

Output-Feedback I&I Adaptive Control for Linear Systems with Time-Varying Parameters

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Abstract—This paper combines the so-called *congelation of variables* method with the *adaptive immersion and invariance (I&I)* approach to control linear single-input-single-output systems with time-varying parameters via output feedback. The system is first reparameterized using a pair of reduced-order filters and the reparameterization error dynamics show that the input is coupled with a time-varying perturbation. By exploiting the input-to-state stability (ISS) of the inverse dynamics, which is regarded as a counterpart in the time-varying setting of the classical minimum-phase property, the coupling between the input and the time-varying perturbation is transformed into a coupling between the output and another time-varying perturbation that can be dominated in the controller design stage. A pair of high-gain filters are then implemented so that the reparameterization error dynamics are ISS. Finally, output regulation is achieved by strengthened damping design, which invokes a small-gain-like argument from a Lyapunov perspective.

I. INTRODUCTION

Since the 1980s, or even earlier, adaptive control has undergone extensive research (see e.g. [1]–[5]), although only a few works have investigated systems with time-varying parameters. Some pioneering works on adaptive control for time-varying systems (see e.g. [6]) exploit *persistence of excitation* to guarantee stability by ensuring that parameter estimates converge to the true parameters. Subsequent works (see e.g. [7], [8]) have removed the restriction of *persistence of excitation* by requiring bounded and slow (in an average sense) parameter variations.

More recent works can be mainly categorized into two trends. One of them is based on the so-called *robust adaptive law*, see [3], which adopts a switching parameter update law called σ -modification. This approach achieves asymptotic tracking when the parameters are constant, otherwise the tracking error is nonzero and related to the rates of the parameter variations, see [9]. In [10] and [11] the parameter variations are modelled in two parts: known parameter variations and unknown variations, so that the residual tracking error only depends on the rates of the unknown parameter variations.

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The other trend exploits *filtered transformations* and *projection operation*, see [12], [13] and [14]. These methods can guarantee asymptotic tracking provided that the parameters are confined within a compact set, their derivatives are $\mathcal{L}_1$ and the disturbance on the state evolution is additive and $\mathcal{L}_2$. Moreover, *a priori* knowledge on parameter variations is not needed and the residual tracking error is independent of the rates of parameter variations.

The methods mentioned above cannot guarantee zero-error regulation when the unknown parameters are persistently varying. To achieve asymptotic state/output regulation when the time-varying parameters are neither known nor asymptotically constant, in [15] and [16] a method called the *congelation of variables* has been proposed and developed on the basis of the *adaptive backstepping* approach and the *adaptive immersion and invariance (I&I)* approach, respectively. In the spirit of the *congelation of variables* method each unknown time-varying parameter is treated as a nominal unknown constant parameter perturbed by the difference between the true parameter and the nominal parameter, which causes a time-varying perturbation term. The controller design is then divided into a classical adaptive control design, with constant unknown parameters, and a damping design via dominance to counteract the time-varying perturbation terms. This method is compatible with most adaptive control schemes using parameter estimates, as it does not change the original parameter update law for time-invariant systems.

In the output-feedback control design with the *congelation of variables* method, the major difficulty is the coupling between the input and the time-varying perturbation, which prevents stabilization via dominance. In [15], this is achieved by forcing the input coefficients to be constants, which eliminates the time-varying perturbation terms. Instead of decoupling the input, [17] manages to transform the coupling between the input and the time-varying perturbation into a coupling between the output and another time-varying perturbation by exploiting a modified minimum-phase property for time-varying systems, which enables the use of the dominance design again. In this paper we follow a similar idea but on the basis of the so-called *adaptive I&I* approach.

II. SYSTEM REPARAMETERIZATION

We consider an n -dimensional single-input-single-output linear system in observable canonical form with relative

degree r , as described by the equations

$$\begin{aligned} \dot{x}_1 &= -a_1(t)x_1 + x_2, \\ &\vdots \\ \dot{x}_r &= -a_r(t)x_1 + x_{r+1} + b_r(t)u, \\ &\vdots \\ \dot{x}_n &= -a_n(t)x_1 + b_n(t)u, \\ y &= x_1, \end{aligned} \quad (1)$$

or, in compact form, by the equations

$$\begin{aligned} \dot{x} &= S_n x - a(t)y + \begin{bmatrix} 0_{(r-1) \times 1} \\ b(t) \end{bmatrix} u, \\ y &= e_1^\top x, \end{aligned} \quad (2)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}$ is the input, $y(t) \in \mathbb{R}$ is the output, S_n is the $n \times n$ upper-shift matrix, and $e_1 = [1, 0, \dots, 0]^\top \in \mathbb{R}^n$. The unknown time-varying parameters are denoted by the vector $\theta(t) = [b^\top(t), a^\top(t)]^\top$, $a(t) = [a_1(t), \dots, a_n(t)]^\top \in \mathbb{R}^n$, $b(t) = [b_r(t), \dots, b_n(t)]^\top \in \mathbb{R}^{n-r+1}$.

We now introduce some assumptions that are useful for the proof of the main results.

Assumption 1 (Bounded parameters): The vector of $2n - r + 1$ unknown time-varying parameters $\theta(t)$ belongs to a compact set Θ , $\forall t \geq 0$. This set is also unknown, but we know its size, that is, we know $\delta_{\Delta_\theta} = \max_{\theta \in \Theta} |\Delta_\theta| \geq \sup_{t \geq 0} |\Delta_\theta|$, where $\Delta_\theta = \theta - \ell_\theta$, with $\ell_\theta \in \Theta$ a constant parameter to be selected. $\diamond$

Since θ is typically defined as a box and ℓ_θ is typically selected as the center of the box, δ_{Δ_θ} can be interpreted as the “radius” of the compact set Θ . The “ $\geq$ ” sign indicates that the knowledge of Θ can be imperfect and conservative, but in this paper we only consider the case in which $\max_{\theta \in \Theta} |\Delta_\theta| = \sup_{t \geq 0} |\Delta_\theta|$ and always refer to the latter.

Assumption 2 (Smooth bounded parameters): The i -th order time derivatives $\theta^{(i)}(t)$ of $\theta(t)$ belong to compact sets $\Theta^{(i)}$, respectively, $i = 1, \dots, \forall t \geq 0$. $\diamond$

This assumption is only needed when implementing a change of coordinates (see Algorithm 1 and Algorithm 2) to guarantee boundedness of (vectors) of time-varying coefficients.

Assumption 3: $|b_r(t)| \geq \varepsilon > 0$ and the sign of $b_r(t)$, i.e. $\text{sgn}(b_r(t))$ is known and does not change. $\diamond$

In the spirit of the *congelation of variables* method the system parameters $\theta(t) = \ell_\theta + \Delta_\theta(t)$ can be regarded as the sum of a vector of constant “congealed” parameters ℓ_θ of a nominal time-invariant system and a vector of differences $\Delta_\theta(t)$ between the actual time-varying parameters and the “congealed” parameters. Therefore we first design a set of filters for system (1) with the nominal constant parameters $\ell_\theta = [\ell_{b_r}, \dots, \ell_{b_n}, \ell_{a_1}, \dots, \ell_{a_n}]^\top$. Note that its input-output relation is described by the differential equation

$$y^{(n)} = [u^{(n-r)}, u^{(n-r-1)}, \dots, u, -y^{(n-1)}, -y^{(n-2)}, \dots, -y] \ell_\theta. \quad (3)$$

Since the time derivatives of u and y are not available for measurement, we apply a stable filter $\Lambda(s) = s^{n-1} + \lambda_{n-1}s^{n-2} + \dots + \lambda_2s + \lambda_1$ to both sides of (3), which yields

$$\frac{s^{n-1}}{\Lambda(s)}[y] = \left[\frac{s^{n-r}}{\Lambda(s)}[u], \dots, \frac{1}{\Lambda(s)}[u], \frac{-s^{n-1}}{\Lambda(s)}[y], \dots, \frac{-1}{\Lambda(s)}[y] \right] \ell_\theta. \quad (4)$$

Noting that $\frac{s^{n-1}}{\Lambda(s)} = 1 - \frac{\lambda_{n-1}s^{n-2} + \dots + \lambda_2s + \lambda_1}{\Lambda(s)}$ yields

$$\begin{aligned} \dot{y} &= s \frac{\lambda_{n-1}s^{n-2} + \dots + \lambda_2s + \lambda_1}{\Lambda(s)}[y] \\ &+ \left[\frac{s^{n-r}}{\Lambda(s)}[u], \dots, \frac{1}{\Lambda(s)}[u], \frac{-s^{n-1}}{\Lambda(s)}[y], \dots, \frac{-1}{\Lambda(s)}[y] \right] \ell_\theta. \end{aligned} \quad (5)$$

Consider the state-space realization of the filters described by

$$\begin{aligned} \dot{\zeta} &= A_\lambda \zeta + e_{n-1}u, \\ \dot{\xi} &= A_\lambda \xi - e_{n-1}y, \end{aligned} \quad (6)$$

where $\zeta(t) \in \mathbb{R}^{n-1}$, $\xi(t) \in \mathbb{R}^{n-1}$, $A_\lambda = S_{n-1} - e_{n-1}\lambda^\top$ is Hurwitz, S_{n-1} is the $(n-1) \times (n-1)$ upper-shift matrix, $e_{n-1} = [0, \dots, 0, 1]^\top \in \mathbb{R}^{n-1}$, $\lambda = [\lambda_1, \dots, \lambda_{n-1}]^\top$. Equation (5) can then be written as

$$\begin{aligned} \dot{y} &= -\lambda^\top \xi + \phi^\top \ell_\theta + \eta_0 \\ &= -\lambda^\top (A_\lambda \xi - e_{n-1}y) + \phi^\top \ell_\theta + \eta_0, \end{aligned} \quad (7)$$

where $\phi = [\zeta_{n-r+1}, \dots, \zeta_1, -\lambda^\top \xi - y, \xi_{n-1}, \dots, \xi_1]^\top$ with the definition that $\zeta_n = -\lambda^\top \zeta + u$. This is also the parameterization used in [5] (Section 4.4.1). For linear time-invariant systems considered in classical adaptive schemes, η_0 is an exponentially decaying error term because of the Hurwitz property of A_λ and it is typically ignored in analysis and design. However, as shown in what follows, when $\theta(t)$ is time-varying the perturbation terms coupled with y and with u appear due to the substitution of ℓ_θ for $\theta(t)$. Rearranging (7) yields

$$\begin{aligned} \eta_0 &= -(\ell_{a_1} + \Delta_{a_1})y + x_2 + \lambda^\top (A_\lambda \xi - e_{n-1}y) \\ &\quad - [\zeta_{n-r+1}, \dots, \zeta_1, -\lambda^\top \xi - y, \xi_{n-1}, \dots, \xi_1] \ell_\theta \\ &= \lambda^\top A_\lambda \xi - \lambda e_{n-1}y + x_2 - \Delta_{a_1}y \\ &\quad - [\zeta_{n-r+1}, \dots, \zeta_1, -\lambda^\top \xi, \xi_{n-1}, \dots, \xi_1] \ell_\theta. \end{aligned} \quad (8)$$

Define $\eta_1 = \eta_0 + \Delta_{a_1}y$ to separate the perturbation term $-\Delta_{a_1}y$ from the expression of η_1 : this allows avoiding differentiating unknown time-varying parameters when deriving the error dynamics and yields

$$\begin{aligned} \dot{\eta}_1 &= \lambda^\top A_\lambda^2 \xi - \lambda^\top A_\lambda e_{n-1}y - \lambda^\top e_{n-1}x_2 + x_3 \\ &\quad - [\zeta_{n-r+2}, \dots, \zeta_2, -\lambda^\top A_\lambda \xi, -\lambda^\top \xi, \xi_{n-1}, \dots, \xi_2] \ell_\theta \\ &\quad + (\lambda^\top e_{n-1} \Delta_{a_1} - \Delta_{a_2})y. \end{aligned} \quad (9)$$

Define $\eta_2 = \dot{\eta}_1 + (-\lambda^\top e_{n-1} \Delta_{a_1} + \Delta_{a_2})y$, which yields

$$\begin{aligned} \dot{\eta}_2 &= \lambda^\top A_\lambda^3 \xi - \lambda^\top A_\lambda^2 e_{n-1}y - \dots - \lambda^\top e_{n-1}x_3 + x_4 \\ &\quad - [\zeta_{n-r+3}, \dots, \zeta_3, -\lambda^\top A_\lambda^2 \xi, \dots, -\lambda^\top \xi, \xi_{n-1}, \dots, \xi_3] \ell_\theta \\ &\quad - (-\lambda^\top A_\lambda e_{n-1} \Delta_{a_1} - \lambda^\top e_{n-1} \Delta_{a_2} + \Delta_{a_3})y. \end{aligned} \quad (10)$$

Repeating the procedures above for $\eta_3, \dots, \eta_{r-2}$ and then defining $\eta_{r-1} = \dot{\eta}_{r-2} + (-\lambda^\top A_\lambda^{r-3} e_{n-1} \Delta_{a_1} - \dots - \lambda^\top e_{n-1} \Delta_{a_{r-2}} + \Delta_{a_{r-1}})y$ yields

$$\begin{aligned} \dot{\eta}_{r-1} = & \lambda^\top A_\lambda^r \xi - \lambda^\top A_\lambda^{r-1} e_{n-1} y - \dots - \lambda^\top e_{n-1} x_r + x_{r+1} \\ & - [-\lambda^\top \zeta, \dots, \zeta_r, -\lambda^\top A_\lambda^{r-1} \xi, \dots, \xi_r] \ell_\theta \\ & - (-\lambda^\top A_\lambda^{r-2} e_{n-1} \Delta_{a_1} - \dots - \lambda^\top e_{n-1} \Delta_{a_{r-1}} + \Delta_{a_r})y \\ & + \Delta_{b_r} u. \end{aligned} \quad (11)$$

Repeating again the procedures for $\eta_r, \dots, \eta_{n-2}$ and, finally, defining $\eta_{n-1} = \dot{\eta}_{n-2} + (-\lambda^\top A_\lambda^{n-3} e_{n-1} \Delta_{a_1} - \dots - \lambda^\top e_{n-1} \Delta_{a_{n-2}} + \Delta_{a_{n-1}})y - (-\lambda^\top A_\lambda^{n-r-2} e_{n-1} \Delta_{b_r} - \dots - \lambda^\top e_{n-1} \Delta_{b_{n-2}} + \Delta_{b_{n-1}})u$ yields

$$\begin{aligned} \dot{\eta}_{n-1} = & \lambda^\top A_\lambda^n \xi - \lambda^\top A_\lambda^{n-1} e_{n-1} y - \dots - \lambda^\top e_{n-1} x_n - \\ & [-\lambda^\top A_\lambda^{n-r} \zeta, \dots, -\lambda^\top \zeta, -\lambda^\top A_\lambda^{n-1} \xi, \dots, -\lambda^\top \xi] \ell_\theta \\ & - (-\lambda^\top A_\lambda^{n-2} e_{n-1} \Delta_{a_1} - \dots - \lambda^\top e_{n-1} \Delta_{a_{n-1}} + \Delta_{a_n})y \\ & + (-\lambda^\top A_\lambda^{n-r-1} e_{n-1} \Delta_{b_r} - \dots - \lambda^\top e_{n-1} \Delta_{b_{n-1}} + \Delta_{b_n})u. \end{aligned} \quad (12)$$

Theorem 1: The reparameterization error η_0 is described by the system

$$\dot{\eta} = A_\lambda \eta + B_\lambda \left(\begin{bmatrix} 0_{(r-1) \times 1} \\ \Delta_b(t) \end{bmatrix} u - \Delta_a(t)y \right), \quad (13)$$

$$\eta_0 = e_1^\top \eta - \Delta_{a_1}(t)y, \quad (14)$$

where $\eta = [\eta_1, \dots, \eta_{n-1}]^\top$ and

$$B_\lambda = \begin{bmatrix} \lambda^\top e_{n-1} & 1 & 0 & \dots & 0 \\ \lambda^\top A_\lambda e_{n-1} & \lambda^\top e_{n-1} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda^\top A_\lambda^{n-2} e_{n-1} & \dots & \dots & \lambda^\top e_{n-1} & 1 \end{bmatrix}. \quad (15)$$

Remark 1: The filters (6) are reduced-order filters ($n-1$ state variables each) because we directly reparameterize the expression of $\dot{y}$ instead of the expression of y . $\diamond$

Due to the perturbation terms coupled with y and with u we cannot directly determine the stability of the dynamics of η and ignore this term, like in classical adaptive control schemes, even if A_λ is Hurwitz. The coupling with y can be dominated by a strengthened filter and controller design, as shown in [15], while the coupling with u is merely avoided by assuming a constant vector of input coefficients b , i.e. $\Delta_b(t) = 0, \forall t \geq 0$. This restriction can be removed by exploiting a minimum-phase property, which merges the coupling with u into the coupling with y .

III. INVERSE DYNAMICS

In this section we adapt some results of [17], which have been developed for a more general class of nonlinear systems, to the present context. Consider the inverse dynamics of system (1), i.e. assuming the system is driven by y and its time-derivatives instead of u , which yields

$$\begin{aligned} x_2 &= y^{(1)} + a_1 y, \\ &\vdots \\ x_r &= y^{(r-1)} + (a_1 y)^{(r-2)} + \dots + a_{r-1} y. \end{aligned} \quad (16)$$

Setting $u = \frac{1}{b_r}(-x_{r+1} + y^{(r)} + (a_1 y)^{(r-1)} + \dots + a_r y)$ yields

$$\begin{aligned} \dot{x}_{r+1} &= -\frac{b_{r+1}}{b_r} x_{r+1} + x_{r+2} \\ &\quad + \frac{b_{r+1}}{b_r} \left(y^{(r)} + (a_1 y)^{(r-1)} + \dots + a_r y \right), \\ &\vdots \\ \dot{x}_n &= -\frac{b_n}{b_r} x_{r+1} \\ &\quad + \frac{b_n}{b_r} \left(y^{(r)} + (a_1 y)^{(r-1)} + \dots + a_r y \right). \end{aligned} \quad (17)$$

We now perform a change of coordinates to eliminate the time derivatives of y , which are not desirable in the design and the analysis. To this end, note that the identities¹

$$s_1 s_2^{(i)} = (-1)^i s_1^{(i)} s_2 + \left(\sum_{j=0}^{i-1} (-1)^j s_1^{(j)} s_2^{(i-1-j)} \right)^{(1)} \quad (18)$$

hold for any pair of smooth signals $s_1(t)$ and $s_2(t)$. Using (18) and defining the new coordinate

$$\begin{aligned} \bar{x}_n &= x_n - \sum_{j=0}^{r-1} (-1)^j \left(\frac{b_n}{b_r} \right)^{(j)} y^{(r-1-j)} \\ &\quad - \sum_{i=1}^{r-1} \sum_{j=0}^{r-i-1} (-1)^{(j)} \left(\frac{b_n}{b_r} \right)^{(j)} (a_i y)^{(r-i-1-j)} \end{aligned} \quad (19)$$

yields

$$\begin{aligned} \dot{\bar{x}}_n &= -\frac{b_n}{b_r} x_{r+1} - a_n y + (-1)^r \left(\frac{b_n}{b_r} \right)^{(r)} y \\ &\quad + \sum_{i=1}^r (-1)^{r-i} \left(\frac{b_n}{b_r} \right)^{(r-i)} a_i y, \end{aligned} \quad (20)$$

which does not contain time derivatives of y . In the spirit of the above procedure, we proceed with the change of coordinates using Algorithm 1 (see next page), which yields the inverse dynamics in the new coordinates described by the equations

$$\dot{\bar{x}} = A_{\bar{b}}(t) \bar{x} + b_y(t)y, \quad (21)$$

$$u = \frac{1}{b_r(t)} \left(-\bar{x}_{r+1} + y^{(r)} + \sum_{j=0}^{r-1} a_{y^{(j)}}(t) y^{(j)} \right), \quad (22)$$

where $\bar{x}(t) = [\bar{x}_{r+1}, \dots, \bar{x}_n]^\top \in \mathbb{R}^{n-r}$, $A_{\bar{b}} = S_{n-r} - \bar{b} e_1^\top$, $\bar{b}(t) = [\frac{b_{r+1}}{b_r}, \dots, \frac{b_n}{b_r}]^\top \in \mathbb{R}^{n-r}$, and S_{n-r} is the $(n-r) \times (n-r)$ upper-shift matrix. $b_y(t)$, $a_{y^{(i)}}(t)$ are unknown due to the unknown $\theta(t)$, but bounded $\forall t \geq 0$ due to Assumption 2.

Assumption 4 (Strong minimum-phase property):

System (1) has a strong minimum-phase property in the sense that the inverse dynamics (21) are input-to-state stable (ISS) with respect to the input y . Moreover, there exists an ISS Lyapunov function $V_{\bar{x}}(\bar{x}) = \gamma_{\bar{x}}^2 \bar{x}^\top P_{\bar{x}} \bar{x}$, with a constant $P_{\bar{x}} = P_{\bar{x}}^\top \succ 0$ and the time derivative of $V_{\bar{x}}$ along the trajectories of the inverse dynamics satisfies the inequality

$$\dot{V}_{\bar{x}} \leq -\bar{x}^\top \bar{x} + \varepsilon_{b_y}^2 \delta_{b_y}^2 y^2, \quad (23)$$

¹The notation $s^{(i)}$ denotes the i -th time derivative of the signal s , assuming it exists.

Algorithm 1 Change of coordinates $x_{r+1}, \dots, x_n$.

Input: $x_{r+1}, \dots, x_n, \dot{x}_{r+1}, \dots, \dot{x}_n$.**Output:** $\bar{x}_{r+1}, \dots, \bar{x}_n, \dot{\bar{x}}_{r+1}, \dots, \dot{\bar{x}}_n$.

- 1: **while** time derivatives of y appear in the expression of $\dot{x}_{r+1}, \dots, \dot{x}_n$ **do** $\triangleright$ This while-loop iterates for r times as it reduces the order of $y^{(r)}$ by one each iteration.
 - 2: **for** $i = n \rightarrow r+2$ **do**
 - 3: Update $\bar{x}_i$ and $\dot{\bar{x}}_i$ using (18).
 - 4: Rewrite $\dot{x}_i$ in terms of $\bar{x}_i$ in the expression of $\dot{x}_{i-1}$ and leave the feedback term $-\frac{b_i}{b_r}x_{r+1}$ unchanged.
 - 5: **end for**
 - 6: Update $\bar{x}_{r+1}$ and $\dot{\bar{x}}_{r+1}$ using (18).
 - 7: Rewrite x_{r+1} in terms of $\bar{x}_{r+1}$ in the expressions of $\dot{x}_{r+1}, \dots, \dot{x}_n$, respectively. $\triangleright$ This brings back the time derivatives of y , but with the order reduced by one.
 - 8: $x_{r+1} \leftarrow \bar{x}_{r+1}, \dots, x_n \leftarrow \bar{x}_n, \dot{x}_{r+1} \leftarrow \dot{\bar{x}}_{r+1}, \dots, \dot{x}_n \leftarrow \dot{\bar{x}}_n$. $\triangleright$ Update the old coordinates before the next iteration.
 - 9: **end while**
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where $\varepsilon_{b_y} > 0$ is constant and $\delta_{b_y} = \sup_{t \geq 0} |b_y(t)|$. $\diamond$

Remark 2: Due to the linearity of the inverse dynamics (21), the exponential stability of the origin of the zero dynamics (which is (21) with $y \equiv 0$) is equivalent to the ISS of the inverse dynamics. For linear time-invariant systems the ISS property can be equivalently replaced by the condition that A_{ℓ_b} is Hurwitz, which is the typical assumption made in classical adaptive control schemes. $\diamond$

IV. FILTER DESIGN

In the previous section we have shown that the control input u can be equivalently written in terms of the inverse dynamics state variable $\bar{x}_{\rho+1}$ and the time derivatives of y of order up to r . Since the time derivatives of y is not desirable in the design, in this section we continue to exploit the low-pass characteristics of the dynamics of the reparameterization error η to eliminate the time derivatives of y . Substituting (22) into (13) yields

$$\dot{\eta} = A_\lambda \eta - B_\lambda \Delta_a(t)y + \bar{\Delta}_b(t) \left(y^{(r)} + \sum_{j=0}^{r-1} a_{y(j)}(t)y^{(j)} - \bar{x}_{r+1} \right), \quad (24)$$

where $\bar{\Delta}_b(t) = B_\lambda \begin{bmatrix} 0_{(r-1) \times 1} \\ \Delta_b(t) \end{bmatrix} \frac{1}{b_r(t)}$, which is unknown, but bounded due to Assumption 1. It should be noted that due to the structure of B_λ , the first $\bar{r}-1$ elements of $\bar{\Delta}_b(t)$ are 0, or equivalently, u is separated from η_1 (or η_0) by $\bar{r}$ integrators, where $\bar{r}$ is defined by

$$\bar{r} = \begin{cases} 1, & \text{if } r = 1; \\ r-1, & \text{if } r \geq 2 \text{ and } \Delta_{b_r} \neq 0; \\ r, & \text{if } r \geq 2 \text{ and } \Delta_{b_r} = 0. \end{cases} \quad (25)$$

To guarantee that no time derivative of y appears in η_0 , at least r integrators between u and η_1 are required, as u

Algorithm 2 Change of coordinates $\eta_1, \dots, \eta_{n-1}$.

Input: $\eta_1, \dots, \eta_{n-1}, \dot{\eta}_1, \dots, \dot{\eta}_{n-1}$.**Output:** $\bar{\eta}_1, \dots, \bar{\eta}_{n-1}, \dot{\bar{\eta}}_1, \dots, \dot{\bar{\eta}}_{n-1}$.

- 1: **while** time derivatives of y appear in the expression of $\dot{\eta}_1, \dots, \dot{\eta}_{n-1}$ **do** $\triangleright$ This while-loop should only iterate for once if Assumption 5 is satisfied.
 - 2: **for** $i = n-1 \rightarrow 2$ **do**
 - 3: Update $\bar{\eta}_i$ and $\dot{\bar{\eta}}_i$ using (18).
 - 4: Rewrite η_i in terms of $\bar{\eta}_i$ in the expression of $\dot{\eta}_{i-1}$.
 - 5: **end for**
 - 6: Update $\bar{\eta}_1$ and $\dot{\bar{\eta}}_1$ using (18).
 - 7: Rewrite $\eta_1, \dots, \eta_{n-1}$ in terms of $\bar{\eta}_1, \dots, \bar{\eta}_{n-1}$, respectively, in the expressions of $\dot{\eta}_{n-1}$. $\triangleright$ This should not bring back any time derivatives of y if Assumption 5 is satisfied.
 - 8: $\eta_1 \leftarrow \bar{\eta}_1, \dots, \eta_{n-1} \leftarrow \bar{\eta}_{n-1}, \dot{\eta}_1 \leftarrow \dot{\bar{\eta}}_1, \dots, \dot{\eta}_{n-1} \leftarrow \dot{\bar{\eta}}_{n-1}$. $\triangleright$ Update the old coordinates before the next iteration.
 - 9: **end while**
-

contains $y^{(r)}$. According to (25), the assumption that follows can fulfil such requirement.

Assumption 5: The relative degree of system (1) is either $r = 1$ or $r \geq 2$, but b_r is constant (which enforces $\Delta_{b_r}(t) = 0$, $\forall t \geq 0$). $\diamond$

Remark 3: The restriction of Assumption 5 is only related to the relative degree of the original system (1) and $\Delta_b(t)$ yet independent of whether we use reduced-order filters or full-order filters (as is implemented in [17]). Using filters of different order can only provide a different non-minimal realization of the system while cannot change the properties related to the relative degree. $\diamond$

Similarly to what is implemented in the previous section we use a change of coordinates to eliminate the time derivatives of y . Applying Algorithm 2 yields the dynamics (24) in the new coordinates

$$\dot{\bar{\eta}} = A_\lambda \bar{\eta} + \bar{b}_y(t)y - \bar{\Delta}_b(t)\bar{x}_{\rho+1}, \quad (26)$$

$$\eta_0 = \bar{\eta}_1 + \bar{a}_y(t)y, \quad (27)$$

where, by Assumption 2, $\bar{b}_y(t)$ and $\bar{a}_y(t)$ are unknown but bounded.

Proposition 1: The reparameterization error dynamics (26) are ISS with respect to the inputs $\bar{x}_{\rho+1}$ and y if the vector of filter gains is given by $\lambda = \frac{1}{2}e_{n-1}^\top P_{\bar{\eta}}$, $P_{\bar{\eta}} = P_{\bar{\eta}}^\top \succ 0$, and $P_{\bar{\eta}}$ satisfies the Riccati inequality

$$S_{n-1}^\top P_{\bar{\eta}} + P_{\bar{\eta}} S_{n-1} - P_{\bar{\eta}} e_{n-1} e_{n-1}^\top P_{\bar{\eta}} + Q_{\bar{\eta}} \preceq 0, \quad (28)$$

where

$$Q_{\bar{\eta}} = \left(\frac{1}{\varepsilon_{P_{\bar{\eta}} \bar{\Delta}_b}^2} + \frac{1}{\varepsilon_{P_{\bar{\eta}} \bar{b}_y}^2} + 1 \right) I_{n-1}, \quad (29)$$

$\varepsilon_{P_{\bar{\eta}} \bar{\Delta}_b} > 0$ and $\varepsilon_{P_{\bar{\eta}} \bar{b}_y} > 0$. Moreover, there exists an ISS Lyapunov function $V_{\bar{\eta}}(\bar{\eta}) = \bar{\eta}^\top P_{\bar{\eta}} \bar{\eta}$ and the time derivative

of $V_{\bar{\eta}}$ along the trajectories of the reparameterization error dynamics (26) satisfies the inequality

$$\dot{V}_{\bar{\eta}} \leq -\bar{\eta}^\top \bar{\eta} + \varepsilon_{P_{\bar{\eta}} \bar{b}_y}^2 \delta_{P_{\bar{\eta}} \bar{b}_y}^2 y^2 + \varepsilon_{P_{\bar{\eta}} \bar{\Delta}_b}^2 \delta_{P_{\bar{\eta}} \bar{\Delta}_b}^2 \bar{x}_{p+1}^2, \quad (30)$$

where $\delta_{P_{\bar{\eta}} \bar{b}_y} = \sup_{t \geq 0} |P_{\bar{\eta}} \bar{b}_y(t)|$ and $\delta_{P_{\bar{\eta}} \bar{\Delta}_b} = \sup_{t \geq 0} |P_{\bar{\eta}} \bar{\Delta}_b(t)|$. $\diamond$

V. CONTROLLER DESIGN

To proceed with the controller design we first rewrite the reparameterization error η_0 in terms of the new coordinate η_1 , which yields

$$\eta_0 = \bar{\eta}_1 + \bar{a}_y(t)y, \quad (31)$$

where $\bar{a}_y(t)$ is unknown, but bounded due to Assumption 2. As discussed in Section II, the input-output relation can be reparameterized using (7) and the filters (6). For the sake of implementing an *I&I* controller conveniently, we rewrite the reparameterized system in the equivalent form²

$$\begin{aligned} \dot{y} &= \vartheta_3 v_1 + \varphi(y, d)^\top \vartheta_1 + \eta_0, \\ \dot{v}_1 &= v_2, \\ &\vdots \\ \dot{v}_{r-2} &= v_{r-1}, \\ \dot{v}_{r-1} &= -\lambda^\top \xi + u, \end{aligned} \quad (32)$$

where $\vartheta_1 = [\ell_{b_{r+1}}, \dots, \ell_{b_n}, \ell_{a_1} - \lambda_{n-1}, \dots, \ell_{a_{n-1}} - \lambda_1, \ell_{a_n}]^\top \in \mathbb{R}^{2n-r}$, $\vartheta_3 = \ell_{b_r} \in \mathbb{R}$, $v_1 = \zeta_{n-r+1}$, $v_2 = \zeta_{n-r+2}, \dots, v_{r-1} = \zeta_{n-1}$, $\varphi(y, d) = [\zeta_{n-r}, \dots, \zeta_1, -\lambda^\top \xi - y, \xi_{n-1}, \dots, \xi_1]^\top$, and d stands for $\xi, \xi_1, \dots, \zeta_{n-r}$. Define $\vartheta_2 = \ell_{b_r}^{-1} \in \mathbb{R}$ and $\vartheta = [\vartheta_1^\top, \vartheta_2, \vartheta_3]^\top \in \mathbb{R}^{2n-r+2}$.

In the spirit of reduced-order observer design, the *adaptive I&I* approach adopts a dynamic (integral) parameter estimate $\hat{\vartheta}$ and a static (proportional) parameter estimate β together for parameter estimation, that is, $\hat{\vartheta} + \beta$ is used for estimating ϑ . Define now the virtual control laws

$$v_1^{**} = \sigma_y + \varphi^\top(\hat{\vartheta}_1 + \beta_1), \quad (33)$$

$$v_1^* = -(\hat{\vartheta}_2 + \beta_2)v_1^{**}, \quad (34)$$

$$\begin{aligned} v_{i+1}^* &= -\sigma_i + \frac{\partial v_i^*}{\partial \hat{\vartheta}} \hat{\vartheta} + \frac{\partial v_i^*}{\partial y} (-\sigma_y + (\hat{\vartheta}_3 + \beta_3)\tilde{v}_1) \\ &\quad + \sum_{j=1}^{i-1} \left(\frac{\partial v_i^*}{\partial v_j} v_{j+1} + \frac{\partial \beta}{\partial \xi} (A_\lambda \xi - e_{n-1}y) \right) \\ &\quad + \sum_{i=1}^{n-r} \frac{\partial \beta}{\partial \xi_i} \zeta_{i+1}, \quad i = 1, \dots, r-1, \end{aligned} \quad (35)$$

and the actual control law

$$u = v_r^* + \lambda^\top \xi, \quad (36)$$

with the virtual control errors $\tilde{v}_i = v_i - v_i^*$, $i = 1, \dots, r-1$, and the update law for the dynamic parameter estimate

$$\begin{aligned} \dot{\hat{\vartheta}} &= -\left(I_{2n-r+2} - \frac{\partial \beta}{\partial \hat{\vartheta}}\right) \left(\frac{\partial \beta}{\partial y} (-\sigma_y + (\hat{\vartheta}_3 + \beta_3)\tilde{v}_1) \right. \\ &\quad \left. + \frac{\partial \beta}{\partial v_1} v_2 + \frac{\partial \beta}{\partial \xi} (A_\lambda \xi - e_{n-1}y) + \sum_{i=1}^{n-r} \frac{\partial \beta}{\partial \xi_i} \zeta_{i+1} \right), \end{aligned} \quad (37)$$

²We only discuss the case in which the relative degree $r \geq 2$, since when $r = 1$ the results can be easily obtained without *backstepping* techniques.

where $\beta = [\beta_1^\top(y, d), \beta_2(y, \hat{\vartheta}_1, d), \beta_3(y, \hat{\vartheta}_1, \hat{\vartheta}_2, d)]^\top$ is the static parameter estimate, with

$$\beta_1 = \gamma_1 \int_0^y \varphi(\chi, d) d\chi, \quad (38)$$

$$\beta_2 = \gamma_2 \text{sgn}(\vartheta_3) \left(\frac{1}{2} \sigma_y y + \int_0^y \varphi^\top(\chi, d) (\hat{\vartheta}_1 + \beta_1(\chi, d)) d\chi \right), \quad (39)$$

$$\beta_3 = \gamma_3 \left(v_1 y - \int_0^y v_1^*(\chi, \hat{\vartheta}_1, \hat{\vartheta}_2, d) d\chi \right), \quad (40)$$

$\gamma_1 > 0$, $\gamma_2 > 0$, $\gamma_3 > 0$ constants and $\sigma_y, \sigma_1, \dots, \sigma_m$ damping terms to be defined.

Remark 4: β_1 is independent of $\hat{\vartheta}$, β_2 is only dependent of $\hat{\vartheta}_1$, and β_3 is only dependent of $\hat{\vartheta}_1, \hat{\vartheta}_2$. Therefore $\frac{\partial \beta}{\partial \hat{\vartheta}}$ is a lower triangular matrix with all-zero diagonal terms and

$$\left(I - \frac{\partial \beta}{\partial \hat{\vartheta}} \right) = \left(I + \frac{\partial \beta}{\partial \hat{\vartheta}} \right)^{-1}. \quad (41)$$

Proposition 2: Consider the system (1), the filters specified by Proposition 1 and the adaptive controller described by (33)-(40) with the damping terms

$$\sigma_y = \left(c_y + 2(r-1)\varepsilon^2(\delta_{\bar{a}_y}^2 + \bar{\varepsilon}^2 \varepsilon_{b_y}^2 \delta_{b_y}^2 + \varepsilon_{P_{\bar{\eta}} \bar{b}_y}^2 \delta_{P_{\bar{\eta}} \bar{b}_y}^2) \right) y, \quad (42)$$

$$\sigma_1 = \left(c_1 + \frac{3}{2\varepsilon^2} \left(\frac{\partial v_1^*}{\partial y} \right)^2 \right) \tilde{v}_1 + (\hat{\vartheta} + \beta_3)y, \quad (43)$$

$$\sigma_i = \left(c_i + \frac{3}{2\varepsilon^2} \left(\frac{\partial v_i^*}{\partial y} \right)^2 \right) \tilde{v}_i + v_{i-1}, \quad i = 2, \dots, r-1, \quad (44)$$

where $c_y > 0$, $c_1 > 0, \dots, c_{r-1} > 0$, $\varepsilon > 0$, $\bar{\varepsilon} > \varepsilon_{P_{\bar{\eta}} \bar{\Delta}_b} \delta_{P_{\bar{\eta}} \bar{\Delta}_b}$. Then all signals in the overall closed-loop system are globally uniformly bounded and $\lim_{t \rightarrow \infty} y(t) = 0$. $\diamond$

Remark 5: In Section 4.2.4 of [5] stability properties of the same system, yet with constant parameters, are discussed. It is shown that when the parameters are constants, the stable dynamics of the reparameterization error η can be directly ignored, and the dynamics of the parameter estimation error z satisfies $\Phi^\top z(t) \in \mathcal{L}_2$. The controller-plant subsystem is cascaded with the parameter estimator, driven by $\Phi^\top z$. However, when there are time-varying parameters in the system, the y - $\tilde{v}$ -dynamics, the z -dynamics, the $\bar{\eta}$ -dynamics, and the $\bar{x}$ -dynamics are all coupled together with loops (instead of cascaded relations), which requires a small-gain like argument (in this paper, from a Lyapunov perspective) to establish the stability of the overall closed-loop system. This comparison reveals the challenges emerging in adaptive control problems when the parameters are time-varying. $\diamond$

VI. SIMULATIONS

Consider the benchmark system described by the equations

$$\begin{aligned} \dot{x}_1 &= -a_1(t) + b_1(t)u, \\ \dot{x}_2 &= -a_2(t) + b_2(t)u, \end{aligned} \quad (45)$$

with time-varying parameters given by $b_1(t) = 1 + 0.2 \sin(5t)$, $b_2(t) = 2 + 1.5 \cos(20t)$, $a_1(t) = 1 + \frac{5 \text{sgn}(y) \varphi_2}{\sqrt{\varphi_2^2 + \varphi_3^2}}$,

$a_2(t) = 1 + \frac{5\text{sgn}(y)\phi_3}{\sqrt{\phi_2^2 + \phi_3^2}}$, where $\phi = [\zeta, -\lambda\xi - y, \xi]^\top$, and ϕ_2, ϕ_3 are the second and the third element of ϕ , respectively. $a_1(t)$ and $a_2(t)$ are state-dependent time-varying parameters designed to destabilize the system. One can easily verify Assumption 4 since the zero dynamics of (45) is exponentially stable due to the fact that $\frac{b_2(t)}{b_1(t)}$ is strictly positive. Note also that system (45) has relative degree $r = 1$, we can therefore design a controller without $\hat{\vartheta}_3$ and β_3 due to the reduced-order filters. Thus the update law (37) is reduced to $\dot{\hat{\vartheta}} = -(I_2 - \frac{\partial\beta}{\partial\hat{\vartheta}})(\frac{\partial\beta}{\partial y}(-\sigma_y + \frac{\partial\beta}{\partial\xi}(-\lambda\xi - y) + \frac{\partial\beta}{\partial\zeta}(-\lambda\zeta + u)))$. Set the controller parameters as $\gamma_1 = \gamma_2 = 0.1$, $\lambda = 100$ and initialize all filter states and parameter estimates to 0. For comparison, consider two controllers: Controller 1, the classical design with a damping term $\sigma_y = y$, and Controller 2, the proposed design with a strengthened damping term $\sigma_y = 7y$. In Fig. 1, the “Baseline” results are obtained

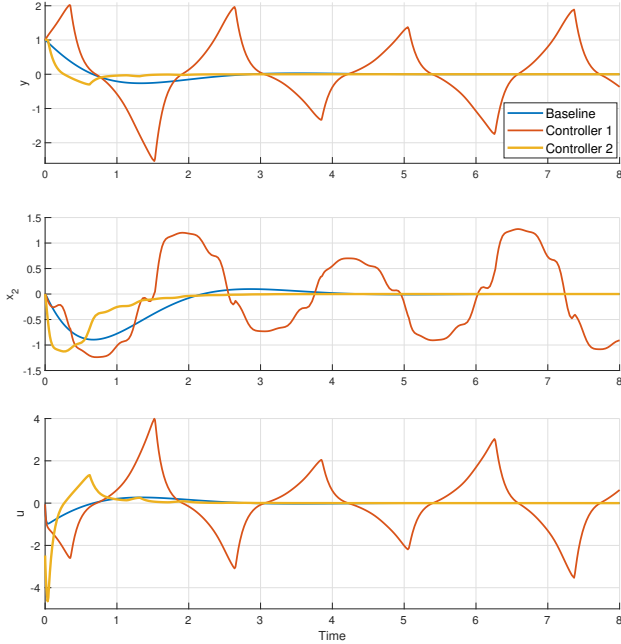


Fig. 1. Time history of the system state and the control signal driven by different controllers.

by implementing Controller 1 under constant parameters $b = [1, 2]^\top$, $a = [1, 1]^\top$ and the other results are obtained by implementing the corresponding controllers under time-varying parameters. One can see that Controller 2 restores the baseline performance in the presence of the time-varying parameters.

VII. CONCLUSIONS AND FUTURE WORK

This paper discusses the combination of the output-feedback *adaptive I&I* approach with the *congelation of variables* method to solve the adaptive output regulation problem when time-varying parameters exist. The system is reparameterized using reduced-order filters, the reparameterization error dynamics are derived and discussed. The coupling

between the input and a time-varying perturbation is replaced by the coupling between the output and another time-varying perturbation by exploiting a so-called strong minimum-phase property, which is defined as ISS of the inverse dynamics of the system. This definition is an extension and modification of the classical minimum-phase property for establishing stability of time-varying systems. High-gain filters are then implemented to guarantee ISS of the reparameterization error dynamics. Finally an *adaptive I&I* controller is designed by synthesizing small-gain-like loops among the controller-plant dynamics, parameter estimation error dynamics, reparameterization error dynamics, and the inverse dynamics, which guarantees output regulation and boundedness of all signals. The simulations show performance improvement on the classical design under parameter variations.

Future work can be carried out to further relax Assumption 5, which is still an obstacle for the proposed method to be implemented for arbitrary linear single-input-single-output systems with time-varying parameters.

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