $x^{n+1/2}$ and $(I+\frac{1}{2}, J+\frac{1}{2})$ of the CV as shown in Fig.4.5 and employing either the Crank-Nicolson or fully-implicit differencing method in the x-direction,

$$\begin{aligned} \frac{\Delta U_{I+\frac{1}{2}, J+\frac{1}{2}}}{\Delta x} = & - \frac{(2 - \psi)}{2} \left[\frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} \right]^{n+1} - \frac{\psi}{2} \left[\frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} \right]^n \\ & + (1-\psi)O(\Delta x) + \psi O(\Delta x^2) \end{aligned} \quad \text{Eq.4.8}$$

where $\psi = 0$ or 1 for fully-implicit or Crank-Nicolson differencing respectively.

Discretizing the crossflow plane derivatives on the r.h.s. of Eq.4.8 by CDS one obtains

$$\begin{aligned} \frac{\Delta U_{I+\frac{1}{2}, J+\frac{1}{2}}}{\Delta x} = & - \frac{(2 - \psi)}{2} \left[\frac{\Delta V_{I+1, J+\frac{1}{2}} - \Delta V_{I, J+\frac{1}{2}}}{\Delta Y_I} \right] \\ & - \frac{(2 - \psi)}{2} \left[\frac{\Delta W_{I+\frac{1}{2}, J+1} - \Delta W_{I+\frac{1}{2}, J}}{\Delta Z_J} \right] + S_P \\ & + (1-\psi)O(\Delta x) + \psi O(\Delta x^2) + O(\Delta Y^2) + O(\Delta Z^2) \end{aligned}$$

Eq.4.9

where S_P denotes the source terms of the continuity equation.

$$S_P = - \left[\frac{V_{I+1, J+\frac{1}{2}}^n - V_{I, J+\frac{1}{2}}^n}{\Delta Y_I} \right] - \left[\frac{W_{I+\frac{1}{2}, J+1}^n - W_{I+\frac{1}{2}, J}^n}{\Delta Z_J} \right]$$

The pressure equation is obtained by substituting the discretized form of the velocities which appear in Eq.4.9 located mid-way on the faces of the CV. The required equation for $\Delta V_{I, J+\frac{1}{2}}$ is obtained by discretizing y-mom at $(I, J+\frac{1}{2})$.

$$\tilde{L}_Y \frac{1}{\Delta X} [\Delta V]_{I, J+\frac{1}{2}} = - \left[\frac{\begin{matrix} n+1/2 \\ P_{I+\frac{1}{2}, J+\frac{1}{2}} - P_{I-\frac{1}{2}, J+\frac{1}{2}} \end{matrix}}{\frac{1}{2} [\Delta Y_I + \Delta Y_{I-1}]} \right] + \tilde{S}_Y^{n+\frac{1}{2}}$$

Eq.4.10

It is important to emphasize that y-mom is re-discretized at $(I, J+\frac{1}{2})$, rather than simply averaging the discretized y-mom at the nodes (I, J) and $(I, J+1)$.

In the same manner, the discretized equations for $\Delta V_{I+1, J+\frac{1}{2}}$, $\Delta W_{I+1/2, J}$ and $\Delta W_{I+1/2, J+1}$ can be written by centring the corresponding momentum equation at $(I+1, J+\frac{1}{2})$, $(I+\frac{1}{2}, J)$ and $(I+\frac{1}{2}, J+1)$ respectively. Substitutions of these equations into Eq.4.9 yield the discretized Poisson pressure equation having a five-point computational stencil centred at $(I+\frac{1}{2}, J+\frac{1}{2})$ with surrounding nodes $(I+1\frac{1}{2}, J+\frac{1}{2})$, $(I-\frac{1}{2}, J+\frac{1}{2})$, $(I+\frac{1}{2}, J+1\frac{1}{2})$ and $(I+\frac{1}{2}, J-\frac{1}{2})$. One can see that satisfaction of the continuity equation is achieved by $\Delta V_{I+1, J+\frac{1}{2}}$, $\Delta V_{I, J+\frac{1}{2}}$, $\Delta W_{I+1/2, J+1}$ and $\Delta W_{I+1/2, J}$ instead of nodal values. It can be easily observed that no decoupling between adjacent pressure nodes occurs, an important

feature lost in "true" collocated grid formulation.

Since momentum equations are centred at the mid-face of the CV, the quantities involved are not all simple nodal values. Except for pressure values, all mid-face velocities are approximated by simple second-order accurate linear interpolations, using the nearest nodal values on each side, for example

$$\phi_{I,J+\frac{1}{2}} = \frac{1}{2} \left[\phi_{I,J+1} + \phi_{I,J} \right]$$

It must be emphasized that such approximation does not impair the accuracy achieved in formulating the momentum equations, that is, still second-order with constant grid spacing.

Consider first y-mom at $(I,J+\frac{1}{2})$, the first-order y-derivative is differenced using the formula

$$\begin{aligned} \frac{\partial \phi}{\partial y}_{I,J+\frac{1}{2}} &= \frac{1}{2} \left[\frac{\partial \phi}{\partial y}_{I,J+1} + \frac{\partial \phi}{\partial y}_{I,J} \right] \\ &+ O(\Delta Y_I - \Delta Y_{I-1}) + O(\Delta Y^2) + O(\Delta Z)^2 \end{aligned}$$

Second-order derivative in the y-direction is differenced by

$$\begin{aligned} \frac{\partial^2 \phi}{\partial y^2}_{I,J+\frac{1}{2}} &= \frac{1}{2} \left[\frac{\partial^2 \phi}{\partial y^2}_{I,J+1} + \frac{\partial^2 \phi}{\partial y^2}_{I,J} \right] \\ &+ O(\Delta Y_I - \Delta Y_{I-1}) + O(\Delta Y^2) + O(\Delta Z)^2 \end{aligned}$$

The second-order spanwise derivative at $(I,J+\frac{1}{2})$ is approximated by

$$\begin{aligned} \frac{\partial^2 \phi}{\partial z^2}_{I, J+\frac{1}{2}} &= \frac{1}{2} \left[\frac{\partial^2 \phi}{\partial z^2}_{I, J+1} + \frac{\partial^2 \phi}{\partial z^2}_{I, J} \right] \\ &+ O(\Delta Z_{I+1} - \Delta Z_I) + O(\Delta Z_I - \Delta Z_{I-1}) + O(\Delta Z)^2 \end{aligned}$$

This form of differencing involves four node values instead of three for the usual second-order derivative. The Reynolds-stress gradients are discretized in exactly the same way as the pressure gradient on the r.h.s. of Eq.4.10. Likewise, similar differenced formulae are adopted for derivatives appearing in z-mom when centred at $(I+\frac{1}{2}, J)$.

The discretized pressure equation is written in the following compact form where F_1, F_2, F_4, G_1, G_2 and G_4 are coefficients operating on pressure nodes.

$$\begin{aligned} G_1 P_{I+\frac{1}{2}, J+\frac{1}{2}} - [F_1 + F_2 + G_1 + G_2] P_{I+\frac{1}{2}, J+\frac{1}{2}} + G_2 P_{I+\frac{1}{2}, J-\frac{1}{2}} \\ = -F_1 P_{I+\frac{1}{2}, J+\frac{1}{2}} - F_2 P_{I-\frac{1}{2}, J+\frac{1}{2}} - [F_4 + G_4] \\ + \frac{\Delta U_{I+\frac{1}{2}, J+\frac{1}{2}}}{\Delta x} - \dot{S}_p^{n+\frac{1}{2}} \end{aligned} \quad \text{Eq.4.11}$$

All pressures are located at the $x^{n+1/2}$ streamwise station, however, the superscript $(n+\frac{1}{2})$ has been dropped in Eq.4.11 for sake of clarity. Likewise, the subscript (I, J) for all the coefficients $F_1, F_2, F_4, G_1, G_2, G_4$ and source term $\dot{S}_p$ has been dropped.

To formulate the pressure equation, the momentum equations were re-written by centring at mid-way on the faces of the CV. An alternative is to approximate mid-face velocities from nodal values and use the discretized momentum equations that have already been written for these nodal velocities during formulation of the crossflow solution. This suffers from the disadvantage of having a computational stencil with nine pressure nodes, which does not reduce to economically solvable block tridiagonal matrices.

4.3.2 Solution Of Pressure Equation

The pressure equation is elliptic and operator-splitting schemes like the ADI technique cannot be employed. Therefore, it has to be solved iteratively. In the present scheme, the solution is obtained by the successive-line-over-relaxation (SLOR) method (see Ferziger, 1981, pp.195-197) by commencing with a guessed pressure field and computing iteratively until the pre-set convergence tolerance is satisfied. Eq.4.11 has a five-point computational stencil centred at $(I+\frac{1}{2}, J+\frac{1}{2})$ surrounded by pressures at $(I+1\frac{1}{2}, J+\frac{1}{2})$, $(I-\frac{1}{2}, J+\frac{1}{2})$, $(I+\frac{1}{2}, J+1\frac{1}{2})$ and $(I+\frac{1}{2}, J-\frac{1}{2})$. It is evident that solution of unknown pressure values located along the $(I+\frac{1}{2})=\text{constant}$ horizontal grid line, with guessed pressures at $(I+1\frac{1}{2}, J+\frac{1}{2})$ and $(I-\frac{1}{2}, J+\frac{1}{2})$, requires solving a tridiagonal matrix.

Using an iteration index, superscripted as ^k, the solution can be written as

$$\begin{aligned}
& G_1 P_{I+\frac{1}{2}, J+\frac{1}{2}}^K - [F_1 + F_2 + G_1 + G_2] P_{I+\frac{1}{2}, J+\frac{1}{2}}^K + G_2 P_{I+\frac{1}{2}, J-\frac{1}{2}}^K \\
& = -F_1 P_{I+\frac{1}{2}, J+\frac{1}{2}}^{K-1} - F_2 P_{I-\frac{1}{2}, J+\frac{1}{2}}^{K'} - [F_4 + G_4] \\
& + \frac{\Delta U_{I+\frac{1}{2}, J+\frac{1}{2}}}{\Delta x} - \bar{S}_p^{n+\frac{1}{2}}
\end{aligned}$$

$$\text{where } P^{K'} = \omega P^K + (1 - \omega)P^{K-1}$$

ω is the relaxation factor. The superscript K refers to values being computed, K' the just computed values which have been relaxed and $K-1$ the values from previous loop.

Unlike the successive-over-relaxation method (see Smith, 1986, pp.262-266) where the computed values at the nodes are immediately used for the next stencil, the SLOR technique advances the solution in a stepwise manner from one horizontal grid line to the next with enhanced convergent speed. The relaxation factor is often an empirically chosen value giving the quickest convergence. It is useful for non-linear problems or as a means to avoid divergence in the solution. ω can either be under-relaxing ($0 < \omega < 1$) or over-relaxing ($1 < \omega < 2$) or even not relaxed ($\omega=1$), all very much flow dependent (see Patankar, 1980, pp.67-68). Other determining factors on ω are the iterative procedure adopted and the mesh resolution. It is also possible to use a different ω value at each iteration.

4.3.3 Pressure Boundary Conditions

An initial pressure field has to be specified at the beginning of the computation to act as guessed values when solving yz-mom in the first iteration outer pressure loop. It must be remembered that the pressure initial plane is not at 1-2-3-4 but half x-step downstream. For slender flows it is sufficient to set the initial pressure field to zero (since $\frac{\partial P}{\partial y} \sim \frac{\partial P}{\partial z} \approx 0$). If a spanwise pressure exists it might be better to specify uniform values only along each vertical grid line (i.e. $\frac{\partial P}{\partial y} \approx 0$ and $\frac{\partial P}{\partial z} \neq 0$) provided one can handle the rather difficult pressure BCs along the lateral boundaries 2-3 and 1-4.

In deriving the pressure equation Eq.4.11, the discretized velocity equations of the form expressed by Eq.4.10 were substituted into the discretized continuity equation Eq.4.9. At a solid boundary, say along 1-2, Eq.4.10 at the node (1,J) is

$$\tilde{L}_y \frac{1}{\Delta x} [\Delta V]_{1,J+\frac{1}{2}} = - \left[\frac{P_{1\frac{1}{2},J+\frac{1}{2}}^{n+1/2} - P_{\frac{1}{2},J+\frac{1}{2}}^{n+1/2}}{\frac{1}{2} [\Delta Y_1 + \Delta Y_{-1}]} \right] + \tilde{S}_1^{n+1/2}$$

which involves P at nodes $(1\frac{1}{2}, J+\frac{1}{2})$ and $(\frac{1}{2}, J+\frac{1}{2})$ for the pressure equation. The no-slip BC, $V_{1,J+1/2} = 0$, implies that the pressure equation does not involve any pressure nodes outside the computational domain, that is those at $(\frac{1}{2}, J+\frac{1}{2})$. However, pressures at $(1\frac{1}{2}, J+\frac{1}{2})$ are still present because $V_{2,J+1/2} \neq 0$. This is equivalent to a zero Neumann BC at the wall.

$$\frac{\partial P}{\partial y}_{\text{WALL}} = 0 \quad (\text{implicit})$$

The accuracy of the above zero Neumann pressure BC is of the order of the viscous diffusion terms. This implicit zero Neumann BC can be implemented in Eq.4.11 simply by setting the coefficient F_2 due to $V_{1,J+1/2}$ equal to zero.

$$\begin{aligned} G_1 P_{1\frac{1}{2},J+\frac{1}{2}} - [F_1 + G_1 + G_2] P_{1\frac{1}{2},J+\frac{1}{2}} + G_2 P_{1\frac{1}{2},J-\frac{1}{2}} \\ = -F_1 P_{2\frac{1}{2},J+\frac{1}{2}} - [F_4 + G_4] + \frac{\Delta U_{1\frac{1}{2},J+\frac{1}{2}}}{\Delta x} - \frac{\partial}{\partial x} \end{aligned}$$

The term F_4 in the above equation now contains only the source term from $V_{2,J+1/2}$.

The use of no-slip velocity BC to obtain Neumann pressure BC is very desirable as it eliminates the difficulty in specifying a proper wall pressure BC and this can be achieved only when the pressure is not located along the wall. In other words, the velocities and pressures MUST be staggered when formulating the pressure equation, a condition satisfied by both PSTAG and the staggered grids. In fact this velocity-pressure staggering was also adopted in the collocated grid formulation of Perić (1985) which prompted Majumdar et al. (1987) to say that both the collocated and staggered grids are basically identical. Abdallah (1987) pointed out that the constraint imposed by Green's theorem on the solution of Poisson equation with Neumann BC is automatically satisfied only in a staggered velocity-pressure

formulation and failure of the "true" collocated grid to satisfy this constraint would lead to endless drifting in an iterative solution.

For lateral boundaries where two-dimensionality exists or can be assumed, the Neumann BC is used.

$$\frac{\partial P}{\partial z} = 0$$

Along 3-4 which is normally the free stream boundary, the slender flow approximations can be used to obtain

$$P = P_{\infty} = 0$$

The pressure and velocities boundaries coincide only at solid boundary and are non-coincident at free surfaces.

4.4 Calculation Of Turbulence Quantities

4.4.1 Discretization Of Transport Equation

The RSTM for 3-D domain requires a solution of seven non-linear partial differential equations in addition to the mean flow equations. Although the equations in the RSTM could be solved simultaneously, it is more costly, in term of computing time, to invert block 7x7 tridiagonal matrices. Besides, simultaneous solutions are not known to exhibit any flexibility when changes to the algorithm need to be made. In PSTAG, the $\overline{u_i u_j}$ - and ϵ -equations are solved sequentially (this can sometimes act like a form of under-relaxation), and since these are all transport

equations, they can be cast in a general form.

$$\underbrace{\frac{\partial}{\partial x_j} U_j \phi}_{\text{"convection"}} = \underbrace{\frac{\partial}{\partial x_j} \left[\Gamma_\phi \frac{\partial \phi}{\partial x_j} \right]}_{\text{"diffusion"}} + \underbrace{S}_{\text{"source"}} \quad \text{Eq.4.12}$$

ϕ denotes the variable of interest (i.e. $\overline{u_i u_j}$ or ϵ), Γ_ϕ the diffusivity (i.e. $\frac{k}{\epsilon} \overline{u_i u_j}$) and S the sources. In the momentum equations, $\Gamma_\phi = \nu = \text{constant}$ but in the $\overline{u_i u_j}$ - and ϵ - equations, Γ_ϕ can be spatial and ϕ dependent. All other terms in the $\overline{u_i u_j}$ - and ϵ - equations which do not fit into either convection or diffusion will be put into S .

In PSTAG the Reynolds-stresses and ϵ are located in a staggered manner similar to the pressures as shown in Fig.4.1, that is, at the centre of the CV whose eight corners are the velocity nodes. Eq.4.12 is differenced fully-implicit in the streamwise direction and again the conservative form is used.

$$\begin{aligned} \frac{(U\phi)^{n+\frac{1}{2}} - (U\phi)^{n-\frac{1}{2}}}{\Delta x} &= -\frac{\partial}{\partial y} (V\phi)^{n+\frac{1}{2}} - \frac{\partial}{\partial z} (W\phi)^{n+\frac{1}{2}} \\ &+ \frac{\partial}{\partial y} \left[\Gamma_\phi \frac{\partial \phi}{\partial y} \right]^{n+\frac{1}{2}} + \frac{\partial}{\partial z} \left[\Gamma_\phi \frac{\partial \phi}{\partial z} \right]^{n+\frac{1}{2}} + S^{n+\frac{1}{2}} \end{aligned}$$

Eq.4.13

The crossflow plane derivatives and source terms are discretized at the $(n+\frac{1}{2})$ station. Second-order derivatives are differenced using CDS whereas UDS is employed for the first-order derivatives. The final discretized transport equation can be written in a form similar to the pressure equation Eq.4.11.

$$\begin{aligned}
& G_{\phi 1} \phi_{I+\frac{1}{2}, J+\frac{1}{2}} + G_{\phi 4} \phi_{I+\frac{1}{2}, J+\frac{1}{2}} + G_{\phi 2} \phi_{I+\frac{1}{2}, J-\frac{1}{2}} \\
& = - F_{\phi 1} \phi_{I+\frac{1}{2}, J+\frac{1}{2}} - F_{\phi 2} \phi_{I-\frac{1}{2}, J+\frac{1}{2}} + S
\end{aligned}$$

Eq.4.14

where $F_{\phi 1}$, $F_{\phi 2}$, $G_{\phi 1}$, $G_{\phi 2}$ and $G_{\phi 4}$ are coefficients of the variables.

Towards the edge of a boundary layer and in the free stream region, the Peclet number in the $\overline{u_i u_j}$ - and ϵ - equations almost always goes to infinity because Γ_ϕ approaches zero. In anticipation of such high grid Peclet number where CDS has been found by many workers to be unsuitable both in terms of numerical accuracy and stability, the UDS was chosen. Although the accuracy of UDS diminishes when the velocity vector is oblique to the grid lines (due to the introduction of false diffusion), it always provides physically realistic results. Since here one is calculating mainly boundary layer type of flows, where velocity vectors are closely aligned with grid lines, UDS is acceptable.

The source term, S in Eq.4.14, can at times contains a large number of terms since it absorbs all the other terms that do not fit into Eq.4.12 either as convective or diffusive. Careful handling of source term discretization can enhance convergence through improved boundedness or diagonal dominance (see Patankar, 1980, pp.48-49 & pp.143-146). The source can often be expressed as

$$S = S_p + \phi S_\phi$$

S_p is the part which is entirely independent of ϕ and S_ϕ is the coefficient of ϕ in S . The advantage of such a form is that when the value of S_ϕ is negative, (ϕS_ϕ) can be brought over to the l.h.s of Eq.4.14. In this way, the matrix to be inverted can be made more strongly positive definite, thus breakdown in the solution of the tridiagonal matrix system can be minimised.

4.4.2 Solution Of Transport Equation

The discretized transport equation in Eq.4.14 at $(I+\frac{1}{2}, J+\frac{1}{2})$ has a five-point stencil with neighbours $(I+1\frac{1}{2}, J+\frac{1}{2})$, $(I-\frac{1}{2}, J+\frac{1}{2})$, $(I+\frac{1}{2}, J+1\frac{1}{2})$ and $(I+\frac{1}{2}, J-\frac{1}{2})$. This five-point cluster, being similar to those in the momentum and pressure equations, is suitable for formulating tridiagonal matrices. The solution of Eq.4.14 by the SLOR method is the same as that employed for the pressure Poisson equation (for details see §4.3.2). Briefly, the calculation sweeps across the computational domain in the y-direction solving for each I=constant grid lines (i.e z-wise) with guessed values on the grid lines $I+1\frac{1}{2}$ and $I-\frac{1}{2}$. Perhaps it is worth mentioning that no relaxation is presently employed in the solution of the $\overline{u_i u_j}$ - and ϵ - equations, though one can always try to find a suitable relaxation factor to enhance convergence.

4.4.3 Turbulence Quantities Boundary Conditions

The RSTM described in chapter 3 is the high Re version and does not resolve across the sublayer to the wall. Therefore, wall

functions become necessary in order to bridge the solution in the "universal" log-law region to the wall. The staggering of turbulence quantities means their first location is not at the wall but at $(1\frac{1}{2}, J+\frac{1}{2})$. Thus, it is essential to locate $I=1\frac{1}{2}$ within the log-law region, which inevitably imposes a minimum grid resolution near the wall. Failure to comply simply implies a wrong physical interpretation of the log-law; this requirement was also taken into careful consideration by Chieng & Launder (1980).

The law of the wall has been reasonably well established for 2-D unseparated turbulent boundary layers. In the present work, the 2-D log-law is being generalised to accommodate for 3-D flows.

$$\frac{Q}{u_\tau} = \frac{1}{\kappa} \ln \left[\frac{Y u_\tau}{\nu} \right] + C \quad \text{Eq.4.15}$$

In the above equation, the resultant x-z plane mean velocity $Q = \sqrt{U^2 + V^2}$ and the total surface shear stress $u_\tau^2 = \sqrt{(\overline{uv})^2 + (\overline{vw})^2}$, with the Q and u_τ vectors assumed to be coincident. From the 3-D boundary layer experimental data of Pontikos (1982), he found Eq.4.15 to be reasonably accurate. A similar form was used by Lilley & Rhode (1982) in a 3-D computation of swirling flows in a combustor (for more elaborate models see Pierce et al., 1983). There are also versions of log-law which take into account the effects of $\frac{dP}{dx}$ (see short review by Granville, 1989) but Bradshaw (1988) pointed out that the modified log-laws do not seem to give significantly better results.

The no-slip wall BC imposes a very steep velocity gradient in the region between the wall and $I=2$, in other words, there is an

abrupt change in the Q velocity profile at node $(2, J+\frac{1}{2})$ because PSTAG does not resolve the sublayer. This tends to underpredict, very slightly only (less than 0.5%), the value of Q at $(2, J+\frac{1}{2})$. For this reason the value of Q in Eq.4.15 is evaluated at $(2\frac{1}{2}, J+\frac{1}{2})$ using a Lagrangian interpolation operating on the Q profile between $(2, J+\frac{1}{2})$, $(3, J+\frac{1}{2})$ and $(4, J+\frac{1}{2})$. Consequently, PSTAG requires the region from $I=1\frac{1}{2}$ to $I=2\frac{1}{2}$ to lie within the log-law zone, that is, from $y^+ \leq 50$ to $y \leq 0.2\delta$. Since u_τ is evaluated from $Q_{2+1/2, J+1/2}$, a correction is made on $Q_{2, J+1/2}$ to compensate for (any) underprediction caused by the no-slip BC.

Having determined u_τ from the law of the wall and assuming a constant stress region from the wall to $I=1\frac{1}{2}$, one obtains

$$\overline{uv} = -u_\tau^2 \cos \theta_\tau$$

$$\overline{vw} = -u_\tau^2 \sin \theta_\tau$$

where $\theta_\tau = \tan^{-1} \left[\frac{W}{U} \right]$. The other near-wall turbulence quantities at $(1\frac{1}{2}, J)$ can also be expressed, in a straightforward manner, as ratios of u_τ^2 .

$$k = C_k u_\tau^2 ; \quad \overline{uu} = C_{uu} u_\tau^2 ; \quad \overline{vv} = C_{vv} u_\tau^2 ;$$

$$\overline{ww} = C_{ww} u_\tau^2 ; \quad \epsilon = \frac{u_\tau^3}{\pi y}$$

Strictly speaking, the above constants of proportionality are flow dependent. However, these have been set to values consistent with those obtained in §3.2.4 from the wall damping function for 2-D flows. This slight simplification is justifiable because

convective transport near the wall in most boundary layer flows is negligible and so these algebraically specified turbulence quantities will not significantly affect the near-wall computations. It remains, therefore, to specify the near-wall value of $\overline{uw}$. In order to be consistent with second-moment closure by not resorting to a ν_t expression for $\overline{uw}$, the zero Neumann BC is adopted for $\overline{uw}$ (i.e. $\frac{\partial \overline{uw}}{\partial y} = 0$).

Along the lateral and free stream boundaries, zero Neumann BCs are implemented for all the turbulence quantities.

$$\frac{\partial}{\partial y} \overline{u_i u_j} = \frac{\partial \epsilon}{\partial y} = \frac{\partial}{\partial z} \overline{u_i u_j} = \frac{\partial \epsilon}{\partial z} = 0$$

4.5 PSTAG Solution Procedure

The set of slender flow equations Eq.2.10 is streamwise parabolic because upstream influence from $\frac{dP}{dx}$ (set a priori), streamwise viscous diffusion and Reynolds-stress gradients is eliminated. The solution algorithm basically marches in the x-direction from the n^{th} to the $(n+1)^{th}$ station where the upstream n^{th} station contains known quantities and the unknown quantities to be calculated are at the downstream $(n+1)^{th}$ station. The staggering of pressure and turbulence quantities between streamwise stations means that the known and unknown stations are at the $(n-\frac{1}{2})^{th}$ and the $(n+\frac{1}{2})^{th}$ stations respectively.

With reference to the block diagram Fig.4.6, at each streamwise station the calculation begins with the solution of x-mom. Due to

the linearization method adopted, x-mom is decoupled from the crossflow momentum equations. The only secondary flow velocities involved in x-mom are the known upstream values, which leaves the primary flow velocity increment as the only unknown quantity. Hence, solution of x-mom by a two-sweep ADI technique will give the downstream primary flow velocity field and the solution is non-iterative. When the $(n+1)^{th}$ primary flow velocity field has been established, the next step is to evaluate the secondary flow and pressure values.

Solutions of the coupled yz-mom and pressure equation are accomplished through two iterative loops referred to as the inner and outer pressure loops. In yz-mom, there are 3 unknowns, namely the secondary velocities (V, W) and pressure. By providing guessed values for the pressures appearing in the crossflow momentum equations, there are then two equations with two unknowns. Therefore the downstream secondary flow field can be computed from yz-mom, in similar manner as for x-mom, by spatial marching in the x- direction using the two-sweep operator-splitting ADI technique.

Once the $(n+1)^{th}$ crossflow velocities have been obtained, the pressures at the $(n+\frac{1}{2})^{th}$ station are evaluated from the pressure equation using an iterative SLOR method in the inner pressure loop until the convergence criterion set for the inner pressure loop is satisfied. The newly calculated pressures are subsequently used as a set of better guessed values for re-calculating yz-mom. Although the secondary flow calculation as such is non-iterative (upstream values of coefficients are used, see §4.2.4), the use of a guessed

pressure field means that it must be repeated each time a new pressure field is obtained from solving the Poisson equation. This iterative solution of the coupled yz-mom and the pressure equation forms the outer pressure loop which terminates once the pressure field calculation satisfies the outer pressure loop convergence criterion. In laminar flow calculations, upon convergence of the outer pressure loop the computation then proceeds to the next streamwise station.

In turbulent flow calculations, the momentum and pressure equations are solved using estimated values of the Reynolds-stresses at the $(n+\frac{1}{2})^{\text{th}}$ station. These Reynolds-stresses are determined from the inner RSTM iteration loop consisting of a sequential solution of the seven transport equations. In the inner RSTM loop, one iteration consists of a solution of all the $\overline{u_i u_j}$ - and ϵ - equations with each equation computed only once and convergence is checked at the end of each iteration loop. Due to the strong coupling between the seven equations, it is not advisable to compute each equation to convergence before proceeding to the next.

The outer RSTM loop provides the necessary linkage between the mean flow field (by updating the momentum equations with the latest $\overline{u_i u_j}$ values) and the turbulence quantities (by using the latest calculated mean velocities in the RSTM solution). At each x-step, convergence in the outer RSTM loop indicates overall convergence. Hence in turbulent flow computations, all equations (i.e. momentum, pressure, $\overline{u_i u_j}$ - & ϵ - equations) are iterated at

each x-step. This has the advantage of reducing the errors introduced by the linearizations of the governing equations, especially the $O(\Delta x)$ accurate x-mom. Algorithms that do not iterate between the mean flow field and turbulence quantities at each x-step (e.g. Dedoussis, 1988) actually calculate the mean flow field based on turbulence quantities that lag one x-step. Perhaps one can argue that since streamwise changes are small in slender flows, the error would not be too significant.

4.6 Comparison Between PSTAG, Collocated & Staggered Grids

4.6.1 Computational Stencil

It is of paramount importance that any numerical solution simulates the transmission of information as described by the governing equations. As an illustration, take an unsteady flow computation: it is entirely wrong to step iteratively between time steps because the time dimension is parabolic. In calculations based on the slender flow equations, one knows that the crossflow plane exhibits an elliptic character, as such the computational stencils are also required to possess this same sort of elliptic influence.

Looking at the PSTAG five-point cluster of yz-mom in Fig.4.7, it can be said that the node at the centre, (I,J) , receives influence from its neighbours in a non-biased manner due to its symmetry. This is a non-trivial aspect because asymmetry would mean the domain of influence is constrained towards certain regions,

implying to a certain extent hyperbolicity. In comparison, Fig.4.8 shows the staggered grid computational stencil for yz-mom centred at (I,J) for evaluating $V_{I+1/2,J}^{n+1}$ and $W_{I,J+1/2}^{n+1}$. It can be seen that the calculation of $V_{I+1/2,J}^{n+1}$ depends on the other surrounding V and W velocities as well as the upstream values which has a similar stencil structure. When observing the neighbouring V^{n+1} velocities, the existence of $V_{I-1/2,J+1}^{n+1}$ shows a biased situation. Influence by the W^{n+1} velocities come more strongly from the right by $W_{I,J+1/2}^{n+1}$ and $W_{I-1/2,J+1/2}^{n+1}$, again exhibiting a biased situation. Likewise, the computation of $W_{I,J+1/2}^{n+1}$ also exhibits a biased influence from $W_{I+1,J-1/2}^{n+1}$ and strong dependence on V^{n+1} on the left. From here it can be deduced that a non-symmetrical stencil will, in general, not be adequate to mimic a flow in which a non-biased type of influence exists. In the computation of a symmetrical flow (see §6.4), the results from the staggered grid are not as good as those obtained by PSTAG.

In a non-iterative solution of yz-mom using operator-splitting schemes such as the ADI technique, the asymmetric computational stencil necessitates an additional as well as cumbersome diagonal sweep. By eliminating this diagonal sweep in PSTAG, the solution converges substantially quicker (by about 50%) as demonstrated by two different test cases in §6.3 and §6.4. One must bear in mind that the diagonal sweep is a consequence of the staggered grid as well as the linearization process needed to obtain a non-iterative scheme. In a fully-iterative scheme, the momentum equations are decoupled and the computational stencils are simply five-point clusters.

The momentum equations are each centred at different locations (i.e. having different CV) in a staggered grid and therefore do not share the same coefficients for the convective terms. On the other hand, PSTAG can take advantage of a common CV for the momentum equations to reduce the computing time needed to evaluate the convective terms. This saving in computing time will become more obvious when convective terms are discretized by more elaborate higher-order differencing schemes (e.g. SUDS or QUICK) or in a multigrid solution (see Barcus et al., 1987). Likewise, all the Reynolds-stresses share common CVs in PSTAG whereas in the staggered grid most of the Reynolds-stresses do not have common CVs. From this argument it seems the collocated grid is best because the CVs for the momentum equations and Reynolds-stresses are common. But the important mean velocity gradients in the generation terms will be evaluated over two mesh spacings (excluding the centre nodal value). The staggering of the Reynolds-stresses in PSTAG, therefore, permits a more natural way of evaluating the strain rates over one mesh spacing.

4.6.2 Implementation Of Boundary Conditions

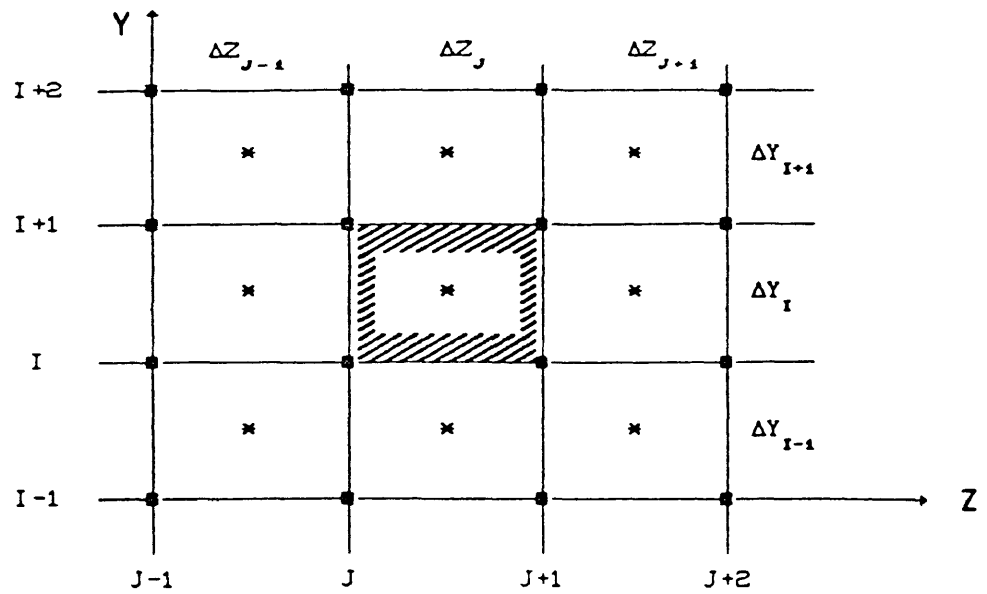
When one specifies the computational domain, the boundaries are aligned along regimes whereby the flow field conditions are known and the Dirichlet and Neumann BCs are probably the most commonly used BCs. However, these BCs are not directly applicable to the diagonal sweep in a staggered grid formulation due to the necessity to relate boundary nodes in a diagonal manner rather

than along grid lines. For example, $\phi_{(IMAX,JMAX)}$ has to be associated with $\phi_{(IMAX+1,JMAX+1)}$ instead of either $\phi_{(IMAX+1,JMAX)}$ or $\phi_{(IMAX,JMAX+1)}$. This does not present a problem when the boundaries can be situated in regions where the flow is near uniformity or where the no-slip condition applies. However, if the only allowable BCs are strictly the Neumann ones then it becomes impossible to relate boundary nodes diagonally which consequently hampers the solution. Another setback (due to the logic needed to control the diagonal sweep) lies in the rather inflexible restriction that the number of grid nodes in the y- and z-direction must be equal.

Since the velocities in PSTAG coincide with the wall, for a five-point stencil (e.g. UDS or CDS) no imaginary nodes have to be created outside the computational domain. For higher order differencing schemes (e.g. SUDS or QUICK) with a nine-point stencil only one imaginary node is required, which can be easily obtained from the no-slip BC. With reference to Fig.4.9, none of the collocated grid velocities coincide with the wall, whereas the staggered grid has only one velocity component at the wall. This is not very cumbersome when one is dealing with a five-point stencil (which requires one imaginary node), but with a nine-point stencil the creation of more imaginary nodes is often difficult, though one can always compromise by reducing to lower order schemes near the wall.

In all three grid arrangements, the no-slip wall BC (conveniently) implies an implicit Neumann pressure BC (see §4.3.3). The pressure

derivative in the momentum equations for PSTAG and staggered grid can be evaluated directly without resorting to any interpolations. This can be better appreciated when comparison is made with the collocated grid where pressure values required at the wall need to be extrapolated from interior values. This is a rather severe disadvantage because it is not only difficult to choose a reliable extrapolation formula, but the pressure field, which is most sensitive near the wall, plays a crucial role in the convergence of the overall solution.



▨ Control volume, CV.

■ Grid node storing U, V & W .

* Grid node at centre of CV storing $P, \overline{u}, \overline{u}_j$ & ϵ .

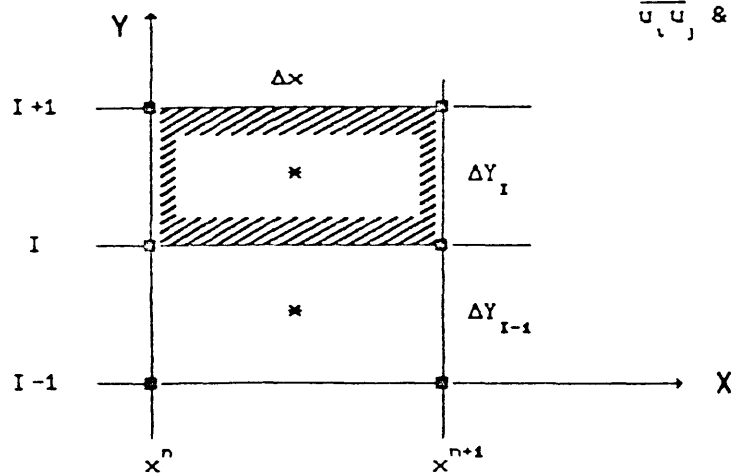


Fig.4.1 Pressure Staggered Grid (PSTAG).

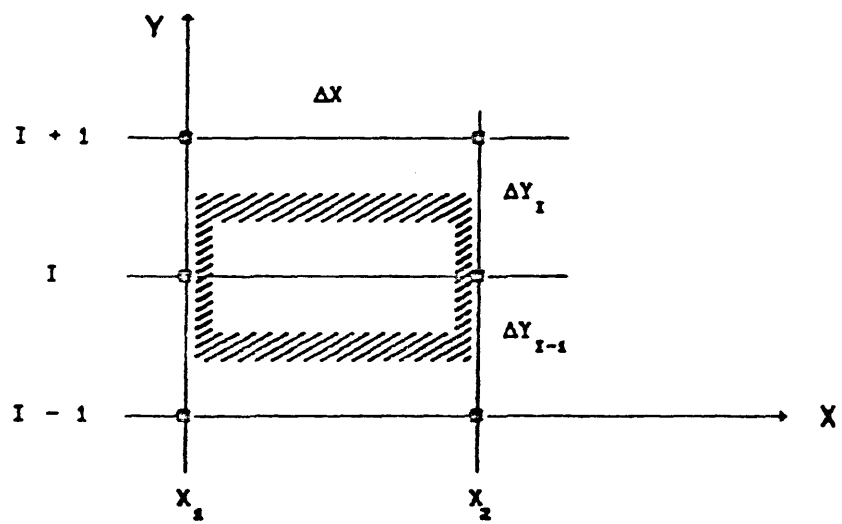
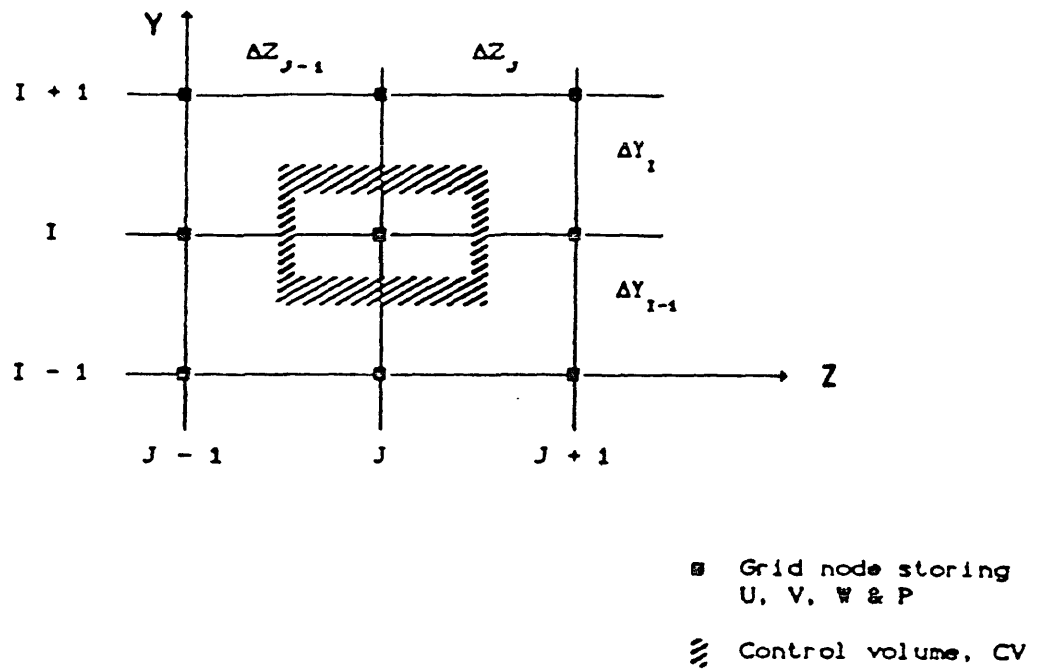


Fig.4.2 Collocated Grid.

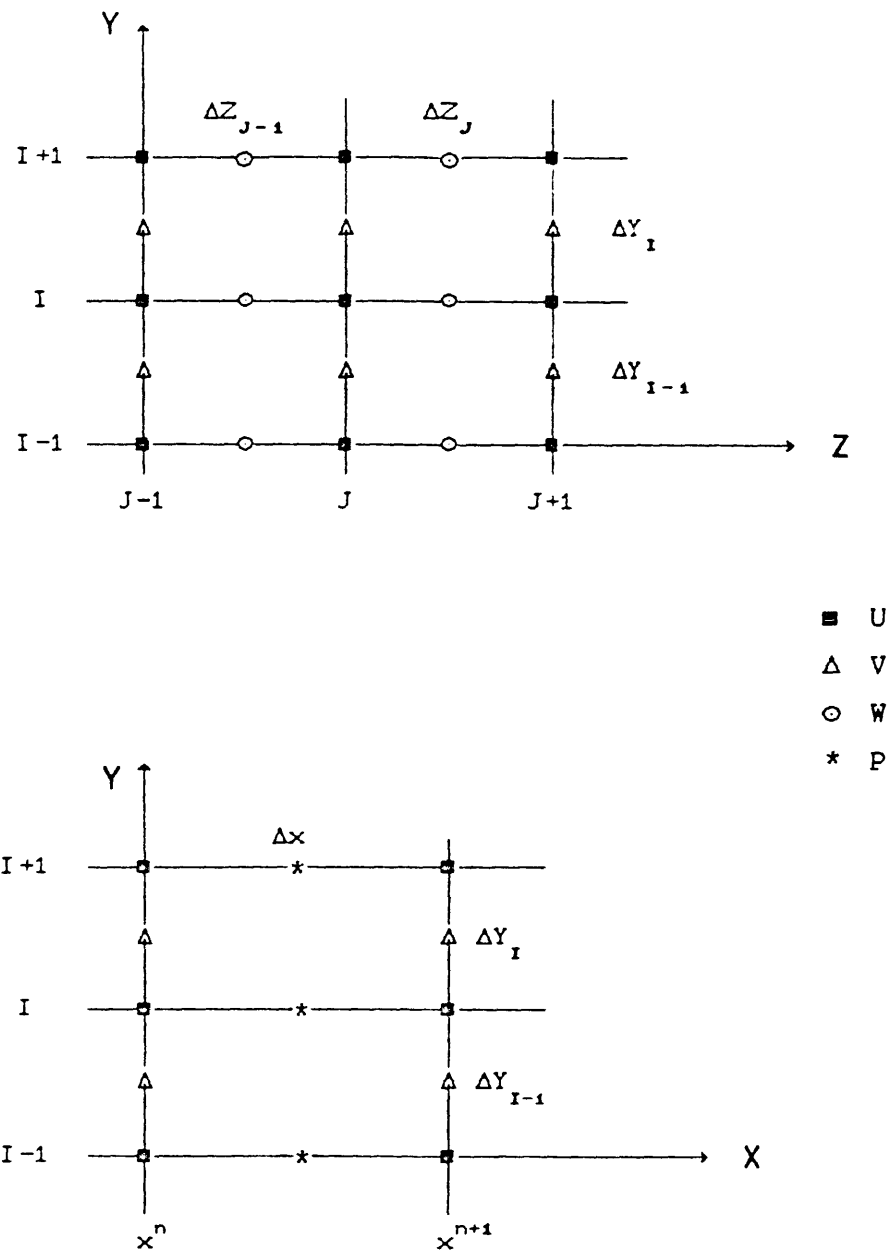


Fig.4.3 Staggered Grid.

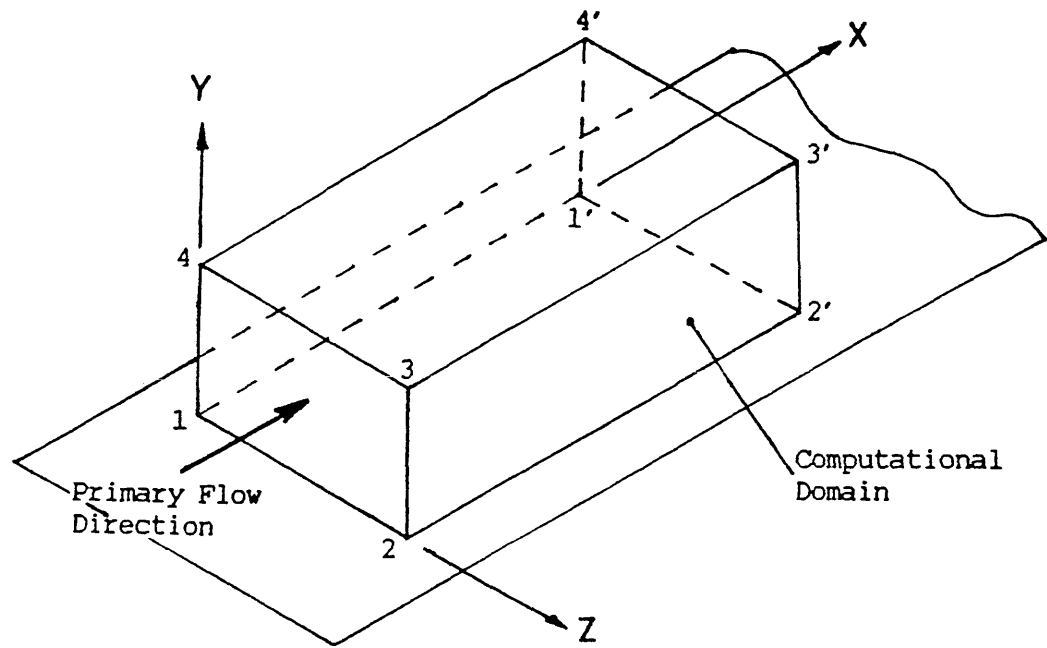


Fig.4.4 Boundaries Of Computational Domain.

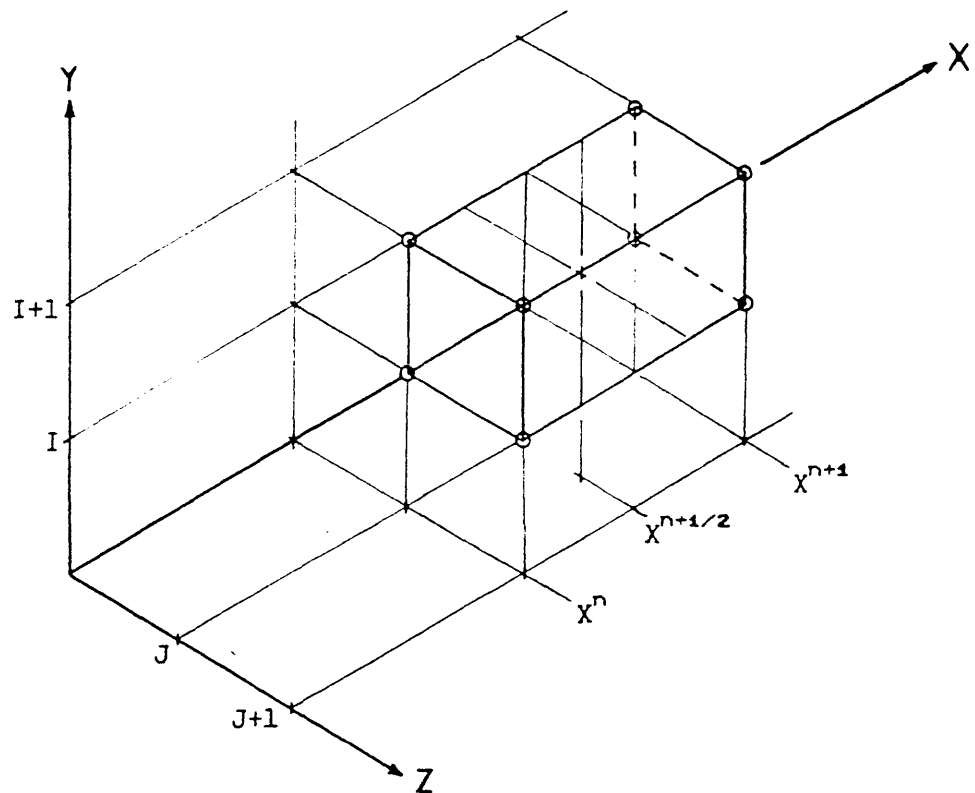


Fig.4.5 Control Volume In Pressure Equation.

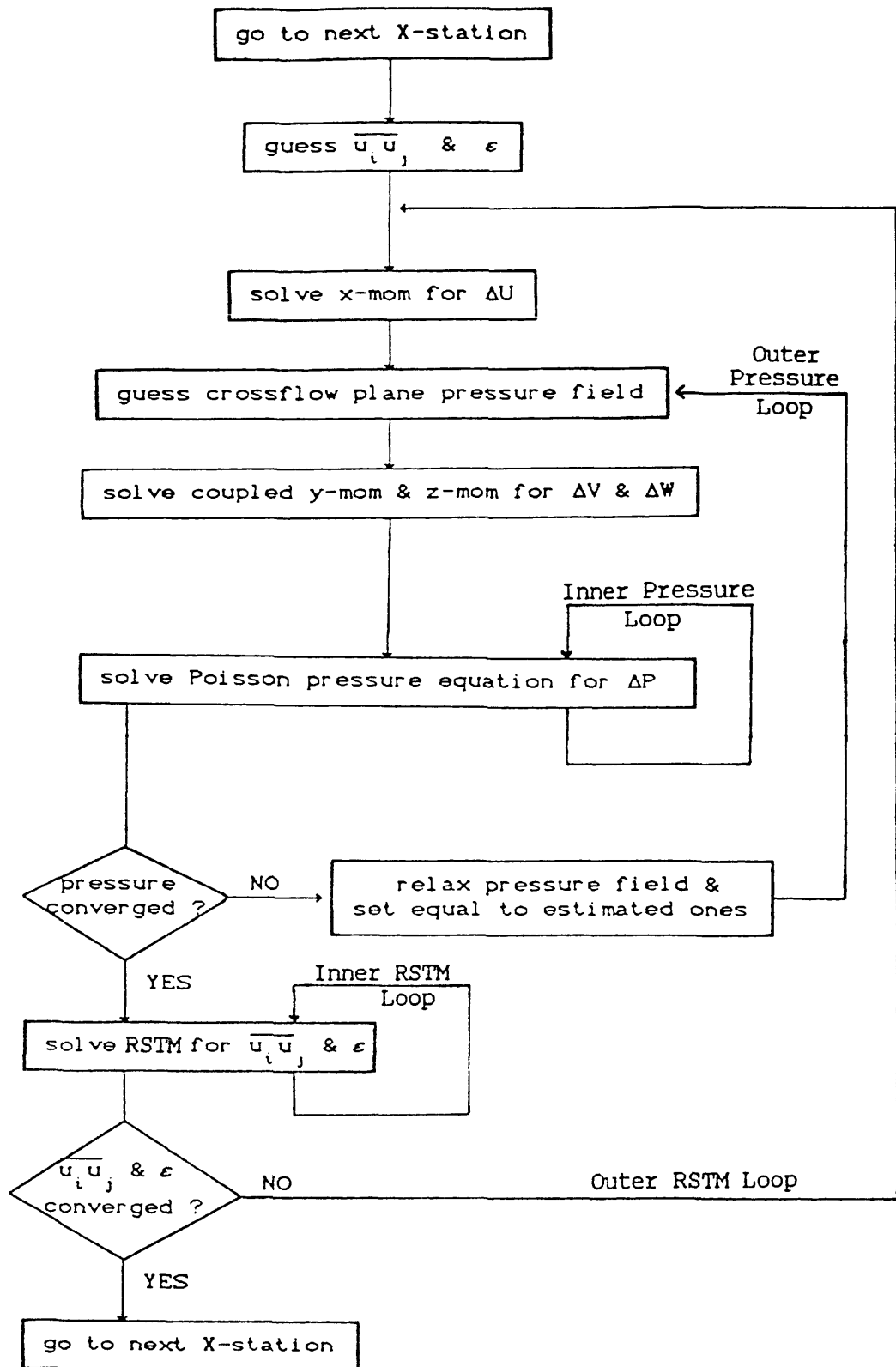


Fig.4.6 Block Diagram Of PSTAG

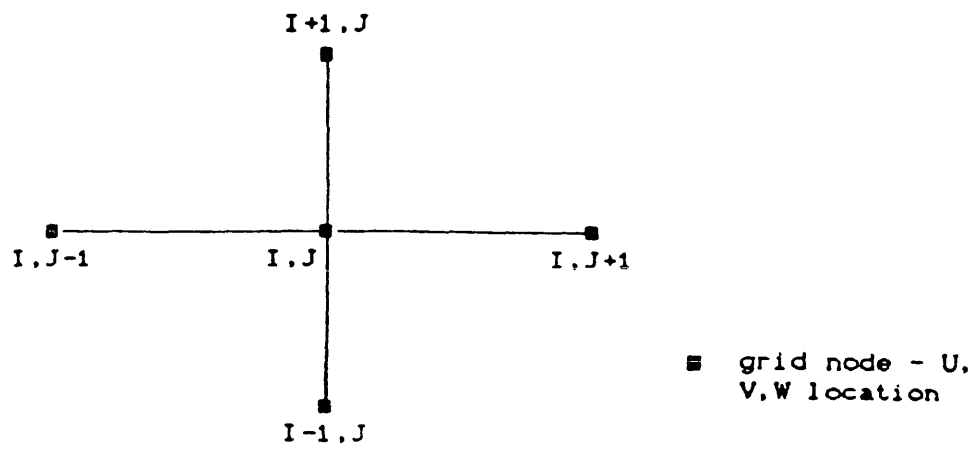


Fig.4.7 PSTAG yz-mom Computational Stencil.

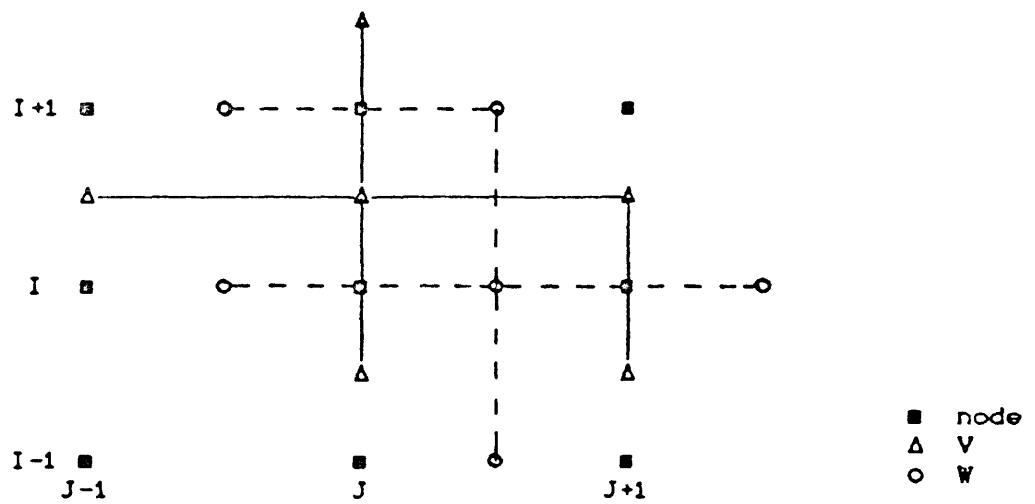
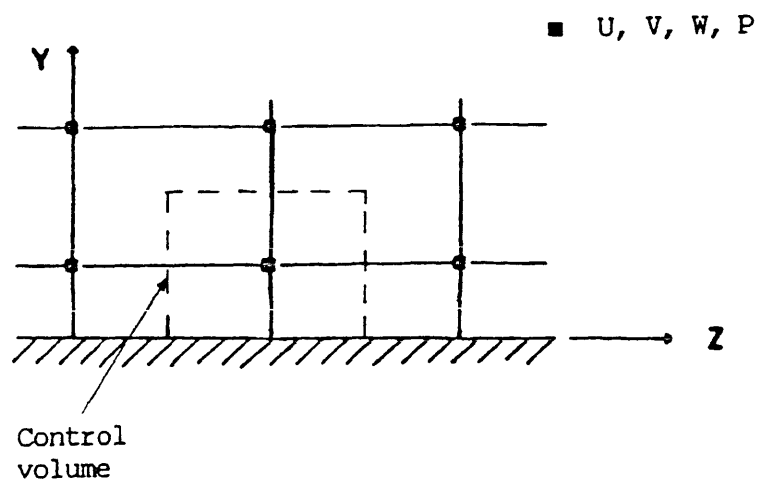
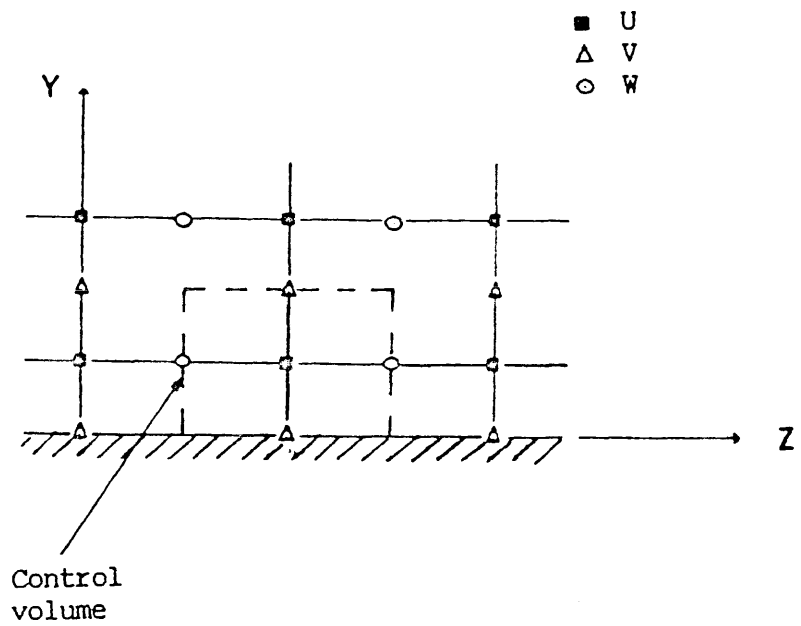


Fig.4.8 Staggered Grid yz-mom Computational Stencil.



Collocated Grid



Staggered Grid

Fig.4.9 Near Wall Grids

5 PROGRAM PSTAG

5.1 The Main Program

The program is written in the FORTRAN language with one version for the VAX minicomputer and another for an 80286/80287-based microcomputer. The structure of the program logic is controlled by the main program PSTAG and can be easily followed by referring to the flowchart in Fig.5.1. NOTE: This program documentation chapter should be read together with the variable namelist in Appendix A3. Variable names are in upper case and subroutine/function names are in BOLD upper case.

The main program begins with the input of the relevant filenames: there is a choice between interactive and non-interactive modes. The five required files are:-

```
UNIT 5  FILE 5 : Parameter file.
UNIT 6  FILE 6 : Initial plane data file.
UNIT 7  FILE 7 : Final plane results.
UNIT 8  FILE 8 : Results at each calculated yz-plane.
UNIT 9  FILE 9 : Status file.
```

The information pertaining to the convergence of the iteration loops is contained in FILE 9. For instance, the number of iterations taken, the maximum absolute error and the location where this error occurs will all be shown in FILE 9.

Once the data files have been opened, INIT (see §5.4) reads in the

data necessary to start the calculations, these data are contained in FILE 5 and FILE 6. The main program then calls GEOMET (see §5.4) to calculate the geometrical coefficients of the mesh. If the flow is turbulent, COTURB (see §5.4) computes the upstream convective terms of the equations in the RSTM. Before the calculation of the downstream flow field begins, the initial plane BCs are written by PRINT (see §5.4) to FILE 8.

5.2 Subroutines For Mean Flow Calculations

At the beginning of each x-step loop, DX is computed and both the counters for the outer RSTM (ITNORO) and pressure (ITNOPO) loops are started. Then GUESS (see §5.4) is called to provide the guessed pressure field and Reynolds-stresses for the calculation of the mean velocities.

Subroutine XMOM

XMOM solves for the U velocity increment $DU(I,J)$ in two sweeps across the computational domain. The first sweep is in the y-direction, and the stencil coefficients for $DU(I-1,J)$, $DU(I,J)$ and $DU(I+1,J)$ are stored in $A11(I-1)$, $B11(I-1)$ & $C11(I-1)$ respectively with the source terms in $R11(I-1)$. The z-sweep stencil coefficients for $DU(I,J-1)$, $DU(I,J)$ and $DU(I,J+1)$ are also stored in $A11(I-1)$, $B11(I-1)$ & $C11(I-1)$ respectively. The BCs are effected through BCN, BCS, BCE & BCW; these variables take the value 0 for zero Dirichlet BC, 1 for zero Neumann BC, and other values depending on the type of BCs. The inversion of the

tridiagonal matrices for each sweep is performed by the line-by-line solver TRDIAG and the solution along a grid line is returned in a 1-D array U1(I). When the solution of DU(I,J) in the whole computational domain has been obtained, the boundary values are updated.

Subroutine CROSFL

The secondary flow velocity increments, that is, DV(I,J) for V increments and DW(I,J) for W increments, are obtained from the yz-mom solution which is performed by CROSFL. The terms in the upstream plane, being source terms, are evaluated as YL(I,J), YR(I,J) & ZR(I,J). The solution of the coupled yz-mom also employs the y-sweep followed by the z-sweep, albeit inverting 2x2 block tridiagonal matrices by calling SOLV2. The BCs are implemented through BCN, BCS, BCE & BCW. The y- and z- sweep stencil coefficients are stored in a similar fashion to XMOM in the arrays A11(I), A12(I), A21(I), A22(I), B11(I), B12(I), B21(I), B22(I), C11(I), C12(I), C21(I) & C22(I). The source-term arrays are R11(I) & R22(I). After SOLV2 has returned the solution of DV(I,J) as U1(I) and DW(I,J) as U2(I), the boundary values are computed.

Subroutine PRESS

The Poisson pressure equation is solved by PRESS which also contains the inner pressure iteration loop. The variable names F1(I,J), F2(I,J), F4(I,J), G1(I,J), G2(I,J) & G4(I,J) correspond to the coefficients given in Eq.4.11, and the source term is

stored in SC(I,J). TRDIAG is used in the line-by-line iterations and the storage variables for the computational stencil are similar to XMOM. At the end of each iteration, the maximum absolute error is computed, stored in ERRPI and compared with the convergence criterion TOLPI. If the solution has not converged and the inner pressure loop iteration counter ITNOPI is within the permissible maximum number of iterations IPERPI, then the newly computed pressures PNEW(I,J) are relaxed through OG and the next iteration loop begins.

Upon convergence in the inner pressure loop, the main program checks for convergence of the outer pressure loop by calling PERR (see §5.4): and the logic is similar to that for the inner loop and is as indicated in the flowchart Fig.5.1. For laminar flows, convergence in the outer pressure loop will transfer control to PRINT (see §5.4) and then UPDATE (see §5.4). When the last x-station has been computed, FINAL (see §5.4) writes the data to FILE 7, otherwise, calculations for the next x-station is started. FILE 7 has the same data format as FILE 6, hence, it can also serve as initial BCs for subsequent calculations.

5.3 Subroutines For The Reynolds - Stress Turbulence Model

The Reynolds-stresses and ϵ are located in a staggered manner in the yz-plane at $x^{n+1/2}$ (see Fig.4.1), hence, MID (see §5.4) is called to compute the rates of strain and mean velocities on this plane.

Subroutines RSMUV, RSMVV, RSMUU, RSMWW, RSMUW, RSMVW, RSMEPL

The turbulent quantities are calculated in the following order: $\overline{uv}$ by RSMUV, $\overline{vv}$ by RSMVV, $\overline{uu}$ by RSMUU, $\overline{ww}$ by RSMWW, $\overline{uw}$ by RSMUW, $\overline{vw}$ by RSMVW and ε by RSMEPL. The use of the log-law means that $\overline{uv}$ has to be solved first since the algebraically specified normal stresses depend on u_τ^2 . In RSMUV, the log-law is solved iteratively for u_τ , with the latest value of u_τ stored in UTNEW and the previous value in UTOLD. The x-z plane resultant mean velocity UUU, its angle to the streamwise axis THETA and y^+ (as ARG) are also evaluated. The resultant mean velocity used in the log-law is not a nodal value, but instead is at $(2\frac{1}{2}, J+\frac{1}{2})$, and this value is linearly interpolated from the surrounding nodal values. Once u_τ is obtained, UV(1,J), VW(1,J), TKE(1,J) & EPL(1,J) are computed.

The general structure of each of these subroutines for the RSTM are basically the same. The generation, "rapid" redistribution, wall damping function and source terms are calculated first and stored in PROD, PHI2, PHIW and SC(I,J) respectively. Next the part of the diffusion term which goes into the source term is computed as DIF1 & DIF2 and added to SC(I,J). Then the coefficients of the five-point computational stencil are evaluated, with the centre node coefficient stored in F4(I,J), west in F1(I,J), east in F2(I,J), south in G1(I,J) and north in G2(I,J). The matrices are solved by TRDIAG. The newly calculated values can be relaxed using the relaxation factors, RFRS for $\overline{u_i u_j}$ and RFEPL for ε . Next the boundary values and maximum absolute error ERRRI are evaluated.

Each inner RSTM loop consists of calling each subroutine once, and since ERRRI is passed between these subroutines it finally contains the largest error in any of them. If ERRRI is greater than the allowable tolerance TOLRI and the inner RSTM loop iteration counter ITNORI does not exceed the permissible number IPERRI, the inner RSTM loop is restarted. Upon convergence in the inner RSTM loop, RSERRO (see §5.4) is called to evaluate the maximum outer RSTM loop absolute error ERRRO. Convergence in the outer RSTM loop signals convergence in the present x-step calculation and UPDATE (see §5.4), followed by PRINT (see §5.4) is then called; otherwise, control is transferred to the beginning of the mean velocity field calculation. When the final x-station calculation has been completed, FINAL (see §5.4) writes the final plane results into FILE 7.

5.4 Auxiliary Subroutines

Detailed description of those auxiliary subroutines that have been mentioned in the previous sections are given here. The subroutines are listed in alphabetical order.

Subroutine COTURB

The convective terms of the $\overline{u_i u_j}$ - and ϵ - equations at the upstream ($x^{n-1/2}$) station are computed as UUCON(I,J), VVCON(I,J), WWCON(I,J), UVCON(I,J), UWCON(I,J), VWCON(I,J), EPLCON(I,J). If one prefers not to allow easy access to the empirical constants in the RSTM via FILE 5, then it is possible to specify them in this

subroutine.

Subroutine FINAL

This subroutine writes the results at the last specified x-station into FILE 7 using the same data format as FILE 6. This is to enable FILE 7 to serve as initial starting data for further calculations downstream.

Subroutine GEOMET

The grid geometrical coefficients are evaluated with the y-direction coefficients stored in DY(I), YCV(I), YACV(I), CYA(I), CYC(I) and the z-direction coefficients stored in DZ(J), ZCV(J), ZACV(J), CZA(J), CZC(J). The respective formulae are given in Appendix A3.

Subroutine GUESS

The guessed pressure field POLD(I,J) for yz-mom in the first outer pressure loop iteration is established using the calculated pressures PNEW(I,J) from the previous x-station. POLD(I,J) is also used in PERR (see §5.4) to determine the maximum absolute pressure error in the outer pressure loop. For turbulent flows, the latest calculated $\overline{u_i u_j}$ and ϵ values are stored in UUOLD(I,J), VVOLD(I,J), WWOLD(I,J), UVOLD(I,J), UWOLD(I,J), VWOLD(I,J) & EPLOLD(I,J) for the sole purpose of checking the outer RSTM loop error. For laminar flows, the turbulent quantities UU(I,J), VV(I,J), WW(I,J),

UW(I,J), UW(I,J), VW(I,J) & EPL(I,J) are set to zero. This is just a precautionary measure, because some computers do not initialise to zero variables that do not have values assigned to them.

Subroutine INIT

INIT reads the following parameters from FILE 5: RE, DPXEXT,KNFLOW, and then computes the reciprocal of the Reynolds number RRE; the RSTM empirical constants C1, C2, CS, CE, CE1, CE1, C1PRI, C2PRI; the x-station locations X(N); the convergence tolerances TOLPI, TOLPO, TOLRI, TOLRO; the relaxation factors OMEGA, OG, RFRS, RFEPL; the permissible number of iterations IPERPI, IPERPO, IPERRI, IPERRO and the flags ICON, IPSI. Next the grid structure from FILE 5 is read and stored in Y(I) and Z(J).

The initial plane mean flow field profiles are read from FILE 6 into UN(I,J), VN(I,J), WN(I,J), PNEW(I,J). If the flag KNFLOW is equal to 1 (i.e. turbulent flows) the Reynolds-stresses and ϵ are read from FILE 6 and stored in UU(I,J), VV(I,J), WW(I,J), UV(I,J), UW(I,J), VW(I,J), EPL(I,J). Basically, FILE 5 and FILE 6 contain the test case specifications and initial BCs. The final part of INIT writes the parameters to FILE 9 (the status file) in a more readable format: a sample can be found in Table 5.1.

Subroutine MID

The locations of $\overline{u_i u_j}$ and ϵ are staggered (like the pressures) in the yz-plane as well as between x-stations. Therefore, MID

computes the rates of strain and mean velocities at these staggered locations at $x^{n+1/2}$. The mean velocity values used in this subroutine are the latest converged values. The strain rates are stored in DUDX(I,J), DVDX(I,J), DWDX(I,J), DUDY(I,J), DVDY(I,J), DWDY(I,J), DUDZ(I,J), DVDZ(I,J) & DWDZ(I,J) and the mean velocities in UNHALF(I,J), VNHALF(I,J) & WNHALF(I,J).

Subroutine PERR

This subroutine computes the maximum error PDIFFV (= POLD(I,J) - PNEW(I,J)) in the outer pressure iteration loop and assigns the absolute value of PDIFFV to ERRPO. PDIFFV and the grid position where this maximum error occurs is written to FILE 9.

Subroutine PRINT

The calculated mean velocities and, for turbulent flows, $\overline{u_i u_j}$ & ϵ at each x-station (including the initial plane) are written into FILE 8. FILE 8 will, therefore, contain a large amount of data and if it is necessary to reduce disk storage space, PRINT can be switch off. It is then recommended to carry out the computations in stages with each stage ending at the x-station of interest and the data can then be retrieved from FILE 7.

Subroutine RSERRO

The maximum absolute error in each $\overline{u_i u_j}$ and ϵ array is evaluated using the function ERR. The UU(I,J) array is the first to be

evaluated and ERR returns the maximum absolute error as DIFF (for example in UU(I,J), $\text{DIFF} = |\text{UU}(\text{I},\text{J}) - \text{UUOLD}(\text{I},\text{J})|$). DIFF and the grid location where it occurs is then written to FILE 9. Using similar logic, the other arrays are dealt with sequentially. The largest of these errors is stored as ERRRO and passed back to the main program to check for outer RSTM loop convergence. The last part of RSERRO overwrites UUOLD(I,J), VVOLD(I,J), WWOLD(I,J), UVOLD(I,J), UWOLD(I,J), VWOLD(I,J) & EPLOLD(I,J) with the latest values of UU(I,J), VV(I,J), WW(I,J), UV(I,J), UW(I,J), VW(I,J) & EPL(I,J) respectively.

Subroutine SOLV2

This is the 2x2 block tridiagonal matrix solution algorithm (see description in §5.2 under the subroutine CROSFL).

Subroutine TRDIAG

This is the block tridiagonal matrix solution algorithm (see description in §5.2 under the subroutine XMOM).

Subroutine UPDATE

UPDATE is called at the end of each x-step to overwrite upstream variables with values from the present x^{n+1} station, because the x^{n+1} station will become the upstream station in the next x-step calculation. UN(I,J), VN(I,J) & WN(I,J) which contain values at the upstream x-station are overwritten with the newly calculated

x^{n+1} values, for example, $UN(I,J) = UN(I,J) + DU(I,J)$. If $KNFLOW=1$ then $UUOLD(I,J)$, $VVOLD(I,J)$, $WWOLD(I,J)$, $UVOLD(I,J)$, $UWOLD(I,J)$, $VWOLD(I,J)$ & $EPLOLD(I,J)$ are also updated.

In **RSMUV**, u_τ is calculated using mean velocities (obtained by linear interpolation) at $(2\frac{1}{2}, J+\frac{1}{2})$ (see §5.3). Here a more accurate u_τ is computed, based on $(2\frac{1}{2}, J+\frac{1}{2})$ mean velocities obtained from a Lagrangian interpolation. The 3-point Lagrangian interpolation is contained in function **UCOR**. Subsequently, the u_τ values are used to correct $UN(2,J)$ & $WN(2,J)$ to ensure that the mean velocities are consistent with the log-law.

Table 5.1

***** TURBULENT FLOW *****

REYNOLDS NO = .740000E+05 EXT. PRES. GRAD. = .0000

CONTINUITY EQUATION DIFFERENCED FULLY IMPLICITLY

Y GRID LOCATION

.00000E+00	.60000E-01	.12300E+00	.18900E+00	.25800E+00
.33000E+00	.40500E+00	.48400E+00	.56700E+00	.65400E+00
.74500E+00	.84000E+00	.94000E+00	.10450E+01	.11550E+01
.12700E+01	.13900E+01	.15160E+01	.16480E+01	.17860E+01
.19300E+01	.21000E+01			

Z GRID LOCATION

.00000E+00	.10000E+00	.20000E+00	.30000E+00	.40000E+00
.50000E+00	.60000E+00	.70000E+00	.80000E+00	.90000E+00
.10000E+01	.11000E+01			

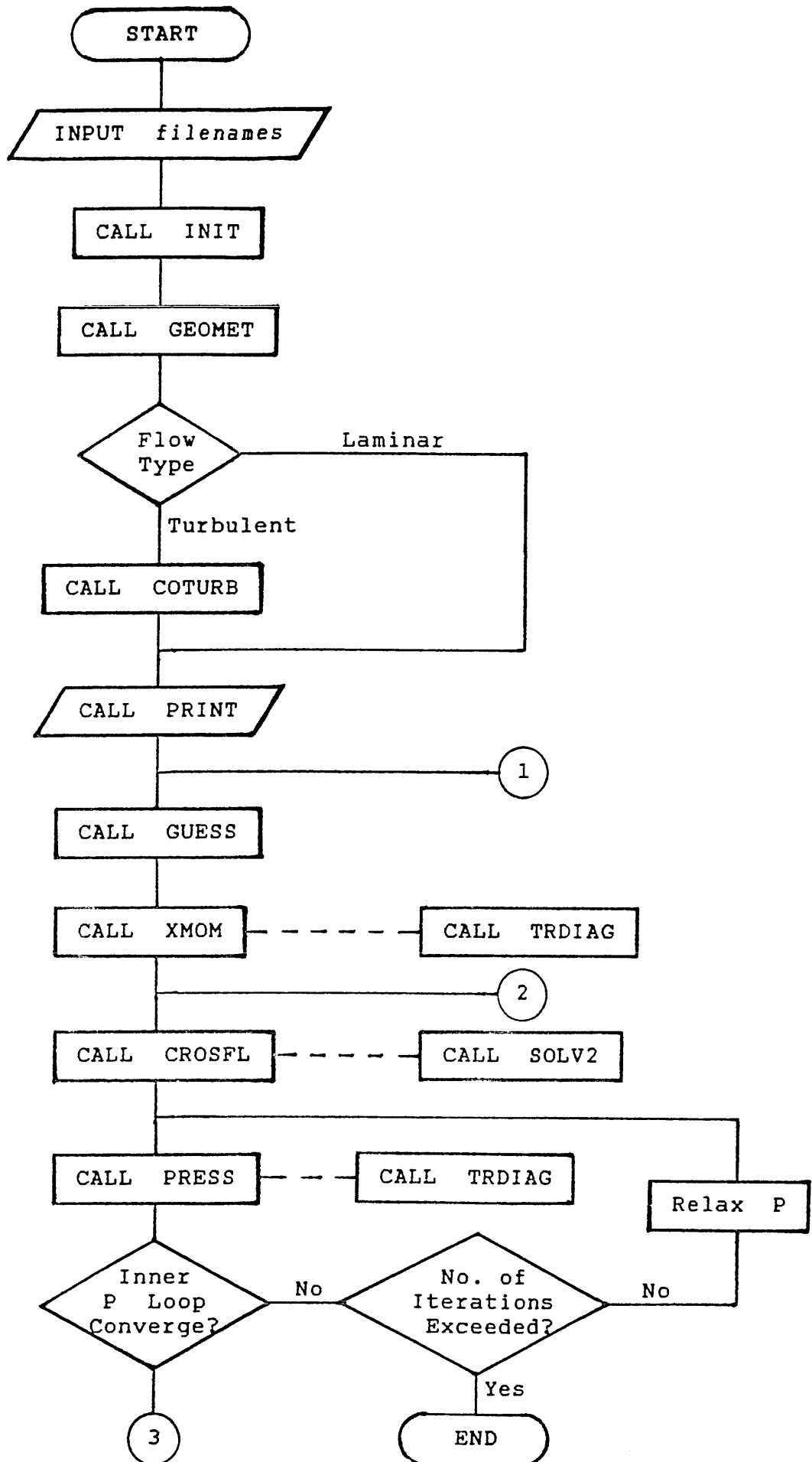
THERE ARE 21 * 11 MEMORY CLUSTERS IN EACH CROSS SECTION OF
COMPUTATION DOMAIN AND 61 STREAMWISE STATIONS.

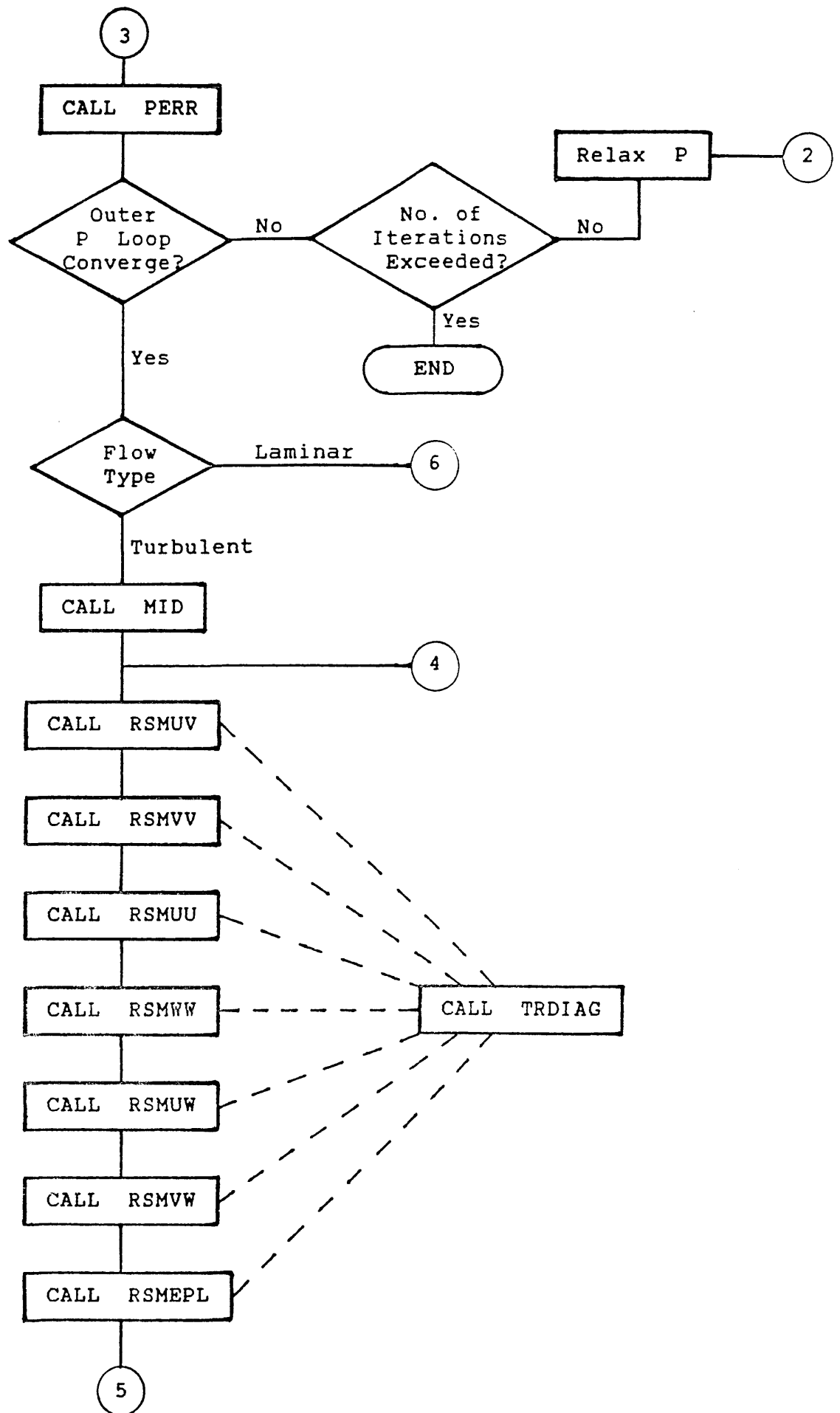
PRESSURE CALCULATIONS (INNER , OUTER LOOP)

RELAXATION FACTOR	=	1.00	1.90
TOLERANCES	=	.10000E-07	.10000E-07
NO. OF ITERATIONS PERMITTED	=	250	100

RSM CALCULATIONS (INNER , OUTER LOOP)

RELAXATION FACTOR	=	1.00	1.00
TOLERANCES	=	.10000E-07	.10000E-07
NO. OF ITERATIONS PERMITTED	=	100	20





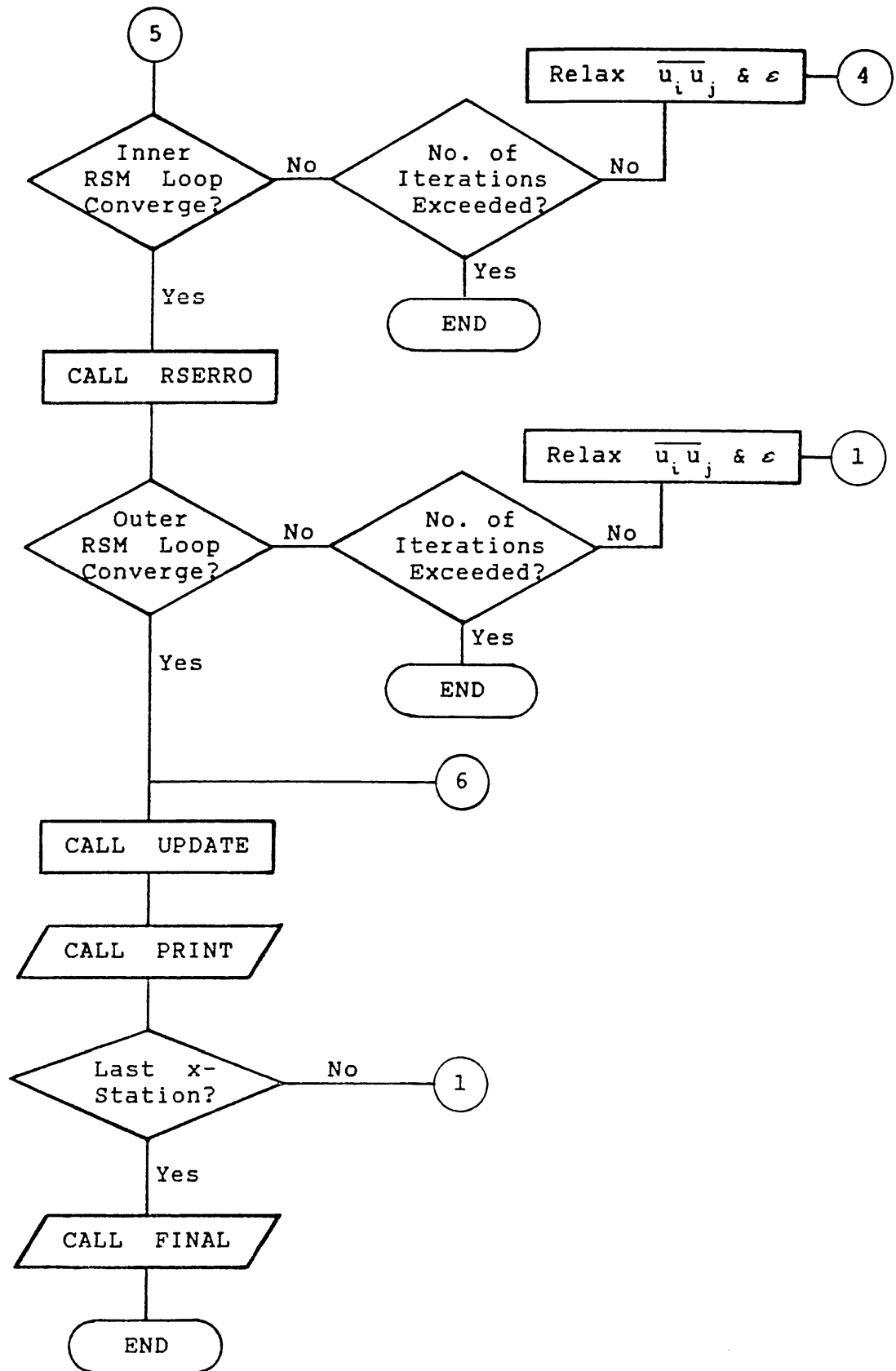


Fig.5.1 Flowchart Of PSTAG.

6 LAMINAR FLOW COMPUTATIONS

6.1 Fully - Developed Flow In A Duct

6.1.1 Problem Background

In laminar flow through a straight duct of rectangular cross-section, boundary layers grow from all four walls until they merge and much further downstream the flow becomes fully-developed. The primary flow velocity field then remains unchanged for all subsequent streamwise positions, that is, streamlines become parallel to the primary flow direction because there is no secondary flow. Since the primary velocity is the only non-zero velocity component, this flow is suitable for evaluating the performance of the x-mom solution. The flow downstream is driven by a constant favourable streamwise pressure gradient and at each crossflow plane, the pressure is constant.

Fully-developed laminar internal flows through different cross-sections such as rectangle (inclusive of square), equilateral triangle and circle have analytic series solutions, all of which can be found in Rouse (1959, pp.219-220). The governing equation of motion is simply the streamwise component momentum equation Eq.2.10a (neglecting convection), since the crossflow momentum and continuity equations are trivial.

$$\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \text{Re} \frac{dP}{dx} \quad \text{Eq.6.1}$$

For a rectangular cross-section of sides a and b as shown in

Fig.6.1, Eq.6.1 is satisfied by the following Fourier equation.

$$U = \frac{\text{Re}}{2} \frac{dP}{dx} z(z-1) + \sum_{m=1}^{\infty} \sin m\pi z \left(A_m \cosh m\pi y + B_m \sinh m\pi y \right)$$

Eq.6.2

where m is an integer, A_m and B_m are constant coefficients depending on the ratio $n (= \frac{b}{a})$. The reference length is the longer side of the duct (i.e. a) and the reference velocity is the duct centreline primary velocity.

In order to determine the coefficient A_m , the no-slip BC along the wall at $y=0$ is used, giving

$$0 = \frac{\text{Re}}{2} \frac{dP}{dx} z(z-1) + \sum_{m=1}^{\infty} A_m \sin m\pi z$$

A_m can then be written in the following form after using Fourier's formula.

$$A_m = - \frac{2}{m^3 \pi^3} \frac{\text{Re}}{a^3} \frac{dP}{dx} (\cos m\pi - 1)$$

Eq.6.3

Using the other no-slip BC at $y=b$ or in normalised form $y=n$ (since $n = \frac{b}{a}$), the coefficient B_m can be found.

$$B_m = - \frac{A_m (\cosh mn\pi - 1)}{\sinh mn\pi}$$

Eq.6.4

Eq.6.2, together with Eq.6.3 for A_m and Eq.6.4 for B_m can totally describe the U velocity contours of a fully-developed laminar duct flow. The normalised U contours are dependent only on the ratio n but not on $\frac{dP}{dx}$.

6.1.2 Initial & Boundary Conditions

The flow field being symmetrical about both the duct centrelines can be computed economically one quadrant at a time: here the lower left hand quadrant was chosen (arbitrarily, of course). The rectangular duct with $n=0.5$ has dimensions $a=0.041\text{m}$ ($= L_{\text{ref}}$) and $b=0.0205\text{m}$. (Note: Unless dimension units are given, all values quoted are in normalised forms.) The uniform mesh has 22×22 nodes with $\Delta Y=0.0122$ and $\Delta Z=0.0244$. The yz -plane BCs as depicted in Fig.6.2 are as follow.

Along the walls - 1-2 & 1-4:

$$U = V = W = 0$$

$$\frac{\partial P}{\partial n} = 0 \quad (\text{implicit BC})$$

Along the duct centrelines - 2-3 & 3-4:

$$\frac{\partial U}{\partial n} = \frac{\partial V}{\partial n} = \frac{\partial W}{\partial n} = 0$$

$$\frac{\partial P}{\partial n} = 0$$

The initial U velocity distribution was established by using the analytic solutions described in §6.1.1, and is plotted as contours in Fig.6.3. The initial plane crossflow velocities and pressure were set to zero.

6.1.3 Results & Discussion

This 3-D test case is ideally suited for investigating the numerical solution in handling the x-mom part of PSTAG because the secondary velocities are zero, hence rendering the yz-mom and pressure equation solutions unnecessary in principle. However, these crossflow solutions were included in the calculation procedure to act as an indicator of the amount of discretization errors (assuming negligible computer roundoff errors) induced by x-mom which could possibly cause numerical instability and/or inaccuracy.

Calculations were performed for three different streamwise pressure gradients of $\frac{dP}{dx} = -0.5, -0.05, -0.00005$. The corresponding duct centreline velocity (or U_{ref}) and Re are given below.

$\frac{dP}{dx}$	-0.5	-0.05	-5×10^{-5}
U_{ref} (m/s)	0.0266	0.266	266
Re	70	700	7×10^5

For convenience, in the discussion to follow, these three cases will be referred to by their Re value.

Although the flows with Re=70 and Re=700 can be laminar in real life, it is highly unlikely that with $Re=7 \times 10^5$ the flow could remain laminar. One must bear in mind that since the objective of this test case is to compare the PSTAG solution of x-mom with the available analytic solution, the inclusion of the physically

unrealistic $Re=7 \times 10^5$ flow would enable x-mom to be tested over a wider range of Re . All computations started in the fully-developed region of the flow and proceeded to a downstream distance of $10L_{ref}$ in 20 x-steps (i.e. $\Delta x=0.5$). The continuity equation was differenced in fully-implicit fashion and pressure convergence tolerance (in terms of the absolute error) for both the inner and outer pressure loops were set to 10^{-8} .

In Fig.6.4, the predicted U velocity contours for $Re=700$ at $x/L_{ref}=5,10$ and the analytic solution are shown. The agreement between the predicted and analytic solutions is good as indicated by the coincidence of the three sets of U contours. A more detailed comparison is made in Fig.6.5 by plotting the U velocity profiles against y at arbitrarily chosen z locations of $z=0.1,0.22,0.34$; again the agreement is good. Essentially, the results depicted are satisfactory, so in order to reaffirm this point, the maximum U velocity error between the computed and analytic solution at each x -station for the three different Re flows is plotted in Fig.6.6. The maximum calculated U errors are $O(0.0001)$ or, conservatively, about 0.01%. With increase in Re , the size of the errors become smaller and with $Re=7 \times 10^5$, the errors are almost zero.

In all three computation runs, the calculated crossflow velocities were found to be of $O(10^{-6})$, which can be considered as sufficiently close to the theoretical value of zero. Non-zero secondary velocities are inevitable because of computer round-off errors and discretization errors. These non-zero V and W

inevitably generated non-zero $\frac{\partial P}{\partial y}$ and $\frac{\partial P}{\partial z}$, whose values are again, correctly, very close to zero. Though these secondary flows are very weak, they nevertheless introduce non-zero viscous diffusion terms in x-mom and therefore could cause x-mom to depart from Eq.6.1. However, when calculations were made with y-mom "switched off" (simply by resetting V to zero at the end of each yz-mom calculation), similar U results as well as convergence rate were obtained. This shows that the x-mom calculations are independent of the yz-mom and pressure equation solutions.

Roache (1982, pp.36-53) analysed the numerical stability criteria for a 2-D unsteady problem in terms of the Courant number and diffusion restriction. The unsteady 2-D flow is analogous to the present 3-D parabolic flow since the time-stepping in the former can be said to be equivalent to the spatial-marching in the latter. The Courant number and diffusion restriction stability criteria are respectively,

$$C_y + C_z \leq 1 \quad \text{Eq.6.5}$$

where $C_y = \frac{V \Delta x}{\Delta Y}$, $C_z = \frac{W \Delta x}{\Delta Z}$, and

$$d_y + d_z \leq \frac{1}{2} \quad \text{Eq.6.6}$$

where $d_y = \frac{\Delta x}{\text{Re } \Delta Y^2}$ and $d_z = \frac{\Delta x}{\text{Re } \Delta Z^2}$.

These stability criteria were established for an explicit scheme for linear differential equations. Thus, they can only serve as a guideline for implicit schemes involving non-linear equations. Eq.6.5 and Eq.6.6 can be combined to yield the more familiar

stability criteria based on the maximum cell Reynolds number Re_{CELL} or the equivalent Peclet number.

$$Re_{CELL,y} , Re_{CELL,z} \leq 2 \quad \text{Eq.6.7}$$

where $Re_{CELL,y} = |V|Re\Delta Z$ and $Re_{CELL,z} = |W|Re\Delta Z$. Generally, in 3-D parabolic problems Re_{CELL} dictates the crossflow plane grid resolution which, once determined, would fix the x-step limit through the diffusion restriction criterion Eq.6.6.

In this test case, with V and W essentially zero, Re_{CELL} will always satisfy Eq.6.7. In other words, the size of ΔY and ΔZ is not crucial but, of course, has to be within physical realism. With $\Delta Y=0.0122$ and $\Delta Z=0.0244$, the most restrictive or maximum allowable x-step is for the case of $Re=70$ where $\Delta x_{LIMIT}=4.16$. Therefore, the use of $\Delta x=0.5$ in all three flows is well within the limit; this partially explains the rapid convergence obtained in all the calculations. Since Δx is the same for all three flows, then from the diffusion restriction criterion, the accuracy of the results will reduce with decrease in Re which is exactly the trend shown in Fig.6.6

6.1.4 Concluding Remarks

The predicted results of the fully-developed laminar flow through a duct has been shown to compare favourably with the analytic solution. The maximum computed streamwise velocity errors being of less than 0.01%. This, at least, demonstrates that the x-mom solution in PSTAG is sufficiently accurate as well as numerically

stable. The inclusion of yz-mom and the pressure equation into the solution, though redundant, resulted in values of the secondary velocities and crossflow pressure gradients that are very nearly zero, correctly showing little or no effects on the primary velocity calculations. This can also be interpreted to be an indication that both yz-mom and the pressure equation solutions are behaving properly.

6.2 Taylor - Green Vortex

6.2.1 Problem Background

Having investigated the x-mom solution in §6.1, it is essential to have a test case to evaluate only the yz-mom and pressure equation. For this purpose, one can use the Taylor-Green vortices described in Taylor (1971). Taylor-Green vortices represent a system of eddies each rotating in the opposite direction to its four surrounding neighbours as shown in Fig.6.7. Each vortex is 2-D in space and decays with time but has the ability to conserve both the size of the square area in which it is rotating as well as its spatial position. Taylor (1971) obtained the analytical solution for this type of laminar vortices by using stream functions to solve the streamwise vorticity equation.

$$V = -e^{-2\nu Kt} \cos y \sin z \quad \text{Eq.6.8}$$

$$W = e^{-2\nu Kt} \sin y \cos z \quad \text{Eq.6.9}$$

The reference length, $L_{\text{ref}} = \frac{d}{\pi}$, where d is the length of the side of a square containing one vortex and the reference velocity is

the primary flow velocity (to be defined later). K is a constant equal to $\left(\frac{\pi}{d}\right)^2$ or L_{ref}^{-2} .

To suit the present study, Eq.6.8 and Eq.6.9 can be transformed from a 2-D time-dependent coordinate to a spatial 3-D coordinate through a Galilean transformation, $t = \frac{x}{U}$. The frame of reference is changed so that the Taylor-Green vortex now decays as it is being convected at a uniform streamwise velocity in the x -direction. Substitution of the Galilean transformation into Eq.6.8 and Eq.6.9 gives

$$V = -e^{-\frac{2x}{Re}} \cos y \sin z \quad \text{Eq.6.10}$$

$$W = e^{-\frac{2x}{Re}} \sin y \cos z \quad \text{Eq.6.11}$$

where $Re = \frac{Ud}{\nu}$. The strength of the vortex is proportional to Re and can be specified quite arbitrarily. Equations Eq.6.10 and Eq.6.11, therefore, constitute the analytic solution of a 3-D Taylor-Green vortex. Since the vortex strength can be specified through Re , for convenience, the uniform primary velocity is set to $U \equiv U_{ref} = 1.0\text{m/s}$. The Taylor-Green has "strong" secondary flow of the same order of magnitude as the primary flow. At the initial plane (i.e. $x=0$), the maximum values of V and W are equal to U_{ref} .

6.2.2 Initial & Boundary Conditions

The computational domain in the yz -plane is bounded on all four sides by zero-shear boundaries and extends from $\frac{\pi}{2} \leq y, z \leq \frac{3}{2}\pi$. The mesh is uniform with 21×21 nodes, giving $\Delta Y = \Delta Z = 0.157$. The BCs

in the yz-plane are shown in Fig.6.8.

Along the lower horizontal boundary 1-2:

$$\frac{\partial U}{\partial y} = V = 0$$

$$\frac{\partial W}{\partial y} = \frac{W_{1,J}}{\Delta Y} (\cos \Delta Y - 1)$$

$$\frac{\partial P}{\partial y} = 0 \quad (\text{implicit BC})$$

Along the upper horizontal boundary 3-4:

$$\frac{\partial U}{\partial y} = V = 0$$

$$\frac{\partial W}{\partial y} = \frac{W_{I_{\max}+1,J}}{\Delta Y} (1 - \cos \Delta Y)$$

$$\frac{\partial P}{\partial y} = 0 \quad (\text{implicit BC})$$

Along the left lateral boundary 1-4:

$$\frac{\partial U}{\partial z} = W = 0$$

$$\frac{\partial V}{\partial z} = \frac{V_{1,1}}{\Delta Z} (\cos \Delta Z - 1)$$

$$\frac{\partial P}{\partial z} = 0 \quad (\text{implicit BC})$$

Along the right lateral boundary 2-3:

$$\frac{\partial U}{\partial z} = W = 0$$

$$\frac{\partial V}{\partial z} = \frac{V_{1, J_{\max}+1}}{\Delta Z} (1 - \cos \Delta Z)$$

$$\frac{\partial P}{\partial z} = 0 \quad (\text{implicit BC})$$

The zero Neumann pressure BCs on all four boundaries are implicit because the necessary no-slip velocity BCs, that is, $V=0$ along 1-2 & 3-4 and $W=0$ along 2-3 & 1-4, are present. Attention is now turned to the non-zero Neumann BCs for V along 2-3 & 1-4 and W along 1-2 & 3-4, which under most circumstances are not always possible to specify accurately or easily. However, in this particular case, we do know the solution beforehand and as such can accurately determine the non-zero Neumann BCs (see Appendix A4 for the derivation). This method is acceptable since, in practice, one must have some prior knowledge of the flow to be predicted and this, of course, includes the BCs. Although the non-zero Neumann BCs indicated above do not look straightforward, they can easily be re-arranged into a more familiar form. For example, $\frac{\partial W}{\partial y}$ along 1-2 can be written as,

$$\frac{W_{2, J}}{W_{1, J}} = \cos \Delta Y$$

which can be implemented easily. To employ these non-zero Neumann BCs in a non-uniform grid, the appropriate grid spacing (i.e. ΔY & ΔZ) must be used, for details see Appendix A4.

An alternative to non-zero Neumann BC is to use the analytic solution to generate Dirichlet BCs at each streamwise station. This is obviously more cumbersome than the Neumann BCs which are

independent of x . Besides, it is a less realistic approach because one rarely knows non-zero Dirichlet BCs beforehand. It should be mentioned that the staggered mesh can, in this case, employ with ease the more convenient zero Neumann BCs for V along 2-3 & 4-1 and for W along 1-2 & 3-4.

On the initial plane, uniform fields of $U=1.0$ and zero pressure were specified with V and W calculated from Eq.6.10 and Eq.6.11 respectively. Although calculations were performed for different vortex strength (i.e. different Re values), the initial secondary velocities shown in Fig.6.9 at $x=0$ are, of course, identical.

6.2.3 Results & Discussion

Computation of the Taylor-Green vortex serves as a useful test case to check the performance of the yz -mom solution, for two main reasons. Firstly, since the primary flow field is uniform at every crossflow plane and remains unchanged as the vortex is convected downstream, therefore, only the yz -mom and pressure equation solutions are important, that is, x -mom calculation is ideally neither necessary nor effective. Secondly, secondary flow is of same order of magnitude as U , therefore, any errors in calculating V and W will be large enough to be reflected in U . Hence the inclusion of x -mom solution. It must be noted that the streamwise pressure gradient was set to zero, so the pressure field calculation actually converged to an arbitrary constant. To verify this point, calculations were also made whereby the whole pressure field at the beginning of each x -step was reset to zero (as

adopted by Dedoussis, 1988). The pressure results obtained with and without resetting the pressure field are identical, hence, it suffices to present the results for the latter case.

To enhance accuracy, the continuity equation was differenced by the Crank-Nicolson method, which though more accurate is also more unstable than the fully-implicit method. Convergence tolerance for both inner and outer pressure loops was set to 10^{-5} of the absolute pressure error. Different calculations were performed for $Re=1$ ($\Delta x=0.05, 0.1$), $Re=20$ ($\Delta x=0.1, 0.2, 0.4$), $Re=40$ ($\Delta x=0.1$) and $Re=80$ ($\Delta x=0.05, 0.1$). In all runs the calculations marched 30 x-steps downstream.

The diffusion restriction criterion and Re_{CELL} values shown below are calculated from Eq.6.6 and 6.7 respectively.

Re	Re_{CELL}	Δx_{LIMIT}
1	0.16	0.0062
20	3.14	0.12
40	6.28	0.25
80	12.56	0.49

Re_{CELL} ($= Re_{CELL,y} = Re_{CELL,z}$ for this particular case) quoted above are based on the initial plane values and as the Taylor-Green vortex is convected downstream, the value of Re_{CELL} would decrease.

The magnitude of the crossflow velocity vectors for $Re=1$ at $x=1$

(using $\Delta x=0.1$) in Fig.6.10 clearly indicates a visible reduction in the vortex strength (by appropriately 7.4 times). The velocity profiles of V and W in the z-direction are plotted in Fig.6.11 and Fig.6.12 respectively. Both plots show very satisfactory agreement between the predicted and analytic values. In order to analyse the results more closely, the maximum error between the computed and analytic velocity values were evaluated. Here, only maximum V errors are presented, the W velocity errors are identical.

Fig.6.13 shows the maximum V errors for $Re=20$ at three different Δx values. The larger the Δx , the larger are the errors. This trend is consistent with the diffusion restriction criterion and is more pronounced in a Crank-Nicolson differencing scheme. Another point to note is that the solution still converged for $\Delta x=0.4$ (which exceeded the Δx_{LIMIT} by almost 4 times) albeit with much larger errors. At the initial stages, the errors with $\Delta x=0.4$ are approximately 4 times greater than the errors with $\Delta x=0.1$. As expected, the errors decay with streamwise distance because Re_{CELL} also decreases.

The effects of Re_{CELL} can be seen in Fig.6.14 for $Re=1,20,40,80$ using the same x-step of $\Delta x=0.1$. A higher Re_{CELL} exhibits bigger errors (except for $Re=1$, whose larger errors at the initial stages can be attributed to the fact that the diffusion limit of $\Delta x_{LIMIT}=0.0062$ has been exceeded by a factor of 16). Although the initial errors of $Re=1$ are comparatively large, they decay very rapidly between $x=0$ and $x=1$ because the Re_{CELL} decreases just as rapidly by approximately 7.4 times. For $Re=20$, the errors peak at

around $x=2.5$ (or $Re_{CELL}=2.44$) and then begin to decrease. $Re=40$ shows a similar trend in that the rate of growth of the errors (i.e. the gradient) is also decreasing. However, the errors have yet to peak, possibly at $x=3$ the $Re_{CELL}=5.4$ is still quite large. In the case of $Re=80$, the calculations failed to converge after 27 x-steps, this is because $Re_{CELL}=11.7$ is still about 6 times the permissible value even though Δx is well within its $\Delta x_{LIMIT}=0.5$. However, by reducing Δx to 0.05 the solution converges satisfactorily. This simply implies that numerical instability in 3-D parabolic problems or analogous forms can be partially remedied by x-step reduction to compensate for coarse mesh.

In Fig.6.15, at the initial stages of the computations, smaller Δx gives smaller errors even though the Re_{CELL} value is the largest for the smallest Δx and smallest for the largest Δx . However, the growth of the errors seems to be dictated more by Re_{CELL} , in other words, the larger the Re_{CELL} the greater the rate of growth. This confirms the general norm that either small x-step or grid refinement can be employed to reduce numerical errors, though each by itself may not be a sufficient condition. Fig.6.16 is intended to demonstrate the degree of high accuracy achievable by PSTAG with a low Re_{CELL} and small Δx ($Re=1$ and $\Delta x=0.05$). By simply reducing Δx from 0.1 to 0.05, the factor exceeding the Δx_{LIMIT} is reduced from 16 to 8, this resulted in a large reduction in the initial errors.

Centreline pressures are plotted in Fig.6.17 for $Re=20$ at various x-steps, though there are some small discrepancies between the

results (due to the effect of different Δx), the general trends are in agreement with one another. The important point here is to note that the pressure is non-oscillatory: the results are therefore considered satisfactory. Further plots of centreline pressures at different Re are shown in Fig.6.18. Results are as expected because with higher Re , or equivalently a stronger vortex, one expects the pressure to be lower (or $\frac{dP}{dx} \propto \frac{1}{Re}$). With all four vortices, the centreline pressures correctly increase with increase streamwise distance and will tend asymptotically to zero as the vortices "diffuse". This is more clearly depicted in the case of $Re=1$ where the secondary flow magnitude at $x=2$ is only 1.8% of the value at the initial plane.

As a comparison with Dedoussis (1988) numerical scheme on a staggered grid, which requires an upper limit of 250 inner pressure loop iterations for convergence, the present scheme requires only 150. One possible explanation is the elimination of the yz-mom diagonal sweep in PSTAG coupled with a more desirable symmetrical computational stencil. Though no computing times were recorded, it is quite evident that PSTAG is quicker than a "similar" staggered grid scheme. A point to be confirmed in §6.3 and §6.4 where computing times were available.

It was found that for those calculations which did not converge, for example $Re=40$ with $\Delta x=0.4$ and $Re=80$ with $\Delta x=0.1$, by setting the primary flow velocity to its theoretical value of 1.0 at each x -step, the solution not only converged but did so very rapidly. It must be added that the errors in U were, in all cases, no more

than 0.01%. From here, it can be deduced that yz-mom errors were indeed transmitted (as stated before) to the x-mom computation and back again, subsequently, these accumulated errors were large enough to cause divergence because of inability to satisfy the constraint $\frac{\partial U}{\partial x} = 0$ imposed by this flow (see Dedoussis, 1988). By following the standard practice of reducing the relaxation factor in the pressure equation, the non-converging behaviour was suppressed. However, no further attempt was made to optimise the pressure relaxation factor for this flow since the intention of the present study is to develop an algorithm for general applications.

6.2.4 Concluding Remarks

The Taylor-Green vortex calculations have shown the crossflow algorithm of PSTAG to be both numerically stable and accurate, just as the x-mom solution is. This at least puts PSTAG on par with the more conventional staggered grid formulations. The stability criteria established for explicit linear systems were found to be not so restrictive for implicit methods. For example, convergence was obtained when the nominal Re_{CELL} limit was exceeded by about 6 times (in $Re=80$ & $\Delta x=0.05$) and the Δx diffusion limit was exceeded by about 16 times (in $Re=1$ & $\Delta x=0.1$). These values are by no means the threshold limit for stability; in other words, no value may be cited as the stability criteria for implicit schemes.

6.3 Boundary Layer On A Flat Plate

6.3.1 Problem Background

The laminar boundary layer on a flat plate can be said to be a standard prerequisite test case when one is evaluating a newly developed algorithm for slender flows. This is a very well documented flow with velocity profiles well prescribed by the Blasius solution, details can be found in many standard text books on fluid dynamics (e.g. Schlichting, 1979, pp.135-140).

The main objective of this test case (and the next one in §6.4) is to enable a direct comparison between PSTAG and a staggered grid formulation. For this purpose the staggered grid numerical scheme IV3 (Imbedded-Vortex 3-D) developed by Dedoussis (1988) is used. As mentioned in chapter 4, the basic solution structure for laminar flow (only) of both PSTAG and IV3 has many similar features. The calculations were performed on an 80286/80287 8 MHz microcomputer in a Xenix environment. Since the original version of IV3 was developed for the Cyber 855 mainframe, care was taken to ensure that similar machine accuracy (of 8-byte word) was achieved when adapting it to the microcomputer.

6.3.2 Initial & Boundary Conditions

The computational domain, shown in Fig.6.19, used a 19x19 non-uniform mesh. In the y-direction, the grid lines are non-uniformly distributed with $\Delta Y_1 = 0.015$ and $k=1.1$. The grid lines in the z-direction are uniformly spaced with $\Delta Z=1.0$. The large number of spanwise nodes is actually unnecessary for a 2-D flow computation

(the minimum number of spanwise nodes is 3, so that second-order derivatives can be evaluated). But due to the rather inflexible program logic of IV3 (see details in §4.6.2) an equal number of grid nodes along the y- and z- axes is preferable. This will also enable one to check on the behaviour of PSTAG and IV3 in maintaining 2-D flows 2-D.

The BCs in the crossflow plane as defined in Fig.6.20 are as follow.

Along the wall - 1-2:

$$U = V = W = 0$$

$$\frac{\partial P}{\partial y} = 0 \text{ (implicit)}$$

Along the lateral boundaries - 2-3 & 1-4:

$$\frac{\partial U}{\partial z} = \frac{\partial V}{\partial z} = \frac{\partial W}{\partial z} = \frac{\partial P}{\partial z} = 0$$

Along the free stream boundary - 3-4:

$$U = U_{\infty} = 1.0$$

$$\frac{\partial V}{\partial y} = \frac{\partial W}{\partial y} = 0$$

$$P = P_{\infty} = 0$$

The initial U & V velocity profiles were obtained from the tabulated Blasius solution given in Schlichting (1979, p.139). The

(arbitrarily chosen) reference length L_{ref} is $1.5\delta_0$ (where δ_0 is the initial plane boundary layer thickness) and U_{ref} is the free stream velocity, thus giving a Re value of 1890. The W velocity and pressure fields in the initial plane were set to zero. Due to the small magnitude of the crossflow velocities (or V), numerical instabilities associated with Re_{CELL} can be discounted. The calculations marched $15L_{ref}$ downstream in 30 x-steps of $\Delta x=0.5$ (this is equal to $22.5\delta_0$). For enhanced accuracy, the continuity equation was differenced using the Crank-Nicolson method, with the inner and outer pressure loop convergence tolerance set to 10^{-8} of the absolute pressure error.

6.3.3 Results & Discussion

The microcomputer CPU times were 10084s for PSTAG and 15454s for IV3; that is, IV3 took approximately 50% more computing time. This improvement in computing speed by PSTAG can be largely attributed to the removal of the diagonal sweep in yz-mom because the basic solution procedure is essentially similar in PSTAG and IV3. Of course, due to the different grid arrangements, there are differences between PSTAG and IV3 in the form of the discretized laminar equations and implementation of BCs.

The U profile predicted by PSTAG, shown in Fig.6.21, agrees very well with the Blasius solution. Good agreement is also obtained by PSTAG for C_f (in Fig.6.22), δ^* (in Fig.6.23) and θ (in Fig.6.24). In both schemes, the calculated value of W, being of the order of 10^{-12} , is acceptably accurate. A check of the predicted results on

the crossflow plane indicates that 2-D flow has been adequately maintained.

In almost all calculation methods, the predictions at the initial stages exhibit some oscillations and it is desirable to have as short a "settling-down" period as possible. The values that are usually most sensitive to oscillations are those near the wall. Hence, in Fig.6.25 the streamwise change in the pressure value at the staggered node next to the wall (i.e at $I=1\frac{1}{2}$) is plotted. The results of PSTAG are much less oscillatory than those of IV3. At around $x=3$, the amplitude of oscillation of IV3 is still large whereas the oscillations in PSTAG seem to have been almost damped out. In fact, this wiggly behaviour of IV3 can still be observed up till $x=12$. In the near-wall V velocity plot in Fig.6.26, one can see an even more pronounced oscillatory behaviour of IV3, persisting up till $x=8$, by which time the wiggles in PSTAG have almost totally subsided. Note also the presence of unrealistic negative V values in IV3 at the initial streamwise region.

The outer pressure iteration loop, in general, requires about five iterations per streamwise step. Since the yz -mom solution is itself non-iterative, it constitutes only a relatively small percentage of the total computing time, the main bulk belonging to the inner pressure loop iterations. The calculated secondary velocities and pressure fields are strongly coupled, so that faster convergence will be attained when a good estimate of one field is used in predicting the other. This is clearly shown in the V velocity results in Fig.6.26 where the less oscillatory

PSTAG behaviour also corresponds to less wiggles in the pressure. Therefore, the predicted secondary velocities (before convergence) must be good estimates to enhance a speedier pressure convergence. From this point of view, the diagonal sweep in yz-mom or the non-symmetric computational stencil in IV3 is undesirable.

6.3.4 Concluding Remarks

In calculating the laminar boundary layer on a flat plate, both PSTAG and IV3 showed the same degree of accuracy. The agreement between the predicted results and the Blasius solution is very good even though the use of a non-uniform grid is only first-order accurate in both schemes. Apart from accuracy, PSTAG performed more satisfactorily than IV3 in the other two criteria of speed and stability. However, one must treat this result cautiously because the staggered grid formulation has been more thoroughly explored. Perhaps, this superior PSTAG results should merely be treated as a demonstration of parity between the two grid arrangements.

6.4 Boundary Layer On A Protruded Flat Plate

6.4.1 Problem Background

In this test case, a laminar boundary layer develops from the protruded leading edge of a flat plate (see Fig.6.27). The protrusion, or spanwise perturbation, at the leading edge is a half-cycle sinusoidal waveform. The flow is symmetrical about the xy-plane at mid-span and the boundary layer thickness variation

across any "initial" crossflow plane can be prescribed by the following equation.

$$\delta(z) = \delta' \{ 0.3 [1 + \sin f(z)] + 1 \} \quad \text{Eq.6.12}$$

where

$$f(z) = \left[\frac{z}{0.375} + \frac{1}{2} \right] \pi \quad \text{for } -0.375 \leq z \leq 0.375$$

$$f(z) = \frac{\pi}{2} \quad \text{for } z < -0.375 \text{ \& } z > 0.375$$

δ' is the initial plane boundary layer thickness at $z=0.375$ (i.e. in the 2-D region). Strictly speaking, Eq.6.12 is meant to prescribe only the initial $\delta(z)$ profile by assuming the flow to be 2-D from the leading edge to the initial plane.

Though there is neither experimental data nor analytical solution available for direct comparison, the purpose of calculating this flow is basically two-fold. Firstly, in computing the whole flow (rather than taking advantage of the symmetry), one can check the symmetry characteristics of PSTAG and IV3. Secondly, the perturbed form of the boundary layer thickness in the crossflow plane resembles the situation of an embedded vortex in a turbulent boundary layer, which is one of the turbulent test cases (see §6.5).

6.4.2 Initial & Boundary Conditions

The mesh in the crossflow plane is uniform with 22x22 grid nodes and $\Delta Y = \Delta Z = 0.05$. The yz-plane BCs shown in Fig.6.28 are similar to

those for the 2-D flat plate boundary layer flow (see §6.3.2), for sake of completeness the BCs are as follows.

Along the wall - 1-2:

$$U = V = W = 0$$

$$\frac{\partial P}{\partial y} = 0 \text{ (implicit)}$$

Along the lateral boundaries - 2-3 & 1-4:

$$\frac{\partial U}{\partial z} = \frac{\partial V}{\partial z} = \frac{\partial W}{\partial z} = \frac{\partial P}{\partial z} = 0$$

Along the free stream boundary - 3-4:

$$U = U_{\infty} = 1.0$$

$$\frac{\partial V}{\partial y} = \frac{\partial W}{\partial y} = 0$$

$$P = P_{\infty} = 0$$

The initial plane, $x=0$, is located some distance from the leading edge and the Re value based on $L_{ref}=36'$ and U_{ref} , the free stream velocity is 3780. The initial U and V profiles in Fig.6.29 were constructed from the Blasius solution (as tabulated in Schlichting, 1979, p.139) using the boundary layer thickness evaluated from Eq.6.12. In this manner, the problem has been simplified by assuming that from the leading edge to the initial plane, the boundary layer has developed in a 2-D way parallel to the y -axis. Of course, the flow should be wholly 3-D because all three vorticity components are non-zero, the streamwise vorticity

induced by non-zero $\frac{\partial v}{\partial z}$ and y-direction vorticity induced by $\frac{\partial u}{\partial z}$. The initial values for W and pressure were set to zero.

PSTAG and IV3 predicted the development of this flow over a distance of 30 x-steps, using $\Delta x=0.5$. The differencing scheme in the continuity equation was the fully-implicit method. The pressure convergence tolerance for both the inner and outer loops were set to a rather tight value of 10^{-9} of the absolute error.

6.4.3 Results & Discussion

The microcomputer CPU times for this test case are 5450s and 8080s for PSTAG and IV3 respectively, again indicating a substantial improvement of approximately 50% by PSTAG. As discussed in §6.3.3, this can be attributed to better predictions of the secondary flow field (before convergence has been attained) through the elimination of the diagonal sweep from IV3. Before discussing the results, it must be emphasized that all the predictions can be analysed only from a qualitative standpoint because there is neither experimental data nor analytical solution available.

The U contours in Fig.6.30 at $x=15$ show that as the flow develops downstream, it becomes less "perturbed". This is indicated by the straightening of the U contours in the spanwise direction. The use of zero Neumann BC along both the lateral boundaries seems to be slightly unrealistic near the boundary layer edge when the perturbed 3-D central portion (say the $U=0.995$ contour) diffuses sideways and reaches the lateral edges of the computational

domain. Although the W velocities are non-zero, these cannot be easily discerned from the secondary flow vectors because W is roughly one order of magnitude smaller than V , except near the wall where both are of the same size. The W vectors plotted in Fig.6.31 show that near the wall, the flow tends to diverge whereas over most of the flow it tends to converge.

Fig.6.32 shows the y -direction U profiles near mid-span ($z=0.025$) and in the 2-D region ($z=0.475$) at $x=15$. The agreement between PSTAG and IV3 is very good indeed, with both schemes having similar values of δ . Generally speaking, both schemes are able to capture the broad character of the flow to the same degree of accuracy. However, slight discrepancies arise in the V profiles (see Fig.6.33); the values predicted by PSTAG are lower. This is again reflected in the W profiles in Fig.6.34. It is interesting to note that the W profiles are in some ways similar to those found in 3-D turbulent boundary layers because of the change in sign of $\frac{\partial W}{\partial y}$ within the layer. From these secondary velocity results, it is not possible to ascertain the quantitative correctness of either scheme, but at least one can say both schemes exhibit qualitative agreement.

Since this is nominally a symmetrical flow, spanwise secondary velocity variations are plotted as a check. It can be seen from Fig.6.35 that the IV3 results are slightly non-symmetrical (though not so obvious in the near-wall region, i.e. at $y=0.5$) whereas PSTAG performs satisfactorily. This becomes more pronounced in Fig.6.36 for W where the difference in peak values from IV3

clearly shows distortion; also on the plane of symmetry the velocities are non-zero. Again PSTAG demonstrates true symmetry and zero velocities at mid-span. For better illustration, some of the V and W velocity values at y=0.5 are tabulated below.

		PSTAG	IV3
V	z=-0.525	1.1589×10^{-3}	1.2161×10^{-3}
V	z=0.525	1.1589×10^{-3}	1.1976×10^{-3}
W	z=-0.525	6.3605×10^{-8}	1.4499×10^{-8}
W _{max}	z=-0.175	7.2461×10^{-5}	9.3994×10^{-5}
W	z=0	0	9.1893×10^{-6}
W _{max}	z=0.175	-7.2461×10^{-5}	-7.9124×10^{-5}
W	z=0.525	-6.3605×10^{-8}	-2.0351×10^{-8}

The ratios of W in the 2-D regions and W_{max} (the peak W magnitude) do indicate a rather significant non-symmetry in the IV3 results, for example the above table shows that the ratios of W_{max} (at z=±0.175) is 1.18 and in the 2-D region is 1.4.

The ability of PSTAG to predict a symmetrical flow without distortion is remarkably good. This can again be attributed to the symmetry in the computational stencil as well as the location of the actual lateral boundaries (i.e. along 2-3 & 1-4). These features, which seem essential for a symmetrical flow calculation, are not present in a staggered grid formulation as discussed in §4.6.2. Of course, for most engineering purposes, true symmetry

may not be an all-important criterion to judge a numerical scheme like IV3.

As stated in §6.3.3, the sensitivity in the near-wall region can often be used as a good stability indicator in a numerical scheme. Streamwise variations of U, V and W at the node next to the wall are plotted in Fig.6.37, Fig.6.38 and Fig.6.39 respectively. The rather close agreement in U is to be expected since the U profiles in Fig.6.32 also showed good agreement. There are slight differences for V and W but overall trends are generally the same. The differences in the V profiles start to appear at about $x=3$ with IV3 indicating a region of $\frac{\partial V}{\partial x} \approx 0$, on the other hand, the PSTAG V profile is much smoother.

In Fig.6.40 for the pressure next to the wall, IV3 shows large initial oscillations which subsequently damp out by $x=9$. (As mentioned before, the calculated pressure field for incompressible flows is in fact a response to the mean velocity field to satisfy the continuity equation.) However, these pressure oscillations in IV3 seem not to correspond to any noticeable oscillatory behaviours in V and/or W in the y- and/or z- direction. Hence, the cause/s of these wiggles is not entirely clear. The PSTAG results are, on the other hand, smoother with some initial fluctuations up to $x=3$ only.

6.4.4 Concluding Remarks

This test case has clearly illustrated the symmetry feature in

PSTAG, obtained through the inherent symmetrical stencil and lateral boundaries. It also shows that a staggered grid scheme would cause some distortions in the flow which can be quite considerable. Putting aside the undetermined quantitative accuracy, from the computing speed and stability viewpoint, PSTAG gave a more satisfactory performance than IV3, as was with the previous test case (see §6.3). The very substantial reduction in computing time (typically 50%) is without doubt a very attractive feature of PSTAG, especially for engineering applications. It must be added that the formulation of a scheme based on the PSTAG grid can be achieved through conventional numerical techniques, that is, without resorting to special techniques uncommon in CFD literature.

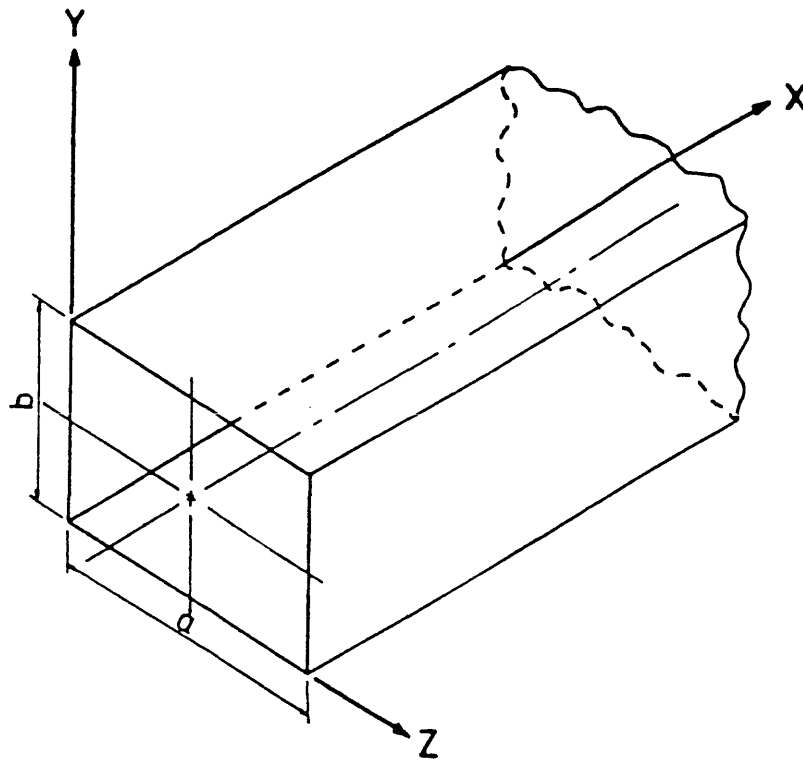


Fig.6.1 Rectangular Duct.

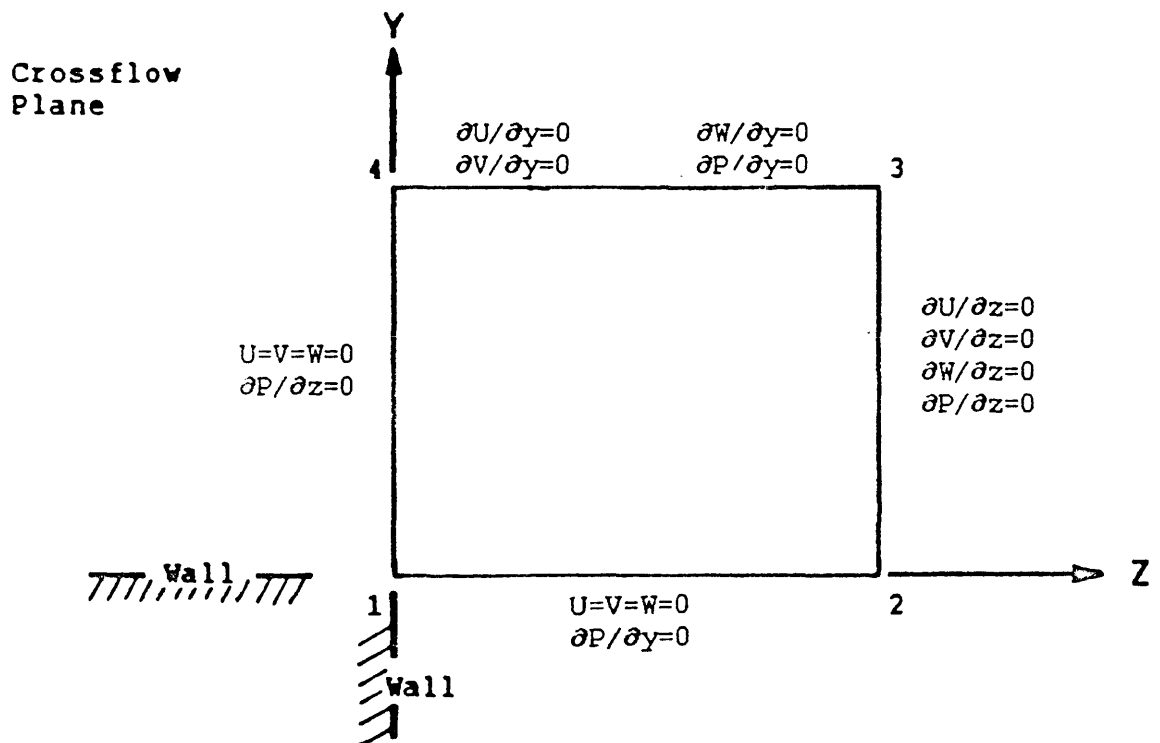


Fig.6.2 Duct Boundary Conditions.

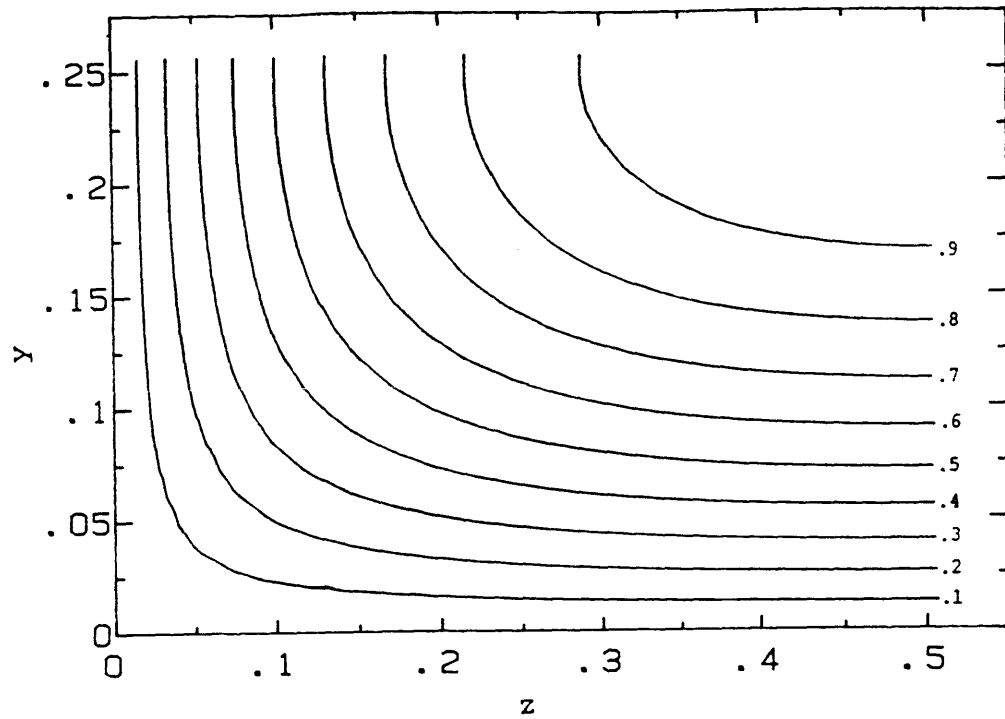


Fig.6.3 Initial Plane U Contours.

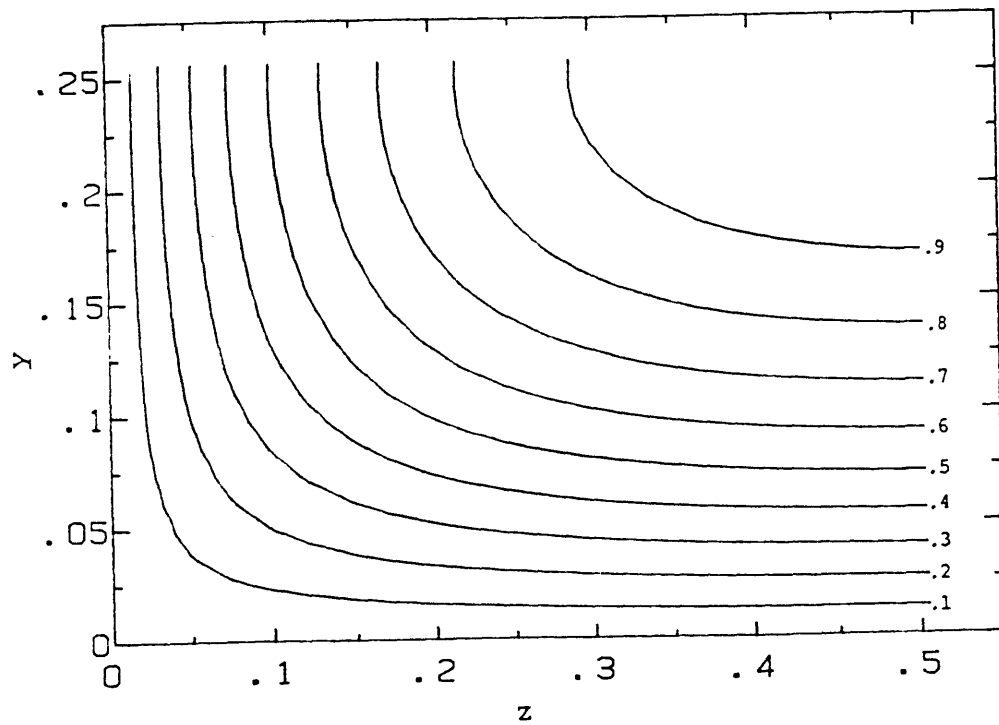


Fig.6.4 Predicted U Contours At $x = 5$ & 10 .

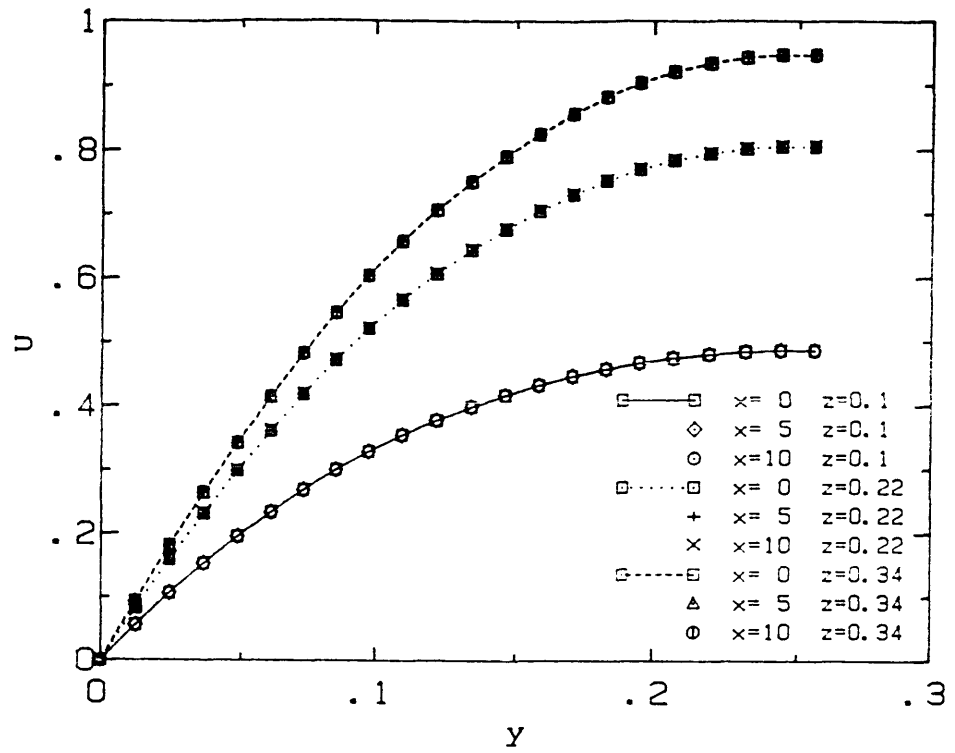


Fig.6.5 Analytic & Predicted y-direction U Profiles for $Re = 700$.

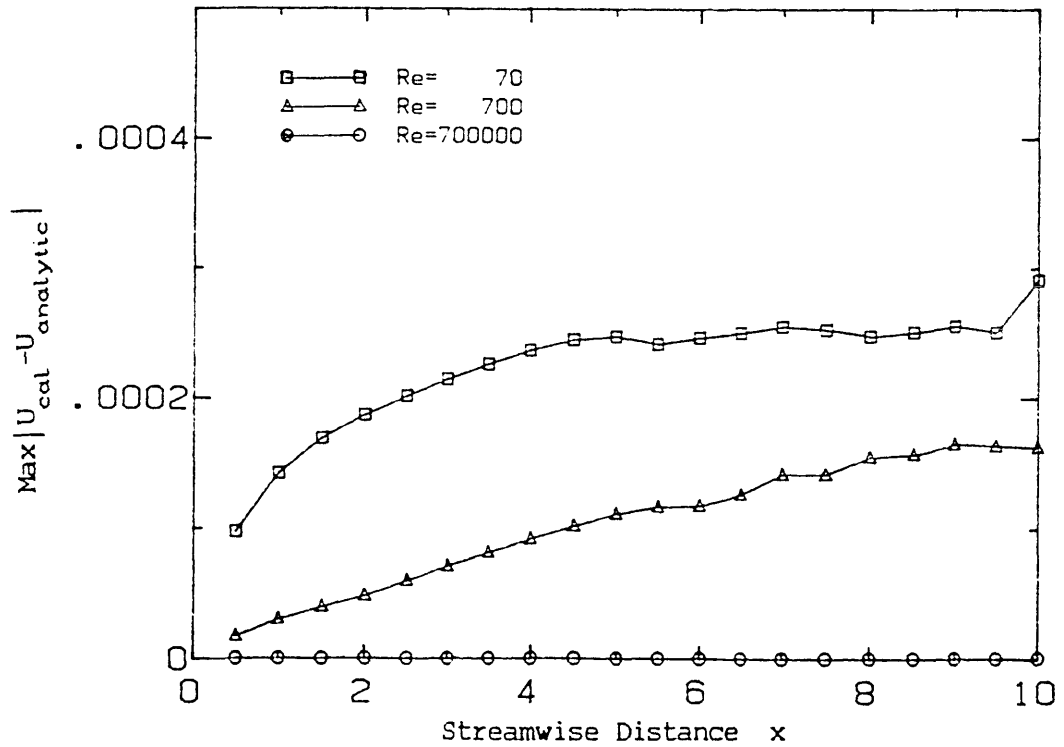


Fig.6.6 Maximum Errors In Predicted U Velocity.

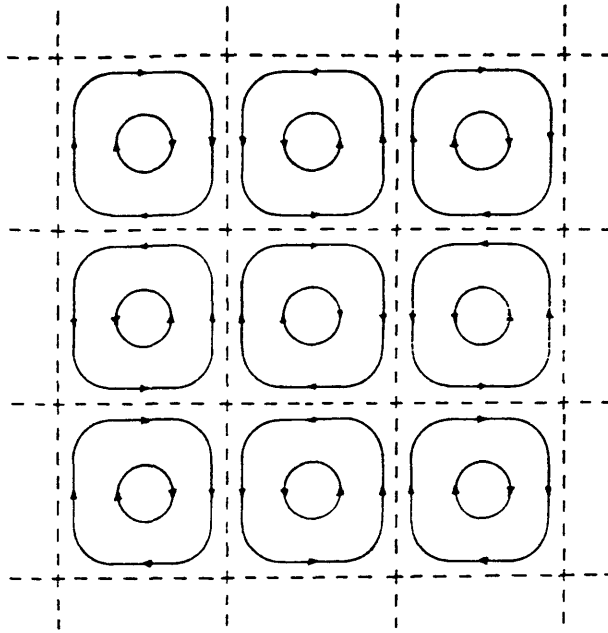


Fig.6.7 System Of Taylor-Green Vortices.

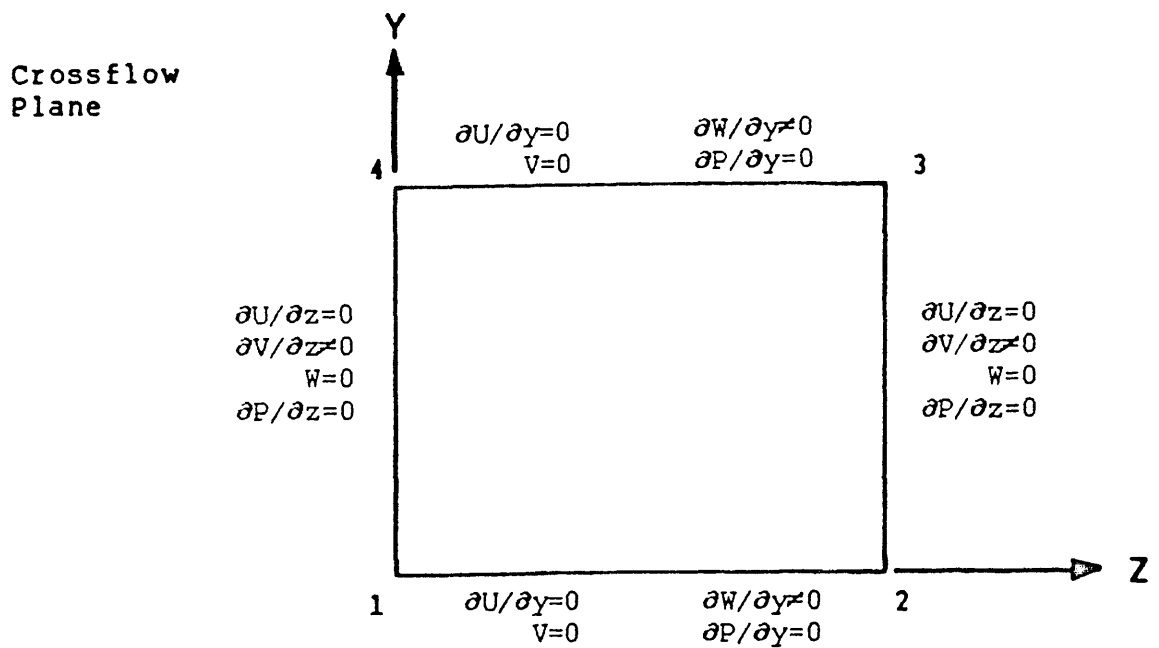


Fig.6.8 Taylor-Green Vortex Boundary Conditions.

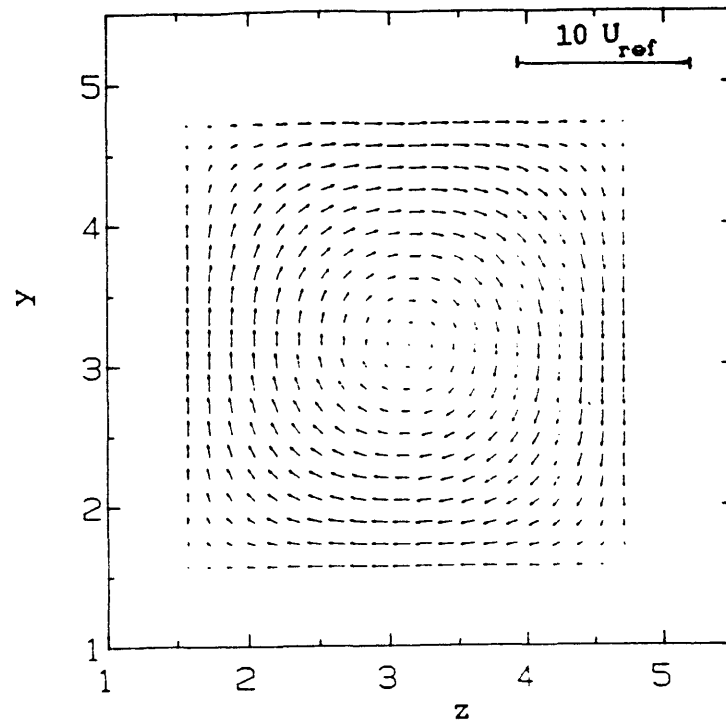


Fig.6.9 Initial Plane Secondary Flow.

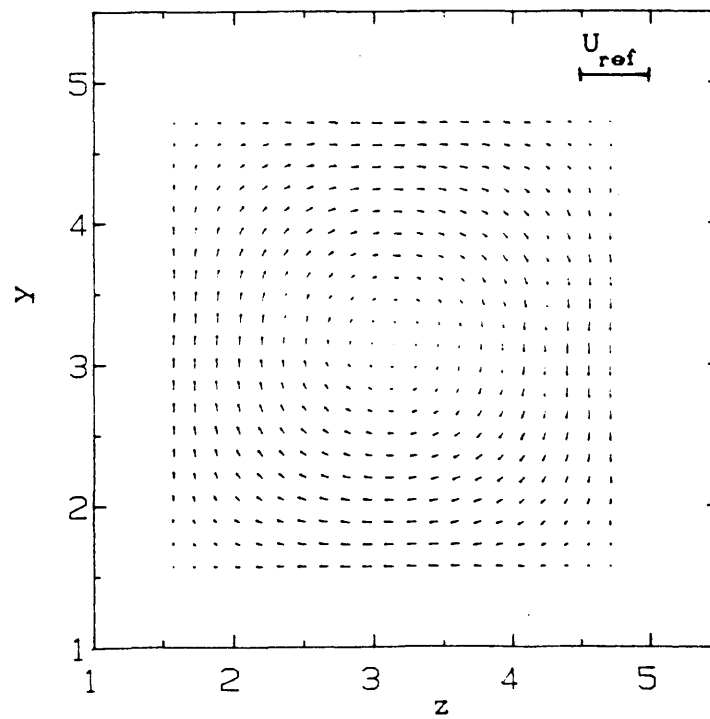


Fig.6.10 Predicted Secondary Flow For $Re = 1$ at $x = 1$.

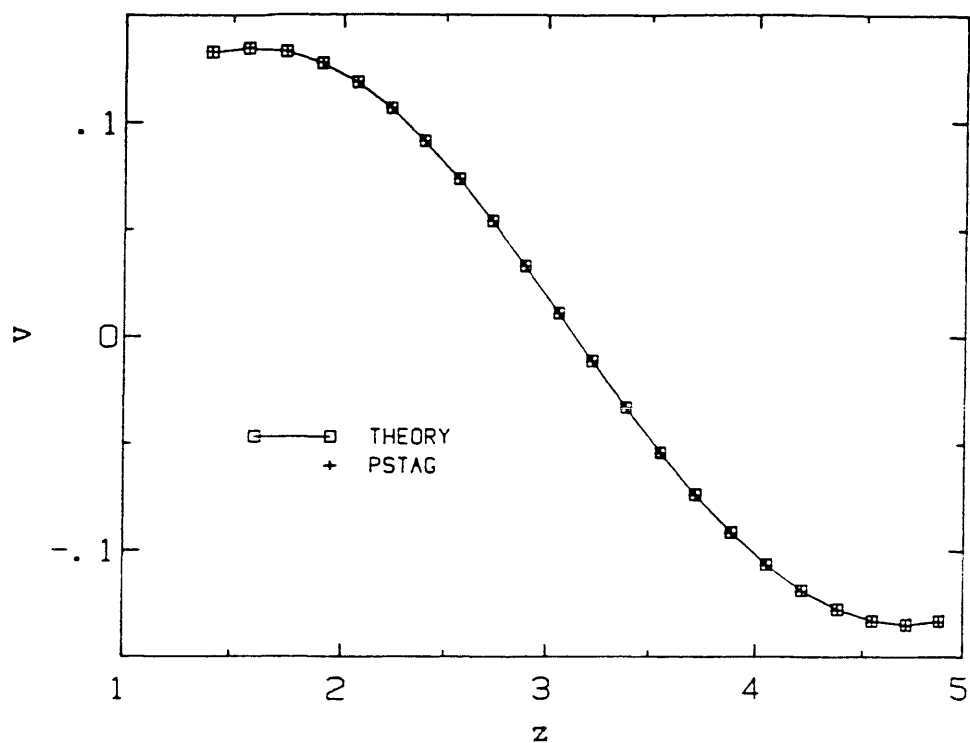


Fig.6.11 Analytic & Predicted Spanwise V Velocity Profiles For $Re = 1$ at $y = \pi$ & $x = 1$.

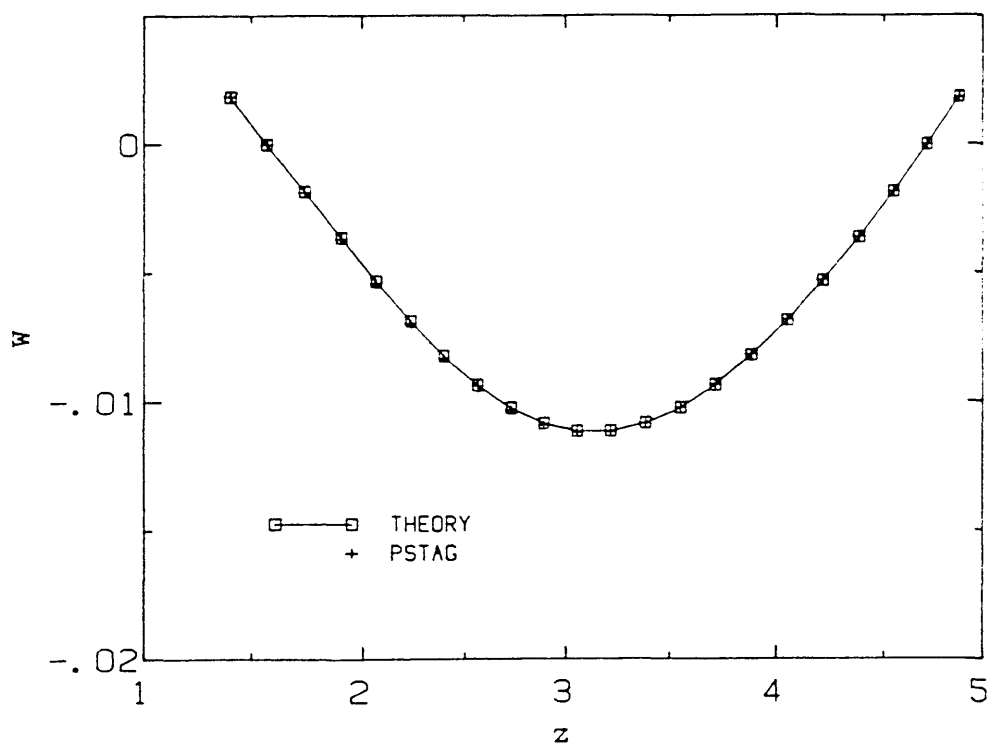


Fig.6.12 Analytic & Predicted Spanwise W Velocity Profiles For $Re = 1$ at $y = \pi$ & $x = 1$.

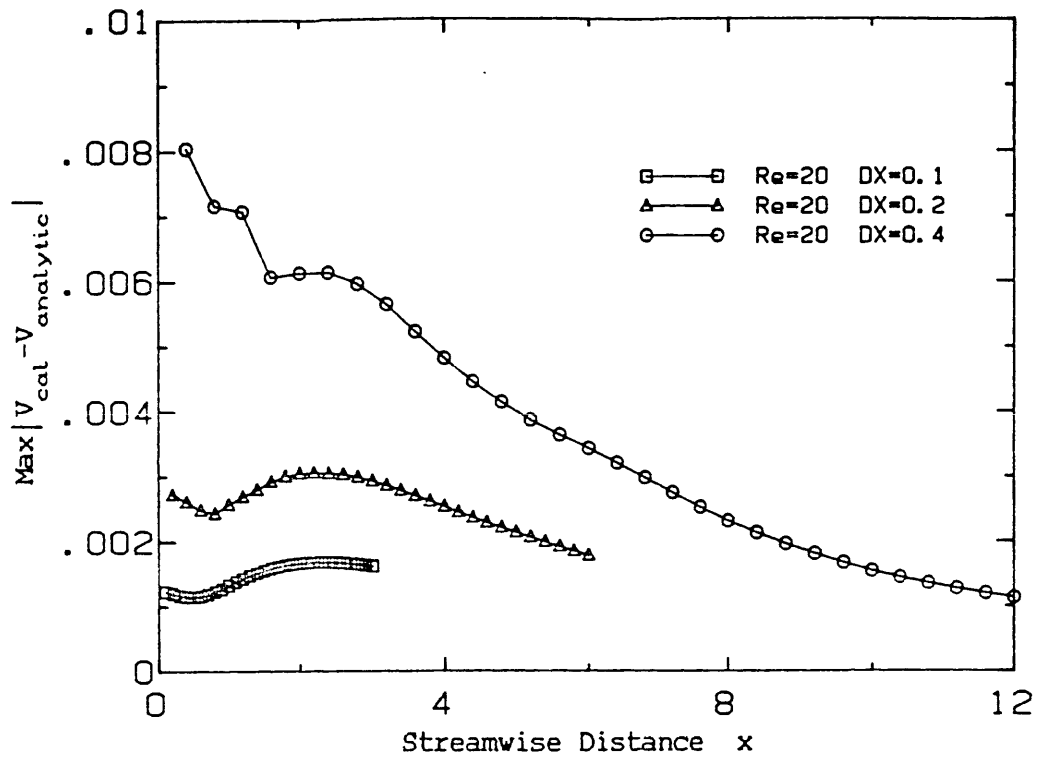


Fig.6.13 Maximum V Velocity Error For Re = 20 with $\Delta x = 0.1, 0.2$ & 0.4 .

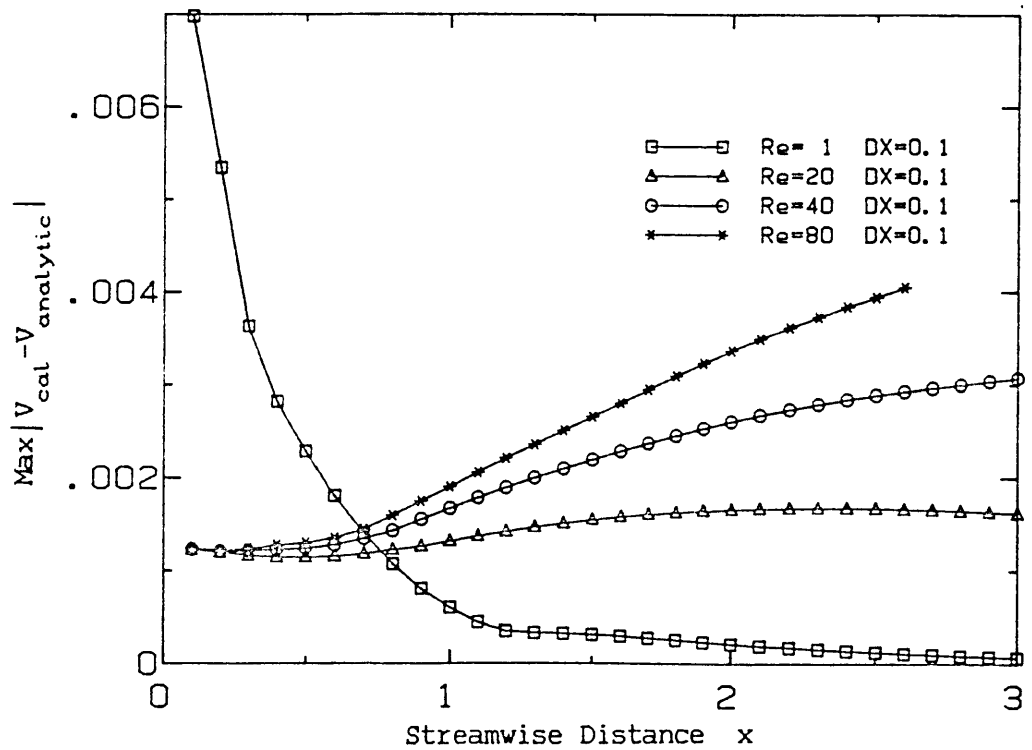


Fig.6.14 Maximum V Velocity Error For Re = 1, 20, 40 & 80 with $\Delta x = 0.1$.

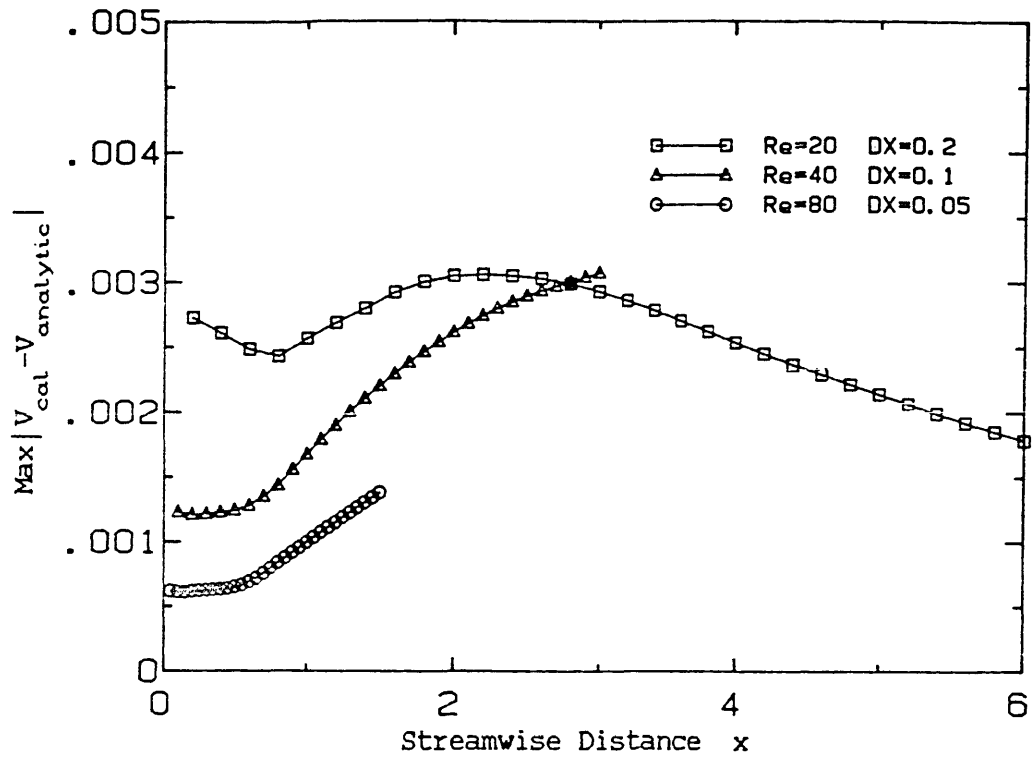


Fig.6.15 Maximum V Velocity Error For $\text{Re} = 20$ ($\Delta x = 0.2$), $\text{Re} = 40$ ($\Delta x = 0.1$) & $\text{Re} = 80$ ($\Delta x = 0.05$).

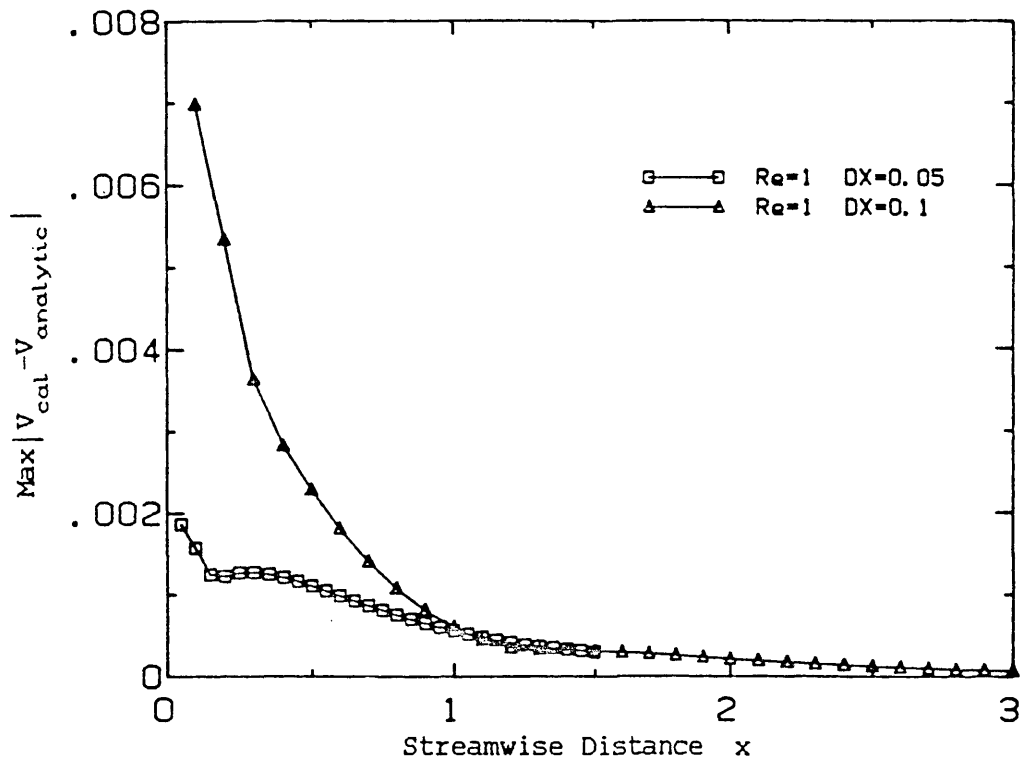


Fig.6.16 Maximum V Velocity Error For $\text{Re} = 1$ with $\Delta x = 0.05$ & 0.1 .

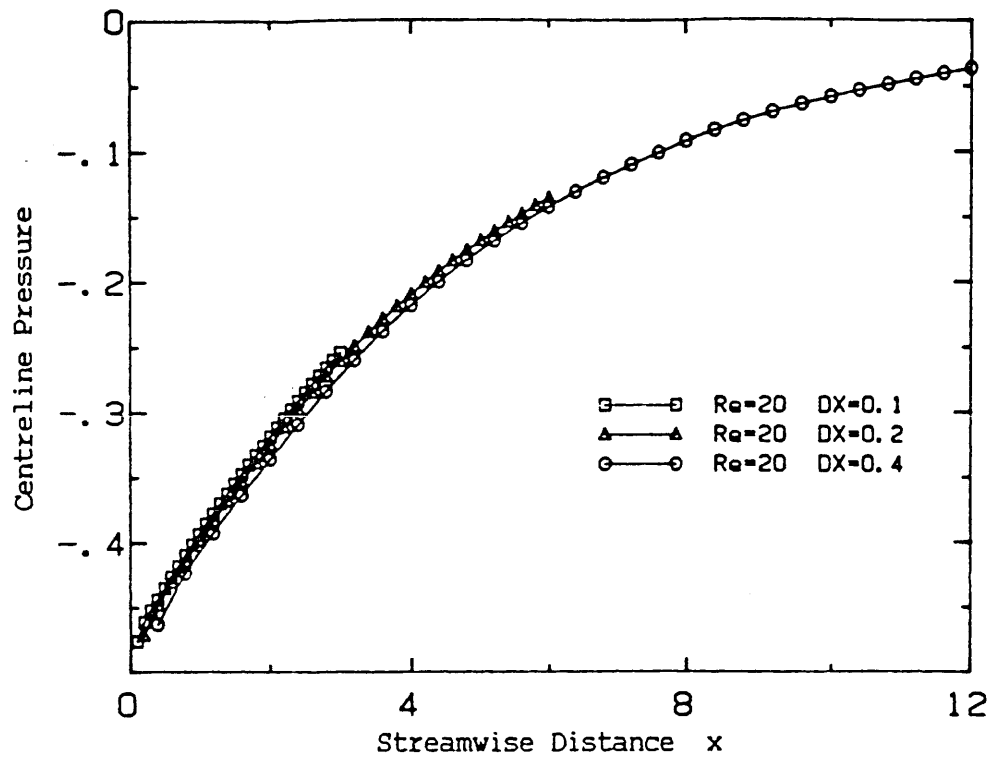


Fig.6.17 Predicted Centreline Pressure For $Re = 20$ with $\Delta x = 0.1$, 0.2 & 0.4 .

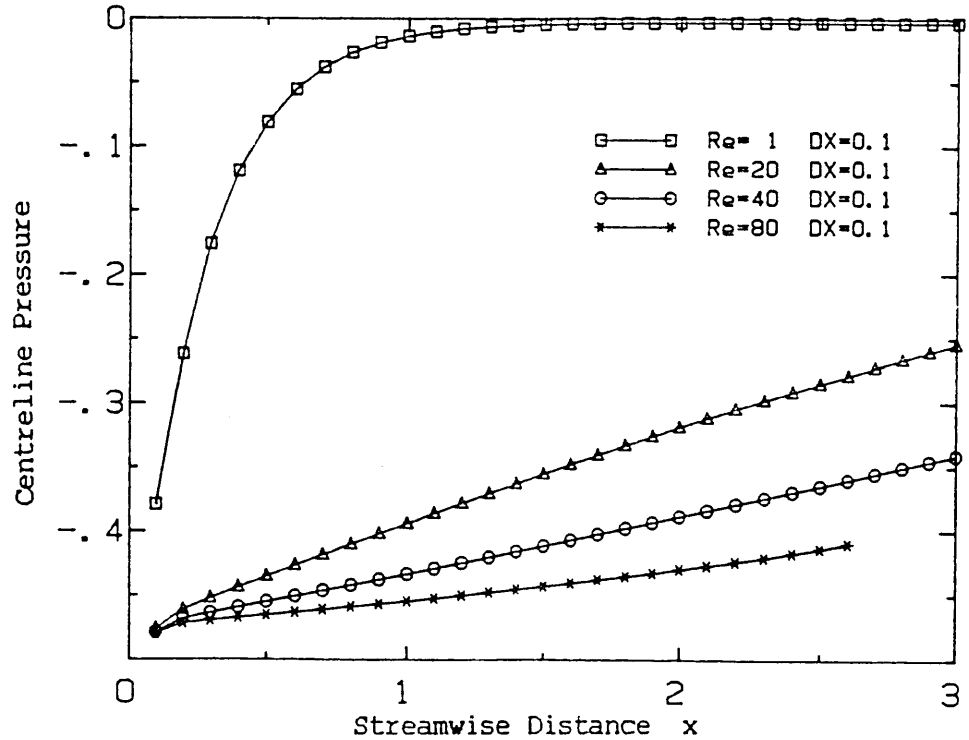


Fig.6.18 Predicted Centreline Pressure For $Re = 1, 20, 40$ & 80 with $\Delta x = 0.1$.

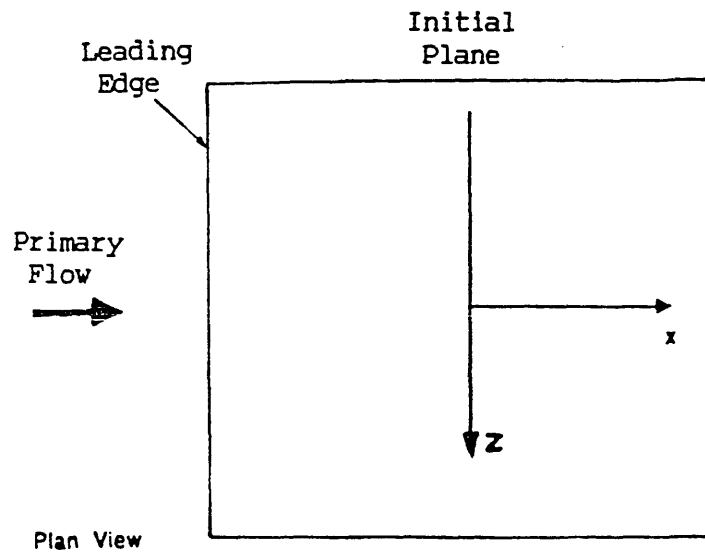


Fig.6.19 Flat Plate.

Crossflow Plane

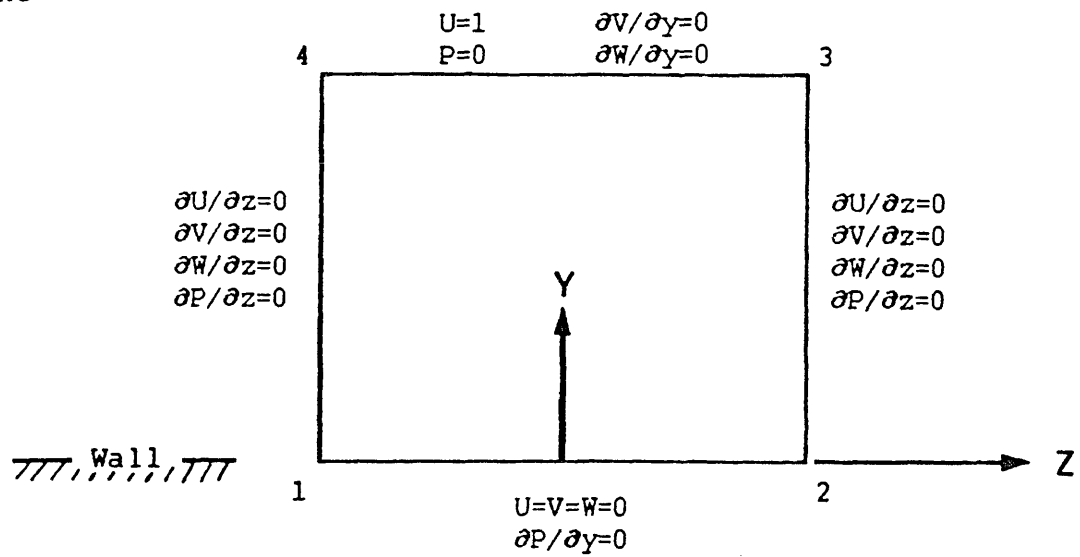


Fig.6.20 Flat Plate Boundary Conditions.

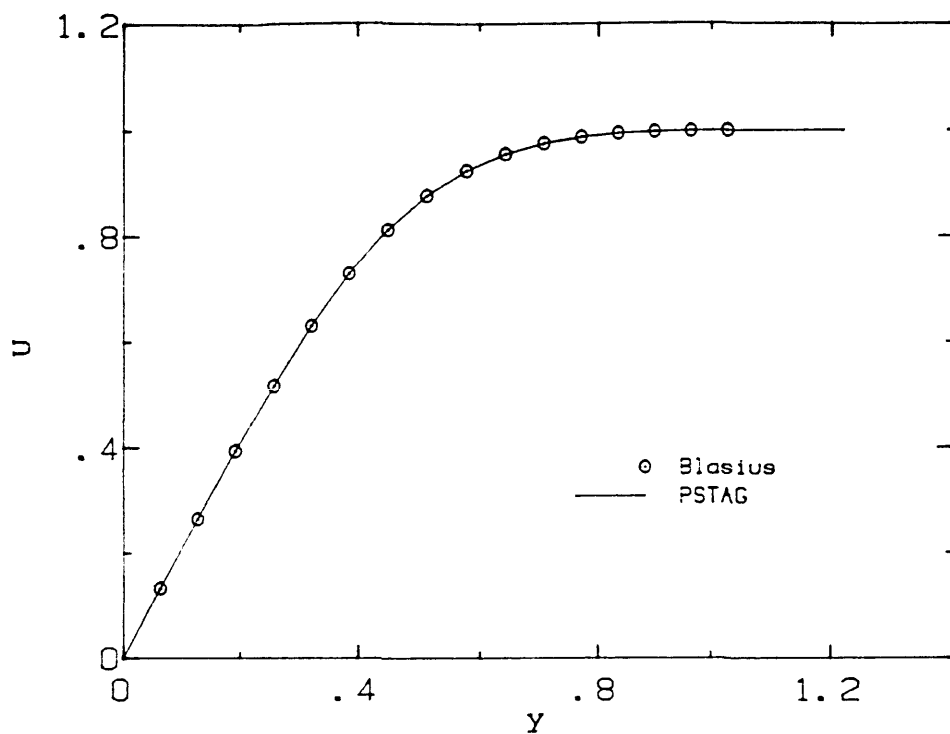


Fig.6.21 Predicted & Blasius U Profile.

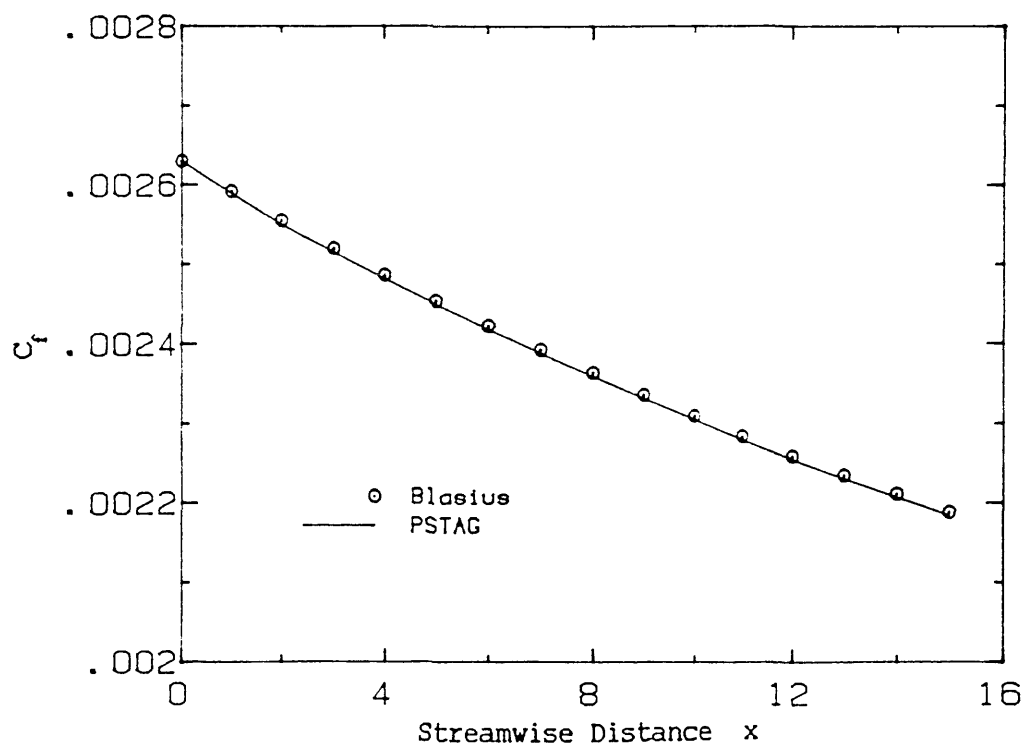


Fig.6.22 Predicted & Blasius Skin Friction Coefficient.

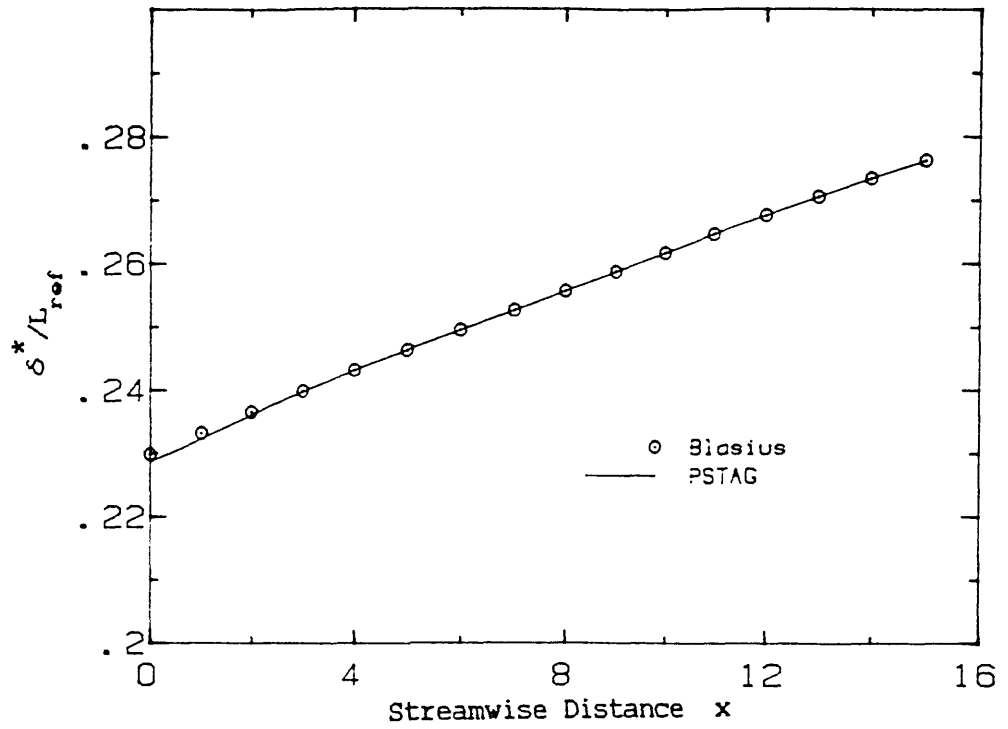


Fig.6.23 Predicted & Blasius Displacement Thickness.

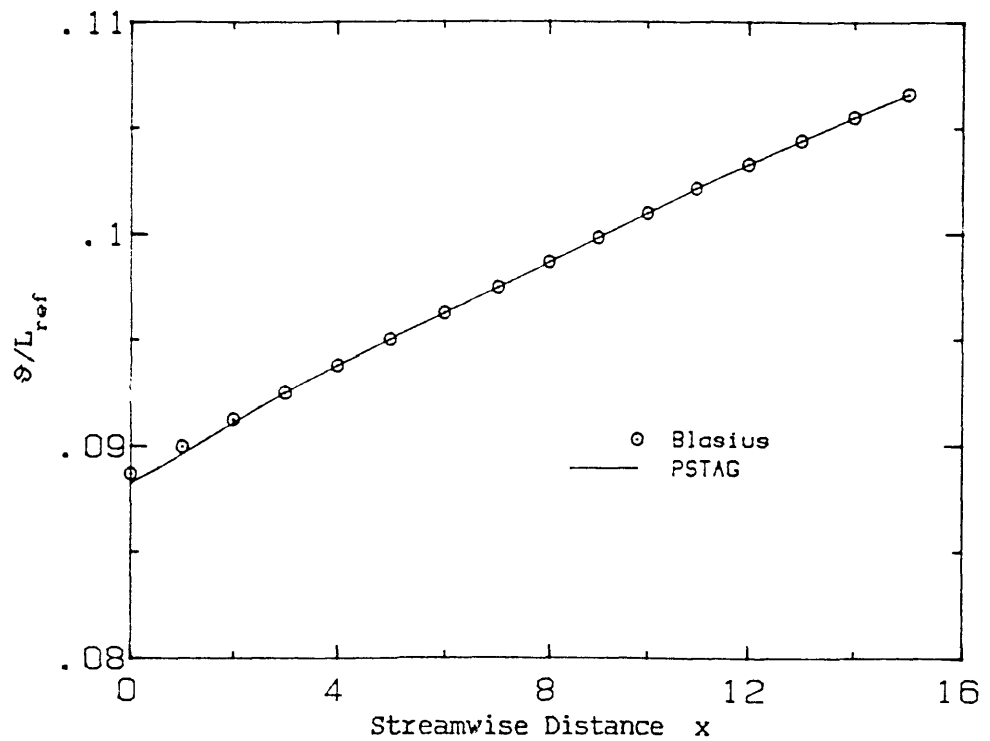


Fig.6.24 Predicted & Blasius Momentum Thickness.

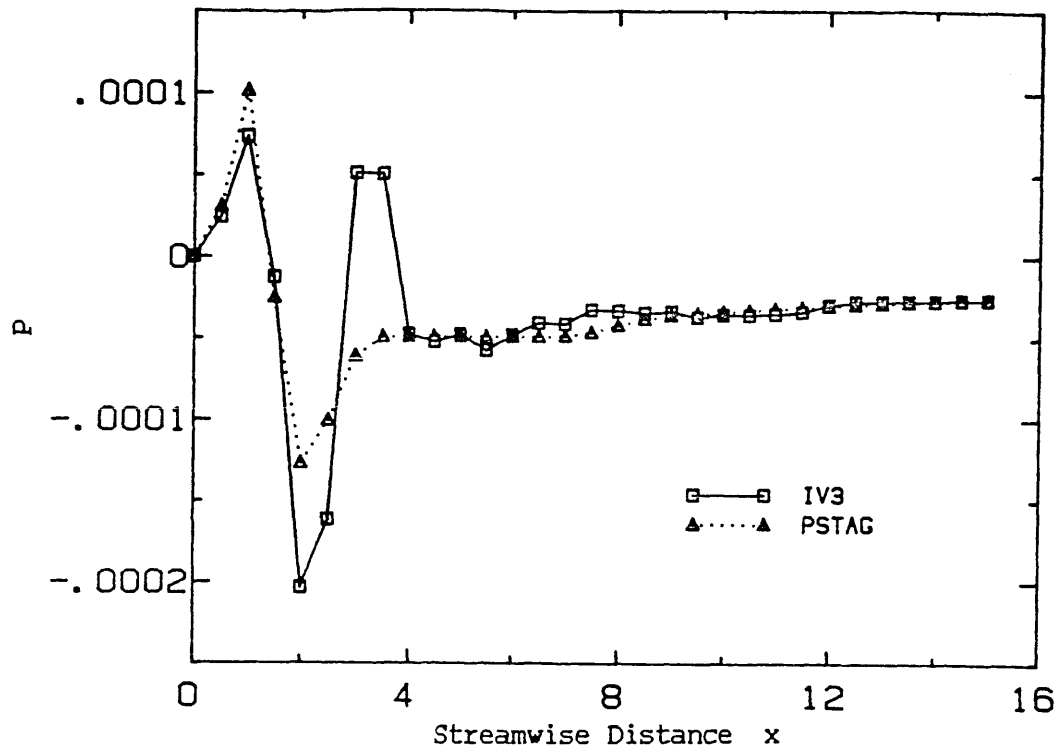


Fig.6.25 Predicted Pressure At Node $Y_{1\frac{1}{2}}$.

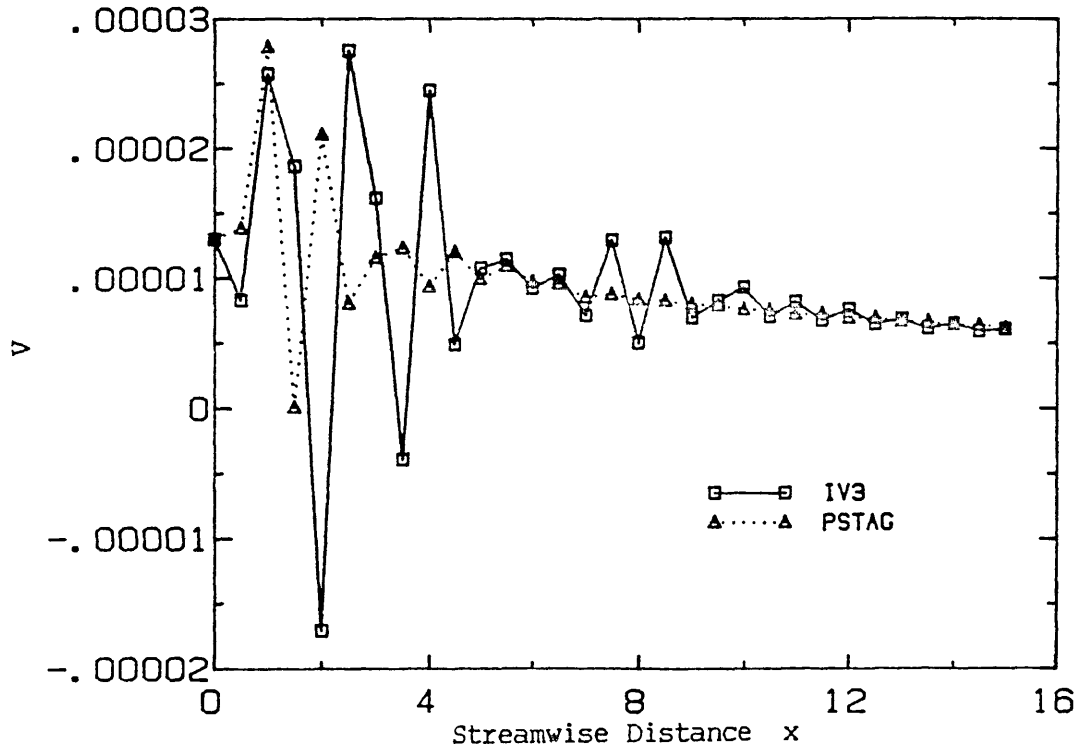


Fig.6.26 Predicted V Velocity At Node Y_2 .

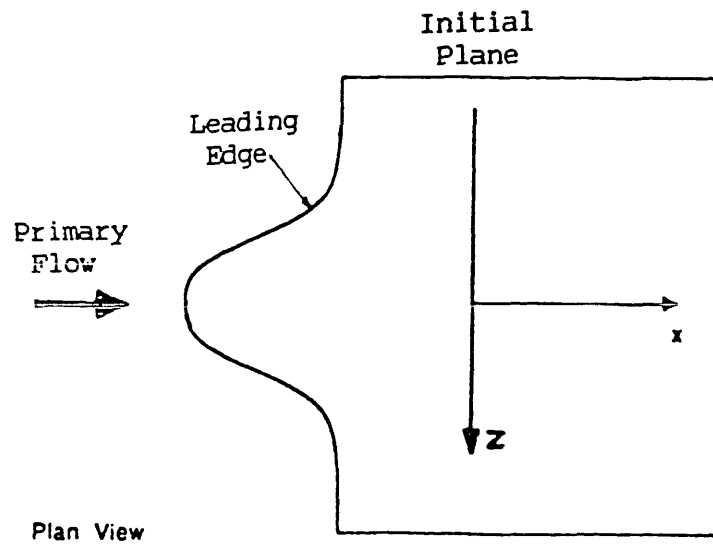


Fig.6.27 Protruded Flat Plate.

Crossflow Plane

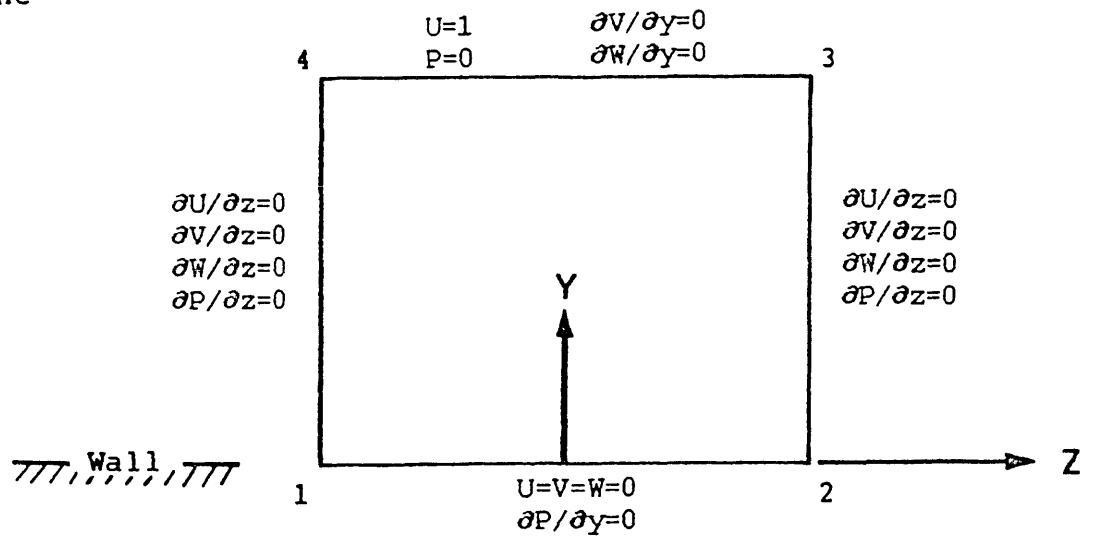


Fig.6.28 Protruded Flat Plate Boundary Conditions.

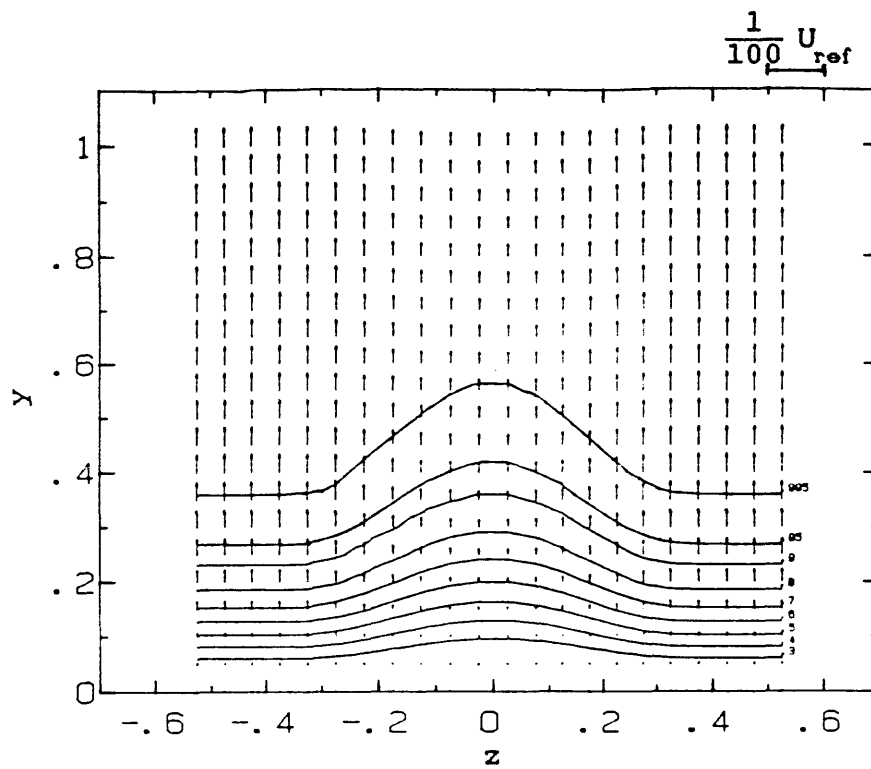


Fig.6.29 Initial Plane U Contours & Secondary Flow Vectors.

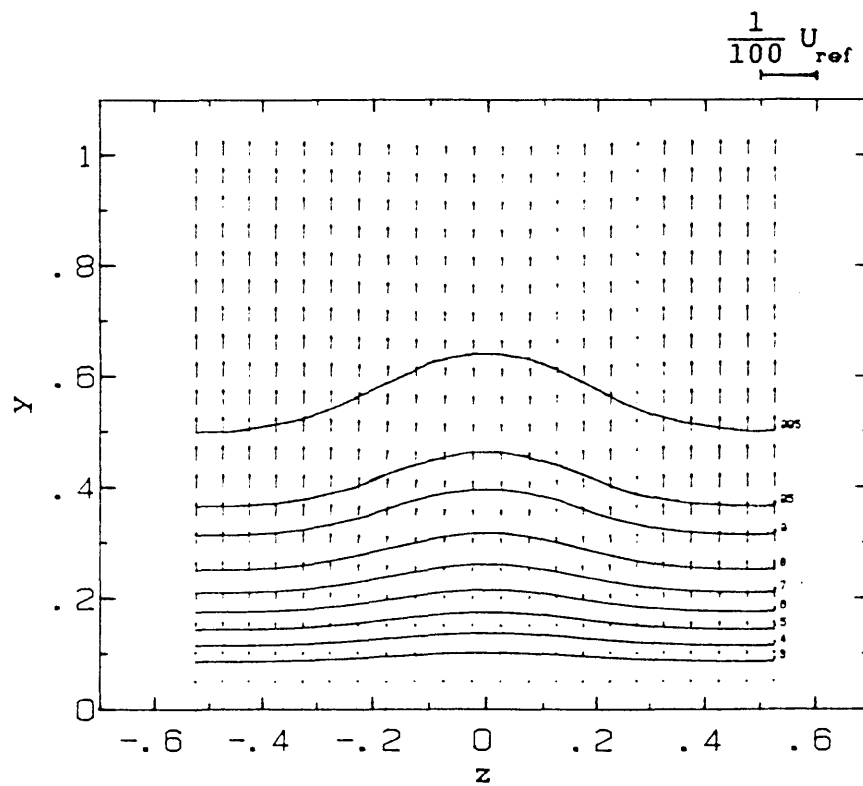


Fig.6.30 Predicted U Contours & Secondary Flow Vectors at $x = 15$.

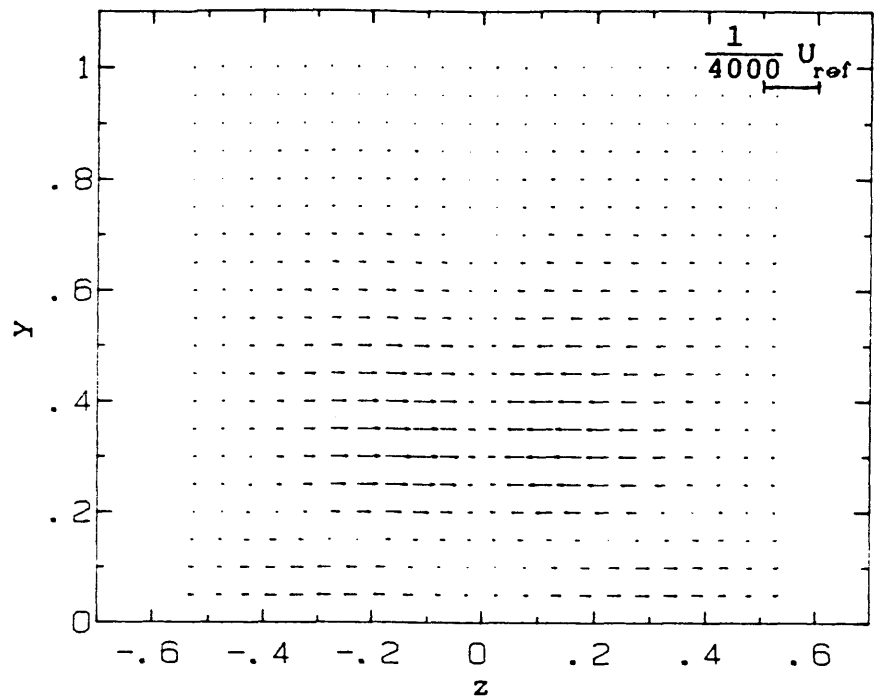


Fig.6.31 Predicted W Velocity Vectors at $x = 15$.

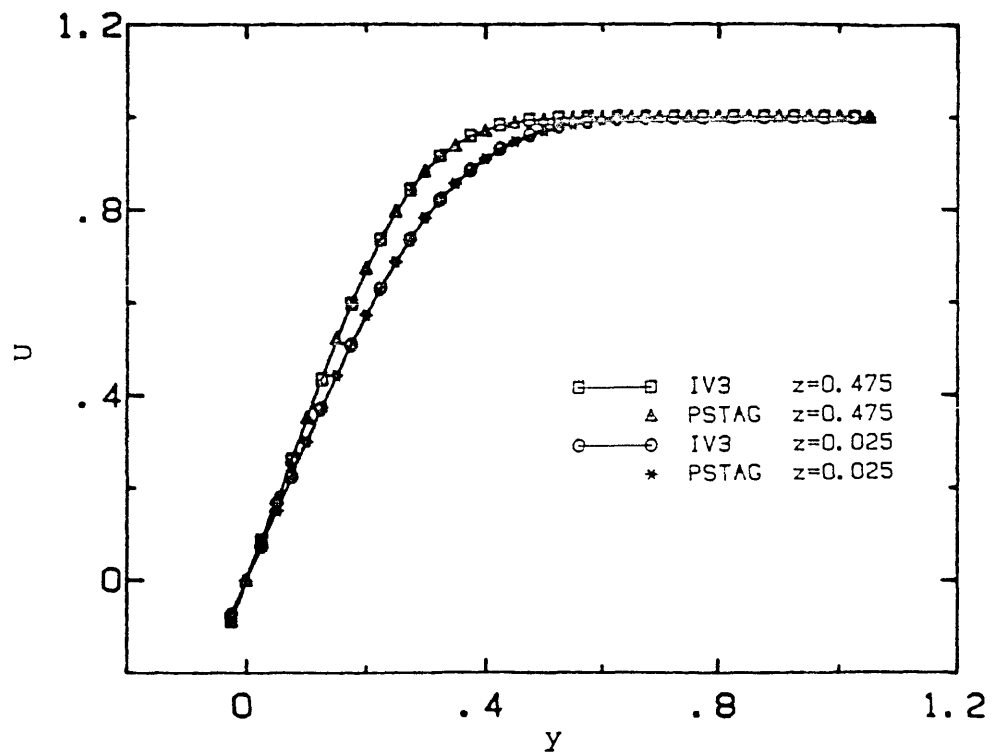


Fig.6.32 Predicted y -direction U Profiles For $z = 0.025$ & 0.475 at $x = 15$.

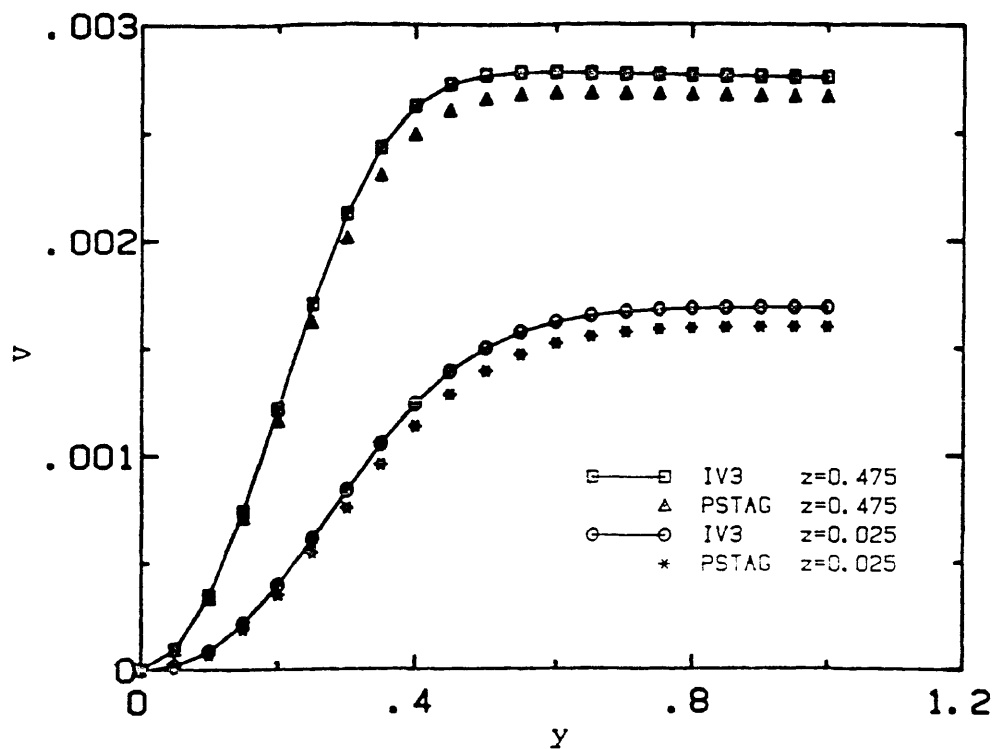


Fig.6.33 Predicted y-direction V Profiles For $z = 0.025$ & 0.475 at $x = 15$.

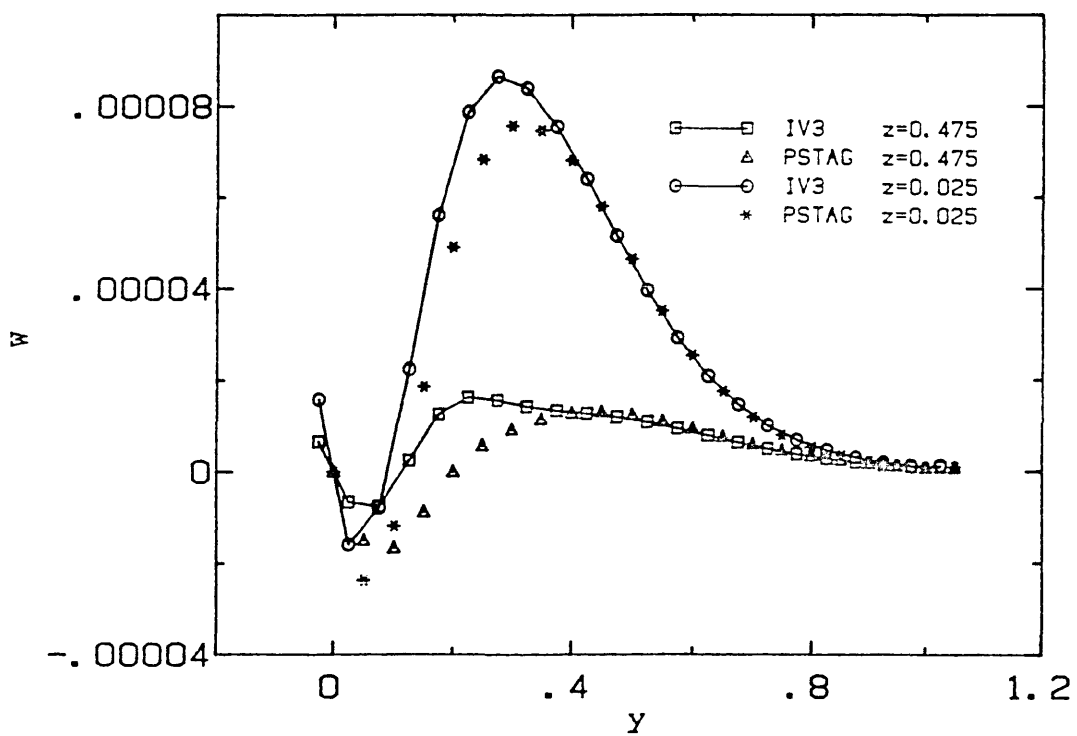


Fig.6.34 Predicted y-direction W Profiles For $z = 0.025$ & 0.475 at $x = 15$.

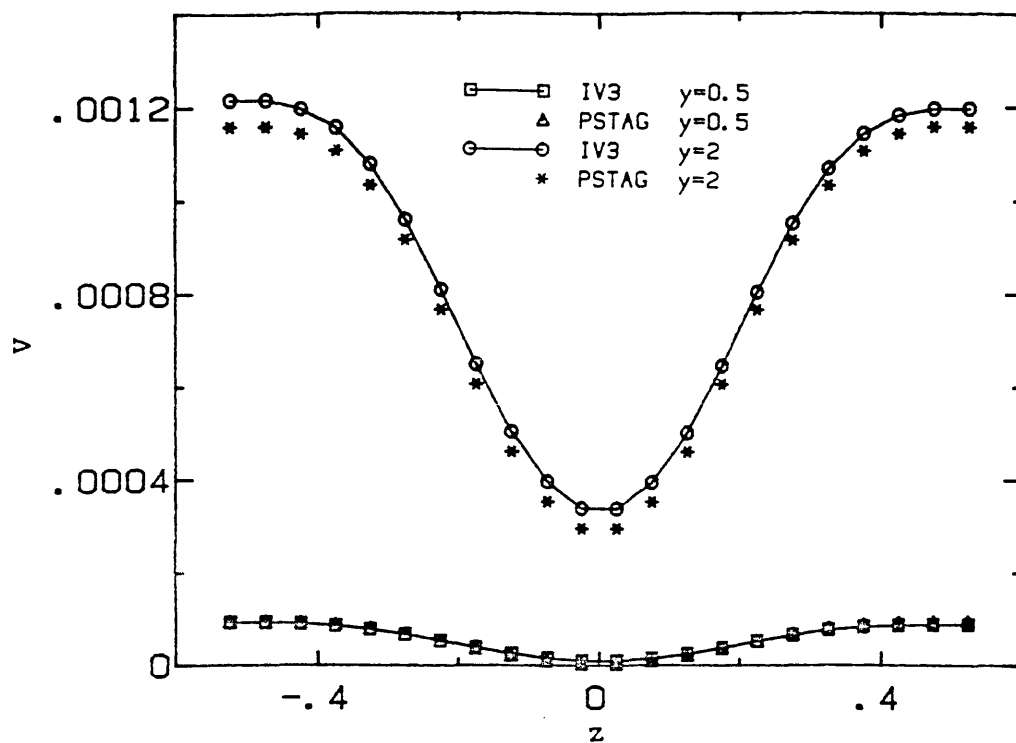


Fig.6.35 Predicted Spanwise V Profiles For $y = 0.5$ & 2 at $x = 15$.

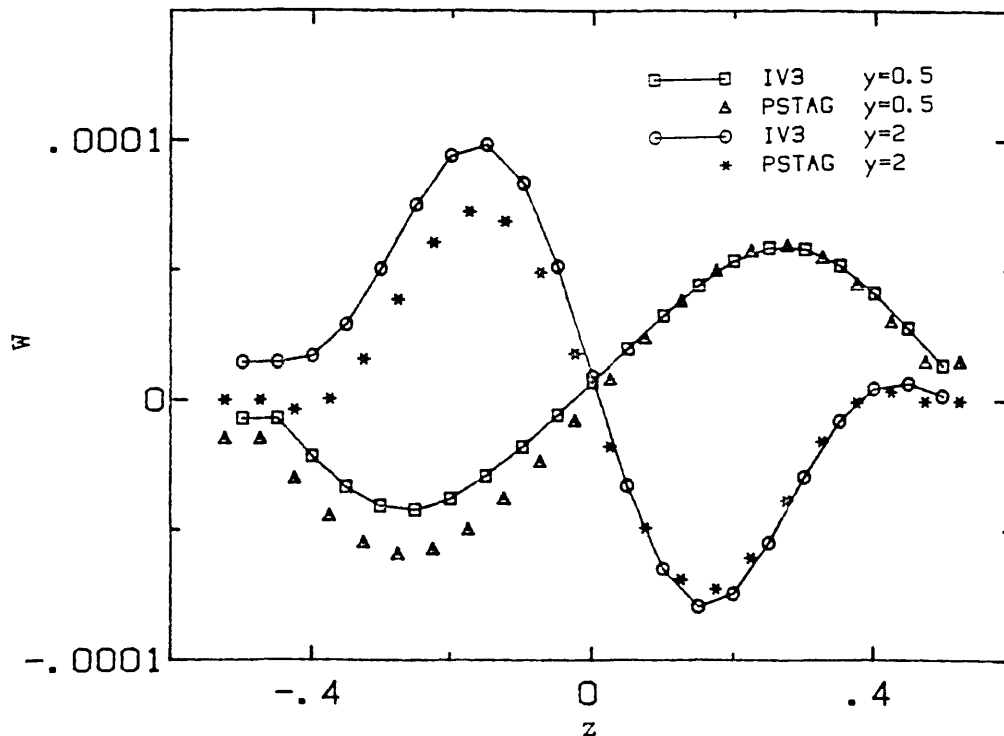


Fig.6.36 Predicted Spanwise W Profiles For $y = 0.5$ & 2 at $x = 15$.

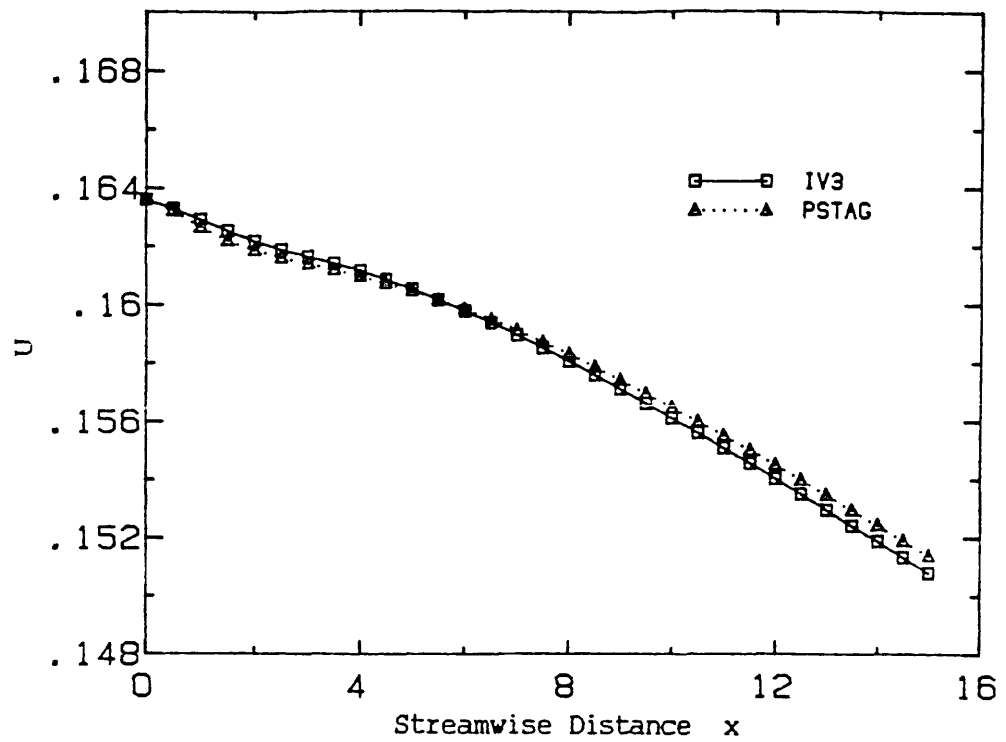


Fig.6.37 Predicted Streamwise Variation Of U at $y = 0.5$ & $z = 0.475$.

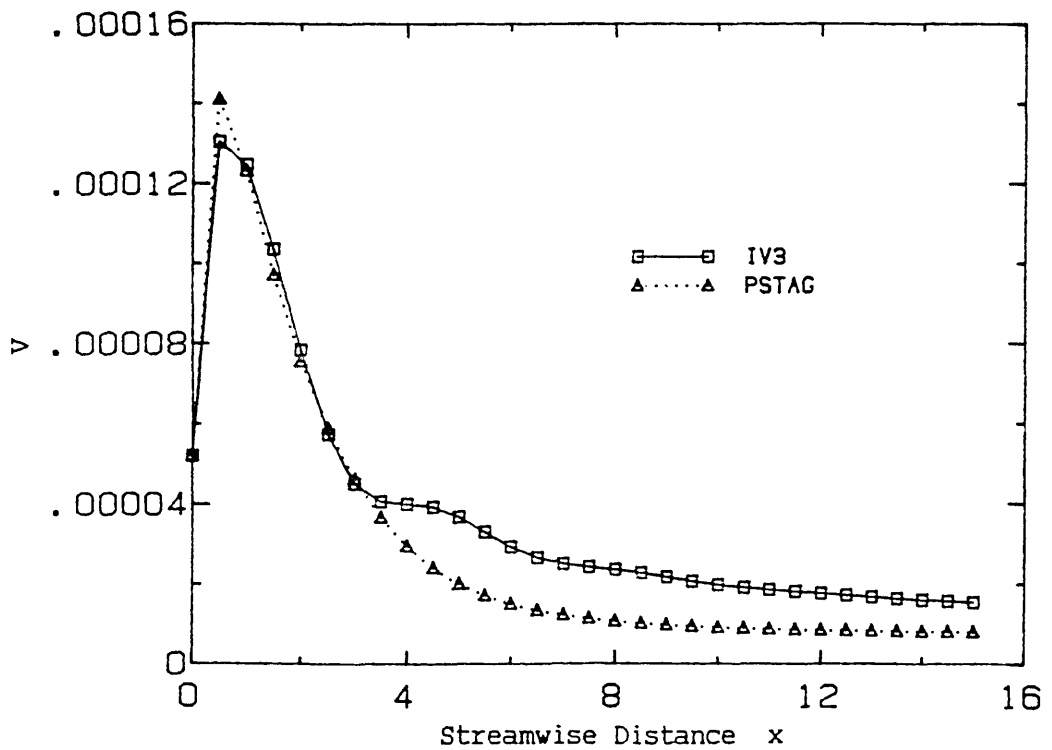


Fig.6.38 Predicted Streamwise Variation Of V at $y = 0.5$ & $z = 0.475$.

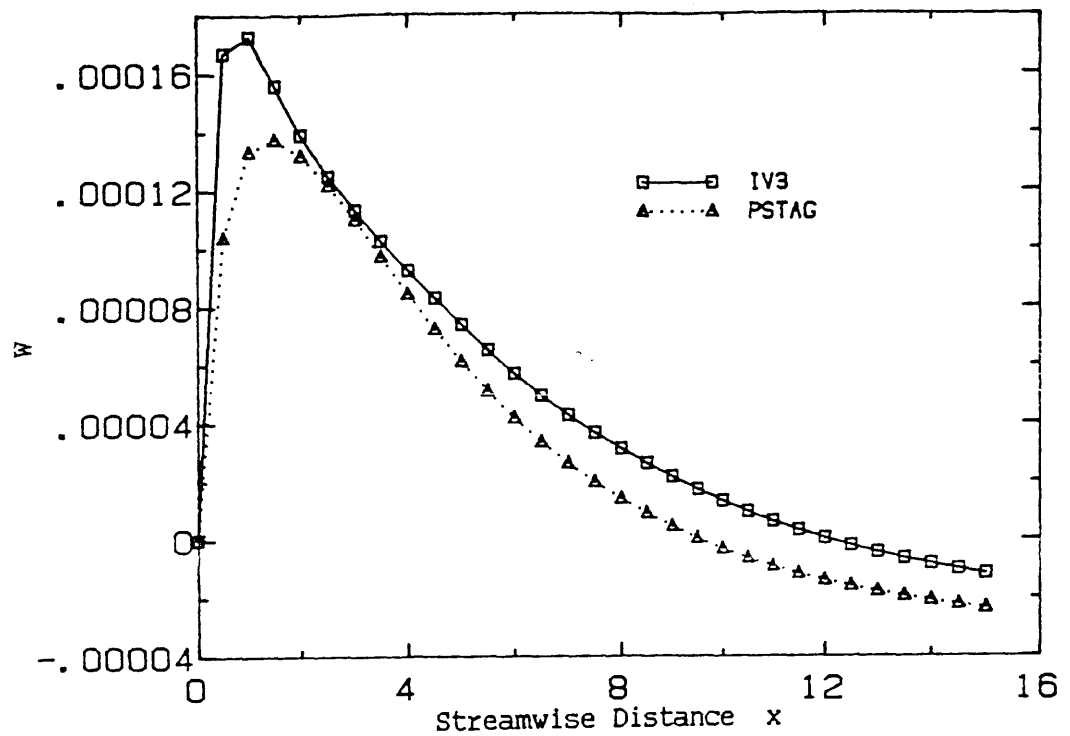


Fig.6.39 Predicted Streamwise Variation Of W at $y = 0.5$ & $z = 0.475$.

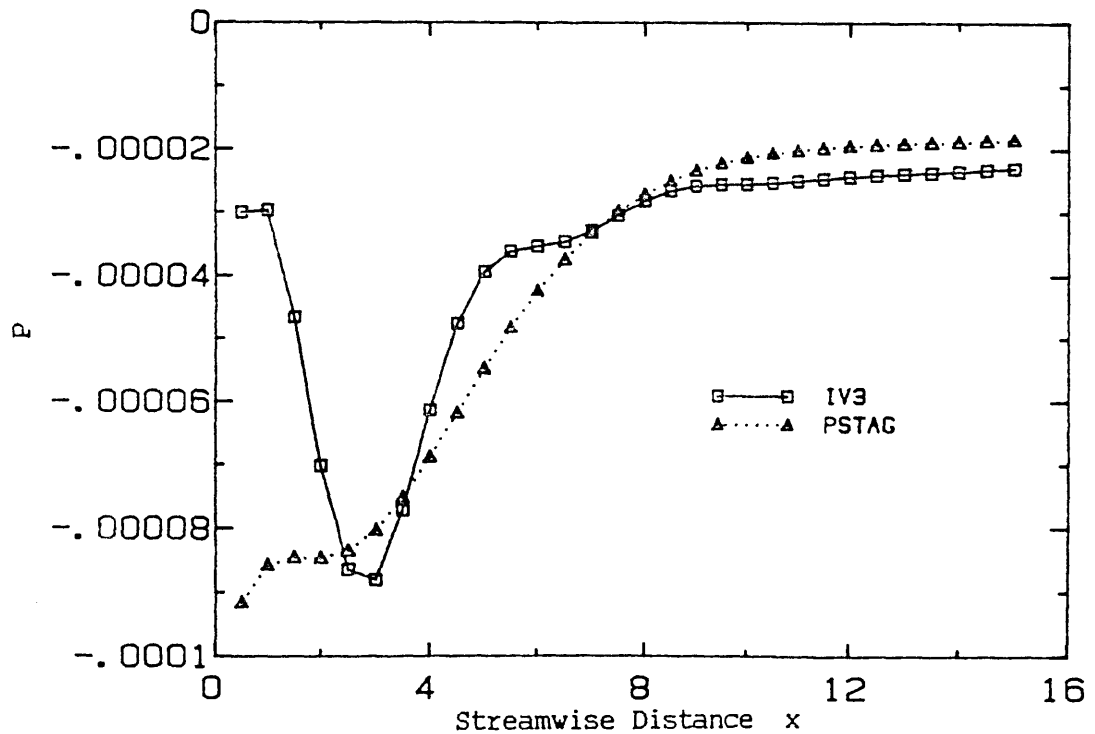


Fig.6.40 Predicted Streamwise Variation Of Pressure at $y = 0.25$ & $z = 0.475$.

7 TURBULENT FLOW COMPUTATIONS

7.1 Boundary Layer On A Flat Plate

7.1.1 Problem Background

The turbulent boundary layer on a flat plate at zero pressure gradient is basically a yardstick for all "infant" numerical algorithms which, of course, includes PSTAG. It is widely acknowledged that any potentially useful numerical scheme or turbulence model must be able to predict this flow satisfactorily. Many workers have already obtained reasonably good predictions for this flow using better-tested numerical schemes incorporating the present form of RSTM. Hence, this test case will serve primarily to gauge the numerical aspects of PSTAG rather than the performance of the RSTM.

The 2-D boundary layer, being part of an important class of practical boundary layers, is a fairly well documented flow. It is worth mentioning that this flow exhibits similarity (a sub-class of self-preservation) and the relevant velocity and length scales are u_τ and δ respectively. The more commonly used empirical formulae for C_f are the von Karman and the power-law equations. The $\frac{1}{7}$ power-law assumption for the primary velocity profile also yields equations for the boundary layer thickness and momentum thickness (see Cebeci & Bradshaw, 1977, pp.186-187).

7.1.2 Initial & Boundary Conditions

In Fig.7.1 is shown diagrammatically the physical and computational domains of this test case. The BCs in the elliptic crossflow plane as defined in Fig.7.2 are as follow.

Along the wall - 1-2:

$$U = V = W = 0$$

$$\frac{\partial P}{\partial y} = 0 \text{ (implicit)}$$

$$\overline{u_i u_j} - \text{wall functions}$$

Along the lateral boundaries - 2-3 & 1-4:

$$\frac{\partial U}{\partial z} = \frac{\partial V}{\partial z} = \frac{\partial W}{\partial z} = \frac{\partial P}{\partial z} = 0$$

$$\frac{\partial}{\partial z} \overline{u_i u_j} = \frac{\partial \epsilon}{\partial z} = 0$$

Along the free stream boundary - 3-4:

$$U = U_\infty = 1.0$$

$$P = P_\infty = 0$$

$$\frac{\partial V}{\partial y} = \frac{\partial W}{\partial y} = 0$$

$$\frac{\partial}{\partial y} \overline{u_i u_j} = \frac{\partial \epsilon}{\partial y} = 0$$

The initial plane was appropriately chosen to coincide with the plane wherein the experimental data of Klebanoff (1954) were taken, that is, 4.328m from the virtual leading edge. The U

velocity and non-zero $\overline{u_i u_j}$ profiles were obtained directly from the experimental data with ϵ equated to the production. The V velocity was calculated using the continuity equation. The initial W and pressure fields were set to zero. Based on $L_{ref}=76.2\text{mm}$ (i.e. the initial plane boundary layer thickness) and the free stream velocity, $U_{ref}=15.24\text{m/s}$, the Re value is 74,000.

7.1.3 Results & Discussion

For this relatively simple 2-D flow, which is not so demanding upon computing time, some numerical experiments on the sensitivity to different mesh sizes were possible. The four different meshes used were three uniform grids of $\Delta Y=0.05$ (43 nodes in the y -direction $\times$ 5 nodes in the z -direction), $\Delta Y=0.07$ (31 $\times$ 5 nodes), $\Delta Y=0.1$ (22 $\times$ 5 nodes) and a non-uniform grid expanding in the y -direction with $\Delta Y_1=0.06$ and $k=1.05$ (22 $\times$ 5 nodes). All the calculations were carried out for 60 $\Delta x=0.5$ steps or, equivalently, 30 initial boundary layer thicknesses. The turbulence model used was Model 2, and as will become clear later, the choice was quite arbitrary since neither Model 1 nor Model 2 showed any distinct superiority over the other model in this particular test case.

The predicted U profiles by all 4 meshes are identical as shown in Fig.7.3 and agreement with both the Klebanoff (1954) experimental data and the Coles (1956) profile are good, this is especially so with the Coles (1956) profile. The predicted U using the rather coarse uniform mesh of $\Delta Y=0.1$ is perhaps surprisingly good, of

course, relative to the local boundary layer thickness this mesh (and all the other meshes) does become more refined as calculations proceed downstream. For example, the number of grid nodes within the boundary layer actually increased from 11 (at $x=0$) to 14 (at $x=30$). Another point to note from this results is the acceptably good accuracy (of first-order) of PSTAG with the non-uniform mesh.

Although grid refinement will enhance the accuracy of the predictions, the use of the log-law to bridge the no-slip wall to the log-law region of the inner boundary layer imposes a minimum distance between the first turbulence quantity storage location at $(1\frac{1}{2}, J)$ and the wall. This is to ensure that $(1\frac{1}{2}, J)$ lies within the log-law region. In PSTAG, the log-law region, which starts at $y^+=60$, therefore gives a minimum ΔY_1 value of about 0.05 for this test case. However, the $\overline{uv}$ predictions in Fig.7.4 for the mesh with $\Delta Y_1=0.06$ (& $k=1.05$) and the finest (possible) mesh with $\Delta Y=0.05$ show that $\overline{uv}|_{2\frac{1}{2}, J}$ is slightly larger than u_τ^2 by 3% and 5% respectively. These "peaks" are due to the fact that the y^+ values at $(1\frac{1}{2}, J)$ (i.e. $y^+=65$ for $\Delta Y=0.05$ and $y^+=80$ for $\Delta Y_1=0.06$), according to Schubauer (1954) near-wall data, correspond to the region where $|\overline{uv}|$ is still slightly less than u_τ^2 . But since the present calculation has assumed $|\overline{uv}|_{1\frac{1}{2}, J}$ to be equal to u_τ^2 instead of $(u_\tau^2 - \text{viscous stress})$ in the $\overline{uv}$ -equation when calculating $|\overline{uv}|_{2\frac{1}{2}, J}$, one would expect $|\overline{uv}|_{2\frac{1}{2}, J}$ to be larger than u_τ^2 . In the log-law region,

$$\text{Viscous stress} = \mu \frac{\partial U}{\partial y} = \frac{\mu}{\kappa y} u_\tau$$

so,
$$\frac{\text{viscous stress}}{\text{wall stress}} = (\kappa y^+)^{-1}$$

The present results do compare well with the above expression which gives ratio of $|\overline{uv}|$ to u_τ^2 as 0.96 at $y^+=65$ and 0.97 at $y^+=80$. With the $\Delta Y=0.07$ mesh (where y^+ at $(1\frac{1}{2}, J)$ is 90), the predicted $\overline{uv}$ profile correctly indicates an almost constant shear stress region near the wall. From this analysis, it seems reasonable to take the predictions using $\Delta Y=0.07$ as being the "most" accurate results.

The other predicted turbulence quantities are shown in Fig.7.5 for $\overline{uu}$, Fig.7.6 for $\overline{vv}$, Fig.7.7 for $\overline{ww}$ and Fig.7.8 for k . Since the ratio of each normal stress to the shear stress remains almost constant across the boundary layer, near-wall "peaks" of similar magnitudes for the $\Delta Y=0.05$ and $\Delta Y_1=0.06$ ($k=1.05$) meshes are to be expected. The overall trends are generally similar in the sense that the magnitude of the predicted values (especially those near the wall) are larger for a finer mesh. This is best illustrated by the non-uniform grid - as the grid spacings become the coarsest towards the boundary layer edge, the predicted values become the smallest. This consistent trend clearly demonstrates a consistency in the numerical scheme itself; this is very important because it enables one to decide on an optimal mesh.

The $\Delta Y=0.07$ mesh with 31×5 nodes requires about 2.5 times more CPU time than the $\Delta Y=0.1$ mesh with 22×5 nodes. From the predicted results, the maximum difference in the predicted values is approximately 6% at $x=30$ which then reduces to 3.5% at $x=60$ (see

Fig.7.9 for k and Fig.7.10. for $\overline{uv}$). It is therefore justifiable to use $\Delta Y=\Delta Z=0.1$ on a uniform 22×22 node mesh to evaluate the two different wall functions (see §3.2.4) for 120 $\Delta x=0.5$ steps. The choice of a large number of spanwise stations was deliberate, in order to check on the numerical capability of PSTAG in maintaining the flow 2-D. In this aspect, the results obtained are good.

Fig.7.11 shows that the predicted mean U profiles by both Model 1 and Model 2 are almost similar and agree well with the data of Klebanoff (1954) and the Coles (1956) profile. The velocity-defect profiles are plotted in Fig.7.12. Since the near-wall proportions of $\overline{uu}$, $\overline{vv}$, $\overline{ww}$ and $\overline{uv}$ to k are different for Model 1 and Model 2, which in turn deviate slightly from the experimental values (see Table 7.1), the initial plane normal stresses were set to levels that are consistent with each model's values. In other words, Model 1 and Model 2 have different initial levels of $\overline{uu}$ and $\overline{ww}$. This was found necessary because the behaviour of $\overline{uv}$ is rather sensitive to these initial BCs (also observed by Younis, 1984).

In Fig.7.13, the magnitude of k is well predicted with both models giving very nearly identical magnitude. Fig.7.14 shows that there are some differences in the calculated $\overline{uu}$ profile in the region $y < 0.15\delta$, this is due to the wall damping function and Model 1 correctly gave a higher $\overline{uu}$ level. The predicted $\overline{vv}$ values by Model 1 are higher than Model 2 as shown in Fig.7.15. In fact the Model 1 $\frac{\overline{vv}}{k}$ ratio is about 0.35, this is both too high as well as not consistent with its near-wall value. The $\overline{ww}$ predictions in Fig.7.16 indicate that the $\frac{\overline{ww}}{k}$ ratio in both models agrees well

with the value in Table 7.1. When compared with the experimental data, the normal stresses can be said to be satisfactorily predicted.

The Model 1 $\overline{uv}$ profile shown in Fig.7.17 has an unusual "peak" of about 3% of u_τ^2 . In fact, when the initial levels of $\overline{uu}$ and $\overline{ww}$ were inconsistent with the derived proportions in Table 7.1, a larger peak of about 5% of u_τ^2 was obtained. The cause of this peaking is not entirely clear but it could possibly be due to the turbulence model itself, or more precisely the wall damping function, because Model 2 does not exhibit this "peak" (at least, not when y^+ at $(1\frac{1}{2}, J)$ is greater than 100). Since $\overline{uv}$ is the most important Reynolds-stress in this flow, perhaps, one can say that the performance of Model 2 is better than Model 1.

In Fig.7.18, the predicted C_f values are satisfactory since they lie between the well established von Karman and power-law formulae. However, both models underpredict the boundary layer thickness at the last calculated x station (at $x=4.572m$) by about 2% (see Fig.7.19). This is not too severe considering the uncertainty that may arise when one defines the boundary layer edge velocity as 0.995 of the free stream velocity. The momentum thickness shown in Fig.7.20 is well predicted.

7.1.4 Concluding Remarks

The mean velocity field and Reynolds-stresses are, in general, satisfactorily predicted. This, at least, indicates that the

present numerical algorithm is suitable for calculating turbulent slender flows. The near-wall turbulence quantities are rather sensitive to grid sizes or more correctly the y^+ value. A y^+ value of 100 at the first turbulence quantity location was found to be most desirable. The comparison between the wall damping functions of Eq.3.9 and Eq.3.10 does not show any overall superior performance in one particular model except for the unusual $\overline{uv}$ "peak" in the inner boundary layer in Model 1.

7.2 Boundary Layer With Adverse Pressure Gradient

7.2.1 Problem Background

The behaviour of a 2-D boundary layer in adverse pressure gradient has been fairly well investigated experimentally and most turbulence models (including lower-order closures) can predict the mean velocity field to reasonable accuracy. The presence of irrotational strain rates $\frac{\partial U}{\partial x}$ and $\frac{\partial V}{\partial y}$ (which are no longer insignificant) has shown up the deficiencies of RSTM, which has worked well for zero pressure gradient boundary layers. These irrotational strain rates are present in the equations for the normal stress as either the generation or pressure-strain terms or both (as in the case of the $\overline{uu}$ & $\overline{vv}$ equations), but, due to the continuity requirement, are absent in the $\overline{uv}$ -equation.

By a simple analysis in §3.3.5, it was shown that the level of ϵ in the wall adjacent region is underpredicted by the "standard" Launder et al. (1975) RSTM when $|\overline{uv}|$ is greater than u_τ^2 . Since $\frac{\partial U}{\partial x}$ and $\frac{\partial V}{\partial y}$ appear only in the generation term of the modelled ϵ -

equation, many workers have suggested means of making these irrotational strain rates contributive towards increasing the near wall ε to the desired level by increasing the ε generation term (see discussion in §1.2.2). The length-scale-limiting model proposed in §3.3.5 (denoted as Model 3) also follows this line of argument by making the coefficient $C_{\varepsilon 1}$ a function of $(|\overline{uv}|/u_\tau^2)$. The primary objective in calculating this flow can be said to be an assessment of the response of the RSTM (including modifications to the ε -equation) to extra rates of strain. It must be mentioned that Hanjalic et al. (1979) also calculated this flow to evaluate their multi-time-scale model.

This flow has been investigated experimentally by Bradshaw (1969) and the present calculations correspond to his boundary layer "C". In his experiment, an equilibrium boundary layer developed into a region of adverse pressure gradient where the free stream velocity varies as $x^{-0.255}$. Within the adverse pressure region, the boundary layer is in transit towards a second but different equilibrium state though in the final measurement station equilibrium was not yet attained.

7.2.2 Initial & Boundary Conditions

The computational domain and physical geometry of this flow are identical to the flat plate situation but for sake of completeness are shown in Fig.7.21. The mesh has 41x4 nodes with $\Delta Y=0.1$ and $\Delta Z=0.01$. With 41 nodes in the y-direction, the computational domain at the initial plane is four times the boundary layer

thickness whereas at the last calculation plane the ratio is only 1.5. Clearly, an adaptive grid system would be more economical in this situation. The arbitrarily chosen number of nodes (i.e. 4) in the z-direction as well as the size of ΔZ is not important because the flow is essentially 2-D. As will be seen in the 3-D boundary layer test case in §7.3, the size of ΔZ does become crucial in ensuring numerical stability when the non-zero W velocities are relatively large even though spanwise gradients are zero.

The BCs in the crossflow plane as defined in Fig.7.22 are as follow.

Along the wall - 1-2:

$$U = V = W = 0$$

$$\frac{\partial P}{\partial y} = 0 \text{ (implicit)}$$

$$\overline{u_i u_j} - \text{wall functions}$$

Along the lateral boundaries - 2-3 & 1-4:

$$\frac{\partial U}{\partial z} = \frac{\partial V}{\partial z} = \frac{\partial W}{\partial z} = \frac{\partial P}{\partial z} = 0$$

$$\frac{\partial}{\partial z} \overline{u_i u_j} = \frac{\partial \epsilon}{\partial z} = 0$$

Along the free stream boundary - 3-4:

$$P = P_\infty(x)$$

$$\frac{\partial U}{\partial y} = \frac{\partial V}{\partial y} = \frac{\partial W}{\partial y} = 0$$

$$\frac{\partial}{\partial y} \overline{u_i u_j} = \frac{\partial \varepsilon}{\partial y} = 0$$

In contrast to zero pressure gradient flow, here, the zero Neumann BC $\frac{\partial U}{\partial y}=0$ is employed along the boundary 3-4. Alternatively, one can also specify the free stream velocity value directly (i.e. Dirichlet BC) by using the Bernoulli's equation and the known $\frac{dP}{dx}$ values. Bearing in mind that PSTAG was developed during the course of this work and therefore needs as many varied forms of testing as possible, it seems more appropriate not to employ the normally more stable Dirichlet BC for U along the free stream boundary.

The initial BCs of U, $\overline{uu}$, $\overline{vv}$, $\overline{ww}$ and $\overline{uv}$ were constructed from the experimental data in the first measurement station and are shown in the (a) plots of Fig.7.23 to Fig.7.27. The reference velocity is the wind tunnel speed ($U_{ref}=37\text{m/s}$) and L_{ref} is the initial boundary layer thickness ($L_{ref}=28.575\text{mm}$), thus giving an Re value of 71,285.

7.2.3 Results & Discussion

This test case was predicted by four different RSTM "setups", all differing in the form of the generation term in the ε -equation. The first "setup" employs Model 2 with the standard empirical constants (i.e. those that were used in §7.1) and will simply be referred to as Model 2. The second and third "setups" are respectively Model 1 and Model 2 but with $C_{\varepsilon 1}=1.6$; this $C_{\varepsilon 1}$ value is a 10% increment from the standard value and was found to be the

most suitable value for this flow. The final "setup" is Model 3. All calculations started from the same initial plane with the same initial BCs. The x-step size being $\Delta x=0.5$ is about 0.35 the local δ at the second x-station (at $x=305\text{mm}$) and 0.18 the local δ at the last x-station. It must be mentioned that identical results were also obtained using $\Delta x=0.25$ for Model 3. Comparison between predictions and experimental data are made at five streamwise stations at intervals of 305mm.

The development of the mean U profiles are plotted in Fig.7.23. Model 1 ($C_{\epsilon 1}=1.6$), Model 2 ($C_{\epsilon 1}=1.6$) and Model 3 give very similar results at all five x-stations, which agree well with the measured data. There are slight underpredictions at $x=305\text{mm}$ in the region $y \leq 20\text{mm}$ (or roughly half the local δ) and overpredictions in the part of the boundary layer near the edge at $x=610\text{mm}$ & 915mm . However, these discrepancies are not severe as these are no more than 3%. The Model 2 predictions show a departure from the other three "setups" as well as the experimental data starting from $x=915\text{mm}$ and continues to predict larger U values till the last station.

The differences in the near-wall U values from the four "setups" are more clearly depicted in the C_f plot in Fig.7.24 because U and u_τ are linked directly by the log-law. At $x=305\text{mm}$, all "setups" overpredict C_f by about 8% but further downstream the overprediction of Model 2 becomes very much larger and at the last station is about 25%. Both Hanjalic & Launder (1980) and Younis (1982) also overpredicted the C_f value at the last station by

about 30% when adopting the standard RSTM constants. The former employed the 2-D form of the Launder et al. (1975) model whereas the latter used the simplified $(\phi_{ij} + \phi_{ji})_2$ model of Naot et al. (1970). The overestimations of C_f by Model 1 ($C_{\varepsilon 1}=1.6$) is in the range 8%-11% and for Model 2 ($C_{\varepsilon 1}=1.6$) is 5-7%. The results of Model 3 is similar to those of Model 2 ($C_{\varepsilon 1}=1.6$) except at the final station where Model 3 overpredicts by only 1.5%. These C_f values do confirm the wide applicability of the simpler form of the law of the wall (albeit over a small range of y^+ , of course) and also indicate the small effects of $\frac{dP}{dx}$ on the inner region of the boundary layer.

These improved C_f results, obtained by modifying the generation term in the ε -equation, compare favourably with the more complicated Hanjalic & Launder (1980) irrotational strain rate model. Although their original paper did not cite the actual C_f values, the plot also shows some slight overpredictions. It must be mentioned that Younis (1982), in using the same irrotational strain model, quoted an overestimation of about 8% at the final measured station, which is quite clearly much larger than the originators' results.

With the final station C_f overpredicted by 25% and the relatively poorer U profiles, the good agreement (in fact it is the best) of δ in Fig.7.25 by Model 2 should not be taken as an indication that the turbulence model parameters are the most suitable. The other three "setups" predicted values of δ that are smaller than the measured ones and the maximum difference, which occurs at $x=305\text{mm}$,

is about 6%. Again the results of Model 3 are relatively more superior, at $x=1525\text{mm}$, δ is underpredicted by 2.6% as compared with 5.9% by Model 2 ($C_{\epsilon_1}=1.6$) and 4.3% by Model 1 ($C_{\epsilon_1}=1.6$).

From the x-momentum equation, one knows that the mean velocity U is dependent upon $\frac{dP}{dx}$ as well as the shear stress gradient $\frac{\partial \overline{uv}}{\partial y}$. As such, it is most appropriate to look closely at the predicted shear stress distribution in Fig.7.26. The Model 2 maximum $\overline{uv}$ values increase with distance downstream from the initial plane to $x=915\text{mm}$ before they begin to decrease. This trend is not in accord with the measured data. The predictions of the other three "setups" lie fairly close to each other, with good agreement in the inner layer but overprediction over most part of the boundary layer. These results are reasonably accurate though the performance of Model 3 is, on the whole, better. It is difficult to make a proper quantitative assessment at the final station because of doubts cast on the experimental data. The measured $\overline{uv}$ show an increase in magnitude from the values in the previous station and is, therefore, not in line with the decreasing trend exhibited so far (this observation also applies to the normal stresses).

Since the ratio of the inner layer shear stress gradient $\frac{\partial \overline{uv}}{\partial y}_{IR}$ to $\frac{dP}{dx}$ (based on Model 3) is an average of 0.29 at $x=305\text{mm}$ and increases to 0.7 at $x=1525\text{mm}$, one can clearly see the importance of the $\overline{uv}$ profile in determining an accurate prediction of U . At all x-stations, Model 2 has the largest $\frac{\partial \overline{uv}}{\partial y}_{IR}$ and consequently gave values of U that are also too large. In the other three

"setups", $\frac{\partial \overline{uv}}{\partial y}_{IR}$ is roughly similar, thus predicting almost identical U . The fact that Model 3 has the best C_f results is simply because it also has the best $\overline{uv}$ profile and naturally the "most" accurate $\frac{\partial \overline{uv}}{\partial y}_{IR}$.

The predicted normal stresses are plotted in Fig.7.27 for $\overline{uu}$, Fig.7.28 for $\overline{vv}$, Fig.7.29 for $\overline{ww}$ and Fig.7.30 for k . The trends in all the normal stresses are similar to those found in k , so for sake of conciseness, discussion will be focussed mainly on the results of k . The predictions of Model 2, other than at $x=305\text{mm}$, are larger in magnitude than the other three "setups" (which themselves lie fairly close to each other), besides, one can also see that the peak value of k shows an unusual increase between $x=0$ to $x=905\text{mm}$. In view of this rather poor quantitative agreement with the experimental data, the following discussion will concentrate only on the other three "setups".

The k profiles in the outer 70% of the boundary layer are reasonably well captured. But within the inner 30% of the boundary layer, there are underpredictions and in some instances, these can be as severe as 40%. There are three possible causes for these underpredictions: (a) the specification of the ratio of the normal stresses to u_τ^2 , (b) the value of ϵ is too high and (c) inadequacy in the turbulence model. Each of these possible causes will now be examined in turn.

The large discrepancies between the measured and predicted normal stress values at the first turbulence quantity location from the

wall simply imply that the normal stress to u_t^2 ratios specified in this calculation (i.e. those indicated in Table 7.1) are lower than the corresponding measured proportions, for example, in Table 7.1 $\frac{\overline{uv}}{k} \approx 0.28$ but from the measured data $\frac{\overline{uv}}{k} \approx 0.2$. Although one can provide the measured proportions (which in this flow vary in the streamwise direction), it was found that this practice did not give significantly different/better profiles (except the wall adjacent values) because this is a region where convection is negligible and the Reynolds-stresses depend strongly on the mean rates of strain. To illustrate this point, one can see that the predicted k profiles by Model 1 and Model 2 in Fig.7.13 are almost identical although the algebraically-specified values adjacent to the wall are different.

As the re-modelling of the ε -equation, via $C_{\varepsilon 1}$, sought to limit the turbulence length scale to κy , it seems quite successful because the computed l_s ($= \frac{-\overline{uv}}{\varepsilon}^{3/2}$) is about 4% less than κy . (On the other hand, the ε -equation with $C_{\varepsilon 1}=1.45$ overpredicts l_s by approximately 30%.) With l_s being sufficiently accurate, it seems unlikely that the ε -equation is the cause of the underprediction of k for $y \leq 0.36$.

Townsend (1961) hypothesized that the inner layer turbulent motion consists of an "active" and an "inactive" part. The "active" motion generates the shear stress and has universal length scales and velocity scales. The "inactive" larger-scale eddies are irrotational and since these do not contribute to the shear stress, the smaller universal scales remained disturbed. The

present calculation is able to accurately simulate the inner layer U and $\overline{uv}$ profiles but underpredicts k . This implies that the RSTM is able to capture the $\overline{uv}$ producing "active" motion but is inadequate in simulating the "inactive" motion.

According to Bradshaw (1967), one of the main producers of this "inactive" motion is the pressure fluctuations generated by the turbulence in the outer layer. As such, no method with a one-point pressure-strain model will be able to predict the "inactive" motion. In other words, there is little doubt that the present pressure-strain model, which works well in zero pressure gradient boundary layers where the "inactive" motion is weak, is not "complete".

7.2.4 Concluding Remarks

From this calculation of an adverse pressure gradient boundary layer, which represents perhaps the simplest of the complex flows, it is evident that turbulence models which have been calibrated for simple flows may not work well for flows with extra rates of strain. The present results, therefore, justify the need to re-model the ϵ -equation, which is accomplished here either by increasing $C_{\epsilon 1}$ from 1.45 to 1.6 or by the length-scale-limiting model, so as to produce the desired effects of reducing l_s near the wall. The performance of the proposed length-scale-limiting model was found to be successful. Another interesting finding is that the RSTM (whether the standard or improved modified form), is incapable of capturing the turbulence generated by the "inactive"

motion in the inner layer.

7.3 "Infinite" Swept Wing With Zero Spanwise Gradients

7.3.1 Problem Background

The turbulent boundary layer on an "infinite" swept wing is basically 3-D and, therefore, necessitates the solution for all three mean velocity components as well as the six Reynolds-stress components. The skew-induced (or Prandtl's first kind) secondary flow is generated by a favourable pressure gradient $\frac{\partial P}{\partial z}$ which deflects the W velocity towards the wing tip. There are two sets of coordinate systems in this flow configuration, one is the wind tunnel axes, denoted by x-y-z, and the other is the swept wing axes, denoted by x'-y-z'. Consequently, this flow can be calculated using either coordinate system. In this section, the x'-y-z' axes, with zero spanwise gradients, is used. (NOTE: The velocities with reference to the x'-y-z' axes will be superscripted ', e.g. U' & u').

There are several essential features inherent in this flow which make it very useful for evaluating the performance of the turbulence model. In addition to the extra (irrotational) rates of strain, $\frac{\partial U}{\partial z}$ and $\frac{\partial V}{\partial y}$, experienced in the previous test case on 2-D adverse pressure gradient boundary layer (see §7.2), there is also $\frac{\partial W}{\partial y}$. Therefore, in addition to $\frac{\partial U}{\partial y}$ and $\frac{\partial W}{\partial y}$ as the most significant strain rates, there are seven other possible extra strain rates, all having the potential to influence the turbulence structure. Bradshaw & Pontikos (1985) hypothesized that $\frac{\partial W}{\partial y}$ "tilts" the large

2-D eddies sideways and thereby reduces the efficiency of the main $\overline{uv}$ producing strain rate $\frac{\partial U}{\partial y}$. Consequently, the magnitude of $\overline{uv}$ and its ratio to k is reduced. Another feature lies in the non-coincidence of the directions of the shear stress vector and mean velocity gradient vector (in fact, the former lags the latter). This undoubtedly makes the RSTM the more appropriate turbulence model to use because isotropic ν_t -models are incapable of reproducing this lag.

The experimental data in this test case were obtained from Pontikos (1982), whose experimental setup up was in many ways similar to the work of van den Berg et al. (1975) and Elsenaar & Boelsma (1974) (where appropriate the latter two papers will be referred collectively as the "NLR experiments"). It is important to mention that the swept wing, referred to here, was simulated by a flat plate instead of an aerofoil, hence there is no need to account for surface curvature effects. The wing leading edge is swept at 35° (see Fig.7.31). In order to achieve and maintain the "infinite" swept wing condition as the boundary layer develops, the gradient w.r.t. the z' -axis was kept as close to zero as possible. In many respects, the Pontikos (1982) experiment more closely resembled an "infinite" swept wing than the NLR work, the free stream velocity vector in the latter work was more severely deflected by crossflow.

7.3.2 Initial & Boundary Conditions

In using the swept wing x' - y' - z' axes as shown in Fig.7.31, the

calculations can be treated as 2-D because spanwise gradients are zero although the demands (i.e. number of equations to be solved) are still 3-D. A uniform 36x4 mesh with $\Delta Y=0.1$ and $\Delta Z'=0.01$ was employed. The choice of $\Delta Y=0.1$ enables y^+ at the first turbulence quantity location away from the wall to fall within the range 50-76. The large W' velocities, of $O(U)$, require the size of $\Delta Z'$ to be kept sufficiently small in order to avoid divergence in the numerical solution due to excessively large Re_{CELL} . The $\Delta Z'$ limit was found to be 0.01. The smallness of $\Delta Z'=0.01$ ($=0.2\text{mm}$) is of course neither practical nor economical when one needs to calculate over a large span, but bearing in mind that since the spanwise gradients are zero, the spanwise mesh distribution is not important. The crossflow plane BCs shown in Fig.7.32 are as follow.

Along the wall - 1-2:

$$U' = V = W' = 0$$

$$\frac{\partial P}{\partial y} = 0 \text{ (implicit)}$$

$$\overline{u_i u_j} - \text{wall functions}$$

Along the lateral boundaries - 2-3 & 1-4:

$$\frac{\partial U'}{\partial z'} = \frac{\partial V}{\partial z'} = \frac{\partial W'}{\partial z'} = \frac{\partial P}{\partial z'} = 0$$

$$\frac{\partial}{\partial z'} \overline{u_i u_j} = \frac{\partial \epsilon}{\partial z'} = 0$$

Along the free stream boundary 3-4:

$$P = P_{\text{ext}} \text{ (specified)}$$

$$\frac{\partial U'}{\partial y} = \frac{\partial V}{\partial y} = \frac{\partial W'}{\partial y} = 0$$

$$\frac{\partial}{\partial y} \overline{u_i u_j} = \frac{\partial \varepsilon}{\partial y} = 0$$

The initial BC for the velocities and Reynolds-stresses (see (a) plots) were obtained from the experimental data at Pontikos (1982) first measured station. Based on L_{ref} , the initial plane boundary layer thickness, of 20mm and U_{ref} , the wind tunnel speed, of 33.7m/s the Re value is 43000. There is a total of four measured x' -stations and the distance between adjacent stations in the x' - y - z' axes is 160mm. The results to be presented in the next section are those with $\Delta x' = 0.25$, that is, 99 x -steps between the first and last station.

7.3.3 Results & Discussion

The turbulence models employed in this test case are Model 1 and Model 2, with the empirical constant $C_{\varepsilon 1} = 1.6$, and Model 3. The prediction of the U' profiles by all three models, shown in Fig.7.33, are similar and agreement with experimental data at all three x' -stations is good although there tend to be overpredictions towards the edge of the boundary layer. The calculated W' velocities are slightly larger but can be said to be also satisfactorily predicted. Under ideal "infinite" swept wing conditions, the boundary layer edge spanwise velocity W'_e is a

constant and in this case of 35° swept $W'_e = 0.5735$. The present calculations were able to correctly maintain W'_e at this constant magnitude whereas in the experimental work there were some variations. Hence, the difference between the predicted and measured values at $x' = 160\text{mm}$ would be much reduced if not for the experimental errors. Since the total boundary layer edge velocity has been correctly calculated, it is inevitable that the discrepancies in W'_e at $x' = 160\text{mm}$ & 480mm would lead to some differences in U'_e too.

The calculated C_f values in Fig.7.35 are quite accurate. Model 2 has the best results and the overprediction at the final station is about 2%: at this station Model 1 and Model 3 overpredict by 6% and 8% respectively. The calculated boundary layer thickness (see Fig.7.36) is too thin and as will become clear later, this leads to mismatch near the boundary layer edge between the predicted and measured profiles of all flow quantities, though this is more evident in the $\overline{u_i u_j}$ profiles. The underprediction of δ can be mainly attributed to the inability of the RSTM to give accurate results for $\overline{u'v}$ and $\overline{vw'}$. The total velocity vector Q and its angle w.r.t. the external streamline, denoted by $\angle(Q)$, are shown in Fig.7.37 and Fig.7.38 respectively. For clarity, only the results by Model 2 are plotted since all three models predicted almost identical mean U' and W' profiles. Due to the differences in the measured and computed U'_e and W'_e , the directions of the measured and calculated total boundary layer edge velocity vectors do not coincide, the largest difference which occurs at $x' = 160\text{mm}$ is, however, small and equal to 1.55° . The calculated $\angle(Q)$ at all

three x' -stations, other than the mismatch caused by δ , agree quite well with the experimental results.

All the Reynolds-stresses in the region from $y \approx 0.7\delta$ to $y = \delta$ are underpredicted. The main cause of this is due to the calculated δ being too small, thereby, resulting in the profile mismatch as mentioned earlier. This point is, perhaps, best illustrated by looking at the calculated results at $x' = 160\text{mm}$. Since the mismatch of δ is about 2mm (or 7%), reasonable agreement with the experimental data is obtained. In view of this mismatch, it is not appropriate to analyse the RSTM in detail within this region though one should not expect the turbulence model to behave too abnormally.

Fig.7.39 shows that although the wall $\overline{u'v}$ values are accurately calculated, over most part of the boundary layer $\overline{u'v}$ is overpredicted and at the final station the error can be as large as 100%. It is obvious that the present form of RSTM is unable to reproduce the reduction in the magnitude of $\overline{u'v}$ caused by three-dimensionality. Looking at the modelled $\overline{u'v}$ -equation (where $\frac{\partial}{\partial z}$ is zero), $\frac{\partial W'}{\partial y}$ exists as a rather insignificant term in the "rapid" pressure-strain model, as such, $\overline{u'v}$ cannot be expected to respond directly to $\frac{\partial W'}{\partial y}$. Even the generation term, which is mainly $-\overline{vv}\frac{\partial U'}{\partial y}$, suggests that the response would be rather slow because the $\overline{vv}$ -equation (just like the $\overline{u'v}$ -equation) does not have $\frac{\partial W'}{\partial y}$ as an important strain rate and $\frac{\partial U'}{\partial y}$ is not strongly affected by $\frac{\partial W'}{\partial y}$.

The computed $\overline{vw'}$ results in Fig.7.40 has a trend similar to $\overline{u'v}$,

good agreement at the wall but generally overprediction. Bradshaw (1987) pointed out that although the $\overline{vw'}$ generation term contains $\frac{\partial \overline{w'}}{\partial y}$, its response to pressure gradient is also slow. Since the generation terms of $\overline{u'v}$ and $\overline{vw'}$ are exact, the inadequacy in the RSTM most probably lies in the pressure-strain model. The most significant term in the "rapid" redistribution model is the first term on the r.h.s. of Eq.3.7 which is in the form of the generation term itself. This is a weakness in the model because the generation terms themselves are unable to respond sufficiently to three-dimensionality. Naturally, the increasingly popular simplified "rapid" pressure-strain model (see §3.2.3) will do no better, if not worse.

At this point, it is most appropriate to discuss the near-wall experimental data of $\overline{u'v}$ and $\overline{vw'}$. Unlike the profiles in a 2-D adverse pressure gradient boundary layer, the Pontikos (1982) data show a drop (or "dip") in the magnitude of the shear stress profiles away from the wall, the "dip" in the $\overline{vw'}$ profile being more severe. On the other hand, the NLR results show an increase in magnitude in the near-wall values even though both experimental setups were in many ways very similar. In a separate experiment on "infinite" swept curved wings by Baskaran et al. (1990), the shape of the shear stress profiles are similar to the NLR measurements (or 2-D adverse pressure gradient flow). The turbulence structure was found to be little affected by mild curvature of either sign, the ratio of initial δ to radius of curvature being 1%. There are, therefore, some uncertainties regarding the near-wall experimental data. If the NLR data are more reliable, then the discrepancies

between the present predictions and measurements would not be so large. It must be added that the three-dimensionality phenomena (i.e. a reduction in the magnitude of the total shear stress $\tau [= \sqrt{\overline{u'v^2} + \overline{vw'}^2}]$ as well as the ratio $\frac{\tau}{k}$) are not captured by the computations. In fact, the ratio of $\frac{\tau}{k}$ is approximately 0.3, a value similar to the 2-D zero pressure gradient flow.

The directions of the Model 2 shear stress vector $\angle(\overline{u'v}, \overline{vw'})$ and mean flow gradient vector $\angle(\frac{\partial U'}{\partial y}, \frac{\partial W'}{\partial y})$ w.r.t. the external streamline are shown in Fig.7.41. The first $\angle(\overline{u'v}, \overline{vw'})$ and $\angle(\frac{\partial U'}{\partial y}, \frac{\partial W'}{\partial y})$ values from the wall are close to the wall $\angle(Q)$ value (indicated by $\odot$), this is because the form of the log-law used here does not take into account any lagging effects. For most part of the boundary layer, the computation is able to show the shear stress vector lagging the mean flow gradient vector, but the lag is smaller than indicated by experiment. The good predictions of $\angle(\frac{\partial U'}{\partial y}, \frac{\partial W'}{\partial y})$ are not surprising because both Q and $\angle(Q)$ have been well predicted.

The predicted normal stresses $\overline{u'u'}$ (Fig.7.42), $\overline{vv}$ (Fig.7.43) and $\overline{w'w'}$ (Fig.7.44) at $x'=160\text{mm}$ agree well with the measured data but at $x'=320\text{mm}$ & 480mm , there are overpredictions in the inner half of the boundary layer. For convenience, these normal stresses will be discussed collectively as k which is shown in Fig.7.45. The region of overprediction of k at $x'=320\text{mm}$ & 480mm corresponds to the region where $\overline{u'v}$ and $\overline{vw'}$ are also overpredicted. Since the production of k is dominated by $-\left[\overline{u'v} \frac{\partial U'}{\partial y} + \overline{vw'} \frac{\partial W'}{\partial y} \right]$, it will be too large. Individually, $-\overline{u'v} \frac{\partial U'}{\partial y}$ and $-\overline{vw'} \frac{\partial W'}{\partial y}$ are the important

generation terms of $\overline{u'u'}$ and $\overline{w'w'}$ respectively, as such, the calculated levels of $\overline{u'u'}$ and $\overline{w'w'}$ are also too high.

Since the role of ϵ , as a sink term, is in a way to counteract the growth of k , the k results may suggest that the near-wall ϵ values are too small or l_ϵ is too large. Contrary to this speculation, the calculated $l_\epsilon (= \frac{\tau^{3/2}}{\epsilon})$ was found to be less than κy . Although the Reynolds-stresses are most strongly overpredicted by Model 3 at $x'=160\text{mm}$ & 320mm , the Model 3 l_ϵ value is much closer to κy than the l_ϵ of Model 1 or Model 2. These findings suggest that the ϵ -equation is probably not the source of the problem but instead the weakness of the RSTM lies in the modelled $\overline{u_i u_j}$ -equations.

The $\overline{u'w'}$ profiles are shown in Fig.7.46 with predictions showing only qualitative agreement and errors can be as large as 100%. Since the $\overline{u'w'}$ generation is approximately $-\left[\overline{u'v} \frac{\partial w'}{\partial y} + \overline{vw'} \frac{\partial u'}{\partial y} \right]$ and with $\overline{u'v}$ and $\overline{vw'}$ overpredicted, the underpredictions for $y < 0.5\delta$ is rather unexpected, thus leading one to deduce, once again, the inadequacy of the pressure-strain model. It is interesting to note that with the magnitude of $\overline{u'w'}$ being the same order as $\overline{vw'}$, the omitted term $\frac{\partial \overline{u'w'}}{\partial x'}$ (based on the slender flow approximations) in the z -momentum equation can sometimes be significant. If this happens to be the case, then the slender flow equations become invalid.

The underprediction of δ and the consequent mismatch between predicted and measured profiles near the boundary layer edge does not allow a close examination of the turbulent diffusion model. It

was pointed out in Bradshaw & Pontikos (1985) that the turbulent diffusion process was found to be somewhat inhibited by the 3-D nature of the flow. This implies that the present diffusion model constant C_g will be too large/strong. As such the actual response of the present gradient diffusion model is still left to be examined.

7.3.4 Concluding Remarks

It is clear that the set of empirical constants which have worked well in the 2-D adverse pressure gradient boundary layer are simply not the optimal ones for this test case. This is because three-dimensionality of the mean flow is known to alter the turbulence structural parameters and the constants in the RSTM can or perhaps should be treated as turbulence structural parameters. Since the exact generation terms of $\overline{u'v}$ and $\overline{vw'}$ are unable to respond to $\frac{\partial w'}{\partial y}$ and, hence, not able to predict the reduction in the shear stress magnitude it is reasonable to say that the pressure-strain model is inadequate. The overpredictions of $\overline{u'v}$ and $\overline{vw'}$ quite inevidently lead to overprediction in the production of k and consequently k itself (and the normal stresses).

Despite the above mentioned shortcomings of the RSTM, it is able to capture the general behaviour of the turbulence quantities rather well, for example, the sign of $\overline{u'w'}$ and $\overline{vw'}$ is right. The RSTM is also able to show the shear stress vector lagging behind the mean velocity gradient vector and, although the lag is underpredicted, the direction of the mean velocity gradient vector

has been accurately calculated.

7.4 "Infinite" Swept Wing With Non-zero Spanwise Gradients

7.4.1 Problem Background

In this test case, the "infinite" swept wing was calculated using the wind tunnel axes x - y - z . Since the flow is the same, a rotation of axes should still give the same predictions. But the problem posed to the calculation method is a different one and in the simplest sense the computational domain is 3-D, in other words, spanwise gradients are non-zero. These non-zero spanwise gradients means the BCs may not be implemented in a straightforward manner (see discussion in §7.4.2).

Unlike the x' - y - z' axes, where $\frac{\partial w'}{\partial y}$ is of one sign (i.e. positive), here $\frac{\partial w}{\partial y}$ changes from positive to negative within the boundary layer and the latter strain rate is roughly five times less than the former strain rate. This, of course, does not imply any change in the physical turbulence structure but one can expect the behaviour (of some) of the $\overline{u_i u_j}$ -equations to be different because of changes in, say, the magnitude of the strain rates. The main objective in employing the x - y - z axes is simply to make comparisons with the x' - y - z' axes results in order to see if any abnormal differences are present.

7.4.2 Initial & Boundary Conditions

The physical flow configuration is, of course, identical to the

previous test case in §7.3 but the orientation of the computational domain is different as depicted in Fig.7.47. Although the crossflow plane BCs shown in Fig.7.48 are similar to those in Fig.7.32, they are, however, approximated forms of the actual BCs (this point will become clear in the discussion to follow).

Along the wall - 1-2:

$$U = V = W = 0$$

$$\frac{\partial P}{\partial y} = 0 \text{ (implicit)}$$

$$\overline{u_i u_j} - \text{wall functions}$$

Along the lateral boundaries - 2-3 & 1-4:

$$\frac{\partial U}{\partial z} = \frac{\partial V}{\partial z} = \frac{\partial W}{\partial z} = \frac{\partial P}{\partial z} = 0$$

$$\frac{\partial}{\partial z} \overline{u_i u_j} = \frac{\partial \epsilon}{\partial z} = 0$$

Along the free stream boundary - 3-4:

$$P = P_o(x, z)$$

$$\frac{\partial U}{\partial y} = \frac{\partial V}{\partial y} = \frac{\partial W}{\partial y} = 0$$

$$\frac{\partial}{\partial y} \overline{u_i u_j} = \frac{\partial \epsilon}{\partial y} = 0$$

Strictly speaking, none of the lateral BCs are of the zero Neumann type as specified above. To further complicate the issue, the

gradient $\frac{\partial}{\partial z}$ varies both in the x- and y- directions. Hence, the correct BCs are the Dirichlet BCs which, in turn, are very difficult to implement because the boundary values also change in both the x- and y- directions. Except for $\frac{\partial P}{\partial z}$, the z-wise gradients of the other quantities are adequately small such that the zero Neumann BCs become reasonably good approximations. It was found that with 22 nodes in the z-direction, the "inaccurate" lateral BCs penetrated only four nodes from either lateral boundaries at the final computation plane and, thereby, leave the central part of the domain unaffected.

In the slender flow equations Eq.2.10, one decouples the crossflow plane pressure field from the streamwise pressure gradient by setting $\frac{dP}{dx}$ a priori and assuming $\frac{\partial P}{\partial y} \sim \frac{\partial P}{\partial z} \approx 0$. The latter assumption is a rather severe constraint in this test case because $\frac{\partial P}{\partial z}$ is of the order of $\frac{dP}{dx}$. Since the crossflow plane pressure field and mean velocity field are strongly linked in the numerical solution, here one can allow the calculated pressure field to respond by itself to the secondary flow.

As in the previous test case, the initial profiles in the (a) plots were constructed from the experimental data of Pontikos (1982). With the initial plane boundary layer thickness as L_{ref} (= 20mm) and the wind tunnel speed as U_{ref} (= 33.7m/s), the Re value is 43000. Calculations were made with Model 1 and Model 2, using a value of $C_{\epsilon 1}=1.6$, and Model 3. The x-distance between adjacent measurement stations is 200mm. The results shown below are for $\Delta x=0.5$.

7.4.3 Results & Discussion

Fig.7.49 shows the U profile at $x=200\text{mm}$ to be well predicted but at the next two stations the results are larger than the measured values by about 10%. The W profiles in Fig.7.50 indicate that the initial profile remains almost unchanged with distance downstream. W is underpredicted because the magnitude of $\frac{\partial P}{\partial z}$ is considerably smaller than the experimental value. This simply implies that the pressure BCs are certainly not the most appropriate ones. More importantly, the W profiles do show the change in sign of $\frac{\partial W}{\partial y}$ and over most part of the boundary layer it is about 30%-70% of the measured gradient, in this respect, one can say the basic essence of the flow is not lost.

The calculated profiles of the total velocity Q in Fig.7.51 are similar to those obtained using the $x'-y-z'$ coordinates (see Fig.7.37). Since Q_0 is determined primarily by $\frac{dP}{dx}$ (because W_0 is small) it is accurately predicted at all three x -stations. Although W is underpredicted, the Q profile adjusts itself to the Q_0 value and therefore causes U to be overpredicted. Due to the magnitude of W being small, the skewing of Q is mild, consequently, $\angle(\frac{\partial U}{\partial y}, \frac{\partial W}{\partial z})$ is also small. The calculated δ is too small, this leads to profile mismatch (mentioned in §7.3.3) towards the edge of the boundary layer. In fact, the magnitude of δ is almost identical to those shown in Fig.7.36. From Fig.7.52, one can see that the C_f values are well predicted, again these are similar to those obtained using the $x'-y-z'$ axes.

The calculated wall $\overline{uv}$ values in Fig.7.53 are in good agreement with the experimental data but the "dips" are not reproduced. The predicted $\overline{vw}$ profiles in Fig.7.54 are at best qualitative, though it must be mentioned that the scatter in the measurements do indicate some degree of uncertainties. There is no doubt that the sign of the computed $\overline{vw}$ values near the wall is correct as this can be easily verified from the measured C_f and the wall shear stress direction. With the generation of $\overline{vw}$ approximately equal to $-\overline{vv}\frac{\partial W}{\partial y}$, the predicted $\overline{vw}$ profiles qualitatively follow the strain rate $\frac{\partial W}{\partial y}$ or more correctly, $\frac{\partial W}{\partial y}$. The reduction in the magnitude of both τ and the ratio $\frac{\tau}{k}$ is not captured and the latter has a value identical to the zero spanwise gradient calculations, that is, $\frac{\tau}{k} \approx 0.3$.

The values of k plotted in Fig.7.55 are "correctly" at the same level as the $x'-y-z'$ axes results in Fig.7.45. The predictions of the normal stresses are shown in Fig.7.56 for $\overline{uu}$, Fig.7.57 for $\overline{vv}$ and Fig.7.58 for $\overline{ww}$. The $\overline{vv}$ profiles are similar to the $x'-y-z'$ axes results. One can see that in the region $y \leq 0.3\delta$, the $\overline{uu}$ predictions are good but, on the other hand, $\overline{ww}$ is strongly overpredicted and this, in turn, contributes mainly to the large k values in the same region. It is interesting to note from Table 7.2 that the ratios of the normal stresses to k predicted by the RSTM is different for different coordinate systems.

From Fig.7.59, one can see that the sign of the shear stress $\overline{uw}$ is correct within most part of the boundary layer but the quantitative agreement is rather poor. In using the zero Neumann

BC near the wall, the calculated value next to the wall has the wrong sign. This can be remedied by employing an eddy-viscosity near-wall algebraic expression because both $\frac{\partial U}{\partial z}$ and $\frac{\partial W}{\partial x}$ are positive. One would find the improvements to be modest, that is, the sign is correct but the magnitude is close to zero and hence does not significantly alter the profiles shown in Fig.7.59.

7.4.4 Concluding Remarks

This test case demonstrates perhaps a major difficulty in employing the slender flow equations in situations where $\frac{\partial P}{\partial z}$ is no longer negligible. The pressure BCs used here are clearly not the most appropriate because the magnitude of $\frac{\partial P}{\partial z}$ is too small, consequently, W is underpredicted and the skewing of the x - z plane total velocity vector becomes mild. With the performance of the RSTM in 3-D boundary layers already being discussed with reference to the x' - y - z' axes in §7.3.3, one can say that the Reynolds-stress results in the x - y - z axes do give confidence in the implementation of the turbulence model in PSTAG (i.e. the numerical aspects which include the discretizations, the method of solution etc).

7.5 Embedded Vortex In A Boundary Layer

7.5.1 Problem Background

Longitudinal vortices embedded in boundary layers can be found in a variety of practical engineering situations. Some examples include horseshoe vortices formed by an obstacle protruding from a

surface (like the turbomachine blade-hub and wing-body junctions), Taylor-Görtler vortices in boundary layers on concave surfaces, helicopter blade tip vortex impinging on the trailing blade and wingtip vortices impinging on downstream boundary layers. Although some of these vortex-boundary layer interactions may not always be desirable, the use of vortex generators (see Stein, 1985) is a case where the vortices can prevent or delay separation on the wing.

The mechanism responsible for formation of longitudinal vortices is basically inviscid or skew-induced (i.e Prandtl's first kind). These vortices can persist for long distance downstream and any attenuation by the Reynolds-stresses are slow. In general, the streamwise velocity contours closely resemble a superposition of a vortex on a pre-existing boundary layer. The measurements of Shabaka et al. (1985) on a single longitudinal vortex embedded in a turbulent boundary layer show that even weak three-dimensionality in the mean flow can cause severe distortion of the Reynolds-stress contours within the boundary layer. The behaviour of ν_t is not simple and with changes in its sign, one cannot expect satisfactory predictions from conventional ν_t -models. In the simplest sense, Tanaka & Suzuki (1978) found it desirable to use anisotropic values of ν_t just to achieve a qualitative agreement on the mean velocity contours. Transport models, like the RSTM, will also face considerable difficulties because the triple product correlations are ill-behaved. To further complicate the issue, structural parameters like $\frac{\tau}{k}$ have large variations. The presence of a wall also means that the vortex behaviour is

unlikely to be similar to a free vortex. All these interesting features undoubtedly make this a good test case in the course of developing a prediction method.

The present work is based on predicting the Shabaka et al. (1985) single vortex-boundary layer flow over a flat plate in zero pressure gradient. It is worth mentioning that the influence of an adverse pressure gradient is to cause an increase in the growth of the vortex core, which leads to greater distortion in the mean velocity and Reynolds-stress contours (see Westphal et al., 1985). Other vortex-boundary layer flows investigated experimentally are streamwise corner vortex, vortex pair with common flow upwards (see Mehta et al., 1981; Bradshaw & Cutler, 1987), and the Eibeck & Eaton (1987) work on modifications in the heat transfer characteristics caused by vortex perturbations.

7.5.2 Initial & Boundary Conditions

The computational domain depicted in Fig.7.60 is a non-uniform mesh with 40x80 nodes. The grid spacing in the y-direction expands by $k=1.05$ with $\Delta Y_1=0.1$ until the size of 0.03 is reached, thereafter, the spacings become uniform at $\Delta Y=0.03$. In the z-direction, the vertical grid lines expands to the left and right of $Z=0.2$ with $k=1.05$ and $\Delta Z=0.015$ until $\Delta Z=0.03$, after which the spacings become uniform at $\Delta Z=0.03$. Since the vortex core grows downstream, it is necessary to shift the closely clustered vertical grid lines towards the negative z-direction in order to capture the downwash side of the vortex more accurately.

By extending the computational domain into regions that are sufficiently far away from the core of the vortex where the flow can be treated as 2-D, one can employ the more convenient forms of BCs in the crossflow plane as given in Fig.7.61.

Along the wall - 1-2:

$$U = V = W = 0$$

$$\frac{\partial P}{\partial y} = 0 \text{ (implicit)}$$

$$\overline{u_i u_j} - \text{wall functions}$$

Along the lateral boundaries - 2-3 & 1-4:

$$\frac{\partial U}{\partial z} = \frac{\partial V}{\partial z} = \frac{\partial W}{\partial z} = \frac{\partial P}{\partial z} = 0$$

$$\frac{\partial}{\partial z} \overline{u_i u_j} = \frac{\partial \epsilon}{\partial z} = 0$$

Along the free stream boundary - 3-4:

$$U = U_\infty = 1.0$$

$$P = P_\infty = 0$$

$$\frac{\partial V}{\partial y} = \frac{\partial W}{\partial y} = 0$$

$$\frac{\partial}{\partial y} \overline{u_i u_j} = \frac{\partial \epsilon}{\partial y} = 0$$

A full set of experimental data for the Shabaka et al. (1985) vortex-boundary layer flow is given by Bradshaw et al. (1982).

Data were extracted from the latter to provide the initial plane BCs (see (a) plots). U_{ref} is the free stream velocity, equal to 30m/s, and L_{ref} is arbitrarily chosen as 100mm, giving a Re value of 190,000. The distance between successive measurement stations is 600mm and the x-step size is 0.025.

7.5.3 Results & Discussion

The downstream development of the vortex is depicted by the U contours in Fig.7.62. It is appropriate to say that the jaggedness in some of the contours is due to the contour plotting program, which joins discrete points without resorting to any form of smoothing. The thinning of the boundary layer on the downwash side and thickening on the upwash side is obvious at $x=600\text{mm}$. The downwash thinning seems to disappear in the predictions at the next two stations but the boundary layer is still thicker in the upwash region. Although the vortex core at $x=1800\text{mm}$ looks rather diffused, the shape is essentially elliptic. This is because the proximity of the wall restrains the core growth. It is now quite certain that meandering of the vortex (i.e. spanwise movement) is not the cause of this ellipticity (see Westphal et al., 1985). At all three x-stations, the height of the core agrees well with the measured values, the unperturbed boundary layer thicknesses at large positive and negative z values are slightly overpredicted, by no more than 10%.

In a private communication with Shabaka (1989), it was acknowledged that the secondary flow velocities have in fact been

corrected to make the secondary velocities zero in the free stream. Thus, the near-wall crossflow velocity vectors have rather large negative values of V (see Fig.7.63a) which are unrealistic in the presence of the wall (due to image vortex). The secondary flow vectors in Fig.7.63, though underpredicted, are of the same order of magnitude as the measurements. At $x=600\text{mm}$, one can clearly make out the main vortex as well as a counter-rotating secondary vortex on the upwash side. The secondary vortex supplies fluid for the upwash motion. Both vortices are also captured at $x=1200\text{mm}$, though some vectors on the downwash side of the main vortex are having positive V . These results do indicate an increase in the height of the core centre. Physically, this lifting of the vortex should be roughly equal to the increase in displacement thickness of the boundary layer. The secondary flow magnitude at $x=1800\text{mm}$ is about half the values at $x=1200\text{mm}$, and it seems the vectors do not typify the presence of a vortex.

Fig.7.64 shows the calculated skin friction coefficient. When compared with Dedoussis (1988) predictions using a mixing-length turbulence model, the present results are obviously superior. The maximum C_f occurs in the region where fluid is brought towards the surface whereas the minimum value is caused by fluid moving away from it. Therefore, in between the maximum and minimum lies the vortex centre. The second C_f peak is due to negative V from the secondary vortex. The location as well as magnitude of minimum C_f are well predicted, this means the shift in the vortex centre in the positive z -direction is reproduced by PSTAG. However, the maximum C_f values are underpredicted. The profiles do show

presence of the second peak, but these are also too low. The lack of good quantitative agreement could well lie in the uncertainties associated with the log-law in 3-D flows.

The predicted normal stresses are shown in Fig.7.65 for $\overline{uu}$, Fig.7.66 for $\overline{vv}$ and Fig.7.67 for $\overline{ww}$, and closely resemble the superposition of a vortex on a boundary layer. The normal stress contours look similar to the U contours with the "smallest value" contours roughly coinciding with the $U=0.99$ contours. In comparing with experimental data, the predictions at $x=600\text{mm}$ are reasonably accurate, but at $x=1200\text{mm}$ & 1800mm the agreement within the vortex core is qualitative, the predicted contours showing less distortions in the core.

In Fig.7.68, the shape of the calculated $\overline{uv}$ contours follow quite closely to those of the normal stress. Again the $\overline{uv}$ contours are not as distorted as the measurements, in addition, the region of positive $\overline{uv}$ on the downwash side is also not captured. As will be discussed later, the lack of distortion in the contours defining the perimeter of the vortex can be attributed to the uncertainties surrounding the present turbulent diffusion model. The contours of the other shear stress, $\overline{uw}$ in Fig.7.69 and $\overline{vw}$ in Fig.7.70, are generally more complicated. The $\overline{uw}$ predictions are remarkably good, ^{*} giving regions of positive $\overline{uw}$ on the downwash side and negative $\overline{uw}$ on the upwash side. As for $\overline{vw}$, the predictions are starkly different from the experimental data, there are four distinct quadrants with opposite quadrant containing $\overline{vw}$ of the same sign. These predictions are quite plausible if one considers

* when compared with Shabata's measurements

the vortex to be axisymmetric because $\overline{vw}$ would then vary as $\sin 2\theta$ (θ being positive in the anti-clockwise direction). In view of the above argument coupled with the difficulties in providing reliable $\overline{vw}$ measurements, one cannot discard the $\overline{vw}$ predictions outright.

Although the longitudinal vorticity ($= \frac{\partial W}{\partial y} - \frac{\partial V}{\partial z}$) is skew-induced rather than stress-induced, the diffusing of the vortex as well as its interaction with the other parts of the flow depend very much on the Reynolds-stresses. From the present calculations, the lack of quantitative agreement on the turbulence stresses (within the vortex core) clearly implies that further work needs to be done to improve the current RSTM. Presumably, the main re-modelling should lie in the pressure-strain and turbulent diffusion models, both terms being roughly of the same order as the convective term near the perimeter of the vortex and will therefore play a significant role there.

In the 3-D boundary layer calculations (see §7.3 & §7.4), we have seen that the inadequate response of the RSTM to the effects of $\frac{\partial W}{\partial y}$ can be largely attributed to the pressure-strain model. So in the vortex-boundary layer flow where $\frac{\partial V}{\partial z} \sim \frac{\partial W}{\partial y}$, one should not expect the pressure-strain model to perform any better, because $\frac{\partial V}{\partial z}$, according to Bradshaw (1988) has large effects on turbulence structure especially in embedded vortex. Re-modelling the pressure-strain term is not going to be an easy task and the same may be said of the triple product correlation. Shabaka et al. (1985) pointed out that the triple product correlation is

extremely complicated and by representing it with a gradient diffusion model, the coefficient (i.e. C_g) does not behave simply and is not always of the same sign. This means the diffusion model in the RSTM with a constant C_g is unlikely to be the correct form. It need be added that before any improved models can be implemented, it is necessary to understand more deeply the physics of 3-D flows.

7.5.4 Concluding Remarks

Although the vortex-boundary layer flow can be easily envisaged as a superposition of a vortex on a boundary layer, one knows that in order to predict the mean flow field quantitatively, the turbulence model must first be able to capture the complicated Reynolds-stress contours. In this respect, the broad characteristics of the contours are well captured by the RSTM but the more intricate distortions within the vortex core are missing. As in the 3-D boundary layer on a swept wing, the pressure-strain model is a highly probable weakness. In addition, anomalies will arise when the triple product is modelled as gradient diffusion because this correlation does not behave simply as to give, say, an empirical coefficient of the same sign throughout the flow.

Table 7.1

	Model 1	Model 2
$\overline{uu} / k$	1.1	1.0
$\overline{vv} / k$	0.3	0.3
$\overline{ww} / k$	0.6	0.7
$\overline{uv} / k$	-0.28	-0.3

Table 7.2

	x'-y-z' axes		x-y-z axes	
	Expt.*	Model 2	Expt.*	Model 2
$\overline{uu} / k$	0.9 - 0.96	0.9	1.1 - 1.2	1.0
$\overline{vv} / k$	0.27 - 0.32	0.32	0.27 - 0.32	0.32
$\overline{ww} / k$	0.64 - 0.85	0.78	0.46 - 0.52	0.68

* Experiment data of Pontikos (1982).

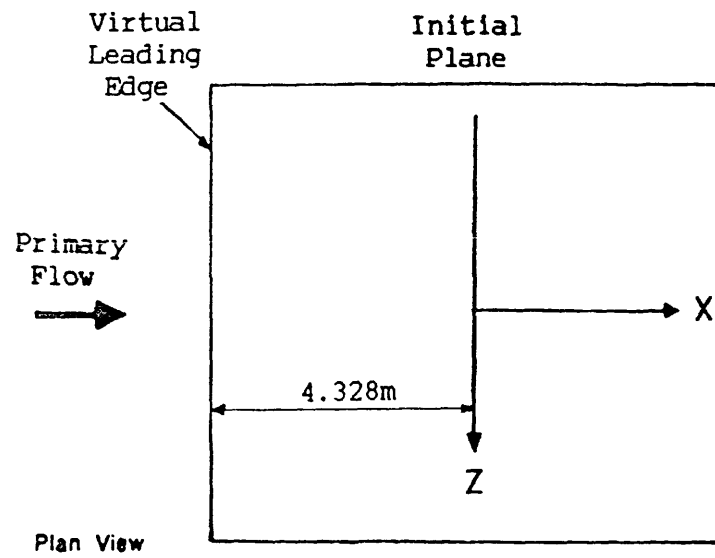


Fig.7.1 Turbulent Boundary Layer Flow Configuration.

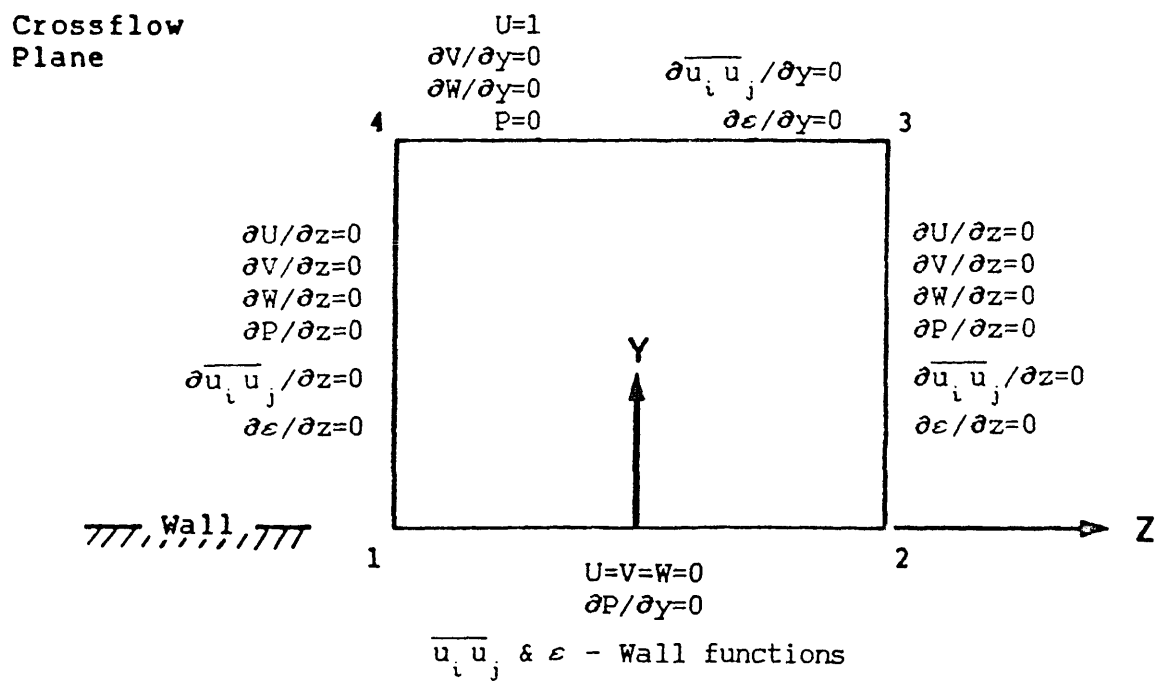


Fig.7.2 Turbulent Boundary Layer Flow Boundary Conditions.

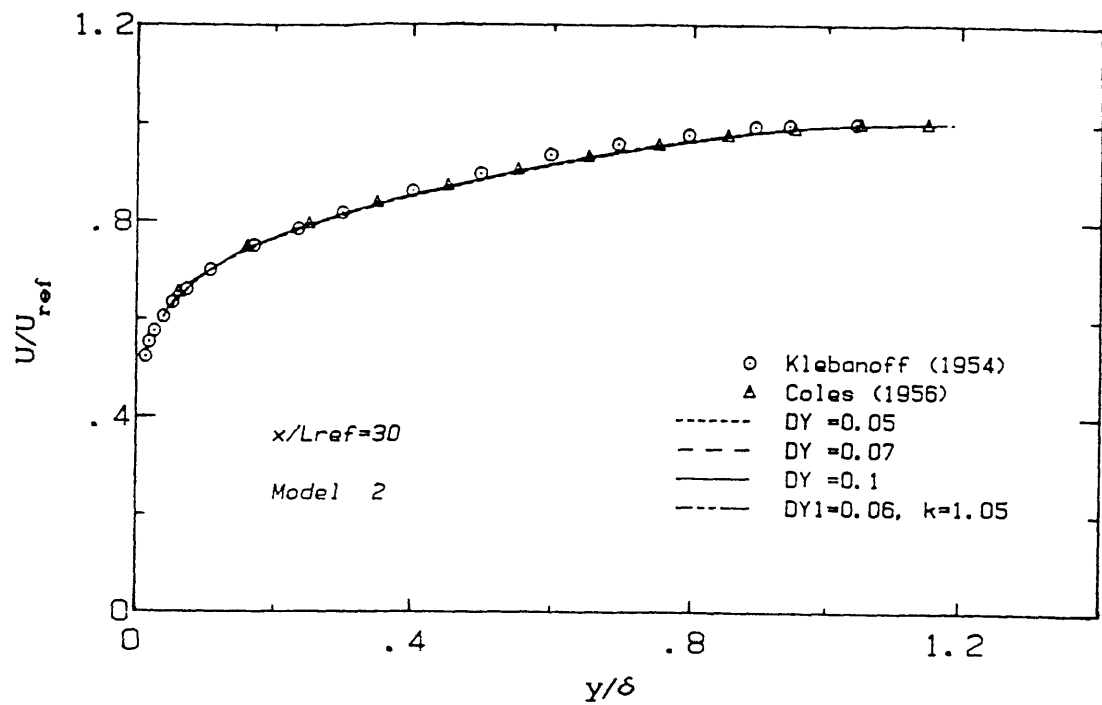


Fig.7.3 Predicted & Measured U Profiles At $x = 30$.

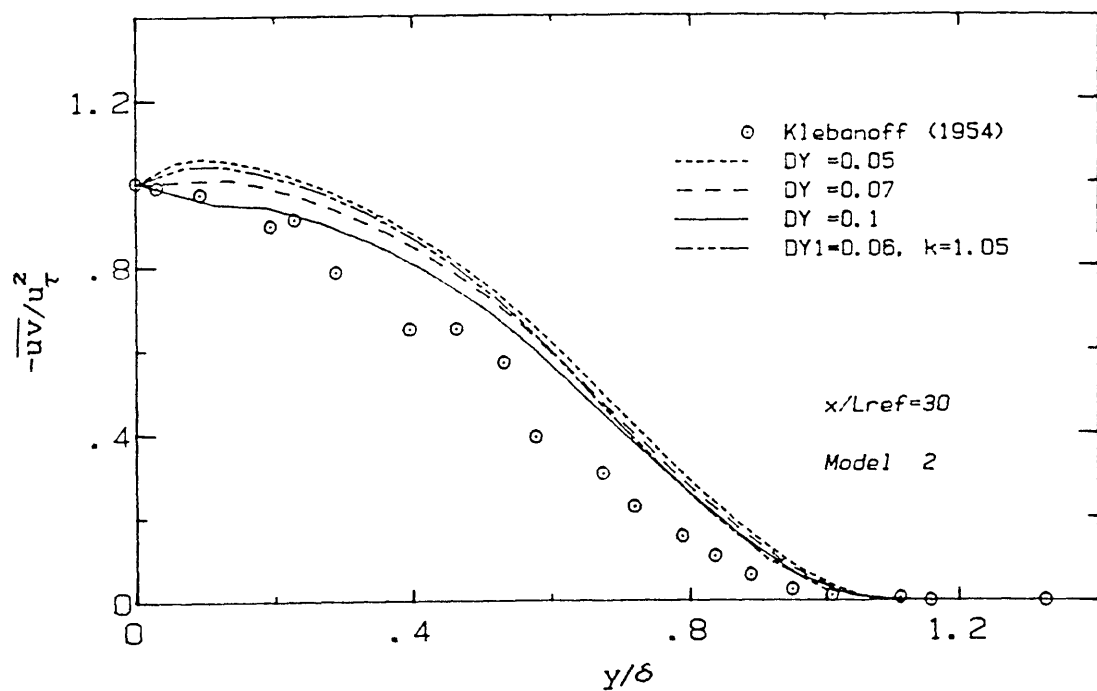


Fig.7.4 Predicted & Measured $\overline{uv}$ Profiles At $x = 30$.

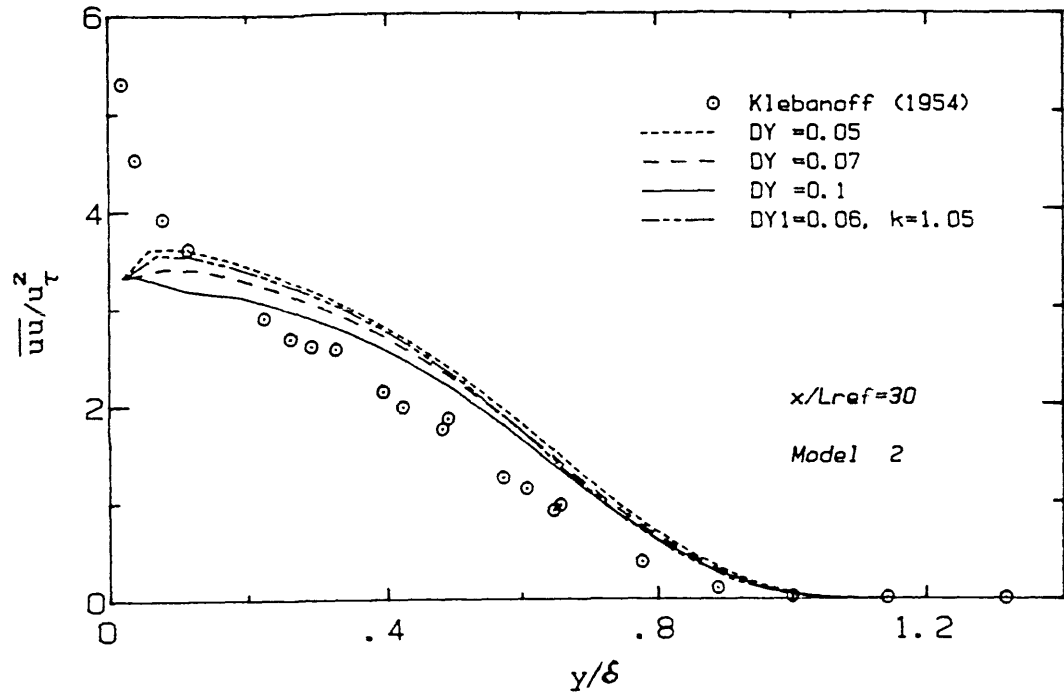


Fig.7.5 Predicted & Measured $\overline{uu}$ Profiles At $x = 30$.

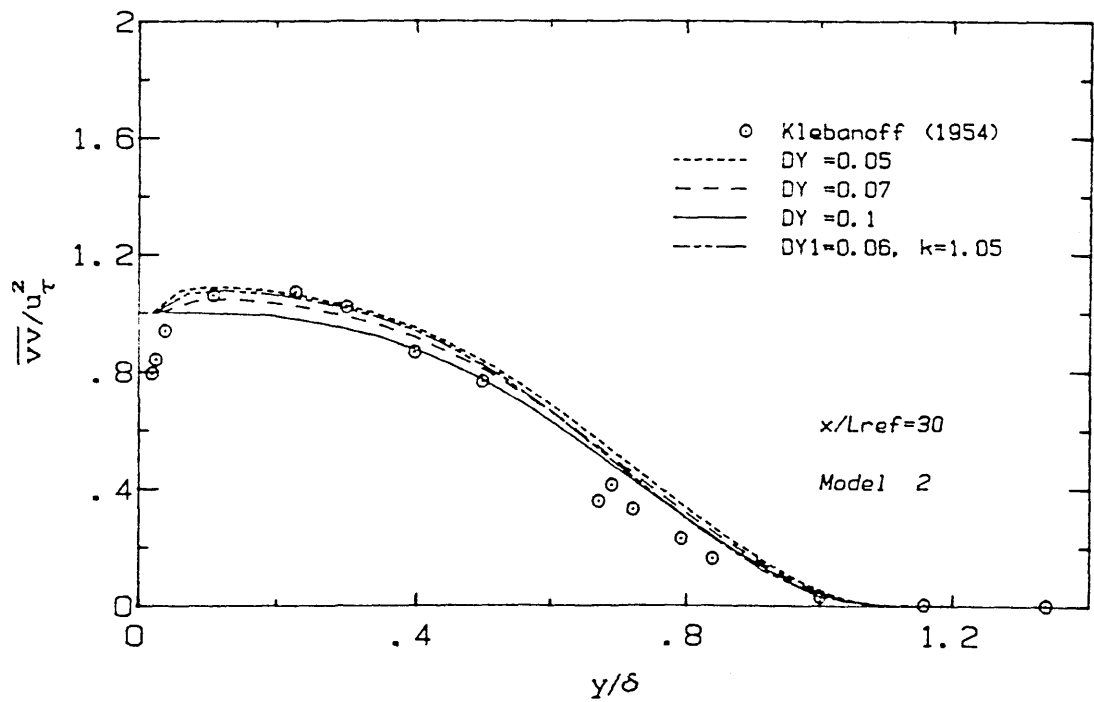


Fig.7.6 Predicted & Measured $\overline{vv}$ Profiles At $x = 30$.

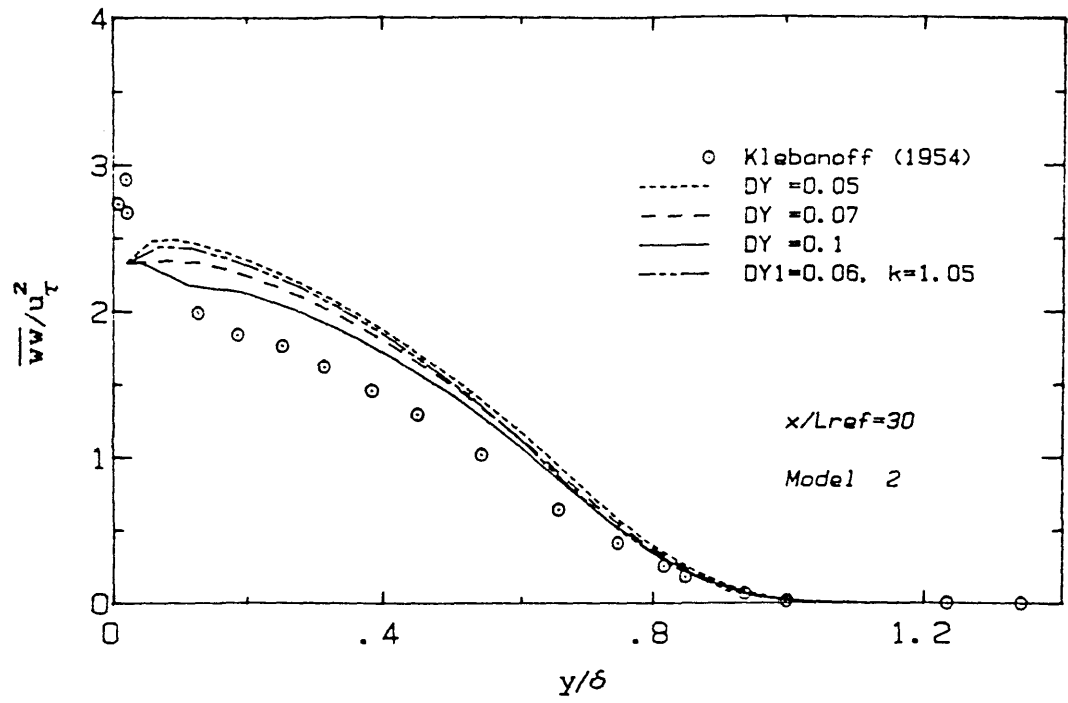


Fig.7.7 Predicted & Measured $\overline{w}w$ Profiles At $x = 30$.

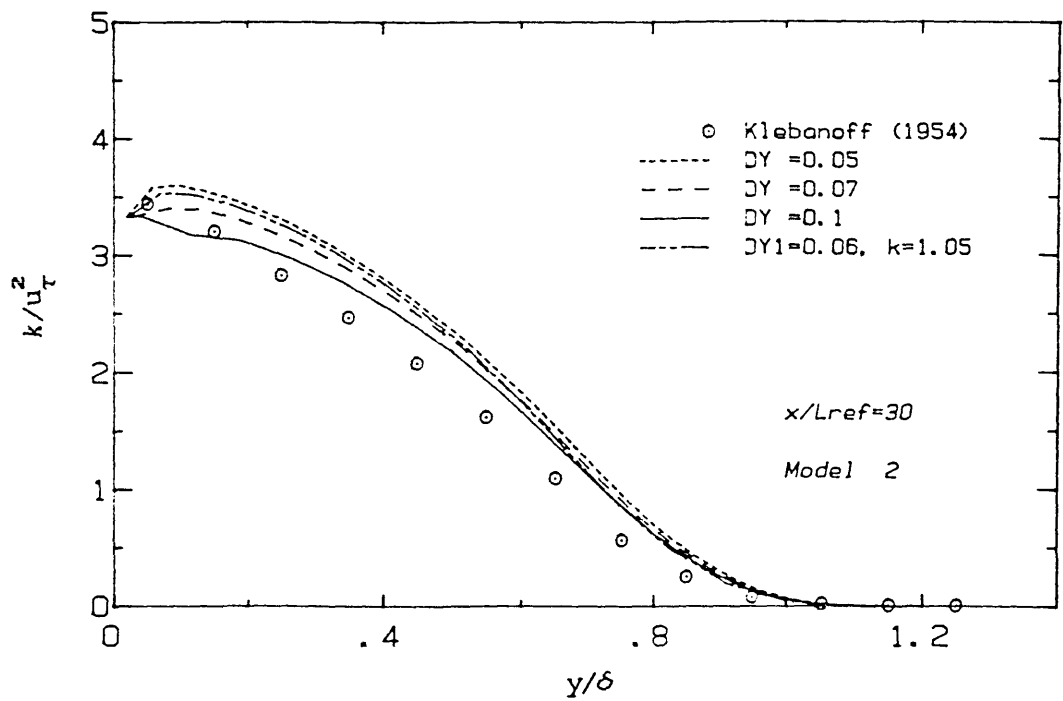


Fig.7.8 Predicted & Measured k Profiles At $x = 30$.

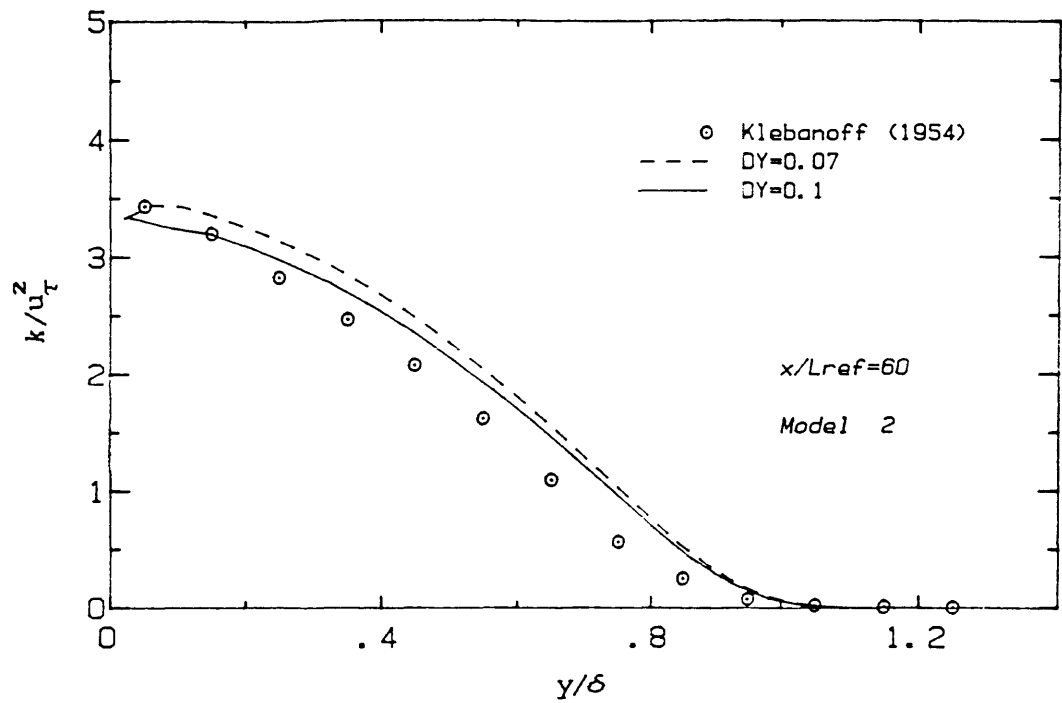


Fig.7.9 Predicted & Measured k Profiles At $x = 60$ with $\Delta Y = 0.07$ & 0.1 .

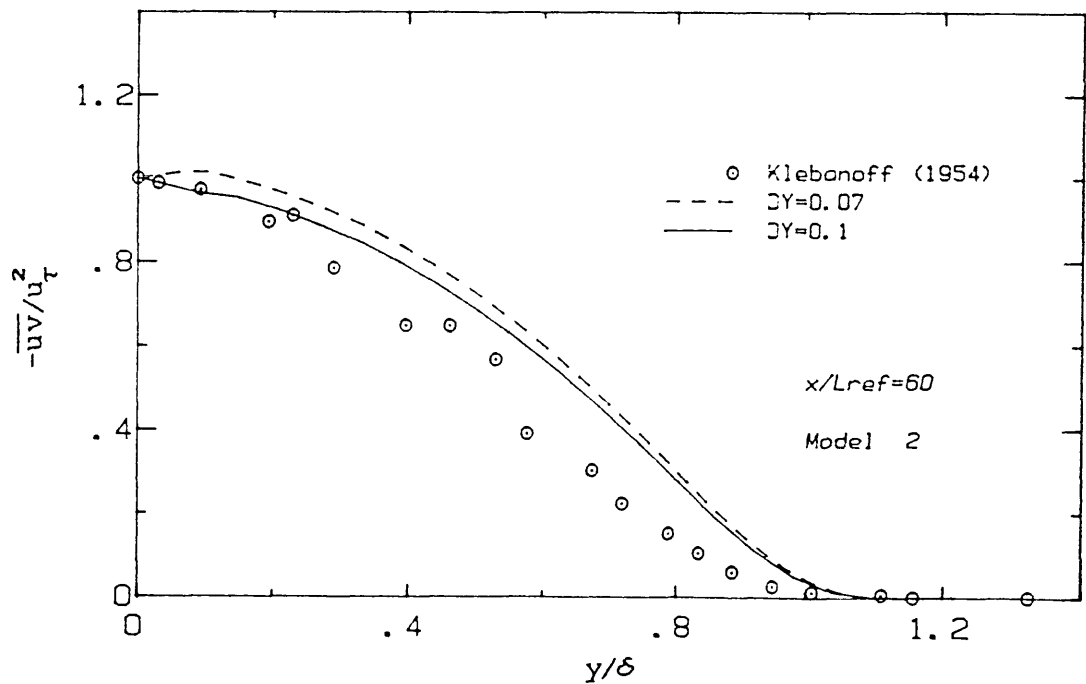


Fig.7.10 Predicted & Measured $\overline{uv}$ Profiles At $x = 60$ with $\Delta Y = 0.07$ & 0.1 .

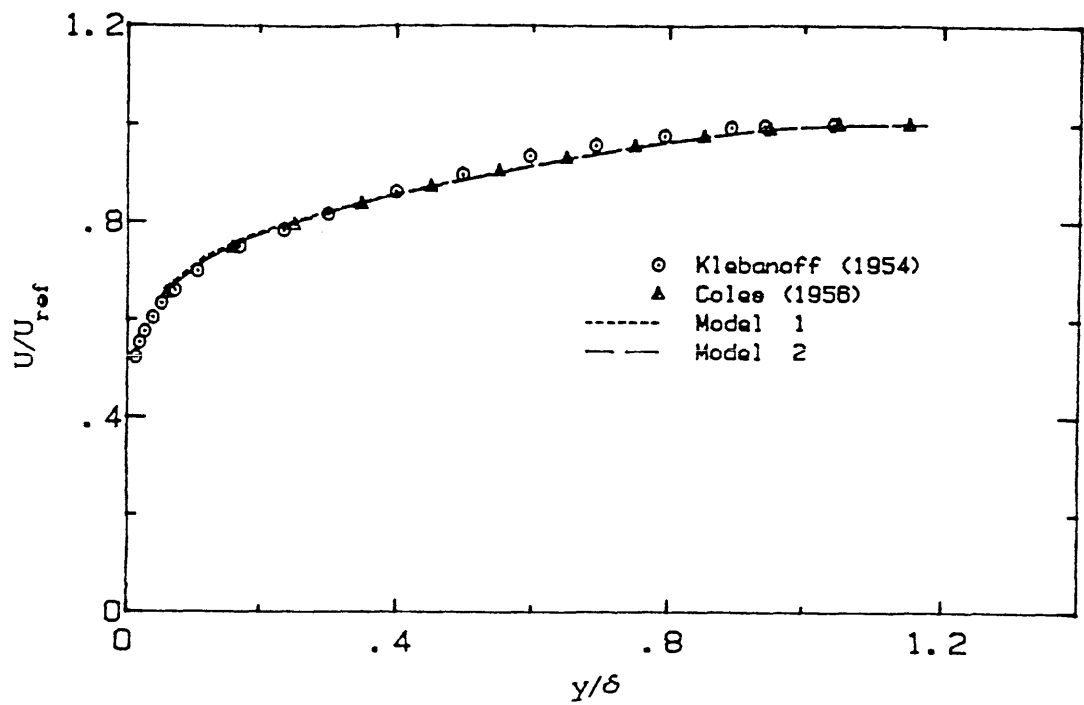


Fig.7.11 Predicted & Measured U Profiles At $x = 60$. Predictions By Model 1 & Model 2.

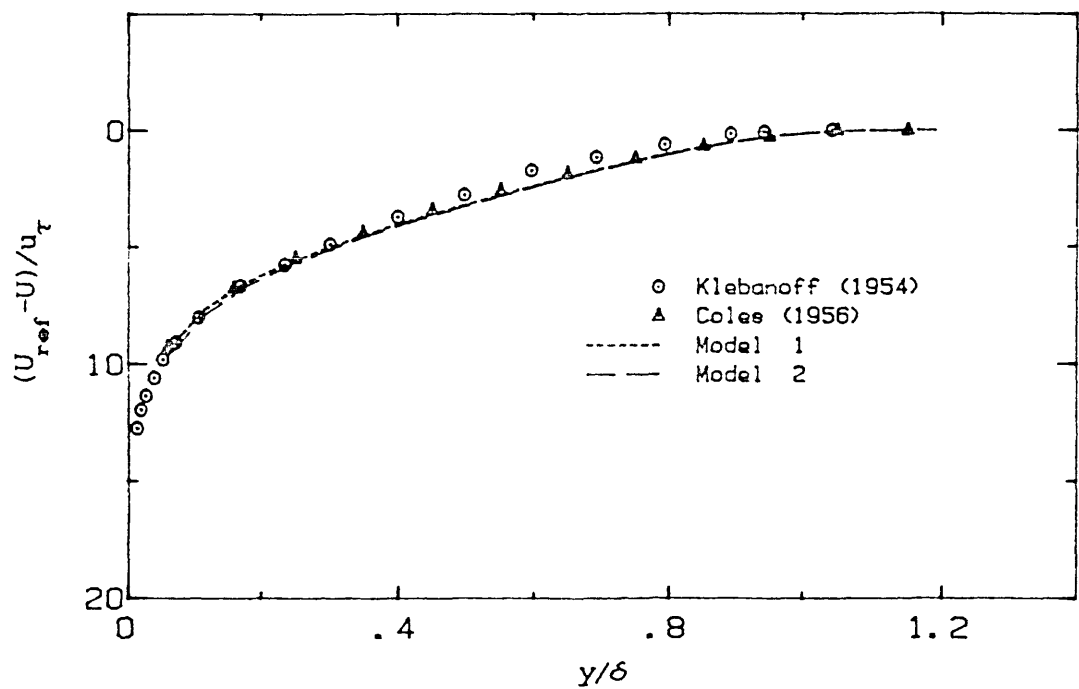


Fig.7.12 Predicted & Measured Velocity-Defect Profiles At $x = 60$. Predictions By Model 1 & Model 2.

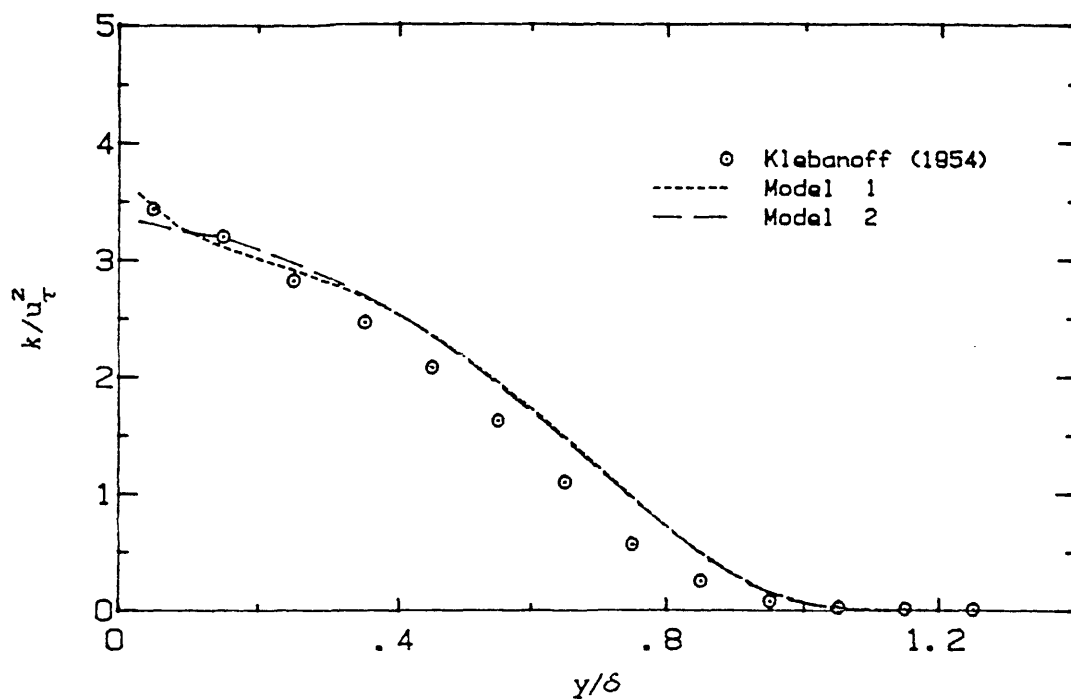


Fig.7.13 Predicted & Measured k Profiles At $x = 60$. Predictions By Model 1 & Model 2.

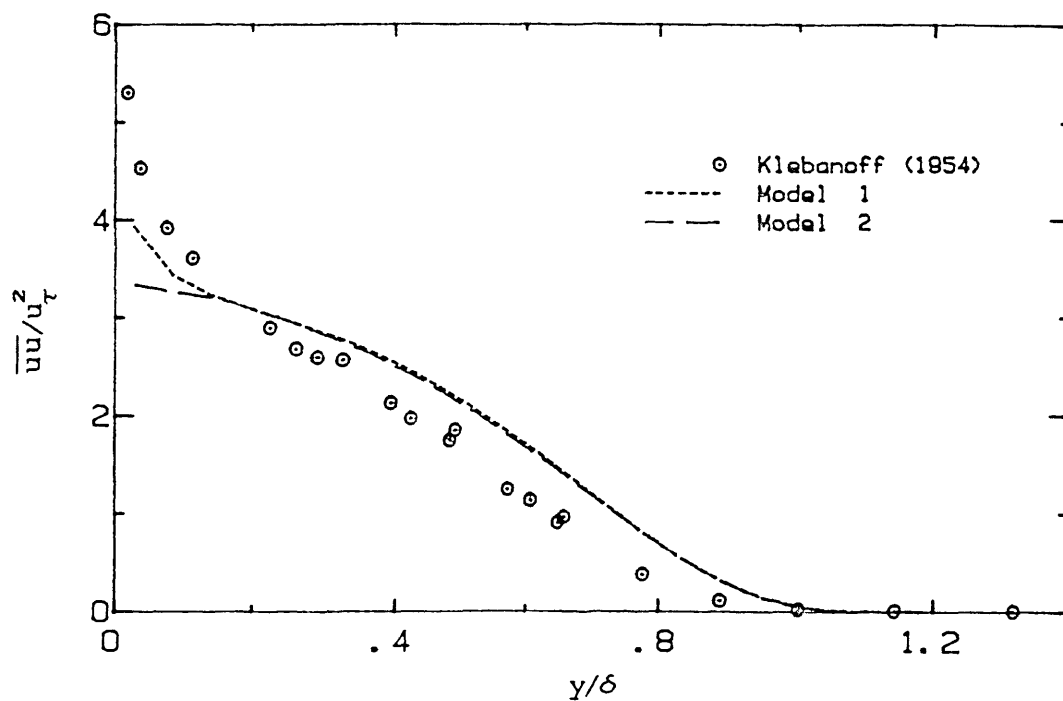


Fig.7.14 Predicted & Measured $\overline{uu}$ Profiles At $x = 60$. Predictions By Model 1 & Model 2.

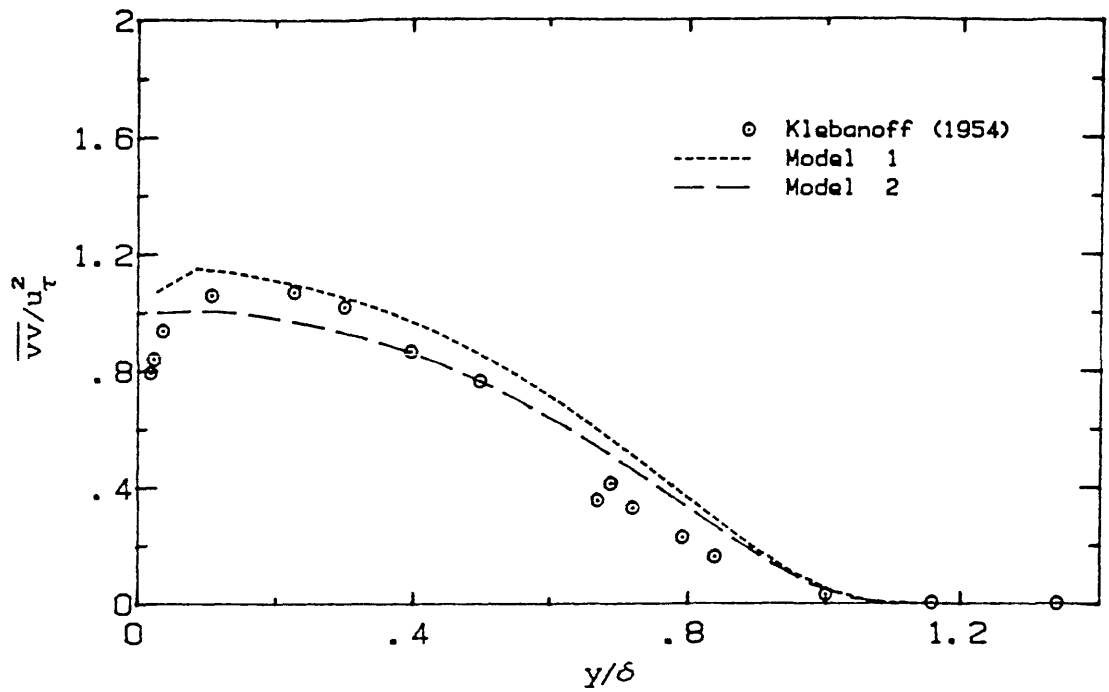


Fig.7.15 Predicted & Measured $\overline{vv}$ Profiles At $x = 60$. Predictions By Model 1 & Model 2.

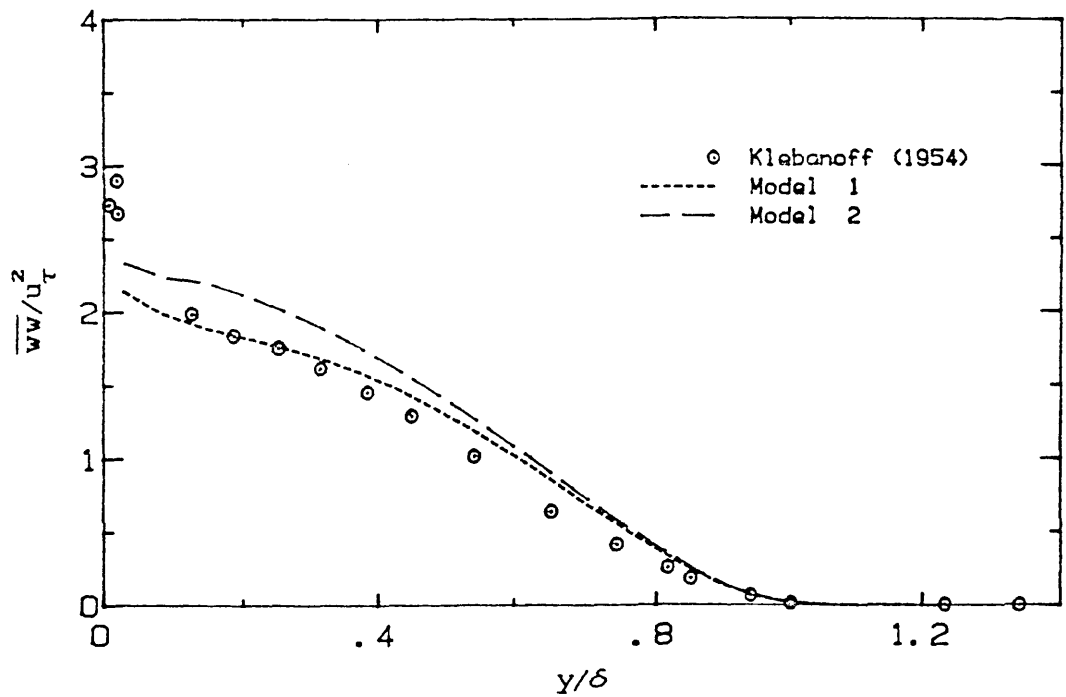


Fig.7.16 Predicted & Measured $\overline{ww}$ Profiles At $x = 60$. Predictions By Model 1 & Model 2.

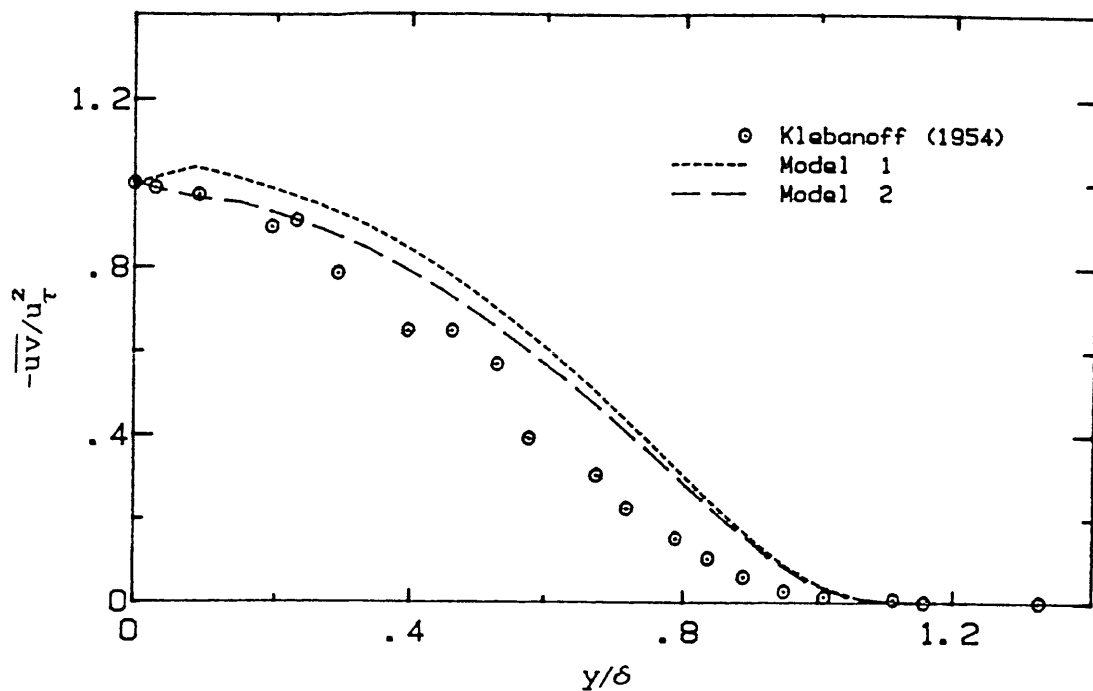


Fig.7.17 Predicted & Measured $\overline{uv}$ Profiles At $x = 60$. Predictions By Model 1 & Model 2.

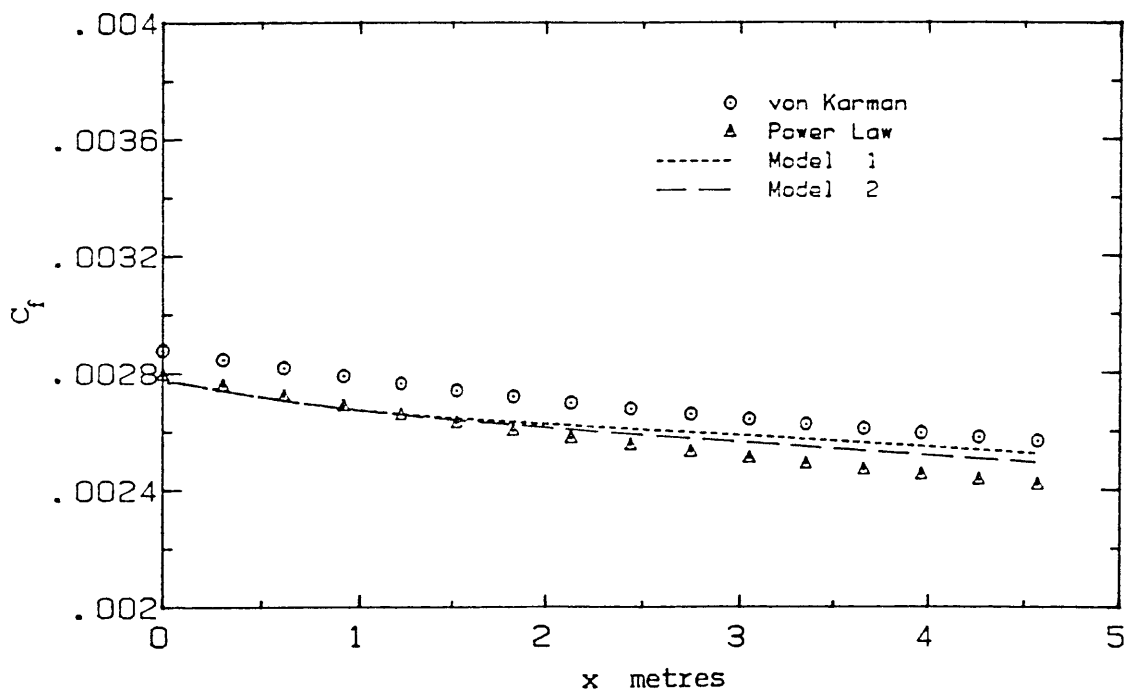


Fig.7.18 Predicted & Analytic Skin Friction Coefficient. Predictions By Model 1 & Model 2.

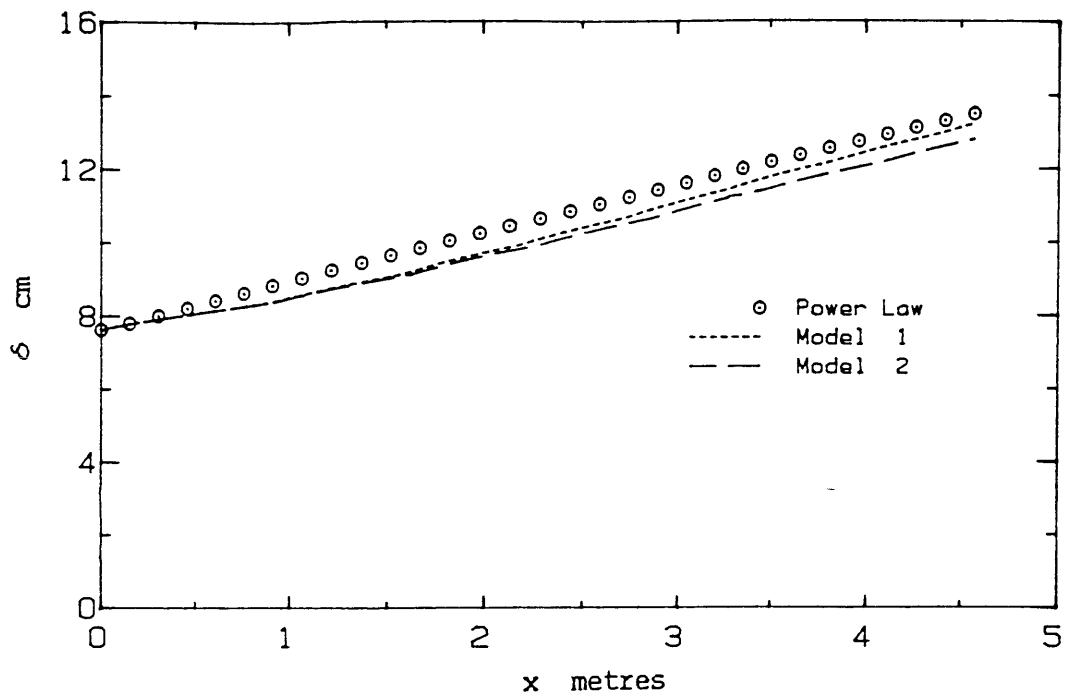


Fig.7.19 Predicted & Analytic Boundary Layer Thickness.
Predictions By Model 1 & Model 2.

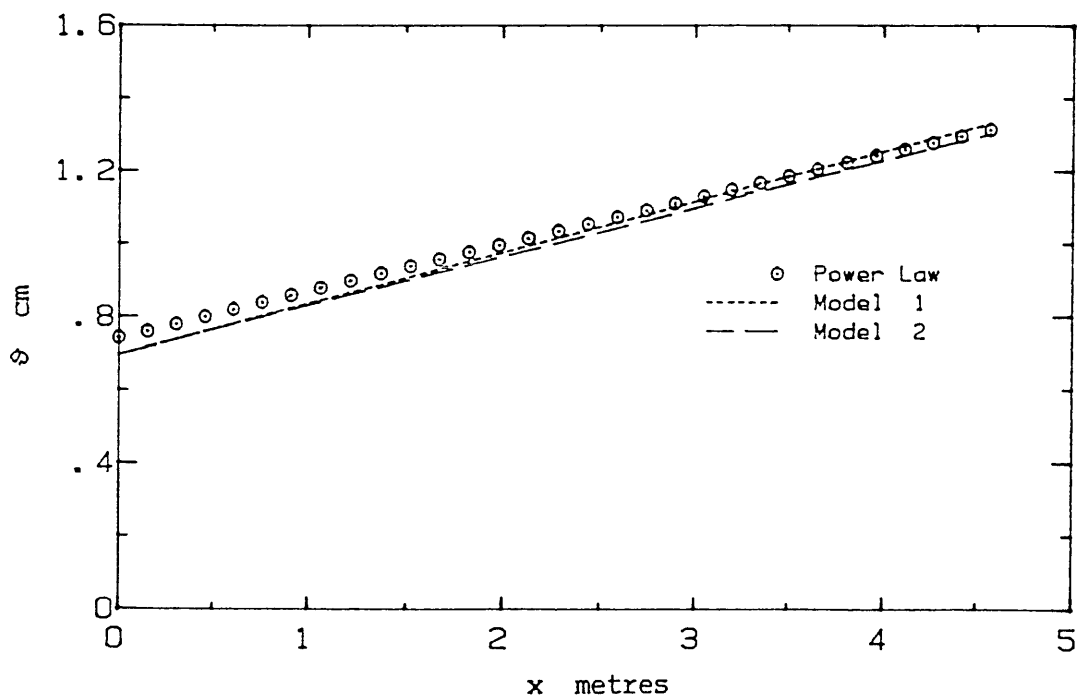


Fig.7.20 Predicted & Analytic Momentum Thickness. Predictions By
Model 1 & Model 2.

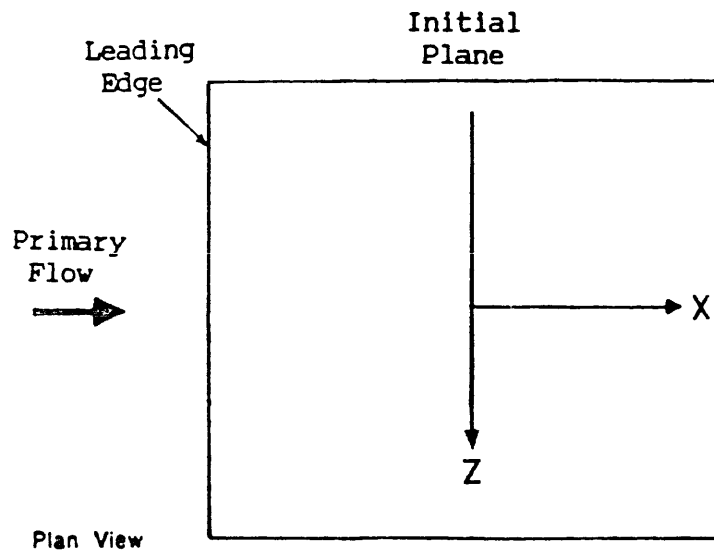


Fig.7.21 Adverse Pressure Gradient Turbulent Boundary Layer.

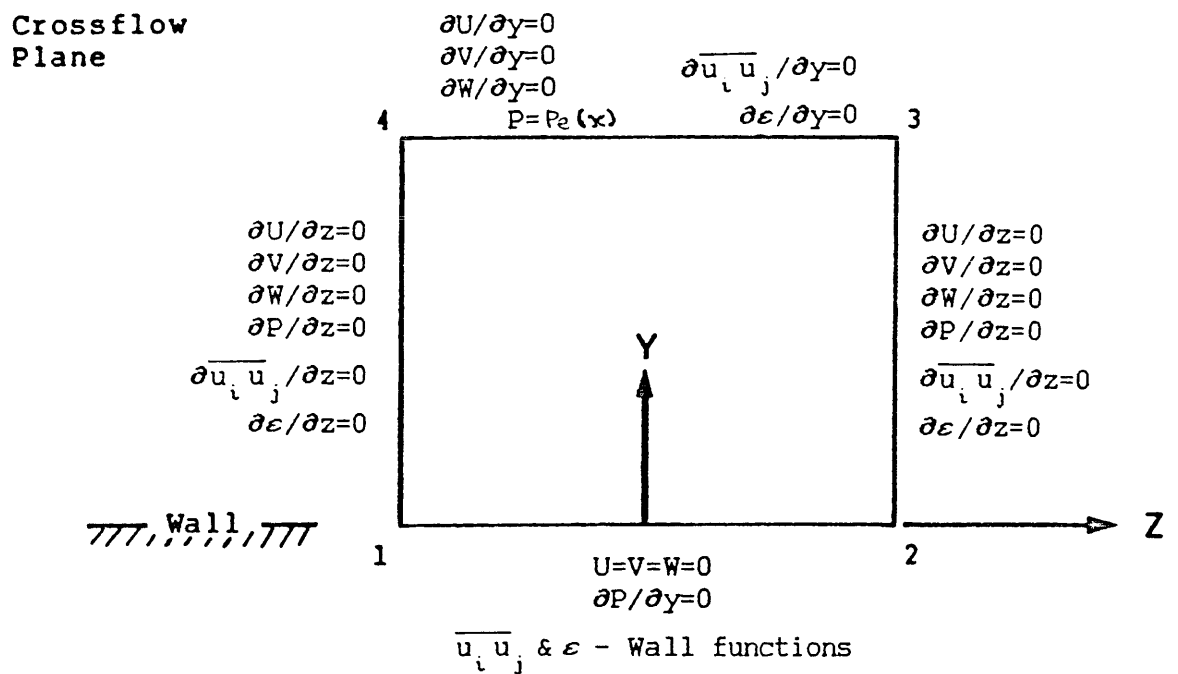
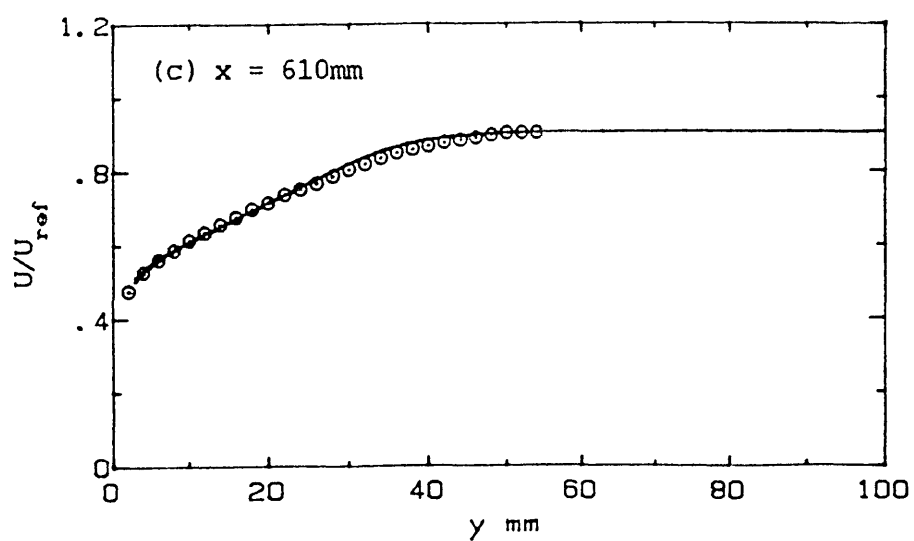
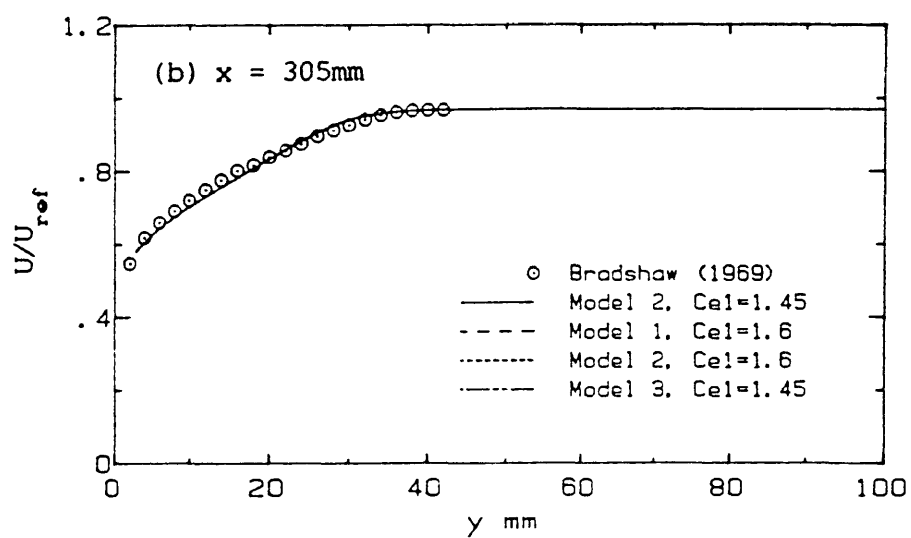
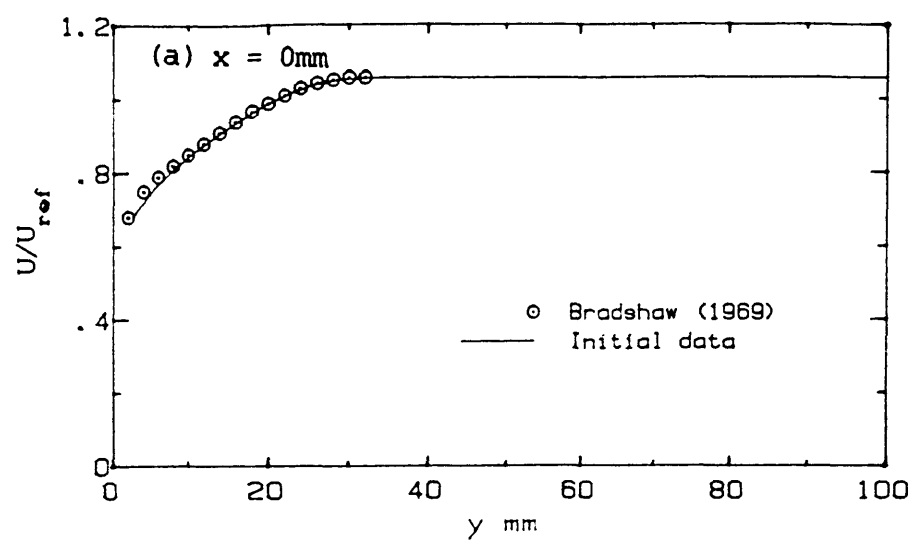


Fig.7.22 Adverse Pressure Gradient Turbulent Boundary Layer Boundary Conditions.



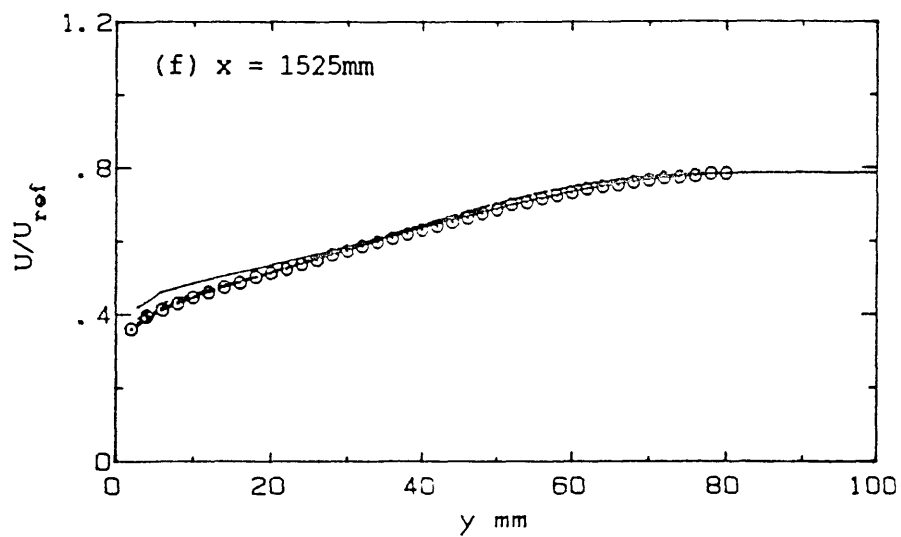
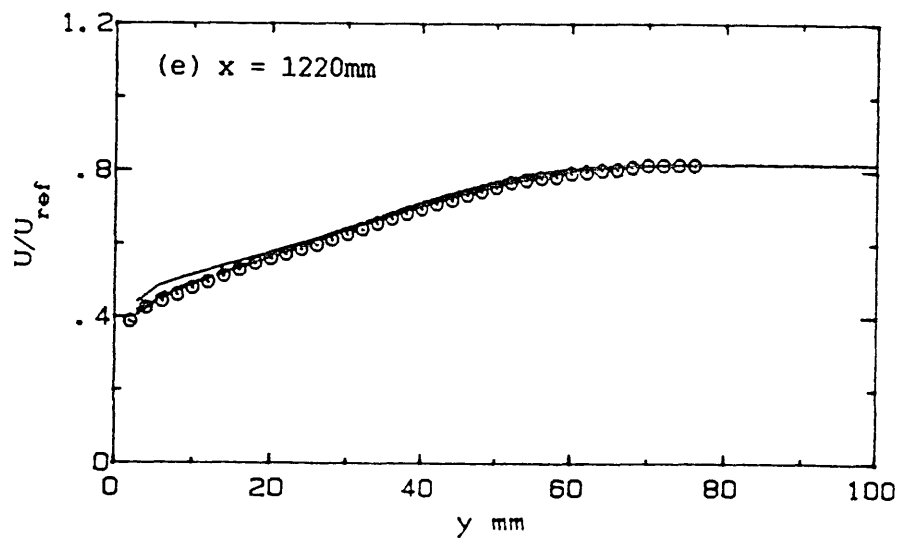
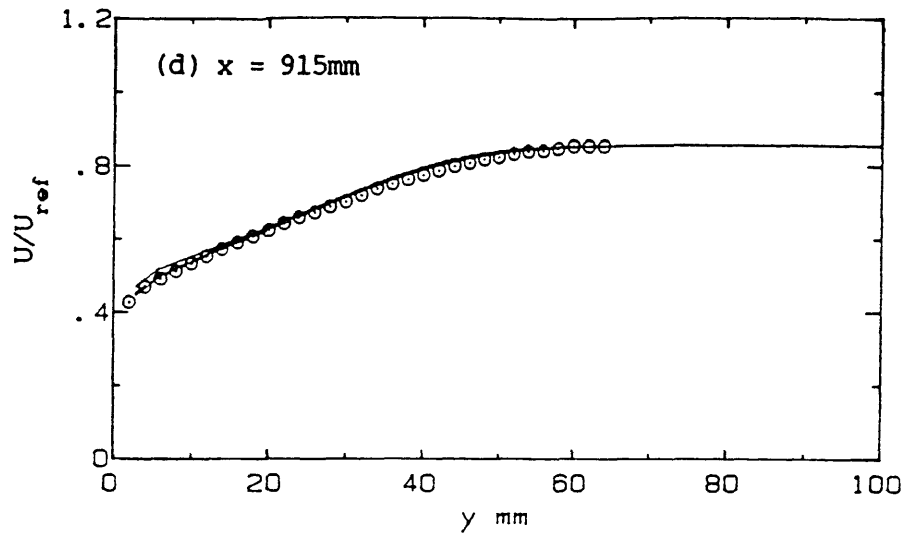


Fig.7.23 Development Of U Profile At $x = 0\text{mm}$, 305mm , 610mm , 915mm , 1220mm & 1525mm . Experimental Data By Bradshaw (1969). Predictions By Model 2, Model 1 ($C_{\epsilon 1}=1.6$), Model 2 ($C_{\epsilon 1}=1.6$) & Model 3.

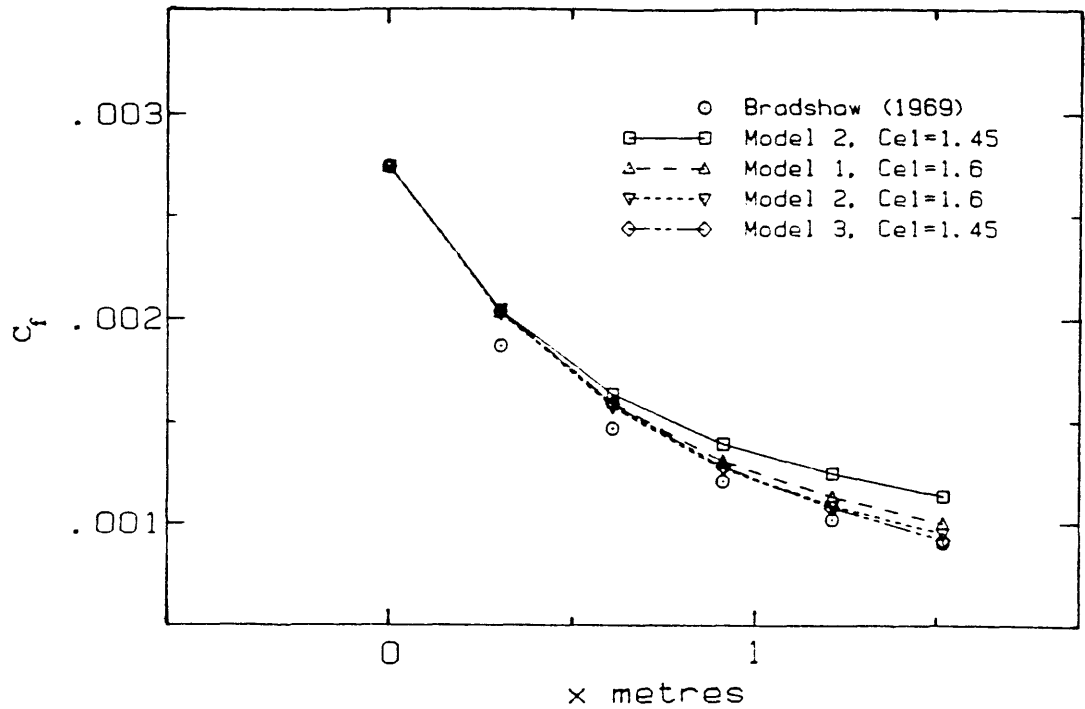


Fig.7.24 Skin Friction Coefficient. Experimental Data By Bradshaw (1969). Predictions By Model 2, Model 1 ($C_{e1}=1.6$), Model 2 ($C_{e1}=1.6$) & Model 3.

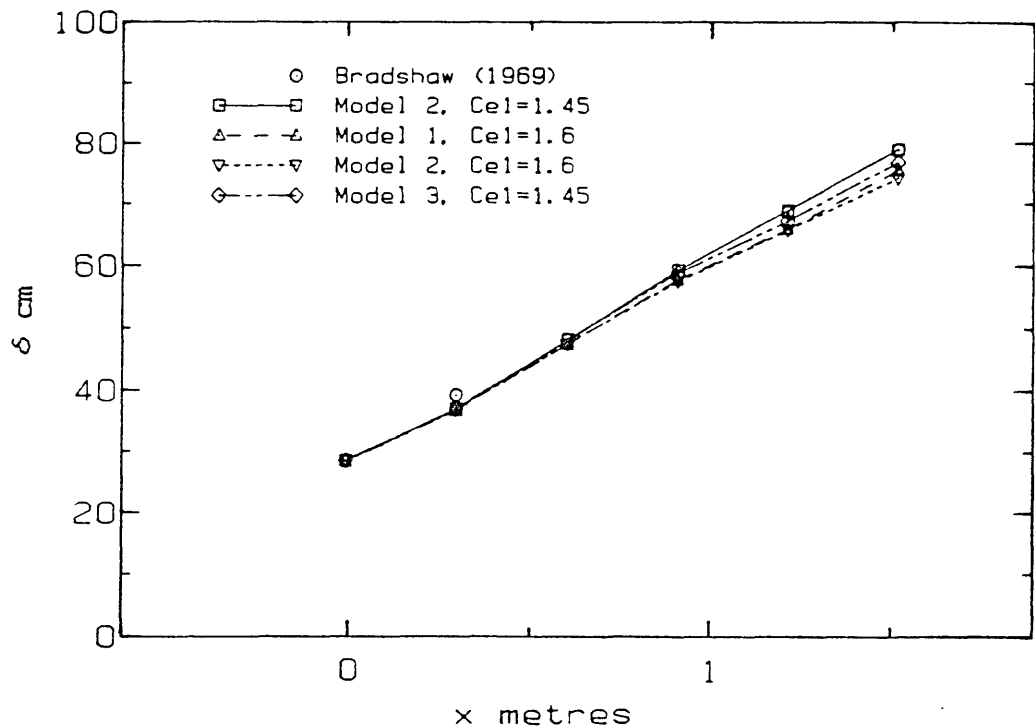
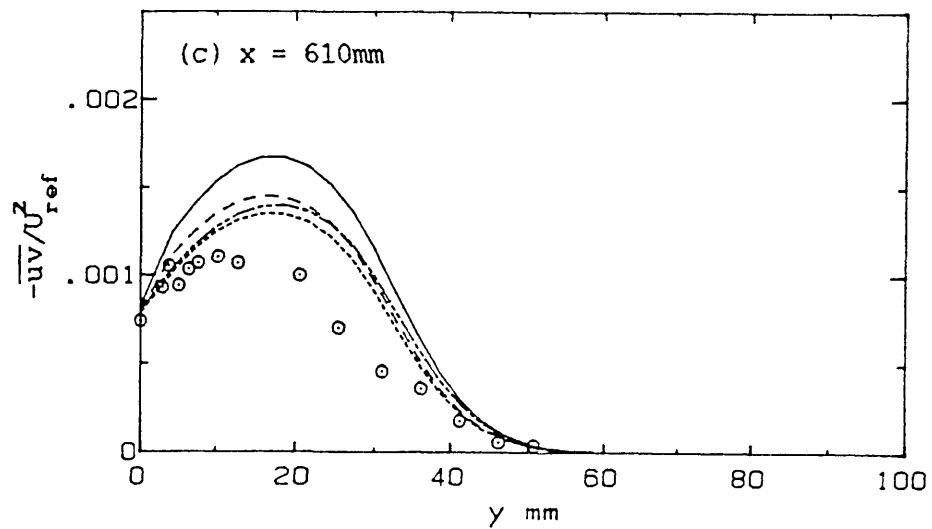
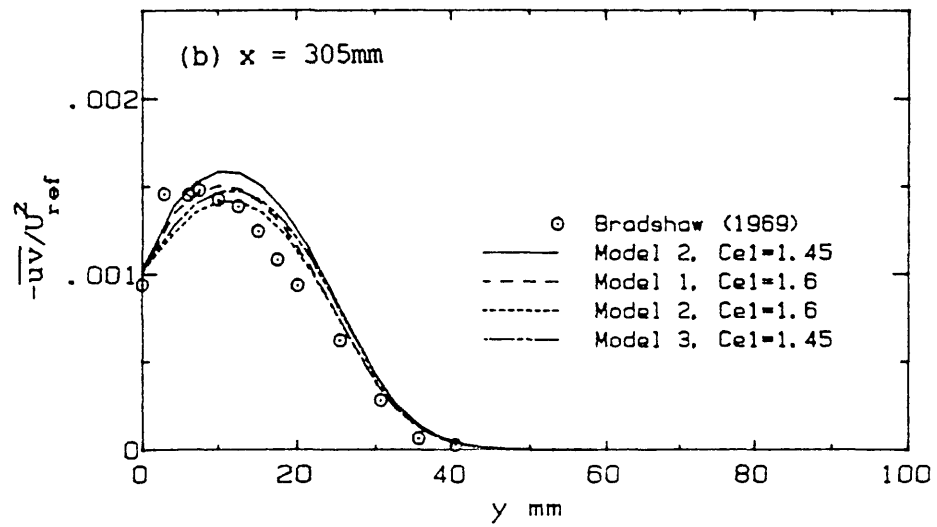
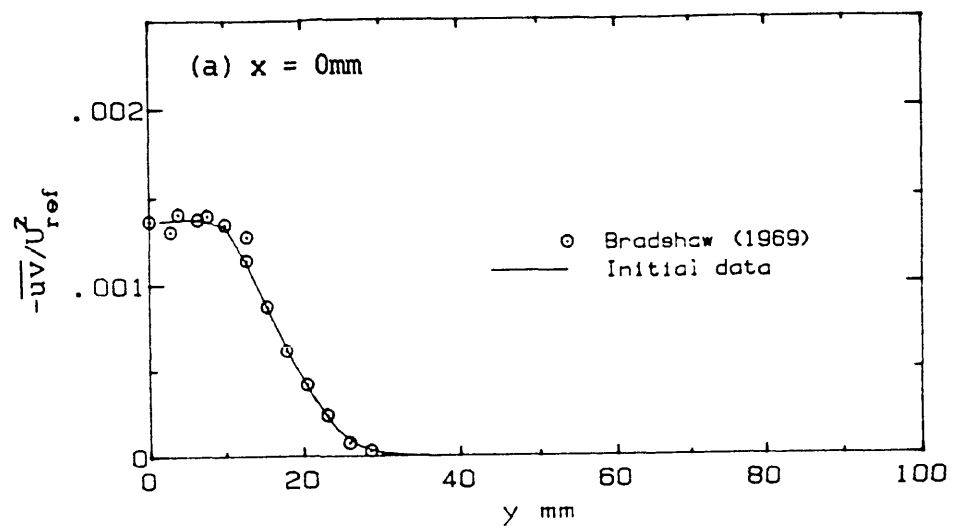


Fig.7.25 Boundary Layer Thickness. Experimental Data By Bradshaw (1969). Predictions By Model 2, Model 1 ($C_{e1}=1.6$), Model 2 ($C_{e1}=1.6$) & Model 3.



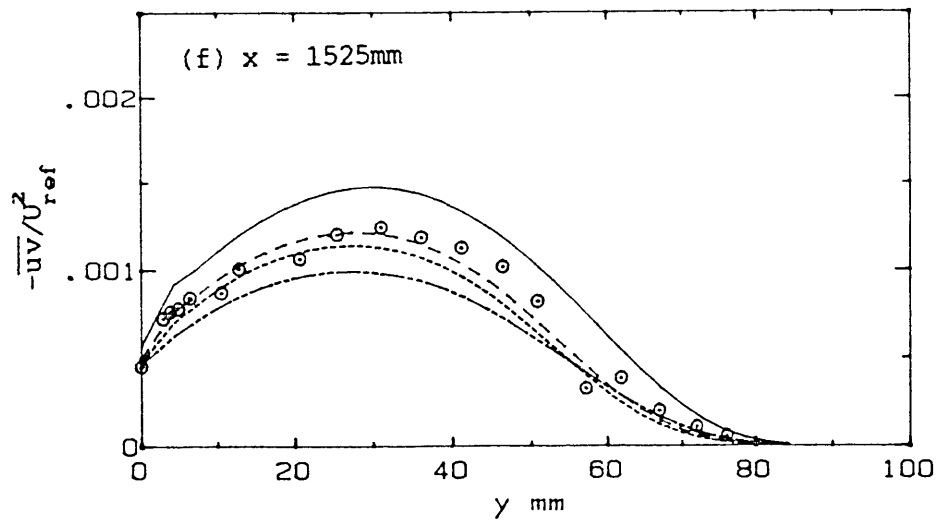
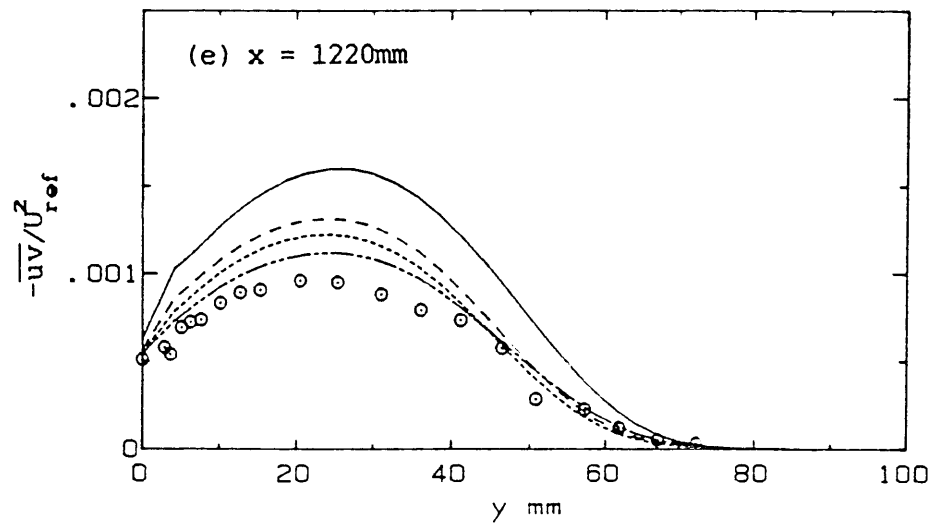
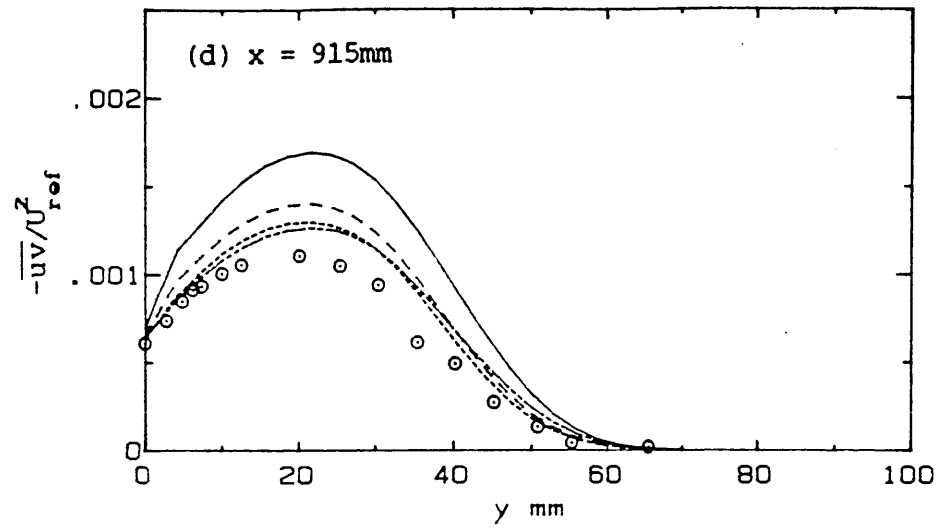
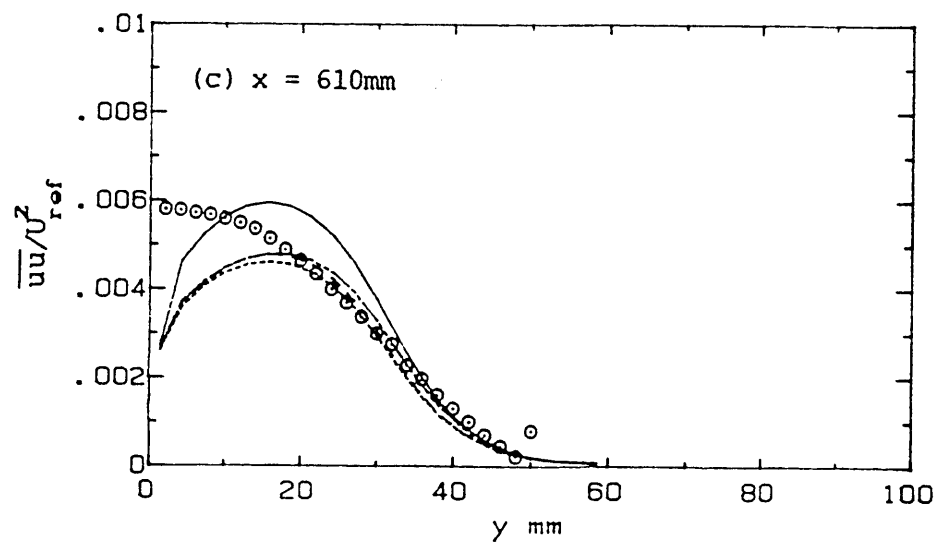
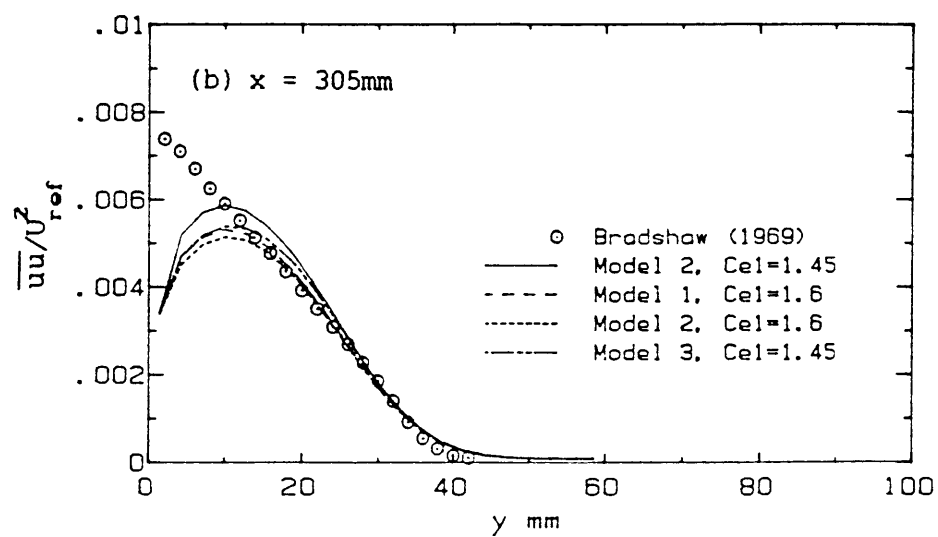
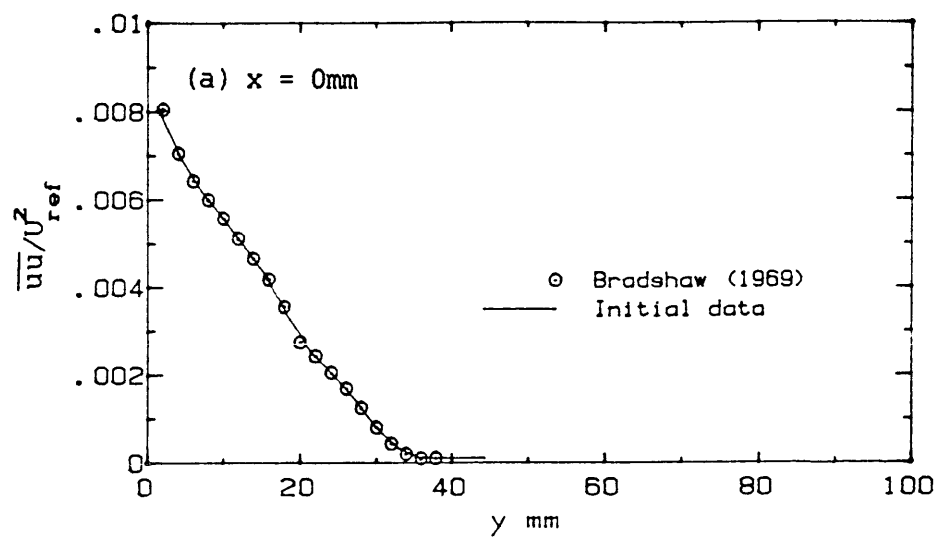


Fig.7.26 Development Of $\overline{uv}$ Profile At $x = 0\text{mm}$, 305mm , 610mm , 915mm , 1220mm & 1525mm . Experimental Data By Bradshaw (1969). Predictions By Model 2, Model 1 ($C_{\epsilon 1}=1.6$), Model 2 ($C_{\epsilon 1}=1.6$) & Model 3.



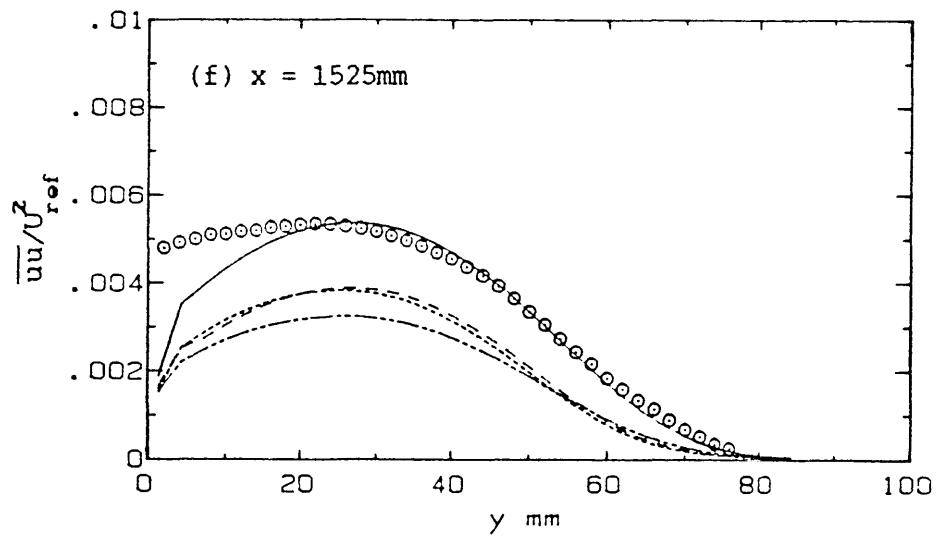
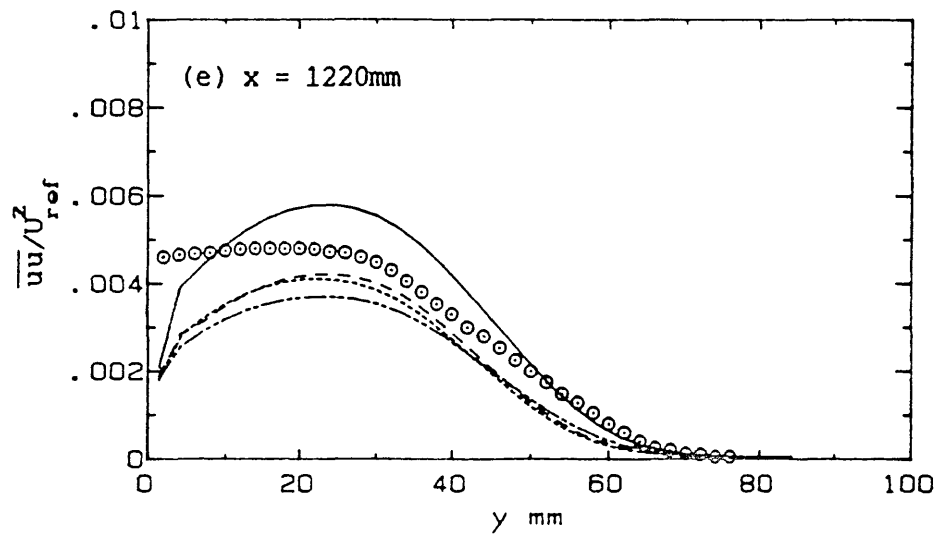
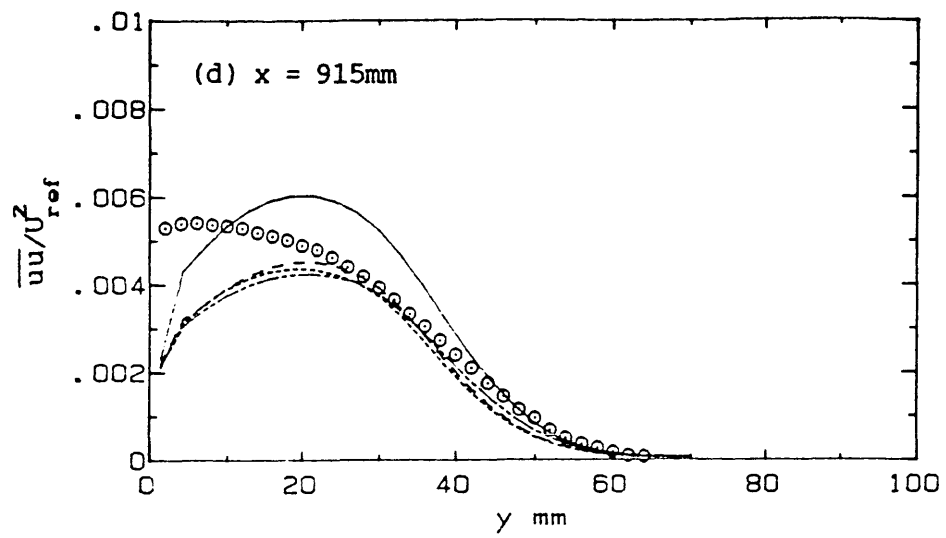
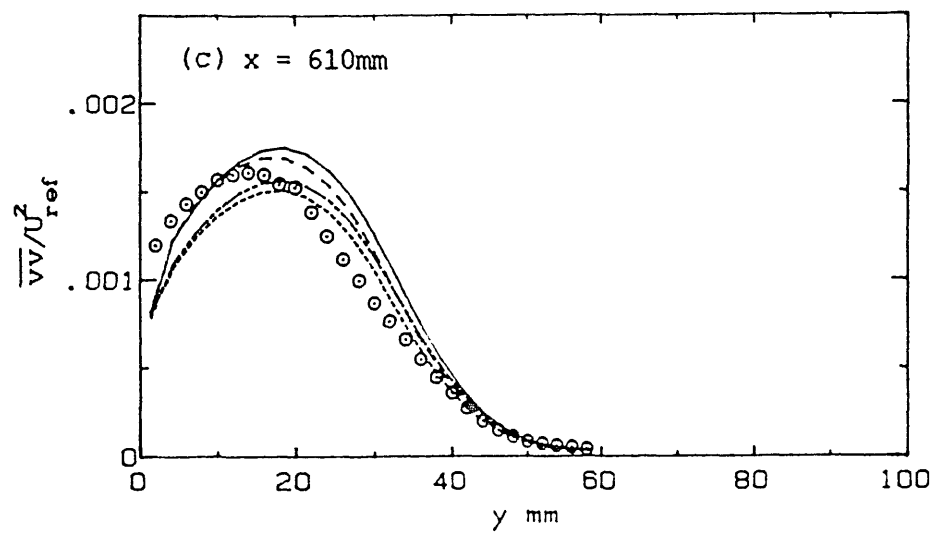
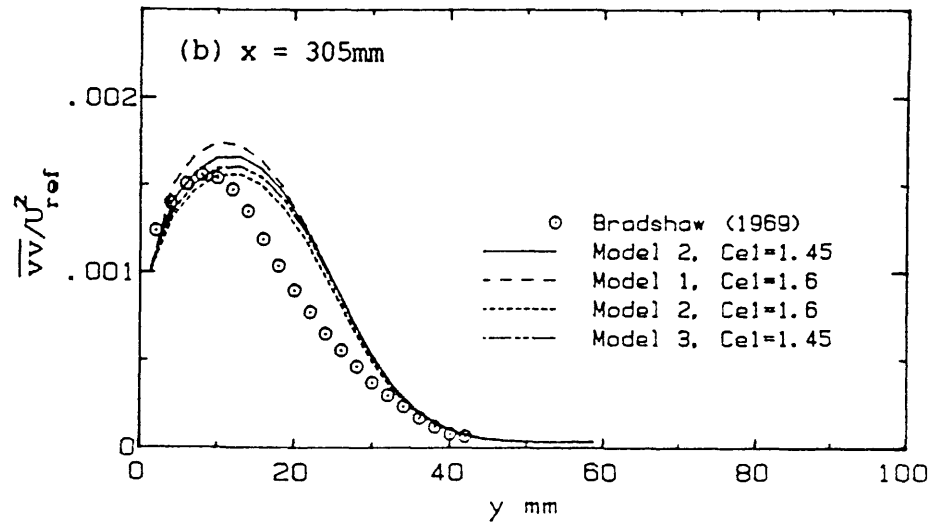
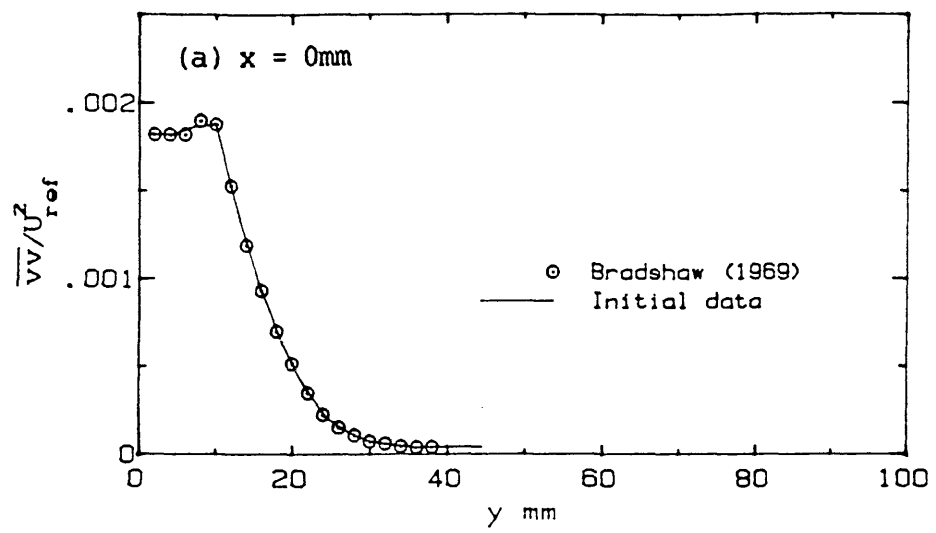


Fig.7.27 Development Of $\overline{uu}$ Profile At $x = 0\text{mm}$, 305mm , 610mm , 915mm , 1220mm & 1525mm . Experimental Data By Bradshaw (1969). Predictions By Model 2, Model 1 ($C_{\epsilon 1}=1.6$), Model 2 ($C_{\epsilon 1}=1.6$) & Model 3.



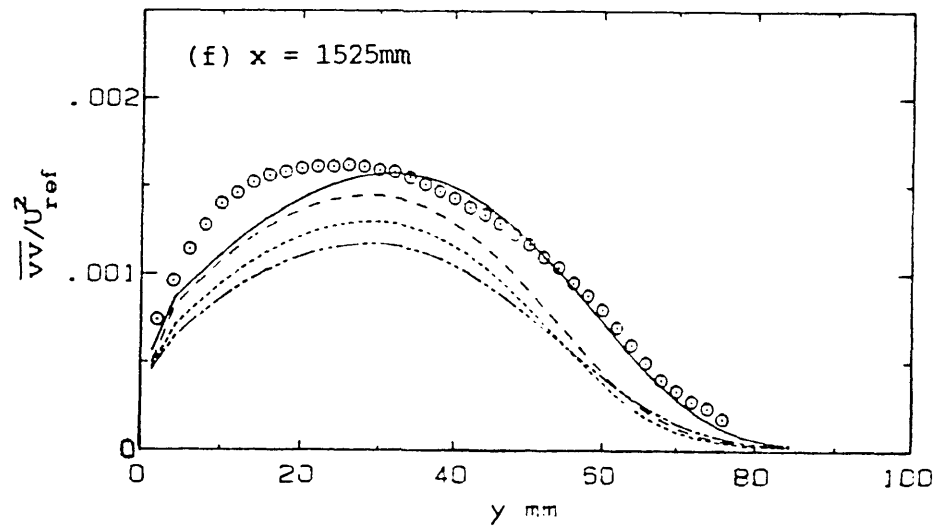
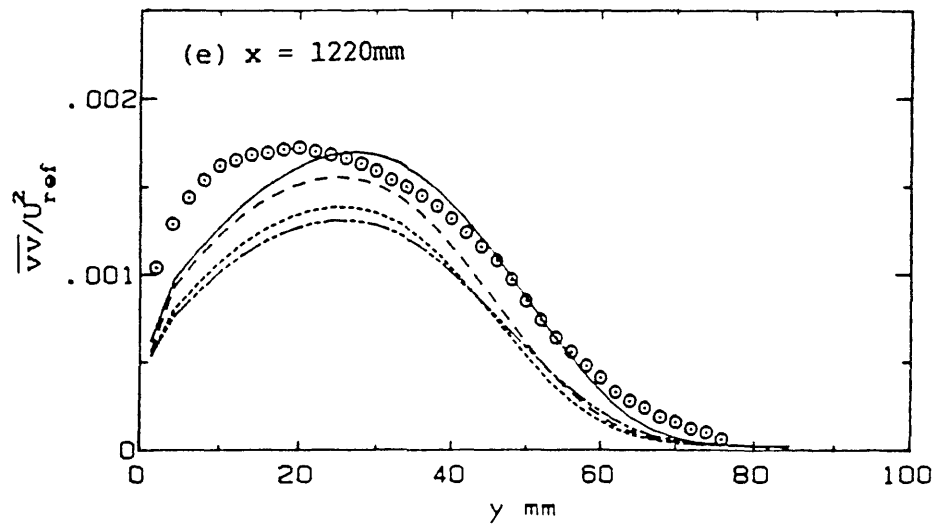
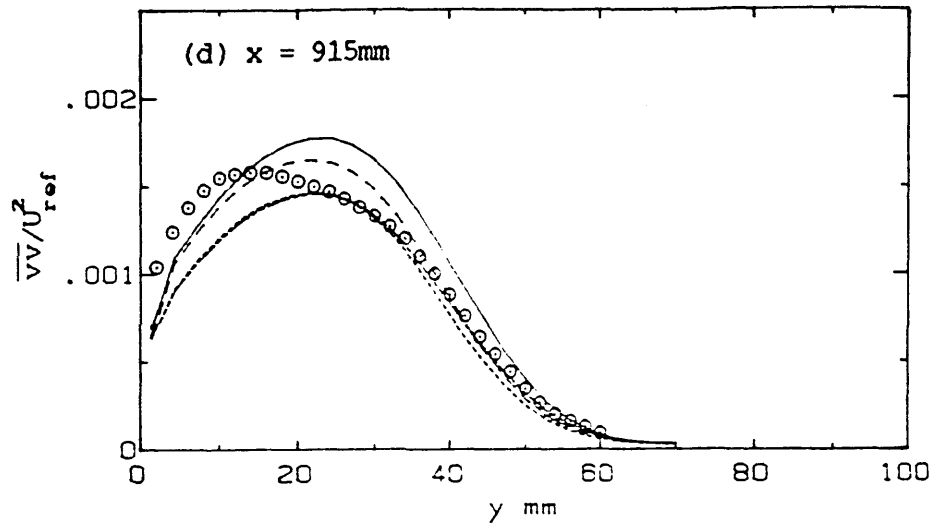
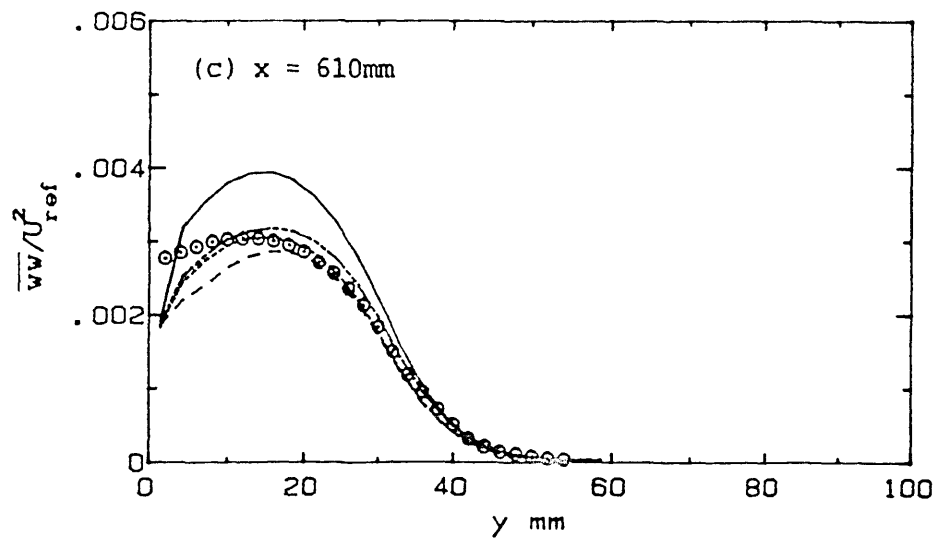
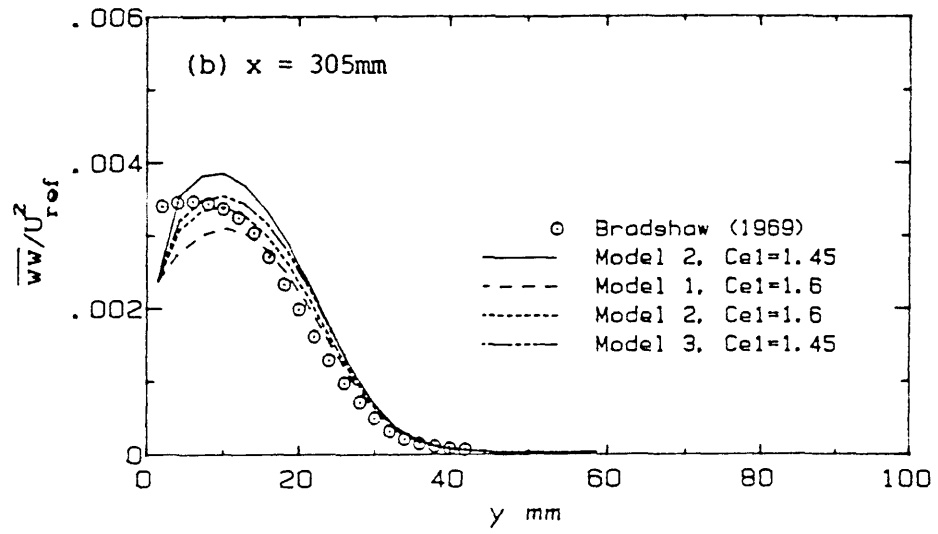
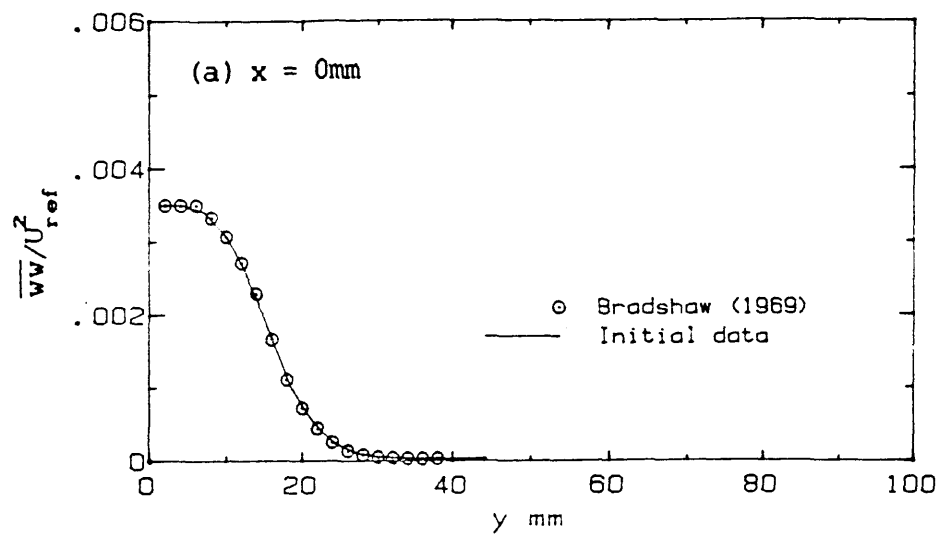


Fig.7.28 Development Of $\overline{vv}$ Profile At $x = 0\text{mm}$, 305mm , 610mm , 915mm , 1220mm & 1525mm . Experimental Data By Bradshaw (1969). Predictions By Model 2, Model 1 ($C_{\epsilon 1}=1.6$), Model 2 ($C_{\epsilon 1}=1.6$) & Model 3.



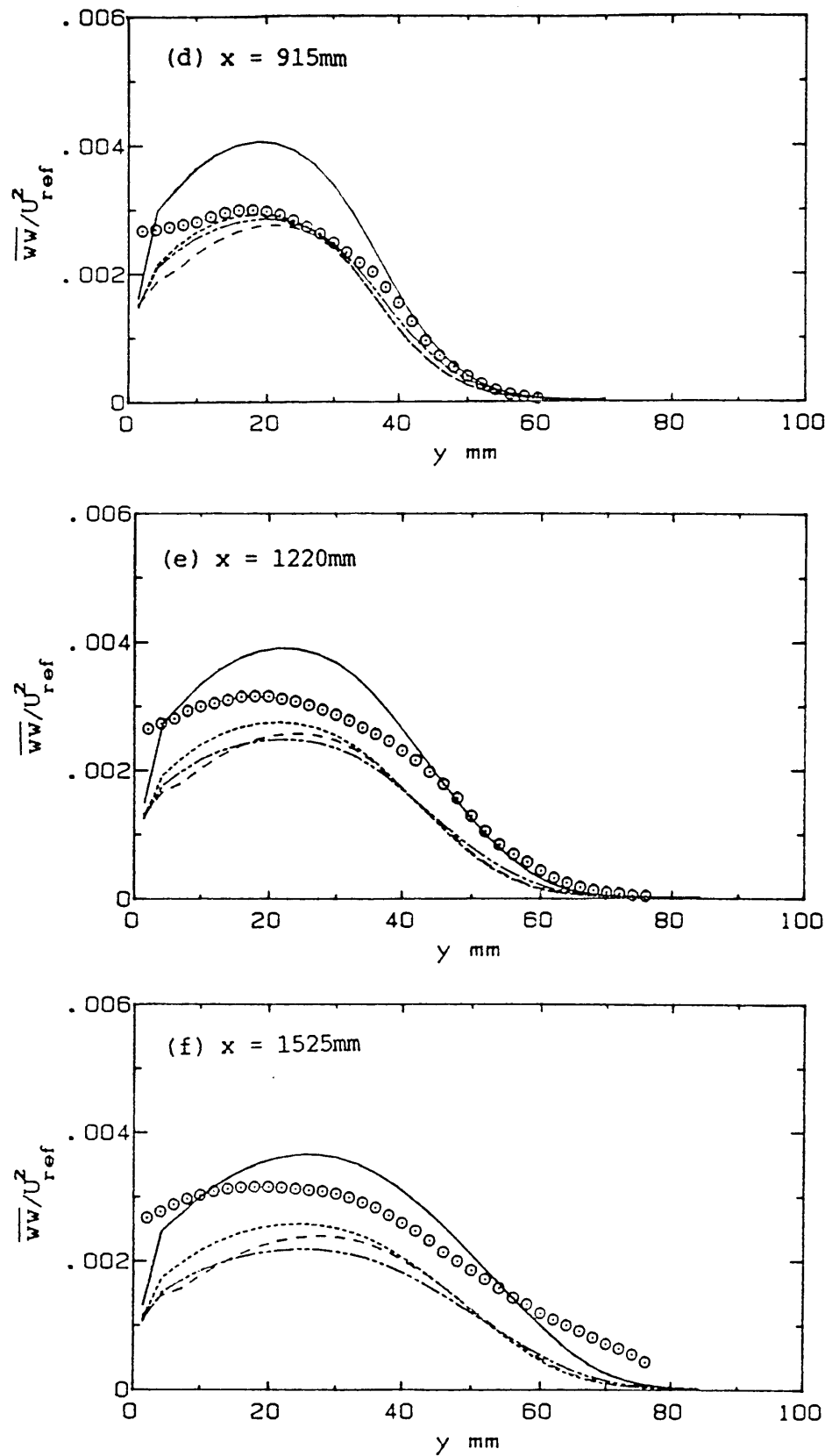
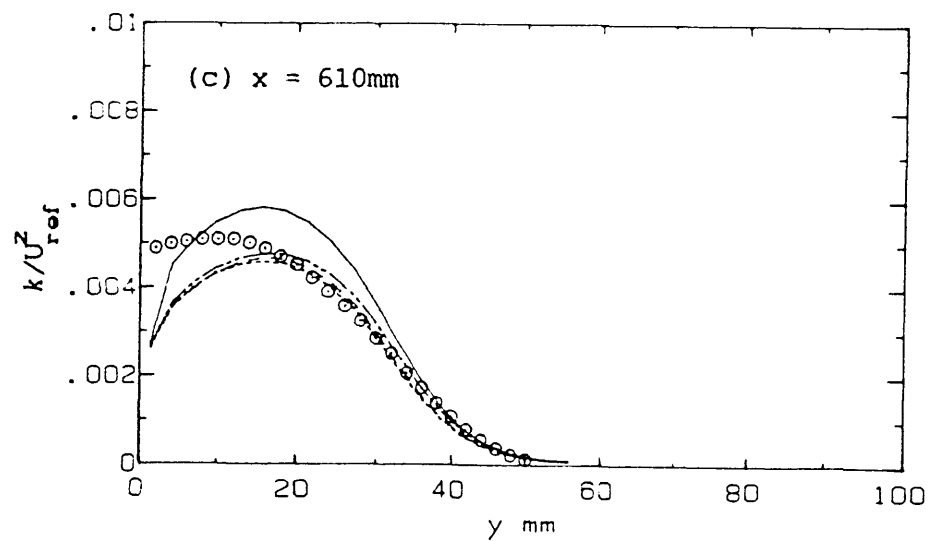
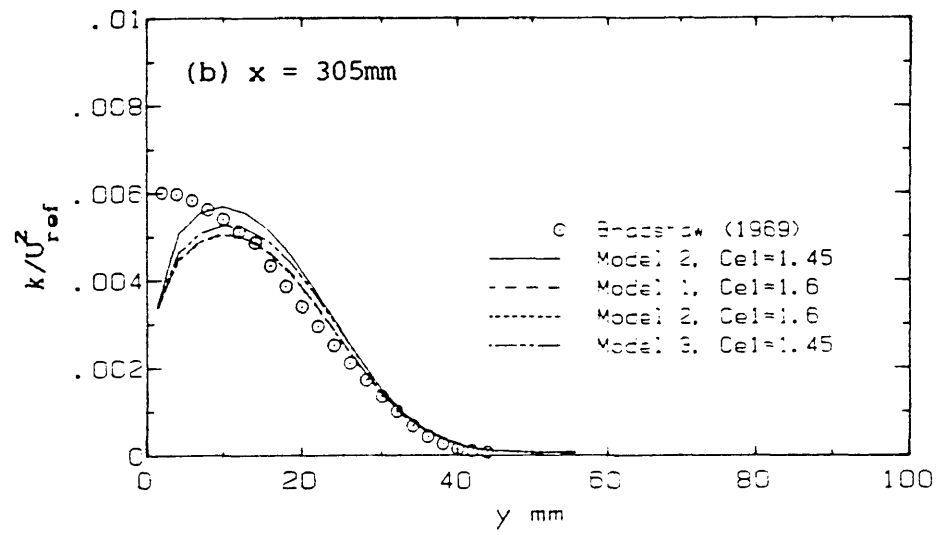
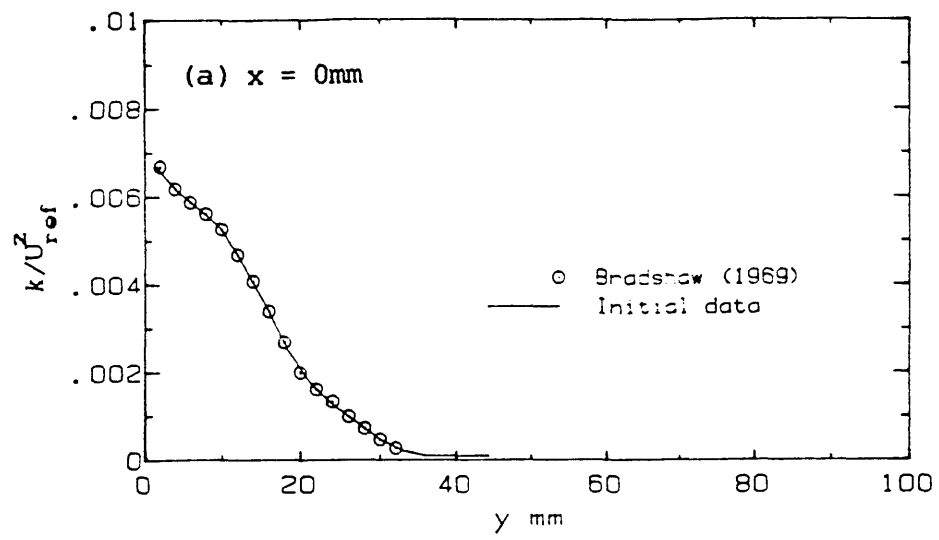


Fig.7.29 Development Of $\overline{w w}$ Profile At $x = 0\text{mm}$, 305mm , 610mm , 915mm , 1220mm & 1525mm . Experimental Data By Bradshaw (1969). Predictions By Model 2, Model 1 ($C_{\epsilon 1}=1.6$), Model 2 ($C_{\epsilon 1}=1.6$) & Model 3.



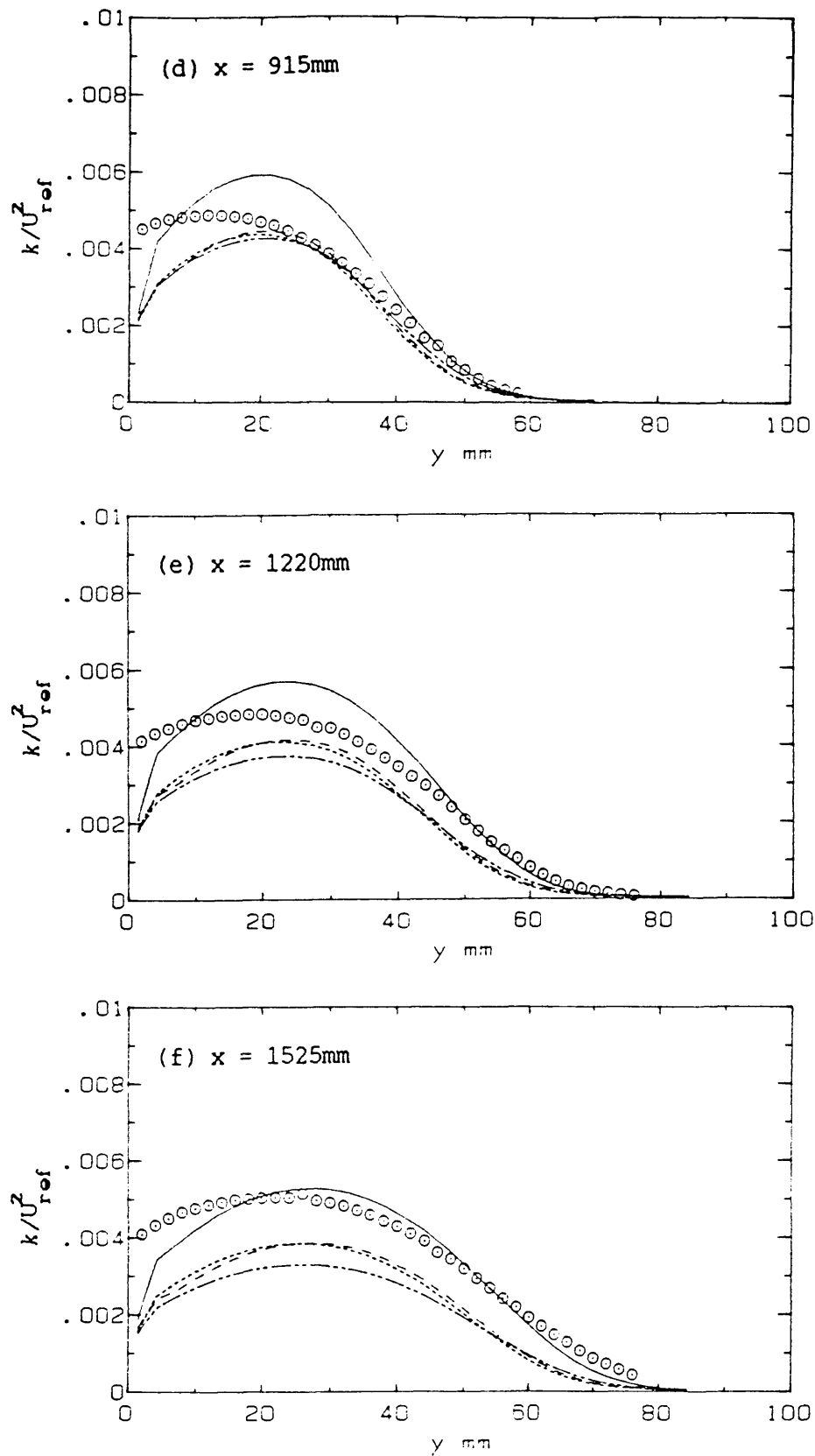


Fig.7.30 Development Of k Profile At $x = 0\text{mm}$, 305mm , 610mm , 915mm , 1220mm & 1525mm . Experimental Data By Bradshaw (1969). Predictions By Model 2, Model 1 ($C_{\epsilon 1}=1.6$), Model 2 ($C_{\epsilon 1}=1.6$) & Model 3.

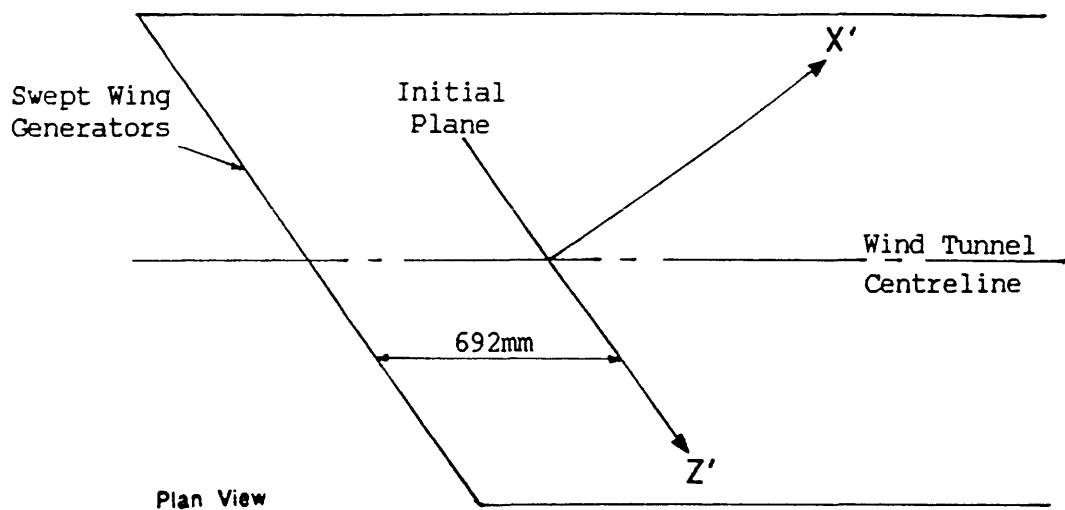


Fig.7.31 "Infinite" Swept Wing In $x'-y-z'$ Axes.

Crossflow
Plane

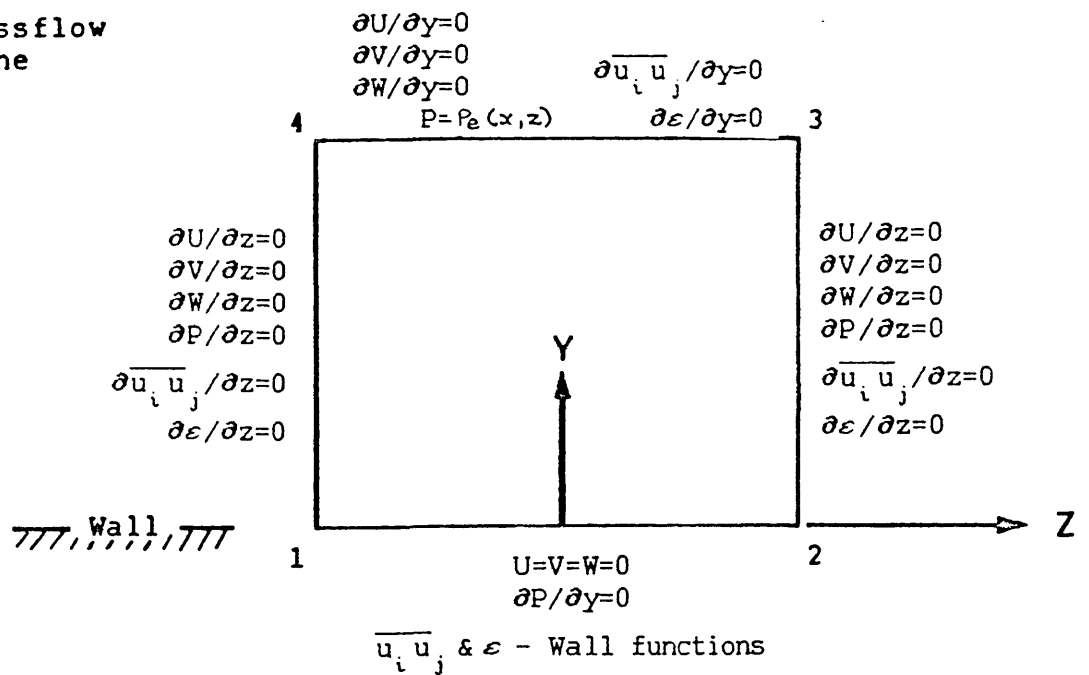
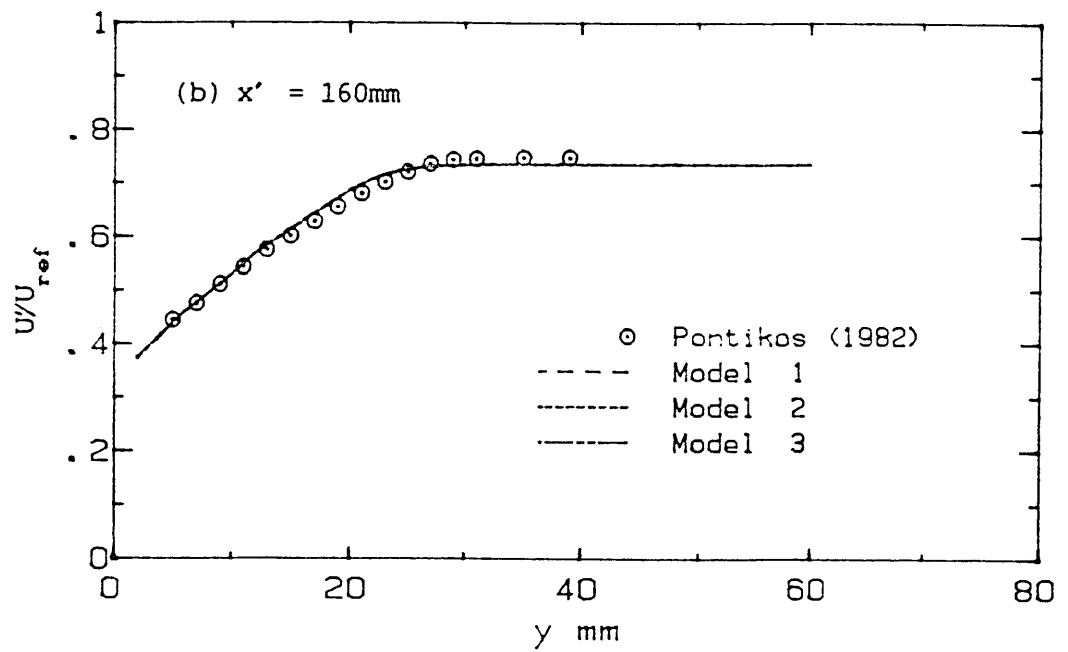
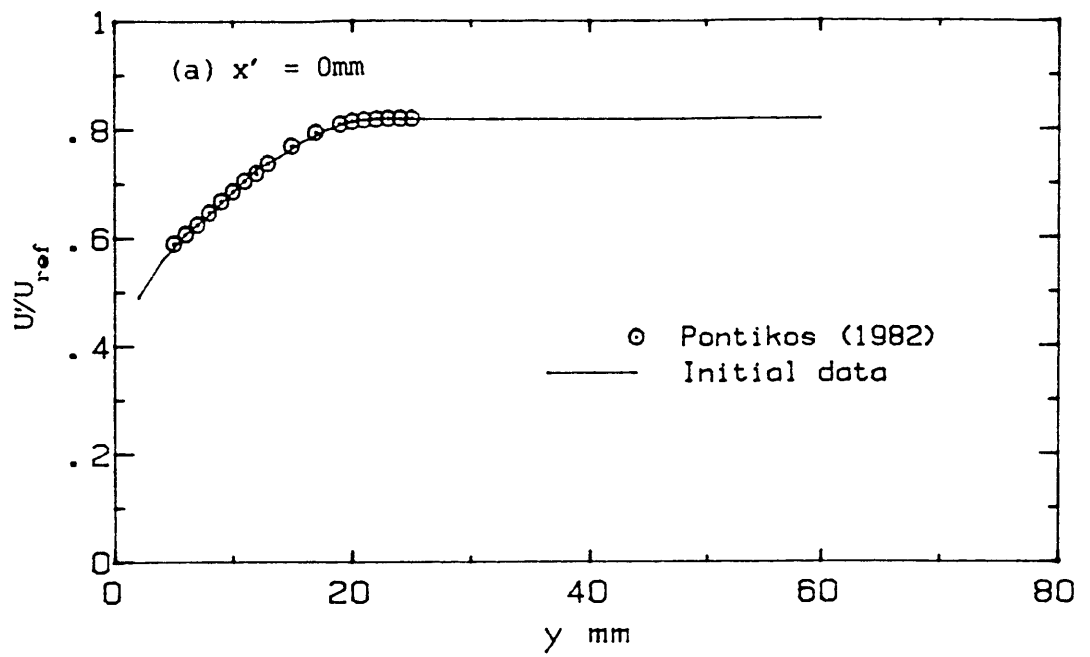


Fig.7.32 Boundary Conditions Of "Infinite" Swept Wing In $x'-y-z'$ Axes.



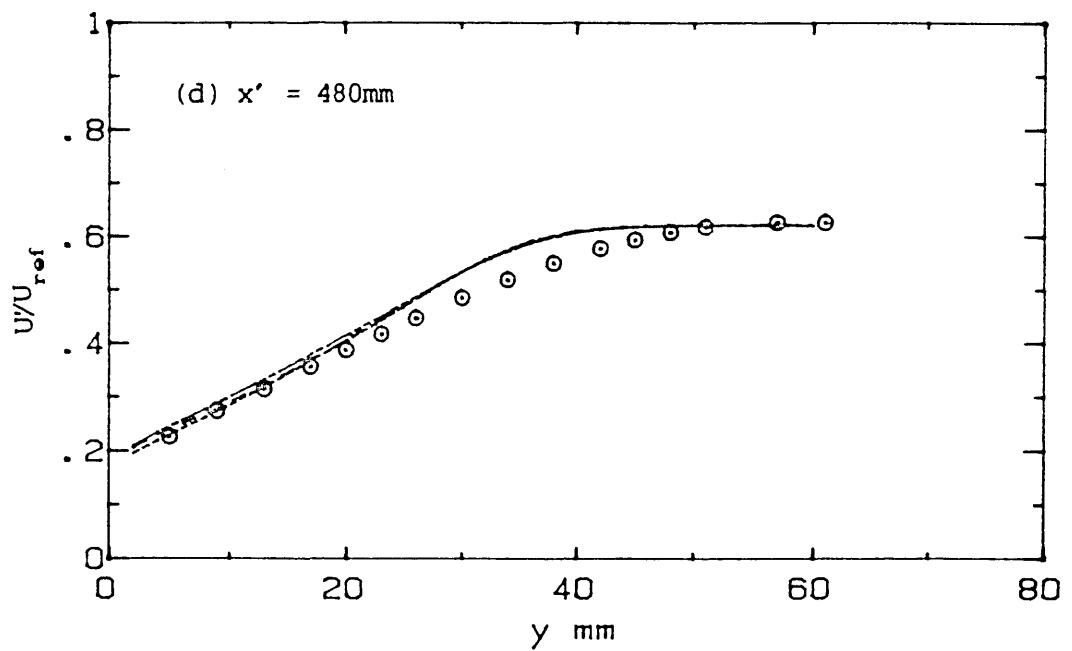
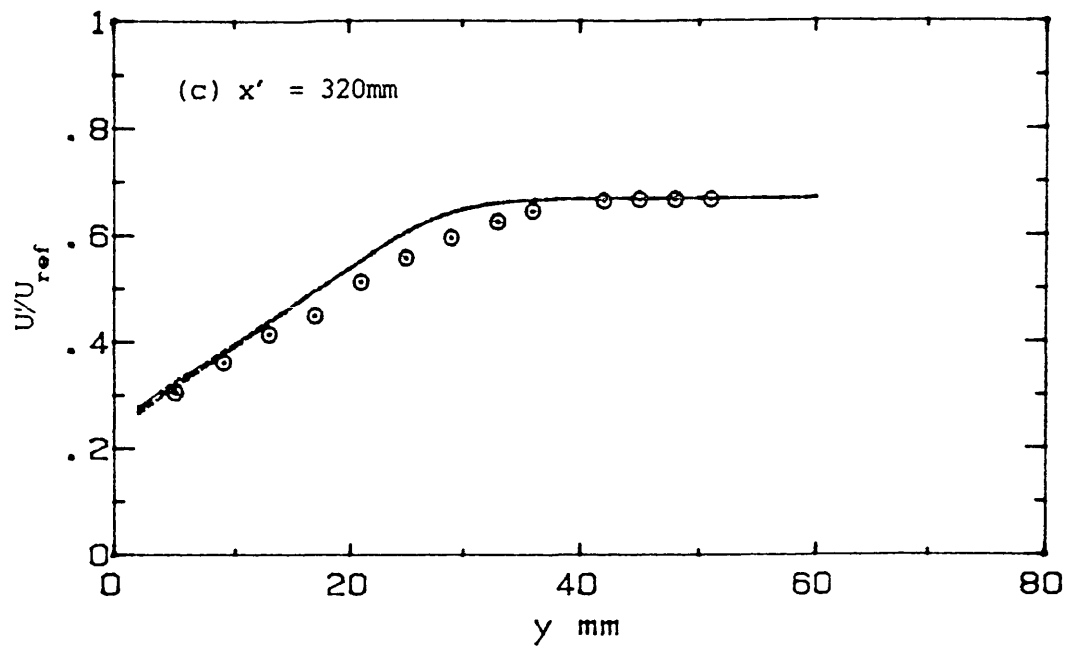
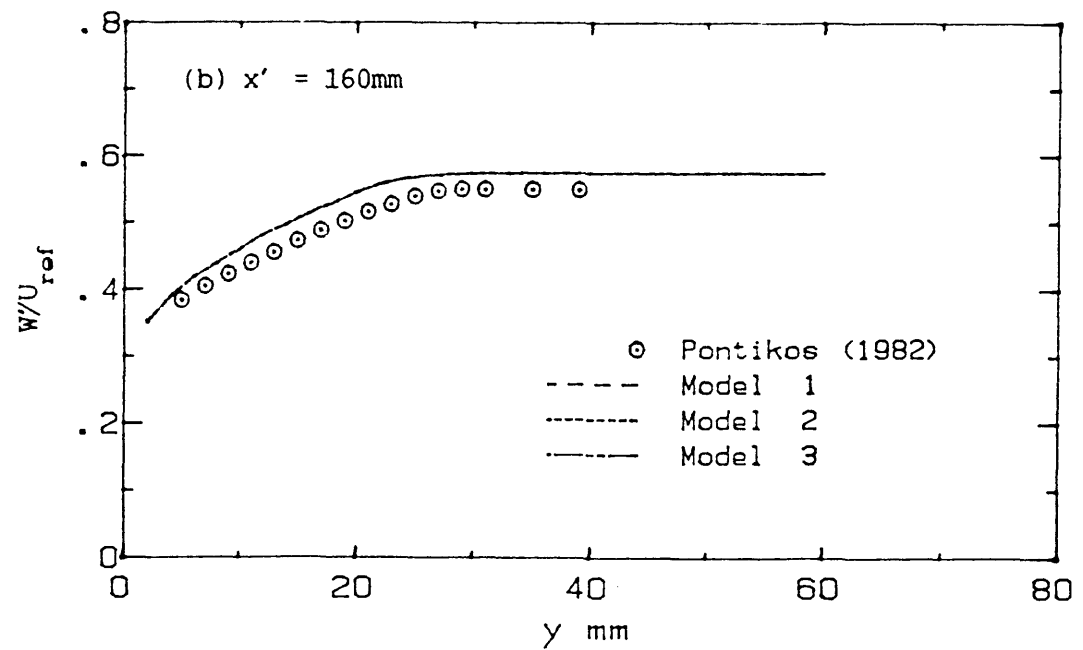
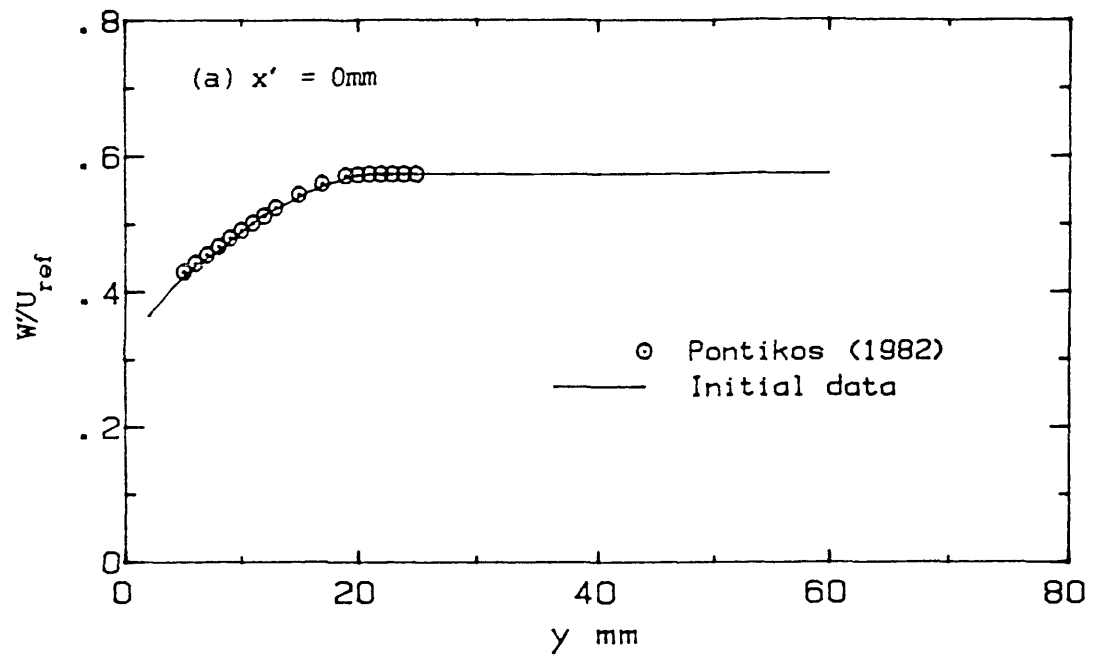


Fig.7.33 Development Of U' Profile At $x' = 0\text{mm}, 160\text{mm}, 320\text{mm}$ & 480mm . Experimental Data By Pontikos (1982). Predictions By Model 1, Model 2 & Model 3.



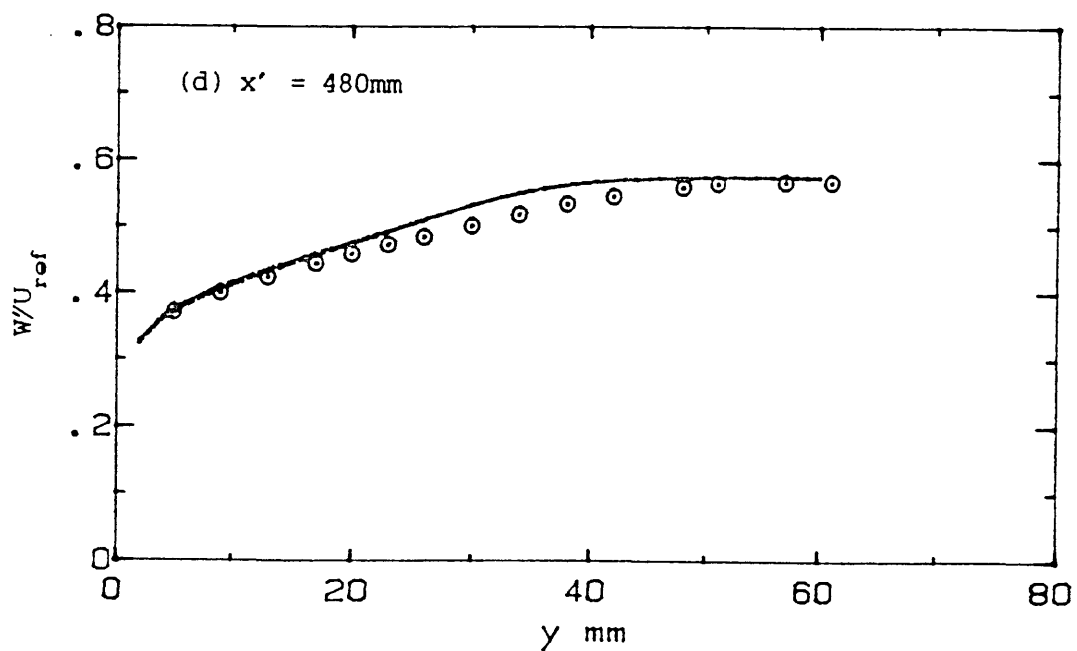
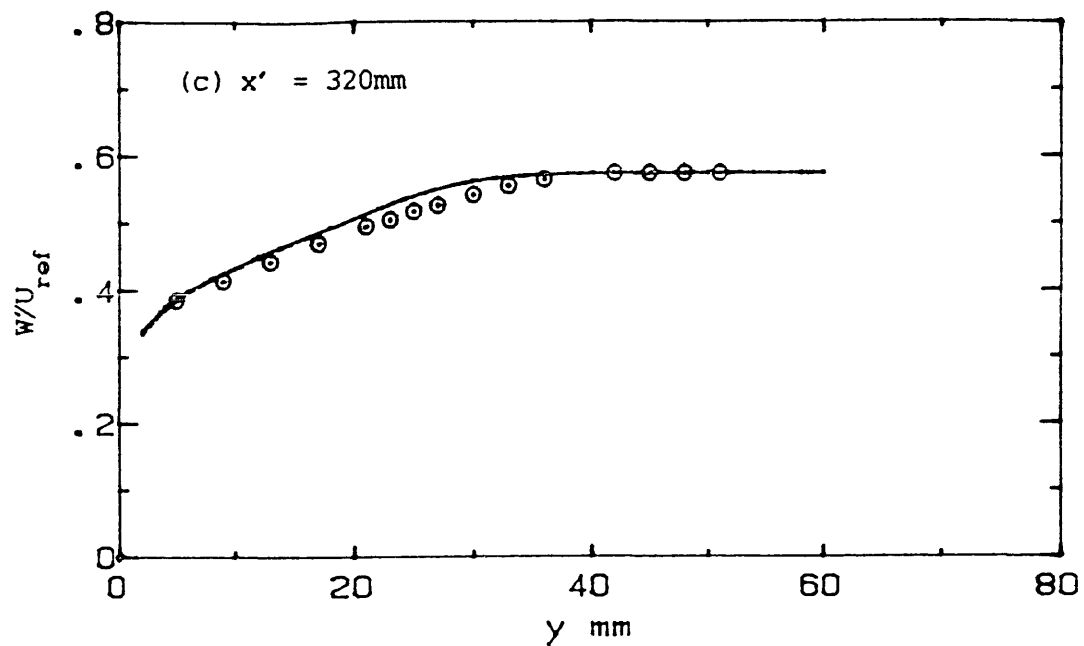


Fig.7.34 Development Of W' Profile At $x' = 0\text{mm}$, 160mm , 320mm & 480mm . Experimental Data By Pontikos (1982). Predictions By Model 1, Model 2 & Model 3.

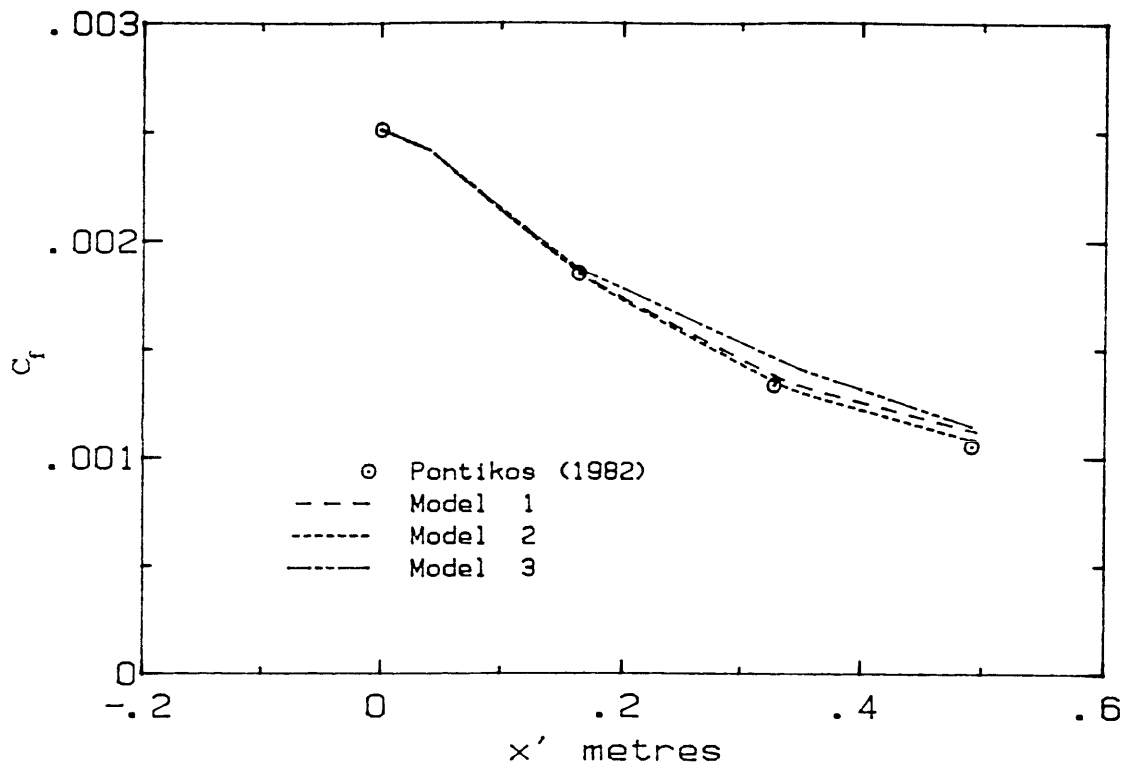


Fig.7.35 Skin Friction Coefficient. Experimental Data By Pontikos (1982). Predictions By Model 1, Model 2 & Model 3.

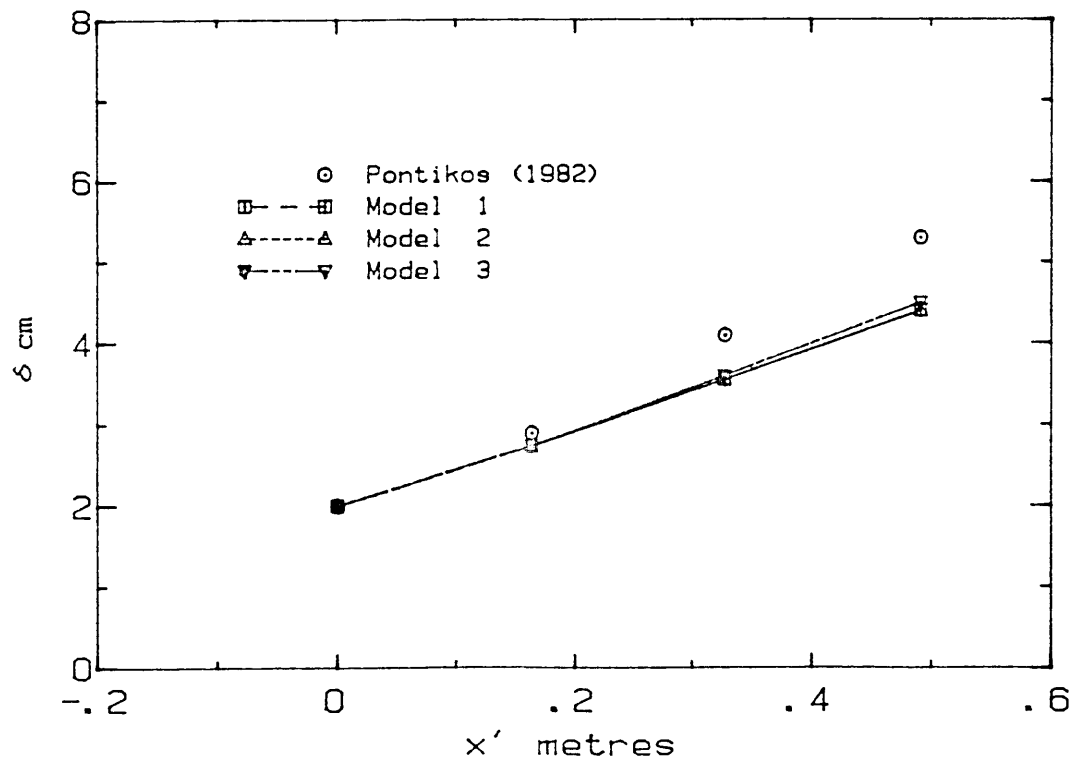


Fig.7.36 Boundary Layer Thickness. Experimental Data By Pontikos (1982). Predictions By Model 1, Model 2 & Model 3.

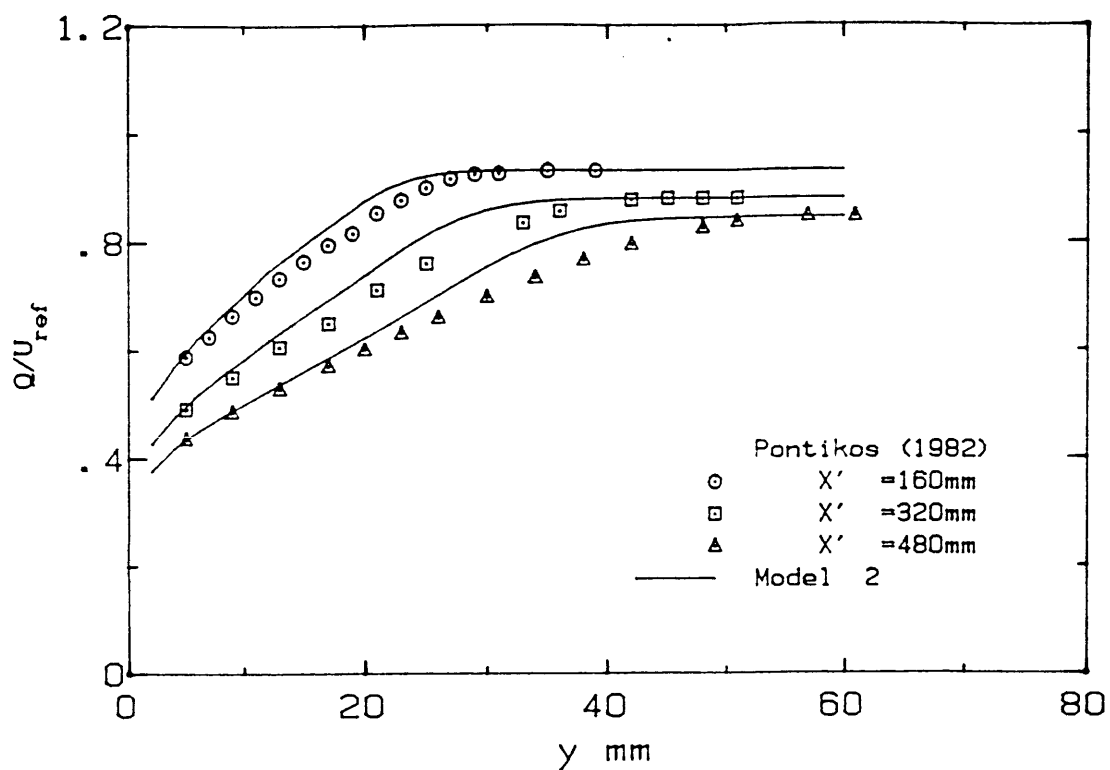


Fig.7.37 Total Velocity Profiles. Experimental Data By Pontikos (1982) & Predictions By Model 2.

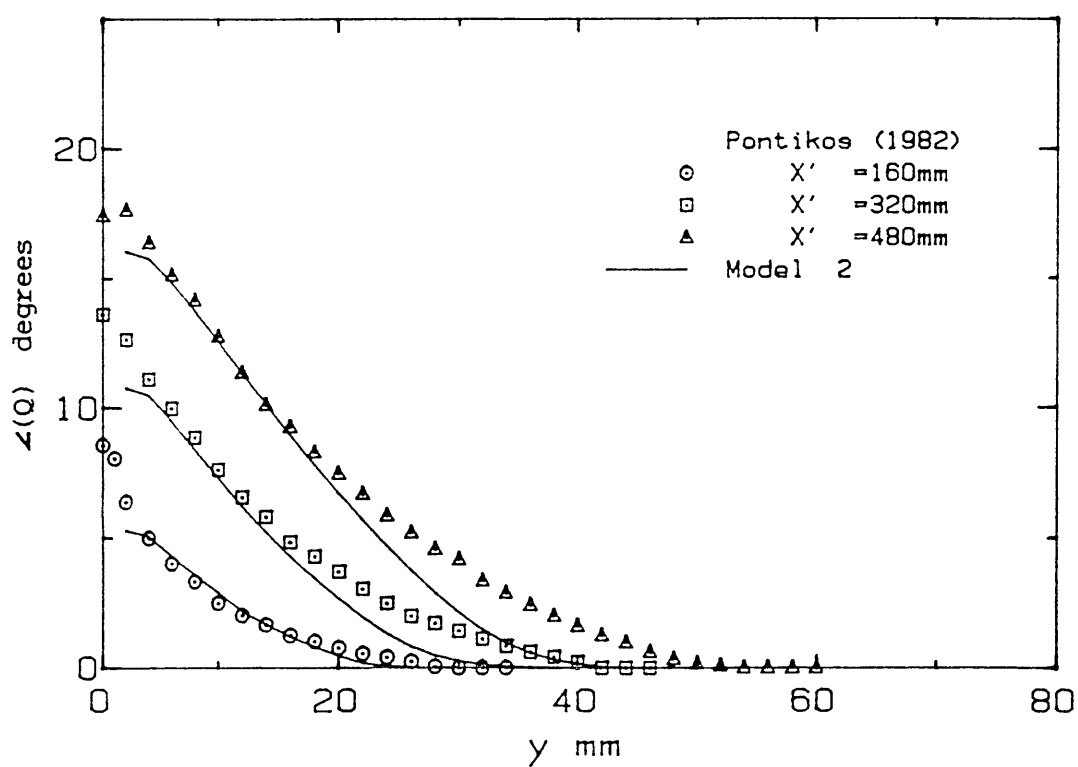
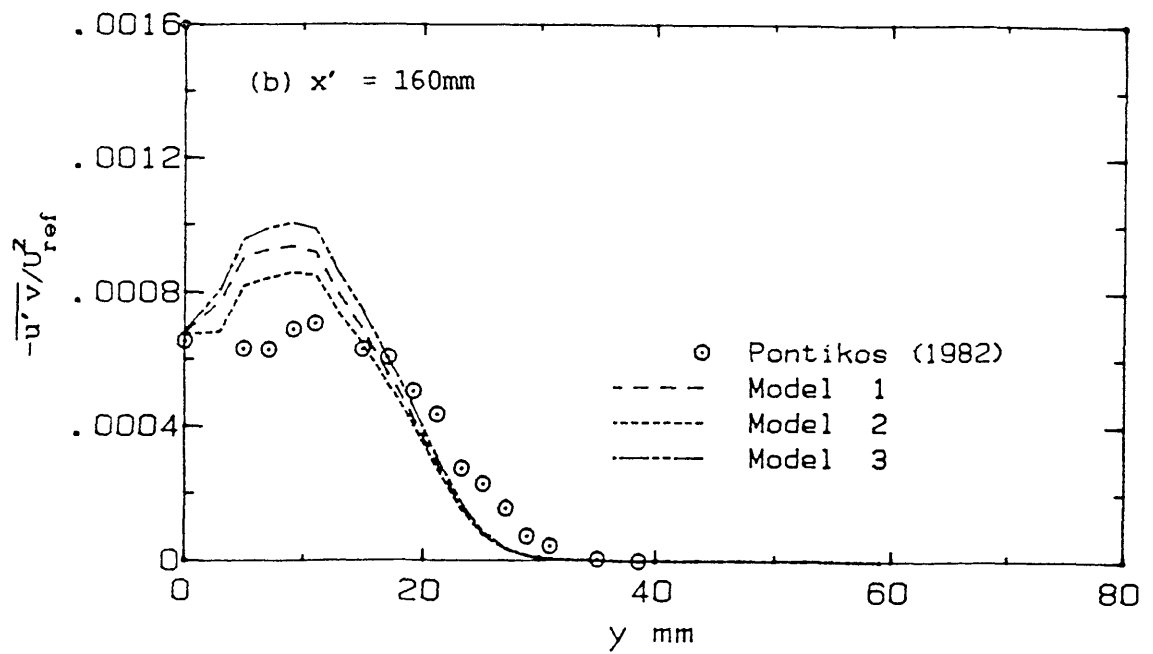
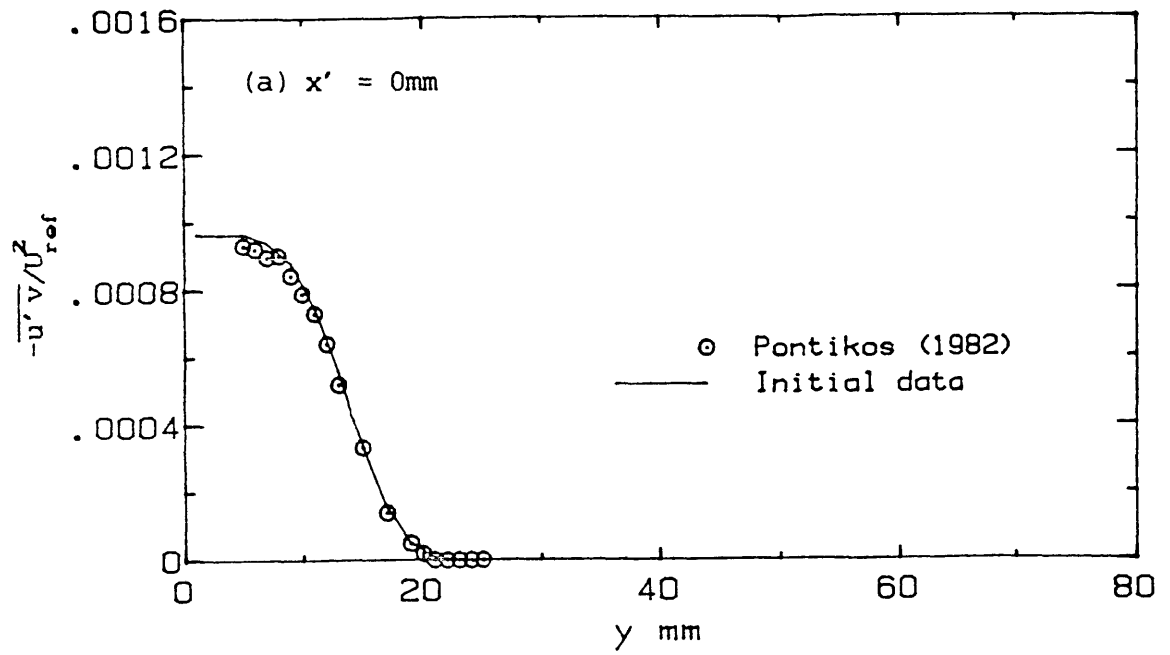


Fig.7.38 The Total Velocity Vector Direction. Experimental Data By Pontikos (1982) & Predictions By Model 2.



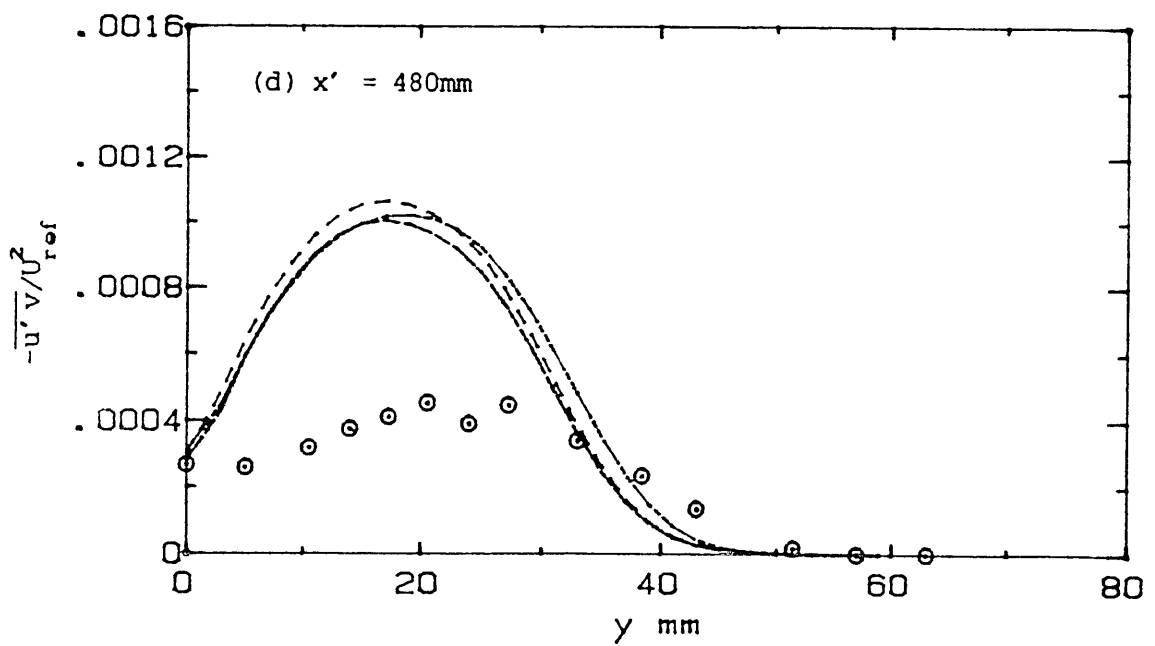
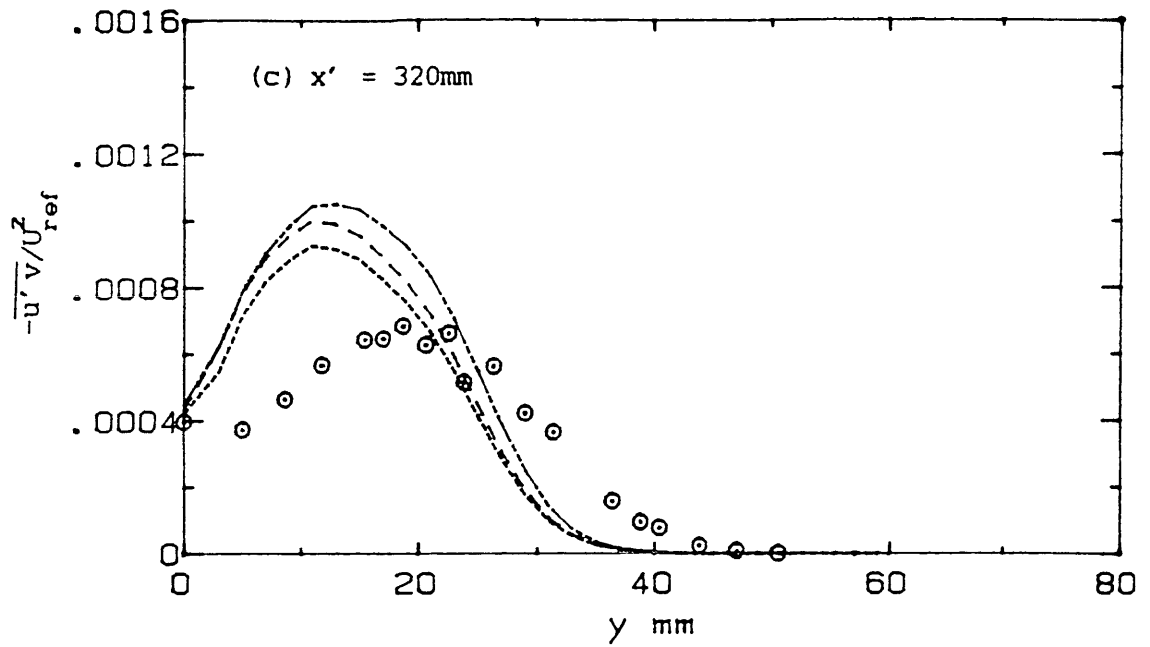
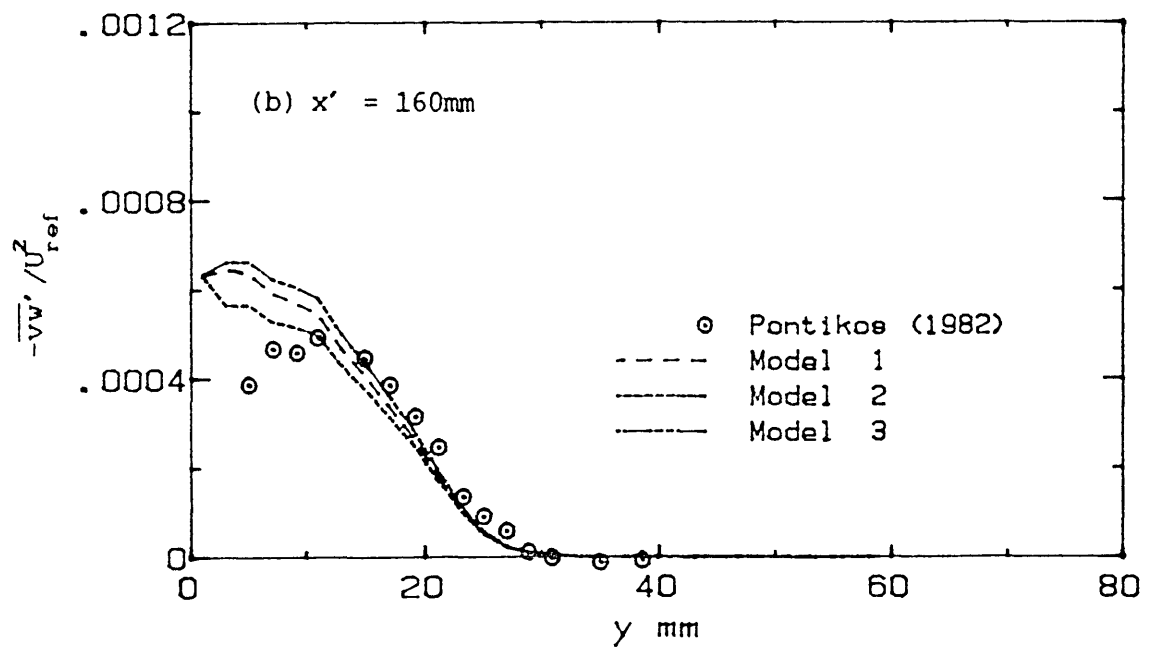
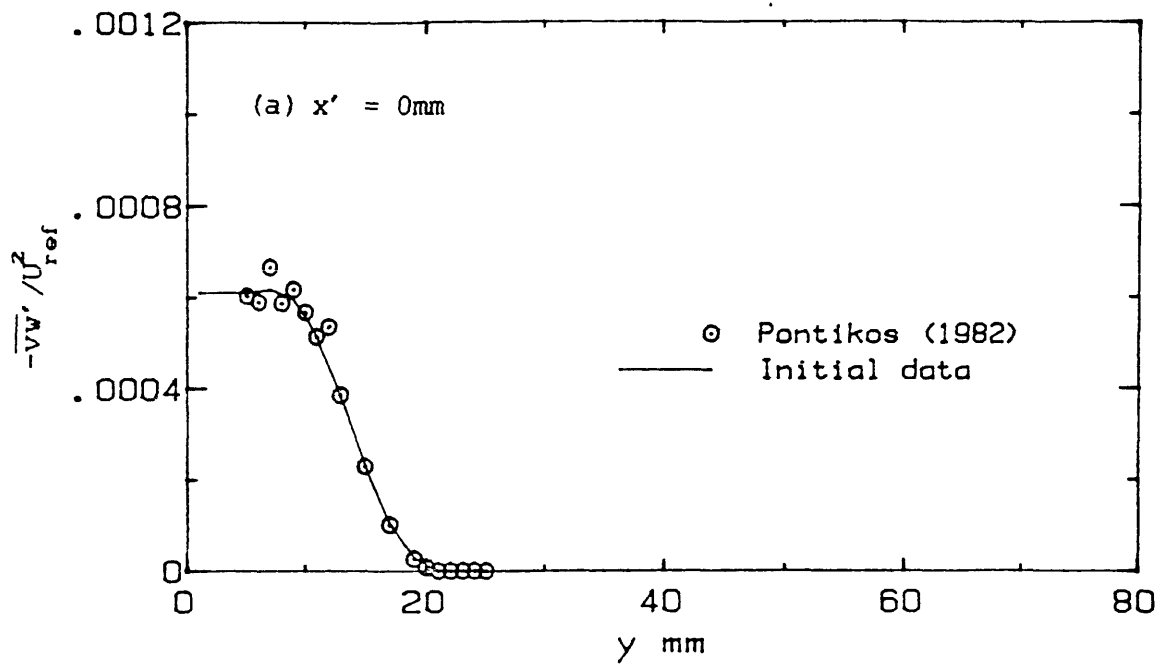


Fig.7.39 Development Of $\overline{u'v}$ Profile At $x' = 0\text{mm}, 160\text{mm}, 320\text{mm}$ & 480mm . Experimental Data By Pontikos (1982). Predictions By Model 1, Model 2 & Model 3.



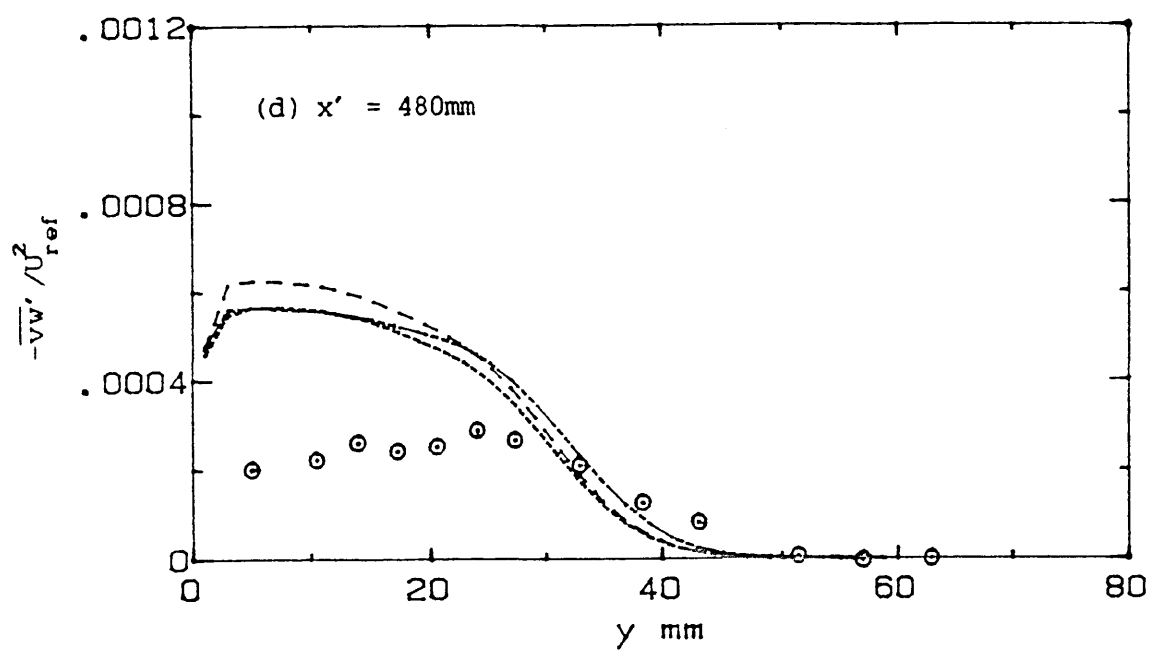
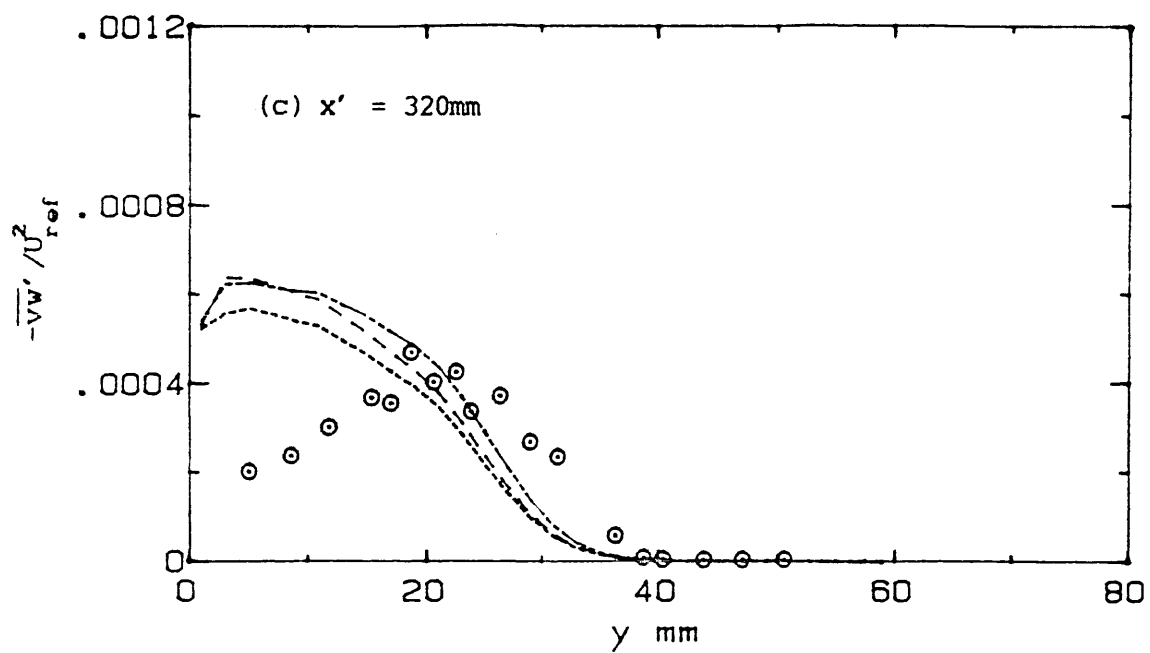


Fig.7.40 Development Of $\overline{v'w'}$ Profile At $x' = 0\text{mm}, 160\text{mm}, 320\text{mm}$ & 480mm . Experimental Data By Pontikos (1982). Predictions By Model 1, Model 2 & Model 3.

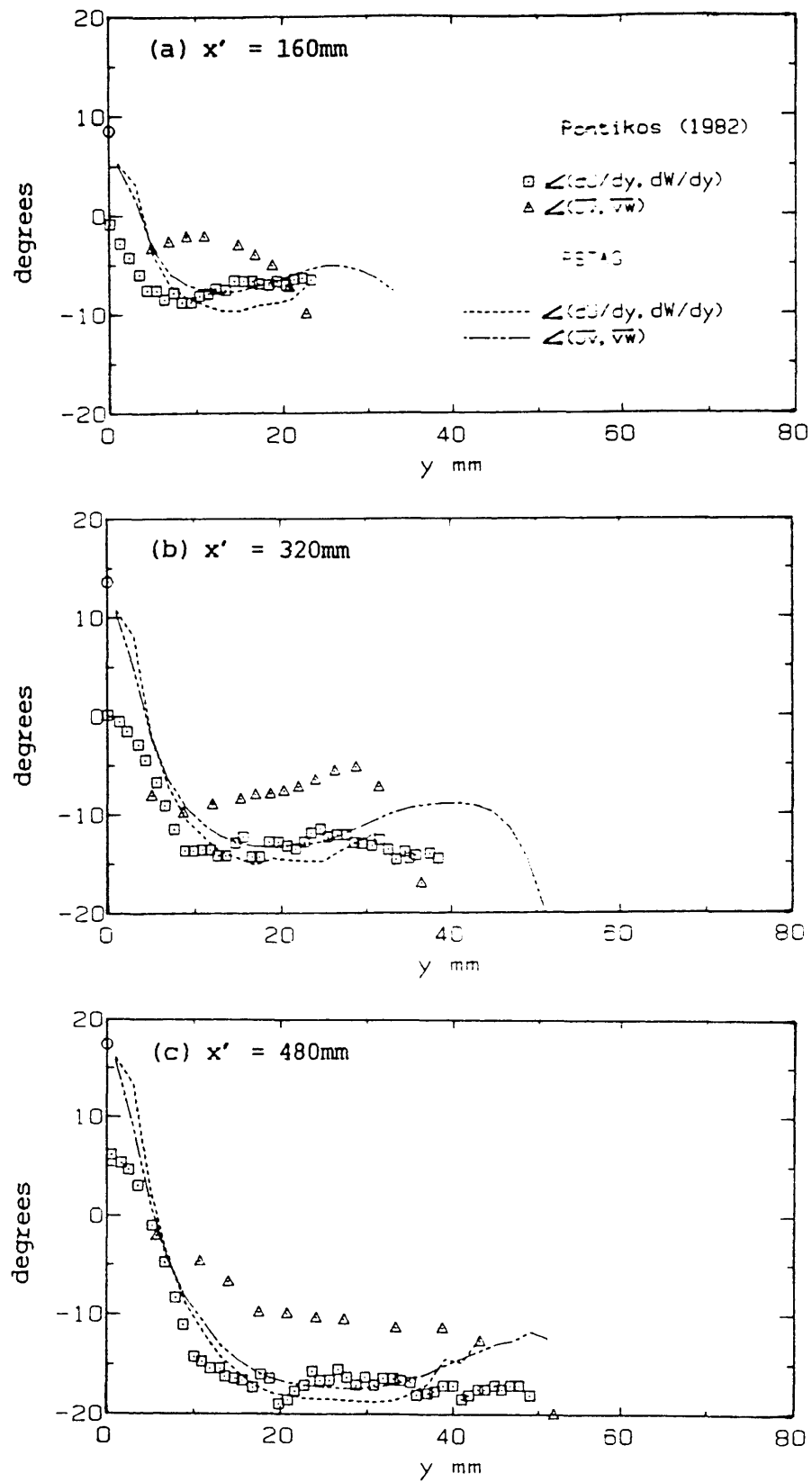
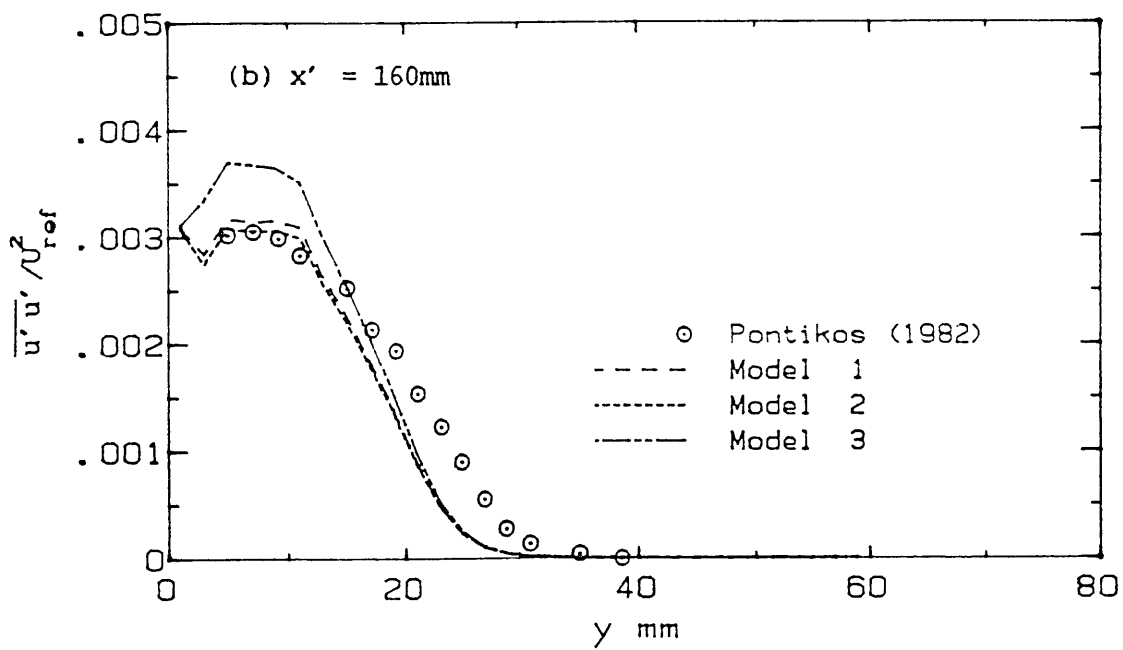
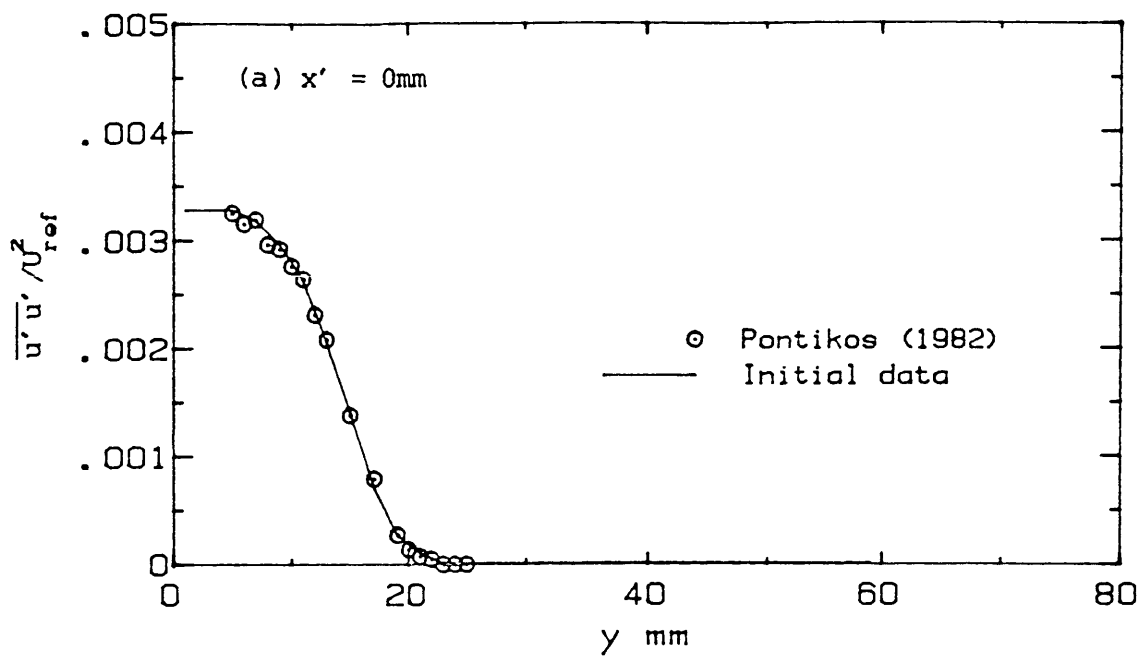


Fig.7.41 Directions Of Shear Stress Vector & Mean Velocity Gradient Vector. Experimental Data By Pontikos (1982) & Predictions By Model 2.



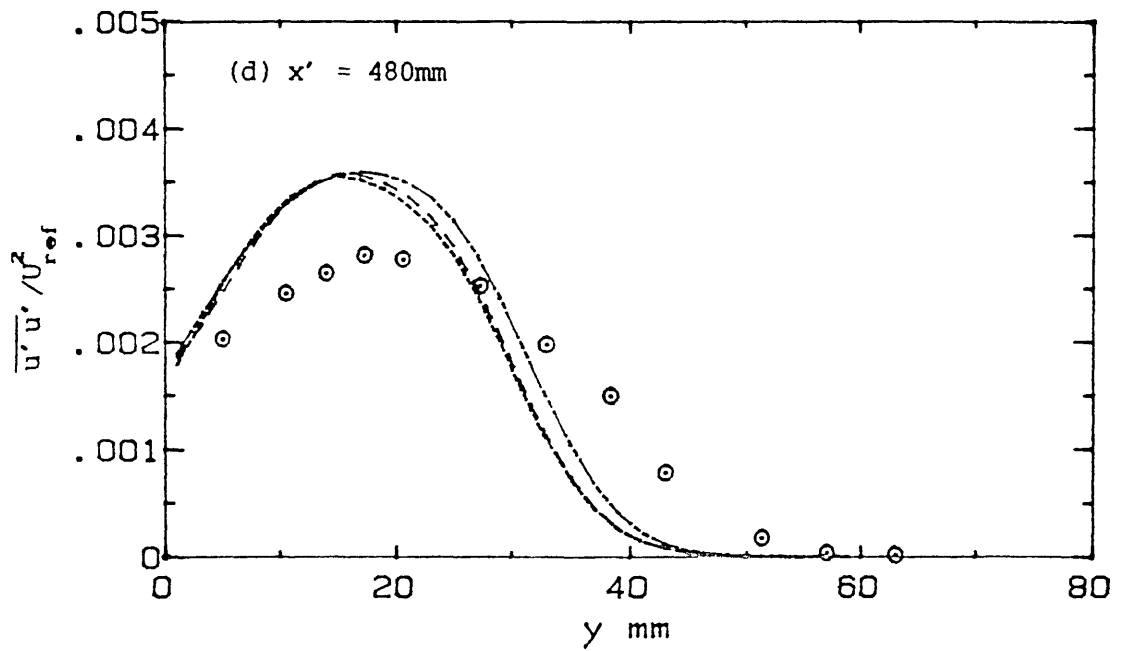
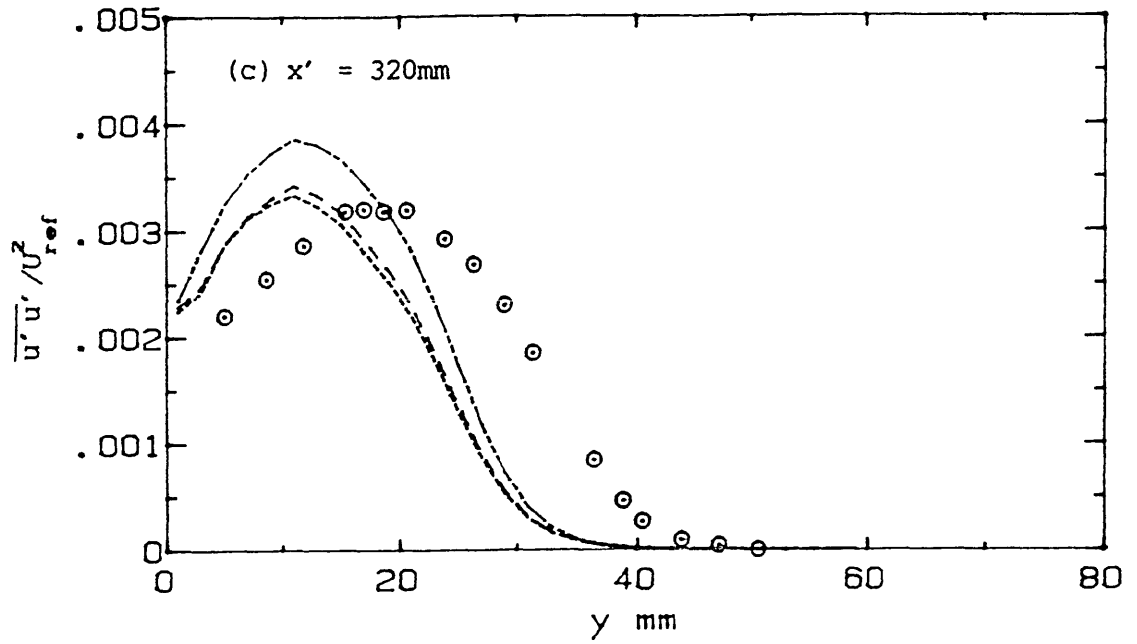
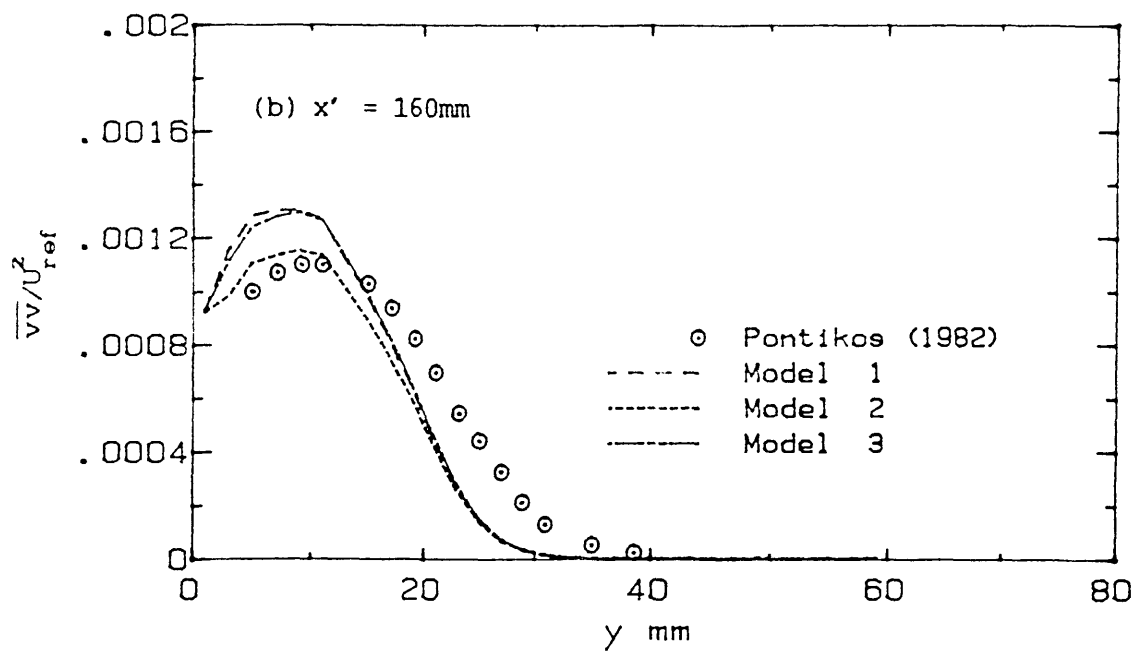
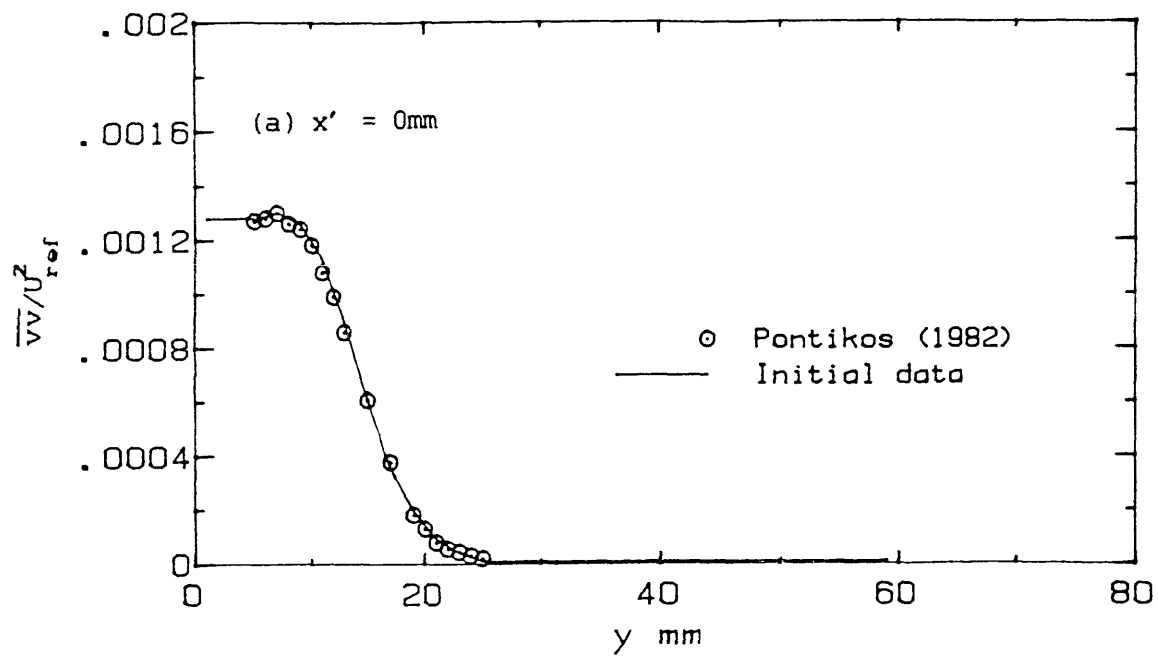


Fig.7.42 Development Of $\overline{u'u'}$ Profile At $x' = 0\text{mm}, 160\text{mm}, 320\text{mm}$ & 480mm . Experimental Data By Pontikos (1982). Predictions By Model 1, Model 2 & Model 3.



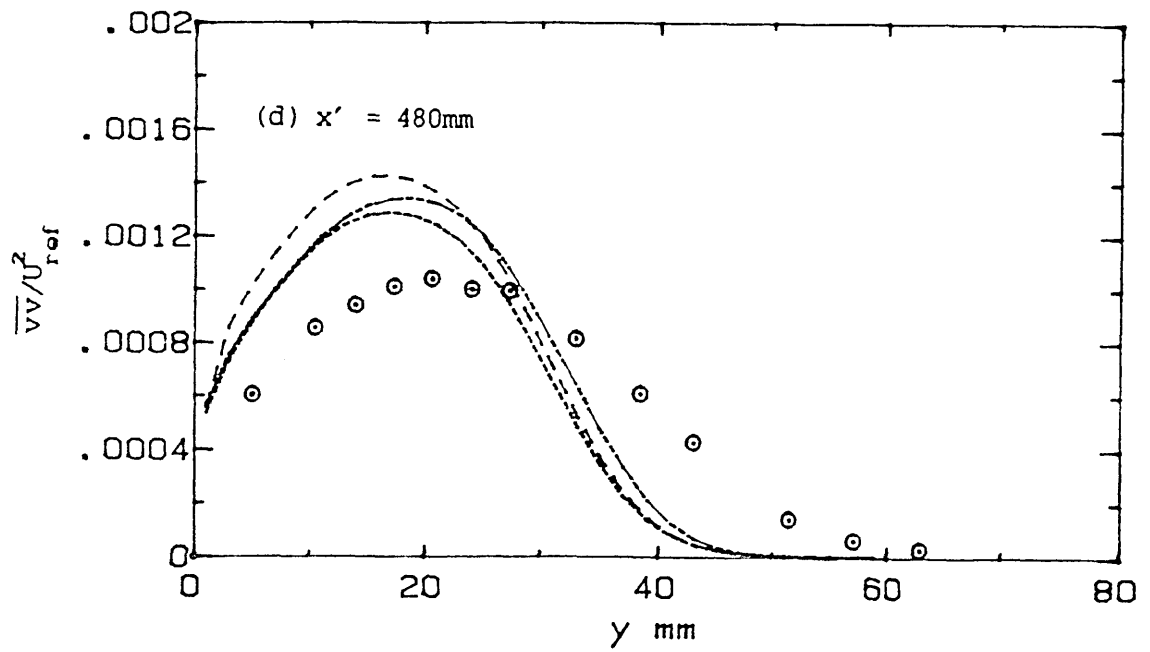
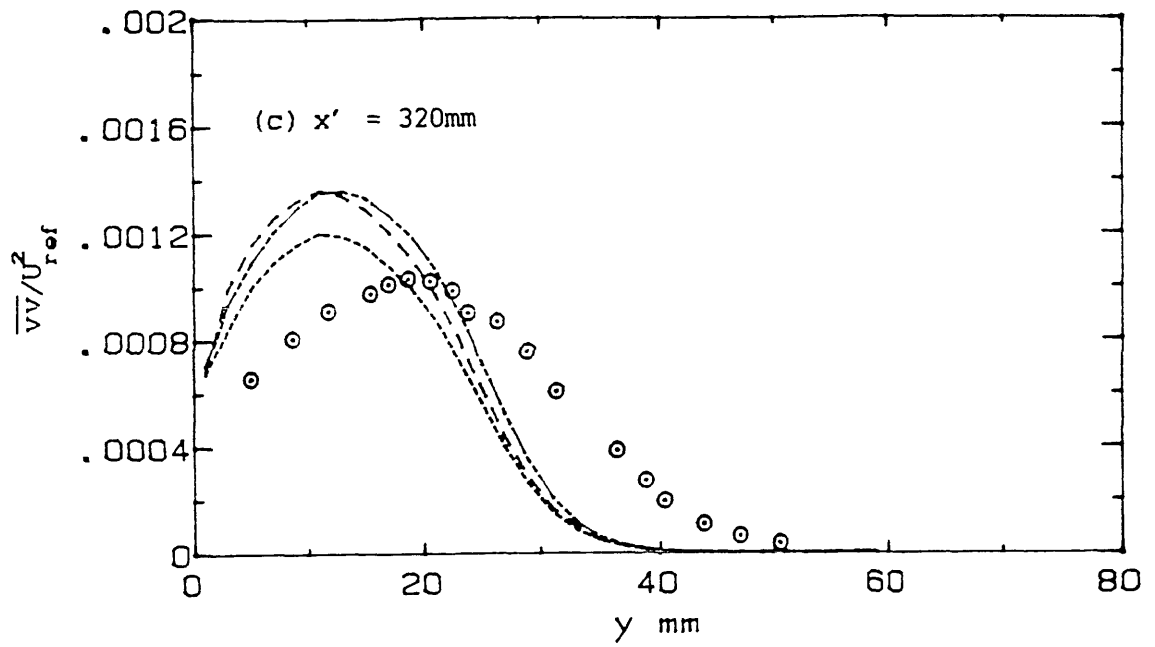
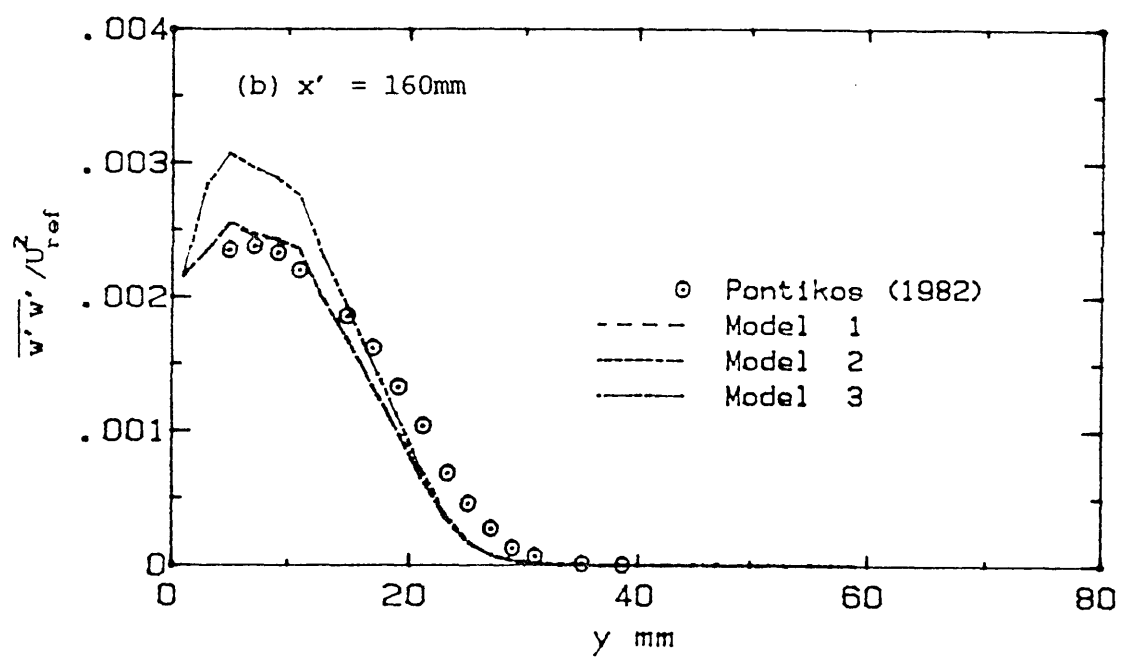
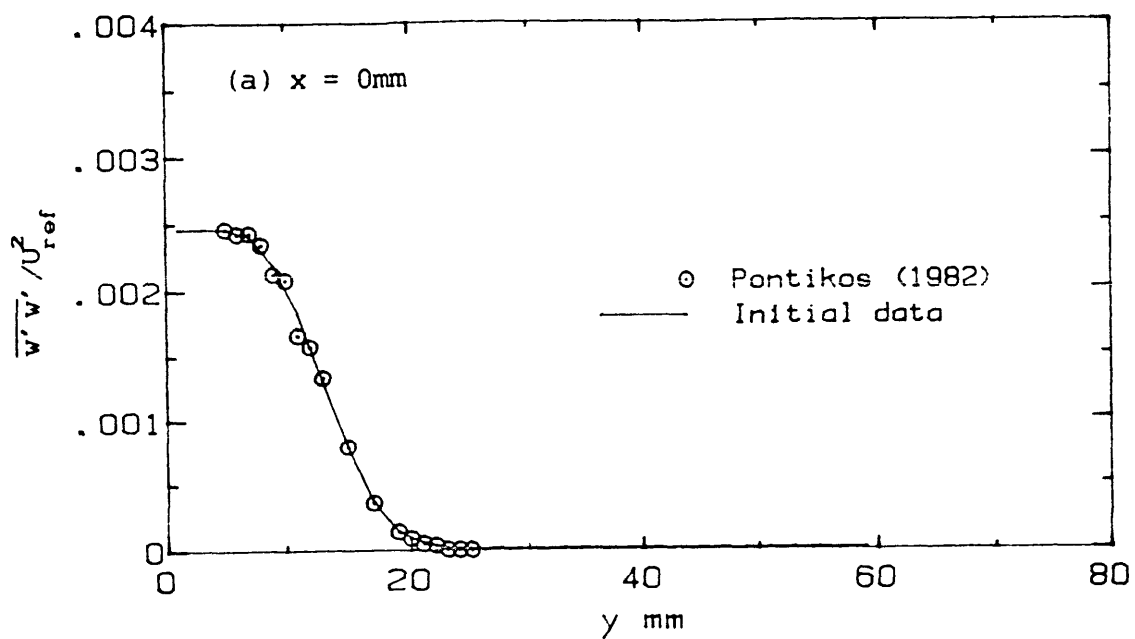


Fig.7.43 Development Of $\overline{vv}$ Profile At $x' = 0\text{mm}, 160\text{mm}, 320\text{mm}$ & 480mm . Experimental Data By Pontikos (1982). Predictions By Model 1, Model 2 & Model 3.



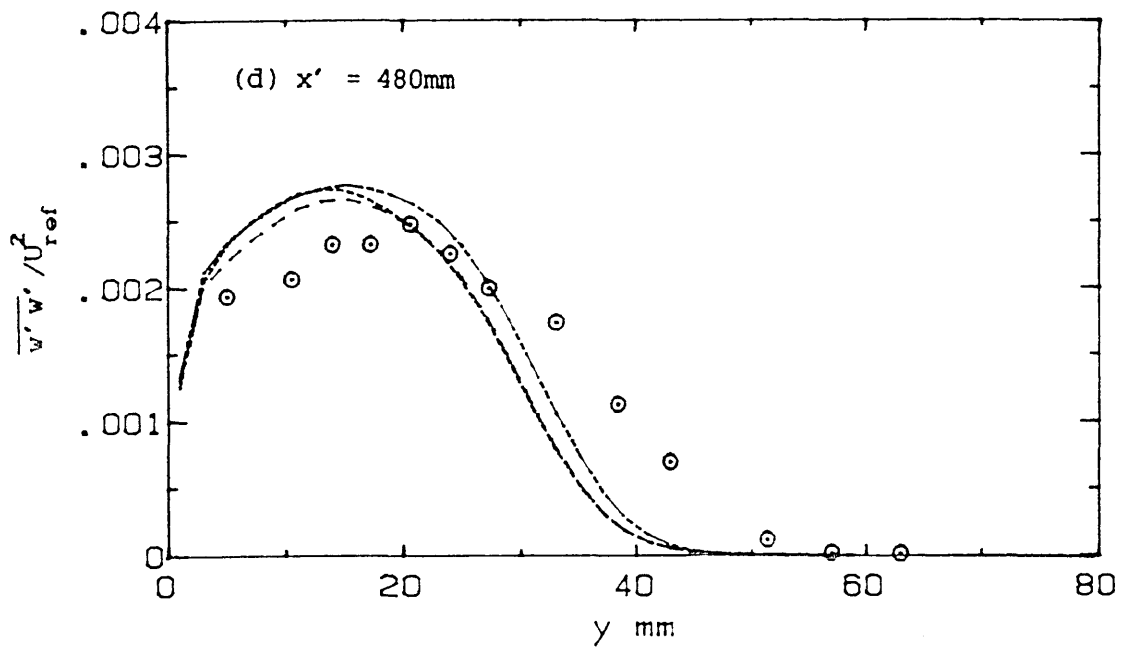
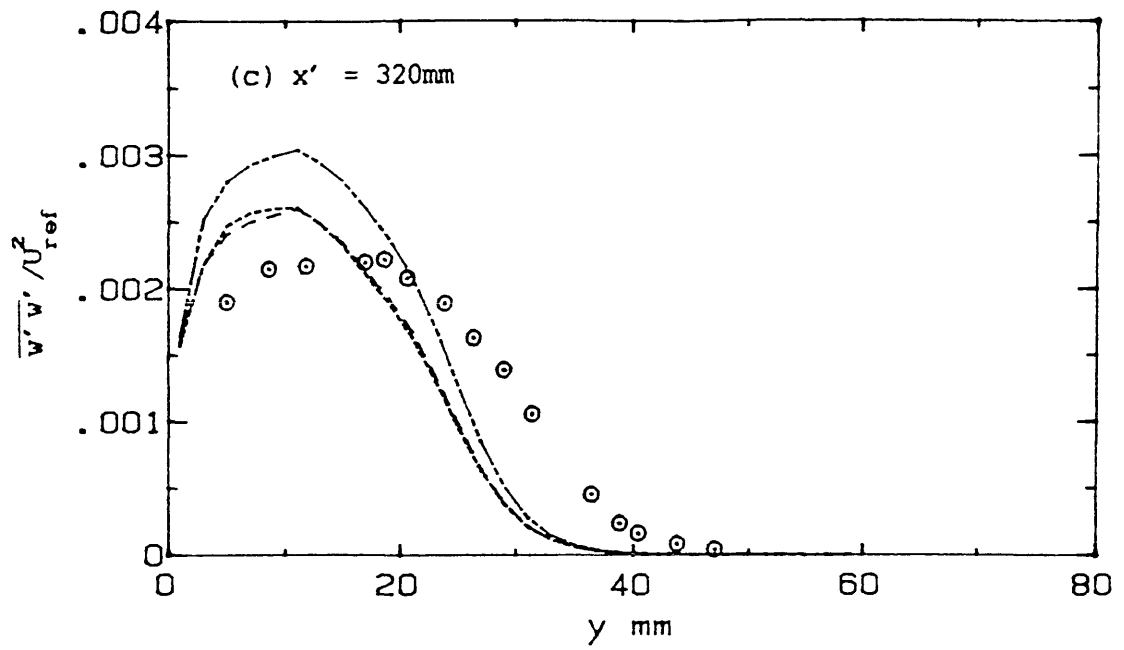
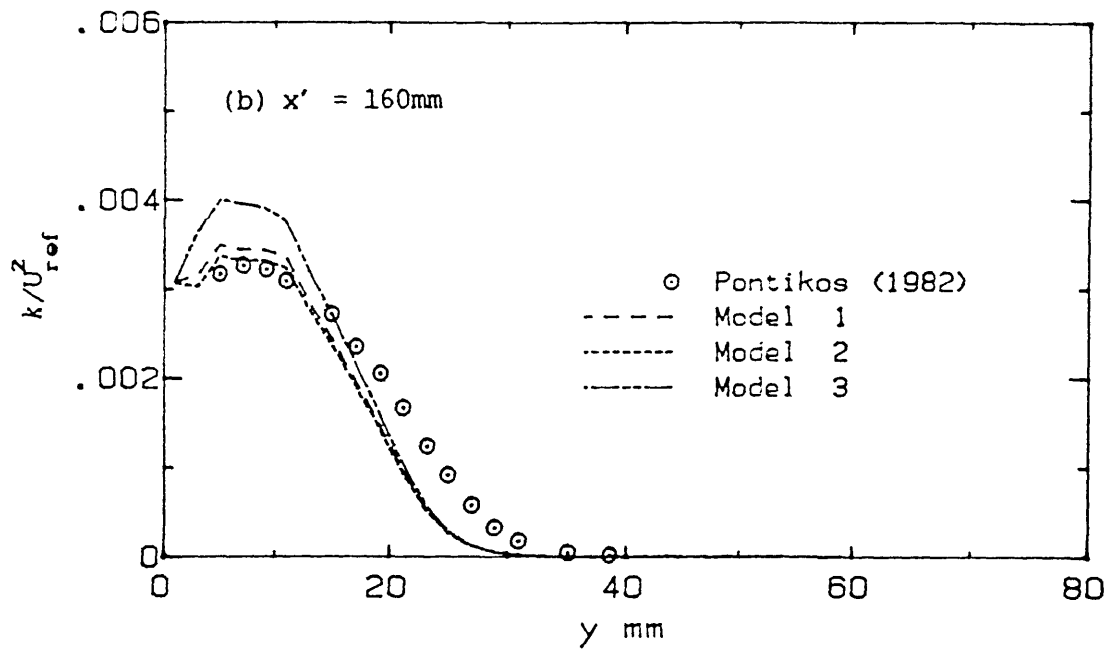
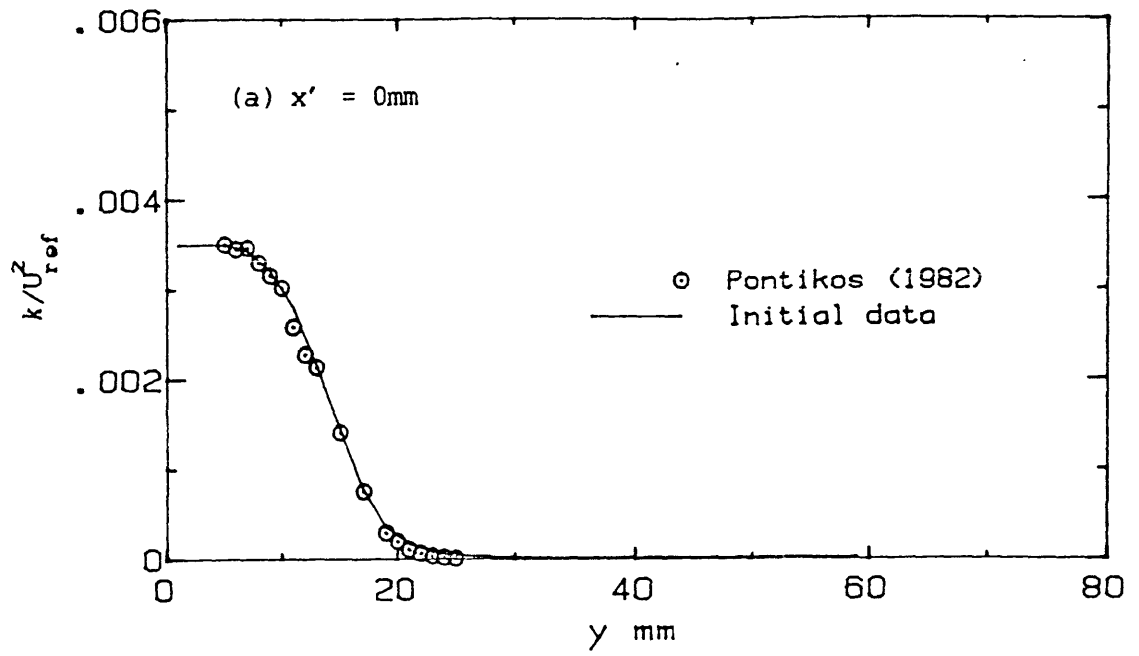


Fig.7.44 Development Of $\overline{w'w'}$ Profile At $x' = 0\text{mm}, 160\text{mm}, 320\text{mm}$ & 480mm . Experimental Data By Pontikos (1982). Predictions By Model 1, Model 2 & Model 3.



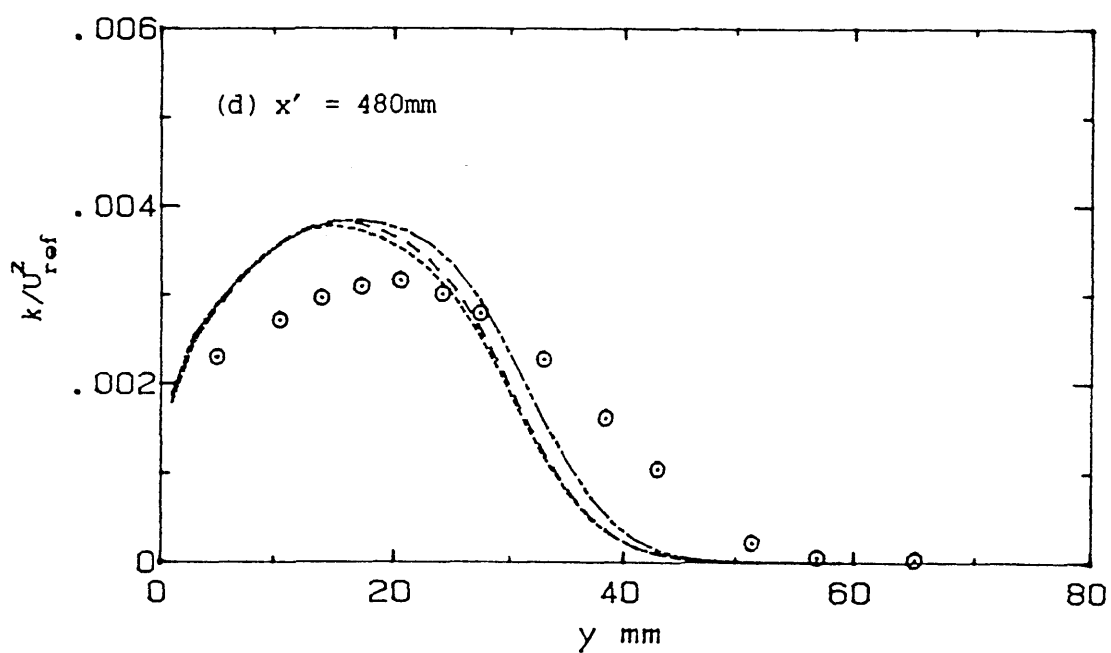
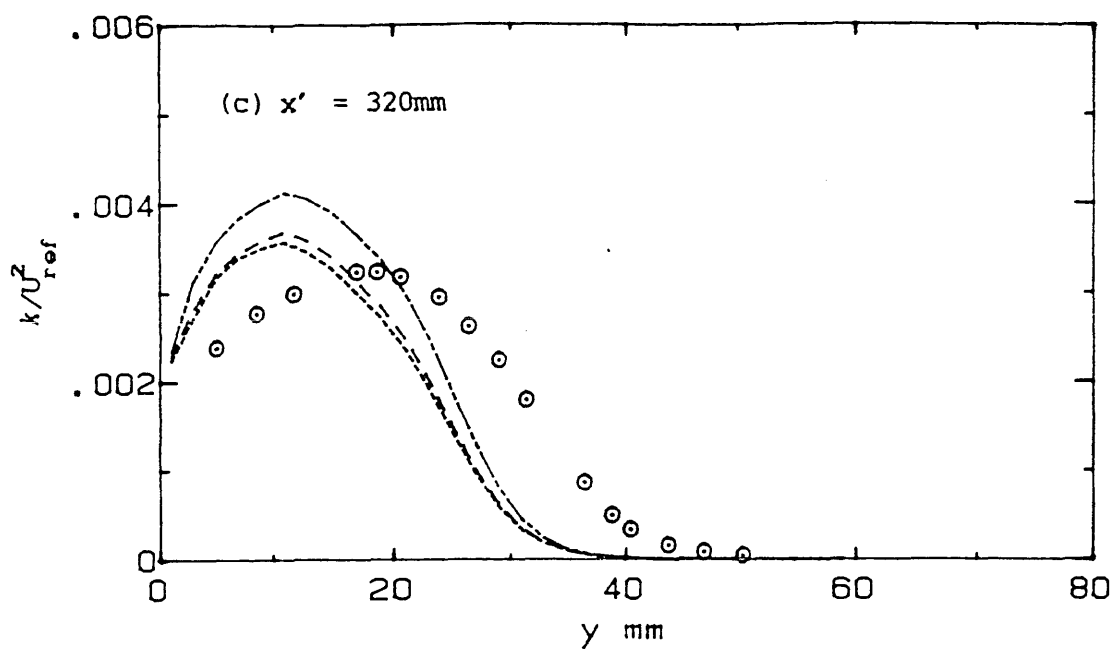
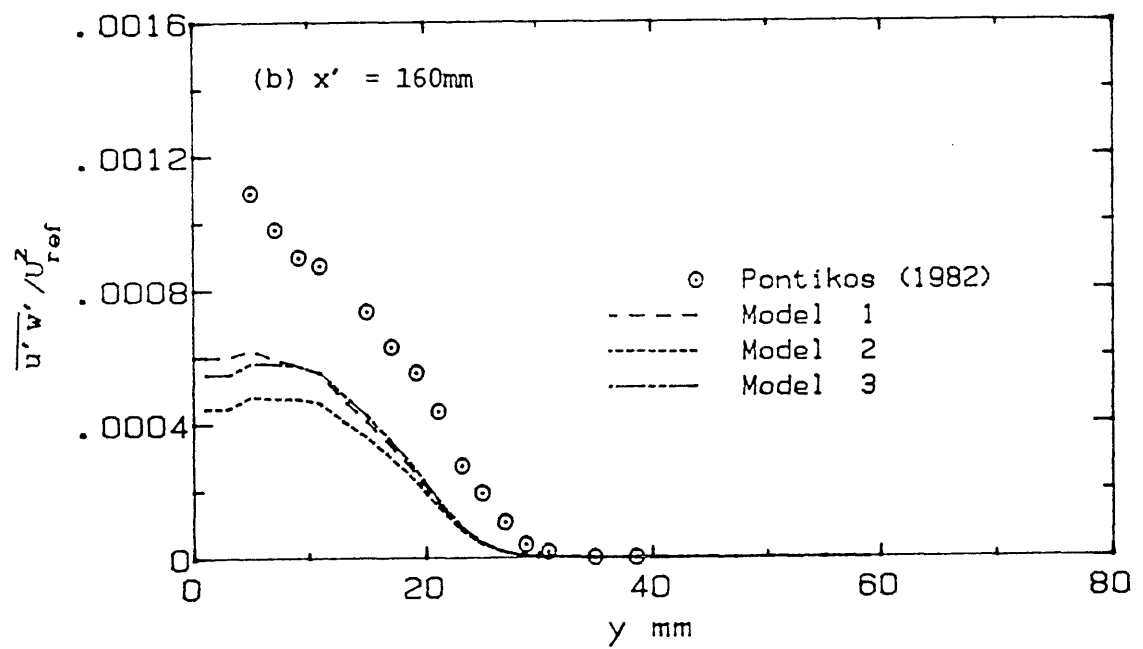
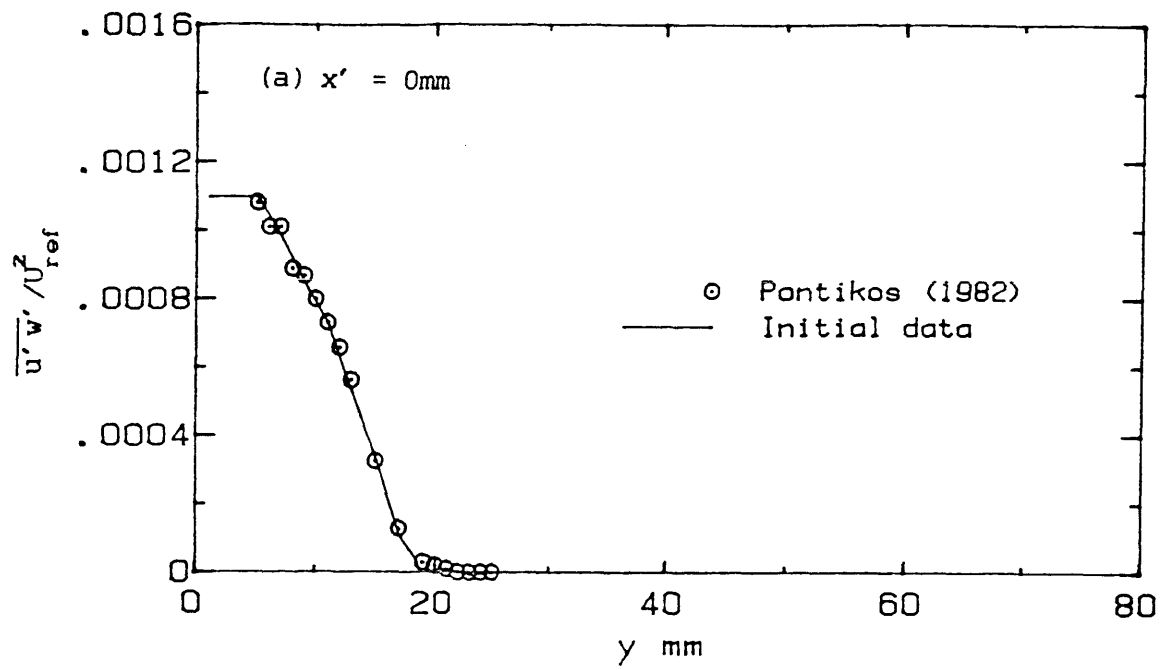


Fig.7.45 Development Of k Profile At $x' = 0\text{mm}$, 160mm , 320mm & 480mm . Experimental Data By Pontikos (1982). Predictions By Model 1, Model 2 & Model 3.



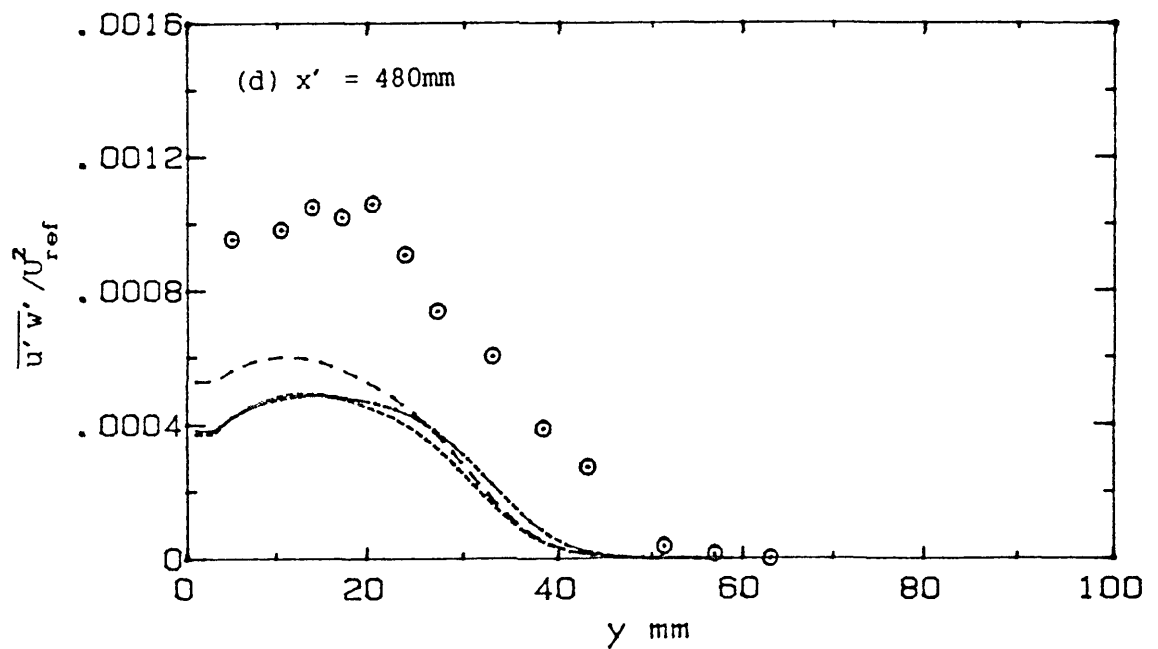
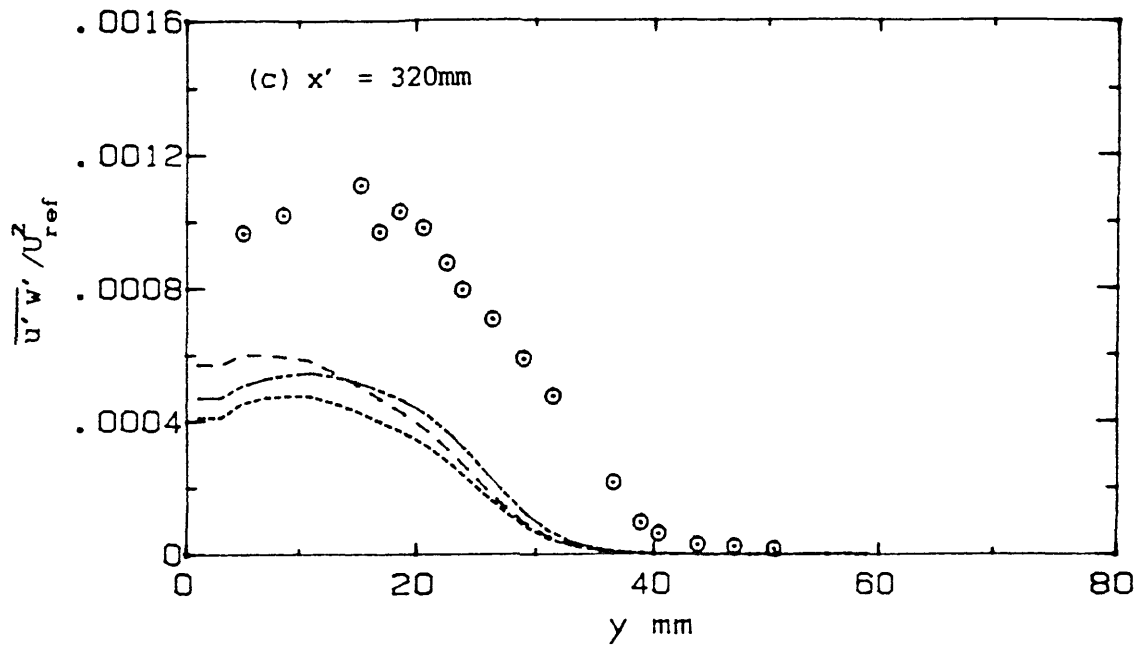


Fig.7.46 Development Of $\overline{u'w'}$ Profile At $x' = 0\text{mm}, 160\text{mm}, 320\text{mm}$ & 480mm . Experimental Data By Pontikos (1982). Predictions By Model 1, Model 2 & Model 3.

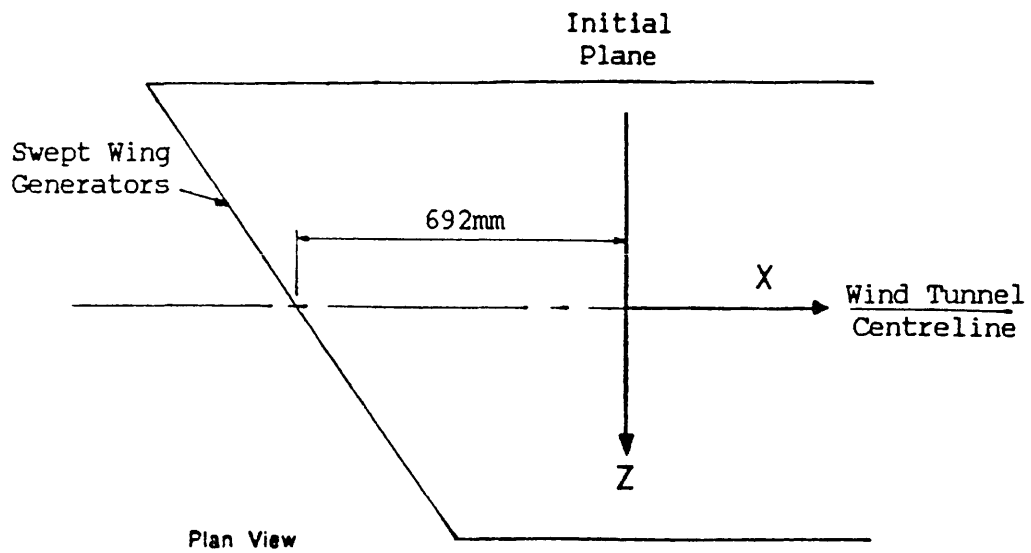


Fig.7.47 "Infinite" Swept Wing In x-y-z Axes.

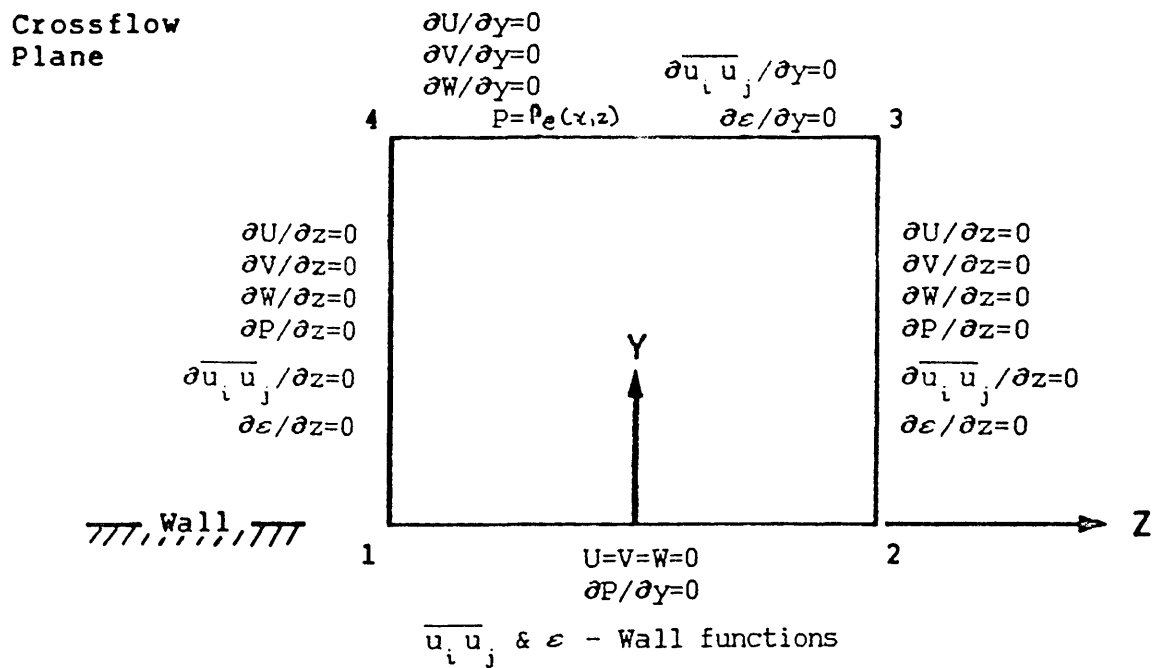
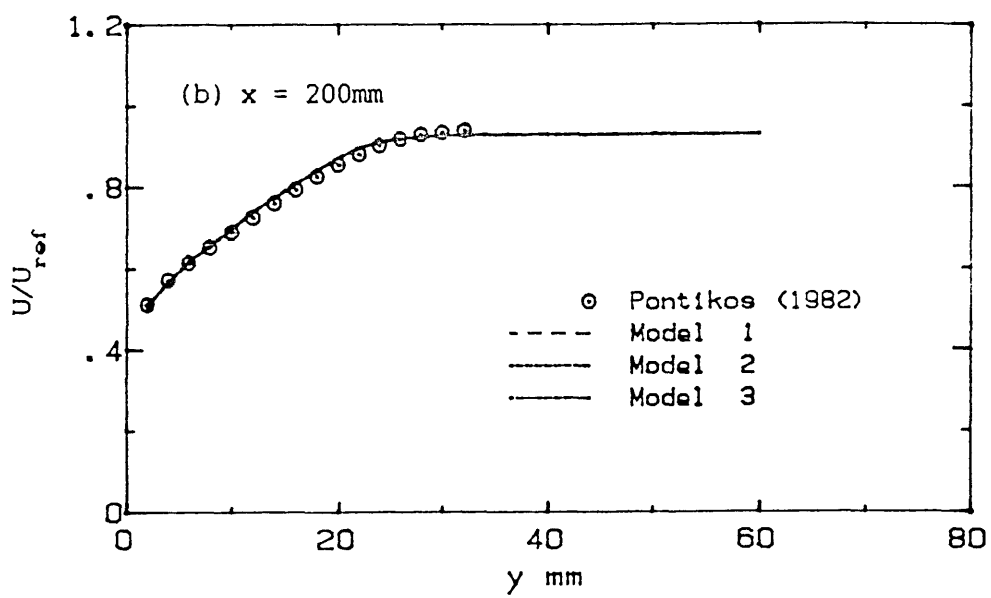
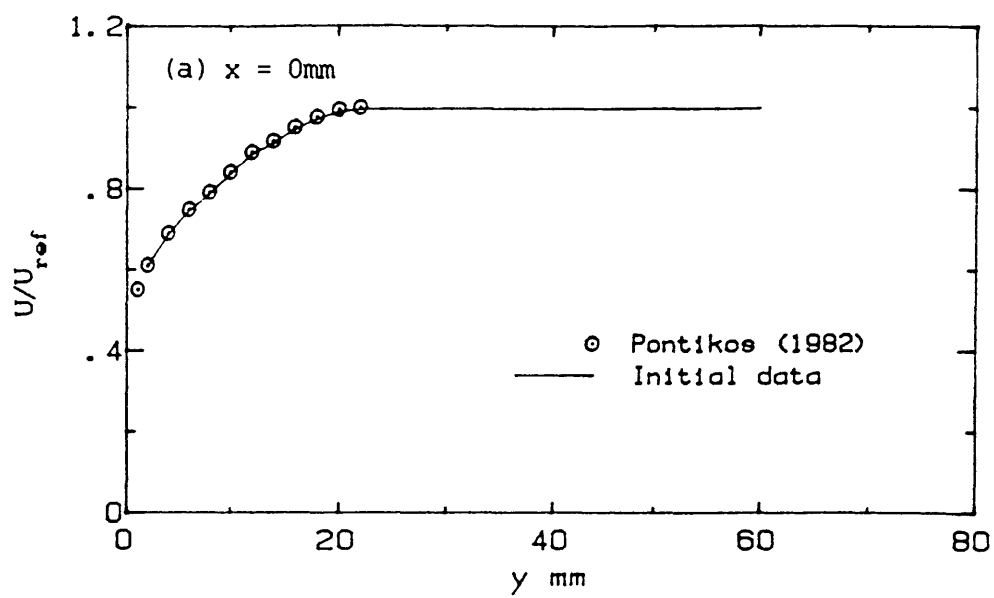


Fig.7.48 Boundary Conditions Of "Infinite" Swept Wing In x-y-z Axes.



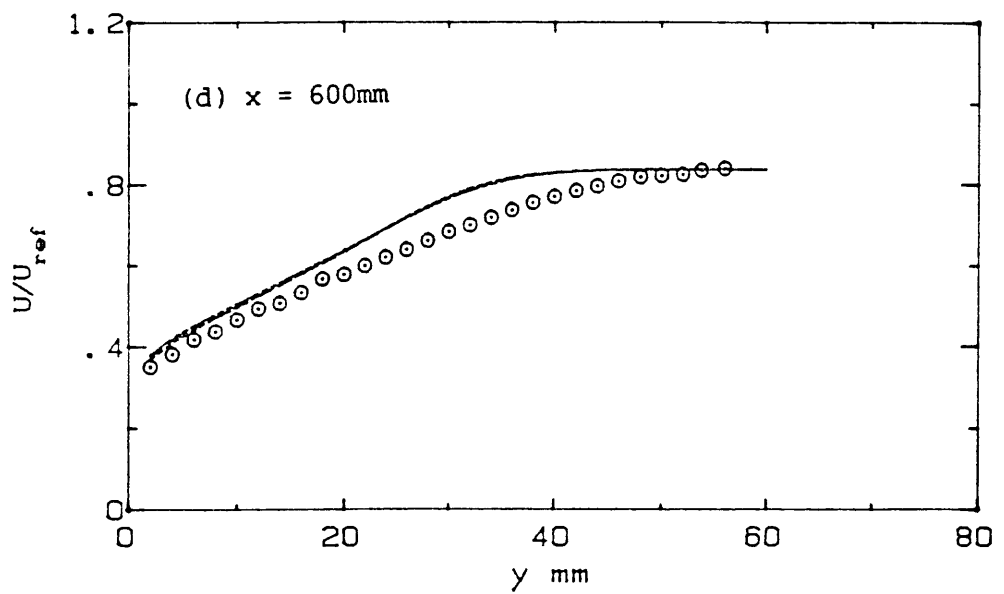
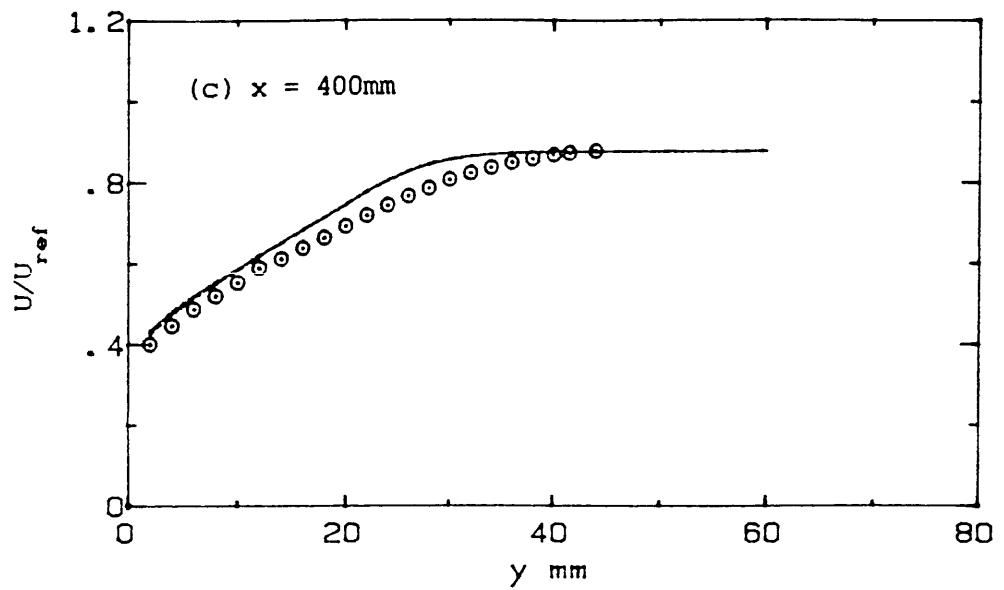
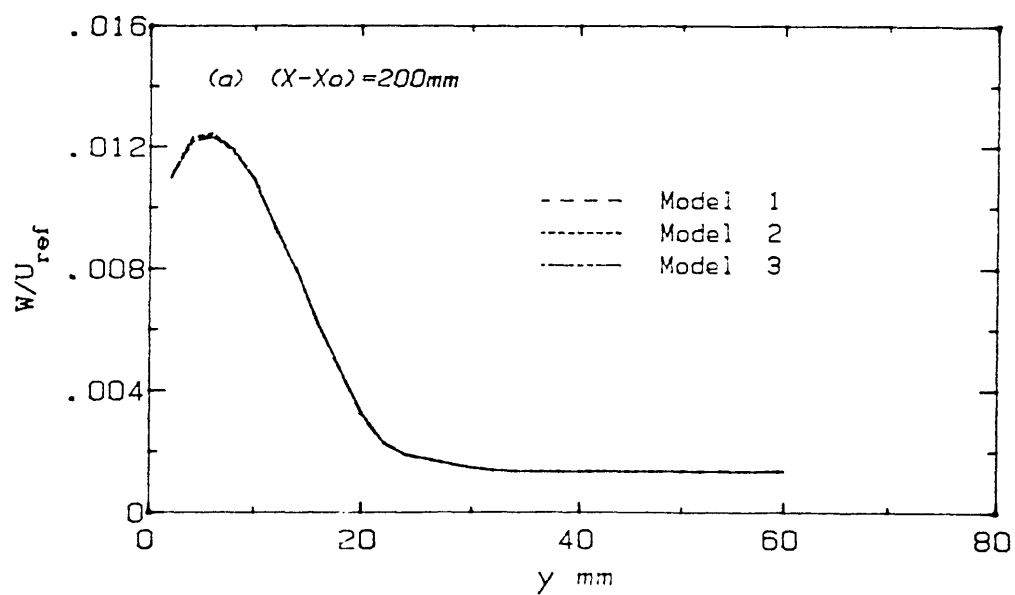
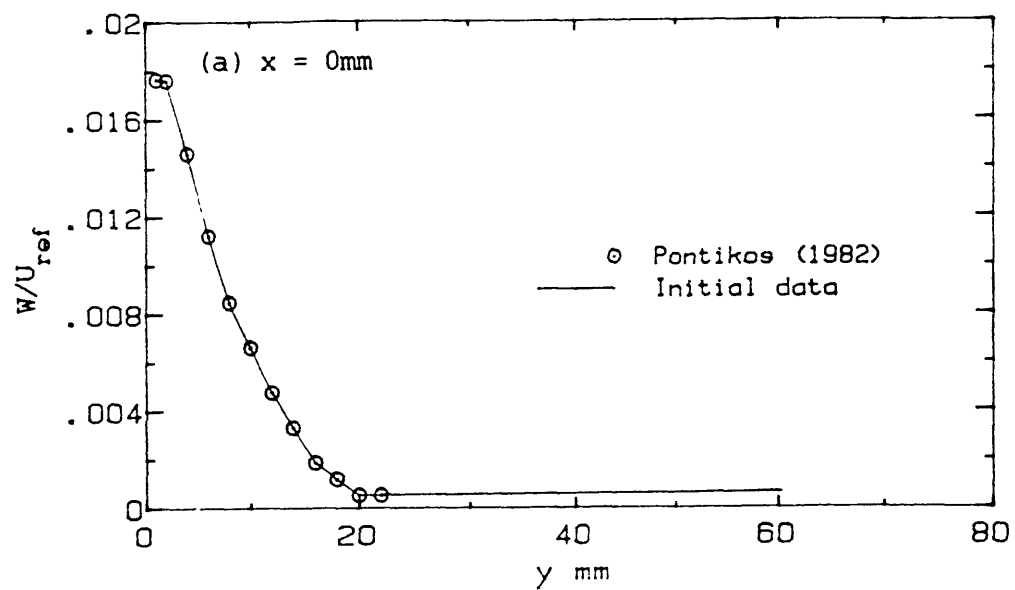


Fig.7.49 Development Of U Profile At $x = 0\text{mm}$, 200mm , 400mm & 600mm . Experimental Data By Pontikos (1982). Predictions By Model 1, Model 2 & Model 3.



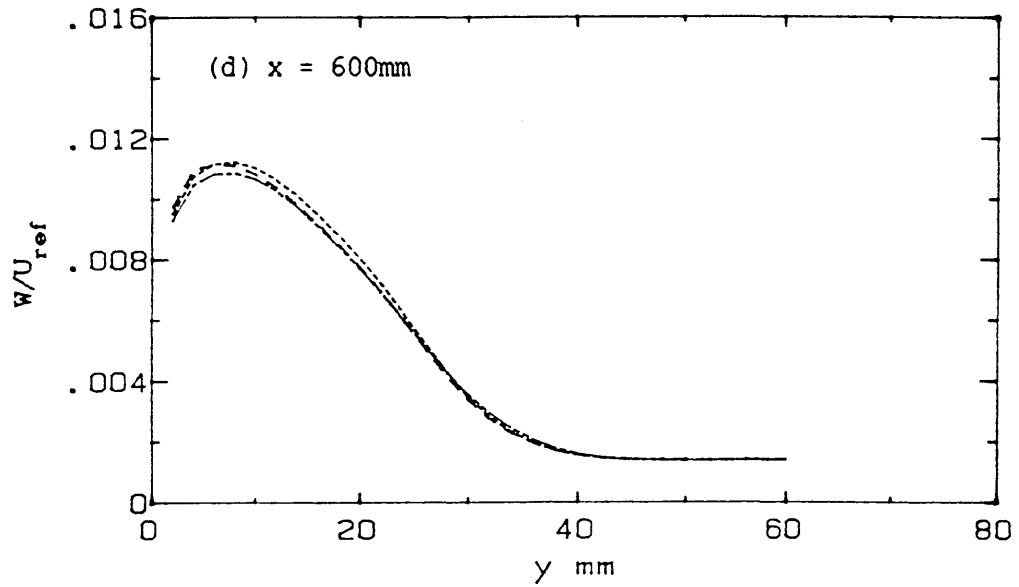
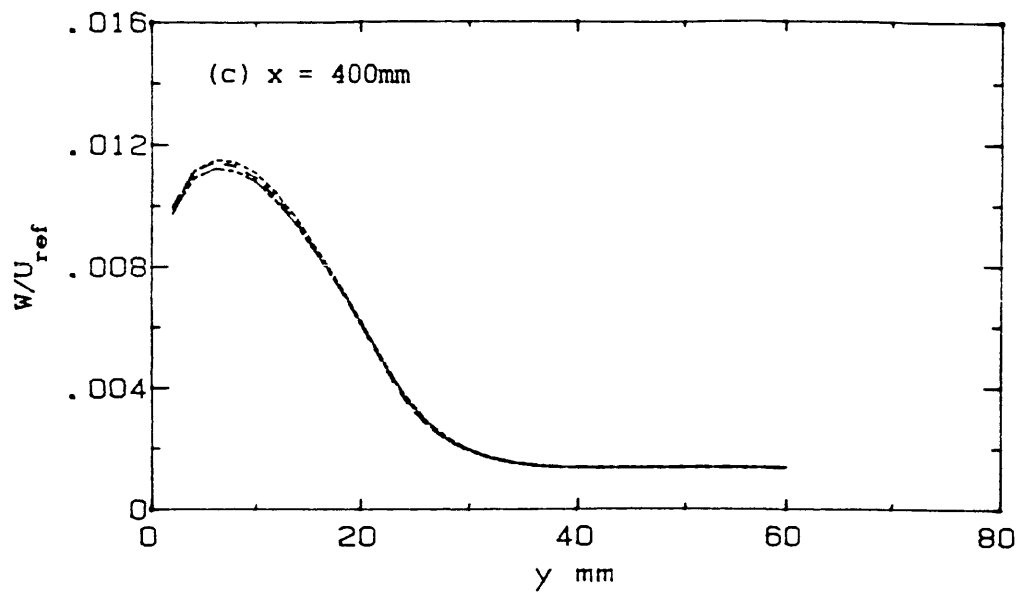


Fig.7.50 Development Of W Profile At $x = 0\text{mm}$, 200mm , 400mm & 600mm . Experimental Data By Pontikos (1982). Predictions By Model 1, Model 2 & Model 3.

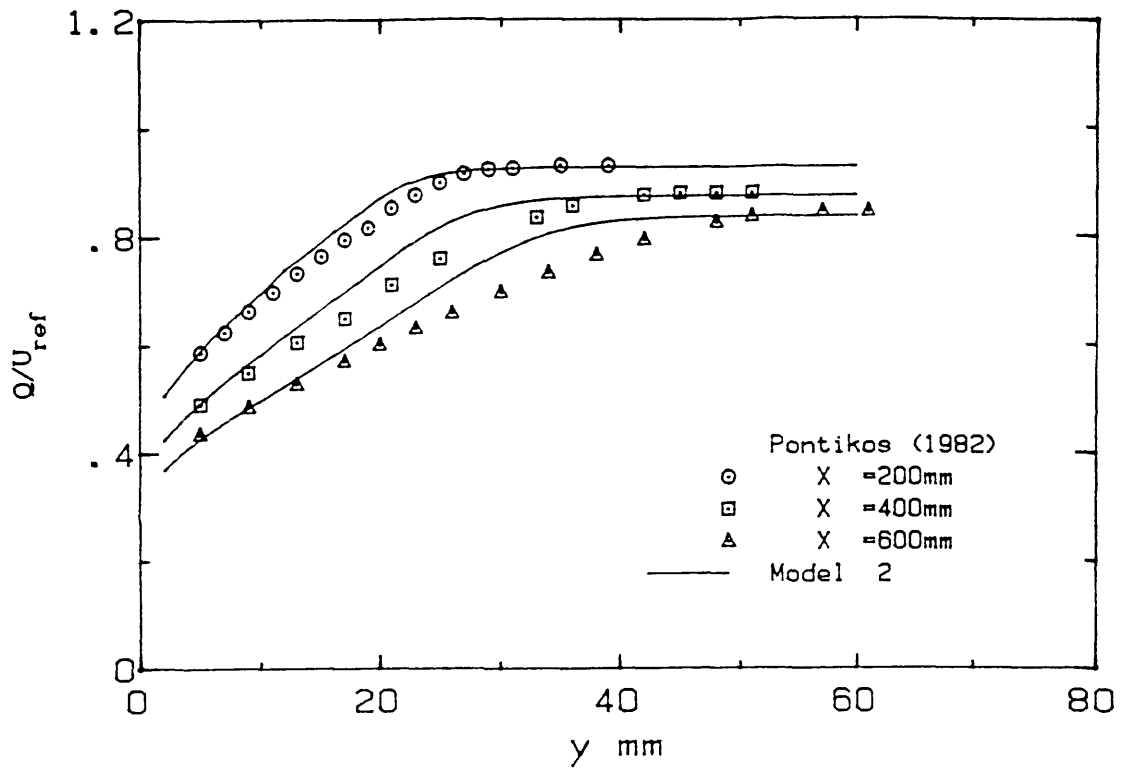


Fig.7.51 Total Velocity Profiles. Experimental Data By Pontikos (1982) & Predictions By Model 2.

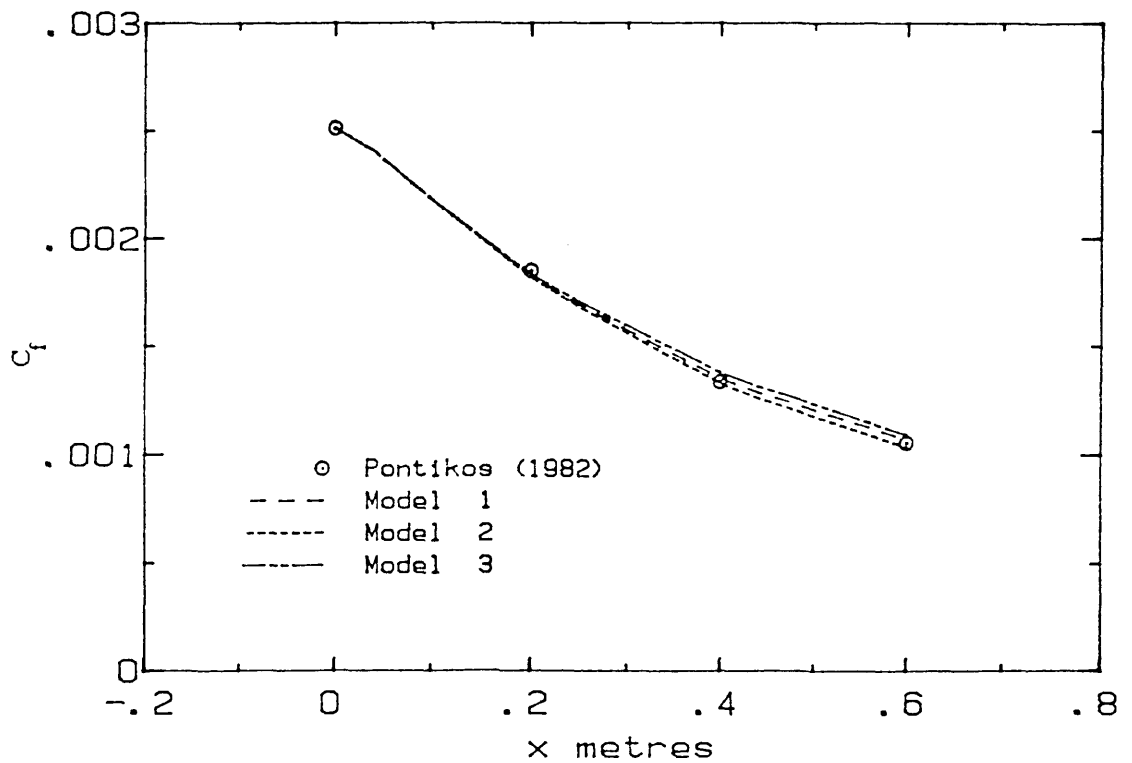
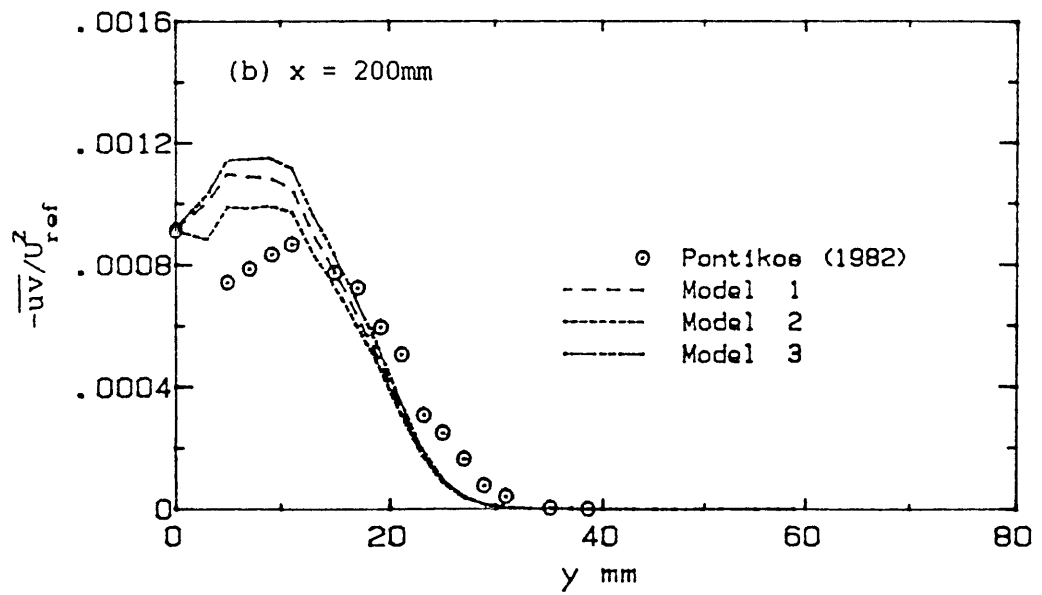
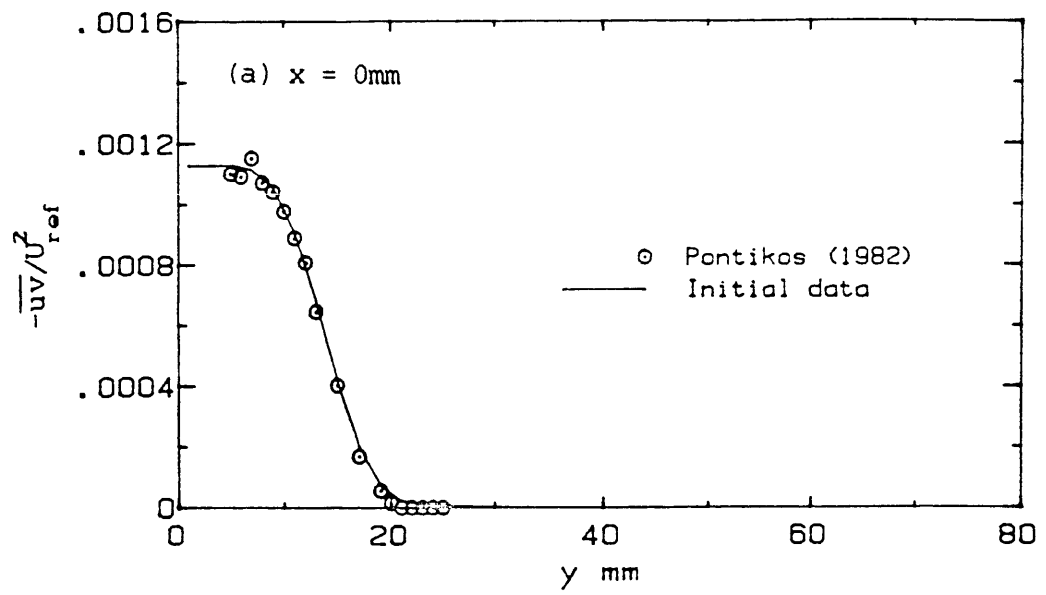


Fig.7.52 Skin Friction Coefficient. Experimental Data By Pontikos (1982). Predictions By Model 1, Model 2 & Model 3.



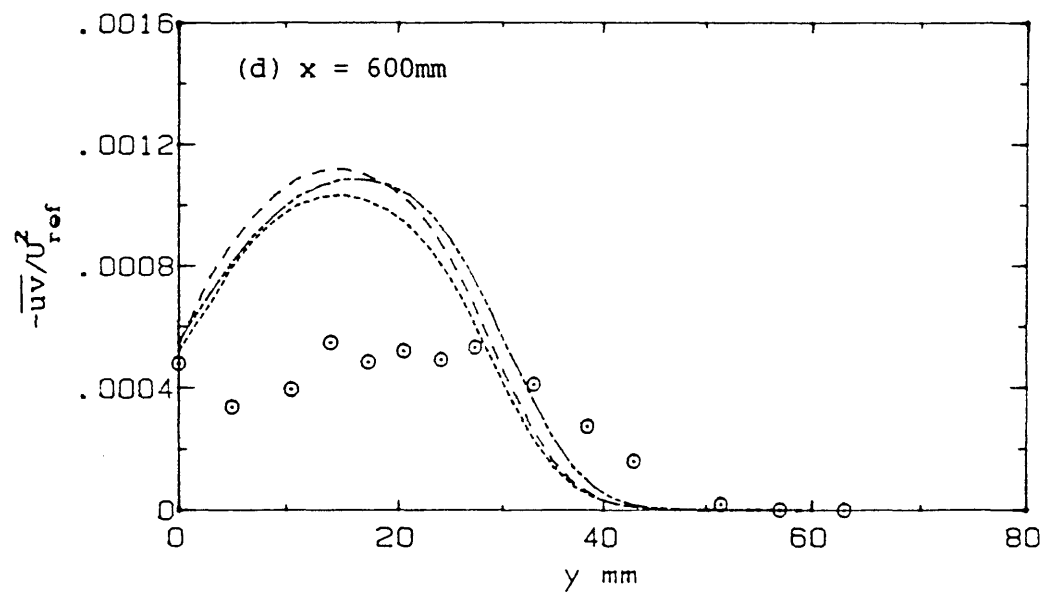
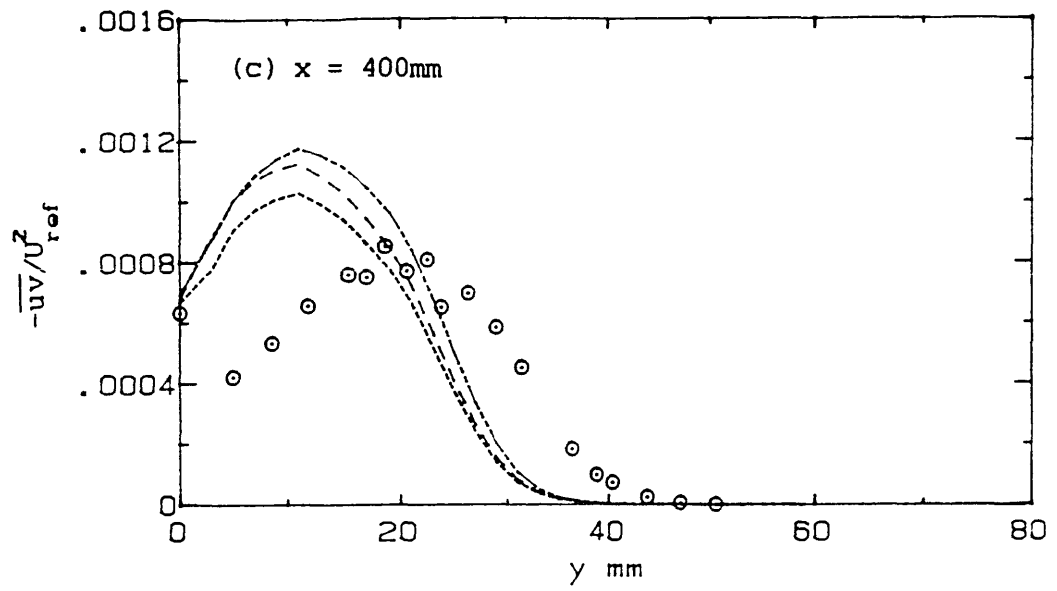
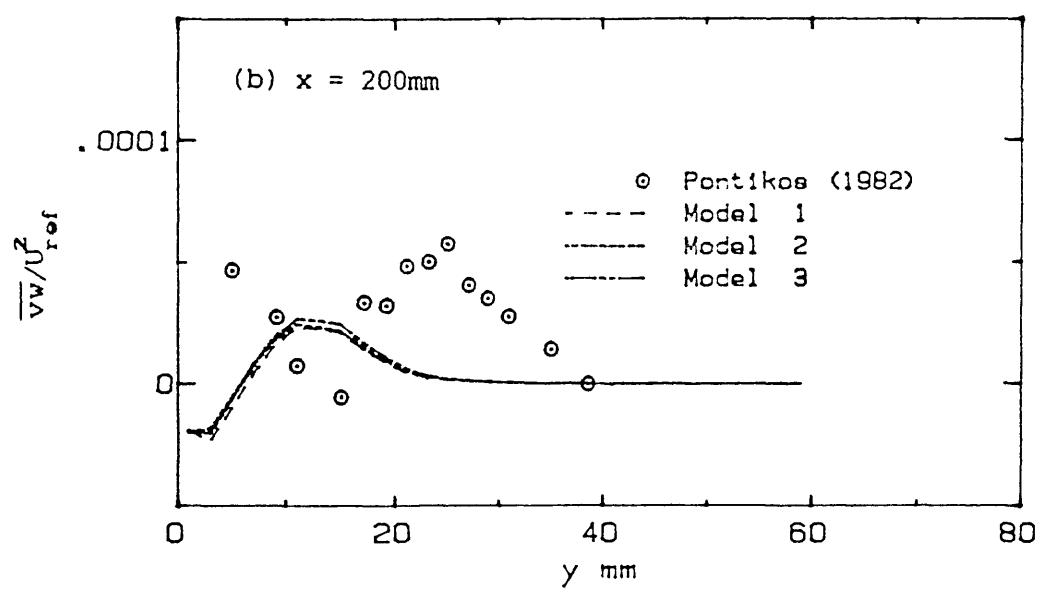
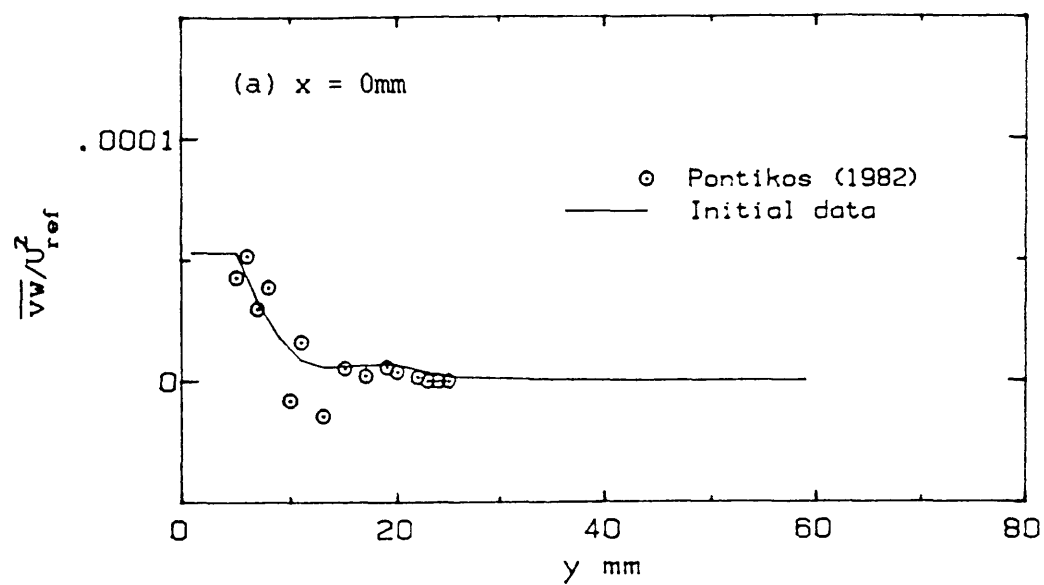


Fig.7.53 Development Of $\overline{uv}$ Profile At $x = 0\text{mm}$, 200mm , 400mm & 600mm . Experimental Data By Pontikos (1982). Predictions By Model 1, Model 2 & Model 3.



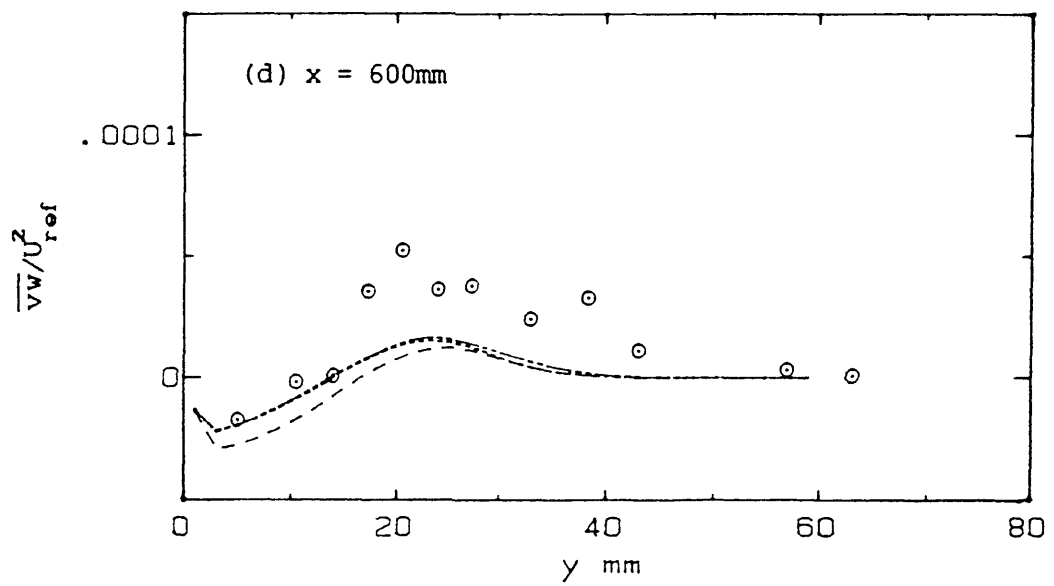
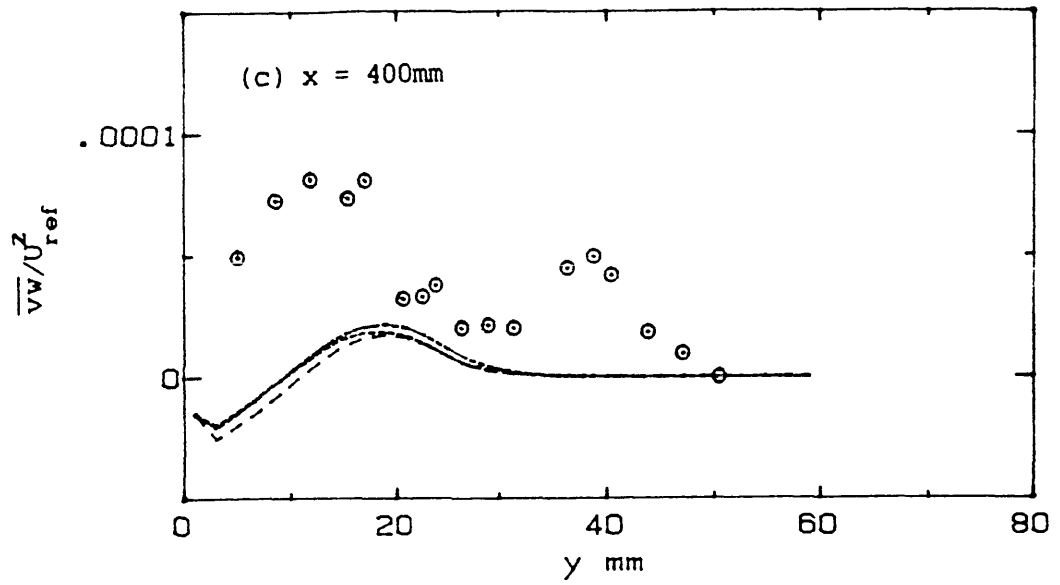
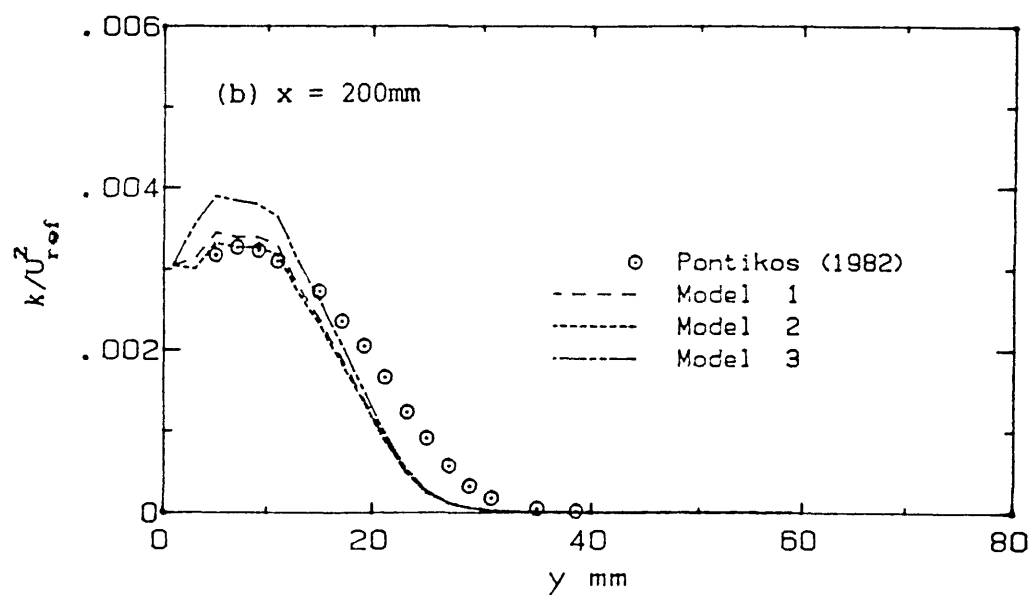
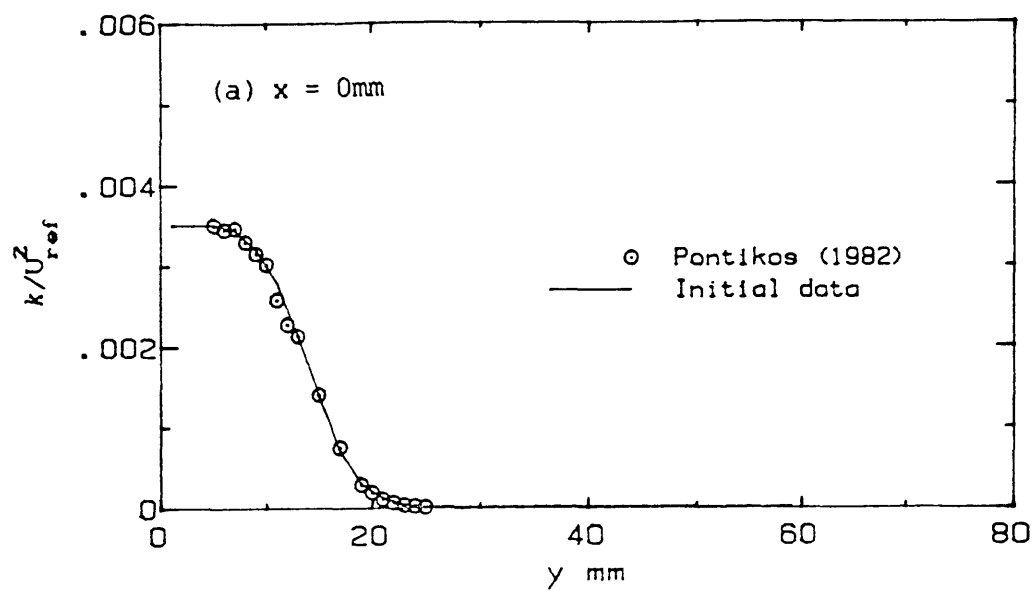


Fig.7.54 Development Of $\overline{vw}$ Profile At $x = 0\text{mm}$, 200mm , 400mm & 600mm . Experimental Data By Pontikos (1982). Predictions By Model 1, Model 2 & Model 3.



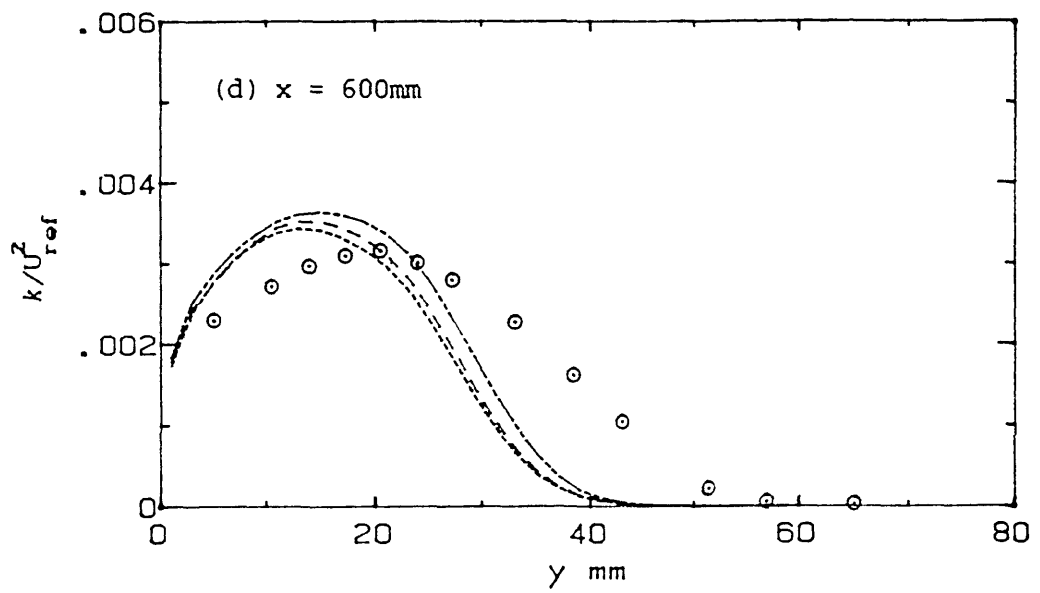
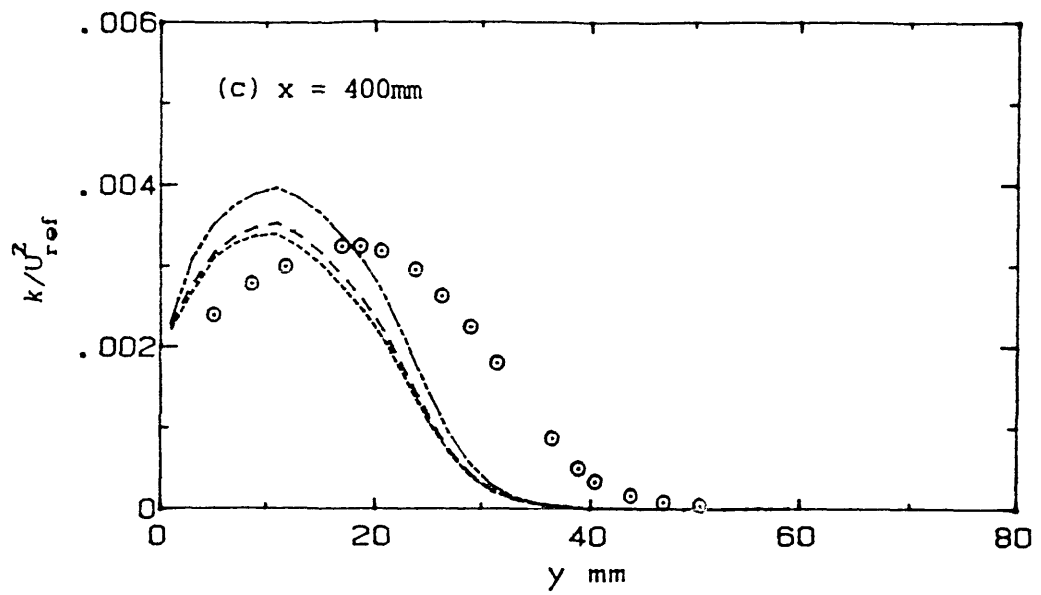
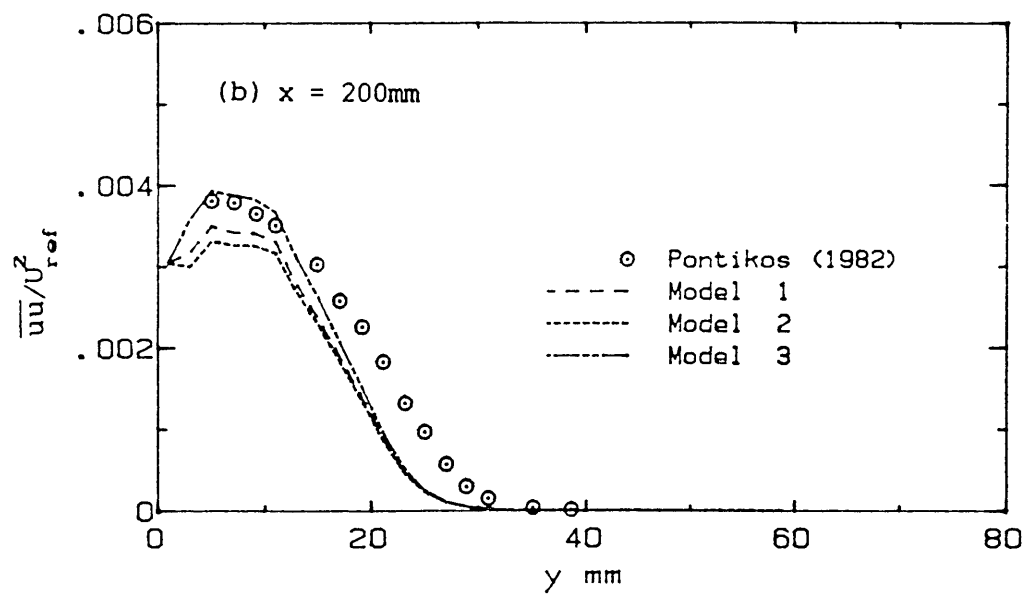
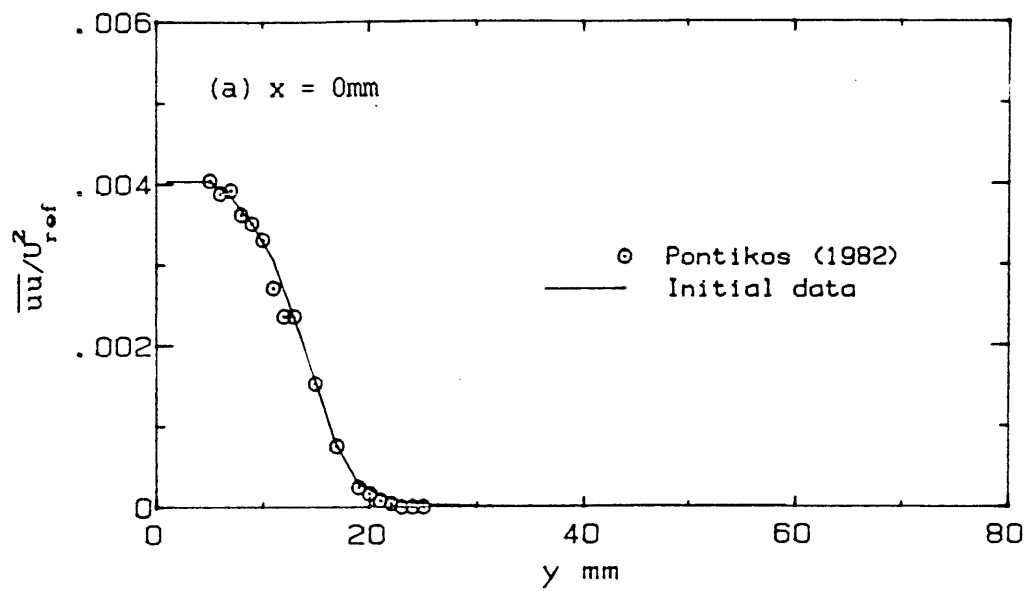


Fig.7.55 Development Of k Profile At $x = 0\text{mm}$, 200mm , 400mm & 600mm . Experimental Data By Pontikos (1982). Predictions By Model 1, Model 2 & Model 3.



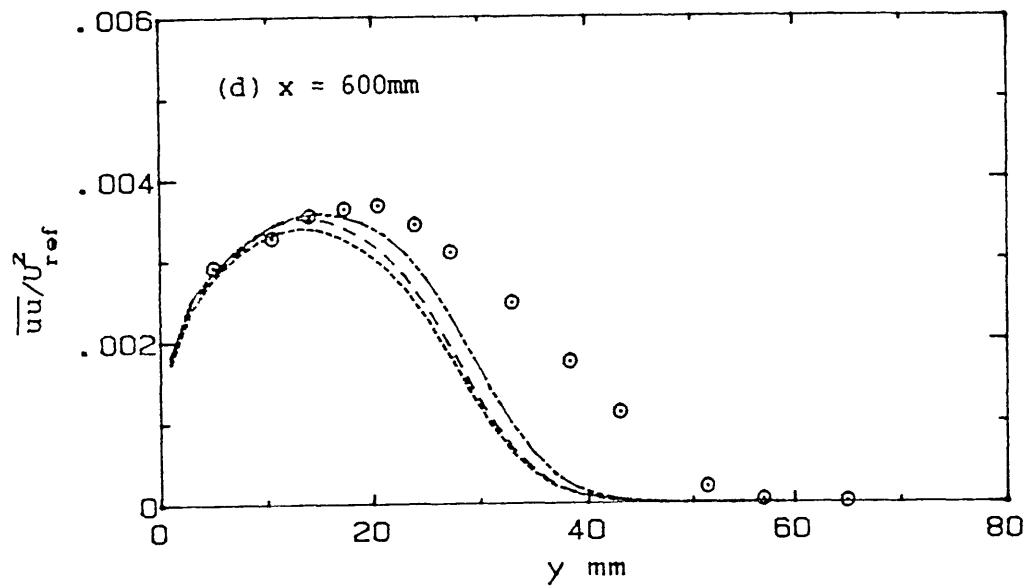
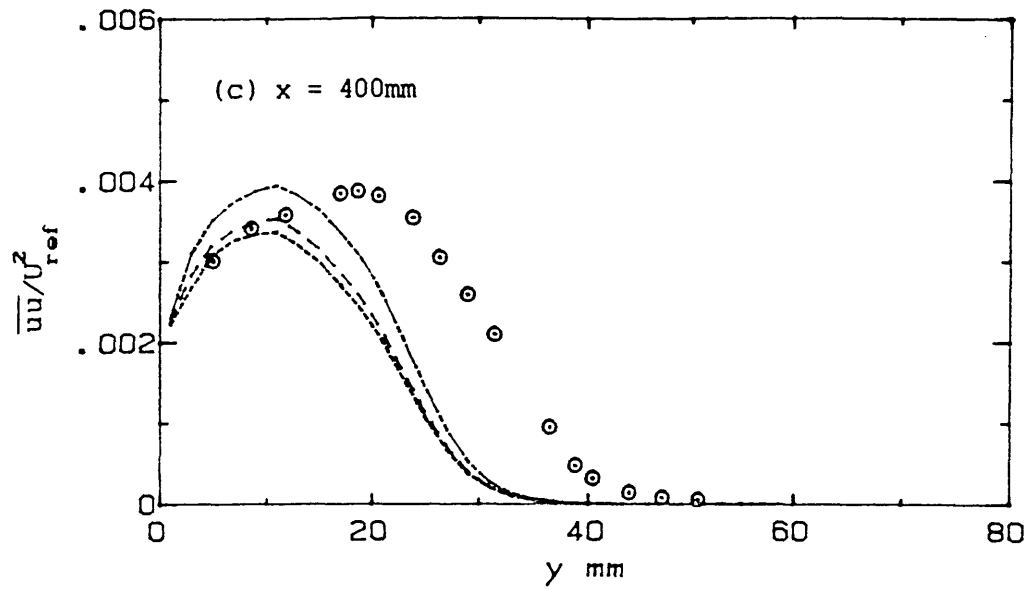
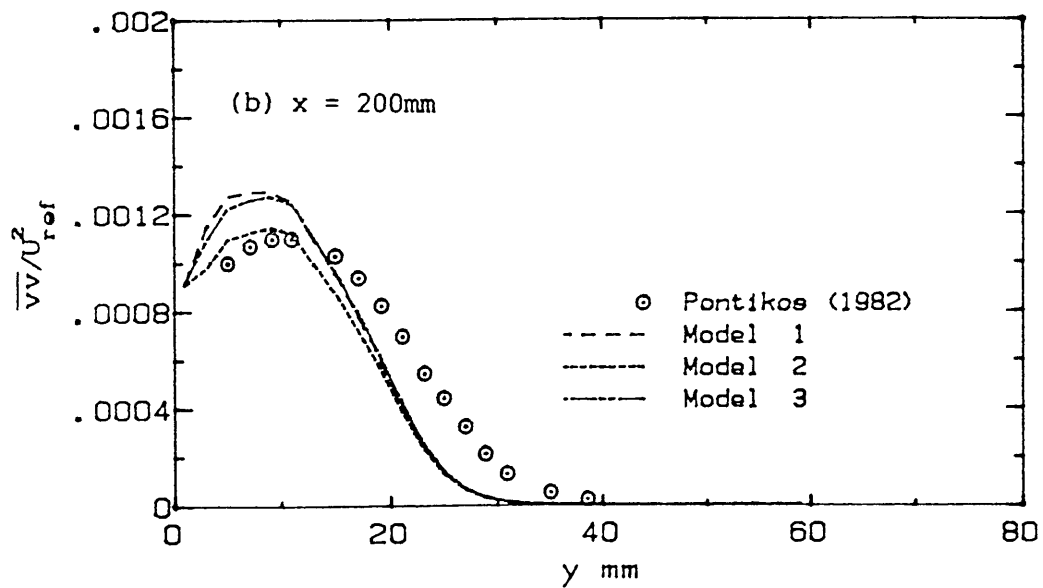
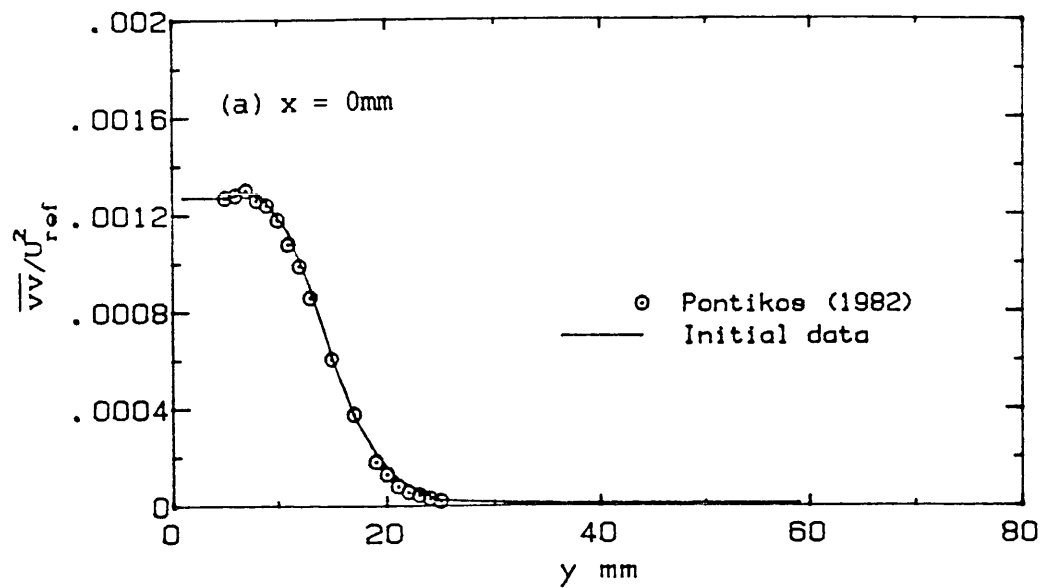


Fig.7.56 Development Of $\overline{uu}$ Profile At $x = 0\text{mm}, 200\text{mm}, 400\text{mm}$ & 600mm . Experimental Data By Pontikos (1982). Predictions By Model 1, Model 2 & Model 3.



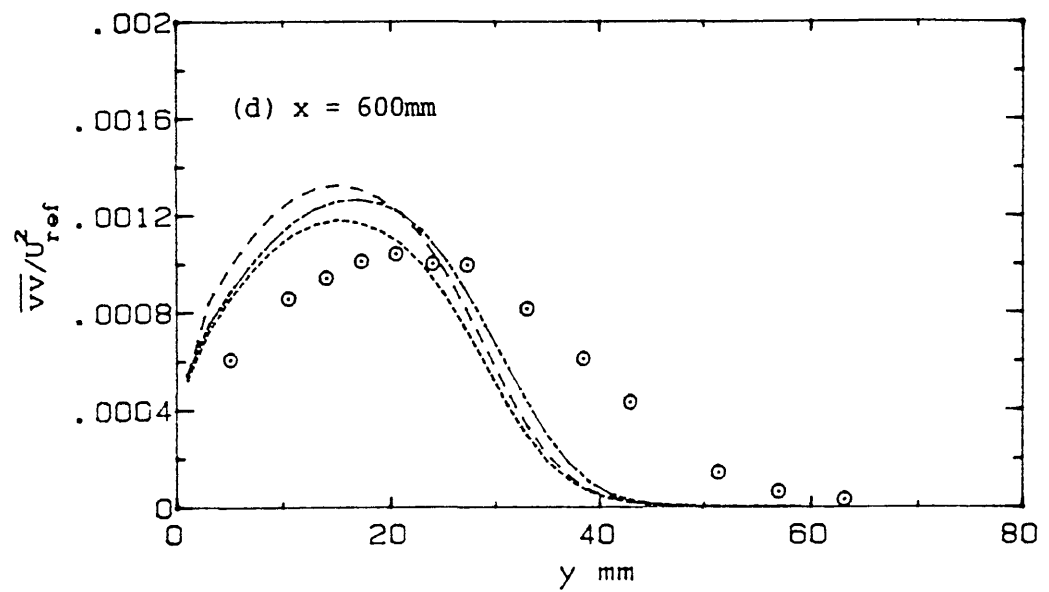
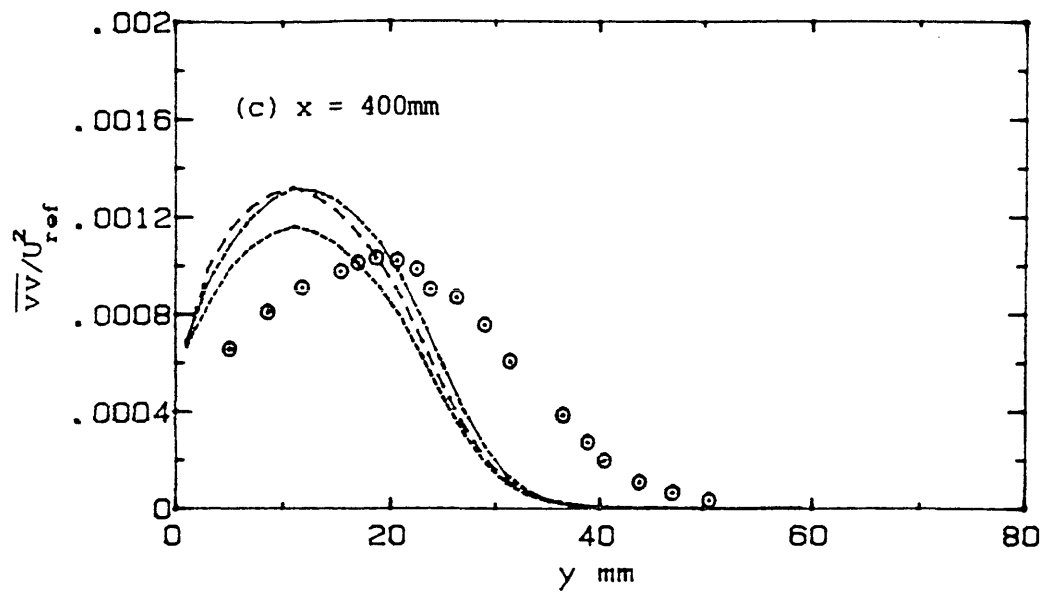
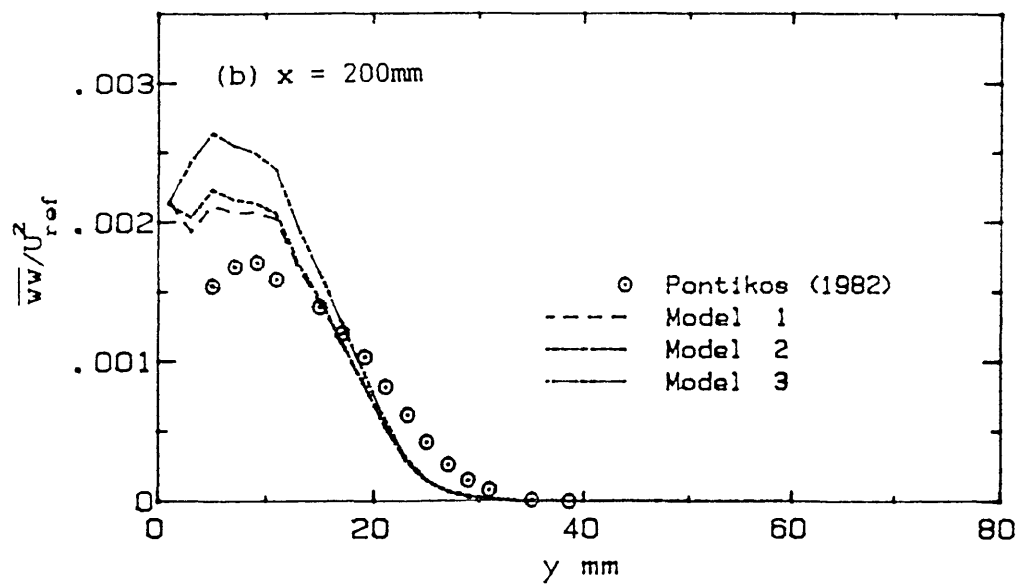
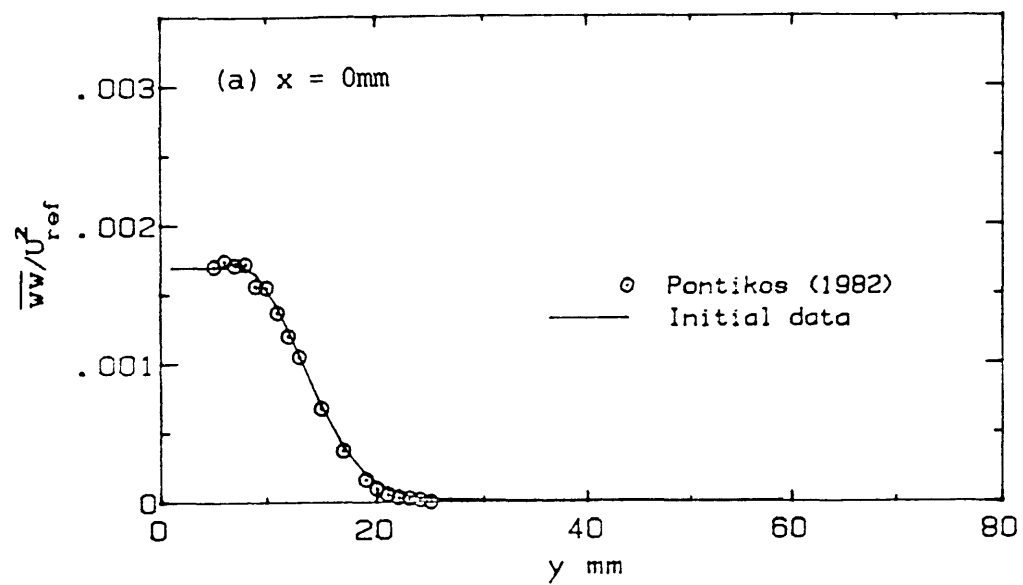


Fig.7.57 Development Of $\overline{vv}$ Profile At $x = 0\text{mm}, 200\text{mm}, 400\text{mm}$ & 600mm . Experimental Data By Pontikos (1982). Predictions By Model 1, Model 2 & Model 3.



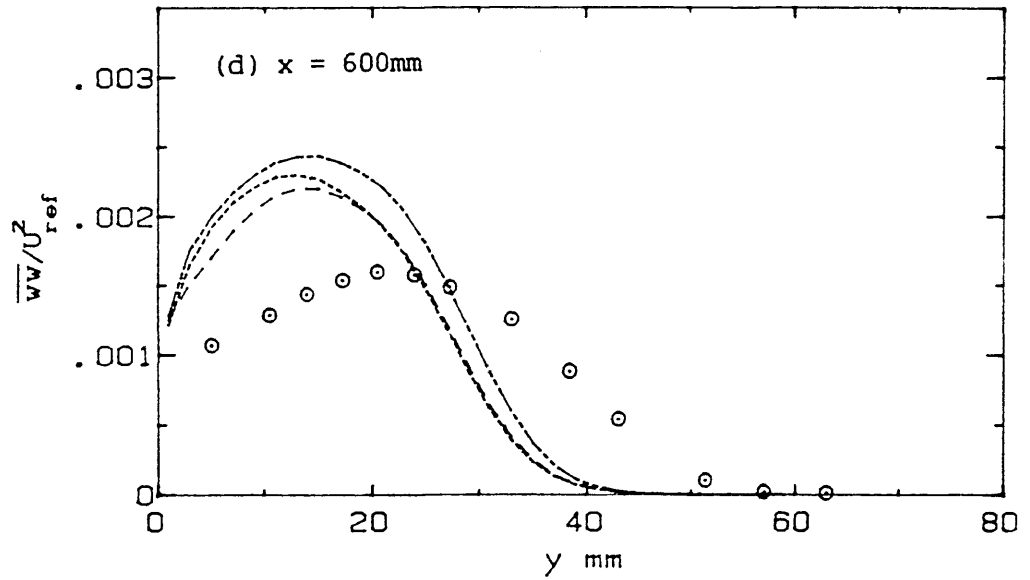
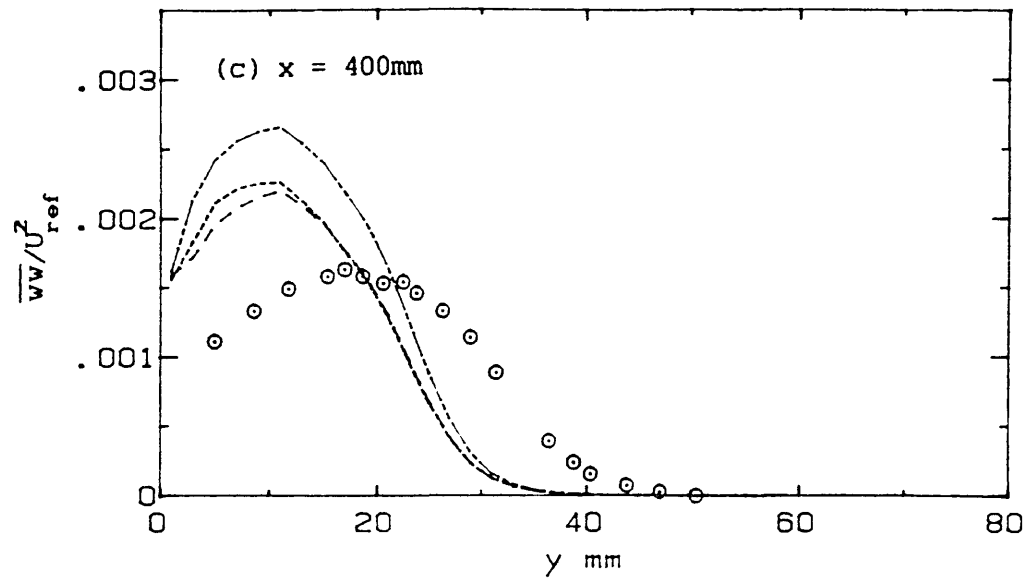
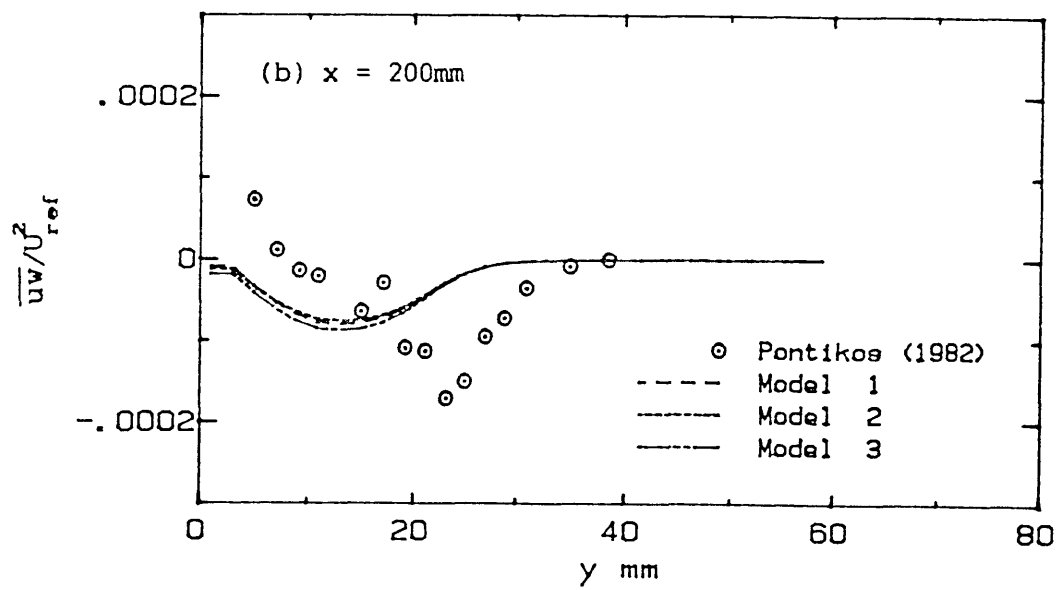
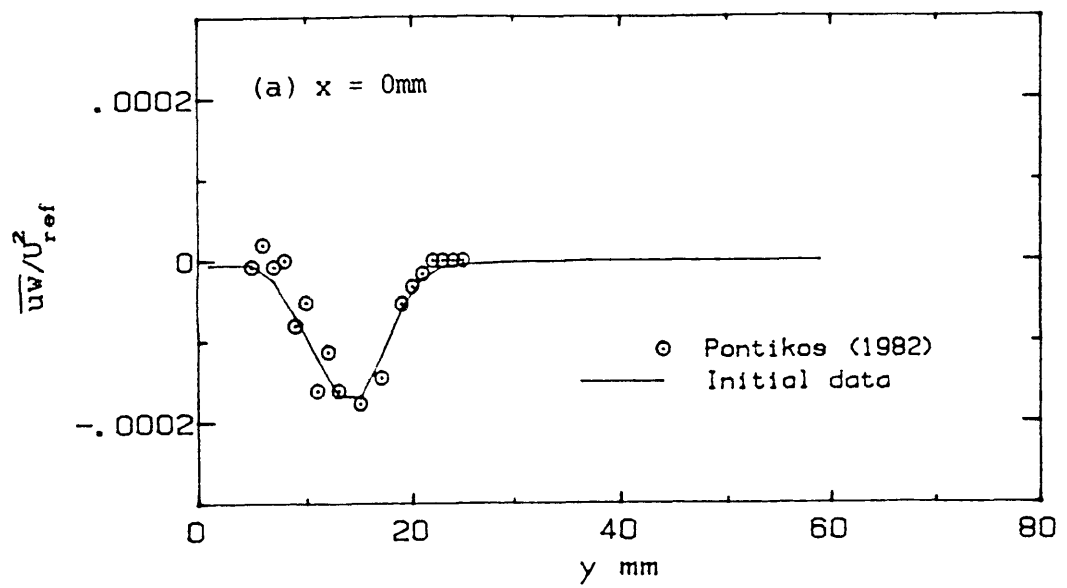


Fig.7.58 Development Of $\overline{ww}$ Profile At $x = 0\text{mm}$, 200mm , 400mm & 600mm . Experimental Data By Pontikos (1982). Predictions By Model 1, Model 2 & Model 3.



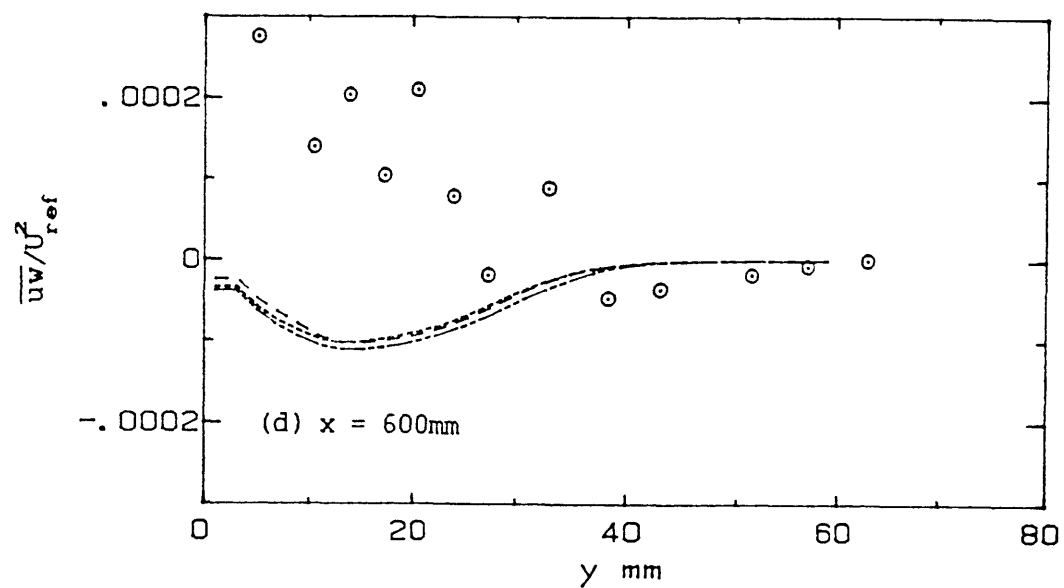
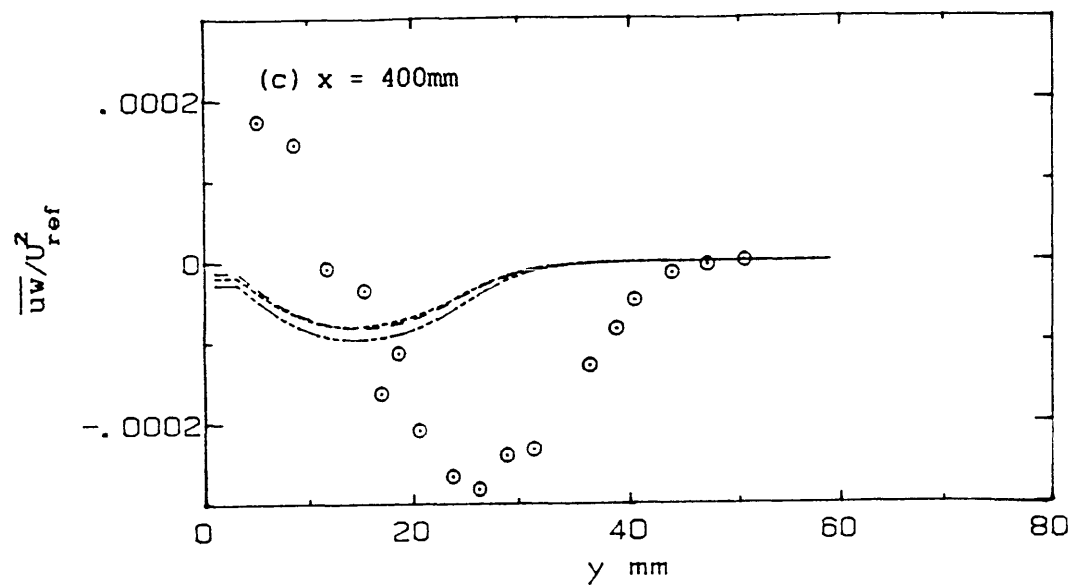


Fig.7.59 Development Of $\overline{uw}$ Profile At $x = 0\text{mm}, 200\text{mm}, 400\text{mm}$ & 600mm . Experimental Data By Pontikos (1982). Predictions By Model 1, Model 2 & Model 3.

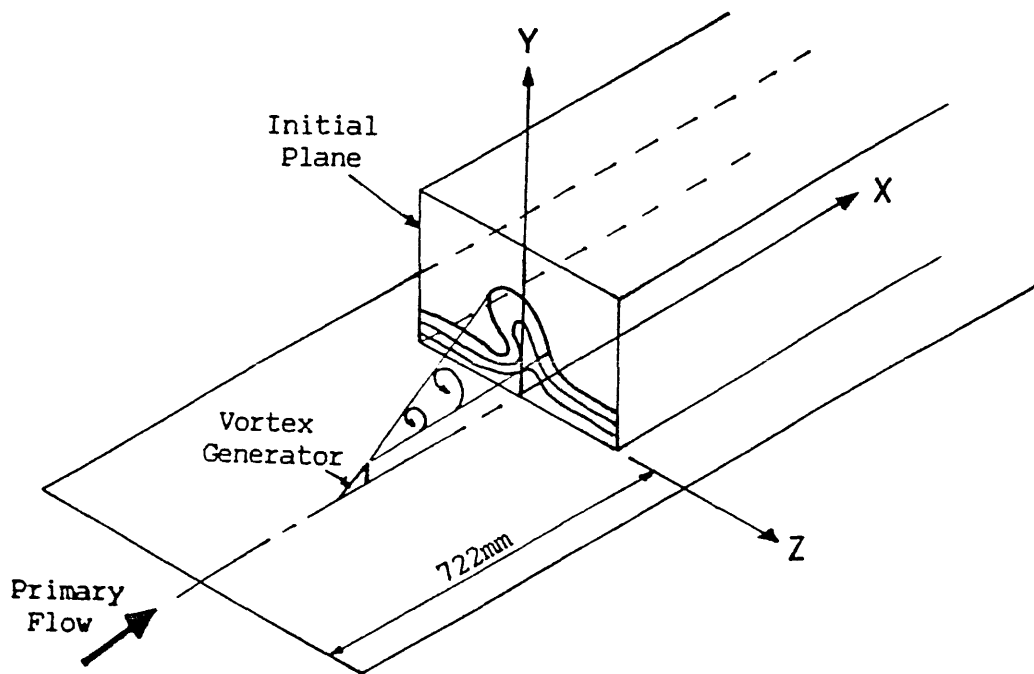


Fig.7.60 Embedded Longitudinal Vortex In Boundary Layer.

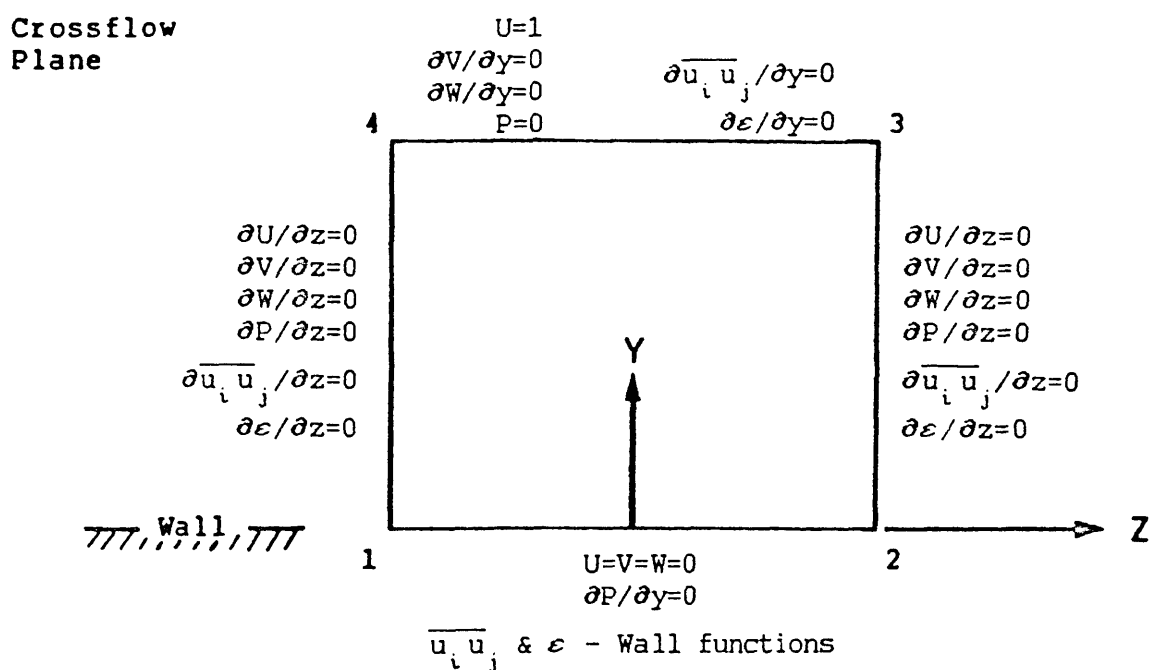
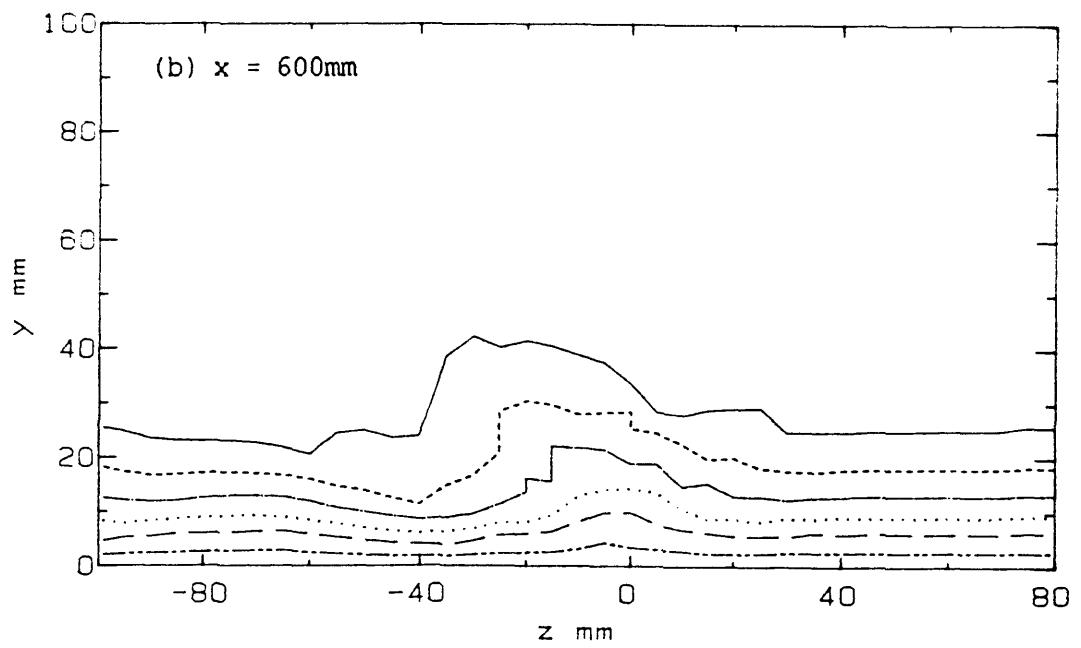
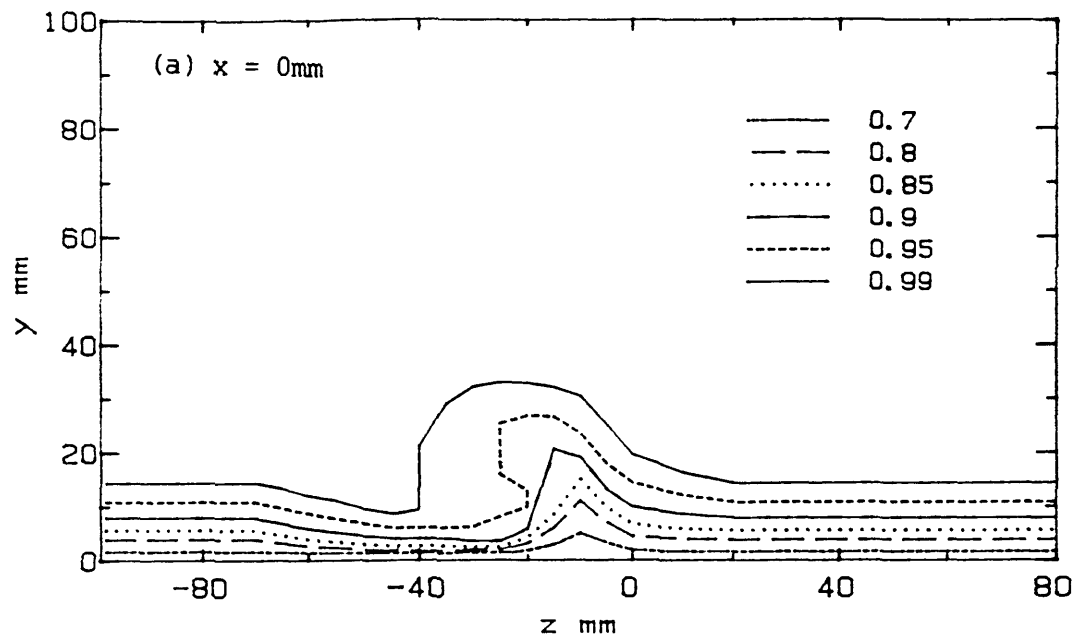


Fig.7.61 Vortex-Boundary Layer Flow Boundary Conditions.



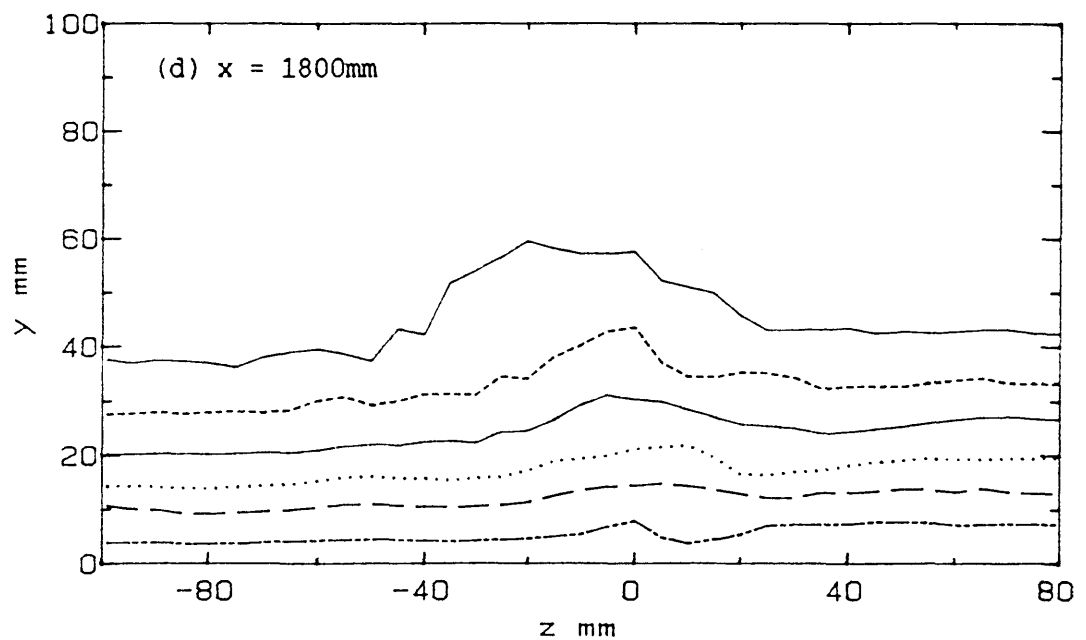
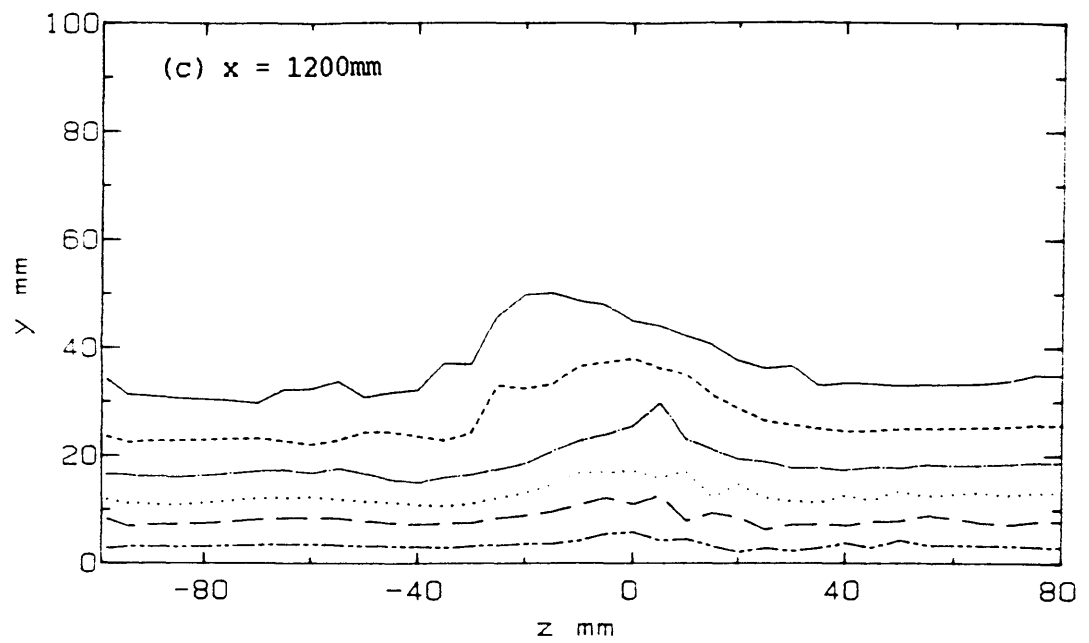
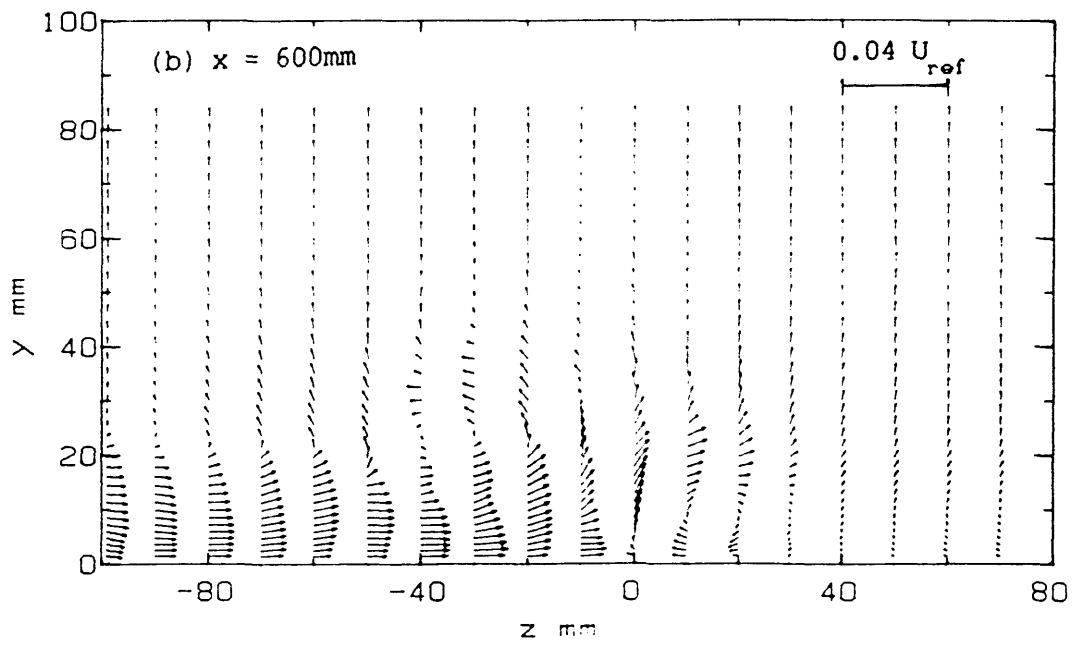
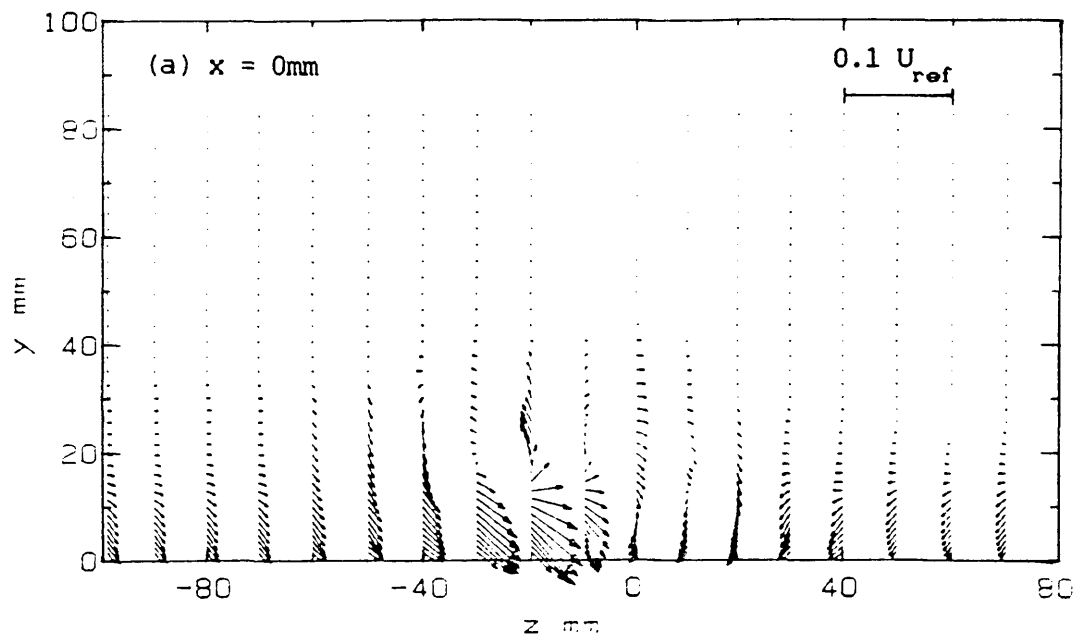


Fig.7.62 U Contours - Experimental Data At $x = 0\text{mm}$ By Shabaka et al. (1985) & Predictions By Model 2 At $x = 600\text{mm}$, 1200mm & 1800mm .



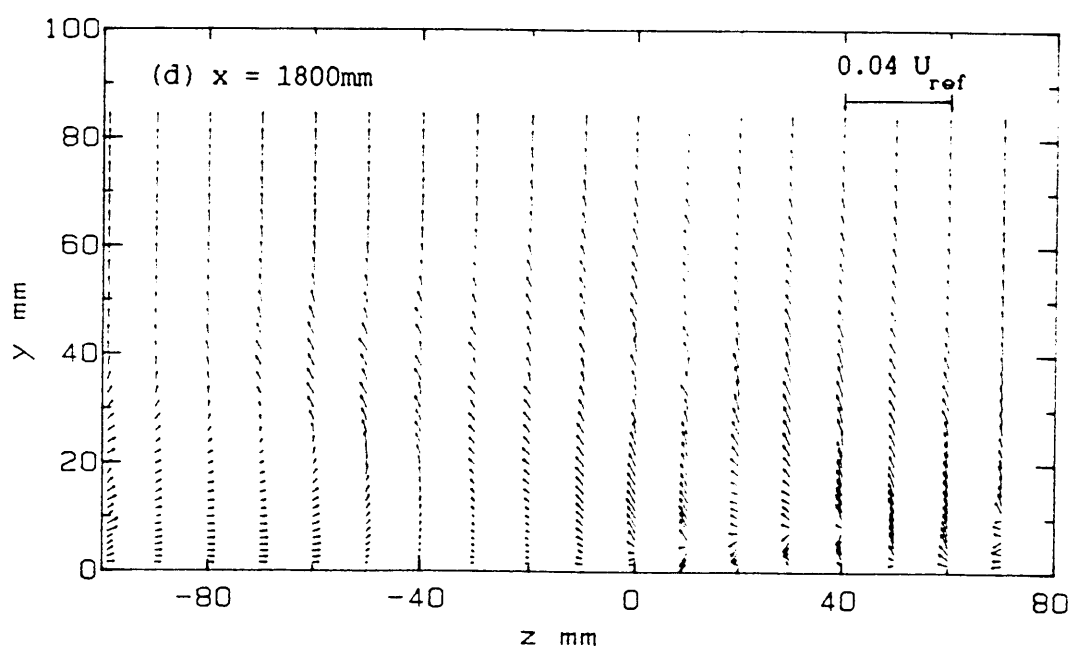
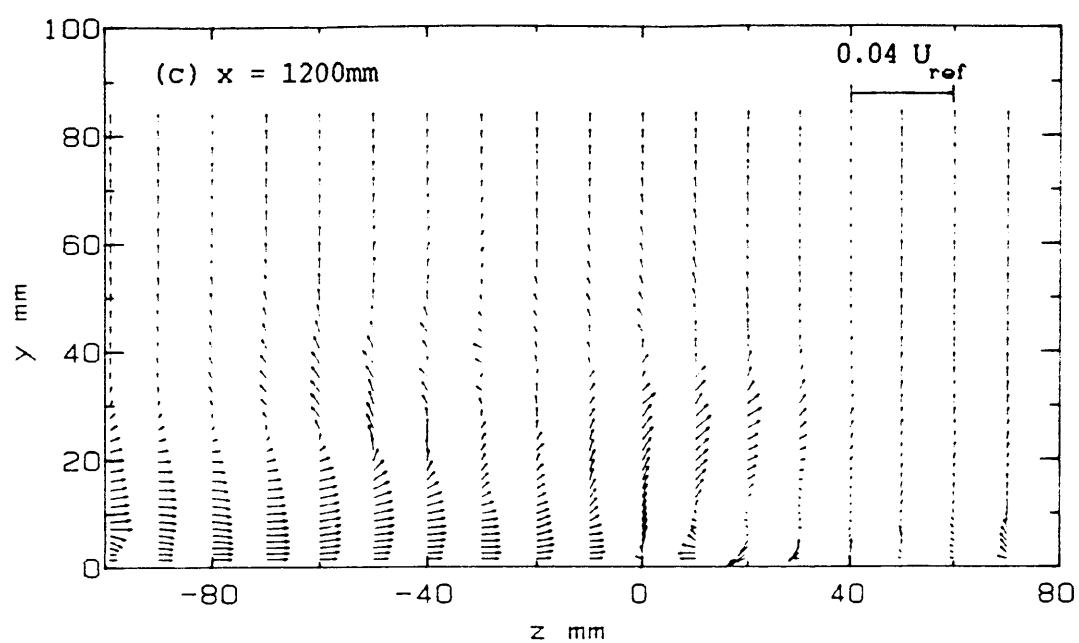


Fig.7.63 Secondary Flow Vectors - Experimental Data At $x = 0\text{mm}$ By Shabaka et al. (1985) & Predictions By Model 2 At $x = 600\text{mm}$, 1200mm & 1800mm .

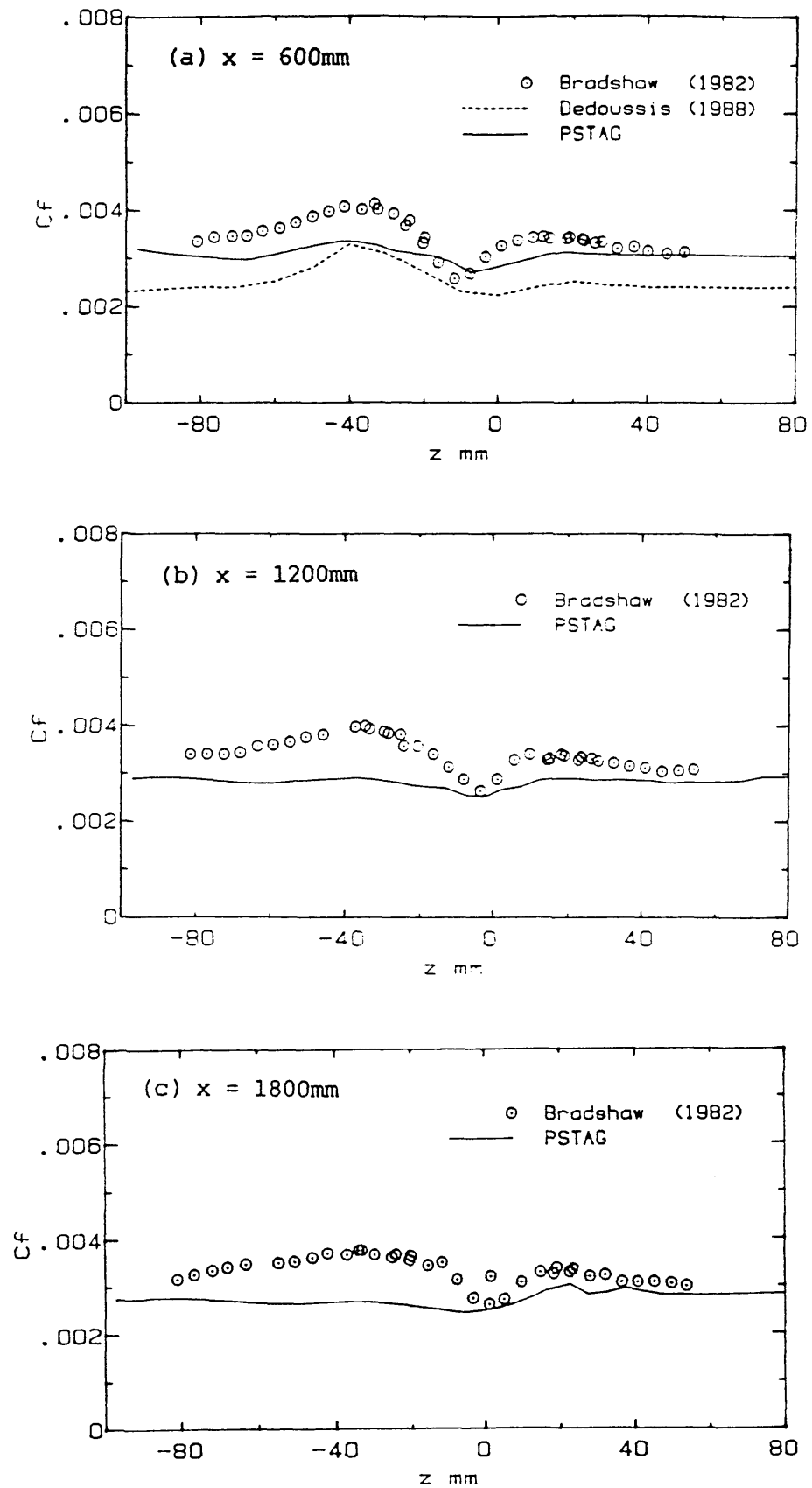
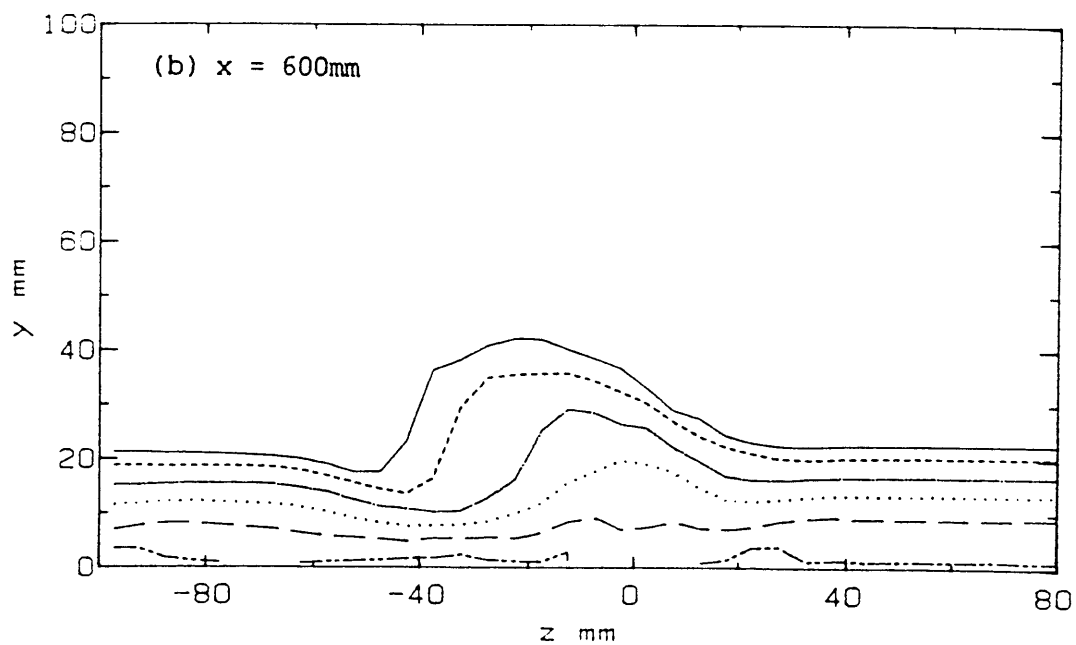
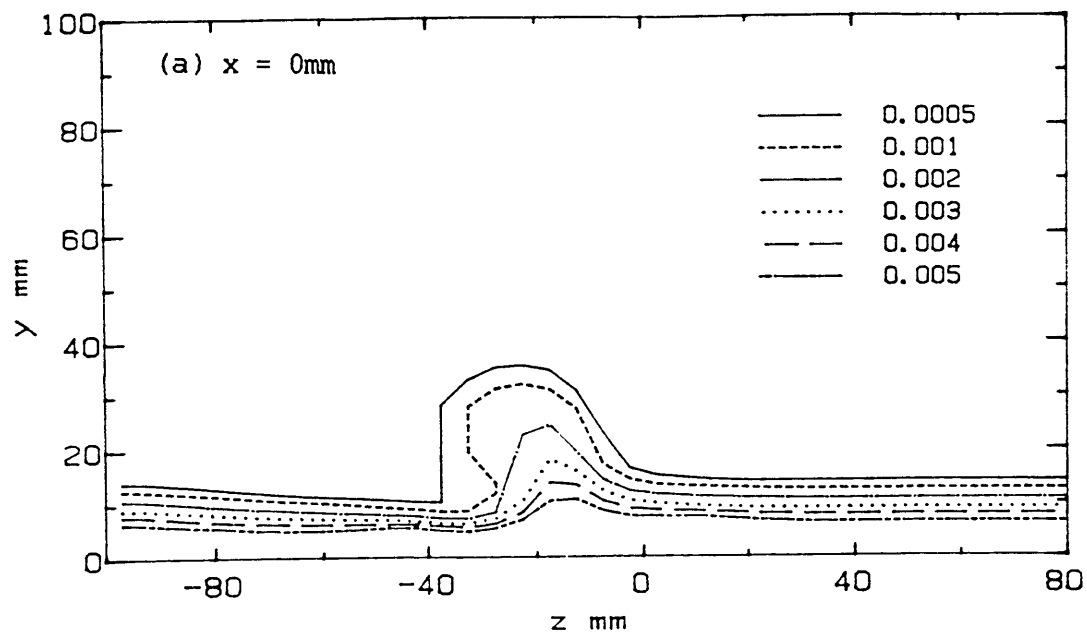


Fig.7.64 Skin Friction Coefficient At $x = 600\text{mm}$, 1200mm & 1800mm . Experimental Data By Bradshaw et al. (1982) & Predictions By Model 2. Dedoussis (1988) Predictions At $x = 600\text{mm}$.



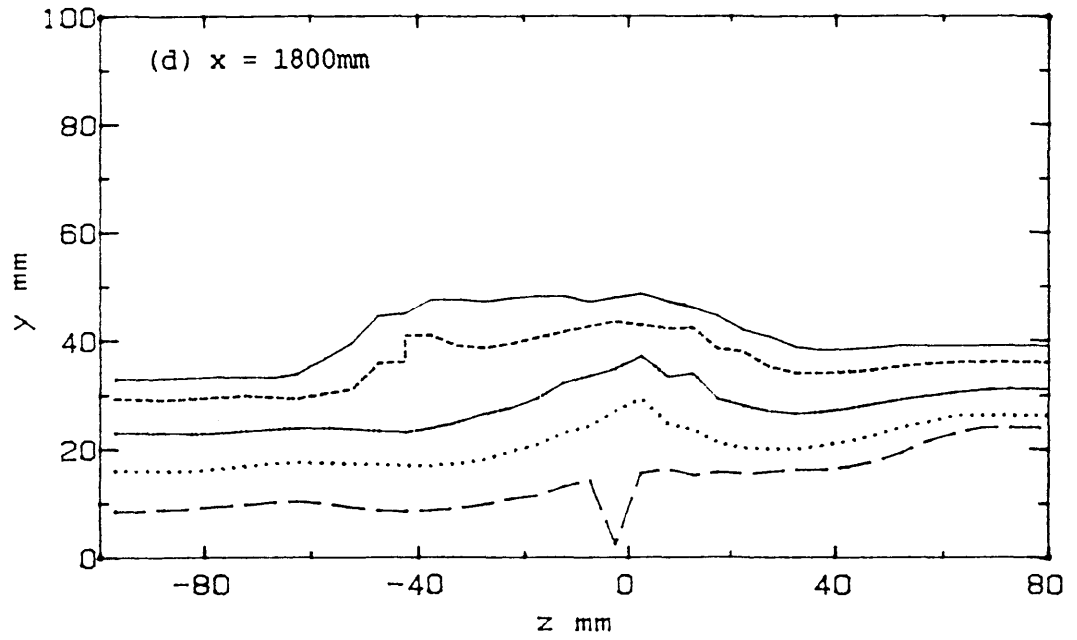
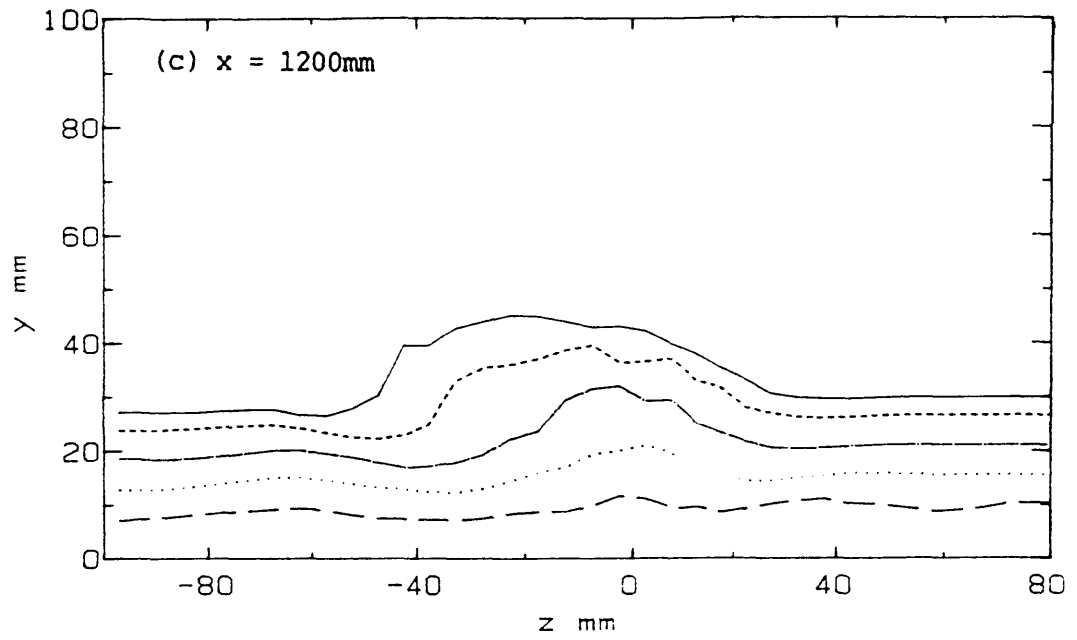
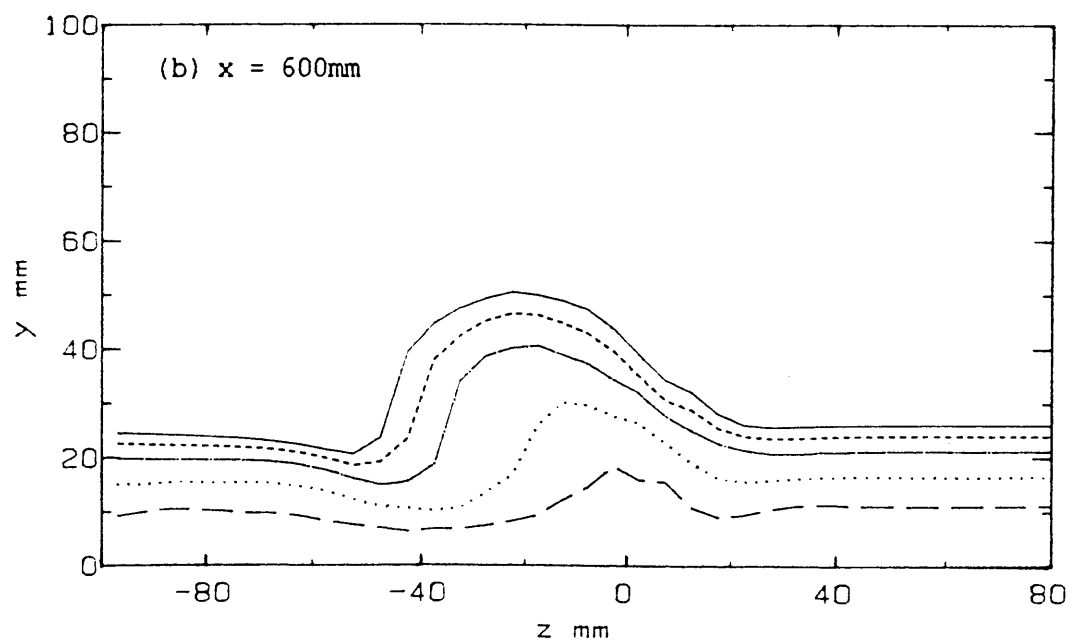
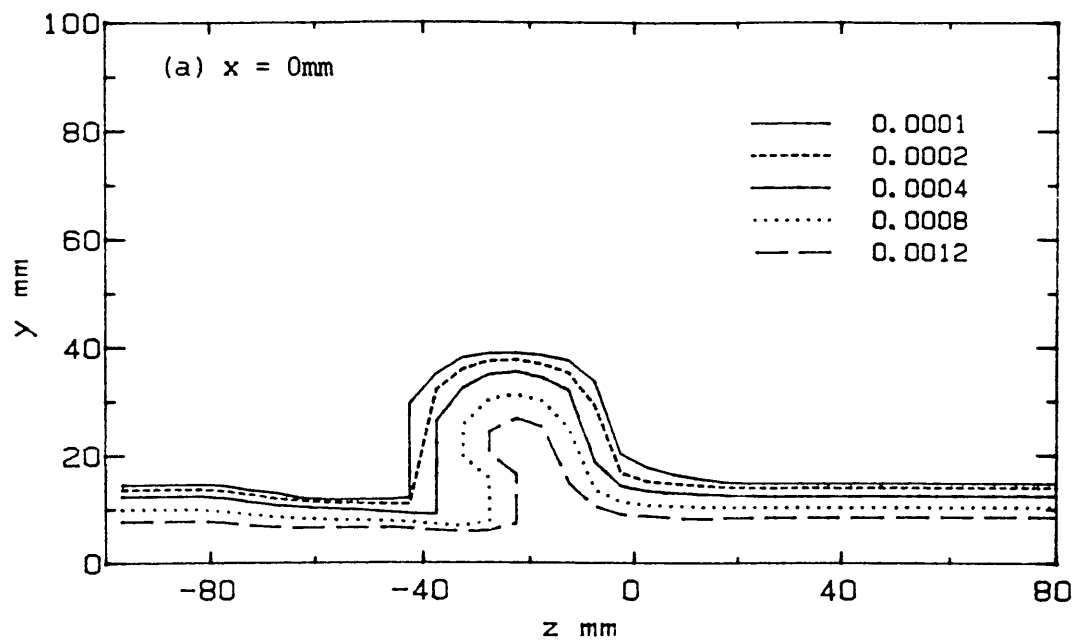


Fig.7.65 $\overline{uu}$ Contours - Experimental Data At $x = 0\text{mm}$ By Shabaka et al. (1985) & Predictions By Model 2 At $x = 600\text{mm}$, 1200mm & 1800mm .



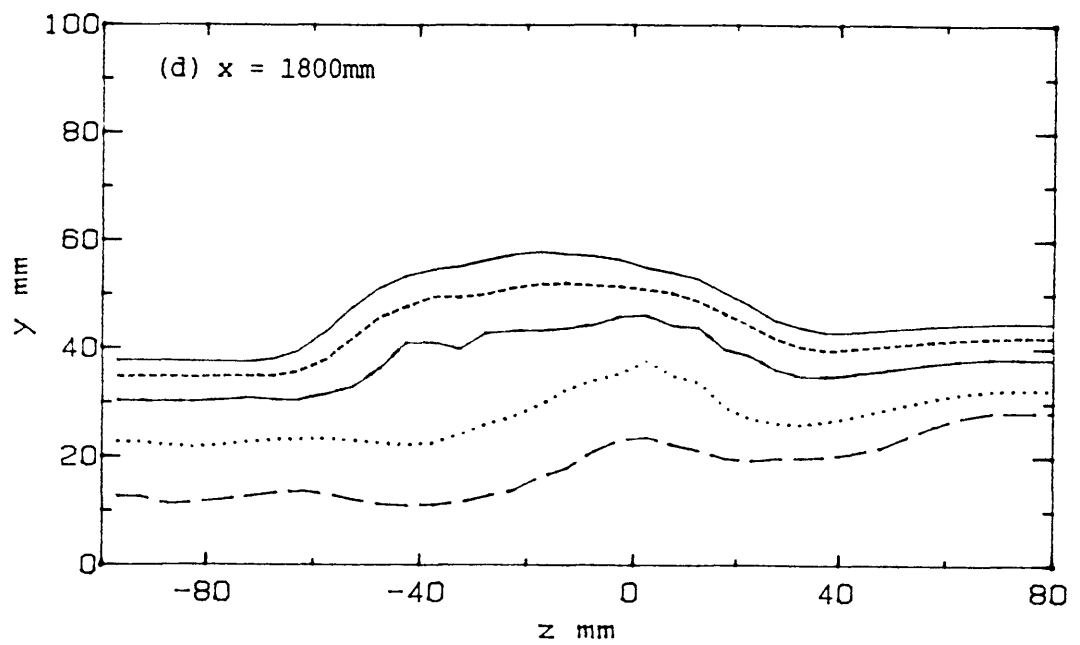
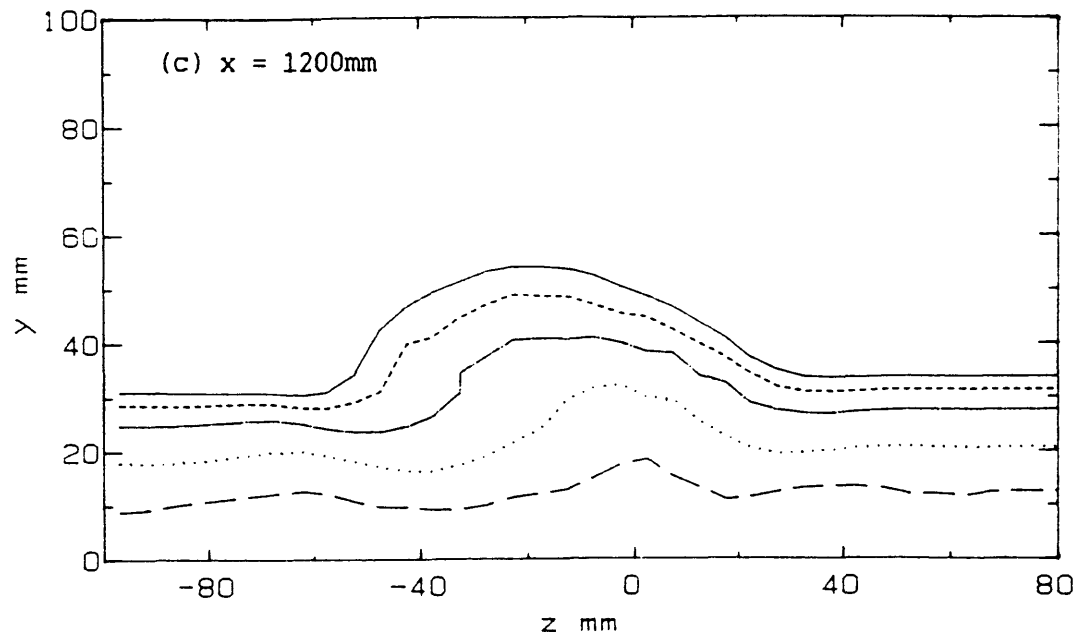
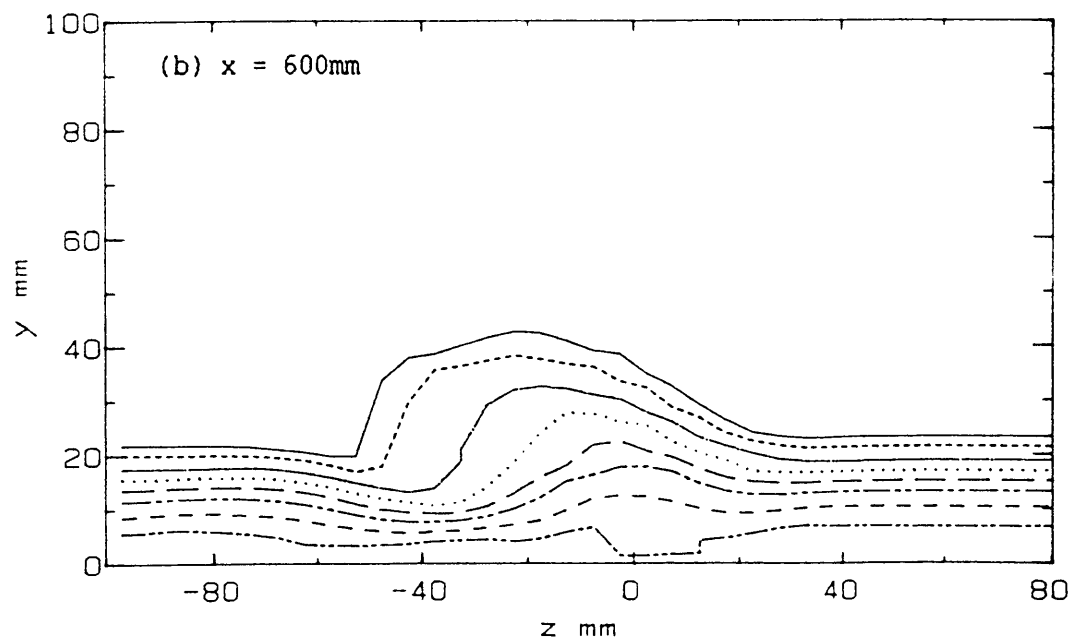
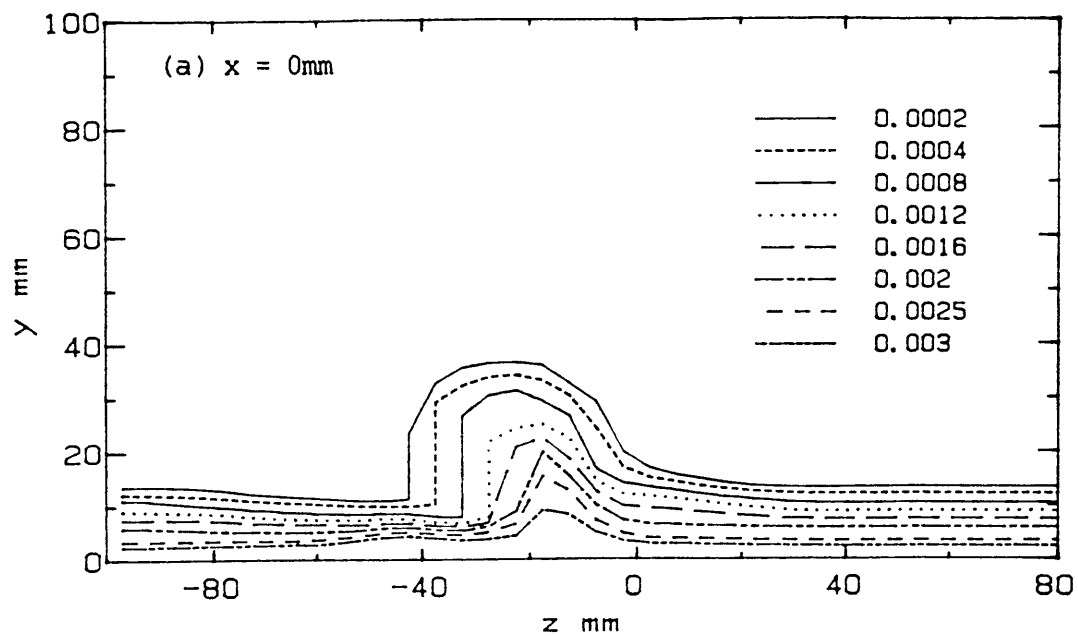


Fig.7.66 $\overline{vv}$ Contours - Experimental Data At $x = 0\text{mm}$ By Shabaka et al. (1985) & Predictions By Model 2 At $x = 600\text{mm}$, 1200mm & 1800mm .



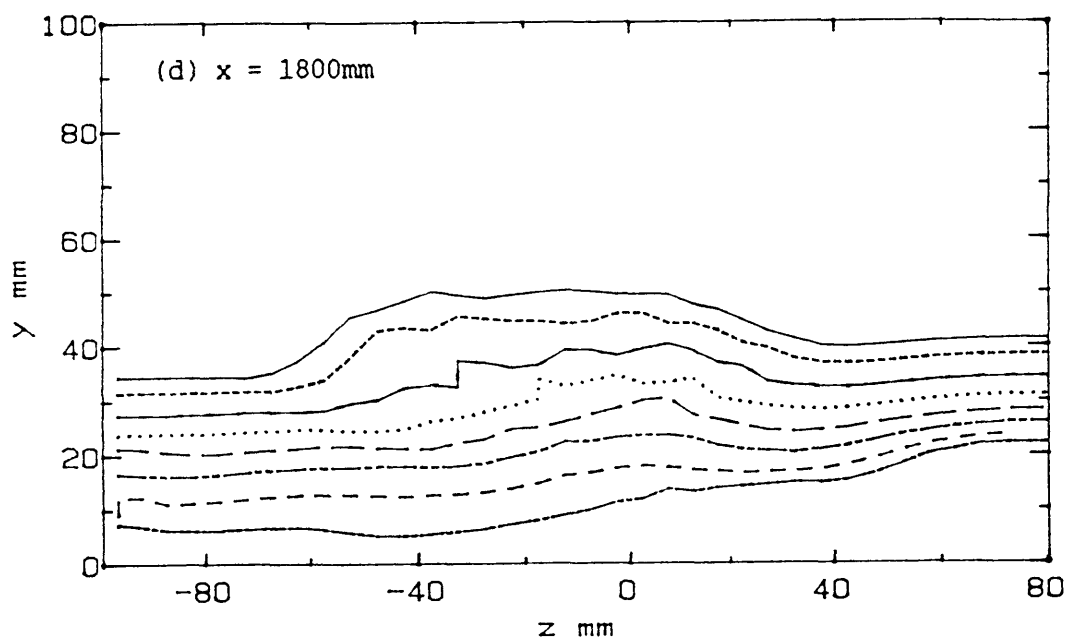
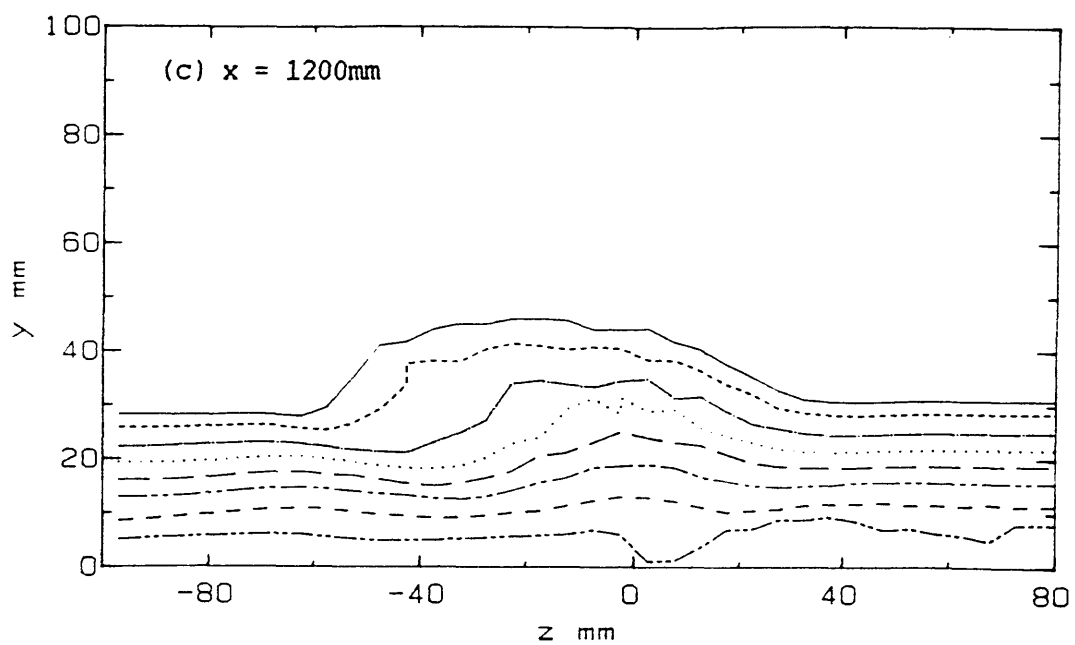
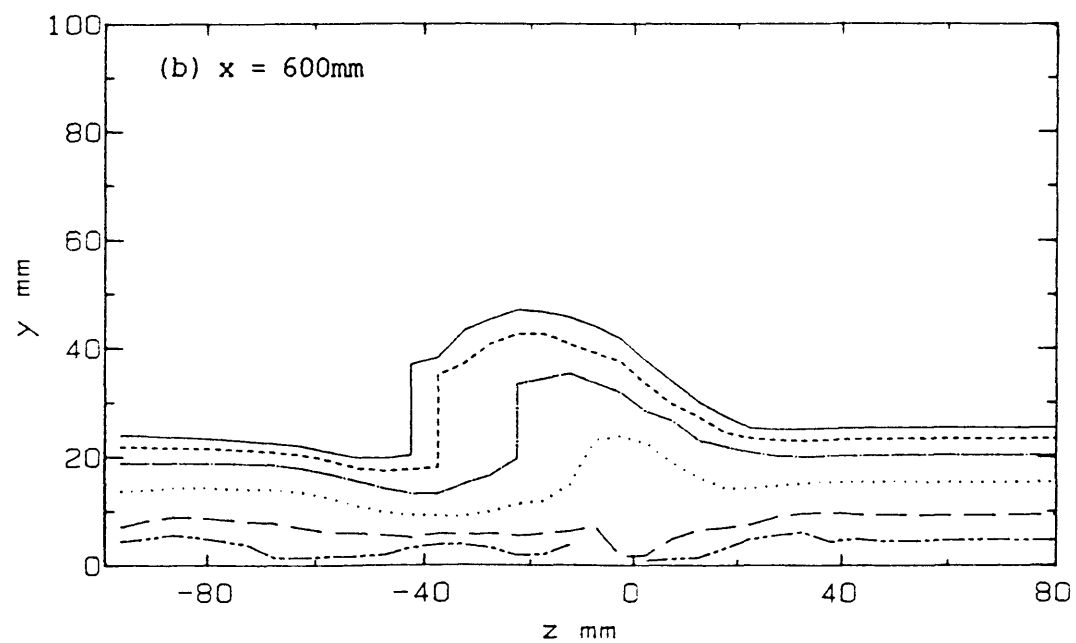
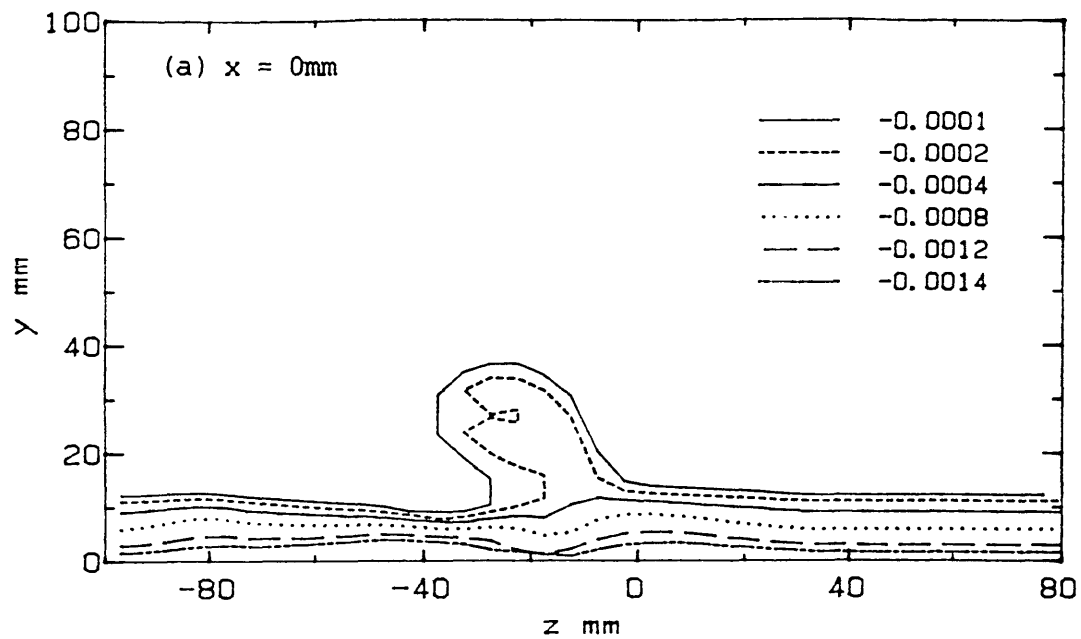


Fig.7.67 $\overline{w}$ Contours - Experimental Data At $x = 0\text{mm}$ By Shabaka et al. (1985) & Predictions By Model 2 At $x = 600\text{mm}$, 1200mm & 1800mm .



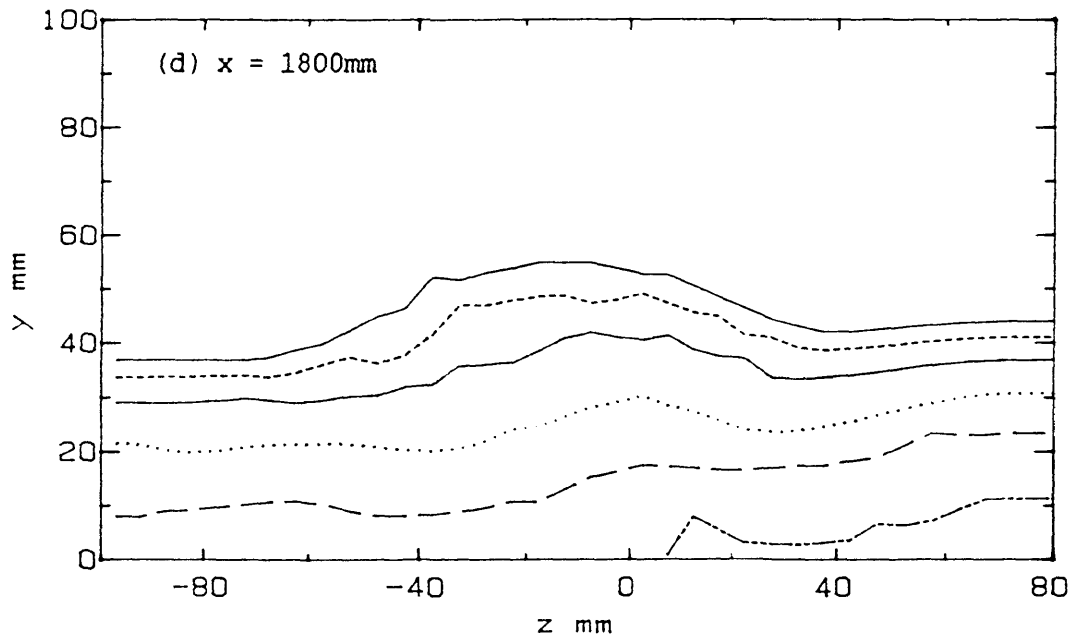
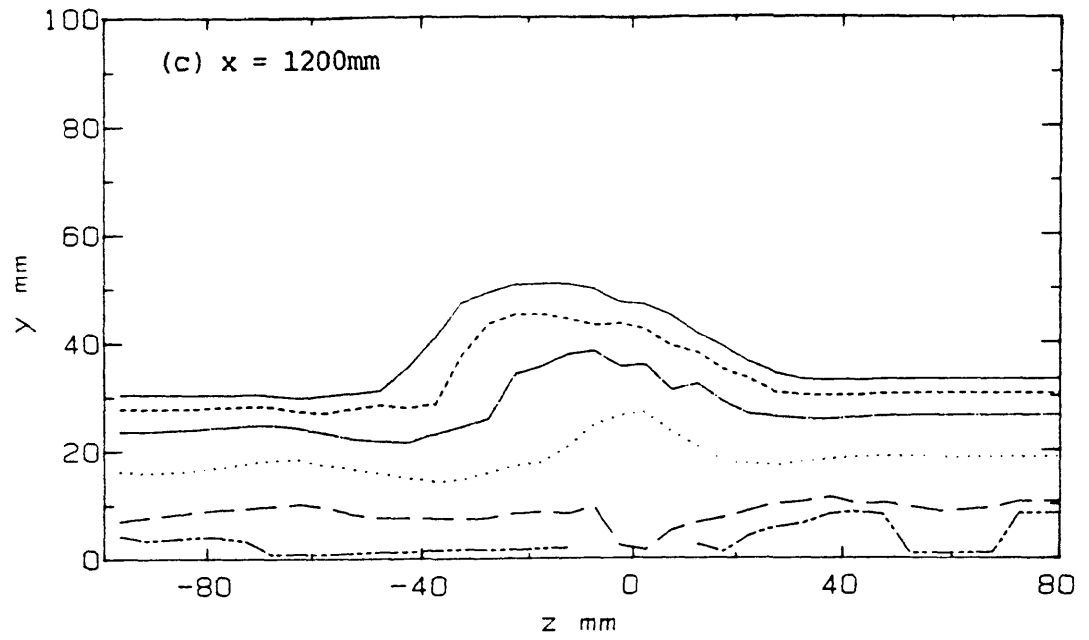
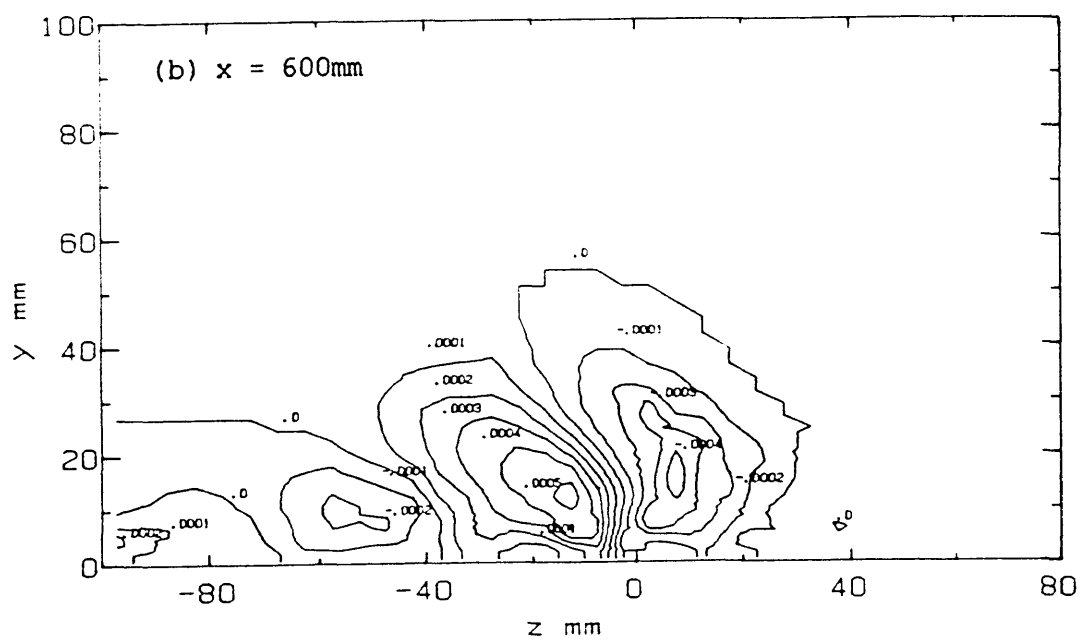
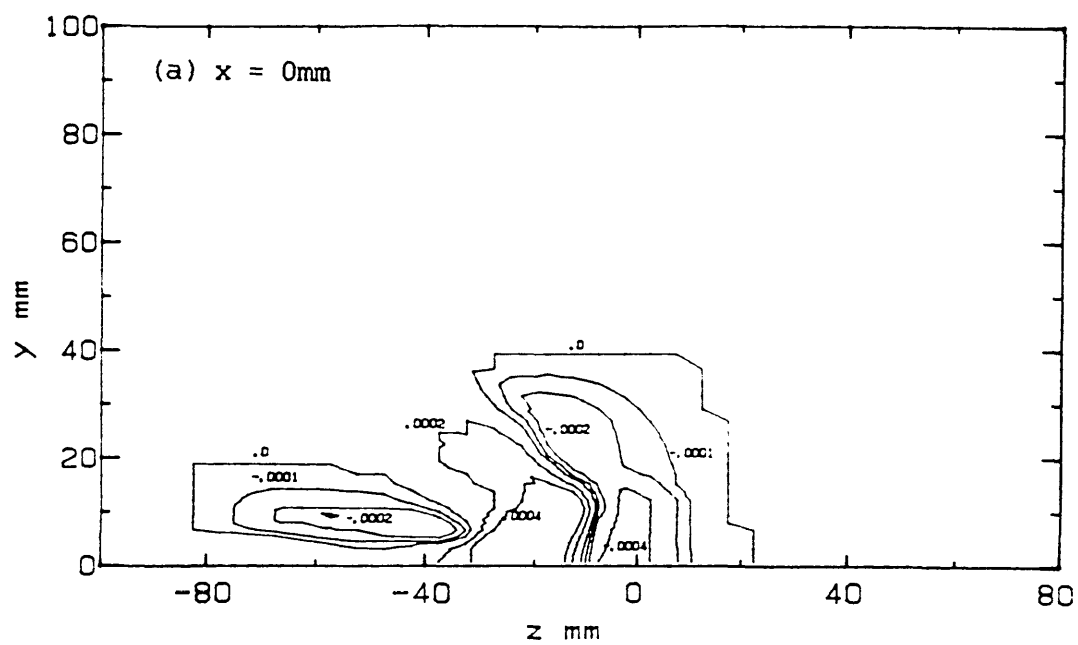


Fig.7.68 $\overline{uv}$ Contours - Experimental Data At $x = 0\text{mm}$ By Shabaka et al. (1985) & Predictions By Model 2 At $x = 600\text{mm}$, 1200mm & 1800mm .



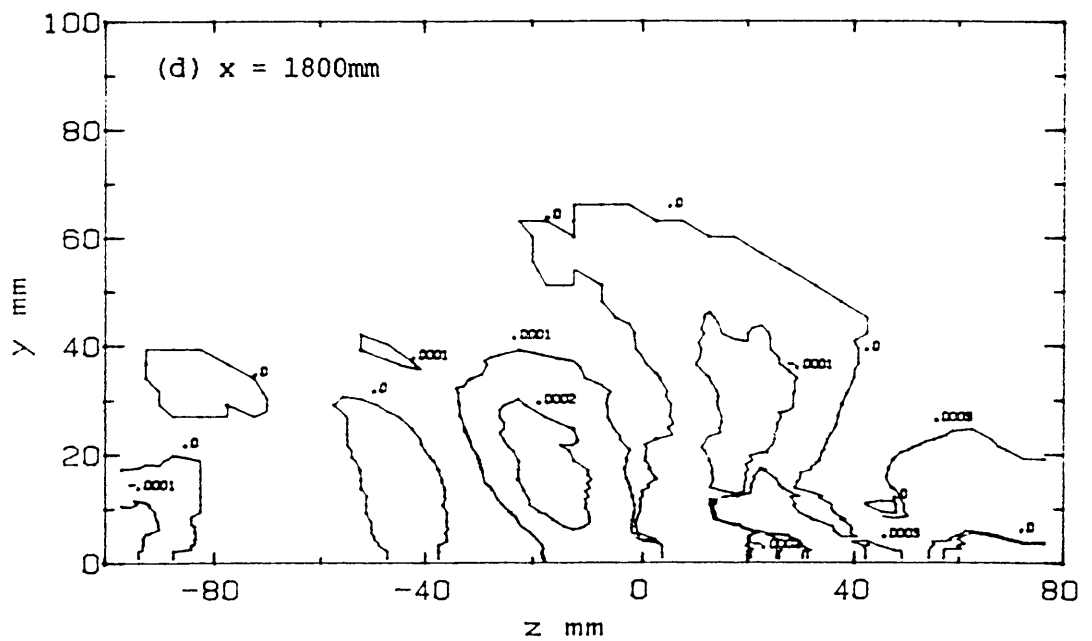
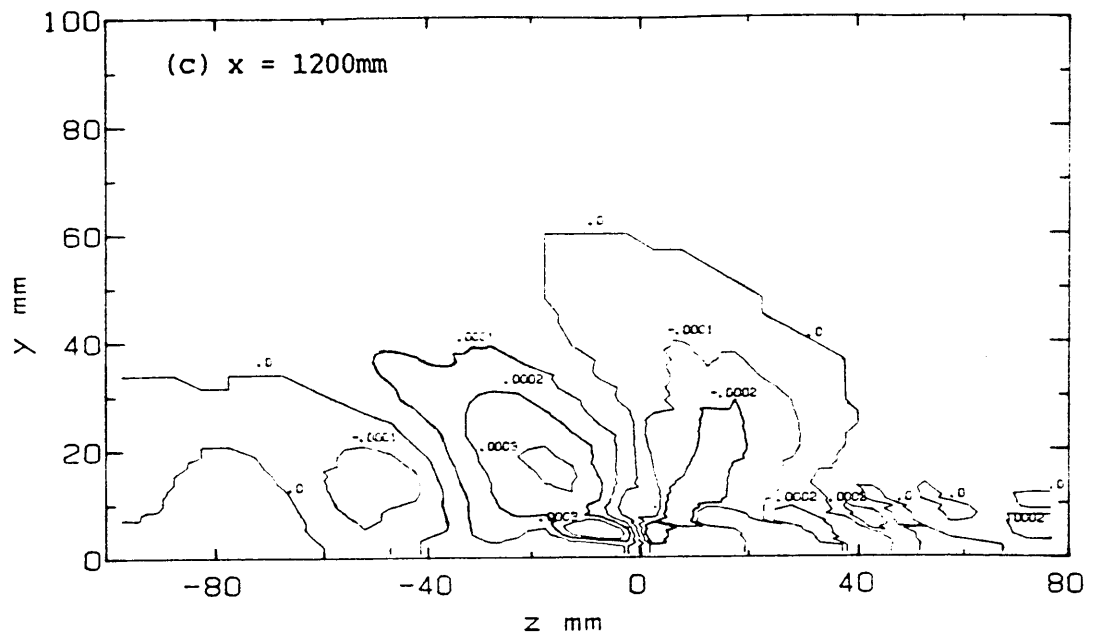
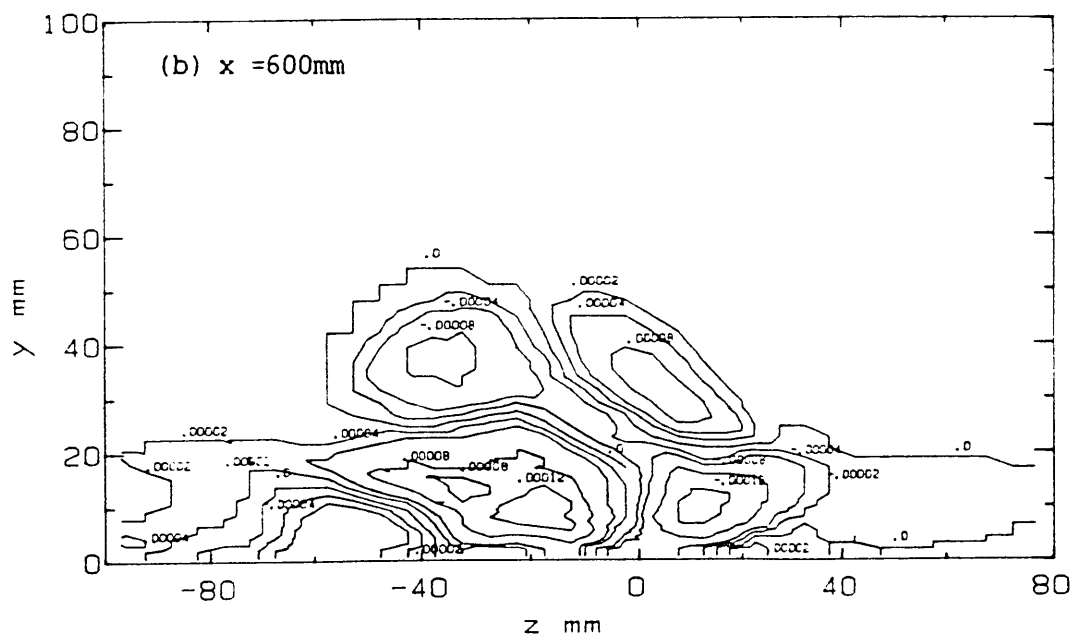
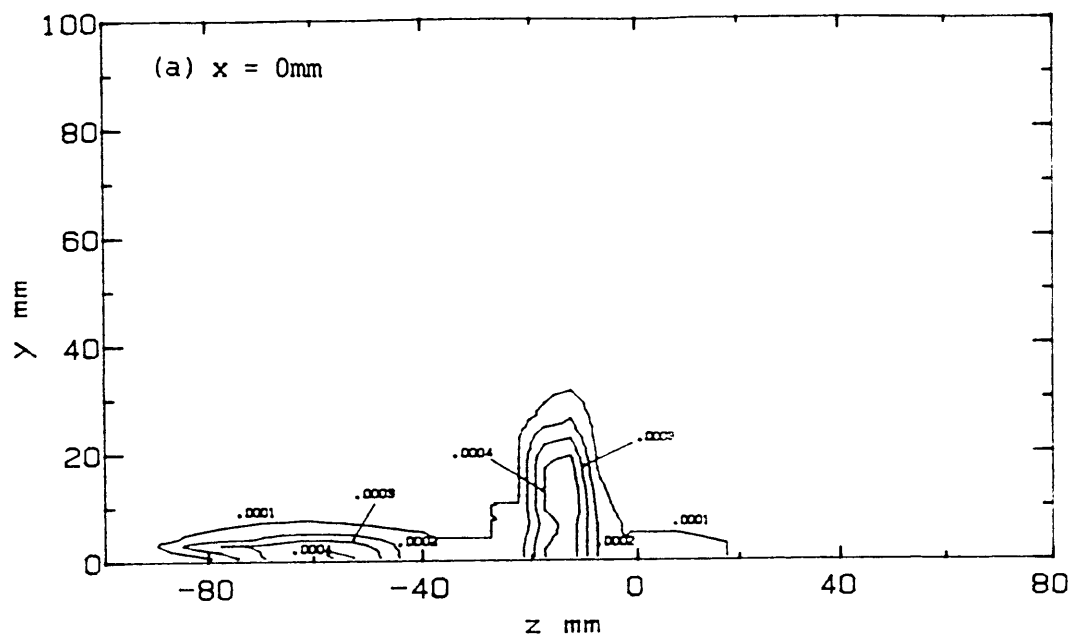


Fig.7.69 $\overline{uw}$ Contours - Experimental Data At $x = 0\text{mm}$ By Shabaka et al. (1985) & Predictions By Model 2 At $x = 600\text{mm}$, 1200mm & 1800mm .



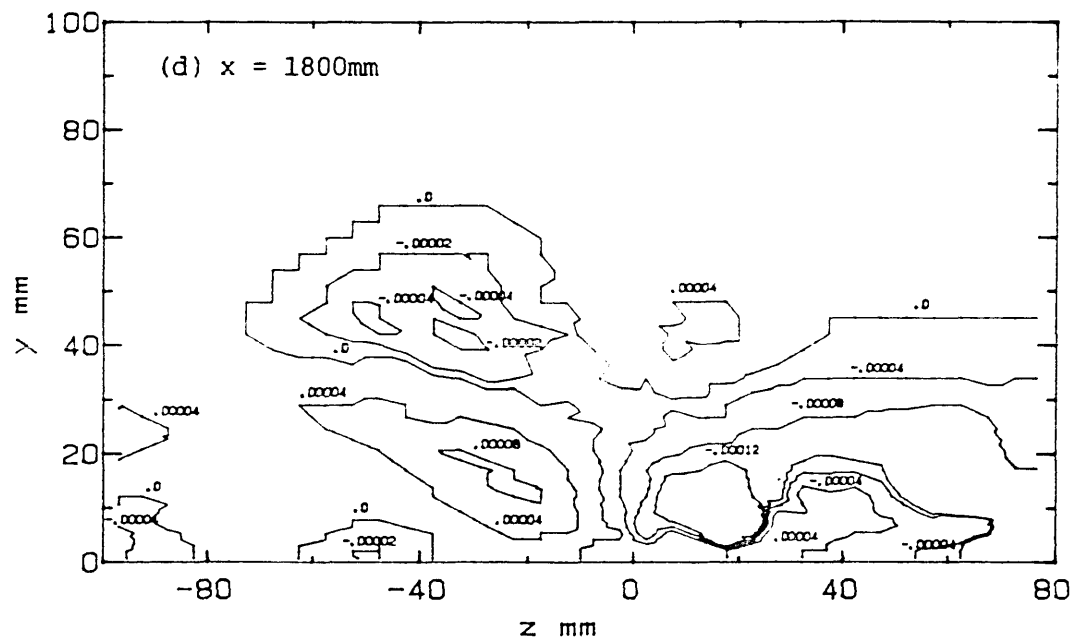
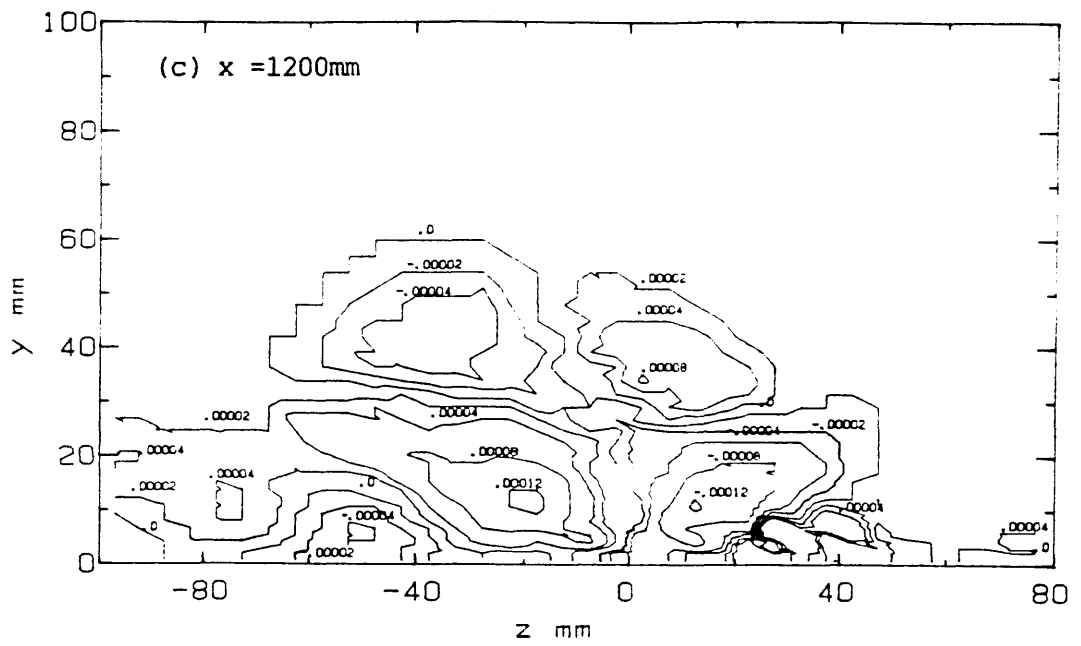


Fig.7.70 $\overline{v_w}$ Contours - Experimental Data At $x = 0\text{mm}$ By Shabaka et al. (1985) & Predictions By Model 2 At $x = 600\text{mm}$, 1200mm & 1800mm .

8 CONCLUSIONS

The present work focuses on the development of a numerical algorithm for steady, incompressible, 3-D "slender" flows which can be either laminar or turbulent. The mean flow field solution uses the three momentum equations and a Poisson pressure equation (which replaces the continuity equation) while the turbulence model solves six $\overline{u_i u_j}$ -equations and the ϵ -equation. The numerical scheme is streamwise parabolic and is based on a pressure staggered grid structure, PSTAG. In PSTAG the mean velocities are stored at the nodes whereas the pressures and turbulence quantities are staggered. In a staggered grid, the mean velocities and turbulence quantities are not all stored at the same grid location whereas in the collocated grid all flow quantities are stored at the nodes.

The conclusions in the numerical aspects are as follow:-

- (a) To eliminate the highly unstable "checker board" solution in the pressure equation, it is necessary during formulation of the pressure equation to stagger the mean velocities w.r.t. the pressures or vice versa. In this respect, there is basically no difference between PSTAG, staggered grids, collocated grids.
- (b) The yz-mom computational stencil in PSTAG is symmetrical whereas in a staggered grid it is not. Consequently, PSTAG does not require the cumbersome diagonal sweep in the ADI

solution method of Dedoussis (1988). This leads to an improvement in computing time of typically 50%.

- (c) The important mean strain rates in a collocated grid are evaluated over two mesh spacings and exclude the mean velocity at the centre node. In PSTAG, the strain rates are calculated more accurately as well as naturally by using the values in adjacent nodes.
- (d) By staggering the pressure, PSTAG does not encounter any problem with regards to pressure BCs. On the other hand, the collocated grid requires extrapolation in order to specify the boundary pressure values for the momentum equations.
- (e) The chief findings in the four laminar flow calculations are,
 - i. Fully-developed flow in a rectangular duct - Since the predictions of the streamwise velocity compare well with the analytic solution, the x-mom algorithm is evidently satisfactory.
 - ii. Taylor-Green vortex flow - With a uniform primary velocity field, this flow tests both the yz-mom and pressure solutions. Again, the results are in good agreement with the analytic solution.
 - iii. Flat plate boundary layer - The computing time of PSTAG is about 50% faster than a staggered grid algorithm that uses similar solution techniques.
 - iv. Flat plate boundary layer with extended leading edge - The unsymmetrical stencil of the staggered grid was found to give a slightly unsymmetrical flow field. Besides being 50% faster, the predictions of PSTAG do not show any distortions.

- (f) Generally, PSTAG can be said to enjoy the advantages of both the collocated and staggered grids. As demonstrated by the present scheme, PSTAG does not resort to unconventional numerical recipes and the discretization techniques, solution procedure etc are methods commonly found in CFD literature. It is of paramount importance that PSTAG exhibits the three desirable features of stability, accuracy and economy.

The conclusions in the turbulence modelling are as follow:-

- (a) Although the RSTM performs well in flows where $\frac{\partial U}{\partial y}$ is the only important strain rate, it cannot be directly used to calculate flows having extra rates of strain without any re-modelling. For instance, the extra strain rates bring about changes in the turbulence structure so the empirical constants in the RSTM must take on different values because these empirical constants are in essence structural parameters.
- (b) In flows where the shear stress gradient near the wall is large (as under the influence of an adverse pressure gradient), many workers have found the turbulent length scale to be overpredicted. An analysis of the constants in the ϵ -equation confirms this. In order to reduce the length scale, it is necessary to re-model the ϵ -equation.
- (c) Using the length scale upper limit as κy , the ϵ -equation was re-modelled by increasing the ϵ generation coefficient $C_{\epsilon 1}$ by 10% from 1.45 to 1.6. A length-scale-limiting model was proposed by making $C_{\epsilon 1}$ a function of τ/τ_w . Both re-modelling

methods successfully prevent the near-wall turbulent length scale from exceeding κy .

(d) The chief findings in the five turbulent flow calculations are,

- i. Flat plate boundary layer - The results show that neither the wall damping function of Launder et al. (1975) nor Gibson & Launder (1978) has any pronounced superiority over the other, although the latter model is more sensitive to the initial conditions, especially near the wall, and the $\overline{uv}$ profile has a slight "peak".
- ii. Boundary layer in adverse pressure gradient - The ϵ -equation re-modelling (see (c) above) has proven successful in giving length scale values which do not exceed κy . The underprediction of the turbulent intensities is due to the inability of the RSTM to capture the "inactive" part of the turbulence.
- iii. "Infinite" swept wing with zero spanwise gradients - The RSTM is unable to reproduce the reductions in either the magnitude of the shear stress or its ratio to the turbulent kinetic energy. It does qualitatively reproduce the lag of the shear stress vector behind the mean velocity gradient vector that is found in experiments. This lag cannot be captured in ν_t -models like the $k-\epsilon$ model.
- iv. "Infinite" swept wing with non- zero spanwise gradients - The predictions are essentially similar to those in the zero spanwise gradient case. However, there is difficulty in specifying appropriate pressure BCs which

can cater for values of $\frac{\partial P}{\partial z}$ that are larger than $\frac{\partial P}{\partial y}$.

v. Vortex-boundary layer interaction - Although there is a lack of good quantitative agreement, the ability of the RSTM to cope with this complex flow where all nine components of strain rates are important is quite remarkable. This should provide more optimistic views in the future use of the model. The fact that ν_t is both anisotropic and ill-behaved means that even transport models for ν_t (e.g. the $k-\epsilon$ model) will not be adequate to predict this flow.

(e) In the calculations of 3-D flows, the pressure redistribution model was found to be a major source of weakness in the turbulence model. Since the dominant term in the "rapid" part actually comprises the generation term, the model is obviously not very effective in providing the means to respond to any strain rates not found in the generation term, for instance, $\frac{\partial \overline{w}}{\partial y}$ exists only as a relatively insignificant term in the $\overline{uv}$ -equation.

(f) The triple-product gradient diffusion model, which works well in 2-D flows, requires closer scrutiny for applications in 3-D boundary layers because of changes in the turbulence structure. Since the empirical constant C_θ is equivalent to a structural parameter, a change in the value of C_θ can be anticipated. The experimental data of Bradshaw & Pontikos (1985) on an "infinite" swept wing do indicate that the turbulence transport by y-direction diffusion is somewhat inhibited by crossflow.

(g) The behaviour of the triple product is more complex in the

vortex-boundary layer case and as pointed out by Shabaka et al. (1985), the gradient diffusivity does not have the same sign throughout the flow. It is therefore possible that the present diffusion model employed for this flow is partly responsible for the predicted Reynolds-stress contours within the vortex core being less distorted than the real ones.

- (h) The 3-D version of the log-law used in this work is simply an extension of the general 2-D form (i.e. without additional term/s to cater for pressure gradient, three-dimensionality etc). The acceptably good C_f predictions clearly indicate the wide range of applicability of the simple log-law formula, though the test cases do not include any strong spanwise pressure gradients.
- (i) Although the range of test cases in this work is only a small proportion of engineering flows, the RSTM is already having difficulties in predicting them using a common set of empirical constants. Thus, one cannot be too optimistic that a universal turbulence model, other than the time-dependent N-S equations, is likely to be within our grasp in the not too distant future. Adjustment of the model "constants" to suit a given type of flow is the only alternative.

The recommendations for further work are as follow:-

- (a) The present numerical scheme, being implicit, has proven stable for Re_{CELL} values up to 12 in laminar flows and, of course, in turbulent flows Re_{CELL} can be much higher. (If one uses apparent ν_t , then Re_{CELL} will probably be about the same order as Re_{CELL} in laminar flow.) It was pointed out in

§7.3.2 and §7.5.2 that to enhance convergence, some refinements in the mesh were necessary. To alleviate this problem, higher-order discretization schemes which are more stable, such as QUICK or SUDS, are recommended. QUICK is preferable to SUDS because the latter requires more computing effort in order to evaluate the precise directions of the velocity vectors.

- (b) Since the region of interest in a flow normally becomes larger with distance downstream, an expanding grid system would enable better utilization of computing resources.
- (c) There is obviously a need for more complex flows calculations to further identify the weakness/es in the RSTM. This information could then be used in the course of modelling the $\overline{u_i u_j}$ - and ϵ - equations.
- (d) Although the present work concludes that the pressure-strain and diffusion models require some form of re-modelling for 3-D flows, the task is definitely not going to be simple. This is especially true for the pressure-strain correlation since it is not measurable directly.

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Appendix A1: Determination Of Empirical Constants In ϵ -Equation

A1.1 Constant $C_{\epsilon 2}$

To determine a suitable value for $C_{\epsilon 2}$, one can turn to unstrained homogeneous grid turbulence and write the equations for

$$\epsilon : \quad U \frac{d\epsilon}{dx} = - C_{\epsilon 2} \frac{\epsilon^2}{k} \quad \text{Eq.A1.1}$$

$$k : \quad U \frac{dk}{dx} = - \epsilon \quad \text{Eq.A1.2}$$

The decay of turbulence kinetic energy can be taken to vary inversely proportional to distance downstream of the grid in a power-law manner (including virtual-origin effects in the definition of x).

$$k = C x^{-n} \quad \text{Eq.A1.3}$$

where C is the constant of proportionality and n is an empirical value. To arrive at an expression for $C_{\epsilon 2}$, first differentiate Eq.A1.3 w.r.t. x .

$$\frac{dk}{dx} = - n C x^{-(n+1)}$$

Substitute the above equation into the l.h.s. of Eq.A1.2.

$$n C U x^{-(n+1)} = \epsilon$$

Then differentiate the above equation w.r.t. x .

$$\frac{d\epsilon}{dx} = -n (n+1) C U x^{-(n+2)}$$

Finally, substitute the above equation into Eq.A1.1 and after some

algebraic manipulations, one can obtain an expression for k .

$$k = C \left\{ C_{\varepsilon 2} \frac{1}{n+1} \right\} x^{-n} \quad \text{Eq.A1.4}$$

By equating the coefficients of Eq.A1.3 and Eq.A1.4, $C_{\varepsilon 2}$ is made a function of n .

$$C_{\varepsilon 2} = 1 + \frac{1}{n} \quad \text{Eq.A1.5}$$

A1.2 Constants C_{ε} & $C_{\varepsilon 1}$

Suitable empirical values for the constants C_{ε} and $C_{\varepsilon 1}$ can be established from the near-wall flow of a 2-D boundary layer where the assumption of local equilibrium holds.

$$- \overline{uv} \frac{\partial U}{\partial y} = u_{\tau}^2 \frac{\partial U}{\partial y} = \varepsilon$$

The above equation can be re-cast by the log-law (or mixing-length hypothesis where $l_m = \kappa y$) as

$$\varepsilon = \frac{u_{\tau}^3}{\kappa y} \quad \text{with} \quad \frac{\partial \varepsilon}{\partial y} = - \frac{u_{\tau}^3}{\kappa y^2}$$

For this flow, by neglecting convection of ε , the modelled form of the ε -equation becomes

$$0 = (C_{\varepsilon 1} - C_{\varepsilon 2}) \frac{\varepsilon^2}{k} + C_{\varepsilon} \frac{\partial}{\partial y} \left[\frac{k}{\varepsilon} \overline{vv} \frac{\partial \varepsilon}{\partial y} \right] \quad \text{Eq.A1.6}$$

and substituting the appropriate forms of ε and $\frac{\partial \varepsilon}{\partial y}$, one obtains

$$(C_{\varepsilon 2} - C_{\varepsilon 1}) \frac{u_{\tau}^6}{\kappa^2 y^2 k} = C_{\varepsilon} \frac{\partial}{\partial y} \left[\overline{vv} k (-y)^{-1} \right] \quad \text{Eq.A1.7}$$

On further algebraic manipulations, an expression relating C_ε and $C_{\varepsilon 1}$ can be found (assuming $C_{\varepsilon 2}$ is known from Eq.A1.5).

$$C_{\varepsilon 1} = C_{\varepsilon 2} - \kappa^2 \frac{b}{a^2} C_\varepsilon \quad \text{Eq.A1.8}$$

where $a = |\overline{uv}|/k$ ($\equiv u_\tau^2/k$) and $b = \overline{vv}/u_\tau^2$.

The turbulence length scale (defined as $\frac{u_\tau^3}{\varepsilon}$) for near-wall flow is roughly $l_s = \kappa y$. Using this length scale, the denominator on the l.h.s. of Eq.A1.7 can be replaced by $(l_s^2 k)$.

$$(C_{\varepsilon 2} - C_{\varepsilon 1}) \frac{u_\tau^6}{l_s^2 k} = C_\varepsilon \frac{\partial}{\partial y} \left[\overline{vv} k (-y)^{-1} \right]$$

or

$$l_s^2 = \frac{C_{\varepsilon 2} - C_{\varepsilon 1}}{C_\varepsilon} \frac{a^2}{b} y^2 \quad \text{Eq.A1.9}$$

Eq.A1.9 is an expression relating the constants on the r.h.s. to the near-wall length scale.

If one considers the near-wall shear stress profile as non-constant in flows with pressure gradients (i.e. $|\overline{uv}| > u_\tau^2$), then

$$\varepsilon = -\overline{uv} \frac{u_\tau}{\kappa y} \quad \text{with} \quad \frac{\partial \varepsilon}{\partial y} = \overline{uv} \frac{u_\tau}{\kappa} y^{-2}$$

assuming, of course, the law of the wall still holds. Substituting the above relations into Eq.A1.6,

$$(C_{\varepsilon 2} - C_{\varepsilon 1}) \frac{\overline{uv}^2 u_\tau^2}{\kappa^2 y^2 k} = C_\varepsilon \frac{\partial}{\partial y} \left[\overline{vv} k (-y)^{-1} \right] \quad \text{Eq.A1.10}$$

Following the same procedure used in deriving Eq.A1.9, that is, by

setting the near-wall value of $l_s = \kappa y$, Eq.A1.10 can be manipulated to give

$$l_{sn}^2 = \frac{C_{\varepsilon 2} - C_{\varepsilon 1}}{C_{\varepsilon}} \frac{a^2}{b} \left[\frac{\overline{uv}^2}{u_{\tau}^4} \right] y^2 \quad \text{Eq.A1.11}$$

l_{sn} denotes the turbulence length scale for a non-uniform shear stress profile.

Appendix A2: Discrete Multi - Dimensional Vector Operators In Momentum Equations

The discrete multi-dimensional vector operator for the streamwise direction $\tilde{L}$ (on the l.h.s. of Eq.4.5) is given below as three components, one for each momentum equation.

$$\underline{\text{x-mom}}: \quad \tilde{L}_x \frac{1}{\Delta x} [\Delta U]_{I,J} \equiv 4 U_{I,J}^n \frac{1}{\Delta x} \Delta U_{I,J}^{n+1}$$

$$\underline{\text{y-mom}}: \quad \tilde{L}_y \frac{1}{\Delta x} [\Delta V]_{I,J} \equiv 2 U_{I,J}^n \frac{1}{\Delta x} \Delta V_{I,J}^{n+1}$$

$$\underline{\text{z-mom}}: \quad \tilde{L}_z \frac{1}{\Delta x} [\Delta W]_{I,J} \equiv 2 U_{I,J}^n \frac{1}{\Delta x} \Delta W_{I,J}^{n+1}$$

The discrete multi-dimensional vector operator in the crossflow plane $\tilde{D}$ (on the r.h.s. of Eq.4.5) is also given below as three components, one for each momentum equation.

$$\begin{aligned} \underline{\text{x-mom}}: \quad & \tilde{D}_x [\Delta U]_{I,J} \\ & \equiv \Delta U_{I,J}^{n+1} \left[-\frac{V_{I+1,J}^n + V_{I,J}^n}{2[\Delta Y_I + \Delta Y_{I-1}]} + \frac{V_{I,J}^n + V_{I-1,J}^n}{2[\Delta Y_I + \Delta Y_{I-1}]} \right. \\ & \quad - \frac{W_{I,J+1}^n + W_{I,J}^n}{2[\Delta Z_J + \Delta Z_{J-1}]} + \frac{W_{I,J}^n + W_{I,J-1}^n}{2[\Delta Z_J + \Delta Z_{J-1}]} \\ & \quad - \frac{1}{\text{Re } \Delta Y_I} \frac{2}{[\Delta Y_I + \Delta Y_{I-1}]} - \frac{1}{\text{Re } \Delta Y_{I-1}} \frac{2}{[\Delta Y_I + \Delta Y_{I-1}]} \\ & \quad \left. - \frac{1}{\text{Re } \Delta Z_J} \frac{2}{[\Delta Z_J + \Delta Z_{J-1}]} - \frac{1}{\text{Re } \Delta Z_{J-1}} \frac{2}{[\Delta Z_J + \Delta Z_{J-1}]} \right] \end{aligned}$$

$$\begin{aligned}
& + \Delta U_{I+1,J}^{n+1} \left[-\frac{V_{I+1,J}^n + V_{I,J}^n}{2[\Delta Y_I + \Delta Y_{I-1}]} + \frac{1}{\text{Re } \Delta Y_I} \frac{2}{[\Delta Y_I + \Delta Y_{I-1}]} \right] \\
& + \Delta U_{I-1,J}^{n+1} \left[\frac{V_{I,J}^n + V_{I-1,J}^n}{2[\Delta Y_I + \Delta Y_{I-1}]} + \frac{1}{\text{Re } \Delta Y_{I-1}} \frac{2}{[\Delta Y_I + \Delta Y_{I-1}]} \right] \\
& + \Delta U_{I,J+1}^{n+1} \left[-\frac{W_{I,J+1}^n + W_{I,J}^n}{2[\Delta Z_J + \Delta Z_{J-1}]} + \frac{1}{\text{Re } \Delta Z_J} \frac{2}{[\Delta Z_J + \Delta Z_{J-1}]} \right] \\
& + \Delta U_{I,J-1}^{n+1} \left[\frac{W_{I,J}^n + W_{I,J-1}^n}{2[\Delta Z_J + \Delta Z_{J-1}]} + \frac{1}{\text{Re } \Delta Z_{J-1}} \frac{2}{[\Delta Z_J + \Delta Z_{J-1}]} \right]
\end{aligned}$$

y-mom:

$$\begin{aligned}
& \tilde{D}_y \begin{bmatrix} \Delta V \\ \Delta W \end{bmatrix}_{I,J} \\
& \equiv \Delta V_{I,J}^{n+1} \left[-\frac{V_{I+1,J}^n + V_{I,J}^n}{[\Delta Y_I + \Delta Y_{I-1}]} + \frac{V_{I,J}^n + V_{I-1,J}^n}{[\Delta Y_I + \Delta Y_{I-1}]} \right. \\
& \quad - \frac{W_{I,J+1}^n + W_{I,J}^n}{2[\Delta Z_J + \Delta Z_{J-1}]} + \frac{W_{I,J}^n + W_{I,J-1}^n}{2[\Delta Z_J + \Delta Z_{J-1}]} \\
& \quad - \frac{1}{\text{Re } \Delta Y_I} \frac{2}{[\Delta Y_I + \Delta Y_{I-1}]} - \frac{1}{\text{Re } \Delta Y_{I-1}} \frac{2}{[\Delta Y_I + \Delta Y_{I-1}]} \\
& \quad \left. - \frac{1}{\text{Re } \Delta Z_J} \frac{2}{[\Delta Z_J + \Delta Z_{J-1}]} - \frac{1}{\text{Re } \Delta Z_{J-1}} \frac{2}{[\Delta Z_J + \Delta Z_{J-1}]} \right] \\
& + \Delta V_{I+1,J}^{n+1} \left[-\frac{V_{I+1,J}^n + V_{I,J}^n}{[\Delta Y_I + \Delta Y_{I-1}]} + \frac{1}{\text{Re } \Delta Y_I} \frac{2}{[\Delta Y_I + \Delta Y_{I-1}]} \right] \\
& + \Delta V_{I-1,J}^{n+1} \left[\frac{V_{I,J}^n + V_{I-1,J}^n}{[\Delta Y_I + \Delta Y_{I-1}]} + \frac{1}{\text{Re } \Delta Y_{I-1}} \frac{2}{[\Delta Y_I + \Delta Y_{I-1}]} \right]
\end{aligned}$$

$$+ \Delta V_{I,J+1}^{n+1} \left[-\frac{W_{I,J+1}^n + W_{I,J}^n}{2[\Delta Z_J + \Delta Z_{J-1}]} + \frac{1}{\text{Re } \Delta Z_J} \frac{2}{[\Delta Z_J + \Delta Z_{J-1}]} \right]$$

$$+ \Delta V_{I,J-1}^{n+1} \left[\frac{W_{I,J}^n + W_{I,J-1}^n}{2[\Delta Z_J + \Delta Z_{J-1}]} + \frac{1}{\text{Re } \Delta Z_{J-1}} \frac{2}{[\Delta Z_J + \Delta Z_{J-1}]} \right]$$

$$+ \Delta W_{I,J}^{n+1} \left[-\frac{V_{I,J+1}^n - V_{I-1,J}^n}{2[\Delta Z_J + \Delta Z_{J-1}]} \right]$$

$$+ \Delta W_{I+1,J}^{n+1} \left[0 \right]$$

$$+ \Delta W_{I-1,J}^{n+1} \left[0 \right]$$

$$+ \Delta W_{I,J+1}^{n+1} \left[-\frac{V_{I,J+1}^n + V_{I,J}^n}{2[\Delta Z_J + \Delta Z_{J-1}]} \right]$$

$$+ \Delta W_{I,J-1}^{n+1} \left[\frac{V_{I,J}^n + V_{I,J-1}^n}{2[\Delta Z_J + \Delta Z_{J-1}]} \right]$$

z-mom:

$$\tilde{D}_z \begin{bmatrix} \Delta V \\ \Delta W \end{bmatrix}_{I,J}$$

$$\equiv \Delta W_{I,J}^{n+1} \left[-\frac{V_{I+1,J}^n + V_{I,J}^n}{2[\Delta Y_I + \Delta Y_{I-1}]} + \frac{V_{I,J}^n + V_{I-1,J}^n}{2[\Delta Y_I + \Delta Y_{I-1}]} \right]$$

$$- \frac{W_{I,J+1}^n + W_{I,J}^n}{[\Delta Z_J + \Delta Z_{J-1}]} + \frac{W_{I,J}^n + W_{I,J-1}^n}{[\Delta Z_J + \Delta Z_{J-1}]}$$

$$- \frac{1}{\text{Re } \Delta Y_I} \frac{2}{[\Delta Y_I + \Delta Y_{I-1}]} - \frac{1}{\text{Re } \Delta Y_{I-1}} \frac{2}{[\Delta Y_I + \Delta Y_{I-1}]}$$

$$\begin{aligned}
& -\frac{1}{\operatorname{Re} \Delta Z_J} \frac{2}{[\Delta Z_J + \Delta Z_{J-1}]} - \frac{1}{\operatorname{Re} \Delta Z_{J-1}} \frac{2}{[\Delta Z_J + \Delta Z_{J-1}]} \Big] \\
& + \Delta W_{I+1,J}^{n+1} \left[-\frac{V_{I+1,J}^n + V_{I,J}^n}{2[\Delta Y_I + \Delta Y_{I-1}]} + \frac{1}{\operatorname{Re} \Delta Y_I} \frac{2}{[\Delta Y_I + \Delta Y_{I-1}]} \right] \\
& + \Delta W_{I-1,J}^{n+1} \left[\frac{V_{I,J}^n + V_{I-1,J}^n}{2[\Delta Y_I + \Delta Y_{I-1}]} + \frac{1}{\operatorname{Re} \Delta Y_{I-1}} \frac{2}{[\Delta Y_I + \Delta Y_{I-1}]} \right] \\
& + \Delta W_{I,J+1}^{n+1} \left[-\frac{W_{I,J+1}^n + W_{I,J}^n}{[\Delta Z_J + \Delta Z_{J-1}]} + \frac{1}{\operatorname{Re} \Delta Z_J} \frac{2}{[\Delta Z_J + \Delta Z_{J-1}]} \right] \\
& + \Delta W_{I,J-1}^{n+1} \left[\frac{W_{I,J}^n + W_{I,J-1}^n}{[\Delta Z_J + \Delta Z_{J-1}]} + \frac{1}{\operatorname{Re} \Delta Z_{J-1}} \frac{2}{[\Delta Z_J + \Delta Z_{J-1}]} \right] \\
& + \Delta V_{I,J}^{n+1} \left[-\frac{W_{I+1,J}^n - W_{I-1,J}^n}{2[\Delta Y_I + \Delta Y_{I-1}]} \right] \\
& + \Delta V_{I+1,J}^{n+1} \left[-\frac{W_{I+1,J}^n + W_{I,J}^n}{2[\Delta Y_I + \Delta Y_{I-1}]} \right] \\
& + \Delta V_{I-1,J}^{n+1} \left[\frac{W_{I,J}^n + W_{I-1,J}^n}{2[\Delta Y_I + \Delta Y_{I-1}]} \right] \\
& + \Delta V_{I,J+1}^{n+1} \left[0 \right] \\
& + \Delta V_{I,J-1}^{n+1} \left[0 \right]
\end{aligned}$$

Appendix A3: Variable Namelist of PSTAG

AA	Temporary storage variable in SOLV2.		
AUX(I,J)	Temporary 2-D storage array.		
A11(I)	}	West or south node coefficients of matrix in TRDIAG & SOLV2.	
A12(I)			
A21(I)			
A22(I)			
BCE	BC of boundary 2-3.	}	
BCN	BC of boundary 3-4.		0 - zero Dirichlet BC.
BCS	BC of boundary 1-2.		1 - zero Neumann BC.
BCW	BC of boundary 1-4.		
B11(I)	}	Centre node coefficient of matrix in TRDIAG & SOLV2.	
B12(I)			
B21(I)			
B22(I)			
CE	C_e in Eq.3.13.		
CE1	C_{e1} in Eq.3.12.		
CE2	C_{e2} in Eq.3.11.		
CS	C_s in Eq.3.3.		
CYA(I)	$= 2 / [YCV(I) * DY(I-1)]$		
CYC(I)	$= 2 / [YCV(I) * DY(I)]$		
CZA(J)	$= 2 / [ZCV(J) * DZ(J-1)]$		
CZC(J)	$= 2 / [ZCV(J) * DZ(J)]$		
C1	C_1 in Eq.3.6.		
C1PRI	C_{1v} in Eq.3.10.		
C2	C_2 in Eq.3.7.		

C2PRI	C_{2v} in Eq.3.7.
C11(I)	$\left\{ \begin{array}{l} \text{North or east node coefficient of matrix in} \\ \text{TRDIAG \& SOLV2.} \end{array} \right.$
C12(I)	
C21(I)	
C22(I)	
D	= $A11(I) / B11(I)$
DET	= Determinant of the 2x2 matrix in SOLV2.
DIFF	Difference between newly calculated and previous iteration loop value.
DIF1	$\left\{ \begin{array}{l} \text{The } \overline{u_i u_j} \text{ and } \varepsilon \text{- equations diffusion term which} \\ \text{forms part of the source term.} \end{array} \right.$
DIF2	
DPXEXT	Streamwise pressure gradient.
DRED	D_{ij} in Eq.3.7.
DU(I,J)	U velocity increment.
DUDX(I,J)	$\frac{\partial U}{\partial x}$ at $x^{n+1/2}$.
DUDY(I,J)	$\frac{\partial U}{\partial y}$ at $x^{n+1/2}$.
DUDZ(I,J)	$\frac{\partial U}{\partial z}$ at $x^{n+1/2}$.
DV(I,J)	V velocity increment.
DVDX(I,J)	$\frac{\partial V}{\partial x}$ at $x^{n+1/2}$.
DVDY(I,J)	$\frac{\partial V}{\partial y}$ at $x^{n+1/2}$.
DVDZ(I,J)	$\frac{\partial V}{\partial z}$ at $x^{n+1/2}$.
DW(I,J)	W velocity increment.
DWDX(I,J)	$\frac{\partial W}{\partial x}$ at $x^{n+1/2}$.
DWDY(I,J)	$\frac{\partial W}{\partial y}$ at $x^{n+1/2}$.
DWDZ(I,J)	$\frac{\partial W}{\partial z}$ at $x^{n+1/2}$.
DX	Distance between the x^{n+1} and x^n stations.
DY(I)	y-direction mesh spacing, equal to $(Y(I+1)-Y(I))$.
DZ(J)	z-direction mesh spacing, equal to $(Z(J+1)-Z(J))$.

ENR(I,J)	Ratio of $\frac{k}{\varepsilon}$ at $x^{n+1/2}$.
EPL(I,J)	ε at $x^{n+1/2}$.
EPLCON(I,J)	ε convective term at upstream $x^{n-1/2}$ station.
EPLOLD(I,J)	ε value at previous outer RSTM loop.
ERR	Temporary error variable storage.
ERRPI	Maximum absolute error in the inner pressure loop.
ERRPO	Maximum absolute error in the outer pressure loop.
ERRRI	Maximum absolute error in the inner RSTM loop.
ERRRO	Maximum absolute error in the outer RSTM loop.
F1(I,J)	South node coefficient in PRESS, or West node coefficient in RSTM subroutines.
F2(I,J)	North node coefficient in PRESS, or East node coefficient in RSTM subroutines.
F4(I,J)	Centre node coefficient in PRESS & RSTM subroutines.
G1(I,J)	West node coefficient in PRESS, or South node coefficient in RSTM subroutines.
G2(I,J)	East node coefficient in PRESS, or North node coefficient in RSTM subroutines.
G4(I,J)	Centre node coefficient in PRESS, or $(\phi_{22})_2$ in Eq.3.7.
ICON	Flag for residual of continuity equation residual: 0 - write to FILE 8; 1 - do not write.
IMAX	(IMAX+1) = No. of y-direction grid nodes.
IPERPI	Maximum no. of iterations permitted in inner pressure loop.

IPERPO	Maximum no. of iterations permitted in outer pressure loop.
IPERRI	Maximum no. of iterations permitted in inner RSTM loop.
IPERRO	Maximum no. of iterations permitted in outer RSTM loop.
IPSI	Continuity equation differencing scheme in Eq.4.8: 0 - fully-implicit; 1 - Crank-Nicolson.
ITNOPI	Inner pressure loop counter.
ITNOPO	Outer pressure loop counter.
ITNORI	Inner RSTM loop counter.
ITNORO	Outer RSTM loop counter.
JMAX	(JMAX+1) = No. of spanwise grid nodes.
KNFLOW	0 - laminar; 1 - turbulent.
NEQ	Size of matrix in TRDIAG & SOLV2.
NMAX	No. of x-stations.
OG	Outer pressure loop relaxation factor.
OMEGA	Inner pressure loop relaxation factor.
PAUX(I,J)	Temporary pressure storage array.
PDIF	Difference between latest and previous pressure.
PDIFV	Maximum pressure error (not the absolute).
PHIW	ϕ_v in Eq.3.9 or Eq.3.10.
PHI2	$(\phi_{ij} + \phi_{ji})_z$ in Eq.3.7.
PNEW(I,J)	Latest calculated pressures.

POLD(I,J)	Guessed pressures.
PR(I,J)	Production of k , P_k in Eq.3.7.
PROD	Generation term P_{ij} in Eq.3.7.
RFEPL	Relaxation factor of ϵ .
RFRS	Relaxation factor of $\overline{u_i u_j}$.
RE	Reynolds no. = $\frac{L_{ref} U_{ref}}{\nu}$
RRE	Reciprocal of the Reynolds no.
R11(I)	} TRDIAG & SOLV2 matrix source term.
R22(I)	
SC(I,J)	Source terms.
TEM	} Temporary storage variable.
TEM1	
TEM2	
TEM3	
TEM4	
TEM5	
TERR	Temporary error storage variable in RSERRO.
THETA	Angle of the resultant U & W velocity w.r.t. the streamwise axis.
TKE(I,J)	k at $x^{n+1/2}$.
TOLPI	Convergence tolerance in inner pressure loop.
TOLPO	Convergence tolerance in outer pressure loop.
TOLRI	Convergence tolerance in inner RSTM loop.
TOLRO	Convergence tolerance in outer RSTM loop.
UN(I,J)	U at x^n .

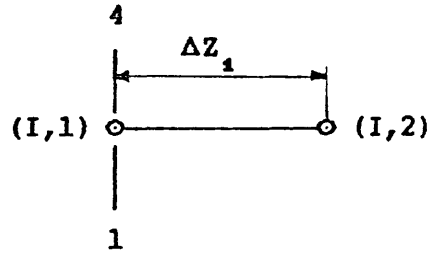
UTNEW	Latest value of u_τ .
UTOLD	Previous iteration loop u_τ value.
UU(I,J)	$\overline{uu}$ at $x^{n+1/2}$.
UUCON(I,J)	$\overline{uu}$ convective term at upstream $x^{n-1/2}$ station.
UULD(I,J)	$\overline{uu}$ value at previous outer RSTM loop.
UUU	Resultant mean velocity in the log-law.
UV(I,J)	$\overline{uv}$ at $x^{n+1/2}$.
UVCON(I,J)	$\overline{uv}$ convective term at upstream $x^{n-1/2}$ station.
UULD(I,J)	$\overline{uv}$ value at previous outer RSTM loop.
UW(I,J)	$\overline{uw}$ at $x^{n+1/2}$.
UWCON(I,J)	$\overline{uw}$ convective term at upstream $x^{n-1/2}$ station.
UWOLD(I,J)	$\overline{uw}$ value at previous outer RSTM loop.
U1(I)	} Solution of matrix.
U2(I)	
VN(I,J)	v at x^n .
VV(I,J)	$\overline{vv}$ at $x^{n+1/2}$.
VVCON(I,J)	$\overline{vv}$ convective term at upstream $x^{n-1/2}$ station.
VVOLD(I,J)	$\overline{vv}$ value at previous outer RSTM loop.
VW(I,J)	$\overline{vw}$ at $x^{n+1/2}$.
VWCON(I,J)	$\overline{vw}$ convective term at upstream $x^{n-1/2}$ station.
VWOLD(I,J)	$\overline{vw}$ value at previous outer RSTM loop.
WN(I,J)	w at x^n .
WW(I,J)	$\overline{ww}$ at $x^{n+1/2}$.
WWCON(I,J)	$\overline{ww}$ convective term at upstream $x^{n-1/2}$ station.
WWOLD(I,J)	$\overline{ww}$ value at previous outer RSTM loop.

X(I)	Locations of x-stations.
Y(I)	Locations of y=constant grid lines.
YACV(I)	$= 2 / \{YCV(I+1) + YCV(I)\}$
YCV(I)	$= DY(I) + DY(I+1)$
YL(I,J)	Upstream x-momentum convective term.
YR(I,J)	y-momentum upstream source term.
Z(J)	Locations of z=constant grid lines.
ZACV(J)	$= 2 / \{ZCV(J+1) + ZCV(J)\}$
ZCV(J)	$= DZ(J) + DZ(J+1)$
ZR(I,J)	z-momentum upstream source term.

Appendix A4: Taylor - Green Vortex Boundary Conditions

A4.1 Non - Zero V Neumann Boundary Condition

The BC for V velocity along the boundaries 2-3 and 1-4 in Fig.6.8 are of the non-zero Neumann type. To derive these BCs, consider the nodes $(I,1)$ and $(I,2)$ on left lateral boundary 4-1 as depicted below.



From Eq.6.10,

$$V_{I,1} = -e^{\frac{2x}{Re}} \cos y \sin \frac{\pi}{2}$$

$$V_{I,2} = -e^{\frac{2x}{Re}} \cos y \sin \left(\frac{\pi}{2} + \Delta Z_1 \right)$$

From the above equations, one can find the ratio of $V_{I,2}$ to $V_{I,1}$.

$$\frac{V_{I,2}}{V_{I,1}} = \frac{\sin \left(\frac{\pi}{2} + \Delta Z_1 \right)}{\sin \frac{\pi}{2}} = \cos \Delta Z_1 \quad \text{Eq.A4.1}$$

With the discretized form of $\frac{\partial V}{\partial z}$ as $\frac{\partial V}{\partial z} = \frac{1}{\Delta Z_1} [V_{I,2} - V_{I,1}]$, one can substitute for $V_{I,2}$ using Eq.A4.1 and after some algebraic manipulations an expression for $\frac{\partial V}{\partial z}$ can be obtained.

$$\frac{\partial V}{\partial z} = \frac{V_{I,1}}{\Delta Z_1} (\cos \Delta Z_1 - 1) \quad \text{Eq.A4.2}$$

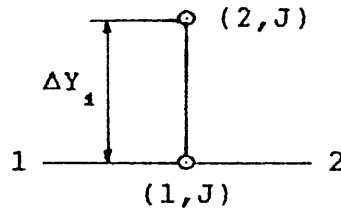
Although Eq.A4.2 is the formal form of a Neumann BC, the implementation of this BC into the computer program is both more

convenient as well as economical by using Eq.A4.1. The other BC along 2-3 is

$$\frac{\partial V}{\partial z} = \frac{V_{I, J_{\max}+1}}{\Delta Z_{J_{\max}}} (1 - \cos \Delta Z_{J_{\max}})$$

A4.2 Non - Zero W Neumann Boundary Condition

Along the boundaries 1-2 and 3-4 in Fig.6.8, the BC for W velocity is also of the non-zero Neumann type. To derive these BCs, consider the bottom boundary 1-2 with nodes (1,J) and (2,J) as shown in the diagram below.



From Eq.6.11,

$$W_{1,J} = -e^{\frac{2x}{Re}} \sin \frac{\pi}{2} \cos z$$

$$W_{2,J} = -e^{\frac{2x}{Re}} \sin \left(\frac{\pi}{2} + \Delta Y_1 \right) \cos z$$

The above equations would give the ratio of $W_{2,J}$ to $W_{1,J}$ as

$$\frac{W_{2,J}}{W_{1,J}} = \frac{\sin \left(\frac{\pi}{2} + \Delta Y_1 \right)}{\sin \frac{\pi}{2}} = \cos \Delta Y_1 \quad \text{Eq.A4.3}$$

$\frac{\partial W}{\partial y}$ in discretized form is $\frac{\partial W}{\partial y} = \frac{1}{\Delta Y_1} [W_{2,J} - W_{1,J}]$ which after some algebraic manipulations, using Eq.A4.1, can be cast as

$$\frac{\partial W}{\partial y} = \frac{W_{1,J}}{\Delta Y_1} (\cos \Delta Y_1 - 1) \quad \text{Eq.A4.4}$$

It is interesting to note that both non-zero Neumann BCs (i.e. Eq.A4.2 & Eq.A4.4) are similar and reduce to exactly the same form for a uniform grid. Along the top boundary 3-4, one obtains

$$\frac{\partial W}{\partial y} = \frac{W_{I_{\max},J+1}}{\Delta Y_{I_{\max}}} (1 - \cos \Delta Y_{I_{\max}})$$

Similarly, the $\frac{\partial W}{\partial y}$ Neumann BC is implemented using the more straightforward form of Eq.A4.3 rather than Eq.A4.4.