Discrete Applied Mathematics 370 (2025) 57–70

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Coherent domains and improved lower bounds for the maximum size of Condorcet domains

Alexander Karpov [a,b], Klas Markström [c,*], Søren Riis [d], Bei Zhou [e,1]

[a] HSE University, Russia
[b] Institute of Control Sciences, Russian Academy of Sciences, Russia
[c] Umeå University, Sweden
[d] Queen Mary University of London, United Kingdom
[e] Imperial College London, United Kingdom

ARTICLE INFO

Article history:
Received 27 August 2024
Received in revised form 21 February 2025
Accepted 2 March 2025
Available online 14 March 2025

Keywords:
Condorcet domains
Preference orders
Majority voting
Catalan numbers

ABSTRACT

In this paper, we study Condorcet domains, sets of linear orders from which majority ranking produces a linear order. We introduce a new class of Condorcet domains, called *coherent* domains, which is natural from both a voting theoretic and combinatorial perspective. After studying the properties of these domains we introduce set-alternating schemes. This is a method for constructing well-behaved coherent domains. Using this we show that, for sufficiently large numbers of alternatives n, there are coherent domains of size more than 2.1973^n. This improves the best existing asymptotic lower bounds for the size of the largest general Condorcet domains.
© 2025 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

1. Introduction

Ranking by majority voting is one of the most well-studied problems in voting theory. The basic set up is that each voter ranks a set of alternatives, i.e. gives a linear order on the alternatives, and then a global ranking is produced by ranking each pair of alternatives according to the majority vote on their pairwise ranking. We here assume that the number of voters is odd. As Condorcet showed in 1785 [8] this may not lead to a linear, or even partial, order on the alternatives. However, if the set of linear orders used by the voters is well-behaved we will get a linear order as the result. A *Condorcet domain* is a set of linear orders such that whenever the voters choose their rankings from the domain, majority voting will lead to a linear order.

The simplest non-trivial model for the possible rankings of a collection of alternatives is perhaps the single-peaked domain. This domain was introduced formally in [5] and corresponds to a society in which all voters have a common political axis, each voter has a preferred position, the "peak", on this axis and on each side of the peak ranks alternatives according to how close to the peak they are. Black [5] showed that on any domain of this type, majority voting will select a unique winner, and in fact produce a transitive ranking of all alternatives. In terms of modeling the behavior of a group of voters the single-peaked domain provides a sensible first approximation of, for example, a society where opinions correlate with a left–right or a liberal-conservative axis. It is however clearly not a universal model of such opinions. A second distinct model is given by the single-dipped domains, in which voters instead have a least preferred position on a

* Corresponding author.
E-mail addresses: akarpov@hse.ru (A. Karpov), klas.markstrom@umu.se (K. Markström), s.riis@qmul.ac.uk (S. Riis), bei.zhou@imperial.ac.uk (B. Zhou).
[1] Authors are listed in alphabetical order.

https://doi.org/10.1016/j.dam.2025.03.007
0166-218X/© 2025 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/).

common axis and rank alternatives higher when they are further away from this position. Up to relabeling of alternatives there is exactly one maximal single-peaked domain for n alternatives, known as Black's single-peaked domain, and one maximal single-dipped domain. Both have size 2^{n-1}, where n is the number of alternatives, and are in fact each other's duals, i.e. each order in one appears reversed in the other.

In real-life situations the collection of opinions is often not strictly determined by a common political axis and modeling requires more flexible options than Black's single-peaked domain, and the single-dipped domain. Arrow [4] introduced one such class by requiring that the restriction to any triple of alternatives must be single-peaked in Black's sense, but letting go of the common political axis. These domains are now known as Arrow's single-peaked domains and have been studied in great detail. Slinko [31] showed that maximal domains of this type have size 2^{n-1}, just as Black's, and they are *minimally rich*, meaning that every alternative is ranked first in some order.

A common generalization of the single-peaked and single-dipped domains is given by the peak-pit domains. These domains follow the set-up given by Arrow and require the restriction to each triple of alternatives to be either single-peaked or single-dipped in Black's sense. The class of peak-pit domains is much larger and structurally diverse than the Arrow's single-peaked domains, giving the modeler the ability to match a wider range of groups or societies. However, this diversity also means that general peak-pit domains do not have the unified social interpretation that the single-peaked domains have. Peak-pit domains have been studied in numerous papers, see e.g. [7,24], and within the class one finds interesting trade-offs between different forms of diversity (e.g. local diversity studied in [19]). It is easy to show that if a peak-pit domain is minimally rich then it is an Arrow's single-peaked domain. However, if we consider domains which are not minimally rich we find examples which are much larger than Arrow's domains, thereby giving more diversity in terms of the total number of distinct opinions. The first such examples were found by [13], and were only larger than Arrow's domains by a linear factor, but by now several constructions [14,20] have given exponentially larger domains. Most such examples have been relatively complicated to describe and analyze.

Our focus is a class of domains which have a societal axis (an underlying linear order on the alternatives) and relative to that axis are defined using only the *never conditions* $1N3$ and $3N1$. The condition $1N3$ for a triple (a, b, c) where $a < b < c$ on the axis implies that the domain never ranks a last of the three, and $3N1$ means that c is never placed first. As we will see this class of domains is mathematically natural, and has a natural social choice interpretation. Given a societal axis the never conditions $1N3$ and $3N1$ can both be seen as expressing a weak form of agreement with the ranking on the axis. Consequently, this can be used to model a relatively homogeneous society where the population mostly agrees with a common societal axis.

There are some exceptional domains in the class. If we have only $1N3$ for all triples we get a unique Arrow's single-peaked domains, which is minimally rich, with only two alternatives ranked last. Dually, the domain with only $3N1$ has only two top alternatives and all alternatives are ranked last in some order. However, for all other domains in this class the set of top ranked alternatives, and last ranked alternatives, are both strict subsets of the alternatives, often quite small. So, in this type of homogeneous societies, or domains, there is little diversity among the highest and lowest ranked alternatives. Strikingly, as our results will show, in terms of domain size some of these seemingly austere societies provide a larger variety of opinions than the single-peaked domains. Though here the source of variation is a diversity of alternatives ranked near the middle of the societal axis, rather than a variety of potential top or bottom alternatives.

Finally, in order to simplify the structure of our domains further, and allow us to give a lower bound for how large coherent domains can become, we restrict our focus to the subclass of *set-alternating* domains. As we will show, this class contains domains larger than any previously known, both when seen as peak-pit domains and more generally as Condorcet domains.

1.1. Related literature

The search for large well-behaved domains has a long history. Abello and Johnson [2] showed that there are Condorcet domains, i.e. domains on which pairwise majority voting leads to a transitive ranking of the candidates, which are larger than 2^{n-1}, the size of Black's single-peaked domain, and asked how large such domains can be. This question was reiterated in the survey [21], which listed several fundamental unsolved problems in mathematical social science. The first of which is "What is the largest size of a set of linear preference orders for n alternatives such that majority voting is transitive when each voter chooses his preferences from this set?". The problem is still unsolved. The most recent surveys are [11,17,28].

Johnson [16], and later Craven [6], conjectured that a Condorcet domain of maximum size contains at most 2^{n-1} linear orders. Here 2^{n-1} is the maximum cardinality in several well-known domain classes, such as the single-peaked, single-dipped, and group-separable domains. After Craven's paper, a new counterexample on five alternatives to the conjecture was given in [12]. The examples in [2] were not well known at the time. Fishburn [13] generalized the new counterexample by introducing an alternating scheme resulting in what is now called Fishburn's domains. These domains mix never-top and never-bottom triples and have a size of order $n \times 2^n$ [15].

Fishburn [13] conjectured that among Condorcet domains such that for any triple of alternatives do not satisfy a never-middle condition the alternating scheme provides domains of maximum cardinality.

The conjecture posed by Fishburn is true for three to five alternatives [13], and six alternatives [14]. In [15] it was claimed to be true for seven alternatives, though without including a proof. This was later proven independently in [3]. For eight alternatives there is a counterexample to Fishburn's conjecture [23]. A version of Fishburn's conjecture where

the domain is also required to have maximum width has also been studied and was eventually refuted in [7] for $n = 42$ alternatives, and in [20] for 34 alternatives. What smallest n is for which the size of Fishburn's alternating domain is exceed in this class remains an open problem.

For a sufficiently large n the lower bound on the size of maximum Condorcet domains has been increased to 2.1708^n [13] and later to 2.1890^n [20].

In this paper we present a sequence of Condorcet domains that for a sufficiently large n have a size exceeding 2.1973^n, improving the existing lower bounds. The value of this result lies not only in improving the lower bound but also in presenting a well-structured domain that does not depend on recursive constructions. Previous cited lower bounds, were based on the replacement scheme that was iteratively applied, starting with a carefully chosen domain on a small number of alternatives.

1.2. Overview of this paper

The structure of the paper is as follows. Section 2 contains notation and definitions used in the paper. We discuss general domains defined by the $1N3$, $3N1$ never conditions in Section 3. Section 4 introduces set-alternating schemes and discusses the structure of the domains these generate. Section 5 contains the analysis of the asymptotic growth rate of some of the largest domains in this class. We conclude in Section 6 with some additional comments and open problems.

2. Preliminaries

Let a finite set $X = [n] = \{1, \dots, n\}$ be the set of alternatives. Let $\mathcal{L}(X)$ be the set of all linear orders over X. Each agent $i \in N$ has a preference order P_i over X (each preference order is a linear order). For brevity, we will write preference orders as strings, e.g. $12 \dots n$ means 1 is the best alternative, n is the worst. We let α denote the order $123 \dots n$.

A subset of preference orders $\mathcal{D} \subseteq \mathcal{L}(X)$ is called a *domain* of preference orders. A domain $\mathcal{D}$ is a *Condorcet domain* if whenever the preferences of all agents belong to the domain, the majority relation of any preference profile with an odd number of agents is transitive. A Condorcet domain $\mathcal{D}$ is *maximal* if every Condorcet domain $\mathcal{D}' \supseteq \mathcal{D}$ (on the same set of alternatives) coincides with $\mathcal{D}$. In our paper we are focused on maximal Condorcet domains unless otherwise specified.

Definition 1. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two domains of equal cardinality on sets of alternatives X_1 and X_2, respectively. We say that domains $\mathcal{D}_1$ and $\mathcal{D}_2$ are *isomorphic* if there are a bijection $\psi : X_1 \to X_2$ and a bijection $\sigma : \mathcal{D}_1 \to \mathcal{D}_2$ such that for each $x \in \mathcal{D}_1$ we have $\sigma(x) = x^\psi$, where x^ψ is an order with permuted alternatives according to ψ. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are isomorphic, we write $\mathcal{D}_1 \cong \mathcal{D}_2$.

The *dual* of a partial order R_1 is the partial order R_2 for which xR_2y if and only if yR_1x. The dual of a domain $\mathcal{D}$ is the domain consisting of the dual of each order in $\mathcal{D}$.

A *societal axis* for a domain is a designated linear order on the set of alternatives. This is usually assumed to correspond to some organizing principle for the alternatives, e.g. a left–right political scale. The societal axis is not necessarily an element of the domain, e.g. for some domains which are not maximal domains.

Sen [30] proved that a domain is a Condorcet domain if the restriction of the domain to any triple of alternatives (a, b, c) satisfies a never condition. A never condition can be of three forms *xNb*, *xNm* *xNt*, referred to as a never bottom, a never middle, and a never top condition respectively. Here x is an alternative from the triple and *xNb*, *xNm*, and *xNt* means that x is not ranked last, second, or first respectively in the restricted domain. Fishburn noted that for domains with a societal axis, never conditions can instead be described as *iNj*, $i, j \in [3]$. *iNj* means that ith alternative from the triple according to societal axis does not fill in jth place within this triple in each order from the domain. For example restriction *abc* to triple $a, b, c \in [n]$, $a < b < c$ satisfies never conditions $1N2$, $1N3$, $2N1$, $2N3$, $3N1$, $3N2$, but violates never conditions $1N1$, $2N2$, $3N3$.

A set N of never conditions is *complete* if it assigns at least one never condition to every triple of alternatives. Sen's result states that every Condorcet domain satisfies some complete set of never conditions. Given a complete set of never condition we let $\mathcal{D}(N)$ denote the set of all linear orders which satisfy the never conditions in N.

A Condorcet domain $\mathcal{D}$ is *connected* if, given any two orders from the domain, the second order can be obtained from the first by a sequence of transpositions of neighboring alternatives such that all orders generated by this sequence belong to the domain. A Condorcet domain has *maximal width* if it contains a pair of completely reversed linear orders. A Condorcet domain $\mathcal{D}$ is *unitary* if it contains the order $\alpha = 123 \dots n$.

Following [31] we say that a Condorcet domain is *copious* if the restriction to any triple has size 4, or equivalently every triple, satisfies a single never condition. A domain is *ample* if the restriction to any pair has size two. Every copious Condorcet domain is ample. We will make us of the following simple lemma:

Lemma 1. *A copious Condorcet domain $\mathcal{D}$ is maximal.*

Proof. Assume that $\mathcal{D}$ is not maximal and that u is an order which can be added to $\mathcal{D}$ to form a larger domain $\mathcal{D}'$. The set of never conditions satisfied by orders in $\mathcal{D}'$ must be a subset of the never conditions satisfied by orders in $\mathcal{D}$. But, since each triple satisfies exactly one never condition in $\mathcal{D}$ this means that $\mathcal{D}'$ cannot satisfy a complete set of never conditions, which contradicts that $\mathcal{D}'$ is a Condorcet domain. □

A domain which satisfies a never condition of the form *xNb* for every triple is called *Arrow's single-peaked domain*. Note that since never-bottom conditions are mapped to never-bottom conditions by isomorphism one may rewrite this as $yN3$ never conditions, using e.g. any order from the domain as a societal axis.

A domain $\mathcal{D}$ is a *peak-pit* domain if, for each triple of alternatives, the restriction of the domain to this triple is either single-peaked ($iN3$ restriction), or single-dipped ($iN1$ restriction).

A domain $\mathcal{D}$ is called a *Fishburn domain* if it satisfies the never conditions of the *alternating scheme* [13]: there exists a linear ordering of alternatives $a_1, \ldots, a_m$ such that for all i, j, k with $1 \leq i < j < k \leq m$ the restriction of the domain to the set $\{a_i, a_j, a_k\}$ is single-peaked with axis $a_i a_j a_k$ if j if is even (odd), and it is single-dipped with axis $a_i a_j a_k$ if j is odd (even). Note that there are either one or two Fishburn domains depending on the parity of n. The dual of a Fishburn domain is also a Fishburn domain.

Fishburn [13] introduced the function

$$f(n) = \max\{|\mathcal{D}| : \mathcal{D} \text{ is a Condorcet domain on a set of } n \text{ alternatives}\}.$$

Similarly Karpov and Slinko [20] introduced the function

$$h(n) = \max\{|\mathcal{D}| : \mathcal{D} \text{ is a peak-pit Condorcet domain on a set of } n \text{ alternatives}\}.$$

and also showed that $f(n) \in \Omega(2.1890^n)$ and $h(n) \in \Omega(2.1045^n)$, where $F(n) \in \Omega(g(n))$ means that $\lim_{n\to\infty} \frac{F(n)}{g(n)} > 0$.

Every domain $\mathcal{D}$ is associated with a graph $G_{\mathcal{D}}$ [27]. The set of linear orders from $\mathcal{D}$ is the set of vertices $V_{\mathcal{D}}$ and for two orders $u, w \in \mathcal{D}$ we draw an edge between them if $\mathcal{D}$ does not contain order x that is between u, w in terms of Kemeny betweenness (which means that x agrees with all binary comparisons on which u, v agrees). Puppe and Slinko [27] proved that for each Condorcet domain $\mathcal{D}$ the graph $G_{\mathcal{D}}$ is a median graph (as defined by [25]). If Condorcet domain $\mathcal{D}$ is connected, then in the associated domain each edge represents one swap of neighboring alternatives.

For set complements we will use the notation $\overline{A} = [n] \setminus A$, where $A \subseteq [n]$.

3. Coherent domains

Our focus in this section is the class of domains which have a societal axis and relative to that axis are defined using only the never conditions $1N3$ and $3N1$. As noted in the previous section we can, as long as we only consider domains up to isomorphic equivalence, assume that the social axis is the order $\alpha = 12\ldots n$. Thus we will from now on assume that domains are unitary unless we specify otherwise.

We call this class of domains the *coherent* domains, motivated by its natural social choice interpretation. Given a societal axis the never conditions $1N3$ and $3N1$ can both be seen as expressing a weak form of agreement with the ranking on the axis. The condition $1N3$ for a triple $a < b < c$ implies that the domain never ranks a last of the three, and $3N1$ means that c is never placed first. Consequently, this can be used to model a relatively homogeneous society where the population mostly agrees with a common societal axis.

Note that given a societal axis we may relabel the alternatives so that each is labeled by their rank along the axis. This gives an isomorphism from the original domain to one in which the societal axis is $12\ldots n$, and the domain is still defined by $1N3$ and $3N1$ never conditions. Hence every coherent domain is isomorphic do a unitary coherent domain and we will in the rest of the paper assume that all domains are unitary, unless we explicitly specify something else.

There are some exceptional domains in the class. If we take the complete set of never conditions N with $1N3$ for all triples, then $\mathcal{D}(N)$ is an Arrow's single-peaked domain, by [32] this domain is copious and minimally rich, with only two alternatives ranked last. Dually, the domain with only $3N1$ has only two top alternatives and all alternatives are ranked last in some order. However, for all other domains in this class the set of top ranked alternatives, and last ranked alternatives, are both strict subsets of the alternatives, often quite small. So, in this type of homogeneous societies, or domains, there is little diversity among the highest and lowest ranked alternatives. Strikingly, as our results will show, in terms of domain size some of these seemingly austere societies provide a larger variety of opinions than the single-peaked domains. Though here the source of variation is a diversity of alternatives ranked near the middle of the societal axis, rather than a variety of potential top or bottom alternatives.

All maximal coherent domains have a well-structured common part, which we will refer to as the *Fibonacci domain*. The Fibonacci domain is the unitary domain of all orders which satisfy both of the two never conditions 1N3 and 3N1 for each triple.

Proposition 1. *Consider the unitary Fibonacci domain:*

1. *For every triple abc, listed according to their rank in α, there are orders in the Fibonacci domain whose restriction to that triple are abc, acb, and bac.*
2. *The size of the Fibonacci domain on n alternatives asymptotically grows like $\frac{5+\sqrt{5}}{10}(\frac{1+\sqrt{5}}{2})^n$.*

Proof. Linear orders that satisfy the two never conditions 1N3 and 3N1 for every triple correspond to permutations that avoid the set of patterns {231, 321, 312}. Such permutations are called known as free permutations and are counted by

the $(n+1)$th Fibonacci number [26]. From the asymptotic growth rate of the Fibonacci sequence we get part 2 of the proposition.

Let abc be a triple as in the proposition. The restriction of α to that triple gives the restriction abc. Let β_1 be the order obtained from α by moving b to the first ranked position, and keeping the relative ranks of all other alternatives as they were. This order satisfies both $3N1$ and $1N3$ for all triples, so it belongs to the domain and has restriction bac to our triple. In the same way the order β_2 obtained from α by moving b to the last ranked position, belongs to the domain and give the restriction acb. □

Going in the other direction we can also provide a concrete upper bound for the maximum possible size of a coherent domain.

Proposition 2. *A maximal coherent domain on n alternatives has size at most* $\frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{n^{1.5}\sqrt{\pi}}$.

Proof. The restriction of a coherent domain to any triple cannot be of the form 321. Hence the domain is a collection of 321-avoiding permutations. It is well known that the number of 321-avoiding permutations of length n is given by the Catalan numbers [22]. □

Lemma 2. *Given* $1 \leq k < n$ *let* $A = [k]$ *and* $B = [n] \setminus A$. *If* $\mathcal{D}$ *is a maximal coherent domain on n alternatives then the restrictions of* $\mathcal{D}$ *to A and B are also maximal coherent domains.*

Proof. Let $\mathcal{D}$ be a unitary maximal coherent domain on n alternatives. Let $\mathcal{D}_1$ be the restriction of $\mathcal{D}$ to $A = \{1, 2, \ldots, k\}$ and $\mathcal{D}_2$ the restriction of $\mathcal{D}$ to $B = \{k+1, \ldots, n\}$.

Assume that at least one of $\mathcal{D}_1$ and $\mathcal{D}_2$ is non-maximal, say $\mathcal{D}_1$. Let a be an order which can be added to $\mathcal{D}_1$ and b any order from $\mathcal{D}_2$. Let c be the order ab. Note that a triple which contains elements from both A and B will never have the highest element ranked last, and never the lowest element ranked first. So, c satisfies both 3N1 and 1N3 for those triples. For triples from B, c satisfies the same never conditions as $\mathcal{D}$. For triples from A c will satisfy at least one never condition from $\mathcal{D}$, since a can be added to $\mathcal{D}_1$. So, c can be added to $\mathcal{D}$, contradicting that $\mathcal{D}$ is maximal.

The same argument works if instead $\mathcal{D}_2$ is not maximal. Hence both $\mathcal{D}_1$ and $\mathcal{D}_2$ must be maximal. □

Note that the lemma implies that the restriction to any interval on the societal axis is a maximal domain.

All copious Condorcet domains are ample, but ample domains are not always copious [3]. For maximal coherent domains this is however an equivalence.

Lemma 3. *An ample maximal coherent domain D is copious.*

Proof. First note that since the set of never conditions satisfied by D are a subset of those for the Fibonacci domain, the Fibonacci domain is a subset of D.

By Proposition 1 the Fibonacci domain has restrictions to any triple which contain the three orders 123, 132, 213. In each of these orders 1 and 3 do not change their relative order. Hence an ample domain must contain one more order for each triple. This gives four orders for each triple, making the domain copious. □

Lemma 4. *A maximal coherent domain on* $n \geq 3$ *alternatives has at least two terminal alternatives and at least two alternatives which are ranked first. The alternatives 1 and 2 are among the first ranked alternatives,* $n-1$ *and n are among the terminal alternatives.*

Proof. The coherent domain contains the Fibonacci domain and the first statement is true for that domain.

The second statement follows by restricting the domain to $[n-1]$ and $[n] \setminus 1$. □

Lemma 5. *Let a be an order from a coherent domain on n alternatives. The values ranked before 1 form a monotone increasing sequence and the values ranked after n form a monotone increasing sequence.*

If n is ranked before 1 then no alternative is ranked between them. If there exists an order containing $n1$ *then it is unique.*

Proof. If a and b are ranked before 1 and $a < b$ on the societal axis then $ba1$ is a 321 pattern, which violates both 1N3 and 3N1. The same argument works for nba.

Note that $na1$ is a 321 pattern for every a. We cannot have two orders which restrict to $xn1$ and $n1x$, since the standard order α restricts to $1xn$ and would form a Latin square with the other two orders. Hence the set of alternatives ranked before $n1$, and the set ranked after $n1$, is unique. Since both sets form increasing sequences we get a unique order. □

Proposition 3. *An ample maximal coherent domain on n alternatives has size at least* 2^{n-1}.

Proof. This statement is true for $n \leq 4$. Suppose it is true for all $k \leq n-1$, let us prove for n.

Assume that the domain $\mathcal{D}$ is an ample maximal coherent domain on n alternatives. For each $k < n$ there are partition of $[n]$ on $A_k = [k]$ and $B_k = [n] \setminus [k]$ and an order $c_k = ab$, where a belongs to the maximal coherent domain on set A_k, b belongs to the maximal coherent domain on set B_k. Because of ampleness, for each k maximal coherent domain on set A_k contains order q with $k1$. For $k' < k$, order q is not equal to restriction of $c_{k'}$ to A_k. Thus, from the induction hypothesis each partition with $1 \leq k < n$ contains at least 2^{n-k-1} orders. Additionally from ampleness, we have an order with $n1$ that does not belong to described partitions. Summing, we get $1 + \sum_{k=1}^{n-1} 2^{n-k-1} = 2^{n-1}$. Thus, domain $\mathcal{D}$ contains at least 2^{n-1} orders. □

Proposition 4. *A maximal coherent domain D is connected.*

Proof. Let us consider a unitary coherent domain $\mathcal{D}$. We first prove that for any order $a = a_1 a_2 \dots a_n$ not equal to α there is a pair of neighboring alternatives $a_i a_{i+1}$, such that $a_i > a_{i+1}$, which we can transpose and still have an order in D.

The order a has at least one pair of neighboring alternatives $a_i a_{i+1}$ such that $a_i > a_{i+1}$, since otherwise the a_i:s would form a monotone increasing sequence, meaning that $a = \alpha$.

Consider the order $b = a_1 \dots a_{i+1} a_i \dots a_n$ obtained by swapping the ranks of a_i and a_{i+1}. The only never conditions which can prohibit b from belonging to D are those for triples containing both a_i and a_{i+1}, as all others are unaffected by the swap.

Let x be an alternative with lower rank than a_i in a. Since a_i is not the smallest of the three, as $a_i > a_{i+1}$, the triple does not violate $1N3$ in b. Neither does it violate $3N1$, since x was ranked first in a as well. If instead x has higher rank than a_{i+1} in a, a symmetrical argument shows that both never conditions are satisfied. Thus, the order b belongs to the domain $\mathcal{D}$.

Each transposition moves some alternative closer to its rank in α, so after a finite number of transpositions we must reach α, for any A □

4. Set-alternating schemes

Next we will focus on domains generated by what we call set-alternating schemes. These domains are special cases of coherent domains and, as we will show, they provide examples of Condorcet domains which, for large enough n, are larger than all previously studied domains.

Let us start our analysis by defining the class of set-alternating schemes.

Definition 2. Given a subset $A \subseteq [n]$ consider the following set $N(A)$ of never conditions. For triples $i < j < k$ with $j \in A$ assign the never condition $1N3$. For triples $i < j < k$ with $j \notin A$ we assign the never condition $3N1$. This is the *set-alternating scheme* defined by A.

We let $\mathcal{D}(A)$ denote the Condorcet domain $\mathcal{D}(N(A))$ and say that $\mathcal{D}(A)$ is the domain generated by A. Let $f_n(A)$ denote the cardinality of $\mathcal{D}(A)$.

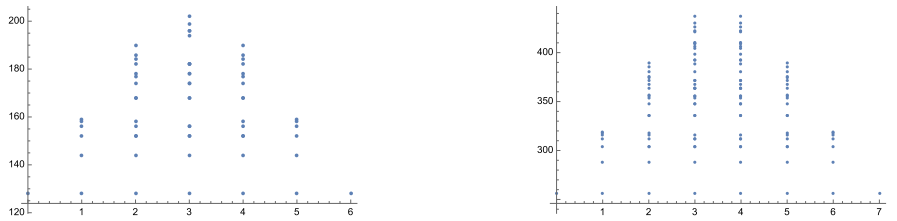
Given that our definition of set-alternating schemes could easily be modified to use any other pair of never conditions, one may ask why we focus on the particular pair $1N3$, $3N1$? If one tests out all pairs of peak-pit never conditions on all sets A for some small n, say $n = 5$, one finds that most pairs do not generate a copious domain for every A, but a small set of pairs do. These pairs fall into four families.

The first are the two domains which assign the same never condition $1N3$ to every triple, or $3N1$ to every triple. These domains are single-peaked and single-caved respectively, and have size at most, in fact exactly, 2^{n-1} by [29] and [31]. Here the domain does not depend on A, but it does depend on the choice of never condition. The second family are those which use two never-bottom conditions, or two never-top conditions. This is a subset of the class of Arrow's single-peaked domains, and their duals. These also have size 2^{n-1} by [31], but here the domain does depend on A. The class of all Arrow's single-peaked domains has been studied in detail in [31]. The third family uses the pair $2N1/2N3$. If we use this pair and take A to be the set of odd numbers, or even numbers, we recover Fishburn's alternating scheme. Generalized versions of Fishburn's domains are discussed in [18,33]. The fourth family, which we will show to be copious for all n, is given by the pair $1N3$, $3N1$ and is the focus of the remainder of the paper.

Let us consider some examples. If $A = [n]$, then all triples are assigned the $1N3$ never condition. This gives an Arrow's single-peaked domain, but it is not Black's single-peaked since the domain does not contain two mutually reversed orders. This is the only Arrow's single-peaked domain given by a set-alternating scheme. It is also the only coherent Arrow's single-peaked domain. By [31] this domain is maximal and has cardinality 2^{n-1}. For $A = \emptyset$ we get a domain which can be related to that for the empty set via the following lemma.

Lemma 6. *Define the* reverse complement *A^* of A to be $A^* = n + 1 - \overline{A}$, where the subtraction means the set of numbers $n + 1 - i$ for all $i \in \overline{A}$.*

Let C_1 be the domain generated by the set A and C_2 the domain generated by A^* then C_2 is obtained from C_1 by first reversing all orders and then reversing the names of the alternatives.



(a) $n = 8$ (b) $n = 9$

Fig. 1. Domain size and set size for all subsets of $\{2, \ldots, n-1\}$.

Proof. Assume that we have a triple (a, b, c) satisfying a Fishburn type never condition *xNp*.

Note that when all orders are replaced by their duals, every never condition *xNp* is replaced by $xN(4-p)$.

Similarly, when the list of names is reversed a never-condition *xNp* is replaced by $(4-x)Np$. Here the triple (a, b, c) is transformed to $(n+1-c, n+1-b, n+1-a)$.

So the joint reversal interchanges $3N1$ and $1N3$, and replaces (a, b, c) by $(n+1-c, n+1-b, n+1-a)$. □

For $n = 4$, $\mathcal{D}(\{2\})$ is isomorphic to Fishburn's domain. In particular, $\mathcal{D}(\{2\})$ is Fishburn's domain relative to the societal axis 2143. For $n > 4$ there is no A corresponding to Fishburn's alternating scheme.

Here we may ask which domain sizes set-alternating schemes more generally generate. In Fig. 1 we plot the pair (set size, domain size) for all subsets of $\{2, \ldots, n-1\}$ for $n = 8$ and 9. We exclude 1 and n from A since they cannot be midpoints of a triple and hence do not affect which domain we generate. As we can expect from Lemma 6 the size distribution is symmetrical. The lowest domain sizes in the figure are of the form 2^{n-1} and in Corollary 2 we show that this is the minimum possible size. We also see many schemes which lead to much larger domains and we can identify the maximum ones.

Definition 3. $\mathcal{D}(B_n)$ is the domain generated by the *odd set-alternating scheme* if

$$B_n = \{2, 3, 5, \ldots, n-3+p_n\},$$

where $p_n = (n \bmod 2)$.

This scheme produces the largest domain sizes for small n and we will later investigate their growth rate. For even n, the set B_n is equal to its reverse complement. For odd n the reverse complement consists of the even numbers $B_n^* = \{2, \ldots, n-3\}$, which by Lemma 6 generates a domain of the same size as B_n. If we extend this set of even numbers by $n-1$ we get a new scheme which we will use in our asymptotic analysis.

Definition 4. $\mathcal{D}(A_n)$ is the domain generated by the *even set-alternating scheme* if

$$A_n = \{2, 4, 6, \ldots, n-2+p_n\},$$

where $p_n = (n \bmod 2)$.

We will refer to the scheme given by taking B_n^* for odd n and $A_n \setminus (n-2)$ for even n as the *truncated even set-alternating scheme*.

Fig. 2 displays the graph associated to the even set-alternating scheme for $n = 6$.

In Table 1 we show the sizes of the domains generated by these schemes and Fishburn's alternating scheme, each computed using the Condorcet Domain Library (CDL) from [36,37].

Regarding the maximum domain size we conjecture the following.

Conjecture 1. *For each $n \geq 5$, the maximum size of a set-alternating domain is the size of the set-alternating domain.*

This conjecture has been verified computationally for $n \leq 24$. As we can see for $n < 16$ Fishburn's alternating scheme leads to larger domains than the odd set-alternating scheme.

Proposition 5. *The domain generated by a set-alternating scheme is a copious peak-pit maximal Condorcet domain.*

Proof. For $n \leq 3$ each maximal set-alternating domain is a maximal copious peak-pit Condorcet domain. Let us assume that it is true for $n-1$ alternatives.

First, let us recall that in a copious domain every triple of alternatives satisfies exactly one never condition. By Lemma 2 a copious domain is maximal, so we only need to show that the domain is copious,

Let C_1 be the restriction of C to $[n] \setminus 1$. The domain C_1 is also a set-alternating domain and hence copious by our induction assumption. If we add 1 as the highest ranked element in each order of C_1 we get a new domain C_1^+ on the set

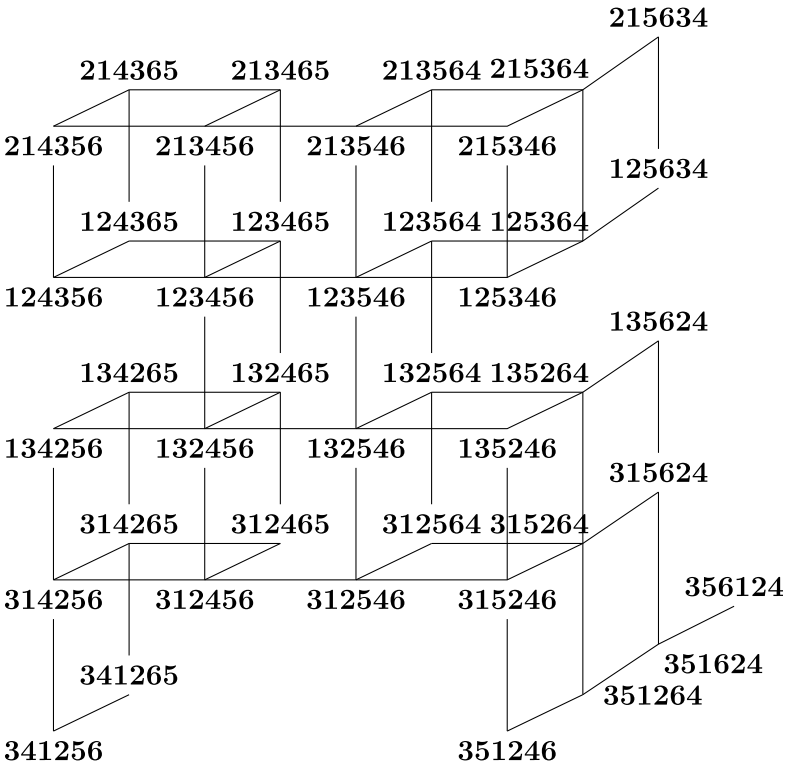


Fig. 2. The graph $G_{\mathcal{D}}$ for the domain $\mathcal{D}(A_6)$ generated by the even set-alternating scheme for $n = 6$.

Table 1
Domain sizes for the *odd* scheme, the *even* scheme, and Fishburn's alternating scheme. The *odd* scheme surpasses Fishburn's alternating scheme for $n \geq 16$.

n	*odd* 1N33N1	*even* 1N33N1	Truncated *even* 1N33N1	Alternating scheme
4	9	9	8	9
5	19	18	19	20
6	42	42	39	45
7	91	84	91	100
8	202	199	190	222
9	437	398	437	488
10	973	950	922	1069
11	2102	1900	2102	2324
12	4690	4554	4464	5034
13	10 122	9108	10 122	10 840
14	22 617	21 884	21 587	23 266
15	48 779	43 768	48 779	49 704
16	**109 104**	105 323	104 322	105 884
17	**235 197**	210 646	235 197	224 720
18	**526 441**	507 398	503 966	475 773
19	**1 134 474**	1 014 796	1 134 474	1004 212
20	**2 540 586**	2 446 022	2 434 088	2115 186

of alternatives $[n]$. Each order in C_1^+ is compatible with both $1N3$ and $3N1$ never conditions for triples $1, i, j$, and all other triples satisfy the never condition given by A. Hence $C_1^+ \subset C$, and since the restriction of C_1^+ to any triple not containing 1 has size four this is true for C as well. This shows that any triple not containing 1 satisfies the condition required for C to be copious.

We can repeat this argument for the restriction of C to $[n] \setminus n$, getting a subdomain C_n+, where n is ranked last in every order, which shows that every triple not containing n has a restriction of size 4. It remains to show that triples of the form $1, j, n$ also give restrictions of size 4.

Since C_1 is copious, and hence ample, C_1^+ restricted to a triple $1, j, n$ contains both $1jn$ and $1nj$. Similarly C_n^+ contains both $1jn$ and $j1n$. So the restriction of C to this triple contains $1jn, 1nj, j1n$.

The domain C contains orders which rank the alternatives from $\overline{A}$ highest, then $n1$, and finally all alternatives from A. If $j \in A$, then the restriction of C to $1, j, n$ contains $n1j$, giving a fourth order and the triples satisfy only the never condition

$1N3$. If $j \notin A$, then the restriction of C to $1, j, n$ contains $jn1$, giving a fourth order and the triple satisfies only the never condition $3N1$.

So, we have shown that the domain C is copious, and thereby a maximal Condorcet domain. □

In addition to potentially large sizes, the set-alternating schemes generate domains with good local structure.

Corollary 1. *Each domain defined by a set-alternating scheme is connected.*

Set-alternating schemes can be used as a straightforward way of producing random examples of maximal Condorcet domains. For a given n we can construct a set A at random by including each value from X_n with probability p, independently, and then construct a domain $\mathcal{D}$ by using the set-alternating scheme given by A. For $p = \frac{1}{2}$ we get the uniform distribution for A.

Question 2. *What is the expected size of the domain $\mathcal{D}$ generated by the set-alternating scheme of a random set A, when A is generated with inclusion probability p?*

Since the domain size here is exponential in n the expected size might deviate significantly from the median size. The same question could be asked for the median.

4.1. Recursive properties of the domain sizes

First of all let us note that whether either of 1 and n is a member of A or not does not affect the size of the generated domain. Simply because neither of these can be the middle element of a triple. Next, let w be the largest element in A, and $A' = A \setminus w$.

In some of our proofs it will be convenient to work with set-alternating domains where the set of alternatives form a different interval than $[n]$. When the interval is X we let $\mathcal{D}_X(A)$ denote the domain, to make clear what the underlying interval is,

Proposition 6. *If $w = n - 1$, then $f_n(A) = 2f_{n-1}(A')$.*

Proof. For all triples $i, n-1, n$, we have the never condition $1N3$. Thus, only alternatives $n-1$ and n can be last in orders from $\mathcal{D}(A)$. Note that adding n, or $n-1$ at the end of an order, which does not use that alternative, does not violate 1N3, 3N1 conditions. Hence, the number of linear orders with n last equals $f_{n-1}(A')$, and the same holds for those with $n-1$ last. □

Proposition 7. *If $1 < w < n - 1$, then $f_n(A) = 2f_{n-1}(A) + f_{n-1}(A') - S$, where $S = \sum_{j=w-1}^{n-2} f_j(A')$.*

Proof. There are $f_{n-1}(A)$ orders, in which n is the last alternative, and there are $f_{n-1}(A)$ orders in which n is the second last alternative.

If there is an order in which two alternatives $a < b$ are ranked after n in some order then the triple a, b, n must satisfy a $1N3$ never condition. So $b \in A$ by the definition of our scheme, and a cannot be ranked last due to the never condition. If $b \neq w$ we get a contradiction, since the triple $b < w < n$ satisfies $1N3$ and so allows only w and n to be ranked last among the three.

Hence, if n is neither ranked last nor second last in an order, then the alternative w is ranked last.

Next we note that there are $f_{n-1}(A')$ orders that satisfy all conditions without alternative w, and we will now consider those orders with w added as the last ranked alternative. If an order of this form violates a never condition it must be of the form $3N1$, since otherwise the middle element of the triple must belong to A and w be the first element, which contradicts the definition of w.

If there is more than one alternative x_j ranked after n, and automatically before w, then the same argument as above shows that the alternative b ranked last before w must belong to A' and the other alternatives x_j must satisfy $x_j < b < w$. So the triples x_i, x_j, w do no violate a $3N1$ condition when w is added last. The triples $x_i < w < n$ have n ranked first, but are assigned a $1N3$ never condition since $w \in A$.

The orders which end with nw satisfy all never conditions and there are $f_{n-2}(A')$ such orders. However, these orders are already included in our count of orders with n ranked second to last. So we need to subtract their number from $f_{n-1}(A')$.

Next we consider orders β in which n is ranked third from the end, and having w ranked last violates a $3N1$ never condition. This implies that there is one element a ranked between n and w and we must have $a > w$, since otherwise this triple would have a $1N3$ never condition. So the possible values for a range from $w+1$ to $n-1$. If $a = w + j$ then we must also have the alternatives from $a+1$ to $n-1$ ranked in increasing order before n, since β satisfies all never conditions not involving w. This means that the end of our order must be of the forms $n(n-1)w$, or $(n-1)n(n-2)w$, and so on up to $(w+2)\ldots(n-1)n(w+1)w$. So there are $f_{a-2}(A')$ orders which end with aw.

This gives $f_{n-1}(A') - \sum_{j=w-1}^{n-2} f_j(A')$ orders from $\mathcal{D}(A)$ with w last, excluding those with n second last.

Adding this to our counts for orders with n last or second last gives our result. □

By Proposition 7 we have $f_n(A) \geq 2f_{n-1}(A)$, giving us a general lower bound on the size of set-alternating domains.

Corollary 2. *We have $f_n(A) \geq 2f_{n-1}(A)$ and $f_n(A) \geq 2^{n-1}$.*

Proof. We have $f_n(A) = 2f_{n-1}(A) + f_{n-1}(A') - \sum_{j=w-1}^{n-2} f_j(A')$, where $2f_{n-1}(A)$ is the number of orders in which n is last or second last, and $f_{n-1}(A') - \sum_{j=w-1}^{n-2} f_j(A')$ is the number of orders in which n is neither last nor second last. Thus, we have $f_{n-1}(A') - \sum_{j=w-1}^{n-2} f_j(A') \geq 0$ and $f_n(A) \geq 2f_{n-1}(A)$. Starting from $f_1 = 1$ we get $f_n(A) \geq 2^{n-1}$. □

We also get some information about sets which generate large domains.

Corollary 3. *If $n-1$ is a member of A then $A_1 = A \setminus n-1$ generates a domain which is at least as large as that for A.*
If 2 is not a member of A then $A_2 = A \cup \{2\}$ generates a domain which is at least as large as that for A.

Proof. Let us prove the first part. First note that $A_1 = A'$. We have $f_n(A) = 2f_{n-1}(A_1)$ and $f_n(A_1) = 2f_{n-1}(A_1) + f_{n-1}(A_1') - \sum_{j=w-1}^{n-2} f_j(A_1')$. Thus, we have $f_n(A) \leq f_n(A_1)$.

For the second part, consider the reverse complement A^*, which generates a domain of the same size as A. If 2 is not a member of A then $n-1 \in A^*$ and as we just showed the domain would have been at least as large if $n-1$ had not been a member of A^*. □

Hence the set of largest, for each n, set-alternating domains, includes domains generated by sets A which contain 2 and does not contain $n-1$.

The following proposition determines the exact sizes for some special cases.

Proposition 8. *For $k = 1$, we have $f_n(\{n-k\}) = 2^{n-1}$ and $f_n(\{k\}) = 2^{n-1}$. For $k > 1$ and $n \leq k+1$, we have $f_n(\{n-k\}) = 2^{n-1}$ and $f_n(\{k\}) = 2^{n-1}$. For $k > 1$ and $n > k+1$, we have $f_n(\{n-k\}) = 5 \cdot 2^{n-3} - 2^{n-k-2}$ and $f_n(\{k\}) = 5 \cdot 2^{n-3} - 2^{k-2}$.*

Proof. For $k = 1$, $f_n(\{n-k\}) = f_n(\{k\}) = 2^{n-1}$ follows from Proposition 6.

For $k > 1$ and $n \leq k+1$, we have $f_n(\{n-k\}) = f_n(\{k\}) = 2^{n-1}$ because it is the size of any Arrow's single-peaked domain.

From Proposition 7 for $k > 1$ and $n > k+1$ we have $f_n(\{n-k\}) = 2f_{n-1}(\{n-k\}) + 2^{n-2} - \sum_{j=n-k-1}^{n-2} 2^{j-1}$. Rearranging, we have $f_n(\{n-k\}) = 2f_{n-1}(\{n-1-(k-1)\}) + 2^{n-k-2}$.

For $k = 2$, we have $f_n(\{n-2\}) = 2^{n-1} + 2^{n-4}$. For $k = 3$, we have $f_n(\{n-3\}) = 2(2^{n-2} + 2^{n-5}) + 2^{n-5} = 2^{n-1} + 3 \cdot 2^{n-5}$. For $k = 4$, we have $f_n(\{n-4\}) = 2(2^{n-2} + 3 \cdot 2^{n-6}) + 2^{n-6} = 2^{n-1} + 7 \cdot 2^{n-6}$. Solving the recurrence, we have $f_n(\{n-k\}) = 2^{n-1} + (2^{k-1} - 1) \cdot 2^{n-k-2}$.

From Proposition 7 for $n > k+1$ we have $f_n(\{k\}) = 2f_{n-1}(\{k\}) + 2^{n-2} - \sum_{j=k-1}^{n-2} 2^{j-1}$. Having initial condition $f_{k+1}(\{k\}) = 2^k$ we get the solution $f_n(\{k\}) = 5 \cdot 2^{n-3} - 2^{k-2}$. □

Utilizing Proposition 7 once more one can find the size of the set-alternating domains with $|A| = 2$. Iteratively applying Proposition 7 one can in principle find the size of any set-alternating domain. Another special case is solved in the following proposition.

Proposition 9. *For even $n \geq 4$, we have*

$$f_n(\{2, 3, \ldots, \frac{n}{2}\}) = 2^{n-1} + \sum_{i=1}^{\frac{n}{2}-1} \sum_{j=1}^{\frac{n}{2}-1} \binom{i+j-2}{i-1} 2^{n-i-j-2}.$$

Proof. An order $c \in L(A \cup B)$ is a shuffle of two orders $a \in \mathcal{L}(A)$, $b \in \mathcal{L}(B)$, where the sets A and B are disjoint, if the restriction of c to A equals a, and the restriction of c to B equals b.

The domain $\mathcal{D}(\{2, 3, \ldots \frac{n}{2}\})$ contains all orders, that have the structure $a + b + c$, where $+$ is concatenation, $a \in D_{\{1,2,3,\ldots \frac{n}{2}-i-1\}}(\{2, 3, \ldots \frac{n}{2} - i\})$, the order b is a shuffle of two the orders $\frac{n}{2} - i + 1 \ldots \frac{n}{2}$ and $\frac{n}{2} + 1 \ldots \frac{n}{2} + j$, and $c \in D_{\{\frac{n}{2}+j+1 \ldots n-1\}}(\{\frac{n}{2} + j + 1 \ldots n\})$. This domain satisfies all never conditions that follow from the definition of the set-alternating domain. Karpov and Slinko [20] proved that domains with this structure are maximal Condorcet domains.

The number of orders which are shuffles of two orders with i and j alternatives is $\binom{i+j}{i}$. If the first alternative belongs to the second order and the last alternative belongs to the first order, then we have $\binom{i+j-2}{i-1}$.

There are $2^{n-i-j-2}\binom{i+j-2}{i-1}$ linear orders which have the described above structure and cannot be described by a smaller shuffle.

Thus, we have $f_n(\{2, 3, \ldots, \frac{n}{2}\}) = 2^{n-2} + \binom{n-2}{\frac{n}{2}-1} + 2\sum_{i=1}^{\frac{n}{2}-1} \binom{i+\frac{n}{2}-2}{i-1} 2^{\frac{n}{2}-i-1} + \sum_{i=1}^{\frac{n}{2}-1} \sum_{j=1}^{\frac{n}{2}-1} \binom{i+j-2}{i-1} 2^{n-i-j-2}$.

Simplifying, we obtain the result. □

For $n = 2, 4, 6, 8, \ldots$ we have 2, 9, 42, 194, 884, 3978, ... Here we get an asymptotic growth rate of

$$2^{n-3}\left(\frac{n}{2}-1\right) - 2^{\frac{n}{2}-2}\mathcal{O}(g^{n/2}),$$

where $g = 1.41...$. This size is within a constant factor of that for the Fishburn domains.

Proposition 6 leads to the following proposition.

Proposition 10. *For $1 < k < n$, we have $f_n(\{k, \ldots, n-1\}) = 2^{n-1}$.*

In the considered cases the set A consists of successive alternatives. All these cases are associated with relatively small domain sizes. Sets A with a large number of alternations $i \in A, i+1 \in \overline{A}$ possibly leads to a greater size. In the next section we will analyze one such case.

5. The asymptotic growth rate of the set-alternating domains

Here we analyze the even set-alternating domains $\mathcal{D}(A_n)$ and their sizes $a(n)$. In proofs we use $A_n = \{1, 2, 4, 6, \ldots, n - 2 + p_n\}$, $p_n = (n \bmod 2)$. Adding 1 to set A_n does not change the definition, but universalizes notation. From now on we let $a(n)$ denote the size of the domain generated by the even set-alternating scheme.

We partition all orders in $\mathcal{D}(A_n)$ on orders that start with elements from the set $\{1, 2\}$, orders that start from the set $\{1, 2, 3, 4\}$, but not from the set $\{1, 2\}$, orders that start from the set $\{1, 2, 3, 4, 5, 6\}$, but not from the set $\{1, 2, 3, 4\}$, etc. Orders from part k start from $[2k]$, but not from $[2(k-1)]$.

Orders from the first part start with 12, 21. There are $2a(n-2)$ such orders. Orders from the second part start with one of 1324, 1342, 3124, 3142, 3412. There are $5a(n-4)$ such orders. For $n = 6$, 18 orders from the upper side of Fig. 2 constitute the first part, 10 orders from the bottom left side of Fig. 2 constitute the second part, 14 orders from the bottom right side of Fig. 2 constitute the third part.

Lemma 7. *In all orders from the kth part of $\mathcal{D}(A_n)$ alternatives from A_{2k} are in ascending order.*

Proof. We will prove the statement by contradiction.

Suppose there is a pair of elements $i > j$ from A_{2k} and an order q from part k of $\mathcal{D}(A_n)$ such that i, j are in descending order in q and there is no $x \in A_{2k}$, $j < x < i$ such that i, x are in descending order in q. We take the closest pair in descending order. Note that i is even and $i < 2k$.

Now we will prove that all alternatives $k \leq i$, $k \neq j$ precede j in order q.

Suppose, that there is element $x \in [2k]$ between i and j in order q. If $j < x < i$, then this triple violates 1N3 and 3N1. If $x > i$, then i, j, x violates 1N3. Thus, the only possibility is $x < j$.

If there is $y > i$ such that it precedes i in order q, then i, j, y violates 1N3. Thus only elements below i precede i in order q.

If there is $z < j$ that stays after j in order q, then i, j, z violates 1N3. If there is z, $j < z < i$ that stays after j in order q, then $z \in A_{2k}$, otherwise triple violates $3N1$. In this case pair z, i is in descending order and closer, than pair j, i. There is no such z.

Thus, all alternatives $k \leq i$, $k \neq j$ precede j in order q, and order q belongs to part $i/2$, or lower. Because $2k \notin A_{2k}$ we have $i/2 < k$. We get a contradiction. □

Lemma 8. *In all orders from the kth part of $\mathcal{D}(A_n)$ alternatives from $\overline{A_{2k}}$ are in ascending order.*

Proof. Because of Lemma 7 elements from A_{2k} are in ascending order in each order from part k.

Let us prove by induction that for each $j \in [k-1]$ element $2j+1$ precede $2j$ in each order from part k.

For $j = 1$, elements $i \in \overline{A_{2k}}$ precede 2 in each order from part k. We have $i = 3$, otherwise triple $2, 3, i$ violates $3N1$. By induction hypothesis for each $l < j$, element $2l+1$ precedes $2l$ in each order from part k. let us prove for j. All $i < 2j$ precede $2j$ in each order from part k. Elements $i \in \overline{A_{2k}}$, $i > 2j$ precede $2j$ in each order from part k. We have $i = 2j+1$, otherwise triple $2j, 2j+1, i$ violates $3N1$.

Suppose there is a pair of elements $a > b$ from $\overline{A_{2k}}$ such that a, b are in descending order in order q from part k of $\mathcal{D}(A)$. Since b precede $b-1$ in order q, triple $b-1, b, a$ violates $3N1$. We get a contradiction. □

Definition 5. A sequence $a_1 a_2 \ldots a_{2k}$ of k elements u and k elements d such that for all $1 \leq j \leq 2k$ we have $|i \in [j] | a_i = u\}| \geq |i \in [j] | a_i = d\}|$ is a *Dyck word*.

Proposition 11 ([9])**.** *The number of Dyck words of size $2k$ is C_k, where C_k is the kth Catalan number.*

Let us define a bijection μ between the set of Dyck words of length $2(k+1)$ and the top $2k$ elements segment of orders in the kth part of $\mathcal{D}(A_n)$.

Table 2
An example.

Top 4 elements	Dyck word
1324	*ududud*
1342	*uduudd*
3124	*uuddud*
3142	*uududd*
3412	*uuuddd*

Definition 6. Our bijection produces Dyck words as follows.

The first element in Dyck words is u. It has no correspondence in orders. Each consequent element in the top $2k$ elements segment of an order from kth part of $\mathcal{D}(A_n)$ corresponds with the consequent element in Dyck word: if the element belongs to A_{2k} then d, if not, then u. The last element in Dyck word is d. It has no correspondence in orders. The length of the constructed Dyck word is $2(k+1)$.

Table 2 presents the bijection applied to the top four elements of the second part of $\mathcal{D}(A_n)$. The following lemmas prove that the bijection is well-defined.

Lemma 9. *The image of each Dyck word of length $2(k+1)$ is a top $2k$ elements segment of an order from kth part of $\mathcal{D}(A_n)$.*

Proof. Let us consider nonboundary subwords $a_i a_j a_l$, $i, j, l \in [2k]$ (boundary elements have indices 0 and $2k+1$). There are eight types such subwords: *ddd*, *ddu*, *dud*, *duu*, *udd*, *udu*, *uud*, *uuu*. The corresponding order should satisfy 1N3 if the median is d, and 3N1 if the median is u. Because the order of elements in A_{2k} and in $\overline{A_{2k}}$ are increasing, images of *ddd*, *dud*, *udu*, *uuu* satisfy 1N3, 3N1, image of *udd* satisfies 1N3, image of *uud* satisfies 3N1.

Image of *ddu* violates 1N3, if

$$|\{r \in [2k] | a_r = u, r < l\}| + 2 < |\{r \in [2k] | a_r = d, r < i\}|.$$

It is impossible because of the definition of the Dyck word.

Image of *duu* violates 3N1, if

$$|\{r \in [2k] | a_r = u, r < j\}| + 2 < |\{r \in [2k] | a_r = d, r < l\}|.$$

It is impossible because of the definition of the Dyck word.

For each $r < k$ we have $|[2r] \cap A_{2k}| = 2 + |[2r] \cap \overline{A_{2k}}|$. The image of the first $2r$ of each Dyck word contains at least one u that corresponds with an element from set $[2n] \setminus [2k]$. The corresponding order belongs to part k. □

Lemma 10. *The image of each top $2k$ elements segment from kth part of $\mathcal{D}(A_n)$ is a Dyck word.*

Proof. Because $|A_{2k}| = |\overline{A_{2k}}|$ we have equal numbers of u and d.

Suppose there is an order q from kth part of $\mathcal{D}(A_n)$ such that for corresponding Dyck work there is $0 \le j \le 2k-1$ with $|\{i \in [j] \cup 0 | a_i = u\}| < |\{i \in [j] \cup 0 | a_i = d\}|$. Therefore there is $0 \le r \le j$ such that $|\{i \in [r] \cup 0 | a_i = u\}| + 1 = |\{i \in [r] \cup 0 | a_i = d\}|$. Thus, order q belongs to part $r/2$, or lower. We get a contradiction. □

We are ready to present our main result.

Proposition 12. *For even n we have*

$$a(n) \sim \frac{\sqrt{2}}{4}\left(\sqrt{2+2\sqrt{2}}\right)^n,$$

for odd n we have $a(n) \sim \frac{\sqrt{\sqrt{2}-1}}{2}\left(\sqrt{2+2\sqrt{2}}\right)^n$.

Proof. For $m = n/2$, even n we define $w(m) = a(n)$. From bijection we have

$$w(m) = \sum_{k=1}^{m} C_{k+1} w(m-k). \tag{1}$$

Starting from $w(0) = 1$ we have a recurrence over even numbers. After rearranging indices one finds that this recurrence is recurrence (32) from [1].

The sequence $w(m)$ is sequence A289684 in [34]. Robert Israel showed that $w(m) \sim \frac{\sqrt{2}}{4}\left(2+2\sqrt{2}\right)^m$. Having $m = n/2$ we get the result for even n. From Proposition 6 for odd n we have $a(n) = 2a(n-1)$ that means $a(n) \sim \frac{1}{2\sqrt{1+\sqrt{2}}}\left(\sqrt{2+2\sqrt{2}}\right)^n$. □

For even n, $a(n)$ is member $n/2$ of the sequence A289684 in [34]. For $n = 1, 2, \ldots,$ we have $1, 2, 4, 9, 18, 42, 84, 199, \ldots$.

Since the set-alternating scheme produces a peak-pit domain we have $f(n), h(n) \in \Omega(\left(\sqrt{2+2\sqrt{2}}\right)^n)$, where $\sqrt{2+2\sqrt{2}} = 2.197368\ldots$.

Eq. (1) has coefficients which are Catalan numbers. Catalan numbers have many interesting properties (see e.g. [35]). Proposition 13 exploits one of the Catalan numbers properties.

Proposition 13. *$a(n)$ is odd if and only if $n = 2^t$, $t > 1$, or $t = 0$.*

Proof. We will prove this by induction. The statement is true for $n \leq 4$. Suppose the statement is true for $n \leq 2^t$. Let us prove for $n \in \{2^t + 1, \ldots, 2^{t+1}\}$.

If n is odd, then by Proposition 6 we have $a(n) = 2a(n-1)$, $a(n)$ is even. For even n we have Eq. (1), where $m = n/2$.

C_{k+1} is odd if and only if $k + 1 = 2^r - 1$, $r > 1$ [10]. $w(m-k)$ is odd if and only if $m - k = 0$, or $m - k = 2^s$, $s > 0$ (it follows from from induction hypothesis).

For $k = m$, both numbers are odd if $m = 2^r - 2$.

For $k = m - 2^s$, both numbers are odd if $m = 2^s + 2^r - 2$.

If we have $m = 2^{r_1} - 2$, then there is exactly one pair r_2, s such that $2^s + 2^{r_2} - 2 = 2^{r_1} - 2$ (solution is $s = r_2 = r_1 - 1$). Thus for all $m = 2^r - 2$, $m > 2$ we have two odd members in Eq. (1), the sum is even.

The expression $m = 2^s + 2^r - 2$ is an equation with unknowns s and r. The number $m + 2$ has a unique partition into two powers of two, if it exists. If the parts are equal numbers, then $m = 2^{r_1} - 2$ for some r_1. In this case we have an even sum. If the parts are different and $s, r > 1$, then there are two ways to define s, r, and the sum is even. If $s = 1$ and $r > 1$, then there is unique $k = m - 2$ that leads to odd member of the sum. There is no other pair of s, t, because $r > 1$. Thus, we have $m = 2 + 2^r - 2$ and the sum is odd. Having $r = t + 1$ we obtain the result. □

The domain generated by the even set-alternating scheme exceeds the domains from Fishburn's alternating scheme in size for all $n \geq 18$, but as we have already seen it is not the largest set-alternating domain. The odd set-alternating scheme leads to larger Condorcet domains. However, using Proposition 7 we can show that the ratio of their sizes is bounded by a constant factor. In particular the growth rate is of the same order as $a(n)$.

Proposition 14. *The odd set-alternating scheme generates domains of size $\Omega(\left(\sqrt{2+2\sqrt{2}}\right)^n)$*

Proof. The restriction of $\mathcal{D}(\{2, 3, 5, 7, \ldots, n-3\}$ to the set $X \setminus \{2, n-1\}$ contains $a(n-2)$ orders. The restriction of $\mathcal{D}(\{2, 3, 5, 7, \ldots, n-2\}$ to the set $X \setminus \{2\}$ contains a(n-1) orders. The restriction of $\mathcal{D}(A_n)$ to the set $X \setminus \{3, n-2\}$ contains $f_{n-2}(\{2, 3, 5, 7, \ldots, n-5\})$ orders. The restriction of $\mathcal{D}(A_n)$ to the set $X \setminus \{3\}$ contains $f_{n-1}(\{2, 3, 5, 7, \ldots, n-3\})$ orders.

Thus, $f_n(\{2, 3, 5, 7, \ldots, n-3\}), f_n(\{2, 3, 5, 7, \ldots, n-2\})$ are between $a(n-2)$ and $a(n+2)$ that proves the statement. □

6. Discussion

In this paper we introduced and studied the class of coherent Condorcet domains and showed that this class contains Condorcet domains which for large numbers of alternatives exceed all previously known examples in size. Our paper leaves open conjectures and questions for the domain classes we have studied. We believe that the coherent domains are of interest both for direct applications, where they can serve as models for homogeneous social choice situations, and for the mathematical study of large general Condorcet domains.

Acknowledgments

The authors would like to thank the anonymous reviewer for their constrictive criticism which made the paper much easier to read. The authors would also like to thank Arkadii Slinko for his useful comments on the manuscript. The work on set-alternating schemes was motivated by some unpublished research notes by Dr Riis. Bei Zhou was funded by the China Scholarship Council (CSC). The Basic Research Program of the National Research University Higher School of Economics, Russia partially supported Alexander Karpov. This research utilized Queen Mary's Apocrita HPC facility, supported by QMUL Research-IT.

Data availability

No data was used for the research described in the article.

References

[1] J. Abate, W. Whitt, Integer sequences from queueing theory, J. Integer Seq. 13 (2010) 97–120.
[2] J.M. Abello, C.R. Johnson, How large are transitive simple majority domains? SIAM Journal on Algebraic Discrete Methods 5 (4) (1984) 603–618.
[3] D. Akello-Egwel, C. Leedham-Green, A. Litterick, K. Markström, S. Riis, Condorcet domains on at most seven alternatives, 2025, Arxiv preprint arxiv:2306.15993.
[4] K.J. Arrow, Social Choice and Individual Values, second ed., vol. 12, Yale University Press, 1963.
[5] D. Black, On the rationale of group decision-making, Journal of Political Economy 56 (1) (1948) 23–34.
[6] J. Craven, Social Choice, Cambridge University Press, 1992.
[7] V.I. Danilov, A.V. Karzanov, G.A. Koshevoy, Condorcet domains of tiling type, Discrete Appl. Math. 160 (7–8) (2012) 933–940.
[8] Le Marquis de Condorcet, Essai sur l'Application de l'Analyse à la Probabilité des Décisions Rendues à la Pluralité des Voix, L'Imprimerie Royale, Paris, 1785.
[9] E. Deutsch, Dyck path enumeration, Discrete Math. 204 (1999) 167–202.
[10] Ö. Eğecioğlu, The parity of the catalan numbers via lattice paths, Fibonacci Quart. 21 (1983) 65–66.
[11] E. Elkind, M. Lackner, D. Peters, Preference restrictions in computational social choice: A survey, 2022, Cs.GT.
[12] P.C. Fishburn, Notes on Craven's conjecture, Soc. Choice Welf. 9 (1992) 259–262.
[13] P. Fishburn, Acyclic sets of linear orders, Soc. Choice Welf. 14 (1) (1996) 113–124.
[14] P.C. Fishburn, Acyclic sets of linear orders: A progress report, Soc. Choice Welf. 19 (2002) 431–447.
[15] A. Galambos, V. Reiner, Acyclic sets of linear orders via the Bruhat orders, Soc. Choice Welf. 30 (2) (2008) 245–264.
[16] R.C. Johnson, Remarks on Mathematical Social Choice, Dept. Economics and Institute for Physical Science and Technology, Univ. Maryland, 1978.
[17] A. Karpov, Structured preferences: A survey, Autom. Remote Control 83 (9) (2022) 1329–1354.
[18] A. Karpov, Preferences over mixed manna, in: B. Goldengorin, S. Kuznetsov (Eds.), Data Analysis and Optimization in Honor of Boris Mirkin's 80th Birthday, Springer, 2023.
[19] A. Karpov, K. Markström, S. Riis, B. Zhou, Local diversity of condorcet domains, 2024, arXiv preprint arXiv:2401.11912.
[20] A. Karpov, A. Slinko, Constructing large peak-pit condorcet domains, Theory and Decision 94 (1) (2023) 97–120.
[21] K.H Kim, F.W Roush, M.D Intriligator, Overview of mathematical social sciences, The American Mathematical Monthly 99 (9) (1992) 838–844.
[22] D. Knuth, The Art of Computer Programming: Sorting and Searching, Computer Science and Information, vol. 3, Addison-Wesley, 1973.
[23] C. Leedham-Green, K. Markström, S. Riis, The largest condorcet domains on 8 alternatives, Soc. Choice Welf. 62 (2024) 109–116.
[24] G. Li, C. Puppe, A. Slinko, Towards a classification of maximal peak-pit condorcet domains, Math. Social Sci. 113 (2021) 191–202.
[25] H.M. Mulder, The structure of median graphs, Discrete Math. 24 (1978) 197–204.
[26] T.K. Petersen, B.E. Tenner, The depth of a permutation, J. Comb. 6 (1–2) (2015) 145–178.
[27] C. Puppe, A. Slinko, Condorcet domains, median graphs and the single-crossing property, Econom. Theory 67 (2019) 285–318.
[28] C. Puppe, A. Slinko, Maximal condorcet domains. A further progress report, Games Econom. Behav. 40 (2024) 143–174.
[29] H. Raynaud, Paradoxical Results from Inada's Conditions for Majority Rule, Institute for Mathematical Studies in the Social Sciences, Stanford University, 1981.
[30] A.K. Sen, A possibility theorem on majority decisions, Econometrica 34 (2) (1966) 491–499.
[31] A. Slinko, Condorcet domains satisfying arrow's single-peakedness, J. Math. Econom. 84 (2019) 166–175.
[32] Arkadii Slinko, Condorcet domains satisfying arrow's single-peakedness, J. Math. Econom. (ISSN: 0304-4068) 84 (2019) 166–175, http://dx.doi.org/10.1016/j.jmateco.2019.08.001, URL https://www.sciencedirect.com/science/article/pii/S0304406819300874.
[33] A. Slinko, A family of condorcet domains that are single-peaked on a circle, Soc. Choice Welf. 63 (2024) 57–67.
[34] N.J.A. Sloane, et al., The on-line encyclopedia of integer sequences, 2003.
[35] R.P. Stanley, Catalan Numbers, Cambridge University Press, 2015.
[36] Bei Zhou, Klas Markström, Søren Riis, CDL: A fast and flexible library for the study of permutation sets with structural restrictions, SoftwareX (ISSN: 2352-7110) 28 (2024) 101951, http://dx.doi.org/10.1016/j.softx.2024.101951, URL https://www.sciencedirect.com/science/article/pii/S2352711024003212.
[37] Bei Zhou, Søren Riis, An efficient heuristic search algorithm for discovering large condorcet domains, 4OR (2025) 1–24.