SHOCK FORMATION IN SMALL-DATA SOLUTIONS TO $3D$ QUASILINEAR WAVE EQUATIONS: AN OVERVIEW

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ABSTRACT. In his 2007 monograph, D. Christodoulou proved a remarkable result giving a detailed description of shock formation, for small H^s-initial conditions (s sufficiently large), in solutions to the relativistic Euler equations in three space dimensions. His work provided a significant advancement over a large body of prior work concerning the long-time behavior of solutions to higher-dimensional quasilinear wave equations, initiated by F. John in the mid 1970's and continued by S. Klainerman, T. Sideris, L. Hörmander, H. Lindblad, S. Alinhac, and others. Our goal in this paper is to give an overview of his result, outline its main new ideas, and place it in the context of the above mentioned earlier work. We also introduce the recent work of J. Speck, which extends Christodoulou's result to show that for two important classes of quasilinear wave equations in three space dimensions, small-data shock formation occurs precisely when the quadratic nonlinear terms fail the classic null condition.

Keywords: characteristic hypersurfaces, compatible current, eikonal function, generalized energy estimates, hyperbolic conservation laws, maximal development, null condition, Raychaudhuri equation, Riccati equation, vectorfield method

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1. INTRODUCTION

1.1. **Motivation and background.** This project was motivated by our desire to understand the work of Christodoulou [11] concerning the formation of shocks in compressible, irrotational, relativistic,[1] $3D$ fluids, starting from small, smooth initial conditions. His work is a landmark result in the venerable area of PDE known as *Systems of Nonlinear Hyperbolic Conservation Laws*. This field originated from considerations concerning the propagation of one-dimensional sound waves through air, by Monge, Poisson, Stokes, Riemann, Rankine, and Hougoniot and was transformed into a systematic theory by Courant [14], Friedrichs [14, 16], O. Oleinik [55], Lax [43], Glimm [17], Bressan [8], and many others.

1.1.1. *Singularities are an unavoidable aspect of the* $1D$ *theory.* The crucial fact, already well understood by Riemann and Stokes, which the theory had to deal with from its beginnings, was the observation that solutions to the equations develop singularities, even when the data are small and smooth. This fact is easy to exhibit in one space dimension and is well-captured by Burgers' equation:

$$\partial_t \Psi + \Psi \partial_x \Psi = 0, \tag{1.1.1}$$

$$\Psi(0, x) = \mathring{\Psi}(x). \tag{1.1.2}$$

[1] The results were later extended to apply to the non-relativistic Euler equations in [9].

In view of the equation, Ψ must be constant along the characteristic curves $x(t,\alpha)$, which are, in this case, solutions to the ODE initial value problems

$$\frac{\partial}{\partial t}x(t,\alpha) = \Psi(t,x(t,\alpha)), \qquad x(0,\alpha) = \alpha \in \mathbb{R}. \tag{1.1.3}$$

Thus, $\Psi\big(t,x(t,\alpha)\big) = \mathring{\Psi}(\alpha)$, $\frac{\partial}{\partial t}x(t,\alpha) = \mathring{\Psi}(\alpha)$, and $\frac{\partial^2}{\partial t\partial\alpha}x(t,\alpha) = \mathring{\Psi}'(\alpha)$. Hence, we have $\frac{\partial}{\partial\alpha}x(t,\alpha) = 1 + t\mathring{\Psi}'(\alpha)$. In particular, $\frac{\partial}{\partial\alpha}x(t,\alpha) = 0$ when $1 + t\mathring{\Psi}'(\alpha) = 0$. It follows that a singularity must form in any solution launched by nontrivial, smooth, compactly supported initial data $\mathring{\Psi}$. An alternative way to see the blow-up is to differentiate Burgers' equation in x and derive the equation

$$\partial_t(\partial_x\Psi) + \Psi\partial_x(\partial_x\Psi) = -(\partial_x\Psi)^2,$$

which is the well-known Riccati equation

$$\frac{d}{dt}y = -y^2, \qquad \text{for } y(t) := \partial_x\Psi(t,x(t,\alpha)). \tag{1.1.4}$$

Note that the L^∞ norm of Ψ is conserved. That is, the blow-up occurs in $\partial_x\Psi$, while Ψ itself remains bounded. It is also easy to check that the time of blow up is no later than[2] $\mathcal{O}(1/\epsilon)$, where $\epsilon = -\min_{\alpha\in\mathbb{R}}\mathring{\Psi}'(\alpha)$ measures the smallness of the initial data.

Though a bit more difficult to prove, the same blow-up results hold true for general classes of systems of quasilinear conservation laws (genuinely nonlinear, strictly hyperbolic) in one space dimension. The first general results for 2×2 strictly hyperbolic systems verifying the genuine nonlinearity condition are due to P. Lax [44], who extended the earlier work of Zabusky [64] on the formation of singularities in nonlinear vibrating strings. The results were later extended to general such systems by F. John [22]. In [38], Klainerman and Majda showed that the genuine nonlinearity condition can be relaxed in the case of $1D$ nonlinear vibrating string equations.

The great achievement of the $1D$ theory of systems of conservation laws was the understanding of how shocks form and how solutions can be extended through shock singularities. This entails a complete description of the shock boundary, as well as a formulation of the equations capable of accommodating such singular solutions. Such machinery is available for general classes of hyperbolic conservation laws (mainly strictly hyperbolic) in one space dimension with general initial data of small bounded variation; see, for example, [58] and [15].

1.1.2. *Present-day limitations of the theory.* A primary goal of the field of conservation laws is replicating the $1D$ success in higher space dimensions, which entails understanding the mechanism of singularity formation as well as how to define generalized solutions extending past sufficiently mild singularities. In higher dimensions, one faces several difficulties including that of finding a suitable definition of generalized solutions and corresponding function spaces.

In $1D$, the continuation of solutions to conservations laws past shock fronts is comfortably achieved, in most cases, by considering the equations in a weak formulation for functions with finite spatial bounded

[2]Throughout the article, we sometimes write $A = \mathcal{O}(B)$ and equivalently $A \lesssim B$ to indicate $A \leq CB$ by some universal constant C; see Footnote 62 on pg. 40 for details.

variation (BV) norm.[3] In higher dimensions, however, the BV norm is incompatible with the simple phenomenon of focusing of perfectly smooth waves, as can be seen for spherically symmetric solutions to the standard wave equation in $\mathbb{R}^{1+n}$, for any $n \geq 2$; see [56]. Instead, the general theory of local well-posedness for systems of hyperbolic conservation laws in higher dimensions is intimately tied to L^2-based H^s Sobolev spaces.[4] This theory has largely been developed using the framework of Friedrichs' symmetric hyperbolic systems [16], and with further contributions by many others such as Sobolev, Schauder, Frankl, and Leray. The theory, however, is quite far from accommodating discontinuous shock fronts[5] and their interactions.

In addition to the problem of defining generalized solutions, one also encounters difficulties with understanding the mechanism of singularity formation. A subtle point is that in higher dimensions there can, in principle, be singularities which differ from shocks in that they do not form from compression. For example, a current venue of investigation is the possibility of vorticity blow-up (the possible mechanism driving it remains an enigma) for the $3D$ compressible Euler equations of fluid dynamics. Furthermore, the phenomenon of dispersion, typical to higher dimensions, may delay or in some cases altogether prevent the formation of singularities for small initial data.

1.1.3. *Quasilinear systems of wave equations.* An obvious way to separate the phenomena of compression and dispersion from the effects of vorticity, in the case of the compressible Euler equations (relativistic or non-relativistic), is to restrict oneself to irrotational flows. For such flows, the Euler equations reduce to a quasilinear wave equation for Φ, the fluid potential.[6] Since the irrotational Euler equations are derivable from a Lagrangian $\mathcal{L}(\partial\Phi)$, the wave equation can be written in the following Euler-Lagrange form relative to standard rectangular coordinates:[7]

$$\partial_\alpha \left\{ \frac{\partial \mathcal{L}(\partial\Phi)}{\partial(\partial_\alpha \Phi)} \right\} = 0. \tag{1.1.5}$$

We explain the connection between equation (1.1.5) and (special) relativistic fluid mechanics in more detail in Subsect. 5.2 and Appendix A. When expanded relative to rectangular coordinates, equation (1.1.5) takes the form

$$(g^{-1})^{\alpha\beta}(\partial\Phi)\partial_\alpha\partial_\beta\Phi = 0. \tag{1.1.6}$$

Remark 1.1. Throughout this article, $\partial f = (\partial_t f, \partial_1 f, \partial_2 f, \partial_3 f)$ denotes the gradient of f relative to the rectangular spacetime coordinates.

For physical choices of the Lagrangian $\mathcal{L}(\partial\Phi)$, (1.1.6) is a wave equation: $(g^{-1})^{\alpha\beta}(\cdot)$ is a non-degenerate symmetric quadratic form of signature $(-,+,+,+)$ depending smoothly on $\partial\Phi$. We can always find an

[3]The theory of smooth solutions for $1D$ hyperbolic equations can be easily developed, starting with Monge [53], using the method of characteristics, in any L^p norm (in particular L^∞ and L^1; the latter being consistent with the BV norm).

[4]The number of derivatives required, s, depends on the number of space dimensions and the strength of the nonlinearity.

[5]The theory can, however, be adapted (within the H^s framework !) to treat, for a short time, one single shock wave, starting with an initial discontinuity across an admissible regular hypersurface in higher dimensions; see [50–52].

[6]Roughly, the gradient of Φ is equal to a rescaled version of the fluid velocity.

[7]We use Einstein's summation convention throughout. Lowercase Greek "spacetime" indices vary over $0, 1, 2, 3$ and lowercase Latin "spatial" indices vary over $1, 2, 3$.

affine change of coordinates on $\mathbb{R}^{1+n}$ to obtain the relationship

$$(g^{-1})^{\alpha\beta}(\partial\Phi = 0) = (m^{-1})^{\alpha\beta}, \tag{1.1.7}$$

where $(m^{-1})^{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$ is the standard inverse Minkowski metric; we will assume henceforth such a coordinate change has been made. In [11], Christodoulou studied a particular class of scalar equations of type (1.1.5) that arise in irrotational relativistic fluid mechanics.[8] Most of the results that we discuss in this introduction, especially those concerning almost global existence, can be extended[9] to apply to the more general class of equations

$$(g^{-1})^{\alpha\beta}(\partial\Phi)\partial_\alpha\partial_\beta\Phi = \mathcal{N}(\Phi, \partial\Phi), \tag{1.1.8}$$

where $(g^{-1})^{\alpha\beta}$ verifies (1.1.7) and $\mathcal{N}$ is smooth in $(\Phi, \partial\Phi)$ and is quadratic or higher-order in $\partial\Phi$ for small $(\Phi, \partial\Phi)$; that is, $\mathcal{N} = \mathcal{O}(|\partial\Phi|^2)$ for small $(\Phi, \partial\Phi)$.

At the beginning of the 20^{th} century, nonlinear wave equations made another dramatic appearance in General Relativity. Relative to the wave coordinates[10] x^α, the Einstein vacuum equations $\mathcal{R}_{\mu\nu} = 0$ (where $\mathcal{R}_{\mu\nu}$ is the Ricci curvature of the dynamic Lorentzian metric g) can be cast as a system of quasilinear wave equations in the components of g, in the form

$$(g^{-1})^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} = \mathcal{N}(g)(\partial g, \partial g), \qquad (\mu, \nu = 0, 1, 2, 3), \tag{1.1.9}$$

where $\mathcal{N}(g)(\partial g, \partial g)$ depends quadratically on ∂g, that is, on all spacetime derivatives of g.

The above considerations have led to the study of general systems of nonlinear wave equations of the form

$$(g^{-1})^{\alpha\beta}(\Psi)\partial_\alpha\partial_\beta\Psi^I = \mathcal{N}^I(\Psi, \partial\Psi), \qquad (I = 1, \dots, K), \tag{1.1.10}$$

where g is a smooth Lorentzian metric depending on the array $\Psi = \{\Psi^I\}_{I=1,\dots,K}$ and $\mathcal{N}$ is smooth in $(\Psi, \partial\Psi)$, at least quadratic in $\partial\Psi$ near $(\Psi, \partial\Psi) = (0, 0)$. Note that (1.1.10) contains equations of type (1.1.6) by simply differentiating the latter and taking $\Psi = \partial\Phi$. Note also that more general systems of the form

$$(g^{-1})^{\alpha\beta}(\Psi, \partial\Psi)\partial_\alpha\partial_\beta\Psi^I = \mathcal{N}^I(\Psi, \partial\Psi), \qquad (I = 1, \dots, K) \tag{1.1.11}$$

can, by differentiation, also be transformed into systems of type (1.1.10). Thus, the systems of the form (1.1.10) encompass the equations which arise in the irrotational compressible Euler equations, both relativistic and non-relativistic under all physically reasonable equations of state, and the Einstein vacuum equations (1.1.9) relative to wave coordinates. Furthermore, while the equations of nonlinear elasticity do not fit[11] into the form of (1.1.10), the important special case of homogeneous and isotropic materials can nevertheless be reduced, by a simple separation between longitudinal and transversal waves, to the same framework; see John's work [27].

[8] More precisely, as we explain in Subsect. 5.2 and Appendix A, the solutions considered in [11] differ from solutions to equations of the form (1.1.5) by choices of normalizations.

[9] The shock formation results seem to be less stable under modifications of the equation; see Remark 2.15.

[10] The coordinate functions themselves verify the covariant wave equation $\Box_g x^\alpha = 0$.

[11] The general form of the equations of elasticity can, upon differentiation, be expressed as a generalization of equation (1.1.11) of the form $(g^{-1})^{\alpha\beta}_{IJ}(\Psi)\partial_\alpha\partial_\beta\Psi^J = \mathcal{N}^I(\Psi, \partial\Psi)$ (with summation over J). Such equations give rise to more complicated geometries. In particular the principal part is no longer the geometric wave operator of a Lorentzian manifold.

1.1.4. *Results in $3D$ prior to Christodoulou's work.* In light of what we know in the one-dimensional case, it makes sense to ask whether the mechanism of shock formation remains the same in higher dimensions. At first glance, we may expect a positive answer simply by observing that plane wave solutions are effectively one dimensional. However, plane waves are non-generic and have infinite energy. The latter flaw can be ameliorated within the past domain of dependence $\mathcal{I}^-(p)$ of an earliest singular point of the plane wave: we can simply cut-off the plane wave data outside of the intersection of $\mathcal{I}^-(p)$ with the initial Cauchy hypersurface $\{t = 0\}$ to construct compactly supported initial data that lead to a shock singularity at p. However, one can show that the cut-off data have large energy and thus do not fit into the framework of small perturbations of the trivial state. It turns out, in fact, that the large-time behavior of data of small size[12] $\mathring{\epsilon}$, in higher dimensions, is radically different from $1D$. This fact was first pointed out by F. John: in [23, 24] he showed for quasilinear wave equations of type (1.1.6) that the dispersion of waves significantly delays the formation of singularities when the data are small.

Starting with John's observation, Klainerman [42] was able to show, for a class of equations including those of form (1.1.8) with $\mathcal{N}$ independent of Φ, that the phenomenon of dispersion is sufficiently strong, in space dimensions greater than or equal to 6, to completely avoid the formation of shocks for small initial data. John and Klainerman [21, 32] were later able to prove an *almost global existence* result[13] showing that in 3 space dimensions, if the data and a certain number their derivatives are of small size $\mathring{\epsilon}$, then singularities cannot form before time $\mathcal{O}(\exp(c\mathring{\epsilon}^{-1}))$. This represents a significant improvement over the existence time $\mathcal{O}(\mathring{\epsilon}^{-1})$ in dimension 1. The result was significantly simplified and extended in [35] using the geometric vectorfield method. See also Theorem 1 below for a sharp version of this result, due independently to John and Hörmander. Moreover, Klainerman [34, 36] was later able to identify a structural condition on the form of the quadratic terms in (1.1.8), called the (classic) *null condition* (see Definition 1.1), which prevents the formation of singularities when the data are sufficiently small in 3 space dimensions. Two distinct proofs of the result were given, one by Klainerman [36] based on the vectorfield method, and the second by Christodoulou [10] based on the conformal method.[14]

In the opposite direction, F. John gave [25] a class of examples in $3D$ of the form[15]

$$\Box_m \Phi = \mathcal{N}(\partial\Phi, \partial^2\Phi),$$

where $\mathcal{N}$ is quadratic in its arguments, the classic null condition fails, and such that a large set[16] of nontrivial compactly supported data lead to finite-time breakdown. Note that John's results are consistent with the almost global existence result of [21]. Unlike, however, the one-dimensional argument that tracks solutions all the way to the first singularity, John's argument shows only that the existence of a

[12]The relevant energy norm depends on a finite number of derivatives.

[13]The proof in [21] used some mild assumptions on the structure of the equation (1.1.8). Suitable assumptions are that the nonlinearity is independent of Φ itself or that the equation can be written in divergence form up to cubic errors.

[14]One should remark that the geometric vectorfield method also yields a direct extension of Klainerman's global existence result [42] to dimensions 4 and 5, while for dimension 2 (where the dispersion is even weaker than in $3D$), versions of the null condition have been identified by Alinhac [1, 2].

[15]Here and throughout, $\Box_m = -\partial_t^2 + \Delta$ is the standard flat d'Alembertian in $\mathbb{R}^{1+n}$, where $n = 3$ at present, and more generally n will be clear from context. John's class includes equations such as $\Box_m\Phi = -(\partial_t\Phi)^2$ and $\Box_m\Phi = -\partial_t\Phi\partial_t^2\Phi$; his proof crucially uses the definiteness of the nonlinearity.

[16]For some nonlinearities, John's results yield blow-up for *all* non-trivial, smooth, compactly supported data.

global C^3 solution would lead to a contradiction.[17] T. Sideris [59] later proved a related but distinct result showing that small initial data for the full compressible Euler equations in $3D$, under some adiabatic equations of state verifying a convexity assumption, also lead to finite time break-down. Sideris' proof was based on virial inequalities and thus provided an explicit upper bound on the solution's lifetime for small data verifying an open condition. Later, Guo and Tahvildar-Zadeh gave a similar proof of breakdown in solutions to the relativistic Euler equations in Minkowski spacetime [18], but their proof required the assumption of large data. We should mention here that the work [9] (following [11]) shows that the convexity assumptions used by Sideris are not necessary and that the first singularity is in fact caused by shock formation.

Though for small initial data in $3D$, the arguments of John and Sideris complement the global existence results that hold when the nonlinearities verify the null condition, they fail to provide a satisfactory answer about the nature of the singularities, an understanding of which is clearly essential if one hopes to continue the solutions beyond them, as can be done in $1D$. The first results in this direction are once more due to F. John [25], who analyzed spherically symmetric solutions of the model equation

$$\Box_m \Phi = -a^2(\partial_t \Phi) \cdot \Delta \Phi, \qquad a(0) = 0, \qquad a'(0) \neq 0 \tag{1.1.12}$$

in $3D$, where the final condition in (1.1.12) guarantees the *failure* of the classic null condition of Definition 1.1. John's work showed that solutions corresponding to all sufficiently small nontrivial spherically symmetric data of compact support necessarily have some second-order derivatives blowing-up near the wave front, while all first derivatives remain bounded.[18] His proof, based on the method of characteristics, makes essential use of the fact that spherically symmetric solutions of the equation verify a simplified equation which is effectively one-dimensional. Although the passage from spherical symmetry to the general case is difficult, we nevertheless shall see that *this simplified case provides the right intuition about the behavior of general small-data shock-forming solutions.*

In the last years of his life, F. John himself tried hard to extend his results to the general case. Although he came close [30, 31], he never was able to follow the solution all the way to the singularity. The first results proving shock formation without symmetry assumptions are due to Alinhac; see [2, 4–6]. His results were highly motivated by John's earlier work [29] (see also Hörmander's work [19]) which provided a lower bound on the solution's lifespan that, as we take the size of the initial data to zero, converges to Alinhac's blow-up time (see Theorem 1 and the right-hand side of (5.1.5)). Alinhac's results provided a major advance in our understanding of blow-up away from spherical symmetry. However, they have some limitations. For example, his proof works only for data that lead to a unique first blow-up point. Hence, for equations invariant under the Euclidean rotations, his results do not apply to some data containing a spherically symmetric sector. A more significant limitation is that his results do not extend in an obvious fashion to provide a complete description of the maximal development of the data; see Subsect. 5.3 for more details. Christodoulou's work [11] eliminates these limitations, opens the way for obtaining a

[17] For John's quasilinear equations, any rigorous proof of *shock formation* would have to establish the precise mechanism for the blow-up of the second derivatives of Φ while also showing that the first derivatives remain bounded.

[18] John's analysis is restricted to a neighborhood of the wave front (the "wave zone"), where one expects (due to dispersion; see next subsection) the first singularity to form. He does not provide information about the entire maximal development of the data. See Figure 2 and the discussion below.

sharp understanding of shock formation in dimension 3, and properly sets up the difficult open problem of continuing the solution beyond the shock.

1.2. **The dispersion of waves.** In this subsection and the next one, we discuss some of the main ideas, especially the role of dispersion, in the development of the theory of the long-time behavior of small-data solutions to nonlinear wave equations of type (1.1.8) and (1.1.10) in $\mathbb{R}^{1+n}$ prior to the work of Christodoulou [11]. We will especially focus on the case of $\mathbb{R}^{1+3}$. In particular, we sketch in this subsection the proofs of the almost global existence result[19] of [21] and the global existence result of [36] for nonlinearities verifying the null condition (see also [10]). In Subsect. 1.3, we study shock formation in detail for spherically symmetric solutions.

1.2.1. *Local well-posedness.* We start by recalling a classical local well-posedness result for the scalar[20] quasilinear wave equation (1.1.8); see, for example, [62]. We denote the initial data for Φ, given on the Cauchy hypersurface $\Sigma_0 := \{t = 0\} \simeq \mathbb{R}^n \subset \mathbb{R}^{1+n}$ by

$$\Phi(0, x) = \mathring{\Phi}(x), \qquad \partial_t \Phi(0, x) = \mathring{\Phi}_0(x). \tag{1.2.1}$$

Proposition 1.1 (Local well-posedness and continuation criteria). *Let $s \geq s_0 = \lfloor \frac{n}{2} \rfloor + 3$ be an integer.*[21]

Local well-posedness. *Then there exists*[22] *a unique classical solution Φ to the equation* (1.1.8) *existing on a nontrivial spacetime slab of the form $[0, T) \times \mathbb{R}^n$ for some $T > 0$. The solution has the following regularity properties:*

$$\|\partial \Phi(t, \cdot)\|_{H^{s-1}(\mathbb{R}^n)} \leq C_s \left(\sum_{a=1}^{n} \|\partial_a \mathring{\Phi}\|_{H^{s-1}(\mathbb{R}^n)} + \|\mathring{\Phi}_0\|_{H^{s-1}(\mathbb{R}^n)} \right) e^{C_s \int_0^t \|\Phi(\tau, \cdot)\|_{W^{2,\infty}} d\tau,} \tag{1.2.2}$$

where C_s depends only on s and $W^{2,\infty}$ is the L^∞ based Sobolev norm, involving up to two derivatives of ϕ.

Continuation criterion. *The solution can be extended beyond $[0, T) \times \mathbb{R}^n$ as long as $\int_0^T \|\Phi(\tau, \cdot)\|_{W^{2,\infty}} d\tau < \infty$. In particular, the time of existence T has a lower bound depending on $\|\mathring{\Phi}\|_{H^{s_0}(\mathbb{R}^n)} + \|\mathring{\Phi}_0\|_{H^{s_0-1}(\mathbb{R}^n)}$.*

Remark 1.2. We note that a similar result holds for the larger class of symmetric hyperbolic systems of Friedrichs [16] and, in particular, for systems of equations of type (1.1.10) relevant to the Einstein field equations. For this latter type, since the quasilinear term depends only on Ψ and not its derivatives, we can close with one fewer derivative, that is, we can set $s_0 = \lfloor \frac{n}{2} \rfloor + 2$ instead.

[19] We give here a version based on the vectorfield method introduced in [35] and not the original of [21].

[20] The results can be extended, with minor modifications, to systems of the form (1.1.11) and that of nonlinear elasticity.

[21] By definition, $\lfloor \frac{n}{2} \rfloor + 3$ is the smallest integer strictly larger than $n/2 + 2$.

[22] In reality, for this theorem to hold (both local well-posedness and the breakdown criterion to follow), we need additional assumptions on the data and the coefficient matrix $(g^{-1})^{\alpha\beta}$ ensuring that the equation is hyperbolic in a suitable sense. For convenience, we ignore this issue.

The a priori energy-type estimate (1.2.2) is really at the heart of the proof. It can be derived by differentiating the original nonlinear equation with respect to $\partial^{\vec{I}}$, for rectangular coordinate derivative multi-indices $\vec{I}$, multiplying the resulting equation by $\partial_t \partial^{\vec{I}} \Phi$, integrating by parts, and using simple commutator estimates; see [42] for example. The local existence result can then be proved by first replacing $\int_0^t \|\Phi(\tau,\cdot)\|_{W^{2,\infty}(\mathbb{R}^n)}\, d\tau$ with the quantity $t \sup_{0\leq\tau\leq t} \|\Phi(\tau,\cdot)\|_{H^{s_0}(\mathbb{R}^n)}$, $0 \leq t \leq T$, in view of the standard Sobolev inequality, and then devising a contraction argument with respect to the norm $\sup_{0\leq\tau\leq T} \|\Phi(\tau,\cdot)\|_{H^s(\mathbb{R}^n)}$ for $s \geq s_0$ and sufficiently small T.

We note in passing that this method is very wasteful and that modern techniques lead to an improved value of the minimal exponent s_0. The new methods avoid the crude use of Sobolev inequalities and rely instead on spacetime estimates such as Strichartz and bilinear estimates. For example, it was shown in [61] that when $n \in \{3,4,5\}$, the general[23] equation $(g^{-1})^{\alpha\beta}(\Psi)\partial_\alpha\partial_\beta\Psi = \mathcal{N}^{\alpha\beta}(\Psi)\partial_\alpha\Psi\partial_\beta\Psi$ is locally well-posed for data $(\Psi, \partial_t\Psi) \in H^s \times H^{s-1}$ whenever $s > (n+1)/2$. The best result in this direction is the recent resolution of the bounded L^2 curvature conjecture, see [33], which for the Einstein-vacuum equations in 3 space dimensions essentially leads to local well-posedness in H^2. That is, for the Einstein equations, this result further improves those of [61] from $s > 2$ to $s = 2$.

1.2.2. *Beyond local existence via the vectorfield method.* As we saw in Proposition 1.1, to go beyond local existence, the main step is to obtain control on the integral in the exponent of (1.2.2). In the proof above, we crudely used the standard Sobolev inequality to bound the integral $\int_0^t \|\Phi(\tau,\cdot)\|_{W^{2,\infty}} d\tau$ and we therefore did not account for the dispersive decay of solutions to wave equations. If we could prove that the well-known uniform dispersive decay rate $(1+t)^{-\frac{n-1}{2}}$ of solutions to the standard linear wave equation $\Box_m \Phi = 0$ also holds also for solutions to the nonlinear wave equation (1.1.8) (and their up-to-second-order derivatives), then the exponential term on the right hand side of (1.2.2) would be integrable for $n \geq 4$ and only logarithmically divergent for $n = 3$. Note that the former estimate implies small-data global existence,[24] while the latter one implies the almost global existence result of [21]. These estimates on decay rates are true as stated, but are nontrivial to prove. The first results in this direction [21,23,24,26,42] were based on the explicit fundamental solution for $\Box_m$ and as such were quite cumbersome and difficult to extend to more complicated situations. The first modern proof, based on the commuting vectorfield method and generalized energy estimates, appeared in [35], though a related method had previously been used in linear theory to derive local decay estimates in the exterior of a convex domain[25] [54]. We now provide a short summary of the commuting vectorfield method as it appears in [35]. The idea is to replace the multi-indexed rectangular spatial derivative operators $\partial^{\vec{I}}$ used in the derivation of (1.2.2) with a larger class of multi-indexed differential operators $\mathscr{Z}^{\vec{I}}_{(Flat)} := Z^{I_1}_{(Flat;1)} \cdots Z^{I_p}_{(Flat;p)}$ that have good commutator properties with the Minkowski wave operator $\Box_m$, where the vectorfields $Z_{(Flat;1)}, \cdots, Z_{(Flat;p)}$ are the

[23] In the particular case of the Einstein-vacuum equations expressed with respect to wave coordinates, the same result was proved earlier in [40].

[24] For $n \geq 5$, this argument can be extended to show small-data global existence in the presence of arbitrary nonlinear terms quadratic in $(\Phi, \partial\Phi, \partial^2\Phi)$ in equation (1.1.8). For $n = 4$, the argument can similarly be extended as long as there are no quadratic terms of the form Φ^2.

[25] The Minkowskian *Morawetz vectorfields* $(t^2+r^2)\partial_t + 2tr\partial_r$ and $f(r)\partial_r$, for appropriate functions $f(r)$, also play fundamental roles in the modern vectorfield method and have their roots in [54].

elements of the following subset of conformal Killing fields[26] of m, expressed relative to rectangular coordinates:

$$\mathscr{Z}_{(Flat)} := \{\partial_t, S_{(Flat)} = t\partial_t + \sum_{a=1}^{n} x^a\partial_a\} \cup \{\partial_i, L_{(Flat;i)} = x^i\partial_t + t\partial_i\}_{1\le i\le n} \cup \{O_{(Flat;ij)} = x^i\partial_j - x^j\partial_i\}_{1\le i<j\le n}, \quad (1.2.3)$$

which forms an $\mathbb{R}$-Lie algebra with the Lie bracket given by the vectorfield commutator. We can then derive energy-type estimates similar to those in (1.2.2), but with $\partial^{\vec{I}}$ replaced by $\mathscr{Z}^{\vec{I}}_{(Flat)}$, and with the H^s norm on the left-hand side of (1.2.2) replaced by the norm[27]

$$|||\Phi|||_{T,s} := \sup_{0\le t\le T}\Big(\sum_{|\vec{I}|\le s}\|\partial\mathscr{Z}^{\vec{I}}_{(Flat)}\Phi(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}\Big)^{1/2}. \quad (1.2.4)$$

The norm (1.2.4) controls $\partial\Phi$ not only in the standard $L^\infty(\mathbb{R}^n)$ norm (through the Sobolev inequality as before), but also the weighted version $(1+t)^{\frac{n-1}{2}}\|\cdot\|_{L^\infty(\mathbb{R}^n)}$, which yields the expected uniform $(1+t)^{-\frac{n-1}{2}}$ rate of decay of $\partial\Phi$ and its lower-order $Z_{(flat)}$ derivatives. A standard way to obtain this control is to use Klainerman-Sobolev inequality [35]:

$$\sup_{0\le t\le T}\sup_{x\in\mathbb{R}^n}(1+t+r)^{\frac{n-1}{2}}(1+|t-r|)^{1/2}|\partial\Phi(t,x)| \le C|||\Phi|||_{T,(n+2)/2}, \quad (1.2.5)$$

where $r = \sqrt{\sum_{a=1}^3 (x^a)^2}$.

In addition, from the boundedness of $|||\Phi|||_{T,s}$, we can derive a further refined account of the dispersive properties of waves which shows that for $t \ge 0$, the derivatives of $\partial\Phi$ in directions tangent to the outgoing Minkowski cones $\{t - r = \text{const}\}$ have better decay properties than derivatives in a transversal direction. To illustrate this fact, we first introduce the standard radial null pair in Minkowski space:

$$L_{(Flat)} := \partial_t + \partial_r, \qquad \underline{L}_{(Flat)} := \partial_t - \partial_r, \quad (1.2.6)$$

with $\partial_r = \frac{x^a}{r}\partial_a$ the standard Euclidean radial derivative. Note that $L_{(Flat)}$ and $\underline{L}_{(Flat)}$ are null vectorfields relative to the Minkowski metric, that is, $m(L_{(Flat)}, L_{(Flat)}) = m(\underline{L}_{(Flat)}, \underline{L}_{(Flat)}) = 0$, and they satisfy $m(L_{(Flat)}, \underline{L}_{(Flat)}) = -2$. The null pair can be completed to a null frame by choosing, at every point in $\mathbb{R}^{1+n}$, $n-1$ vectorfields $e_1, \dots, e_{n-1}$ orthogonal to $e_n := L_{(Flat)}, e_{n+1} := \underline{L}_{(Flat)}$ such that $m(e_i, e_j) = \delta_{ij}$ for $i, j = 1, \dots, n-1$. As we sketch below, assuming that we have control over $|||\Phi|||_{T,(n+4)/2}$, we are

[26]The vectorfield method has also been extended to apply to some equations that are not invariant under the full Lie algebra of conformal symmetries of Minkowski spacetime, but are instead invariant under only a subalgebra; see, for example, [41, 60, ?tS1997, ?tS2000].

[27] Note that the norm $|||\Phi|||_{T,s}$ does not directly control Φ itself or its $\mathscr{Z}_{(Flat)}$ derivatives in L^2. Various approaches for controlling these terms are described in [37, 49, 62].

also able to obtain uniform control of the following derivatives ($a = 1, \cdots, n-1$):

$$\begin{cases} \sup_{0\leq t\leq T}\sup_{x\in\mathbb{R}^n}(1+t+r)^{\frac{n+1}{2}}(1+|t-r|)^{1/2}|e_a(\partial\Phi)(t,x)|, \\ \sup_{0\leq t\leq T}\sup_{x\in\mathbb{R}^n}(1+t+r)^{\frac{n+1}{2}}(1+|t-r|)^{1/2}|L_{(Flat)}(\partial\Phi)(t,x)|, \\ \sup_{0\leq t\leq T}\sup_{x\in\mathbb{R}^n}(1+t+r)^{\frac{n-1}{2}}(1+|t-r|)^{3/2}|\underline{L}_{(Flat)}(\partial\Phi)(t,x)|. \end{cases} \tag{1.2.7}$$

In other words, the derivatives of $\partial\Phi$ in the directions $e_1, \ldots, e_{n-1}, L_{(Flat)}$, which span the tangent space of the outgoing Minkowski cones $\{t - r = \text{const}\}$, have better uniform decay rates than $\underline{L}_{(Flat)}\partial\Phi$. The gain of decay rates can be obtained by expressing the vectorfields $e_1, \ldots, e_{n-1}, L_{(Flat)}, \underline{L}_{(Flat)}$ in terms of the vectorfields in $\mathscr{Z}_{(Flat)}$ and estimating the coefficients. The decay estimates for $L_{(Flat)}(\partial\Phi)$ and $\underline{L}_{(Flat)}(\partial\Phi)$ come from combining (1.2.5) with the algebraic identities

$$(t+r)L_{(Flat)} = S_{(Flat)} + \frac{1}{r}\sum_{i=1}^{n} x^i L_{(Flat;i)}, \tag{1.2.8a}$$

$$(t-r)\underline{L}_{(Flat)} = S_{(Flat)} - \frac{1}{r}\sum_{i=1}^{n} x^i L_{(Flat;i)}. \tag{1.2.8b}$$

The decomposition for $e_a(\partial\Phi)$ is similar, but slightly more involved.

The above discussion can also be used to provide clear motivation for the null condition and the corresponding small-data global existence results [10, 36] in $3D$. The null condition (see Subsubsect. 1.2.3) is designed to capture the fact that some quadratic terms exhibit better decay properties than others. For example, we can consider the bilinear forms

$$\mathscr{Q}_0(\Phi,\Psi) := (m^{-1})^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Psi, \qquad \mathscr{Q}_{\alpha\beta}(\Phi,\Psi) := \partial_\alpha\Phi\partial_\beta\Psi - \partial_\beta\Phi\partial_\alpha\Psi. \tag{1.2.9}$$

If $\mathscr{Q}$ is any of the bilinear forms (1.2.9), then by using vectorfield algebra as in (1.2.8a)-(1.2.8b), it is straightforward to derive the pointwise estimate[28]

$$|\mathscr{Q}(\Phi,\Psi)| \leq \frac{C}{1+t+r}\sum_{Z_{(Flat)},Z'_{(Flat)}\in\mathscr{Z}_{(Flat)}} \left|Z_{(Flat)}\Phi\right|\left|Z'_{(Flat)}\Psi\right|. \tag{1.2.10}$$

When $n = 3$, the gain of the critically important factor $(1+t+r)^{-1}$ helps one avoid logarithmic divergences in L^2 estimates involving $\mathscr{Q}(\Phi,\Psi)$. In contrast, for a general quadratic form such as, for example, $\partial_t\Phi\partial_t\Psi$ or $\nabla\Phi\cdot\nabla\Psi := \sum_{a=1}^{n}\partial_a\Phi\partial_a\Psi$, the factor $(1+t+r)^{-1}$ in (1.2.10) must be replaced with $(1+|t-r|)^{-1}$, which yields no gain in the wave zone $\{t \sim r\}$.

1.2.3. *The classic null condition.* The considerations described in Subsubsect. 1.2.2 lead to the classic null condition for equations of type (1.1.8) and for (1.1.10) in the scalar case, which we will now discuss. We first consider equation (1.1.8). Since we are studying only the behavior of small solutions, we rewrite the equation as a perturbation of the linear wave equation $\Box_m\Phi = 0$, that is, in the form

$$-\partial_t^2\Phi + \Delta\Phi + \mathcal{A}^{\alpha\beta}(\partial\Phi)\partial^2_{\alpha\beta}\Phi = \mathcal{N}(\Phi,\partial\Phi), \tag{1.2.11}$$

[28] If we only used rectangular coordinates derivatives, then we could only derive the weaker estimate $\mathscr{Q}(\Phi,\Psi) \leq C|\partial\Phi||\partial\Psi|$.

where $\mathcal{A}^{\mu\nu} = \mathcal{O}(|\partial\Phi|)$ and $\mathcal{N} = \mathcal{O}(|\partial\Phi|^2)$ for small $(\Phi, \partial\Phi)$. Taylor expanding further $\mathcal{A}$ and $\mathcal{N}$, we have

$$\mathcal{A}^{\mu\nu}(\partial\Phi) = \mathcal{A}^{\mu\nu\sigma}\partial_\sigma\Phi + \mathcal{O}(|\partial\Phi|^2), \tag{1.2.12}$$

$$\mathcal{N}(\Phi, \partial\Phi) = \mathcal{N}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \mathcal{O}(|\Phi||\partial\Phi|^2 + |\partial\Phi|^3), \tag{1.2.13}$$

where the constants $\mathcal{A}^{\mu\nu\sigma}$ and $\mathcal{N}^{\mu\nu}$ are

$$\mathcal{A}^{\mu\nu\sigma} := \frac{\partial}{\partial(\partial_\sigma\Phi)}\mathcal{A}^{\mu\nu}(\partial\Phi)|_{\partial\Phi=0}, \tag{1.2.14}$$

$$\mathcal{N}^{\mu\nu} := \frac{\partial^2}{\partial(\partial_\mu\Phi)\partial(\partial_\nu\Phi)}\mathcal{N}(\Phi, \partial\Phi)|_{(\Phi,\partial\Phi)=(0,0)}. \tag{1.2.15}$$

Similarly, under the assumptions $(g^{-1})^{\mu\nu}(\Psi = 0) = (m^{-1})^{\mu\nu}$ and that $\mathcal{N}(\Psi, \partial\Psi) = \mathcal{O}(|\partial\Psi|^2)$ for small $(\Psi, \partial\Psi)$, we can rewrite (1.1.10) as a perturbation of the linear wave equation, where $\mathcal{A}^{\mu\nu}(\partial\Phi)$ in (1.2.11) is replaced by $\mathcal{A}^{\mu\nu}(\Psi)$, $\mathcal{N}(\Phi, \partial\Phi)$ is replaced by $\mathcal{N}(\Psi, \partial\Psi)$, $\mathcal{A}^{\mu\nu\sigma}$ is replaced by $\mathcal{A}'^{\mu\nu} := \frac{d}{d\Psi}\mathcal{A}^{\mu\nu}(\Psi)|_{\Psi=0}$, and $\mathcal{N}^{\mu\nu}$ is replaced by $\mathcal{N}^{\mu\nu} := \frac{\partial^2}{\partial(\partial_\mu\Psi)\partial(\partial_\nu\Psi)}\mathcal{N}(\Psi, \partial\Psi)|_{(\Psi,\partial\Psi)=(0,0)}$.

Definition 1.1 (**Classic null condition**)**.** We say that the nonlinearities in equation (1.2.11) verify the classic null condition if for every covector $\ell = (\ell_0, \ell_1, \ell_2, \ell_3)$ satisfying $(m^{-1})^{\alpha\beta}\ell_\alpha\ell_\beta := -\ell_0^2 + \ell_1^2 + \ell_2^2 + \ell_3^2 = 0$, we have the identities

$$\mathcal{A}^{\mu\nu\sigma}\ell_\mu\ell_\nu\ell_\sigma = \mathcal{N}^{\mu\nu}\ell_\mu\ell_\nu = 0.$$

Similarly, in the case of (1.1.10) with $I = 1$ (the case of a single scalar equation), we say that the nonlinearities verify the classic null condition if for every Minkowski-null covector ℓ, we have the identities

$$\mathcal{A}'^{\mu\nu}\ell_\mu\ell_\nu = \mathcal{N}^{\mu\nu}\ell_\mu\ell_\nu = 0\,. \tag{1.2.16}$$

Remark 1.3. Definition 1.1 can be extended for systems of wave equations; see Remark 2.12 or [34].

We now provide two standard examples.

- For the scalar equation (1.2.11), it is straightforward to check that the quadratic semilinear term $\mathcal{N}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi$ verifies the classic null condition if and only if it is a constant multiple of the null form $\mathscr{Q}_0(\Phi, \Phi)$ from (1.2.9).
- Similarly, for equation (1.1.10) in the scalar case under the assumption $(g^{-1})^{\mu\nu}(\Psi = 0) = (m^{-1})^{\mu\nu}$, one can show that the quadratic quasilinear terms verify the classic null condition if and only if $\mathcal{A}'^{\mu\nu}$ is a multiple of $(m^{-1})^{\mu\nu}$.

The proof of global existence for equations of type (1.2.11) verifying the null condition follows a similar pattern as the proof of the almost global existence in [21] by taking into account the favorable factor $(1 + t + r)^{-1}$ in (1.2.10). Another important feature of the proof, which is by now a familiar aspect of the literature, is that the highest energy norm is not bounded but is instead allowed to grow like a small power of t as $t \to \infty$. Despite the possible slow top-order energy growth, the resulting global solutions to equations verifying the classic null condition in fact enjoy the same type of peeling properties (1.2.7) as solutions to the linear wave equation in Minkowski spacetime (at least as far as the low-order derivatives of Φ are concerned). In the small-data shock-formation problem, we also encounter a similar top-order

growth phenomenon, but it is much more severe when the characteristic hypersurfaces intersect (in fact, the top-order energies are allowed to blow-up); see Prop. 3.4.

1.2.4. *John's conjecture and an overview of Alinhac's proof of it for non-degenerate small data.* In $3D$, when the quadratic nonlinearities fail the classic null condition, we expect that small-data global existence fails to hold (recall that John showed [25] that in many cases, one does have a breakdown, though the mechanism is not revealed by the proof). Nonetheless, we still have the almost global existence result of John and Klainerman mentioned earlier and also a sharper version, due to John and Hörmander, which we state as Theorem 1. We first recall that the Radon transform of a function f on $\mathbb{R}^3$ can be defined for points $q \in \mathbb{R}$, $\theta \in \mathbb{S}^2 \subset \mathbb{R}^3$ as

$$\mathcal{R}[f](q,\theta) := \int_{P_{q,\theta}} f(y)\, d\sigma_{q,\theta}(y), \tag{1.2.17}$$

where $P_{q,\theta} := \{y \in \mathbb{R}^3 \mid e(\theta, y) = q\}$ is the plane with unit normal θ that passes through $q\theta \in \mathbb{R}^3$, $d\sigma(y)$ denotes the area form induced on the plane $P_{q,\theta}$ by the Euclidean metric e on $\mathbb{R}^3$, and $e(\theta, y)$ is the Euclidean inner product of θ and y. We also introduce the following function $\digamma[(\mathring{\Phi}, \mathring{\Phi}_0)] : \mathbb{R} \times \mathbb{S}^2 \to \mathbb{R}$, which also depends on the initial data pair $(\Phi|_{t=0}, \partial_t\Phi|_{t=0}) = (\mathring{\Phi}, \mathring{\Phi}_0)$:

$$\digamma[(\mathring{\Phi}, \mathring{\Phi}_0)](q,\theta) := -\frac{1}{4\pi}\frac{\partial}{\partial q}\mathcal{R}[\mathring{\Phi}](q,\theta) + \frac{1}{4\pi}\mathcal{R}[\mathring{\Phi}_0](q,\theta). \tag{1.2.18}$$

Remark 1.4 (Friedlander's radiation field). The function $\digamma[(\mathring{\Phi}, \mathring{\Phi}_0)]$ from (1.2.18) is Friedlander's radiation field for the solution to the *linear wave equation* corresponding to the data $(\mathring{\Phi}, \mathring{\Phi}_0)$. See Subsect. 5.5 for an extended discussion of the role that $\digamma[(\mathring{\Phi}, \mathring{\Phi}_0)]$ plays in determining when and where blow-up occurs.

Theorem 1. [19; 29, **John and Hörmander**] *Consider the initial value problem*

$$(g^{-1})^{\alpha\beta}(\partial\Phi)\partial_\alpha\partial_\beta\Phi = 0,$$
$$(\Phi|_{t=0}, \partial_t\Phi|_{t=0}) = \lambda(\mathring{\Phi}, \mathring{\Phi}_0)$$

for a quasilinear wave equation in $\mathbb{R}^{1+3}$ verifying (1.1.7) *with compactly supported smooth initial data, for which the classical null condition does* ***not*** *hold. Then the classical lifespan $T_{(Lifespan);\lambda}$ of the solution verifies*

$$\liminf_{\lambda\downarrow 0} \lambda \ln T_{(Lifespan);\lambda} \geq \frac{1}{\sup_{(q,\theta)\in\mathbb{R}\times\mathbb{S}^2} \frac{1}{2}{}^{(+)}\aleph(\theta)\frac{\partial^2}{\partial q^2}\digamma[(\mathring{\Phi}, \mathring{\Phi}_0)](q,\theta)}, \tag{1.2.19}$$

where ${}^{(+)}\aleph$ is the future null condition failure factor for the equation; see (4.2.2) *for an explicit formula.*

The natural conjecture, which was envisioned by F. John,[29] is that Theorem 1 is sharp and that small-data solutions in fact blow up at times near (1.2.19). Moreover, the blow-up should be due to the crossing of characteristics, similar to the case of the $1D$ Burgers' equation. A restricted version of this conjecture,

[29] In [30], John also contemplated the possibility that away from spherical symmetry, singularity formation might be avoided.

applicable to initial data satisfying some non-degeneracy conditions, was first proved by Alinhac; see Theorem 3 and the discussion in Subsect. 5.3.

It is easy to see that the right-hand side of (1.2.19) must be non-negative for compactly supported data. The importance of Alinhac's work is further enhanced by the next proposition, which shows that the right-hand side of (1.2.19) is strictly positive whenever the data are compactly supported and nontrivial. Thus, Alihnac's work shows that in the $\lambda \downarrow 0$ limit, for nontrivial data verifying his non-degeneracy conditions, *shocks will always form*. Moreover, as we describe in Subsect. 5.5.3, Alinhac's non-degeneracy conditions on the data turn out to be unnecessary. However, as we describe in Subsect. 5.3, his proof cannot be extended to recover this fact; the proof requires the full power of Christodoulou's framework.

Proposition 1.2. [29, pg. 98] *Let* $\mathring{\Phi}, \mathring{\Phi}_0 \in C_c^\infty(\mathbb{R}^3)$. *Assume that* ${}^{(+)}\aleph \not\equiv 0$ *and that*

$$\sup_{(q,\theta)\in\mathbb{R}\times\mathbb{S}^2} {}^{(+)}\aleph(\theta)\frac{\partial^2}{\partial q^2}\digamma[(\mathring{\Phi},\mathring{\Phi}_0)](q,\theta) = 0. \tag{1.2.20}$$

Then $(\mathring{\Phi},\mathring{\Phi}_0) = (0,0)$.

1.3. **A sharp description of small-data shock formation for spherically symmetric solutions in** $3D$**.** We are now ready to describe, in the simplified setting of spherical symmetry, how failure of the classic null condition can cause small-data solutions to equations of type (1.2.11) to form shock-type singularities. It turns out that in the small-data regime, the main mechanism of shock formation is the same both in and out of spherical symmetry. Hence, in the spherically symmetric case, we provide a detailed proof of singularity formation in the higher-order derivatives and regularity of the lower-order derivatives within an appropriate wave zone using the framework[30] of Christodoulou [11].

1.3.1. *Geometric formulation of the problem.* Following F. John [28], we examine the model equation (1.1.12). In particular, we focus here on the simplest case[31] $\Box_m\Phi = -\partial_t\Phi\Delta\Phi$, which takes the following form relative to standard spherical coordinates on Minkowski spacetime:

$$\partial_t^2(r\Phi) = (1+\partial_t\Phi)\,\partial_r^2(r\Phi). \tag{1.3.1}$$

In (1.3.1), $\Phi(t,x) = \Phi(t,r)$ and $r := \sqrt{\sum_{a=1}^3 (x^a)^2}$. We expect that shock formation corresponds to the blow-up of some second derivatives of Φ, while Φ itself and its first derivatives remain bounded. Hence, we can equivalently consider the equation

$$-\partial_t^2\,(r\Psi) + (1+\Psi)\,\partial_r^2\,(r\Psi) = -r\frac{(\partial_t\Psi)^2}{1+\Psi}\,. \tag{1.3.2}$$

for $\Psi := \partial_t\Phi$ induced from (1.3.1), and show that Ψ remains bounded while some of its first derivatives blow up. Our analysis takes place in a small strip $\mathcal{M}_{t,U_0}$ (contained in the "wave zone") defined just below; see also Figure 1.

[30]The proof given here is a bit sharper than the one given by John [28] in that it exhibits the precise blow-up mechanism due to the intersection of the characteristic hypersurfaces, similar to that of Burgers' equation (see Subsubsect. 1.1.1). The additional precision afforded by Christodoulou's framework is essential for extending the result beyond spherical symmetry.

[31]It is straightforward to see that this equation fails the classic null condition of Definition 1.1.

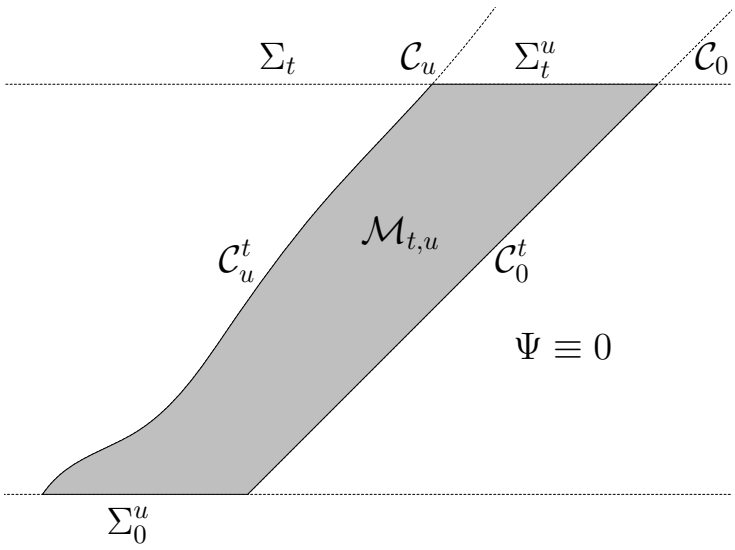


FIGURE 1. Regions of interest, where $u \in [0, U_0]$

To define the region $\mathcal{M}_{t,U_0}$, we note that the initial value problem for (1.3.2) with initial data

$$\Psi(0,r) := \mathring{\Psi}(r), \qquad \partial_t \Psi(0,r) := \mathring{\Psi}_0(r) \tag{1.3.3}$$

can be solved using the method of characteristics. The characteristic vectorfields[32] are

$$L := \partial_t + \sqrt{1+\Psi}\partial_r \quad \text{and} \quad \underline{L} := \partial_t - \sqrt{1+\Psi}\partial_r\,. \tag{1.3.4}$$

Corresponding to the "outgoing" vectorfield L, we can define an *eikonal function* $u(t,r)$ satisfying[33]

$$Lu(t,r) = 0,$$

with $u = const$ defining the outgoing characteristics $\mathcal{C}_u$. The function $u(t,r)$ is uniquely determined once we fix its value along the hypersurface $\{t=0\}$; we initialize u by prescribing

$$u|_{t=0} := 1 - r. \tag{1.3.5}$$

Finite speed of propagation [25, 31] for the wave equation implies that the solution along $\mathcal{C}_u$ only depends on the data at points $r \geq 1-u$.

We assume for convenience that $(\mathring{\Psi}, \mathring{\Psi}_0)$ are supported in $\{r \leq 1\}$. This implies that $\Psi \equiv 0$ when $u \leq 0$. We thus define, in spherical coordinates, the region of interest

$$\mathcal{M}_{t,U_0} := \{(t',r) \mid 0 \leq t' < t \text{ and } 0 \leq u(t',r) \leq U_0\}. \tag{1.3.6}$$

[32]These are the replacements for $L_{(Flat)}$ and $\underline{L}_{(Flat)}$ defined in (1.2.6), and are adapted to the true quasilinear geometry of the equation. Note that we chose them to be normalized (see Footnote 33 for notation) $Lt = \underline{L}t = 1$.

[33]If X is a vectorfield and f a function, we write $Xf = X^\alpha \partial_\alpha f$ for the X-directional derivative of f.

On $\mathcal{M}_{t,U_0}$, the solution depends only on the data belonging to the annulus $r \in [1-U_0, 1]$. For convenience in notation, we also define

$$\begin{aligned} \Sigma_t &:= \{(t,r) \mid r \geq 0\}, & \Sigma_t^{u'} &:= \{(t,r) \mid 0 \leq u(t,r) \leq u'\}, \\ \mathcal{C}_{u'} &:= \{(t',r) \mid 0 \leq t' \text{ and } u(t',r) = u'\}, & \mathcal{C}_{u'}^t &:= \{(t',r) \mid 0 \leq t' \leq t \text{ and } u(t',r) = u'\}. \end{aligned} \tag{1.3.7a}$$

Definition 1.2 (Inverse foliation density). The quantity μ defined by

$$\mu^{-1} := \partial_t u(t,r) = -\sqrt{1+\Psi}\partial_r u, \tag{1.3.8}$$

is called the *inverse foliation density*. It is also known as the null lapse.

By the choice of initial data for u, on Σ_0, we see $\mu = 1 + \mathcal{O}(\Psi)$. The quantity μ^{-1} plays a fundamental role in the analysis of shock formation. It measures the density of the leaves[34] $\mathcal{C}_u$ with respect to the time coordinate t, and shock formation (intersection of characteristics) corresponds to $\mu \to 0$. As long as μ remains positive, the two functions t, u are independent and form a coordinate system of $\mathcal{M}_{t,U_0}$. As we will see, there are advantages to using the "geometric" coordinates (t,u) in place of (t,r). We note that

$$L = \frac{\partial}{\partial t}|_u, \qquad \mu \underline{L} u = 2. \tag{1.3.9}$$

Straightforward computations reveal that (1.3.2) can be expressed in the two equivalent forms

$$L\underline{L}(r\Psi) = \frac{1}{4}\frac{r}{(1+\Psi)}\left[(\underline{L}\Psi)^2 + 3(L\Psi)(\underline{L}\Psi)\right] - \frac{1}{2}\frac{1}{\sqrt{1+\Psi}}\Psi(L\Psi), \tag{1.3.10a}$$

$$\underline{L}L(r\Psi) = \frac{1}{4}\frac{r}{(1+\Psi)}\left[(L\Psi)^2 + 3(L\Psi)(\underline{L}\Psi)\right] + \frac{1}{2}\frac{1}{\sqrt{1+\Psi}}\Psi(\underline{L}\Psi), \tag{1.3.10b}$$

which can be used to derive estimates along the characteristic directions. Note that (1.3.10b) follows from (1.3.10a) and the commutator relations

$$[L, \underline{L}] = -\frac{\partial_t \Psi}{\sqrt{1+\Psi}}\partial_r, \qquad [L, \underline{L}]\Psi = \frac{(\underline{L}\Psi)^2 - (L\Psi)^2}{4(1+\Psi)}.$$

Remark 1.5. Examining the semilinear terms in equations (1.3.10a)-(1.3.10b), we see that some of the nonlinearities in equation (1.3.1) fail the classic null condition of Definition 1.1. In particular, both[35] $(L\Psi)^2$ and $(\underline{L}\Psi)^2$ fail[36] the classic null condition of Definition 1.1. However, in view of the forward peeling properties (1.2.7), we expect that in the relevant future region $\mathcal{M}_{t,U_0}$, $L\Psi$ decays faster than $\underline{L}\Psi$. Hence, the only term that behaves poorly, from the point of view of *linear decay*, is the term $\frac{1}{4}\frac{r}{1+\Psi}(\underline{L}\Psi)^2$ on the right-hand side of (1.3.10a). This term is in fact the source of the small-data shock formation: we will use the estimate $r \approx t$ (within $\mathcal{M}_{t,U_0}$) to show that this term drives a Riccati-type blow-up along the integral curves of L. We rigorously prove a refined version of this claim in Prop. 1.3 and Cor. 1.4. Furthermore, as we will see, the terms $\frac{1}{\sqrt{1+\Psi}}\Psi(L\Psi)$ and $\frac{1}{\sqrt{1+\Psi}}\Psi(\underline{L}\Psi)$ are negligible error terms.

[34] Later, we will introduce the eikonal function and the inverse foliation density in full generality, without the assumption of spherical symmetry. Here it is sufficient to understand them in the context of the method of characteristics.

[35] Note that in the limit $\Psi \to 0$ we have $L \to L_{(Flat)}$ and similarly for $\underline{L}$.

[36] As we will see, the remaining terms in (1.3.10a)-(1.3.10b) can be treated as negligible errors.

1.3.2. *Rescaling in the transversal direction.* The crucial observation[37] of Christodoulou is that we can derive an equivalent system of equations for new μ-weighted quantities for which the problematic term $(\underline{L}\Psi)^2$ does not appear. As we show below in Prop. 1.3, the rescaled system can then be treated by straightforward dispersive-type methods (reminiscent of the peeling properties (1.2.7)), in the spirit of small-data (spherically symmetric) global existence results. This rescaling by the factor of μ takes place only in the $\mathcal{C}_u$-transversal direction $\underline{L}$ and has a simple interpretation, at least in spherical symmetry, in terms of the method of characteristics. More precisely, it is straightforward to show that relative to the (t,u) coordinates, we have the identity $\mu\underline{L} = \mu\frac{\partial}{\partial t} + 2\frac{\partial}{\partial u}$. We note that expressing Ψ as a function of (t,u) is analogous to our earlier representation of a solution to the Burgers' equation (1.1.1) in the characteristic (also known as Lagrangian) coordinates (t,α) (see (1.1.3)). Just as solutions of Burgers' equation remain regular[38] *in the coordinates* (t,α), the shock-forming solutions of (1.3.2) remain regular in the coordinates (t,u); the singularity manifests itself only when we change variables back to the (t,r) coordinates because the Jacobian of the change of variables map[39] contains factors of μ^{-1}.

To reveal the rescaled structure, we first note that from the definition (1.3.8), we have

$$L\mu^{-1} = L\partial_t u(t,r) = \partial_t L u(t,r) - [\partial_t, L]\, u(t,r).$$

From this equation and the identities $Lu(t,r) = 0$, $\underline{L}u(t,r) = 2\mu^{-1}$ and $[\partial_t, L] = \frac{1}{2}\frac{1}{\sqrt{1+\Psi}}\partial_t\Psi\partial_r$, we deduce that

$$L\mu = -\frac{1}{4}\frac{1}{(1+\Psi)}\mu\left(L\Psi + \underline{L}\Psi\right). \tag{1.3.11}$$

Hence, we can rewrite (1.3.10a)-(1.3.10b) as

$$L\left(\mu\underline{L}\left(r\Psi\right)\right) = \frac{1}{2}\frac{r}{(1+\Psi)}\left[(L\Psi)\mu\underline{L}\Psi\right] - \frac{1}{2}\frac{\mu}{\sqrt{1+\Psi}}\Psi L\Psi, \tag{1.3.12a}$$

$$\mu\underline{L}L\left(r\Psi\right) = \frac{1}{4}\frac{r}{(1+\Psi)}\left[\mu\left(L\Psi\right)^2 + 3(L\Psi)\mu\underline{L}\Psi\right] + \frac{1}{2}\frac{1}{\sqrt{1+\Psi}}\Psi\mu\underline{L}\Psi. \tag{1.3.12b}$$

A key point is that all products on the right-hand sides of (1.3.12a)-(1.3.12b) are expected to decay at an integrable-in-time rate. In summary, we have formulated a system of equations that on the one hand is expected to remain regular and exhibit dispersive properties, and on the other hand is tailored to see the blow-up of precisely the $\underline{L}$ derivative of the solution as $\mu \to 0$.

1.3.3. *A sharp classical lifespan result and proof of shock formation.* The rescaling by μ has introduced a partial decoupling of (1.3.2) into the wave equations (1.3.12a)-(1.3.12b), which we expect to remain regular, and a transport equation (1.3.11) for the inverse foliation density μ, which we expect to drive the blow-up of (t,r) coordinate derivatives of Ψ. This allows us to attack the problem of shock formation as a two-step process, which we now outline.

[37] F. John implicitly used a similar strategy in his original argument [28] and in his earlier work [22] in $1D$. The idea of using Burgers' equation as a guide and studying the system in characteristic coordinates is also used, in a somewhat different form, by Alinhac [1–5].

[38] Relative to the coordinates (t,α), the solution Ψ to Burgers' equation (1.1.1) verifies $\frac{\partial}{\partial t}\Psi = 0$ and $\frac{\partial}{\partial\alpha}\Psi = \mathring{\Psi}'(\alpha)$, where $\mathring{\Psi} := \Psi|_{t=0}$.

[39] Equivalently, if $\mu\underline{L}\Psi$ remains non-zero when $\mu \searrow 0$, we must have $|\underline{L}\Psi| \nearrow \infty$.

(1) First, we prove "global-existence-type" estimates and establish a breakdown criterion for the system in the small data regime. In particular, we will show that classical solutions can be continued as long as μ remains away from 0. Furthermore, we will show that when $\mu \to 0$, some of the coordinate derivatives of Ψ must blow-up. We prove these claims in Prop. 1.3.
(2) Next, using the global-existence-type estimates from Step (1), we can rigorously justify our intuition that the $(\underline{L}\Psi)^2$ terms in (1.3.10a) drives a Riccati-type blow-up. To this end, we study the transport equation (1.3.11) and show that the right hand side has enough positivity to drive μ to zero in finite time, provided that we sufficiently shrink the amplitude of the initial data. See Figure 2 for a picture illustrating the formation of the shock, and Cor. 1.4 for the statement.

Remark 1.6. Following Christodoulou [11], we will also use this two-step process in the non-spherically symmetric case. A related approach was also used by Christodoulou to study the formation of trapped surfaces in general relativity [12]. The main difficulty in the analysis of the full problem is precisely establishing an analog to the sharp classical lifespan Prop. 1.3 outside of spherical symmetry. Once the "global-existence-type" estimates (that is, analogs of (1.3.15)-(1.3.17) below) are established, it is relatively easy to prove a version of the shock-formation results of Cor. 1.4.

We now provide the relevant definition of the solution's lifespan in the shock formation problem.

Definition 1.3 (**Outgoing classical lifespan**)**.** We define $T_{(Lifespan);U_0}$, the *outgoing classical lifespan* of the solution with parameter U_0, to be the supremum over all times $t > 0$ such that Ψ is a C^2 solution (relative to the coordinates (t, r)) to equation (1.3.2) in the strip $\mathcal{M}_{t,U_0}$ (see Definition (1.3.6)).

We now state the main sharp classical lifespan result for spherically symmetric solutions.

Proposition 1.3 (**A sharp classical lifespan result for equation** (1.3.1))**.** *Let* $\mathring{\epsilon} := \|\mathring{\Psi}\|_{C^2} + \|\mathring{\Psi}_0\|_{C^1}$ *denote the size of the spherically symmetric data* (1.3.3)*, supported in* $\{r \leq 1\}$*, for the wave equation* (1.3.2)*. Let* $0 \leq U_0 < 1$ *be a fixed parameter. Then there exists a constant* $\epsilon_0 > 0$ *such that if* $\mathring{\epsilon} \leq \epsilon_0$*, then we have the following conclusions. First, the outgoing classical lifespan* $T_{(Lifespan);U_0}$ *of Definition 1.3 is characterized by*

$$T_{(Lifespan);U_0} = \sup\{t > 0 \mid \inf_{\Sigma_t^{U_0}} \mu > 0\}. \tag{1.3.13}$$

In addition, there exists a constant $C_{(Lower-Bound)} > 0$ *such that*

$$T_{(Lifespan);U_0} > \exp\left(\frac{1}{C_{(Lower-Bound)}\mathring{\epsilon}}\right). \tag{1.3.14}$$

Furthermore, there exists a constant $C > 0$ *such that on* $\mathcal{M}_{T_{(Lifespan);U_0},U_0}$*, we have*

$$\begin{aligned} &|r^3L^2\Psi| \leq C\mathring{\epsilon}, \qquad |r^2L(\mu\underline{L}\Psi)| \leq C\mathring{\epsilon}, \qquad |r^2\mu\underline{L}L\left(r\Psi\right)| \leq C\mathring{\epsilon}, \\ &|r^2L\Psi| \leq C\mathring{\epsilon}, \qquad |r\mu\underline{L}\Psi| \leq C\mathring{\epsilon}, \qquad |r\Psi| \leq C\mathring{\epsilon}, \\ &|\mu - 1| \leq C\mathring{\epsilon}\ln(e+t), \qquad |1 - r + t - u| \leq C\mathring{\epsilon}\ln(e+t). \end{aligned} \tag{1.3.15}$$

Finally, there exists a constant $c > 0$ *such that at any point with* $\mu < 1/4$*, we have*

$$L\mu \leq -c\frac{1}{(1+t)\ln(e+t)}. \tag{1.3.16}$$

and

$$|\mu \underline{L}\Psi| \geq c\frac{1}{(1+t)\ln(e+t)}. \tag{1.3.17}$$

In particular, it follows from (1.3.17) *that* $\underline{L}\Psi$ *blows up like* μ^{-1} *at points where* μ *vanishes.*

Remark 1.7 (**The sharp "constant").** As was first shown by John [29] and Hörmander [19] in Theorem 1, the sharp "constant" $C_{(Lower-Bound)}$ in (1.3.14) depends on the profile of the data and the structure of the nonlinearities; see also equation (5.1.5).

Remark 1.8. The estimate (1.3.16) is a quantified version of the following rough idea: the only way μ can shrink along the integral curves of L is for $L\mu$ to be significantly negative. An interesting aspect is the "point of no return" nature of this estimate: once $\mu < \frac{1}{4}$ (recall that at $t = 0$, its value is approximately 1), μ *must continue to shrink until it eventually vanishes and a shock forms*. The specific value $\frac{1}{4}$ is not significant: the actual point of no return depends on ϵ_0 and $\frac{1}{4}$ is just a convenient number.

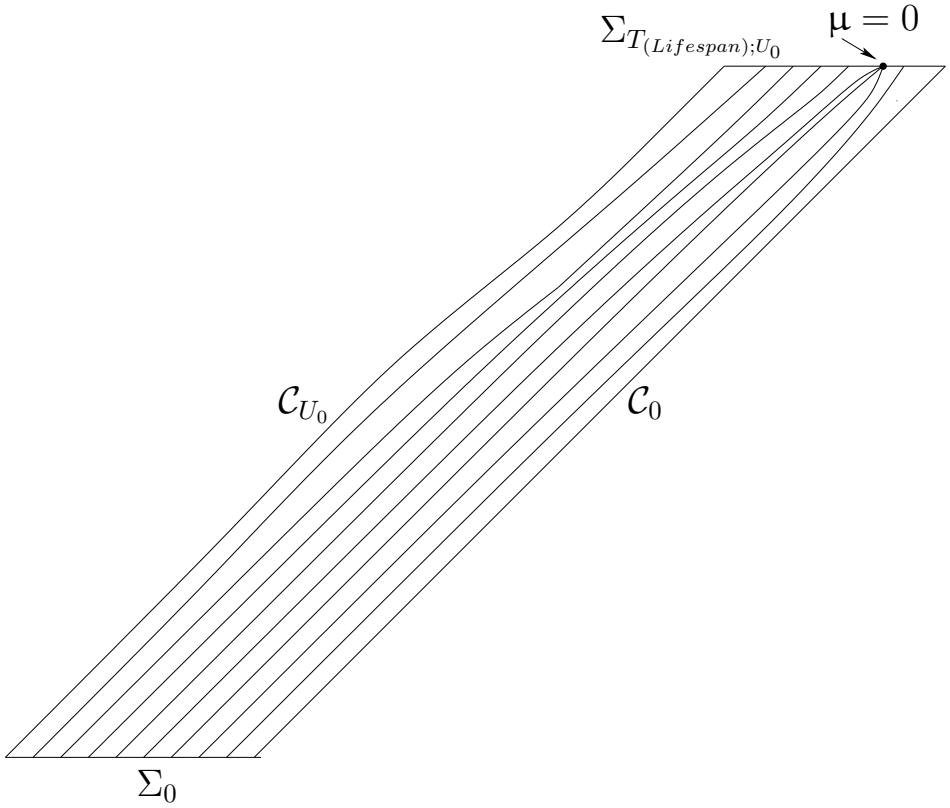


FIGURE 2. Shock formation in spherically symmetric solutions to equation (1.3.2)

Remark 1.9 (**A preview on the Heuristic Principle**)**.** Later, when investigating the general non-spherically symmetric case, we will encounter dispersive estimates in the spirit of (1.3.15), complemented with estimates for the angular derivatives. Such expected estimates, which we refer to as the "Heuristic Principle," provide the basic intuition behind our approach in the non-symmetric case.

With the help of Prop. 1.3, we can easily derive the following shock-formation result for spherically symmetric solutions.

Remark 1.10. For technical reasons, in the corollary, we start with "initial" data at time $-1/2$ supported in $\{r \leq 1/2\}$. We will explain this assumption in more detail at the end of the proof of the corollary; see Footnote 42.

Corollary 1.4 (Shock formation for rescaled spherically symmetric data). *Let $(\check{\Psi}, \check{\Psi}_0) \in C^2 \times C^1$ be nontrivial spherically symmetric "initial" data on $\Sigma_{-1/2}$ that vanish for $r \geq 1/2$. Let $(\mathring{\Psi} := \Psi|_{\Sigma_0}, \mathring{\Psi}_0 := \partial_t \Psi|_{\Sigma_0})$ denote the data induced on Σ_0 by the solution[40] Ψ. Note that $(\mathring{\Psi}, \mathring{\Psi}_0)$ vanish for $r \geq 1$. Then we can chose a $U_0 \in (0,1)$ such that if we rescale the initial data to be $(\lambda\check{\Psi}, \lambda\check{\Psi}_0)$ for sufficiently small $\lambda > 0$, then $(\mathring{\Psi}, \mathring{\Psi}_0)$ is small enough such that the results of Prop. 1.3 apply and furthermore, Ψ has a lifespan $T_{(Lifespan);U_0} < \infty$ due to μ vanishing in finite time. That is, a shock forms in finite time.*

Remark 1.11 (Maximal development of the data). An important merit of the proofs of Prop. 1.3 and Cor. 1.4 is that with some additional effort, they can be extended to reveal information beyond the hypersurface $\Sigma_{T_{(Lifespan);U_0}}$. That is, they can be extended to reveal a portion of the maximal development of the data up to the boundary; see Remark 4.1 and Figure 7.

Remark 1.12. Roughly, the maximal development is the largest possible spacetime domain on which there exists a unique classical solution determined by completely the data; see, for example, [?wW2013].

We now provide the proofs of the proposition and the corollary.

Proof of Prop. 1.3. It suffices to prove (1.3.15)-(1.3.17) on $\mathcal{M}_{T_{(Lifespan)},U_0}$. For by the identity (1.3.4), if μ remains uniformly bounded from above and from below away from 0, then the estimates (1.3.15) imply that $|\Psi|$, $|\partial_t \Psi|$, and $|\partial_r \Psi|$ remain uniformly bounded; it is a standard fact that such bounds allow us to extend the solution's lifespan (in a strip of u-width U_0).

Since our analysis is based on integrating along characteristics, we will work relative to the geometric coordinate system (t,u), where u is the eikonal function constructed above.[41] We use a continuity argument: let $\mathcal{B} \subset \left[0, T_{(Lifespan);U_0}\right)$ be the subset consisting of those times T such that the estimates (1.3.15) of the proposition hold on $\mathcal{M}_{T,U_0}$, but with $C\mathring{\epsilon}$ replaced by $\sqrt{\mathring{\epsilon}}$. We remark that for $T \in \mathcal{B}$, we have $r \approx 1 + t$ on $\mathcal{M}_{T,U_0}$. For $\epsilon_0 < 1$ sufficiently small, $\mathcal{B}$ is a connected, non-empty, relatively closed subset of $\left[0, T_{(Lifespan)}\right)$. To show that $\mathcal{B}$ is relatively open, we improve the bootstrap assumptions with a series of estimates that we now derive.

We start by deriving an estimate for $L(r\Psi)$. To this end, we insert the bootstrap assumptions into the right-hand side of equation (1.3.12b) to deduce that

$$|\mu \underline{L} L(r\Psi)| \leq C\mathring{\epsilon} \frac{1}{(1+t)^2}. \tag{1.3.18}$$

Starting from the spacetime point where we are trying to estimate $L(r\Psi)$, we integrate inequality (1.3.18) "to the past" along the integral curves of $\mu\underline{L}$ parametrized with respect to the affine parameter u (note that

[40] If that data on $\Sigma_{-1/2}$ are sufficiently small, then the solution will persist until time 0.

[41] Note that this "dynamic" coordinate depends on the solution itself; this is a feature of the quasilinear nature of the equations.

$\mu \underline{L}(u) = 2$). The curve either first intersects Σ_0, where $L(r\Psi) = \mathcal{O}(\mathring{\epsilon})$, or it first intersects the initial cone $\mathcal{C}_0$, along which the solution vanishes. Since the strip of interest has eikonal function width $U_0 < 1$, since $\frac{dt}{du} = \mu$ along the integral curve, and since $\mu(t,u) \leq C\ln(e+t)$, we see that the change in t along the integral curve is $t_2 - t_1 = \int_{t_1}^{t_2} dt \leq C\int_0^{U_0} \mu\, du \leq C\ln(e+t_2)$, where $t_1 \leq t_2$ denote the times corresponding to the terminal and initial points of the curve. In either of the two above cases, we therefore conclude from (1.3.18) that

$$|L(r\Psi)| \leq C\mathring{\epsilon}\frac{1}{(1+t)^2}. \tag{1.3.19}$$

Next, integrating inequality (1.3.19) from $t = 0$ along the integral curves of $L = \frac{\partial}{\partial t}$ and using the smallness of the data, we deduce that $|r\Psi| \leq C\mathring{\epsilon}$. In view of the bootstrap assumption corresponding to (1.3.15), we have $r \approx t$ inside our region, and therefore

$$|\Psi| \leq C\mathring{\epsilon}\frac{1}{1+t} \tag{1.3.20}$$

as desired. Next, inserting the estimate (1.3.20) into (1.3.19) and using that $|Lr| = \left|\sqrt{1+\Psi}\right| < 2$, we find that

$$|L\Psi| \leq C\mathring{\epsilon}\frac{1}{(1+t)^2} \tag{1.3.21}$$

as desired. Next, inserting the estimates (1.3.20)-(1.3.21) and the bootstrap assumptions for μ into the right-hand side of (1.3.12a), we find that

$$|L(\mu\underline{L}(r\Psi))| \leq C\mathring{\epsilon}\frac{1}{(1+t)^2}. \tag{1.3.22}$$

Integrating (1.3.22) along $\mathcal{C}_u$ from $t = 0$ and using the small-data assumption, we find that $|\mu\underline{L}(r\Psi)| \leq C\mathring{\epsilon}$. Using the bootstrap assumptions, we deduce that $\mu|\underline{L}r| = |\mu\sqrt{1+\Psi}| \leq C\ln(e+t)$ and hence, thanks to (1.3.20), that

$$|\mu\underline{L}\Psi| \leq C\mathring{\epsilon}\frac{1}{1+t} \tag{1.3.23}$$

as desired. Next, we insert the estimates (1.3.20), (1.3.21), and (1.3.23) and the bootstrap assumption for μ into the right-hand side of equation (1.3.11), thereby deducing that

$$|L\mu| \leq C\mathring{\epsilon}\frac{1}{1+t}. \tag{1.3.24}$$

Integrating (1.3.24) along $\mathcal{C}_u$ from $t = 0$ where $|\mu - 1| \leq C\mathring{\epsilon}$, we establish that

$$|\mu - 1| \leq C\mathring{\epsilon}\ln(e+t) \tag{1.3.25}$$

as desired. Next, we note the identity $L(1 - r + t - u(t,r)) = 1 - \sqrt{1+\Psi}$. Hence, by (1.3.20), we have

$$|L(1 - r + t - u)| \leq C\mathring{\epsilon}\frac{1}{1+t}. \tag{1.3.26}$$

Integrating (1.3.26) from $t = 0$, where $u = 1 - r$, we find that

$$|1 - r + t - u| \leq C\mathring{\epsilon} \ln(e + t) \tag{1.3.27}$$

as desired. Next, using the identity

$$rL(\mu \underline{L}\Psi) = L\left(\mu \underline{L}(r\Psi)\right) - \mu \underline{L}\Psi + \frac{1}{2}\mu \frac{1}{\sqrt{1+\Psi}} \Psi L\Psi + \mu\sqrt{1+\Psi} L\Psi + (L\mu)\sqrt{1+\Psi}\Psi \tag{1.3.28}$$

and the previously proven estimates, we deduce that

$$|L(\mu \underline{L}\Psi)| \leq C\mathring{\epsilon} \frac{1}{(1+t)^2} \tag{1.3.29}$$

as desired. We now show that

$$\left|\mu \underline{L} L^2 (r\Psi)\right| \leq C\mathring{\epsilon} \frac{1}{(1+t)^2}. \tag{1.3.30}$$

To this end, we commute equation (1.3.12b) with L to derive an equation of the form $\mu \underline{L} L^2 (r\Psi) = \cdots$. To bound the magnitude of L applied to the right-hand side of (1.3.12b) by $\leq$ the right-hand side of (1.3.30), we use the bootstrap assumptions and the previously proven estimates. Similarly, to bound the commutator term $[L, \mu \underline{L}] L(r\Psi) = -\mu \frac{1}{4(1+\Psi)} (L\Psi) L^2(r\Psi) - \frac{1}{4(1+\Psi)} (\mu \underline{L}\Psi) L^2(r\Psi)$ by $\leq$ the right-hand side of (1.3.30), we use the bootstrap assumptions and the previously proven estimates. We have thus proved (1.3.30). Next, by arguing as in our proof of (1.3.19), we deduce from (1.3.30) that

$$\left|L^2 (r\Psi)\right| \leq C\mathring{\epsilon} \frac{1}{(1+t)^2}. \tag{1.3.31}$$

From the identity $rL^2\Psi = L^2 (r\Psi) - 2L\Psi$ and the estimates (1.3.21) and (1.3.31), we deduce that

$$\left|L^2 \Psi\right| \leq C\mathring{\epsilon} \frac{1}{(1+t)^3}. \tag{1.3.32}$$

We have thus improved the bootstrap assumptions, having shown that $\sqrt{\mathring{\epsilon}}$ can be replaced with $C\mathring{\epsilon}$, as stated in the estimates (1.3.15) of the proposition.

We now prove inequality (1.3.17). First, we multiply the evolution equation (1.3.11) by r, apply L, and use the previously proven estimates, including (1.3.24), (1.3.22), and (1.3.32), to deduce that

$$|L(rL\mu)| \leq C\mathring{\epsilon} \frac{\ln(e+t)}{(1+t)^2}. \tag{1.3.33}$$

Integrating (1.3.33) from s to t along the integral curves of L and using $r(s,u) = 1 - u + s + \mathcal{O}(\mathring{\epsilon} \ln(e+s))$, we find that for $0 \leq s \leq t$, we have $|[rL\mu](t,u) - [rL\mu](s,u)| \leq C\mathring{\epsilon} \ln(e+s)(1+s)^{-1}$ and hence that

$$L\mu(s,u) = \frac{1}{r(s,u)} [rL\mu](t,u) + \mathcal{O}\left(\mathring{\epsilon} \frac{\ln(e+s)}{(1+s)^2}\right). \tag{1.3.34}$$

Integrating (1.3.34) from $s = 0$ to $s = t$ and using $|\mu(0,u) - 1| \leq C\mathring{\epsilon}$, we find that for $0 \leq s \leq t$, we have

$$\mu(s,u) = 1 + \ln\left(\frac{1-u+s}{1-u}\right) [rL\mu](t,u) + \mathcal{O}(\mathring{\epsilon}). \tag{1.3.35}$$

On the other hand, using the previously proven estimates and equation (1.3.11), we deduce that

$$[rL\mu](s,u) = -\frac{1}{4}[r\mu\underline{L}\Psi](s,u) + \mathcal{O}\left(\mathring{\epsilon}\frac{\ln(e+s)}{1+s}\right). \tag{1.3.36}$$

Combining (1.3.35) and (1.3.36), we see that

$$\mu(s,u) = 1 - \frac{1}{4}\ln\left(\frac{1-u+s}{1-u}\right)[r\mu\underline{L}\Psi](t,u) + \mathcal{O}(\mathring{\epsilon}). \tag{1.3.37}$$

It follows from (1.3.37) that if $\mu(t,u) < 1/4$ and $\mathring{\epsilon}$ is sufficiently small, then

$$\mu\underline{L}\Psi(t,u) \geq c\frac{1}{(1+t)\ln(e+t)}, \tag{1.3.38}$$

which is the desired estimate (1.3.17). The desired estimate (1.3.16) then follows from inserting the estimate (1.3.17) into (1.3.36).

□

Sketch of a proof of Cor. 1.4. We must show that μ vanishes in finite time along at least one of the characteristics $\mathcal{C}_u$. Throughout most of the proof, we work with the data $(\mathring{\Psi}, \mathring{\Psi}_0)$ induced on Σ_0^1 by the solution. The parameter $\mathring{\epsilon}$ appearing throughout this proof is by definition $\mathring{\epsilon} := \|\mathring{\Psi}\|_{C^2} + \|\mathring{\Psi}_0\|_{C^1}$, as in the statement of Prop. 1.3. We assume that $\mathring{\epsilon}$ is small enough that the results of the proposition apply. As a first step, we insert the estimates (1.3.15) into the evolution equation (1.3.12a) and integrate along $\mathcal{C}_u$ to deduce that

$$|\mu\underline{L}\,(r\Psi)\,(t,u) - \mu\underline{L}\,(r\Psi)\,(0,u)| \leq C\mathring{\epsilon}^2. \tag{1.3.39}$$

Using the estimate $|\mu\underline{L}r| \leq C\ln(e+t)$, the estimate $|\Psi|(t,u) \leq C\mathring{\epsilon}(1+t)^{-1}$, the estimate $r(t,u) \approx 1+t$, and the fact that at $t=0$, $\mu - 1 = \mathcal{O}(\mathring{\epsilon})$ and $\underline{L} = \underline{L}_{(Flat)} + \mathcal{O}(\mathring{\epsilon})\partial_r$, we deduce from (1.3.39) that

$$[r\mu\underline{L}\Psi](t,u) = [\underline{L}_{(Flat)}\,(r\Psi)](0,u) + \mathcal{O}(\mathring{\epsilon}^2) + \mathcal{O}\left(\mathring{\epsilon}\frac{\ln(e+t)}{1+t}\right). \tag{1.3.40}$$

Combining (1.3.40) with equation (1.3.35), we deduce that

$$\mu(t,u) = 1 + \mathcal{O}(\mathring{\epsilon}) + \ln\left(\frac{1-u+t}{1-u}\right)[\underline{L}_{(Flat)}\,(r\Psi)](0,u) + \mathcal{O}(\mathring{\epsilon}^2)\ln(e+t) + \mathcal{O}\left(\mathring{\epsilon}\frac{\ln^2(e+t)}{1+t}\right). \tag{1.3.41}$$

From (1.3.41), we conclude that if $\underline{L}_{(Flat)}\,(r\Psi)\,(0,u)$ is sufficiently negative for some $u \in (0, U_0]$ to overwhelm the $\mathcal{O}(\mathring{\epsilon}^2)$ term, then μ will vanish in finite time. The negativity of $\underline{L}_{(Flat)}\,(r\Psi)\,(0,u)$ at some $u_* \in (0,1)$ is an easy consequence of the assumption that the data given along $\Sigma_{-1/2}$ are nontrivial and compactly supported in the Euclidean ball of radius $1/2$ centered at the origin. This fact is roughly a spherically symmetric analog of Prop. 1.2; see [28] or [63, Lemmas 22.2.1 and 22.2.2] for more details.[42] We now run the above analysis with U_0 less than one but greater than u_*. By shrinking the amplitude of

[42] As can be discerned from [28] or [63, Lemmas 22.2.1 and 22.2.2], if we had started with data on Σ_0 supported in $\{r \leq 1\}$, then the shock might "want to form" at a value of u larger than 1, that is, in a region where our eikonal function is not defined. It is for this reason that we started with the data on $\Sigma_{-1/2}$ supported in $\{r \leq 1/2\}$.

the data as stated in the corollary, we can guarantee that $\mathcal{O}(\mathring{\epsilon}^2)$ in (1.3.41) is an "error term" compared to $\underline{L}_{(Flat)}(r\Psi)(0,u_*)$. This guarantees finite-time shock formation in $\mathcal{M}_{T_{(Lifespan);U_0},U_0}$. □

1.4. **Systems of equations of the form** (1.1.10)**.** As we mentioned earlier, scalar equations of the form (1.1.8) can be re-expressed in terms of systems of equations of type (1.1.10), where Ψ is the array $\Psi = \partial\Phi$. Given the fact that the shocks we are studying correspond to singularities of $\partial^2\Phi$ while $\Psi = \partial\Phi$ remains bounded, it is easy to convince ourselves that *for this purpose*, the system of the type (1.1.10) is not more difficult to treat than the simplified case of the scalar equation

$$(g^{-1})^{\alpha\beta}(\Psi)\partial_\alpha\partial_\beta\Psi = \mathcal{N}(\Psi,\partial\Psi), \tag{1.4.1}$$

with $g(\Psi) = m + \mathcal{O}(|\Psi|)$ and $\mathcal{N}(\Psi,\partial\Psi) = \mathcal{O}(|\partial\Psi|^2)$ in a neighborhood of $(\Psi,\partial\Psi) = (0,0)$ (and as usual, m is the Minkowski metric). We shall thus concentrate our attention on this scalar model, even though some important concepts, such as the "weak null condition" (see just below), are more broadly applicable to the full system.

In Subsect. 1.3, for F. John's equation (1.3.2) in spherical symmetry, we saw that the shock formation is essentially driven by some semilinear terms that lead to a Riccati-type blow-up. A remarkable fact about scalar equations of type (1.4.1) is that in $3D$, small data blow-up cannot occur[43] without semilinear terms. Indeed, as was first pointed out by H. Lindblad, if we drop the nonlinear term on the right-hand side of (1.4.1) (in the scalar case), then the remaining quasilinear equation admits global solutions (even in 3D) for all sufficiently small initial conditions. In [45], Lindblad proved this for spherically symmetric solutions of the model equation

$$-\partial_t^2\Psi + c^2(\Psi)\Delta\Psi = 0, \qquad\qquad c^2(0) = 1. \tag{1.4.2}$$

The result was later extended by S. Alinhac [7] and H. Lindblad [46] to equations of the form[44]

$$(g^{-1})^{\alpha\beta}(\Psi)\partial_\alpha\partial_\beta\Psi \quad = 0, \tag{1.4.3}$$

with $g(\Psi) = m + \mathcal{O}(|\Psi|)$. This result carries over to systems of the form (1.1.10) for which the nonlinear terms $\mathcal{N}^I(\Psi,\partial\Psi)$ verify the classic null condition. In [47], H. Lindblad and I. Rodnianski further extended the result to a larger class of nonlinearities $\mathcal{N}$ that verify the *weak null condition*.[45] Moreover, they showed that the weak null condition is verified by the Einstein vacuum equations in the wave coordinate gauge [49].

Hence, if we are interested in describing the phenomenon of small-data shock formation, we must consider either scalar equations of type (1.4.1) with nonlinearities $\mathcal{N}$ which do not verify the classic null condition, or more generally, systems of the type (1.1.10) which do not verify the weak null condition. A

[43] In contrast, we note that this statement is false for metrics $g = g(\partial\Phi)$; see Remark 4.2.

[44] Lindblad considered the general case (1.4.3) while Alinhac addressed the specific equation $-\partial_t^2\Psi + (1+\Psi)^2\Delta\Psi = 0$.

[45] A system (in 3 space dimensions) verifies the weak null condition if the corresponding *asymptotic system* admits global solutions; see [47] for an explanation of how to form the asymptotic system. The weak null condition manifests as a structural condition on the nonlinearity $\mathcal{N}$ for *systems* of wave equations, allowing, in addition to quadratic terms that verify the classic null condition, also weakly coupled quadratic terms. For example, systems such as $\Box_m\Phi = \partial\Phi\cdot\partial\Psi$, $\Box_m\Psi = 0$ (or even $\Box_m\Psi = \mathcal{N}(\Psi,\partial\Psi)$ with $\mathcal{N}$ verifying the classic null condition) verify the weak null condition.

convenient way to generate scalar equations that fail the classic null condition is to rewrite (1.4.1) in the geometric form

$$\Box_{g(\Psi)}\Psi = \mathcal{N}(\Psi, \partial\Psi), \tag{1.4.4}$$

with $\Box_g$ the standard covariant wave operator associated to the metric $g = g(\Psi)$. Note that the term $\mathcal{N}(\Psi, \partial\Psi)$ in (1.4.4) is of course different from the corresponding term in (1.4.1) in view of the difference between the operators $(g^{-1})^{\alpha\beta}\partial_\alpha\partial_\beta$ and $\Box_g$. We already stress here that in order to prove a shock formation result, we must make assumptions on the semilinear term $\mathcal{N}(\Psi, \partial\Psi)$ in equation (1.4.4). For example, one could choose $\mathcal{N}(\Psi, \partial\Psi)$ so that (1.4.4) is equivalent to equation (1.4.3), in which case there would be small-data global existence. In Subsubsect. 2.1.7, we describe some sufficient assumptions on the nonlinearities that lead to small-data shock formation. Note that in the particular case when the right hand side of (1.4.4) is trivial, that is, in the case of the equation $\Box_{g(\Psi)}\Psi = 0$, the non-geometric form of the equation (equation (1.4.1)) is such that the corresponding right hand side does *not* verify, except in trivial cases, the null condition; see the discussion in Subsubsect. 2.1.6. As a simple example to keep in mind, consider the equation $\Box_{g(\Psi)}\Psi = 0$ in the case of the metric

$$-dt^2 + c^2(\Psi)\sum_{a=1}^{3}(dx^a)^2. \tag{1.4.5}$$

The non-geometric form of the equation is:

$$(g^{-1})^{\alpha\beta}(\Psi)\partial_\alpha\partial_\beta\Psi = -(g^{-1})^{\alpha\beta}(\Psi)\partial_\alpha \ln c(\Psi)\partial_\beta\Psi + 2\partial_t \ln c(\Psi)\partial_t\Psi. \tag{1.4.6}$$

The first term on the right-hand side of (1.4.6) verifies the classic null condition, and if not for the second term, the methods of Alinhac [7] and Lindblad [46] would lead to small-data global existence. However, the term $2\partial_t \ln c(\Psi)\partial_t\Psi$ does not verify the classic null condition and causes the finite-time shock formation. We remark that equation (1.4.6) admits spherically symmetric solutions whose finite time blow-up can be analyzed by employing essentially the same strategy that F. John used to study equation (1.1.12), or by using the sharper strategy described in Subsect. 1.3.

Finally, we note, that although one can establish small-data global existence for Lindblad's equation, and more generally for systems of type (1.1.10) verifying the weak null condition, the resulting solutions sometimes verify weaker peeling properties than the ones (1.2.7) corresponding to the linear wave equation. Alinhac refers to the distorted asymptotic behavior as "blow-up at infinity;" see, for example, [7]. In the case of Lindblad's scalar equation, this effect can only be generated by the quasilinear (principal) part of the equation and is in fact due to the nontrivial asymptotic behavior of the *null (characteristic) hypersurfaces*, which are levels sets of a solution u to the following *eikonal equation*[46]

$$(g^{-1})^{\alpha\beta}(\Psi)\partial_\alpha u\partial_\beta u = 0. \tag{1.4.7}$$

Solutions to (1.4.7) are analogs of the coordinate u constructed in Subsect. 1.3 in spherical symmetry. They will play a major role in all of the remaining discussion in this article.

[46] In the next subsection, we discuss the eikonal equation in more detail.

1.5. **Why is the proof of shock formation so much harder in the general case?** The short answer is simply this: because the spherically symmetric problem is truly $1+1$ dimensional and therefore one can rely almost exclusively on the method of characteristics, a method which is in itself insufficient in higher dimensions. After his blow-up work in spherical symmetry [28], F. John tried to extend it by treating the general case as a perturbation of the spherically symmetric one. In particular, in treating the general case, he used radial characteristic curves corresponding to the truncated problem in which angular derivatives are set equal to 0, which are not true characteristics for the actual equation. At first glance, this seems reasonable since, in view of the peeling properties (1.2.7), we may expect that the angular derivatives decay faster and thus, for large values of t, the radial behavior dominates. The problem with such a strategy is that it is not so easy to verify that the angular derivatives are indeed negligible. Actually, Christodoulou's work [11] and the third author's work [63] allow for the possibility that the standard angular derivatives of the solution along the Euclidean spheres are non-negligible at late time and in fact *they can blow up* when the shock forms! The reason is that they can contain a small component that is transversal to the actual characteristic hypersurfaces, and it is exactly this transversal derivative of the solution that blows up.

1.5.1. *Eikonal functions in* $3D$. In Subsect. 1.2, we outlined how to derive the decay properties of solutions to higher-dimensional nonlinear wave equations using a version of the vectorfield method that relies on the Killing and conformal Killing vectorfields $\mathscr{Z}_{(Flat)}$ of Minkowski spacetime. These vectorfields are well-adapted to $u_{(Flat)} := t - r$, which is an eikonal function of the Minkowski metric (whose level sets are characteristics for the linear wave equation). In general, these vectorfields are not suitable for studying quasilinear wave equations, whose characteristics may be very different from those of solutions to the linear wave equation. We were able to use the Minkowskian vectorfields $\mathscr{Z}_{(Flat)}$ in the proofs of small-data global and almost-global existence theorems, essentially because we worked within spacetime regions where we can use a bootstrap procedure based on the peeling estimates (1.2.7) to control the difference between the actual characteristics and the Minkowskian ones.

In contrast, in the shock formation problem, one is studying spacetime regions where the true characteristics are catastrophically diverging from those of the linear wave equation. Therefore, there is no reason to hope that we can derive good peeling estimates by commuting with vectorfields in $\mathscr{Z}_{(Flat)}$. In the study of the linear wave equation the peeling estimates (1.2.7) are adapted to the outgoing Minkowskian null cones. These cones are level sets of the eikonal function $u_{(Flat)}$, which solves the eikonal equation $(m^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$. To study shock formation, it is natural then to replace $u_{(Flat)}$ by an appropriate outgoing solution of the eikonal equation of the dynamic metric $g(\partial\Phi)$ of equation (1.1.8):

$$(g^{-1})^{\alpha\beta}(\partial\Phi)\partial_\alpha u \partial_\beta u = 0 \tag{1.5.1}$$

or (1.4.7) in the case of equations of type (1.4.4). The hope is that u will serve as a good coordinate, as it did in Subsect. 1.3 in spherical symmetry.

Note that in the particular case of John's equation (1.3.2), or the equation $\Box_{g(\Psi)}\Psi = 0$ with Lindblad's metric (1.4.5), the eikonal equation takes the form

$$(\partial_t u)^2 = c^2(\Psi)|\nabla u|^2. \tag{1.5.2}$$

Suppose now that u is a solution to the eikonal equation with $\partial_t u > 0$ and such that at each fixed time t, the level sets are embedded 2-spheres. We say that u is outgoing if the spatial gradient ∇u is inward-pointing, and incoming otherwise.[47] Note that if Ψ is spherically symmetric, then we can also choose a pair of eikonal functions u and $\underline{u}$, respectively outgoing and incoming, to be spherically symmetric, that is, $(\partial_t u)^2 = c^2(\Psi)(\partial_r u)^2$ and likewise for $\underline{u}$. These symmetric eikonal functions are completely determined by the radial characteristics that played a crucial role in F. John's work [28] and in our argument in Subsect. 1.3. In the general non-spherically symmetric case, we will use a non-degenerate (i.e., $\partial u \neq \mathbf{0}$) outgoing eikonal function u to construct the adapted vectorfields needed to derive peeling estimates. Starting in Subsubsect. 1.5.2, we describe the many technical difficulties that accompany the use of an eikonal function in the general case.

1.5.2. *A preview on the vectorfield method tied to an eikonal function u.* From now on, we shall primarily discuss equations of type (1.4.4) under assumptions[48] that lead to small-data shock formation. Following the strategy described in Subsubsect. 1.3.2, we aim to derive peeling estimates, similar to those in (1.2.7), for a rescaled problem, with the aid of vectorfields Z adapted to an eikonal function that have good commuting properties with the covariant wave operator $\Box_g$. Of course, we cannot expect vanishing commutators as in the flat case; we can only hope to control the error terms generated by the commutation.

To begin, we note the following general formula for the commutator between $\Box_g$ and an arbitrary vectorfield Z :

$$\Box_g(Z\Psi) \ = Z(\Box_g\Psi) - {}^{(Z)}\pi \cdot \mathscr{D}^2\Psi + (\mathscr{D}^{(Z)}\pi) \cdot \mathscr{D}\Psi, \tag{1.5.3}$$

where $\mathscr{D}$ denotes the Levi-Civita connection corresponding to the metric g, ${}^{(Z)}\pi$ denotes the deformation tensor of the vectorfield Z, that is,

$${}^{(Z)}\pi_{\alpha\beta} := \mathcal{L}_Z g_{\alpha\beta} = \mathscr{D}_\alpha Z_\beta + \mathscr{D}_\beta Z_\alpha, \tag{1.5.4}$$

where $\mathcal{L}_Z$ denotes Lie differentiation with respect to Z. The term $(\mathscr{D}^{(Z)}\pi) \cdot \mathscr{D}\Psi$ schematically denotes tensorial products between first covariant derivatives of ${}^{(Z)}\pi$ and the first derivatives of Ψ, and similarly for the term ${}^{(Z)}\pi \cdot \mathscr{D}^2\Psi$.

In Subsect. 3.1, we will describe the commutator vectorfields $\mathscr{Z}$ needed in the shock-formation problem. For illustrative purposes, we discuss here a subset of the commutators that we use: the rotations O. The simplest way to define good rotation vectorfields O tied to the eikonal function u is to use Christodoulou's strategy [11] by projecting, using the metric g, the Euclidean rotation vectorfields[49] $O_{(Flat;ij)}$ onto the intersection of the level sets of u with Σ_t (the hypersurfaces of constant t in Minkowski space). The projection operator can be constructed with the help of the null geodesic vectorfield $L_{(Geo)} := -(g^{-1})^{\alpha\beta}(\Psi)\partial_\beta u \partial_\alpha$ corresponding to u. Thus, the projection operator depends on Ψ and the first rectangular derivatives of u. It is then easy to see that the deformation tensor ${}^{(O)}\pi$ must depend on the first derivatives of Ψ and the Hessian $H := \mathscr{D}^2 u$. Therefore, the term $\mathscr{D}^{(O)}\pi$ appearing on the right

[47] In Minkowski spacetime $t - r$ is outgoing and $t + r$ is incoming; the terminology refers to the direction of travel of the level sets as time flows forward. We typically denote outgoing solutions by u and incoming ones by $\underline{u}$.

[48] See Subsubsect. 2.1.7.

[49] We recall from (1.2.3) that the Euclidean rotations are defined relative to the standard rectangular coordinates.

hand side of the equation

$$\Box_{g(\Psi)}(O\Psi) \;=\; O(\Box_{g(\Psi)}\Psi) + {}^{(O)}\pi \cdot \mathscr{D}^2\Psi + (\mathscr{D}^{(O)}\pi)\cdot \mathscr{D}\Psi \tag{1.5.5}$$

depends on the second derivatives of Ψ and the *third derivatives* of u. Hence, to close L^2 estimates at a consistent level of derivatives, we need to make sure that we can estimate the third derivatives of u in terms of two derivatives of Ψ. Note that by equation (1.4.7), u depends on Ψ. At first glance of equation (1.4.7), one might believe in the heuristic relationship $\partial u \sim \Psi$ and hence $\partial^3 u \sim \partial^2\Psi$, which is the desired degree of differentiability. However, as we explain below, only a weakened version, just barely sufficient for our purposes, of these relationships is true. Furthermore, the weakened version is quite difficult to prove.

To flesh out the difficulty, we first note that one can derive a Riccati-type matrix evolution equation for H of the schematic form

$$L_{(Geo)}H + H^2 = \mathcal{R}, \tag{1.5.6}$$

where $\mathcal{R}$ depends[50] on up-to-second-order derivatives of Ψ and up-to-second-order derivatives of u. Ignoring for now the Riccati-type term H^2, which actually plays a crucial role in the blow-up mechanism, we note that the obvious way to estimate H is by integrating the curvature term $\mathcal{R}$ along the integral curves of $L_{(Geo)}$. The obstacle is that this argument only allows one to conclude that H has the same degree of differentiability, in directions transversal to L, as $\mathcal{R}$. In particular, using this argument, we can only estimate H in terms of two derivatives of Ψ, which makes $\mathscr{D}^{(O)}\pi$ dependent on *three* derivatives of Ψ. Thus, the term $(\mathscr{D}^{(O)}\pi)\cdot\mathscr{D}\Psi$ is far from being a lower-order term as one would hope. It in fact seems to be an *above top-order term* that obstructs closure of the estimates. This appears to make equation (1.5.5) useless and casts doubt on the desired differentiability $\partial^3 u \sim \partial^2\Psi$. This difficulty in deriving good differentiability properties of u may be the reason that F. John was not able to extend the vectorfield method to study non-spherically symmetric blow-up. As we explain in Subsubsect. 3.4.3, this loss can be overcome by carefully exploiting some special tensorial structures present in the components of $\mathscr{D}^{(O)}\pi$ and the components of equation (1.5.6), and by using elliptic estimates. Some of these special structures are closely tied to the fact that our commutators Z are adapted to the eikonal function u; see Remark 3.7.

1.5.3. *Connections between the proof of shock formation and the proof of the stability of the Minkowski space.* The first successful use of null (characteristic) hypersurfaces in a global nonlinear evolution problem appeared in the proof of the nonlinear stability of the Minkowski space [13]. The properties of an exact, carefully constructed eikonal function u were crucial for building approximate Killing and conformal Killing vectorfields to replace[51] those appearing in Subsubsect. 1.2.2. These vectorfields were then used to derive generalized energy estimates and the peeling properties of the Riemann curvature tensor $\mathcal{R}$ of the metric g, much like the linear peeling properties (1.2.7). The non-vanishing nature of the commutation of these carefully constructed vectorfields with Einstein's field equations is measured by their deformation tensors (1.5.4), which in turn depend on the properties of various higher-order derivatives of the eikonal function u. The Hessian H of u verifies an equation of type (1.5.6) and hence its regularity and decay properties again depends on those of the curvature tensor. The apparently loss of derivatives

[50] $\mathcal{R}$ is in fact the Riemann curvature tensor of the metric $g(\Psi)$ contracted twice with the vectorfield $L_{(Geo)}$.

[51] In [48, 49], Lindblad and Rodnianski were able to prove a weaker (based on weaker peeling estimates) version of the stability of Minkowski space using the Minkowskian vectorfields instead of ones adapted to the dynamic geometry.

mentioned above also appears and is overcome via a renormalization procedure and elliptic estimates. As we shall see, a similar procedure allows one to avoid derivative loss in the shock formation problem, but it is more difficult to implement. Although the basic ideas in the proof of [13] are simple and compelling, the proof required a complicated and laborious bootstrap argument in which one uses[52] the expected properties of the curvature tensor to derive estimates for various derivatives of the eikonal function and, based on them, precise estimates for the deformation tensors of the adapted vectorfields mentioned above. These vectorfields are then used to derive generalized energy estimates for various components of the curvature tensor, which are L^2 analogs of the peeling estimates. The main error terms, which appear in these curvature estimates, are controlled by a procedure similar to, but much more subtle, than the one we have described in Subsubsect. 1.2.3 for wave equations with nonlinearities that verify the classic null condition. Just as in the case of these wave equations, in the Einstein equations, the nonlinear terms are such that the most dangerous error terms that could in principle appear in the generalized energy estimates are *not present* due to the special structure of the equations relative to the dynamic coordinates t and u.

2. THE MAIN IDEAS BEHIND THE ANALYSIS OF SHOCK-FORMING SOLUTIONS IN $3D$

We now outline some of the new difficulties encountered and the key ingredients that Christodoulou used to overcome them [11] when extending the proof of shock formation from the spherically symmetry case (see Subsect. 1.3) to the general case.

(1) **(Dynamic geometric objects, dependent on the solution)** As in the proof of the stability of Minkowski spacetime [13], the proof of shock formation uses a true outgoing eikonal function u corresponding to the dynamic metric and a collection of vectorfields dynamically adapted to it.

(2) **(Inverse foliation density and shock formation)** As in the case of spherical symmetry, shock formation is caused by the degeneracy of u as measured by the density of its level surfaces relative to the Minkowskian time coordinate t, captured by the inverse foliation density μ going to 0 in finite time (see Definition 2.1 below).

(3) **(Peeling and sharp classical lifespan in rescaled frame)** At the heart of Christodoulou's entire approach lies a sharp classical lifespan result according to which solutions can be extended as long as μ does not vanish.[53] To derive such a result, one needs to re-express the evolution equations as a coupled system between the *nonlinear wave equation*, expressed relative to a μ-rescaled vectorfield frame, together with a *nonlinear transport equation* describing the evolution of the eikonal function u (and hence, by extension, of μ). In this formulation, the μ-rescaled wave equation no longer exhibits the dangerous slow-decaying quadratic term analogous to the term $(\underline{L}\Psi)^2$ from spherical symmetry (see Remark 1.5). To prove the desired sharp classical lifespan result, we need to show that the lower-order derivatives of the solution behave according to the linear peeling estimates (1.2.7). To establish such peeling estimates, one needs to rely on appropriate energy estimates for derivatives of the solution with respect to the u-adapted vectorfields. The main technical

[52]The description given here is vastly simplified. There is another layer of complexity connected to the choice of the time function t, which like u, is dynamically constructed. Note that unlike the case of general relativity, in [11] and in [63] there is a preferred physical time function t from the background Minkowski spacetime.

[53]The reader should keep in mind the simpler case of spherical symmetry discussed above (see Prop. 1.3).

difficulty one needs to overcome is that the energy norms of the highest derivatives can degenerate with respect to μ^{-1}, as we discuss in point (4).

(4) **(Generalized energy estimates)** To establish the desired energy-type estimates, we need to commute the wave equation a large number of times with the adapted vectorfields,[54] a procedure which not only generates a huge number of error terms, but also seems to lead to a loss of derivatives. To overcome this apparent loss of derivatives at the top order (see Subsubsect. 1.5.2), Christodoulou uses renormalizations and $2D$ elliptic estimates, in the spirit[55] of [13]. The price one pays for renormalizing is the introduction of a factor of μ^{-1} at the top order. This leads to μ^{-1}-degenerate high-order L^2 estimates; see Prop. 3.4 and Subsubsect. 3.4.3. *Establishing these degenerate high-order L^2 estimates and showing that the degeneracy does not propagate down to the lower levels are the main new advances of* [11].

In Sect. 2, we describe the implementation of points (1) and (2). In connection with point (3), we also state the *Heuristic Principle*, which is a collection of peeling estimates that play an important role in controlling error terms in the proof. The proof of the Heuristic Principle is based on the generalized energy estimates mentioned in point (4) and Sobolev embedding. Because the derivation of generalized energy estimates is the most difficult aspect of the proof, we dedicate all of Sect. 3 to outlining the central ideas. This step is where the proof deviates the most from the spherically symmetric case. In Sect. 4, we summarize the sharp classical lifespan theorem,[56] which is the main ingredient needed to show that a shock actually forms. We also outline its proof and indicate the role of the estimates described in the previous sections. In Sect. 5, we compare the results of Christodoulou to those of Alinhac and discuss some of the new results in [63].

2.1. **Basic geometric notions and set-up of the problem without symmetry assumptions.** Motivated by the discussion at the beginning of Subsect. 1.4, from now until Subsect. 4.2, we consider the model *scalar* wave equation of the form (1.4.4) under the assumption

$$g_{\alpha\beta}(\Psi = 0) = m_{\alpha\beta}, \tag{2.1.1}$$

where $m_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$ denotes the Minkowski metric. Furthermore, we assume that the semilinear terms on the right-hand side are quadratic in $\partial\Psi$ with coefficients depending on Ψ :

$$\Box_{g(\Psi)}\Psi = \mathcal{N}(\Psi)(\partial\Psi, \partial\Psi). \tag{2.1.2}$$

Above, $\Box_{g(\Psi)} := (g^{-1})^{\alpha\beta}\mathscr{D}_\alpha\partial_\beta$ denotes the covariant wave operator of $g(\Psi)$ and $\mathscr{D}$ denotes the Levi-Civita connection[57] of $g(\Psi)$, given in coordinates by

$$\mathscr{D}_\alpha\partial_\beta\Psi = \partial_\alpha\partial_\beta\Psi - {}^{(\Psi)}\Gamma_{\alpha\ \beta}^{\ \lambda}\partial_\lambda\Psi. \tag{2.1.3}$$

[54] Christodoulou did not give explicit bounds on the number of commutations needed to close the estimates in [11]. In [63], the third author used 24 commutations. This may be further optimized.

[55] In his work [11], Christodoulou recognizes that similar renormalization procedures can be done in the context of nonlinear wave equations of type (1.1.5) and (2.1.2). A similar observation had previously been used in [39] in the context of quasilinear wave equations similar to (2.1.2) to derive a low regularity local well-posedness result.

[56] We state the version of the theorem from [63], which applies to the scalar equations $\Box_{g(\Psi)}\Psi = 0$.

[57] Throughout, $(g^{-1})^{\alpha\beta}(\Psi)$ denotes the inverse metric. That is, $g_{\alpha\beta}(g^{-1})^{\beta\gamma} = \delta_\alpha^\gamma$.

Above, $\Gamma = {}^{(\Psi)}\Gamma$ denotes a Christoffel symbol of $g(\Psi)$,

$$\begin{aligned} {}^{(\Psi)}\Gamma_{\alpha\ \beta}^{\ \lambda} &:= \frac{1}{2}(g^{-1})^{\lambda\sigma}(\Psi)\left\{\partial_\alpha(g_{\sigma\beta}(\Psi)) + \partial_\beta(g_{\alpha\sigma}(\Psi)) - \partial_\sigma(g_{\alpha\beta}(\Psi))\right\} \\ &= \frac{1}{2}(g^{-1})^{\lambda\sigma}\left\{G_{\sigma\beta}\partial_\alpha\Psi + G_{\alpha\sigma}\partial_\beta\Psi - G_{\alpha\beta}\partial_\sigma\Psi\right\}, \end{aligned} \tag{2.1.4}$$

where

$$G_{\mu\nu} = G_{\mu\nu}(\Psi) := \frac{d}{d\Psi}g_{\mu\nu}(\Psi). \tag{2.1.5}$$

Thus, the left-hand side of (2.1.2) can be written as

$$\Box_{g(\Psi)}\Psi = (g^{-1})^{\alpha\beta}\partial_\alpha\partial_\beta\Psi - \frac{1}{2}(g^{-1})^{\alpha\beta}(g^{-1})^{\lambda\sigma}\left\{G_{\sigma\beta}\partial_\alpha\Psi + G_{\alpha\sigma}\partial_\beta\Psi - G_{\alpha\beta}\partial_\sigma\Psi\right\}\partial_\lambda\Psi. \tag{2.1.6}$$

Without loss of generality[58] we make the following assumption, which simplifies some of the calculations:

$$(g^{-1})^{00}(\Psi) \equiv -1. \tag{2.1.7}$$

To state and prove the main theorems, we assume for convenience that the initial data are supported in the Euclidean unit ball. As in the case of spherical symmetry, we fix a constant $U_0 \in (0,1)$. We will study the solution in a spacetime region that is evolutionarily determined by the portion of the nontrivial part of the data lying in the region $\Sigma_0^{U_0}$, which is the annular subset of Σ_0 bounded between the inner sphere of Euclidean radius $1-U_0$ and the outer sphere of Euclidean radius 1. The spacetime region of interest is bounded by the inner null cone $\mathcal{C}_{U_0}$ and the outer null cone $\mathcal{C}_0$, where $\mathcal{C}_0$ is "flat" (i.e. Minkowskian) because the solution Ψ completely vanishes in its exterior; see Figure 3. The region is an analog of the spherically symmetric region $\mathcal{M}_{t,U_0}$ encountered in Subsect. 1.3.

[58] It is straightforward to see that one component of the metric can always be fixed by a conformal rescaling. This conformal rescaling generates an additional semilinear term that verifies the future strong null condition of Subsubsect. 2.1.7; as we later explain, such terms have negligible effect on the dynamics.

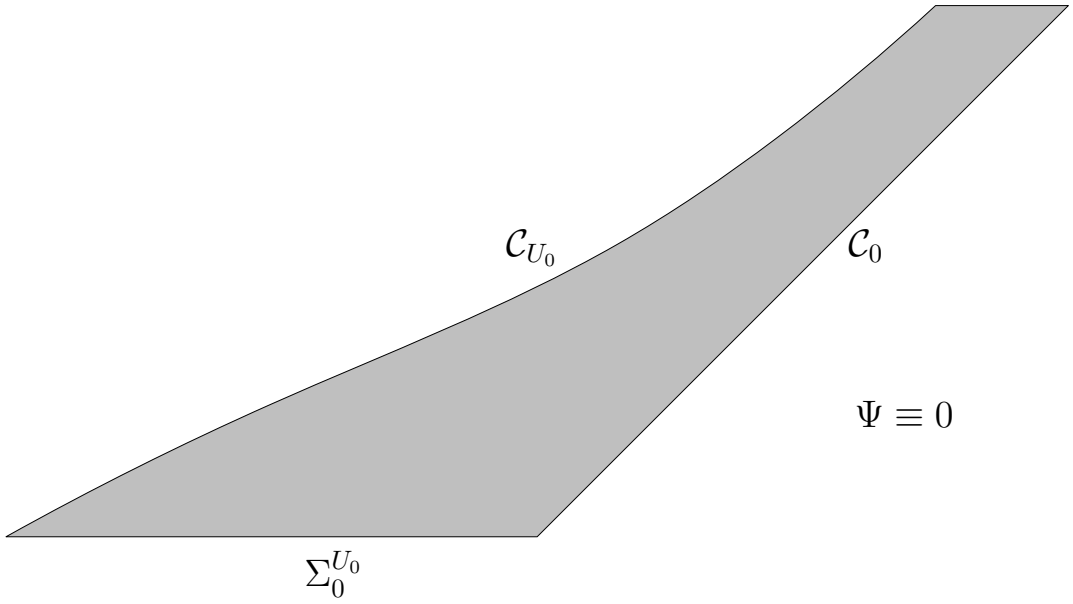


FIGURE 3. The region of study at a fixed angle.

To prove that small-data shock formation occurs in solutions to (2.1.2), we must make assumptions on the structure of $G_{\mu\nu}$ as well as the ($g-$null) structure of $\mathcal{N}(\Psi)(\partial\Psi, \partial\Psi)$. We shall give precise conditions in Subsubsect. 2.1.7, after we introduce some basic notions.

2.1.1. *The eikonal function, adapted frames, and the Heuristic Principle.* We start with an outgoing eikonal function, that is, a solution u of the eikonal equation

$$(g^{-1})^{\alpha\beta}(\Psi)\partial_\alpha u \partial_\beta u = 0, \tag{2.1.8}$$

with $\partial_t u > 0$, subject to the initial condition

$$u|_{t=0} = 1 - r, \tag{2.1.9}$$

where r denotes the Euclidean radial coordinate on $\mathbb{R}^3$. We stress that (2.1.8) can be viewed as a *nonlinear transport* equation to be solved in conjunction with the wave equation (2.1.2). This uniquely-defined u is a perturbation of the flat eikonal function $u_{(Flat)} = 1 - r + t$, where t is the Minkowski time coordinate. We associate to u its gradient vectorfield

$$L^\nu_{(Geo)} := -(g^{-1})^{\nu\alpha}\partial_\alpha u. \tag{2.1.10}$$

Since $\mathscr{D}g = 0$, it follows that $L_{(Geo)}$ is null and geodesic, that is, $g(L_{(Geo)}, L_{(Geo)}) = 0$ and

$$\mathscr{D}_{L_{(Geo)}} L_{(Geo)} = 0. \tag{2.1.11}$$

Relative to the rectangular coordinates x^ν, (2.1.11) can be expressed as

$$L^\alpha_{(Geo)}\partial_\alpha L^\nu_{(Geo)} = -{}^{(\Psi)}\Gamma_{\alpha\ \beta}^{\ \nu} L^\alpha_{(Geo)} L^\beta_{(Geo)}. \tag{2.1.12}$$

The level sets of u, which we denote by $\mathcal{C}_u$, are outgoing $g-$null hypersurfaces. They intersect the flat hypersurfaces Σ_t of constant Minkowskian time in topological spheres, which we denote by $S_{t,u}$. We denote by $\not{g}$ the Riemannian metric on $S_{t,u}$ induced by g.

As in the case of spherical symmetry, shock formation is intimately tied to the degeneration of the inverse foliation density μ of the null hypersurfaces $\mathcal{C}_u$ (as measured with respect to Σ_t).

Definition 2.1 (**Inverse foliation density**)**.** We define the inverse foliation density μ by

$$\frac{1}{\mu} := -(g^{-1})^{\alpha\beta}\partial_\alpha t \partial_\beta u. \tag{2.1.13}$$

Remark 2.1. For the background solution $\Psi \equiv 0$ we have $\mu \equiv 1$.

The proof of shock formation outside of spherical symmetry will follow the same general strategy as implemented in the proof of Proposition 1.3. In particular, we will prove a sharp lifespan theorem along with "global-existence-type" estimates. We will derive these latter estimates relative to a frame in which $\mathcal{C}_u-$transversal directional derivatives are rescaled by μ and the $\mathcal{C}_u-$tangential directional derivatives are near their Minkowskian counterparts. We summarize this strategy in the following rough statement.

Heuristic Principle I. *If we work with properly* μ*-rescaled quantities we can effectively transform the shock formation problem into a sharp long-time existence problem in which various rescaled quantities exhibit dispersive behavior and decay similarly to the peeling properties* (1.2.7)*. See also Remark 2.4.*

We remark again here that, as in Christodoulou's work [11], the Heuristic Principle is only expected to hold strictly for lower-order derivatives of the solution; it turns out that the argument requires accommodating possible degeneracies, in terms of the L^2-norm control, of the higher-order μ-rescaled derivatives, near the time of first shock formation.

We introduce the following null vectorfield, which is a rescaled version of (2.1.10):

$$L := \mu L_{(Geo)}. \tag{2.1.14}$$

It follows from definitions (2.1.10) and (2.1.13) that $Lt = 1$ and hence $L^0 = 1$. The vectorfield L is the replacement of the one of (1.3.4) encountered in spherical symmetry: in the particular case of the metric associated to John's equation $-\partial_t^2\Phi + (1+\partial_t\Phi)\Delta\Phi$, for spherically symmetric solutions $\Psi := \partial_t\Phi$ (see Subsect. 1.3), the vectorfield $\mu L_{(Geo)}$ coincides with the vectorfield $L = \partial_t + \sqrt{1+\Psi}\partial_r$. Consistent with the Heuristic Principle mentioned above and with our experience in spherical symmetry, we expect that L remains close to the vectorfield $L_{(Flat)} = \partial_t + \partial_r$.

We are now in a position to define a good set of coordinates, in analogy with the coordinates (t,u) that we used in proving Prop. 1.3 in spherical symmetry. Specifically, to obtain a sharp picture of the dynamics, we use *geometric coordinates*

$$(t, u, \vartheta^1, \vartheta^2). \tag{2.1.15}$$

In (2.1.15), t is the Minkowski time coordinate, u is the eikonal function, and $(\vartheta^1, \vartheta^2)$ are local angular coordinates on the spheres $S_{t,u}$, propagated from the initial Euclidean sphere $S_{0,0}$ by first solving the transport equation

$$-\partial_r \vartheta^A = 0, \qquad (A = 1, 2) \tag{2.1.16}$$

to propagate them to the $S_{0,u}$ and then solving the transport equation

$$L\vartheta^A = 0, \qquad (A = 1, 2) \tag{2.1.17}$$

to propagate them to the $S_{t,u}$. In particular, relative to geometric coordinates, we have

$$L = \frac{\partial}{\partial t}, \tag{2.1.18}$$

a relation that we use throughout our analysis.

Remark 2.2 (μ **is connected to the Jacobian determinant**)**.** We note here another important role played by μ : it is not too difficult to show that the Jacobian determinant of the change of variables map $(t, u, \vartheta^1, \vartheta^2) \xrightarrow{\Upsilon} (t, x^1, x^2, x^3)$ from geometric to rectangular coordinates is proportional to μ; see [63, Lemma 2.17.1].

In particular, for small-data solutions, one can show that the Jacobian determinant $\det d\Upsilon$ vanishes precisely at the points where μ vanishes. Hence, solutions that are regular relative to the geometric coordinates can in fact have rectangular derivatives that blow-up at the locations where μ vanishes because of the degeneracy of the change of variables map.

In addition to the geometric coordinates, we will also use a vectorfield frame adapted to the shock-forming solutions. Three of the frame vectors are L, X_1, X_2, where the X_A are the angular coordinate vectorfields along the $S_{t,u}$, that is, $X_1 = \frac{\partial}{\partial \vartheta^1}|_{t,u,\vartheta^2}$, and similarly for X_2. These three vectors are tangent to the $\mathcal{C}_u$. To complete the frame, we introduce the transversal vectorfields

$$R, \qquad \breve{R} := \mu R, \tag{2.1.19}$$

where R is uniquely defined by requiring it to be tangent to Σ_t, $g-$orthogonal to $S_{t,u}$, inward pointing, and normalized by $g(R, R) = 1$. The analog of R in Subsect. 1.3 is $-\sqrt{1+\Psi}\partial_r$. We are using R as a convenient replacement for the null vectorfield $\underline{L}$ from spherical symmetry (see (1.3.4)). Even though R is not null, it is transversal to the $\mathcal{C}_u$, which is the property of greatest relevance. Under the assumption (2.1.7) we have

$$g(\breve{R}, \breve{R}) = \mu^2, \quad g(L, R) = -1, \quad g(L, \breve{R}) = -\mu.$$

Consistent with the Heuristic Principle, one can show that

$$R = -\partial_r + \text{Err},$$

where Err is small and decaying in time. Hence, the vectorfield $\breve{R}$ vanishes exactly at the first shock singularity point, where μ vanishes. Observe further that $g(L, \breve{R}) = -\mu \implies \breve{R}u = 1$; hence relative to the geometric coordinates t, u, ϑ, we have

$$\breve{R} = \frac{\partial}{\partial u} + \text{a small } S_{t,u}-\text{tangent angular deviation}.$$

Having defined the above vectorfields, we can now define the frame that we use to analyze solutions; see Figure 4.

Definition 2.2 (**Rescaled frame**)**.** We define the rescaled frame as follows:

$$\left\{L, \breve{R}, X_1 = \frac{\partial}{\partial \vartheta^1}, X_2 = \frac{\partial}{\partial \vartheta^2}\right\}. \tag{2.1.20}$$

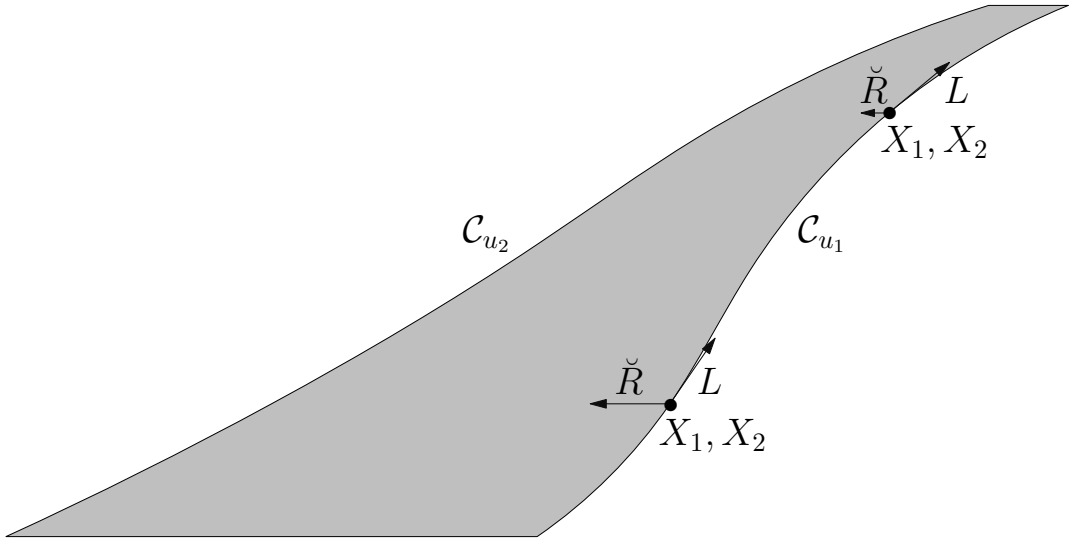


FIGURE 4. The rescaled frame at two distinct points. X_1, X_2 are orthogonal to the plane of the page and are omitted.

At each point where $\mu > 0$, the frame (2.1.20) has span equal to span$\{\frac{\partial}{\partial x^\alpha}\}_{\alpha=0,1,2,3}$. In fact, relative to an arbitrary coordinate system, we have the following decompositions:

$$\begin{aligned}(g^{-1})^{\alpha\beta} &= -L^\alpha L^\beta - (L^\alpha R^\beta + R^\alpha L^\beta) + (\not{g}^{-1})^{AB} X_A^\alpha X_B^\beta \\ &= -L^\alpha L^\beta - \mu^{-1}(L^\alpha \breve{R}^\beta + \breve{R}^\alpha L^\beta) + (\not{g}^{-1})^{AB} X_A^\alpha X_B^\beta,\end{aligned} \tag{2.1.21}$$

where $\not{g}_{AB} := g(X_A, X_B)$, $A, B = 1, 2$.

In most of our analysis, we find it convenient to work with $\breve{R}$, but in some of our analysis, it is better to use instead the following vectorfield:

$$\breve{\underline{L}} := \mu L + 2\breve{R}. \tag{2.1.22}$$

The analog of $\breve{\underline{L}}$ in Subsect. 1.3 is $\mu\underline{L}$, where $\underline{L}$ is defined in (1.3.4). Note that $\breve{\underline{L}}$ is the uniquely defined null vectorfield that is orthogonal to the spheres $S_{t,u}$ and verifies

$$g(L, \breve{\underline{L}}) = -2\mu.$$

2.1.2. *The wave operator relative to the rescaled frame and the $S_{t,u}$ tensorfield χ.* Now that we have a good frame for analyzing solutions, it is important to understand how the covariant wave operator looks when expressed relative to it. Some rather tedious but straightforward calculations reveal that we can decompose

$$\mu\Box_{g(\Psi)}\Psi = -L\overbrace{(\mu L\Psi + 2\breve{R}\Psi)}^{\underline{\breve{L}}\Psi} + \mu\not{\Delta}\Psi - \mathrm{tr}_{\not{g}}\chi\breve{R}\Psi + \mathrm{Err}, \tag{2.1.23}$$

where $\not{\Delta}$ is the Laplacian of the metric $\not{g}$ on the spheres $S_{t,u}$ and the error terms are small and decaying according to the Heuristic Principle; see [63, Proposition 4.3.1] for more details.

In equation (2.1.23), χ is a symmetric type $\binom{0}{2}$ tensorfield on $S_{t,u}$ that verifies

$$\chi_{AB} = g(\mathscr{D}_A L, X_B), \tag{2.1.24}$$

and $\mathrm{tr}_{\not{g}}\chi := (\not{g}^{-1})^{AB}\chi_{AB}$. Equivalently, $\chi_{AB} = \mathcal{L}_L \not{g}_{AB} = \frac{\partial}{\partial t}|_{u,\vartheta}\not{g}_{AB}$, where $\mathcal{L}_L$ denotes Lie differentiation with respect to L. In the case of the background solution $\Psi \equiv 0$, we have $\mathrm{tr}_{\not{g}}\chi = 2r^{-1}$, where r is the Euclidean radial coordinate on Σ_t. For the perturbed solutions under consideration, we have $\mathrm{tr}_{\not{g}}\chi = 2\varrho^{-1} + \mathrm{Err}$, where

$$\varrho := 1 - u + t. \tag{2.1.25}$$

Since $L\varrho = 1$, we therefore deduce from (2.1.23) that

$$\varrho\mu\Box_{g(\Psi)}\Psi = -L\left\{\mu L(\varrho\Psi) + 2\breve{R}(\varrho\Psi)\right\} + \varrho\mu\not{\Delta}\Psi + \mathrm{Err}. \tag{2.1.26}$$

As we will see, the form of the equation (2.1.26) is important for showing that μ can go to 0 in finite time.

Remark 2.3. Note that (2.1.26) corresponds, roughly, to equation (1.3.12a) in the context of John's spherically symmetric wave equation.

Remark 2.4 (**Rescaling by μ "removes" the dangerous semilinear term**)**.** Note that we have brought a factor of μ under the outer L differentiation in equations (2.1.23) and (2.1.26). We have already seen the importance of this "rescaling by μ" in spherical symmetry: it removes the dangerous quadratic semilinear term that decays slowly; see equation (1.3.12a). Although a similar remark applies away from spherical symmetry, it is more difficult to see that the remaining error terms Err are indeed such that, according to the Heuristic Principle, they enjoy better time decay. The reason is that they involve, in addition to the first derivatives of Ψ, the second derivatives of the eikonal function. Hence, to show that these error terms are negligible with respect to time decay, one must control the asymptotic behavior of eikonal function. For example, one of the error terms in equation (2.1.26) is $\varrho\left\{\mathrm{tr}_{\not{g}}\chi - \frac{2}{\varrho}\right\}\breve{R}\Psi$. Using the Heuristic Principle decay estimates, one can show that the sup-norm of the product $\varrho\left\{\mathrm{tr}_{\not{g}}\chi - \frac{2}{\varrho}\right\}\breve{R}\Psi$ is $\leq C\varepsilon\ln(e+t)(1+t)^{-2}$, which is the same decay rate[59] as that of the two error term products on the right-hand side of equation (1.3.12a) in spherical symmetry.

[59]If there were a $(\underline{L}\Psi)^2$ term on the right-hand side of (1.3.12a), then, because of the presence of the factor r, its decay rate would not be integrable in t.

We also note that, from the definition of the covariant derivative $\mathscr{D}$, we have

$$\chi_{AB} = g_{ab}(X^c_A \partial_c L^a) X^b_B + X^a_A X^b_B L^\gamma \Gamma_{ab\gamma}, \tag{2.1.27}$$

where the lowercase Latin indices are relative to spatial rectangular spatial coordinates, the lowercase Greek indices are relative to rectangular spacetime coordinates, and the uppercase Latin indices correspond to the two $S_{t,u}$ frame vectors. Hence, (2.1.27) shows that χ is an auxiliary quantity expressible in terms of the frame derivatives of Ψ and the frame derivatives of the rectangular components L^i. However, because of its importance, to be clarified below, it is convenient to think of χ as an independent quantity.

2.1.3. *The evolution equation for* μ. We now derive a transport equation for μ, analogous to the equation (1.3.11) in our analysis in spherical symmetry.

Lemma 2.1 (**Transport equation for** μ)**.** *The quantity* μ *defined in* (2.1.13) *verifies the following transport equation:*

$$L\mu = \frac{1}{2} G_{LL} \breve{R}\Psi + \mu \mathrm{Err}, \qquad G_{LL} := G_{\alpha\beta}(\Psi) L^\alpha L^\beta, \tag{2.1.28}$$

where the term Err *is an error term involving* $\mathcal{C}_u-$*tangential derivatives of* Ψ.

Proof. Recalling that $\mu = (L^0_{(Geo)})^{-1}$, relative to the rectangular coordinates, we consider the 0 component of equation (2.1.12):

$$L_{(Geo)} L^0_{(Geo)} = -(g^{-1})^{0\gamma\,(\Psi)}\Gamma_{\alpha\gamma\beta}\, L^\alpha_{(Geo)}\, L^\beta_{(Geo)}. \tag{2.1.29}$$

Multiplying (2.1.29) by μ^3, using the definition (2.1.14) of L, the decomposition (2.1.21), the identities $(\not{g}^{-1})^{0\gamma} = 0$, $L^0 = 1$, $R^0 = 0$, $\breve{R} = \mu R$, and equation (2.1.4), we deduce that

$$\begin{aligned} \mu^2 L L^0_{(Geo)} &= \frac{1}{2}\left\{\mu L^\gamma + \breve{R}^\gamma\right\}\left(G_{\gamma\beta}\partial_\alpha\Psi + G_{\alpha\gamma}\partial_\beta\Psi - G_{\alpha\beta}\partial_\gamma\Psi\right) L^\alpha L^\beta \\ &= \frac{1}{2}\mu G_{LL} L\Psi + \mu G_{LR} L\Psi - \frac{1}{2} G_{LL}\breve{R}\Psi. \end{aligned} \tag{2.1.30}$$

Equation (2.1.28) now follows from equation (2.1.30), the relation $\mu^2 L L^0_{(Geo)} = \mu^2 L(\frac{1}{\mu}) = -L\mu$, and incorporating the $\mathcal{C}_u-$tangent derivatives into the term Err. □

Remark 2.5 (**The coupled system**)**.** We stress the following important point: the basic equations that we need to study are the wave equation (2.1.2) coupled to the evolution equations (2.1.12) for the rectangular components of $L_{(Geo)}$. Though the standard form (2.1.2) is useful for deriving energy estimates, the equivalent form (2.1.23) is fundamental for understanding the behavior of the rescaled quantities.

2.1.4. *Initial conditions and relevant regions.* We start by recalling the basic setup described in Subsect. 2.1, and in particular, gather our notations in one place. Our setup here is closely related to the one we used in Subsubsect. (1.3.1) in spherical symmetry. We recall that we are studying solutions to the wave equation (2.1.2) subject to the following small initial conditions on $\Sigma_0 = \{t = 0\}$:

$$\mathring{\Psi} := \Psi|_{\Sigma_0}, \qquad \mathring{\Psi}_0 := \partial_t \Psi|_{\Sigma_0}, \tag{2.1.31}$$

where $(\mathring{\Psi}, \mathring{\Psi}_0)$ are supported in the Euclidean unit ball $\{r \leq 1\}$, with $r = \sqrt{\sum_{a=1}^{3}(x^a)^2}$ the standard Euclidean distance to the origin on Σ_0. As we mentioned before, U_0 is a real number verifying

$$0 \leq U_0 < 1. \tag{2.1.32}$$

We study the future-behavior of the solution in the region that corresponds to the portion of the data lying in an annular region of inner Euclidean spherical radius $1 - U_0$ and outer Euclidean spherical radius 1 (that is, the thickness of the region is U_0):

$$\Sigma_0^{U_0} := \{x \in \Sigma_0 \mid 1 - U_0 \leq r(x) \leq 1\}; \tag{2.1.33}$$

see Figure 3. The reason that we assume $U_0 < 1$ is simply to avoid potential problems with degeneration of our coordinates at the origin. We define the size of the data as follows:

$$\mathring{\epsilon} = \mathring{\epsilon}[(\mathring{\Psi}, \mathring{\Psi}_0)] := \|\mathring{\Psi}\|_{H^{25}(\Sigma_0^1)} + \|\mathring{\Psi}_0\|_{H^{24}(\Sigma_0^1)}. \tag{2.1.34}$$

In (2.1.34), H^N is the standard Euclidean Sobolev space involving order $\leq N$ rectangular spatial derivatives along Σ_0^1. To prove small-data shock formation, we assume that $\mathring{\epsilon}$ is sufficiently small (see Footnote 54 on pg. 30) together with some other open conditions that we explain below.

The following regions of spacetime depend on our eikonal function u and are analogs of regions that we encountered in Subsubsect. (1.3.1) in spherical symmetry.

Definition 2.3 (**Subsets of spacetime**)**.** We define the following spacetime subsets:

$$\Sigma_{t'} := \{(t, x^1, x^2, x^3) \in \mathbb{R}^4 \mid t = t'\}, \tag{2.1.35a}$$

$$\Sigma_{t'}^{u'} := \{(t, x^1, x^2, x^3) \in \mathbb{R}^4 \mid t = t',\ 0 \leq u(t, x^1, x^2, x^3) \leq u'\}, \tag{2.1.35b}$$

$$\mathcal{C}_{u'}^{t'} := \{(t, x^1, x^2, x^3) \in \mathbb{R}^4 \mid u(t, x^1, x^2, x^3) = u'\} \cap \{(t, x^1, x^2, x^3) \in \mathbb{R}^4 \mid 0 \leq t \leq t'\}, \tag{2.1.35c}$$

$$S_{t',u'} := \mathcal{C}_{u'}^{t'} \cap \Sigma_{t'}^{u'} = \{(t, x^1, x^2, x^3) \in \mathbb{R}^4 \mid t = t',\ u(t, x^1, x^2, x^3) = u'\}, \tag{2.1.35d}$$

$$\mathcal{M}_{t',u'} := \cup_{u \in [0,u']} \mathcal{C}_u^{t'} \cap \{(t, x^1, x^2, x^3) \in \mathbb{R}^4 \mid t < t'\}. \tag{2.1.35e}$$

We refer to the Σ_t and Σ_t^u as "constant time slices," the $\mathcal{C}_u^t$ as "outgoing null cones," and the $S_{t,u}$ as "spheres." We sometimes use the notation $\mathcal{C}_u$ in place of $\mathcal{C}_u^t$ when we are not concerned with the truncation time t.

Remark 2.6. Note that $\mathcal{M}_{t,u}$ is, by definition, "open at the top."

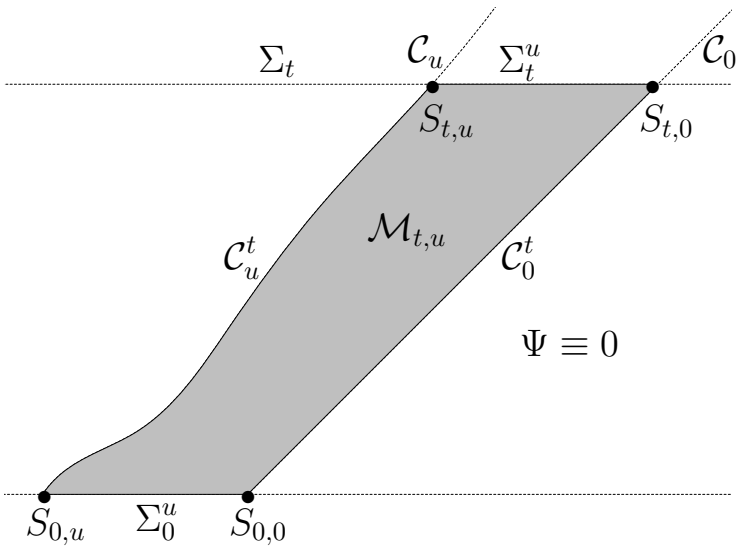


FIGURE 5. Spacetime subsets

Just as in the case of the spherically symmetric shock formation discussed in Subsect. 1.3, all of the interesting dynamics takes place in the region $\mathcal{M}_{t,u}$ for t sufficiently large.

2.1.5. *A more precise version of the Heuristic Principle.* We are now ready to give a more precise version of the Heuristic Principle of Subsubsect. 2.1.1; see [63, Section 11.4] for more details.

Heuristic Principle II. *Consider a small-data solution Ψ to equation* (2.1.2)*, where the size of the data is defined in* (2.1.34)*. Then under the structural conditions on the nonlinearities explained below in Subsubsect. 2.1.7, the directional derivatives of a solution Ψ with respect to the rescaled frame $\{L, \breve{R}, X_1, X_2\}$ (see* (2.1.20)*) decay, in the relevant $\mathcal{M}_{t,u}$ region, in a manner analogous to the linear peeling properties* (1.2.7) *all the way to the formation of the first shock. More precisely,*[60] *the following estimates hold:*

$$|L\Psi|,\ |\not{\nabla}\Psi| \leq \frac{\varepsilon}{(1+t)^2}, \tag{2.1.36a}$$

$$|\Psi|,\ |\breve{R}\Psi| \leq \frac{\varepsilon}{1+t}, \tag{2.1.36b}$$

where ε is a small constant that is controlled by the size of the data, and $|\not{\nabla}\Psi|$ is the size of the angular gradient[61] *of Ψ as measured by $\not{g}$, that is, the size of the gradient of Ψ viewed as a function on the $S_{t,u}$. A similar statement holds for a limited number of higher directional derivatives, where each additional L and $\not{\nabla}$ differentiation leads to a gain in decay of $(1+t)^{-1}$. However, unlike in the linear case, the very high derivatives are allowed to have degenerate behavior in μ; see Prop. 3.4.*

[60]Note that terms of the form $(1+|u|)^{1/2}$ are omitted in these estimates. These factors are in fact $\mathcal{O}(1)$ inside the relevant $\mathcal{M}_{t,u}$ region.

[61]Note that the frame vectorfields X_1 and X_2 span the tangent space of the $S_{t,u}$ and hence a bound for $|\not{\nabla}\Psi|$ also implies a bound for the directional derivatives $|X_1\Psi|$ and $|X_2\Psi|$.

Note that (2.1.36b) is equivalent to the following estimate for $R\Psi = \mu^{-1}\breve{R}\Psi \sim -\partial_r\Psi$:

$$|R\Psi| \leq \frac{1}{\mu}\frac{\varepsilon}{(1+t)}. \tag{2.1.37}$$

Actually, for the shock-forming solutions of interest, a key ingredient in the proof is showing that for an open set of data, we have a *lower bound* of the form[62] $|\breve{R}\Psi| \gtrsim \varepsilon(1+t)^{-1}$ (see inequality (2.2.1)), so that

$$|R\Psi| \gtrsim \frac{1}{\mu}\frac{\varepsilon}{(1+t)}. \tag{2.1.38}$$

The inequalities (2.1.37) and (2.1.38) imply that $R\Psi$ blows up *exactly at the points where* μ *vanishes.* Note that we have already encountered an analog of the lower bound (2.1.38) in spherical symmetry; see inequalities (1.3.17) and (1.3.40).

Remark 2.7. The decay estimates (2.1.36a), (2.1.36b) can be used to derive estimates for the components of the covariant Hessian $H = \mathscr{D}^2 u$ of the eikonal function relative to the rescaled frame.[63] Indeed, recall that H verifies the transport equation (1.5.6), and hence, ignoring for now factors of μ^{-1}, we have the schematic equation

$$LH + H^2 = \mathcal{R} \tag{2.1.39}$$

where as before, $\mathcal{R} = \mathcal{R}(\Psi)$ is a curvature component that can be algebraically expressed in terms of the up-to-second-order derivatives of Ψ and the up-to-second-order derivatives of u. Using the precise decay estimates for various components of $\mathcal{R}(\Psi)$ relative to the rescaled frame, which follow from (2.1.36a) and (2.1.36b), together with some bootstrap assumptions on the first derivatives of u, we can derive precise decay estimates for the rescaled frame components of H. The behavior of some of these components (more precisely those involving $\breve{R}$) differs significantly from that of the corresponding components of the Hessian of the flat eikonal function $1 + t - r$.

2.1.6. *The failure of the classic null condition.* The classic null condition is defined for scalar equations of the form (1.1.10) (with $I = 1$, see Definition 1.1), scalar wave equations of the form (1.1.8), and for general systems of wave equations of the form (1.1.10) (see Remark 2.12 or [34]). Here we study the particular case of scalar equations of the form (2.1.2) in detail. For simplicity, we assume in the present subsubsection that the semilinear term $\mathcal{N}(\Psi)(\partial\Psi, \partial\Psi)$ on the right-hand side of (2.1.2) is equal to 0. That is, we discuss here only the equation

$$\Box_{g(\Psi)}\Psi = 0.$$

In Subsubsect. 2.1.7, we will address the case in which $\mathcal{N}(\Psi)(\partial\Psi, \partial\Psi)$ is non-zero. We now show that when $\mathcal{N}(\Psi)(\partial\Psi, \partial\Psi) = 0$ in (2.1.2), the classic null condition holds if and only if the following scalar-valued function ${}^{(+)}\aleph$ completely vanishes.

[62] We often write $A \lesssim B$ whenever there exists a uniform constant $C > 0$ such that $A \leq CB$. Similarly, we often write $A \gtrsim B$ whenever there exists a uniform constant $C > 0$ such that $A \geq CB$.

[63] Since μ and L^i are first derivatives of u, this is essentially equivalent to deriving estimates for the first derivatives of μ and L^i. In practice, we directly estimate the derivatives of μ and L^i by studying the transport equations that they verify; see Remark 2.5.

Definition 2.4. We define the *future null condition failure factor* ${}^{(+)}\aleph$ by

$${}^{(+)}\aleph := \underbrace{G_{\alpha\beta}(\Psi = 0)}_{\text{constants}} L^{\alpha}_{(Flat)} L^{\beta}_{(Flat)}, \tag{2.1.40}$$

where $L_{(Flat)} = \partial_t + \partial_r$ and $G_{\alpha\beta}(\Psi) = \frac{d}{d\Psi} g_{\alpha\beta}(\Psi)$ (see (2.1.5)).

Remark 2.8. Note that relative to standard spherical coordinates (t, r, θ) on Minkowski spacetime, ${}^{(+)}\aleph$ can be viewed as a function that depends only on θ.

Remark 2.9. In the region $\{t \geq 0\}$, ${}^{(+)}\aleph$ is the *coefficient* of the most dangerous (in terms of *linear* decay rate) quadratic terms in the wave equation $\Box_{g(\Psi)}\Psi = 0$, when the equation is expressed relative to the Minkowskian frame (2.1.44) introduced below. Roughly, as in the case of F. John's equation in spherical symmetry, these dangerous terms are the ones that drive future shock formation. However, when carrying out detailed analysis, the correct frame to use is the dynamic one given in (2.1.20).

Remark 2.10 (Connection between ${}^{(+)}\aleph$ and small-data shock formation). When $\mathcal{N}(\Psi)(\partial\Psi, \partial\Psi) \equiv 0$, small-data shock formation occurs whenever ${}^{(+)}\aleph$ is nontrivial; see Theorem 5.

Remark 2.11 (Past null condition failure factor). We could also study shock formation in the region $\{t \leq 0\}$. In this case, the relevant function is not ${}^{(+)}\aleph$, but is instead the past null condition failure factor ${}^{(-)}\aleph$, defined by replacing the vectorfield $L_{(Flat)}$ in equation (2.1.40) with $-\partial_t + \partial_r$. Note that $-\partial_t + \partial_r$ is outward pointing as we head to the past. The point is that quadratic terms that have a slow decay rate as $t \to \infty$ can have a faster decay rate as $t \to -\infty$ and vice versa; the function ${}^{(-)}\aleph$ is the coefficient of the slowest decaying quadratic terms as $t \to -\infty$. Note also that ${}^{(+)}\aleph$ completely vanishes if and only if ${}^{(-)}\aleph$ completely vanishes. In fact, the functions ${}^{(+)}\aleph$ and $-{}^{(-)}\aleph$ have the same range.

To show that the complete vanishing of ${}^{(+)}\aleph$ is equivalent to the classic null condition being verified (see Definition 1.1), we first Taylor expand the right-hand side of (2.1.6) around $(\Psi, \partial\Psi, \partial^2\Psi) = (0, \mathbf{0}, \mathbf{0})$ and find that the quadratic nonlinear terms are, up to constant factors,

$$G_{\alpha\beta}(0)(m^{-1})^{\alpha\kappa}(m^{-1})^{\beta\lambda}\Psi\partial_\kappa\partial_\lambda\Psi, \tag{2.1.41}$$

$$G_{\kappa\lambda}(0)(m^{-1})^{\kappa\lambda}(m^{-1})^{\alpha\beta}\partial_\alpha\Psi\partial_\beta\Psi, \tag{2.1.42}$$

$$G_{\kappa\lambda}(0)(m^{-1})^{\alpha\kappa}(m^{-1})^{\beta\lambda}\partial_\alpha\Psi\partial_\beta\Psi. \tag{2.1.43}$$

Clearly the term (2.1.42) always verifies the classic null condition. By definition, the term (2.1.41) verifies the classic null condition if and only if $G_{\alpha\beta}(0)(m^{-1})^{\alpha\kappa}(m^{-1})^{\beta\lambda}\ell_\kappa\ell_\lambda = 0$ for all Minkowski-null covectors ℓ. It is straightforward to see that equivalently,[64] the term (2.1.41) verifies the classic null condition if and only if ${}^{(+)}\aleph \equiv 0$. Similarly, the term (2.1.43) verifies the classic null condition if and only if ${}^{(+)}\aleph$ is trivial.

We now discuss three relevant examples.

- It is easy to see that ${}^{(+)}\aleph$ completely vanishes if and only if the constant tensorfield $G_{\alpha\beta}(\Psi = 0)$ is proportional to the Minkowski metric $m_{\alpha\beta} = \text{diag}(-1, 1, 1, 1)$. Thus, for the equations $\Box_{g(\Psi)}\Psi = 0$, a necessary and sufficient condition for the nonlinearities to verify the classic null condition is

[64] Note that given any future-directed Minkowski-null vector ℓ^α, there exists a spacetime point such that the Minkowski-null vector $L^\alpha_{(Flat)}$ in (2.1.40) is parallel to ℓ^α.

that up to cubic terms, $g(\Psi) = (1 + f(\Psi))m$, where $f(0) = 0$. Consequently, for the equations $\Box_{g(\Psi)}\Psi = 0$, the classic null condition is very restrictive and is satisfied only in trivial cases.

- Consider the equation $\Box_{g(\Psi)}\Psi = 0$ in the case of F. John's metric $-(dt)^2+(1+\Psi)^{-1}\sum_{a=1}^{3}(dx^a)^2$, as in equation[65] (1.3.2). We compute that $G_{ij}(\Psi = 0) = -1$ if $i = j \in \{1, 2, 3\}$, and all other rectangular components of $G(\Psi = 0)$ vanish. Using also that $L^{\alpha}_{(Flat)} = (1, x^1/r, x^2/r, x^3/r)$ relative to the rectangular coordinates, where $r = \sqrt{\sum_{a=1}^{3}(x^a)^2}$, we find that ${}^{(+)}\aleph \equiv -1$. This example is a good model of the kinds of equations that Christodoulou studied in [11], where the analog of ${}^{(+)}\aleph$ is constant.
- If $g_{\alpha\beta}(\Psi) = m_{\alpha\beta} + \Psi(\delta^1_\alpha\delta^2_\beta + \delta^2_\alpha\delta^1_\beta)$, then[66] $G_{12}(\Psi = 0) = G_{21}(\Psi = 0) = 1$, and all other rectangular components of $G(\Psi = 0)$ vanish. Hence, we find that ${}^{(+)}\aleph = 2L^1_{(Flat)}L^2_{(Flat)} = 2x^1x^2/r^2$ in this case. Note that ${}^{(+)}\aleph$ can be viewed as a function on $\mathbb{S}^2 \subset \mathbb{R}^3$.

We now discuss the classic null condition for equations $\Box_{g(\Psi)}\Psi = 0$ from a slightly different point of view, one which explains the connection between the non-vanishing of ${}^{(+)}\aleph$ and the presence of dangerous quadratic terms and which is connected to our analysis of shock formation outside of spherically symmetry. Specifically, we will write the equation relative to rectangular coordinates and then decompose the quadratic parts of the nonlinearities relative to the following Minkowskian frame:

$$\{L_{(Flat)}, R_{(Flat)}, X_{(Flat);1}, X_{(Flat);2}\}, \tag{2.1.44}$$

where $L_{(Flat)} = \partial_t + \partial_r$, $R_{(Flat)} = -\partial_r$, and the $X_{(Flat);A}$ are angular vectorfields tangent to the Euclidean spheres of constant $r-$value in Σ_t. We stress that we use the frame (2.1.44) for illustrative purposes only. It is not suitable for studying solutions near the shock, where we should instead use the dynamic frame (2.1.20). Nonetheless, the main idea to keep in mind is that the *forward* linear peeling properties (1.2.7) suggest that relative to the frame (2.1.44), the most dangerous quadratic terms in equation $\Box_{g(\Psi)}\Psi = 0$ in the region $\{t \geq 0\}$ are the ones proportional to $\Psi R_{(Flat)}(R_{(Flat)}\Psi)$ and $(R_{(Flat)}\Psi)^2$; these terms have the slowest t decay rates. Note that this assertion can be relevant only in the region $\{t \geq 0\}$ and should be altered if it is to apply to the region $\{t \leq 0\}$.

To carry out the decomposition, we first note that in analogy with (2.1.21), relative to the frame (2.1.44), the inverse Minkowski metric can be decomposed as

$$(m^{-1})^{\alpha\beta} = -L^{\alpha}_{(Flat)}L^{\beta}_{(Flat)} - (L^{\alpha}_{(Flat)}R^{\beta}_{(Flat)} + R^{\alpha}_{(Flat)}L^{\beta}_{(Flat)}) + (\not{m}^{-1})^{AB}X^{\alpha}_{(Flat);A}X^{\beta}_{(Flat);B}. \tag{2.1.45}$$

Next, decomposing the quadratic part (2.1.41) of the quasilinear term relative to the frame (2.1.44) and using in particular (2.1.45), we find that the component proportional to $\Psi R_{(Flat)}(R_{(Flat)}\Psi)$, is, up to constant factors,

$${}^{(+)}\aleph\Psi R_{(Flat)}(R_{(Flat)}\Psi). \tag{2.1.46}$$

[65] It is easy to show that equation (1.3.2) is equivalent to the covariant wave equation $\Box_{g(\Psi)}\Psi = \mathcal{N}(\Psi)(\partial\Psi, \partial\Psi)$, where $\mathcal{N}(\Psi)(\partial\Psi, \partial\Psi) = -\frac{1}{2}(1 + \Psi)^{-1}(g^{-1})^{\alpha\beta}\partial_\alpha\Psi\partial_\beta\Psi$. Note that $\mathcal{N}(\Psi)(\partial\Psi, \partial\Psi)$ verifies the classic null condition. Hence, from the point of view of investigating failure of the classic null condition, we can study the equation $\Box_{g(\Psi)}\Psi = 0$ instead of (1.3.2).

[66] Here, $\delta^{\bullet}_{\bullet}$ denotes the standard Kronecker delta.

Similarly, decomposing (2.1.42) and (2.1.43), we find that the term proportional to $(R_{(Flat)}\Psi)^2$ is, up to constant factors,

$$^{(+)}\aleph(R_{(Flat)}\Psi)^2. \tag{2.1.47}$$

Hence, for the equation $\Box_{g(\Psi)}\Psi = 0$, $^{(+)}\aleph \equiv 0$ is equivalent to the absence, relative to the frame (2.1.44), of the dangerous quadratic terms $\Psi R_{(Flat)}(R_{(Flat)}\Psi)$ and $(R_{(Flat)}\Psi)^2$.

Remark 2.12. In the case of *general systems* of the form (2.1.2) with $\Psi = \{\Psi^I\}_{I=1,\dots,N}$, the correct definition of $^{(+)}\aleph$ has to be changed, in view of possible cancellations between components. For example, if Φ verifies the scalar equation $g^{\alpha\beta}(\partial\Phi)\partial_\alpha\partial_\beta\Phi = 0$ and $\Psi := (\Psi_0, \Psi_1, \Psi_2, \Psi_3)$, where $\Psi_\lambda := \partial_\lambda\Phi$, then the relevant definition of $^{(+)}\aleph$ is as follows:

$$^{(+)}\aleph := m_{\kappa\lambda}G^\kappa_{\alpha\beta}(\Psi = 0)L^\alpha_{(Flat)}L^\beta_{(Flat)}L^\lambda_{(Flat)}, \tag{2.1.48}$$

where

$$G^\lambda_{\alpha\beta} = G^\lambda_{\alpha\beta}(\Psi) := \frac{\partial}{\partial\Psi_\lambda}g_{\alpha\beta}(\Psi). \tag{2.1.49}$$

If, for the equation $g^{\alpha\beta}(\partial\Phi)\partial_\alpha\partial_\beta\Phi = 0$, we repeat the Minkowskian frame decomposition carried out above for the equation $\Box_{g(\Psi)}\Psi = 0$, we find that up to constant factors, $^{(+)}\aleph$ as defined in (2.1.48) is precisely the coefficient of the dangerous quadratic term $(R_{(Flat)}\Phi)\cdot R_{(Flat)}(R_{(Flat)}\Phi)$. We shall return to this issue in Subsect. 4.2.

2.1.7. *Structural assumptions on the nonlinearities in equation* (1.4.4). We are now ready to make assumptions on the metric g and the semilinear term $\mathcal{N}(\Psi)(\partial\Psi, \partial\Psi)$ from equation (2.1.2) for which we can derive a small-data shock-formation result in the region $\{t \geq 0\}$.

(1) To produce a shock, we assume that the metric g verifies the condition $^{(+)}\aleph \not\equiv 0$ (see Definition 2.4).
(2) We assume that for Ψ sufficiently small, the semilinear term $\mathcal{N}(\Psi)(\partial\Psi, \partial\Psi)$ on the right-hand side of (2.1.2) has the following structure when it is decomposed relative to the non-rescaled dynamic frame $\{L, R, X_1, X_2\}$:

$$\text{No terms in the expansion of } \mathcal{N}(\Psi)(\partial\Psi, \partial\Psi) \text{ involve the factor } (R\Psi)^2. \tag{2.1.50}$$

Remark 2.13. A term $\mathcal{N}(\Psi)(\partial\Psi, \partial\Psi)$ verifying the above assumptions should be viewed as a negligible error term that does not interfere with the shock formation processes.

Remark 2.14. To further explain the relevance of the condition $^{(+)}\aleph \not\equiv 0$ in the shock-formation problem, it pays to redo, relative to the non-rescaled dynamic frame $\{L, R, X_1, X_2\}$, the analysis that identified the dangerous terms (2.1.46) and (2.1.47). In doing so, we find that the dangerous terms are, up to constant factors, $G_{LL}\Psi R(R\Psi)$ and $G_{LL}(R\Psi)^2$, where

$$G_{LL} := G_{\alpha\beta}(\Psi)L^\alpha L^\beta. \tag{2.1.51}$$

The connection with $^{(+)}\aleph$ is: the decay estimates of the Heuristic Principle can be used to show that G_{LL} is well-approximated by $^{(+)}\aleph$ along the integral curves of L. Hence, if $^{(+)}\aleph \not\equiv 0$, then the dangerous terms have the strength needed to drive shock formation.

Remark 2.15 (**Future strong null condition**). Note that the condition (2.1.50) for $\mathcal{N}(\Psi)(\partial\Psi,\partial\Psi)$ *cannot be extended to include arbitrary cubic or higher order terms* in $\partial\Psi$. Though such terms are harmless in the context of proving small-data global existence (for example, when the classic null condition is verified), this is no longer the case if we expect $R\Psi$ to become singular, since in that case cubic terms can become dominant whenever $R\Psi$ is large. A useful version of the null condition for all higher order terms in $\partial\Psi$ can only allow terms which are *linear* with respect to the directional derivative $R\Psi$. Such a condition may be called the *future strong null condition*, where we explain the "future" aspect of it in Remark 2.16. We stress that *the future strong null condition is a true nonlinear condition tied to the dynamic metric* g, as opposed to the classic null condition, which is based on Taylor expanding a nonlinearity around 0 and keeping only the quadratic part.

Remark 2.16 (**Asymmetry between the future and the past**). Note that $\mathcal{N}(\Psi)(\partial\Psi,\partial\Psi) := (L\Psi)^2$ verifies the future strong null condition even though the flat analog term $(L_{(Flat)}\Psi)^2$ fails the classic null condition. Hence, strictly speaking, it is not correct to view the future strong null condition as more restrictive than the classic null condition. The relevant point is that as $t \to \infty$, $(L\Psi)^2$ is expected to decay sufficiently quickly, while the same behavior for $(L\Psi)^2$ is not expected as $t \to -\infty$. We could also formulate a "past strong null condition." We would simply need to replace the dynamic frame $\{L, R, X_1, X_2\}$ used in the statement (2.1.50) with an analogous dynamic frame whose first vector is $g-$null and outgoing as $t \downarrow -\infty$.

2.2. **The role of** μ**.** Here, we continue our rough description of shock formation and show that $\mu \to 0$ precisely corresponds to the formation of a shock and the blow-up of the directional derivative $R\Psi$. For simplicity, we focus on solutions that are nearly spherically symmetric, at least in the sense of lower-order derivatives. The argument we give here closely follows the argument given in spherical symmetry in Subsect. 1.3 (see in particular the proof of Cor. 1.4).

As a first step, we use the wave equation (2.1.26) to infer the existence of an open set of initial data such that, *for sufficiently large* t, a lower bound of the form

$$\breve{R}\Psi(t,u,\vartheta) \gtrsim \mathring{\epsilon}\frac{1}{1+t} \tag{2.2.1}$$

holds along some integral curve of L (that is, at fixed u and ϑ), with $\mathring{\epsilon}$ the size of the data. Alternatively, for a different open set of data, we could derive the bound $\breve{R}\Psi(t,u,\vartheta) \lesssim -\mathring{\epsilon}(1+t)^{-1}$ (again for fixed u,ϑ). To derive these bounds, we use the fact that the last two terms on the right-hand side of (2.1.26) are small error terms that decay at an integrable-in-time rate, thereby deducing that

$$L\left\{\mu L(\varrho\Psi) + 2\breve{R}(\varrho\Psi)\right\} = \text{Err}. \tag{2.2.2}$$

Hence, we can integrate (2.2.2) along the integral curves of L to deduce

$$\left\{\mu L(\varrho\Psi) + 2\breve{R}(\varrho\Psi)\right\}(t,u,\vartheta) \approx f_{data}(u,\vartheta), \tag{2.2.3}$$

where $f_{data}(u,\vartheta)$ is equal to $\mu L(\varrho\Psi)+2\breve{R}(\varrho\Psi)$ evaluated at $(0,u,\vartheta)$. Expanding the left-hand side (2.2.3) via the Leibniz rule and appealing to the Heuristic Principle decay estimates, we see that all terms except

for $2\varrho\breve{R}\Psi$ decay. Hence, we find that for suitably large times, we have

$$\breve{R}\Psi(t,u,\vartheta) \approx \frac{1}{2}\frac{1}{\varrho(t,u)} f_{data}(u,\vartheta) \approx \frac{1}{1+t} f_{data}(u,\vartheta). \tag{2.2.4}$$

We have therefore derived the desired bounds.

Remark 2.17 (Remarks on the linear term $\mu\varrho\not{\Delta}\Psi$). The linearly small product $\mu\varrho\not{\Delta}\Psi$ is present in the term Err in equation (2.2.2) (see equation (2.1.23)). At $t = 0$, the term $\mu\varrho\not{\Delta}\Psi$ can be large compared to $f_{data}(u,\vartheta)$. Hence, in order for the above proof of (2.2.4) to work, we must assume that the initial angular derivatives of Ψ are even smaller than the other derivatives. However, using a more refined argument based on Friedlander's radiation field, one can significantly enlarge the set of small data for which it is possible to prove a lower bound of the form (2.2.1); see Subsect. 5.5.

Next, we insert the bound (2.2.4) into the evolution equation (2.1.28) for μ and ignore the error terms, which are small and decaying sufficiently fast by the Heuristic Principle. Although the factor G_{LL} in equation (2.1.28) is not constant along the integral curves of L, the Heuristic Principle decay estimates can be used to show that G_{LL} is well-approximated (relative to the geometric coordinates) by

$$\overset{(+)}{\mathring{\aleph}}(t,u,\vartheta) = \overset{(+)}{\mathring{\aleph}}(\vartheta) := \overset{(+)}{\aleph}(t=0,u=0,\vartheta). \tag{2.2.5}$$

The good feature of $\overset{(+)}{\mathring{\aleph}}$ is that it (by definition) depends only[67] on the geometric angular coordinates ϑ and hence is constant along the integral curves of L. Hence, for suitably large times, we have

$$L\mu(t,u,\vartheta) \approx \frac{1}{2}\overset{(+)}{\mathring{\aleph}}(\vartheta)\breve{R}\Psi(t,u,\vartheta) \approx \frac{1}{2}\overset{(+)}{\mathring{\aleph}}(\vartheta)\frac{1}{1+t} f_{data}(u,\vartheta). \tag{2.2.6}$$

Integrating (2.2.6) along the integral curves of L and using the small-data assumption that μ is initially near 1, we deduce that

$$\mu(t,u,\vartheta) \approx 1 + \frac{1}{2}\overset{(+)}{\mathring{\aleph}}(\vartheta)\ln(1+t) f_{data}(u,\vartheta). \tag{2.2.7}$$

Clearly, if the data are such that for some angle ϑ, $\overset{(+)}{\mathring{\aleph}}(\vartheta) f_{data}(u,\vartheta)$ is negative, then (2.2.7) implies that μ will become 0 at a time of order $\exp(C\mathring{\epsilon}^{-1})$.

We now remind the reader of the following simple consequence of the above discussion: in view of lower bound (2.2.4) and the relation $R\Psi = \mu^{-1}\breve{R}\Psi$, where $R \sim -\partial_r$ has close to Euclidean-unit-length, it follows that some rectangular spatial derivative of Ψ blows up precisely when μ vanishes.

In the work [11], Christodoulou studied quasilinear wave equations for which the analog of $\overset{(+)}{\aleph}$ was constant-valued, as in the case of John's equation, which we discussed in the first example given just below Definition 2.4. This property simplified some of his analysis and, as we describe in Subsect. 5.5, it played a central role in his identification of a class of small data that lead to shock formation for his equations. In general, $\overset{(+)}{\aleph}$ can be highly angularly dependent and in particular, there can be angular directions along which the function $\overset{(+)}{\mathring{\aleph}}$ from (2.2.5) vanishes. Along the integral curves of L corresponding to such angular directions, μ is not expected to change very much during the solution's classical lifespan.

[67] Recall that $\overset{(+)}{\aleph}$ was defined in (2.1.40) and that at $t = 0$, $u = 1 - r$ and the geometric angular coordinates coincide with the standard Euclidean angular coordinates. Since $\overset{(+)}{\aleph}$ can be viewed as a function depending only on the standard Euclidean angular coordinates, it follows that indeed, the right-hand side of (2.2.5) depends only on ϑ.

3. GENERALIZED ENERGY ESTIMATES

In this section, we discuss the most difficult aspect of proving small-data shock formation away from spherical symmetry: the derivation of generalized energy estimates that hold up to top order. Our discussion in this section applies to the nonlinear wave equation $\Box_{g(\Psi)}\Psi = \mathcal{N}(\Psi)(\partial\Psi, \partial\Psi)$ (that is, (2.1.2)) in the region $\{t \geq 0\}$ under the structural conditions on $\mathcal{N}$ stated in Subsubsect. 2.1.7.

3.1. **The basic strategy for deriving generalized energy estimates.** The discussion in the previous sections suggests the following strategy for proving shock formation in solutions to equation (2.1.2).

(1) With the help of the eikonal function u, one should construct *commutator vectorfields* Z that have good commuting properties with our nonlinear wave equation (2.1.2). It turns out that a suitable collection of commutators is the set

$$\mathscr{Z} := \{\varrho L, \breve{R}, O_{(1)}, O_{(2)}, O_{(3)}\}, \tag{3.1.1}$$

which has span equal to $\text{span}\{\partial_\alpha\}_{\alpha=0,1,2,3}$ at each spacetime point where $\mu > 0$. The rotational vectorfields $O_{(l)}$ are constructed by projecting the standard Euclidean rotation vectorfields $O^j_{(Flat,l)} := \epsilon_{laj}x^a$ onto the spheres $S_{t,u}$, where ϵ_{ijk} is the fully antisymmetric symbol normalized by $\epsilon_{123} = 1$; see [63, Chapter 5] for more details. Note that all vectorfields in $\mathscr{Z}$ depend on the first derivatives of u.

(2) Derive generalized energy estimates for a sufficiently large number of Z-derivatives of Ψ. The work of the third author showed (see Prop. 3.4) that it suffices to commute the nonlinear wave equation (2.1.2) up to 24 times with the commutation vectorfields belonging to $\mathscr{Z}$. Typically, in the flat case, such estimates are derived by contracting the energy-momentum tensorfield (see (3.2.2)) against the following two *multiplier vectorfields*:

$$\partial_t = \frac{1}{2}\left\{L_{(Flat)} + \underline{L}_{(Flat)}\right\}, \qquad \widetilde{K}_{(Flat)} = \frac{1}{2}\left\{(t+r)^2 L_{(Flat)} + (t-r)^2 \underline{L}_{(Flat)}\right\},$$

where $L_{(Flat)} := \partial_t + \partial_r$ and $\underline{L}_{(Flat)} := \partial_t - \partial_r$ are the standard radial null pair, as in (1.2.6). We remark that ∂_t is Killing while $\widetilde{K}_{(Flat)}$ is conformally Killing in Minkowski space. If one is interested only in the region[68] where $0 < u_{(Flat)} < 1$, we can replace the Morawetz vectorfield $\widetilde{K}_{(Flat)}$ by $r^2 L_{(Flat)}$. The vectorfields that we use in the shock-formation problem in the region $\mathcal{M}_{t,u}$ (see (2.1.35e)) are the following dynamic versions, which are essentially the same as the vectorfields used in [11]:

$$T := (1+\mu)L + \breve{\underline{L}} = (1+2\mu)L + 2\breve{R}, \tag{3.1.2a}$$

$$\widetilde{K} := \varrho^2 L. \tag{3.1.2b}$$

T is a $g-$timelike vectorfield that is designed to yield generalized energy quantities that are useful both in regions where μ is large and where it is small. $\widetilde{K}$ is a $g-$null vectorfield whose role we will explain below. These vectorfields are neither Killing nor conformal Killing,[69] even when $\Psi \equiv 0$. Nonetheless, the energy estimate error terms corresponding to their deformation tensors (see the

[68] Recall that $u_{(Flat)} := 1 - r + t$ is an eikonal function corresponding to the Minkowski metric, so in this regime $r \approx 1 + t$.

[69] That is, their deformations tensors (1.5.4) neither vanish nor are proportion to the metric.

right-hand side of (3.2.7)) are controllable in the region $\mathcal{M}_{t,u}$. Actually, as we will see in Lemma 3.3, one of the deformation tensor terms corresponding to $\widetilde{K}$ has a favorable sign and is important for controlling other error terms.

(3) As long as we can suitably bound the generalized energy quantities, based on the commutation vectorfields (3.1.1) and multiplier vectorfields (3.1.2a), (3.1.2b), we can also derive, via Sobolev embedding, decay estimates for the low-order derivatives of Ψ, consistent with our Heuristic Principle; see [63, Corollary 17.2.2] for the details.

(4) The deformation tensors (see (1.5.4)) of the commutator vectorfields Z can be expressed in terms of the covariant Hessian $H = \mathscr{D}^2 u$, which verifies a transport equation of the form

$$LH + H^2 = \mathcal{R}, \tag{3.1.3}$$

where, as we have mentioned, $\mathcal{R}$ depends on the up-to-second-order derivatives of Ψ and the up-to-second-order derivatives of u. As we explained in Subsubsect. 1.5.2, every time we commute $\Box_{g(\Psi)}$ with a vectorfield Z, we generate terms of the form $(\mathscr{D}^{(Z)}\pi) \cdot \mathscr{D}\Psi$ which can be traced back, via the transport equation (3.1.3), to one more derivative of Ψ than we are able to control by an energy estimate at the same level. However, it is essential to note that even though we lose a derivative, we do not introduce any factors of μ^{-1}, which would blow-up as we approach the expected singularity.[70] In other words, we can derive estimates for the components of the derivatives of H relative to the rescaled frame $\{L, \breve{R}, X_1, X_2\}$ that are regular with respect to μ, but only at the expense of losing a derivative. This is key to understanding Christodoulou's strategy: at the top level we combat derivative loss through renormalization (see the next item), which has as a trade-off the introduction of a factor of μ^{-1}; at the lower derivative levels we can avoid this factor since the derivative loss can be absorbed. This trade-off is where understanding the dynamic geometry is most important.

(5) To control the top derivatives of Ψ when the loss of a derivative, due to (3.1.3), can no longer be ignored, we use a renormalization procedure,[71] which recovers the loss of derivatives mentioned above at the expense of introducing a dangerous factor of μ^{-1}; see Subsubsect. 3.4.3. This new difficulty of having to derive a priori estimates in the presence of the singular factor μ^{-1} is handled by Christodoulou with the help of a subtle Gronwall-type inequality, which we provide as Lemma 3.5.

3.2. **Energy estimates via the multiplier method.** Before specializing to equation (3.4.1), we first recall the multiplier method for deriving generalized energy estimates for solutions to

$$\mu \Box_g \Psi = \mathfrak{F}. \tag{3.2.1}$$

[70] To see this in detail, one must decompose (3.1.3) relative to the rescaled frame $\{L, \breve{R}, X_1, X_2\}$. At one derivative level lower, the μ-regular behavior can be seen in the transport equation (2.1.28) for μ, where there are no factors of μ^{-1} present.

[71] The procedure involves combining (3.1.3) with the wave equation $\Box_{g(\Psi)}\Psi = 0$ and using elliptic estimates on the surfaces $S_{t,u}$. This is similar to the approach taken in [13] and [39].

3.2.1. *A version of the divergence theorem via the multiplier method.* One key ingredient is the *energy-momentum tensorfield*

$$Q_{\mu\nu}[\Psi] = Q_{\mu\nu} := \mathscr{D}_\mu \Psi \mathscr{D}_\nu \Psi - \frac{1}{2} g_{\mu\nu} \mathscr{D}^\alpha \Psi \mathscr{D}_\alpha \Psi. \tag{3.2.2}$$

It is straightforward to check that for solutions to (3.2.1), we have

$$\mu \, \mathscr{D}_\alpha Q^{\alpha\nu} = \mathfrak{F} \mathscr{D}^\nu \Psi. \tag{3.2.3}$$

Furthermore, for any pair of future-directed vectorfields V and W verifying $g(V,V), g(W,W) \leq 0$, we have the well-known inequality[72] which plays a role in the construction of coercive L^2 quantities:

$$Q_{\alpha\beta} V^\alpha W^\beta \geq 0. \tag{3.2.4}$$

The following currents are useful for bookkeeping during integration by parts. Specifically, to any auxiliary "multiplier" vectorfield X, we associate the following *compatible current* vectorfield.

Definition 3.1 (**Compatible current**).

$$^{(X)}J^\nu[\Psi] := Q^\nu{}_\alpha[\Psi] X^\alpha. \tag{3.2.5}$$

By (3.2.3), for solutions Ψ to (3.2.1), we have

$$\mu \mathscr{D}^{(X)}_\alpha J^\alpha = \frac{1}{2} \mu Q^{\alpha\beta}[\Psi] {}^{(X)}\pi_{\alpha\beta} + (X\Psi)\mathfrak{F}, \tag{3.2.6}$$

where $^{(X)}\pi_{\alpha\beta} = \mathscr{D}_\alpha X_\beta + \mathscr{D}_\beta X_\alpha$ is the deformation tensor of X, as in (1.5.4)

To derive generalized energy estimates, we apply the divergence theorem on the region $\mathcal{M}_{t,u}$ (see Figure 6) to obtain the following energy-flux identity for solutions to (3.2.1).

Lemma 3.1. [63, **Lemma 9.2.1; Divergence theorem**] *For solutions Ψ to $\mu \Box_g \Psi = \mathfrak{F}$ that vanish along*[73] *$\mathcal{C}_0$, we have*

$$\begin{aligned} \int_{\Sigma_t^u} \mu Q[\Psi](X,N) \, d\underline{\varpi} + \int_{\mathcal{C}_u^t} Q[\Psi](X,L) \, d\overline{\varpi} = \int_{\Sigma_0^u} \mu Q[\Psi](X,N) \, d\underline{\varpi} \\ - \int_{\mathcal{M}_{t,u}} (X\Psi)\mathfrak{F} \, d\varpi - \frac{1}{2} \int_{\mathcal{M}_{t,u}} \mu Q[\Psi] \cdot {}^{(X)}\pi \, d\varpi, \end{aligned} \tag{3.2.7}$$

where $Q[\Psi] \cdot {}^{(X)}\pi := Q^{\alpha\beta} {}^{(X)}\pi_{\alpha\beta}$.

In (3.2.7), $N = L + R$ is the future-directed unit-normal to Σ_t^u, $Q(X,N) = g(^{(X)}J, N)$ and $Q(X,L) = g(^{(X)}J, L)$. Furthermore,

$$d\underline{\varpi} := \sqrt{\det \not{g}} \, d\vartheta du', \qquad d\overline{\varpi} := \sqrt{\det \not{g}} \, d\vartheta dt', \qquad d\varpi := \sqrt{\det \not{g}} \, d\vartheta du' dt' \tag{3.2.8}$$

are rescaled volume forms on Σ_t^u, $\mathcal{C}_u^t$, and $\mathcal{M}_{t,u}$. As before, $\not{g}$ is the Riemannian metric induced by g on the spheres $S_{t,u}$ and the determinant is taken relative to the geometric angular coordinates $(\vartheta^1, \vartheta^2)$. We

[72] In general relativity, inequality (3.2.4) is often referred to as the *dominant energy condition*.

[73] Recall that the vanishing of Ψ along $\mathcal{C}_0$ is an easy consequence of our assumptions on the support of the data.

call the above volume forms "rescaled" because the canonical volume forms induced by g on Σ_t^u and $\mathcal{M}_{t,u}$ are $\mu\, d\underline{\varpi}$ and $\mu\, d\varpi$.

Note that by the property (3.2.4), the first two integrands on the left-hand side of (3.2.7) are non-negative for both of the multiplier vectorfields $X = T$ and $X = \widetilde{K}$; see Prop. 3.2 for a more precise account of the coerciveness of these terms.

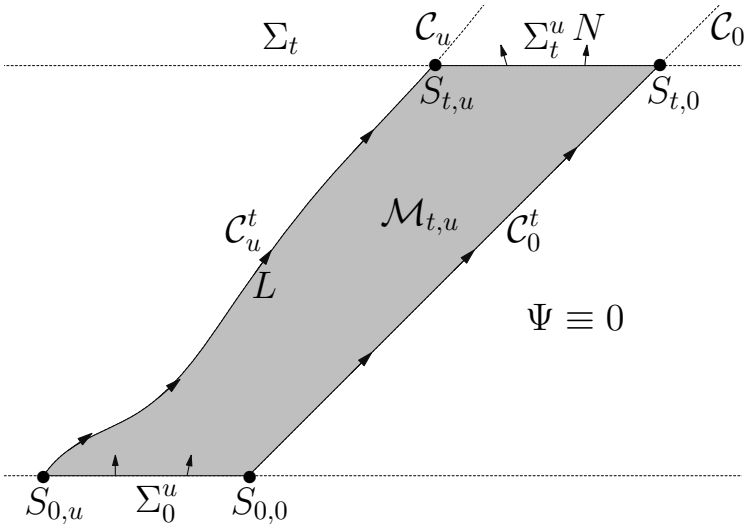


FIGURE 6. The divergence theorem on $\mathcal{M}_{t,u}$

Remark 3.1 (**Correction terms**). Actually, in the case $X = \widetilde{K}$, we need to modify the current (3.2.5) by adding to it the correction current ${}^{(Correction)}J^\nu[\Psi] := \frac{1}{2}\big\{\varrho^2 \mathrm{tr}_{\not g}\chi \Psi \mathscr{D}^\nu \Psi - \frac{1}{2}\Psi^2 \mathscr{D}^\nu[\varrho^2 \mathrm{tr}_{\not g}\chi]\big\}$. We need this correction current because ${}^{(\widetilde{K})}\pi_{\alpha\beta}$ fails to vanish even in the case of Minkowski spacetime, and it turns out that the corresponding error terms are not controllable. The use of corrections ("Lagrangian term") is standard and is often used even in the case of semilinear wave equations; see, for example, [37].

Remark 3.2. The divergence theorem identity (3.2.7) does not account for the effect of adding the correction current ${}^{(Correction)}J^\nu[\Psi]$ described in Remark 3.1. To adjust (3.2.7) so that it is correct after the modification, one needs to include some additional integrals in the identity (3.2.7), and in particular, the last integral needs to be replaced with $-\frac{1}{2}\int_{\mathcal{M}_{t,u}} \mu Q^{\alpha\beta}\left\{{}^{(\widetilde{K})}\pi_{\alpha\beta} - \varrho^2 \mathrm{tr}_{\not g}\chi g_{\alpha\beta}\right\} d\varpi$.

3.2.2. *Energies and fluxes.* With the help of the above currents and the corresponding divergence identities, we can now define the energies and fluxes, which are the main quantities that we use to control Ψ and its derivatives in L^2.

Definition 3.2 (**Energies and fluxes**). Let $N := L + R$ denote the future-directed unit normal to Σ_t. We define the energy $\mathbb{E}[\Psi](t,u)$ and the cone flux $\mathbb{F}[\Psi](t,u)$ corresponding to the multiplier vectorfield T (see (3.1.2a)) in terms of the *rescaled* volume forms (3.2.8) as follows:

$$\mathbb{E}[\Psi](t,u) := \int_{\Sigma_t^u} \mu Q[\Psi](T,N)\, d\underline{\varpi}, \tag{3.2.9a}$$

$$\mathbb{F}[\Psi](t,u) := \int_{\mathcal{C}_u^t} Q[\Psi](T,L)\, d\overline{\varpi}. \tag{3.2.9b}$$

We can also define similar quantities $\widetilde{\mathbb{E}}[\Psi](t,u)$, $\widetilde{\mathbb{F}}[\Psi](t,u)$ corresponding to the Morawetz multiplier $\widetilde{K}$ (see (3.1.2a)), but we have to take into account the lower-order terms mentioned in Remark 3.2.

The following proposition reveals the coercive nature of the energies and fluxes. Roughly speaking, its proof is based on carefully decomposing the energy-momentum tensor (3.2.2) relative to[74] the rescaled frame $\{L, \breve{R}, X_1, X_2\}$.

Proposition 3.2. [63, **Lemma 13.1.1; Coerciveness of the energies and fluxes**] *Under suitable smallness bootstrap assumptions, the energies and fluxes from Definition 3.2 have the following coerciveness properties:*

$$\begin{aligned}\mathbb{E}[\Psi](t,u) &\geq \|\breve{R}\Psi\|^2_{L^2(\Sigma_t^u)} + C^{-1}\|\mu \not{\nabla}\Psi\|^2_{L^2(\Sigma_t^u)} \\ &\quad + C^{-1}\|\Psi\|^2_{L^2(S_{t,u})}, + C^{-1}\|\Psi\|^2_{L^2(\Sigma_t^u)} + C^{-1}\|\sqrt{\mu}L\Psi\|^2_{L^2(\Sigma_t^u)} + C^{-1}\|\mu L\Psi\|^2_{L^2(\Sigma_t^u)},\end{aligned} \tag{3.2.10a}$$

$$\mathbb{F}[\Psi](t,u) \geq C^{-1}\|L\Psi\|^2_{L^2(\mathcal{C}_u^t)} + C^{-1}\|\sqrt{\mu}L\Psi\|^2_{L^2(\mathcal{C}_u^t)} + C^{-1}\|\sqrt{\mu}\not{\nabla}\Psi\|^2_{L^2(\mathcal{C}_u^t)}, \tag{3.2.10b}$$

$$\widetilde{\mathbb{E}}[\Psi](t,u) \geq C^{-1}(1+t)^2 \left\| \sqrt{\mu}\left(L\Psi + \frac{1}{2}\mathrm{tr}_{\not g}\chi\Psi\right)\right\|^2_{L^2(\Sigma_t^u)} + \frac{1}{2}\|\varrho\sqrt{\mu}\not{\nabla}\Psi\|^2_{L^2(\Sigma_t^u)}, \tag{3.2.11a}$$

$$\widetilde{\mathbb{F}}[\Psi](t,u) \geq C^{-1}\left\|(1+t')\left(L\Psi + \frac{1}{2}\mathrm{tr}_{\not g}\chi\Psi\right)\right\|^2_{L^2(\mathcal{C}_u^t)}. \tag{3.2.11b}$$

The L^2 norms above are relative to the rescaled volume forms $d\underline{\varpi}$, and $d\overline{\varpi}$ (see (3.2.8)*), which do not degenerate as $\mu \to 0$. Furthermore, $\varrho(t,u) = 1 - u + t$.*

Remark 3.3. Note that we have provided the explicit constant "1" in the term $\|\breve{R}\Psi\|^2_{L^2(\Sigma_t^u)}$ on the right-hand side of (3.2.10a) and similarly for the second term on the right-hand side of (3.2.11a). These constants are important because they affect the number of derivatives we need to close the estimates; see, for example, the derivation of inequality (3.4.17) from inequality (3.4.16).

3.2.3. *The role of μ weights in the energies and fluxes.* Observe that the energies $\mathbb{E}$ and $\widetilde{\mathbb{E}}$ from Prop. 3.2 control only $\mu-$weighted versions of $L\Psi$ and $\not{\nabla}\Psi$. Hence, for μ near 0, they provide only very weak control over $L\Psi$ and $\not{\nabla}\Psi$. However, when bounding various error integrals on the right-hand side of (3.2.7), we encounter *non $\mu-$weighted factors of $L\Psi$* and $\not{\nabla}\Psi$, which cannot be controlled directly by $\mathbb{E}$ and $\widetilde{\mathbb{E}}$. We give an example of such an error term and describe how to handle it in Subsubsect. 3.4.2. To handle the non $\mu-$weighted factors of $L\Psi$ when μ is small, we will need to rely on the null-fluxes $\mathbb{F}$ and $\widetilde{\mathbb{F}}$ from Prop. 3.2, which provide control over $L\Psi$ *without any* μ *weights*.

[74] Actually, in the proof, it is convenient to decompose relative to the rescaled null frame $\{L, \breve{\underline{L}}, X_1, X_2\}$, where $\breve{\underline{L}} := \mu L + 2\breve{R}$.

3.2.4. *The need for the Morawetz spacetime integral.* Note that Prop. 3.2 does not provide any quantity that yields control of the non $\mu-$weighted factors of $\nabla\!\!\!/\Psi$ when μ is small. To obtain such control, we use a much more interesting and subtle estimate, first derived by Christodoulou in [11], which we now discuss.

The main idea is that in the case of the Morawetz multiplier $\widetilde{K}$, there is a subtly coercive term hiding in the last integral on the right-hand side of (3.2.7). That is, a careful decomposition of the integrand $-\frac{1}{2}\mu Q^{\alpha\beta}\left\{{}^{(\widetilde{K})}\pi_{\alpha\beta} - \varrho^2 \mathrm{tr}_{g\!\!\!/}\chi g_{\alpha\beta}\right\}$ (see Remark 3.2) reveals the presence of an important *negative* spacetime integral $-\widetilde{\mathbb{K}}[\Psi]$ on the right-hand side of (3.2.7). The corresponding positive spacetime integral has the following structure.

Definition 3.3 (Coercive Morawetz spacetime integral).

$$\widetilde{\mathbb{K}}[\Psi](t,u) := \int_{\mathcal{M}_{t,u}} \varrho^2 [L\mu]_- |\nabla\!\!\!/\Psi|^2 \, d\varpi. \tag{3.2.12}$$

Here $[L\mu]_- = |L\mu|$ when $L\mu < 0$ and $[L\mu]_- = 0$ otherwise.

The coerciveness of the Morawetz integral is provided by the following lemma.

Lemma 3.3. [63, **Lemma 13.2.1; Quantified coerciveness of the Morawetz spacetime integral**] *The Morawetz integral $\widetilde{\mathbb{K}}[\Psi]$ from Definition 3.3 verifies the following lower bound:*

$$\widetilde{\mathbb{K}}[\Psi](t,u) \geq \frac{1}{C}\int_{\mathcal{M}_{t,u}} \mathbf{1}_{\{\mu \leq 1/4\}} \frac{1+t'}{\ln(e+t')} |\nabla\!\!\!/\Psi|^2(t',u',\vartheta) \, d\varpi. \tag{3.2.13}$$

The main idea behind the proof of Lemma 3.3 is simple: just insert an estimate very similar[75] to (1.3.16) (derived in spherical symmetry) into (3.2.12). The important points concerning $\widetilde{\mathbb{K}}[\Psi]$ are:

- $-\widetilde{\mathbb{K}}[\Psi](t,u)$ appears on the right-hand side of (3.2.7) (see Remark 3.2) and hence we can bring $\widetilde{\mathbb{K}}[\Psi](t,u)$ to the left and obtain additional spacetime control over $|\nabla\!\!\!/\Psi|^2$.
- It *contains no* μ *weights*, so it is significantly coercive even in regions where μ is near 0.
- The integrand features favorable factors of t'.

3.2.5. *Overview of the L^2 hierarchy and the $\mu_\star^{-1}$ degeneracy.* We are almost ready to provide an overview of the main a priori energy-flux estimates. The estimates involve the following important quantity, which captures the "worst-case" scenario for μ being small along Σ_t^u.

Definition 3.4 (A modified minimum value of μ). We define the function $\mu_\star(t,u)$ as follows:

$$\mu_\star(t,u) := \min\{1, \min_{\Sigma_t^u} \mu\}. \tag{3.2.14}$$

Now that we have defined all of the quantities of interest, we can now state a proposition that provides the a priori energy-flux-Morawetz estimates that hold on spacetime domains of the form $\mathcal{M}_{t,u}$. There is no analog of this proposition in spherical symmetry because in the symmetric setting, we did not need to derive L^2 estimates.

[75] Indeed in the small-data regime, the same estimate (1.3.16) holds even in the non-spherical symmetric case (see (3.3.7)).

Proposition 3.4. [63, **Lemma 19.2.3; Rough statement of the hierarchy of a priori energy-flux-Morawetz estimates**] *Assume that $\Box_{g(\Psi)}\Psi = 0$. Assume that the data are of size $\mathring{\epsilon}$, defined by* (2.1.34)*. Then there exist large constants $C > 0$ and $A_* > 0$ such that if $\mathring{\epsilon}$ is sufficiently small, then the following energy-flux-Morawetz estimates hold for the quantities from Definitions 3.2 and 3.3 for $0 \leq M \leq 7$:*

$$\mathbb{E}^{1/2}[\mathscr{Z}^{\leq 15}\Psi](t,u) + \mathbb{F}^{1/2}[\mathscr{Z}^{\leq 15}\Psi](t,u) \leq C\mathring{\epsilon}, \tag{3.2.15a}$$

$$\widetilde{\mathbb{E}}^{1/2}[\mathscr{Z}^{\leq 15}\Psi](t,u) + \widetilde{\mathbb{F}}^{1/2}[\mathscr{Z}^{\leq 15}\Psi](t,u) + \widetilde{\mathbb{K}}^{1/2}[\mathscr{Z}^{\leq 15}\Psi](t,u) \leq C\mathring{\epsilon}\ln^2(e+t), \tag{3.2.15b}$$

$$\mathbb{E}^{1/2}[\mathscr{Z}^{16+M}\Psi](t,u) + \mathbb{F}^{1/2}[\mathscr{Z}^{16+M}\Psi](t,u) \leq C\mathring{\epsilon}\mu_\star^{-.75-M}, \tag{3.2.15c}$$

$$\widetilde{\mathbb{E}}^{1/2}[\mathscr{Z}^{16+M}\Psi](t,u) + \widetilde{\mathbb{F}}^{1/2}[\mathscr{Z}^{16+M}\Psi](t,u) + \widetilde{\mathbb{K}}^{1/2}[\mathscr{Z}^{16+M}\Psi](t,u) \leq C\mathring{\epsilon}\ln^2(e+t)\mu_\star^{-.75-M}(t,u), \tag{3.2.15d}$$

$$\mathbb{E}^{1/2}[\mathscr{Z}^{24}\Psi](t,u) + \mathbb{F}^{1/2}[\mathscr{Z}^{24}\Psi](t,u) \leq C\mathring{\epsilon}\ln^{A_*}(e+t)\mu_\star^{-8.75}(t,u), \tag{3.2.15e}$$

$$\widetilde{\mathbb{E}}^{1/2}[\mathscr{Z}^{24}\Psi](t,u) + \widetilde{\mathbb{F}}^{1/2}[\mathscr{Z}^{24}\Psi](t,u) + \widetilde{\mathbb{K}}^{1/2}[\mathscr{Z}^{24}\Psi](t,u) \leq C\mathring{\epsilon}\ln^{A_*+2}(e+t)\mu_\star^{-8.75}(t,u). \tag{3.2.15f}$$

In the above estimates, $\mathscr{Z}^{\leq k}$ denotes an arbitrary differential operator of order $\leq k$ corresponding to repeated differentiation with respect to the commutation vectorfields in $\mathscr{Z}$ (see (3.1.1)*).*

Remark 3.4 (**The $\mu_\star^{-1}$ hierarchy**)**.** An important feature of Prop. 3.4 to notice is that the top-order quantities are allowed to blow up like $\mu_\star^{-8.75}$ as $\mu_\star$ tends to 0. The power -8.75 is a consequence of some delicate structural features of the equations. We explain this below (see in particular Remark 3.9). Another important feature is that as we descend below the top-order, we see improvements in the $\mu_\star^{-1}$ blow-up rate until we reach a level in which the quantities no longer blow-up. The non-degenerate estimates can be used to show that the lower-order derivatives of Ψ extend as continuous functions, relative to the geometric coordinates (t,u,ϑ), to the constant-time hypersurface of first shock formation. The precise features of this hierarchy play a fundamental role in guiding the analysis.

3.3. **Details on the behavior of μ.** In order to explain how to derive the energy estimate hierarchy of Prop. 3.4, we first need to provide some sharp information on the behavior of μ. In the next three lemmas, we state the most relevant properties of μ and sketch some of their proofs. See [63, Chapter 12] for more details. We emphasize once more that one needs very detailed control on the blow-up behavior of μ^{-1} to close the energy estimates and that this is a major difference from the spherically symmetric case.

The first lemma provides the main Gronwall estimate that leads to the degeneracy of the top-order energy estimates (3.2.15e)-(3.2.15f). The reader should think that (3.3.1) is the type of inequality appearing when trying to close the energy estimates at the top order. The lemma is a drastically simplified version of [63, Lemma 19.2.3].

Lemma 3.5 (**A Gronwall estimate used at top order**)**.** *Let $B > 0$ be a constant. There exist a small constant $0 < \sigma \ll 1$ and large constants $C > 0$ and $A > 0$ such that for $u \in [0, U_0]$, solutions $f(t)$ to the*

inequality

$$f(t) \leq C\mathring{\epsilon} + B\int_{t'=0}^{t}\left(\sup_{\Sigma_{t'}^{u}}\left|\frac{L\mu}{\mu}\right|\right) f(t')\,dt' \tag{3.3.1}$$

verify the estimate

$$f(t) \leq C\mathring{\epsilon}\ln^{A}(e+t)\mu_{\star}^{-(B+\sigma)}(t,u). \tag{3.3.2}$$

The second lemma is used to show that the below-top-order energy estimates are less degenerate than the top-order ones. The main idea is that we can gain powers of $\mu_\star$ by integrating in time.

Lemma 3.6. [63, **Proposition 12.3.1; Gaining powers of $\mu_\star$ by time integration**] *Let $B > 1$ be a constant. Then for $u \in [0, U_0]$, we have*

$$\int_{t'=0}^{t}\frac{1}{(1+t')^{3/2}}\mu_{\star}^{-B}(t',u)\,dt' \leq C\mu_{\star}^{1-B}(t,u). \tag{3.3.3}$$

Furthermore,

$$\int_{t'=0}^{t}\frac{1}{(1+t')^{3/2}}\mu_{\star}^{-3/4}(t',u)\,dt' \leq C. \tag{3.3.4}$$

The third lemma plays a supporting role in establishing the previous two lemmas. In addition, the estimate (3.3.7) is the ingredient used to show that the Morawetz spacetime integral is coercive in the regions where μ is small; see Lemma 3.3.

Lemma 3.7. [63, **Sections 12.1 and 12.2; Some key properties of μ.**] *Consider a fixed point (t,u,ϑ) and let $\delta_{t,u,\vartheta} := \varrho\,(t,u)\,L\mu(t,u,\vartheta)$. Then for $0 \leq s \leq t$, we have*[76]

$$L\mu(s,u,\vartheta) \sim \frac{1}{\varrho(s,u)}\delta_{t,u,\vartheta}, \tag{3.3.5}$$

$$\mu(s,u,\vartheta) \sim 1 + \delta_{t,u,\vartheta}\ln\left(\frac{\varrho(s,u)}{\varrho(0,u)}\right). \tag{3.3.6}$$

Let $[L\mu]_- = |L\mu|$ when $L\mu < 0$ and $[L\mu]_- = 0$ otherwise. Then at any point (t,u,ϑ) with $\mu(t,u,\vartheta) < 1/4$, we have

$$[L\mu]_-(t,u,\vartheta) \geq c\frac{1+t}{\ln(e+t)}. \tag{3.3.7}$$

Discussion of the proof of Lemma 3.7. Thanks to the decay estimates of the Heuristic Principle (see Subsubsect. 2.1.5), Lemma 3.7 can be proved by using essentially the same arguments that we used above in spherical symmetry; see Prop. 1.3 and its proof. The additional terms present away from spherical symmetry involve $\mathcal{C}_u-$tangential derivatives of Ψ, and hence they decay very rapidly and make only a negligible contribution to the inequalities. □

[76] The notation $A \sim B$ indicates, in an imprecise fashion, that A is well-approximated by B.

Discussion of the proof of Lemma 3.5. By the standard Gronwall inequality, we deduce

$$f(t) \leq C\mathring{\epsilon} \exp\left(B \int_{s=0}^{t} \sup_{\Sigma_s^u} \left| \frac{L\mu}{\mu} \right| \right) ds. \tag{3.3.8}$$

We now need to pass from (3.3.8) to (3.3.2). The detailed proof is somewhat difficult because of the presence of the sup on the right.

To reveal the main ideas behind the proof, we first use (3.3.5) and (3.3.6) to deduce that for $0 \leq s \leq t$, we have

$$\frac{L\mu}{\mu}(s,u,\vartheta) \sim \frac{\delta_{t,u,\vartheta}}{\varrho(s,u)\left\{1 + \delta_{t,u,\vartheta} \ln\left(\frac{\varrho(s,u)}{\varrho(0,u)}\right)\right\}}. \tag{3.3.9}$$

The important point in (3.3.9) is that *the same constant* $\delta_{t,u,\vartheta}$ appears in the numerator and denominator. For the sake of illustration, let us simplify the analysis by assuming that $\delta_{t,u',\vartheta} \leq 0$ for $u' \in [0,u], \vartheta \in \mathbb{S}^2$.[77] Using the fact that for a fixed $a > 0$, the function $f(x) = \frac{x}{1+ax}$ is increasing on the domain $x \in (-a^{-1}, 0]$, we deduce (recall $\delta_{t,u',\vartheta}$ is non-positive)

$$\frac{L\mu}{\mu}(s,u,\vartheta) \geq \frac{\min_{u' \in [0,u], \vartheta \in \mathbb{S}^2} \delta_{t,u',\vartheta}}{\varrho(s,u)\left\{1 + \min_{u' \in [0,u], \vartheta \in \mathbb{S}^2} \delta_{t,u',\vartheta} \ln\left(\frac{\varrho(s,u)}{\varrho(0,u)}\right)\right\}} + \text{Err}. \tag{3.3.10}$$

We finally set

$$\delta_t := \left| \min_{u' \in [0,u], \vartheta \in \mathbb{S}^2} \delta_{t,u',\vartheta} \right|$$

and conclude

$$\sup_{\Sigma_s^u} \left| \frac{L\mu}{\mu} \right| \leq \frac{\delta_t}{\varrho(s,u)\left\{1 - \delta_t \ln\left(\frac{\varrho(s,u)}{\varrho(0,u)}\right)\right\}} + \text{Err}. \tag{3.3.11}$$

Note also that, in view of Definition 3.4 and (3.3.6), we deduce that for $0 \leq s \leq t$, we have

$$\mu_\star(s,u) \sim 1 - \delta_t \ln\left(\frac{\varrho(s,u)}{\varrho(0,u)}\right). \tag{3.3.12}$$

We now integrate (3.3.11) ds from $s = 0$ to t and use (3.3.12) to deduce that

$$\begin{aligned} \int_{s=0}^{t} \sup_{\Sigma_s^u} \left| \frac{L\mu}{\mu} \right| ds &\sim \int_{s=0}^{t} \frac{\delta_t}{\varrho(s,u)\left\{1 - \delta_t \ln\left(\frac{\varrho(s,u)}{\varrho(0,u)}\right)\right\}} ds \\ &= \ln\left|1 - \delta_t \ln\left(\frac{\varrho(t,u)}{\varrho(0,u)}\right)\right| \sim \ln\left|\mu_\star^{-1}(t,u)\right| \end{aligned} \tag{3.3.13}$$

(recall that $\mu_\star(t,u) \leq 1$ by definition). The desired estimate[78] (3.3.2) now easily follows from (3.3.8). □

[77] This is indeed what holds for *some* (u', ϑ) close to the formation of the shock. The fact that in reality it does not hold for all (u', ϑ), even close to the time of shock formation, leads to additional technical complications which we suppress here.

[78] Up to the correction factors σ and $\ln^A(e+t)$.

Discussion of the proof of Lemma 3.6. The main idea is that the integrals in (3.3.3) and (3.3.4) are easy to estimate once we have obtained sharp information about the behavior of $\mu_\star$. For example, if we assume for simplicity that $\delta_{t,u',\vartheta} \leq 0$ for $u' \in [0,u], \vartheta \in \mathbb{S}^2$, then the estimate (3.3.12) holds. We can then estimate the integrals by using (3.3.12), splitting them into a small-time portion and a large-time portion, and optimizing the splitting time.

□

3.4. **Details on the top-order energy estimates.** We now explain some of the main ideas behind the proof of Prop. 3.4. Throughout Subsect. 3.4, $\mathring{\epsilon}$ denotes the small size of the data and ε denotes the small amplitude size corresponding to the Heuristic Principle estimates of Subsubsect. 2.1.5, which we use as bootstrap assumptions. For convenience, we focus on only the difficult top-order energy estimate (3.2.15e). To illustrate the main ideas, we might as well commute the equation with a single rotational vectorfield O, pretend that we are at the highest level of derivatives, and show how to avoid the derivative loss. We remark that we must also avoid, using similar arguments, the derivative loss when we commute with $\breve{R}$ and ϱL, though the difficulties are somewhat less severe in the case of ϱL. To proceed, we consider the wave equation verified by $O\Psi$. That is, we commute the equation $\mu\Box_{g(\Psi)}\Psi = 0$ with $O \in \{O_{(1)}, O_{(2)}, O_{(3)}\}$ to deduce the equation

$$\mu\Box_{g(\Psi)}O\Psi = (\breve{R}\Psi)O\mathrm{tr}_{\not g}\chi - (O\mu)\not{\Delta}\Psi + \cdots. \tag{3.4.1}$$

Remark 3.5. On the right hand side of (3.4.1) $\cdots$ denotes a long list of additional error terms that turn out to be much easier to control than the first one that is explicitly listed; see [63, Proposition 6.2.2, Lemma 8.1.2, Proposition 8.2.1] for more details. A rigorous derivation of (3.4.1) would involve lengthy computations; in an effort to avoid distracting the reader, we will simply take (3.4.1) for granted. We furthermore remark that related but distinct difficulties arise when we commute with $\breve{R}$ or ϱL, but for simplicity, we discuss only the case of O.

Remark 3.6. Note that in (3.4.1), we are working with the $\mu-$weighted wave operator $\mu\Box_{g(\Psi)}$. It turns out that $\mu\Box_{g(\Psi)}$ has better commutation properties with the vectorfields in $\mathscr{Z}$ (see (3.1.1)) than the unweighted operator $\Box_{g(\Psi)}$. The important property of $\mu\Box_{g(\Psi)}$ is that we do not introduce any factors of μ^{-1} when we repeatedly commute it with vectorfields in $\mathscr{Z}$; generally, we would not be able to control such factors. The moral reason behind the good properties of $\mu\Box_{g(\Psi)}$ can be discerned from the decomposition (2.1.23). To see this, we recall that, relative to the geometric coordinates, we have $L = \frac{\partial}{\partial t}$ and $\breve{R} = \frac{\partial}{\partial u} +$ angular error term. We can therefore rewrite (2.1.23) as

$$\mu\Box_{g(\Psi)}\Psi = -\frac{\partial}{\partial t}\left\{\mu\frac{\partial}{\partial t}\Psi + 2\frac{\partial}{\partial u}\Psi\right\} + \mu\not{\Delta}\Psi + \mathrm{Err}. \tag{3.4.2}$$

The right-hand side of (3.4.2) now suggests that, for example, the differential operator $\breve{R} \in \mathscr{Z}$ can be commuted through the equation without introducing any dangerous factors of μ^{-1}.

Remark 3.7 (On the importance of terms that are not present). One crucial property of the commutation vectorfield set $\mathscr{Z}$ is that after commuting the through the operator $\mu\Box_{g(\Psi)}$ one time, we never produce terms of the form $\not{\nabla}\breve{R}\mu$ or $\breve{R}\breve{R}\mu$. This is important because we have no means to bound the top-order derivatives of these terms. In contrast, as we will see, there is a procedure based on modified quantities

and elliptic estimates that allows us to bound the top-order derivatives of the term $O\mathrm{tr}_{\not g}\chi$ on the right-hand side of (3.4.1) (see Subsubsect. 3.4.3). This discrepancy occurs even though $\not\nabla\breve{R}\mu$, $O\mathrm{tr}_{\not g}\chi$, and $\breve{R}\breve{R}\mu$ are all third-order derivatives of the eikonal function u.

Our goal is to show how to estimate solutions to (3.4.1) without losing derivatives. In particular, we sketch a proof of how to derive a "top-order" estimate for the rotation commutation vectorfields O (described at the beginning of Sect. 3) of the form

$$\mathbb{E}^{1/2}[O\Psi](t,u)+\mathbb{F}^{1/2}[O\Psi](t,u)\leq C\mathring{\epsilon}\ln^{A}(e+t)\mu_{\star}^{-B}(t,u),$$

in the spirit of (3.2.15e), and we highlight the role played by Lemma 3.5. To begin, we use (3.4.1), (3.2.7), (3.2.9a), and (3.2.9b) to deduce that[79]

$$\mathbb{E}[O\Psi](t,u)+\mathbb{F}[O\Psi](t,u)\leq C\mathring{\epsilon}-\int_{\mathcal{M}_{t,u}}(2\breve{R}\Psi)(O\mathrm{tr}_{\not g}\chi)\breve{R}O\Psi\,d\varpi+\int_{\mathcal{M}_{t,u}}(O\mu)(LO\Psi)\not\Delta\Psi\,d\varpi+\cdots. \tag{3.4.3}$$

To deduce (3.4.3), we have used the divergence identity (3.2.7) with $X=T$ (see (3.1.2a)), $O\Psi$ in the role of Ψ, and $\mathfrak{F}=(\breve{R}\Psi)O\mathrm{tr}_{\not g}\chi-(O\mu)\not\Delta\Psi+\cdots$ from the right-hand side of (3.4.1). Furthermore, we have replaced the integrand $(TO\Psi)\mathfrak{F}$ from (3.2.7) with the expression

$$2(\breve{R}\Psi)(\breve{R}O\Psi)O\mathrm{tr}_{\not g}\chi-(O\mu)(LO\Psi)\not\Delta\Psi+\ldots$$

with $\ldots$ denoting terms that are easier to treat.[80]

Remark 3.8. We have suppressed the error-term $\int_{\mathcal{M}_{t,u}}(O\mu)(\breve{R}O\Psi)\not\Delta\Psi\,d\varpi$ by relegating it to the $\ldots$ term on the right-hand side of (3.4.3). One might expect that this integral is more difficult to estimate than the second one written on the right-hand side of (3.4.3) because it involves the transversal derivative factor $\breve{R}O\Psi$ in place of $LO\Psi$. However, the $LO\Psi-$involving error integral is actually slightly more difficult to estimate because we have to use the cone fluxes and the Morawetz spacetime integral to bound it; see inequality (3.4.15). In contrast, the arguments given in Subsubsect. 3.4.2 can easily be modified to show that the $\breve{R}O\Psi-$involving error integral can be bounded in magnitude by

$$\begin{aligned}&\lesssim\mathring{\epsilon}\int_{t'=0}^{t}\int_{\Sigma_{t'}^{u}}\frac{\ln(e+t')}{1+t'}|\breve{R}O\Psi||OO\Psi|\,d\underline{\varpi}\,dt' \qquad (3.4.4)\\ &\lesssim\mathring{\epsilon}\int_{t'=0}^{t}\frac{1}{\mu_{\star}^{1/2}(t',u)}\frac{\ln(e+t')}{(1+t')^{2}}\mathbb{E}[O\Psi]\widetilde{\mathbb{E}}[O\Psi]\,dt',\end{aligned}$$

where we have used Cauchy-Schwarz on $\Sigma_{t'}^{u}$ and Prop. 3.2 to pass to the final inequality. Thanks to favorable powers of t' present in the integrand on the right-hand side of (3.4.4), we can handle the singular factor $\mu_{\star}^{-1/2}$ with inequality (3.3.4), and it is therefore easy to show that the right-hand side of (3.4.4) is a cubic error term. In contrast, if we tried to handle the $LO\Psi-$involving error integral in the same way,

[79]The remaining error integrals $\cdots$ on the right-hand side of (3.4.3) are easier to estimate than the explicitly indicated ones, so we ignore them here.

[80]The remaining terms in T involve $\mathcal{C}_u-$tangential derivatives of Ψ.

we would find the worse factor $\mu_\star^{-1}(t',u)$ in the integrand. This strategy will fail because $\mu_\star^{-1}(t',u)$ is too singular to be handled by inequality (3.3.4).

3.4.1. *Raychaudhuri-type identity.* We now highlight the main technical hurdle in proving Prop. 3.4, which we already mentioned in Subsubsect. 1.5.2: the only way by which we can estimate the factor $O\mathrm{tr}_{\not g}\chi$ on the right-hand side of (3.4.3) is by exploiting an important transport equation which is the exact analog of the well-known *Raychaudhuri equation* [57] in General Relativity. The Raychaudhuri-type equation satisfied by $\mathrm{tr}_{\not g}\chi$ is (see, for example, the proof of [63, Corollary 10.2.1])

$$L\mathrm{tr}_{\not g}\chi + \frac{1}{2}(\mathrm{tr}_{\not g}\chi)^2 + |\hat{\chi}|^2 = -\mathrm{Ric}_{LL} + \frac{L\mu}{\mu}\mathrm{tr}_{\not g}\chi, \tag{3.4.5}$$

where Ric is the Ricci curvature of g and $\hat{\chi}$ is the trace-free part of χ. The Ricci tensor (see [63, Corollary 10.1.3]) can be decomposed through a tedious but straightforward calculation, which yields $\mathrm{Ric}_{LL} := \mathrm{Ric}_{\alpha\beta}L^\alpha L^\beta = -\frac{1}{2}G_{LL}\not\Delta\Psi + \cdots$. Since the term $\frac{1}{2}(\mathrm{tr}_{\not g}\chi)^2 + |\hat{\chi}|^2$ is also lower-order,[81] we arrive at the transport equation

$$L\mathrm{tr}_{\not g}\chi = \frac{1}{2}G_{LL}\not\Delta\Psi + \cdots, \tag{3.4.6}$$

where $\cdots$ denotes easier terms which can be ignored.

The main difficulty is that after we commute (3.4.6) with O, we obtain the equation

$$LO\mathrm{tr}_{\not g}\chi = \frac{1}{2}G_{LL}\not\Delta O\Psi + \cdots,$$

which depends on *three derivatives* of Ψ, whereas the left-hand side of (3.4.3) only yields control over *two derivatives* of Ψ (see Prop. 3.2). Hence, it seems that we are *losing derivatives* in our estimates for $O\mathrm{tr}_{\not g}\chi$. In Subsubsect. 3.4.3, we explain how to overcome this difficulty.

3.4.2. *The energy estimates ignoring derivative loss.* Before we address how to circumvent the loss in derivatives mentioned above, we first address how the proof of Prop. 3.4 would work if we did not have to worry about it. Our discussion will highlight the role of the Morawetz integral (3.2.13) in the proof. To begin, we imagine that (3.4.1), that is, the equation

$$\mu\Box_{g(\Psi)}O\Psi = (\breve{R}\Psi)O\mathrm{tr}_{\not g}\chi - (O\mu)\not\Delta\Psi + \cdots, \tag{3.4.7}$$

is the top-order equation and that we are trying to bound the right-hand side of (3.4.3) back in terms of the left so that we can apply Gronwall's inequality. We will use Prop. 3.2 to connect various L^2 norms back to $\mathbb{E}$, $\mathbb{F}$, etc. For the time being, we ignore the difficult error integral on the right-hand side of (3.4.3) and instead focus on the second one

$$\int_{\mathcal{M}_{t,u}} (O\mu)(LO\Psi)\not\Delta\Psi \, d\varpi, \tag{3.4.8}$$

[81] In reality, the analysis is somewhat more complicated. Specifically, we need to use elliptic estimates to bound the top-order derivatives of $|\hat{\chi}|^2$; see Remark 3.10.

in which we do not have to worry about derivative loss. To bound this integral, we use the following pointwise estimate:

$$|O\mu| \lesssim \varepsilon \ln(e+t). \tag{3.4.9}$$

The estimate (3.4.9) is easy to derive by commuting the evolution equation (2.1.28) for μ with O, using the Heuristic Principle estimates (see Subsubsect. 2.1.5) to bound the right-hand side, and then integrating the resulting inequality along the integral curves of $L = \frac{\partial}{\partial t}$; see [63, Proposition 11.27.1] for the details. We also use the following property of our rotation vectorfields, which is familiar from the case of Minkowski spacetime (see [63, Lemma 11.12.1] for a proof):

$$|\not{\Delta}\Psi| \lesssim \frac{1}{1+t} \sum_{l=1}^{3} |\not{\nabla} O_{(l)}\Psi|. \tag{3.4.10}$$

In view of (3.4.9) and (3.4.10), we see that the error integral (3.4.8) can be bounded as follows, where we split it into the region where $\mu \leq 1/4$ and the region where $\mu > 1/4$:

$$\lesssim \varepsilon \int_{\mathcal{M}_{t,u}} \mathbf{1}_{\{\mu \leq 1/4\}} \frac{\ln(e+t')}{1+t'} |LO\Psi||\not{\nabla} O\Psi| \, d\varpi + \varepsilon \int_{\mathcal{M}_{t,u}} \mathbf{1}_{\{\mu > 1/4\}} \frac{\ln(e+t')}{1+t'} |LO\Psi||\not{\nabla} O\Psi| \, d\varpi. \tag{3.4.11}$$

The main difficulty is present in the first integral in (3.4.11). Indeed, the first integral lacks a μ weight and involves the angular derivative term $|\not{\nabla} O\Psi|$. Hence, the angular derivative coerciveness of the energy-flux quantities, which is provided by Prop. 3.2, is not sufficient to control it. As in [11], to overcome the difficulty, we use the strength of the Morawetz integral; see inequality (3.2.13). More precisely, by dividing the time interval $[0,t]$ into suitable subintervals and using Cauchy-Schwarz, it is not difficult to show (see the proof of [63, Lemma 19.3.3]) that the first integral on the right-hand side of (3.4.11) is

$$\lesssim \varepsilon \int_{\mathcal{M}_{t,u}} \mathbf{1}_{\{\mu \leq 1/4\}} |LO\Psi|^2 \, d\varpi + \varepsilon \sup_{\tau \in [0,t)} \frac{1}{(1+\tau)^{1/2}} \int_{\mathcal{M}_{\tau,u}} \mathbf{1}_{\{\mu \leq 1/4\}} \frac{1+t'}{\ln(e+t')} |\not{\nabla} O\Psi|^2 \, d\varpi. \tag{3.4.12}$$

Using (3.2.10b), we deduce that the first term on the right-hand side of (3.4.12) is bounded by

$$\varepsilon \int_{\mathcal{M}_{t,u}} \mathbf{1}_{\{\mu \leq 1/4\}} |LO\Psi|^2 \, d\varpi \lesssim \varepsilon \int_{u'=0}^{u} \mathbb{F}[O\Psi](t,u') \, du'. \tag{3.4.13}$$

In addition, the second term on the right-hand side of (3.4.12) is bounded by

$$\varepsilon \sup_{\tau \in [0,t)} \frac{1}{(1+\tau)^{1/2}} \int_{\mathcal{M}_{\tau,u}} \mathbf{1}_{\{\mu \leq 1/4\}} \frac{1+t'}{\ln(e+t')} |\not{\nabla} O\Psi|^2 \, d\varpi \lesssim \varepsilon \sup_{\tau \in [0,t)} \frac{1}{(1+\tau)^{1/2}} \widetilde{\mathbb{K}}[O\Psi](\tau,u), \tag{3.4.14}$$

where we have used the key Morawetz estimate (3.2.13).

We then insert these estimates into the right-hand side of (3.4.3), ignore the (difficult) first error integral, and find that

$$\mathbb{E}[O\Psi](t,u) + \mathbb{F}[O\Psi](t,u) \leq C\mathring{\mathfrak{e}} + \varepsilon \int_{u'=0}^{u} \mathbb{F}[O\Psi](t,u') \, du' + \varepsilon \sup_{\tau \in [0,t)} \frac{1}{(1+\tau)^{1/2}} \widetilde{\mathbb{K}}[O\Psi](\tau,u) + \cdots . \tag{3.4.15}$$

Clearly, the first integral on the right-hand side of (3.4.15) is treatable with Gronwall's inequality (recall that $0 < u < 1$). Furthermore, the second integral $\varepsilon \sup_{\tau \in [0,t)} \frac{1}{(1+\tau)^{1/2}} \widetilde{\mathbb{K}}[O\Psi](\tau,u)$ can be treated as a

harmless cubic term, even if the Morawetz integral $\widetilde{\mathbb{K}}[O\Psi](t,u)$ grows logarithmically in time, consistent with (3.2.15b). Hence, assuming data of small size $\mathring{\epsilon}$, we have provided some indication of how to derive an a priori estimate of the form $\mathbb{E}^{1/2}[O\Psi](t,u)+\mathbb{F}^{1/2}[O\Psi](t,u)\lesssim\mathring{\epsilon}$ if we did not have to worry about the dangerous error integral $-\int_{\mathcal{M}_{t,u}}(2\breve{R}\Psi)(O\mathrm{tr}_{\not g}\chi)\breve{R}O\Psi\,d\varpi$. As we now discuss, this dangerous integral leads to a much worse a priori estimate.

3.4.3. *Avoiding top-order derivative loss via a Raychaudhuri-type identity.* We now confront the main difficulty in deriving the top-order energy estimate (3.2.15e): the potential derivative loss in the $O\mathrm{tr}_{\not g}\chi$ term in the error integral

$$-\int_{\mathcal{M}_{t,u}}(2\breve{R}\Psi)(O\mathrm{tr}_{\not g}\chi)\breve{R}O\Psi\,d\varpi$$

on the right-hand side of (3.4.3). We are still imagining, for the sake of illustration, that the second-order derivatives of Ψ are top-order. The main point of the procedure outlined below is to replace this error integral with

$$-4\int_{\mathcal{M}_{t,u}}\frac{L\mu}{\mu}(\breve{R}O\Psi)^2\,d\varpi+\cdots, \tag{3.4.16}$$

where the $\cdots$ integrals are similar in nature or easier. We can then use the coerciveness property (3.2.10a) and the co-area formula $\int_{\mathcal{M}_{t,u}}\cdots\,d\varpi=\int_{t'=0}^{t}\int_{\Sigma_{t'}^u}\cdots\,d\underline{\varpi}dt'$ to bound (3.4.16) in magnitude by

$$\leq 4\int_{t'=0}^{t}\left(\sup_{\Sigma_{t'}^u}\left|\frac{L\mu}{\mu}\right|\right)\mathbb{E}[O\Psi](t,u)\,dt'. \tag{3.4.17}$$

Thus, recalling (3.4.3), we find that

$$\mathbb{E}[O\Psi](t,u)+\mathbb{F}[O\Psi](t,u)\leq C\mathring{\epsilon}+4\int_{t'=0}^{t}\left(\sup_{\Sigma_{t'}^u}\left|\frac{L\mu}{\mu}\right|\right)\mathbb{E}[O\Psi](t',u)\,dt'+\cdots, \tag{3.4.18}$$

where the constant "4" on the right-hand side of (3.4.18) is a "structural constant," the $\cdots$ terms are similar and nature or easier, and $\mathring{\epsilon}$ is the size of the data.

We can now appeal to Lemma 3.5 to derive an a priori estimate for $\mathbb{E}[O\Psi](t,u)+\mathbb{F}[O\Psi](t,u)$, that is, we have

$$\mathbb{E}^{1/2}[O\Psi](t,u)+\mathbb{F}^{1/2}[O\Psi](t,u)\leq C\mathring{\epsilon}\ln^A(e+t)\mu_\star^{-B}(t,u).$$

Remark 3.9 (**The importance of the structural constants**)**.** Note that the structural constant "4" that appears in (3.4.18) is *independent of the number of times that we commute the wave equation with vectorfield operators.* This observation is important, for the structural constant affects the power of $\mu_\star^{-1}$ appearing in the top-order energy estimates and hence the number of derivatives we need to close the estimates.

It remains for us to explain the procedure used above, which allowed us to replace the derivative-losing error integral

$$-\int_{\mathcal{M}_{t,u}}(2\breve{R}\Psi)(O\mathrm{tr}_{\not g}\chi)\breve{R}O\Psi\,d\varpi$$

with (3.4.16). The procedure is based on the following *renormalized Raychaudhuri equation*[82] which we explain below.

3.4.4. *Renormalized Raychaudhuri equation.* We begin by recalling equation (3.4.6):

$$L\mathrm{tr}_{\not g}\chi = \frac{1}{2}G_{LL}\not\Delta\Psi + \cdots . \tag{3.4.19}$$

To avoid the derivative loss, we need to take advantage of the wave equation in the form (see (2.1.22) and (2.1.23))

$$0 = \mu\Box_{g(\Psi)}\Psi = -L\underline{\check{L}}\Psi + \mu\not\Delta\Psi + l.o.t.$$

Hence, using the wave equation, we can replace, up to a crucially important factor of μ^{-1} and $l.o.t.$, the term $\frac{1}{2}G_{LL}\not\Delta\Psi$ in (3.4.19) with a *perfect L derivative of* $\frac{1}{2}G_{LL}\underline{\check{L}}\Psi$ and then bring this perfect L derivative over to the *left-hand side* of (3.4.19). Furthermore, one can show that the remaining second derivatives of Ψ in the $\cdots$ terms on the right-hand side of (3.4.19) are also perfect L derivatives, and thus we can bring those terms to the left as well. In total, at the expense of a factor of μ^{-1}, we can renormalize away all of the terms in equation (3.4.19) that lose derivatives relative to Ψ, thereby obtaining an equation for a "modified" version of $\mathrm{tr}_{\not g}\chi$ of the form $L(Modified) = l.o.t.$

Hence, the important structure used by Christodoulou in [11] can be restated as follows: for solutions to $\Box_{g(\Psi)}\Psi = 0$, the Ric_{LL} term in the Raychaudhuri equation (3.4.5) is, up to lower-order terms, a perfect L derivative of the first derivatives of Ψ. To close our estimates, what we really need are higher-order[83] versions of this identity. In particular, we can commute the Raychaudhuri-type identity with O to obtain a transport equation equation for a "modified" version of $O\mathrm{tr}_{\not g}\chi$ that does not lose derivatives relative to Ψ. We make this precise in the following definition, where ${}^{(O)}\mathscr{X}$ is the "modified" quantity.

Definition 3.5 (**Modified version of** $O\mathrm{tr}_{\not g}\chi$**).** We define the modified quantity ${}^{(O)}\mathscr{X}$ as follows:

$${}^{(O)}\mathscr{X} := \mu O\mathrm{tr}_{\not g}\chi + O\mathfrak{X}, \tag{3.4.20}$$

$$\mathfrak{X} := -G_{LL}\check{R}\Psi + \mu\not G_L^{\ A}\not\nabla_A\Psi - \frac{1}{2}\mu\not G_A^{\ A}L\Psi - \frac{1}{2}\mu G_{LL}L\Psi. \tag{3.4.21}$$

In (3.4.21), $\not G_L^{\ A}$ is the $S_{t,u}-$tangent vectorfield formed by projecting the vectorfield with rectangular components $G_\alpha^{\ \nu}L^\alpha$ onto the $S_{t,u}$.

In total, the strategy described above allows us to show that ${}^{(O)}\mathscr{X}$ verifies a transport equation of the following delicate form.

Lemma 3.8. [63, **Proposition 10.2.3; Transport equation for the modified quantity**] *The quantity* ${}^{(O)}\mathscr{X}$ *defined in* (3.4.20) *verifies the transport equation*

$$L{}^{(O)}\mathscr{X} - \left\{2\frac{L\mu}{\mu} - \mathrm{tr}_{\not g}\chi\right\}{}^{(O)}\mathscr{X} = \left\{\frac{1}{2}\mathrm{tr}_{\not g}\chi - 2\frac{L\mu}{\mu}\right\}O\mathfrak{X} + \mathrm{Err}, \tag{3.4.22}$$

where Err *depends on at most two derivatives of* Ψ*, is regular in* μ*, and decaying in* t*.*

[82]The same idea was also used earlier, in a different context, in [39].

[83]In fact, we need only top-order versions of the identity.

We stress again that the advantage of (3.4.22) over the unmodified equation $LO\mathrm{tr}_{\not g}\chi = \frac{1}{2}G_{LL}\not\Delta O\Psi + \cdots$ is that the right-hand side of equation (3.4.22) *does not depend on the third derivatives of* Ψ. Hence, equation (3.4.22) can be used to derive L^2 estimates for ${}^{(O)}\mathscr{X}$ that *do not lose derivatives relative to* Ψ.

Remark 3.10 (**The need for elliptic estimates**)**.** Hiding in the terms Err in (3.4.22) lies another technical headache that we will briefly mention but not dwell on. Specifically, there is a quadratically small term, roughly of the form $\mu\hat{\chi}\cdot\not\mathcal{L}_O\hat{\chi}$, that formally involves the same number of χ derivatives as the modified quantity ${}^{(O)}\mathscr{X}$ (that is, one) but that cannot be directly estimated back in terms of ${}^{(O)}\mathscr{X}$. Here, $\hat{\chi}$ is the trace-free part of the $S_{t,u}$ tensor (2.1.24) and $\not\mathcal{L}_O$ denotes Lie differentiation with respect to O followed by projection onto the $S_{t,u}$. The term $\not\mathcal{L}_O\hat{\chi}$ involves three derivatives of the eikonal function u and as we have described, it will lead to derivative loss if not properly handled. To derive suitable L^2 estimates for this term, we have to derive a family of elliptic estimates on the spheres $S_{t,u}$. The main ideas behind this strategy can be traced back to Christodoulou-Klainerman's proof of the stability of Minkowski spacetime [13]. Similar strategies were also employed in [39] and [11]. The main point is that the elliptic estimates allow us to estimate $\|\mu\not\mathcal{L}_O\hat{\chi}\|_{L^2(S_{t,u})}$ back in terms of $\|\mu O\mathrm{tr}_{\not g}\chi\|_{L^2(S_{t,u})}$ plus errors, and that $\mu O\mathrm{tr}_{\not g}\chi$ can be controlled in L^2 by using the L^2 estimates for ${}^{(O)}\mathscr{X}$ and the up-to-second-order L^2 estimates for Ψ.

Remark 3.11. In the detailed proof, we must invert the transport equation (3.4.22) and obtain suitable L^2 estimates for ${}^{(O)}\mathscr{X}$. However, this is not an easy task; see the proof of [63, Lemma 19.4.1] for the details. The main reason is that the factors $\left\{2\frac{L\mu}{\mu} - \mathrm{tr}_{\not g}\chi\right\}$ and $\left\{\frac{1}{2}\mathrm{tr}_{\not g}\chi - 2\frac{L\mu}{\mu}\right\}$ in (3.4.22) have a drastic effect on the behavior of ${}^{(O)}\mathscr{X}$ and require a careful analysis.

We now return to the question of how to replace the derivative-losing error integral

$$-\int_{\mathcal{M}_{t,u}}(2\breve{R}\Psi)(O\mathrm{tr}_{\not g}\chi)\breve{R}O\Psi\,d\varpi$$

with (3.4.16). We first use the identity $O\mathrm{tr}_{\not g}\chi = \mu^{-1}{}^{(O)}\mathscr{X} - \mu^{-1}O\mathfrak{X}$ to replace the derivative-losing term $O\mathrm{tr}_{\not g}\chi$ with terms that do not lose derivatives. As we described in Remark 3.11, the most difficult analysis corresponds to the error integral generated by the piece $\mu^{-1}{}^{(O)}\mathscr{X}$. We do not want to burden the reader with the large number of technical complications that arise in the analysis of this error integral. Instead, we focus on the error integral generated by the other piece, namely $\int_{\mathcal{M}_{t,u}}2(\breve{R}\Psi)\mu^{-1}(O\mathfrak{X})\breve{R}O\Psi\,d\varpi$. The difficult part of this error integral comes from the top-order part of O applied to the first term $-G_{LL}\breve{R}\Psi$ on the right-hand side of (3.4.21). That is, we focus on the following error integral:

$$-2\int_{\mathcal{M}_{t,u}}\frac{1}{\mu}(\breve{R}\Psi)G_{LL}(\breve{R}O\Psi)^2\,d\varpi. \tag{3.4.23}$$

Though the integral (3.4.23) does not lose derivatives, it is nonetheless difficult to bound. If we were to try to bound it (3.4.23) by simply inserting the Heuristic Principle-type estimates $|\breve{R}\Psi| \lesssim \varepsilon(1+t)^{-1}$ and $|G_{LL}| \lesssim 1$, then we would not be able to derive the desired a priori energy estimate (3.2.15e); we would find that there is a loss that spoils the estimates and allows for the power of $\mu_\star^{-1}$ on the right-hand side of (3.2.15e) to grow like $C\varepsilon\ln(e+t)$, thereby completely ruining the L^2 hierarchy of Prop. 3.4.

Christodoulou overcame this difficulty by observing the following *critically important structure*: by using the transport equation $L\mu = \frac{1}{2}G_{LL}\breve{R}\Psi + \text{Err}$ (see (2.1.28)), we can rewrite (3.4.23) as

$$-4\int_{\mathcal{M}_{t,u}} \frac{L\mu}{\mu}(\breve{R}O\Psi)^2\, d\varpi + \cdots, \tag{3.4.24}$$

which is precisely the integral (3.4.16) that we successfully treated above. We have thus sketched the main ideas behind the procedure that allows us to avoid losing derivatives.

Remark 3.12 (Difficult top-order error integrals that arise during the Morawetz multiplier estimates). In order to derive the top-order estimate (3.2.15f) corresponding to the Morawetz multiplier $\widetilde{K} = \varrho^2 L$, we use Christodoulou's strategy [11], which is quite different than the one we use to derive the estimate (3.2.15e) corresponding to the timelike multiplier T. The main idea is that since $\widetilde{K}$ is proportional to L, we can integrate by parts in the divergence theorem identity (3.2.7) (see Remark 3.2) in order to trade, in the analog of the error integral (3.4.3), the O derivative on $\text{tr}_{\not g}\chi$ for an L derivative. The gain is that whenever the top-order derivative of an eikonal function quantity such as $\text{tr}_{\not g}\chi$ involves an L derivative, we do not have to worry about losing derivatives because we have a "direct expression" for these quantities based on the fact that they verify a transport equation in the direction of L. Hence, for the Morawetz multiplier estimates, we can avoid working with fully modified quantities such as (3.4.20)-(3.4.21), and we do not have to invoke any elliptic estimates on $S_{t,u}$ (see Remark 3.10). However, moving the L derivative generates some very difficult Σ_t^u error integrals that lead to top-order $\mu_\star^{-1}$ degeneracy, similar to the degeneracy we encountered in Lemma 3.5. In deriving the Morawetz multiplier estimates, although we do not need to use fully modified quantities of the form (3.4.20)-(3.4.21), we do need to define and use related partially modified versions of both $\text{tr}_{\not g}\chi$ and $\not\nabla\mu$ in order to avoid certain error integrals that have unfavorable $t-$growth. We do not want to further burden the reader with these technical details here, so we do not pursue this issue further.

3.5. **Descending below top order.** If we applied the above strategy of Subsubsect. 3.4.3 at all derivative levels, then *all of the energy-flux-Morawetz estimates would degenerate in the same way as* (3.2.15e)-(3.2.15f) *with respect to* $\mu_\star^{-1}$. In particular, we would not recover the non-degenerate estimates (3.2.15a)-(3.2.15b). This would in turn prevent us from recovering the decay estimates of the Heuristic Principle, which are based on (3.2.15a)-(3.2.15b) and Sobolev embedding. Hence, we would not be able to show that the terms we have deemed small errors are in fact small, and the entire proof would break down.

To overcome this difficulty, we note that since we are below top order, we can allow the loss in derivatives in the difficult error integral. In particular, there is no need to use the complicated procedure that led to the difficult top-order integral (3.4.16). In avoiding this procedure, we are rewarded with a less degenerate power of $\mu_\star^{-1}$, which comes from Lemma 3.6 and the availability of favorable powers of t.

As before, in the following discussion, $\mathring{\epsilon}$ denotes the small size of the data. To illustrate our strategy in some detail, let us imagine that three derivatives of Ψ in L^2 (which corresponds to $\mathbb{E}[\mathscr{Z}^2\Psi]$, etc.) represents the top-order. We also imagine, consistent with (3.2.15e) and (3.2.15f), that the top-order energy-flux-Morawetz quantities are bounded by

$$\mathbb{E}^{1/2}[\mathscr{Z}^2\Psi](t,u) + \mathbb{F}^{1/2}[\mathscr{Z}^2\Psi](t,u) \lesssim \mathring{\epsilon}\ln^A(e+t)\mu_\star^{-B}(t,u), \tag{3.5.1}$$

$$\widetilde{\mathbb{E}}^{1/2}[\mathscr{Z}^2\Psi](t,u) + \widetilde{\mathbb{F}}^{1/2}[\mathscr{Z}^2\Psi](t,u) + \widetilde{\mathbb{K}}^{1/2}[\mathscr{Z}^2\Psi](t,u) \lesssim \mathring{\epsilon}\ln^{A+2}(e+t)\mu_\star^{-B}(t,u) \tag{3.5.2}$$

for positive constants A and B. We will use these estimates to show how to derive a bound for the just-below-top-order quantities $\mathbb{E}^{1/2}[O\Psi](t,u)+\mathbb{F}^{1/2}[O\Psi](t,u)$ with a *smaller power of* $\mu_\star^{-1}$.

One important ingredient is that the weighted quantity $\varrho^2 O\mathrm{tr}_{\not g}\chi$ verifies a transport equation with a good structure, and this allows us to recover *good $t-$weighted estimates*[84] for $\|O\mathrm{tr}_{\not g}\chi\|_{L^2(\Sigma_t^u)}$ at the expense of a loss of derivatives. More precisely, a careful analysis of equation (3.4.6) reveals that we can commute it with $\varrho^2 O$ and use (3.4.10) to deduce

$$\big|L(\varrho^2 O\mathrm{tr}_{\not g}\chi)\big| = \big|\varrho^2 \not\Delta O\Psi\big| + \cdots \lesssim (1+t)\,|\not\nabla OO\Psi| + \cdots, \tag{3.5.3}$$

where $\not\nabla OO\Psi$ is a top-order term. Using the coerciveness property (3.2.11a) (note the appearance of another factor of $\mu_\star^{-1/2}(t,u)$!), (3.5.3), and the top-order estimate (3.5.2), and ignoring the $\cdots$ terms, we deduce

$$\big\|L(\varrho^2 O\mathrm{tr}_{\not g}\chi)\big\|_{L^2(\Sigma_t^u)} \lesssim \mu_\star^{-1/2}(t,u)\widetilde{\mathbb{E}}[OO\Psi](t,u) \lesssim \mathring{\epsilon}\ln^{A+2}(e+t)\mu_\star^{-B-1/2}(t,u). \tag{3.5.4}$$

Recalling that $L=\frac{\partial}{\partial t}$ and taking into account the fact that the spherical area form inherent in the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ is, in a pointwise sense, $\sim \varrho^2 \sim (1+t)^2$, it is not too difficult (see [63, Lemma 11.30.6]) to integrate (3.5.4) to deduce

$$\big\|O\mathrm{tr}_{\not g}\chi\big\|_{L^2(\Sigma_t^u)} \lesssim \mathring{\epsilon}\frac{\ln^{A+2}(e+t)}{1+t}\int_{t'=0}^{t}\frac{1}{1+t'}\mu_\star^{-B-1/2}(t',u)\,dt' + \cdots. \tag{3.5.5}$$

Applying Lemma 3.6 to inequality (3.5.5), we gain a power of $\mu_\star$ through the time integration:

$$\big\|O\mathrm{tr}_{\not g}\chi\big\|_{L^2(\Sigma_t^u)} \lesssim \mathring{\epsilon}\frac{\ln^{A+3}(e+t)}{1+t}\mu_\star^{-B+1/2}(t',u) + \cdots. \tag{3.5.6}$$

We now bound the first integral on the right-hand side of (3.4.3), that is, the integral

$$-\int_{\mathcal{M}_{t,u}}(2\breve{R}\Psi)(O\mathrm{tr}_{\not g}\chi)\breve{R}O\Psi\,d\varpi,$$

by using the estimate (3.5.6), the Heuristic Principle estimate (see Subsubsect. 2.1.5) $|\breve{R}\Psi|\lesssim \mathring{\epsilon}(1+t)^{-1}$, the coerciveness property (3.2.10a), and Cauchy-Schwarz. It therefore follows from (3.4.3) that

$$\begin{aligned}\sup_{s\in[0,t]}\mathbb{E}[O\Psi](s,u)+\mathbb{F}[O\Psi](s,u) &\leq C\mathring{\epsilon}+C\mathring{\epsilon}^2\int_{t'=0}^{t}\frac{1}{(1+t')^{3/2}}\mu_\star^{-B+1/2}(t',u)\mathbb{E}^{1/2}[O\Psi](t',u)\,dt'+\cdots \\ &\leq C\mathring{\epsilon}^2\sup_{s\in[0,t]}\mathbb{E}^{1/2}[O\Psi](s,u)\int_{t'=0}^{t}\frac{1}{(1+t')^{3/2}}\mu_\star^{-B+1/2}(t',u)\,dt'.\end{aligned} \tag{3.5.7}$$

Using Lemma 3.6 to bound the time integral on the right-hand side of (3.5.7), and in particular taking advantage of the good time decay in the integrand (3.5.7), we deduce from (3.5.7) that

$$\mathbb{E}^{1/2}[O\Psi](t,u)+\mathbb{F}^{1/2}[O\Psi](t,u)\lesssim \mathring{\epsilon}\mu_\star^{-B+3/2}(t',u)+\cdots. \tag{3.5.8}$$

The inequality (3.5.8) has thus yielded the desired gain in $\mu_\star$ compared to the top-order bound (3.5.1).

[84] Recall that $\varrho(t,u)\approx 1+t$ in the region of interest.

Remark 3.13. Inequality (3.5.8) is mildly misleading in the sense that there are some worse error terms that only allow us to gain a single power of $\mu_\star$, rather than the $3/2$ suggested by (3.5.8).

We have thus explained the main ideas of how to descend one level below the top order in the energy estimate hierarchy of Prop. 3.4. One can continue the descent, each time using Lemma 3.6 to gain a power of $\mu_\star$. Furthermore, the estimate (3.3.4) explains why we can eventually descend to the estimates (3.2.15a)-(3.2.15b), which no longer degenerate at all, *even as a shock forms*!

4. The Sharp Classical Lifespan Theorem in $3D$ and Generalizations

In this section, we provide a detailed statement of the general *sharp classical lifespan result* from [63], which applies to equations of the type $\Box_{g(\Psi)}\Psi = 0$. The theorem is an analog of the main theorem from Christodoulou's work, namely [11, Theorem 13.1 on pg. 888], which applied to a related class of quasilinear wave equations that arise in relativistic fluid mechanics; see Subsect. 5.2. We also provide a brief overview of its proof, which complements our discussion of generalized energy estimates from Sect. 3. We then sketch how to extend the result to apply to equations of the form $(g^{-1})^{\alpha\beta}(\partial\Phi)\partial_\alpha\partial_\beta\Phi = 0$. As in the spherically symmetric case, the result is the main ingredient used in proving that a shock actually forms in solutions launched by an open set of data (see Sect. 5).

4.1. **The sharp classical lifespan theorem.**

The sharp classical lifespan theorem below is a direct analog of Prop. 1.3, which applied to spherically symmetric solutions.

Theorem 2. [63, **Theorem 21.1.1; Sharp classical lifespan theorem**] *Let* $(\mathring{\Psi} := \Psi|_{\Sigma_0}, \mathring{\Psi}_0 := \partial_t\Psi|_{\Sigma_0})$ *be initial data for the covariant scalar wave equation*[85] *(in* 3 *space dimensions)*

$$\Box_{g(\Psi)}\Psi = 0.$$

Assume that the data are supported in the Euclidean unit ball Σ_0^1*. Let* $\mathring{\epsilon} = \mathring{\epsilon}[(\mathring{\Psi}, \mathring{\Psi}_0)] := \|\mathring{\Psi}\|_{H^{25}(\Sigma_0^1)} + \|\mathring{\Psi}_0\|_{H^{24}(\Sigma_0^1)}$ *be the size of the data. Let* $0 < U_0 < 1$ *be a fixed constant, and* Ψ *the corresponding solution restricted to a nontrivial region of the form* $\mathcal{M}_{T,U_0}$ *(see Definition* (2.1.35e) *and Figure 6). If* $\mathring{\epsilon}$ *is sufficiently small, then the outgoing lifespan* $T_{(Lifespan);U_0}$*, as defined in Subsubsect. 1.3.3, is determined as follows:*

$$T_{(Lifespan);U_0} := \sup\{t \mid \inf_{s\in[0,t)} \mu_\star(s, U_0) > 0\}, \tag{4.1.1}$$

where $\mu_\star(t,u) := \min\{1, \min_{\Sigma_t^u}\mu\}$ *(see Definition* (2.1.35b)*). Furthermore, there exists a constant* $C_{(Lower-Bound)} > 0$ *such that*

$$T_{(Lifespan);U_0} > \exp\left(\frac{1}{C_{(Lower-Bound)}\mathring{\epsilon}}\right). \tag{4.1.2}$$

In addition, the following statements hold true in $\mathcal{M}_{T_{(Lifespan);U_0},U_0}$.

[85] The theorem extends without any significant alterations to equations of the form $\Box_{g(\Psi)}\Psi = \mathcal{N}(\Psi)(\partial\Psi,\partial\Psi)$ whenever $\mathcal{N}$ verifies the future strong null condition of Remark 2.15. For simplicity we also assume (2.1.7). As we have noted earlier, this assumption is easy to eliminate.

(1) ***Energy estimates.*** *The energy estimate hierarchy of Prop. 3.4 is verified for* $(t,u) \in [0,U_0] \times [0,T_{(Lifespan);U_0})$*. A similar* L^2 *hierarchy holds for the scalar-valued functions* $\mu - 1$*,* $L^i_{(Small)} := L^i - \frac{x^i}{\varrho}$*, and* $R^i_{(Small)} := R^i + \frac{x^i}{\varrho}$*, and for the* $S_{t,u}$ *tensorfield* $\chi^{(Small)} := \chi - \frac{\not{g}}{\varrho}$*, where* $\varrho(t,u) := 1 - u + t$*.*

(2) ***Heuristic Principle.*** *The Heuristic Principle estimates stated in Subsubsect. 2.1.5 are valid for* Ψ *and its low-order derivatives with respect to the commutation set* $\mathscr{Z} := \{\varrho L, \breve{R}, O_{(1)}, O_{(2)}, O_{(3)}\}$ *(see* (3.1.1)*).*

Related C^0 *estimates hold for the low-order derivatives of the scalar-valued functions* $\mu - 1$*,* $L^i_{(Small)} := L^i - \frac{x^i}{\varrho}$*, and* $R^i_{(Small)} := R^i + \frac{x^i}{\varrho}$*, and for the* $S_{t,u}$ *tensorfield* $\chi^{(Small)} := \chi - \frac{\not{g}}{\varrho}$*. In particular, if* $T_{(Lifespan);U_0} < \infty$*, then these quantities extend to* $\Sigma^{U_0}_{T_{(Lifespan);U_0}}$ *as many-times classically differentiable functions of the geometric coordinates* (t,u,ϑ)*.*

(3) ***Rectangular coordinates.*** *If* $T_{(Lifespan);U_0} < \infty$*, then the change of variables map* $\Upsilon : [0,T_{(Lifespan);U_0}) \times [0,U_0] \times \mathbb{S}^2 \to \mathcal{M}_{T_{(Lifespan);U_0},U_0}$ *from geometric to rectangular coordinates extends continuously to* $[0,T_{(Lifespan);U_0}] \times [0,U_0] \times \mathbb{S}^2$*. Furthermore,* Υ *has a positive Jacobian determinant and is globally invertible on* $[0,T_{(Lifespan);U_0}) \times [0,U_0] \times \mathbb{S}^2$*. In addition, if* $T_{(Lifespan);U_0} < \infty$*, then the Jacobian determinant of* Υ *vanishes precisely on the set of points* $p \in \Sigma^{U_0}_{T_{(Lifespan);U_0}}$ *with* $\mu(p) = 0$*.*

(4) ***Lower bound for*** $\breve{R}\Psi = \mu R\Psi$***.*** *There exists a constant* $c > 0$ *such that if* $\mu(t,u,\vartheta) \leq 1/4$*, and* $G_{LL}(t,u,\vartheta) = \frac{d}{d\Psi} g_{\alpha\beta}(\Psi) L^\alpha L^\beta(t,u,\vartheta) \neq 0$ *then,*

$$L\mu(t,u,\vartheta) \leq -\frac{c}{(1+t)\ln(e+t)}, \tag{4.1.3}$$

$$|R\Psi|(t,u,\vartheta) \geq \frac{c}{\mu(t,u,\theta)(1+t)\ln(e+t)} \frac{1}{|G_{LL}(t,u,\vartheta)|}. \tag{4.1.4}$$

Moreover, the vectorfield R *verifies the Euclidean estimate*[86]

$$|R - (-\partial_r)|_e \lesssim \mathring{\epsilon} \ln(e+t)(1+t)^{-1}. \tag{4.1.5}$$

At all points $p \in \Sigma^{U_0}_{T_{(Lifespan);U_0}}$ *where* $\mu(p) = 0$*, the derivative* $R\Psi$ *blows up like* μ^{-1}*.*

Remark 4.1 (**Maximal development of the data**)**.** We stress the following important feature, made possible by Christodoulou's framework: with some additional effort, the results of Theorem 4 can be extended to a larger region, beyond the hypersurface $\Sigma_{T_{(Lifespan);U_0}}$, to reveal a portion of the maximal development of the data; see Subsect. 5.2 and in particular Christodoulou's Theorem 4. This extra information can be obtained because the results of Theorem 4 are sufficiently sharp.

Discussion of the proof of Theorem 2. The basic strategy begins with assuming, as bootstrap assumptions, that the Heuristic Principle C^0 decay estimates (see Subsubsect. 2.1.5) hold for Ψ and its low-order derivatives on a region of the form $\mathcal{M}_{T,U_0}$ for which $\mu > 0$. By "derivatives," we mean derivatives with respect to the commutation vectorfields $\mathscr{Z} := \{\varrho L, \breve{R}, O_{(1)}, O_{(2)}, O_{(3)}\}$ (see (3.1.1)). This mirrors the start to our proof of Proposition 1.3, in spherical symmetry. Using these bootstrap assumptions for Ψ and the

[86]Here, $|V|^2_e := \delta_{ab} V^a V^b$ and ∂_r is the standard Euclidean radial derivative.

smallness of the initial data, we derive analogous C^0 estimates for $\mu - 1$ and its low-order derivatives by using the transport equation (2.1.28) (note that $\mu - 1$ vanishes in the case $\Psi \equiv 0$). We also derive C^0 estimates for the quantities $L^i - x^i/\varrho$ as well as χ from the simple transport equations which they satisfy; see (2.1.28), (2.1.24), and (2.1.27). It is essential to note that all of these low-order estimates are regular relative to μ. In particular, the estimate (4.1.5) can be proved during this stage of the argument. Furthermore, assuming that one knows that the quantity in (4.1.1) is the classical lifespan of the solution in the region of interest (below, we describe how to establish this fact), the estimate (4.1.2) can easily be derived by using the transport equation (2.1.28) to prove that μ must remain positive up to a time of order $\exp\left(\frac{1}{C_{(Lower-Bound)}\mathring{\epsilon}}\right)$.

Next, we derive generalized energy estimates for Ψ on the region $\mathcal{M}_{T,U_0}$. The main ideas behind these estimates were discussed in Sect. 3. To control the error terms, it is convenient to rely not only on the low derivative assumptions discussed below, but on a full set of bootstrap assumptions, including L^2 assumptions consistent with Prop. 3.4. Clearly, in deriving the generalized energy estimates, we must bound the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ of the high derivatives of the eikonal function quantities such as μ, L^i, and χ. Most of these estimates can be derived using the transport equations mentioned in the previous paragraph. However, to bound the top derivatives of χ in the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$, we avoid derivative loss by using the modified quantities described in Subsubsect. 3.4.3 and elliptic estimates (see Remark 3.10). Similarly, one must carefully avoid top-order derivative loss stemming from the terms $\not{\Delta}\mu$, which appear upon commuting the wave equation with the transversal derivative $\breve{R}$.

After deriving the generalized energy estimates, we improve the Heuristic Principle bootstrap assumptions by assuming small-data and using Sobolev embedding on the spheres $S_{t,u}$. In particular, we use the low-order L^2 estimates (3.2.15a) of Prop. 3.4, which do not degenerate at all relative to μ^{-1}.

We now give the main idea explaining why the classical lifespan of Ψ in regions of the form $\mathcal{M}_{T,U_0}$ is given by (4.1.1). The main point is that if $\inf_{\mathcal{M}_{T,U_0}} \mu > 0$, then the rescaled frame $\{L, \breve{R}, X_1, X_2\}$ is uniformly comparable to the rectangular coordinate vectorfield frame $\{\frac{\partial}{\partial x^\alpha}\}_{\alpha=0,1,2,3}$ on $\mathcal{M}_{T,U_0}$. Hence, the above C^0 bounds, which show that Ψ and its derivatives relative to the rescaled frame remain uniformly bounded on $\mathcal{M}_{T,U_0}$, imply that the first rectangular derivatives of Ψ also remain uniformly bounded on $\mathcal{M}_{T,U_0}$. Therefore, by standard techniques,[87] we can extend the solution to a larger region of the form $\mathcal{M}_{T+\Delta,U_0}$.

Next, as we noted in Remark 2.2, the Jacobian determinant of the change of variables map Υ is proportional to μ. This is the main observation needed to prove the statements concerning Υ.

Finally, the estimates (4.1.3) and (4.1.4) are analogs of the estimates (1.3.16) and (1.3.17) proved in spherical symmetry. The additional terms present away from spherical symmetry involve $\mathcal{C}_u-$tangential derivatives of Ψ. Hence, they decay very rapidly and make only a negligible contribution to the estimates. □

[87] By "standard techniques," we mean an adapted version of the continuation criterion of Proposition 1.1. Note that since in the present context, the metric depends only on Ψ, it suffices to control Ψ in $W^{1,\infty}$, instead of $W^{2,\infty}$ as in the proposition.

4.2. **Extending the sharp classical lifespan theorem to a related class of equations.** Below we sketch how to extend Theorem 2 to apply to non-covariant quasilinear equations of the form

$$(g^{-1})^{\alpha\beta}(\partial\Phi)\partial_\alpha\partial_\beta\Phi = 0. \tag{4.2.1}$$

Analogously, the small-data shock-formation theorem (Theorem 5 below) can be extended to apply to equations of type (4.2.1), provided the nonlinearities fail the classic null condition (we assume, of course, that $g_{\alpha\beta} = m_{\alpha\beta} + \mathcal{O}(|\partial\Phi|)$ is a perturbation of the Minkowski metric). In particular, recall from Remark 2.12 that the correct analog of the future null condition failure factor ${}^{(+)}\aleph$ is

$${}^{(+)}\aleph := m_{\kappa\lambda}G^\kappa_{\alpha\beta}(\partial\Phi = 0)L^\alpha_{(Flat)}L^\beta_{(Flat)}L^\lambda_{(Flat)}, \tag{4.2.2}$$

where

$$G^\lambda_{\alpha\beta} = G^\lambda_{\alpha\beta}(\partial\Phi) := \frac{\partial}{\partial(\partial_\lambda\Phi)}g_{\alpha\beta}(\partial\Phi). \tag{4.2.3}$$

When ${}^{(+)}\aleph \equiv 0$, the nonlinearities verify Klainerman's classic null condition [34], and the methods of [36] and [10] yield small-data global existence. When ${}^{(+)}\aleph$ is nontrivial, Theorem 5 below can be extended to show that small-data future shock formation occurs. See also Sect. 5.1 for a discussion of Alinhac's related small-data shock formation theorem.

Remark 4.2. Note that there is a major difference between equations of type (4.2.1) and the *scalar equations* of the form $(g^{-1})^{\alpha\beta}(\Psi)\partial_\alpha\partial_\beta\Psi = 0$. As we described in Subsect. 1.4, the latter type of equations exhibit small-data global existence even when the classic null condition fails.

Examples

- In the case of the equation $\Box_m\Phi = \partial_t((m^{-1})^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi)$, we compute that $G^\lambda_{\alpha\beta} = 2\delta^\lambda_\alpha m_{\beta 0}$. Hence, ${}^{(+)}\aleph = m_{\kappa\lambda}\delta^\kappa_\alpha L^\alpha_{(Flat)}L^\beta_{(Flat)}L^\lambda_{(Flat)} = -m_{\alpha\lambda}L^\alpha_{(Flat)}L^0_{(Flat)}L^\lambda_{(Flat)} = 0$. Therefore, the nonlinearities in this equation verify the classic null condition.
- In the case of the equation $\Box_m\Phi = 2\partial_t\Phi\partial_t^2\Phi$, we compute that $G^\lambda_{\alpha\beta} = m_{\alpha 0}m_{\beta 0}\delta^\lambda_0$. Hence, ${}^{(+)}\aleph = m_{\alpha 0}m_{\beta 0}\delta^\lambda_0 L^\alpha_{(Flat)}L^\beta_{(Flat)}L^\lambda_{(Flat)} \equiv 1$. Therefore, the nonlinearities in this equation fail the classic null condition.

4.2.1. *Connections to equations of the form $\Box_g\Psi = \mathcal{N}$.* The main idea of extending the theorem is to differentiate (4.2.1) with rectangular coordinate derivatives ∂_ν and to set

$$\Psi_\nu := \partial_\nu\Phi, \tag{4.2.4}$$

$$\vec{\Psi} := (\Psi_0, \Psi_1, \Psi_2, \Psi_3), \tag{4.2.5}$$

thereby arriving at a coupled system that can be put into the form

$$\Box_{g(\vec{\Psi})}\Psi_\nu = \mathcal{N}(\vec{\Psi})(\partial\vec{\Psi}, \partial\Psi_\nu), \tag{4.2.6}$$

where $\Box_{g(\vec{\Psi})}$ is the *covariant wave operator* corresponding to $g(\vec{\Psi})$. The semilinear term $\mathcal{N}(\vec{\Psi})(\partial\vec{\Psi},\partial\Psi_\nu)$ generated from the commutation verifies the future strong null condition[88] of Remark 2.15. Hence, as in our study of the scalar equation (2.1.2) under the structural assumptions of Subsubsect. 2.1.7, the dangerous quadratic terms, whose presence is heralded by ${}^{(+)}\aleph \not\equiv 0$, can only hide in the operator $\Box_{g(\vec{\Psi})}$. With the term $\mathcal{N}(\vec{\Psi})(\partial\vec{\Psi},\partial\Psi_\nu)$ having little effect on the dynamics, we can effectively analyze the system (4.2.6) by studying each scalar equation for Ψ_ν using methods similar to the ones we used to analyze the scalar equation $\Box_{g(\Psi)}\Psi = 0$. In particular, the Heuristic Principle estimates (2.1.36a)-(2.1.36b) hold for each scalar component Ψ_ν.

4.2.2. *The evolution equation for* μ *and its connection to the top-order* L^2 *estimates.* It is instructive to examine the dangerous quadratic terms present in the system (4.2.6) from a different point of view by deriving the evolution equation for μ (that is, an analog of equation (2.1.28)) in the present case of the metric $g(\vec{\Psi})$. Specifically, arguing as in our proof of (2.1.28) and exploiting the identity $\partial_\alpha\Psi_\beta = \partial_\beta\Psi_\alpha$, we find that

$$\begin{aligned} L\mu(t,u,\vartheta) &= -\frac{1}{2}[G^L_{LL}R^a\breve{R}\Psi_a](t,u,\vartheta) + \mu\mathrm{Err} \\ &= -\frac{1}{2}{}^{(+)}\mathring{\aleph}(\vartheta)[R^a\breve{R}\Psi_a](t,u,\vartheta) + \mu\mathrm{Err}, \end{aligned} \tag{4.2.7}$$

where[89]

$${}^{(+)}\mathring{\aleph}(\vartheta) := {}^{(+)}\aleph(t=0,u=0,\vartheta), \tag{4.2.8}$$

$G^L_{LL} := G^\lambda_{\alpha\beta}(\vec{\Psi})L^\alpha L^\beta L_\lambda$ and the error terms Err in (4.2.7) are small and decaying according to (2.1.36a)-(2.1.36b). In deriving the second line in (4.2.7), we have used the fact that we can prove an estimate of the form $G^L_{LL}(t,u,\vartheta) = {}^{(+)}\mathring{\aleph}(\vartheta) + \mathcal{O}(\mathring{\epsilon})$, where $\mathring{\epsilon}$ is the size of the data. Hence, the term $-\frac{1}{2}{}^{(+)}\mathring{\aleph}(\vartheta)[R^a\breve{R}\Psi_a](t,u,\vartheta)$ is the dangerous one that can cause μ to vanish in finite time.

To analyze solutions $\vec{\Psi}$, we can derive energy estimates for each scalar component Ψ_ν by commuting each equation (4.2.6) with vectorfield operators $\mathscr{Z}^N$ and deriving energy identities of the form (3.2.7) for $\mathscr{Z}^N\Psi_\nu$. The energy identities for the Ψ_ν are of course coupled, but the analysis of each component is essentially the same as it is for the scalar equation $\Box_{g(\Psi)}\Psi = 0$. The only new ingredients that we need are estimates of the following form, *which must hold for each* $\nu = 0,1,2,3$:

$$\left|G^L_{LL}\breve{R}\Psi_\nu\right| \leq 2\,|L\mu| + \mathrm{Err}. \tag{4.2.9}$$

[88]More precisely, we have

$$\mathcal{N}(\vec{\Psi})(\partial\vec{\Psi},\partial\Psi) = (g^{-1})^{\alpha\alpha'}(g^{-1})^{\beta\beta'}G^\mu_{\alpha'\beta'}\left\{\partial_\beta\Psi_\alpha\partial_\mu\Psi - \partial_\mu\Psi_\alpha\partial_\beta\Psi\right\} + (g^{-1})^{\alpha\beta}\frac{1}{\sqrt{|\mathrm{det}g|}}\frac{\partial\sqrt{|\mathrm{det}g|}}{\partial\Psi_\lambda}\partial_\alpha\Psi_\lambda\partial_\beta\Psi,$$

where $G^\lambda_{\alpha\beta}(\vec{\Psi})$ is defined in (4.2.3) and the determinant is taken relative to the rectangular coordinates; see [63, Lemma A.1.2] for the details.

[89]As we described in Subsect. 2.2, ${}^{(+)}\mathring{\aleph}$ is a good approximation to ${}^{(+)}\aleph$ that has the advantage of being constant along the integral curves of L.

The estimates (4.2.9) are are less straightforward to derive compared to the case of the scalar equation $\Box_{g(\Psi)}\Psi = 0$; their derivation uses the symmetry condition $\partial_\alpha \Psi_\beta = \partial_\beta \Psi_\alpha$; see Appendix A of [63]. These estimates are used to replace inequality (3.4.18) for each scalar component Ψ_ν of our system. More precisely, the estimates (4.2.9) are analogs of the algebraic replacement $L\mu = \frac{1}{2} G_{LL} \breve{R}\Psi + \text{Err}$ that we used to derive (3.4.24) from (3.4.23). As such, they play essential roles in allowing us to close the top-order L^2 estimates for the Ψ_ν.

5. The Shock-Formation Theorems and Comparisons

In this final section, we first state Alinhac's and Christodoulou's shock formation theorems. We then compare and contrast their approaches and explain the advantages of Christodoulou's framework. In particular, we highlight the conceptual and technical gains that stem from using a true eikonal function throughout the proof and working with quantities that are properly rescaled by μ : relative to the rescaled quantities, the problem becomes a traditional one in which one establishes long-time well- posedness. We then state the shock formation theorem of [63]. Finally, we compare and contrast the various results.

5.1. Alinhac's shock formation theorem.

In this section, we state Alinhac's shock formation results in 3 space dimensions. We summarize the most important aspects of his shock formation results in the following theorem.[90] We try to stay true to the original formulation when stating his theorems (some of the coordinate systems appearing in the theorem could actually be eliminated). The results are a partial summary of Theorems 2 and 3 of [2] in the case of 3 space dimensions.

Theorem 3 (Alinhac). *Consider the following initial value problem expressed relative to Minkowski-rectangular coordinates:*

$$(g^{-1})^{\alpha\beta}(\partial\Phi)\partial_\alpha\partial_\beta\Phi = 0, \tag{5.1.1}$$

$$(\Phi|_{t=0}, \partial_t\Phi|_{t=0}) = \lambda(\mathring{\Phi}, \mathring{\Phi}_0), \tag{5.1.2}$$

where $\lambda(\mathring{\Phi}, \mathring{\Phi}_0)$ is a one-parameter family of smooth, compactly supported initial data indexed by $\lambda > 0$. Let Φ_λ be the solution corresponding to the data. Assume that (1.1.7) *holds and that Klainerman's classic null condition fails for the nonlinearities in* (5.1.1)*, that is, that the function ${}^{(+)}\aleph(\theta)$ from* (4.2.2) *is non-vanishing at some Euclidean angle $\theta \in \mathbb{S}^2$. Recall that Friedlander's radiation field is the function $\digamma[(\mathring{\Phi}, \mathring{\Phi}_0)] : \mathbb{R} \times \mathbb{S}^2 \to \mathbb{R}$ defined by*

$$\digamma[(\mathring{\Phi}, \mathring{\Phi}_0)](q, \theta) := -\frac{1}{4\pi}\frac{\partial}{\partial q}\mathcal{R}[\mathring{\Phi}](q, \theta) + \frac{1}{4\pi}\mathcal{R}[\mathring{\Phi}_0](q, \theta), \tag{5.1.3}$$

where the Radon transform $\mathcal{R}$ is defined in (1.2.17)*. Assume that the function*

$$\frac{1}{2}{}^{(+)}\aleph(\theta)\frac{\partial^2}{\partial q^2}\digamma[(\mathring{\Phi}, \mathring{\Phi}_0)](q, \theta) \tag{5.1.4}$$

[90] Despite the title of the article [2], it addresses both the cases of 2 and 3 space dimensions.

has a unique, strictly positive, non-degenerate maximum at (q_*, θ_*). If λ is sufficiently small and positive, then the classical lifespan $T_{(Lifespan);\lambda}$ of the solution is finite and verifies

$$\lim_{\lambda \downarrow 0} \lambda \ln T_{(Lifespan);\lambda} = \frac{1}{\frac{1}{2}^{(+)}\aleph(\theta_*)\frac{\partial^2}{\partial q^2}\digamma[(\mathring{\Phi}, \mathring{\Phi}_0)](q_*, \theta_*)}. \tag{5.1.5}$$

In addition, there exists a first blow-up point $p_{(Blow-up);\lambda}$ with rectangular coordinates $p_{(Blow-up);\lambda} = (T_{(Lifespan);\lambda}, x^1_\lambda, x^2_\lambda, x^3_\lambda)$ and a constant $C > 0$ depending on $(\mathring{\Phi}, \mathring{\Phi}_0)$ such that whenever λ is sufficiently small and positive, the following statements hold true.

*C^1 **behavior relative to rectangular coordinates**. Φ_λ is a C^1 function of the rectangular coordinates $\{x^\alpha\}$ and for $t \leq T_{(Lifespan);\lambda}$, we have*

$$|\Phi_\lambda| + \sum_{\alpha=0}^{3} |\partial_\alpha \Phi_\lambda| \leq C\lambda \frac{1}{1+t}. \tag{5.1.6}$$

***Blow-up of second rectangular derivatives**. One can obtain the behavior of the second rectangular derivatives of Φ_λ in the past domain of dependence of a neighborhood of $p_{(Blow-up);\lambda}$ in $\Sigma_{T_{(Lifespan);\lambda}}$. More precisely, strictly away from $p_{(Blow-up);\lambda}$, Φ_λ is a C^2 function of the rectangular coordinates with second-order derivatives that verify a bound of the form* (5.1.6)*, where the constant C depends on the distance to $p_{(Blow-up);\lambda}$. In contrast to the regular behavior* (5.1.6)*, the following blow-up behavior occurs:*

$$C^{-1}\left(t \ln \frac{T_{(Lifespan);\lambda}}{t}\right)^{-1} \leq \sum_{\alpha,\beta=0}^{3} \left\|\partial_\alpha \partial_\beta \Phi_\lambda\right\|_{C^0(\Sigma_t)} \leq C\left(t \ln \frac{T_{(Lifespan);\lambda}}{t}\right)^{-1}. \tag{5.1.7}$$

***Detailed description near the first blow-up point**. We define the rescaled time variable $\tau := \lambda \ln t$ and in particular set $\tau_{(Lifespan);\lambda} = \lambda \ln T_{(Lifespan);\lambda}$. Let $u_{(Flat)} = 1 + t - r$ be a flat eikonal function of the Minkowski metric. There exists a true eikonal function u for the dynamic metric $g(\partial\Phi_\lambda)$ defined near $p_{(Blow-up);\lambda}$. $u_{(Flat)}$ and u respectively induce time-rescaled flat coordinates $(\tau, u_{(Flat)}, \theta)$ and geometric coordinates (τ, u, θ), where θ is the Euclidean angle. The first blow-up point can be written uniquely in the time-rescaled flat coordinates as $p_{(Blow-up);\lambda} = (\tau_{(Lifespan);\lambda}, u_{(Flat);\lambda}, \theta_\lambda)$.*

Relative to the time-rescaled geometric coordinates, we have the following conclusions. There exists a value u_λ, a neighborhood $\Omega \subset \{(\tau, u, \theta) \mid \tau \leq \tau_{(Lifespan);\lambda}, u \in \mathbb{R}, \theta \in \mathbb{S}^2\}$ of $(\tau_{(Lifespan);\lambda}, u_\lambda, \theta_\lambda)$, and functions $v, w, \zeta \in C^3(\Omega)$ with the following properties.

(1) *The functions v, w, ζ can be related to the solution Φ_λ by interpreting ζ as the change of variables from (τ, u, θ) to $u_{(Flat)}$, v as the solution Φ_λ expressed in the time-rescaled geometric coordinates, and w as the rescaled first transversal derivative of v. More precisely, we have*

$$\zeta(\tau_{(Lifespan);\lambda}, u_\lambda, \theta_\lambda) = u_{(Flat);\lambda}, \tag{5.1.8a}$$

$$v(\tau, u, \theta) = \lambda^{-1} \underbrace{(1 + e^{\tau/\lambda} - \zeta(\tau, u, \theta))}_{r} \Phi_\lambda(\tau, u_{(Flat)} = \zeta(\tau, u, \theta), \theta), \tag{5.1.8b}$$

$$\frac{\partial}{\partial u} v = w \frac{\partial}{\partial u} \zeta. \tag{5.1.8c}$$

(2) *The change-of-variables function ζ satisfies*
   - $\frac{\partial}{\partial u}\zeta \geq 0$*, with equality exactly at* $(\tau_{(Lifespan);\lambda}, u_\lambda, \theta_\lambda)$ *and nowhere else.*
   - *At the point* $(\tau_{(Lifespan);\lambda}, u_\lambda, \theta_\lambda)$*, we have* $\frac{\partial^2}{\partial\tau\partial u}\zeta < 0$*,* $\frac{\partial^2}{\partial\theta\partial u}\zeta = \frac{\partial^2}{\partial u^2}\zeta = 0$*, and the Hessian with respect to* u, θ *of* $\frac{\partial}{\partial u}\zeta$ *is positive definite.*

(3) *The derivative* $\frac{\partial}{\partial u}w$ *does not vanish at* $(\tau_{(Lifespan);\lambda}, u_\lambda, \theta_\lambda)$.

We make the following clarifying remarks concerning Alinhac's theorem.

- Consider the inverse change of variables to ζ. That is, let η be defined by $\eta(\tau, \zeta(\tau, u, \theta), \theta) = u$. Then η is, relative to rectangular coordinates, a solution to the eikonal equation: $(g^{-1})^{\alpha\beta}(\partial\Phi_\lambda)\partial_\alpha\eta\partial_\beta\eta = 0$.
- Note that by the chain rule and the change of variables $u_{(Flat)} = \zeta(\tau, u, \theta)$ we have, with τ, θ fixed,
$$\frac{\partial\zeta}{\partial u}\frac{\partial}{\partial u_{(Flat)}} = \frac{\partial}{\partial u}.$$
Hence, by (5.1.8b)-(5.1.8c), we have
$$\frac{\partial}{\partial u_{(Flat)}}(r\Phi_\lambda)(\tau, \zeta(\tau, u, \theta), \theta) = \lambda w(\tau, u, \theta), \tag{5.1.9a}$$
$$\frac{\partial^2}{\partial u^2_{(Flat)}}(r\Phi_\lambda)(\tau, \zeta(\tau, u, \theta), \theta) = \lambda\frac{\frac{\partial}{\partial u}w}{\frac{\partial}{\partial u}\zeta}(\tau, u, \theta). \tag{5.1.9b}$$
Hence, from (5.1.9a)-(5.1.9b) and the conclusions of the theorem, it follows that the transversal second derivative $\frac{\partial^2}{\partial u^2_{(Flat)}}(r\Phi_\lambda)$ blows up at $p_{(Blow-up);\lambda}$ thanks to the vanishing of $\frac{\partial}{\partial u}\zeta$, while the first derivative $\frac{\partial}{\partial u_{(Flat)}}(r\Phi_\lambda)$ does not blow-up.
- The quantity $\frac{\partial}{\partial u}\zeta$ should be compared to the quantity μ discussed throughout this paper. The statements concerning the first derivatives of $\frac{\partial}{\partial u}\zeta$ given in the theorem above are natural: the non-degeneracy condition $\frac{\partial^2}{\partial\tau\partial u}\zeta < 0$ is the exact analogue of (3.3.7) (see also (1.3.16) in spherical symmetry); the conditions concerning $\frac{\partial^2}{\partial u^2}\zeta$ and $\frac{\partial^2}{\partial\tau\partial u}\zeta$ are in fact *necessary* if $\tau_{(Lifespan);\lambda}$ is the first (rescaled) blow-up time and $(\tau_{(Lifespan);\lambda}, u_\lambda, \theta_\lambda)$ is the unique first blow-up point.

5.2. **Christodoulou's results.** In [11], Christodoulou proved, for a class of quasilinear wave equations arising in irrotational relativistic fluid mechanics (see also [9] for a generalization to the non-relativistic Euler equations), theorems that are analogous to the sharp classical lifespan theorem (Theorem 2) and the small-data shock-formation theorem (Theorem 5) of the third author. Actually, Christodoulou's work went somewhat beyond these two results in the following two senses.

(1) His shock-formation theorem was extended to apply to a class of small fluid equation data for which there is non-zero vorticity. However, most of his main results, including the shock-formation aspect of his work, applied only to a region in which the fluid is irrotational (vorticity-free), in which case the fluid equations reduce to the aforementioned scalar quasilinear wave equation. Hence, we will not elaborate on Christodoulou's treatment of the full relativistic Euler equations, but instead focus only on describing his results for irrotational flows.

(2) After identifying the constant-time hypersurface region $\Sigma^{U_0}_{T_{(Lifespan)};U_0}$ where the first shock-point occurs, he goes further by characterizing the nature of the maximal future development, including the boundary, of the data lying in the exterior of the sphere $S_{0,U_0} \subset \Sigma^{U_0}_0$. Christodoulou's full description of the maximal development is made possible by the sharp estimates he proved in his sharp classical lifespan theorem [11, Theorem 13.1 on pg. 888], analogous to Theorem 2 stated above, and which forms the most difficult part of the analysis.

We now describe Christodoulou's results [11] in more detail. There are some inessential complications that arise in the formulation of the problem compared to our study of the equations $\Box_{g(\Psi)}\Psi = 0$ and that of Alinhac because Christodoulou's background solutions are not $\Phi = 0$, but rather $\Phi = kt$, where k is a non-zero constant. These are the solutions that correspond to the nontrivial constant states in relativistic fluid mechanics in Minkowski spacetime, and the resulting complications are simply issues of normalization and not serious ones. To avoid impeding the flow of the paper, we describe Christodoulou's equations in detail and address the normalization issue in Appendix A. Here, we summarize the most important aspects of his work. The results stated below as Theorem 4 are a conglomeration of [11, Theorem 13.1 on pg. 888, Theorem 14.1 on pg. 903, Proposition 15.3 on pg. 974, and the Epilogue on pg. 977]. The quantities that appear in the theorem are essentially the same as the quantities we have studied in Sects. 2-4, up to the differences in normalization we describe in Appendix A.

Theorem 4 (Christodoulou). *Let* $\mathfrak{o} = -(m^{-1})^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi$ *be as defined in* (A.0.12)*, where* m *is the Minkowski metric. Assume that the Lagrangian* $\mathcal{L}(\mathfrak{o})$ *verifies the positivity conditions* (A.0.13) *in a neighborhood of* $\mathfrak{o} = k^2$*, where* k *is a non-zero constant, but that* $\mathcal{L}(\mathfrak{o})$ *is not the exceptional Lagrangian* (A.0.29)*. Consider the following Cauchy problem for the quasilinear (Euler-Lagrange) wave equation corresponding to* $\mathcal{L}(\mathfrak{o})$*, expressed relative to rectangular coordinates:*

$$\partial_\alpha\left(\frac{\partial\mathcal{L}(\mathfrak{o})}{\partial(\partial_\alpha\Phi)}\right) = 0, \tag{5.2.1}$$

$$(\Phi|_{t=0}, \partial_t\Phi|_{t=0}) = (\mathring{\Phi}, \mathring{\Phi}_0). \tag{5.2.2}$$

Assume that the data are small perturbations of the data corresponding to the non-zero constant-state solution $\Phi = kt$ *and that the perturbations are compactly supported in the Euclidean unit ball. Let* $U_0 \in (0, 1/2)$ *and let*

$$\mathring{\epsilon} = \mathring{\epsilon}[(\mathring{\Phi}, \mathring{\Phi}_0)] := \|\mathring{\Phi}_0 - k\|_{H^N(\Sigma^{U_0}_0)} + \sum_{i=1}^{3}\|\partial_i\mathring{\Phi}\|_{H^N(\Sigma^{U_0}_0)} \tag{5.2.3}$$

denote the size of the data, where N *is a sufficiently large integer.*[91]

Sharp classical lifespan*. If* $\mathring{\epsilon}$ *is sufficiently small, then a sharp classical lifespan theorem in analogy with Theorem 2 holds.*

[91] A numerical value of N was not provided in [11].

Small-data shock formation. *We define the following data-dependent functions of $u|_{\Sigma_0} = 1 - r$ (see Appendix A for definitions of α, η, etc.):*

$$\mathcal{E}[(\mathring{\Phi}, \mathring{\Phi}_0)](u) \tag{5.2.4}$$
$$:= \sum_{\Psi \in \{\partial_t \Phi - k, \partial_1 \Phi, \partial_2 \Phi, \partial_3 \Phi\}} \int_{\Sigma_0^u} \left\{ \alpha^{-2} \mu (\eta_0^{-1} + \alpha^{-2} \mu)(L\Psi)^2 + (\breve{\underline{L}}\Psi)^2 + (\eta_0^{-1} + 2\alpha^{-2}\mu)\mu |\nabla\!\!\!/ \Psi|^2 \right\} d\underline{\varpi},$$

$$\mathcal{S}[(\mathring{\Phi}, \mathring{\Phi}_0)](u) := \int_{S_{0,u}} r \left\{ (\mathring{\Phi}_0 - k) - \eta_0 \partial_r \mathring{\Phi} \right\} dv_{\not g} + \int_{\Sigma_0^u} \left\{ 2(\mathring{\Phi}_0 - k) - \eta_0 \partial_r \mathring{\Phi} \right\} d^3x, \tag{5.2.5}$$

where $d\underline{\varpi}$ is defined in (3.2.8)*, $dv_{\not g}$ denotes the Euclidean area form on the sphere $S_{0,u}$ of Euclidean radius $r = 1 - u$, and d^3x denotes the standard flat volume form on $\mathbb{R}^3$. Assume that (see* (A.0.21) *for the definition of H)*

$$\ell := \frac{dH}{d\sigma}(\sigma = k^2) > 0. \tag{5.2.6}$$

There exist constants $C > 0$ and $C' > 0$, independent of $U \in (0, U_0]$, such that if $\mathring{\epsilon}$ is sufficiently small and if for some $U \in (0, U_0]$ we have

$$\mathcal{S}[(\mathring{\Phi}, \mathring{\Phi}_0)](U) \leq -C\mathring{\epsilon}\mathcal{E}^{1/2}[(\mathring{\Phi}, \mathring{\Phi}_0)](U) < 0, \tag{5.2.7}$$

then a shock forms in the solution[92] *Φ and the first shock in the maximal development of the portion of the data in the exterior of $S_{0,U} \subset \Sigma_0^U$ originates in the hypersurface region $\Sigma^U_{T_{(Lifespan);U}}$ (see Definition 2.3), where*

$$T_{(Lifespan);U} < \exp\left(C' \frac{U}{\left| k^3 \ell \mathcal{S}[(\mathring{\Phi}, \mathring{\Phi}_0)](U) \right|} \right). \tag{5.2.8}$$

A similar result holds if $\ell < 0$; in this case, we delete the "$-$" sign in (5.2.7) *and change "$\leq$" and "$<$" to "$\geq$" and "$>$."*

Description of the boundary of the maximal development. *For shock-forming solutions,*[93] *the boundary $\mathcal{B}$ of the maximal development of the data in the exterior of $S_{0,U} \subset \Sigma_0^U$ is a disjoint union $\mathcal{B} = (\partial_- \mathcal{H} \cup \mathcal{H}) \cup \underline{\mathcal{C}}$, where $\partial_- \mathcal{H} \cup \mathcal{H}$ is the singular part (where μ vanishes) and $\underline{\mathcal{C}}$ is the regular part (where μ extends continuously to a positive value). The solution and its rectangular derivatives extend continuously in rectangular coordinates to the regular part. Each component of $\partial_- \mathcal{H}$ is a smooth $2-$dimensional embedded submanifold of Minkowski spacetime, spacelike with respect to the dynamic metric*[94] *h (see* (A.0.15)*). The corresponding component of $\mathcal{H}$ is a smooth, embedded, $3-$dimensional submanifold in Minkowski spacetime ruled by curves with vanishing arc length as measured by h and with past endpoints*

[92] That is, Φ and its first rectangular derivatives remain bounded, while some second-order rectangular derivative blows up due to the vanishing of μ.

[93] Some of the results stated here depend on some non-degeneracy assumptions on the solution that are expected to hold generically, such as $\frac{\partial^2}{\partial u^2}\mu > 0$ at the shock points.

[94] We follow the conventions of [11] and denote the dynamic metric by $h = h(\partial\Phi)$ in this section.

on $\partial_- \mathcal{H}$. The corresponding component $\underline{\mathcal{C}}$ is the incoming null hypersurface corresponding to $\partial_- \mathcal{H}$, and it is ruled by incoming $h-$null geodesics with past endpoints on $\partial_- \mathcal{H}$.

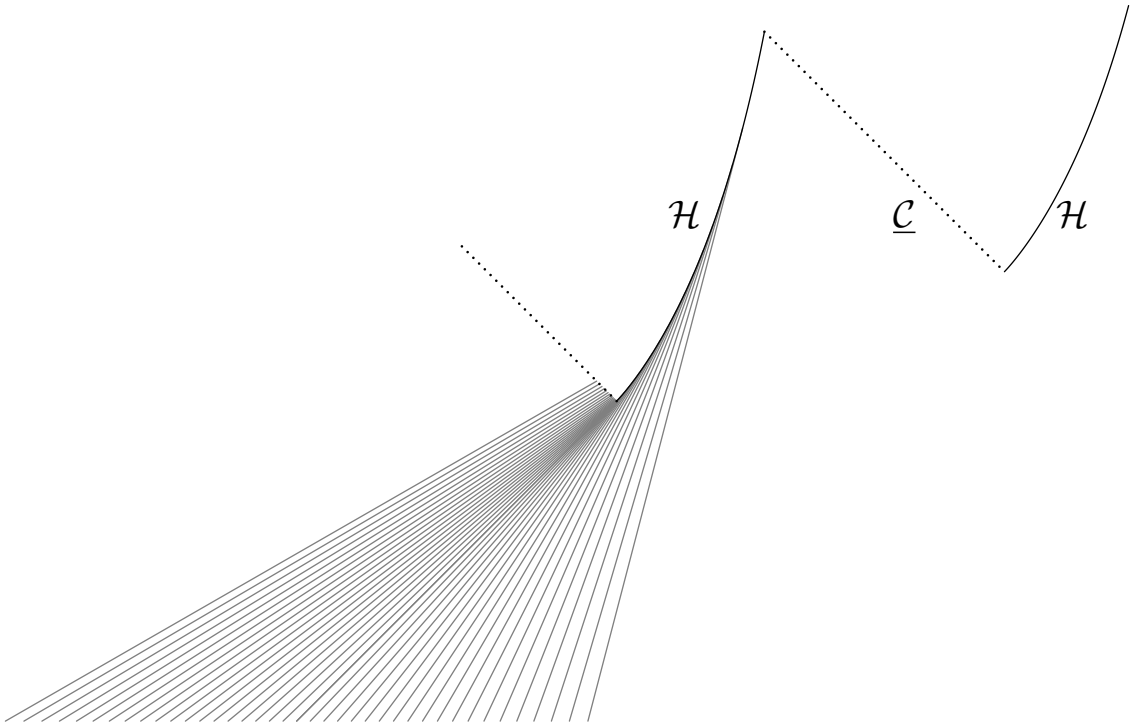


FIGURE 7. The geometry of the maximum development in cross-sectional view. The gray lines indicate the level sets of our eikonal function u, near the first blow-up point. The dotted lines form the regular boundary $\underline{\mathcal{C}}$. The black curves indicate the singular boundary $\mathcal{H}$, whose lower endpoints are $\partial_- \mathcal{H}$.

We make the following remarks concerning Christodoulou's theorem.

- Most aspects of Theorem 2 can be proved by using the strategy outlined in the discussion of the proof of Theorem 4 (see also Subsect. 4.2).
- The full description of the boundary of the maximal development, especially in view of the goal of extending the solution past the shock front, involves discussions both relative to Minkowski spacetime and relative to the eikonal foliation corresponding to u, which degenerates along $\partial_- \mathcal{H} \cup \mathcal{H}$. We invite interested readers to consult [11, Ch.15] and will not discuss these issues further except to note that the full description requires studying the solution at times t beyond the time of first blow-up and studying the blow-up sets $\mu \to 0$ along the Σ_t, which have positive dimension.
- The quantity (5.2.6) is the exact analog of the future null condition failure factor ${}^{(+)}\aleph$ from (2.1.40). Note that unlike the general classes of equations considered in Theorems 2, 3, and 5, the quantity (5.2.6) is not angularly dependent.
- As we make clear in Subsect. 5.3, Christodoulou's condition for shock formation, though compelling, is not sharp. On the other hand, Alinhac's condition for shock formation, based on John's conjecture, is sharp in a sense that we make precise. For example, it is easy to see

that there exist spherically symmetric data for which Christodoulou's quantity (5.2.5) verifies[95] $\mathcal{S}[(\mathring{\Phi}, \mathring{\Phi}_0)](U) \geq 0$ for all U. For such data, the shock formation condition (5.2.7) cannot be satisfied. However, Cor. 1.4 can be extended to show that such data, when nontrivial, lead to finite-time shock formation, and Theorem 5 shows that this shock formation is in fact stable under general small perturbations. Hence, the condition (5.2.7) does not detect all shock forming data.

5.3. **Comparison of Alinhac's and Christodoulou's frameworks.** The frameworks of Alinhac and Christodoulou share some fundamental features, including the following:

- Shock formation is caused by the crossing of characteristics, as in Burgers' equations.
- Shock-forming solutions remain regular relative to adapted coordinates constructed out of a true eikonal function.
- Establishing good peeling properties plays an important role in the analysis.

However, they also differ in one significant way. The main advantage of Christodoulou's framework is that it allows one to extend the solution beyond the hypersurface $\Sigma_{T_{(Lifespan)}}$ where the first singularity occurs. In fact, his methods reveal a large portion of the maximal development of the data (see Remark 1.12 and Figure 7). The extension is made possible by the precise form of the dispersive estimates and the formulation of the well-posedness theorem (see Theorems 2 and 4) in terms of the sharp breakdown criterion $\mu \to 0$.

In contrast, Alinhac's results are valid only up the hypersurface $\Sigma_{T_{(Lifespan)}}$ where first singularity occurs, and only for data for which there is a unique first singularity point; see his non-degeneracy assumptions on the data stated just below (5.1.4). In particular, his results do not apply to the spherically symmetric data that we treated in Subsubsect. 1.3.1. This should be further contrasted with another strength of Christodoulou's framework, which is that it can be extended to show the *stability* (under general small perturbations) of John's spherically symmetric shock-formation result; see Theorem 5 by the third author. It is natural to wonder whether or not Alinhac's approach can be easily modified to recover all of the detailed features revealed by Christodoulou's framework. Unfortunately, as we describe below, the answer seems to be "no." In total, only Christodoulou's framework is suitable for setting up the important problem that we discussed in the Introduction: extending our understanding of $1D$ conservation laws to higher dimensions, including extending the solution beyond the shock.

We now highlight two merits of Alinhac's results. First, his proofs are relatively short and he was the first to show that indeed, failure of the null condition in equation (5.1.1) leads, for a set of small data, to finite-time shock formation. A second merit is that his condition on the data for shock-formation, stated just below (5.1.4), is explicitly connected to the limiting lifespan of the solution via equation (5.1.5). That is, he proved a restricted version of John's conjecture, limited only by his non-degeneracy assumptions on the data. We also note that in [30] (see also [31]), John made notable progress towards proving his conjecture by showing that the second derivatives of Φ start to grow near the limiting time. However, he never proved actual blow-up. This discussion suggests that the John-Hörmander lifespan lower bound is essentially sharp and that if the John-Hörmander quantity (5.1.4) is *non-positive* in a region, then the solution should exist beyond the standard almost global existence time in a related spacetime region. In Subsect. 5.5, we will in fact sketch a proof of this statement.

[95] Simply take data with $\mathring{\Phi}_0 - k \geq 0$ and $\partial_r \mathring{\Phi} \equiv 0$.

We now describe a few aspects of Alinhac's proof and explain the origin of its limitations. His proof is relatively short, primarily because he was able to disregard many of the intricate geometric structures present in Christodoulou's framework. As we have seen in Sect. 3, Christodoulou's framework leads to a complicated interplay between derivative loss and μ-degeneration of the generalized energy estimates. Having disregarded these features, Alinhac's approach led to linearized equations that lose derivatives relative to the background. More precisely, he set up an iteration scheme[96] to construct the blow-up solution Φ_λ together with the smooth functions v, w, and ζ, as well as the coordinates of the first blow-up time (in particular, $\tau_{(Lifespan);\lambda)}$) of Theorem 3. At each step in the iteration, his effective eikonal function corresponds to the current iterate of ζ. Hence, the ζ iterate does not correspond to a true eikonal function of the nonlinear solution. For similar reasons, his adapted vectorfields (which also vary from iterate to iterate) have small components that are transversal to the true characteristics, which led to derivative loss in the estimates relative to the previous iterate. These derivative losses turns out to be sufficiently tame,[97] and Alinhac was therefore able to handle them with a Nash-Moser scheme.

Alinhac's iteration scheme, however, fundamentally depends on a condition that he calls "(H)"; see [4, pg. 15]. Roughly speaking, condition (H) demands that each iterate has a corresponding μ that vanishes at exactly one point on its constant-time hypersurface of first blow-up; this turns out to be guaranteed when his non-degeneracy assumption on the data, stated immediately after (5.1.4), hold. On the other hand, when the maximum of the John-Hörmander quantity (5.1.4) is attained at multiple points, or perhaps even along a submanifold, condition (H) fails for the zeroth iterate and the scheme cannot continue. It is for this reason that Alinhac's framework does not recover the stability of spherically symmetric blow-up; compare with Remark 5.1. Furthermore, the condition (H) also poses a barrier to recovering the geometry of the maximal development, as Christodoulou did in his Theorem 4: Christodoulou showed that to the future of the first blow-up point, the subset of Σ_t where $\mu \to 0$ generically has dimension at least two and thus falls beyond the scope of Alinhac's iteration scheme.

5.4. **The shock-formation theorem of** [63]**.** We now state the small-data shock formation theorem from [63] for solutions to the covariant wave equation $\Box_{g(\Psi)}\Psi = 0$ in 3 space dimensions. We also briefly discuss its proof. As we have described above, the theorem extends without any significant alterations to equations of the form $\Box_{g(\Psi)}\Psi = \mathcal{N}(\Psi)(\partial\Psi, \partial\Psi)$ whenever the semilinear term $\mathcal{N}(\Psi)(\partial\Psi, \partial\Psi)$ verifies the future strong null condition of Remark 2.15 (or, if we are studying shock formation to the past, the past strong null condition of Remark 2.16); see also Remark 2.13.

Theorem 5. [63, **Theorem 22.3.1; Shock formation for nearly radial data**] *Let* $(\check{\Psi} := \Psi|_{\Sigma_{-1/2}}, \check{\Psi}_0 := \partial_t\Psi|_{\Sigma_{-1/2}})$ *be "initial" data (at time* $-1/2$*) for the covariant scalar wave equation*

$$\Box_{g(\Psi)}\Psi = 0.$$

Assume that Klainerman's classic null condition fails for the nonlinearities, that is, that the future null condition failure factor ${}^{(+)}\aleph$ *from Definition 2.4 does not completely vanish. Assume that the data are*

[96] The initial guess is "$\Phi_{\lambda=0}$", which formally solves a Burgers-type equation along each outgoing null geodesic.

[97] Interestingly, although Alinhac did not use the elliptic estimates of Remark 3.10 in his work, he *did* need to use an analog of the renormalized Raychaudhuri equation of Subsubsect. 3.4.4 in his derivation of tame L^2 estimates for his linearized equations.

nontrivial, spherically symmetric,[98] *supported in the Euclidean ball of radius* $1/2$ *centered at the origin and that* $(\check{\Psi}, \check{\Psi}_0) \in H^{25} \times H^{24}$*. Then (perhaps shrinking the amplitude of the data if necessary), a shock-formation result in analogy with Cor. 1.4 holds for the corresponding solution. Furthermore, for each shock-forming spherically symmetric (small) data pair, the shock-formation processes are* ***stable under general small perturbations*** *(without symmetry assumptions) of the data belonging to* $H^{25} \times H^{24}$ *and the Euclidean ball of radius* $1/2$*.*

Furthermore, all of the conclusions of Theorem 2 hold for the solution. In particular, its lifespan is finite precisely because μ *vanishes at one or more points and at such points, some rectangular derivative* $\partial_\nu \Psi$ *blows up.*

Remark 5.1 (**The stability of spherically symmetric blow up**)**.** An immediate corollary is that F. John's blow-up result in spherical symmetry (see Subsubsect. 1.3.1) is stable under small arbitrary perturbations. It turns out, however, that for technical reasons, it is easier to prove that shock formation occurs for spherically symmetric initial data, *even for equations that are not invariant under the Euclidean rotations.* Theorem 5 asserts that these shock formation processes are also stable under general small perturbations.

Discussion of the proof. Thanks to the difficult estimates of Theorem 2, Theorem 5 can be proved without much difficulty. We need only to show that μ vanishes in finite time. In fact, for the nearly spherically data under consideration, Theorem 5 can be proved by using arguments very similar to the ones we used in proving Cor. 1.4, given in spherical symmetry. Although there are additional terms present away from spherical symmetry, the low-order Heuristic Principle estimates of Theorem 2 can be used to show that they decay sufficiently fast and do not affect the shock formation processes in a substantial manner. See Subsect. 2.2 for some additional details.

□

5.5. **Additional connections between the results.** We now discuss some additional connections between the shock-formation results of Christodoulou, those of Alinhac, and those of [63]. Throughout this subsection $\mathring{\epsilon}$ denotes the small size of the data.

5.5.1. *Only one term can drive* μ *to* 0. The sufficient conditions on the initial data from Theorems 3, 4, and 5 that lead to finite-time shock formation are not obviously related. However, as we have noted in the previous subsections as well as our discussion of Theorem 2, shock formation is essentially driven by one term and one term only, at least in the context the three theorems mentioned above. In the case of Theorem 5, to analyze the behavior of μ, one uses the following estimate for solutions to the equations $\Box_{g(\Psi)}\Psi = 0$ (see (2.2.6)):

$$L\mu(t,u,\vartheta) = \frac{1}{2}{}^{(+)}\mathring{\aleph}(\vartheta)\breve{R}\Psi(t,u,\vartheta) + \cdots, \qquad t \geq \mathring{\epsilon}^{-1}. \tag{5.5.1}$$

In the case of the equations treated in Alinhac's Theorem 3 or in Christodoulou's Theorem 4, one uses equation (4.2.7). Thus, to guarantee shock formation, one must carry out the following two steps.

(1) Show that the term $\frac{1}{2}{}^{(+)}\mathring{\aleph}(\vartheta)\breve{R}\Psi(t,u,\vartheta)$ from (5.5.1) (or its analog in the case of the other equations) becomes negative with a sufficiently strong lower bound on its absolute value.

[98] Note that we are not assuming that the equation itself is invariant under Euclidean rotations. Hence, spherically symmetric data do not generally launch spherically symmetric solutions.

(2) Derive upper bounds for the remaining terms showing that they are dominated by the negative term.

The various conditions on the initial data stated in the three shock-formation theorems are all included for these two purposes.

In the case of Theorem 5, which applies to nearly spherically symmetric data, we explained the claim made in the previous sentence in Subsect. 2.2. In the case of Christodoulou's Theorem 4, his arguments are explained on [11, pgs. 893-903]. In Subsubsect. 5.5.2 we provide additional details on Christodoulou's arguments and explain how his conditions on the data can be modified to apply to some other equations not studied in his monograph. In Subsubsect. 5.5.3, we flesh out the connection between Alinhac's condition on the data for shock formation and the two steps described above. Furthermore, we show how to use Christodoulou's framework to relax Alinhac's non-degeneracy assumptions on the data, thus yielding a full resolution of John's conjecture; see Subsubsect. 1.2.3. We finish in Subsubsect. 5.5.4 by describing the various shock formation results from a unified perspective.

5.5.2. *Extending Christodoulou's shock-formation condition to other equations.* Under some structural assumptions on the nonlinearities, it is possible to modify Christodoulou's condition (5.2.7) so that it applies to the scalar equations $\Box_{g(\Psi)}\Psi = 0$ from Theorems 2 and 5. Such a condition provides a set of shock-generating data that differs from the nearly spherically symmetric data of Theorem 5. Specifically, his condition can be modified without difficulty to apply whenever the future null condition failure factor[99] ${}^{(+)}\aleph(\theta)$ from (2.1.40) takes on a strictly positive or negative sign for $\theta \in \mathbb{S}^2$. The reason is that Christodoulou's analysis is based on averaging over the spheres $S_{t,u}$, and his condition guarantees that the analog of $\pm\breve{R}\Psi$ eventually verifies a lower bound of the form $\gtrsim \mathring{\epsilon}(1+t)^{-1}$ (as in (2.2.1)) along *some unknown* integral curve of L. Hence, when ${}^{(+)}\aleph$ has a definite sign, the analog of Christodoulou's condition, with the correct sign, ensures that the product $\frac{1}{2}{}^{(+)}\mathring{\aleph}(\vartheta)\breve{R}\Psi(t,u,\vartheta)$ from equation (5.5.1) becomes sufficiently negative along the unknown integral curve of L; this is sufficient to guarantee that μ vanishes in finite time. Clearly, because nothing is known about the integral curve, the definite sign[100] of ${}^{(+)}\aleph(\theta)$ for all $\theta \in \mathbb{S}^2$ plays an essential role in this argument. Similarly, thanks to the observations of Subsect. 4.2, Christodoulou's condition (5.2.7) can be modified without difficulty to apply to the non-covariant equation $(g^{-1})^{\alpha\beta}(\partial\Phi)\partial_\alpha\partial_\beta\Phi = 0$ whenever the future null condition failure factor ${}^{(+)}\aleph = {}^{(+)}\aleph(\theta)$ from (4.2.2) takes on a strictly positive or negative sign for $\theta \in \mathbb{S}^2$.

5.5.3. *Eliminating Alinhac's non-degeneracy assumptions on the data.* With the more precise estimates from Christodoulou's framework, we can eliminate the non-degeneracy conditions on the data that Alinhac used to prove shock formation (see just below equation (5.1.4)). Here, we sketch a proof that small-data finite-time shock formation occurs in solutions to equation (5.1.1) if we sufficiently shrink the amplitude of the data and if John's condition holds:

$$\text{the John-Hörmander quantity (5.1.4) is positive at one point } (q_*, \theta_*). \tag{5.5.2}$$

[99] Recall that relative to standard spherical coordinates (t, r, θ) on Minkowski spacetime, we have ${}^{(+)}\aleph = {}^{(+)}\aleph(\theta)$.

[100] Recall that at $t = 0$, the Euclidean angular coordinate θ is equal to the geometric angular coordinate ϑ, and hence ${}^{(+)}\mathring{\aleph}(\vartheta) = {}^{(+)}\aleph(\theta = \vartheta)$ (see (2.2.5) and (4.2.8)).

This shows that the lifespan lower-bound of Theorem 1 is sharp in the small-data limit. An analogous sharp condition can also be stated in the cases of Christodoulou's equations (5.2.1) and the equations $\Box_{g(\Psi)}\Psi = 0$. Furthermore, we recall that by Prop. 1.2, *the condition* (5.5.2) *always holds for nontrivial compactly supported data*. We begin our sketch by first studying Alinhac's equations using Christodoulou's framework and showing how the condition (5.5.2) can be exploited. For convenience, we assume here that the data for Alinhac's equations are supported in the Euclidean unit ball Σ_0^1, and we study the solution only in regions of the form $\mathcal{M}_{t,U_0}$ (see (2.1.35e)), where $0 < U_0 < 1$ is a fixed constant.

We first explain how the behavior of the term $-\frac{1}{2}{}^{(+)}\mathring{\aleph}(\vartheta)[R^a\breve{R}\Psi_a](t,u,\vartheta)$ on the right-hand side of the relevant evolution equation (4.2.7) for μ is connected to the John-Hörmander quantity

$$\frac{1}{2}{}^{(+)}\aleph\frac{\partial^2}{\partial q^2}F[(\mathring{\Phi},\mathring{\Phi}_0)], \tag{5.5.3}$$

the (data-dependent) function appearing in Theorems 1 and 3. The first important observation is that at time $\mathring{\epsilon}^{-1}$, long before any singularity can form, we have the following estimate (whose proof we will sketch below) relative to standard spherical coordinates (t,r,θ) on Minkowski spacetime, valid in the constant-time hypersurface subset $\Sigma^{U_0}_{\mathring{\epsilon}^{-1}}$:

$$\left|-\frac{1}{2}\varrho^{(+)}\aleph R^a\breve{R}\Psi_a(t=\frac{1}{\mathring{\epsilon}},r,\theta)+\frac{1}{2}r^{(+)}\aleph\frac{\partial^2}{\partial q^2}F[(\mathring{\Phi},\mathring{\Phi}_0)](q=r-\frac{1}{\mathring{\epsilon}},r,\theta)\right|\leq C\mathring{\epsilon}^2\ln\left(\frac{1}{\mathring{\epsilon}}\right). \tag{5.5.4}$$

Hence, switching to geometric coordinates (t,u,ϑ), using (5.5.4), the estimate ${}^{(+)}\aleph(t,u,\vartheta)\approx{}^{(+)}\mathring{\aleph}(\vartheta)$ mentioned in Subsect. 2.2, and assuming that $\mathring{\epsilon}$ is sufficiently small, we see that the assumption (5.5.2) implies that there exists a point p belonging to a region[101] $\Sigma^{U_0}_{\mathring{\epsilon}^{-1}}$ such that at p, the term $-\frac{1}{2}{}^{(+)}\mathring{\aleph}(\vartheta)[R^a\breve{R}\Psi_a](t,u,\vartheta)$ from equation (4.2.7) is dominant, negative, and of order $c\mathring{\epsilon}\varrho^{-1}(t=\mathring{\epsilon}^{-1},u)\approx c\mathring{\epsilon}(1+t)^{-1}$. Hence, at time $t=\mathring{\epsilon}^{-1}$, this term causes μ to begin decaying along the integral curve of L emanating from p, at the rate $-c\mathring{\epsilon}\ln(1+t)$. Furthermore, since the Heuristic Principle estimates (see (2.1.36a)) imply that the geometric angular derivatives of the Ψ_ν have significantly died off by time $\mathring{\epsilon}^{-1}$, we can use ideas similar to the ones used to prove (2.2.4) to deduce that the product $-\frac{1}{2}{}^{(+)}\mathring{\aleph}\varrho R^a\breve{R}\Psi_a$ (note the factor of ϱ) is approximately constant along the integral curves of L for times beyond $\mathring{\epsilon}^{-1}$. In particular, along the integral curve emanating from p, the product $-\frac{1}{2}{}^{(+)}\mathring{\aleph}\varrho R^a\breve{R}\Psi_a$ remains order $-c\mathring{\epsilon}$ for times beyond $\mathring{\epsilon}^{-1}$. Hence, by equation (4.2.7), we see that along that integral curve (which corresponds to fixed u and ϑ), we have[102]

$$L\mu(t,u,\vartheta)\approx -c\mathring{\epsilon}(1+t)^{-1}. \tag{5.5.5}$$

Since $L=\frac{\partial}{\partial t}$, it easily follows from (5.5.5) that μ must vanish in finite time and a shock forms. We stress that in contrast to our proof of (2.2.4), we did not assume here that the angular derivatives of the data are even smaller than the small radial derivatives. Previously, we had made this assumption (see Remark 2.17) so that we could treat the linear term $\varrho\mu\not{\Delta}\Psi$ on the right-hand side of (2.1.26) as negligible starting from

[101] It could happen that p does not belong to a subset $\Sigma^{U_0}_{\mathring{\epsilon}^{-1}}$ with $0<U_0<1$. In this case, we would have to rework some of our constructions in order to allow us to study regions with $U_0>1$. Alternatively, we could start with data given at time $-1/2$ and supported in the Euclidean ball of radius $1/2$ centered at the origin, as in Theorem 5.

[102] A careful proof of (5.5.5) would involve possibly shrinking the amplitude of the data by data $\to \lambda\cdot$data (for λ sufficiently small) to ensure that the term $-\frac{1}{2}{}^{(+)}\mathring{\aleph}(\vartheta)[R^a\breve{R}\Psi_a](t,u,\vartheta)$ from equation (4.2.7) dominates all of the other terms.

$t = 0$; in general, we have to wait for the angular derivatives of the solution to die off before this term becomes negligible and in this sense, inequality (5.5.4) is important because it accounts for the nontrivial influence of the angular derivatives of the *data* on the product $\varrho R^a \breve{R}\Psi_a$ at time $\mathring{\epsilon}^{-1}$.

We now provide arguments that lead to a sketch of a proof of (5.5.4) and more. We begin by considering data $(\mathring{\Phi}, \mathring{\Phi}_0)$ for Alinhac's wave equation (5.1.1), but we now solve the Cauchy problem for the *linear* wave equation with that data:

$$\Box_m \Phi_{(Linear)} = 0, \tag{5.5.6}$$

$$\Phi_{(Linear)}|_{t=0} = \mathring{\Phi}, \qquad \partial_t \Phi_{(Linear)}|_{t=0} = \mathring{\Phi}_0. \tag{5.5.7}$$

We now recall that the function $\mathcal{F}[(\mathring{\Phi}, \mathring{\Phi}_0)]$ from (5.1.3) is Friedlander's radiation field for the linear wave equation (5.5.6). That is, the $r-$weighted solution $r\Phi_{(Linear)}$ to (5.5.6) is, relative to standard spherical coordinates (t, r, θ) on Minkowski spacetime, asymptotic to $\mathcal{F}[(\mathring{\Phi}, \mathring{\Phi}_0)](q = r-t, r, \theta)$. Related statements hold for various derivatives of $\Phi_{(Linear)}$. In particular, with

$$\Sigma_t' := \Sigma_t \cap \left\{ r > \frac{t}{2} > 1 \right\}, \tag{5.5.8}$$

we have the following standard estimate (see, for example, [20]):

$$\left| r\partial_r^2 \Phi_{(Linear)}(t, r, \theta) - \frac{\partial^2}{\partial q^2}|_{t,\theta} \mathcal{F}[(\mathring{\Phi}, \mathring{\Phi}_0)](q = r - t, r, \theta) \right| \leq C \frac{\mathring{\epsilon}}{1+t}, \qquad \text{along } \Sigma_t', \tag{5.5.9}$$

where $\mathring{\epsilon}$ is the (small) size of $(\mathring{\Phi}, \mathring{\Phi}_0)$.

To deduce (5.5.4), we must connect the estimate (5.5.9) back to the nonlinear problem (5.1.1). To this end, we solve both the nonlinear wave equation (5.1.1) and the linear wave equation (5.5.6) with the same data $(\mathring{\Phi}, \mathring{\Phi}_0)$. The difference $\Phi - \Phi_{(Linear)}$ solves the inhomogeneous linear wave equation with trivial data and with a source equal to the quadratic term $\left\{ (m^{-1})^{\alpha\beta} - (g^{-1})^{\alpha\beta}(\partial\Phi) \right\} \partial_\alpha \partial_\beta \Phi$. It therefore follows from the standard Minkowskian vectorfield method, as developed in [35], that $\left\| r\Phi_{(Linear)} \right\|_{C^0(\Sigma'_{\mathring{\epsilon}^{-1}})} \leq C\mathring{\epsilon}$, $\|r\Phi\|_{C^0(\Sigma'_{\mathring{\epsilon}^{-1}})} \leq C\mathring{\epsilon}$, and $\left\| r(\Phi - \Phi_{(Linear)}) \right\|_{C^0(\Sigma'_{\mathring{\epsilon}^{-1}})} \leq C\mathring{\epsilon}^2 \ln \mathring{\epsilon}^{-1}$. Furthermore, when the data are sufficiently regular, similar estimates hold for a limited number of higher (t, r, θ) coordinate derivatives of Φ and $\Phi_{(Linear)}$. In addition, it is not difficult to show using (4.2.7) that along $\Sigma_{\mathring{\epsilon}^{-1}}^{U_0}$, $|\mu - 1|$ is no larger than $C\mathring{\epsilon} \ln \mathring{\epsilon}^{-1}$. One can also show that along $\Sigma_{\mathring{\epsilon}^{-1}}^{U_0}$, R^a is equal to $-x^a/r$ plus an error term that is no larger than $C\mathring{\epsilon}^2 \ln \mathring{\epsilon}^{-1}$, and similarly, $\varrho - r$ is no larger than $C\mathring{\epsilon} \ln \mathring{\epsilon}^{-1}$. It follows that along $\Sigma_{\mathring{\epsilon}^{-1}}^{U_0}$, the nonlinear solution $r\partial_r^2 \Phi$ is equal to $\varrho R^a \breve{R}\Psi_a$ (recall that $\Psi_a = \partial_a \Phi$) up to an error term of size $\leq C\mathring{\epsilon}^2 \ln \mathring{\epsilon}^{-1}$. In total, we have the following estimate:

$$\left\| r\partial_r^2 \Phi_{(Linear)} - \varrho R^a \breve{R}\Psi_a \right\|_{C^0(\Sigma_{\mathring{\epsilon}^{-1}}^{U_0})} \leq C\mathring{\epsilon}^2 \ln \left(\frac{1}{\mathring{\epsilon}} \right). \tag{5.5.10}$$

Combining (5.5.9) and (5.5.10), we arrive at (5.5.4).

The above discussion suggests that it should be possible to show that in the relevant region, John's condition (5.5.2) is automatically implied by the shock formation criteria of Theorem 5 or Christodoulou's criteria; we do not investigate this possibility here. It would be interesting to know whether or not all

(nontrivial) compactly supported data that are small enough for Theorem 2 to apply must necessarily lead to shock formation. The proof outlined above is limited in the sense that the argument requires one to perhaps shrink the amplitude of the data in order to deduce shock formation. A hint that such a result might hold true, at least for some nonlinearities, lies in John's results [25]: for the class of equations that he addressed, finite-time breakdown of an unknown nature occurs for all such data, even without the smallness assumption.

The above discussion can also easily be extended to prove the following interesting consequence: if the John-Hörmander quantity $\frac{1}{2}{}^{(+)}\aleph(\theta)\frac{\partial^2}{\partial q^2}\digamma[(\mathring{\Phi},\mathring{\Phi}_0)](q,\theta)$ appearing in (5.1.4) is negative on a Minkowskian annular region $(q,\theta)\in[q_1,q_2:=0]\times\mathbb{S}^2\subset\Sigma_0^{U_0}$, and if $\mathring{\epsilon}$ is sufficiently small, then the corresponding solution to equation (5.1.1) exists beyond the standard lifespan lower bound $\exp\left(\frac{1}{C\mathring{\epsilon}}\right)$, in a spacetime region bounded by an inner $g-$null cone $\mathcal{C}_{u_1}$ and the outer $g-$null cone $\mathcal{C}_0$, where $u_1\approx 1-q_1$. The reason is simple: under the assumptions, up to small errors, the term $-\frac{1}{2}{}^{(+)}\mathring{\aleph}(\vartheta)[R^a\breve{R}\Psi_a](t,u,\vartheta)$ from the evolution equation (4.2.7) for μ is positive and thus works *against* shock formation. More precisely, an argument similar to the one outlined above, based on the estimate (5.5.4) and equation (4.2.7), leads to the conclusion that $\mu\geq 1-C\mathring{\epsilon}^2\ln\left(\frac{1}{\mathring{\epsilon}}\right)\ln(e+t)$, where the factor $C\mathring{\epsilon}^2\ln\left(\frac{1}{\mathring{\epsilon}}\right)$ is from the right-hand side of (5.5.4). Hence, since μ cannot vanish in the region before the time

$$\sim\exp\left(\frac{1}{C\mathring{\epsilon}^2\ln\left(\frac{1}{\mathring{\epsilon}}\right)}\right), \tag{5.5.11}$$

we see from the analog of Theorem 2 for equation (5.1.1) (as outlined in Subsect. 4.2) that blow-up cannot occur in the region of interest before the time (5.5.11). Similar results hold for Christodoulou's equations (5.2.1) and for the equations $\Box_{g(\Psi)}\Psi=0$.

5.5.4. *A unified perspective on shock-formation involving three phases.* A convenient way to summarize the above results for shock-forming data, combining the methods of Alinhac [2] and Christodoulou [11] and incorporating the perspective of John [30], is to divide the shock-formation evolution (for sufficiently small data) into the following three phases. Our use of the terminology "phases" is motivated by John's work [30], in which he was able to follow the solution nearly to the singularity, long enough to see some growth in the higher derivatives of the solution (John's "third phase"), but not long enough to see the actual singularity form.

Phase (i): On the time interval $[0,\mathring{\epsilon}^{-1}]$, the linear evolution dominates, the $\mathcal{C}_u-$tangential derivatives die off, and the important transversal derivative term behaves according to Friedlander's radiation field, as in (5.5.4).

Phase (ii): This is the period after time $\mathring{\epsilon}^{-1}$ where μ remains bounded away from 0 by a fixed amount. Starting at around time $\mathring{\epsilon}^{-1}$, if the $\mathcal{C}_u-$transversal derivative term has, at least at some points, the "right" sign and is large enough in magnitude, then μ begins to decay along the corresponding integral curves of L. Once the decay starts, it does not stop. As long as μ stays some fixed distance away from 0, the geometric L^2 estimates, such as those of Prop. 3.4, are essentially equivalent to standard L^2 estimates that could be derived via the vectorfield commutator and multiplier methods applied with Minkowski conformal Killing fields, as outlined in Subsect. 1.2.

In particular, we do not need the precision of a true eikonal function in order to understand the behavior of the solution in this phase.

Phase (iii): μ is now very close to 0 at some spacetime point. We need the full precision of the eikonal function to follow the dynamics all the way to shock formation. To close the geometric L^2 estimates without derivative loss, we need to use Christodoulou's strategy as described in Subsubsect. 3.4.3. For the shock-generating data that verify Alinhac's assumptions (which in particular ensure that at the time of first breakdown, there is only one shock point), if we are willing to allow the linearized equations to lose derivatives relative to the previous iterate, then we can also close the L^2 estimates using his Nash-Moser scheme. However, Alinhac's methods are not designed to reveal the complete structure of the maximal development of the data, including the boundary.

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APPENDIX A. SOME DETAILS ON THE WAVE EQUATIONS STUDIED IN [11]

In [11], Christodoulou considered Lagrangians of the form $\mathcal{L}(\sigma)$, where

$$\sigma := -(m^{-1})^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi, \tag{A.0.12}$$

and as usual, $(m^{-1})^{\alpha\beta} = \text{diag}(-1,1,1,1)$ is the standard inverse Minkowski metric expressed relative to rectangular coordinates. In particular, in order for the corresponding Euler-Lagrange (wave) equation to have a fluid interpretation, Christodoulou considered Lagrangians $\mathcal{L}(\sigma)$ in a regime where the following five positivity assumptions hold:

$$\sigma,\ \mathscr{L}(\sigma),\ \frac{d\mathscr{L}}{d\sigma},\ \frac{d}{d\sigma}\left(\mathscr{L}/\sqrt{\sigma}\right),\ \frac{d^2\mathscr{L}}{d\sigma^2} > 0. \tag{A.0.13}$$

The assumptions (A.0.13) imply that Φ can be interpreted as a potential function for a physically reasonable irrotational relativistic fluid with desirable properties such as having a characteristic speed (of sound) strictly in between 0 and 1 (speed of light). The corresponding Euler-Lagrange equation, which is the main equation that he studies, is

$$\partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\Phi)}\right) = -2\partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial\sigma}(m^{-1})^{\alpha\beta}\partial_\beta\Phi\right) = 0. \tag{A.0.14}$$

As we mentioned in Subsect. 5.2, Christodoulou studied perturbations of solutions of the form $\Phi = kt$, where k is a non-zero constant. These are the solutions that correspond to the nontrivial constant states in

relativistic fluid mechanics in Minkowski spacetime. When expanded relative to rectangular coordinates, (A.0.14) becomes[103]

$$(h^{-1})^{\alpha\beta}\partial_\alpha\partial_\beta\Phi = 0, \tag{A.0.15}$$

where the *reciprocal acoustical metric* h^{-1} is defined by

$$(h^{-1})^{\alpha\beta} = (h^{-1})^{\alpha\beta}(\partial\Phi) := (m^{-1})^{\alpha\beta} - F(m^{-1})^{\alpha\kappa}(m^{-1})^{\beta\lambda}\partial_\alpha\Phi\partial_\beta\Phi, \tag{A.0.16}$$

$$F = F(\upsigma) := \frac{2}{G}\frac{dG}{d\upsigma}, \tag{A.0.17}$$

$$G = G(\upsigma) := 2\frac{d\mathcal{L}}{d\upsigma}. \tag{A.0.18}$$

The characteristic speed of the background solution $\Phi = kt$ is not 1 as in our work and that of Alinhac, but rather

$$\upeta_0 = \upeta(\upsigma = k^2), \tag{A.0.19}$$

where $\upeta > 0$ is the function defined by

$$\upeta^2 = \upeta^2(\upsigma) = 1 - \upsigma H, \tag{A.0.20}$$

$$H = H(\upsigma) := \frac{F}{1+\upsigma F}. \tag{A.0.21}$$

More precisely, $\upeta$ is the *speed of sound*, and by virtue of (A.0.13), it is straightforward to show that $0 < \upeta < 1$. Also, $(h^{-1})^{00}$ is not assumed to be equal to -1 as in our work, but rather there is a lapse function $\upalpha$ defined by

$$\upalpha^{-2} = \upalpha^{-2}(\partial\Phi) := -(h^{-1})^{00}(\partial\Phi). \tag{A.0.22}$$

The proper analog of our background inverse Minkowski metric is in fact the flat inverse metric

$$(h^{-1})^{\alpha\beta}(\partial_t\Phi = k, \partial_1\Phi = \partial_2\Phi = \partial_3\Phi = 0), \tag{A.0.23}$$

which is equivalent to

$$h(\partial_t\Phi = k, \partial_1\Phi = \partial_2\Phi = \partial_3\Phi = 0) = -\upeta_0^2 dt^2 + \sum_{a=1}^{3}(dx^a)^2. \tag{A.0.24}$$

The eikonal function corresponding to the background solution is

$$u_{(Flat)} = 1 - r + \upeta_0 t, \tag{A.0.25}$$

where r is the standard Euclidean radial coordinate. The inverse foliation density corresponding to the background solution is

$$\upmu_{(Flat)} = \upeta_0. \tag{A.0.26}$$

The outgoing and ingoing null vectorfields corresponding to the background solution are

$$L_{(Flat)} = \partial_t + \upeta_0\partial_r, \qquad \underline{\breve{L}}_{(Flat)} = \upeta_0^{-1}\partial_t - \partial_r. \tag{A.0.27}$$

[103]We follow the conventions of [11] and denote the dynamic metric by $h = h(\partial\Phi)$ in this section.

The analog of the future null condition failure factor (4.2.2) is

$$\frac{dH}{d\sigma}(\sigma = k^2). \tag{A.0.28}$$

Note that unlike the general case of (4.2.2), the quantity in (A.0.28) is a constant. It was shown in [11] that $\frac{dH}{d\sigma}(\sigma = k^2)$ vanishes when $k \neq 0$ if and only if, up to trivial normalization constants,

$$\mathcal{L}(\sigma) = 1 - \sqrt{1 - \sigma}. \tag{A.0.29}$$

The Lagrangian (A.0.29) is therefore exceptional in the sense that the quadratic nonlinearities that arise in expanding its wave equation (A.0.14) around the background $\Phi = kt$ verify the null condition.

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