FIELD THEORY MODELS AND SPONTANEOUS SYMMETRY BREAKING

by

SARBEN SARKAR

Physics.

A thesis presented for the Degree of Doctor of Philosophy of the University of London.

Abstract:

The thesis is divided into four chapters (apart from the introduction).

In chapter one a general formal foundation of local theories using the Higgs-Kibble mechanism is given in the unitary gauge. The general term that is added to the Lagrangian owing to the presence of a gauge group is found both when the gauge group has and has not a non-trivial unbroken subgroup. The conjecture made by S. Weinberg about the form of this term when there is a non-trivial unbroken subgroup is found to be false in general. However, when the unbroken subgroup is abelian, the conjecture is shown to be correct. A brief discussion of possible theories of currents following from these theories is given.

In chapter two a specific model of weak and electromagnetic interactions suggested by B. Lee and Prentki and Zumino is used in studying K_{l_3} , K_{l_2} and π_{l_2} decays.This model uses the gauge principles given in chapter one. We find that the model does not fit the data well even though it has many arbitrary parameters. Comparisons with current algebra are also made.

In chapter three non-polynomial theories for which the fields are only functions of time are primarily investigated. The usual contour integral methods are shown to be unnecessarily ambiguous. The conditions for the validity of the Kinking theorem are found not to be entirely consistent. The validity of the extrapolation

of such results to the one time and three space field theories is discussed.

In chapter four the zero space dimension $\lambda\psi^4$ model is studied. A non-perturbative solution in the one-loop approximation for the self-energy functional is obtained. This amplitude is found to be non-analytic in λ. The possible implications for higher dimensional field theories as well as the validity of perturbation theory are discussed.

Preface:

The work presented in this thesis was carried out in the Department of Physics, Imperial College, London University, under the supervision of Dr. C. J. Isham. I would like to thank him and Dr. R. Delbourgo for their interest and encouragement. The material given in this thesis is original, except where otherwise stated in the text, and has not been submitted in this or any other University for any other degree. Moreover, Dr. C. J. Isham has shown much sympathetic understanding and this has helped to double my enthusiasm for my research work. I am also grateful to Professors P.T. Matthews, T.W.B. Kibble, and A. Salam for giving me every possible assistance. I am particularly indebted to Professor A. Salam for useful discussions and for his hospitality at the International Centre for Theoretical Physics, Trieste. All other members of the Theory Group have helped to shape my outlook for the better. Finally I wish to thank the Science Research Council for financial support.

Contents:

		Page
Abstract		2
Preface		4
Contents		5
Introduction		6
Chapter 1	Spontaneous symmetry breaking in gauge theories	23
Chapter 2	Gauge model calculations	54
Chapter 3	Non-polynomial field theories in low dimensions	71
Chapter 4	Non-perturbative calculations	90
Appendix 1		109
Appendix 2		115
Appendix 3		118
Appendix 4		120
References		122

Introduction:

In the approach that we shall adopt, a physical theory is given by a lagrangian and an appropriate quantisation procedure. A recurrent theme in particle physics is the invariance of the lagrangian under group transformations of the fields. Often, the parameters of the group transformation are not dependent on space-time, but theories in which this is not the case are known as gauge theories. The importance of these theories is best understood from specific examples.

A free Dirac theory of electrons is invariant under a phase transformation which is independent of space-time. Such a theory requires the phase at all points to be changed equally and simultaneously. Dynamically there does not seem to be any reason for this. However, if we allow a mediating field, it is possible to have the phase of the electron field varying from one space-time point to another. The mediating field is known as the electromagnetic field. In fact the line integral of this field between two points measures the change in phase of the electron field induced by a general gauge transformation. The gauge group of electrodynamics is isomorphic to rotations in two dimensions. Yang and Mills[1] were the first to generalise this idea to the case of isotopic spin. The concept of isotopic spin invariance arose from the observation that particles such as the proton and the neutron are very similar, differing only in their charge, and that in their

strong interactions they behave identically. If we consider that all interactions apart from the strong interactions are switched off it should be possible to adopt arbitrary local definitions of proton and neutron states at different points. As in the case of quantum electrodynamics (Q.E.D.) a local gauge invariance can be achieved by the introduction of a mediating field (with actually three components). Nowadays a Yang-Mills group is taken to denote the infinite dimensional group which is obtained by taking the continuous direct product of an arbitrary Lie group with itself over space-time.

Although the idea of Yang and Mills is very simple and beautiful it still only allows us to write down a lagrangian. There remains the major problem of making calculations with such a lagrangian. We will consider two examples which will naturally lead to the study of the main items in the thesis:

$$\mathcal{L}_1 = -1/4\,(G_{\mu\nu})^2 - (D_\mu K)^\dagger (D_\mu K) - mK^\dagger K - 1/2\,\lambda(K^\dagger K)^2$$

where W^a is a massless Yang-Mills isospin one triplet of vector mesons and K is an isospin a half scalar field first introduced in this context by Higgs[2] and Kibble[3]. This is very close to the original model of Yang and Mills except that we are dealing with scalars and have a quartic scalar interaction. We have the definitions

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g \epsilon_{abc} W^b_\mu W^c_\nu$$

and $D = \partial_\mu - 1/2\ ig\ W^a_\mu \tau^a$ (where τ^a, a = 1,2.3 are Pauli matrices). Explicitly $\mathcal{L}_1$ is invariant under the group transformation

$$W^a_\mu \longrightarrow W'^a_\mu = W^a_\mu + g\epsilon_{abc} \Lambda^b W^c_\mu - \partial_\mu \Lambda^a$$
$$K \longrightarrow K' = (1 - i/2\ g \Lambda^a \tau^a) K$$

The essential difference between $\mathcal{L}_1$ and Q.E.D., mathematically, is the non-abelian nature of the gauge group. Physically, since experiments do not find evidence for massless spin one particles other than the photon, if $\mathcal{L}_1$ is to be capable of development into a physical theory all but one component of the W boson should be massive. In fact the masslessness problem in $\mathcal{L}_1$ can be avoided and this was the major achievement of Higgs and Kibble, Hagen and Guralnik[4]. If we write

$$K = 1/\sqrt{2}\ (M + \sqrt{2}k + i \phi^a \tau^a) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where the fields M, and (ϕ^a ; a = 1,2,3) are hermitian and require a non-invariance of the vacuum under the symmetry by demanding

$$\langle 0 | M | 0 \rangle = 0$$

then the term linear in M must vanish i.e.

$$-\sqrt{2}\, mk - \sqrt{2}\, \lambda k^3 = 0$$

This implies that $k = 0$ or $k = \sqrt{-m/\lambda}$.

Now if m is positive , and k is non-zero then we must have the mass of the W boson imaginary. If we want k non-zero and m negative then $\mathcal{L}_1$ in terms of the new fields M and ϕ^a becomes

$$-1/4\ G^a_{\mu\nu}\ G^a_{\mu\nu} - 1/2\ (gk/\sqrt{2})^2\ W^2 - 1/2\ (\partial_\mu M)^2$$
$$- 1/2\ (\ 4\ \frac{\lambda}{g^2}\ \frac{g^2}{2}\ k^2)M^2 - 1/2\ (\partial_\mu \phi^a)\ (\partial^\mu \phi_a) + \ldots$$

This is in its most naive form , the mechanism of Higgs and Kibble. Before considering the quantisation aspects we will show another naturally occurring gauge symmetry. In fact general relativity can be written in a Dirac γ-matrix basis which emphasises the SL(2,C) gauge symmetry. The Einstein lagrangian is

$$\mathcal{L}_2 = \frac{i\ \mathrm{Tr}\ [L^\mu, L^\nu]\ B_{\mu\nu}}{\sqrt{-\det \frac{1}{4}\ \mathrm{Tr}\ L^\mu L^\nu}}$$

where L^μ and B^μ are fields which transform as

$$L^\mu(x) \longrightarrow L'^\mu(x) = (\ \bar{S}(x)\ )^{-1}\ L^\mu(x)\ S^{-1}(x)$$
$$B^\mu(x) \longrightarrow B'^\mu(x) = S(x)\ B_\mu\ S^{-1}(x) - i\ S(x)\partial_\mu S^{-1}(x)$$

and $S(x)$ is exp i $(\ \epsilon_{ab}(x)\ \frac{\sigma^{ab}}{2}\ )$. We define $\bar{S}$ to be $\gamma_0 S^\dagger \gamma_0$.

As usual $B_{\mu\nu}$ is the invariant field tensor

$$\partial_\mu B_\nu - \partial_\nu B_\mu + i\ [B_\mu, B_\nu]$$

i.e. $B_{\mu\nu}$ transforms under SL(2,C) as

$$B_{\mu\nu} \longrightarrow B'_{\mu\nu} = S\ B_{\mu\nu} S^{-1}$$

The striking new feature here[5] is the naturally occurring non-polynomiality in the lagrangian. Methods have to be invented to deal with this.

We will mainly consider cases in this thesis for which $\mathcal{L}_1$ and $\mathcal{L}_2$ are typical. In common with Q.E.D. we have the problems of

gauge quantisation [6]. The problems of quantising are already present in the case of the free electromagnetic field. The equations of motion are given by the vanishing of the four divergence of the field tensor. Owing to the abelian gauge freedom there is an infinity of solutions for the photon field. The established methods for dealing with this problem all break the gauge symmetry. One method keeps the lagrangian in tact but the divergence of the photon field is fixed. This in effect chooses a gauge. Alternatively a piece $\delta\mathcal{L}$ can be added to the lagrangian to make it no longer gauge invariant. Fermi used this second procedure and for $\delta\mathcal{L}$ took minus half the square of the four divergence of the photon field. This approach involves an indefinite metric[6]. States containing an odd number of time-like photons have a negative norm. The actual details are non-trivial and the above techniques are of somewhat historical interest owing to the method of Faddeev and Popov[7] which leans heavily on formal arguments based on the Feynman path integral. The question of unitarity and an indefinite metric are closely connected. Feynman[8] was motivated in his quantisation of Yang-Mills fields by the need to make his amplitudes unitary rather than gauge invariant. Fictitious particles which flowed only in closed loops were found to be necessary. The fictitiousness of the particles is seen from the extra factor of (-1) associated with the closed loop. Just as in the standard treatment in Q.E.D. a gauge breaking is introduced into the lagrangian via a Lorentz type condition or via adding a non-invariant piece to the lagrangian. A treatment of

the main features in the method of Faddeev and Popov will be given in Appendix 1.

The remaining problems are those of renormalisation. If a lagrangian involving a general field ψ has the expansion

$$\mathcal{L}(\psi) = \sum_{n=0}^{\infty} L_n/n! \quad \psi^n$$

where L_n are independent of ψ then we can consider three separate cases. We can have

i) $L_n = 0$ for n greater than 4.

ii) $L_n \neq 0$ for a finite number of values of n larger than four.

iii) $L_n \neq 0$ for an infinite set of values of n.

Cases (i) and (ii) are both polynomial lagrangians whereas case three is not. In fact case (ii) is not renormalisable and no useful theory exists for it. Case (i) gives the simplest situation. However when local non-abelian gauge symmetries are present the renormalisability is not easy to prove; owing to the work of 't Hooft[9] and Lee and Zinn-Justin[10] $\mathcal{L}_1$ (and generalisations of this) have been shown to be renormalisable.

Case (iii) seems at first sight to be the least tractable. However a remarkable amount of progress has been made on these theories by Delbourgo, Isham, Salam and Strathdee[11]. A very important example is given by

$$\mathcal{L}(\phi) = G : \exp(\kappa\phi(x)) - 1 :$$

Formally $\quad <T(:\exp \kappa\phi(x) - 1 : : \exp \kappa\phi(0) - 1 :)>_0$

equals $\exp(\kappa^2 D_F(x)) - 1$ ($\mathcal{D}(x)$ say) where $D_F(x)$ denotes the free Feynman propagator for a scalar field $\phi(x)$. If reasonable field theories are to emerge for such interactions it is necessary to have adequate distribution theoretic interpretations for $\mathcal{D}(x)$. Following Speer[12], Isham, Salam and Strathdee and Mitter[13] we define

$$\mathcal{D}(x,\lambda) \equiv \sum_{n=1}^{\infty} \left(\frac{\kappa^2}{4\pi^2}\right)^n \frac{1}{n!} \left(x^2 + i0 \right)^{n(\lambda-1)}$$

Although formally the $\lambda=0$ limit gives $\mathcal{D}(x)$ we have to be more careful. Jaffe has introduced a new class of test function spaces which generalises the space of Schwarz in such a way that in momentum space the test functions decrease exponentially (approximately) while in co-ordinate space the Jaffe spaces have intersections with the space of infinitely differentiable functions with compact support which are dense in the latter space. In fact it can be shown that $\mathcal{D}(x, \lambda)$ is a Jaffe distribution when $0 < \mathrm{Re}\,\lambda < 1$, $0 < a < \mathrm{Im}\,\lambda < 1$ and a is a fixed constant. Speer was the first to introduce analytic renormalisation in conventional perturbation theory. However the actual idea of defining ' ill-defined ' quantities by analytic continuation goes back to M. Riesz[14]. The $\lambda=0$ limit is normally singular and Speer devised a general evaluator on a function of several variables which in essence is the operation of taking the constant term of the Laurent expansion of the function. The essence of the non-polynomial method is the extension of the Speer evaluator to the case when the Laurent expansion does not exist. For an

arbitrary function

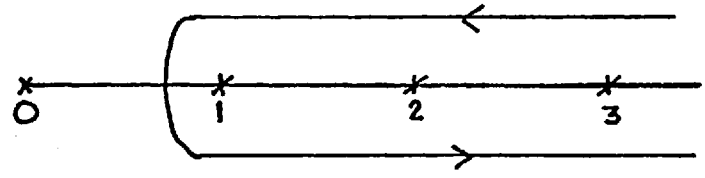
$$A(\lambda) \equiv A_L(\lambda) + A_{NL}(\lambda)$$

where A_L has a Laurent expansion at $\lambda=0$ but A_{NL} does not. The decomposition is obviously non-unique and the methods of Isham, Salam and Strathdee, Volkov[15] and Mitter give a prescription for obtaining an A_L. Given such an A_L the method of Speer can be used to obtain a value at $\lambda=0$ for $A(\lambda)$. Now taking the Fourier transform $\mathcal{F}$ of $\mathcal{D}(x,\lambda)$ we have

$$\mathcal{F}(\mathcal{D}(x,\lambda)) = -\frac{\pi^3 2^4}{2i}\int_\Gamma dz \; \frac{\cos\pi z \left(2^{2\lambda}\frac{x^2}{4\pi^2}\right)^z}{\sin\pi z \sin\pi(1-\lambda)z} J(p,\lambda)$$

where $J(p,\lambda) = \dfrac{(p^2-i0)^{(1-\lambda)z-2}}{\Gamma((1-\lambda)z-1)\,\Gamma((1-\lambda)z)\,\Gamma(z+1)}$

and the contour Γ is given by



(Here we have used that

$$\mathcal{F}\left(\frac{1}{4\pi^2}(x^2+i0)^{n(1-\lambda)}\right) = \pi^2\left(\frac{p^2-i0}{4}\right)^{n(1-\lambda)-2}\frac{\Gamma(2-n(1-\lambda))}{\Gamma(n(1-\lambda))}$$

both sides of the equation being interpreted as Schwarz distributions). As $\lambda \to 0$ Γ is pinched at $z = n$, $n = 2, 3, 4, \ldots$ $(\mathcal{F}(\mathcal{D}))_{NL}$ is extracted by locally deforming the contour Γ so as to enclose the pinch singularities. Actually when we take the

limit $\lambda \to 0$ we find that

$$\text{Re}\,(i\mathcal{F}D) \sim 0 \qquad \text{as} \quad p^2 \to \infty$$

$$\text{Im}\,(i\mathcal{F}D) \sim \exp\,(p^2)^{1/3} \quad \text{as} \quad p^2 \to \infty$$

There is a rapid fall off of the real part[16]. Now from the work of Bogoliubov, Shirkov and Jaffe[17] we know that the T^x products of the interaction lagrangian are ambiguous up to operators with point support. To second order in the coupling constant the coefficient of such an operator is a Jaffe distribution of the form

$$\sum_{n=0}^{\infty} c_n \Box^n \delta^{(4)}(x)$$

with c_n real and $\sum c_n z^n$ an entire function of order less than a half.(This last requirement is due to locality). Our superpropagator does not contain this type of ambiguity because of the fast decay at infinity.Lehmann and Pohlmeyer[16] noticed this and called this property ' minimal singularity '; so far for $L_{int} = :\exp(\lambda\phi) - 1:$ it has been possible to construct Greens functions displaying minimal singularity for second and third order Greens functions only. It is useful at this stage to consider what is desirable in a quantum field theory. Following Wightman we will require

(a) a separable Hilbert space, $\mathcal{H}$, of states

(b) covariance of the fields under inhomogeneous SL(2,C) transformations

(c) the spectrum of the generators of translations lie in or on the plus cone

(d) local commutativity of fields

(e) asymptotic completeness, i.e. $\mathcal{H} = \mathcal{H}^{in} = \mathcal{H}^{out}$

Usual local quantum field theory has fields which are Schwarz distributions. From our review of second order perturbation calculations using the exponential lagrangian it seems that we should be considering Jaffe fields. Working from the axioms (a) to (e) it is possible to derive for this general class of fields a connection between spin and statistics, existence of PCT symmetry, crossing symmetry, the proof of dispersion relations, and the asymptotic condition. It sometimes happens that a non-polynomial lagrangian arises from a polynomial lagrangian by an analytic redefinition of fields. Indeed a non-localisable field constructed as a strong limit of local fields can be shown to be PCT equivalent to a given local field provided certain conditions hold on the local fields of the approximating sequence. Fields which are PCT equivalent will be S equivalent if they have the same in-fields.

$\mathcal{L}_2$ unlike the simple non-polynomial field theory considered above contains a gauge symmetry. Both these facets are intimately connected with each other and the nature of the problem of derivatives is best exemplified by the following example. (A proper distribution theoretic formulation of the argument is easy but cumbersome).

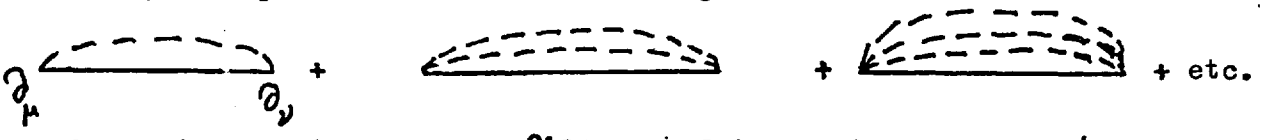
$$\mathcal{L} = 1/2 \ (\partial_\mu \chi \partial^\mu \chi) : v(\phi) : + 1/2 \ \partial_\mu \phi \, \partial^\mu \phi$$

where $v(0)$ is 1 and χ and ϕ are independent scalar fields.

We will define $\Pi_{\mu\nu}(x,y)$ as

$$\Pi_{\mu\nu}(x,y) \equiv \langle 0 | T\,(\partial_\mu \chi(x) : v(\phi(x)) - 1 : \partial_\nu \chi(y) : v(\phi(y)) - 1 :) | 0 \rangle$$

This just represents the series of diagrams



where '———' represents χ lines and '– – –' represents ϕ lines.

Now

$$\Pi_{\mu\nu}(x,y) = \partial_\mu \partial_\nu D_F(x-y) \langle 0 | T(: v(\phi(x))-1 :: v(\phi(y))-1 :) | 0 \rangle$$

In fact using standard techniques we have (taking $v(\phi) = \sum_{n=0}^{\infty} \phi^n \lambda^{\frac{1}{2}n} v(n)$)

$$\mathcal{F}\Pi_{\mu\nu}(p^2) = \frac{1}{2\pi i} \int_\Gamma dx\, e^{ip\cdot x} \int dz\, \Gamma(-z)\, (-\lambda)^z (-(v(z))^2) D_F^z(x) \partial_\mu \partial_\nu D_F(x)$$

We shall use the derivative formula of Salam which is obtained by taking time imaginary and performing the usual derivative operations in the resulting Symanzik region of space time. The formula is

$$D_F^z \partial_\mu \partial_\nu D_F = \frac{2}{(z+1)(z+2)} \left\{ \partial_\mu \partial_\nu - \frac{1}{4} g_{\mu\nu} \partial^2 \right\} D_F^{z+1}$$

There are of course analogous derivative formulae for other products of derivatives and these are always arranged so that the order of derivatives on both sides of a relation is the same. Using the above formula we have

$$\Pi_{\mu\nu}(p) = (1/2\pi i) \int e^{ip\cdot x} \int_\Gamma dz\, \frac{\Gamma(-z)\, (-\lambda)^z (-(v(z))^2)}{(z+1)(z+2)} \cdot 2 (\partial_\mu \partial_\nu - \tfrac{1}{4} g_{\mu\nu} \partial^2)\, D^{z+1}$$

Performing the co-ordinate space integration we have

$$\mathcal{F}\Pi_{\mu\nu}(p) = \frac{2\pi}{i}\int dz\ (-\lambda)^z\,\Gamma(-z)\ (-2\,(v(z))^2)\frac{\Gamma(1-z)}{(z+1)(z+2)\,\Gamma(z+1)}$$
$$\cdot(\tfrac{1}{4}p^2)^z\,(-p_\mu p_\nu/p^2 + \tfrac{1}{4}g_{\mu\nu})$$

It is to be expected that as $\Gamma \longrightarrow \Gamma'$ (where Γ' is given by displacing Γ by one unit to the left so that 0 is now enclosed by the contour) the residue at z=0 should correspond to a simple χ- propagation. This is particularly clearly seen from our diagrammatic interpretation for $\Pi_{\mu\nu}$. However we find that

$$-i(2\pi)^{-1}\,\mathrm{Res}(p_\mu p_\nu \Pi^{\mu\nu})_{z=0} = \tfrac{3}{4}\,p^2 \quad \text{instead of} \quad 1.p^2.$$

It may be argued with some justification that the derivative formula is manifestly absurd since it implies

$$\Box D_F = 0$$

D_F is well-known to be an inhomogeneous equation, and it is easy to modify the theory in an ad hoc way to correct this. However this ad hoc procedure has not been generalised to arbitrary products of derivatives. Given a derivative prescription it is still necessary that only a formally gauge invariant set of diagrams has been included in the definition of the superpropagators. Some progress in the chiral case has been made [18]. In principle a gauge invariant set of diagrams can be defined for more complicated theories such as gravitation. However the problems of obtaining a compact expression on which superpropagator techniques may be usefully applied are present. Even given a formally gauge

invariant set of graphs it is necessary to regularise them in a gauge invariant manner. The non-polynomial method has built in a regularisation somewhat similar to Speer's. The latter regularisation is known not to be gauge invariant in general and so we expect a similar sort of trouble in non-polynomial theories.

In this thesis we are interested primarily in questions involved in quantising lagrangians which are non-polynomial in form or involve gauge fields (or both). The subject is very technical but we are stimulated to this study by the undoubted elegance of gauge field theories. The scheme of the thesis is as follows :

In chapter one we consider spontaneously broken local symmetries. No explicit symmetry breaking is ever introduced. Indeed in the arguments given by 't Hooft and others concerning renormalisability of massive Yang-Mills fields gauge invariance played the dominant role. The basic ideas involved in giving vector bosons masses is discussed at a dynamical level. Recently Weinberg[19] has considered the quantisation of these local theories in the general case in the unitary gauge.He made a very interesting conjecture about the non-local gauge compensating term (characteristic of gauge theories) which has to be added to the lagrangian. In essence he required this term to be independent of the generators of the unbroken subgroup. This will all be transparent when the technical apparatus has been set up. Unfortunately Weinberg's method of attack was not flexible enough to prove his conjecture.

We[19] used the very powerful method of Faddeev and Popov to examine under what conditions the conjecture of Weinberg is valid. In addition the results of Weinberg for the case when the gauge group has no unbroken subgroup are easily obtained using F-P. A brief discussion of the currents in the theory follows. Currents have not been the centre of attention in these spontaneous symmetry breaking models since the possibility of higher order corrections which are calculable is the main difference between these models and effective lagrangians used in the past. We find difficulties inherent in the spontaneous symmetry breaking nature of our lagrangian which seems to make the method of generating Sugawara type theories using the method of Bardakci, Halpern and Frishman not directly applicable. This approach has recently been advocated for the Higgs-Kibble type theories. We have not found any consistent way of modifying the above procedure to make it satisfactory. It is amusing to note the presence of q-number Schwinger terms in some current commutators, a feature which also occurs in the work of Barnes and Isham[21]. This perhaps should not be too surprising since non-linear realisations are closely connected with spontaneous symmetry breaking.

In chapter two models of unified theories of weak and electromagnetic interactions are discussed. We give some simple criteria for the usefulness of these models. In particular we examine the Lee-Prentki-Zumino[22] model (LPZ). There are many other models, of course, but this is one of the simplest off-springs of the

Salam-Weinberg[23] model which avoids the obvious experimental pitfalls. Indeed until recently no better models were known. The models which are now available are often more complicated and so the predictive power is not great. In LPZ we find that we are unable to calculate[24] f_π and f_K mutually consistently. Some other features are found to support current algebra predictions and yet others to contradict current algebra results. We found that the leptonic processes were not very useful in constraining the LPZ model. Only the semi-leptonic processes seem to imply constraints. Actually the purely leptonic process $\bar{\nu}_e$ e $\longrightarrow$ $\bar{\nu}_e$ e gives a very modest limit on the mass of the new scalar mesons which occur in the theory. Many calculations exist now in the literature using other models but we only believe the order of magnitudes of the calculations.

In chapter three we consider one dimensional non-polynomial field theories. At first sight this may seem an extreme simplification but it is really surprising how much of the established features of the four dimensional features persist. The degree of rigour and simplicity in interpretation of tadpoles allows in our opinion a useful probe into the thorny topic of derivative interactions[26]. Various forms of the non-polynomial interactions are considered. The behaviour at infinity of the interactions as well as the analytic properties of superpropagators in the minor coupling constant both point to exponential interactions as being the best behaved.

It is necessary to be somewhat cautious about extrapolating the results of low dimensional field theories to higher dimensional ones. In particular the exact nature of the λ analyticity in the minor coupling constant does change as the dimension of the space changes. The most interesting aspect of this study is of course the possibility of a rigorous treatment of derivatives. In the fully relativistic case these derivatives are very badly understood owing partly to the first type and second type tadpoles that occur. We have little to guide us in our choice of values of these tadpoles; the work in lower dimensional theories will not help us to find the values of the tadpoles but we hope that it will show up the degree to which the ad hoc assumptions about tadpoles may be accurate. We were able to scrutinise the Kinking theorem[11] and it was found to be not quite consistent. We suspect that this is true also in the four dimensional case but at the moment the conceptual difficulties (in the non-polynomial framework where counterterms are not permitted) are too great to extend the considerations we have given.

In chapter four we look at perturbation theory itself and consider whether it may not be a suitable tool. In quantum gravity we would like to know the fluctuations about the classical solution for the metric; but this is too difficult at the moment. Using classical solutions given by Raczka[27] we find a closed form for the self-energy functional for the one dimensional theory $\lambda q^4(t)$. in the one loop approximation. Since the solution is non-perturbative

the expressions that we obtain are very complicated. It is clearly necessary to understand the nature of the solution of a simple case like this before the more realistic and complex cases are treated. It is very interesting to see emerging a non-analyticity in λ even for such a simple divergence free theory[28]. Our experience[25] is that coupling constant analyticity becomes more singular with higher dimensions of space. In this connection we will discuss an amusing and perhaps profound observation made by Salam and Weisskopf about the role of λ -analyticity and the nature of the Schwarzschild singularity. This investigation supports the view that coupling constant perturbation theory may not be a valid procedure to follow.

Four appendices have been included. The first deals with Faddeev-Popov quantisation and Feynman path integrals. The second gives some of the interactions in the LPZ model as well as a brief consideration of the order of magnitude of higher order calculations. The third appendix gives a simple exposition of the Kinking theorem due to I.S.S.[11]. The final appendix is concerned with the calculation of the classical action involved in the considerations in chapter four.

Chapter 1 Spontaneous symmetry breaking in gauge theories

We would first like briefly to examine the dynamical reasons for the apparent breakdown of Goldstone's theorem[4] in the Higgs-Kibble mechanism. We cannot really solve any non-trivial field theories in four dimensions; fortunately the free field gives a non-trivial illustration of the Higgs-Kibble mechanism. We will thus consider (in a first order formalism)

$$\text{i)}\quad \mathcal{L}_G = \phi^\mu \partial_\mu \phi + \tfrac{1}{2}\phi^\mu \phi_\mu - \tfrac{1}{2}\mu_o^2 \phi^2$$

$$\text{ii)}\quad \mathcal{L}_{HK} = -\tfrac{1}{2} F_{\mu\nu}(\partial^\mu A^\nu - \partial^\nu A^\mu) + \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + \phi^\mu \partial_\mu \phi + \tfrac{1}{2}\phi^\mu \phi_\mu - g\phi^\mu A_\mu$$

$\mathcal{L}_G$ denotes $\mathcal{L}_{\text{Goldstone}}$ whereas $\mathcal{L}_{HK}$ denotes $\mathcal{L}_{\text{Higgs-Kibble}}$. Actually the developments of the Higgs-Kibble mechanism were due to many people. In fact it was P.W. Anderson[28] who suggested the mechanism first for non-relativistic field theories. Now $\mathcal{L}_G$ is invariant under a constant shift in the field when $\mu_o = 0$. The conserved current for this symmetry is ϕ^μ_G. Similarly $\mathcal{L}_{HK}$ is invariant under the field transformations

$$\phi(x) \longrightarrow \phi'(x) = \phi(x) + g\,\varepsilon(x)$$
$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \varepsilon(x)$$

In the proof of the Goldstone theorem the commutator

$$C^\mu(x) = \langle 0 | [\phi^\mu(x), \phi(0)] | 0 \rangle$$

plays an important part. The broken symmetry condition is just

$$\int_{R^3} C^0(x)\ d^3x \neq 0$$

The current conservation requirement in the Goldstone theorem hypotheses imposes

$$\partial_0 \int_{R^3} C^0(x)\ d^3x = 0$$

Given these two conditions the proof of the Goldstone theorem goes through. For $\mathcal{L}_G$ we have

$$C_G^\mu(x) = -\partial^\mu \Delta(x,0)$$

where

$$\Delta(x,\mu^2) = 2\pi \int \frac{d^4p}{(2\pi)^4}\ e^{ip\cdot x}\ \varepsilon(p^0)\ \delta(p^2-\mu^2)$$

Explicit calculation gives

$$\int_{R^3} C_G^0(x)\ d^3x = -i\cos\mu_0 t = -i \text{ when } \mu_0 = 0$$

So the Goldstone theorem hypotheses apply if and only if $\mu_0 = 0$. This is of course obvious from simpler considerations in this case!

We will choose to quantise $\mathcal{L}_{HK}$ in the Coulomb gauge. This is not a covariant gauge and it is conceivable that quantisation using a manifestly covariant gauge may give drastically different conclusions. Actually in the Lorentz gauge Goldstone bosons appear! However these massless bosons occur in the lagrangian with the wrong sign for the kinetic energy and so they are unphysical. A method of introducing the Coulomb gauge condition is to add to the lagrangian a lagrange multiplier term $-\lambda \sum_{k=1}^{3} \partial_k A^k$

The field equations are

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\phi_\mu = -\partial_\mu \phi + g A_\mu$$

$$\partial_\mu \phi^\mu = 0$$

$$\partial_\mu F^{\mu\nu} = g\phi^\nu + \delta_\kappa^{\ \nu}\, \partial^\kappa \lambda$$

$$\partial_k A^k = 0$$

The equations which do not involve time are the constraint equations and we see that

$$-\nabla^2 A^0 = -g\phi^0 = \partial_k F^{0k}$$

The dynamical variables ϕ , ϕ^0 , and the transverse components of F^{0k} and A^k are independent. The canonical commutation relation of interest is

$$\left[\phi^0(\underline{x},t),\ \phi(\underline{y},t)\right] = -i\delta(\underline{x}-\underline{y})$$

and so we see that $ig\phi^\mu$ is the conserved current which generates the first type gauge transformations in $\mathcal{L}_{HK}$. We write

$$g V_\mu = g A_\mu - \partial_\mu \phi_{HK}$$

and then $\mathcal{L}_{HK}$ becomes $\mathcal{L}'_{HK}$

$$\mathcal{L}'_{HK} = -\tfrac{1}{2} F^{\mu\nu}(\partial_\mu V_\nu - \partial_\nu V_\mu) + \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} - \tfrac{1}{2} g^2 V^\mu V_\mu$$
$$-\lambda \partial_k \left(V^k - g^{-1} \partial^k \phi \right)$$

Moreover

$$C^\mu_{HK}(x) = ig^3 \left\langle 0 \middle| \left[V^\mu(x),\ (\nabla^2)^{-1} \partial_k V^k \right] \middle| 0 \right\rangle$$

From the form of $\mathcal{L}'_{HK}$ we see that we have a free field and so

we know the commutator involved in C^{μ}_{HK} for all values of x. In fact

$$i\ \left[V^{\mu}(x),\ V_{\nu}(y)\right] = (\delta^{\mu}_{\nu} - g^{-2}\partial^{\mu}\partial_{\nu})\Delta(x-y;g^2)$$

An explicit calculation gives

$$\int_{R^3} C^{o}_{HK}(x)\ d^3x = g\cos gt$$

It is this time dependence which spoils the proof of Goldstone's theorem.

We will now turn to the main interest of this chapter : the setting up of a general formalism for local Higgs-Kibble type theories. We shall consider a compact Lie group G of dimension N and generators $\left\{J_{\alpha} \middle| \alpha = 1,\ldots,\ N\right\}$. Our theory contains spin zero fields, $\phi_p(x)$, transforming under a representation R_0 of G, spin half fields $\psi_n(x)$ transforming under a representation $R_{\frac{1}{2}}$ of G and finally spin one fields $A_{\alpha\mu}(x)$ transforming under the adjoint representation of G. (Frequently we shall suppress the representation indices such as p, n, and α). The most general lagrangian compatible with conventional renormalisability and symmetry under G is

$$\begin{aligned}\mathcal{L} = {} & \tfrac{1}{4} F_{\alpha\mu\nu} F_{\alpha}^{\mu\nu} - \tfrac{1}{2} F_{\alpha}^{\mu\nu}(\partial_{\mu}A_{\alpha\nu} - \partial_{\nu}A_{\alpha\mu} - E_{\alpha\beta\gamma}A_{\beta\mu}A_{\gamma\nu}) \\ & + \tfrac{1}{2} D^{\mu}\phi_p\ D_{\mu}\phi^p - D^{\mu}\phi_p\ (\partial_{\mu}\phi^p + i\,(\tau^{o}_{\alpha}\phi)^p A_{\alpha\mu}) \\ & - \bar{\psi}_n \gamma^{\mu}(\partial_{\mu}\psi_n + i\,(\tau^{\frac{1}{2}}_{\alpha}\psi)^n A_{\alpha\mu}) - \bar{\psi}_n\ m_o \psi_n \\ & - P(\phi) - \bar{\psi}(F,\phi)\psi\end{aligned}$$

where $\tau^{\frac{1}{2}}_{\alpha}$ and τ^{o}_{α} are matrices representing the action of

J_α on $\psi(x)$ and $\phi(x)$ respectively and

$$\left[J_\alpha , J_\beta \right] = i \, E_{\alpha\beta\gamma} J_\gamma$$

The equations of motion give in particular

$$F_{\alpha\mu\nu} = \partial_\mu A_{\alpha\nu} - \partial_\nu A_{\alpha\mu} - E_{\alpha\beta\gamma} A_{\beta\mu} A_{\gamma\nu}$$

$$(D_\mu \phi)_p = \partial_\mu \phi_p + i \, (\tau^0_\alpha)_{pq} \, \phi_q \, A_{\alpha\mu}$$

and $$(D_\mu \psi)_n = \partial_\mu \psi_n + i \, (\tau^{\frac{1}{2}}_\alpha)_{nm} \, \psi_m \, A_{\alpha\mu}$$

For the sake of completeness[19] we will now give some conditions that have to be satisfied by $P(\phi)$, F and m_0. However the details of these quantities do not play a large part in our considerations. The criterion of hermiticity requires $P(\phi)$ to be a real function and renormalisability in the conventional sense demands that $P(\phi)$ is at most a fourth degree polynomial. Naturally we must require that $P(\phi)$ is invariant under the group action of G on ϕ. This gives simply

$$\frac{\partial P(\phi)}{\partial \phi_p} \, (\tau^0_\alpha)_{pq} \, \phi_q = 0$$

Similarly G invariance of $\bar{\psi}(F, \phi)\psi$ implies that

$$\left[\tau^{\frac{1}{2}}_\alpha , \gamma_0 F_p \right] = - (\tau^0_\alpha)_{pq} \gamma_0 F_q$$

while hermiticity requires

$$F_p^\dagger = - \gamma_0 F_p \gamma_0$$

Identical reasoning gives $m_0^\dagger = - \gamma_0 m_0 \gamma_0$. The various gauge

coupling constants are absorbed in τ^0_α, $\tau^{\frac{1}{2}}_\alpha$, J_α and $E_{\alpha\beta\gamma}$. Without loss of generality we can choose J_α and $A_{\alpha\mu}$ hermitian which in turn requires $E_{\alpha\beta\gamma}$ real. Moreover when G is semi-simple or of the form (semi-simple group) x (abelian group) we can demand that[19]

$$E_{\alpha\beta\gamma} = - E_{\alpha\gamma\beta} = - E_{\beta\alpha\gamma} = - E_{\gamma\beta\alpha}$$

We introduce the inner product

$$\langle x,y \rangle = x_p y_p$$

where x and y have the same representation under G . The inner product is group invariant if $\langle gu,v\rangle$ equals $\langle u,g^{-1}v\rangle$ for any g in G. A necessary condition for group invariance of the inner product is

$$(\tau^0_\alpha)_{qr} = - (\tau^0_\alpha)_{rq}$$

We choose τ^0 so that this is true and also we take $\phi(x)$ hermitian. For a and b transforming under R^0 we have

$$\langle a, \tau^0_\alpha b \rangle = -i \frac{d}{ds} \langle a, e^{is\tau^0_\alpha} b \rangle \Big|_{s=0}$$

In order to work in classical terms but still have a proper quantum treatment we shall use the formalism of Jona-Lasinio[30]. To the ordinary lagrangian $\mathcal{L}(\varphi, \partial_\mu \varphi, \ldots)$ is added a Schwinger source term $K(x)\varphi(x)$. We define the generating functional for connected Greens functions $W[K]$.

$$\exp i\, W[K] \equiv \langle 0, \infty \mid 0, -\infty \rangle_K$$

where $|0, \pm\infty\rangle$ are the $\left\{\begin{matrix}\text{out}\\ \text{in}\end{matrix}\right\}$ vacuum states. Following standard procedure we make the functional Legendre transformation[29]

$$\Gamma[\varphi_c] \equiv W[K] - \int d^4x\ K(x)\varphi_c(x)$$

where $\varphi_c(x) = \delta W/\delta K(x) = \dfrac{\langle 0,\infty|\varphi(x)|0,-\infty\rangle_K}{\langle 0,\infty\,|\,0,-\infty\rangle_K}$

It is clear that $\dfrac{\delta\Gamma}{\delta K(x)} = 0$ and that

$$\delta\Gamma/\delta\varphi_c(x) = -K(x)$$

In field theory we are often interested in connected Feynman diagrams which cannot be disconnected by cutting a single internal line. These are called one particle irreducible Feynman diagrams. It turns out that Γ can be interpreted in terms of such Feynman diagrams. Now

$$\frac{\delta^2\Gamma}{\delta\varphi_c(x)\delta K(y)} = \delta(x-y) = \int d^4z\ \frac{\delta^2 W}{\delta K(y)\delta K(z)}\ \frac{\delta K(z)}{\delta\varphi_c(x)}$$

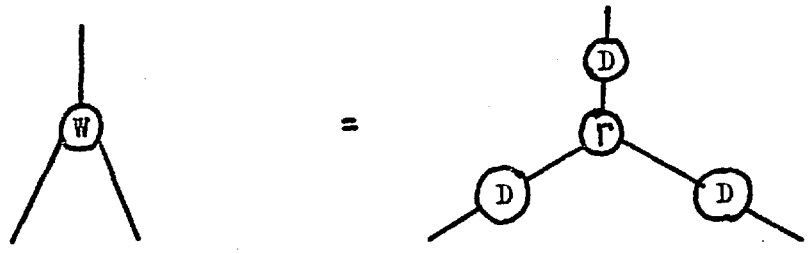
But $\delta K(z)/\delta\varphi_c(x) = \delta^2\Gamma/\varphi_c(z)\varphi_c(x)$ and the second functional derivative of W is just the propagator; hence

$\delta^2\Gamma/\delta\varphi_c(z)\delta\varphi_c(x)$ is the inverse propagator. Similarly

$$\frac{\delta^3\Gamma}{\delta\varphi_c(x)\delta\varphi_c(y)\delta\varphi_c(z)} = -\iiint d^4\alpha\ d^4\beta\ d^4\gamma \frac{\delta^2\Gamma}{\delta\varphi_c(x)\delta\varphi_c(\alpha)} \cdot S$$

where $S = \dfrac{\delta^3 W}{\delta K(\alpha)\delta K(\beta)\delta K(\gamma)} \cdot \dfrac{\delta^2\Gamma}{\delta\varphi_c(\beta)\delta\varphi_c(y)} \cdot \dfrac{\delta^2\Gamma}{\delta\varphi_c(\gamma)\delta\varphi_c(z)}$

Diagramatically we can interpret this last equation as



where D denotes propagator. Further study of vertices would confirm that Γ is indeed the generating functional of one particle irreducible vertices. We can expand Γ as

$$\Gamma[\varphi_c] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \, \Gamma^{(n)}(x_1, \ldots, x_n)\, \varphi_c(x_1) \ldots \varphi_c(x_n)$$

where $\Gamma^{(n)}$ is the sum of all one particle irreducible diagrams with n external lines. This is not the most useful form of the expansion. In fact we can consider $\Gamma^{(n)}$ in momentum space and expand about the point of zero external momenta; then

$$\Gamma = \int d^4x \; \left(- V(\varphi_c(x)) + \tfrac{1}{2} (\partial_\mu \varphi_c)^2 \, U(\varphi_c) + \ldots\right)$$

The '$\partial_\mu \varphi_c$' term is absent because of Lorentz invariance. $V(\varphi_c)$ is known as the effective potential. Since

$$\exp i \int d^4x \; \left(- V(\varphi_c) + \tfrac{1}{2} (\partial_\mu \varphi_c)^2 \, U(\varphi_c) + \ldots\right)$$

$$= \exp i \left(W[K] - \int d^4x \; K(x) \varphi_c(x)\right)$$

we have $V(\varphi_c)$ equal to the classical potential in the tree graph approximation. Spontaneous symmetry breaking occurs if

$$\langle 0, \infty | \varphi(x) | 0, -\infty \rangle \neq 0 \quad \text{when} \quad K = 0$$

i.e. $\delta\Gamma/\delta\varphi_c(x)$ is zero for some non-zero φ_c.

If we are interested in translationally invariant vacuum expectation values $\langle 0,\infty | \partial_\mu \varphi(x) | 0,-\infty \rangle$ is zero and so the vanishing of $\delta\Gamma/\delta\varphi_c$ for non-zero φ_c is equivalent to the vanishing of $dV/d\varphi_c$. The unitary gauge condition can now be formulated in terms of the effective potential. This is somewhat better than Weinberg's formulation since he uses the ordinary potential in his definitions and so strictly is working to a tree graph (or semiclassical) approximation. The gauge condition is given by

$$\langle \tau^0_\alpha \lambda , \phi(x) \rangle = 0 , \quad \alpha = 1,\ldots,N \quad \text{for all} \quad x$$

where λ satisfies (1.2)

$$\frac{\partial P_{eff}(\lambda)}{\partial \lambda_q} = 0$$

Here $P_{eff}(\phi)$ is the effective potential which reduces to $P(\phi)$ in the tree graph approximation. It is useful at this point to present an argument given by Weinberg which proves the existence of the unitary gauge.

We suppose that $\langle \tau^0_\alpha \lambda , \phi(x) \rangle \neq 0$ for all α and consider a gauge transformation T belonging to R_0 which transforms $\phi(x)$ into $T(x)\phi(x)$. G is of course assumed to be a compact and local Lie group. We define

$$F(T) = \langle \lambda , T\phi \rangle$$

F(T), for fixed x, is a real bounded differentiable function defined in the closed manifold of G and so has both maxima and minima. We suppose F(T) has an extremum at T equal to T_E ; then

$(F (T_E + \delta T) - F (T_E))$ is zero to first order in δT. Hence if

$$\delta T = i \epsilon_\alpha \tau^0_\alpha T$$

where ϵ_α are arbitrary infinitesimal parameters. Now

$$i \epsilon_\alpha \langle \lambda , \tau^0_\alpha T_E \phi \rangle = -i \epsilon_\alpha \langle \tau^0_\alpha \lambda , T_E \phi \rangle$$

The condition for an extremum together with locality gives

$$\langle \tau^0_\alpha \lambda , T_E \phi \rangle = 0 \quad \text{for all } x, \text{ and for all } \alpha .$$

This proves the existence of the unitary gauge.

One of the aims of our work has been to build up an analogue of Kibble's beautiful counting theorem in the context of quantising gauge lagrangians. Weinberg [19] made conjectures about these analogues but was unable to prove them.

Kibble's Theorem[3]:

We let S be the isotropy group of λ. We denote the dimension of the representation R^0 by n and the dimension of the isotropy group by M. Of course M is less than or equal to N where N is the dimension of G. Before introducing the gauge vector fields there are (N-M) massless scalar fields which may be placed in one to one correspondence with the broken components of the symmetry group, and (n+M-N) massive scalar fields. When the Higgs-Kibble mechanism is introduced, (n+M-N) massive scalar fields remain, but the remaining (N-M) massless scalar fields linearly combine with (N-M) of the vector fields to yield (N-M) massive vector fields. M massless vector fields remain.

The proof of this theorem will not be explicitly given but it is to be found implicitly in our formalism. First we will give concrete examples of this theorem. We will take $U(n)$[31] for G and consider ϕ to belong to the complex fundamental representation. If the vacuum expectation value of ϕ is λ the massless vector fields are n x n hermitian matrices γ satisfying

$$\gamma \cdot \lambda = 0$$

and also $$\lambda^{\dagger} \cdot \gamma = 0$$

These equations give (2n- 1) conditions on γ and so out of ($n + 2\frac{n}{2}(n-1)$),i.e. n^2, gauge mesons of U(n) $(n-1)^2$ are photons. We can explicitly construct the γ corresponding to the photons. We let $\{\lambda_{(i)} \mid i = 2, 3, \ldots, n\}$ be the basis of the (n-1) dimensional space orthogonal to λ . The matrices

$$\gamma^{(i,j)} = \lambda_{(i)}\lambda_{(j)}$$

satisfy $\gamma \cdot \lambda = 0$; these γ can be made hermitian by symmetrisation. Essentially the same process can be used to produce a theory with $(n-m)^2$ photons for any m less than n. In particular for m equal to (n-1) we have the physically useful case of just one photon. Instead of one complex fundamental representation, n-1fundamental complex representations are used.

We can understand now the unitary gauge condition. The Goldstone bosons have directions of the vectors $(\tau^{o}_{\alpha}\lambda)$ and in the unitary gauge we are eliminating the Goldstone modes. The fields σ_a and B_{α} (which are the Goldstone bosons) are defined by

$$\phi \equiv \sigma_a n_a + B_\alpha (\tau^0_\alpha \lambda)$$

where $\langle n_a, \tau^0_\alpha \lambda \rangle = 0$ for all α

$$\langle n_a, n_b \rangle = \delta_{ab}$$

$$n_a \langle n_a, X \rangle = X \text{ if } \langle X, \tau^0_\alpha \lambda \rangle = 0$$

Equation (1.2) implies that

$$\langle \tau^0_\alpha \lambda , \tau^0_\beta \lambda \rangle B_\beta = 0$$

Defining the matrix

$$m^2_{\alpha\beta} = - \langle \tau^0_\alpha \lambda, \tau^0_\beta \lambda \rangle$$

we have as a necessary and sufficient condition for the vanishing of $m^2_{\alpha\beta} X_\beta$ that $\langle \tau^0_\alpha \lambda , X_\beta \tau^0_\beta \lambda \rangle$ vanishes. This in turn requires $X_\beta \tau^0_\beta \lambda$ to be zero. We will use the term 'unbroken subgroup' for the isotropy group of λ. Until further notice we will assume that we have no non-trivial unbroken subgroup. Hence in this case $X_\beta = 0$ if and only if $m^2_{\alpha\beta} X_\beta$ vanishes. This shows that (m^2) has an inverse (m^{-2}).

We shall follow the Faddeev-Popov method of quantisation; the action is constant on the orbits of G and we extract an infinite factor Δ proportional to $\prod_x d\Omega(x)$ before developing a perturbation series and Ω belongs to G. So

$$\Delta(\phi) \int \prod_{\alpha=1}^{N} \delta(\langle \tau^0_\alpha \lambda, \phi \rangle^\Omega) \, d\Omega = \text{constant} \qquad (1.3)$$

Now $$\delta\phi = i\Omega^\gamma \tau^0_\gamma \, (\sigma_a n_a + B_\alpha \tau^0_\alpha \lambda) \qquad (1.4)$$

From the unitary gauge condition we have

$$\langle \tau^{\circ}_{\alpha} \lambda , \lambda \rangle = 0$$

i.e. $$\lambda = \langle \lambda, n_a \rangle n_a$$

We define $\sigma'_a = \sigma_a - \langle n_a , \lambda \rangle$ so that σ'_a has vanishing vacuum expectation value. $C^{a\gamma}_{\alpha}$ and $D^{\gamma}_{\beta\alpha}$ are defined as follows :

$$\begin{aligned} \tau^{\circ}_{\gamma} n_a &= \langle \tau^{\circ}_{\gamma} n_a , n_b \rangle n_b + C^{a\gamma}_{\alpha} \tau^{\circ}_{\alpha} \lambda \\ \tau^{\circ}_{\gamma} (\tau^{\circ}_{\alpha} \lambda) &= \langle \tau^{\circ}_{\gamma} (\tau^{\circ}_{\alpha} \lambda) , n_b \rangle n_b + D^{\gamma}_{\beta\alpha} \tau^{\circ}_{\beta} \lambda \end{aligned} \qquad (1.5)$$

Moreover it is easy to see that

$$D^{\gamma}_{\beta\alpha} - D^{\alpha}_{\beta\gamma} = i E_{\gamma\alpha\beta}$$

(using $[\tau^{\circ}_{\gamma} , \tau^{\circ}_{\alpha}] \lambda = i E_{\gamma\alpha\beta} \tau^{\circ}_{\beta} \lambda$)

Using these quantities we can express equation (1.3) as

$$\delta \phi = i \Omega_{\gamma} \Big(\Big\{ \sigma_a \langle \tau^{\circ}_{\gamma} n_a , n_b \rangle + B_{\alpha} \langle \tau^{\circ}_{\gamma} (\tau^{\circ}_{\alpha} \lambda) , n_b \rangle \Big\} n_b + (\tau^{\circ}_{\beta} \lambda) \Big\{ C^{a\gamma}_{\beta} \sigma_a + B_{\alpha} D^{\gamma}_{\beta\alpha} \Big\} \Big)$$

Hence

$$\delta \sigma'_b = i \Omega_{\gamma} (\sigma'_a + \langle n_a , \lambda \rangle) \langle \tau^{\circ}_{\gamma} n_a , n_b \rangle + i \Omega_{\gamma} B_{\alpha} \langle \tau^{\circ}_{\gamma} (\tau^{\circ}_{\alpha} \lambda) , n_b \rangle$$

Similarly $$\delta B_{\beta} = i \Omega_{\gamma} (C^{a\gamma}_{\beta} (\sigma'_a + \langle n_a , \lambda \rangle) + B_{\alpha} D^{\gamma}_{\beta\alpha})$$

Owing to the non-singularity of (m^2)

$$(m^2) B = 0 \text{ if and only if } B = 0$$

In evaluating (1.3) we are then constrained to the hypersurface B = 0 and so we need include only infinitesimal transformations in the integration. It is first convenient to define

$$\theta_{\beta\alpha} = -\langle \tau^{0}_{\beta}\lambda, \tau^{0}_{\alpha}\phi \rangle$$
$$= C^{a\alpha}_{\delta}\, \sigma_{a}\, (m^{2})_{\beta\delta} + B_{\gamma}\, D^{\alpha}_{\delta\gamma}\, (m^{2})_{\beta\delta}$$

Hence $\delta B_{\nu} = i\, (m^{-2})_{\kappa\nu}\, \Omega_{\gamma}\, \theta^{\gamma}{}_{\kappa}$

Now $$\int \prod_{\alpha=1}^{N} \delta\, (\, m^{-2}\, \Omega^{\gamma}\, \theta_{\gamma\beta}\,)\, d\Omega$$

$$= \left| \text{Det}\quad m^{-2}\theta\ \delta_{(x-x')} \right|^{-1}$$

$$= \exp i\, (\, i\, \delta(0)\ \log \det\ m^{-2}\theta\ )$$

We thus easily obtain Weinberg's result[19] that the correct Feynman rules are given by replacing T- products by T^{X}- products and using the negative of the interaction lagrangian as the interaction Hamiltonian provided a term

$$i\, \delta(0)\ \log \det\ (\, m^{-2}\theta(x)\,) \qquad (1.6)$$

is added to the hamiltonian. Although we have not explicitly obtained the Feynman rules these are easily computed from their definition as the Greens functions of the free part of the total action. In fact the vector meson Green's function is of the form

$$\langle\, T^{X}\, \{\, A_{\alpha\mu}(x)\ A_{\beta\nu}(0)\}\rangle = (\, g_{\mu\nu}\, m^{2} - \partial_{\mu}\partial_{\nu})\, m^{-2}\, \Delta(x)$$

where $\Delta(x)$ is the massive propagator.

Now we shall see that the above theory admits a canonical quantisation. In a path integral formalism this means that an arbitrary Green function $T(\ Q(\text{fields})\ )$ where $Q(\text{fields})$ is an arbitrary function of the non-Goldstone fields (but not of the conjugate field momenta) can be expressed as

$$\begin{aligned} &\langle T\,(Q)\rangle \\ &= \int d(\text{fields})\ d(\text{conjugate momenta})\ \delta(\text{gauge condition}) \\ &\qquad \exp\Big(\frac{i}{\hbar}\int dx\ \big(\ L_{\text{covariant}} + \delta(o)\ \text{term}\big)\Big) \end{aligned} \tag{1.7}$$

$$\begin{aligned} &= \int d(\text{fields})\ d(\text{conjugate momenta to fields})\ Q(\text{fields}) \\ &\quad \exp\Big(\frac{i}{\hbar}\int dx\ \big(\sum((\text{conjugate momentum})\times\frac{d}{dt}(\text{field co-ordinate})) \\ &\qquad - (\text{a Hamiltonian})\big)\Big) \end{aligned} \tag{1.7'}$$

In (1.1) we have

$$\frac{\partial \mathcal{L}}{\partial(\partial_o A_{\alpha o})} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_o A_{\alpha k})} = -\tfrac{1}{2}\ (\ F_\alpha^{ok} - F_\alpha^{ko}\ ) = -\ F_\alpha^{ok} \equiv \mathcal{E}_\alpha^k$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_o \sigma_a)} = -\ \langle D^o\phi\ ,\ n_a\rangle \equiv \pi_a$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_o B_\beta)} = -\ \langle D^o\phi\ ,\ \tau_\beta^{\hat{\omega}}\lambda\rangle \equiv \chi_\beta$$

Also $$\frac{\partial \mathcal{L}}{\partial(\partial_o \psi_n)} = -\bar{\psi}_n \gamma^o = i\psi^\dagger_n = \pi_n$$

So

$$\mathcal{L} = \tfrac{1}{2}(\mathcal{E}^k_\alpha)^2 + \mathcal{E}^k_\alpha(\partial_o A_{\alpha k} - \partial_k A_{\alpha o} - E_{\alpha\beta\gamma} A_{\beta o} A_{\gamma k}) + \tfrac{1}{2}(\pi_\alpha)^2$$

$$+ \tfrac{1}{4} F_{\alpha ab} F_\alpha^{ab} - \tfrac{1}{2}(m^{-2})_{\gamma\gamma'} \chi_\gamma \chi_{\gamma'} - \tfrac{1}{2}\langle D^k\phi, D_k\phi\rangle$$

$$- \langle -\pi_a n_a + (m^{-2})_{\beta\gamma}\chi_\gamma \tau^o_\beta \lambda, \partial_o \sigma_b n_a + i\sigma_b A_{\alpha o}\tau^o_\alpha n_b$$

$$+ iA_{\alpha o} B_\gamma \tau^o_\alpha \tau^o_\gamma \lambda + \partial_o B_\gamma \tau^o_\gamma \lambda\rangle$$

$$+ \cdots$$

We add $\partial_k(\mathcal{E}^k_\alpha A_{\alpha o})$ to $\mathcal{L}$ so that the term $-\mathcal{E}^k_\alpha \partial_k A_{\alpha o}$ becomes $(\partial_k \mathcal{E}^k_\alpha) A_{\alpha o}$.

Now we integrate over B, χ and then $A_{\alpha o}$ in (1.7). We are interested specifically in

$$\int d\chi \exp\left[i/\hbar\left(-\tfrac{1}{2}(m^{-2})_{\gamma\gamma'}\chi_\gamma\chi_{\gamma'} - i(m^{-2})_{\beta\gamma}\chi_\gamma\langle\tau^o_\beta\lambda, \tau^o_\alpha n_b\rangle\sigma_b A_{\alpha o}\right)\right]$$

Now

$$(m^{-2})_{\gamma\gamma'}\chi_\gamma\chi_{\gamma'} + 2(m^{-2})_{\beta\gamma}\langle\tau^o_\beta\lambda, \tau^o_\alpha n_b\rangle i\sigma_b A_{\alpha o}\chi_\gamma$$

$$= \left((m^{-1})_{\gamma\delta}\chi_\gamma + (m^{-1})_{\delta\beta}\langle\tau^o_\beta\lambda, \tau^o_\alpha n_b\rangle i\sigma_b A_{\alpha o}\right)^2 \qquad (1.8)$$

$$+ (m^{-2})_{\beta\beta'}\langle\tau^o_\beta\lambda, \tau^o_\alpha n_b\rangle\langle\tau^o_{\beta'}\lambda, \tau^o_{\alpha'} n_{b'}\rangle \sigma_b\sigma_{b'} A_{\alpha o} A_{\alpha' o}$$

where $m^{-1}.\ m^{-1}$ equals m^{-2} ; m^{-1} exists since m^{-2} is real and symmetric with eigenvalues the reciprocals of the squares of the vector meson masses (and no mass is zero). In the case we are

dealing with

$$1 = n_a n_a^T - (\tau^o_\alpha \lambda)\,(m^{-2})_{\alpha\beta}\,(\tau^o_\beta \lambda)^T$$

From equation (1.5) we see that

$$\langle \tau^o_\alpha n_b \,,\, \tau^o_{\alpha'} n_{b'} \rangle$$
$$= \langle \tau^o_\alpha n_b \,,\, n_a \rangle \langle \tau^o_{\alpha'} n_{b'} \,,\, n_a \rangle - C^{b\alpha}_{\gamma}\, C^{b'\alpha'}_{\gamma'}\, (m^2)_{\gamma\gamma'}$$

Hence

$$(m^{-2})_{\beta\beta'} \langle \tau^o_\alpha n_b \,,\, \tau^o_\beta \lambda \rangle \langle \tau^o_{\beta'} \lambda \,,\, \tau^o_{\alpha'} n_{b'} \rangle$$
$$= \langle \tau^o_\alpha n_b \,,\, n_a \rangle \langle n_a \,,\, \tau^o_{\alpha'} n_{b'} \rangle - \langle \tau^o_\alpha n_b \,,\, \tau^o_{\alpha'} n_{b'} \rangle$$
$$= C^{b\alpha}_{\gamma}\, C^{b'\alpha'}_{\gamma'}\, (m^2)_{\gamma\gamma'}$$

From equation (1.7) we can see that the Gaussian integration for χ gives a factor $|\text{Det } m^{-1}|^{-1}$ whereas the Gaussian integration for A_o gives a factor $|\text{Det } m^1 \theta\, m^{-2}|^{-1}$. The product of these factors is $|\text{Det } m^{-2}\theta|$ and this perfectly cancels the $\delta(o)$ term in (1.7); so a hamiltonian exists for which (1.7') is equivalent to (1.7).

We can get a clearer idea of this formalism by considering the very familiar case G= SU(2). In the usual formalism the scalar fields are given as a complex doublet

$$K_a = X_a + i\, Y_a \qquad a = 1,2$$

where

$$X_a = \tfrac{1}{2}\,(K_a + K_a^\dagger) \quad \text{and} \quad Y_a = (-\tfrac{1}{2})i\,(K_a - K_a^\dagger)$$

and $\lambda = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ after appropriate normalisations. In a four component formalism we deal with

$$\phi = \begin{bmatrix} X_1 \\ X_2 \\ Y_1 \\ Y_2 \end{bmatrix} \qquad \text{and} \qquad \lambda = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now $\delta K_a = i \Omega^\alpha (\tau^\alpha/2)_a{}^b K_b$

where $\{\tau_\alpha ; \alpha = 1,2,3\}$ is the set of Pauli matrices and are the infinitesimal transformation parameters. It is easy to see that

$$\delta X_a = \tfrac{1}{2} i \Omega^\alpha ((\tau^\alpha/2)_a{}^b K_b - K_b^\dagger (\tau^\alpha/2)^b{}_a)$$

$$\delta Y_a = \tfrac{1}{2} \Omega^\alpha ((\tau^\alpha/2)_a{}^b K_b + K_b^\dagger (\tau^\alpha/2)^b{}_a)$$

$$\tau^o_\alpha \phi = \tfrac{1}{2} \begin{bmatrix} (\tau^\alpha/2)_1{}^b K_b - K_b^\dagger (\tau^\alpha/2)^b{}_1 \\ (\tau^\alpha/2)_2{}^b K_b - K_b^\dagger (\tau^\alpha/2)^b{}_2 \\ -i ((\tau^\alpha/2)_1{}^b K_b + K_b^\dagger (\tau^\alpha/2)^b{}_1) \\ -i ((\tau^\alpha/2)_2{}^b K_b + K_b^\dagger (\tau^\alpha/2)^b{}_2) \end{bmatrix}$$

Hence

$$\tau^o_1 \lambda = \tfrac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -i \end{bmatrix} \qquad \tau^o_2 \lambda = \tfrac{1}{2} \begin{bmatrix} 0 \\ i \\ 0 \\ 0 \end{bmatrix}$$

$$\tau^o_3 \lambda = -\tfrac{1}{2} \begin{bmatrix} 0 \\ 0 \\ i \\ 0 \end{bmatrix}$$

This requires that

$$n = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence $C_{\alpha}^{1\gamma} = \delta_{\alpha}^{\gamma}$

$$(m^2)_{\alpha\beta} = \tfrac{1}{2}\,\delta_{\alpha\beta}$$

It is also easy to see that (after a little algebra)

$$D^1_{\alpha 2} = - D^2_{\alpha 1} = \delta_{\alpha 3}\ (1/2i)$$

$$D^1_{\alpha 3} = - D^3_{\alpha 1} = \delta_{\alpha 2}\ (1/2i)$$

$$D^2_{\alpha 3} = - D^3_{\alpha 2} = \delta_{\alpha 1}\ (i/2)$$

and $D^{\gamma}_{\alpha\beta} = 0$ otherwise.

Weinberg has obtained the same results using canonical quantisation. This approach is much longer than the one we have adopted and its extension to the most interesting case when $0 < M < N$ is not obvious. Indeed Weinberg seemed to be in difficulties in the latter case and was unable to come to firm conclusions. The whole philosophy of the Weinberg approach is entirely different from ours. His justification for (1.6) depended on differences such as

$$\langle T\,(\, A_{\alpha\mu}(x)\,,\ A_{\beta\nu}(y)\rangle - \langle T^{X}\,(\, A_{\alpha\mu}(x),\, A_{\beta\nu}(y)\,)\rangle$$

$$= - i\ m^{-2}_{\alpha\beta}\ \delta^{\circ}_{\mu}\ \delta^{\circ}_{\nu}\ \delta^{4}(\,x-y\,)$$

These differences are present always in derivative interactions

and are not the prerogative of gauge theories. In fact Charap[32] and Gerstein, Jackiw, Lee and Weinberg[32] found for the interaction

$$\mathcal{H} = \tfrac{1}{2}\,\hat{\varphi}_a\; g^{-1}{}_{ab}(\varphi)\,\hat{\varphi}_b - \mathcal{L}$$

$$\mathcal{L} = \tfrac{1}{2}\,\partial_\mu\varphi^a\; g_{ab}(\varphi)\,\partial_\mu\varphi^b$$

where $\hat{\varphi}_a = g_{ab}(\varphi)\,\partial_0\varphi_b$

that we can use the T^* part of the propagator if we use the interaction hamiltonian

$$\mathcal{H}_I = -\tfrac{1}{2}\partial^\mu\varphi_a\;\bar{g}_{ab}(\varphi)\partial_\mu\varphi_b + \frac{i}{2}\,\delta^4(0)\;\mathrm{Tr}\log(1+\bar{g}(\varphi))$$

where $g_{ab} = \delta_{ab} + \bar{g}_{ab}$.

Weinberg summed the effect of non-covariant parts of the propagators using the method of Lee and Yang.[19]

We will now consider the case when G contains an unbroken subgroup. If the group G has a non-trivial unbroken subgroup S with M independent generators the formalism described above breaks down. Following Weinberg we introduce the complete set of eigenvectors $\bar{c}_n$ of (m^2). Since (m^2) is real and symmetric such a set exists

$$m^2\;\bar{c}_n = (m^{(n)})^2\;\bar{c}_n$$

$$\langle \bar{c}_p, \bar{c}_q \rangle = \delta_{pq}$$

$$(\bar{c}_n)_\alpha\,(\bar{c}_n)_\beta = \delta_{\alpha\beta}$$

Of course (m^2) is a singular matrix now. We define

$$\bar{A}_{n\mu} = (\bar{c}_n)_\alpha A_{\alpha\mu}$$

$$\bar{\tau}^0_n = (\bar{c}_n)_\alpha \tau^0_\alpha$$

$$\bar{\Omega}_n = (\bar{c}_n)_\alpha \Omega_\alpha$$

$$\bar{J}_n = (\bar{c}_n)_\alpha J_\alpha$$

If $(m^{(n)})^2$ is zero then $(\bar{c}_n)_\alpha J_\alpha$ is a generator of S. Conversely, if $c_\beta J_\beta$ is a generator of S then c_β is an eigenvector of (m^2) with zero eigenvalue. Both these statements can easily be proved. For convenience we let $\{\bar{J}_n ;\ n = 1,\ldots,M\}$ be the independent set of generators of S, so that $m^2 \bar{c}_n$ is zero for n less than or equal to M.

$$\langle \tau^0_\alpha \lambda ,\ \tau^0_\beta \lambda \rangle = \sum_{n,m \geq M+1} (\bar{c}_n)_\alpha (\bar{c}_m)_\beta (\hat{m}^2)_{nm}$$

where $(\hat{m}^2)$ is an $(N-M) \times (N-M)$ matrix

$$(\hat{m}^2)_{nm} = \langle \bar{\tau}^0_n \lambda ,\ \bar{\tau}^0_m \lambda \rangle \quad \text{with} \quad n,m \geq M+1$$

Now $$\phi = \sigma_a n_a + \sum_{m=M+1}^{N} \bar{B}_m \bar{\tau}^0_m \lambda$$

$$\langle \tau^0_\alpha \lambda ,\ \phi \rangle = \bar{c}_{n\alpha} \bar{B}_m (\hat{m}^2)_{nm}$$

So $\langle \tau^0_\alpha \lambda ,\ \phi \rangle = 0$ if and only if $\{\bar{B}_m = 0 ,\ m = M+1,\ldots,N\}$ (as $\hat{m}^2$ is non-singular).

In order to specify a gauge we need M more independent conditions. Hence a gauge with no Goldstone bosons is given by

$$\bar{B}_m = 0 \qquad m = M+1,\ldots,N$$
$$\partial_\mu \bar{A}^\mu_n(x) = K_n(x) \qquad n = 1,\ldots,M$$

where $K_n(x)$ are arbitrary functions.

Now

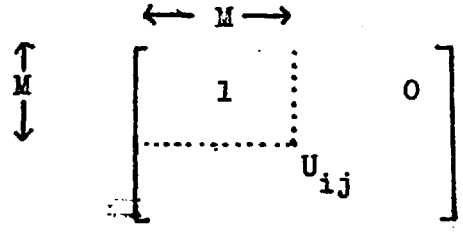
$$\delta \bar{B}_m = i\, \bar{c}_{m\alpha}\, \bar{c}_{n\gamma} \bar{\Omega}_n \left(C^{a\gamma}_{\alpha} \sigma_a + \bar{c}_{p\kappa}\, \bar{B}_p\, D^{\gamma}_{\alpha\kappa} \right)$$

$$\delta\, \partial.\bar{A}_n = \bar{c}_{n\alpha}\, \partial_\mu \left(\left(E_{\alpha\beta\gamma}\, \bar{c}_{l\beta}\, \bar{c}_{m\gamma}\, \bar{A}^\mu_l - \bar{c}_{m\alpha}\, \partial_\mu \right) \bar{\Omega}_m \right)$$

We consider the change of variables

$$\begin{aligned} \omega_m &= \bar{\Omega}_m && m = 1,\ldots,M \\ &= \bar{c}_{n\gamma}\, \bar{c}_{m\alpha}\, C^{a\gamma}_{\alpha}\, \sigma_a\, \bar{\Omega}_n && m = M+1,\ldots,N \end{aligned} \qquad (1.9)$$

We can visualise the transformation as a NxN matrix



where $U_{nm} = \bar{c}_{n\gamma}\, \bar{c}_{m\alpha}\, C^{a\gamma}_{\alpha}\, \sigma_a\, \delta(x-x')$. In the cases that we shall consider we will see that (1.9) is a non-singular transformation. The determinant of the transformation is

$$\mathrm{Det}\ \left(\bar{c}_{n\gamma}\, \bar{c}_{m\alpha}\, C^{a\gamma}_{\alpha}\, \sigma_a\, \delta(x-x') \right)_{n,m \geq M+1}$$

The Faddev-Popov factor necessary for the quantisation of the lagrangian is

$$\int d\bar{\Omega} \prod_{m=M+1}^{N} \delta(\bar{B}_m^{\bar{\Omega}}) \prod_{n=1}^{M} (\partial \cdot \bar{A}_n^{\bar{\Omega}}) \tag{1.10}$$

$$= \int d\omega \left| \mathrm{Det}\left(\bar{c}_{n\gamma}\, \bar{c}_{m\alpha}\, c^{a\gamma}_{\alpha}\, \sigma_a\, \delta(x-x') \right)_{n,m \geq M+1} \right|^{-1}$$

$$\cdot \prod_{m=M+1}^{N} \delta(\omega_m) \prod_{n=1}^{M} \delta\left(\bar{c}_{n\alpha} \partial_\mu \left(E_{\alpha\beta\gamma}\, \bar{c}_{l\beta}\, \bar{c}_{q\gamma}\, \bar{A}^{\mu}_{l} \bar{\Omega}_q - \bar{c}_{q\alpha} \partial_\mu \bar{\omega}_q \right) \right)$$

Weinberg's conjecture essentially requires that the integrations over ω introduce only field independent factors. We shall first give a general class of counterexamples to this conjecture. Since we will find it convenient to work with $\bar{J}_m$ it is perhaps worth mentioning that the structure function with respect to these generators is $\bar{\bar{E}}_{pqr}$ where

$$\bar{\bar{E}}_{pqr} = \bar{c}_{p\alpha}\, \bar{c}_{q\beta}\, \bar{c}_{r\gamma}\, E_{\alpha\beta\gamma}$$

We suppose that S is a subgroup of G such that

$$\left[\bar{J}_p , \bar{J}_q \right] = 0 \quad \text{if either p or q is larger than M (but not both)} \tag{1.11}$$

The interpretation of this is simple.

If G is isomorphic to the direct product of S and some group the condition is satisfied; however this is not a necessary condition for the validity of (1.11). Actually (1.11) implies that

$$\left[\bar{J}_m , \left[\bar{J}_p , \bar{J}_q \right] \right]$$

vanishes for the domain $\left\{ q,p \geq M+1 , \; m < M+1 \right\}$ via the Jacobi

identity; so $[\bar{J}_p , \bar{J}_q]$ belongs to the centre of S (or outside S) for p and q larger than M. This implies that

$$G \cong (S / (\text{centre of } S)) \times (\text{some group})$$

If S is semisimple then the centre of S is trivial so that G is indeed of the form S x (some group) . Now (1.11) implies that

$$\bar{c}_{p\alpha} \bar{c}_{q\beta} \bar{c}_{r\gamma} E_{\alpha\beta\gamma} = 0$$

if either p or q (but not both) is larger than M . Using the antisymmetry of the structure constants we have

$$\bar{c}_{p\alpha} \bar{c}_{q\beta} \bar{c}_{r\gamma} E_{\alpha\beta\gamma} = 0 \qquad \text{if } r \geq M+1,\ q \leq M \text{ or } r \leq M,\ q \geq M+1$$

Also $\bar{\tau}^o_n (\bar{\tau}^o_m \lambda) = \bar{\tau}^o_m (\bar{\tau}^o_n \lambda)$

when $n \leq M$ and $m > M$.

So $-\langle \bar{\tau}^o_n n_b , \bar{\tau}^o_m \lambda \rangle = \langle n_b , \bar{\tau}^o_n \bar{\tau}^o_m \lambda \rangle = 0$

Hence since

$$\bar{\tau}^o_n n_b = \bar{c}_{n\gamma} \langle \tau^o_\gamma n_b , n_a \rangle n_a + \bar{c}_{n\gamma} c^{b\gamma}_{\alpha} \tau^o_\alpha \lambda$$

we have $\bar{c}_{n\gamma} c^{b\gamma}_{\alpha} \bar{c}_{m\alpha} = 0$ for $n \leq M$, $m > M$ (1.12)

Now $\{ \delta \bar{B}_m , m = M+1,\ldots,N \}$ is an independent set of variables; so from (1.11) we see that

$$\{ \omega_m = 0 , m = M+1,\ldots,N \} \text{ if and only if } \{ \bar{\Omega}_m = 0,\ m = M+1,\ldots,N \} \qquad (1.13)$$

So $$\int d\bar{\Omega} \prod_{m=M+1}^{N} \delta(\bar{B}_m^{\bar{\Omega}}) \prod_{n=1}^{M} \delta(\partial \cdot \bar{A}_n^{\bar{\Omega}})$$

$$= \left|\mathrm{Det}\ (\bar{c}_{n\gamma}\, \bar{c}_{m\alpha}\, c_{\alpha}^{a\gamma}\, \sigma_a\, \delta(x-x')_{n,m\geq M+1}\right|^{-1}$$
$$\int \prod_{m=1}^{M} d\omega_m \prod_{n=1}^{M} \delta(\bar{c}_{n\alpha}\partial_\mu(E_{\alpha\beta\gamma}\bar{c}_{\ell\beta}\bar{A}_\ell^\mu \sum_{q=1}^{M} \bar{c}_{q\gamma}\omega_q - \bar{c}_{q\alpha}\partial_\mu\omega_q))$$

This is characteristic of the Faddev-Popov ghosts. In order to be explicit we arrange $\bar{c}_{n\alpha}$ to be $\delta_{n\alpha}$ for all n and take M=3 and N=4 (e.g. G = SU(2)xU(1) but with S = SU(2)). The above integral then becomes

$$\int d\omega \prod_{n=1}^{3} \delta(\partial_\mu(\sum_{q=1}^{3} E_{n\ell q} A_\ell^\mu \omega_q - \partial_\mu \omega_n))$$

We make the transformation

$$s_1 = -\partial^2\omega_1 + \partial_\mu(A_2^\mu\omega_3 - A_3^\mu\omega_2) E_{123}$$
$$s_2 = -\partial^2\omega_2 + \partial_\mu(A_3^\mu\omega_1 - A_1^\mu\omega_3) E_{231}$$
$$s_3 = -\partial^2\omega_3 + \partial_\mu(A_1^\mu\omega_2 - A_2^\mu\omega_1) E_{312}$$

The Jacobian for the transformation is

$$\begin{vmatrix} -\partial^2\delta(x-x') & -A_3'\cdot\partial\delta(x-x') & A_2'\cdot\partial\delta(x-x') \\ A_3'\cdot\partial\delta(x-x') & -\partial^2\delta(x-x') & -A_1'\cdot\partial\delta(x-x') \\ -A_2'\cdot\partial\delta(x-x') & A_1'\cdot\partial\delta(x-x') & -\partial^2\delta(x-x') \end{vmatrix}$$

since $\partial \cdot A_i = 0$ $i = 1,2,3$ and $A' = AE$.

This Jacobian is just the Faddev-Popov ghost contribution.(Bars have been omitted in the above).

If we now demand that S is abelian while (1.11) still holds we have

$$\bar{E}_{pqr} = 0 \quad \text{if} \quad q \leq M$$

This gives

$$\int d\bar{\Omega} \prod_{m=M+1}^{N} \delta(\omega_m) \prod_{n=1}^{M} \delta(\partial \cdot \bar{A}_n^{\bar{\Omega}}) \qquad (1.14)$$

$$= \left| \text{Det} \; \bar{c}_{n\gamma} \bar{c}_{m\alpha} C_{\alpha}^{a\gamma} \sigma_a \delta(x-x') \right|^{-1}_{n,m \geq M+1} \left| \det \partial^2 \delta(x-x') \right|^{-1}$$

Hence Weinberg's conjecture holds here.Now, however, we shall show that, if S is abelian, Weinberg is right irrespective of the validity of (1.11). From (1.5) we have

$$\bar{\tau}_m^o n_a = \langle \bar{\tau}_m^o n_a, n_b \rangle n_b + \bar{c}_{n\alpha} \bar{c}_{m\gamma} C_{\alpha}^{a\gamma} \bar{\tau}_n^o \lambda$$

For $n \leq M$, $\bar{c}_{n\alpha} \bar{c}_{m\gamma} C_{\alpha}^{a\gamma}$ is an undetermined quantity and can be arranged to be zero i.e. $U_{mn} = 0$, for $m > M$, $n \leq M$. Hence by the argument used before we have again (1.13). Since S is just abelian $\bar{E}_{pqr}$ vanishes if $p,q \leq M$. Now

$$\bar{E}_{nlq} \bar{A}_l^{\mu} \bar{\Omega}_l = \sum_{q=M+1}^{N} \bar{E}_{nlq} \bar{A}_l^{\mu} \bar{\Omega}_q \qquad (n \leq M)$$

As before the factor $\prod_{m=M+1}^{N} \delta(\omega_m)$ in (1.10) together with (1.13) makes this sum vanish; hence (1.14) again is valid. Since we only have one photon, S = U(1) is a useful case to consider. If in addition we take R_o to be the adjoint representation, then the dimension of the space orthogonal to $\left\{ \tau_{\alpha}^o \lambda \,\middle|\, \alpha = 1,\ldots,N \right\}$ is one.

Now

$$\bar{\tau}^{o}_{m}\ n_1 \;=\; \bar{\tau}^{o}_{m}\lambda\ /\langle\lambda\,,\ n_1\rangle$$

giving

$$\bar{\tau}^{o}_{m}\ n_1 \;=\; \begin{cases} 0 & m\leq M \\ \dfrac{\delta_{nm}\bar{\tau}_{m}\lambda}{\langle\lambda\,,\ n_1\rangle} & m>M \end{cases}$$

So $\bar{c}_{n\alpha}\,\bar{c}_{m\gamma}\,C^{1\gamma}_{\alpha} \;=\; \delta_{nm}\ /\langle\lambda,\ n_1\rangle$ for $m,n>M$. Hence in this fairly general case we have found the Faddev-Popov factor in terms of σ and λ only. In (1.14) the determinant is that of an (N-M)x(N-M) matrix. Usually it is arranged that

$$\bar{c}_{n\alpha} = \delta_{n\alpha} \quad \text{for} \quad n > M$$

We then have for the Faddev- Popov factor

$$\left|\ \mathrm{Det}\ \left(\ C^{an}_{m}\ \sigma_{a}\ \delta(x-x')\ \right)_{n,m\geq M+1}\right|^{-1}$$

which precisely means that the factor is evaluated taking account only of the broken generators. This is essentially Weinberg's conjecture and is a counting theorem in the context of quantisation.

Unlike most effective lagrangians the class of lagrangians given above are renormalisable.This class provides unified theories of weak and electromagnetic interactions (which will be discussed in the next chapter). It is however interesting briefly to look at the currents in the theory. This aspect has been largely ignored and most of the effort has gone into field theory perturbative

calculations. Yang-Mills theories can give (via a formal limit due to Bardakci, Frishman and Halpern [20] (BFH)) Sugawara theories with c-number Schwinger terms[34]. It has been suggested[35] that in the Higgs-Kibble theories it may be possible (by taking essentially the same formal limit) to obtain new Sugawara theories which contain q-number Schwinger terms. Sugawara theories with q-number Schwinger terms have been produced of course before by Barnes and Isham[21]. We however found difficulties with the BFH limit owing to the fact that the masses in the theory are in part or completely generated by spontaneous symmetry breaking. The limit procedure requires the bare vector boson mass and the bare gauge coupling constant to tend to zero in such a way that their ratio tends to a constant. In conventional massive Yang-Mills theories we have a strict field current identity. The canonical commutation relations of the field theory translate simply into equal time commutation relations for the currents. In the limit of vanishing bare mass and coupling constant we find the ' canonical conjugate' of the current commuting with the current. Now any operator, which commutes with all the canonically independent dynamical variables, is just proportional to the identity operator. The theory of currents which is obtained after taking the limit is a non-canonical one.

If we take $G = U(3)_L \times U(3)_R$ and follow our general prescription for writing down the lagrangian displaying the

Higgs-Kibble mechanism we have

$$\mathcal{L} = -\tfrac{1}{4}\,\mathrm{Tr}\,(\,F_{\mu\nu}{}^{L}\,F^{\mu\nu}{}_{L} + F_{\mu\nu}{}^{R}\,F^{\mu\nu}{}_{R}\,)$$

$$-\tfrac{1}{2}\,\mathrm{Tr}\,(\,(D_\mu\phi_L)^\dagger\,D_\mu\phi_L + (D_\mu\phi_R)^\dagger\,D_\mu\phi_R\,)$$

$$+ m_o{}^2\,\mathrm{Tr}\,(\,\phi_L^\dagger\,\phi_L + \phi_R^\dagger\,\phi_R\,) + P(\phi_L) + P(\phi_R)$$

The scalars are 3x3 complex matrices which transform as a $(3,\bar{3})$ representation and the vector gauge particles are 3x3 hermitian matrices. The vacuum expectation value of ϕ is taken to be real, and the anti-hermitian part of ϕ gives the Goldstone bosons. Consequently $\mathcal{L}$ in the unitary gauge is obtained by considering ϕ to be hermitian, and replacing ϕ by $(\phi + \lambda)$. It was pointed out by Bardakci and Halpern [31] that $\mathcal{L}_{\text{unitary}}$ is invariant under a global group

$$A \longrightarrow A'_\mu = S\,A_\mu\,S^{-1}$$

$$\phi \longrightarrow \phi' = S\,\phi\,S^{-1}$$

where $S = \exp i\,(\,\frac{\lambda}{2}\,.\,\varepsilon\,)$

and $\{\lambda_i\,;\ i = 0,\ldots,8\}$ is the set of Gell-Mann matrices and $\{\epsilon^i\,;\ i = 0,\ldots,8\}$ is the set of group parameters. This invariance is of course an extremely natural one and similar global symmetry groups exist for $U(n)_L \times U(n)_R$. The field current (as opposed to the Noether current) is independent of $P(\phi)$ and is given by

$$j_\mu = \tfrac{1}{2} i \left[\phi, \partial_\mu \phi\right] + g(\phi + \lambda) A_\mu (\phi + \lambda)$$

We see already the effect of the Higgs-Kibble particles. As $\phi \longrightarrow 0$ we regain the strict field current identity. The Noether current is of the form $j_\mu + g^{-1} \partial_\mu F^{\mu\nu}$ and so even after using the equations of motion it has terms quadratic in A and involving a derivative of A. Now the mass of A is $g\lambda$. The BFH limit requires that $g \longrightarrow 0$ but that λ remains non-zero. These are not inconsistent requirements. We define

$$\tilde{F}^{\delta}_{\mu\nu} = \partial_\mu j^{\delta}_{\nu} - \partial_\nu j^{\delta}_{\mu} - (\partial_\mu Z\ Z^{-1})^{\delta\beta}\ j^{\beta}_{\nu} + (\partial_\nu Z\ Z^{-1})^{\delta\beta}\ j^{\beta}_{\mu} + E^{\alpha\beta\gamma} \hat{Z}^{\delta\alpha} (\hat{Z}^{-1})^{\beta\varepsilon} (\hat{Z}^{-1})^{\gamma\epsilon} j^{\varepsilon}_{\mu} j^{\epsilon}_{\nu}$$

where $\hat{Z}^{\delta\beta} = 2\,\mathrm{Tr}\left(\frac{\lambda^\delta}{2}\frac{\lambda^\alpha}{2}\frac{\lambda^\beta}{2}\frac{\lambda^\gamma}{2}\right)(\phi^\alpha + \lambda^\alpha)(\phi^\beta + \lambda^\gamma)$

In fact $\tilde{F}^{\delta}_{\mu\nu} = g\ \hat{Z}^{\delta\alpha} F^{\alpha}_{\mu\nu}$. Using the canonical commutation relations

$$\left[A_{\alpha i}(\underline{x}, t),\ -\tfrac{1}{2} F_{\beta}{}^{0j}(\underline{x}', t)\right] = i \delta_{\alpha\beta} \delta_{ij} \delta^3(\underline{x}-\underline{x}')$$

$$\left[\phi_\alpha(\underline{x}, t),\ -\mathrm{Tr}\left(\frac{\lambda^\beta}{2} D_0 \phi(\underline{x}', t)\right)\right] = i\ \delta^3(\underline{x}-\underline{x}')\ \delta_\alpha{}^\beta$$

we have

$$\left[\tilde{F}_{a\,0k}(\underline{x}, t),\ j_{br}(\underline{x}', t)\right]$$

$$= g^2 \left[\hat{Z}^{ac}(\underline{x}) F^{c}_{0k}(\underline{x}, t),\ \hat{Z}^{bd}(\underline{x}', t)\, V_r^d(\underline{x}', t)\right]$$

$$= g^2\ \hat{Z}^{ac}(\underline{x}, t)\ \hat{Z}^{bd}(\underline{x}', t)\ \left[F^{c}_{0k}(\underline{x}, t),\ V_r{}^{d}(\underline{x}', t)\right]$$

$$\longrightarrow 0 \qquad \text{as} \quad g \to 0$$

This is the essential idea in the BFH procedure. However we have

scalar particles and it is necessary that $\tilde{F}^{\alpha}_{ok}$ should commute with all the ϕ dynamical variables (viz. ϕ and $D_o\phi$) for it to be a constant . The BFH prescription for obtaining pure non-canonical theories (i.e. theories in which the ' field ' variables do not have canonical conjugate variables) requires a new field ϕ' such that

$$\phi = \phi' / g \quad \text{with} \quad g \longrightarrow 0$$

Formally ϕ' has no canonically conjugate field. Unfortunately then the last term in the expression for $\tilde{F}^{\delta}_{\mu\nu}$ becomes infinite and not a particularly meaningful theory would emerge. This type of problem seems to be present in whatever way we modify the BFH procedure. Finally a brute force calculation of commutators does however reveal q number Schwinger terms which have canonical dimension two. This fits in with the ideas of asymptotic scale invariance. The actual details of the current commutators is very complicated and not too illuminating. We feel that any model of currents which will emerge from the Higgs-Kibble type theories is likely to be very involved and ugly compared to the Sugawara model.

Chapter 2 Gauge model calculations

The ideas of Higgs[2], Kibble[3] and others have been used by Weinberg, Salam and Ward[23] to construct weak interaction theories. Since these first efforts many models of weak interactions have been invented. It would be true to say that the main impetus to the growth of such theories is the elegance inherent in spontaneous symmetry breaking. With the benefit of hind-sight we shall indicate how these theories may have been discovered by physicists who were not field theorists absorbed in gauges and spontaneous symmetry breaking.

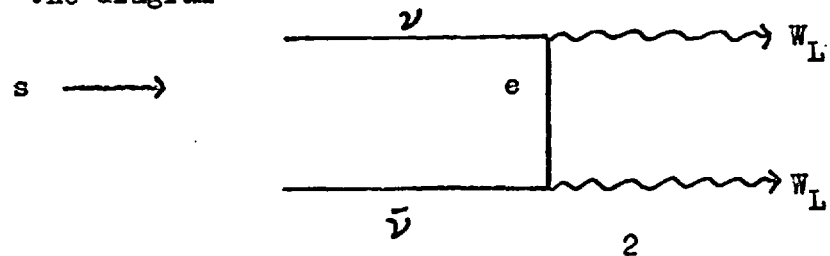
The old conventional weak interactions required[36]

$$\mathcal{L} = g_W \left(J_\rho \, W^\rho + \text{h.c.} \right)$$

where $$J_\rho = \bar{\mu} \gamma_\rho (1 - \gamma_5) \nu_\mu + \bar{e} \gamma_\rho (1 - \gamma_5) \nu_e$$

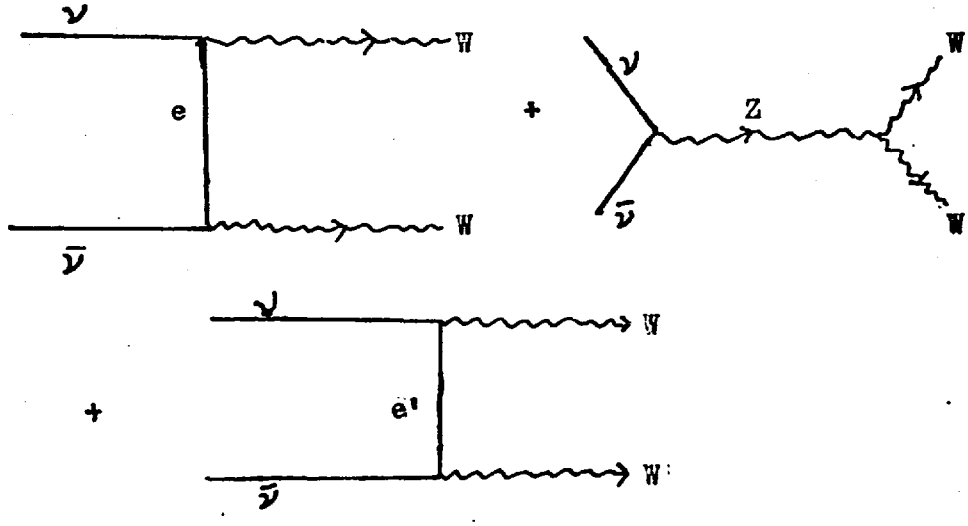
$$+ \bar{p} \gamma_\rho (1 - \gamma_5) n \cos\theta_c + \bar{p} \gamma_\rho (1 - \gamma_5) \lambda \sin\theta$$

and θ_c is the Cabibbo angle; only charged W vector bosons are involved. However for such theories it has been shown that for scattering of neutrinos into longitudinally polarised W mesons the diagram

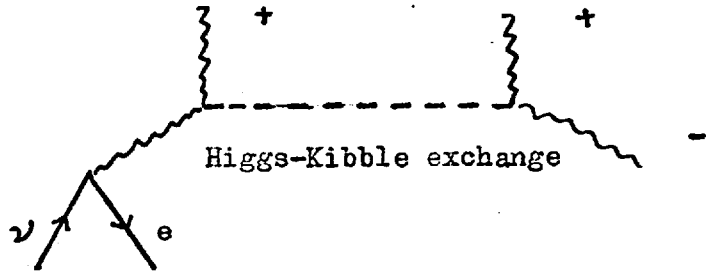


gives a cross-section which behaves as $\frac{g_W^2}{M_W^4} s$

for s very much larger than m_W^2 (the square of the W meson mass). Unitarity demands that $\sigma(\bar{\nu}\nu \longrightarrow W_L W_L)$ is bounded by (constant / s) . We have something much stronger than a tolerable log s violation of unitarity ; however if we introduce new exchanges [37] in the s-channel or the t (or u) channel with suitable couplings, e.g.



we may be able to cancel the bad high energy behaviour. Here Z is a neutral vector boson and e' is a charged heavy lepton. Bjorken, Khriplovitch and Vainstein have adopted this approach systematically and have found relations between the couplings which are characteristic of Yang-Mills theories. A S-matrix view-point has been given by Quinn and J.C. Taylor [38] for the necessity of the Higgs-Kibble scalar. For $\nu e \longrightarrow 3W_L$ it is known that asymptotically for large s $\sigma \sim s^{-\frac{1}{2}}$. In order for $\nu\bar{e} \longrightarrow \nu\bar{e}$ to be finite it has been shown that a condition is $\sigma \sim s^{-1}$ as $s \longrightarrow \infty$. However if the diagram

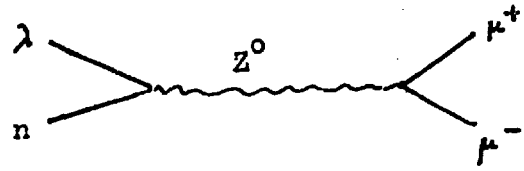


is taken into account then $s^{-\frac{1}{2}}$ type behaviour is cancelled off to allow $\sigma \sim s^{-1}$. These considerations indicate that the new models of weak interactions may be more than picturesque mathematical schemes.

The method of model construction is now well understood. The gauge group is frequently $SU(2) \times U(1)$ and the differences between models is often based on different choices of representations. The couplings are chosen such that the classical interaction Hamiltonian of the Higgs-Kibble fields is a minimum when a neutral Higgs-Kibble field ϕ^0 has a non-vanishing value $\langle \phi_0 \rangle$[39]. Neutral components are chosen for symmetry breakdown so that we maintain electric charge conservation. All couplings are gauge invariant and renormalisable; moreover there should be only one massless vector boson (which will be identified with the photon) as a result of the Higgs-Kibble phenomenon, i.e. the unbroken subgroup is U(1).

Naturally not all models constructed in the above way will be meaningful; there are experimental constraints. The original model of Salam and Weinberg allows $\nu_\mu + e \longrightarrow \nu_\mu + e$ in first order; however it can still be made to fit the Reines-Gurr[40] data,

but it is in difficulties because it allows a process λ,n being quarks



and gives rise to unacceptably large values for $K \longrightarrow 2\mu$. In view of this we shall concentrate on another model due to Lee, and Prentki and Zumino (LPZ)[22]. We now establish the model as well as some notation :

$$G = SU(2) \times U(1)$$

The generators J^a, a= 0,1,2,3 of G satisfy

$$[J^k , J^l] = i \epsilon^{klm} J^m$$

$$[J^k , J^0] = 0$$

We introduce

$$E_L = \begin{bmatrix} e' \\ \nu_e \\ e \end{bmatrix}_L \qquad M_L = \begin{bmatrix} \mu' \\ \nu_\mu \\ \mu \end{bmatrix}_L$$

Our conventions for the Dirac matrices are such that

$$\frac{1 - \gamma_5}{2} \nu_e = (\nu_e)_R = 0$$

E_L and M_L transform under the fundamental SU(2) representation; e'_R, μ'_R, e_R, and μ_R are SU(2) singlets. The covariant derivative of $\psi(x)$ a field transforming under G is given by

$$D_\mu \psi = \partial_\mu \psi + ig X^k_\mu j^k \psi + i \frac{g'}{2} X^0_\mu j^0 \psi$$

where the gauge fields transform as

$$\delta X^{k}_{\mu} = \epsilon^{klm} X^{l}_{\mu} \Omega^{m} - \frac{1}{g} \partial_{\mu} \Omega^{k}$$

$$\delta X^{0}_{\mu} = - \frac{2}{g'} \partial_{\mu} \Omega^{0}$$

and j^k are suitable representations of J^k. We define the charge operator Q as

$$Q = J_3 + \tfrac{1}{2} J^0 .$$

So $$A_{\mu} = \left(\frac{1}{g} X^{3}_{\mu} + \frac{1}{g'} X^{0}_{\mu} \right) \left(\frac{1}{g^2} + \frac{1}{g'^2} \right)^{-\frac{1}{2}}$$

couples to the charge operator and is taken to be the electromagnetic field. In order that the leptons have the correct electric charge we demand that

$$j^0 \, e_R = -2 \, e_R \text{ and } j^0 E_L = 0$$

Of course identical equations hold for the muon multiplets. The linear combination of X^3 and X^0 orthogonal to A is denoted by Z. Explicitly

$$Z_{\mu} = \left(\frac{1}{g'} X^{3}_{\mu} - \frac{1}{g} X^{0}_{\mu} \right) \left(\frac{1}{g^2} + \frac{1}{g'^2} \right)^{-\frac{1}{2}}$$

Apart from the neutral vector mesons we have the charged mesons which are given by

$$W^{\mp} = (1/2)^{\frac{1}{2}} \left(X_1 \pm X_2 \right)$$

The " ' " leptons are known as the 'heavy' leptons. In these theories every particle present which has not been found experimentally is regarded as being heavy. Consequently it is important to try to find experimental constraints on all the heavy particles

occurring in these theories. We let $\{C_i ;\ i = 1,2,3\}$ be the set of 3x3 matrices which form a representation of the SU(2) algebra, i.e. $[C_i/2 ,\ C_j/2] = i \in_{ijk} C_k/2$. Only interactions with the electron multiplet will be explicitly exhibited. Exactly similar couplings apply for the muon multiplet. LPZ have constructed the following invariant couplings :

$$\text{i)} \quad \bar{E}_L \frac{i}{2} \gamma^\mu \left(\overleftrightarrow{\partial_\mu} + g\, C_j\, X^j_\mu \right) E_L$$

$$\text{ii)} \quad \bar{e}'_R \frac{i}{2} \gamma^\mu \left(\overleftrightarrow{\partial_\mu} + 2g \tan\theta\, X^0_\mu \right) e'_R$$

$$\text{iii)} \quad \bar{e}_R \frac{i}{2} \gamma^\mu \left(\overleftrightarrow{\partial_\mu} - 2g \tan\theta\, X^0_\mu \right) e_R$$

Here we have introduced the Salam-Ward-Weinberg mixing angle which is defined to be θ

$$g' = g \tan\theta$$

LPZ have extended the above considerations to hadrons by introducing integrally charged quarks. As usual right handed quarks are taken to be singlets. Explicitly

$$h'_L = \begin{bmatrix} p^+ \\ n_c \\ r^- \end{bmatrix}_L \qquad h^2_L = \begin{bmatrix} p'^+ \\ \lambda_c \\ r'^- \end{bmatrix}_L$$

These are triplets and

$$n_c = n \cos\theta_c + \lambda \sin\theta_c$$

$$\lambda_c = -n \sin\theta_c + \lambda \cos\theta_c$$

Denoting p,n,λ by q the meson octet is given by $\bar{q}q$.

(For baryons it is necessary to add a G singlet quark which is also an SU(3) singlet). New quarks r,r' and p' have been introduced and are known as charmed quarks. The invariant couplings are clearly very similar to those given before.

In contrast to the Salam-Ward-Weinberg model two triplets of Higgs-Kibble scalars are introduced.

$$\gamma = \begin{bmatrix} \gamma^+ \\ \gamma^0 \\ \gamma^- \end{bmatrix} = \begin{bmatrix} (\gamma^1 - i\gamma^2)/(2)^{\frac{1}{2}} \\ \gamma^3 \\ (\gamma^1 + i\gamma^2)/(2)^{\frac{1}{2}} \end{bmatrix}$$

Here γ^i is hermitian. The second triplet of scalars is

$$\chi = \begin{bmatrix} \chi^{++} \\ \chi^{+} \\ \chi^{0} \end{bmatrix} = \begin{bmatrix} (\chi^1 - i\chi^2)/(2)^{\frac{1}{2}} \\ \chi_3 \\ (\chi^1 + i\chi^2)/(2)^{\frac{1}{2}} \end{bmatrix}$$

with $\chi_j = \alpha_j + i\beta_j$ and $\alpha_j^\dagger = \alpha_j$, $\beta_j^\dagger = \beta_j$.

The most general Higgs-Kibble potential P which is compatible with renormalisability is

$$a I_1 + bI_1^2 + cI_2 + dI_2^2 + fI_4^x I_4 + h I_3^x I_3$$

where $I_1 = (\vec{\gamma})^2$ $\quad I_3 = \gamma^x \chi$

$I_2 = \chi^x \chi$ $\quad I_4 = \chi \varepsilon \chi$

and

$$\varepsilon = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

I_1 and I_2 have charge zero, and I_3 has charge one and I_4 has charge two. It is important to have all possible invariant couplings since a gauge theory is renormalisable in the strict sense if it contains all possible terms of dimension four or less which are gauge invariant. If this condition is satisfied, the lagrangian contains all the necessary counterterms for renormalisation. It has been shown by LPZ that

$$P = a(\vec{\gamma})^2 + b(\vec{\gamma}^{\,2})^2 + c(\vec{\alpha}^{\,2} + \vec{\beta}^{\,2}) + d(\vec{\alpha}^{\,2} + \vec{\beta}^{\,2})^2$$
$$+ f((\vec{\alpha}^{\,2} - \vec{\beta}^{\,2})^2 + 4(\vec{\alpha}\cdot\vec{\beta})^2) + h((\vec{\alpha}\cdot\vec{\gamma})^2 + (\vec{\beta}\cdot\vec{\gamma})^2)$$

Now a and c are taken to be negative whereas b,d,f and h are taken positive. P has a minimum for $\vec{\alpha}\cdot\vec{\beta} = \vec{\alpha}\cdot\vec{\gamma} = \vec{\beta}\cdot\vec{\gamma} = 0$, and $\vec{\gamma}^{\,2} = -a/2b \equiv u^2$, as well as $\vec{\alpha}^{\,2} = \vec{\beta}^{\,2} = -c/4d \equiv v^2/4$.

It is possible to make a SU(2) gauge transformation so that

$$\gamma = \begin{bmatrix} 0 \\ u + \gamma^0 \\ 0 \end{bmatrix} \qquad \chi = \begin{bmatrix} \chi^{++} \\ \chi^{-} \\ \frac{v + \varrho^0}{\sqrt{2}} \end{bmatrix}$$

Now a U(1) gauge transformation can be made to make ϱ^0 real. We have only touched on the bare bones of the model but the subject with its many ramifications is large. We find that

$$m^2_{W^{\pm}} = g^2(v^2/2 + u^2) \qquad m^2_Z = (g^2 + g'^2)v^2$$

$$m^2_{\chi^{\pm}} = 2(c - cv^2/(2)^{\frac{1}{2}} + hu^2) \qquad m^2_{\chi^{++}} = 2(c - \frac{cv^2}{\sqrt{2}} + 2fv)$$

$$m^2_{\varrho^0} = 4(c/2 - \frac{3}{2}\frac{c}{\sqrt{2}}v^2)$$

Before proceeding with calculations we would like to mention the choice of gauges[41]. We will not be performing higher order loop calculations and so the unitary gauge is sufficient for our purpose. 't Hooft and Veltman have shown that spontaneously broken gauge symmetries are renormalisable for a gauge in which the massive vector boson propagator is

$$\Delta_{\mu\nu}(p,\xi) = -i\left[g_{\mu\nu} - (1-1/\xi)\ p_\mu p_\nu / \left(p^2 - \frac{1}{\xi}M^2\right)\right]\frac{1}{(p^2-M^2)}$$

The choice

$\xi = \infty$ gives the Landau or renormalisable gauge

$\xi = 1$ gives the 't Hooft - Feynman gauge

$\xi = 0$ gives the unitary gauge

The equation

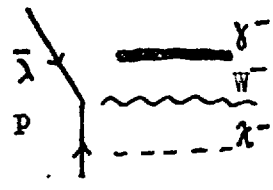
$$\lim_{\epsilon\to 0}\ \left(\left(\text{ calculation }\right)_{\xi=\epsilon>0}\right) = \left(\text{ calculation }\right)_{\xi=0}$$

has not been satisfactorily established. However for all $\xi > 0$ the above type of equation has been verified (and indeed proved generally).

Many models are found to be inconsistent[42] with the data on $K_L \to \bar{\mu}\mu$ (where our conventions are such that

$$|K_L\rangle = \left((1+\epsilon)|K^0\rangle - (1-\epsilon)|\bar{K}^0\rangle\right) / \left(2(1+|\epsilon|^2)\right)^{\frac{1}{2}}\ ).$$

The current experimental upper limit on the branching ratio $\Gamma(K_L \to \bar{\mu}\mu)/\Gamma(K_L \to \text{all})$ is 1.8×10^{-9}. In LPZ we have vertices such as

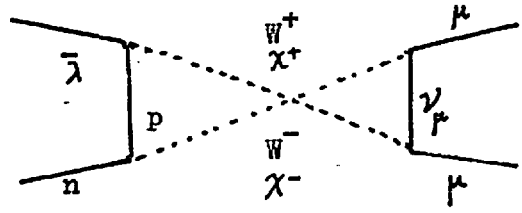




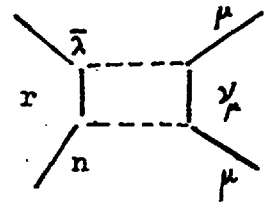
However the LPZ model has been designed to eliminate the $\Delta S = 1$ neutral current and so



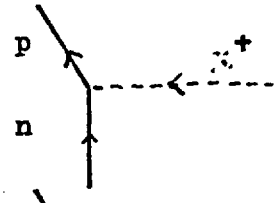
does not occur. We still have diagrams of the form



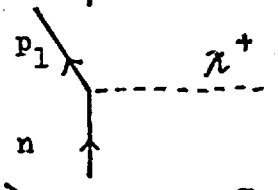
and



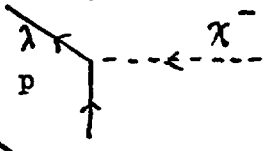
It is here that the 'charmed' diagrams come into effect[43]. In fact the vertex couplings are



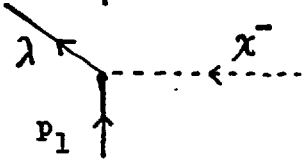
$$-\sqrt{2}\,\frac{m_p}{v}\,\frac{1+\gamma_5}{2}\cos\theta_c$$



$$\sqrt{2}\,\frac{m_{p_1}}{v}\,\frac{1+\gamma_5}{2}\sin\theta_c$$



$$-\sqrt{2}\,\frac{m_p}{v}\,\frac{1-\gamma_5}{2}\sin\theta_c$$



$$-\sqrt{2}\,\frac{m_{p_1}}{v}\,\frac{1-\gamma_5}{2}\cos\theta_c$$

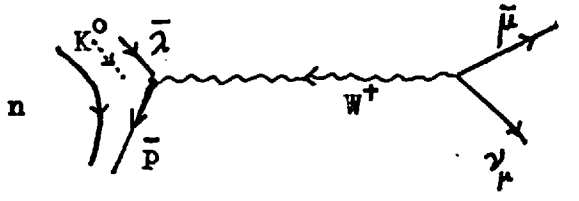
These show that the charmed quark contributions can be used to make the decay $K_L \longrightarrow \bar{\mu}\mu$ as small as necessary by adjusting $|m_{p_1} - m_p|$ and similar charmed-uncharmed mass differences. Exact

SU(4) invariance would give $\Gamma(K_L \longrightarrow \bar{\mu}\mu) = 0$ to all orders! This cancellation mechanism has been invented by Glashow-Maiani-Iliopoulos. Some models have now been eliminated by considering K_1-K_2 mass differences[42]. The data for $K_L \longrightarrow \bar{\mu}\mu$ and K_1-K_2 mass differences give incompatible bounds on the quark mass differences. However since strong interaction effects are appreciable in the $K_1 - K_2$ mass difference such considerations do not rule out the model completely. We would like to discuss in fact how plausible any quantitative calculations are for models such as LPZ for semi-leptonic processes. Unfortunately after much calculation we found that the $\bar{\nu}_e$ - e scattering data gives the following bound

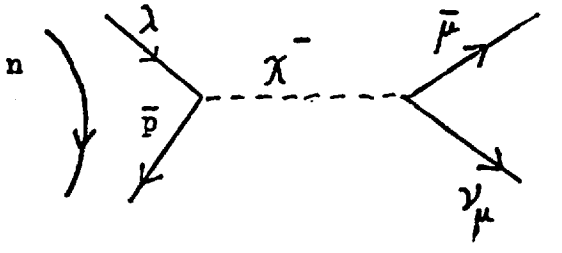
$$a^2 m_e^2 / m_\chi^2 < 0.055$$

where $a = (m_W / m_Z) \sec\theta$. This is not very restrictive and actually gives a lower bound on the mass of the charged Higgs-Kibble scalar of 3 MeV.

We now consider K_{l3} and K_{l2} decays [44] since the data is fairly good. In addition to vector boson exchanges



we have Higgs-Kibble scalar exchanges (which is a new feature)



The matrix element for $K^0 \longrightarrow \pi^- \mu \nu_\mu$ is written as

$$\mathcal{M}(K^0 \longrightarrow \pi^- \mu \nu_\mu) \equiv \frac{iG}{\sqrt{2}} \sin\theta_c \left[f_+(t)\,(k+q)^\beta + f_-(t)\,(k-q)^\beta \right] V_\beta^{\text{lepton}}$$

where $V_\beta^{\text{lepton}} = \langle \nu_\mu | \gamma_\beta (1-\gamma_5) | u_{\nu_\mu} \rangle$, k^β and q^β are the momenta of the kaon and pion respectively and $t = (k-q)^2$ and θ_c is the Cabibbo angle. Now in LPZ as we have seen

$$\mathcal{M}(K^0 \longrightarrow \pi^- \mu \nu_\mu) = \mathcal{M}(\text{vector exchange}) + \mathcal{M}(\text{scalar exchange})$$

From our previous discussion the renormalisable gauge calculations would bring in γ^+ and γ^- as well as Faddeev-Popov ghosts. Anyway it is plausible that owing to the Glashow-Iliopoulos-Maiani mechanism the higher order corrections should be small.

$$\mathcal{M}(\text{vector exchange}) = -i\frac{G}{4}\sin\theta_c \left(\frac{(m_K+m_\pi)}{2m_\pi m_K} (k+q)^\alpha + \frac{(m_\pi - m_K)}{2m_\pi m_K} (k-q)^\alpha \right)$$

$$\cdot \langle \nu_\mu | \gamma_\alpha (1-\gamma_5) | u_{\nu_\mu} \rangle$$

$$\mathcal{M}(\text{scalar exchange}) = -i\,\frac{G\sin\theta_c}{2\sqrt{E_\pi E_K}} \left(m_K^2 + m_\pi^2 - t \right)^{\frac{1}{2}} \frac{(E_K m_\pi + m_\pi m_K)}{m_K (t - m_\pi^2)} a^2$$

$$\cdot \langle \nu_\mu | (1-\gamma_5) | u_{\nu_\mu} \rangle$$

We know that the lepton scalar part can be written as $\frac{(k-q)^\alpha}{m_\mu} \langle \nu_\mu | \gamma_\alpha (1-\gamma_5) | u_{\nu_\mu} \rangle$

This the only vector that we can form from k^α and q^α only. These processes involve the annihilation of a quark and an antiquark. Van Royen and Weisskopf [45] (in terms of older quark models) have considered this for K_{l2} and π_{l2}. They write the creation operator $a^x_\phi(0)$ for a meson ϕ in terms of the creation operators $b^x_r(\underline{p})$

and $c^{x}_{s}(\underline{p})$ for the quark and antiquark with unitary spin components r and s as follows

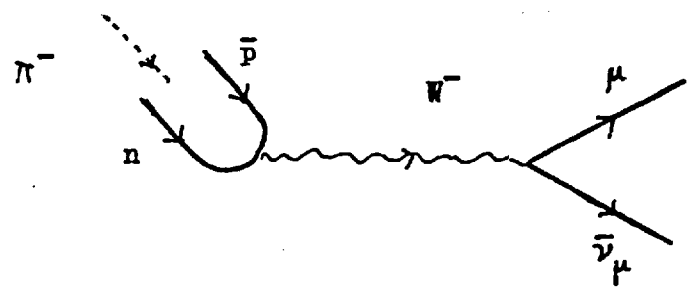
$$a^{x}_{\phi}(o) = \sum_{r,s} \int d^3p \;\; \tilde{f}(\phi, \underline{p})\; d(r,s)\; b^{x}_{r}(\underline{p})\; c^{x}_{s}(-\underline{p})$$

where $f(\phi, \underline{r})$ is an unknown quantity. It is in fact the space wave function of the quark-antiquark bound state corresponding to ϕ , and

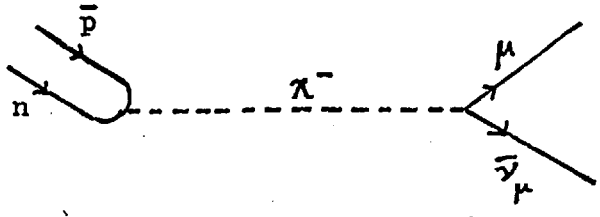
$$\tilde{f}(\phi, \underline{p}) = \frac{1}{(2\pi)^{3/2}} \int f(\phi, \underline{r})\; e^{-i\underline{p}\cdot\underline{r}}\; d^3\underline{r}$$

Weisskopf and Van Royen then proceed to show that the decay widths are proportional to $|f(\phi, 0)|^2$. Indeed this is not just a refinement but is necessary in order to obtain decay rates with the correct dimensions (rather than the dimension of the cross-sections). The normalisation of the meson fields is not the same as the normalisation of the product of two fermion fields. When two quarks make a meson the matrix element is multiplied by $\sqrt{nE}^{-1}$ for each external meson line, where E is the meson energy and n is chosen so that the result obeys certain conditions such as $f_{+}(0) = 1$ in the SU(3) symmetry limit. We have also assumed that the p-quark has a mass $\frac{1}{2}m_{\pi}$ and that the λ-quark has a mass $\frac{1}{2}m_{K}$[46]. The n-quark has a mass $\left\{\begin{matrix}\frac{1}{2}m_{\pi} \\ \frac{1}{2}m_{K}\end{matrix}\right\}$ when accompanied by a $\left\{\begin{matrix}p \\ \lambda\end{matrix}\right\}$ quark. The quarks each have momentum $\frac{1}{2}p^{\mu}$ where p^{μ} is the meson momentum.

In the LPZ model π_{l_2} and K_{l_2} processes are given by



and



We have similar diagrams with λ and n interchanged (for K_{l2} of course). The values that we obtain for $f_+(t)$ and $f_-(t)$ are:

$$f_+(t) = \frac{(m_K^2 + m_\pi^2 - t)^{\frac{1}{2}}}{2\sqrt{2}\, m_K m_\pi} (m_K + m_\pi) \qquad (2.1)$$

$$f_-(t) = \frac{(m_K^2 + m_\pi^2 - t)^{\frac{1}{2}}}{2\sqrt{2}} \left\{ \frac{(m_\pi - m_K)}{m_\pi m_K} + \frac{a^2((m_K+m_\pi)^2 - t)}{m_K (t - m_{K^*}^2)} \right\} \qquad (2.2)$$

The experimental situation is approximately as follows:

$$f_\pm(q^2) \simeq f_\pm(0)\left(1 + \lambda_\pm \frac{q^2}{m_\pi^2}\right)$$

The experiments seem to give the mean value

$$\lambda_+ \approx 0.04$$

We also have

$$\frac{f_-(0)}{f_+(0)} = -.85 \pm .15 \qquad (2.3)$$

$$\frac{f_K}{f_\pi}\frac{1}{f_+(0)} \cos\theta_c \approx 1.26$$

(2.1) and (2.2) do not give the right slopes away from t=0 but it may be that the quark-antiquark wave-function is t-dependent. We now consider $f_+(0)$ and $f_-(0)$. We find that $f_+(0)$ is 1.710

which is badly in disagreement with experiment. However we may reasonably suppose that we have a scaling factor refinement due to quark-antiquark bound states. It is amusing here to note that the Ademollo-Gatto theorem[47] is not violated. From the Gell-Mann-Okubo mass formula we can write

$$m_\pi^2 = m_o^2 + \epsilon$$
$$m_K^2 = m_o^2 - \tfrac{1}{2}\epsilon$$
$$m_\eta^2 = m_o^2 - \epsilon$$

and this gives

$$f_+(0) = \frac{(2m_o^2 + \frac{1}{2}\epsilon)^{\frac{1}{2}}(2m_o + \frac{1}{4}\epsilon/m_o)}{2\sqrt{2}\, m_o^2(1+\frac{1}{2}\epsilon/m_o^2)(1-\frac{1}{4}\epsilon/m_o^2)}$$
$$= 1 + O(\epsilon^2)$$

In the usual theory of these decays using current algebra and octet dominance of sum rules[44] (since mesonic resonances occur only in SU(3) octet and singlet states) it has been found that

$$f_+(t) < 1 \quad \text{for} \quad t \leq 0$$

i.e. the symmetry breaking contribution is negative. From the considerations regarding the Ademollo-Gatto theorem and the above we see here a situation where the model incorporates some current algebra but not all.

Now (2.3) gives us the condition

$$2 \times 10^5 < \frac{m_x^2}{a^2} < 7 \times 10^5$$

We now try to calculate f_K / f_π from the model. It is necessary to multiply the contributions from the second order diagrams for $\pi^+ \longrightarrow \bar{\mu}\nu$ and $K^+ \longrightarrow \bar{\mu}\nu$ by the quark-antiquark wave function. In the case of a purely vector interaction the factors are found by comparison with experiment to be $((m_\pi)^{3/2} / (2)^{\frac{1}{2}})$ for the pion and $(m_\pi m_K^{\frac{1}{2}}) / (2)^{\frac{1}{2}}$ for the kaon. The same factors may be used in our case provided the contributions from the scalar exchange is small. For pion decay we have

$$f_\pi = m_\pi \left(1 + \frac{a^2 m_\pi^2}{m_\pi^2 - m_\chi^2} \right)$$

and the scalar part satisfies $\left| \frac{a^2 m_\chi^2}{m_\pi^2 - m_\chi^2} \right| < 0.09$ and so the assumption about the smallness of the scalar part is not contradicted. However for the kaon decay we obtain the result

$$f_K = m_\pi \left(1 + \frac{a^2 m_K^2}{m_K^2 - m_\chi^2} \right)$$

and so the scalar part satisfies

$$-1.2 < \frac{a^2 m_K^2}{m_K^2 - m_\chi^2} < -0.25$$

This gives $f_K / f_\pi < .75$ which is very badly in disagreement with experiment. Hence it seems that we have no consistent way of calculating K-decays. In view of this it seems preferable to realise the elegant idea of Salam and Weinberg concerning

renormalisable unified theories of weak and electromagnetic interactions by not using quarks. Actually now such models are in existence. Bars, Yoshimura, and Halpern[34] have constructed such a model with the physical mesons causing spontaneous symmetry breaking (with $G = U(3)_L \times U(3)_R \times SU(2) \times U(1)$). The simplicity of the original models has gone however. The real direction for progress lies in ideas such as using radiative corrections as a source of spontaneous symmetry breaking. It is essential to increase the predictive power of these theories by cutting down on the large number of parameters which presently they seem to require.

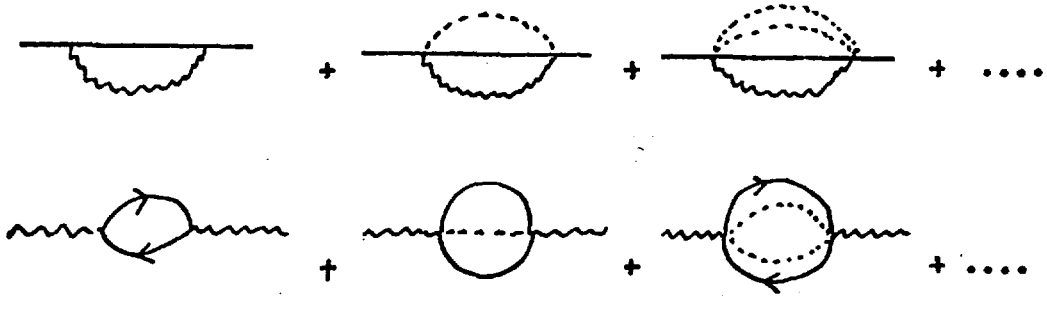
Chapter 3 Non-polynomial field theories in low dimensions

There are many difficulties still remaining in non-polynomial field theories. It is a topic full of subtleties. Slight changes in form of interactions often result in large changes in behaviour of matrix elements. The aim of the founders of the theory[48] is to have a perfectly finite theory without having renormalisation constants (which are essentially undetermined). Before delving into technicalities it is perhaps wise to see some of the predictions of the theory. Computations by I.S.S.[49] to second order in the electromagnetic coupling constant show that

$$\delta m/m \approx \frac{3}{4\pi} \log (1/\varkappa m)^2 \approx 2/11$$

$$\delta e/e \approx -\frac{\alpha}{6\pi} \log (1/\varkappa m)^2 \approx -1/25$$

where $\varkappa$,which is the square root of 16π times the Newtonian constant G, equals 0.5×10^{-18} GeV $^{-1}$. These results were computed by summing the following typical set of graphs



where ——→—— , ~~~~~ , and ..... denote electron,photon, and graviton lines respectively. A very

similar method has been used by Duff, Huskins and Rothery [50] to compute the gravity modified electromagnetic mass difference of the charged and neutral pion using the Lee and Nish phenomenological lagrangian. Their result was $m_{\pi^+} - m_{\pi^0} \sim 6.8$ MeV. The experimental answer is 4.7 MeV. Applications to weak interactions also exist.There are objections to these calculations. The most important of these (or perhaps the most pertinent as regards non-polynomial field theories) is that the calculations may not be gauge invariant.

A lot of progress can be made if we compensate for the increased complexity of non-polynomial field theories over polynomial theories by reducing the number of dimensions d of space. Glimm and Jaffe[51] have been particularly successful in studying d= 1 theories. In particular for the ϕ^4 interaction they have been able to solve the theory to verify most of the Wightman axioms. Lower dimensional theories are often useful in pointing out errors in methodology which are often obscured by distribution theoretic ambiguities in cases[52] for which d= 3.

We first consider a free field theory

$$L = \tfrac{1}{2} (\partial\varphi)^2$$

The dimension d is not explicitly relevant here and canonical terms will be ignored. We consider the following field redefinitions

i) $\varphi = 1/\lambda \quad (\exp\lambda\chi - 1)$

ii) $\varphi = 1/\lambda \log(1 + \lambda\chi)$

The redefinition (i) implies that

$$L \longrightarrow L' = 1/2 \quad (\partial\chi)^2 \exp (2\lambda\chi)$$

Similarly (ii) implies that

$$L \longrightarrow L' = 1/2 \ (\partial\chi)^2 (1 + \lambda\chi)^{-2}$$

In this way we naturally obtain derivative interactions as well as different forms of non-polynomiality. Indeed this might lead us to think that there is not much difference between the two forms of non-polynomiality.

We shall investigate the following interactions

a) $L_{int} = \lambda : q (1 - g q^2)^{-1} :$

a') $L_{int} = : (1-q)^{-\nu} : \quad (\nu \neq -n)$

b) $L_{int} = \exp(-\lambda q)$

b') $L_{int} = : \exp(-\lambda q) :$

c) $L_{int} = \exp(-\lambda q^2)$

c') $L_{int} = : \exp(-\lambda q^2) :$

d) $H_{int} = \frac{1}{2} : p^2 \exp(-\lambda q) :$

d') $H_{int} = \frac{1}{2} : p^2 V(q) :$

where $V(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \ v(n) \ x^n$ and $v(o) = 1.$

In (d) and (d') we could consider (taking α and β as real constants)

e)
$$\propto p^2 \begin{Bmatrix} e^{-\lambda q} \\ \\ V(q) \end{Bmatrix} + \propto \begin{Bmatrix} e^{-\lambda q} \\ \\ V(q) \end{Bmatrix} p^2 + \beta\, p \begin{Bmatrix} e^{-\lambda q} \\ \\ V(q) \end{Bmatrix} p$$
e')

but on grounds of simplicity we avoid these cases. It is worth mentioning that the interactions in (b) to (e') are somewhat unconventional. Normally, in (c) to (c'), only

$$L_{int} - 1 - \tfrac{1}{2} q^2 \left(\partial^2 L_{int}/\partial q^2\right)_{q=0}$$

would be regarded as an interaction. Similarly in (d) and (d')

$$H_{int} - \tfrac{1}{2} p^2$$

would be the conventional interaction. λ is known as the minor coupling constant.

We have the usual relations

$$\left[q(t),\, p(t)\right] = i$$

$$\left[a(t),\, a^{\dagger}(t)\right] = 1 \quad \text{where} \quad a^{\dagger} = \frac{1}{\sqrt{2\omega}} \left(p + i\omega q \right)$$

Solutions for the free field (i.e. $H_o = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2 q^2$)

are $q(t) = \alpha_o\, e^{i\omega t} + \beta_o\, e^{-i\omega t}$

$$p(t) = i\omega \left(\alpha_o\, e^{i\omega t} - \beta_o e^{-i\omega t} \right)$$

$$\left[\beta_o\, , \alpha_o\right] = 1/2\omega$$

We define $a^{\dagger}(t) = \sqrt{2\omega}\; i\alpha_o\; e^{i\omega t}$

$$a(t) = -\sqrt{2\omega}\; i\beta_o\; e^{-i\omega t}$$

So for free fields we have

$$\left[p(t)\ ,\ \bar{p}(0)\right] = -i\omega \sin \omega t$$

$$\left[q(t)\ ,\ q(0)\right] = -i/\omega \sin \omega t$$

$$\left[\bar{p}(t)\ ,\ q(0)\right] = -i \cos \omega t$$

The Fock states $|n\rangle$ satisfy

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle\ ,\quad a^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

and $$|n\rangle \equiv \frac{1}{\sqrt{n!}}\ (a^{\dagger})^{n}\ |0\rangle$$

Clearly

$$D^{0}(t) = \langle 0|T(\ q(t)\ q(0))|0\rangle = \theta(t)\langle 0|\ q(t)\ q(0)|0\rangle + \theta(-t)\langle 0|q(0)\ q(t)|0\rangle$$

$$= \frac{1}{2\omega}\ \exp\ (-i\omega|t|)$$

Similarly we find that

$$\langle 0|\ T\ (\ p(t)\ q(0)\ )|0\rangle = -\tfrac{1}{2}\,i\ e^{-i\omega|t|} + i\,\theta(-t)e^{-i\omega|t|}$$

$$\langle 0|\ T\ (\ q(t)\ p(0)\ )|0\rangle = \tfrac{1}{2}i\ e^{-i\omega|t|} - i\,\theta(-t)\ e^{-i\omega|t|}$$

$$\langle 0|\ T\ (\ p(t)\ p(0)\ )|0\rangle = \tfrac{1}{2}\omega\, e^{-i\omega|t|}$$

We shall also consider (b') for d= 1,2.

$$D^{(d)}(x) \equiv \frac{1}{i(2\pi)^{d+1}} \int d^{(d+1)}k\ \frac{\exp ik.x}{-k - i\epsilon}$$

for $d = 1, 2$ and $\epsilon > 0$. We look at these last two cases to point out the dangers of indiscriminate extrapolation of the ' d = 0' behaviour to ' d = 3' behaviour.

$$D^{(1)}(t,x) = \frac{1}{i(2\pi)^{2}} \int dk_{o}\ dk_{1}\ \frac{\exp\ i\ (\ k_{o}t - k_{1}x)}{-k_{o}^{2} + k_{1}^{2} - i\epsilon}$$

On performing the contour integral

$$D^{(1)}(x,t) = \theta(t)\,(\,-\tfrac{1}{2}i\,\theta(-x-t)) - \theta(-t)(\,\tfrac{1}{2}i\,\theta(t-x)\,)$$

The momentum space superpropagator corresponding to

$$\exp\ (-\frac{i\lambda^2}{2}\ (\theta(t)\,\theta(-x-t) + \theta(-t)\,\theta(t-x))\ )$$

is

$$\int e^{i(st+kx)}\ e^{\lambda^2 D^{(1)}(x,t)}\ dx\ dt$$

$$= \exp(-\tfrac{1}{2}i\lambda^2)\ (\int_{\frac{3}{4}\pi}^{\frac{5}{4}\pi} d\theta \int_0^\infty dr\ r\ e^{ir(s\sin\theta\ +\ k\cos\theta\ )}$$

$$+\ \lambda\text{-independent quantity}$$

Hence the d= 1 superpropagator is an entire function in λ. The case d = 2 is somewhat cumbersome.

$$D^{(2)}(x) = \frac{1}{i(2\pi)^3} \int \frac{e^{i(k_o t - k_1 x_1 - k_2 x_2)} d^3k}{-k^2 - i\epsilon}$$

This reduces to

$$D^{(2)}(x) = \tfrac{1}{2}i \int_0^{2\pi} d\theta\ (\ (x_1^2 + x_2^2)^{\frac{1}{2}} \cos\theta + |t|\ )^{-1}$$

where $x = (t, x_1, x_2)$

It is known that [53]

$$\int \frac{dx}{a + b\cos x} = \frac{2}{(a^2 - b^2)^{\frac{1}{2}}}\ \theta(a^2 - b^2)\ \tan^{-1}\left(\frac{(a^2-b^2)^{\frac{1}{2}}\tan\frac{1}{2}x}{a+b}\right)$$

$$+\ \frac{1}{(b^2 - a^2)^{\frac{1}{2}}}\ \theta(b^2-a^2)\log\ \frac{(b^2-a^2)^{\frac{1}{2}}\tan\frac{1}{2}x + a + b}{(b^2-a^2)^{\frac{1}{2}}\tan\frac{1}{2}x - a - b}$$

Hence in the Symanzik region

$$D^{(2)}(x_1,x_2,t) \propto i/(\hat{x}^2)^{\frac{1}{2}}$$

where $\hat{x}^2 = t^2 + x_1^2 + x_2^2$

Using $$\tilde{f}(q^2) \equiv \int d^3\underline{x}\ e^{\pm i\underline{q}\cdot\underline{x}}\ f(\underline{x}^2)$$

$$= \frac{4\pi}{|\underline{q}|}\int_0^\infty dr\ r\ \sin r|\underline{q}|\ f(r^2)$$

The momentum space superpropagator for $d = 2$ is

$$\int \exp i(\lambda^2(\hat{x}^2)^{-\frac{1}{2}})\exp(ip.x)\ e^{-\epsilon(\hat{x}^2)^{\frac{1}{2}}}\ dx$$

$$= \frac{4\pi}{|\underline{p}|}\int_0^\infty dr\ r\sin r|\underline{p}|\exp(i\lambda^2/r)\ e^{-\epsilon r} \quad \text{where } \epsilon > 0$$

A very simple argument can be given to show that this is not analytic at $\lambda = 0$. This goes as follows:

$\tilde{D}^{(2)}$ exists at $\lambda = 0$. We let $\lambda = a+ib$ where a and b are real; then

$$\tilde{D}^{(2)} = \frac{4\pi}{|\underline{p}|}\int_0^\infty dr\ r\sin r|\underline{p}|\exp i\left[(a^2-b^2+2abi)/r\right]\exp(-\epsilon r)$$

For ab negative this integrand blows up at $r = 0$ whereas for ab positive this is not the case. This demonstrates non-analyticity at $\lambda = 0$. More explicitly the nature of the non-analyticity is seen from

$$\int_0^\infty \exp\left(-\frac{\beta}{4x} - \gamma x\right) dx = \sqrt{\frac{\beta}{\gamma}}\,K_1(\sqrt{\beta\gamma})$$

where $\mathrm{Re}\,\beta \geq 0$, and $\mathrm{Re}\,\gamma > 0$ and

$$K_n(z) = \tfrac{1}{2}\sum_{k=0}^{n-1}(-1)^k(n-k-1)!(\tfrac{1}{2}z)^{2k-n}/k! + (-1)^{n+1}\sum_{k=0}^{\infty}(\tfrac{1}{2}z)^{n+2k}\,K(z,n,k)$$

Here $K(z,n,k) = (\log\frac{1}{2}z - \frac{1}{2}\psi(k+1) - \frac{1}{2}\psi(n+k+1))/k!(n+k)!$

Also $\int_0^\infty dr\ \exp(-((-i|p| + \epsilon)r - i\lambda^2/r)) = (4\lambda^2/|p|)^{\frac{1}{2}} K_1(-4\lambda^2|p|)$

Using these relations and after some more manipulation we have

$$\tilde{D}^{(2)} = \frac{1}{2|p|}\Big((-4\lambda^2/|p|)^{\frac{1}{2}} K_1(4\lambda^2|p|) - (4\lambda^2/|p|)^{\frac{1}{2}} K_1(-4\lambda^2|p|)$$

$$+ \frac{1}{(|p|)^3}\ \Big((4\lambda^2/|p|)^{-\frac{1}{2}} + (-4\lambda^2/|p|)^{-\frac{1}{2}} \Big)$$

$$-\tfrac{1}{2}|p| \sum_{k=0}^{\infty} \frac{(2\lambda^2|p|)^{2k}}{k!\,(k+1)!}\ \Big((4\lambda^2/|p|)^{3/2} + (-4\lambda^2/|p|)^{3/2} \Big)$$

$$-|p|\,(4\lambda^2/|p|)^{3/2} \sum_{k=0}^{\infty} \frac{(2k+1)(-2\lambda^2|p|)^{2k}}{2\,k!\,(k+1)!}\Big(\log(-2\lambda^2|p|) - \tfrac{1}{2}\psi(k+1) - \tfrac{1}{2}\psi(k+2)\Big)$$

$$-|p|(-4\lambda^2/|p|)^{\frac{3}{2}} \sum_{k=0}^{\infty} \frac{(2k+1)(2\lambda^2|p|)^{2k}}{2\,k!\,(k+1)!}\Big(\log(2\lambda^2|p|) - \tfrac{1}{2}\psi(k+1) - \tfrac{1}{2}\psi(k+2)\Big)$$

This rather ugly expression shows that $D^{(2)}$ has a dominant singularity of the form $(\lambda^2)^{-\frac{1}{2}}$; hence it is important to note that the analyticity properties of the 'd=2' case is quite different from the d= 1 case. Our interest in the study of lower dimensional theories is based on obtaining qualitative rather than quantitative insights into behaviours of superpropagators in d=3 theories.

From now on we will concentrate on the d=0 case. It is useful to note that

$$L(q) = :L'(q):$$

where $$L'(q) = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} d\alpha\ e^{-\frac{1}{2}\alpha^2}\ L(q + \alpha D^{\frac{1}{2}}(0))$$

This easily leads to

$$e^{-\lambda q} = e^{\lambda^2/4\omega} : e^{-\lambda q} : \tag{3.1}$$

$$e^{-\lambda q^2} = (1 + \lambda/\omega)^{-\frac{1}{2}} : \exp\left(-\tfrac{1}{2} \frac{\lambda}{(\frac{1}{2}+\lambda D(0))} q^2\right) : \tag{3.2}$$

The inverse relation is

$$: e^{-\lambda q^2} : = (1- 2\lambda D(0))^{-\frac{1}{2}} \exp\left(-\lambda q^2 (1-2\lambda D(0))^{-1}\right) \tag{3.3}$$

In particular $:q^n: = H_n(\omega^{\frac{1}{2}} q) / (2\omega^{\frac{1}{2}})^n$

So
$$L_{int}^{(a)} = \lambda \sum_{l=0}^{\infty} g^l \, H_{2l+1}(\omega^{\frac{1}{2}} q) / (2\omega^{\frac{1}{2}})^{2l+1}$$

$$= \int_0^{\infty} dt\, e^{-t} \sum_{l=0}^{\infty} g^l t^{2l+1} \frac{H_{2l+1}(\omega^{\frac{1}{2}} q)}{(2l+1)! (2\omega^{\frac{1}{2}})^{2l+1}}$$

Setting $f(q,t) = \sum_{l=0}^{\infty} g^l (t/2\omega^{\frac{1}{2}})^{2l+1} H_{2l+1}(\omega^{\frac{1}{2}} q)/(2l+1)!$

we have
$$\frac{1}{t} \frac{df}{dt} + g^{\frac{1}{2}} f = \sum_{l=0}^{\infty} \frac{1}{l!} \left(\tfrac{1}{2} t (g/\omega)^{\frac{1}{2}} \right)^l H_l(\omega^{\frac{1}{2}} q)$$

$$= \exp\left(-gt^2/4\omega + g^{\frac{1}{2}} tq \right)$$

This gives
$$f(q,t) = \frac{1}{2g^{\frac{1}{2}}} e^{-gt^2/4\omega} \left(e^{g^{\frac{1}{2}} tq} - e^{-g^{\frac{1}{2}} tq} \right)$$

and
$$L_{int}^{(a)}(q) = \frac{1}{2g^{\frac{1}{2}}} \int_0^{\infty} e^{-(gt^2/4\omega) - t} \left(e^{g^{\frac{1}{2}} tq} - e^{-g^{\frac{1}{2}} tq}\right) dt$$

$$\sim \pm (\pi\omega)^{\frac{1}{2}} g^{-1} \exp \omega \left(1/g^{\frac{1}{2}} \mp q \right)^2$$

as $q \to \pm \infty$

Similarly

$$L_{int}^{(a')}(q) \sim \frac{1}{\Gamma(\nu)} (\pi/\omega) (2\omega q)^{\nu} e^{\omega(1-q)^2}$$

Now it is well-known that the harmonic oscillator wave functions are $L_2(R)$ functions $u_n(x)$ given by

$$u_n(t) = (\omega^{\frac{1}{2}}/\pi^{\frac{1}{2}} 2^n n!)^{\frac{1}{2}} H_n(\omega^{\frac{1}{2}}t) e^{-\frac{1}{2}\omega^2 t^2}$$

for all non-negative n. Hence the Wightman functions corresponding to (a) and (a') blow up whereas this is not the case for (b) to (d'); so in this simple framework we can already see the difference between local and non-local interactions , or, strictly speaking, between the analogues in $d = 0$ of local and non-local interactions in fully relativistic theories.

From the equations (3.1) to (3.3) certain properties of the n-point functions(regarded as functions in λ) can be deduced. The superpropagators arising from interactions (a) and (a') have the same analyticity in λ . For (b) and (b') the superpropagators have an essential singularity at $\lambda = 1/2D(0)$ together with a possible branch point.

When ω has a small positive imaginary part

$$\int_{-\infty}^{\infty} e^{-in\omega|t|} e^{ist} dt = -2n\omega/i(s^2-n^2\omega^2) \quad , n \neq 0$$

Hence

$$\int_{-\infty}^{\infty} e^{ist} \langle 0 | T(:e^{-\lambda q(t)}: :e^{-\lambda q(0)}:) | 0 \rangle dt$$

$$= 2\pi\delta(s) + i\lambda^2 \sum_{n=1}^{\infty} \frac{1}{(n-1)!} (\lambda^2/2\omega)^{n-1} (1/(s^2-n^2\omega^2)) \qquad (3.4)$$

Naturally the behaviour of (3.4) as a function of λ (and not of s) is of interest; so s is kept away from the regions of its poles. From the ratio test (3.4) is absolutely convergent and so the superpropagator is entire in λ .

Similarly

$$\int_{-\infty}^{\infty} dt\, e^{ist} \langle 0 \mid T\,(:e^{-\lambda q^2(t)}:: e^{-\lambda q^2(0)}:) \mid 0 \rangle \qquad (3.5)$$

$$= 2\pi\delta(s) + \sum_{n=1}^{\infty} (\lambda^n/n!)^2\, (2n)!\, (1/2\omega)^{2n} (-4n\omega/i(s^2-4n^2\omega^2))$$

Again by the ratio test (3.5) is analytic in the region $|\lambda|<|\omega|$. From equations (3.3) to (3.5) the following can be deduced :

(A) Interactions (b) and (b') both give two point functions which are entire in λ.

(B) The superpropagators in (c) and (c') have the same λ-analyticity in the interior of the disc $|\lambda| = \omega$ which has on its border an essential singularity.

We have already managed to narrow down the choice of non-polynomial lagrangians. The rationals are not pleasant and now (b) seems to have more symmetrical and elegant analyticity properties than (c) . Indeed it has been found by Lehmann and Trute [18] that localisability picks out the exponential parametrisation of fields in the non-linear chiral theories. Salam has also advocated this parametrisation in gravity. It is fairly obvious also that derivatives do not affect the analyticity properties. Explicitly we have

$$\langle 0 \mid T\left(:\tfrac{1}{2}p^2(t)\exp(-\lambda q(t)):\,:\tfrac{1}{2}p^2(0)\exp(-\lambda q(0)):\right) \mid 0\rangle$$

$$= \tfrac{1}{2}\left(\overbrace{p(t)\;p(0)}\right)^2 \exp\left(\lambda^2\,\overbrace{q(t)\;q(0)}\right)$$

$$+ \lambda^2\,\overbrace{p(t)\;p(0)}\;\overbrace{p(t)\;q(0)}\;\overbrace{q(t)\;q(0)}\;\exp\left(\lambda^2\,\overbrace{q(t)\;q(0)}\right)\left(\overbrace{q(t)\;p(0)}\right)$$

$$+ \tfrac{1}{4}\lambda^4\left(\overbrace{p(t)\;q(0)}\right)^2\left(\overbrace{q(t)\;p(0)}\right)^2 \exp\left(\lambda^2\,\overbrace{q(t)\;q(0)}\right)$$

where ' $\overbrace{\quad}$ ' denotes the usual contraction symbol in Wick's theorem. A comparison with (3.4) reveals essentially the same convergence conditions and so the same analyticity content.

We would now like to see whether the Kinking theorem is rigorously true in d=0 theories. The d=3 version is extremely elegant but there is no doubt that the proof of the theorem is not rigorous (since we do not know how to treat derivatives satisfactorily). It is of course true that the theorem is most useful in d=2 and d=3 theories where the free propagator is singular (in the sense of being infinite on the light cone). The kinked Greens functions are generally finite whereas the conventional Greens functions are not.

We consider a modified version of (d') in which V is a function of $\hat{q}$; $\hat{q}$ fields are independent of q and p. The case (d') itself is interesting but actually the analogue of it in 'd=3' field theories has not been explicitly examined in the context of kinking.

$$K^{(2)} = \frac{1}{2!}\,i^2\int\!\!\int dt_1 dt_2 \,\langle 0 \mid T(q_a :\tfrac{1}{2}p_1^2\, V(\hat{q}_1^2)::\tfrac{1}{2}p_2^2\, V(\hat{q}_2^2):q_b\,) \mid 0\rangle$$

where $q_a = q(t_a)$, $p_a = p(t_a)$ etc.

$$G^{(2)} = \frac{1}{2!} i^2 \iint dt_1\, dt_2 \langle 0 |\, T(q_a : \tfrac{1}{2}p_1^2 u(\hat{q}_1^2) :: \tfrac{1}{2}p_2^2 u(\hat{q}_2^2) :\, q_b) \,| 0 \rangle$$

where $u = V - 1$.

So

$$K^{(2)} - G^{(2)} = -\tfrac{1}{8} \iint dt_1\, dt_2 \;\langle 0 |\, T(q_a : p_1^2 :: p_2^2 :\, q_b) \,| 0 \rangle$$

The theorem of Kinking requires that

$$(K^{(2)} - G^{(2)})_{\text{connected}} = G^{(0)} + (\text{extra terms})$$

where $G^{(0)} = \langle 0 |\, T(q_a q_b) \,| 0 \rangle$. The extra terms in the above expression occur because we are treating the analogue of a massive theory, and will be given later. (Tadpoles are automatically zero since we have normal ordered interactions).

We wish to see if a rigorous treatment (which is possible in the ' d=0 ' case) demonstrates that the assumptions made in the Kinking theorem are consistent.

It is easy to see that

$$\tfrac{1}{2} \langle 0 |\, T(\, q_a :\, p_1^2 :: p_2^2 : q_b \,) \,| 0 \rangle$$

$$= \overbracket{q_a q_b}\; (\overbracket{p_1 p_2})^2 + 2 \overbracket{q_a p_1}\; \overbracket{p_2 q_b}\; \overbracket{p_1 p_2} + 2\, \overbracket{q_a p_2}\; \overbracket{p_1 q_b}\; \overbracket{p_1 p_2}$$

From these identities we have

$$2(K^{(2)} - G^{(2)})_{\text{connected}}$$

$$= -\tfrac{1}{2} \iint dt_1\, dt_2 \left\{ \tfrac{1}{8}\omega\, e^{-i\omega(|t_1|+|t_2|+|t_{12}|)} - \tfrac{1}{4}\omega\theta(t_1)\; e^{-i\omega(|t_1|+|t_2|+|t_{12}|} \right.$$

$$- \tfrac{1}{4}\omega\ \theta(-t_2)\ e^{-i\omega(|t_1| + |t_2| + |t_{12}|)}$$

$$+ \tfrac{1}{2}\omega\ \theta(t_1)\ \theta(-t_2)\ e^{-i\omega(|t_1| + |t_2| + |t_{12}|)}$$

$$+ \text{(terms with } t_1 \text{ and } t_2 \text{ interchanged)} \Big\}$$

(where for simplicity we have taken $t_a = t_b = 0$)

$$= -\iint dt_1 dt_2 \left(-\tfrac{1}{8}\omega\ e^{-i\omega(|t_1|+|t_2|+|t_{12}|)} + \tfrac{1}{2}\omega\theta(t_1)\theta(-t_2)e^{-i\omega(|t_1|+|t_2|+|t_{12}|)}\right.$$

Since we are interested in $(K^{(2)} - G^{(2)})_{\text{connected}}$ we have dropped the term

$$\int\int e^{-i\omega|t_{ab}|}\ e^{-2i\omega|t_{12}|}\ dt_1 dt_2$$

We are faced with the evaluation of

$$\iint e^{-i\omega(|t_1| + |t_{12}| + |t_2|)}\ dt_1\ dt_2 \qquad (3.6)$$

and

$$\iint \theta(t_1)\ \theta(-t_2)\ e^{-i\omega(|t_1| + |t_{12}| + |t_2|)}\ dt_1 dt_2 \qquad (3.7)$$

We make the transformation

$$t_1 = r \sin\theta \qquad 0 \le r < \infty$$

$$t_2 = -r\cos\theta \qquad 0 \le \theta < 2\pi$$

The Jacobian for the transformation is r . We have the following regions

$$(t_1 > 0,\ t_2 < 0,\ t_{12} > 0) \equiv (r > 0,\ 0 < \theta < \pi\tfrac{1}{2})$$

$$(t_1 > 0,\ t_2 > 0,\ t_{12} > 0) \equiv (r > 0,\ \tfrac{1}{2}\pi < \theta < \tfrac{3}{4}\pi)$$

$$(t_1 > 0,\ t_2 > 0,\ t_{12} < 0) \equiv (r > 0,\ \tfrac{3}{4}\pi < \theta < \pi)$$

$$(t_{12} < 0,\ t_2 > 0,\ t_1 < 0) \equiv (r > 0,\ \pi < \theta < 3/2\ \pi)$$

$$(t_{12} < 0,\ t_1 < 0,\ t_2 < 0) \equiv (r > 0,\ \tfrac{3}{2}\pi < \theta < \tfrac{7}{4}\pi)$$

$$(t_{12} > 0\ ,\ t_1 < 0,\ t_2 < 0) \equiv (r > 0,\ 7/4\ \pi < \theta < 2\pi)$$

So

$$\iint e^{-i\omega(|t_{12}|+|t_1|+|t_2|)}\,dt_1 dt_2$$
$$= -\int_0^\infty dr\, r\left(\int_0^{\frac{1}{2}\pi} e^{-2i\omega r(\sin\theta+\cos\theta)}\,d\theta + \int_{\frac{1}{2}\pi}^{\frac{3}{4}\pi} e^{-2i\omega r\sin\theta}\,d\theta\right.$$
$$+\int_{\frac{3}{4}\pi}^{\pi} e^{2i\omega r\cos\theta}\,d\theta + \int_{\pi}^{\frac{3}{2}\pi} e^{-2i\omega r(-\cos\theta-\sin\theta)}\,d\theta$$
$$\left.+\int_{\frac{3}{2}\pi}^{7/4\pi} e^{2i\omega r\sin\theta}\,d\theta + \int_{\frac{7}{4}\pi}^{2\pi} e^{-2i\omega r\cos\theta}\,d\theta\right)$$

A function in terms of which we will be able to express (3.6) and (3.7) is $F(\lambda, u)$

$$F(\lambda,u) \equiv \int_0^u e^{-i\lambda x}/(1-x^2)^{+\frac{1}{2}}\,dx$$
$$= -i \sum_{r=0}^{\infty} \gamma(2r+1, i\lambda u)\,\Gamma(\tfrac{1}{2})/(\Gamma(r+1)\,\Gamma(\tfrac{1}{2}-r))$$

where we have used

$$\int_0^u x^{\nu-1} e^{-\mu x}\,dx = \mu^{-\nu}\gamma(\nu,\mu u) \quad \text{for } \mathrm{Re}\,\nu > 0$$

and $$\gamma(\alpha, x) \equiv \int_0^x e^{-t} t^{\alpha-1}\,dt \quad \text{for } \mathrm{Re}\,\alpha > 0$$

In fact $F(\lambda,u)$ is finite. This is seen from the following crude inequalities.

When $\mathrm{Im}\,\alpha = 0$, $\alpha > 0$, and $x > 0$ (or x imaginary)

$$|\gamma(\alpha,x)| < |x^\alpha/\alpha|$$

Moreover $$\left|\frac{\Gamma(\frac{1}{2})}{\Gamma(r+1)\Gamma(\frac{1}{2}-r)}\right| = (-1)^r \frac{\Gamma(\frac{1}{2})}{\Gamma(r+1)\,\Gamma(\frac{1}{2}-r)} \quad \text{when } r \geqslant 0$$

So $$|F(\lambda,u)| < |\lambda u| \sum_{r=0}^{\infty} \frac{|\lambda u|^{2r}}{(2r+1)} (-1)^r \Gamma(\tfrac{1}{2})/(\Gamma(r+1)\,\Gamma(\tfrac{1}{2}-r))$$

i.e. $\quad |F(\lambda,u)| < |\lambda u|\ (\ 1 - |\lambda u|^2\ )^{-\frac{1}{2}}$

Any singularities that $F(\lambda,u)$ has are integrable. Putting $\lambda = 2\omega r$ we have

$$\iint e^{-i\omega(|t_{12}|+|t_1|+|t_2|)}\ dt_1\ dt_2$$
$$= \frac{1}{4\omega^2}\int_0^\infty d\lambda\ \lambda\Big[2\int_0^{\frac{1}{2}\pi} e^{-i\lambda(\sin\theta+\cos\theta)}\ d\theta + 2\int_0^{\frac{1}{4}\pi} e^{-i\lambda\cos\theta}\ d\theta$$
$$+ 2\int_0^{\frac{1}{4}\pi} e^{-i(2)^{-\frac{1}{2}}\lambda(\sin\theta+\cos\theta)}\ d\theta\Big]$$

We make the substitutions

$$x = (\sin\theta + \cos\theta)\ (\ 2\ )^{-\frac{1}{2}}$$
$$y = \cos\theta$$

Then (3.6) becomes

$$\frac{1}{2\omega^2}\int_0^\infty d\lambda\ \lambda\Big(2\ (\ F(\ 2^{\frac{1}{2}}\lambda,\ 1\ ) - F\ (\ 2^{\frac{1}{2}}\lambda,\ 2^{-\frac{1}{2}})\Big)$$
$$+ 2\ (\ F(\lambda,\ 1\ ) - F(\lambda,\ 2^{-\frac{1}{2}})\ )\Big)$$

If the Kinking theorem is to be true then the analytic continuations in ω on both sides of the equation should be true. Allowing ω to become $\omega - i\epsilon$, $(\epsilon > 0)$ i.e. $\lambda \longrightarrow \lambda - i\epsilon\lambda/\omega$, we have

$$\int_0^\infty d\lambda\ \lambda\int_u^{u'} e^{-ix(\lambda - i\epsilon\lambda/\omega)}\ (1-x^2)^{-\frac{1}{2}}\ dx$$
$$(\text{ with } u' > u > 0\ )$$
$$= -\int_u^{u'} dx\ (\ \epsilon x/\omega + ix)^{-2}\ (1-x^2)^{-\frac{1}{2}}$$
$$\longrightarrow -\int_u^{u'} x^{-2}\ (1-x^2)^{-\frac{1}{2}}\ dx \quad \text{as } \epsilon \longrightarrow 0$$

which is finite.

Putting $x = \sin\beta$

$$\int x^{-2} (1 - x^2)^{-\frac{1}{2}} \, dx = \int d\beta \, \sin^{-2}\beta = - \cot\beta$$

Hence (3.6) is

$$\omega^{-2} \left(\frac{3}{2} \left[\cot\beta \right]_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \right) = - \frac{3}{2\omega^2}$$

Similarly (3.7) is

$$\frac{1}{4\omega^2} \int_0^{\infty} d\lambda \, \lambda \int_{2^{-\frac{1}{2}}}^{1} dx \quad (1 - x^2)^{-\frac{1}{2}} \, e^{-i\lambda 2^{\frac{1}{2}} x} = - \frac{1}{4\omega^2}$$

Therefore

$$(K^{(2)} - G^{(2)})_{\text{connected}}$$

$$= -\tfrac{1}{2}\omega (-\tfrac{1}{4}\omega^{-2}) + \tfrac{1}{8}\omega \left(- \frac{3}{2\omega^2} \right) = - \frac{1}{16\omega}$$

Now the Kinking theorem would require $(K^{(2)} - G^{(2)})_{\text{connected}}$ to be given by

$$i^2 \iint dt_1 \, dt_2 \quad \frac{d^2}{dt_1^2} \, D^{(0)}(-t_1) \, D^{(0)}(t_1 - t_2) \, \frac{d^2}{dt_2^2} \, D^{(0)}(t_2)$$

$$= i^2 \iint dt_1 dt_2 \, (-\omega^2 \, D^{(0)}(-t_1) - i\delta(t_1)) \, D^{(0)}(t_1 - t_2)$$

$$\cdot (-\omega^2 \, D^{(0)}(t_2) - i\,\delta(t_2))$$

(since $\left(\frac{d^2}{dt^2} + \omega^2 \right) D^{(0)}(t) = -i\,\delta(t)$)

$$= - \iint dt_1 dt_2 \, \frac{\omega^4}{(2\omega)^3} \, e^{-i\omega (|t_1| + |t_2| + |t_{12}|)} + \frac{1}{2\omega}$$

$$+ \int dt_1 \frac{i(i\omega)^2}{(2\omega)^2} \, e^{-i\omega(2|t_1|)} - \int dt_2 \, \frac{i\omega^2}{(2\omega)^2} \, e^{-i\omega(2|t_2|)}$$

$$= \frac{3}{16\omega} + \frac{1}{2\omega} - \frac{1}{4\omega} - \frac{1}{4\omega} = \frac{3}{16\omega}$$

This seems to show that the Kinking theorem is not valid in this case; there are two possible reactions. One would state that the ' d =0' analogues are massive theories and it is well-known that massive theories do not continue smoothly into massless theories. If we adopt this view point kinking may be valid in massless but not massive theories. The more pessimistic view point is that the assumptions of the Kinking theorem may not be mutually consistent (irrespective of whether the field theory is massive or massless). The d-number is important in these considerations. For $d = 0$ we can be rigorous and use T-products rather than T^X products and moreover second type tadpoles pose no problems. As soon as we consider $d = 3$ field theories we have meaningless quantities occurring (i.e. values of distributions at fixed arguments).Since many of the valuable well-established features in $d=3$ theories seem to have analogues in $d = 0$ theories we feel that the Kinking theorem should not be regarded at the moment of being to the same degree fundamental.

There is nothing in principle to prevent the extension of the programme with one dimensional theories to interactions invariant under internal symmetry groups. There are none of the problems of maintaining unitarity and gauge invariance since no regularisations are required; however the Sommerfeld-Watson summation technique will have to be employed. One of the weaknesses of this method

lies in the inherent ambiguity in defining interpolating functions. The nature of this ambiguity is beautifully summed up by a theorem due to Fuchs[53] which states the following :

If $f(z)$ is regular for Re $z \geq 0$ and of exponential type characterised by c, and if $f(\lambda_n) = 0$ with $\lambda_n > 0$, $\lambda_{n+1} - \lambda_n > \delta > 0$ then $f(z) \equiv 0$ if

$$\limsup_{r \to \infty} r^{-2c/\pi} \Phi(r) = \infty.$$

where

$$\Phi(r) = \exp 2 \sum_{\lambda_n < r} \lambda_n^{-1}$$

Moreover if the limit superior condition is not true then there is an $f(z)$ (satisfying all the hypotheses of the theorem) which is not identically zero.

Of course the ambiguity in the interpolating functions does not matter if all the residues are summed; however one of the techniques in use is the opening out of the contour of the momentum space [54] superpropagator and evaluating the superpropagator from one or two residues. In $d=3$ theories some success has been had by using the minimal singularity ansatz but for $d = 0$ theories this is not a relevant concept; the basic trouble is that ' opening out of the contour ' is not a smooth mathematical operation. Hence the ambiguities remain an embarrassment for lower dimensional theories.This seems to indicate that the technique may be at fault , i.e. the ambiguities are not purely of a distribution theoretic nature.

Chapter 4 Non-perturbative calculations

We have concentrated so far on perturbative calculations i.e. on expanding the S-matrix in a power series in the coupling constant. In gauge theories we are often interested in questions which make us suspect whether perturbative treatments are meaningful. In quantum gravity it is interesting to understand quantum fluctuations about the classical solutions for the metric field[55]. Intuitively we can see why non-perturbative treatments are required. The classical solution can be built up by summing all tree graphs. This has been shown explicitly for the Schwarzschild solution; the classical solution is already a non-perturbative one. The natural extension of this programme is to sum up all diagrams with one loop. We shall not try quantum gravity but rather the one-dimensional model

$$\mathcal{L}_{int} = \lambda q^4(t) \qquad (4.1)$$

This obviously does not have gauge symmetries and also no problems of renormalisation arise but it is a non-linear interaction and gives rise to a Hamiltonian with a positive definite spectrum for $-\lambda > 0$. One dimensional models of gravity are far from absurd and have been seriously suggested by Klauder[56]. A non-trivial example of such a model is given by

$$I = \int p(t)\, q(t) - p(t) q^2(t) - V(p(t)) \; dt$$

with I the action and $p(t) > 0$.

The first problem is to find a non-perturbative summation technique for summing one loop contributions. The answer to this has been indicated by Feynman. Given an action $S[\varphi]$ we will associate an action $S_{1\ loop}[\varphi]$ which will give all Greens functions in the one loop approximation.

$$S_{1\ loop}[\varphi] = S[\hat{\varphi}] + \tfrac{1}{2} S_{,\,ij}\ \phi^i\ \phi^j$$

and $\hat{\varphi}$ is a classical solution. We have used the notation $,\,ij\ldots k$ to denote the functional derivative $\frac{\delta}{\delta\phi_i}\cdots\frac{\delta}{\delta\phi_k}$. Moreover

$$\varphi = \hat{\varphi} + \phi$$

where φ and ϕ are both quantum fields. We proceed to a non-perturbative calculation via the Feynman path integral[57].

That a non-perturbative solution may well be the only meaningful solution is indicated by the following argument. Power series converge within entire circles and for λ negative the potential goes to $-\infty$ as q approaches ∞. The character of the bound states changes completely as $\lambda < 0$ or $\lambda > 0$. Hence we may suspect that amplitudes are not analytic at $\lambda = 0$. Actually λ-analyticity can have also quite an amusing direct physical interpretation. Salam has stressed that in gravity modified quantum electrodynamics the effect of gravity is to remove ultraviolet infinities and instead leave a factor $\log \lambda$ (where λ is the gravitational coupling constant). However no summation in perturbation theory was attempted. Now Weisskopf has interpreted the infinity suppression in gravity

modified quantum electrodynamics as connected with the limiting frequency of a standing photon wave in the curved space time around an electron. Such a photon has the wavelength given by the Schwarzschild radius and it is not unreasonable to regard this as the definition of the latter. From general relativity this limiting frequency is $1/3m_e$ where m_e is the electron mass. Using Weisskopf's formula for $\delta m_e / m_e$ we get something proportional to $\log(1/3m_e\lambda)$. In this way we have some sort of connection between the Schwarzschild solution and the analyticity of our S-matrix elements. We can perhaps get some idea about the effect of higher order quantum corrections on the Schwarzschild solution from this. If for example the λ behaviour becomes $(\log\lambda)\lambda^{-\frac{1}{2}}$ owing to quantum loop corrections the Schwarzschild radius behaves like $\lambda \exp(-\lambda^{-\frac{1}{2}})$.

We consider

$$\mathcal{L} = -\tfrac{1}{2}\mu^2 (q(\tau))^2 + \tfrac{1}{2}(\dot{q}(\tau))^2 + \tfrac{1}{4}\lambda (q(\tau))^4$$

The equation of motion is

$$\ddot{q}(\tau) + \mu^2 q(\tau) = \lambda (q(\tau))^3$$

A class of solutions is given by[26]

$$q_c(\tau) = c\, e(\Omega\tau)$$

with $$\Omega^2 = \mu^2 (1 - \lambda c^2)$$

$$k^2 = -2\lambda c^2 / \Omega^2$$

and $$e(y) = cn(y,k) + i\, sn(y,k)$$

The Jacobian elliptic functions are defined by

$$u = \int_0^{sn(u,k)} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}$$

and $$cn(u,k) = \sqrt{1 - sn^2(u,k)}$$

$$dn(u,k) = \sqrt{1 - k^2\, sn^2(u,k)}$$

As $\lambda \longrightarrow 0$, and $k \longrightarrow 0$ $e(y) \longrightarrow \exp(iy)$

We define $\eta(\tau)$ by[58]

$$q(\tau) = q_c(\tau) + \hbar^{\frac{1}{2}} \eta(\tau)$$

Hence

$$S_{1\ loop}[q(\tau)] = S[q_c(\tau)] + \hbar \int_{t_a}^{t_b} d\tau \left(\tfrac{1}{2}(\dot{\eta}(\tau))^2 - \tfrac{1}{2}\mu^2(\eta(\tau))^2 + \tfrac{3}{2}\lambda q_c^2(\tau)\, \eta^2(\tau)\right)$$

since

$$S = \int_{t_a}^{t_b} d\tau \left(-\tfrac{1}{2}\mu^2 (q(\tau))^2 + \tfrac{1}{2}(\dot{q}(\tau))^2 + \tfrac{1}{4}\lambda (q(\tau))^4\right)$$

$S[q_c(\tau)]$ is somewhat more complicated.

$$S[q_c] = \frac{-1}{\Omega} \int_{\Omega t_b}^{\Omega t_a} d(\Omega\tau)\left(\tfrac{1}{2} C^2\Omega^2(\dot{e}(\Omega\tau))^2 - \tfrac{1}{2}\mu^2 C^2 (e(\Omega\tau))^2 + \tfrac{\lambda}{4} C^4 (e(\Omega\tau))^4\right)$$

In fact

$$\Omega^{-1}\ S[q_c] = F(u)\Big|_{u = \Omega t_b} - F(u)\Big|_{u = \Omega t_a}$$

where

$$F(u) = E(u)\left(\frac{1}{6} c^2\Omega^2 (1-2/k^2) - \mu^2 c^2/k^2 + 2\lambda c^4(k^2-2)/3k^4 \right)$$

$$+u\left(\tfrac{1}{2}c^2\Omega^2 \frac{2k'^2}{3k^2} - \tfrac{1}{2}\mu^2c^2(1-2/k^2)+(14k'^2+k^2-3k^2k'^2+2)\lambda c^4/12k^4\right)$$

$$+ \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \left(\tfrac{1}{2}c^2\Omega^2(-2/3)+2\lambda c^4/3k^2\right)$$

$$+\operatorname{dn}^3 u \left(\tfrac{1}{2} c^2\Omega^2 2i/ (3k^2)^2\right) + \operatorname{dn}^2 u \left(- i\lambda c^4/k^4\right)$$

$$+ \operatorname{dn} u \left(i\mu^2c^2/k^2 + i\lambda c^4 (k^2+2k'^2)/k^4\right)$$

and $$E(u) = \int_0^u \operatorname{dn}^2 v \, dv \quad ; \quad k' = (1-k^2)^{\frac{1}{2}}$$

The details of how we calculated E(u) are given in appendix 4. Now the Feynman integral in these one dimensional models can be related formally to Wiener integrals.

$$\int \mathcal{D}_F(q) \; e^{i\frac{1}{2}\epsilon \langle \dot{q},\dot{q}\rangle} \; F\{q\} = \int \mathcal{D}_W(q) \; e^{-\frac{\epsilon}{2}\langle \dot{q},\dot{q}\rangle} \; F\{i^{\frac{1}{2}}q\}$$

where $\mathcal{D}_W(q)$ is Wiener measure[58] and $\mathcal{D}_F(q)$ is Feynman measure. ($F\{q\}$ is just some integrable function).We will first consider in a little detail the measure in Feynman integrals. There are various approaches and all can be related to the commutative Hilbert space integral approach.We shall choose the most physical of these approaches. An alternative view-point is via contraction semigroups and Trotter's formula[59].

We put

$$K(\underline{b},\underline{a}) = \int_\Omega \exp\tfrac{i}{\hbar} \; S[\underline{q} , \underline{b} , \underline{a}] \; \mathcal{M}(q) \; \mathcal{D}_F(q)$$

The Stone-Weierstrass theorem states that every function which

is continuous on Ω can be uniformly approximated by functions continuous in a finite number of variables.

$$K(b;a) = \lim_{\substack{\epsilon \to 0 \\ N\epsilon = t_b - t_a}} \int \cdots \int \exp \frac{i}{\hbar} S(\underline{q}_b, \ldots, \underline{q}_k, \ldots, \underline{q}_a) \prod_{k=1}^{N} \delta(\underline{q}_b - \underline{q}_N) m_{k,k-1} d\underline{q}_k$$

where the interval $[t_a, t_b]$ has been divided into N equal intervals, and $q(t_a + k\epsilon)$ and $t_a + k\epsilon$ by q_k and t_k respectively. Now we have

$$\text{i)} \quad K(\underline{b};\underline{a}) = \int K(\underline{b};\underline{c})\, K(\underline{c};\underline{a})\, d\underline{q}_c$$

$$\text{ii)} \int_{-\infty}^{\infty} K(\underline{q}_1\, t_1 \,;\, \underline{q}_2\, t_2)\, K(\underline{q}_2\, t_2 \,;\, \underline{q}_1'\, t_1)\, d\underline{q}_2 = \delta(\underline{q}_1' - \underline{q}_1)$$

Both these relations are intuitively obvious probabilistic requirements. Sometimes (i) is known as the semigroup property and (ii) is called the unitarity property[60].

For infinitesimal $(\underline{q}_1 - \underline{q}_2)$ and $(t_1 - t_2)$

$$K(2;1) = m_{2,1} \exp \frac{i}{\hbar} \bar{S}(\underline{q}_2, \underline{q}_1)$$

where $\bar{S}$ is the classical action. This is Feynman's definition.

Now $$\bar{S}(q_2,q_1) = \bar{S}(q_2,q_2) + (q_1 - q_2)^{\mu} \partial \bar{S} / \partial q_1^{\mu} \Big|_{\underline{q}_1 = \underline{q}_2} + 0(\underline{q}_1 - \underline{q}_2)^2$$

(ii) implies that

$$\int_{-\infty}^{\infty} \exp\left[\frac{i}{\hbar}(\underline{q}_1' - \underline{q}_1) \cdot \underline{p}_{2,1}\right] |m_{2,1}|^2 \, d\underline{q}_2 = \delta(\underline{q}_1' - \underline{q}_1)$$

where $$-p_{\mu}\, 2,1(\underline{q}_2,\underline{q}_1) = \partial \bar{S}(\underline{q}_2,\underline{q}_1) / \partial q_1^{\mu}$$

Making the change of variable

$$\underline{q}_2 \longrightarrow \underline{p}_{2,1} / 2n\hbar$$

we have

$$d\,\underline{q}_2 = \left[\frac{\partial(\,\underline{p}_{2,1}/2\pi\hbar\,)}{\partial\,\underline{q}_2}\right]^{-1} \frac{d\,\underline{p}_{2,1}}{2\pi\,\hbar}$$

and this gives

$$|m_{2,1}| = \left[\frac{\partial(\,\underline{p}_{2,1}/2\pi\,\hbar)}{\partial\,\underline{q}_2}\right]^{\frac{1}{2}}$$

The phase of $m_{2,1}$ can be found from the semi-group property.

For the free field

$$\bar{S}\ (b\ ,\ a\ ) = \tfrac{1}{2}\ \frac{\mu(q_b - q_a)^2}{t_b - t_a}$$

and $m_{k,k-1}$ is independent of k.

$$\int K(b;c)\ K(c;a)\ dq_c = (m_{\Delta t})^2 \int dq_c\ \exp\frac{i}{\hbar}\tfrac{1}{2}\mu\,\frac{1}{\Delta t}\left((q_b-q_c)^2+(q_c-q_a)^2\right)$$

$$= (\,m_{\Delta t})^2\ (i\pi\hbar\Delta t/\mu)^{\frac{1}{2}}\exp\frac{i}{\hbar}\tfrac{1}{2}\mu\frac{(q_b-q_a)^2}{2\,\Delta t}$$

where $t_b - t_c = t_c - t_a = \Delta t$.

So

$m_{\Delta t} = (\,\mu/i\pi\,h\,\Delta t)^{\frac{1}{2}}$ and this gives the phase of the measure. Hence

$$m_{k,k-1} = (1/2\pi\hbar i)^{\frac{1}{2}}\ \left|\mathrm{Det}\ \frac{\partial^2\ \bar{S}}{\partial\,q_k^{\mu}\,\partial q_{k-1}^{\nu}}\right|$$

Now that we know what the Feynman measure is we can proceed with our calculations. For $S_{1\ \mathrm{loop}}$ path integrals m is a constant factor and so the measure is particularly simple.

We are interested in

$$K(b,a)$$
$$= \int \mathcal{D}_F(\eta)\ \exp\frac{i}{\hbar}\left(S[q_c] + \hbar \int_{t_a}^{t_b} \tfrac{1}{2}(\dot{\eta}(\tau))^2 + \eta^2(\tau)(\tfrac{3}{2}\lambda q_c^2(\tau) - \tfrac{1}{2}\mu^2)\ d\tau \right.$$

$$= \exp\frac{i}{\hbar} S[q_c] \int \mathcal{D}_W(\eta)\ \exp(-\tfrac{1}{2}\int_{t_a}^{t_b} (\dot{\eta}(\tau))^2 d\tau) \exp(\ \tfrac{1}{2}\ (-3\lambda q_c^2(\tau) + \mu^2)\ d\tau\ )$$

Now it is known that [58]

$$\int \mathcal{D}_W(\chi)\ e^{-\frac{1}{2}\epsilon\langle \dot{\chi}, \dot{\chi}\rangle}\ \exp\ (\ \tfrac{1}{2}\int_0^t d\tau\ Q\chi^2\ )$$
$$= (\pi\ f(0))^{-\frac{1}{2}}$$

where

$$f''(\tau) + Q(\tau)\ f(\tau) = 0$$

$$f(t) = 0 \quad \text{and} \quad f'(t) = -1$$

(when $\chi(0)$ and $\chi(t)$ are both fixed). For the case of interest

$$f''(\tau) + (\mu^2 - 3\lambda c^2 e^2(\Omega\tau))f(\tau) = 0 \qquad (4.2)$$

In order to solve this equation[61] we have to consider certain mathematical facts about elliptic functions. Sn and cn[62] are both single valued functions with real and imaginary periods . Sn u has the following properties:

i) sn u is an odd elliptic function of order 2 .
(The order of a doubly-periodic function is the sum of the multiplicities of all the poles which the function has within a fundamental parallelogram).

ii) sn u has a simple pole of residue 1/k at every point congruent to iK' (mod 4K, 2iK')

iii) sn u has a simple pole of residue -1/k at points congruent to $2K+iK'$ (mod$4K, 2iK'$)

Similarly cn u is an even elliptic function of order two, has simple poles at every point congruent to iK'(mod$4K, 2K+2iK'$) with residue $-i/k$ together with simple poles at every point congruent to $2K+iK'$ (mod$4K, 2K+2iK'$) with residue i/k. The quantities K and K' are given by

$$K = \int_0^{\frac{1}{2}\pi} (1 - k^2\sin^2\theta)^{-\frac{1}{2}} \, d\theta$$

$$K' = \int_0^{\frac{1}{2}\pi} (1-k'^2\sin^2\theta)^{-\frac{1}{2}} \, d\theta$$

Our real interest is e(u). From the above we can deduce that e(u) has a period parallelogram determined by $(\omega, \omega + 2\omega')$ (where $\omega = 4K$ and $\omega' = 2K'$). Within the parallelogram e(u) has poles given by $(\omega + i\frac{3}{2}\omega')$ and $(\frac{3}{2}\omega + i\frac{3}{2}\omega')$ with residues $2i/k$ and $-2i/k$ respectively; so

$$e(u) = \frac{2i/k}{u-\omega-i\frac{3}{2}\omega'} - \frac{2i/k}{u-\frac{3}{2}\omega-i\frac{3}{2}\omega'} + A(u)$$

(where A(u) is analytic and free from singularities within the parallelogram). Consequently within the fundamental parallelogram the double pole terms of $e^2(u)$ are given by

$$-\frac{4}{k^2}(u-(\omega+i\tfrac{3}{2}\omega'))^{-2} - \frac{4}{k^2}(u-(\tfrac{3}{2}\omega+i\tfrac{3}{2}\omega'))^{-2}$$

and the single pole terms are given by

$$\frac{2i}{k} A(u) \left(u - \omega - i \frac{3}{2}\omega' \right)^{-1} - \frac{2i}{k} A(u) \left(u - \frac{3}{2}\omega - i \frac{3}{2}\omega' \right)^{-1}$$

$$+ \frac{8}{k^2} \left(u - \omega - i \frac{3}{2}\omega' \right)^{-1} \left(u - \omega - i \frac{3}{2}\omega' \right)^{-1}$$

Before we can progress further it is necessary to introduce further concepts in elliptic functions. The structure of elliptic functions (and in fact all doubly periodic functions)is very restricted. We depend heavily on the following type of theorem:

" Two functions, which are doubly periodic in the same period and which have the same zeros and the same infinities each in the same degrees,respectively, are in a constant ratio."

It is convenient to introduce Weierstrass's σ -function

$$\sigma(z) = z \prod_{-\infty}^{\infty}{}' \prod_{-\infty}^{\infty} \left(1 - z/k \right) \exp\left(\frac{z}{k} + \frac{1}{2} \frac{z^2}{k^2} \right)$$

where $\kappa = 2m\alpha + 2m'\alpha'$, the ratio $\alpha' : \alpha$ not being purely real; the infinite product is extended over all terms that are given by assigning to m and m' all positive and negative integral values from $+\infty$ to $-\infty$ excepting only simultaneous zero values. The double-infinite product converges uniformly and unconditionally for all values of z in the finite part of the plane; hence $\sigma(z)$ is an odd function but not a doubly periodic one. Indeed

$$\sigma(z + 2m\alpha) = (-1)^m \, e^{2\eta(mz + m^2\alpha)} \, \sigma(z)$$

$$\sigma(z + 2m'\alpha') = (-1)^{m'} \, e^{2\eta'(m'z' + m'^2\alpha')} \, \sigma(z)$$

where $\eta' = \dot{\sigma}(\alpha')/\sigma(\alpha')$ and $\eta = \dot{\sigma}(\alpha)/\sigma(\alpha)$

Now if $g(t)$ is a function with its set of infinities given by $\{ c_i \mid 1 \leq i \leq n \}$ in a parallelogram of periods containing t and with a set of zeros given by $\{ \gamma_j \mid 1 \leq j \leq n \}$ (repeated zeros and infinities appearing the relevant number of times in the above sets) we associate a function

$$\hat{g}(t) = \frac{\sigma(t-\gamma_1)\sigma(t-\gamma_2)\ldots\sigma(t-\gamma_n)}{\sigma(t-c_1)\sigma(t-c_2)\ldots\sigma(t-c_n)} \; e^{\rho t}$$

where ρ is a constant.

$$\hat{g}(t+2\alpha) = \hat{g}(t)\; e^{2\eta(\sum c - \sum \gamma) + 2\rho\alpha}$$

$$\hat{g}(t+2\alpha') = \hat{g}(t)\; e^{2\eta'(\sum c - \sum \gamma) + 2\rho\alpha'}$$

By assumption $g(t+2\alpha) = \nu g(t)$

$$g(t+2\alpha') = \nu' g(t)$$

If we choose

$$a = \sum c - \sum \gamma$$

$$\nu = \exp(2\eta a + 2\rho\alpha)$$

$$\nu' = \exp(2\eta' a + 2\rho\alpha')$$

then $\hat{g}(t)/g(t)$ is a constant by the theorem stated earlier. These considerations will form the basis of our solution for the Feynman path integral. Now $f(\tau)$ is a doubly periodic function (at least of the second kind). This is clearly seen from (4.2). We consider

$$f(\tau) = \frac{B}{(\tau - d)^m}\left(1 + p(\tau - d) + q(\tau - d)^2 + \ldots\right)$$

This gives

$$\frac{1}{f}\frac{d^2 f}{d\tau^2} = \frac{m + m^2}{(\tau - d)^2} - \frac{2mp}{(\tau - d)} + \text{positive powers of } (\tau - d)$$

Hence we require

$$\frac{m + m^2}{(\tau - d)^2} - \frac{2mp}{(\tau - d)} + \ldots = 3\lambda c^2 e^2(\Omega\tau) - \mu^2$$

However we already know the singularity structure of the right hand side of this equation. Hence one possibility is

$$d = \frac{\omega + i\,\frac{3}{2}\omega'}{\Omega} \equiv d_1$$

$$m^2 + m = -12\,\lambda c^2 / k^2\Omega^2$$

$$-2mp = \frac{6i}{k\Omega} A(\omega + i\,\tfrac{3}{2}\omega')\,\lambda c^2 - \frac{12\lambda c^2}{k^2\Omega^2}(\omega/\Omega)^{-1}$$

It is necessary that m is a positive integer if we are to have doubly periodic solutions. Now since $m^2 + m + 12\lambda c^2/k^2\Omega^2$ vanishes we have m equal to $\frac{1}{2}(-1 \pm (1 - 48\lambda c^2/k^2\Omega^2)^{\frac{1}{2}})$; thus we need $(1 - 48\,\lambda c^2/k^2\Omega^2)^{\frac{1}{2}} - 1$ to be a positive integer. This can be guaranteed since k is a variable parameter and can be taken imaginary. Another possibility for d etc.is given by

$$d = \frac{1}{\Omega}\left(\tfrac{3}{2}\omega + i\,\tfrac{3}{2}\omega'\right) = d_2$$

$$m + m^2 = -12\lambda c^2 / k^2\Omega^2$$

$$-2mp = -\frac{6i\lambda c^2}{k\Omega} A(\tfrac{3}{2}\omega + i\,\tfrac{3}{2}\omega') + \frac{12\lambda c^2}{k^2\Omega^2}(\omega/\Omega)^{-1}$$

There is a very useful theorem which states the following:

Every doubly-periodic function has as many irreducible zeros as it has irreducible infinities.

This theorem shows us that we must consider

$$f(\tau) = \frac{\sigma(\tau-\gamma_1)\,\sigma(\tau-\gamma_2)\ldots\sigma(\tau-\gamma_{2m})}{(\sigma(\tau-d_1))^m\,(\sigma(\tau-d_2))^m}\exp(\varrho\tau)$$

We know that

$$\log\sigma(z) = \log z - \sum_{n=2}^{\infty}(2n)^{-1}z^{2n}\sum\varkappa^{-2n}$$

in a neighbourhood of $z=0$ where $\sum$ denotes summation for all $\varkappa$ $(\neq 0)$. We define auxiliary functions

$$\zeta(z) = \sigma'(z)/\sigma(z) \quad \text{and} \quad \wp(z) = -\,d\zeta(z)/dz$$

In fact $$\wp(z+2\alpha) = \wp(z+2\alpha') = \wp(z)$$

Hence in a neighbourhood of $z=0$

$$\zeta(z) = z^{-1} - g_2z^3/60 - g_3z^5/140 - \ldots - z^{2n-1}\sum\varkappa^{-2n} - \ldots$$

$$\wp(z) = z^{-2} + g_2z^2/20 + g_3z^4/28 - \ldots + (2n-1)z^{2n-2}\sum\varkappa^{-2n} + \ldots$$

Now $$\frac{1}{f}\frac{df}{d\tau} = \varrho + \sum_{i=1}^{2m}\zeta(\tau-\gamma_i) - m\zeta(\tau-d_1) - m\zeta(\tau-d_2)$$

$$\frac{1}{f}\frac{d^2f}{d\tau^2} = \left(\frac{1}{f}\frac{df}{d\tau}\right)^2 - \sum_{i=1}^{2m}\wp(\tau-\gamma_i) - m\wp(\tau-d_1) - m\wp(\tau-d_2)$$

The next steps should now be clear. We obtain an expansion of $\frac{1}{f}\frac{d^2f}{d\tau^2}$ in the vicinity of $\tau = 0$ and then compare the coefficients of this expansion with the coefficients of the known expansion $(3\lambda c^2\,e^2(\Omega\tau)-\mu^2)$

about the origin. For the ease of computations we will use the identities which follow

$$\wp(u+v) = -\wp(u) - \wp(v) + \frac{1}{4}\left(\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right)^2$$

$$\zeta(u+v) = \zeta(u) + \zeta(v) + \frac{1}{2}\left(\frac{\wp'(v) - \wp'(u)}{\wp(v) - \wp(u)}\right)$$

We shall consider the case $m=1$. This will illustrate the main features adequately.

$$f(\tau) = \frac{\sigma(\tau-\gamma_1)\,\sigma(\tau-\gamma_2)}{\sigma(\tau-d_1)\,\sigma(\tau-d_2)} \exp(\varrho\tau)$$

It is now easy to see that

$$\frac{1}{f}\frac{df}{d\tau} = V + X\tau + Y\tau^2 + Z\tau^3 + \dots$$

where

$$X = \wp(-d_1) + \wp(-d_2) - \wp(-\gamma_1) - \wp(-\gamma_2)$$

$$Y = \tfrac{1}{2}\left(\wp(-\gamma_1) + \wp(-\gamma_2) - \wp(-d_1) - \wp(-d_2)\right)$$

$$Z = \wp^2(-d_1) + \wp^2(-d_2) - \wp^2(-\gamma_1) - \wp^2(-\gamma_2)$$

$$V = \varrho + \zeta(-\gamma_1) + \zeta(-\gamma_2) - \zeta(-d_1) - \zeta(-d_2)$$

Moreover in a neighbourhood of $\tau = 0$

$$3\lambda c^2 e^2(\Omega\tau) - \mu^2 = (3\lambda c^2 - \mu^2) + 6i\lambda c^2\Omega\tau - 6\lambda c^2\Omega^2\tau^2 + \dots$$

and

$$\frac{1}{f}\frac{d^2f}{d\tau^2} = (V^2 + 2XV) + 2t(2VY + X^2 + XV) + 2t^2(3VZ + 3XY + VY + \tfrac{1}{2}X^2) + \dots$$

Hence the three conditions which determine ϱ, γ_1, and γ_2 are

$$\begin{aligned} 3\lambda c^2 - \mu^2 &= V^2 + 2XV \\ 3i\lambda c^2\Omega &= 2VY + X^2 + XV \\ -3\lambda c^2\Omega^2 &= 3VZ + 3XY + VY + \tfrac{1}{2}X^2 \end{aligned} \qquad (4.3)$$

From general theory we know that equation (4.3) gives two different sets of values for $\{\varrho , \gamma_1 , \gamma_2\}$. The two sets are denoted by $\{\varrho^{(j)} , \gamma_1^{(j)} , \gamma_2^{(j)} ; j = 1,2\}$. The solution of (4.2) is given by

$$f(\tau) = D_1 \sigma(\tau-\gamma_1^{(1)})\sigma(\tau-\gamma_2^{(1)})(\sigma(\tau-d_1))^{-1}(\sigma(\tau-d_2))^{-1}\exp(\varrho^{(1)}\tau)$$
$$+ D_2\sigma(\tau-\gamma_1^{(2)})\sigma(\tau-\gamma_2^{(2)})(\sigma(\tau-d_1))^{-1}(\sigma(\tau-d_2))^{-1}\exp(\varrho^{(2)}\tau)$$

where

$$e^{\varrho^{(1)}\tau} D_1 = \frac{\sigma(\tau-d_1)\sigma(\tau-d_2)\,(\varrho^{(2)}-\varrho^{(1)}+\zeta_1(\tau-\gamma_1^{(2)})+\zeta_1(\tau-\gamma_2^{(2)})-\zeta_1(\tau-\gamma_1^{(1)})-\zeta_1(\tau-\gamma_2^{(1)}))^{-1}}{\sigma(\tau-\gamma_1^{(1)})\sigma(\tau-\gamma_2^{(1)})}$$

$$e^{\varrho^{(2)}\tau} D_2 = -\frac{\sigma(\tau-d_1)\sigma(\tau-d_2)(\varrho^{(2)}-\varrho^{(1)}+\zeta_1(\tau-\gamma_1^{(2)})-\zeta_1(\tau-\gamma_1^{(1)})+\zeta_1(\tau-\gamma_2^{(2)})-\zeta_1(\tau-\gamma_2^{(1)}))^{-1}}{\sigma(\tau-\gamma_1^{(2)})\sigma(\tau-\gamma_2^{(2)})}$$

Hence we have

$$K(b,a) = \pi^{-\frac{1}{2}}\exp\tfrac{i}{\hbar}S[q_c]\,\frac{(\sigma(d_1)\sigma(d_2))^{\frac{1}{2}}}{(D_1\sigma(\gamma_1^{(1)})\sigma(\gamma_2^{(1)})+D_2\sigma(\gamma_1^{(2)})\sigma(\gamma_2^{(2)}))^{\frac{1}{2}}}$$

This is an extremely complex expression. We shall try to understand some of its implications. First we shall elaborate a little more on the classical solution used.

$$q_c(\tau) = (\lambda(1-2/k^2\mu^2))^{-\frac{1}{2}}\, e((2/(2-k^2\mu^2))^{\frac{1}{2}}\mu\tau)$$

As $\lambda \longrightarrow 0$ we have $q_c(\tau)$ approaching $(-2\beta/\mu^2)^{-\frac{1}{2}}\exp(i\mu\tau)$ if (λ/k^2) tends to a non-zero value β ; this is just the free field solution.

We are mainly interested in our analysis in the λ -analyticity properties of K(b;a). The factor dependent on $S[q_c]$ is non-analytic

in λ . This can be seen from F(u) (remembering that c^2 is $\lambda^{-1}k^2/(k^2-2/\mu^2)$ and Ω^2 is $-2/(k^2-2/\mu^2)$). The other factor contributing to K should represent non-classical effects. We may well expect similar non-analyticity in λ to be present. The Weierstrassian σ -function has no singularities in the finite part of the complex plane and this applies also of course to the exponential function; so singularities occur when the arguments have singularities (i.e. when γ_1, γ_2 and ρ have singularities). The factor $(\sigma(d_1)\, \sigma(d_2))^{\frac{1}{2}}$ is unimportant in the sense that it is solely a function of k which is a parameter which labels the set of classical solutions that we have used. Now γ_1, γ_2 and ρ are given by equation (4.3). It is the case however that λc^2 is independent of λ and so the singularities of γ_1, γ_2 and ρ will be in μ and k . It is quite remarkable that the non-classical terms do not contribute to the non-analyticity in λ here (at least in the approximation that we have used). Of course if we choose k to be dependent on λ this will no longer hold.

We can make our solution more explicit by actually calculating γ_1, γ_2 and ρ . This is hard to do exactly but we can take γ_1 and γ_2 small and replace the second equation in (4.3) for example by

$$3i\lambda c^2\Omega \approx (1/\gamma_1^2 + 1/\gamma_2^2)^2$$

This is approximate but still meaningful since we have the values of K and K' at our disposal (in the sense that they are just functions of k which we can vary); basically we can choose the d_1 and d_2 such that γ_1 and γ_2 are indeed small. The other equations in (4.3) seem also to suggest a cut structure in k and μ. Our framework is particularly appropriate to quantum gravity; in fact DeWitt has shown that functional differentiation with respect to the background field of the functional that we have just constructed gives all amplitudes.

Naturally we would like to extend our discussion to the case $\mathcal{L}_{int} = \lambda\varphi^4(x)$ in four dimensions. The equation of motion is

$$(\Box + m^2) \varphi (x, \beta) = \lambda \varphi^3 (x, \beta)$$

Putting $\sigma = p^\mu x_\mu$ we have

$$(d^2/d\sigma^2 + m^2/p^2) \varphi = \lambda \varphi^3 / p^2$$

Hence a solution is $\varphi(x,\beta) = \varphi_0(\beta)\ e(\Omega\sigma)$ where φ_0 and Ω are suitably chosen. We are interested in the quantum fluctuation around this solution. We have so far been considering

$$q: \quad R \longrightarrow M$$

where M is a space of operators. However now we have the case

$$\varphi: \quad V^4 \longrightarrow M$$

where V^4 is Minkowski space. The Feynman integral is written as

$$K(\underline{b}, \underline{a}) = \int \exp \frac{i}{\hbar} S[\varphi, \underline{b}, \underline{a}] \, \mathcal{D}\varphi$$

where $$\mathcal{D}\varphi = \mathcal{M}[\varphi] \prod_{\mu, x} d\varphi_{\mu}(x)$$

As before we cannot proceed without determining $\mathcal{M}$ [57], and it is the form of $\mathcal{M}$ which makes an important difference in the four dimensional case. We shall choose the vacuum for both the initial and final configuration.

$$\langle 0, \infty \mid 0, -\infty \rangle = \int \exp \frac{i}{\hbar} S[\hat{\varphi} + \phi] \, \mathcal{D}\phi$$

Now the equation of motion is

$$S_{,i}[\hat{\varphi} + \phi] = 0$$

But we also have

$$\int \frac{\delta}{\delta \phi^i} \exp \left(\frac{i}{\hbar} S[\hat{\varphi} + \phi] + \log \mathcal{M} \right) d\phi = 0$$

Hence $$0 = \langle 0, \infty \mid T \left(\frac{i}{\hbar} S_{,i}[\hat{\varphi} + \phi] + (\log \mathcal{M})_{,i} \right) \mid 0, -\infty \rangle$$

From this we make the assumption

$$T \left(\frac{i}{h} S_{,i}[\hat{\varphi} + \phi] + (\log \mathcal{M})_{,i} \right) = 0$$

So following DeWitt we require

$$\begin{aligned} T((\log \mathcal{M})_{,i}) &= \frac{i}{\hbar} S_{,i}[\hat{\varphi} + \phi] - \frac{i}{\hbar} T S_{,i}[\hat{\varphi} + \phi] \\ &= \frac{i}{2\hbar} S_{,ijk} (\phi_j \phi_k - T \phi_j \phi_k) + \dots \\ &= -\frac{1}{2\hbar} S_{,ijk} G^{+jk} + \dots = -\tfrac{1}{2} (\log \det G^{+})_{,i} + \dots \end{aligned}$$

where G^{+jk} is the advanced propagator. Here we have used the

relations

$$\delta \log \det G = - \mathrm{tr} \left(G \delta G^{-1} \right)$$

$$\left(\log \det G \right)_{,i} = - G^{jk} \left(G^{-1}_{kj} \right)_{,i}$$

We have so far managed the analogues of the steps made in the one dimensional theory. DeWitt and others have even formally obtained

$$\langle 0, \infty \mid 0, -\infty \rangle_{1 \text{ loop}} = \left(\det (1 - g_0 U) / \det (1 - g_0^{+} U) \right)^{\frac{1}{2}} \qquad (4.6)$$

$$\left(\text{where } U_{,ij} = S_{,ij} - S_{,ij} [\hat{\varphi} = 0] \right)$$

by taking $\mathcal{M}[\hat{\varphi}] = (\det g^{+})^{-\frac{1}{2}}$; here g^{+} denotes the advanced Greens function evaluated at $\phi = 0$. Unfortunately unlike in the one dimensional case we have no non-perturbative method of evaluating the determinant in (4.6). Fredholm methods developed in this field theory context by Schwinger just recover the perturbation expansions when applied to (4.6).It is very rarely in interacting field theories (irrespective of the space dimension) that a non-perturbative method can be made to work. Since perturbation theory seems unlikely to be valid it is crucial that new summation techniques are vigorously sought.

Appendix 1 Functional integrals

This section is based in part on private communication and notes from J. Tarski.

Functional integrals provide a convenient formalism for describing Greens functions and hence S matrix elements. Current methods of quantising gauge fields rely solely on the functional integral. Moreover, alternative definitions of functional integrals allow for the possibility of new forms of quantisation.

Usually functional integrals are regarded as objects which are manipulated to give formal relations (e.g. between unrenormalised Greens functions in different gauges or forms for tadpole cancelling terms). Their structure is usually disregarded. However there exists a rigorous definition of the commutative Hilbert space integral. The history integral of Feynman can be regarded as an analytic continuation of a Hilbert space integral with modified weight.

For a real separable Hilbert space $\mathcal{H}$

$$F : \mathcal{H} \longrightarrow \mathbb{C}$$

is a tame functional if

a) there exists a finite dimensional subspace E such that $F(x) = F(x_E)$ where x_E is the orthogonal projection of x on E.

b) F is Lebesgue measurable on E

We define $\mathcal{T}$ to be the set of tame functionals. For a tame

functional F E is known as the base of F. If the dimension of E is n we will define the measure

$$d\,\mu_E(x) = (2\pi\sigma)^{-\frac{1}{2}n}\, dx^1 \ldots dx^n \exp\left[-\sum_{j=1}^{n} (x^j)^2/2\sigma\right]$$

with σ positive. Of course

$$\int_E d\,\mu_E(x) = 1$$

It is easy to see that if we define the functional integral of a tame functional as

$$I_{\mathcal{H}}(F) = I_E(F) \equiv \int_{\mathcal{H}} d\,\mu_E(x) \quad F(p_E x)$$

this is independent of the base E.

Definition :

$$\mathcal{T}_\alpha = \left\{F \text{ in } \mathcal{T}:\ I_{\mathcal{H}}(|F|^\alpha) < \infty\right\},\quad \alpha \geq 1$$

$\|F\|_\alpha = \left[I_{\mathcal{H}}(|F|^\alpha)\right]^{1/\alpha}$ is a norm on $\mathcal{T}_\alpha$. $\mathcal{T}_\alpha$ can be completed in the usual way.

It is necessary to extend the operator $I_{\mathcal{H}}$ for non-tame functionals. A basic system $\{E\}$ in $\mathcal{H}$ is given by $\left\{E_i \,|\, \dim E_i = i, E_i \subset \mathcal{H}, \bigcup_i E_i = \mathcal{H}\right\}$

For any functional F

$$F_{E_i}(x) \equiv F(x_{E_i})$$

F is α-invariant if for any two basic systems $\{A\}$ and $\{B\}$

i) F_{A_p}, F_{B_q} in $\mathcal{T}_\alpha$ for all p,q sufficiently large

ii) $\lim_{p,q \to \infty} \|F_{A_p} - F_{B_q}\|_\alpha = 0$

If the space of α - invariant functionals is denoted by I_α then

since

$$\|FG\|_{\alpha} \leq \|F\|_{\beta} \|G\|_{\gamma} \qquad \text{where} \quad \alpha^{-1} = \beta^{-1} + \gamma^{-1}$$

F belonging to I_{α} implies that F belongs to I_{β} for all β satisfying $1 \leq \beta \leq \alpha$. The essential property about I_{α} functionals is that the projection approximating functions are Cauchy and so F is in the completion of $\mathcal{T}_{\alpha}$.

From the above it is clear that $I_{\mathcal{H}}$ can be defined for F in I_{α} and in particular for $\alpha = 1$. I_{α} is non-trivial since $\mathcal{T}_{\alpha} \subset I_{\alpha}$. (It is often found convenient to restrict I_{α} further by demanding that conditions (i) and (ii) are satisfied for arbitrary systems of projections $\{\mathcal{A}_n\}$ and $\{\mathcal{B}_m\}$ which tend to unity as n and m become large. All tame functionals belong to this restricted space)

$$I_{\mathcal{H}}(F) \equiv \lim_{n \to \infty} I_{\mathcal{H}}(F_{E_n}) \equiv \int \mathcal{D}(x)\, e^{-\langle x,x\rangle/2\sigma} F(x)$$

(The concept of α-invariance makes the above definition independent of E). The lattice or polygonal approximation that is often used for functional integrals can be seen to fit into this scheme. The lattice is merely a finite dimensional projection. A lattice type defines a basic system. The Feynman path integral is of the form

$$\int \mathcal{D}_F(x)\; e^{\frac{1}{2}i\mu\langle \dot{x},\dot{x}\rangle}\; F(x)$$
$$= \int \mathcal{D}_W(x)\; e^{-\frac{1}{2}\mu\langle \dot{x},\dot{x}\rangle}\; F(i^{\frac{1}{2}}x)$$

where $\mathcal{D}_F(x)$ is imaginary but $\mathcal{D}_W(x)$ is real. Hence it looks like a commutative Hilbert space integral. However for a history integral

a rigorous interpretation is not known except for the trivial system of the free field. Feynman's postulate is that

$$\langle 0^{in} | 0^{out} \rangle = \int \exp \frac{i}{\hbar} S[\phi] \prod_x d\phi(x)$$

where $|0^{\substack{in \\ out}}\rangle$ are the $\substack{in \\ out}$ vacuums. $S[\phi]$ is the action functional. Elements of the S-matrix can be expressed in terms of Greens functions.

$$\langle T(F(\phi)) \rangle_0 = \int F(\phi) \exp\left(\frac{i}{\hbar} S[\phi]\right) \prod_x d\phi(x)$$

F is an arbitrary functional. On the right of the above expression ϕ is a classical field whereas on the left it is a quantum field. In gauge-invariant theories the action is constant on the orbits of the gauge group.

i.e. $S[\phi^\Omega] = S[\phi]$ where ϕ^Ω is a result of applying the element of the gauge group to the field ϕ. The idea of Faddeev and Popov is to extract the (infinite) factor arising in expressions for the Greens functions owing to this gauge degeneracy. There are two ways of progressing. One shall be called the Landau-F-P method and the other the Fermi-F-P method.

The Landau-F-P way is as follows:

We define a hypersurface $\left(\phi \mid f(\phi) = 0\right)$ in the space of fields such that the equation $f(\phi^\Omega) = 0$ has only one solution in $\Omega(x)$ for any $\phi(x)$. A functional associated with this hypersurface is defined via the condition

$$\int \Delta_f[\phi] \prod_x \delta(f[\phi^\Omega(x)]) d\Omega(x) = \text{constant}$$

From the definition it is necessary that $\Delta_f[\phi^\Omega] = \Delta_f \phi$

Now F-P postulate that

$$\int F(\phi)\, \exp \frac{i}{h} S[\phi]\, \Delta_f[\phi] \prod_x \delta(\, f[\phi(x)]\,)\ d\phi(x)$$

should be used in developing a perturbation theory for gauge fields.

The F-F-P procedure is only slightly different. A piece is added to the action which is not gauge invariant, i.e.

$$S(\phi) \longrightarrow \hat{S}[\phi] = S[\phi] + \Delta S[\phi]$$

such that $\Delta S[\phi^\Omega] \neq \Delta S[\phi]$. As before

$$\Delta_{\Delta S}[\phi] \int \exp \frac{i}{h} \Delta S[\phi^\Omega] \prod_x d\Omega(x) = \text{constant}$$

$\Delta_{\Delta S}[\phi]$ is a gauge invariant quantity. F-P postulate that

$$\int F(\phi)\, \Delta_{\Delta S}[\phi]\, \exp\left(i(S+\Delta S)\right) \prod_x d\phi(x)$$

should be used in developing perturbation expansions of the Greens functions.

As a simple example quantum electrodynamics will be considered. The lagrangian will be quantised in the Landau gauge i.e $\partial_\mu A^\mu \equiv 0$ This requires choosing the hypersurface given by

$$f[A] = \partial_\mu A^\mu$$

Hence $\Delta_f A$ is proportional to

$$\left(\int \prod_x \delta(\partial . A(x) - \partial^2 \Lambda(x)\,)\ d\Lambda(x)\,\right)^{-1}$$

Here $\Delta_f A$ is A independent and so can be taken as unity .

So the Greens functions are now calculable.

In particular

$$\langle T\,(\,\psi(x)\,\bar{\psi}(y)\,)\rangle_{\text{Landau}}$$

$$= \text{constant} \cdot \int \psi(x)\,\bar{\psi}(y)\;\exp\tfrac{i}{\hbar}S\;\prod_x \delta(\partial.A)\;dA\;d\psi\,d\bar{\psi}$$

We could proceed similarly using the F-F-P method. Here $\Delta S\,[A]$ is taken to be $-\frac{1}{2\beta}\int(\partial.A(x))^2\,dx$ and $\Delta_{\Delta_S^{(A)}}$ is found to be A independent.

Appendix 2 Details of the LPZ model

We will let I denote interaction. In particular I(Y) will be the part of the interaction lagrangian which involves Y alone. It turns out that

$$
\begin{aligned}
I(E,W,A,Z) = ig\,(& -\bar{\nu}_e \gamma^\mu e'_L W^-_\mu + \bar{e}_L \gamma^\mu \nu_e W^-_\mu - \bar{e}'_L \gamma^\mu \nu_e W^+_\mu \\
& + \bar{\nu}_e \gamma^\mu e_L W^+_\mu) \\
& + ig \sin\theta \,((\bar{e}'_L \gamma^\mu e'_L - \bar{e}_L \gamma^\mu e_L) A_\mu + (e'_R \gamma^\mu e'_R - \bar{e}_R \gamma^\mu e_R) A_\mu) \\
& + ig \cos\theta \,((\bar{e}'_L \gamma^\mu e'_L - \bar{e}_L \gamma^\mu e_L) Z_\mu) \\
& + ig \tan\theta \sin\theta \,(\bar{e}_R \gamma^\mu e_R - \bar{e}'_R \gamma^\mu e'_R) Z_\mu
\end{aligned}
$$

$$
\begin{aligned}
I(\text{quarks},W,A,Z) = ig\,(& -(\bar{n}_L \cos\theta_c + \bar{\lambda}_L \sin\theta_c) \gamma^\mu p_L W^-_\mu + \bar{r}_L \gamma^\mu (n_L \cos\theta_c \\
& + \lambda_L \sin\theta_c) W^-_\mu - \bar{p}_L \gamma^\mu (n_L \cos\theta_c + \lambda_L \sin\theta_c) W^+_\mu \\
& + (\bar{n}_L \cos\theta_c + \bar{\lambda}_L \sin\theta_c) \gamma^\mu r_L W^+_\mu) \\
& + igsin\theta \,((\bar{p}_L \gamma^\mu p_L - \bar{r}_L \gamma^\mu r_L + \bar{p}'_L \gamma^\mu p'_L - \bar{r}'_L \gamma^\mu r'_L) A_\mu) \\
& + igsin\theta \,(\bar{p}_R \gamma^\mu p_R - \bar{r}_R \gamma^\mu r_R + \bar{p}'_R \gamma^\mu p'_R - \bar{r}'_R \gamma^\mu r'_R) A_\mu \\
& + igcos\theta \,((\bar{p}_L \gamma^\mu p_L - \bar{r}_L \gamma^\mu r_L + \bar{p}'_L \gamma^\mu p'_L - \bar{r}'_L \gamma^\mu r'_L) Z_\mu) \\
& + igtan\theta \sin\theta \,(\bar{r}_R \gamma^\mu r_R - \bar{p}_R \gamma^\mu p_R + \bar{r}'_R \gamma^\mu r'_R - \bar{p}'_R \gamma^\mu p'_R) Z_\mu \\
& + ig \,(-(-\bar{n}_L \sin\theta_c + \bar{\lambda}_L \cos\theta_c) \gamma^\mu p'_L W^-_\mu + \bar{r}'_L \gamma^\mu (-n_L \sin\theta_c \\
& + \lambda_L \cos\theta_c) W^-_\mu + (-\bar{n}_L \sin\theta_c + \bar{\lambda}_L \cos\theta_c) \gamma^\mu r'_L W^+_\mu \\
& - \bar{p}'_L \gamma^\mu (-n_L \sin\theta_c + \lambda_L \cos\theta_c) W^+_\mu)
\end{aligned}
$$

$$I(E, \chi, \gamma)$$

$$= -\sqrt{2}\,\frac{m_e}{v}\,(\bar{e}_R \chi^{--} e'_L + \bar{e}_R \chi^{-} \nu_e + \bar{e}_R \frac{\rho_o + v}{\sqrt{2}} e_L + \bar{e}'_L \chi^{++} e_R + \bar{\nu}_e \chi^{+} e_R + \bar{e}_L \frac{\rho_o + v}{\sqrt{2}} e_R)$$

$$-\sqrt{2}\,\frac{m'_e}{v}\,(\bar{e}'_R \chi^{++} e_L + \bar{e}'_R \pi^{+} \nu_e + \bar{e}'_R \frac{\rho_o + v}{\sqrt{2}} e'_L + \bar{e}_L \chi^{--} e'_R + \bar{\nu}_e \chi^{-} e'_R + \bar{e}'_L \frac{\rho_o + v}{\sqrt{2}} e'_R)$$

$$I(\text{Quarks}, \chi, \gamma)$$

$$= -\sqrt{2}\,\frac{m_r}{v}\,(\bar{r}_R \chi^{--} p_L + \bar{r}_R \chi^{-} n_{c_L} + \bar{r}_R \frac{\rho_o + v}{\sqrt{2}} r_L + \bar{p}_L \chi^{++} r_R + \bar{n}_{c_L} \chi^{+} r_R + \bar{r}_L \frac{\rho_o + v}{\sqrt{2}} r_R)$$

$$-\sqrt{2}\,\frac{m_p}{v}\,(\bar{p}_R \chi^{++} r_L + \bar{p}_R \chi^{+} n_{c_L} + \bar{p}_R \frac{\rho_o + v}{\sqrt{2}} p_L + \bar{r}_L \chi^{--} p_R + \bar{n}_{c_L} \chi^{-} p_R + \bar{p}_L \frac{\rho_o + v}{\sqrt{2}} p_R)$$

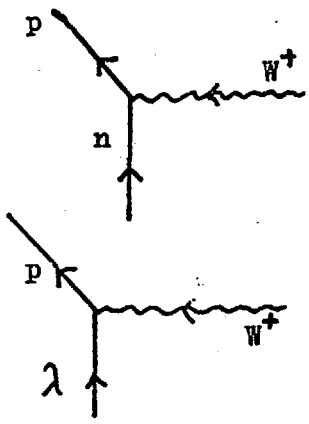
$$-\sqrt{2}\,\frac{m_{p'}}{v}\,(\bar{p}'_R \chi^{++} r'_L + \bar{p}'_R \chi^{+} \lambda_{c_L} + \bar{p}'_R \frac{\rho_o + v}{\sqrt{2}} p'_L + \bar{r}'_L \chi^{--} p'_R + \bar{\lambda}_{c_L} \chi^{-} p'_R + \bar{p}'_L \frac{\rho_o + v}{\sqrt{2}} p'_R)$$

$$-\sqrt{2}\,\frac{m_{r'}}{v}\,(\bar{r}'_R \chi^{--} p'_L + \bar{r}'_R \chi^{-} \lambda_{c_L} + \bar{r}'_R \frac{\rho_o + v}{\sqrt{2}} r'_L + \bar{p}'_R \chi^{++} r'_R + \bar{\lambda}_{c_L} \chi^{+} r'_R + \bar{r}'_L \frac{\rho_o + v}{\sqrt{2}} r'_R)$$

$$-\sqrt{2}\,\frac{m_n}{u}\,(\bar{n}_{c_R}(u + \gamma^{o}) n_{c_L} + \bar{n}_{c_L}(u + \gamma^{o}) n_{c_R} + \bar{\lambda}_{c_R}(u + \gamma^{o}) \lambda_{c_L} + \bar{\lambda}_{c_L}(u + \gamma^{o}) \lambda_{c_R})$$

There are many more important couplings but we do not need them.

We now give specific couplings



$$-g\cos\theta_c\,\frac{(1-\gamma_5)}{2}\,\gamma^\mu$$

$$-g\sin\theta_c\,\frac{(1-\gamma_5)}{2}\,\gamma^\mu$$

We shall now give the general order of magnitude of higher order effects on dimensional grounds

$$G = g^2/m_W^2 \quad , \quad e^{-2} = g^{-2} + g'^{-2} \quad \text{and} \quad \alpha = e^2$$

In higher order processes large momentum contributions are more important than the low momentum contributions. Hence the contributions from photon and massive vector boson propagators are of the same order; the T matrix for weak processes will have the expansion

$$T \simeq \frac{G}{\sqrt{2}}\left[a_0 + a_1\left(\frac{\alpha}{2\pi}\right) + a_2\left(\frac{\alpha}{2\pi}\right)^2 + \ldots\right]$$

$$\sim \frac{G}{\sqrt{2}}\left[b_0 + b_1\,(G\,M^2) + b_2\,(GM^2)^2 + \ldots\right]$$

where M is usually the mass of the vector boson W; so we would expect that higher order effects are larger in models with large M^2.

Appendix 3 : The Kinking Theorem

The idea of Kinking was first suggested by Salam and later the Kinking theorem was proved by Isham, Salam and Strathdee[11]. The new concept introduced is that of Kinked Greens functions which are finite (after the standard non-polynomial regularisation). We shall consider the system in ' d=3 ' field theories described by

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\chi\,\partial_\mu\chi\,V(\varphi) + \tfrac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi$$

where V(0) is 1 and χ , and φ are independent scalar fields.

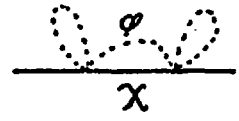
$$G(x,y) = \langle T\,(\,\chi(x)\,\chi(y)\,)\rangle_{\text{Heisenberg picture}}$$

$$= \langle T^X(\,\chi(x)\chi(y)\,\exp i\,\tfrac{1}{2}\int(\partial_\mu\chi)^2(V(\varphi)-1)\rangle_{\text{Dirac picture}}$$

It is one of the assumptions of the Kinking theorem that the replacement of a T product by a T^X product is valid provided tadpole contributions are dropped; this is an assumption which has not been proved in the framework of perturbation theory. We define $G^{(N)}(x,y)$ to be the Nth term in the expansion for G(x,y) obtained by expanding the exponential involved. We define the Kinked Greens function K(x,y) as

$$K(x,y) = \langle T^X(\chi(x)\chi(y)\,\exp i\int \tfrac{1}{2}(\partial_\mu\chi)^2V(\varphi)\;\;)\rangle$$

$K_N(x,y)$ is the Nth term in a perturbation expansion for K(x,y).

e.g. $G^{(2)} =$  +  = 

if tadpoles vanish

I.S.S. have succinctly expressed the Kinking theorem as

$$G^{(N)}(x,y) = \sum_{r=0}^{N} \binom{N}{r} K^{(r)}$$

We will not reproduce here the general proof given by I.S.S.; instead we shall look at a special case

$$\begin{aligned}
&K^{(2)}(x,y) - G^{(2)}(x,y) \\
&= (i)^2 \iint dz_1\, dz_2 \quad \partial_\mu^{(z_1)} D(x-z_1)\ \partial_\mu^{(z_1)}\ \partial_\nu^{(z_2)}\ D(z_1-z_2) \\
&\qquad\qquad \cdot\ \partial_\nu^{(z_2)} D(z_2 - y) \\
&= (i)^2 \iint dz_1 dz_2 \quad \Box^{(z_1)} D(x-z_1)\ D(z_1-z_2)\ \Box^{(z_2)} D(z_2-y) \\
&= (i)^2(-i)^2 \iint dz_1 dz_2\ \delta(x-z_1)\ D(z_1-z_2)\ \delta(z_2-y) \\
&= D(x-y) = G^{(0)}(x,y) = K^{(0)}(x,y)
\end{aligned}$$

Appendix 4 : The calculation of $S[q_c]$.

$$S[q_c] = \Omega^{-1} \int_{\Omega t_a}^{\Omega t_b} d(\Omega t)\left(\tfrac{1}{2}c^2\Omega^2(\dot{e}(\Omega t))^2 - \tfrac{1}{2}\mu^2c^2(e(\Omega t))^2 + \tfrac{1}{4}\lambda c^4 (e(\Omega t))^4\right)$$

We shall often use relations of the form[62]

$$\int sn\ u\ cn\ u\ du = -(dn\ u)/k^2 \qquad (A4.1)$$

$$\int sn\ u\ cn\ u\ dn^m u\ du = \frac{-1}{(m+1)k^2}\ dn^{m+1}u \qquad (A4.2)$$

$$\int cn^2u\ dn^2u\ du = \frac{1}{3k^2}\left((1+k^2)E(u) - k'^2u + k^2 sn\ u\ cn\ u\ dn\ u\right) \qquad (A4.3)$$

and $$\int \frac{sn^nu\ cn^pu}{dn^{2m}u}\ du \qquad (\text{for } n+p-1-2m \neq 0) \qquad (A4.4)$$

$$= \frac{1}{k^2}\int sn^nu\ cn^{p-2}u\ nd^{2m-1}u\ du - k'^2\int sn^ru\ cn^{p-2}u\ nd^{2m}u\ du$$

Now $\dot{e}(u) = -\ sn\ u\ dn\ u + i\ cn\ u\ dn\ u$

Hence

$$\int_{u_a}^{u_b} du\ (\dot{e}(u))^2 = \int_{u_a}^{u_b} du\ (sn^2u\ dn^2u - cn^2u\ dn^2u - 2i\ sn\ u\ cn\ u\ dn^2u)$$

$$= \frac{1}{3k^2}\ B(u)\Big|_{u_a}^{u_b}$$

where $$B(u) = \frac{1}{3k^2}\left((k^2-2)\ E(u) + 2k'^2u - 2k^2 sn\ u\ cn\ u\ dn\ u + \frac{2i}{3k^2}\ dn^3\ u\right)$$

In evaluating the remaining terms in the expression for $S[q_c]$

we shall just expand the integrands as products of sn u and cn u and then use relations such as given in (A4.1) to (A4.4).

$$\int_{u_a}^{u_b} du\ (e(u))^2 = \left[u - \frac{2}{k^2}(u-E(u)) - 2i(dn\ u)/k^2\right]_{u=u_a}^{u=u_b}$$

Similarly

$$\int du\ (e(u))^4$$
$$= 8\ \frac{k^2-2}{3k^4}\ E(u) + \left(\frac{14k'^2+k^2-3k^2k'^2+2}{3k^4}\right) u + \frac{8}{3k^2}\ sn\ u\ cn\ u\ dn\ u$$
$$- \frac{4i}{k^4}\ dn^2 u + 4i\left(\frac{k^2+2k'^2}{k^4}\right)\ dn\ u$$

where we have evaluated $\int sn\ u\ cn^3 u\ du$ using (A4.4) with m = 0 n = 1 and p = 3. Finally collecting together terms we have

$$S\left[q_c\right] = \Omega^{-1}\left[F(u)\right]_{u=\Omega t_a}^{u=\Omega t_b}$$

where F(u) is given by

$$E(u)\left(\tfrac{1}{2}c^2\Omega^2\ \frac{k^2-2}{3k^2} - \frac{\mu^2 c^2}{k^2} + 2\left(\frac{k^2-2}{3k^4}\right)\lambda c^4\right.$$
$$+ u\left(\tfrac{1}{2}c^2\Omega^2 \cdot \frac{2k'^2}{3k^2} - \tfrac{1}{2}\mu^2c^2\left(1 - \frac{2}{k^2}\right) + \left(\frac{14k'^2+k^2-3k^2k'^2+2}{3k^4}\right)\frac{\lambda}{4}c^4\right)$$
$$+ sn\ u\ cn\ u\ dn\ u\left(\tfrac{1}{2}c^2\Omega^2(-2/3) + 2\lambda c^4/3k^2\right)$$
$$+ dn^3 u\left(\tfrac{1}{2}c^2\Omega^2 \cdot 2i/(3k^2)^2\right) + dn^2 u\left(-\lambda i c^4/k^4\right)$$
$$+ dn\ u\left(i\frac{\mu^2 c^2}{k^2} + i\lambda c4\left(\frac{k^2+2k'^2}{k^4}\right)\right.$$

References :

1) C. N. Yang and R.L Mills, Phys. Rev. 96,('54) 191

2) P. Higgs, Phys. Lett. 12,('64) 132

3) T. W. B. Kibble, Phys. Rev. 155,('67) 1554

4) G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble,
' Advances in Particle Physics, Vol.2 (ed. R. L. Cool and
R. E. Marshak) (Interscience Publishers, 1968) 567

5) C. J. Isham, Abdus Salam, and J. Strathdee, Lettere al
Nuovo Cimento, Vol. 5, No. 15, ('72) 969

6) S. Gupta, Proc. Phys. Soc. (London) A63,('51) 681
K. Bleuler, Helv. Phys. Acta 23, ('50) 567

7) Abdus Salam and J. Strathdee, Nuovo Cim. 11A, ('72) 397
L.D. Faddeev and V.N. Popov, Phys. Lett. B25, ('67) 30

8) R.P. Feynman, Acta Phys. Polon. 24, ('63) 697

9) G.'t Hooft, Nucl. Phys. B35, ('71) 167

10) B.W. Lee and J. Zinn-Justin, Phys. Rev.D5, ('72) 3155

11) R. Delbourgo, R. Koller and A. Salam, Annals of Physics 66,('71) 1
C.J. Isham, A. Salam, and J. Strathdee, Phys. Rev.D5,('72) 2548

12) E.R. Speer, J. Math. Phys. 9, ('68) 1404

13) P.K. Mitter, Comm. Math. Phys. 20, ('71) 251

14) M. Riesz, Acta Math. 81, ('49) 1

15) M. Volkov, Ann. of Phys. 49, ('68) 202

16) H. Lehmann and K. Pohlmeyer, Comm. Math. Phys. 20,('71) 101

17) A. Jaffe, Phys. Rev. 158, ('67) 1454

18) G. Ecker and J. Honerkamp, Phys. Lett. B42, ('72) 253

H. Lehmann and H. Trute, Nucl. Phys. B 52,('73) 280

19) S. Weinberg, ' General Theory of Broken Local Symmetries' C.T.P. 308 ('72)

S. Sarkar, ' Local Higgs-Kibble type theories in the unitary gauge' ICTP/72/14 (to be published in Nucl. Phys. B)

20) K. Bardakci, M.B. Halpern, and Y. Frishman, Phys.Rev.170, ('68) 1353

21) K.J. Barnes and C.J. Isham, Nucl. Phys. B15 ('70) 333

22) J. Prentki and B. Zumino, Nucl. Phys. B47 ('72) 99

B.W. Lee, Phys. Rev. D6 ('72) 1118

23) Abdus Salam and J.C. Ward, Phys. Lett. 13 ('64) 168

Abdus Salam, Proc. of the 8th Nobel Symposium (ed. N. Svartholm)(Almqvist and Wicksel, Stockholm '68)

S. Weinberg, Phys. Rev. Letters 19, ('67) 1264

24) D.A. Ross and S. Sarkar, Lett. al Nuovo Cimento, Vol.6 No.1 ('73) 1

25) B.W. Lee, Perspectives on Theory of Weak Interactions,NAL-THY -92 ('72)

26) S. Sarkar, Nucl. Phys. B49,('72) 173

27) R. Raczka, Lettere al Nuovo Cim., Vol.3,('72) 575

28) S. Sarkar, One loop corrections to tree diagrams,ICTP/71/51 (to be published in Phys. Rev. D)

29) P.W. Anderson, Phys. Rev. 130, ('63) 439

30) G. Jona-Lasinio, Nuovo Cim. 34, ('64) 1790

31) K. Bardakci and M.B. Halpern, Phys. Rev. D6 ('72) 696

32) J. M. Charap, Phys. Rev. D8 ('70) 1554

I.S. Gerstein, R. Jackiw, B.W. Lee, and S. Weinberg, Phys. Rev. D3, ('71) 2486

33) H. Sugawara, Phys. Rev. 170, ('68) 1659

34) J. Schwinger, Phys. Rev. Lett. 3, ('59) 296

35) I. Bars, M.B. Halpern, and M. Yoshimura, Phys. Rev. Lett.29,('72) 969

36) E.C.G. Sudarshan and R.E. Marshak, Phys. Rev. 109, ('58) 1860

R.P. Feynman and M. Gell-Mann, Phys. Rev. 109, ('58) 193

37) I.B. Khriplovich and A.I. Vainshtein (submitted to Nuovo Cimento Letters)

38) H. Quinn (unpublished)

J.C. Taylor, Rutherford Laboratory Report RPP/T/29

39) J.D. Bjorken and C.H. Llewellyn Smith, Spontaneously broken gauge theories of weak interactions and heavy leptons, ('72) SLAC-PUB-1107

40) H.H. Chen and B.W. Lee, Phys. Rev. D5,('72) 1874

F. Reines and J.S. Gurr, Phys. Rev. Lett. 24, ('70) 1448

41) G.'t Hooft and M. Veltman, Nucl. Phys. B50, ('72) 318

42) B.W. Lee, J.R. Primack, and S.B. Treiman, Some Physical Constraints on Gauge Models of Weak Interactions, ('72) NAL-THY-74

43) S.L. Glashow, J. Iliopoulos, and L. Maiani, Phys.Rev. D2,('70) 1285

44) S. Fubini and G. Furlan, Physics 1, ('65) 229

M.K. Gaillard and L.M. Chounet, CERN Report 70-14

45) R. Van Royen and V.F. Weisskopf, Nuovo Cim. 50, ('67) 617

46) J.J.J. Kokkedee, The Quark Model, (W.A. Benjamin 1969, Amsterdam)

47) M. Ademollo and R. Gatto, Phys. Rev. Letters 13 ('65) 264

48) Abdus Salam, Non-polynomial lagrangians, renormalisation, and gravity, (Gordon and Breach 1971, New York)

49) C.J. Isham, Abdus Salam, and J. Strathdee, Phys. Rev. D3 ('71) 1805

50) M.J. Duff, J. Huskins and A. Rothery, Phys. Rev. D4 ('71) 1851

51) A.M. Jaffe, Proc. of the International School of Physics Enrico Fermi Course XLV (ed. R. Jost) (Academic Press, New York 1969)

52) S. Fels, Phys. Rev. D1 ('70) 2370

53) I.S. Gradshteyn and I.M. Ryzhik, Tables of integrals, series and products (Academic Press, New York, 1965)
R. P. Boas, Entire functions (Academic Press, New York, 1965)

54) Abdus Salam and J. Strathdee, Phys. Rev. D1 ('70) 3296

55) M.J. Duff, Quantum tree graphs and the Schwarzschild solution ICTP/71/30 (to be published in Phys. Rev. D)

56) J.R. Klauder, Soluble models of Quantum Gravitation, in Proceedings of the Relativity Conference in the Midwest Cincinnati '69 in ' Relativity ' (ed. M. Carmeli et al.) (Plenum Press, New York, 1970)

57) B.S. DeWitt, Dynamical Theory of Groups and Fields, in ' Relativity, Groups and Topology ' (ed. B.S. and C.M. DeWitt) (Blackie, London 1964)

58) I.M. Gelfand and A.M. Yaglom, Journal Math. Phys. 1, ('60) 48

59) E. Nelson, Journal Math. Phys. 5, ('64) 352

60) R.P. Feynman and A.R. Hibbs, Quantum mechanics and path integrals (Mc Graw- Hill Book Co., New York, 1965)

61) A.R. Forsyth, Theory of Differential Equations , Part 3 Cambridge University Press 1902

62) P.F. Byrd and M.D. Friedman, Handbook of Elliptic Integrals for engineers and physicists (Springer-Verlag,Berlin, 1954)