RANK r DT THEORY FROM RANK 1

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ABSTRACT. Fix a Calabi-Yau 3-fold X satisfying the Bogomolov-Gieseker conjecture of Bayer-Macrì-Toda, such as the quintic 3-fold.

We express Joyce's generalised DT invariants counting Gieseker semistable sheaves of any rank r on X in terms of those counting sheaves of rank 1. By the MNOP conjecture they are therefore determined by the Gromov-Witten invariants of X.

Let X be a smooth projective Calabi-Yau 3-fold satisfying the Bogomolov-Gieseker conjecture of Bayer-Macrì-Toda [BMT]. We show that the higher rank or "nonabelian" DT theory of X is governed by its rank one "abelian" theory. ("Nonabelian" and "abelian" refer to the gauge groups $U(r)$ and $U(1)$ respectively.) This can be thought of as a 6-dimensional analogue of the correspondence between nonabelian Donaldson theory and abelian Seiberg-Witten theory for smooth 4-manifolds.

Combined with the MNOP conjecture [MNOP], now proved for most Calabi-Yau 3-folds [PP], this expresses any DT invariant $\mathsf{J}(v)$ entirely in terms of the Gromov-Witten invariants of X. Here $\mathsf{J}(v) \in \mathbb{Q}$ denotes Joyce-Song's generalised DT invariant [JS] counting Gieseker semistable sheaves on X of numerical K-theory class v of rank $r > 0$.

Theorem 1. *Let $(X, \mathcal{O}_X(1))$ be a Calabi-Yau 3-fold satisfying the conjectural Bogomolov-Gieseker inequality of* [BMT]. *Then for fixed v of any rank $r > 0$,*

$$\mathsf{J}(v) \;=\; F\big(\mathsf{J}(\alpha_1), \mathsf{J}(\alpha_2), \dots\big) \tag{1}$$

is a universal polynomial in invariants $\mathsf{J}(\alpha_i)$, with all α_i of rank 1. If X also satisfies the MNOP conjecture then we can replace the $\mathsf{J}(\alpha_i)$ by the Gromov-Witten invariants of X.[1]

The coefficients of F depend only on $H^*(X, \mathbb{Q})$ as a graded ring with pairing, $\mathrm{ch}(v)$, the Chern classes of X, and the class $H := c_1(\mathcal{O}_X(1))$ used to define Gieseker stability. There are countably many terms in the formula (1) but only finitely many are nonzero.

Theorem 2. *If $H^1(\mathcal{O}_X) = 0$ then Theorem 1 also holds for classes v of rank $r = 0$.*

Joyce-Song pairs. We fix any smooth complex projective threefold $(X, \mathcal{O}_X(1))$ (not necssarily Calabi-Yau). A *Joyce-Song stable pair* (F, s) of class v on X consists of

- a rank r semistable sheaf F of class v, a fixed $n \gg 0$, and
- $s \in H^0(F(n))$ which factors through no semi-destabilising subsheaf of F.

[1] In particular if X_1, X_2 are symplectomorphic and both satisfy the BG and MNOP conjectures then their DT theories coincide. It has long been expected that DT invariants should be symplectic invariants.

For us "semistable" will refer to a specific weak stability condition of [BMT, BMS] (in contrast to the Gieseker stability used by Joyce-Song). When $r \geq 1$ this is enough to ensure that s is *injective*, so it makes sense to consider its cokernel E,

$$0 \longrightarrow \mathcal{O}_X(-n) \xrightarrow{s} F \longrightarrow E \longrightarrow 0, \tag{2}$$

of class $v_n := v - [\mathcal{O}_X(-n)]$. This allows us to relate the counting of the semistable sheaves F of rank r to the counting of sheaves E of rank $r-1$. We can then use wall crossing to try to move from weakly semistable E, F to Gieseker semistable E, F. In this way we expressed rank $r \geq 1$ DT invariants $\mathsf{J}(v)$ in terms of invariants J of ranks $\leq r-1$ for

- ideal sheaves of curves (so $r = 1$) in [FT1],
- arbitrary sheaves F of rank $r = 1$ in [FT2], and
- arbitrary sheaves F of arbitrary rank $r > 1$ in [FT3].

This does not give Theorem 1, however. Instead, by inducting on $r = \mathrm{rank}(v)$, it expresses $\mathsf{J}(v)$ in terms of invariants of *ranks 1 and 0*. Or, applying it one more time to the rank 1 invariants, it expresses $\mathsf{J}(v)$ in terms of counts of Gieseker semistable sheaves of *rank 0 and pure dimension 2* [FT3, Theorem 1]. Thus to prove Theorem 1 it therefore remains[2] to prove Theorem 2 for *rank 0 dimension 2* classes v, i.e. those with $\mathrm{ch}_1(v).H^2 > 0$.

In this paper we go one step further, from rank 0 to -1. We replace rank 0 sheaves F by the cones $E \in \mathcal{D}(X)$ on their Joyce-Song pairs. These are nontrivial complexes as s has both kernel and cokernel. The exact sequence (2) is replaced by the exact triangle

$$\mathcal{O}_X(-n) \xrightarrow{s} F \longrightarrow E \longrightarrow \mathcal{O}_X(-n)[1]. \tag{3}$$

By further wall crossing, dualising and shifting by [1] we can then relate the semistable complexes E to rank 1 ideal sheaves to prove Theorem 2 for rank 0 dimension 2 classes.

Wall crossing. Most of the work is in Section 2, which applies to *any* smooth projective 3-fold satisfying the Bogomolov-Gieseker conjecture. We use the weak stability conditions of [BMT, BMS], finding the walls of instability for objects of class

$$v_n \;=\; [E] \;=\; [F] - [\mathcal{O}_X(-n)] \;=\; v - [\mathcal{O}_X(-n)].$$

The most important we call its *Joyce-Song wall* ℓ_{JS} on which the slopes of E, F and $\mathcal{O}_X(-n)[1]$ coincide. Below ℓ_{JS} the exact triangle (3) destabilises E, while above it such complexes become semistable. The same holds when we replace $\mathcal{O}_X(-n)[1]$ by $T(-n)[1]$ for any line bundle T with torsion first Chern class,

$$T \;\in\; \mathrm{Pic}_0(X) \;:=\; \big\{L \in \mathrm{Pic}(X) \,:\, c_1(L) \,=\, 0 \,\in\, H^2(X,\mathbb{Q})\big\}. \tag{4}$$

We show that all other walls of instability for the class $[E]$ are similar: whenever E is destabilised below the wall it is by a triangle $E_0 \to E \to E_1$ with E_0 a rank 0 sheaf and E_1 a rank -1 complex (with cohomology in degrees -1 and 0 only), while above the wall it is extensions in the opposite direction that are unstable.

[2] We may assume $H^1(\mathcal{O}_X) = 0$ in Theorem 1; otherwise the $\mathrm{Jac}(X)$ action on moduli spaces of sheaves forces $\mathsf{J}(v) = 0$ for all classes v of rank > 0. We focus on Theorem 2 for classes v of dim $= 2$. For the $\dim v = 1$ case see Section 4. For $\dim v = 0$ [JS, Equation 6.19] expresses $\mathsf{J}(v)$ in terms of $e(X) = c_3(X)$.

Wall crossing formula. From Section 3 we assume $K_X \cong \mathcal{O}_X$ and $H^1(\mathcal{O}_X) = 0$. Thus Joyce-Song's generalised DT invariants are defined and satisfy a wall crossing formula.

The wall crossing formula calculates the change in the invariants counting semistable objects in class $[E]$, as we cross any of the walls described above, in terms of the counting invariants of the classes $[E_0]$ and $[E_1]$ (and of the classes of semistable factors of E_0, E_1 of the same slope). For instance suppose for simplicity that, on crossing the Joyce-Song wall ℓ_{JS}, linear combinations of v and $\big[\mathcal{O}_X(-n)[1]\big]$ are the only classes whose slope crosses that of v_n, and that v is primitive. Then the wall crossing formula is

$$\mathsf{J}_{b,w_+}(v_n) \;=\; \mathsf{J}_{b,w_-}(v_n) + (-1)^{\chi(v(n))-1}\chi(v(n)) \cdot \#H^2(X,\mathbb{Z})_{\mathrm{tors}} \cdot \mathsf{J}_{b,w_+}(v), \tag{5}$$

where $(b, w_\pm)$ are points just above and below the wall ℓ_{JS} in the space of weak stability conditions, the $\mathsf{J}_{b,w}$ are the corresponding invariants counting semistable objects, and $\#H^2(X,\mathbb{Z})_{\mathrm{tors}}$ is the invariant counting the objects $T(-n)[1]$ for $T \in \mathrm{Pic}_0(X)$. Finally $(-1)^{\chi(v(n))-1}\chi(v(n))$ counts the extensions (3) by taking the signed topological Euler characteristic of the space $\mathbb{P}\big(H^0(F(n))\big)$ of all of them.

From small to large volume. The Bogomolov-Gieseker inequality gives a region in the space of weak stability conditions in which there are no semistable objects E. Conversely, by work of Toda [To3], there is a region ("the large volume chamber") in which the rank -1 objects E (or E_1) are semistable if and only if the rank 1 complex $E^\vee \otimes (\det E)^{-1}[1]$ is a *stable pair* in the sense of [PT]. Using the wall crossing formula over the walls described above we move from the first of these regions (where the invariants vanish) to the second (where they equal stable pair invariants).

Just as (5) relates $\mathsf{J}_{b,w}(v)$ to the rank -1 invariants $\mathsf{J}_{b,w}(v_n)$, the eventual result is an analogous — but more complicated — formula relating invariants counting rank 0 sheaves such as F and E_0 to stable pair invariants counting rank -1 complexes such as E and E_1. The crucial fact, proved in Propositions 2.4 and 2.5, is that the degree $\mathrm{ch}_1.H^2$ of the support of these rank 0 sheaves is $\leq \mathrm{ch}_1(v).H^2$ with equality only on ℓ_{JS}. So working by induction on $\mathrm{ch}_1.H^2$ we may assume the lower order terms — counting rank 0 sheaves E_0 with $\mathrm{ch}_1.H^2 < \mathrm{ch}_1(v).H^2$ — have already been expressed in terms of stable pair invariants. We thus conclude that so too is the invariant counting the sheaves F (3) in class v.

By a further wall crossing already carried out in [Br3, To1], stable pair invariants can be expressed in terms of the rank 1 DT invariants counting ideal sheaves. And a final wall crossing from [FT3, Section 5] expresses the counts $\mathsf{J}(v)$ of Gieseker semistable sheaves of rank 0 and dimension 2 in terms of those which are semistable in the large volume chamber.

Our main technique for finding walls of instability is the Bogomolov-Gieseker conjecture of [BMT, BMS]. This is now proved for many 3-folds, including some Calabi-Yau 3-folds [BMS, Ko, Li1, MP]. The restricted set of weak stability conditions handled in [Ko, Li1] are sufficient for our purposes, as we check carefully on the quintic 3-fold in Section 2.2. The MNOP conjecture is also proved for the quintic in [PP]. So all DT invariants of the quintic 3-fold are given by universal formulae in its Gromov-Witten invariants.

Outlook. Having dealt with the counting of sheaves, one could try to extend our result to counts of arbitrary Bridgeland semistable complexes of sheaves. After a possible shift to ensure rank ≥ 0 this reduces to another wall crossing problem — assuming the space of stability conditions is connected — to reach stability conditions dominated by the weak stability conditions considered in this paper and thus return to the problem of counting sheaves.

Applying Joyce's new wall crossing formula for Fano 3-folds [Jo, Theorem 7.69] to the results of Section 2 should hopefully result in a Fano version of Theorem 1 with insertions.

Kontsevich and Soibelman's refinements of $\mathsf{J}(v)$ [KS1, KS2] should be well defined now their integrality conjecture has been proved [DM] and orientation data has been shown to exist on Calabi-Yau 3-folds [JU]. By using their wall crossing formula [KS1] in place of [JS] it should be possible to prove an analogue of Theorem 1 for these invariants too.

Theorem 1 is an abstract existence theorem for a universal expression F; it is not a practical route to concrete formulae. For explicit formulae for related invariants in special cases see [Ob] (for reduced DT invariants of $K3 \times E$), [FMR, Equation 1.6], [DNPZ, Equation 5.30], [AK] (for K-theoretic counts of quotients of a trivial bundle on noncompact toric Calabi-Yau 3-folds) and [To3, Fe] (for explicit forms of Theorem 2 expressing counts of dimension 2 sheaves — with restricted Chern classes — in terms of curve counts).

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1. Weak stability conditions

Let $(X, \mathcal{O}(1))$ be a smooth polarised complex projective threefold with bounded derived category of coherent sheaves $\mathcal{D}(X)$ and Grothendieck group $K(\mathrm{Coh}(X))$. Dividing by the kernel of the Mukai pairing gives the numerical Grothendieck group

$$K(X) := \frac{K(\mathrm{Coh}(X))}{\ker \chi(\ ,\ )}. \tag{6}$$

Notice that $K(X)$ is torsion-free, isomorphic to its image in $H^*(X,\mathbb{Q})$ under the Chern character. Denoting $H = c_1(\mathcal{O}_X(1))$, for any $v \in K(X)$ we set[3]

$$\begin{aligned} \mathrm{ch}_H(v) &:= \left(\mathrm{ch}_0(v),\ \tfrac{1}{H^3}\,\mathrm{ch}_1(v).H^2,\ \tfrac{1}{H^3}\,\mathrm{ch}_2(v).H,\ \tfrac{1}{H^3}\,\mathrm{ch}_3(v)\right) \in \mathbb{Q}^4, \\ \mathrm{ch}_H^{\leq 2}(v) &:= \left(\mathrm{ch}_0(v),\ \tfrac{1}{H^3}\,\mathrm{ch}_1(v).H^2,\ \tfrac{1}{H^3}\,\mathrm{ch}_2(v).H\right) \in \mathbb{Q}^3. \end{aligned}$$

Scaling the usual definition by H^3, we define the μ_H-slope of a coherent sheaf E to be

$$\mu_H(E) := \begin{cases} \frac{\mathrm{ch}_1(E).H^2}{\mathrm{ch}_0(E)H^3} & \text{if } \mathrm{ch}_0(E) \neq 0, \\ +\infty & \text{if } \mathrm{ch}_0(E) = 0. \end{cases}$$

Associated to this slope every sheaf E has a Harder-Narasimhan filtration. Its graded pieces have slopes whose maximum we denote by $\mu_H^+(E)$ and minimum by $\mu_H^-(E)$.

For any $b \in \mathbb{R}$, let $\mathcal{A}_b \subset \mathcal{D}(X)$ denote the abelian category of complexes

$$\mathcal{A}_b = \left\{E^{-1} \xrightarrow{d} E^0 \,:\, \mu_H^+(\ker d) \leq b,\ \mu_H^-(\mathrm{coker}\, d) > b\right\}. \tag{7}$$

In particular, setting $\mathrm{ch}^{bH}(E) := \mathrm{ch}(E)e^{-bH}$, each $E \in \mathcal{A}_b$ satisfies

$$\mathrm{ch}_1(E).H^2 - bH^3\,\mathrm{ch}_0(E) = \mathrm{ch}_1^{bH}(E).H^2 = \mathrm{ch}_1^{bH}(\mathrm{coker}\, d) - \mathrm{ch}_1^{bH}(\ker d) \geq 0, \tag{8}$$

with ≥ 0 replaced by > 0 when $\mathrm{coker}\, d = \mathcal{H}^0(E)$ has dimension ≥ 2. By [Br2, Lemma 6.1] $\mathcal{A}_b$ is the heart of a t-structure on $\mathcal{D}(X)$. We denote its positive cone by

$$C(\mathcal{A}_b) := \left\{\sum\nolimits_i a_i[E_i] \,:\, a_i \in \mathbb{N},\ E_i \in \mathcal{A}_b\right\} \subset K(X). \tag{9}$$

For any $w > \frac{1}{2}b^2$, we have on $\mathcal{A}_b$ the slope function

$$\nu_{b,w}(E) = \begin{cases} \frac{\mathrm{ch}_2(E).H - w\,\mathrm{ch}_0(E)H^3}{\mathrm{ch}_1^{bH}(E).H^2} & \text{if } \mathrm{ch}_1^{bH}(E).H^2 \neq 0, \\ +\infty & \text{if } \mathrm{ch}_1^{bH}(E).H^2 = 0. \end{cases} \tag{10}$$

By [BMT][4] $\nu_{b,w}$ defines a Harder-Narasimhan filtration on $\mathcal{A}_b$, and so a *weak stability condition* on $\mathcal{D}(X)$.

Definition 1.1. Fix $w > \frac{1}{2}b^2$. Given an injection $F \hookrightarrow E$ in $\mathcal{A}_b$ we call F a *destabilising subobject of* E if and only if

$$\nu_{b,w}(F) \geq \nu_{b,w}(E/F), \tag{11}$$

and *strictly destabilising* if $>$ holds. We say $E \in \mathcal{D}(X)$ is $\nu_{b,w}$-(semi)stable if and only if

- $E[k] \in \mathcal{A}_b$ for some $k \in \mathbb{Z}$, and
- $E[k]$ contains no (strictly) destabilising subobjects.

[3] Note ch_H meant something different in [FT3].

[4] We use notation from [FT2]; in particular the rescaling [FT2, Equation 6] of [BMT]'s slope function.

It is important to note that under these conventions an object can be *destabilised* even if it is *semistable* — in fact, if and only if it is strictly semistable. That is, our "destabilised" is what others might call "semi-destabilised" (as we did on page 1).

Also note that we cannot replace (11) by $\nu_{b,w}(F) \geq \nu_{b,w}(E)$; this is implied by (11) but does not imply it. So for instance the sequence $I_p \hookrightarrow \mathcal{O}_X \twoheadrightarrow \mathcal{O}_p$, for p a point of X, does not destabilise $\mathcal{O}_X$ even though $\nu_{b,w}(I_p) = \nu_{b,w}(\mathcal{O}_X)$.

Remark 1.2. Given $(b, w) \in \mathbb{R}^2$ with $w > \frac{1}{2}b^2$, the argument in [Br1, Propostion 5.3] describes $\mathcal{A}_b$. It is the extension-closure of the set of $\nu_{b,w}$-stable two-term complexes $E = \{E^{-1} \to E^0\}$ in $\mathcal{D}(X)$ satisfying the following conditions on the denominator and numerator of $\nu_{b,w}$ (10):

(a) $\mathrm{ch}_1^{bH}(E).H^2 \geq 0$, and
(b) $\mathrm{ch}_2(E).H - w\,\mathrm{ch}_0(E)H^3 \geq 0$ if $\mathrm{ch}_1^{bH}(E).H^2 = 0$.

By [BMS, Theorem 3.5] any $\nu_{b,w}$-semistable object $E \in \mathcal{D}(X)$ satisfies

$$\Delta_H(E) \;:=\; \big(\mathrm{ch}_1(E).H^2\big)^2 - 2(\mathrm{ch}_2(E).H)\,\mathrm{ch}_0(E)H^3 \;\geq\; 0. \tag{12}$$

Therefore, if we plot the (b, w)-plane simultaneously with the image of the projection map

$$\begin{aligned} \Pi\colon\; K(X) \smallsetminus \big\{E\colon \mathrm{ch}_0(E) = 0\big\} \;&\longrightarrow\; \mathbb{R}^2, \\ E \;&\longmapsto\; \left(\frac{\mathrm{ch}_1(E).H^2}{\mathrm{ch}_0(E)H^3}\,,\, \frac{\mathrm{ch}_2(E).H}{\mathrm{ch}_0(E)H^3}\right), \end{aligned}$$

as in Figure 1, then $\nu_{b,w}$-semistable objects E lie outside the open set

$$U \;:=\; \Big\{(b, w) \in \mathbb{R}^2\colon w > \tfrac{1}{2}b^2\Big\} \tag{13}$$

while (b, w) lies inside U. Shifting if necessary we may assume $E \in \mathcal{A}_b$, then by (8) we see

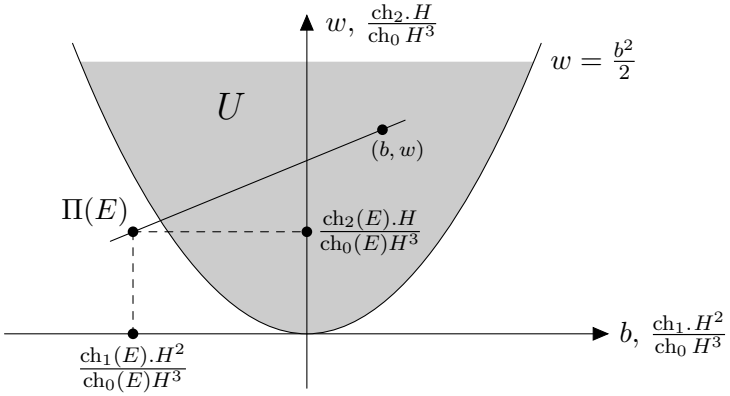


FIGURE 1. (b, w)-plane and the projection $\Pi(E)$ of a $\nu_{b,w}$-semistable object $E \in \mathcal{A}_b$ with $\mathrm{ch}_0(E) < 0$

$\Pi(E)$ lies on or to the left of the vertical line through (b,w) if $\mathrm{ch}_0(E)<0$, to the right if $\mathrm{ch}_0(E)>0$, and at infinity if $\mathrm{ch}_0(E)=0$. The slope $\nu_{b,w}(E)$ of E is the gradient of the line connecting (b,w) to $\Pi(E)$ (or $\mathrm{ch}_2(E).H/\mathrm{ch}_1(E).H^2$ if $\mathrm{rank}(E)=0$).

Any object $E\in\mathcal{D}(X)$ gives the space of weak stability conditions a wall and chamber structure by [BMS, Proposition 12.5], as rephrased in [FT1, Proposition 4.1] for instance.

Proposition 1.3 (**Wall and chamber structure**). *Fix $v\in K(X)$ with $\Delta_H(v)\geq 0$ and $\mathrm{ch}_H^{\leq 2}(v)\neq 0$. There exists a set of lines $\{\ell_i\}_{i\in I}$ in $\mathbb{R}^2$ such that the segments $\ell_i\cap U$ (called "walls of instability") are locally finite and satisfy*

(a) *If $\mathrm{ch}_0(v)\neq 0$ then all lines ℓ_i pass through $\Pi(v)$.*
(b) *If $\mathrm{ch}_0(v)=0$ then all lines ℓ_i are parallel of slope $\frac{\mathrm{ch}_2(v).H}{\mathrm{ch}_1(v).H^2}$.*
(c) *The $\nu_{b,w}$-(semi)stability of any $E\in\mathcal{D}(X)$ of class v is unchanged as (b,w) varies within any connected component (called a "chamber") of $U\setminus\bigcup_{i\in I}\ell_i$.*
(d) *For any wall $\ell_i\cap U$ there is a map $f\colon F\to E$ in $\mathcal{D}(X)$ such that*
   - *for any $(b,w)\in\ell_i\cap U$, the objects E, F lie in the heart $\mathcal{A}_b$,*
   - *E is $\nu_{b,w}$-semistable of class v with $\nu_{b,w}(E)=\nu_{b,w}(F)=\mathrm{slope}\,(\ell_i)$ constant on the wall $\ell_i\cap U$, and*
   - *f is an injection $F\hookrightarrow E$ in $\mathcal{A}_b$ which strictly destabilises E for (b,w) in one of the two chambers adjacent to the wall ℓ_i.* □

We can improve on this local finiteness by showing we have finiteness as $w\to\infty$. This gives, for each fixed $v\in K(X)$, a *large volume chamber* $\subseteq U$ in which there are no walls for v, so the $\nu_{b,w}$-(semi)stability of objects of class v is independent of $w\gg 0$. It is the subset of U above the uppermost walls described below, or all of U if there are no walls.

When $\mathrm{rank}(v)\neq 0$ we temporarily denote the half of U to the left or right of $\Pi(v)$ by

$$U_<:=\left\{(b,w)\in U:\ b<\tfrac{\mathrm{ch}_1(v).H^2}{\mathrm{ch}_0(v)H^3}\right\}\ \text{ and }\ U_>:=\left\{(b,w)\in U:\ b>\tfrac{\mathrm{ch}_1(v).H^2}{\mathrm{ch}_0(v)H^3}\right\}.$$

Proposition 1.4 (**Large volume chamber**). *Fix $v\in K(X)$ with $\Delta_H(v)\geq 0$.*

If $\mathrm{rank}(v)=0$ (respectively $\mathrm{rank}(v)\neq 0$) and there is a wall of instability for v in U (respectively $U_<$ or $U_>$), then there is an uppermost such wall.

Proof. By the results of Proposition 1.3 it is sufficient to show that for a fixed $b_0\in\mathbb{Q}$ not equal to $\frac{\mathrm{ch}_1(v).H^2}{\mathrm{ch}_0(v)H^3}$ there are only finitely many points (b_0,w) which lie on walls of instability for the class v.

A point (b_0,w) on a wall of instability for v gives a decomposition $v=v_1+v_2$ with

$$0\ \leq\ \mathrm{ch}_1^{b_0H}(v_i).H^2\ \leq\ \mathrm{ch}_1^{b_0H}(v).H^2\ \text{ and }\ 0\ \leq\ \Delta_H(v_i)\ <\ \Delta_H(v),\tag{14}$$

the first by (7) and the second by [BMS, Corollary 3.10] or [FT3, Lemma 3.2]. We will show this means there are only finitely many points $\big(\mathrm{ch}_0(v_i),\,\mathrm{ch}_1^{bH}(v_i).H^2,\,\mathrm{ch}_2^{bH}(v_i).H\big)\in\mathbb{Q}^3$ corresponding to such decompositions. There are therefore only finitely many $\Pi(v_i)\in\mathbb{Q}^2$, which by Proposition 1.3 means only finitely many walls, as required.

Since $b_0\in\frac{1}{N}\mathbb{Z}$ for some $N\in\mathbb{N}$, the bounds (14) imply there are only finitely many values $\mathrm{ch}_1^{b_0H}(v_i).H^2\in\frac{1}{N}\mathbb{Z}$ can take, and only finitely many integers $\Delta_H(v_i)$ can take.

Furthermore, rewriting Δ_H as

$$\Delta_H \;=\; \big(\mathrm{ch}_1^{b_0H}.H^2\big)^2 - 2\big(\mathrm{ch}_2^{b_0H}.H\big)\,\mathrm{ch}_0\,H^3,$$

we see that

$$(\mathrm{ch}_2^{b_0H}(v_i).H)\,\mathrm{ch}_0(v_i) \text{ takes only finitely many values.} \tag{15}$$

If $\mathrm{ch}_0(v_i)=0$ for $i=1,2$ then both slopes $\nu_{b,w}(v_i)$ are constant in w, contradicting the existence of a wall. So without loss of generality $\mathrm{ch}_0(v_1)\neq 0$, which by (15) and bounded denominators means that $\mathrm{ch}_2^{b_0H}(v_1).H$ takes only finitely many values. The same therefore also applies to

$$\mathrm{ch}_2^{b_0H}(v_2).H \;=\; \mathrm{ch}_2^{b_0H}(v).H - \mathrm{ch}_2^{b_0H}(v_1).H.$$

Finally if $\mathrm{ch}_2^{b_0H}(v_i)=0$ for $i=1,2$ then a simple calculation shows both $\Pi(v_i)$ lie on the line $w=b_0b-\frac{1}{2}b_0^2$. Therefore so does $\Pi(v)$, so by Proposition 1.3 this is the line of instability. But it is tangent to ∂U at $(b_0,\frac{1}{2}b_0^2)$, so does not pass through U. Thus $\mathrm{ch}_2^{b_0H}(v_1)\neq 0$ without loss of generality and by (15) there are only finitely many values for $\mathrm{ch}_0(v_1)$ and $\mathrm{ch}_0(v_2)=\mathrm{ch}_0(v)-\mathrm{ch}_0(v_1)$. □

In this paper, we always assume X satisfies the conjectural Bogomolov-Gieseker inequality of Bayer-Macrì-Toda [BMT]. In the form of [BMS, Conjecture 4.1], rephrased in terms of the rescaling [FT2, Equation 6], it is the following.

Conjecture 1.5 (Bogomolov-Gieseker inequality). *For any $(b,w)\in U$ and $\nu_{b,w}$-semistable $E\in\mathcal{D}(X)$, we have the inequality*

$$B_{b,w}(E) \;:=\; (2w-b^2)\Delta_H(E) + 4\big(\mathrm{ch}_2^{bH}(E).H\big)^2 - 6\big(\mathrm{ch}_1^{bH}(E).H^2\big)\,\mathrm{ch}_3^{bH}(E) \;\geq\; 0. \tag{16}$$

Multiplying out and cancelling we find that $B_{b,w}$ is actually linear in (b,w):

$$\tfrac{1}{2}B_{b,w}(E) \;=\; \big(C_1^2-2C_0C_2\big)w + \big(3C_0C_3-C_1C_2\big)b + (2C_2^2-3C_1C_3), \tag{17}$$

where $C_i := \mathrm{ch}_i(E).H^{3-i}$. The coefficient of w is ≥ 0 by (12). When it is >0 the Bogomolov-Gieseker inequality (16) says that E can be $\nu_{b,w}$-semistable only above the line $\ell_f(E)$ defined by the equation $B_{b,w}(E)=0$. When $\mathrm{ch}_0(E)\neq 0\neq \mathrm{ch}_1(E).H^2$ we can rearrange to see $\ell_f(E)$ is the line through the points $\Pi(E)$ and

$$\Pi'(E) \;:=\; \left(\frac{2\,\mathrm{ch}_2(E).H}{\mathrm{ch}_1(E).H^2}\,,\;\frac{3\,\mathrm{ch}_3(E)}{\mathrm{ch}_1(E).H^2}\right). \tag{18}$$

2. Walls

Throughout the whole of this Section we fix a rank 0 class $\mathsf{v}\in K(X)$ with

$$\mathrm{ch}_H(\mathsf{v}) \;=\; (0,\,c,\,s_0,\,d_0), \tag{19}$$

where $c>0$. Then we pick $n_0\gg 0$ as follows. There are finitely many explicit inequalities of the form $O\big(n_0^i\big)<O(n_0^{i+1})$ in this Section, starting at (24) and ending with (45). We fix

$$n_0 \;=\; n_0(c,s_0,d_0) \;\gg\; 0 \tag{20}$$

sufficiently large that all of these inequalities hold. Then set $v_{n_0} := \mathsf{v} - [\mathcal{O}_X(-n_0)]$, so

$$\mathrm{ch}_H(v_{n_0}) \;=\; \Big(-1,\; c+n_0,\; s_0-\tfrac{1}{2}n_0^2,\; d_0+\tfrac{1}{6}n_0^3\Big). \tag{21}$$

This Section studies stability conditions $\nu_{b,w}$ with b strictly to the *right* of $\Pi(v_{n_0})$, i.e.

$$b \;>\; -(c+n_0), \quad \text{so that} \quad \nu_{b,w}(v_{n_0}) \;<\; +\infty. \tag{22}$$

By (18) all walls for class v_{n_0} lie on or above the line $\ell_f = \ell_f(v_{n_0})$ which passes through

$$\Pi(v_{n_0}) = \Big(-(n_0+c),\; -s_0+\tfrac{1}{2}n_0^2\Big) \quad \text{and} \quad \Pi'(v_{n_0}) = \left(\frac{2s_0-n_0^2}{n_0+c},\; \frac{3d_0+\frac{1}{2}n_0^3}{n_0+c}\right).$$

The equation of ℓ_f then works out to be

$$4w \;=\; \left(-n_0+\frac{n_0(6s_0+c^2)+4(cs_0+3d)}{2cn_0+c^2+2s_0}\right)b + n_0^2 + \frac{n_0^2(6s_0-c^2)+4(3dn_0+3dc-2s_0^2)}{2cn_0+c^2+2s_0}\,. \tag{23}$$

Since $c \in \frac{1}{H^3}\mathbb{Z}$ this gives, for $n_0 \gg 0$,

$$\mathrm{slope}(\ell_f) \;>\; -\frac{n_0}{4} - |s_0|H^3. \tag{24}$$

Setting $w = \frac{1}{2}b^2$ in (23) gives a quadratic equation with roots the values $b_1^f < b_2^f$ of b at the two intersection points $\ell_f \cap \partial U$,

$$b_1^f \;=\; -n_0 + \tfrac{1}{3}c + O\big(\tfrac{1}{n_0}\big) \quad \text{and} \quad b_2^f \;=\; \tfrac{1}{2}n_0 - \tfrac{1}{12}c + \tfrac{3s_0}{2c} + O\big(\tfrac{1}{n_0}\big). \tag{25}$$

The large distance $\frac{3}{2}n_0+O(1)$ between these points enables us to restrict the form of certain $\nu_{b,w}$-semistable objects, such as those of class v_{n_0}, and their semistable factors. So fix data

- a $\nu_{b,w}$-semistable object $E \in \mathcal{A}_b$ of rank -1 and $\mathrm{ch}_1(E).H^2 \le (n_0+c)H^3$,
- a sequence $E' \hookrightarrow E \twoheadrightarrow E''$ in $\mathcal{A}_b$ with all objects of the same $\nu_{b,w}$-slope, where
- (b,w) lies to the right of $\Pi(E)$, and the line ℓ joining them is on or above $\ell_f \cap U$.[5]

We allow either of E', E'' to be 0, in which case we could take the other to be E of class v_{n_0}.

Lemma 2.1. *There is an ordering E_0, E_1 of E', E'' such that*

- *E_0 is a sheaf of rank 0; if it is nonzero then $\mathrm{ch}_1(E_0).H^2 > 0$,*
- *$\mathrm{rank}\big(\mathcal{H}^{-1}(E_1)\big) = 1$, $\mathrm{rank}\big(\mathcal{H}^0(E_1)\big) = 0$ and $\mathrm{ch}_1(E_1).H^2 \ge \big(n_0 - \frac{c}{3}\big)H^3$.*

Proof. Since $\mathrm{rank}(E) = -1$, one of the objects E', E'' has rank < 0; call it E_1. The other E_0 has rank ≥ 0. Set $r_i := \mathrm{rank}\big(\mathcal{H}^{-1}(E_i)\big) \ge 0$. Since ℓ lies on or above $\ell_f \cap U$, it lies inside U for b in the interval (b_1^f, b_2^f). Thus $E_i \in \mathcal{A}_b$ by Proposition 1.3. By (7) this gives

$$\mathrm{ch}_1\big(\mathcal{H}^{-1}(E_i)\big).H^2 \;\le\; \Big[-n_0+\tfrac{1}{3}c+O\big(\tfrac{1}{n_0}\big)\Big]r_iH^3 \tag{26}$$

$$\text{and} \quad \mathrm{ch}_1\big(\mathcal{H}^0(E_i)\big).H^2 \;\ge\; \Big[\tfrac{1}{2}n_0-\tfrac{1}{12}c+\tfrac{3s_0}{2c}+O\big(\tfrac{1}{n_0}\big)\Big](\mathrm{ch}_0(E_i)+r_i)H^3. \tag{27}$$

[5] Throughout the paper this phrase will mean that w is larger on ℓ than on ℓ_f whenever $b \in \big[b_1^f, b_2^f\big]$ (25).

In particular, subtracting gives

$$\mathrm{ch}_1(E_i).H^2 \;\geq\; \Big[\tfrac{1}{2}n_0 + O(1)\Big]\,\mathrm{ch}_0(E_i)H^3 + \Big[\tfrac{3}{2}n_0 + O(1)\Big]r_iH^3. \tag{28}$$

Adding over $i = 0, 1$ gives $\mathrm{ch}_1(E).H^2$, so

$$(n_0 + c)H^3 \;\geq\; -\Big[\tfrac{1}{2}n_0 + O(1)\Big]H^3 + \Big[\tfrac{3}{2}n_0 + O(1)\Big](r_0 + r_1)H^3. \tag{29}$$

Taking the coefficient of n_0 gives $1 \geq r_0 + r_1$. But $r_1 \geq 1$, so in fact $r_0 = 0$ and $r_1 = 1$. From the exact sequence

$$0 \to \mathcal{H}^{-1}(E') \longrightarrow \mathcal{H}^{-1}(E) \longrightarrow \mathcal{H}^{-1}(E'') \longrightarrow \mathcal{H}^0(E') \longrightarrow \mathcal{H}^0(E) \longrightarrow \mathcal{H}^0(E'') \to 0 \tag{30}$$

we deduce $\mathrm{rank}\big(\mathcal{H}^{-1}(E)\big) = 1$ while $\mathcal{H}^0(E_0)$, $\mathcal{H}^0(E_1)$, $\mathcal{H}^0(E)$ and $\mathcal{H}^{-1}(E_0)$ are all sheaves of rank 0. By the definition (7) of $\mathcal{A}_b$ the latter implies $\mathcal{H}^{-1}(E_0) = 0$, so E_0 is a sheaf of rank 0. Moreover $\nu_{b,w}(E_0) = \nu_{b,w}(E) < +\infty$, so $\mathrm{ch}_1(E_0).H^2 > 0$ if $E_0 \neq 0$.

Finally substituting $i = 1$, $r_i = 1$ and $\mathrm{ch}_0(E_1) + r_i = 0$ into (26, 27) gives

$$\mathrm{ch}_1(E_1).H^2 \;=\; \big[\mathrm{ch}_1\big(\mathcal{H}^0(E_1)\big) - \mathrm{ch}_1\big(\mathcal{H}^{-1}(E_1)\big)\big].H^2 \;>\; \big(n_0 - \tfrac{1}{3}c\big)H^3 + O\big(\tfrac{1}{n_0}\big).$$

Since n_0, $\mathrm{ch}_1(E_1).H^2 \in \mathbb{Z}$ and $c \in \frac{1}{H^3}\mathbb{Z}$, we conclude $\mathrm{ch}_1(E_1).H^2 \geq (n_0 - \frac{1}{3}c)H^3$. □

It turns out E_0 is already semistable in its large volume chamber.

Lemma 2.2. *There are no walls for E_0 which lie on or above $\ell_f \cap U$.*

Proof. Lemma 2.1 gives $\mathrm{ch}_1(E_1).H^2 \geq \big(n_0 - \frac{c}{3}\big)H^3$, so

$$\mathrm{ch}_1(E_0).H^3 \;\leq\; \tfrac{4}{3}cH^3. \tag{31}$$

Given a wall of instability $\ell_0 \cap U$ for E_0 which lies along or above $\ell_f \cap U$, the difference in b-values $b_1 < b_2$ of the points of $\ell_0 \cap \partial U$ satisfies $b_2 - b_1 \geq b_2^f - b_1^f = \frac{3}{2}n_0 + O(1)$ by (25). Repeating the arguments (26, 27, 28, 29) for destabilising factors F_i of E_0, (29) becomes

$$\mathrm{ch}_1(E_0).H^2 \;\geq\; b_2\,\mathrm{rank}(E_0)H^3 + (b_2 - b_1)(r_0 + r_1)H^3,$$

where $\mathrm{rank}(E_0) = 0$ and $r_i := \mathrm{rank}\,\big(\mathcal{H}^{-1}(F_i)\big)$. So by (31),

$$\tfrac{4}{3}cH^3 \;\geq\; \big(\tfrac{3}{2}n_0 + O(1)\big)(r_0 + r_1)H^3. \tag{32}$$

Thus $r_0 = 0 = r_1$ and both F_i have rank 0. But then their $\nu_{b,w}$ slopes (10) — and that of E_0 — are constant in (b, w), contradicting the existence of the wall ℓ_0. □

As we cross walls of instability for v_{n_0} above ℓ_f we will have to deal with semistable factors of various types, and their own wall crossing. One type will be classes

$$w_n \;=\; \mathrm{w} - [\mathcal{O}_X(-n)] \;\text{ which are close to }\; v_{n_0} \;=\; \mathsf{v} - [\mathcal{O}_X(-n_0)]$$

in an appropriate sense. In fact it is important to note we will allow fractional $n \in \frac{1}{H^3}\mathbb{Z}$ and $\mathrm{w} \in K(X) \otimes \mathbb{Q}$ so long as $w_n \in K(X)$ is integral.

Definition 2.3. We say that $w_n \in K(X)$ is *close to* v_{n_0} if

- $\Pi(w_n)$ lies on or above ℓ_f and

- $\mathrm{ch}_H(w_n) = (0,c,s,d) - \big(1, -n, \frac{1}{2}n^2, -\frac{1}{6}n^3\big)$,

where $n \in \frac{1}{H^3}\mathbb{Z}$ and s, d satisfy the following bounds.

- $n \in \big[n_0 - \frac{c}{3},\, n_0\big]$. That is, if we set $\delta_n := n_0 - n \in \frac{1}{H^3}\mathbb{Z}$ then $\delta_n \in \big[0, \frac{c}{3}\big]$.
- $-n_0\delta_n - |s_0| \leq s \leq -\frac{3}{4}n_0\delta_n + \big(c + |s_0|H^3\big)\delta_n + s_0$.
- $d \geq \frac{15}{32}n_0^2\delta_n - n_0\delta_n\big(|s_0|H^3 + c\big) - \delta_n\big(s_0H^3\big)^2 + d_0$.

In particular setting $\delta_n = 0$ shows that v_{n_0} is close to v_{n_0}, which is good.

For such a class we let $\ell(w_n)$ be the line parallel to ℓ_f through the point $\Pi(w_n)$ as in Figure 2. We will only ever need to understand walls of instability inside the region

$$U(w_n) \;:=\; \Big\{(b,w) \in U \;:\; (b,w) \text{ is on or above } \ell(w_n) \text{ and to the right of } \Pi(w_n)\Big\}.$$

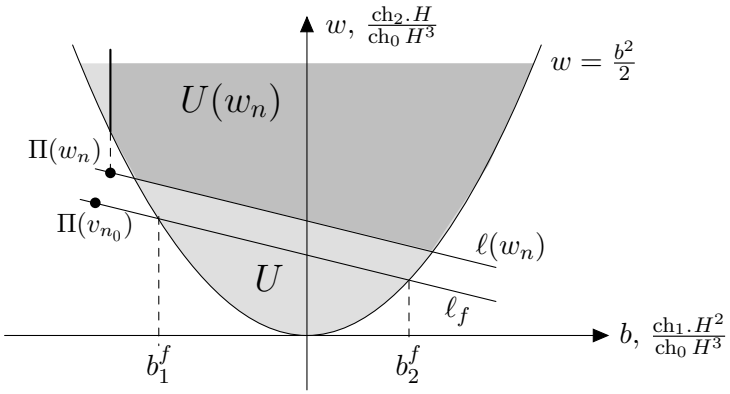


FIGURE 2. The line $\ell(w_n)$ and the region $U(w_n) \subset U$

We now analyse the possible wall crossing of objects of class v_{n_0} and, more generally, classes w_n close to v_{n_0}. So for the rest of this Section we fix the following data,

- a class w_n close to v_{n_0} in the sense of Definition 2.3,
- a weak stability condition $(b,w) \in U(w_n)$,
- a $\nu_{b,w}$-semistable object E of class w_n, and
- a $\nu_{b,w}$-destabilising sequence $E' \hookrightarrow E \twoheadrightarrow E''$ in $\mathcal{A}_b$.

We then let ℓ denote the line through (b,w) and $\Pi(w_n)$. Since $\mathrm{ch}_1(w_n).H^2 = (n+c)H^3 \leq (n_0+c)H^3$ we may apply Lemma 2.1. This renames E', E'' (in some order) as E_0, E_1 of ranks $0, -1$ respectively.

Since ℓ lies on or above $\ell_f \cap U$, and all possible walls for E_0 are parallel to ℓ, we conclude from Lemma 2.2 that there are no walls for E_0 on or above $\ell \cap U$.

We write the classes of the destabilising factors E_0, E_1 as

$$\mathrm{ch}_H(E_0) \;=\; \big(0,\, c-c',\, s-s',\, d-d'\big), \quad \mathrm{ch}_H(E_1) \;=\; \big(-1,\, n+c',\, -\tfrac{1}{2}n^2+s',\, \tfrac{1}{6}n^3+d'\big), \tag{33}$$

for some $c', s', d' \in \mathbb{Q}$ with $c - c' > 0$. The key result in this paper is the following.

Proposition 2.4. *Either* $c' \in (0,c)$ *or* $c'=0$ *and* $\delta_n = 0$.

Proof. Assume $c' \leq 0$. By Lemma 2.1, $n+c' \geq n_0 - \frac{1}{3}c$, so we get the bounds

$$-\delta_n + c' \;\geq\; -\tfrac{1}{3}c \quad \text{and} \quad c' \;\in\; \left[-\tfrac{1}{3}c,\, 0\right] \cap \tfrac{1}{H^3}\mathbb{Z}. \tag{34}$$

We can also bound s'. By Proposition 1.3 the wall ℓ passes through $\Pi(E_1)$ and $\Pi(w_n)$, so has equation

$$w - \tfrac{1}{2}n^2 + s \;=\; \frac{s-s'}{c-c'}(b+n+c). \tag{35}$$

Since ℓ lies above or on $\ell(w_m)$, which has slope $-\frac{n_0}{4}+O(1)$ by (23), we get the bound

$$s' \;\leq\; s + (c-c')\left(\frac{n_0}{4} + O(1)\right). \tag{36}$$

The Bogomolov inequality (12) for E_1 is $0 \leq \Delta_H(E_1) = (n+c')^2 + 2\big(s' - \frac{1}{2}n^2\big)$, which is

$$s' \;\geq\; -\big(n + \tfrac{1}{2}c'\big)c' \;\overset{(34)}{\geq}\; 0.$$

Combined with (36), (34) and the bound $s \leq s_0$ from Definition 2.3 this bounds

$$0 \;\leq\; s' \;\leq\; \tfrac{1}{3}cn_0 + O(1). \tag{37}$$

Next we apply the Bogomolov-Gieseker inequality to $\mathrm{ch}_H(E_0)$ and $\mathrm{ch}_H(E_1)$ (33). First consider E_0. By Lemma 2.1 it is a *sheaf of rank* 0 which is $\nu_{b,w}$-semistable on $\ell \cap U$, so [FT3, Lemma B.3] applies. There we showed it is $\nu_{b,w}$-semistable at a judicious point (b,w) of U where the Bogomolov-Gieseker inequality (16) bounds its ch_3 by

$$d - d' \;\leq\; \frac{(s-s')^2}{2(c-c')} + \frac{1}{24}(c-c')^3 \tag{38}$$

$$\overset{(36)}{\leq}\; \frac{(c-c')}{2}\left(-\frac{n_0}{4}\right)^2 + O(1) \;=\; \frac{(c-c')}{32}n_0^2 + O(1). \tag{39}$$

Next we do the same for $\mathrm{ch}_H(E_1)$ at the point

$$b \;=\; 0, \quad w \;=\; \frac{n^2}{2} - s + (n+c)\frac{s-s'}{c-c'}$$

of the line ℓ (35). Since ℓ is above $\ell_f \cap U$ this lies in U. Arranged in powers of n (17) gives

$$\frac{n^3}{2}c' + n^2\left(\frac{c'^2}{2} + 2c'\frac{s-s'}{c-c'} - s'\right) + n\left(-2c's + \frac{s-s'}{c-c'}(2c'c + c'^2 + 2s')\right) + 2s'^2 +$$

$$(c'^2 + 2s')\left(\frac{sc'-cs'}{c-c'}\right) \;\geq\; 3d'(n+c') \;\overset{(38)}{\geq}\; 3(n+c')\left[d - \frac{(s-s')^2}{2(c-c')} - \frac{1}{24}(c-c')^3\right]. \tag{40}$$

By (34), (37) and the bounds on n, s, d in Definition 2.3, each term above is $\leq O(n_0^3)$. Discarding $O(n_0^2)$ terms shows the following is nonnegative,

$$\frac{n_0^3}{2}c' + n_0^2\left(-s' + 2c'\frac{s-s'}{c-c'}\right) + 2n_0 s'\frac{s-s'}{c-c'} - 3n_0 d + \frac{3n_0}{2}\frac{(s-s')^2}{c-c'}$$

$$= \frac{n_0^3}{2}c' + \frac{n_0}{c-c'}\Big[-n_0 c's' - n_0 cs' - ss' + 2n_0 c's - \tfrac{1}{2}(s')^2 + \tfrac{3}{2}s^2 - 3dc + 3dc'\Big]. \tag{41}$$

By $-s \leq n_0\delta_n + |s_0|$ from Definition 2.3, the first three terms in the square bracket are

$$-n_0 c's' - n_0 cs' - ss' \overset{(37)}{\leq} n_0 s'\big(-c' - c + \delta_n + O\big(\tfrac{1}{n_0}\big)\big) \overset{(34)}{\leq} \big(-\tfrac{2}{3}n_0 c + O(1)\big)s' \overset{(37)}{\leq} 0,$$

so removing them — and the fifth bracketed term $-\frac{1}{2}(s')^2 \leq 0$ — from (41) we deduce

$$\tfrac{1}{2}c'n_0^3 + \frac{n_0}{c-c'}\Big[2n_0 sc' + \tfrac{3}{2}s^2 - 3dc + 3dc'\Big] \;\geq\; 0. \tag{42}$$

Since $d \geq \frac{1}{3}n_0^2\delta_n + O(n_0)$ by Definition 2.3, the coefficient of c' inside the bracket is

$$2n_0 s + 3d \;\geq\; -2n_0^2\delta_n + n_0^2\delta_n + O(n_0) \;=\; -n_0^2\delta_n + O(n_0).$$

Using this and $0 \geq c' \geq \delta_n - \frac{1}{3}c$ (34) in the inequality (42), and discarding $O(n_0^2)$ terms,

$$\begin{aligned} 0 \;&\leq\; \tfrac{1}{2}c'n_0^3 + \frac{n_0}{c-c'}\Big[-n_0^2\delta_n\big(\delta_n - \tfrac{1}{3}c\big) + \tfrac{3}{2}n_0^2\delta_n^2 - cn_0^2\delta_n\Big] \\ &=\; \tfrac{1}{2}c'n_0^3 + \frac{n_0^3\delta_n}{c-c'}\Big[\tfrac{1}{2}\delta_n - \tfrac{2}{3}c\Big]. \end{aligned}$$

By Definition 2.3, $\delta_n \leq \frac{c}{3}$ so the term in square brackets is < 0. Thus both terms are ≤ 0 and the only way the inequality can hold is if $c' = 0 = \delta_n$. □

Proposition 2.5. *If $w_n = v_{n_0}$ and $c' = 0$ then $E_1 \cong T(-n_0)[1]$ for some $T \in \mathrm{Pic}_0(X)$ (4).*

Proof. When $c' = 0$ and $w_n = v_{n_0}$ the inequality (40) becomes

$$-s'n_0^2 + \frac{2n_0 s'(s_0 - s')}{c} \;\geq\; 3n_0\left(d_0 - \frac{(s_0-s')^2}{2c} - \frac{1}{24}c^3\right).$$

Discarding terms of $O(n_0)$, this becomes

$$n_0 s'\left(-n_0 - \frac{s'}{2c} - \frac{s_0}{c}\right) \;\geq\; O(n_0).$$

Since $s' \geq 0$ (37) this forces $s' = 0$, which is $2\Delta_H(E_1) = 0$. By [BMS, Corollary 3.10] or [FT3, Lemma 3.2] this implies there is no wall for E_1 in U. Thus E_1 is $\nu_{b,w}$-semistable for any $(b, w) \in U$, so we may apply [FT3, Lemma 2.6] to conclude it is $T(-n_0)[1]$ for some $T \in \mathrm{Pic}_0(X)$. □

The first case of the next result will allow us to do an induction on $\mathrm{ch}_1.H^2$, which decreases on passing from E to E_1 since $\mathrm{ch}_1(E_1).H^2 < \mathrm{ch}_1(w_n).H^2$ by (33). (We will deal with the second case afterwards.) Let $b_1 < b_2$ be the b-values of the intersection points $\ell \cap \partial U$.

Proposition 2.6. *If $c' > 0$, then either*

- *$(b, w) \in U([E_1])$ and $[E_1]$ is close to v_{n_0}, or*
- *$\mathrm{ch}_1(E_1).H^2 + b_1 H^3 < \min(c,\, b_2 - b_1)H^3$. This always holds when $c' < \delta_n + \frac{2}{3}c$.*

Proof. We consider two cases depending on the size of c'.

Case (i): $c' < \delta_n + \frac{2}{3}c$. Since $c', \delta_n, c \in \frac{1}{H^3}\mathbb{Z}$ this is $c' \leq \delta_n + \frac{2}{3}c - \frac{1}{3H^3}$, so

$$n + c' \;=\; n_0 - \delta_n + c' \;\leq\; n_0 + \tfrac{2}{3}c - \tfrac{1}{3H^3}. \tag{43}$$

Since $(b, w) \in U(w_n)$ lies on or above ℓ_f by Definition 2.3, we have

$$b_1 \;\leq\; b_1^f \;=\; -n_0 + \tfrac{1}{3}c + O\big(\tfrac{1}{n_0}\big) \quad\text{and}\quad b_2 - b_1 \;\geq\; b_2^f - b_1^f \;=\; \tfrac{3}{2}n_0 + O(1) \tag{44}$$

by (25). Combining the first with (43) gives

$$\tfrac{1}{H^3}\,\mathrm{ch}_1(E_1).H^2 + b_1 \;=\; n + c' + b_1 \;\leq\; n_0 + \tfrac{2}{3}c - \tfrac{1}{3H^3} - n_0 + \tfrac{1}{3}c + O\big(\tfrac{1}{n_0}\big) \;<\; c. \tag{45}$$

Since (44) shows $c < b_2 - b_1$, this means $\mathrm{ch}_1(E_1).H^2 + b_1 H^3 < \min(c,\, b_2 - b_1)H^3$.

Case (ii): $c' \geq \delta_n + \frac{2}{3}c$. Since $\delta_n \geq 0$ and $c - c' > 0$ this gives $c' \in \big[\frac{2}{3}c, c\big)$. Let $\ell_{\mathrm{JS}}(w_n)$ denote the *Joyce-Song wall* for w_n connecting

$$\Pi(w_n) \;=\; \big(-n - c, -s + \tfrac{1}{2}n^2\big) \;\text{ to }\; \Pi\big(\mathcal{O}_X(-n)\big) \;=\; \big(-n, \tfrac{1}{2}n^2\big). \tag{46}$$

This is easily calculated to intersect ∂U at $b_1' = -n$ and $b_2' = n + 2\frac{s}{c}$, so that $\mathrm{ch}_1(w_n).H^2 + b_1' H^3 = cH^3 = \min(c, b_2' - b_1')H^3$. If ℓ lies strictly above ℓ_{JS} it follows that

$$\mathrm{ch}_1(w_n).H^2 + b_1 H^3 \;<\; \mathrm{ch}_1(w_n).H^2 + b_1' H^3 \;=\; \min(c, b_2' - b_1')H^3 \;\leq\; \min(c, b_2 - b_1)H^3,$$

giving the second case of the Proposition.

So we now assume ℓ lies on or below $\ell_{\mathrm{JS}}(w_n)$ and try to show that $[E_1]$ is close to v_{n_0} by rewriting $\mathrm{ch}_H(E_1)$ as

$$\mathrm{ch}_H(E_1) \;=\; \big(0, c, \tilde{s}, \tilde{d}\big) - \mathrm{ch}_H\big(\mathcal{O}_X(-n')\big) \;=\; \Big(-1,\; n' + c,\; -\tfrac{1}{2}n'^2 + \tilde{s},\; \tfrac{1}{6}n'^3 + \tilde{d}\Big),$$

where $n' := n - c + c' \in \frac{1}{H^3}\mathbb{Z}$ and $\tilde{s}, \tilde{d} \in \mathbb{Q}$ are defined by

$$-\tfrac{1}{2}(n - c + c')^2 + \tilde{s} \;=\; -\tfrac{1}{2}n^2 + s' \quad\Longrightarrow\quad \tilde{s} \;=\; s' - (c - c')(n_0 - \delta_n) + \tfrac{1}{2}(c - c')^2, \tag{47}$$

$$\tfrac{1}{6}(n - c + c')^3 + \tilde{d} \;=\; \tfrac{1}{6}n^3 + d' \quad\Longrightarrow\quad \tilde{d} \;=\; d' + \tfrac{1}{2}n^2(c - c') - \tfrac{1}{2}n(c - c')^2 + \tfrac{1}{6}(c - c')^3. \tag{48}$$

Setting $\delta_n' := n_0 - n' = \delta_n + c - c' \in \frac{1}{H^3}\mathbb{Z}$, the inequality $c' \geq \delta_n + \frac{2}{3}c$ gives $\delta_n' \in \big(0, \frac{c}{3}\big]$. This is one of the bounds required by Definition 2.3; we need to show that $\tilde{s}, \tilde{d}$ satisfy the other two.

By (24) the slopes of both ℓ_f and $\ell(w_n)$ of Figure 2 are $> -\frac{1}{4}n_0 - |s_0|H^3$. Since ℓ lies between $\ell(w_n)$ and $\ell_{\mathrm{JS}}(w_n)$ (46), and all three pass through $\Pi(w_n)$, their gradients are ordered

$$-\frac{n_0}{4} - |s_0|H^3 \;<\; \frac{s - s'}{c - c'} \;\leq\; \frac{s}{c}, \tag{49}$$

which implies

$$\frac{c'}{c}s \;\leq\; s' \;<\; \left(\tfrac{1}{4}n_0 + |s_0|H^3\right)(c-c') + s. \tag{50}$$

Together with (47) and $s \geq -n_0\delta_n - |s_0|$ from Definition 2.3, this bounds $\tilde{s}$ below by

$$\begin{aligned}\tilde{s} &\geq \frac{c'}{c}s - (c-c')(n_0-\delta_n) + \tfrac{1}{2}(c-c')^2\\ &> \left(-n_0\delta_n - |s_0|\right) - (c-c')n_0\\ &= -n_0\delta_n' - |s_0|.\end{aligned}$$

Similarly using (47), (50) and $s \leq -\frac{3}{4}n_0\delta_n + \left(c + |s_0|H^3\right)\delta_n + s_0$ from Definition 2.3, we bound $\tilde{s}$ above by

$$\begin{aligned}\tilde{s} &< \left(\tfrac{1}{4}n_0 + |s_0|H^3\right)(c-c') + s - (c-c')(n_0-\delta_n) + \tfrac{1}{2}(c-c')^2\\ &= -\tfrac{3}{4}n_0(c-c') + (c-c')\left(|s_0|H^3 + \delta_n + \tfrac{1}{2}(c-c')\right) + s\\ &< -\tfrac{3}{4}n_0(c-c') + (c-c')\left(|s_0|H^3 + c\right) - \tfrac{3}{4}n_0\delta_n + \delta_n(c + |s_0|H^3) + s_0\\ &= -\tfrac{3}{4}n_0\delta_n' + \delta_n'\left(|s_0|H^3 + c\right) + s_0,\end{aligned}$$

where in passing from the second line to the third we used that δ_n and $c-c'$ both lie in $\left[0, \frac{c}{3}\right]$, so $\delta_n + \frac{1}{2}(c-c') < c$. Thus $\tilde{s}$ satisfies Definition 2.3.

Now we turn to $\tilde{d}$. Combining (49) with the upper bound $s \leq s_0$ of Definition 2.3 gives $\left(\frac{s-s'}{c-c'}\right)^2 \leq \left(-\frac{1}{4}n_0 - |s_0|H^3\right)^2$. With (38) this gives

$$d' \;\geq\; d - \tfrac{1}{2}(c-c')\left(\tfrac{1}{4}n_0 + |s_0|H^3\right)^2 - \tfrac{1}{24}(c-c')^3.$$

Together with (48) this gives

$$\tilde{d} \geq d - \tfrac{1}{2}(c-c')\left(\tfrac{1}{4}n_0 + |s_0|H^3\right)^2 - \tfrac{1}{24}(c-c')^3 + \tfrac{1}{2}(c-c')n^2 - \tfrac{1}{2}(c-c')^2 n + \tfrac{1}{6}(c-c')^3.$$

Substituting $n = n_0 - \delta_n$ and recalling that $\delta_n + \frac{1}{2}(c-c') < c$,

$$\begin{aligned}\tilde{d} &\geq d - \tfrac{1}{2}(c-c')\left(\tfrac{1}{4}n_0 + |s_0|H^3\right)^2 + \tfrac{1}{2}(c-c')n_0^2 - \delta_n(c-c')n_0 - \tfrac{1}{2}(c-c')^2 n_0\\ &> d + \tfrac{15}{32}(c-c')n_0^2 - n_0(c-c')\left(\tfrac{1}{4}|s_0|H^3 + \delta_n + \tfrac{1}{2}(c-c')\right) - \tfrac{1}{2}(c-c')(s_0H^3)^2\\ &> d + \tfrac{15}{32}(c-c')n_0^2 - n_0(c-c')\left(|s_0|H^3 + c\right) - (c-c')(s_0H^3)^2.\end{aligned}$$

Comparing with $d \geq \frac{15}{32}n_0^2\delta_n - n_0\delta_n\left(|s_0|H^3 + c\right) - \delta_n\left(s_0H^3\right)^2 + d_0$ from Definition 2.3 and using $\delta_n' = \delta_n + c - c'$ this finally gives

$$\tilde{d} \;>\; \frac{15}{32}n_0^2\delta_n' - n_0\delta_n'\left(|s_0|H^3 + c\right) - \delta_n'\left(s_0H^3\right)^2 + d_0.$$

Thus $\tilde{d}$ satisfies Definition 2.3. Finally since the slope of ℓ is $\geq$ that of ℓ_f it is also $\geq$ that of $\ell([E_1])$. So (b,w) lies on or above $\ell([E_1])$ and therefore in $U([E_1])$ as claimed. □

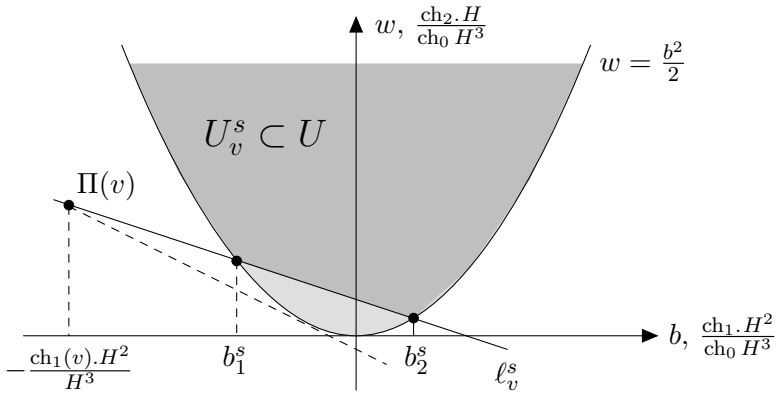


FIGURE 3. The safe area $U_v^s \subset U$ of a rank -1 class v with $\Delta_H(v) > 0$

We can phrase the second possibility of Proposition 2.6 by saying that (b, w) lies in E_1's *safe area* $U^s_{[E_1]} \subset U$ defined as follows (see also Figure 3).

Let $v \in K(X)$ be any class of rank -1 and $\Delta_H(v) \geq 0$. There is a unique line passing through $\Pi(v)$ which is tangent to $\partial U = \left\{w = \frac{1}{2}b^2\right\}$ at a point to the right of $\Pi(v)$ (or at $\Pi(v)$ if $\Delta_H(v) = 0$). Rotating the line anticlockwise about $\Pi(v)$, it intersects ∂U in two points with b-values $b_1^s \leq b_2^s$ such that $b_2^s - b_1^s$ moves monotonically through $[0, \infty)$, while the horizontal distance $b_1^s + \mathrm{ch}_1(v).H^2/H^3$ from b_1^s to $\Pi(v)$ decreases to 0. Thus there is a unique line ℓ_v^s passing through $\Pi(v)$ such that

$$\mathrm{ch}_1(v).H^2 + b_1^s H^3 \ = \ \min(c,\, b_2^s - b_1^s)H^3. \tag{51}$$

For lines through $\Pi(v)$ strictly above ℓ_v^s, with b-values $b_1' < b_2'$ at the two intersections points with ∂U, the equality is replaced by the inequality

$$\mathrm{ch}_1(v).H^2 + b_1' H^3 \ < \ \min(cH^3,\, b_2' - b_1')H^3, \tag{52}$$

while below we get the opposite inequality $>$. A similar definition was used in [FT3, Section 3] for rank ≥ 0 classes. Here we include the extra cH^3 term in order to work above the Joyce-Song wall $\ell_{\mathrm{JS}}(w_n)$ for w_n, which we observed in (46) satisfies (51), so that

$$\ell_{\mathrm{JS}}(w_n) \ = \ \ell^s_{w_n}.$$

Definition 2.7. For a class $v \in K(X)$ of rank -1 with $\Delta_H(v) \geq 0$ the safe area $U_v^s \subset U$ is the set of (b, w) which lie strictly above ℓ_v^s and to the right of $\Pi(v)$, i.e. $bH^3 > -\mathrm{ch}_1(v).H^2$.

If we manage to wall cross into the safe area, we stay there, in the following sense.

Lemma 2.8. *Suppose* $\mathrm{rank}(v) = -1$, $(b, w) \in U_v^s$ *and* E *is* $\nu_{b,w}$*-semistable of class* v*. Then any* $\nu_{b,w}$*-destabilising sequence* $E' \hookrightarrow E \twoheadrightarrow E''$ *satisfies*

- *one of the factors* E_0 *is a rank zero sheaf with* $\mathrm{ch}_1(E_0).H^2 \in (0, cH^3)$*, and*
- *the other factor* E_1 *has rank* -1 *with* (b, w) *inside its safe area* $U^s_{[E_1]}$*.*

Proof. The proof is similar to that of Lemma 2.1. One of the objects E', E'' has rank < 0; call it E_1. The other E_0 has rank ≥ 0. Let $r_i := \operatorname{rank}\big(\mathcal{H}^{-1}(E_i)\big) \geq 0$. Then since $E_i \in \mathcal{A}_b$ for $b \in [b_1^s, b_2^s]$,

$$\begin{aligned} \operatorname{ch}_1\big(\mathcal{H}^{-1}(E_i)\big).H^2 &\leq b_1^s r_i H^3 \\ \text{and} \quad \operatorname{ch}_1\big(\mathcal{H}^0(E_i)\big).H^2 &\geq b_2^s(\operatorname{ch}_0(E_i) + r_i)H^3, \end{aligned}$$

with the first inequality strict for $i = 1$. Subtracting gives the inequality

$$\operatorname{ch}_1(E_i).H^2 \;\geq\; b_2^s \operatorname{ch}_0(E_i)H^3 + (b_2^s - b_1^s)r_i H^3, \tag{53}$$

which is again strict when $i = 1$. Adding over $i = 0, 1$ implies

$$\operatorname{ch}_1(E).H^2 - b_2^s \operatorname{ch}_0(E)H^3 \;>\; (b_2^s - b_1^s)(r_0 + r_1)H^3. \tag{54}$$

Since $\operatorname{ch}_0(E) = -1$ the left hand side is

$$\operatorname{ch}_1(E).H^2 + b_1^s H^3 + (b_2^s - b_1^s)H^3 \overset{(52)}{<} 2(b_2^s - b_1^s)H^3,$$

so by (54) this gives $r_0 + r_1 < 2$. Therefore we can repeat the argument (30) to show that E_0 is a sheaf of rank zero and E_1 is a complex of rank -1 with rank zero $\mathcal{H}^0(E_1)$.

Thus $\operatorname{ch}_1(E_0).H^2 \geq 0$, and it cannot be zero since $\nu_{b,w}(E_0) = \nu_{b,w}(E) < +\infty$. So

$$\operatorname{ch}_1(E_0).H^2 \;>\; 0 \quad \text{and} \quad \operatorname{ch}_1(E_1).H^2 \;<\; \operatorname{ch}_1(E).H^2. \tag{55}$$

The line through $\Pi(E)$ and (b, w) passes through the safe area $U^s_{[E]}$ so satisfies (52). Hence by the second inequality of (55) it also satisfies (52) for $v = [E_1]$. Since it also goes through $\Pi(E_1)$, we conclude that it lies in the safe area $U^s_{[E_1]}$.

Finally $\operatorname{ch}_1^{b_1^s H}(E).H^2 \leq cH^3$ by (51) and $\operatorname{ch}_1^{b_1^s H}(E_1).H^2 > 0$ by (53), so

$$\operatorname{ch}_1(E_0).H^2 \;=\; \operatorname{ch}_1^{b_1^s H}(E_0).H^2 \;=\; \operatorname{ch}_1^{b_1^s H}(E).H^2 - \operatorname{ch}_1^{b_1^s H}(E_1).H^2 \;<\; cH^3. \qquad \square$$

2.1. **Summary.** We summarise what we have proved so far. (1) and (2) are the content of Lemma 2.1, (3) is Proposition 2.5, (4) is Proposition 2.4 and (5) is Proposition 2.6.

(1) Any $\nu_{b,w}$-semistable object E of class v_{n_0} is a 2-term complex with $\operatorname{rank}\big(\mathcal{H}^{-1}(E)\big) = 1$ and $\operatorname{rank}\big(\mathcal{H}^0(E)\big) = 0$.
(2) If E is destabilised on a wall ℓ the possible semistable factors are a rank -1 complex E_1 with $\operatorname{ch}_1(E_1).H^2 < \operatorname{ch}_1(E).H^2$, and rank 0 sheaves E_0.
(3) If $\operatorname{ch}_1(E_0).H^2 = cH^3$ then our wall is $\ell_{\mathrm{JS}}(v_{n_0})$ (46) and $E_1 \cong T(-n_0)[1]$.
(4) Otherwise $\operatorname{ch}_1(E_0).H^2 \in (0, cH^3)$.
(5) As we increase w, moving above ℓ, the above results (1), (2), (4) for E also hold for E_1 (and its rank -1 destabilising factors, etc). This is because either
   (a) $[E_1]$ is close to v_{n_0} in the sense of Definition 2.3 and $(b, w) \in U([E_1])$ by Figure 4, so the results for E apply to E_1 immediately, or
   (b) (b, w) is in the safe area for E_1, where the results follow from Lemma 2.8.

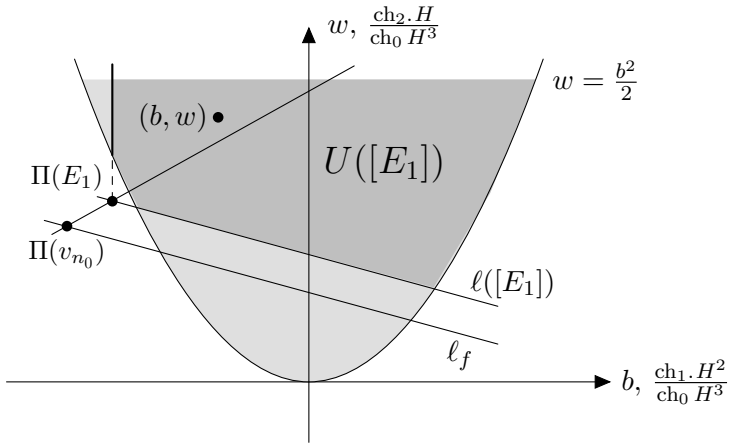


FIGURE 4. Diagram showing why $(b,w)\in U([E_1])$ in 5(a) of the Summary

In particular $\mathrm{ch}_1.H^2$ drops any time we cross a wall and pass from a rank -1 complex to a rank -1 semistable factor (and then cross further walls for that, and pass to its rank -1 semistable factors, etc). This process terminates in finitely many steps because $\mathrm{ch}_1.H^2$ cannot drop below $n_0-\frac{c}{3}$ by Lemma 2.1.

We note in passing that, with a little further effort, the results of this Section can be used to prove that on crossing a wall, those semistable objects which become unstable have Harder-Narasimhan filtrations of length 2 only. The semistable factors have ranks 0 and -1. However the rank 0 factor E_0 can be strictly semistable, with further decompositions into semistable factors, all with $\mathrm{ch}_H^{\leq 2}$ proportional to $\mathrm{ch}_H^{\leq 2}(E_0)$. All of these factors enter into the wall crossing formula, which therefore does not simplify much. So we omit the details of this result and do not use it.

2.2. **The quintic threefold.** Let $X\subset\mathbb{P}^4$ be a smooth quintic 3-fold. Chunyi Li [Li1, Theoreom 2.8] has proved the Bogomolov-Gieseker inequality (16) for a restricted subset of weak stability conditions on X: those (b,w) satisfying

$$w\ >\ \tfrac{1}{2}b^2+\tfrac{1}{2}\big(b-\lfloor b\rfloor\big)\big(\lfloor b\rfloor+1-b\big). \tag{56}$$

In particular Conjecture 1.5 holds for any $w>\frac{1}{2}b^2$ when $b\in\mathbb{Z}$.

We check that knowing this restricted form of the Bogomolov-Gieseker inequality (16) is sufficient for our purposes. In this Section we have applied (16) to

(a) objects E of class v_{n_0} to find ℓ_f (23), and
(b) the rank -1 destabilising objects E_1 of an object of class w_n close to v_{n_0} in (40).

Now $(b,w)\in\ell_f\cap U$ in case (a), while in (b) we worked at a point $(b,w)\in U(w_m)$ which is therefore above $\ell_f\cap U$. But by (25) we know b takes an integer value along $\ell_f\cap U$, so by (56) we can apply (16) at $b\in\mathbb{Z}$ to prove the same results in cases (a) and (b).

The only other place we applied (16) is in (38), where we invoked [FT2, Lemma B.3] for E_0. Setting $v = [E_0]$, the proof of that Lemma applied (16) along a line ℓ_v which intersects ∂U at points $a_v < b_v$ with $b_v - a_v = \mathrm{ch}_1(v).H^2/H^3$. This is ≥ 1 because $\mathrm{Pic}(X) = \mathbb{Z}.H$, so b takes an integer value on the closure of $\ell_v \cap U$, so again (56) holds at a point of $\ell_v \cap U$.

3. Wall crossing

We now assume the Calabi-Yau condition $K_X \cong \mathcal{O}_X$. If $h^1(\mathcal{O}_X) > 0$ the $\mathrm{Jac}(X)$ action on moduli spaces of sheaves forces $\mathsf{J}(v) = 0$ whenever $\mathrm{rank}(v) > 0$, so Theorem 1 already holds. So in this Section we further assume that $H^1(\mathcal{O}_X) = 0$ and prove Theorem 2 for the class v (19) of rank 0 and dimension 2, by applying wall crossing to the class v_{n_0} of (21).

Wall crossing formula. The work of Joyce-Song [JS] can be used to define generalised DT invariants counting $\nu_{b,w}$-semistable objects of class $v \in K(X)$ with $\nu_{b,w}(v) < +\infty$,

$$\mathsf{J}_{b,w}(v) \ \in \ \mathbb{Q}. \tag{57}$$

In [FT3, Section 4 and Appendix C] we described these invariants and showed they are well defined so long as $\nu_{b,w}(v) < +\infty$. When $w \gg 0$, so that (b, w) lies in v's large volume chamber of Proposition 1.4, we denote (57) by $\mathsf{J}_{b,\infty}(v)$.

In [FT3, Section 4] we also showed the Joyce-Song wall crossing formula applies to the $\mathsf{J}_{b,w}(v)$ under the same $\nu_{b,w}(v) < +\infty$ condition.

Suppose ℓ is the line through $\Pi(v)$ and a point $(b, w_0) \in U$ such that $\nu_{b,w_0}(v) < +\infty$.[6] By the local finiteness of walls of Proposition 1.3 we may choose

$$(b, w_\pm) \in U \text{ just above and below the wall } \ell, \tag{58}$$

in the sense that $(b, w_\pm) \notin \ell$ and between (b, w_-) and (b, w_+) there are no walls for v, *nor any of its finitely many semistable factors*, except for ℓ. Then the wall crossing formula is

$$\mathsf{J}_{b,w_-}(v) \ = \ \mathsf{J}_{b,w_+}(v) \ + \sum_{\substack{m\geq 2,\ \alpha_1,\ldots,\alpha_m \in C(\mathcal{A}_b),\\ \sum_{i=1}^m \alpha_i = v,\ \nu_{b,w_0}(\alpha_i) = \nu_{b,w_0}(v)\,\forall i}} C_{+,-}(\alpha_1,\ldots,\alpha_m) \prod_{i=1}^m \mathsf{J}_{b,w_+}(\alpha_i). \tag{59}$$

Here $C(\mathcal{A}_b)$ is the positive cone (9), and the $C_{+,-}(\alpha_1,\ldots,\alpha_n) \in \mathbb{Q}$ are universal coefficients depending only on the Mukai pairings $\chi(\alpha_i, \alpha_j)$ and the relative sizes of the set of slopes $\{\nu_{b,w_\pm}(\alpha_i)\}$. The term

$$C_{+,-}(\alpha_1,\ldots,\alpha_m) \prod_{i=1}^m \mathsf{J}_{b,w_+}(\alpha_i)$$

is zero unless there is a ν_{b,w_0}-semistable object E of class v with ν_{b,w_0}-semistable factors of classes $\alpha_1,\ldots,\alpha_m$. (Even most of these terms vanish — although (59) is a countable sum, only finitely many terms in it are nonzero.) The formula reflects the different Harder-Narasimhan filtrations of E on the two sides of the wall, and then further filtrations of the

[6] If $\mathrm{rank}(v) = 0$ and $\mathrm{ch}_1(v).H^2 > 0$ then ℓ is the line through (b, w_0) of gradient $\mathrm{ch}_2(v).H/\mathrm{ch}_1(v).H^2$.

semistable Harder-Narasimhan factors by semi-destabilising subobjects. A similar formula holds if we swap (b, w_-) with (b, w_+).

The coefficients $C_{+,-}$ are given by complicated formulae. The only explicit expression we will need is that when $m = 2$, $\nu_{b,w_0}(\alpha_1) = \nu_{b,w_0}(\alpha_2)$, $\nu_{b,w_+}(\alpha_1) > \nu_{b,w_+}(\alpha_2)$ and $\nu_{b,w_-}(\alpha_1) < \nu_{b,w_-}(\alpha_2)$ then

$$C_{+,-}(\alpha_1, \alpha_2) + C_{+,-}(\alpha_2, \alpha_1) \;=\; (-1)^{\chi(\alpha_1,\alpha_2)-1}\chi(\alpha_1, \alpha_2). \tag{60}$$

For economy of notation we incorporate the sign into the Mukai pairing by setting

$$\bar{\chi}(v, w) \;:=\; (-1)^{\chi(v,w)-1}\chi(v, w), \qquad \bar{\chi}(w(d)) \;:=\; (-1)^{\chi(w(d))-1}\chi(w(d)),$$

for $v, w \in K(X)$. The second is the special case of the first where $v = [\mathcal{O}_X(-d)]$.

The base case. We are now ready to apply the wall crossing formula to the class v_{n_0} (21). We will induct on the value of c, so we begin by fixing any class v (19) with

$$\tfrac{1}{H^3}\operatorname{ch}_1(\mathsf{v}).H^2 \;=\; c_{\min} \;\in\; \tfrac{1}{H^3}\mathbb{N},$$

where $c_{\min}$ is the minimal value of $D.H^2/H^3$ over effective divisors $D \neq 0$.

We choose $n_0 \gg 0$ and always work with $(b, w) \in U$ to the right of $\Pi(v_{n_0})$ — i.e. with $b > -(c_{\min} + n_0)$ — to ensure that $\nu_{b,w}(v_{n_0}) < +\infty$. For (b, w) below ℓ_f (23) all objects of class v_{n_0} are unstable. For (b, w) on or above $\ell_f \cap U$ we have $(b, w) \in U(v_{n_0})$, where we have the analysis of possible destabilising sequences in Summary 2.1. The rank 0 factor E_0 has $\operatorname{ch}_1(E_0).H^2 = (c - c')H^3$, where $c' \in [0, c_{\min})$. Thus $c' = 0$ and the other factor E_1 is $T(-n_0)[1]$ for some $T \in \operatorname{Pic}_0(X)$.

Thus we are on the Joyce-Song wall $\ell_{\mathrm{JS}}(v_{n_0})$, and this is the *only wall* in U for v_{n_0}. Let $(b, w_\pm)$ be points just above and below $\ell_{\mathrm{JS}}(v_{n_0})$. Since c is minimal E_0 is *stable* with no semistable factors. So using (60) the wall crossing formula simplifies to

$$\mathsf{J}_{b,w_-}(v_{n_0}) \;=\; \mathsf{J}_{b,w_+}(v_{n_0}) - \bar{\chi}(v(n_0)) \cdot \mathsf{J}_{b,w_+}\big[T(-n)[1]\big] \cdot \mathsf{J}_{b,w_+}(\mathsf{v}).$$

Here $\mathsf{J}_{b,w_-}(v_{n_0}) = 0$ and $\mathsf{J}_{b,w_+}\big[T(-n)[1]\big] = \#H^2(X, \mathbb{Z})_{\mathrm{tors}}$ is the invariant counting line bundles $T \in \operatorname{Pic}_0(X)$. There are no further walls for v_{n_0}, and there are no walls at all in $U(v_n)$ for v by Lemma 2.2, so the formula simplifies further to give

$$\mathsf{J}_{\infty}(v_{n_0}) \;=\; \bar{\chi}(\mathsf{v}(n_0)) \cdot \#H^2(X, \mathbb{Z})_{\mathrm{tors}} \cdot \mathsf{J}_{b,\infty}(\mathsf{v}). \tag{61}$$

As before $J_{b,\infty}$ denotes $J_{b,w}$ for $w \gg 0$, while from now on $J_\infty(\alpha)$, for a rank -1 class α, will always denote $J_{b,\infty}(\alpha)$ for $b > \mu_H(\alpha)$ — i.e. for b to the right of $\Pi(\alpha)$. Rearranging (61) proves a special case of the following.

Theorem 3.1. *For any rank 0 class $\mathsf{v} \in K(X)$ with $\operatorname{ch}_1(\mathsf{v}).H^2 > 0$ there is a universal formula $\mathsf{J}_{b,\infty}(\mathsf{v}) = F\big(\mathsf{J}_\infty(\alpha_1), \mathsf{J}_\infty(\alpha_2), \dots\big)$ with all α_i of rank -1.*

Here F is a polynomial with countably many terms, but only finitely many of them nonzero. It may change from line to line — we only care that its coefficients depend on just the cohomology and Chern classes of X.

The induction step. We will prove Theorem 3.1 by an ascending induction on $c = \mathrm{ch}_1(\mathsf{v}).H^2/H^3$. The universal formula (61) proves the base case $c = c_{\min}$. So we can inductively assume that we have proved Theorem 3.1 for all rank 0 classes with $\mathrm{ch}_1.H^2 < cH^3$ and next prove it for a fixed class v with $\mathrm{ch}_1(\mathsf{v}).H^2 = cH^3$.

So we choose $n_0 \gg 0$ as in (20), set $v_{n_0} = \mathsf{v} - [\mathcal{O}_X(-n_0)]$ and take (b, w) to the right of $\Pi(v_{n_0})$. As we wall cross for the class v_{n_0} we will produce rank -1 classes of smaller $\mathrm{ch}_1.H^2$, which we can also express in terms of invariants $\mathsf{J}_\infty(\beta_i)$ with $\mathrm{rank}(\beta_i) = -1$ by a separate induction, as follows.

Proposition 3.2. *Fix a class β of rank -1 with $\mathrm{ch}_1(\beta).H^2/H^3 \in \left[n_0 - \frac{c}{3},\, n_0 + c\right)$ and a point $(b, w) \in U$ to the right of $\Pi(\beta)$, on or above ℓ_f, not[7] on a wall of instability for β. Suppose that either*

(a) $(b, w) \in U^s_\beta$ is in the safe area of β, or
(b) β is close to v_{n_0} and $(b, w) \in U([\beta])$.

Then there is a universal formula $\mathsf{J}_{b,w}(\beta) = F\big(\mathsf{J}_\infty(\beta_1), \mathsf{J}_\infty(\beta_2), \dots\big)$ with $\mathrm{rank}(\beta_i) = -1$.

Proof. We prove this by an ascending induction on $\mathrm{ch}_1(\beta).H^2/H^3$. If it takes the minimal value $\geq n_0 - \frac{c}{3}$ attained by integral classes then there is no wall for β on or above ℓ_f by Lemma 2.1, so the claim is true and the induction starts.

The claim is trivial when (b, w) lies in the large volume chamber for β of Proposition 1.4. We take this as the base case of a descending induction on the walls ℓ for β above ℓ_f. Since there are only finitely many of them, we may assume the claim holds at a point (b, w_+) just above ℓ, in the sense of (58). Then we need only deduce from this, and the wall crossing formula (59), that it also holds at a point (b, w_-) just below ℓ.

We analyse the formula (59) for $v = \beta$. For each term of the sum, Proposition 2.6 and Lemma 2.8 show that precisely one α_i has rank -1; it then satisfies either (a) or (b) above with $\mathrm{ch}_1(\alpha_i).H^2 < \mathrm{ch}_1(\beta).H^2$, so by our induction on $\mathrm{ch}_1(\beta).H^2/H^3$ the invariant $\mathsf{J}_{b,w_+}(\alpha_i)$ can be written as universal expressions in invariants J_∞ of rank -1 classes.

The other terms α_j in (59) are all of rank zero with $\mathrm{ch}_1(\alpha_j).H^2 < cH^3$. They satisfy $\mathsf{J}_{b,w_+}(\alpha_i) = \mathsf{J}_{b,\infty}(\alpha_i)$ by Lemma 2.2. Thus our induction on c writes these as universal expressions in invariants J_∞ of rank -1 classes too. □

Starting with (b, w) just below ℓ_f — the lower boundary of $U(v_{n_0})$ — all objects of class v_{n_0} are strictly unstable and we have the universal formula

$$\mathsf{J}_{b,w}\big(v_{n_0}\big) \;=\; 0. \tag{62}$$

Now we move upwards towards the large volume chamber $w \gg 0$ for v_{n_0}, crossing finitely many walls $\ell \cap U$ for v_{n_0} as we go and modifying the formula accordingly.

Let (b, w_-), (b, w_0), (b, w_+) denote points just below, on and above ℓ in the sense of (58). By induction on the walls we may assume that we have a universal formula for $\mathsf{J}_{b,w_-}(v_{n_0})$

[7] This condition is not strictly necessary; on changing the universal constants $C_{+,-}$ the wall "crossing" formula (59) still holds when one of $(b, w_\pm)$ lies *on* the wall. Since we only care about crossing walls to reach the large volume chamber we do not concern ourselves with the values of the invariants $\mathsf{J}_{b,w}$ on walls.

in terms of invariants J_∞ of rank -1 classes, plus the extra term

$$\bar{\chi}(\mathsf{v}(n_0)) \cdot \#H^2(X,\mathbb{Z})_{\mathrm{tors}} \cdot \mathsf{J}_{b,\infty}(\mathsf{v}) \tag{63}$$

from (61) *if and only if* (b,w_-) *is above* $\ell_{\mathrm{JS}}(v_{n_0})$. The induction step is to use the wall crossing formula across ℓ — with (b,w_+) and (b,w_-) swapped in (59) — to deduce a similar formula for $\mathsf{J}_{b,w_+}(v_{n_0})$.

Since ℓ is on or above ℓ_f we have $(b,w_0) \in U(v_{n_0})$, so Summary 2.1 applies. We conclude that in each term of the sum on the right hand side of (59) precisely one α_i is of rank -1 and satisfies either

(i) condition (a) or (b) of Proposition 3.2, or
(ii) $[\alpha_i] = \big[\mathcal{O}_X(-n)[1]\big]$.

The other α_j are all of rank zero with $\mathrm{ch}_1(\alpha_j).H^2 \leq cH^3$. Equality can only hold in case (ii); then $\ell = \ell_{\mathrm{JS}}$, the corresponding term in (59) has $m=2$ and the terms α_1, α_2 are $\mathcal{O}_X(-n)$ and v in some order, giving the contribution

$$\bar{\chi}(\mathsf{v}(n_0)) \cdot \#H^2(X,\mathbb{Z})_{\mathrm{tors}} \cdot \mathsf{J}_{b,\infty}(\mathsf{v})$$

of (63). Thus every other term J_{b,w_-} in the sum can be written in terms of invariants J_∞ of rank -1 classes by Proposition 3.2 and the induction assumption. Therefore our universal formula for $\mathsf{J}_{b,w_-}(v_{n_0})$ induces one for $\mathsf{J}_{b,w_+}(v_{n_0})$, with the extra term (63) if (b,w_0) is on ℓ_{JS}. This completes the induction on the walls, allowing us to move to the large volume chamber for v_{n_0}, where the universal formula now takes the form

$$\mathsf{J}_\infty(v_{n_0}) \;=\; F\big(\mathsf{J}_\infty(\alpha_1), \mathsf{J}_\infty(\alpha_1), \dots\big) + \bar{\chi}(\mathsf{v}(n_0)) \cdot \#H^2(X,\mathbb{Z})_{\mathrm{tors}} \cdot \mathsf{J}_{b,\infty}(\mathsf{v}),$$

for some rank -1 classes α_i. Rearranging proves Theorem 3.1 for our class v and therefore, by induction, all v.

Proof of Theorem 1. As explained in the Introduction, to prove Theorem 1 it is sufficient to prove Theorem 2 for rank 0 classes $\mathsf{v} \in K(X)$ with $\mathrm{ch}_1(\mathsf{v}).H^2 > 0$. For these we now have Theorem 3.1.

By results of Toda [To3] explained in Appendix A, the invariants $\mathsf{J}_\infty(\alpha_i)$ in Theorem 3.1 are stable pair invariants up to multiplying by the factor $\#H^2(X,\mathbb{Z})_{\mathrm{tors}}$. In turn we can replace these by DT invariants $\mathsf{J}(\beta_j)$ counting ideal sheaves[8] in rank 1 classes β_j, by the wall crossing formula [Br3, Theorem 1.1]. This proves a version of Theorem 1 for v, but with $\mathsf{J}(\mathsf{v})$ replaced by $\mathsf{J}_{b,\infty}(\mathsf{v})$ on the left hand side,

$$\mathsf{J}_{b,\infty}(\mathsf{v}) \;=\; F\big(\mathsf{J}(\beta_1), \mathsf{J}(\beta_2), \dots\big). \tag{64}$$

Finally we wall cross from large volume (or "tilt") stability to Gieseker stability for v. This was already carried out in [FT3, Sections 4 and 5]; we briefly review the details.

For any $\alpha \in K(X)$ we denote its Hilbert polynomial by

$$P_\alpha(t) \;:=\; \chi(\alpha(t)) \;=\; a_d t^d + a_{d-1}t^{d-1} + \cdots + a_0,$$

[8]These ideal sheaves are all Gieseker stable, and — after tensoring by elements of $\mathrm{Pic}_0(X)$ — account for all Gieseker semistable sheaves in the same class.

for $d \leq 3$ such that $a_d \neq 0$. From this we define the reduced Hilbert polynomial $p_\alpha(t)$ and its truncation $\widetilde{p}_\alpha(t)$ by

$$p_\alpha(t) \;:=\; \tfrac{1}{a_d}P(t) \quad \text{and} \quad \widetilde{p}_\alpha(t) \;:=\; p_\alpha(t) - \tfrac{a_0}{a_d} \;=\; t^d + \cdots + \tfrac{a_1}{a_d}t.$$

There is a total order $\prec$ on $\{\text{monic polynomials}\} \sqcup \{0\}$ defined by $p \prec q$ if and only if

(i) $\deg p > \deg q$, or
(ii) $\deg p = \deg q$ and $p(t) < q(t)$ for $t \gg 0$.

(The strange direction of the inequality (i) ensures that sheaves of lower dimension have *greater* slope. Our convention is that $\deg 0 = 0$, so 0-dimensional classes α have $\widetilde{p}_\alpha = 0 \succ \widetilde{p}_\beta$ whenever β is not of dimension 0.) Then $E \in \mathrm{Coh}(X)$ is called *Gieseker* (respectively *tilt*) (semi)stable if for all non-trivial exact sequences $0 \to A \to E \to B \to 0$ in $\mathrm{Coh}(X)$ we have

$$p_{[A]} \;(\preceq)\; p_{[B]} \quad \big(\text{respectively } \widetilde{p}_{[A]} \;(\preceq)\; \widetilde{p}_{[B]}\big). \tag{65}$$

Here $(\preceq)$ means $\prec$ for stability and $\preceq$ for semistability. In particular (i) ensures that Gieseker and tilt semistable sheaves are pure.

Fix a class $\alpha \in K(X)$ with $\mathrm{rank}(\alpha) > 0$, or $\mathrm{rank}(\alpha) = 0$ and $\mathrm{ch}_1(\alpha).H^2 > 0$. Then an argument from [Br2, Proposition 14.2] shows that an object $E \in \mathcal{A}_b$ of class α is $\nu_{b,w}$-(semi)stable in the large volume chamber (i.e. for $w \gg 0$) if and only if $E \in \mathrm{Coh}(X)$ is a tilt (semi)stable sheaf. So the invariants counting tilt semistable sheaves are the $\mathsf{J}_{b,\infty}(\alpha)$. Joyce-Song define an invariant $\mathsf{J}(\alpha) \in \mathbb{Q}$ [JS, Definition 5.15] counting Gieseker semistable sheaves of class α; when there are no strictly semistables this reproduces the integer invariant defined in [Th].

So we can work in $\mathrm{Coh}(X)$ to pass from the invariants $\mathsf{J}_{b,\infty}(\alpha)$ counting tilt semistable sheaves to the invariants $\mathsf{J}(\alpha)$ counting Gieseker semistable sheaves. Since tilt stability dominates Gieseker stablity in the sense of [JS, Definition 3.12] we get a wall crossing formula from [JS, Theorems 3.14 and 5.14]. It takes the form

$$\mathsf{J}(\alpha) \;=\; \mathsf{J}_{b,\infty}(\alpha) + \sum_{\substack{m\geq 2,\ \alpha_1,\ldots,\alpha_m \in C(\mathcal{A}),\\ \sum_{i=1}^m \alpha_i = \alpha,\ \widetilde{p}_{\alpha_i}(t) = \widetilde{p}_\alpha(t)\ \forall i}} C(\alpha_1,\ldots,\alpha_m) \prod_{i=1}^m \mathsf{J}_{b,\infty}(\alpha_i); \tag{66}$$

see [FT3, Equation 78]. The universal coefficients $C(\alpha_1,\ldots,\alpha_m)$ depend only on the Mukai pairings $\chi(\alpha_i,\alpha_j)$ and the $\prec$-ordering of the reduced Hilbert polynomials $\{p_{\alpha_i}(t)\}_{i=1}^m$.

The definition (65) of $\widetilde{p}(t)$ ensures that if $\mathrm{rank}(\alpha) = 0$, $\mathrm{ch}_1(\alpha).H^2 > 0$ then the same is true of the α_i that appear in (66). Therefore we can substitute $\alpha = \mathsf{v}$ into (66) and replace all the terms $\mathsf{J}_{b,\infty}(\alpha)$, $\mathsf{J}_{b,\infty}(\alpha_i)$ on the right hand side by their expressions (64) in terms of rank 1 DT invariants $\mathsf{J}(\beta_i)$. This proves Theorem 1.

4. Dimension one sheaves

We again take $K_X = \mathcal{O}_X$ and $H^1(\mathcal{O}_X) = 0$. In this Section we prove that our technique of using Joyce-Song stable pairs and wall crossing works for dimension 1 classes as well, proving Theorem 2 in this case.

This is just for completeness, as Toda gives a much better, completely explicit formula for the DT invariants $\mathsf{J}(0,0,\beta,m)$ in terms of stable pair invariants in [To4, Lemma 3.15].

Fix $n \in \mathbb{Z}$ and work in the full subcategory

$$\mathcal{A} \;:=\; \Big\langle \mathcal{O}_X(-n),\; \mathrm{Coh}_{\leq 1}(X)[-1] \Big\rangle_{\mathrm{ext}} \;\subset\; \mathcal{D}(X),$$

which is proved in [To1, Lemma 3.5] to be an abelian category. Letting $K(\mathcal{A})$ denote its numerical Grothendieck group, define the map

$$\begin{aligned} \mathrm{cl}\colon K(\mathcal{A}) \;&\longrightarrow\; \mathbb{Z}^{\oplus 3}, \\ \big[\mathcal{O}_X(-n)^{\oplus r}\big] + \big[F[-1]\big] \;&\longmapsto\; \big(r,\, \mathrm{ch}_2(F).H,\, \mathrm{ch}_3(F)\big) \;=\; (r,c,s), \end{aligned}$$

for $F \in \mathrm{Coh}_{\leq 1}(X)$. Equivalently, for $E \in \mathcal{A} \subset \mathcal{D}(X)$,

$$\mathrm{cl}(E) \;=\; \Big(\mathrm{ch}_0(E),\; -\mathrm{ch}_2(E).H + \tfrac{1}{2}n^2H^3\,\mathrm{ch}_0(E),\; -\mathrm{ch}_3(E) - \tfrac{1}{6}n^3H^3\,\mathrm{ch}_0(E)\Big).$$

Toda [To1] constructs a one dimensional family of weak stability conditions ν_θ on $\mathcal{A}$, parameterised by $\theta \in \mathbb{R}$. The slope function $\nu_\theta\colon K(\mathcal{A}) \to (-\infty,+\infty]$ factors through $\mathbb{Z}^3$ as

$$\nu_\theta(F) \;=\; \nu_\theta(r,c,s) \;:=\; \begin{cases} \theta & \text{if } r \neq 0, \\ \frac{s}{c} & \text{if } r = 0,\; c \neq 0, \\ +\infty & \text{if } r = c = 0. \end{cases}$$

As usual we say $E \in \mathcal{A}$ is ν_θ-(semi)stable if and only if for any nontrivial short exact sequence $A \hookrightarrow E \twoheadrightarrow B$ in $\mathcal{A}$, we have

$$\nu_\theta(A)\; (\leq)\; \nu_\theta(B).$$

Here $(\leq)$ means $\leq$ for semistability and $<$ for stability. By [To2, Section 5.3], $\{\nu_\theta\}_{\theta\in\mathbb{R}}$ is a continuous family of weak stability conditions satisfying the support property.

The $\mathcal{A}$-subobjects of $F[-1] \in \mathrm{Coh}_{\leq 1}(X)$ are the subsheaves of F (shifted by $[-1]$), and the ν_θ-slope s/c of a dimension 1 sheaf is, up to scale, the constant term of its reduced Hilbert polynomial. Therefore, for $F \in \mathrm{Coh}_{\leq 1}(X) \subset \mathcal{A}$, the following are equivalent,

- the ν_θ-(semi)stability of $F[-1]$ for one fixed θ,
- the ν_θ-(semi)stability of $F[-1]$ for all $\theta \in \mathbb{R}$,
- the Gieseker (semi)stability of F.

We now fix $c \in \mathbb{Z}_{>0},\; s \in \mathbb{Z}$ and consider rank 1 objects $E \in \mathcal{A}$ with class

$$\mathrm{cl}(E) \;=\; (1,c,s).$$

Since we have a one parameter family of stability conditions $\{\nu_\theta\}_{\theta\in\mathbb{R}}$, the Joyce-Song wall for the class $(1,c,s)$ is now a single point — the stability condition $\nu_{\mathrm{JS}} = \nu_{\theta_{\mathrm{JS}}}$ for which $[E]$ and $[E]-[\mathcal{O}_X(-n)]$ have the same slope. It is

$$\theta_{\mathrm{JS}} \;=\; \frac{s}{c} \quad \text{so that} \quad \nu_{\mathrm{JS}}(1,c,s) \;=\; \nu_{\mathrm{JS}}(0,c,s) \;=\; \nu_{\mathrm{JS}}(1,0,0) \;=\; \theta_{\mathrm{JS}}.$$

Lemma 4.1. *There are only finitely many walls $\theta_{\mathrm{JS}} = \theta_0 < \theta_1 < \cdots < \theta_k < +\infty$ to the right of θ_{JS} on which objects E of class $(1,c,s)$ can be strictly ν_θ-semistable. For θ in any chamber (θ_i,θ_{i+1}) the semistable objects do not change and are all stable.*

Proof. Note that the classes $(r,c,s)\in\mathbb{Z}^3$ of objects of $\mathcal{A}$ have $r,c\geq 0$, and if $c=0$ then $s\geq 0$, because the same is true in $\mathrm{Coh}_{\leq 1}(X)$. Note also that if an object E of class $(1,c,s)$ is strictly ν_θ-semistable then it has a destabilising $\mathcal{A}$-short exact sequence $E'\hookrightarrow E\twoheadrightarrow E''$ with the ranks of E', E'' being 0 and 1 in some order. We call them E_0, E_1, where $\operatorname{rank}E_i=i$, and set

$$\mathrm{cl}(E_0)\;=\;(0,\,c_0,\,s_0)\quad\text{so that}\quad\mathrm{cl}(E_1)\;=\;(1,\,c-c_0,\,s-s_0).$$

Since $\nu_\theta(E_0)=\nu_\theta(E)=\theta<+\infty$ we have $c_0>0$.

Since $\theta>\theta_{\mathrm{JS}}$ is to the right of the Joyce-Song wall,

$$\nu_\theta(E_0)\;=\;\frac{s_0}{c_0}\;=\;\theta\;>\;\frac{s}{c}\,. \tag{67}$$

But $c-c_0\geq 0$, and if $c-c_0=0$ then $s-s_0\geq 0$. The latter contradicts (67), so

$$0\;<\;c_0\;<\;c. \tag{68}$$

Thus c_0 can take only finitely many values, while $s_0\in\mathbb{Z}$ is bounded below by (67). If we can find an upper bound for s_0 this will prove the finiteness of the possible destabilising classes $\mathrm{cl}(E_0)=(0,c_0,s_0)\in\mathbb{Z}^3$ and walls $\theta=\frac{s_0}{c_0}$ to the right of θ_{JS}.

Since $E_1\in\mathcal{A}$ it is an iterated extension of objects in $\langle\mathcal{O}_X(-n)\rangle$ and $\mathrm{Coh}_{\leq 1}(X)[-1]$. Since it has rank 1 we use precisely one term $\mathcal{O}_X(-n)$ from the former, so collecting the other terms presents E_1 as a (non-unique) $\mathcal{A}$-extension

$$E_1'\hookrightarrow E_1\twoheadrightarrow F[-1]\quad\text{where}\quad F'[-1]\hookrightarrow E_1'\twoheadrightarrow\mathcal{O}_X(-n)$$

for some $F,F'\in\mathrm{Coh}_{\leq 1}(X)$. That is, E_1 is an $\mathcal{A}$-extension

$$\big[\mathcal{O}_X(-n)\xrightarrow{s}F'\big]\hookrightarrow E_1\twoheadrightarrow F[-1]. \tag{69}$$

Taking the long exact sequence of cohomology sheaves $\mathcal{H}^i$ shows that $\mathcal{H}^0(E_1)$ is the kernel of $\mathcal{O}_X(-n)\to F'$ and $\mathcal{H}^1(E_1)$ is an extension of F and the cokernel of $\mathcal{O}_X(-n)\to F'$. Thus $\mathcal{H}^0(E_1)$ and $\mathcal{H}^1(E_1)[-1]$ lie in $\mathcal{A}$ and the following is an $\mathcal{A}$-short exact sequence,

$$\mathcal{H}^0(E_1)\hookrightarrow E_1\twoheadrightarrow\mathcal{H}^1(E_1)[-1].$$

In other words *we may assume s is onto in* (69). Then writing $\mathrm{cl}(F)=(0,c',s')$, we have

$$0\;\leq\;c'\;=\;c-c_0-\mathrm{ch}_2(F').H\;\leq\;c-c_0. \tag{70}$$

If $c'>0$ then the ν_θ-semistability of E_1 gives

$$\frac{s'}{c'}\;\geq\;\nu_\theta(E_1)\;=\;\theta\;>\;\frac{s}{c}\,,$$

while if $c'=0$ then $s'\geq 0$. So in either case we find $s'\geq\max\left(0,\frac{s}{c}\right)$ and therefore

$$s-s_0\;=\;s'+\mathrm{ch}_3(F')\;\geq\;\max\left(0,\tfrac{s}{c}\right)+\mathrm{ch}_3(F'). \tag{71}$$

Quotients F' of a fixed sheaf $\mathcal{O}_X(-n)$ with bounded $\mathrm{ch}_2(F').H$ (70) have $\mathrm{ch}_3(F')$ bounded below; see [HL, Lemma 1.7.9] for instance. Thus (71) bounds s_0 above, as required. □

We note for later that these walls θ_i are also to the right of the Joyce-Song wall $\theta_{\mathrm{JS}}(E_1) = \frac{s-s_0}{c-c_0}$ of the rank 1 semistable factor E_1. This follows from (67) and the see-saw inequality,

$$\frac{s_0}{c_0} \;=\; \theta \;>\; \frac{s}{c} \;>\; \frac{s-s_0}{c-c_0}\,. \tag{72}$$

We now pick $n \gg 0$ so that $H^1(F(n)) = 0$ for any Gieseker semistable sheaf

$$F \in \mathrm{Coh}_{\leq 1}(X) \;\text{ with }\; 0 \leq \mathrm{ch}_2(F).H \leq c \;\text{ and }\; \frac{\mathrm{ch}_3(F)}{\mathrm{ch}_2(F).H} \geq \theta_{\mathrm{JS}} - 1. \tag{73}$$

Theorem 4.2. *Consider ν_θ-semistable objects E of class $(1, c, s)$.*

- *For $\theta \in (\theta_{\mathrm{JS}} - 1, \theta_{\mathrm{JS}})$ there are none,*
- *For $\theta \in [\theta_{\mathrm{JS}}, \infty)$ they are complexes $E = [\mathcal{O}_X(-n) \to F]$ for some $F \in \mathrm{Coh}_{\leq 1}(X)$,*
- *For $\theta \gg 0$ the complex $E(n)$ is a stable pair, and all stable pairs arise in this way.*

That is, F is pure and $\dim \mathcal{H}^1(E) = 0$ when $\theta \gg 0$.

Proof. Take $\theta > \theta_{\mathrm{JS}} - 1$ and write any ν_θ-semistable object E of class $(1, c, s)$ in the form

$$\big[\mathcal{O}_X(-n) \xrightarrow{s_0} F'\big] \hookrightarrow E \twoheadrightarrow F''[-1] \tag{74}$$

as in (69), with $F', F'' \in \mathrm{Coh}_{\leq 1}(X)$. Let $F'' \twoheadrightarrow F^{\min}$ be the the final Gieseker semistable quotient of its Harder-Narasimhan filtration — i.e. the quotient sheaf of F'' of minimum $\mathrm{ch}_3/\mathrm{ch}_2.H$ slope. Thus $F^{\min}[-1]$ is the $\mathcal{A}$-quotient of $F''[-1]$ of minimal ν_θ-slope, and is ν_θ-semistable for any $\theta \in \mathbb{R}$.

The semistability of E and its $\mathcal{A}$-quotient $E \twoheadrightarrow F^{\min}[-1]$ give the slope inequality

$$\theta_{\mathrm{JS}} - 1 \;<\; \theta \;\leq\; \frac{\mathrm{ch}_3(F^{\min})}{\mathrm{ch}_2(F^{\min}).H} \;\leq\; \frac{\mathrm{ch}_3(F^i)}{\mathrm{ch}_2(F^i).H}\,,$$

for all Harder-Narasimhan factors F^i of F''. Therefore (73) gives $H^1(F^i(n)) = 0$ for all i, so $H^1(F''(n)) = 0 = \mathrm{Ext}^2(F'', \mathcal{O}_X(-n))$ by Serre duality. Thus $b \circ a = 0$ in the following diagram of exact triangles, enabling us to pick a (not necessarily unique) lift c of b and take (co)cones to complete it to

$$\begin{array}{ccccc}
F''[-2] & \longrightarrow & F'[-1] & \longrightarrow & F[-1] \\
\| & & \downarrow & & \downarrow \\
F''[-2] & \xrightarrow{a} & \big[\mathcal{O}_X(-n) \xrightarrow{s_0} F'\big] & \longrightarrow & E \\
 & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\
 & & \mathcal{O}_X(-n) & = & \mathcal{O}_X(-n).
\end{array}$$

Since F is an extension of F' and F'' it lies in $\mathrm{Coh}_{\leq 1}(X)$, so the right hand column now shows any ν_θ-semistable object E can be written $\mathcal{O}_X(-n) \xrightarrow{s} F$ when $\theta > \theta_{\mathrm{JS}} - 1$. But for $\theta \in (\theta_{\mathrm{JS}} - 1, \theta_{\mathrm{JS}})$ the $\mathcal{A}$-injection $F[-1] \hookrightarrow \big[\mathcal{O}_X(-n) \to F\big] = E$ destabilises E because

$$\nu_\theta\big(F[-1]\big) \;=\; \frac{s}{c} \;>\; \theta \;=\; \nu_\theta(E).$$

Finally take $\theta > \theta_k$ beyond the final wall for $(1, c, s)$. The $\mathcal{A}$-surjection $E \twoheadrightarrow \mathcal{H}^{-1}(E) = \operatorname{coker} s[-1]$ and the ν_θ-semistability of E gives

$$\theta \;=\; \nu_\theta(E) \;\leq\; \frac{\operatorname{ch}_3(\operatorname{coker} s)}{\operatorname{ch}_2(\operatorname{coker} s).H}\,.$$

The lack of further walls means this holds for all $\theta > \theta_k$, so we conclude that $\operatorname{ch}_2(\operatorname{coker} s).H$ vanishes, so $\operatorname{coker} s$ has dimension 0. Furthermore any dimension 0 subsheaf $G \hookrightarrow F$ would define a destabilising $\mathcal{A}$-injection $G[-1] \hookrightarrow E$, so F is also pure and $E(n)$ is a stable pair.

Conversely we claim that twisting any stable pair by $\mathcal{O}_X(-n)$ gives a ν_θ-semistable object for $\theta > \theta_k$. So fix $E = \big[\mathcal{O}_X(-n) \xrightarrow{s} F\big]$ of class $(1, c, s)$ with F pure and $\dim \operatorname{coker} s = 0$, and consider a ν_θ-destabilising $\mathcal{A}$-exact sequence

$$A \hookrightarrow E \twoheadrightarrow B. \tag{75}$$

If $\operatorname{rank}(A) = 1$ then $\operatorname{rank}(B) = 0$ so $B = \mathcal{H}^1(B)[-1] \in \operatorname{Coh}_{\leq 1}(X)[-1]$. The long exact sequence of cohomology sheaves of (75) shows $\mathcal{H}^1(B)$ is a quotient of $\operatorname{coker} s$ and hence 0-dimensional. Thus $\nu_\theta(B) = +\infty > \theta = \nu_\theta(E)$ is not destabilising.

If $\operatorname{rank}(B) = 1$ then $A \in \operatorname{Coh}_{\leq 1}(X)[-1]$ and its slope $\nu_\theta(A)$ is constant in θ. But there are no walls for E beyond θ_k, so (75) destabilises for all $0 \ll \theta = \nu_\theta(E) \leq \nu_\theta(A)$. It follows that $\nu_\theta(A) = +\infty$ and $A = Q[-1]$ for some sheaf Q of dimension 0. Hence $\operatorname{Hom}(Q[-1], \mathcal{O}_X(-n)) = 0$ and

$$\begin{array}{ccccc} & & Q[-1] & & \\ & \overset{\exists}{\swarrow} & \downarrow & \searrow^{0} & \\ F[-1] & \longrightarrow & E & \longrightarrow & \mathcal{O}_X(-n) \end{array}$$

defines a nonzero map $Q[-1] \to F[-1]$, contradicting the purity of F. □

Wall crossing. By now the wall crossing argument, to express the one dimensional generalised DT invariants $\mathsf{J}(0, 0, \beta, m)$ in terms of universal formulae of stable pair invariants, is familiar. We give a brief summary.

We choose a Chern character $(0, 0, \beta, m)$ and work with complexes E of

$$\operatorname{ch}(E) \;=\; \operatorname{ch}(\mathcal{O}_X(-n)) - (0, 0, \beta, m) \;\text{ and so }\; \operatorname{cl}(E) \;=\; (1, c, s) \;:=\; (1,\, \beta.H,\, m).$$

We wall cross from the chamber $\theta \in (\theta_{\mathrm{JS}} - 1, \theta_{\mathrm{JS}})$, where there are no semistable objects, to the large volume chamber $\theta \gg 0$ where the semistable objects E are all stable pairs, with counting invariant $P_{\beta,\, m+n\beta.H}(X)$. As observed in the proof of Lemma 4.1, on any wall we have destabilising factors E_0, E_1 of ranks 0, 1 respectively, both with smaller values of c (68) and with E_0 a Gieseker semistable sheaf.

When $c = c_{\min} > 0$ takes the minimal possible value this means the only possibility is that $\operatorname{cl}(E_0) = (0, c_{\min}, s)$, so $\operatorname{cl}(E_1) = (1, 0, 0)$ and the wall is θ_{JS}. It follows that E_0 is Gieseker stable and the wall crossing formula is the simplest one (61). Thus we find

$$P_{\beta,\, m+n\beta.H}(X) \;=\; (-1)^{m+n\beta.H-1}(m + n\beta.H)\, \mathsf{J}(0, 0, \beta, m), \tag{76}$$

expressing $\mathsf{J}(0, 0, \beta, m)$ in terms of stable pair invariants.

Now take general $c > 0$. On each wall the semistable factors E_0, E_1 either have the classes $(1,0,0)$, $(0,c,s)$ again, or have *strictly smaller* c. In the first case we are on θ_{JS} and again pick up the term on the right hand side of (76), plus higher order terms counting semistable factors of E_0; since these have smaller c we may assume by induction they are written in terms of stable pair invariants.

In the second case both E_0, E_1 have strictly smaller $c > 0$, and θ is to the right of the Joyce-Song wall of E_1 by (72). By induction we may assume both $\mathsf{J}(0,0,\beta',m')$ and $\mathsf{J}_\theta(1,\beta',m')$ can be written in terms of stable pair invariants for all $\beta'.H < c$ and θ to the right of the Joyce-Song wall of $(1,\beta',m')$, since the base case $\beta'.H = c_{\min}$ was shown above. After finitely many walls, we reach the large volume chamber where we get the stable pair invariant $P_{\beta,\, m+n\beta.H}(X)$ equated with the right hand side of (76) plus a function of stable pair invariants with smaller c. Rearranging — and rewriting the stable pair invariants in terms of counts of ideal sheaves using the DT/PT wall crossing proved in [Br3, To1] — proves Theorem 2 for dimension 1 classes.

Finally we note that the working of [JS, Section 6.3] shows that the same technique of using Joyce-Song pairs also works for dimension 0 sheaves.

Appendix A. Stable pairs

We explain here some results of Toda. Though they are all proved in [To3, Section 3] they are not stated in the generality we need, so we give a brief account (with slightly different proofs) for completeness.

Definition A.1. A pair (F,s) consisting of a 1-dimensional sheaf F and a section $s \in H^0(F)$ is called a *stable pair* [PT] if

- F is a *pure* 1-dimensional sheaf, i.e. it has no 0-dimensional subsheaves, and
- $s\colon \mathcal{O}_X \to F$ has zero-dimensional cokernel.

We often abuse notation and call the 2-term complex $\mathcal{O}_X \xrightarrow{s} F$ (with $\mathcal{O}_X$ in degree 0) a stable pair. We also use the notation $E^\vee := R\mathcal{H}om(E,\mathcal{O}_X)$ for derived dual.

Lemma A.2. *Suppose $E \in \mathcal{A}_b$ is a $\nu_{b,w}$-semistable object of rank -1 with $\nu_{b,w}(E) < +\infty$ and $w \gg 0$. Set $L := (\det E)^{-1}$. Then*

$$E^\vee \otimes L[1] \ \textit{ is a stable pair.}$$

Proof. By [BMS, Lemma 2.7(c)(iii)] we have $\mathcal{H}^0(E) \in \mathrm{Coh}_{\leq 1}(X)$ while $\mathcal{H}^{-1}(E)$ is a μ_H-semistable sheaf. In particular $\mathcal{H}^{-1}(E)$ is torsion free, and rank 1 by Lemma 2.1. Thus it is $L \otimes I_Z$ for some subscheme $Z \subset X$ of dimension ≤ 1 and line bundle L — the double dual of $\mathcal{H}^{-1}(E)$, which also equals $\det\big(E[1]\big)$ since Z and $\mathcal{H}^0(E)$ have dimension ≤ 1.

Since $\mu_H(L) = \mu_H\big(\mathcal{H}^{-1}(E)\big)$ the shifts by [1] of both sheaves are in $\mathcal{A}_b$. This gives two $\mathcal{A}_b$ short exact sequences

$$L \otimes \mathcal{O}_Z \hookrightarrow \mathcal{H}^{-1}(E)[1] \twoheadrightarrow L[1] \qquad \text{and} \qquad \mathcal{H}^{-1}(E)[1] \hookrightarrow E \twoheadrightarrow \mathcal{H}^0(E) \tag{77}$$

and so the $\mathcal{A}_b$-injection $L \otimes \mathcal{O}_Z \hookrightarrow E$. Since $\nu_{b,w}(L \otimes \mathcal{O}_Z) = +\infty$ this contradicts the $\nu_{b,w}$-semistability of E unless $Z = \emptyset$. Thus $\mathcal{H}^{-1}(E) = L$.

The derived dual of the second exact sequence of (77) is the exact triangle

$$\mathcal{H}^0(E)^\vee \longrightarrow E^\vee \longrightarrow L^{-1}[-1].$$

Setting $F := \mathcal{H}^0(E)^\vee \otimes L[2]$ and $I^\bullet := E^\vee \otimes L[1]$ this gives

$$I^\bullet \longrightarrow \mathcal{O}_X \longrightarrow F. \tag{78}$$

Since $\dim \mathcal{H}^0(E) \leq 1$, we know $\mathcal{H}^0(F) = \mathcal{E}xt^2(\mathcal{H}^0(E)\otimes L^{-1}, \mathcal{O}_X)$ is a pure 1-dimensional sheaf, $\mathcal{H}^1(F) = \mathcal{E}xt^3(\mathcal{H}^0(E)\otimes L^{-1}, \mathcal{O}_X)$ is a 0-dimensional sheaf, and the other $\mathcal{H}^i(F)$ vanish. The long exact sequence of cohomology sheaves of (78) then shows that

$$\mathcal{H}^{\geq 3}(I^\bullet) = 0 \ \text{ and } \ \mathcal{H}^2(I^\bullet) = \mathcal{H}^1(F) \ \text{ is 0-dimensional.} \tag{79}$$

Now we use the $\nu_{b,w}$-semistability of $E \in \mathcal{A}_b$ and $\nu_{b,w}(E) < +\infty$ to deduce

$$\mathrm{Hom}(G, E) = 0 = \mathrm{Hom}(E^\vee, G^\vee) \ \text{ for any } \ G \in \mathrm{Coh}_{\leq 1}(X). \tag{80}$$

Setting $G = \mathcal{O}_x$ for any point $x \in X$ gives

$$0 = \mathrm{Hom}\big(I^\bullet \otimes L^{-1}[-1], \mathcal{O}_x[-3]\big) \cong \mathrm{Hom}(I^\bullet[2], \mathcal{O}_x) = \mathrm{Hom}\big(\mathcal{H}^2(I^\bullet), \mathcal{O}_x\big),$$

where the last equality follows from $\mathcal{H}^{\geq 3}(I^\bullet) = 0$ (79). Combined with the second part of (79) this gives $\mathcal{H}^2(I^\bullet) = 0$, so in fact

$$\mathcal{H}^{\geq 2}(I^\bullet) = 0 \ \text{ and } \ F \ \text{ is a pure 1-dimensional sheaf.} \tag{81}$$

Now let G be any pure 1-dimensional sheaf. Then $G^\vee \otimes L[2]$ is also a pure 1-dimensional sheaf, which we may also subsitute into (80) in place of G. This gives

$$0 = \mathrm{Hom}\big(E^\vee, (G^\vee \otimes L[2])^\vee\big) = \mathrm{Hom}(I^\bullet[1], G) \overset{(81)}{=} \mathrm{Hom}\big(\mathcal{H}^1(I^\bullet), G\big). \tag{82}$$

But by (78) $\mathcal{H}^1(I^\bullet)$ is a quotient of a pure 1-dimensional sheaf F, so by (82) it must be 0-dimensional. Thus by (78), $I^\bullet$ is the cocone of a map $\mathcal{O}_X \to F$ with 0-dimensional cokernel, i.e. it is a stable pair. □

For the converse let (F, s) be a stable pair and $I^\bullet = \{\mathcal{O}_X \xrightarrow{s} F\}$ the associated complex.

Lemma A.3. *For any line bundle L take $b > L.H^2/H^3$ and $w \gg 0$. Then*

$$(I^\bullet)^\vee \otimes L[1] \in \mathcal{A}_b \ \text{ and is } \ \nu_{b,w}\text{-stable.}$$

Proof. The dual $\otimes L[1]$ of the exact triangle $I^\bullet \to \mathcal{O}_X \to F$ gives

$$L[1] \longrightarrow L \otimes (I^\bullet)^\vee[1] \longrightarrow L \otimes F^\vee[2]. \tag{83}$$

Since F is a pure 1-dimensional sheaf, so is $L \otimes F^\vee[2]$. So the exact triangle shows the rank -1 complex $E := L \otimes (I^\bullet)^\vee[1]$ lies in $\mathcal{A}_b$ (7) when $b > \mu_H(L) = L.H^2/H^3$.

By the wall and chamber structure of Proposition 1.3 and the primitivity of $\mathrm{ch}(E)$ it is sufficient to prove that E is $\nu_{b,w}$-semistable for a fixed $b > \max\big(0, \mu_H(L)\big)$ and $w \gg 0$. Since $\mathrm{rank}(E) = -1$ the slope $\nu_{b,w}(E)$ is linear and increasing in w. So take w satisfying

(i) $\nu_{b,w}(E) > 0$,
(ii) $\nu_{b,w}(E) > \mathrm{ch}_2(E).H + \frac{1}{2H^3}\max\big((bH^3)^2, (L.H^2)^2\big)$, and

(iii) $w > \frac{1}{2}\big(\mathrm{ch}_1^{bH}(E).H^2\big)^2 + b\,\mathrm{ch}_1^{bH}(E).H^2 + \frac{1}{2}b^2$,

and suppose for a contradiction that E is strictly $\nu_{b,w}$-unstable. Taking the first term of its $\nu_{b,w}$-Harder-Narasimhan filtration gives an $\mathcal{A}_b$-exact sequence $E' \hookrightarrow E \twoheadrightarrow E''$ with $\nu_{b,w}$-semistable E' and

$$\nu_{b,w}(E') > \nu_{b,w}(E) \geq \nu_{b,w}(E''). \tag{84}$$

Thus $\mathcal{H}^0(E'')$ is a quotient of the 1-dimensional sheaf $\mathcal{H}^0(E)$, so it is supported in dimension ≤ 1. If $\mathcal{H}^{-1}(E'')$ also has rank 0 then it vanishes by (7), so $\mathrm{rank}(E'') = 0$ and $\nu_{b,w}(E'') = +\infty$, contradicting (84). We conclude $\mathrm{rank}(E'') < 0$ and $\mathrm{rank}(E') \geq 0$.

Suppose first that $\mathrm{rank}(E') > 0$. Then $\mathrm{rank}(\mathcal{H}^0(E')) > 0$ so (8) gives $\mathrm{ch}_1^{bH}(E').H^2 > 0$. By (i) and (84) we also have $\nu_{b,w}(E') > 0$. So their product is also positive,

$$\begin{aligned}
0 &< \mathrm{ch}_2(E').H - w\,\mathrm{ch}_0(E')H^3 \\
&\overset{(12)}{\leq} \frac{(\mathrm{ch}_1(E').H^2)^2}{2\,\mathrm{ch}_0(E')H^3} - w\,\mathrm{ch}_0(E')H^3 \\
&= \frac{(\mathrm{ch}_1^{bH}(E').H^2)^2}{2\,\mathrm{ch}_0(E')H^3} + b\,\mathrm{ch}_1^{bH}(E').H^2 + \tfrac{1}{2}b^2\,\mathrm{ch}_0(E')H^3 - w\,\mathrm{ch}_0(E')H^3 \\
&\leq \tfrac{1}{2}(\mathrm{ch}_1^{bH}(E).H^2)^2 + b\,\mathrm{ch}_1^{bH}(E).H^2 + \tfrac{1}{2}b^2\,\mathrm{ch}_0(E')H^3 - w\,\mathrm{ch}_0(E')H^3.
\end{aligned}$$

Therefore $w < \frac{1}{2}(\mathrm{ch}_1^{bH}(E).H^2)^2 + b\,\mathrm{ch}_1^{bH}(E).H^2 + \frac{1}{2}b^2$, which contradicts (iii).

So $\mathrm{rank}(E') = 0$ and $\mathrm{rank}(E'') = -1$. We claim that E' is a sheaf. If not then since $\mathcal{H}^{-1}(E')$ and $\mathcal{H}^{-1}(E'')$ are torsion free (7) and $\mathrm{rank}\big(\mathcal{H}^{-1}(E)\big) = 1$, the exact sequence

$$0 \longrightarrow \mathcal{H}^{-1}(E') \xrightarrow{\alpha} \mathcal{H}^{-1}(E) \xrightarrow{\beta} \mathcal{H}^{-1}(E'') \longrightarrow \mathcal{H}^0(E') \longrightarrow \mathcal{H}^0(E) \longrightarrow \mathcal{H}^0(E'') \longrightarrow 0$$

shows the map α is an isomorphism and β is 0. Since $\mathcal{H}^0(E)$ has dimension ≤ 1 the rest of this sequence then shows that $\mu_H\big(\mathcal{H}^{-1}(E'')\big) = \mu_H\big(\mathcal{H}^0(E')\big)$, which is impossible by the definition (7) of $\mathcal{A}_b$.

So E' is a rank 0 sheaf. The inclusion $\mathcal{H}^{-1}(E) \hookrightarrow \mathcal{H}^{-1}(E'')$ implies the rank 1 torsion free sheaf $\mathcal{H}^{-1}(E'')$ is $L'' \otimes I_{Z''}$ for some Z'' of dim ≤ 1 and line bundle L'' with

$$L.H^2 \leq L''.H^2 \overset{(7)}{\leq} bH^3.$$

The Hodge index theorem $\big((L'')^2.H\big)(H^3) \leq (L''.H^2)^2$ and $\mathrm{ch}_2\big(\mathcal{H}^0(E'')\big).H \geq 0$ then give

$$\mathrm{ch}_2(E'').H \geq -\mathrm{ch}_2(L'' \otimes I_{Z''}).H \geq -\tfrac{1}{2}(L'')^2.H \geq -\tfrac{1}{2H^3}\max\big((bH^3)^2, (L.H^2)^2\big).$$

If $\mathrm{ch}_1(E').H^2 > 0$ then we get the following contradiction to (ii) and (84),

$$\nu_{b,w}(E') = \frac{\mathrm{ch}_2(E').H}{\mathrm{ch}_1(E').H^2} \leq \mathrm{ch}_2(E').H \leq \mathrm{ch}_2(E).H + \tfrac{1}{2H^3}\max\big((bH^3)^2, (L.H^2)^2\big).$$

So in fact the rank 0 sheaf E' is supported in dimension ≤ 1.

From (83) we get the following diagram in $\mathcal{A}_b$,

$$\begin{array}{ccccc} L[1] & \hookrightarrow & L\otimes(I^\bullet)^\vee[1] & \twoheadrightarrow & L\otimes F^\vee[2] \\ \uparrow & & \uparrow & \nearrow_{\gamma} & \uparrow \\ \ker\gamma & \hookrightarrow & E' & \twoheadrightarrow & \operatorname{im}\gamma. \end{array}$$

Since $\mathcal{A}_b$-subobjects of sheaves are sheaves, both $\ker\gamma$ and $\operatorname{im}\gamma$ are sheaves. Thus the exact sequence of cohomology sheaves of the lower row shows $\ker\gamma$ is a *subsheaf* of E', so it has dimension ≤ 1. The $\mathcal{A}_b$-cokernel of the left hand injection has torsion free $\mathcal{H}^{-1}$ sitting in an exact sequence of sheaves $0\to L\to\mathcal{H}^{-1}\to\ker\gamma\to 0$, which forces $\ker\gamma=0$.

Thus we have the $\mathcal{A}_b$-injection $\gamma\colon E'\hookrightarrow L\otimes F^\vee[2]$. Hence $\mathcal{H}^{-1}(\operatorname{coker}\gamma)$ is both torsion free and a subsheaf of E'; it is therefore zero and $E'\hookrightarrow L\otimes F^\vee[2]$ is also an injection of sheaves. We deduce E' is a pure 1-dimensional sheaf.

The dual $\otimes L[1]$ of $E'\hookrightarrow E\twoheadrightarrow E''$ gives

$$(E''\otimes L^{-1})^\vee[1]\longrightarrow I^\bullet\longrightarrow(E')^\vee\otimes L[1]. \tag{85}$$

Since $E''\in\mathcal{A}_b$ we have $E''\otimes L^{-1}\in\mathcal{A}_{b'}$, where $b'=b-L.H^2/H^3$. And $\nu_{b,w}^{\max}(E'')<+\infty$ by the definition of E'' via the $\nu_{b,w}$-Harder-Narasimhan filtration, so $\nu_{b',w}^{\max}(E''\otimes L^{-1})<+\infty$. Therefore [BMT, Proposition 5.1.3(b)] applies to show $\mathcal{H}^{\geq 2}\big((E''\otimes L^{-1})^\vee[1]\big)=0$. Then the exact sequence of cohomology sheaves of (85) gives a surjection

$$\operatorname{coker}(s)\twoheadrightarrow\mathcal{H}^2\big((E')^\vee\otimes L\big)$$

from a 0-dimensional sheaf to a nonzero 1-dimensional sheaf, a contradiction. □

The bijection of sets defined by Lemmas A.2 and A.3 can be upgraded to an isomorphism of moduli stacks. There is a fine moduli space P of stable pairs of fixed Chern character. It admits a universal stable pair on $P\times X$, flat over P. Taking its derived dual and twisting by $L[1]$ defines a perfect complex over $P\times X$. By Lemma A.3 its restriction to any $\{\text{point}\}\times X$ is a $\nu_{b,w}$-stable complex for $bH^3>L.H^2$ and $w\gg 0$.

These complexes all have the same class $v\in K(X)$. There is an algebraic moduli stack of finite type $\mathcal{M}_{b,w}(v)$ of $\nu_{b,w}$-semistable objects in this class [FT3, Theorem C.5].

The perfect complex is classified by a map $P\to\mathcal{M}_{b,w}(v)$ which factors through $P/\mathbb{C}^*$. (Here the $\mathbb{C}^*$ acts trivially on P but scales the complex.) Twisting further by elements of $\operatorname{Pic}_0(X)$ defines a map

$$P/\mathbb{C}^*\times\operatorname{Pic}_0(X)\longrightarrow\mathcal{M}_{b,w}(v) \tag{86}$$

which is a bijection by Lemmas A.2 and A.3. But deformations of stable pairs (F,s) and the complexes $(I^\bullet)^\vee\otimes L[1]$ coincide to all orders [PT, Theorem 2.7], so the bijection induces an isomorphism $P\times\operatorname{Pic}_0(X)\to M_{b,w}(v)$ on coarse moduli spaces. Since the stabiliser groups are $\mathbb{C}^*$ on both sides, we deduce that (86) is an isomorphism of stacks.

Hence, for $bH^3>L.H^2$, the invariants $\mathsf{J}_{b,\infty}(v)$ equal the stable pair invariants of [PT] multiplied by $\#H^2(X,\mathbb{Z})_{\text{tors}}$.

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