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EXISTENCE AND EQUILIBRATION OF GLOBAL WEAK SOLUTIONS TO KINETIC MODELS FOR DILUTE POLYMERS I: FINITELY EXTENSIBLE NONLINEAR BEAD-SPRING CHAINS

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We show the existence of global-in-time weak solutions to a general class of coupled FENE-type bead-spring chain models that arise from the kinetic theory of dilute solutions of polymeric liquids with noninteracting polymer chains. The class of models involves the unsteady incompressible Navier–Stokes equations in a bounded domain in $\mathbb{R}^d$, $d = 2$ or 3 , for the velocity and the pressure of the fluid, with an elastic extra-stress tensor appearing on the right-hand side in the momentum equation. The extra-stress tensor stems from the random movement of the polymer chains and is defined by the Kramers expression through the associated probability density function that satisfies a Fokker–Planck-type parabolic equation, a crucial feature of which is the presence of a centre-of-mass diffusion term. We require no structural assumptions on the drag term in the Fokker–Planck equation; in particular, the drag term need not be corotational. With a square-integrable and divergence-free initial velocity datum $\underline{u}_0$ for the Navier–Stokes equation and a nonnegative initial probability density function ψ_0 for the Fokker–Planck equation, which has finite relative entropy with respect to the Maxwellian M , we prove, *via* a limiting procedure on certain regularization parameters, the existence of a global-in-time weak solution $t \mapsto (\underline{u}(t), \psi(t))$ to the coupled Navier–Stokes–Fokker–Planck system, satisfying the initial condition $(\underline{u}(0), \psi(0)) = (\underline{u}_0, \psi_0)$, such that $t \mapsto \underline{u}(t)$ belongs to the classical Leray space and $t \mapsto \psi(t)$ has bounded relative entropy with respect to M and $t \mapsto \psi(t)/M$ has integrable Fisher information (w.r.t. the measure $d\mu := M(\underline{q}) d\underline{q} d\underline{x}$) over any time interval $[0, T]$, $T > 0$. If the density of body forces $\underline{f}$ on the right-hand side of the Navier–Stokes momentum equation vanishes, then a weak solution constructed as above is such that $t \mapsto (\underline{u}(t), \psi(t))$ decays exponentially in time to $(\underline{0}, M)$ in the $L^2 \times L^1$ norm, at a rate that is independent of $(\underline{u}_0, \psi_0)$ and of the centre-of-mass diffusion coefficient.

Keywords: Kinetic polymer models, FENE chain, Navier–Stokes–Fokker–Planck system.

1. Introduction

This paper establishes the existence of global-in-time weak solutions to a large class of bead-spring chain models with finitely extensible nonlinear elastic (FENE) type spring potentials, — a system of nonlinear partial differential equations that arises from the kinetic theory of dilute polymer solutions. The solvent is an incompressible, viscous, isothermal Newtonian fluid confined to a bounded open Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3 , with boundary $\partial\Omega$. For the sake of simplicity of presentation, we shall suppose that Ω has a ‘solid boundary’ $\partial\Omega$; the velocity field $\underline{u}$ will then satisfy the no-slip boundary condition $\underline{u} = \underline{0}$ on $\partial\Omega$. The polymer chains, which are suspended in the solvent, are assumed not to interact with each other. The conservation of momentum and mass equations for the solvent then have the form of the incompressible Navier–Stokes equations in which the elastic *extra-stress* tensor $\underline{\tau}$ (i.e. the polymeric part of the Cauchy stress tensor) appears as a source term:

Given $T \in \mathbb{R}_{>0}$, find $\underline{u} : (\underline{x}, t) \in \bar{\Omega} \times [0, T] \mapsto \underline{u}(\underline{x}, t) \in \mathbb{R}^d$ and $p : (\underline{x}, t) \in \Omega \times (0, T] \mapsto p(\underline{x}, t) \in \mathbb{R}$ such that

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla_{\underline{x}}) \underline{u} - \nu \Delta_{\underline{x}} \underline{u} + \nabla_{\underline{x}} p = \underline{f} + \nabla_{\underline{x}} \cdot \underline{\tau} \quad \text{in } \Omega \times (0, T], \quad (1.1a)$$

$$\nabla_{\underline{x}} \cdot \underline{u} = 0 \quad \text{in } \Omega \times (0, T], \quad (1.1b)$$

$$\underline{u} = \underline{0} \quad \text{on } \partial\Omega \times (0, T], \quad (1.1c)$$

$$\underline{u}(\underline{x}, 0) = \underline{u}_0(\underline{x}) \quad \forall \underline{x} \in \Omega. \quad (1.1d)$$

It is assumed that each of the equations above has been written in its nondimensional form; $\underline{u}$ denotes a nondimensional velocity, defined as the velocity field scaled by the characteristic flow speed U_0 ; $\nu \in \mathbb{R}_{>0}$ is the reciprocal of the Reynolds number, i.e. the ratio of the kinematic viscosity coefficient of the solvent and $L_0 U_0$, where L_0 is a characteristic length-scale of the flow; p is the nondimensional pressure and $\underline{f}$ is the nondimensional density of body forces.

In a *bead-spring chain model*, consisting of $K + 1$ beads coupled with K elastic springs to represent a polymer chain, the extra-stress tensor $\underline{\tau}$ is defined by the *Kramers expression* as a weighted average of ψ , the probability density function of the (random) conformation vector $\underline{q} := (q_1^T, \dots, q_K^T)^T \in \mathbb{R}^{Kd}$ of the chain (cf. (1.7) below), with $\underline{q}_i$ representing the d -component conformation/orientation vector of the i th spring. The Kolmogorov equation satisfied by ψ is a second-order parabolic equation, the Fokker–Planck equation, whose transport coefficients depend on the velocity field $\underline{u}$. The domain D of admissible conformation vectors $D \subset \mathbb{R}^{Kd}$ is a K -fold Cartesian product $D_1 \times \dots \times D_K$ of balanced convex open sets $D_i \subset \mathbb{R}^d$, $i = 1, \dots, K$; the term *balanced* means that $\underline{q}_i \in D_i$ if, and only if, $-\underline{q}_i \in D_i$. Hence, in particular, $\underline{0} \in D_i$, $i = 1, \dots, K$. Typically D_i is the whole of $\mathbb{R}^d$ or a bounded open d -dimensional ball centred at the origin $\underline{0} \in \mathbb{R}^d$ for each $i = 1, \dots, K$. When $K = 1$, the model is referred to as the *dumbbell model*.

Let $\mathcal{O}_i \subset [0, \infty)$ denote the image of D_i under the mapping $\underline{q}_i \in D_i \mapsto \frac{1}{2}|\underline{q}_i|^2$,

and consider the *spring potential* $U_i \in C^2(\mathcal{O}_i; \mathbb{R}_{\geq 0})$, $i = 1, \dots, K$. Clearly, $0 \in \mathcal{O}_i$. We shall suppose that $U_i(0) = 0$ and that U_i is monotonic increasing and unbounded on $\mathcal{O}_i$ for each $i = 1, \dots, K$. The elastic spring-force $\underline{F}_i : D_i \subseteq \mathbb{R}^d \rightarrow \mathbb{R}^d$ of the i th spring in the chain is defined by

$$\underline{F}_i(\underline{q}_i) = U'_i(\tfrac{1}{2}|\underline{q}_i|^2) \underline{q}_i, \quad i = 1, \dots, K. \quad (1.2)$$

Example 1.1. *In the Hookean dumbbell model $K = 1$, and the spring force is defined by $\underline{F}(\underline{q}) = \underline{q}$, with $\underline{q} \in D = \mathbb{R}^d$, corresponding to $U(s) = s$, $s \in \mathcal{O} = [0, \infty)$. This model is physically unrealistic as it admits an arbitrarily large extension.* $\diamond$

We shall therefore assume in what follows that D is a Cartesian product of K bounded open balls $D_i \subset \mathbb{R}^d$, centred at the origin $\underline{0} \in \mathbb{R}^d$, $i = 1, \dots, K$, with $K \geq 1$.

We shall further suppose that for $i = 1, \dots, K$ there exist constants $c_{ij} > 0$, $j = 1, 2, 3, 4$, and $\gamma_i > 1$ such that the (normalized) Maxwellian M_i , defined by

$$M_i(\underline{q}_i) = \frac{1}{\mathcal{Z}_i} e^{-U_i(\frac{1}{2}|\underline{q}_i|^2)}, \quad \mathcal{Z}_i := \int_{D_i} e^{-U_i(\frac{1}{2}|\underline{q}_i|^2)} d\underline{q}_i,$$

and the associated spring potential U_i satisfy

$$c_{i1} [\text{dist}(\underline{q}_i, \partial D_i)]^{\gamma_i} \leq M_i(\underline{q}_i) \leq c_{i2} [\text{dist}(\underline{q}_i, \partial D_i)]^{\gamma_i} \quad \forall \underline{q}_i \in D_i, \quad (1.3a)$$

$$c_{i3} \leq [\text{dist}(\underline{q}_i, \partial D_i)] U'_i(\tfrac{1}{2}|\underline{q}_i|^2) \leq c_{i4} \quad \forall \underline{q}_i \in D_i. \quad (1.3b)$$

The Maxwellian in the model is then defined by

$$M(\underline{q}) := \prod_{i=1}^K M_i(\underline{q}_i) \quad \forall \underline{q} := (\underline{q}_1^T, \dots, \underline{q}_K^T)^T \in D := \bigtimes_{i=1}^K D_i. \quad (1.4)$$

Observe that, for $i = 1, \dots, K$,

$$M(\underline{q}) \nabla_{\underline{q}_i} [M(\underline{q})]^{-1} = -[M(\underline{q})]^{-1} \nabla_{\underline{q}_i} M(\underline{q}) = \nabla_{\underline{q}_i} U_i(\tfrac{1}{2}|\underline{q}_i|^2) = U'_i(\tfrac{1}{2}|\underline{q}_i|^2) \underline{q}_i. \quad (1.5)$$

Since $[U_i(\tfrac{1}{2}|\underline{q}_i|^2)]^2 = (-\log M_i(\underline{q}_i) + \text{Const.})^2$, it follows from (1.3a,b) that (if $\gamma_i > 1$, as has been assumed here,)

$$\int_{D_i} \left[1 + [U_i(\tfrac{1}{2}|\underline{q}_i|^2)]^2 + [U'_i(\tfrac{1}{2}|\underline{q}_i|^2)]^2 \right] M_i(\underline{q}_i) d\underline{q}_i < \infty, \quad i = 1, \dots, K. \quad (1.6)$$

Example 1.2. *In the FENE (finitely extensible nonlinear elastic) dumbbell model $K = 1$ and the spring force is given by $\underline{F}(\underline{q}) = (1 - |\underline{q}|^2/b)^{-1} \underline{q}$, $\underline{q} \in D = B(\underline{0}, b^{\frac{1}{2}})$, corresponding to $U(s) = -\frac{b}{2} \log(1 - \frac{2s}{b})$, $s \in \mathcal{O} = [0, \frac{b}{2})$. Here $B(\underline{0}, b^{\frac{1}{2}})$ is a bounded open ball in $\mathbb{R}^d$ centred at the origin $\underline{0} \in \mathbb{R}^d$ and of fixed radius $b^{\frac{1}{2}}$, with $b > 0$. Direct calculations show that the Maxwellian M and the elastic potential U of the FENE model satisfy the conditions (1.3a,b) with $K = 1$ and $\gamma := \frac{b}{2}$ provided that $b > 2$. Thus, (1.6) also holds for $K = 1$ and $b > 2$.* $\diamond$

The governing equations of the general FENE-type bead-spring chain model with centre-of-mass diffusion are (1.1a–d), where the extra-stress tensor $\underline{\tau}$ is defined by

4 John W. Barrett and Endre Süli

the *Kramers expression*:

$$\tau(\underline{x}, t) = k \left(\sum_{i=1}^K \int_D \psi(\underline{x}, \underline{q}, t) \underline{q}_i \underline{q}_i^T U_i' \left(\frac{1}{2} |\underline{q}_i|^2 \right) d\underline{q} - \rho(\underline{x}, t) I \right), \quad (1.7)$$

with the density of polymer chains located at $\underline{x}$ at time t given by

$$\rho(\underline{x}, t) = \int_D \psi(\underline{x}, \underline{q}, t) d\underline{q}. \quad (1.8)$$

The probability density function ψ is a solution of the Fokker–Planck equation

$$\begin{aligned} \frac{\partial \psi}{\partial t} + (\underline{u} \cdot \underline{\nabla}_x) \psi + \sum_{i=1}^K \underline{\nabla}_{\underline{q}_i} \cdot \left(\underline{\sigma}(\underline{u}) \underline{q}_i \psi \right) \\ = \varepsilon \Delta_x \psi + \frac{1}{2\lambda} \sum_{i=1}^K \sum_{j=1}^K A_{ij} \underline{\nabla}_{\underline{q}_i} \cdot \left(M \underline{\nabla}_{\underline{q}_j} \left(\frac{\psi}{M} \right) \right) \quad \text{in } \Omega \times D \times (0, T], \end{aligned} \quad (1.9)$$

with $\underline{\sigma}(\underline{v}) \equiv \underline{\nabla}_x \underline{v}$, where $(\underline{\nabla}_x \underline{v})(\underline{x}, t) \in \mathbb{R}^{d \times d}$ and $\{\underline{\nabla}_x \underline{v}\}_{ij} = \frac{\partial v_i}{\partial x_j}$. The dimensionless constant $k > 0$ featuring in (1.7) is a constant multiple of the product of the Boltzmann constant k_B and the absolute temperature T . In (1.9), $\varepsilon > 0$ is the centre-of-mass diffusion coefficient defined as $\varepsilon := (\ell_0/L_0)^2/(4(K+1)\lambda)$ with $\ell_0 := \sqrt{k_B T/H}$ signifying the characteristic microscopic length-scale and $\lambda := (\zeta/4H)(U_0/L_0)$, where $\zeta > 0$ is a friction coefficient and $H > 0$ is a spring-constant. The dimensionless parameter $\lambda \in \mathbb{R}_{>0}$, called the Weissenberg number (and usually denoted by Wi), characterizes the elastic relaxation property of the fluid, and $A = (A_{ij})_{i,j=1}^K$ is the symmetric positive definite *Rouse matrix*.

Definition 1.1. The collection of equations and structural hypotheses (1.1a–d)–(1.9) will be referred to throughout the paper as model (P_ε) , or as the *general FENE-type bead-spring chain model with centre-of-mass diffusion*.

A noteworthy feature of equation (1.9) in the model (P_ε) compared to classical Fokker–Planck equations for bead-spring models in the literature is the presence of the $\underline{x}$ -dissipative centre-of-mass diffusion term $\varepsilon \Delta_x \psi$ on the right-hand side of the Fokker–Planck equation (1.9). We refer to Barrett & Süli⁷ for the derivation of (1.9) in the case of $K = 1$; see also the article by Schieber³⁴ concerning generalized dumbbell models with centre-of-mass diffusion, and the recent paper of Degond & Liu¹⁴ for a careful justification of the presence of the centre-of-mass diffusion term through asymptotic analysis. In standard derivations of bead-spring models the centre-of-mass diffusion term is routinely omitted on the grounds that it is several orders of magnitude smaller than the other terms in the equation. Indeed, when the characteristic macroscopic length-scale $L_0 \approx 1$, (for example, $L_0 = \text{diam}(\Omega)$), Bhawe, Armstrong & Brown¹¹ estimate the ratio ℓ_0^2/L_0^2 to be in the range of about 10^{-9} to 10^{-7} . However, the omission of the term $\varepsilon \Delta_x \psi$ from (1.9) in the case of a heterogeneous solvent velocity $\underline{u}(\underline{x}, t)$ is a mathematically counterproductive model

reduction. When $\varepsilon \Delta_x \psi$ is absent, (1.9) becomes a degenerate parabolic equation exhibiting hyperbolic behaviour with respect to $(\underline{x}, t)$. Since the study of weak solutions to the coupled problem requires one to work with velocity fields $\underline{u}$ that have very limited Sobolev regularity (typically $\underline{u} \in L^\infty(0, T; \underline{L}^2(\Omega)) \cap L^2(0, T; \underline{H}_0^1(\Omega))$), one is then forced into the technically unpleasant framework of hyperbolically degenerate parabolic equations with rough transport coefficients (cf. Ambrosio¹ and DiPerna & Lions¹⁶). The resulting difficulties are further exacerbated by the fact that, when D is bounded, a typical spring force $\underline{F}(\underline{q})$ for a finitely extensible model (such as FENE) explodes as $\underline{q}$ approaches ∂D ; see Example 1.2 above. For these reasons, here we shall retain the centre-of-mass diffusion term in (1.9). In order to emphasize that the positive centre-of-mass diffusion coefficient ε is *not* a mathematical artifact but the outcome of the physical derivation of the model, in Section 2 and thereafter the variables $\underline{u}$ and ψ have been labelled with the subscript ε .

We continue with a brief literature survey. Unless otherwise stated, the centre-of-mass diffusion term is absent from the model considered in the cited reference (i.e. ε is set to 0); also, in all references cited $K = 1$, i.e. a simple dumbbell model is considered rather than a bead-spring chain model.

An early contribution to the existence and uniqueness of local-in-time solutions to a family of dumbbell type polymeric flow models is due to Renardy³³. While the class of potentials $\underline{F}(\underline{q})$ considered by Renardy³³ (cf. hypotheses (F) and (F') on pp. 314–315) does include the case of Hookean dumbbells, it excludes the practically relevant case of the FENE dumbbell model (see Example 1.2 above). More recently, E, Li & Zhang¹⁹ and Li, Zhang & Zhang²⁵ have revisited the question of local existence of solutions for dumbbell models. A further development in this direction is the work of Zhang & Zhang⁴⁰, where the local existence of regular solutions to FENE-type dumbbell models has been shown. All of these papers require high regularity of the initial data. Constantin¹² considered the Navier–Stokes equations coupled to nonlinear Fokker–Planck equations describing the evolution of the probability distribution of the particles interacting with the fluid. Otto & Tzavaras³² investigated the Doi model (which is similar to a Hookean model (cf. Example 1.1 above), except that $D = S^2$) for suspensions of rod-like molecules in the dilute regime. Jourdain, Lelièvre & Le Bris²¹ studied the existence of solutions to the FENE dumbbell model in the case of a simple Couette flow. By using tools from the theory of stochastic differential equations, they showed the existence of a unique local-in-time solution to the FENE dumbbell model for $d = 2$ when the velocity field $\underline{u}$ is unidirectional and of the particular form $\underline{u}(x_1, x_2) = (u_1(x_2), 0)^T$.

In the case of Hookean dumbbells ($K = 1$), and assuming $\varepsilon = 0$, the coupled microscopic-macroscopic model described above yields, formally, taking the second moment of $\underline{q} \mapsto \psi(\underline{q}, \underline{x}, t)$, the fully macroscopic, Oldroyd-B model of viscoelastic flow. Lions & Masmoudi²⁸ have shown the existence of global-in-time weak solutions to the Oldroyd-B model in a simplified corotational setting (i.e. with $\sigma(\underline{u}) = \underline{\nabla}_x \underline{u}$ replaced by $\frac{1}{2}(\underline{\nabla}_x \underline{u} - (\underline{\nabla}_x \underline{u})^T)$) by exploiting the propagation in time of the com-

pactness of the solution (i.e. the property that if one takes a sequence of weak solutions that converges weakly and such that the corresponding sequence of initial data converges strongly, then the weak limit is also a solution) and the DiPerna–Lions¹⁶ theory of renormalized solutions to linear hyperbolic equations with nonsmooth transport coefficients. It is not known if an identical global existence result for the Oldroyd-B model also holds in the absence of the crucial assumption that the drag term is corotational. With $\varepsilon > 0$, the coupled microscopic-macroscopic model above yields, taking the appropriate moments in the case of Hookean dumbbells, a dissipative version of the Oldroyd-B model. In this sense, the Hookean dumbbell model has a macroscopic closure: it is the Oldroyd-B model when $\varepsilon = 0$, and a dissipative version of Oldroyd-B when $\varepsilon > 0$ (cf. Barrett & Süli⁷). Barrett & Boyaval⁵ have proved a global existence result for this dissipative Oldroyd-B model in two space dimensions. In contrast, the FENE model is not known to have an exact closure at the macroscopic level, though Du, Yu & Liu¹⁷ and Yu, Du & Liu³⁹ have recently considered the analysis of approximate closures of the FENE dumbbell model. Lions & Masmoudi²⁹ proved the global existence of weak solutions for the *corotational* FENE dumbbell model, once again corresponding to the case of $\varepsilon = 0$ and $K = 1$, and the Doi model, also called the rod model; see also the work of Masmoudi³⁰. Recently, Masmoudi³¹ has extended this analysis to the noncorotational case.

Previously, El-Kareh & Leal²⁰ had proposed a steady macroscopic model, with added dissipation in the equation satisfied by the conformation tensor, defined as $\tilde{A}(\underline{x}) := \int_D \underline{q} \underline{q}^T U'(\frac{1}{2}|\underline{q}|^2) \psi(\underline{x}, \underline{q}) d\underline{q}$, in order to account for Brownian motion across streamlines; the model can be thought of as an approximate macroscopic closure of a FENE-type micro-macro model with centre-of-mass diffusion.

Barrett, Schwab & Süli⁶ showed the existence of global weak solutions to the coupled microscopic-macroscopic model (1.1a–d), (1.9) with $\varepsilon = 0$, $K = 1$, an $\underline{x}$ -mollified velocity gradient in the Fokker–Planck equation and an $\underline{x}$ -mollified probability density function ψ in the Kramers expression, admitting a large class of potentials U (including the Hookean dumbbell model and general FENE-type dumbbell models); in addition to these mollifications, $\underline{u}$ in the $\underline{x}$ -convective term $(\underline{u} \cdot \underline{\nabla}_{\underline{x}}) \psi$ in the Fokker–Planck equation was also mollified. Unlike Lions & Masmoudi²⁸, the arguments in Barrett, Schwab & Süli⁶ did *not* require that the drag term $\underline{\nabla}_{\underline{q}} \cdot (\underline{\sigma}(\underline{u}) \underline{q} \psi)$ in the Fokker–Planck equation was corotational in the FENE case.

In Barrett & Süli⁷, we derived the coupled Navier–Stokes–Fokker–Planck model with centre-of-mass diffusion stated above, in the case of $K = 1$. We established the existence of global-in-time weak solutions to a mollification of the model for a general class of spring-force-potentials including in particular the FENE potential. We justified also, through a rigorous limiting process, certain classical reductions of this model appearing in the literature that exclude the centre-of-mass diffusion term from the Fokker–Planck equation on the grounds that the diffusion coefficient is small relative to other coefficients featuring in the equation. In the case of a corotational drag term we performed a rigorous passage to the limit as the mollifiers

in the Kramers expression and the drag term converge to identity operators.

In Barrett & Süli⁸ we showed the existence of global-in-time weak solutions to the general class of noncorotational FENE type dumbbell models (including the standard FENE dumbbell model) with centre-of-mass diffusion, in the case of $K = 1$, with microscopic cut-off (cf. (1.11) and (1.12) below) in the drag term

$$\nabla_q \cdot (\underline{\sigma}(u) \underline{q} \psi) = \nabla_q \cdot \left[\underline{\sigma}(u) \underline{q} M \left(\frac{\psi}{M} \right) \right]. \quad (1.10)$$

In this paper we prove the existence of global-in-time weak solutions to the model *without* cut-off or mollification, in the general case of $K \geq 1$. Since the argument is long and technical, we give a brief overview of the main steps of the proof here.

Step 1. Following the approach in Barrett & Süli⁸ and motivated by recent papers of Jourdain, Lelièvre, Le Bris & Otto²² and Lin, Liu & Zhang²⁶ (see also Arnold, Markowich, Toscani & Unterreiter⁴, and Desvillettes & Villani¹⁵) concerning the convergence of the probability density function ψ to its equilibrium value $\psi_\infty(\underline{x}, \underline{q}) := M(\underline{q})$ (corresponding to the equilibrium value $u_\infty(\underline{x}) := \underline{0}$ of the velocity field) in the absence of body forces $\underline{f}$, we observe that if ψ/M is bounded above then, for $L \in \mathbb{R}_{>0}$ sufficiently large, the drag term (1.10) is equal to

$$\nabla_q \cdot \left[\underline{\sigma}(u) \underline{q} M \beta^L \left(\frac{\psi}{M} \right) \right], \quad (1.11)$$

where $\beta^L \in C(\mathbb{R})$ is a cut-off function defined as

$$\beta^L(s) := \min(s, L). \quad (1.12)$$

More generally, in the case of $K \geq 1$, in analogy with (1.11), the drag term with cut-off is defined by $\sum_{i=1}^K \nabla_{q_i} \cdot \left(\underline{\sigma}(u) \underline{q}_i M \beta^L \left(\frac{\psi}{M} \right) \right)$. It then follows that, for $L \gg 1$, any solution ψ of (1.9), such that ψ/M is bounded above, also satisfies

$$\begin{aligned} & \frac{\partial \psi}{\partial t} + (u \cdot \nabla_x) \psi + \sum_{i=1}^K \nabla_{q_i} \cdot \left(\underline{\sigma}(u) \underline{q}_i M \beta^L \left(\frac{\psi}{M} \right) \right) \\ &= \varepsilon \Delta_x \psi + \frac{1}{2\lambda} \sum_{i=1}^K \sum_{j=1}^K A_{ij} \nabla_{q_i} \cdot \left(M \nabla_{q_j} \left(\frac{\psi}{M} \right) \right) \quad \text{in } \Omega \times D \times (0, T]. \end{aligned} \quad (1.13)$$

We impose the following boundary and initial conditions:

$$\begin{aligned} & M \left[\frac{1}{2\lambda} \sum_{j=1}^K A_{ij} \nabla_{q_j} \left(\frac{\psi}{M} \right) - \underline{\sigma}(u) \underline{q}_i \beta^L \left(\frac{\psi}{M} \right) \right] \cdot \frac{\underline{q}_i}{|\underline{q}_i|} = 0 \\ & \quad \text{on } \Omega \times \partial D_i \times \left(\bigtimes_{j=1, j \neq i}^K D_j \right) \times (0, T], \quad \text{for } i = 1, \dots, K, \end{aligned} \quad (1.14a)$$

$$\varepsilon \nabla_x \psi \cdot \underline{n} = 0 \quad \text{on } \partial \Omega \times D \times (0, T], \quad (1.14b)$$

$$\psi(\cdot, \cdot, 0) = M(\cdot) \beta^L(\psi_0(\cdot, \cdot)/M(\cdot)) \geq 0 \quad \text{on } \Omega \times D, \quad (1.14c)$$

where $\underline{q}_i$ is normal to ∂D_i , as D_i is a bounded ball centred at the origin, and $\underline{n}$ is normal to $\partial\Omega$; ψ_0 is nonnegative, defined on $\Omega \times D$, with $\int_D \psi_0(\underline{x}, \underline{q}) d\underline{q} = 1$ for a.e. $\underline{x} \in \Omega$, and assumed to have finite relative entropy with respect to the Maxwellian M ; i.e. $\int_{\Omega \times D} \psi_0(\underline{x}, \underline{q}) \log(\psi_0(\underline{x}, \underline{q})/M(\underline{q})) d\underline{q} d\underline{x} < \infty$. Clearly, if there exists $L > 0$ such that $0 \leq \psi_0 \leq LM$, then $M \beta^L(\psi_0/M) = \psi_0$. Henceforth $L > 1$ is assumed.

Definition 1.2. The coupled problem (1.1a–d), (1.7), (1.8), (1.13), (1.14a–c) will be referred to as model $(P_{\varepsilon,L})$, or as the *general FENE-type bead-spring chain model with centre-of-mass diffusion and microscopic cut-off*, with cut-off parameter $L > 1$.

In order to highlight the dependence on ε and L , in subsequent sections the solution to (1.13), (1.14a–c) will be labelled $\psi_{\varepsilon,L}$. Due to the coupling of (1.13) to (1.1a) through (1.7), the velocity and the pressure will also depend on ε and L and we shall therefore denote them in subsequent sections by $u_{\varepsilon,L}$ and $p_{\varepsilon,L}$.

The cut-off β^L has a convenient property: the couple (u_∞, ψ_∞) , defined by $u_\infty(\underline{x}) := \underline{0}$ and $\psi_\infty(\underline{x}, \underline{q}) := M(\underline{q})$, is still an equilibrium solution of (1.1a–d) with $\underline{f} = \underline{0}$, (1.7), (1.8), (1.13), (1.14a–c) for all $L > 0$. Thus, unlike the truncation of the (unbounded) potential proposed in El-Kareh & Leal²⁰, the introduction of the cut-off function β^L into the Fokker–Planck equation (1.9) does not alter the equilibrium solution (u_∞, ψ_∞) of the original Navier–Stokes–Fokker–Planck system. In addition, the boundary conditions for ψ on $\partial\Omega \times D \times (0, T]$ and $\Omega \times \partial D \times (0, T]$ ensure that $\int_D \psi(\underline{x}, \underline{q}, t) d\underline{q} = \int_D \psi(\underline{x}, \underline{q}, 0) d\underline{q}$ for a.e. $\underline{x} \in \Omega$ and a.e. $t \in \mathbb{R}_{\geq 0}$.

Step 2. Ideally, one would like to pass to the limit $L \rightarrow \infty$ in problem $(P_{\varepsilon,L})$ to deduce the existence of solutions to (P_ε) . Unfortunately, such a direct attack at the problem is (except in the special case of $d = 2$, or in the absence of convection terms from the model,) fraught with technical difficulties. Instead, we shall first (semi)discretize problem $(P_{\varepsilon,L})$ by an implicit Euler scheme with respect to t , with step size Δt ; this then results in a time-discrete version $(P_{\varepsilon,L}^{\Delta t})$ of $(P_{\varepsilon,L})$. By using Schauder’s fixed point theorem, we will show in Section 3 the existence of solutions to $(P_{\varepsilon,L}^{\Delta t})$. In the course of the proof, for technical reasons, a further cut-off, now from below, is required, with a cut-off parameter $\delta \in (0, 1)$, which we shall let pass to 0 to complete the proof of existence of solutions to $(P_{\varepsilon,L}^{\Delta t})$ in the limit of $\delta \rightarrow 0_+$ (cf. Section 3). Ultimately, of course, our aim is to show existence of weak solutions to the general FENE-type bead-spring chain model with centre-of-mass diffusion, (P_ε) , and that demands passing to the limits $\Delta t \rightarrow 0_+$ and $L \rightarrow \infty$; this then brings us to the next step in our argument.

Step 3. We shall link the time step Δt to the cut-off parameter $L > 1$ by demanding that $\Delta t = o(L^{-1})$, as $L \rightarrow \infty$, so that the only parameter in the problem $(P_{\varepsilon,L}^{\Delta t})$ is the cut-off parameter (the centre-of-mass diffusion parameter ε being fixed). By using special energy estimates, based on testing the Fokker–Planck equation in $(P_{\varepsilon,L}^{\Delta t})$ with the derivative of the relative entropy with respect to the Maxwellian of the general FENE-type bead-spring chain model, we show that $u_{\varepsilon,L}^{\Delta t}$ can be bounded, independent of L . Specifically $u_{\varepsilon,L}^{\Delta t}$ is bounded in the norm of the classical Leray space, independent of L ; also, the L^∞ norm in time of the relative

entropy of $\psi_{\varepsilon,L}^{\Delta t}$ and the L^2 norm in time of the Fisher information of $\hat{\psi}_{\varepsilon,L}^{\Delta t} := \psi_{\varepsilon,L}^{\Delta t}/M$ are bounded, independent of L . We then use these L -independent bounds on the relative entropy and the Fisher information to derive L -independent bounds on the time-derivatives of $u_{\varepsilon,L}^{\Delta t}$ and $\hat{\psi}_{\varepsilon,L}^{\Delta t}$ in very weak, negative-order Sobolev norms.

Step 4. The collection of L -independent bounds from Step 3 then enables us to extract a weakly convergent subsequence of solutions to problem $(P_{\varepsilon,L}^{\Delta t})$ as $L \rightarrow \infty$. We then apply a general compactness result in seminormed sets due to Dubinskii¹⁸, which furnishes strong convergence of a subsequence of solutions $(u_{\varepsilon,L_k}^{\Delta t}, \psi_{\varepsilon,L_k}^{\Delta t})$ to $(P_{\varepsilon,L}^{\Delta t})$ with $\Delta t = o(L^{-1})$ as $L \rightarrow \infty$, in $L^2(0, T; L^2(\Omega)) \times L^p(0, T; L^1(\Omega \times D))$ for any $p > 1$. A crucial observation is that the set of functions with finite Fisher information is not a linear space; therefore, typical Aubin–Lions–Simon type compactness results (see, for example, Simon³⁵) do not work in our context; however, Dubinskii’s compactness theorem, which applies to seminormed sets in the sense of Dubinskii, does, enabling us to pass to the limit with the microscopic cut-off parameter L in the model $(P_{\varepsilon,L}^{\Delta t})$, with $\Delta t = o(L^{-1})$, as $L \rightarrow \infty$, to finally deduce the existence of a weak solution to the general FENE-type bead-spring chain model with centre-of-mass diffusion, (P_{ε}) .

The paper is structured as follows. We begin, in Section 2, by stating $(P_{\varepsilon,L})$, the coupled Navier–Stokes–Fokker–Planck system with centre-of-mass diffusion and microscopic cut-off for a general class of FENE-type spring potentials. In Section 3 we establish the existence of solutions to the time-discrete problem $(P_{\varepsilon,L}^{\Delta t})$. In Section 4 we derive a set of L -independent bounds on $u_{\varepsilon,L}^{\Delta t}$ in the classical Leray space, together with L -independent bounds on the relative entropy of $\psi_{\varepsilon,L}^{\Delta t}$ and Fisher information of $\hat{\psi}_{\varepsilon,L}^{\Delta t}$. We then use these L -independent bounds on spatial norms to obtain L -independent bounds on very weak norms of time-derivatives of $u_{\varepsilon,L}^{\Delta t}$ and $\hat{\psi}_{\varepsilon,L}^{\Delta t}$. Section 5 is concerned with the application of Dubinskii’s theorem to our problem; and the extraction of a strongly convergent subsequence, which we shall then use in Section 6 to pass to the limit with the cut-off parameter L in problem $(P_{\varepsilon,L}^{\Delta t})$, with $\Delta t = o(L^{-1})$, as $L \rightarrow \infty$, to deduce the existence of a weak solution $(u_{\varepsilon}, \psi_{\varepsilon} := M\hat{\psi}_{\varepsilon})$ to problem (P_{ε}) , the general FENE-type bead-spring chain model with centre-of-mass diffusion. Finally, in Section 7, we show using a logarithmic Sobolev inequality and the Csiszár–Kullback inequality that, when $f \equiv 0$, global weak solutions $t \mapsto (u_{\varepsilon}(t), \psi_{\varepsilon}(t))$ thus constructed decay exponentially in time to $(0, M)$, at a rate that is independent of the initial data for the Navier–Stokes and Fokker–Planck equations and of the centre-of-mass diffusion coefficient ε .

We shall operate within Maxwellian-weighted Sobolev spaces, which provide the natural functional-analytic framework for the problem; we refer to the Appendix of the extended version of this paper⁹ for technical details on these.

For an analogous set of existence and equilibration results for weak solutions of Hookean-type bead-spring chain models for dilute polymers, we refer to Part II of the present paper¹⁰.

2. The polymer model $(\mathbf{P}_{\varepsilon,L})$

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with a Lipschitz-continuous boundary $\partial\Omega$, and suppose that the set $D := D_1 \times \cdots \times D_K$ of admissible conformation vectors $\underline{q} := (q_1^T, \dots, q_K^T)^T$ in (1.9) is such that D_i , $i = 1, \dots, K$, is an open ball in $\mathbb{R}^d$, $d = 2$ or 3 , centred at the origin with boundary ∂D_i and radius $\sqrt{b_i}$, $b_i > 2$; let

$$\partial D := \partial D := \bigcup_{i=1}^K \left[\partial D_i \times \left(\times_{j=1, j \neq i}^K D_j \right) \right]. \quad (2.1)$$

Collecting (1.1a–d), (1.7), and (1.9), we then consider the following initial-boundary-value problem, dependent on the parameter $L > 1$. As has been already emphasized in the Introduction, the centre-of-mass diffusion coefficient $\varepsilon > 0$ is a physical parameter and is regarded as being fixed, although we systematically highlight its presence in the model through our subscript notation.

$(\mathbf{P}_{\varepsilon,L})$ Find $\underline{u}_{\varepsilon,L} : (\underline{x}, t) \in \bar{\Omega} \times [0, T] \mapsto \underline{u}_{\varepsilon,L}(\underline{x}, t) \in \mathbb{R}^d$ and $p_{\varepsilon,L} : (\underline{x}, t) \in \Omega \times (0, T] \mapsto p_{\varepsilon,L}(\underline{x}, t) \in \mathbb{R}$ such that

$$\frac{\partial \underline{u}_{\varepsilon,L}}{\partial t} + (\underline{u}_{\varepsilon,L} \cdot \nabla_x) \underline{u}_{\varepsilon,L} - \nu \Delta_x \underline{u}_{\varepsilon,L} + \nabla_x p_{\varepsilon,L} = \underline{f} + \nabla_x \cdot \underline{\tau}(\psi_{\varepsilon,L}) \quad \text{in } \Omega \times (0, T], \quad (2.2a)$$

$$\nabla_x \cdot \underline{u}_{\varepsilon,L} = 0 \quad \text{in } \Omega \times (0, T], \quad (2.2b)$$

$$\underline{u}_{\varepsilon,L} = \underline{0} \quad \text{on } \partial\Omega \times (0, T], \quad (2.2c)$$

$$\underline{u}_{\varepsilon,L}(\underline{x}, 0) = \underline{u}_0(\underline{x}) \quad \forall \underline{x} \in \Omega, \quad (2.2d)$$

where $\psi_{\varepsilon,L} : (\underline{x}, \underline{q}, t) \in \bar{\Omega} \times \bar{D} \times [0, T] \mapsto \psi_{\varepsilon,L}(\underline{x}, \underline{q}, t) \in \mathbb{R}$, and $\underline{\tau}(\psi_{\varepsilon,L}) : (\underline{x}, t) \in \Omega \times (0, T] \mapsto \underline{\tau}(\psi_{\varepsilon,L})(\underline{x}, t) \in \mathbb{R}^{d \times d}$ is the symmetric extra-stress tensor defined as

$$\underline{\tau}(\psi_{\varepsilon,L}) := k \left(\sum_{i=1}^K \underline{C}_i(\psi_{\varepsilon,L}) \right) - k \rho(\psi_{\varepsilon,L}) \underline{I}. \quad (2.3)$$

Here $k \in \mathbb{R}_{>0}$, $\underline{I}$ is the unit $d \times d$ tensor,

$$\underline{C}_i(\psi_{\varepsilon,L})(\underline{x}, t) := \int_D \psi_{\varepsilon,L}(\underline{x}, \underline{q}, t) U'_i(\tfrac{1}{2} |\underline{q}_i|^2) \underline{q}_i \underline{q}_i^T d\underline{q}, \quad \text{and} \quad (2.4a)$$

$$\rho(\psi_{\varepsilon,L})(\underline{x}, t) := \int_D \psi_{\varepsilon,L}(\underline{x}, \underline{q}, t) d\underline{q}. \quad (2.4b)$$

The Fokker–Planck equation with microscopic cut-off satisfied by $\psi_{\varepsilon,L}$ is:

$$\begin{aligned} \frac{\partial \psi_{\varepsilon,L}}{\partial t} + (\underline{u}_{\varepsilon,L} \cdot \nabla_x) \psi_{\varepsilon,L} + \sum_{i=1}^K \nabla_{\underline{q}_i} \cdot \left[\sigma(\underline{u}_{\varepsilon,L}) \underline{q}_i M \beta^L \left(\frac{\psi_{\varepsilon,L}}{M} \right) \right] \\ = \varepsilon \Delta_x \psi_{\varepsilon,L} + \frac{1}{2\lambda} \sum_{i=1}^K \sum_{j=1}^K A_{ij} \nabla_{\underline{q}_i} \cdot \left(M \nabla_{\underline{q}_j} \left(\frac{\psi_{\varepsilon,L}}{M} \right) \right) \quad \text{in } \Omega \times D \times (0, T]. \end{aligned} \quad (2.5)$$

Here, for a given $L > 1$, $\beta^L \in C(\mathbb{R})$ is defined by (1.12), $\underline{\sigma}(\underline{v}) \equiv \nabla_x \underline{v}$, and

$A \in \mathbb{R}^{K \times K}$ is symmetric positive definite with smallest eigenvalue $a_0 \in \mathbb{R}_{>0}$. (2.6)

We impose the following boundary and initial conditions:

$$M \left[\frac{1}{2\lambda} \sum_{j=1}^K A_{ij} \nabla_{q_j} \left(\frac{\psi_{\varepsilon,L}}{M} \right) - \sigma(u_{\varepsilon,L}) q_i \beta^L \left(\frac{\psi_{\varepsilon,L}}{M} \right) \right] \cdot \frac{q_i}{|q_i|} = 0$$

$$\text{on } \Omega \times \partial D_i \times \left(\bigtimes_{j=1, j \neq i}^K D_j \right) \times (0, T], \quad i = 1, \dots, K, \quad (2.7a)$$

$$\varepsilon \nabla_x \psi_{\varepsilon,L} \cdot \underline{n} = 0 \quad \text{on } \partial\Omega \times D \times (0, T], \quad (2.7b)$$

$$\psi_{\varepsilon,L}(\cdot, \cdot, 0) = M(\cdot) \beta^L(\psi_0(\cdot, \cdot)/M(\cdot)) \geq 0 \quad \text{on } \Omega \times D, \quad (2.7c)$$

where $\underline{n}$ is the unit outward normal to $\partial\Omega$. The boundary conditions for $\psi_{\varepsilon,L}$ on $\partial\Omega \times D \times (0, T]$ and $\Omega \times \partial D \times (0, T]$ have been chosen so as to ensure that

$$\int_D \psi_{\varepsilon,L}(x, q, t) dq = \int_D \psi_{\varepsilon,L}(x, q, 0) dq \quad \forall (x, t) \in \Omega \times (0, T]. \quad (2.8)$$

Henceforth, we shall write $\widehat{\psi}_{\varepsilon,L} := \psi_{\varepsilon,L}/M$, $\widehat{\psi}_0 := \psi_0/M$. Thus, for example, (2.7c) in terms of this compact notation becomes: $\widehat{\psi}_{\varepsilon,L}(\cdot, \cdot, 0) = \beta^L(\widehat{\psi}_0(\cdot, \cdot))$ on $\Omega \times D$.

The notation $|\cdot|$ will be used to signify one of the following. When applied to a real number x , $|x|$ will denote the absolute value of the number x ; when applied to a vector $\underline{v}$, $|\underline{v}|$ will stand for the Euclidean norm of the vector $\underline{v}$; and, when applied to a square matrix A , $|A|$ will signify the Frobenius norm, $[\text{tr}(A^T A)]^{\frac{1}{2}}$, of the matrix A , where, for a square matrix B , $\text{tr}(B)$ denotes the trace of B .

3. Existence of a solution to the discrete-in-time problem

Let

$$\underline{H} := \{\underline{w} \in \underline{L}^2(\Omega) : \nabla_x \cdot \underline{w} = 0\} \quad \text{and} \quad \underline{V} := \{\underline{w} \in \underline{H}_0^1(\Omega) : \nabla_x \cdot \underline{w} = 0\}, \quad (3.1)$$

where the divergence operator $\nabla_x \cdot$ is to be understood in the sense of distributions on Ω . Let $\underline{V}'$ be the dual of $\underline{V}$. Let $\underline{S} : \underline{V}' \rightarrow \underline{V}$ be such that $\underline{S}\underline{v}$ is the unique solution to the Helmholtz–Stokes problem

$$\int_{\Omega} \underline{S}\underline{v} \cdot \underline{w} d\underline{x} + \int_{\Omega} \nabla_x (\underline{S}\underline{v}) : \nabla_x \underline{w} d\underline{x} = \langle \underline{v}, \underline{w} \rangle_V \quad \forall \underline{w} \in \underline{V}, \quad (3.2)$$

where $\langle \cdot, \cdot \rangle_V$ denotes the duality pairing between $\underline{V}'$ and $\underline{V}$. We note that

$$\langle \underline{v}, \underline{S}\underline{v} \rangle_V = \|\underline{S}\underline{v}\|_{H^1(\Omega)}^2 \quad \forall \underline{v} \in \underline{V}', \quad (3.3)$$

and $\|\underline{S} \cdot\|_{H^1(\Omega)}$ is a norm on $\underline{V}'$. More generally, let $\underline{V}_{\sigma}$ denote the closure of the set of all divergence-free $\underline{C}_0^{\infty}(\Omega)$ functions in the norm of $\underline{H}_0^1(\Omega) \cap \underline{H}^{\sigma}(\Omega)$, $\sigma \geq 1$, equipped with the Hilbert space norm, denoted by $\|\cdot\|_{V_{\sigma}}$, inherited from $\underline{H}^{\sigma}(\Omega)$, and let $\underline{V}'_{\sigma}$ signify the dual space of $\underline{V}_{\sigma}$, with duality pairing $\langle \cdot, \cdot \rangle_{V_{\sigma}}$. As Ω is a bounded Lipschitz domain, we have that $\underline{V}_1 = \underline{V}$ (cf. Temam³⁷, Ch. 1, Thm. 1.6). Similarly, $\langle \cdot, \cdot \rangle_{H_0^1(\Omega)}$ will denote the duality pairing between $(\underline{H}_0^1(\Omega))'$ and $\underline{H}_0^1(\Omega)$.

The norm on $(H_0^1(\Omega))'$ will be that induced from taking $\|\nabla_x \cdot\|_{L^2(\Omega)}$ to be the norm on $H_0^1(\Omega)$.

For later purposes, we recall the following well-known Gagliardo–Nirenberg inequality. Let $r \in [2, \infty)$ if $d = 2$, and $r \in [2, 6]$ if $d = 3$ and $\theta = d(\frac{1}{2} - \frac{1}{r})$. Then, there is a constant $C = C(\Omega, r, d)$, such that, for all $\eta \in H^1(\Omega)$:

$$\|\eta\|_{L^r(\Omega)} \leq C \|\eta\|_{L^2(\Omega)}^{1-\theta} \|\eta\|_{H^1(\Omega)}^\theta. \quad (3.4)$$

Let $\mathcal{F} \in C(\mathbb{R}_{>0})$ be defined by $\mathcal{F}(s) := s(\log s - 1) + 1$, $s > 0$. As $\lim_{s \rightarrow 0+} \mathcal{F}(s) = 1$, the function $\mathcal{F}$ can be considered to be defined and continuous on $[0, \infty)$, where it is a nonnegative, strictly convex function with $\mathcal{F}(1) = 0$. We assume the following:

$$\begin{aligned} \partial\Omega \in C^{0,1}; \quad u_0 \in H; \quad \widehat{\psi}_0 := \frac{\psi_0}{M} \geq 0 \quad \text{a.e. on } \Omega \times D \quad \text{with} \\ \mathcal{F}(\widehat{\psi}_0) \in L_M^1(\Omega \times D) \quad \text{and} \quad \int_D M(q) \widehat{\psi}_0(x, q) \, dq = 1 \quad \text{for a.e. } x \in \Omega; \\ \gamma_i > 1, \quad i = 1, \dots, K \quad \text{in } (1.3a, b); \quad \text{and} \quad f \in L^2(0, T; (H_0^1(\Omega))'). \end{aligned} \quad (3.5)$$

Here, $L_M^p(\Omega \times D)$, for $p \in [1, \infty)$, denotes the Maxwellian-weighted L^p space over $\Omega \times D$ with norm

$$\|\widehat{\varphi}\|_{L_M^p(\Omega \times D)} := \left\{ \int_{\Omega \times D} M |\widehat{\varphi}|^p \, dq \, dx \right\}^{\frac{1}{p}}.$$

Similarly, we introduce $L_M^p(D)$, the Maxwellian-weighted L^p space over D . Letting

$$\|\widehat{\varphi}\|_{H_M^1(\Omega \times D)} := \left\{ \int_{\Omega \times D} M \left[|\widehat{\varphi}|^2 + |\nabla_x \widehat{\varphi}|^2 + |\nabla_q \widehat{\varphi}|^2 \right] \, dq \, dx \right\}^{\frac{1}{2}}, \quad (3.6)$$

we then set

$$\widehat{X} \equiv H_M^1(\Omega \times D) := \left\{ \widehat{\varphi} \in L_{\text{loc}}^1(\Omega \times D) : \|\widehat{\varphi}\|_{H_M^1(\Omega \times D)} < \infty \right\}. \quad (3.7)$$

It is shown in Appendix C in the extended version of this paper⁹ that

$$C^\infty(\overline{\Omega \times D}) \text{ is dense in } \widehat{X}. \quad (3.8)$$

We have from Sobolev embedding that

$$H^1(\Omega; L_M^2(D)) \hookrightarrow L^s(\Omega; L_M^2(D)), \quad (3.9)$$

where $s \in [1, \infty)$ if $d = 2$ or $s \in [1, 6]$ if $d = 3$. Similarly to (3.4) we have, with r and θ as there, that there is a constant C , depending only on Ω , r and d , such that

$$\|\widehat{\varphi}\|_{L^r(\Omega; L_M^2(D))} \leq C \|\widehat{\varphi}\|_{L^2(\Omega; L_M^2(D))}^{1-\theta} \|\widehat{\varphi}\|_{H^1(\Omega; L_M^2(D))}^\theta \quad \forall \widehat{\varphi} \in H^1(\Omega; L_M^2(D)). \quad (3.10)$$

In addition, we note that the embeddings

$$H_M^1(D) \hookrightarrow L_M^2(D), \quad (3.11a)$$

$$H_M^1(\Omega \times D) \equiv L^2(\Omega; H_M^1(D)) \cap H^1(\Omega; L_M^2(D)) \hookrightarrow L_M^2(\Omega \times D) \equiv L^2(\Omega; L_M^2(D)) \quad (3.11b)$$

are compact if $\gamma_i \geq 1$, $i = 1, \dots, K$, in (1.3a,b); see Appendix D in the extended version of this paper⁹.

Let $\hat{X}'$ be the dual space of $\hat{X}$ with $L_M^2(\Omega \times D)$ being the pivot space. Then, similarly to (3.2), let $\mathcal{G} : \hat{X}' \rightarrow \hat{X}$ be such that $\mathcal{G}\hat{\eta}$ is the unique solution of

$$\begin{aligned} \int_{\Omega \times D} M \left[(\mathcal{G}\hat{\eta}) \hat{\varphi} + \nabla_q (\mathcal{G}\hat{\eta}) \cdot \nabla_q \hat{\varphi} + \nabla_x (\mathcal{G}\hat{\eta}) \cdot \nabla_x \hat{\varphi} \right] dq dx \\ = \langle M\hat{\eta}, \hat{\varphi} \rangle_{\hat{X}} \quad \forall \hat{\varphi} \in \hat{X}, \end{aligned} \quad (3.12)$$

where $\langle M\cdot, \cdot \rangle_{\hat{X}}$ is the duality pairing between $\hat{X}'$ and $\hat{X}$. Then, as in (3.3),

$$\langle M\hat{\eta}, \mathcal{G}\hat{\eta} \rangle_{\hat{X}} = \|\mathcal{G}\hat{\eta}\|_{\hat{X}}^2 \quad \forall \hat{\eta} \in \hat{X}', \quad (3.13)$$

and $\|\mathcal{G}\cdot\|_{\hat{X}}$ is a norm on $\hat{X}'$.

We recall the Aubin–Lions–Simon compactness theorem, see, e.g., Temam³⁷ and Simon³⁵. Let $\mathcal{B}_0$, $\mathcal{B}$ and $\mathcal{B}_1$ be Banach spaces, $\mathcal{B}_i$, $i = 0, 1$, reflexive, with a compact embedding $\mathcal{B}_0 \hookrightarrow \mathcal{B}$ and a continuous embedding $\mathcal{B} \hookrightarrow \mathcal{B}_1$. Then, for $\alpha_i > 1$, $i = 0, 1$, the embedding

$$\{ \eta \in L^{\alpha_0}(0, T; \mathcal{B}_0) : \frac{\partial \eta}{\partial t} \in L^{\alpha_1}(0, T; \mathcal{B}_1) \} \hookrightarrow L^{\alpha_0}(0, T; \mathcal{B}) \quad (3.14)$$

is compact.

Throughout we will assume that (3.5) hold, so that (1.6) and (3.11a,b) hold. We note for future reference that (2.4a) and (1.6) yield that, for $\hat{\varphi} \in L_M^2(\Omega \times D)$,

$$\int_{\Omega} |C_i(M\hat{\varphi})|^2 dx = \int_{\Omega} \left| \int_D M \hat{\varphi} U'_i q_i q_i^T dq \right|^2 dx \leq C \left(\int_{\Omega \times D} M |\hat{\varphi}|^2 dq dx \right), \quad (3.15)$$

for $i = 1, \dots, K$, where C is a positive constant.

We establish a simple integration-by-parts formula.

Lemma 3.1. *Let $\hat{\varphi} \in H_M^1(D)$ and suppose that $B \in \mathbb{R}^{d \times d}$ is a square matrix such that $\text{tr}(B) = 0$; then,*

$$\int_D M \sum_{i=1}^K (B q_i) \cdot \nabla_{q_i} \hat{\varphi} dq = \int_D M \hat{\varphi} \sum_{i=1}^K q_i q_i^T U'_i(\frac{1}{2}|q_i|^2) : B dq. \quad (3.16)$$

Proof. By Theorem C.1 in Appendix C of Barrett & Süli⁹, the set $C^\infty(\overline{D})$ is dense in $H_M^1(D)$; hence, for any $\hat{\varphi} \in H_M^1(D)$ there exists a sequence $\{\hat{\varphi}_n\}_{n \geq 0} \subset C^\infty(\overline{D})$ converging to $\hat{\varphi}$ in $H_M^1(D)$. As $M \in C^2(\overline{D})$ and vanishes on ∂D , the same is true of each of the functions $M\hat{\varphi}_n$, $n \geq 1$. On replacing $\hat{\varphi}$ by $\hat{\varphi}_n$ on both sides of (3.16), the resulting identity is easily verified by using the classical divergence theorem for smooth functions, noting (1.5), that $M\hat{\varphi}_n$ vanishes on ∂D and that $\text{tr}(B) = 0$. Then, (3.16) itself follows by letting $n \rightarrow \infty$, recalling the definition of the norm in $H_M^1(D)$ and hypothesis (1.6). $\square$

We now formulate our discrete-in-time approximation of problem $(P_{\varepsilon, L})$ for fixed parameters $\varepsilon \in (0, 1]$ and $L > 1$. For any $T > 0$ and $N \geq 1$, let $N \Delta t = T$ and

$t_n = n \Delta t$, $n = 0, \dots, N$. To prove existence of a solution under minimal smoothness requirements on the initial datum u_0 (recall (3.5)), we introduce $\underline{u}^0 = \underline{u}^0(\Delta t) \in \underline{V}$ such that

$$\int_{\Omega} \left[\underline{u}^0 \cdot \underline{v} + \Delta t \nabla_x \underline{u}^0 : \nabla_x \underline{v} \right] dx = \int_{\Omega} \underline{u}_0 \cdot \underline{v} dx \quad \forall \underline{v} \in \underline{V}; \quad (3.17)$$

and so

$$\int_{\Omega} [|\underline{u}^0|^2 + \Delta t |\nabla_x \underline{u}^0|^2] d\mathbf{x} \leq \int_{\Omega} |\underline{u}_0|^2 d\mathbf{x} \leq C. \quad (3.18)$$

In addition, we have that $\underline{u}^0$ converges to $\underline{u}_0$ weakly in $\underline{H}$ in the limit of $\Delta t \rightarrow 0_+$. For $p \in [1, \infty)$, let

$$\hat{Z}_p := \{ \hat{\varphi} \in L_M^p(\Omega \times D) : \hat{\varphi} \geq 0 \text{ a.e. on } \Omega \times D \text{ and } \int_D M(q) \hat{\varphi}(x, q) dq \leq 1 \text{ for a.e. } x \in \Omega \}. \quad (3.19)$$

Analogously to defining $\underline{u}^0$ for a given initial velocity field $\underline{u}_0$, we shall assign a certain ‘smoothed’ initial datum, $\hat{\psi}^0 = \hat{\psi}^0(\Delta t)$, to the initial datum $\hat{\psi}_0$. The definition of $\hat{\psi}^0$ is delicate; it will be given in Section 6. All we need to know for now is that there exists a $\hat{\psi}^0$, independent of the cut-off parameter L , such that:

$$\hat{\psi}^0 \in \hat{Z}_1; \quad \left\{ \begin{array}{l} \mathcal{F}(\hat{\psi}^0) \in L_M^1(\Omega \times D); \\ \sqrt{\hat{\psi}^0} \in H_M^1(\Omega \times D); \end{array} \right. \quad \int_{\Omega \times D} M \mathcal{F}(\hat{\psi}^0) dq dx \leq \int_{\Omega \times D} M \mathcal{F}(\hat{\psi}_0) dq dx. \quad (3.20)$$

The proofs of these properties will be given in Lemma 6.2 in Section 6. It follows from (3.20) and (1.12) that $\beta^L(\hat{\psi}^0) \in \hat{Z}_2$; in fact, $\beta^L(\hat{\psi}^0) \in L^\infty(\Omega \times D) \cap H_M^1(\Omega \times D)$.

Our discrete-in-time approximation of $(P_{\varepsilon, L})$ is then defined as follows.

(P $_{\varepsilon, L}^{\Delta t}$) Let $\underline{u}_{\varepsilon, L}^0 := \underline{u}^0 \in \underline{V}$ and $\hat{\psi}_{\varepsilon, L}^0 := \beta^L(\hat{\psi}^0) \in \hat{Z}_2$. Then, for $n = 1, \dots, N$, given $(\underline{u}_{\varepsilon, L}^{n-1}, \hat{\psi}_{\varepsilon, L}^{n-1}) \in \underline{V} \times \hat{Z}_2$, find $(\underline{u}_{\varepsilon, L}^n, \hat{\psi}_{\varepsilon, L}^n) \in \underline{V} \times (\hat{X} \cap \hat{Z}_2)$ such that

$$\begin{aligned} \int_{\Omega} \left[\frac{\underline{u}_{\varepsilon, L}^n - \underline{u}_{\varepsilon, L}^{n-1}}{\Delta t} + (\underline{u}_{\varepsilon, L}^{n-1} \cdot \nabla_x) \underline{u}_{\varepsilon, L}^n \right] \cdot \underline{w} dx + \nu \int_{\Omega} \nabla_x \underline{u}_{\varepsilon, L}^n : \nabla_x \underline{w} dx \\ = \langle \underline{f}^n, \underline{w} \rangle_{H_0^1(\Omega)} - k \sum_{i=1}^K \int_{\Omega} C_i(M \hat{\psi}_{\varepsilon, L}^n) : \nabla_x \underline{w} dx \quad \forall \underline{w} \in \underline{V}, \end{aligned} \quad (3.21a)$$

$$\begin{aligned} \int_{\Omega \times D} M \frac{\hat{\psi}_{\varepsilon, L}^n - \hat{\psi}_{\varepsilon, L}^{n-1}}{\Delta t} \hat{\varphi} dq dx \\ + \int_{\Omega \times D} M \sum_{i=1}^K \left[\frac{1}{2\lambda} \sum_{j=1}^K A_{ij} \nabla_{q_j} \hat{\psi}_{\varepsilon, L}^n - [\sigma(\underline{u}_{\varepsilon, L}^n) q_i] \beta^L(\hat{\psi}_{\varepsilon, L}^n) \right] \cdot \nabla_{q_i} \hat{\varphi} dq dx \\ + \int_{\Omega \times D} M \left[\varepsilon \nabla_x \hat{\psi}_{\varepsilon, L}^n - \underline{u}_{\varepsilon, L}^{n-1} \hat{\psi}_{\varepsilon, L}^n \right] \cdot \nabla_x \hat{\varphi} dq dx = 0 \quad \forall \hat{\varphi} \in \hat{X}; \end{aligned} \quad (3.21b)$$

where, for $t \in [t_{n-1}, t_n)$, and $n = 1, \dots, N$,

$$\tilde{f}^{\Delta t, +}(\cdot, t) = \tilde{f}^n(\cdot) := \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \tilde{f}(\cdot, t) dt \in (\tilde{H}_0^1(\Omega))' \subset \tilde{V}'. \quad (3.22)$$

It follows from (3.5) and (3.22) that

$$\tilde{f}^{\Delta t, +} \rightarrow \tilde{f} \text{ strongly in } L^2(0, T; (\tilde{H}_0^1(\Omega))') \text{ as } \Delta t \rightarrow 0_+. \quad (3.23)$$

Note that as the test function $\tilde{w}$ in (3.21a) is chosen to be divergence-free, the term containing the density ρ in the definition of $\tilde{\tau}$ (cf. (2.3)) is eliminated from (3.21a).

In order to prove the existence of a solution to $(P_{\varepsilon, L}^{\Delta t})$, we require the following convex regularization $\mathcal{F}_\delta^L \in C^{2,1}(\mathbb{R})$ of $\mathcal{F}$ defined, for any $\delta \in (0, 1)$ and $L > 1$, by

$$\mathcal{F}_\delta^L(s) := \begin{cases} \frac{s^2 - \delta^2}{2\delta} + s(\log \delta - 1) + 1 & \text{for } s \leq \delta, \\ \mathcal{F}(s) \equiv s(\log s - 1) + 1 & \text{for } \delta \leq s \leq L, \\ \frac{s^2 - L^2}{2L} + s(\log L - 1) + 1 & \text{for } L \leq s. \end{cases} \quad (3.24)$$

Hence,

$$[\mathcal{F}_\delta^L]'(s) = \begin{cases} \frac{s}{\delta} + \log \delta - 1 & \text{for } s \leq \delta, \\ \log s & \text{for } \delta \leq s \leq L, \\ \frac{s}{L} + \log L - 1 & \text{for } L \leq s, \end{cases} \quad (3.25a)$$

$$[\mathcal{F}_\delta^L]''(s) = \begin{cases} \delta^{-1} & \text{for } s \leq \delta, \\ s^{-1} & \text{for } \delta \leq s \leq L, \\ L^{-1} & \text{for } L \leq s. \end{cases} \quad (3.25b)$$

We note that

$$\mathcal{F}_\delta^L(s) \geq \begin{cases} \frac{s^2}{2\delta} & \text{for } s \leq 0, \\ \frac{s^2}{4L} - C(L) & \text{for } s \geq 0; \end{cases} \quad (3.26)$$

and that $[\mathcal{F}_\delta^L]''(s)$ is bounded below by $1/L$ for all $s \in \mathbb{R}$. Finally, we set

$$\beta_\delta^L(s) := ([\mathcal{F}_\delta^L]'')^{-1}(s) = \max\{\beta^L(s), \delta\}, \quad (3.27)$$

and observe that $\beta_\delta^L(s)$ is bounded above by L and bounded below by δ for all $s \in \mathbb{R}$. Note also that both β^L and β_δ^L are Lipschitz continuous on $\mathbb{R}$, with Lipschitz constants equal to 1.

3.1. Existence of a solution to $(P_{\varepsilon, L}^{\Delta t})$

It is convenient to rewrite (3.21a) as

$$b(\underline{u}_{\varepsilon, L}^n, \underline{w}) = \ell_b(\hat{\psi}_{\varepsilon, L}^n)(\underline{w}) \quad \forall \underline{w} \in \tilde{V}; \quad (3.28)$$

where, for all $\underline{w}_i \in \tilde{H}_0^1(\Omega)$, $i = 1, 2$,

$$b(\underline{w}_1, \underline{w}_2) := \int_{\Omega} \left[\underline{w}_1 + \Delta t (\underline{u}_{\varepsilon, L}^{n-1} \cdot \nabla_x) \underline{w}_1 \right] \cdot \underline{w}_2 dx + \Delta t \nu \int_{\Omega} \nabla_x \underline{w}_1 : \nabla_x \underline{w}_2 dx, \quad (3.29a)$$

and, for all $w \in H_0^1(\Omega)$ and $\hat{\varphi} \in L_M^2(\Omega \times D)$,

$$\ell_b(\hat{\varphi})(w) := \Delta t \langle f^n, w \rangle_{H_0^1(\Omega)} + \int_{\Omega} \left[u_{\varepsilon,L}^{n-1} \cdot w - \Delta t k \sum_{i=1}^K C_i(M \hat{\varphi}) : \nabla_x w \right] dx. \quad (3.29b)$$

We note that, for all $v \in V$ and all $w_1, w_2 \in H^1(\Omega)$, we have that

$$\int_{\Omega} \left[(v \cdot \nabla_x) w_1 \right] \cdot w_2 dx = - \int_{\Omega} \left[(v \cdot \nabla_x) w_2 \right] \cdot w_1 dx \quad (3.30)$$

and hence $b(\cdot, \cdot)$ is a continuous nonsymmetric coercive bilinear functional on $H_0^1(\Omega) \times H_0^1(\Omega)$. In addition, thanks to (3.15), $\ell_b(\hat{\varphi})(\cdot)$ is a continuous linear functional on $H_0^1(\Omega)$ for any $\hat{\varphi} \in L_M^2(\Omega \times D)$.

For $r > d$, let

$$Y^r := \left\{ v \in L^r(\Omega) : \int_{\Omega} v \cdot \nabla_x w dx = 0 \quad \forall w \in W^{1, \frac{r}{r-1}}(\Omega) \right\}. \quad (3.31)$$

It is also convenient to rewrite (3.21b) as

$$a(\hat{\psi}_{\varepsilon,L}^n, \hat{\varphi}) = \ell_a(u_{\varepsilon,L}^n, \beta^L(\hat{\psi}_{\varepsilon,L}^n))(\hat{\varphi}) \quad \forall \hat{\varphi} \in \hat{X}, \quad (3.32)$$

where, for all $\hat{\varphi}_1, \hat{\varphi}_2 \in \hat{X}$,

$$\begin{aligned} a(\hat{\varphi}_1, \hat{\varphi}_2) := & \int_{\Omega \times D} M \left(\hat{\varphi}_1 \hat{\varphi}_2 + \Delta t \left[\varepsilon \nabla_x \hat{\varphi}_1 - u_{\varepsilon,L}^{n-1} \hat{\varphi}_1 \right] \cdot \nabla_x \hat{\varphi}_2 \right. \\ & \left. + \frac{\Delta t}{2\lambda} \sum_{i=1}^K \sum_{j=1}^K A_{ij} \nabla_{q_i} \hat{\varphi}_1 \cdot \nabla_{q_i} \hat{\varphi}_2 \right) dq dx, \end{aligned} \quad (3.33a)$$

and, for all $v \in H^1(\Omega)$, $\hat{\eta} \in L^\infty(\Omega \times D)$ and $\hat{\varphi} \in \hat{X}$,

$$\ell_a(v, \hat{\eta})(\hat{\varphi}) := \int_{\Omega \times D} M \left[\hat{\psi}_{\varepsilon,L}^{n-1} \hat{\varphi} + \Delta t \sum_{i=1}^K [\sigma(v) q_i] \hat{\eta} \cdot \nabla_{q_i} \hat{\varphi} \right] dq dx. \quad (3.33b)$$

It follows from (3.31) and (3.9) that, for $r > d$,

$$\int_{\Omega \times D} M v \hat{\varphi} \cdot \nabla_x \hat{\varphi} dq dx = 0 \quad \forall v \in Y^r, \quad \forall \hat{\varphi} \in \hat{X}. \quad (3.34)$$

Hence $a(\cdot, \cdot)$ is a continuous coercive bilinear functional on $\hat{X} \times \hat{X}$. In addition, we have that, for all $v \in H^1(\Omega)$, $\hat{\eta} \in L^\infty(\Omega \times D)$ and $\hat{\varphi} \in \hat{X}$,

$$\begin{aligned} |\ell_a(v, \hat{\eta})(\hat{\varphi})| & \leq \|\hat{\psi}_{\varepsilon,L}^{n-1}\|_{L_M^2(\Omega \times D)} \|\hat{\varphi}\|_{L_M^2(\Omega \times D)} \\ & + \Delta t \left(\int_D M |q|^2 dq \right)^{\frac{1}{2}} \|\hat{\eta}\|_{L^\infty(\Omega \times D)} \|\nabla_x v\|_{L^2(\Omega)} \|\nabla_q \hat{\varphi}\|_{L_M^2(\Omega \times D)}. \end{aligned} \quad (3.35)$$

Therefore, by noting that $\hat{\psi}_{\varepsilon,L}^{n-1} \in \hat{Z}_2$ and recalling (1.4), it follows that $\ell_a(v, \hat{\eta})(\cdot)$ is a continuous linear functional on $\hat{X}$ for all $v \in H^1(\Omega)$ and $\hat{\eta} \in L^\infty(\Omega \times D)$.

In order to prove existence of a solution to (3.21a,b), i.e. (3.28) and (3.32), we consider a regularized system for a given $\delta \in (0, 1)$:

Find $(\underline{u}_{\varepsilon,L,\delta}^n, \hat{\psi}_{\varepsilon,L,\delta}^n) \in \mathcal{V} \times \hat{X}$ such that

$$b(\underline{u}_{\varepsilon,L,\delta}^n, \underline{w}) = \ell_b(\hat{\psi}_{\varepsilon,L,\delta}^n)(\underline{w}) \quad \forall \underline{w} \in \mathcal{V}, \quad (3.36a)$$

$$a(\hat{\psi}_{\varepsilon,L,\delta}^n, \hat{\varphi}) = \ell_a(\underline{u}_{\varepsilon,L,\delta}^n, \beta_\delta^L(\hat{\psi}_{\varepsilon,L,\delta}^n))(\hat{\varphi}) \quad \forall \hat{\varphi} \in \hat{X}. \quad (3.36b)$$

The existence of a solution to (3.36a,b) will be proved by using a fixed-point argument. Given $\hat{\psi} \in L_M^2(\Omega \times D)$, let $(\underline{u}^*, \hat{\psi}^*) \in \mathcal{V} \times \hat{X}$ be such that

$$b(\underline{u}^*, \underline{w}) = \ell_b(\hat{\psi})(\underline{w}) \quad \forall \underline{w} \in \mathcal{V}, \quad (3.37a)$$

$$a(\hat{\psi}^*, \hat{\varphi}) = \ell_a(\underline{u}^*, \beta_\delta^L(\hat{\psi}))(\hat{\varphi}) \quad \forall \hat{\varphi} \in \hat{X}. \quad (3.37b)$$

The Lax–Milgram theorem yields the existence of a unique solution to (3.37a,b), and so the overall procedure (3.37a,b) is well defined.

Lemma 3.2. *Let $G : L_M^2(\Omega \times D) \rightarrow \hat{X} \subset L_M^2(\Omega \times D)$ denote the nonlinear map that takes the function $\hat{\psi}$ to $\hat{\psi}^* = G(\hat{\psi})$ via the procedure (3.37a,b). Then G has a fixed point. Hence there exists a solution $(\underline{u}_{\varepsilon,L,\delta}^n, \hat{\psi}_{\varepsilon,L,\delta}^n) \in \mathcal{V} \times \hat{X}$ to (3.36a,b).*

Proof. Clearly, a fixed point of G yields a solution of (3.36a,b). In order to show that G has a fixed point, we apply Schauder's fixed-point theorem; that is, we need to show that: (i) $G : L_M^2(\Omega \times D) \rightarrow L_M^2(\Omega \times D)$ is continuous; (ii) G is compact; and (iii) there exists a $C_\star \in \mathbb{R}_{>0}$ such that

$$\|\hat{\psi}\|_{L_M^2(\Omega \times D)} \leq C_\star \quad (3.38)$$

for every $\hat{\psi} \in L_M^2(\Omega \times D)$ and $\kappa \in (0, 1]$ satisfying $\hat{\psi} = \kappa G(\hat{\psi})$.

Let $\{\hat{\psi}^{(p)}\}_{p \geq 0}$ be such that

$$\hat{\psi}^{(p)} \rightarrow \hat{\psi} \quad \text{strongly in } L_M^2(\Omega \times D) \quad \text{as } p \rightarrow \infty. \quad (3.39)$$

It follows immediately from (3.27) and (3.15) that

$$M^{\frac{1}{2}} \beta_\delta^L(\hat{\psi}^{(p)}) \rightarrow M^{\frac{1}{2}} \beta_\delta^L(\hat{\psi}) \quad \text{strongly in } L^r(\Omega \times D) \quad \text{as } p \rightarrow \infty, \quad (3.40a)$$

for all $r \in [1, \infty)$ and, for $i = 1, \dots, K$,

$$\tilde{C}_i(M \hat{\psi}^{(p)}) \rightarrow \tilde{C}_i(M \hat{\psi}) \quad \text{strongly in } L^2(\Omega) \quad \text{as } p \rightarrow \infty. \quad (3.40b)$$

In order to prove (i) above, we need to show that

$$\hat{\eta}^{(p)} := G(\hat{\psi}^{(p)}) \rightarrow G(\hat{\psi}) \quad \text{strongly in } L_M^2(\Omega \times D) \quad \text{as } p \rightarrow \infty. \quad (3.41)$$

We have from the definition of G (see (3.37a,b)) that, for all $p \geq 0$,

$$a(\hat{\eta}^{(p)}, \hat{\varphi}) = \ell_a(\underline{v}^{(p)}, \beta_\delta^L(\hat{\psi}^{(p)}))(\hat{\varphi}) \quad \forall \hat{\varphi} \in \hat{X}, \quad (3.42a)$$

where $\underline{v}^{(p)} \in \mathcal{V}$ satisfies

$$b(\underline{v}^{(p)}, \underline{w}) = \ell_b(\hat{\psi}^{(p)})(\underline{w}) \quad \forall \underline{w} \in \mathcal{V}. \quad (3.42b)$$

Choosing $\widehat{\varphi} = \widehat{\eta}^{(p)}$ in (3.42a) yields, on noting the simple identity

$$2(s_1 - s_2)s_1 = s_1^2 + (s_1 - s_2)^2 - s_2^2 \quad \forall s_1, s_2 \in \mathbb{R}, \quad (3.43)$$

(2.6), (3.34) and (3.27) that, for all $p \geq 0$,

$$\begin{aligned} & \int_{\Omega \times D} M \left[|\widehat{\eta}^{(p)}|^2 + |\widehat{\eta}^{(p)} - \widehat{\psi}_{\varepsilon, L}^{n-1}|^2 + \frac{a_0 \Delta t}{2\lambda} |\nabla_q \widehat{\eta}^{(p)}|^2 + 2\varepsilon \Delta t |\nabla_x \widehat{\eta}^{(p)}|^2 \right] dq dx \\ & \leq \int_{\Omega \times D} M |\widehat{\psi}_{\varepsilon, L}^{n-1}|^2 dq dx + C(L, \lambda, a_0^{-1}) \Delta t \int_{\Omega} |\nabla_x v^{(p)}|^2 dx. \end{aligned} \quad (3.44)$$

Choosing $w \equiv v^{(p)}$ in (3.42b), and noting (3.43), (3.30), (3.15) and (3.39) yields, for all $p \geq 0$, that

$$\begin{aligned} & \int_{\Omega} \left[|v^{(p)}|^2 + |v^{(p)} - u_{\varepsilon, L}^{n-1}|^2 \right] dx + \Delta t \nu \int_{\Omega} |\nabla_x v^{(p)}|^2 dx \\ & \leq \int_{\Omega} |u_{\varepsilon, L}^{n-1}|^2 dx + C \Delta t \|f^n\|_{(H_0^1(\Omega))'}^2 + C \Delta t \int_{\Omega \times D} M |\widehat{\psi}^{(p)}|^2 dq dx \leq C. \end{aligned} \quad (3.45)$$

Combining (3.44) and (3.45), we have for all $p \geq 0$ that

$$\|\widehat{\eta}^{(p)}\|_{\widehat{X}} + \|v^{(p)}\|_{H^1(\Omega)} \leq C(L, (\Delta t)^{-1}). \quad (3.46)$$

It follows from (3.46), (3.9) and the compactness of the embedding (3.11b) that there exists a subsequence $\{(\widehat{\eta}^{(p_k)}, v^{(p_k)})\}_{p_k \geq 0}$ and functions $\widehat{\eta} \in \widehat{X}$ and $v \in V$ such that, as $p_k \rightarrow \infty$,

$$\widehat{\eta}^{(p_k)} \rightarrow \widehat{\eta} \quad \text{weakly in } L^s(\Omega; L_M^2(D)), \quad (3.47a)$$

$$M^{\frac{1}{2}} \nabla_x \widehat{\eta}^{(p_k)} \rightarrow M^{\frac{1}{2}} \nabla_x \widehat{\eta} \quad \text{weakly in } L^2(\Omega \times D), \quad (3.47b)$$

$$M^{\frac{1}{2}} \nabla_q \widehat{\eta}^{(p_k)} \rightarrow M^{\frac{1}{2}} \nabla_q \widehat{\eta} \quad \text{weakly in } L^2(\Omega \times D), \quad (3.47c)$$

$$\widehat{\eta}^{(p_k)} \rightarrow \widehat{\eta} \quad \text{strongly in } L_M^2(\Omega \times D), \quad (3.47d)$$

$$v^{(p_k)} \rightarrow v \quad \text{weakly in } H^1(\Omega); \quad (3.47e)$$

where $s \in [1, \infty)$ if $d = 2$ or $s \in [1, 6]$ if $d = 3$. We deduce from (3.42b), (3.29a,b), (3.47e) and (3.40b) that the functions $v \in V$ and $\widehat{\psi} \in \widehat{X}$ satisfy

$$b(v, w) = \ell_b(\widehat{\psi})(w) \quad \forall w \in V. \quad (3.48)$$

It follows from (3.42a), (3.33a,b), (3.47a-e) and (3.40a) that $\widehat{\eta}, \widehat{\psi} \in \widehat{X}$ and $v \in V$ satisfy

$$a(\widehat{\eta}, \widehat{\varphi}) = \ell_a(v, \beta_\delta^L(\widehat{\psi}))(\widehat{\varphi}) \quad \forall \widehat{\varphi} \in C^\infty(\overline{\Omega \times D}). \quad (3.49)$$

Then, noting that $a(\cdot, \cdot)$ is a bounded bilinear functional on $\widehat{X} \times \widehat{X}$, that $\ell_a(v, \beta_\delta^L(\widehat{\psi}))(\cdot)$ is a bounded linear functional on $\widehat{X}$, and recalling (3.8), we deduce that (3.49) holds for all $\widehat{\varphi} \in \widehat{X}$. Combining this $\widehat{X}$ version of (3.49) and (3.48), we have that $\widehat{\eta} = G(\widehat{\psi}) \in \widehat{X}$. Therefore the whole sequence

$$\widehat{\eta}^{(p)} \equiv G(\widehat{\psi}^{(p)}) \rightarrow G(\widehat{\psi}) \quad \text{strongly in } L_M^2(\Omega \times D),$$

as $p \rightarrow \infty$, and so (i) holds.

Since the embedding $\widehat{X} \hookrightarrow L_M^2(\Omega \times D)$ is compact, it follows that (ii) holds. It therefore remains to show that (iii) holds.

As regards (iii), $\widehat{\psi} = \kappa G(\widehat{\psi})$ implies that $\{\widehat{v}, \widehat{\psi}\} \in \mathcal{V} \times \widehat{X}$ satisfies

$$b(\widehat{v}, \widehat{w}) = \ell_b(\widehat{\psi})(\widehat{w}) \quad \forall \widehat{w} \in \mathcal{V}, \quad (3.50a)$$

$$a(\widehat{\psi}, \widehat{\varphi}) = \kappa \ell_a(\widehat{v}, \beta_\delta^L(\widehat{\psi}))(\widehat{\varphi}) \quad \forall \widehat{\varphi} \in \widehat{X}. \quad (3.50b)$$

Choosing $\widehat{w} \equiv \widehat{v}$ in (3.50a) yields, similarly to (3.45), that

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \left[|\widehat{v}|^2 + |\widehat{v} - \widehat{u}_{\varepsilon, L}^{n-1}|^2 - |\widehat{u}_{\varepsilon, L}^{n-1}|^2 \right] d\widehat{x} + \Delta t \nu \int_{\Omega} |\widehat{\nabla}_x \widehat{v}|^2 d\widehat{x} \\ &= \Delta t \left[\langle \widehat{f}^n, \widehat{v} \rangle_{H_0^1(\Omega)} - k \sum_{i=1}^K \int_{\Omega} \mathcal{C}_i(M \widehat{\psi}) : \widehat{\nabla}_x \widehat{v} d\widehat{x} \right]. \end{aligned} \quad (3.51)$$

Choosing $\widehat{\varphi} = [\mathcal{F}_\delta^L]'(\widehat{\psi})$ in (3.50b) and noting the convexity of $\mathcal{F}_\delta^L$, (3.27) and that $\widehat{v}$ is divergence-free, yield

$$\begin{aligned} & \int_{\Omega \times D} M \left[\mathcal{F}_\delta^L(\widehat{\psi}) - \mathcal{F}_\delta^L(\kappa \widehat{\psi}_{\varepsilon, L}^{n-1}) + \Delta t \left[\varepsilon \widehat{\nabla}_x \widehat{\psi} - \widehat{u}_{\varepsilon, L}^{n-1} \widehat{\psi} \right] \cdot \widehat{\nabla}_x ([\mathcal{F}_\delta^L]'(\widehat{\psi})) \right] d\widehat{q} d\widehat{x} \\ &+ \frac{\Delta t}{2\lambda} \sum_{i=1}^K \sum_{j=1}^K A_{ij} \int_{\Omega \times D} M \widehat{\nabla}_{q_j} \widehat{\psi} \cdot \widehat{\nabla}_{q_i} ([\mathcal{F}_\delta^L]'(\widehat{\psi})) d\widehat{q} d\widehat{x} \\ &\leq \kappa \Delta t \sum_{i=1}^K \int_{\Omega \times D} M \sigma(\widehat{v}) q_i \cdot \widehat{\nabla}_{q_i} \widehat{\psi} d\widehat{q} d\widehat{x} \\ &= \kappa \Delta t \sum_{i=1}^K \int_{\Omega} \mathcal{C}_i(M \widehat{\psi}) : \sigma(\widehat{v}) d\widehat{x}, \end{aligned} \quad (3.52)$$

where in the transition to the final inequality we applied (3.16) with $B := \sigma(\widehat{v})$ (on account of it being independent of the variable $\widehat{q}$), together with the fact that $\text{tr}(\sigma(\widehat{v})) = \widehat{\nabla}_x \cdot \widehat{v} = 0$, and recalled (2.4a). Next, on noting (3.27) and that $\widehat{u}_{\varepsilon, L}^{n-1} \in \mathcal{V}$, it follows that

$$\begin{aligned} \int_{\Omega \times D} M \widehat{u}_{\varepsilon, L}^{n-1} \widehat{\psi} \cdot \widehat{\nabla}_x ([\mathcal{F}_\delta^L]'(\widehat{\psi})) d\widehat{q} d\widehat{x} &= \int_{\Omega \times D} M \widehat{u}_{\varepsilon, L}^{n-1} \frac{\widehat{\psi}}{\beta_\delta^L(\widehat{\psi})} \cdot \widehat{\nabla}_x \widehat{\psi} d\widehat{q} d\widehat{x} \\ &= \int_{\Omega \times D} M \widehat{u}_{\varepsilon, L}^{n-1} \cdot \widehat{\nabla}_x (G_\delta^L(\widehat{\psi})) d\widehat{q} d\widehat{x} = 0, \end{aligned} \quad (3.53)$$

where $G_\delta^L \in C^{0,1}(\mathbb{R})$ is defined by

$$G_\delta^L(s) := \begin{cases} \frac{1}{2\delta} s^2 + \frac{\delta-L}{2} & \text{if } s \leq \delta, \\ s - \frac{L}{2} & \text{if } s \in [\delta, L], \\ \frac{1}{2L} s^2 & \text{if } s \geq L; \end{cases} \quad (3.54)$$

20 *John W. Barrett and Endre Süli*

and so $[G_\delta^L]'(s) = s/\beta_\delta^L(s)$. Combining (3.51) and (3.52), and noting (3.53), (3.25b) and (2.6) yields that

$$\begin{aligned} & \frac{\kappa}{2} \int_{\Omega} \left[|v|^2 + |v - u_{\varepsilon,L}^{n-1}|^2 \right] dx + \kappa \Delta t \nu \int_{\Omega} |\nabla_x v|^2 dx + k \int_{\Omega \times D} M \mathcal{F}_\delta^L(\hat{\psi}) dq dx \\ & + k L^{-1} \Delta t \int_{\Omega \times D} M \left[\varepsilon |\nabla_x \hat{\psi}|^2 + \frac{a_0}{2\lambda} |\nabla_q \hat{\psi}|^2 \right] dq dx \\ & \leq \kappa \Delta t \langle f^n, v \rangle_{H_0^1(\Omega)} + \frac{\kappa}{2} \int_{\Omega} |u_{\varepsilon,L}^{n-1}|^2 dx + k \int_{\Omega \times D} M \mathcal{F}_\delta^L(\kappa \hat{\psi}_{\varepsilon,L}^{n-1}) dq dx \\ & \leq \frac{\kappa \Delta t \nu}{2} \int_{\Omega} |\nabla_x v|^2 dx + \frac{\kappa \Delta t}{2\nu} \|f^n\|_{(H_0^1(\Omega))'}^2 \\ & \quad + \frac{\kappa}{2} \int_{\Omega} |u_{\varepsilon,L}^{n-1}|^2 dx + k \int_{\Omega \times D} M \mathcal{F}_\delta^L(\kappa \hat{\psi}_{\varepsilon,L}^{n-1}) dq dx. \end{aligned} \quad (3.55)$$

It is easy to show that $\mathcal{F}_\delta^L(s)$ is nonnegative for all $s \in \mathbb{R}$, with $\mathcal{F}_\delta^L(1) = 0$. Furthermore, for any $\kappa \in (0, 1]$, $\mathcal{F}_\delta^L(\kappa s) \leq \mathcal{F}_\delta^L(s)$ if $s < 0$ or $1 \leq \kappa s$, and also $\mathcal{F}_\delta^L(\kappa s) \leq \mathcal{F}_\delta^L(0) \leq 1$ if $0 \leq \kappa s \leq 1$. Thus we deduce that

$$\mathcal{F}_\delta^L(\kappa s) \leq \mathcal{F}_\delta^L(s) + 1 \quad \forall s \in \mathbb{R}, \quad \forall \kappa \in (0, 1]. \quad (3.56)$$

Hence, the bounds (3.55) and (3.56), on noting (3.26), give rise to the desired bound (3.38) with C_* dependent only on $\delta, L, k, a_0, \nu, f$ and $\hat{\psi}_{\varepsilon,L}^{n-1}$. Therefore (iii) holds, and so G has a fixed point, proving existence of a solution to (3.36a,b). $\square$

Choosing $w \equiv u_{\varepsilon,L,\delta}^n$ in (3.36a) and $\hat{\varphi} \equiv [\mathcal{F}_\delta^L]'(\hat{\psi}_{\varepsilon,L,\delta}^n)$ in (3.36b), and combining, then yields, as in (3.55), with $C(L)$ a positive constant, independent of δ and Δt ,

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \left[|u_{\varepsilon,L,\delta}^n|^2 + |u_{\varepsilon,L,\delta}^n - u_{\varepsilon,L}^{n-1}|^2 \right] dx + k \int_{\Omega \times D} M \mathcal{F}_\delta^L(\hat{\psi}_{\varepsilon,L,\delta}^n) dq dx \\ & + \Delta t \left[\frac{\nu}{2} \int_{\Omega} |\nabla_x u_{\varepsilon,L,\delta}^n|^2 dx + k L^{-1} \varepsilon \int_{\Omega \times D} M |\nabla_x \hat{\psi}_{\varepsilon,L,\delta}^n|^2 dq dx \right. \\ & \quad \left. + \frac{k L^{-1} a_0}{2\lambda} \int_{\Omega \times D} M |\nabla_q \hat{\psi}_{\varepsilon,L,\delta}^n|^2 dq dx \right] \\ & \leq \frac{\Delta t}{2\nu} \|f^n\|_{(H_0^1(\Omega))'}^2 + \frac{1}{2} \int_{\Omega} |u_{\varepsilon,L}^{n-1}|^2 dx + k \int_{\Omega \times D} M \mathcal{F}_\delta^L(\hat{\psi}_{\varepsilon,L}^{n-1}) dq dx \\ & \leq C(L). \end{aligned} \quad (3.57)$$

We are now ready to pass to the limit $\delta \rightarrow 0_+$, to deduce the existence of a solution $\{(u_{\varepsilon,L}^n, \hat{\psi}_{\varepsilon,L}^n)\}_{n=1}^N$ to $(P_{\varepsilon,L}^{\Delta t})$, with $u_{\varepsilon,L}^n \in V$ and $\hat{\psi}_{\varepsilon,L}^n \in \hat{X} \cap \hat{Z}_2$, $n = 1, \dots, N$.

Lemma 3.3. *There exists a subsequence (not indicated) of $\{(u_{\varepsilon,L,\delta}^n, \hat{\psi}_{\varepsilon,L,\delta}^n)\}_{\delta>0}$, and functions $u_{\varepsilon,L}^n \in V$ and $\hat{\psi}_{\varepsilon,L}^n \in \hat{X} \cap \hat{Z}_2$, $n \in \{1, \dots, N\}$, such that, as $\delta \rightarrow 0_+$,*

$$u_{\varepsilon,L,\delta}^n \rightharpoonup u_{\varepsilon,L}^n \quad \text{weakly in } V, \quad (3.58a)$$

$$u_{\varepsilon,L,\delta}^n \rightarrow u_{\varepsilon,L}^n \quad \text{strongly in } L^r(\Omega), \quad (3.58b)$$

where $r \in [1, \infty)$ if $d = 2$ and $r \in [1, 6)$ if $d = 3$; and

$$M^{\frac{1}{2}} \widehat{\psi}_{\varepsilon, L, \delta}^n \rightarrow M^{\frac{1}{2}} \widehat{\psi}_{\varepsilon, L}^n \quad \text{weakly in } L^2(\Omega \times D), \quad (3.59a)$$

$$M^{\frac{1}{2}} \nabla_q \widehat{\psi}_{\varepsilon, L, \delta}^n \rightarrow M^{\frac{1}{2}} \nabla_q \widehat{\psi}_{\varepsilon, L}^n \quad \text{weakly in } L^2(\Omega \times D), \quad (3.59b)$$

$$M^{\frac{1}{2}} \nabla_x \widehat{\psi}_{\varepsilon, L, \delta}^n \rightarrow M^{\frac{1}{2}} \nabla_x \widehat{\psi}_{\varepsilon, L}^n \quad \text{weakly in } L^2(\Omega \times D), \quad (3.59c)$$

$$M^{\frac{1}{2}} \widehat{\psi}_{\varepsilon, L, \delta}^n \rightarrow M^{\frac{1}{2}} \widehat{\psi}_{\varepsilon, L}^n \quad \text{strongly in } L^2(\Omega \times D), \quad (3.59d)$$

$$M^{\frac{1}{2}} \beta_\delta^L(\widehat{\psi}_{\varepsilon, L, \delta}^n) \rightarrow M^{\frac{1}{2}} \beta^L(\widehat{\psi}_{\varepsilon, L}^n) \quad \text{strongly in } L^s(\Omega \times D), \quad (3.59e)$$

for all $s \in [2, \infty)$ and, for $i = 1, \dots, K$,

$$C_i(M \widehat{\psi}_{\varepsilon, L, \delta}^n) \rightarrow C_i(M \widehat{\psi}_{\varepsilon, L}^n) \quad \text{strongly in } L^2(\Omega). \quad (3.59f)$$

Further, $(u_{\varepsilon, L}^n, \widehat{\psi}_{\varepsilon, L}^n)$ solves (3.21a,b) for $n = 1, \dots, N$. Hence there exists a solution $\{(u_{\varepsilon, L}^n, \widehat{\psi}_{\varepsilon, L}^n)\}_{n=1}^N$ to $(P_{\varepsilon, L}^{\Delta t})$, with $u_{\varepsilon, L}^n \in \mathcal{V}$ and $\widehat{\psi}_{\varepsilon, L}^n \in \widehat{X} \cap \widehat{Z}_2$ for all $n = 1, \dots, N$.

Proof. The weak convergence results (3.58a), (3.59a) and that $\widehat{\psi}_{\varepsilon, L}^n \geq 0$ a.e. on $\Omega \times D$ follow immediately from the first two bounds on the left-hand side of (3.57), on noting (3.26). The strong convergence (3.58b) for $u_{\varepsilon, L, \delta}^n$ follows from (3.58a), on noting that $\mathcal{V} \subset H_0^1(\Omega)$ is compactly embedded in $L^r(\Omega)$ for the stated values of r .

It follows immediately from the bound on the fifth term on the left-hand side of (3.57) that (3.59b) holds for some limit $g \in L^2(\Omega \times D)$, which we need to identify. However, for any $\eta \in \mathcal{C}_0^1(\Omega \times D)$, it follows from (1.5) and the compact support of η on D that $[\nabla_q \cdot (M^{\frac{1}{2}} \eta)]/M^{\frac{1}{2}} \in L^2(\Omega \times D)$, and hence the above convergence implies, noting (3.59a), that

$$\begin{aligned} \int_{\Omega \times D} g \cdot \eta \, dq \, dx &\leftarrow - \int_{\Omega \times D} M^{\frac{1}{2}} \widehat{\psi}_{\varepsilon, L, \delta}^n \frac{\nabla_q \cdot (M^{\frac{1}{2}} \eta)}{M^{\frac{1}{2}}} \, dq \, dx \\ &\rightarrow - \int_{\Omega \times D} M^{\frac{1}{2}} \widehat{\psi}_{\varepsilon, L}^n \frac{\nabla_q \cdot (M^{\frac{1}{2}} \eta)}{M^{\frac{1}{2}}} \, dq \, dx = - \int_{\Omega \times D} \widehat{\psi}_{\varepsilon, L}^n \nabla_q \cdot (M^{\frac{1}{2}} \eta) \, dq \, dx \end{aligned} \quad (3.60)$$

as $\delta \rightarrow 0_+$. Equivalently, on dividing and multiplying by $M^{\frac{1}{2}}$ under the integral sign in the left-most term in (3.60), we have that

$$\int_{\Omega \times D} M^{-\frac{1}{2}} g \cdot M^{\frac{1}{2}} \eta \, dq \, dx = - \int_{\Omega \times D} \widehat{\psi}_{\varepsilon, L}^n \nabla_q \cdot (M^{\frac{1}{2}} \eta) \, dq \, dx \quad \forall \eta \in \mathcal{C}_0^1(\Omega \times D).$$

Observe that $\eta \in \mathcal{C}_0^1(\Omega \times D) \mapsto M^{\frac{1}{2}} \eta \in \mathcal{C}_0^1(\Omega \times D)$ is a bijection of $\mathcal{C}_0^1(\Omega \times D)$ onto itself; thus, the equality above is equivalent to

$$\int_{\Omega \times D} M^{-\frac{1}{2}} g \cdot \chi \, dq \, dx = - \int_{\Omega \times D} \widehat{\psi}_{\varepsilon, L}^n (\nabla_q \cdot \chi) \, dq \, dx \quad \forall \chi \in \mathcal{C}_0^1(\Omega \times D).$$

Since $C_0^\infty(\Omega \times D) \subset C_0^1(\Omega \times D)$, the last identity also holds for all $\eta \in C_0^\infty(\Omega \times D)$. As $M^{\frac{1}{2}} \in L^\infty(D)$ and $M^{-\frac{1}{2}} \in L_{\text{loc}}^\infty(D)$, it follows that $M^{-\frac{1}{2}}g \in L_{\text{loc}}^2(\Omega \times D)$ and $\widehat{\psi}_{\varepsilon,L}^n \in L_{\text{loc}}^2(\Omega \times D)$. By identification of a locally integrable function with a distribution we deduce that $M^{-\frac{1}{2}}g$ is the distributional gradient of $\widehat{\psi}_{\varepsilon,L}^n$ w.r.t. q :

$$M^{-\frac{1}{2}}g = \nabla_q \widehat{\psi}_{\varepsilon,L}^n \quad \text{in } \mathcal{D}'(\Omega \times D).$$

As $M^{-\frac{1}{2}}g \in L_{\text{loc}}^2(\Omega \times D)$, whereby also $\nabla_q \widehat{\psi}_{\varepsilon,L}^n \in L_{\text{loc}}^2(\Omega \times D)$, it follows that

$$g = M^{\frac{1}{2}} \nabla_q \widehat{\psi}_{\varepsilon,L}^n \in L_{\text{loc}}^2(\Omega \times D).$$

However, the left-hand side belongs to $L^2(\Omega \times D)$, which then implies that the right-hand side also belongs to $L^2(\Omega \times D)$. Thus we have shown that

$$g = M^{\frac{1}{2}} \nabla_q \widehat{\psi}_{\varepsilon,L}^n \in L^2(\Omega \times D), \quad (3.61)$$

and hence the desired result (3.59b), as required. A similar argument proves (3.59c) on noting (3.59a), and the fourth bound in (3.57).

The strong convergence result (3.59d) for $\widehat{\psi}_{\varepsilon,L,\delta}^n$ follows directly from (3.59a–c) and (3.11b). Finally, (3.59e,f) follow from (3.59d), (3.27), (2.4a) and (3.15).

It follows from (3.58a,b), (3.59b–f), (3.29a,b), (3.33a,b), (3.35) and (3.8) that we may pass to the limit $\delta \rightarrow 0_+$ in (3.36a,b) to obtain that $(u_{\varepsilon,L}^n, \widehat{\psi}_{\varepsilon,L}^n) \in V \times \widehat{X}$ with $\widehat{\psi}_{\varepsilon,L}^n \geq 0$ a.e. on $\Omega \times D$ solves (3.28), (3.32); i.e. it solves (3.21a,b).

Next we prove the integral constraint on $\widehat{\psi}_{\varepsilon,L}^n$. First, for $m = n - 1, n$, let

$$\rho_{\varepsilon,L}^m(x) := \int_D M(q) \widehat{\psi}_{\varepsilon,L}^m(x, q) dq, \quad x \in \Omega. \quad (3.62)$$

For $m = n - 1, n$, as $\widehat{\psi}_{\varepsilon,L}^m \in \widehat{X}$, we deduce from the Cauchy–Schwarz inequality and Fubini’s theorem that $\rho_{\varepsilon,L}^m \in H^1(\Omega)$ and $\rho_{\varepsilon,L}^{n-1} \in L^2(\Omega)$. We introduce also the following closed linear subspace of $\widehat{X} = H_M^1(\Omega \times D)$:

$$H^1(\Omega) \otimes 1(D) := \left\{ \widehat{\varphi} \in H_M^1(\Omega \times D) : \widehat{\varphi}(\cdot, q^*) = \widehat{\varphi}(\cdot, q^{**}) \quad \forall q^*, q^{**} \in D \right\}. \quad (3.63)$$

Then, on choosing $\widehat{\varphi} = \varphi \in H^1(\Omega) \otimes 1(D)$ in (3.21b), we deduce from (3.62) and Fubini’s theorem that, for all $\varphi \in H^1(\Omega)$,

$$\int_\Omega \frac{\rho_{\varepsilon,L}^n - \rho_{\varepsilon,L}^{n-1}}{\Delta t} \varphi dx + \int_\Omega \left[\varepsilon \nabla_x \rho_{\varepsilon,L}^n - u_{\varepsilon,L}^{n-1} \rho_{\varepsilon,L}^n \right] \cdot \nabla_x \varphi dx = 0, \quad (3.64)$$

with

$$0 \leq \rho_{\varepsilon,L}^0 := \int_D M \widehat{\psi}_{\varepsilon,L}^0 dq = \int_D M \beta^L(\widehat{\psi}^0) dq \leq \int_D M \widehat{\psi}^0 dq \leq 1, \quad \text{a.e. on } \Omega. \quad (3.65)$$

By introducing the function $z_{\varepsilon,L}^m := 1 - \rho_{\varepsilon,L}^m$, $m = n-1, n$, and noting that $z_{\varepsilon,L}^n \in H^1(\Omega)$ and $z_{\varepsilon,L}^{n-1} \in L^2(\Omega)$, we deduce from (3.64), and as $u_{\varepsilon,L}^{n-1}$ is divergence-free on Ω with zero trace on $\partial\Omega$, that

$$\int_{\Omega} \frac{z_{\varepsilon,L}^n - z_{\varepsilon,L}^{n-1}}{\Delta t} \varphi \, dx + \int_{\Omega} \left[\varepsilon \nabla_x z_{\varepsilon,L}^n - u_{\varepsilon,L}^{n-1} z_{\varepsilon,L}^n \right] \cdot \nabla_x \varphi \, dx = 0, \quad (3.66)$$

for all $\varphi \in H^1(\Omega)$. Let us now define by $[x]_{\pm} := \frac{1}{2}(x \pm |x|)$ the positive and negative parts, $[x]_+$ and $[x]_-$, of a real number x , respectively. As $\widehat{\psi}_{\varepsilon,L}^{n-1} \in \widehat{Z}_2$, we then have that $[z_{\varepsilon,L}^{n-1}]_- = 0$ a.e. on Ω . Taking $\varphi = [z_{\varepsilon,L}^n]_-$ as a test function in (3.66), noting that this is a legitimate choice since $[z_{\varepsilon,L}^n]_- \in H^1(\Omega)$, decomposing $z_{\varepsilon,L}^m$, $m = n-1, n$, into their positive and negative parts, and noting that $u_{\varepsilon,L}^{n-1}$ is divergence-free on Ω and has zero trace on $\partial\Omega$, we deduce that

$$\|[z_{\varepsilon,L}^n]_-\|^2 + \Delta t \varepsilon \|\nabla_x [z_{\varepsilon,L}^n]_-\|^2 = 0,$$

where $\|\cdot\|$ denotes the $L^2(\Omega)$ norm. Hence, $[z_{\varepsilon,L}^n]_- = 0$ a.e. on Ω . In other words, $z_{\varepsilon,L}^n \geq 0$ a.e. on Ω , which then gives that $\rho_{\varepsilon,L}^n \leq 1$ a.e. on Ω , i.e. $\widehat{\psi}_{\varepsilon,L}^n \in \widehat{Z}_2$ as required. As $(u_{\varepsilon,L}^0, \widehat{\psi}_{\varepsilon,L}^0) \in \mathcal{V} \times \widehat{Z}_2$, performing the above existence proof at each time level t_n , $n = 1, \dots, N$, yields a solution $\{(u_{\varepsilon,L}^n, \widehat{\psi}_{\varepsilon,L}^n)\}_{n=1}^N$ to $(P_{\varepsilon,L}^{\Delta t})$. $\square$

4. Entropy estimates

Next, we derive bounds on the solution of $(P_{\varepsilon,L}^{\Delta t})$, independent of L . Our starting point is Lemma 3.3, concerning the existence of a solution to the problem $(P_{\varepsilon,L}^{\Delta t})$. The model $(P_{\varepsilon,L}^{\Delta t})$ includes ‘microscopic cut-off’ in the drag term of the Fokker–Planck equation, where $L > 1$ is a (fixed, but otherwise arbitrary,) cut-off parameter. Our ultimate objective is to pass to the limits $L \rightarrow \infty$ and $\Delta t \rightarrow 0_+$ in the model $(P_{\varepsilon,L}^{\Delta t})$, with L and Δt linked by the condition $\Delta t = o(L^{-1})$, as $L \rightarrow \infty$. To that end, we need to develop various bounds on sequences of weak solutions of $(P_{\varepsilon,L}^{\Delta t})$ that are uniform in the cut-off parameter L and thus permit the extraction of weakly convergent subsequences, as $L \rightarrow \infty$, through the use of a weak compactness argument. The derivation of such bounds, based on the use of the relative entropy associated with the Maxwellian M , is our main task in this section.

Let us introduce the following definitions, in line with (3.22):

$$u_{\varepsilon,L}^{\Delta t}(\cdot, t) := \frac{t - t_{n-1}}{\Delta t} u_{\varepsilon,L}^n(\cdot) + \frac{t_n - t}{\Delta t} u_{\varepsilon,L}^{n-1}(\cdot), \quad t \in [t_{n-1}, t_n], \quad n = 1, \dots, N, \quad (4.1a)$$

$$u_{\varepsilon,L}^{\Delta t,+}(\cdot, t) := u^n(\cdot), \quad u_{\varepsilon,L}^{\Delta t,-}(\cdot, t) := u^{n-1}(\cdot), \quad t \in (t_{n-1}, t_n], \quad n = 1, \dots, N. \quad (4.1b)$$

We shall adopt $u_{\varepsilon,L}^{\Delta t,(\pm)}$ as a collective symbol for $u_{\varepsilon,L}^{\Delta t}$, $u_{\varepsilon,L}^{\Delta t,\pm}$. The corresponding notations $\widehat{\psi}_{\varepsilon,L}^{\Delta t,(\pm)}$, $\widehat{\psi}_{\varepsilon,L}^{\Delta t}$, and $\widehat{\psi}_{\varepsilon,L}^{\Delta t,\pm}$ are defined analogously; recall (3.18) and (3.20).

We note for future reference that

$$u_{\varepsilon,L}^{\Delta t} - u_{\varepsilon,L}^{\Delta t,\pm} = (t - t_n^{\pm}) \frac{\partial u_{\varepsilon,L}^{\Delta t}}{\partial t}, \quad t \in (t_{n-1}, t_n), \quad n = 1, \dots, N, \quad (4.2)$$

where $t_n^+ := t_n$ and $t_n^- := t_{n-1}$, with an analogous relationship in the case of $\widehat{\psi}_{\varepsilon,L}^{\Delta t}$.

Using the above notation, (3.21a) summed for $n = 1, \dots, N$ can be restated in a form that is reminiscent of a weak formulation of (1.1a–d): find $\underline{u}_{\varepsilon,L}^{\Delta t(\pm)} \in \underline{V}$, $t \in (0, T]$, such that

$$\begin{aligned} & \int_0^T \int_{\Omega} \frac{\partial \underline{u}_{\varepsilon,L}^{\Delta t}}{\partial t} \cdot \underline{w} \, dx \, dt + \int_0^T \int_{\Omega} \left[\left[(\underline{u}_{\varepsilon,L}^{\Delta t,-} \cdot \nabla_x) \underline{u}_{\varepsilon,L}^{\Delta t,+} \right] \cdot \underline{w} + \nu \nabla_x \underline{u}_{\varepsilon,L}^{\Delta t,+} : \nabla_x \underline{w} \right] \, dx \, dt \\ &= \int_0^T \left[\langle f^{\Delta t,+}, \underline{w} \rangle_{H_0^1(\Omega)} - k \sum_{i=1}^K \int_{\Omega} C_i(M \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}) : \nabla_x \underline{w} \, dx \right] \, dt, \end{aligned} \quad (4.3)$$

for all $\underline{w} \in L^1(0, T; \underline{V})$, subject to the initial condition $\underline{u}_{\varepsilon,L}^{\Delta t}(\cdot, 0) = \underline{u}^0 \in \underline{V}$.

Analogously, (after a minor re-ordering of terms on the left-hand side for presentational reasons,) (3.21b) summed through $n = 1, \dots, N$ can be restated as follows: find $\widehat{\psi}_{\varepsilon,L}^{\Delta t(\pm)}(t) \in \widehat{Z}_2$, $t \in (0, T]$, such that

$$\begin{aligned} & \int_0^T \int_{\Omega \times D} M \frac{\partial \widehat{\psi}_{\varepsilon,L}^{\Delta t}}{\partial t} \widehat{\varphi} \, dq \, dx \, dt \\ &+ \int_0^T \int_{\Omega \times D} M \left[\varepsilon \nabla_x \widehat{\psi}_{\varepsilon,L}^{\Delta t,+} - \underline{u}_{\varepsilon,L}^{\Delta t,-} \widehat{\psi}_{\varepsilon,L}^{\Delta t,+} \right] \cdot \nabla_x \widehat{\varphi} \, dq \, dx \, dt \\ &+ \frac{1}{2\lambda} \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \sum_{j=1}^K A_{ij} \nabla_{q_j} \widehat{\psi}_{\varepsilon,L}^{\Delta t,+} \cdot \nabla_{q_i} \widehat{\varphi} \, dq \, dx \, dt \\ &- \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \left[\sigma(\underline{u}_{\varepsilon,L}^{\Delta t,+}) q_i \right] \beta^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}) \cdot \nabla_{q_i} \widehat{\varphi} \, dq \, dx \, dt = 0, \end{aligned} \quad (4.4)$$

for all $\widehat{\varphi} \in L^1(0, T; \widehat{X})$, subject to the initial condition $\widehat{\psi}_{\varepsilon,L}^{\Delta t}(\cdot, \cdot, 0) = \beta^L(\widehat{\psi}^0(\cdot, \cdot)) \in \widehat{Z}_2$. We emphasize that (4.3) and (4.4) are an equivalent restatement of problem $(P_{\varepsilon,L}^{\Delta t})$, for which existence of a solution has been established (cf. Lemma 3.3).

Similarly, with analogous notation for $\{\rho_{\varepsilon,L}^n\}_{n=0}^N$, (3.64) summed for $n = 1, \dots, N$ can be restated as follows: Given $\underline{u}_{\varepsilon,L}^{\Delta t(\pm)}(t) \in \underline{V}$, $t \in (0, T]$, solving (4.3), find $\rho_{\varepsilon,L}^{\Delta t(\pm)}(t) \in \mathcal{K} := \{\eta \in H^1(\Omega) : \eta \in [0, 1] \text{ a.e. on } \Omega\}$, $t \in (0, T]$, such that

$$\begin{aligned} & \int_0^T \int_{\Omega} \frac{\partial \rho_{\varepsilon,L}^{\Delta t}}{\partial t} \varphi \, dx \, dt + \int_0^T \int_{\Omega} \left[\varepsilon \nabla_x \rho_{\varepsilon,L}^{\Delta t,+} - \underline{u}_{\varepsilon,L}^{\Delta t,-} \rho_{\varepsilon,L}^{\Delta t,+} \right] \cdot \nabla_x \varphi \, dx \, dt = 0 \\ & \quad \forall \varphi \in L^1(0, T; H^1(\Omega)), \end{aligned} \quad (4.5)$$

subject to the initial condition $\rho_{\varepsilon,L}^{\Delta t}(\cdot, 0) = \int_D M(q) \beta^L(\widehat{\psi}^0(\cdot, q)) \, dq \in \mathcal{K}$; cf. (3.62) and recall that $\widehat{\psi}_{\varepsilon,L}^0 = \beta^L(\widehat{\psi}^0)$. Once again, on noting (3.62) and (3.64), we have established the existence of a solution to (4.5) and that

$$\rho_{\varepsilon,L}^{\Delta t(\pm)}(\underline{x}, t) = \int_D M(q) \widehat{\psi}_{\varepsilon,L}^{\Delta t(\pm)}(\underline{x}, q, t) \, dq \quad \text{for a.e. } (\underline{x}, t) \in \Omega \times (0, T). \quad (4.6)$$

In conjunction with β^L , defined by (1.12), we consider the following cut-off version $\mathcal{F}^L$ of the entropy function $\mathcal{F} : s \in \mathbb{R}_{\geq 0} \mapsto \mathcal{F}(s) = s(\log s - 1) + 1 \in \mathbb{R}_{\geq 0}$:

$$\mathcal{F}^L(s) := \begin{cases} s(\log s - 1) + 1, & 0 \leq s \leq L, \\ \frac{s^2 - L^2}{2L} + s(\log L - 1) + 1, & L \leq s. \end{cases} \quad (4.7)$$

Note that

$$(\mathcal{F}^L)'(s) = \begin{cases} \log s, & 0 < s \leq L, \\ \frac{s}{L} + \log L - 1, & L \leq s, \end{cases} \quad (4.8)$$

and

$$(\mathcal{F}^L)''(s) = \begin{cases} \frac{1}{s}, & 0 < s \leq L, \\ \frac{1}{L}, & L \leq s. \end{cases} \quad (4.9)$$

Hence,

$$\beta^L(s) = \min(s, L) = [(\mathcal{F}^L)''(s)]^{-1}, \quad s \in \mathbb{R}_{\geq 0}, \quad (4.10)$$

with the convention $1/\infty := 0$ when $s = 0$, and

$$(\mathcal{F}^L)''(s) \geq \mathcal{F}''(s) = s^{-1}, \quad s \in \mathbb{R}_{>0}. \quad (4.11)$$

We shall also require the following inequality, relating $\mathcal{F}^L$ to $\mathcal{F}$:

$$\mathcal{F}^L(s) \geq \mathcal{F}(s), \quad s \in \mathbb{R}_{\geq 0}. \quad (4.12)$$

For $0 \leq s \leq 1$, (4.12) trivially holds, with equality. For $s \geq 1$, it follows from (4.11), with s replaced by a dummy variable σ , after integrating twice over $\sigma \in [1, s]$, and noting that $(\mathcal{F}^L)'(1) = \mathcal{F}'(1)$ and $(\mathcal{F}^L)(1) = \mathcal{F}(1)$.

4.1. L -independent bounds on the spatial derivatives

We are now ready to embark on the derivation of the required bounds, uniform in the cut-off parameter L , on norms of $u_{\varepsilon,L}^{\Delta t,+}$ and $\hat{\psi}_{\varepsilon,L}^{\Delta t,+}$. As far as $u_{\varepsilon,L}^{\Delta t,+}$ is concerned, this is a relatively straightforward exercise. We select $\varpi = \chi_{[0,t]} u_{\varepsilon,L}^{\Delta t,+}$ as test function in (4.3), with t chosen as t_n , $n \in \{1, \dots, N\}$, and $\chi_{[0,t]}$ denoting the characteristic function of the interval $[0, t]$. We then deduce, with $t = t_n$ and noting (3.18), that

$$\begin{aligned} & \|u_{\varepsilon,L}^{\Delta t,+}(t)\|^2 + \frac{1}{\Delta t} \int_0^t \|u_{\varepsilon,L}^{\Delta t,+}(s) - u_{\varepsilon,L}^{\Delta t,-}(s)\|^2 ds + \nu \int_0^t \|\nabla_x u_{\varepsilon,L}^{\Delta t,+}(s)\|^2 ds \\ & \leq \|u_0\|^2 + \frac{1}{\nu} \int_0^t \|f^{\Delta t,+}(s)\|_{(H_0^1(\Omega))'}^2 ds \\ & \quad - 2k \int_0^t \int_{\Omega \times D} M(q) \sum_{i=1}^K q_i q_i^T U_i'(\tfrac{1}{2}|q_i|^2) \hat{\psi}_{\varepsilon,L}^{\Delta t,+} : \nabla_x u_{\varepsilon,L}^{\Delta t,+} dq dx ds, \end{aligned} \quad (4.13)$$

where, again, $\|\cdot\|$ denotes the L^2 norm over Ω .

Having dealt with $u_{\varepsilon,L}^{\Delta t,+}$, we now embark on the less straightforward task of deriving bounds on norms of $\hat{\psi}_{\varepsilon,L}^{\Delta t,+}$ that are uniform in the cut-off parameter L . The

appropriate choice of test function in (4.4) for this purpose is $\widehat{\varphi} = \chi_{[0,t]} (\mathcal{F}^L)'(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+})$ with $t = t_n$, $n \in \{1, \dots, N\}$; this can be seen by noting that with such a $\widehat{\varphi}$, at least formally, the final term on the left-hand side of (4.4) can be manipulated to become identical to the final term in (4.13), but with opposite sign. While Lemma 3.3 guarantees that $\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(\cdot, \cdot, t)$ belongs to $\widehat{Z}_2$ for all $t \in [0, T]$, and is therefore nonnegative a.e. on $\Omega \times D \times [0, T]$, there is unfortunately no reason why $\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}$ should be strictly positive on $\Omega \times D \times [0, T]$, and therefore the expression $(\mathcal{F}^L)'(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+})$ may in general be undefined; the same is true of $(\mathcal{F}^L)''(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+})$, which also appears in the algebraic manipulations. We shall circumvent this problem by working with $(\mathcal{F}^L)'(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha)$ instead of $(\mathcal{F}^L)'(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+})$, where $\alpha > 0$; since $\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}$ is known to be nonnegative from Lemma 3.3, $(\mathcal{F}^L)'(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha)$ and $(\mathcal{F}^L)''(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha)$ are well-defined. After deriving the relevant bounds, which will involve $\mathcal{F}^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha)$ only, we shall pass to the limit $\alpha \rightarrow 0_+$, noting that, unlike $(\mathcal{F}^L)'(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+})$ and $(\mathcal{F}^L)''(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+})$, the function $(\mathcal{F}^L)(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+})$ is well-defined for any *nonnegative* $\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}$. Thus, we take any $\alpha \in (0, 1)$, whereby $0 < \alpha < 1 < L$, and we choose $\widehat{\varphi} = \chi_{[0,t]} (\mathcal{F}^L)'(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha)$, with $t = t_n$, $n \in \{1, \dots, N\}$, as test function in (4.4). As the calculations are quite involved, for the sake of clarity of exposition we shall manipulate the terms in (4.4) one at a time and will then merge the resulting bounds on the individual terms with (4.3) to obtain a single energy inequality for the pair $(u_{\varepsilon,L}^{\Delta t,+}, \widehat{\psi}_{\varepsilon,L}^{\Delta t,+})$. For the sake of brevity, some of the more elementary transitions are omitted; we refer the reader to our extended paper⁹ for details.

We start by considering the first term in (4.4). Clearly $\mathcal{F}^L(\cdot + \alpha)$ is twice continuously differentiable on the interval $(-\alpha, \infty)$ for any $\alpha > 0$. Thus, by Taylor series expansion with remainder of the function

$$s \in [0, \infty) \mapsto \mathcal{F}^L(s + \alpha) \in [0, \infty),$$

we have, for any $c \in [0, \infty)$, that

$$(s - c) (\mathcal{F}^L)'(s + \alpha) = \mathcal{F}^L(s + \alpha) - \mathcal{F}^L(c + \alpha) + \frac{1}{2}(s - c)^2 (\mathcal{F}^L)''(\theta s + (1 - \theta)c + \alpha),$$

with $\theta \in (0, 1)$. Hence, on noting that $t \in [0, T] \mapsto \widehat{\psi}_{\varepsilon,L}^{\Delta t}(\cdot, \cdot, t) \in \widehat{X}$ is piecewise linear relative to the partition $\{0 = t_0, t_1, \dots, t_N = T\}$ of the interval $[0, T]$,

$$\begin{aligned} T_1 &:= \int_0^T \int_{\Omega \times D} M \frac{\partial \widehat{\psi}_{\varepsilon,L}^{\Delta t}}{\partial s} \chi_{[0,t]} (\mathcal{F}^L)'(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha) \, d\mathbf{q} \, d\mathbf{x} \, ds \\ &= \int_{\Omega \times D} M \mathcal{F}^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(t) + \alpha) \, d\mathbf{q} \, d\mathbf{x} - \int_{\Omega \times D} M \mathcal{F}^L(\beta^L(\widehat{\psi}^0) + \alpha) \, d\mathbf{q} \, d\mathbf{x} \\ &\quad + \frac{1}{2\Delta t} \int_0^t \int_{\Omega \times D} M (\mathcal{F}^L)''(\theta \widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + (1 - \theta) \widehat{\psi}_{\varepsilon,L}^{\Delta t,-} + \alpha) (\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} - \widehat{\psi}_{\varepsilon,L}^{\Delta t,-})^2 \, d\mathbf{q} \, d\mathbf{x} \, ds. \end{aligned}$$

Noting from (4.9) that $(\mathcal{F}^L)''(s + \alpha) \geq 1/L$ for all $s \in [0, \infty)$ and all $\alpha > 0$, this

then implies, with $t = t_n$, $n \in \{1, \dots, N\}$, that

$$\begin{aligned} T_1 &\geq \int_{\Omega \times D} M \mathcal{F}^L(\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(t) + \alpha) \, d\mathbf{q} \, d\mathbf{x} - \int_{\Omega \times D} M \mathcal{F}^L(\beta^L(\hat{\psi}^0) + \alpha) \, d\mathbf{q} \, d\mathbf{x} \\ &\quad + \frac{1}{2\Delta t L} \int_0^t \int_{\Omega \times D} M (\hat{\psi}_{\varepsilon,L}^{\Delta t,+} - \hat{\psi}_{\varepsilon,L}^{\Delta t,-})^2 \, d\mathbf{q} \, d\mathbf{x} \, ds. \end{aligned} \quad (4.14)$$

The denominator in the prefactor of the last integral motivates us to link Δt to L so that $\Delta t L = o(1)$ as $\Delta t \rightarrow 0_+$ (or, equivalently, $\Delta t = o(L^{-1})$ as $L \rightarrow \infty$), in order to drive the integral multiplied by the prefactor to 0 in the limit of $L \rightarrow \infty$, once the product of the two has been bounded above by a constant, independent of L .

Next we consider the second term in (4.4), using repeatedly that $\nabla_x \cdot \mathbf{u}_{\varepsilon,L}^{\Delta t,-} = 0$ and that $\mathbf{u}_{\varepsilon,L}^{\Delta t,-}$ has zero trace on $\partial\Omega$:

$$\begin{aligned} T_2 &:= \int_0^T \int_{\Omega \times D} M \left[\varepsilon \nabla_x \hat{\psi}_{\varepsilon,L}^{\Delta t,+} - \mathbf{u}_{\varepsilon,L}^{\Delta t,-} \cdot \hat{\psi}_{\varepsilon,L}^{\Delta t,+} \right] \cdot \nabla_x \chi_{[0,t]} (\mathcal{F}^L)'(\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha) \, d\mathbf{q} \, d\mathbf{x} \, ds \\ &= \varepsilon \int_0^t \int_{\Omega \times D} M \nabla_x (\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha) \cdot \nabla_x (\mathcal{F}^L)'(\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha) \, d\mathbf{q} \, d\mathbf{x} \, ds \\ &\quad - \int_0^t \int_{\Omega \times D} M \mathbf{u}_{\varepsilon,L}^{\Delta t,-} (\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha) \cdot \nabla_x (\mathcal{F}^L)'(\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha) \, d\mathbf{q} \, d\mathbf{x} \, ds, \end{aligned}$$

where in the last line we subtracted 0 in the form of

$$\alpha \int_0^t \int_{\Omega \times D} M \mathbf{u}_{\varepsilon,L}^{\Delta t,-} \cdot \nabla_x (\mathcal{F}^L)'(\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha) \, d\mathbf{q} \, d\mathbf{x} \, ds = 0.$$

Hence, similarly to (3.53),

$$\begin{aligned} T_2 &= \varepsilon \int_0^t \int_{\Omega \times D} M (\mathcal{F}^L)''(\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha) |\nabla_x (\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha)|^2 \, d\mathbf{q} \, d\mathbf{x} \, ds \\ &\quad - \int_0^t \int_{\Omega \times D} M \mathbf{u}_{\varepsilon,L}^{\Delta t,-} (\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha) \cdot [(\mathcal{F}^L)''(\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha) \nabla_x (\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha)] \, d\mathbf{q} \, d\mathbf{x} \, ds \\ &= \varepsilon \int_0^t \int_{\Omega \times D} M (\mathcal{F}^L)''(\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha) |\nabla_x \hat{\psi}_{\varepsilon,L}^{\Delta t,+}|^2 \, d\mathbf{q} \, d\mathbf{x} \, ds \\ &\quad - \int_0^t \int_{\Omega \times D} M \mathbf{u}_{\varepsilon,L}^{\Delta t,-} \cdot \nabla_x [G^L(\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha)] \, d\mathbf{q} \, d\mathbf{x} \, ds, \end{aligned}$$

where G^L denotes the (locally Lipschitz continuous) function defined on $\mathbb{R}$ by $s - \frac{L}{2}$ if $s \leq L$ and $\frac{1}{2L}s^2$ otherwise. On noting that the integral involving G^L vanishes, (4.11) yields the lower bound

$$T_2 \geq \varepsilon \int_0^t \int_{\Omega \times D} M (\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha)^{-1} |\nabla_x (\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha)|^2 \, d\mathbf{q} \, d\mathbf{x} \, ds. \quad (4.15)$$

Next, we consider the third term in (4.4). Thanks to (2.6) we have, again with

28 *John W. Barrett and Endre Süli*

$t = t_n$ and $n \in \{1, \dots, N\}$:

$$\begin{aligned} T_3 &:= \frac{1}{2\lambda} \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \sum_{j=1}^K A_{ij} \nabla_{q_j} \hat{\psi}_{\varepsilon,L}^{\Delta t,+} \cdot \nabla_{q_i} \chi_{[0,t]} (\mathcal{F}^L)' (\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha) \, dq \, d\tilde{x} \, ds \\ &\geq \frac{a_0}{2\lambda} \int_0^t \int_{\Omega \times D} M (\mathcal{F}^L)'' (\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha) |\nabla_q \hat{\psi}_{\varepsilon,L}^{\Delta t,+}|^2 \, dq \, d\tilde{x} \, ds. \end{aligned} \quad (4.16)$$

We are now ready to consider the final term in (4.4), with $t = t_n$, $n \in \{1, \dots, N\}$:

$$\begin{aligned} T_4 &:= - \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K [\mathcal{G}(\hat{u}_{\varepsilon,L}^{\Delta t,+}) q_i] \beta^L (\hat{\psi}_{\varepsilon,L}^{\Delta t,+}) \cdot \nabla_{q_i} \chi_{[0,t]} (\mathcal{F}^L)' (\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha) \, dq \, d\tilde{x} \, ds \\ &= - \int_0^t \int_{\Omega \times D} M \sum_{i=1}^K q_i q_i^T U'_i(\tfrac{1}{2}|q|^2) \hat{\psi}_{\varepsilon,L}^{\Delta t,+} : \nabla_x \hat{u}_{\varepsilon,L}^{\Delta t,+} \, dq \, d\tilde{x} \, ds \\ &\quad + \int_0^t \int_{\Omega \times D} M \sum_{i=1}^K \left[(\nabla_x \hat{u}_{\varepsilon,L}^{\Delta t,+}) q_i \right] \left[1 - \frac{\beta^L (\hat{\psi}_{\varepsilon,L}^{\Delta t,+})}{\beta^L (\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha)} \right] \cdot \nabla_{q_i} \hat{\psi}_{\varepsilon,L}^{\Delta t,+} \, dq \, d\tilde{x} \, ds, \end{aligned} \quad (4.17)$$

where in the transition to the second equality we applied (3.16) with $B := \nabla_x \hat{u}_{\varepsilon,L}^{\Delta t,+}$ (on account of it being independent of the variable q), together with the fact that $\text{tr}(\nabla_x \hat{u}_{\varepsilon,L}^{\Delta t,+}) = \nabla_x \cdot \hat{u}_{\varepsilon,L}^{\Delta t,+} = 0$. Summing (4.14)–(4.16) and (4.17) yields, with $t = t_n$ and $n \in \{1, \dots, N\}$, the following inequality:

$$\begin{aligned} &\int_{\Omega \times D} M \mathcal{F}^L (\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(t) + \alpha) \, dq \, d\tilde{x} + \frac{1}{2\Delta t L} \int_0^t \int_{\Omega \times D} M (\hat{\psi}_{\varepsilon,L}^{\Delta t,+} - \hat{\psi}_{\varepsilon,L}^{\Delta t,-})^2 \, dq \, d\tilde{x} \, ds \\ &\quad + \varepsilon \int_0^t \int_{\Omega \times D} M \frac{|\nabla_x \hat{\psi}_{\varepsilon,L}^{\Delta t,+}|^2}{\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha} \, dq \, d\tilde{x} \, ds \\ &\quad + \frac{a_0}{2\lambda} \int_0^t \int_{\Omega \times D} M (\mathcal{F}^L)'' (\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha) |\nabla_q \hat{\psi}_{\varepsilon,L}^{\Delta t,+}|^2 \, dq \, d\tilde{x} \, ds \\ &\leq \int_{\Omega \times D} M \mathcal{F}^L (\beta^L (\hat{\psi}^0) + \alpha) \, dq \, d\tilde{x} \\ &\quad + \int_0^t \int_{\Omega \times D} M \sum_{i=1}^K q_i q_i^T U'_i(\tfrac{1}{2}|q|^2) \hat{\psi}_{\varepsilon,L}^{\Delta t,+} : \nabla_x \hat{u}_{\varepsilon,L}^{\Delta t,+} \, dq \, d\tilde{x} \, ds \\ &\quad - \int_0^t \int_{\Omega \times D} M \sum_{i=1}^K \left[(\nabla_x \hat{u}_{\varepsilon,L}^{\Delta t,+}) q_i \right] \left[1 - \frac{\beta^L (\hat{\psi}_{\varepsilon,L}^{\Delta t,+})}{\beta^L (\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha)} \right] \cdot \nabla_{q_i} \hat{\psi}_{\varepsilon,L}^{\Delta t,+} \, dq \, d\tilde{x} \, ds. \end{aligned} \quad (4.18)$$

Comparing (4.18) with (4.13) we see that after multiplying (4.18) by $2k$ and adding the resulting inequality to (4.13) the final term in (4.13) is cancelled by $2k$ times the second term on the right-hand side of (4.18). Hence, for any $t = t_n$, with

$n \in \{1, \dots, N\}$, we deduce that

$$\begin{aligned}
& \|u_{\varepsilon,L}^{\Delta t,+}(t)\|^2 + \frac{1}{\Delta t} \int_0^t \|u_{\varepsilon,L}^{\Delta t,+} - u_{\varepsilon,L}^{\Delta t,-}\|^2 ds + \nu \int_0^t \|\nabla_x u_{\varepsilon,L}^{\Delta t,+}(s)\|^2 ds \\
& + 2k \int_{\Omega \times D} M \mathcal{F}^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(t) + \alpha) dq d\tilde{x} + \frac{k}{\Delta t L} \int_0^t \int_{\Omega \times D} M (\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} - \widehat{\psi}_{\varepsilon,L}^{\Delta t,-})^2 dq d\tilde{x} ds \\
& + 2k \varepsilon \int_0^t \int_{\Omega \times D} M \frac{|\nabla_x \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}|^2}{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha} dq d\tilde{x} ds \\
& + \frac{a_0 k}{\lambda} \int_0^t \int_{\Omega \times D} M (\mathcal{F}^L)''(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha) |\nabla_q \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}|^2 dq d\tilde{x} ds \\
& \leq \|u_0\|^2 + \frac{1}{\nu} \int_0^t \|f^{\Delta t,+}(s)\|_{(H_0^1(\Omega))'}^2 ds + 2k \int_{\Omega \times D} M \mathcal{F}^L(\beta^L(\widehat{\psi}^0) + \alpha) dq d\tilde{x} \\
& - 2k \int_0^t \int_{\Omega \times D} M \sum_{i=1}^K [(\nabla_x u_{\varepsilon,L}^{\Delta t,+}) q_i] \left[1 - \frac{\beta^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+})}{\beta^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha)} \right] \cdot \nabla_{q_i} \widehat{\psi}_{\varepsilon,L}^{\Delta t,+} dq d\tilde{x} ds.
\end{aligned} \tag{4.19}$$

It remains to bound the last term on the right-hand side of (4.19). Noting that β^L is Lipschitz continuous, with Lipschitz constant equal to 1, and $\beta^L(s + \alpha) \geq \alpha$ for $s \geq 0$, we have that

$$\begin{aligned}
0 & \leq \left(1 - \frac{\beta^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+})}{\beta^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha)} \right) \frac{1}{\sqrt{(\mathcal{F}^L)''(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha)}} = \frac{\beta^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha) - \beta^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+})}{\sqrt{\beta^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha)}} \\
& \leq \frac{\beta^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha) - \beta^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+})}{\sqrt{\alpha}} \leq \begin{cases} \sqrt{\alpha} & \text{when } \widehat{\psi}_{\varepsilon,L}^{\Delta t,+} \leq L, \\ 0 & \text{when } \widehat{\psi}_{\varepsilon,L}^{\Delta t,+} \geq L. \end{cases}
\end{aligned}$$

With this bound we now focus our attention on the last term in the inequality (4.19). Let $\underline{b} := (b_1, \dots, b_K)$ and $b := |\underline{b}|_1 := b_1 + \dots + b_K$; as $|q_i| \leq \sqrt{b_i}$, $i = 1, \dots, K$, we have that $|q| \leq b^{1/2}$ for all $q \in D$. For $t = t_n$, $n \in \{1, \dots, N\}$, we then have that⁹

$$\begin{aligned}
& \left| -2k \int_0^t \int_{\Omega \times D} M \sum_{i=1}^K [(\nabla_x u_{\varepsilon,L}^{\Delta t,+}) q_i] \left[1 - \frac{\beta^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+})}{\beta^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha)} \right] \cdot \nabla_{q_i} \widehat{\psi}_{\varepsilon,L}^{\Delta t,+} dq d\tilde{x} ds \right| \\
& \leq \frac{a_0 k}{2\lambda} \left(\int_0^t \int_{\Omega \times D} M (\mathcal{F}^L)''(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha) |\nabla_q \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}|^2 dq d\tilde{x} ds \right) \\
& \quad + \alpha \frac{2\lambda b k}{a_0} \left(\int_0^t \int_{\Omega} |\nabla_x u_{\varepsilon,L}^{\Delta t,+}|^2 d\tilde{x} ds \right).
\end{aligned} \tag{4.20}$$

Substitution of (4.20) into (4.19) and use of (4.11) to bound $(\mathcal{F}^L)''(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha)$ from below by $\mathcal{F}''(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha) = (\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha)^{-1}$ and (4.12) to bound $\mathcal{F}^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha)$ by

30 *John W. Barrett and Endre Süli*

$\mathcal{F}(\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha)$ from below finally yields, for all $t = t_n$, $n \in \{1, \dots, N\}$, that

$$\begin{aligned} & \|\hat{u}_{\varepsilon,L}^{\Delta t,+}(t)\|^2 + \frac{1}{\Delta t} \int_0^t \|\hat{u}_{\varepsilon,L}^{\Delta t,+} - \hat{u}_{\varepsilon,L}^{\Delta t,-}\|^2 ds + \nu \int_0^t \|\nabla_x \hat{u}_{\varepsilon,L}^{\Delta t,+}(s)\|^2 ds \\ & + 2k \int_{\Omega \times D} M \mathcal{F}(\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(t) + \alpha) dq d\tilde{x} + \frac{k}{\Delta t L} \int_0^t \int_{\Omega \times D} M (\hat{\psi}_{\varepsilon,L}^{\Delta t,+} - \hat{\psi}_{\varepsilon,L}^{\Delta t,-})^2 dq d\tilde{x} ds \\ & + 2k \varepsilon \int_0^t \int_{\Omega \times D} M \frac{|\nabla_x \hat{\psi}_{\varepsilon,L}^{\Delta t,+}|^2}{\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha} dq d\tilde{x} ds + \frac{a_0 k}{2\lambda} \int_0^t \int_{\Omega \times D} M \frac{|\nabla_q \hat{\psi}_{\varepsilon,L}^{\Delta t,+}|^2}{\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha} dq d\tilde{x} ds \\ & \leq \|u_0\|^2 + \frac{1}{\nu} \int_0^t \|f^{\Delta t,+}(s)\|_{(H_0^1(\Omega))'}^2 ds + 2k \int_{\Omega \times D} M \mathcal{F}^L(\beta^L(\hat{\psi}^0) + \alpha) dq d\tilde{x} \\ & \quad + \alpha \frac{2\lambda b k}{a_0} \int_0^t \|\nabla_x \hat{u}_{\varepsilon,L}^{\Delta t,+}(s)\|^2 ds. \end{aligned} \quad (4.21)$$

The only restriction we have imposed on α so far is that it belongs to the open interval $(0, 1)$; let us now restrict the range of α further by demanding that, in fact,

$$0 < \alpha < \min\left(1, \frac{a_0 \nu}{2\lambda b k}\right). \quad (4.22)$$

Then, the last term on the right-hand side of (4.21) can be absorbed into the third term on the left-hand side, giving, for $t = t_n$ and $n \in \{1, \dots, N\}$,

$$\begin{aligned} & \|\hat{u}_{\varepsilon,L}^{\Delta t,+}(t)\|^2 + \frac{1}{\Delta t} \int_0^t \|\hat{u}_{\varepsilon,L}^{\Delta t,+} - \hat{u}_{\varepsilon,L}^{\Delta t,-}\|^2 ds + \left(\nu - \alpha \frac{2\lambda b k}{a_0}\right) \int_0^t \|\nabla_x \hat{u}_{\varepsilon,L}^{\Delta t,+}(s)\|^2 ds \\ & + 2k \int_{\Omega \times D} M \mathcal{F}(\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(t) + \alpha) dq d\tilde{x} + \frac{k}{\Delta t L} \int_0^t \int_{\Omega \times D} M (\hat{\psi}_{\varepsilon,L}^{\Delta t,+} - \hat{\psi}_{\varepsilon,L}^{\Delta t,-})^2 dq d\tilde{x} ds \\ & + 2k \varepsilon \int_0^t \int_{\Omega \times D} M \frac{|\nabla_x \hat{\psi}_{\varepsilon,L}^{\Delta t,+}|^2}{\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha} dq d\tilde{x} ds + \frac{a_0 k}{2\lambda} \int_0^t \int_{\Omega \times D} M \frac{|\nabla_q \hat{\psi}_{\varepsilon,L}^{\Delta t,+}|^2}{\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha} dq d\tilde{x} ds \\ & \leq \|u_0\|^2 + \frac{1}{\nu} \int_0^t \|f^{\Delta t,+}(s)\|_{(H_0^1(\Omega))'}^2 ds + 2k \int_{\Omega \times D} M \mathcal{F}^L(\beta^L(\hat{\psi}^0) + \alpha) dq d\tilde{x}. \end{aligned} \quad (4.23)$$

We now focus our attention on the final integral on the right-hand side of (4.23):

$$\mathbf{T}_5(\alpha) := \int_{\Omega \times D} M \mathcal{F}^L(\beta^L(\hat{\psi}^0) + \alpha) dq d\tilde{x} = \int_{\mathfrak{A}_{L,\alpha} \cup \mathfrak{B}_{L,\alpha}} M \mathcal{F}^L(\beta^L(\hat{\psi}^0) + \alpha) dq d\tilde{x},$$

where

$$\begin{aligned} \mathfrak{A}_{L,\alpha} &:= \{(x, q) \in \Omega \times D : 0 \leq \beta^L(\hat{\psi}^0(x, q)) \leq L - \alpha\}, \\ \mathfrak{B}_{L,\alpha} &:= \{(x, q) \in \Omega \times D : L - \alpha < \beta^L(\hat{\psi}^0(x, q)) \leq L\}. \end{aligned}$$

We begin by noting that

$$\int_{\mathfrak{A}_{L,\alpha}} M \mathcal{F}^L(\beta^L(\hat{\psi}^0) + \alpha) dq d\tilde{x} = \int_{\mathfrak{A}_{L,\alpha}} M \mathcal{F}(\beta^L(\hat{\psi}^0) + \alpha) dq d\tilde{x}.$$

For the integral over $\mathfrak{B}_{L,\alpha}$ we have

$$\int_{\mathfrak{B}_{L,\alpha}} M \mathcal{F}^L(\beta^L(\hat{\psi}^0) + \alpha) \, d\mathbf{q} \, d\mathbf{x} \leq \frac{3}{2} \alpha |\Omega| + \int_{\mathfrak{B}_{L,\alpha}} M \mathcal{F}(\beta^L(\hat{\psi}^0) + \alpha) \, d\mathbf{q} \, d\mathbf{x}.$$

Thus we have shown that

$$T_5(\alpha) \leq \frac{3}{2} \alpha |\Omega| + \int_{\Omega \times D} M \mathcal{F}(\beta^L(\hat{\psi}^0) + \alpha) \, d\mathbf{q} \, d\mathbf{x}. \quad (4.24)$$

Now, there are two possibilities:

- Case 1.* If $\beta^L(\hat{\psi}^0) + \alpha \leq 1$, then $0 \leq \beta^L(\hat{\psi}^0) \leq 1 - \alpha$. Since $L > 1$ it follows that $0 \leq \beta^L(s) \leq 1$ if, and only if, $\beta^L(s) = s$. Thus we deduce that in this case $\beta^L(\hat{\psi}^0) = \hat{\psi}^0$, and therefore $0 \leq \mathcal{F}(\beta^L(\hat{\psi}^0) + \alpha) = \mathcal{F}(\hat{\psi}^0 + \alpha)$.
- Case 2.* Alternatively, if $1 < \beta^L(\hat{\psi}^0) + \alpha$, then, on noting that $\beta^L(s) \leq s$ for all $s \in [0, \infty)$, it follows that $1 < \beta^L(\hat{\psi}^0) + \alpha \leq \hat{\psi}^0 + \alpha$. However the function $\mathcal{F}$ is strictly monotonic increasing on the interval $[1, \infty)$, which then implies that $0 = \mathcal{F}(1) < \mathcal{F}(\beta^L(\hat{\psi}^0) + \alpha) \leq \mathcal{F}(\hat{\psi}^0 + \alpha)$.

The conclusion we draw is that, either way, $0 \leq \mathcal{F}(\beta^L(\hat{\psi}^0) + \alpha) \leq \mathcal{F}(\hat{\psi}^0 + \alpha)$. Hence,

$$T_5(\alpha) \leq \frac{3}{2} \alpha |\Omega| + \int_{\Omega \times D} M \mathcal{F}(\hat{\psi}^0 + \alpha) \, d\mathbf{q} \, d\mathbf{x}. \quad (4.25)$$

Substituting (4.25) into (4.23) thus yields, for $t = t_n$ and $n \in \{1, \dots, N\}$,

$$\begin{aligned} & \|u_{\varepsilon,L}^{\Delta t,+}(t)\|^2 + \frac{1}{\Delta t} \int_0^t \|u_{\varepsilon,L}^{\Delta t,+} - u_{\varepsilon,L}^{\Delta t,-}\|^2 \, ds + \left(\nu - \alpha \frac{2\lambda b k}{a_0} \right) \int_0^t \|\nabla_x u_{\varepsilon,L}^{\Delta t,+}(s)\|^2 \, ds \\ & + 2k \int_{\Omega \times D} M \mathcal{F}(\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(t) + \alpha) \, d\mathbf{q} \, d\mathbf{x} + \frac{k}{\Delta t L} \int_0^t \int_{\Omega \times D} M (\hat{\psi}_{\varepsilon,L}^{\Delta t,+} - \hat{\psi}_{\varepsilon,L}^{\Delta t,-})^2 \, d\mathbf{q} \, d\mathbf{x} \, ds \\ & + 2k \varepsilon \int_0^t \int_{\Omega \times D} M \frac{|\nabla_x \hat{\psi}_{\varepsilon,L}^{\Delta t,+}|^2}{\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha} \, d\mathbf{q} \, d\mathbf{x} \, ds + \frac{a_0 k}{2\lambda} \int_0^t \int_{\Omega \times D} M \frac{|\nabla_q \hat{\psi}_{\varepsilon,L}^{\Delta t,+}|^2}{\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha} \, d\mathbf{q} \, d\mathbf{x} \, ds \\ & \leq \|u_0\|^2 + \frac{1}{\nu} \int_0^t \|f^{\Delta t,+}(s)\|_{(H_0^1(\Omega))'}^2 \, ds + 3\alpha k |\Omega| + 2k \int_{\Omega \times D} M \mathcal{F}(\hat{\psi}^0 + \alpha) \, d\mathbf{q} \, d\mathbf{x}. \quad (4.26) \end{aligned}$$

The key observation at this point is that the right-hand side of (4.26) is completely independent of the cut-off parameter L .

We shall tidy up the bound (4.26) by passing to the limit $\alpha \rightarrow 0_+$. The first α -dependent term on the right-hand side of (4.26) trivially converges to 0 as $\alpha \rightarrow 0_+$; concerning the second α -dependent term, Lebesgue's dominated convergence theorem implies that

$$\lim_{\alpha \rightarrow 0_+} \int_{\Omega \times D} M \mathcal{F}(\hat{\psi}^0 + \alpha) \, d\mathbf{q} \, d\mathbf{x} = \int_{\Omega \times D} M \mathcal{F}(\hat{\psi}^0) \, d\mathbf{q} \, d\mathbf{x}.$$

Similarly, we can easily pass to the limit on the left-hand side of (4.26). By applying Fatou's lemma to the fourth, sixth and seventh term on the left-hand side of (4.26)

32 *John W. Barrett and Endre Süli*

we get, for $t = t_n$, $n \in \{1, \dots, N\}$, that

$$\begin{aligned}
\liminf_{\alpha \rightarrow 0+} \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(t) + \alpha) &\geq \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(t)) \, d\mathbf{q} \, d\mathbf{x}, \\
\liminf_{\alpha \rightarrow 0+} \int_0^t \int_{\Omega \times D} M \frac{|\nabla_x \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}|^2}{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha} \, d\mathbf{q} \, d\mathbf{x} \, ds &\geq \int_0^t \int_{\Omega \times D} M \frac{|\nabla_x \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}|^2}{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}} \, d\mathbf{q} \, d\mathbf{x} \, ds \\
&= 4 \int_0^t \int_{\Omega \times D} M |\nabla_x \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}}|^2 \, d\mathbf{q} \, d\mathbf{x} \, ds, \\
\liminf_{\alpha \rightarrow 0+} \int_0^t \int_{\Omega \times D} M \frac{|\nabla_q \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}|^2}{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha} \, d\mathbf{q} \, d\mathbf{x} \, ds &\geq \int_0^t \int_{\Omega \times D} M \frac{|\nabla_q \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}|^2}{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}} \, d\mathbf{q} \, d\mathbf{x} \, ds \\
&= 4 \int_0^t \int_{\Omega \times D} M |\nabla_q \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}}|^2 \, d\mathbf{q} \, d\mathbf{x} \, ds.
\end{aligned}$$

Thus, after passage to the limit $\alpha \rightarrow 0+$, on recalling (3.20), we have, for all $t = t_n$, $n \in \{1, \dots, N\}$, that

$$\begin{aligned}
&\|u_{\varepsilon,L}^{\Delta t,+}(t)\|^2 + \frac{1}{\Delta t} \int_0^t \|\mathbf{u}_{\varepsilon,L}^{\Delta t,+} - \mathbf{u}_{\varepsilon,L}^{\Delta t,-}\|^2 \, ds + \nu \int_0^t \|\nabla_x \mathbf{u}_{\varepsilon,L}^{\Delta t,+}(s)\|^2 \, ds \\
&\quad + 2k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(t)) \, d\mathbf{q} \, d\mathbf{x} + \frac{k}{\Delta t L} \int_0^t \int_{\Omega \times D} M (\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} - \widehat{\psi}_{\varepsilon,L}^{\Delta t,-})^2 \, d\mathbf{q} \, d\mathbf{x} \, ds \\
&\quad + 8k\varepsilon \int_0^t \int_{\Omega \times D} M |\nabla_x \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}}|^2 \, d\mathbf{q} \, d\mathbf{x} \, ds \\
&\quad + \frac{2a_0 k}{\lambda} \int_0^t \int_{\Omega \times D} M |\nabla_q \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}}|^2 \, d\mathbf{q} \, d\mathbf{x} \, ds \\
&\leq \|u_0\|^2 + \frac{1}{\nu} \int_0^t \|f^{\Delta t,+}(s)\|_{(H_0^1(\Omega))'}^2 \, ds + 2k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}^0) \, d\mathbf{q} \, d\mathbf{x} \tag{4.27} \\
&\leq \|u_0\|^2 + \frac{1}{\nu} \int_0^T \|f(s)\|_{(H_0^1(\Omega))'}^2 \, ds + 2k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_0) \, d\mathbf{q} \, d\mathbf{x} =: [\mathbf{B}(u_0, f, \widehat{\psi}_0)]^2, \tag{4.28}
\end{aligned}$$

where, in the last line, we used (3.20) to bound the third term in (4.27), and that $t \in [0, T]$ together with the definition (3.22) of $f^{\Delta t,+}$ to bound the second term.

We select $\varphi = \chi_{[0,t]} \rho_{\varepsilon,L}^{\Delta t,+}$ as test function in (4.5), with t chosen as t_n and $n \in \{1, \dots, N\}$. Then, similarly to (4.13), we deduce, with $t = t_n$, that

$$\begin{aligned}
\|\rho_{\varepsilon,L}^{\Delta t,+}(t)\|^2 + \frac{1}{\Delta t} \int_0^t \|\rho_{\varepsilon,L}^{\Delta t,+}(s) - \rho_{\varepsilon,L}^{\Delta t,-}(s)\|^2 \, ds + 2\varepsilon \int_0^t \|\nabla_x \rho_{\varepsilon,L}^{\Delta t,+}(s)\|^2 \, ds \\
\leq \left\| \int_D \beta^L(\widehat{\psi}^0) \, d\mathbf{q} \right\|_{\sim}^2 \leq |\Omega|, \tag{4.29}
\end{aligned}$$

where we have noted (3.43), (3.30) and that $\beta^L(\widehat{\psi}^0) \in \widehat{\mathcal{Z}}_2$.

4.2. L -independent bounds on the time-derivatives

Next, we derive L -independent bounds on the time-derivatives of the functions $u_{\varepsilon,L}^{\Delta t}$, $\widehat{\psi}_{\varepsilon,L}^{\Delta t}$ and $\rho_{\varepsilon,L}^{\Delta t}$. We begin by bounding the time-derivative of $\widehat{\psi}_{\varepsilon,L}^{\Delta t}$ using (4.28); we shall then bound the time-derivatives of $\rho_{\varepsilon,L}^{\Delta t}$ and $u_{\varepsilon,L}^{\Delta t}$ in a similar manner.

4.2.1. L -independent bound on the time-derivative of $\widehat{\psi}_{\varepsilon,L}^{\Delta t}$

It follows from (4.4) that

$$\begin{aligned} & \left| \int_0^T \int_{\Omega \times D} M \frac{\partial \widehat{\psi}_{\varepsilon,L}^{\Delta t}}{\partial t} \widehat{\varphi} \, dq \, dx \, dt \right| \leq \left| \int_0^T \int_{\Omega \times D} M \nabla_x \widehat{\psi}_{\varepsilon,L}^{\Delta t,+} \cdot \nabla_x \widehat{\varphi} \, dq \, dx \, dt \right| \\ & + \left| \int_0^T \int_{\Omega \times D} M u_{\varepsilon,L}^{\Delta t,-} \widehat{\psi}_{\varepsilon,L}^{\Delta t,+} \cdot \nabla_x \widehat{\varphi} \, dq \, dx \, dt \right| \\ & + \left| \frac{1}{2\lambda} \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \sum_{j=1}^K A_{ij} \nabla_{q_j} \widehat{\psi}_{\varepsilon,L}^{\Delta t,+} \cdot \nabla_{q_i} \widehat{\varphi} \, dq \, dx \, dt \right| \\ & + \left| \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \left[\sigma(u_{\varepsilon,L}^{\Delta t,+}) q_i \right] \beta^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}) \cdot \nabla_{q_i} \widehat{\varphi} \, dq \, dx \, dt \right| \\ & =: S_1 + S_2 + S_3 + S_4 \quad \forall \widehat{\varphi} \in L^1(0, T; \widehat{X}). \end{aligned} \quad (4.30)$$

We proceed to bound each of the terms $S_1, \dots, S_4$, bearing in mind (cf. the last sentence in the statement of Lemma 3.3) that

$$\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} \geq 0 \quad \text{a.e. on } \Omega \times D \times [0, T], \quad \int_D M(q) \, dq = 1, \quad (4.31a)$$

$$0 \leq \int_D M(q) \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(x, q, t) \, dq \leq 1 \quad \text{for a.e. } (x, t) \in \Omega \times D. \quad (4.31b)$$

We shall use throughout the rest of this section test functions $\widehat{\varphi}$ such that

$$\widehat{\varphi} \in L^2(0, T; H^1(\Omega; L^\infty(D)) \cap L^2(\Omega; W^{1,\infty}(D))). \quad (4.32)$$

We begin by considering S_1 ; noting (4.31a,b) and (4.28), we have that

$$S_1 \leq \sqrt{\frac{\varepsilon}{2k}} \mathbf{B}(u_0, \widehat{f}, \widehat{\psi}_0) \left(\int_0^T \int_{\Omega} \|\nabla_x \widehat{\varphi}\|_{L^\infty(D)}^2 \, dx \, dt \right)^{\frac{1}{2}}. \quad (4.33)$$

Next, we consider the term S_2 :

$$S_2 \leq C_P(\Omega) \left(\int_0^T \int_{\Omega} |\nabla_x u_{\varepsilon,L}^{\Delta t,-}|^2 \, dx \, dt \right)^{\frac{1}{2}} \left(\int_0^T \int_{\Omega} \|\nabla_x \widehat{\varphi}\|_{L^\infty(D)}^2 \, dx \, dt \right)^{\frac{1}{2}}, \quad (4.34)$$

where $C_P(\Omega)$ denotes the (positive) constant appearing in the Poincaré inequality $\|u_{\varepsilon,L}^{\Delta t,-}\| \leq C_P(\Omega) \|\nabla_x u_{\varepsilon,L}^{\Delta t,-}\|$ on Ω , with $u_{\varepsilon,L}^{\Delta t,-} \in V \subset H_0^1(\Omega)$. On recalling the

definitions of $u_{\varepsilon,L}^{\Delta t,+}$ and $u_{\varepsilon,L}^{\Delta t,-}$ from (4.1b), and noting (3.18), we have that

$$\begin{aligned} \int_0^T \int_{\Omega} |\nabla_x u_{\varepsilon,L}^{\Delta t,-}|^2 dx dt &= \Delta t \|\nabla_x u^0\|^2 + \int_0^{T-\Delta t} \|\nabla_x u_{\varepsilon,L}^{\Delta t,+}\|^2 dt \\ &\leq \|u_0\|^2 + \int_0^T \|\nabla_x u_{\varepsilon,L}^{\Delta t,+}\|^2 dt \leq (1 + \frac{1}{\nu}) [\mathbf{B}(u_0, f, \hat{\psi}_0)]^2. \end{aligned} \quad (4.35)$$

Therefore,

$$S_2 \leq C_P(\Omega) (1 + \frac{1}{\nu})^{\frac{1}{2}} \mathbf{B}(u_0, f, \hat{\psi}_0) \left(\int_0^T \int_{\Omega} \|\nabla_x \hat{\varphi}\|_{L^\infty(D)}^2 dx dt \right)^{\frac{1}{2}}. \quad (4.36)$$

Alternatively, directly from the second line of (4.34), we have that

$$S_2 \leq \sqrt{T} \left(\text{ess.sup}_{t \in [0,T]} \int_{\Omega} |u_{\varepsilon,L}^{\Delta t,-}|^2 dx \right)^{\frac{1}{2}} \left(\int_0^T \int_{\Omega} \|\nabla_x \hat{\varphi}\|_{L^\infty(D)}^2 dx dt \right)^{\frac{1}{2}}. \quad (4.37)$$

Similarly as above,

$$\begin{aligned} \text{ess.sup}_{t \in [0,T]} \int_{\Omega} |u_{\varepsilon,L}^{\Delta t,-}|^2 dx &= \text{ess.sup}_{t \in [0,T]} \|u_{\varepsilon,L}^{\Delta t,-}(t)\|^2 \\ &= \max \left(\|u_0\|^2, \text{ess.sup}_{t \in (0, T-\Delta t]} \|u_{\varepsilon,L}^{\Delta t,+}(t)\|^2 \right) \\ &\leq \max \left(\|u_0\|^2, \text{ess.sup}_{t \in [0,T]} \|u_{\varepsilon,L}^{\Delta t,+}(t)\|^2 \right) \leq [\mathbf{B}(u_0, f, \hat{\psi}_0)]^2. \end{aligned} \quad (4.38)$$

Combining (4.36), (4.37) and (4.38), we have that

$$S_2 \leq \min \left(C_P(\Omega) (1 + \frac{1}{\nu})^{\frac{1}{2}}, \sqrt{T} \right) \mathbf{B}(u_0, f, \hat{\psi}_0) \left(\int_0^T \int_{\Omega} \|\nabla_x \hat{\varphi}\|_{L^\infty(D)}^2 dx dt \right)^{\frac{1}{2}}. \quad (4.39)$$

We are ready to consider S_3 ; we have, by (4.28), that

$$S_3 \leq \frac{|A|}{\sqrt{2a_0 k \lambda}} \mathbf{B}(u_0, f, \hat{\psi}_0) \left(\int_0^T \int_{\Omega} \|\nabla_q \hat{\varphi}\|_{L^\infty(D)}^2 dx dt \right)^{\frac{1}{2}}. \quad (4.40)$$

Finally, for term S_4 , recalling the notation $b := |b|_1$ (cf. the paragraph before (4.20)) together with the inequality $\beta^L(s) \leq s$ for $s \in \mathbb{R}_{\geq 0}$ and (4.28), we have that

$$S_4 \leq \sqrt{\frac{b}{\nu}} \mathbf{B}(u_0, f, \hat{\psi}_0) \left(\int_0^T \int_{\Omega} \|\nabla_q \hat{\varphi}\|_{L^\infty(D)}^2 dx dt \right)^{\frac{1}{2}}. \quad (4.41)$$

Upon substituting the bounds on the terms S_1 to S_4 into (4.30), with $\hat{\varphi} \in L^2(0, T; H^1(\Omega; L^\infty(D)) \cap L^2(\Omega; W^{1,\infty}(D)))$, and noting that the latter space is contained in $L^1(0, T; \hat{X})$ we deduce from (4.30) that

$$\begin{aligned} &\left| \int_0^T \int_{\Omega \times D} M \frac{\partial \hat{\psi}_{\varepsilon,L}^{\Delta t}}{\partial t} \hat{\varphi} dq dx dt \right| \\ &\leq C_* \mathbf{B}(u_0, f, \hat{\psi}_0) \left(\int_0^T \int_{\Omega} [\|\nabla_x \hat{\varphi}\|_{L^\infty(D)}^2 + \|\nabla_q \hat{\varphi}\|_{L^\infty(D)}^2] dx dt \right)^{\frac{1}{2}}, \end{aligned} \quad (4.42)$$

for any $\widehat{\varphi} \in L^2(0, T; H^1(\Omega; L^\infty(D)) \cap L^2(\Omega; W^{1,\infty}(D)))$, where C_* denotes a positive constant (that can be computed by tracking the constants in (4.33)–(4.41)), which depends solely on ε , ν , $C_P(\Omega)$, T , $|A|$, a_0 , k , K , λ , K and b .

We now consider the time-derivative of $\rho_{\varepsilon,L}^{\Delta t}$. It follows from (4.5), (4.29), (4.31b) and (4.38) that

$$\begin{aligned} \left| \int_0^T \int_\Omega \frac{\partial \rho_{\varepsilon,L}^{\Delta t}}{\partial t} \varphi \, dx \, dt \right| &\leq \int_0^T \int_\Omega \left| \left[\varepsilon \nabla_x \rho_{\varepsilon,L}^{\Delta t,+} - u_{\varepsilon,L}^{\Delta t,-} \rho_{\varepsilon,L}^{\Delta t,+} \right] \cdot \nabla_x \varphi \right| \, dx \, dt \\ &\leq \left[\varepsilon \left(\int_0^T \|\nabla_x \rho_{\varepsilon,L}^{\Delta t,+}\|^2 \, dt \right)^{\frac{1}{2}} + \text{ess.sup}_{t \in [0,T]} \|\rho_{\varepsilon,L}^{\Delta t,+}\|_{L^\infty(\Omega)} \left(\int_0^T \|u_{\varepsilon,L}^{\Delta t,-}\|^2 \, dt \right)^{\frac{1}{2}} \right] \\ &\quad \times \left(\int_0^T \|\nabla_x \varphi\|^2 \, dt \right)^{\frac{1}{2}} \\ &\leq \left[\varepsilon \left(\frac{|\Omega|}{2} \right)^{\frac{1}{2}} + B(u_0, f, \widehat{\psi}_0) \right] \left(\int_0^T \|\nabla_x \varphi\|^2 \, dt \right)^{\frac{1}{2}} \quad \forall \varphi \in L^2(0, T; H^1(\Omega)). \end{aligned} \quad (4.43)$$

4.2.2. L -independent bound on the time-derivative of $u_{\varepsilon,L}^{\Delta t}$

In this section we shall derive an L -independent bound on the time-derivative of $u_{\varepsilon,L}^{\Delta t}$. Our starting point is (4.3), from which we deduce that

$$\begin{aligned} \left| \int_0^T \int_\Omega \frac{\partial u_{\varepsilon,L}^{\Delta t}}{\partial t} \cdot w \, dx \, dt \right| &\leq \left| \int_0^T \int_\Omega \left[(u_{\varepsilon,L}^{\Delta t,-} \cdot \nabla_x) u_{\varepsilon,L}^{\Delta t,+} \right] \cdot w \, dx \, dt \right| + \nu \left| \int_0^T \int_\Omega \nabla_x u_{\varepsilon,L}^{\Delta t,+} : \nabla_x w \, dx \, dt \right| \\ &\quad + \left| \int_0^T \langle f^{\Delta t,+}, w \rangle_{H_0^1(\Omega)} \, dt \right| + k \left| \sum_{i=1}^K \int_0^T \int_\Omega C_i(M \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}) : \nabla_x w \, dx \, dt \right| \\ &=: U_1 + U_2 + U_3 + U_4 \quad \forall w \in L^1(0, T; V). \end{aligned} \quad (4.44)$$

On recalling from the discussion following (3.3) the definition of V_σ , we shall assume henceforth that

$$w \in L^2(0, T; V_\sigma), \quad \sigma \geq \frac{1}{2}d, \quad \sigma > 1.$$

Clearly, $L^2(0, T; V_\sigma) \subset L^1(0, T; V)$. By Lemma 4.1 in Ch. 3 of Temam³⁷ and using (4.38) and (4.28), we have, with $\sigma \geq \frac{1}{2}d$, $\sigma > 1$, and $c(\Omega, d)$ a constant that only depends on Ω and d , that

$$U_1 \leq c(\Omega, d) \sqrt{\frac{1}{\nu}} [B(u_0, f, \widehat{\psi}_0)]^2 \left(\int_0^T \|w\|_{V_\sigma}^2 \, dt \right)^{\frac{1}{2}}. \quad (4.45)$$

36 *John W. Barrett and Endre Süli*

For the term U_2 we have,

$$U_2 \leq \sqrt{\nu} B(u_0, f, \widehat{\psi}_0) \left(\int_0^T \|\nabla_x w\|^2 dt \right)^{\frac{1}{2}}. \quad (4.46)$$

Concerning the term U_3 , on noting the definition of the norm $\|\cdot\|_{(H_0^1(\Omega))'}$ and that thanks to (3.22) we have $\|f^{\Delta t,+}\|_{L^2(0,T;(H_0^1(\Omega))')} \leq \|f\|_{L^2(0,T;(H_0^1(\Omega))')}$, it follows that

$$U_3 \leq \sqrt{\nu} B(u_0, f, \widehat{\psi}_0) \left(\int_0^T \|\nabla_x w\|^2 dt \right)^{\frac{1}{2}}. \quad (4.47)$$

Before we embark on the estimation of the term U_4 we observe that

$$\begin{aligned} U_4 &= k \left| \int_0^T \int_{\Omega} \left[\int_D M \widehat{\psi}_{\varepsilon,L}^{\Delta t,+} \sum_{i=1}^K q_i q_i^T U_i' \left(\frac{1}{2} |q_i|^2 \right) : \nabla_x w \, dq \right] dx \, dt \right| \\ &= k \left| \int_0^T \int_{\Omega} \left[\int_D M \sum_{i=1}^K (\nabla_x w) q_i \cdot \nabla_{q_i} \widehat{\psi}_{\varepsilon,L}^{\Delta t,+} \, dq \right] dx \, dt \right|, \end{aligned} \quad (4.48)$$

where we used the integration-by-parts formula (3.16) to transform the expression in the square brackets in the first line into the expression in the square brackets in the second line. Thus we have that

$$U_4 \leq 2k \sqrt{b} \int_0^T \int_{\Omega} |\nabla_x w| \left(\int_D M |\nabla_q \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}}|^2 \, dq \right)^{\frac{1}{2}} dx \, dt, \quad (4.49)$$

where we used the Cauchy–Schwarz inequality and (4.31b). Hence, by (4.28),

$$U_4 \leq \sqrt{\frac{2\lambda b k}{a_0}} B(u_0, f, \widehat{\psi}_0) \left(\int_0^T \|\nabla_x w\|^2 dt \right)^{\frac{1}{2}}. \quad (4.50)$$

Collecting the bounds on the terms U_1 to U_4 and inserting them into (4.44) yields

$$\left| \int_0^T \int_{\Omega} \frac{\partial u_{\varepsilon,L}^{\Delta t}}{\partial t} \cdot w \, dx \, dt \right| \leq C_{**} \max \left([B(u_0, f, \widehat{\psi}_0)]^2, B(u_0, f, \widehat{\psi}_0) \right) \left(\int_0^T \|w\|_{V_{\sigma}}^2 dt \right)^{\frac{1}{2}}, \quad (4.51)$$

for any $w \in L^2(0, T; V_{\sigma})$, $\sigma \geq \frac{1}{2}d$, $\sigma > 1$, where C_{**} denotes a positive constant (that can be computed by tracking the constants in (4.45)–(4.50)), which depends solely on Ω , d , ν , k , K , λ , a_0 and b .

5. Dubinskiĭ's compactness theorem

Having developed a collection of L -independent bounds in Sections 4.1 and 4.2, we now describe the theoretical tool that will be used to set up a weak compactness argument using these bounds: Dubinskiĭ's compactness theorem in seminormed sets.

Let $\mathcal{A}$ be a linear space over the field $\mathbb{R}$ of real numbers, and suppose that $\mathcal{M}$ is a subset of $\mathcal{A}$ such that

$$(\forall \varphi \in \mathcal{M}) (\forall c \in \mathbb{R}_{\geq 0}) \quad c\varphi \in \mathcal{M}. \quad (5.1)$$

In other words, whenever φ is contained in $\mathcal{M}$, the ray through φ from the origin of the linear space $\mathcal{A}$ is also contained in $\mathcal{M}$. Note in particular that while any set $\mathcal{M}$ with property (5.1) must contain the zero element of the linear space $\mathcal{A}$, the set $\mathcal{M}$ need not be closed under summation. The linear space $\mathcal{A}$ will be referred to as the *ambient space* for $\mathcal{M}$.

Suppose further that each element φ of a set $\mathcal{M}$ with property (5.1) is assigned a certain real number, denoted $[\varphi]_{\mathcal{M}}$, such that:

- (i) $[\varphi]_{\mathcal{M}} \geq 0$; and $[\varphi]_{\mathcal{M}} = 0$ if, and only if, $\varphi = 0$; and
- (ii) $(\forall c \in \mathbb{R}_{\geq 0}) [c\varphi]_{\mathcal{M}} = c[\varphi]_{\mathcal{M}}$.

We shall then say that $\mathcal{M}$ is a *seminormed set*.

A subset $\mathcal{B}$ of a seminormed set $\mathcal{M}$ is said to be *bounded* if there exists a positive constant K_0 such that $[\varphi]_{\mathcal{M}} \leq K_0$ for all $\varphi \in \mathcal{B}$.

A seminormed set $\mathcal{M}$ contained in a normed linear space $\mathcal{A}$ with norm $\|\cdot\|_{\mathcal{A}}$ is said to be *embedded in $\mathcal{A}$* , and we write $\mathcal{M} \hookrightarrow \mathcal{A}$, if the inclusion map $i : \varphi \in \mathcal{M} \mapsto i(\varphi) = \varphi \in \mathcal{A}$ (which is, by definition, injective and positively 1-homogeneous, i.e. $i(c\varphi) = ci(\varphi)$ for all $c \in \mathbb{R}_{\geq 0}$ and all $\varphi \in \mathcal{M}$) is a bounded operator, i.e.

$$(\exists K_0 \in \mathbb{R}_{>0}) (\forall \varphi \in \mathcal{M}) \quad \|i(\varphi)\|_{\mathcal{A}} \leq K_0[\varphi]_{\mathcal{M}}.$$

The symbol $i(\cdot)$ is usually omitted from the notation $i(\varphi)$, and $\varphi \in \mathcal{M}$ is simply identified with $\varphi \in \mathcal{A}$. Thus, a bounded subset of a seminormed set is also a bounded subset of the ambient normed linear space the seminormed set is embedded in.

The embedding of a seminormed set $\mathcal{M}$ into a normed linear space $\mathcal{A}$ is said to be *compact* if from any infinite, bounded set of elements of $\mathcal{M}$ one can extract a subsequence that converges in $\mathcal{A}$; we shall write $\mathcal{M} \hookrightarrow \mathcal{A}$ to denote that $\mathcal{M}$ is compactly embedded in $\mathcal{A}$.

Suppose that T is a positive real number, φ maps the nonempty closed interval $[0, T]$ into a seminormed set $\mathcal{M}$, and $p \in \mathbb{R}$, $p \geq 1$. We denote by $L^p(0, T; \mathcal{M})$ the set of all functions $\varphi : t \in [0, T] \mapsto \varphi(t) \in \mathcal{M}$ such that

$$\left(\int_0^T [\varphi(t)]_{\mathcal{M}}^p dt \right)^{1/p} < \infty;$$

$L^p(0, T; \mathcal{M})$ is then a seminormed set in the ambient linear space $L^p(0, T; \mathcal{A})$, with

$$[\varphi]_{L^p(0, T; \mathcal{M})} := \left(\int_0^T [\varphi(t)]_{\mathcal{M}}^p dt \right)^{1/p}.$$

We denote by $L^\infty(0, T; \mathcal{M})$ and $[\varphi]_{L^\infty(0, T; \mathcal{M})}$ the usual modifications of these definitions when $p = \infty$.

For two normed linear spaces, $\mathcal{A}_0$ and $\mathcal{A}_1$, we shall continue to denote by $\mathcal{A}_0 \hookrightarrow \mathcal{A}_1$ that $\mathcal{A}_0$ is (continuously) embedded in $\mathcal{A}_1$.

Theorem 5.1 (Dubinskii¹⁸). *Suppose that $\mathcal{A}_0$ and $\mathcal{A}_1$ are Banach spaces, $\mathcal{A}_0 \hookrightarrow \mathcal{A}_1$, and $\mathcal{M}$ is a seminormed subset of $\mathcal{A}_0$ such that $\mathcal{M} \hookrightarrow \mathcal{A}_0$. Consider the set*

$$\mathcal{Y} := \left\{ \varphi : [0, T] \rightarrow \mathcal{M} : [\varphi]_{L^p(0, T; \mathcal{M})} + \left\| \frac{d\varphi}{dt} \right\|_{L^{p_1}(0, T; \mathcal{A}_1)} < \infty \right\},$$

where $1 \leq p \leq \infty$, $1 \leq p_1 \leq \infty$, $\|\cdot\|_{\mathcal{A}_1}$ is the norm of $\mathcal{A}_1$, and $d\varphi/dt$ is understood in the sense of $\mathcal{A}_1$ -valued distributions on the open interval $(0, T)$. Then, $\mathcal{Y}$, with

$$[\varphi]_{\mathcal{Y}} := [\varphi]_{L^p(0, T; \mathcal{M})} + \left\| \frac{d\varphi}{dt} \right\|_{L^{p_1}(0, T; \mathcal{A}_1)},$$

is a seminormed set in $L^p(0, T; \mathcal{A}_0) \cap W^{1, p_1}(0, T; \mathcal{A}_1)$, and $\mathcal{Y} \hookrightarrow L^p(0, T; \mathcal{A}_0)$.

We note that in Dubinskii¹⁸ the author writes $\mathbb{R}$ instead of our $\mathbb{R}_{\geq 0}$ in (5.1) and property (ii). The proof of Thm. 1 in Dubinskii's work, stated as Theorem 5.1 above, reveals however that the result remains valid with our weaker notion of *seminormed set*, as (5.1) and property (ii) are only ever used in the proof with $c \geq 0$. In the next section, we shall apply Dubinskii's theorem by selecting

$$\mathcal{A}_0 = L_M^1(\Omega \times D) \quad \text{with norm} \quad \|\widehat{\varphi}\|_{\mathcal{A}_0} := \int_{\Omega \times D} M(\underline{q}) |\widehat{\varphi}(\underline{x}, \underline{q})| d\underline{x} d\underline{q}$$

and

$$\mathcal{M} = \left\{ \widehat{\varphi} \in \mathcal{A}_0 : \widehat{\varphi} \geq 0 \quad \text{with} \right. \\ \left. \int_{\Omega \times D} M(\underline{q}) \left(\left| \nabla_{\underline{x}} \sqrt{\widehat{\varphi}(\underline{x}, \underline{q})} \right|^2 + \left| \nabla_{\underline{q}} \sqrt{\widehat{\varphi}(\underline{x}, \underline{q})} \right|^2 \right) d\underline{x} d\underline{q} < \infty \right\},$$

and, for $\widehat{\varphi} \in \mathcal{M}$, we define

$$[\widehat{\varphi}]_{\mathcal{M}} := \|\widehat{\varphi}\|_{\mathcal{A}_0} + \int_{\Omega \times D} M(\underline{q}) \left(\left| \nabla_{\underline{x}} \sqrt{\widehat{\varphi}(\underline{x}, \underline{q})} \right|^2 + \left| \nabla_{\underline{q}} \sqrt{\widehat{\varphi}(\underline{x}, \underline{q})} \right|^2 \right) d\underline{x} d\underline{q}.$$

Note that $\mathcal{M}$ is a seminormed subset of the ambient space $\mathcal{A}_0$. Finally, we put

$$\mathcal{A}_1 := M^{-1}(H^s(\Omega \times D))' := \{\widehat{\varphi} : M\widehat{\varphi} \in (H^s(\Omega \times D))'\},$$

equipped with the norm $\|\widehat{\varphi}\|_{\mathcal{A}_1} := \|M\widehat{\varphi}\|_{(H^s(\Omega \times D))'}$, and take $s > 1 + \frac{1}{2}(K+1)d$. Our choice of $\mathcal{A}_1$ is motivated by the fact that, thanks to the Sobolev embedding theorem on $\Omega \times D \subset \mathbb{R}^{d \times Kd} \cong \mathbb{R}^{(K+1)d}$, the final factor on the right-hand side of (4.42) can be further bounded from above by a constant multiple of $\|\widehat{\varphi}\|_{L^2(0, T; H^s(\Omega \times D))}$, with $s > 1 + \frac{1}{2}(K+1)d$. For such s it then follows, again from the Sobolev embedding theorem that, for any $\widehat{\varphi} \in \mathcal{A}_0$,

$$\|\widehat{\varphi}\|_{\mathcal{A}_1} = \sup_{\chi \in H^s(\Omega \times D)} \frac{|(M\widehat{\varphi}, \chi)|}{\|\chi\|_{H^s(\Omega \times D)}} \leq \sup_{\chi \in H^s(\Omega \times D)} \frac{\|\widehat{\varphi}\|_{\mathcal{A}_0} \|\chi\|_{L^\infty(\Omega \times D)}}{\|\chi\|_{H^s(\Omega \times D)}} \leq K_0 \|\widehat{\varphi}\|_{\mathcal{A}_0},$$

where K_0 is any positive constant that is greater than or equal to the constant K_s , the norm of the continuous linear operator corresponding to the Sobolev embedding $(H^s(\Omega \times D) \hookrightarrow H^{s-1}(\Omega \times D) \hookrightarrow L^\infty(\Omega \times D))$, $s > 1 + \frac{1}{2}(K+1)d$. Hence, $\mathcal{A}_0 \hookrightarrow \mathcal{A}_1$.

Trivially, $\mathcal{M} \hookrightarrow \mathcal{A}_0$. We shall show that in fact $\mathcal{M} \hookrightarrow \mathcal{A}_0$. Suppose, to this end, that $\mathcal{B}$ is an infinite, bounded subset of $\mathcal{M}$. We can assume without loss of generality that $\mathcal{B}$ is the infinite sequence $\{\widehat{\varphi}_n\}_{n \geq 1} \subset \mathcal{M}$ with $[\widehat{\varphi}_n]_{\mathcal{M}} \leq K_0$ for all $n \geq 1$, where K_0 is a fixed positive constant. We define $\widehat{\rho}_n := \sqrt{\widehat{\varphi}_n}$ and note that $\widehat{\rho}_n \geq 0$ and $\widehat{\rho}_n \in H_M^1(\Omega \times D)$ for all $n \geq 1$, with

$$\|\widehat{\rho}_n\|_{H_M^1(\Omega \times D)}^2 = [\widehat{\varphi}_n]_{\mathcal{M}} \leq K_0 \quad \forall n \geq 1.$$

Since $H_M^1(\Omega \times D)$ is compactly embedded in $L_M^2(\Omega \times D)$ (see Appendix D in the extended version of this paper⁹ for a proof of this), we deduce that the sequence $\{\widehat{\rho}_n\}_{n \geq 1}$ has a subsequence $\{\widehat{\rho}_{n_k}\}_{k \geq 1}$ that is convergent in $L_M^2(\Omega \times D)$; denote the limit of this subsequence by $\widehat{\rho}$; $\widehat{\rho} \in L_M^2(\Omega \times D)$. Then, since a subsequence of the sequence $\{\widehat{\rho}_{n_k}\}_{k \geq 1}$ also converges to $\widehat{\rho}$ a.e. on $\Omega \times D$ and each $\widehat{\rho}_{n_k}$ is nonnegative on $\Omega \times D$, the same is true of $\widehat{\rho}$. Now, define $\widehat{\varphi} := \widehat{\rho}^2$, and note that $\widehat{\varphi} \in L_M^1(\Omega \times D)$. Clearly,

$$\begin{aligned} \|\widehat{\varphi}_{n_k} - \widehat{\varphi}\|_{L_M^1(\Omega \times D)} &= \int_{\Omega \times D} M(\widehat{\rho}_{n_k} + \widehat{\rho}) |\widehat{\rho}_{n_k} - \widehat{\rho}| \, d\mathbf{x} \, d\mathbf{q} \\ &\leq \|\widehat{\rho}_{n_k} + \widehat{\rho}\|_{L_M^2(\Omega \times D)} \|\widehat{\rho}_{n_k} - \widehat{\rho}\|_{L_M^2(\Omega \times D)} \\ &\leq \left(\|\widehat{\rho}_{n_k}\|_{L_M^2(\Omega \times D)} + \|\widehat{\rho}\|_{L_M^2(\Omega \times D)} \right) \|\widehat{\rho}_{n_k} - \widehat{\rho}\|_{L_M^2(\Omega \times D)}. \end{aligned}$$

As $\{\widehat{\rho}_{n_k}\}_{k \geq 1}$ converges to $\widehat{\rho}$ in $L_M^2(\Omega \times D)$, and is therefore also a bounded sequence in $L_M^2(\Omega \times D)$, it follows from the last inequality that $\{\widehat{\varphi}_{n_k}\}_{k \geq 1}$ converges to $\widehat{\varphi}$ in $L_M^1(\Omega \times D) = \mathcal{A}_0$. This implies that $\mathcal{M}$ is compactly embedded in $\mathcal{A}_0$; hence the triple $\mathcal{M} \hookrightarrow \mathcal{A}_0 \hookrightarrow \mathcal{A}_1$ satisfies the conditions of Theorem 5.1.

Remark 5.1. In fact, there is a deep connection between $\mathcal{M}$ and the set of functions with finite relative entropy on D ; this can be seen by noting the *logarithmic Sobolev inequality*:

$$\int_D M(\mathbf{q}) |\widehat{\rho}(\mathbf{q})|^2 \log \frac{|\widehat{\rho}(\mathbf{q})|^2}{\|\widehat{\rho}\|_{L_M^2(D)}^2} \, d\mathbf{q} \leq \frac{2}{\kappa} \int_D M(\mathbf{q}) |\nabla_{\mathbf{q}} \widehat{\rho}(\mathbf{q})|^2 \, d\mathbf{q} \quad \forall \widehat{\rho} \in H_M^1(D), \quad (5.2)$$

with a constant $\kappa > 0$; the inequality (5.2) is known to hold whenever M satisfies the *Bakry–Émery condition*: $\text{Hess}(-\log M(\mathbf{q})) \geq \kappa \text{Id}$ on D , asserting the logarithmic concavity of the Maxwellian on D , with the last inequality understood in the sense of symmetric $Kd \times Kd$ matrices. The inequality (5.2) follows from inequality (1.3) in Arnold et al.², with the Maxwellian M extended by 0 to the whole of $\mathbb{R}^{Kd}$ to define a probability measure on $\mathbb{R}^{Kd}$ supported on $D = D_1 \times \cdots \times D_K$.

The validity of the Bakry–Émery condition for the FENE Maxwellian, for example, is an easy consequence of the fact that

$$\begin{aligned} \text{Hess}(-\log M(\mathbf{q})) &= \text{Hess} \left(\sum_{i=1}^K U_i \left(\frac{1}{2} |\mathbf{q}_i|^2 \right) \right) \\ &= \text{diag} \left(\text{Hess} \left(U_1 \left(\frac{1}{2} |\mathbf{q}_1|^2 \right) \right), \dots, \text{Hess} \left(U_K \left(\frac{1}{2} |\mathbf{q}_K|^2 \right) \right) \right), \end{aligned} \quad (5.3)$$

for all $\underline{q} := (q_1^T, \dots, q_K^T)^T \in D_1 \times \dots \times D_K = D$, and the following lower bounds (cf. Knezevic & Süli²³, Sec. 2.1) on the $d \times d$ Hessian matrices that are the diagonal blocks of the $Kd \times Kd$ Hessian matrix $\text{Hess}(-\log M(\underline{q}))$:

$$\xi_i^T \left(\text{Hess} \left(U_i \left(\frac{1}{2} |q_i|^2 \right) \right) \right) \xi_i \geq (1 - |q_i|^2/b)^{-1} |\xi_i|^2 \geq |\xi_i|^2,$$

for all $q_i \in D_i$ and all $\xi_i \in \mathbb{R}^d$, $i = 1, \dots, K$. Hence,

$$\xi^T \text{Hess}(-\log M(\underline{q})) \xi \geq |\xi|^2$$

for all $\underline{q} \in D$ and all $\xi \in \mathbb{R}^{Kd}$, yielding

$$\text{Hess}(-\log M(\underline{q})) \geq \text{Id} \quad \forall \underline{q} \in D;$$

i.e. $\kappa = 1$.

More generally, we see from (5.3) that if $q_i \in D_i \mapsto U_i(\frac{1}{2}|q_i|^2)$ is strongly convex on D_i for each $i = 1, \dots, K$, then M satisfies the Bakry–Émery condition on D .

On writing $\widehat{\varphi}(\underline{q}) := |\widehat{\rho}(\underline{q})|^2 (\geq 0)$ in (5.2), we then have that

$$\int_D M(\underline{q}) \widehat{\varphi}(\underline{q}) \log \frac{\widehat{\varphi}(\underline{q})}{\|\widehat{\varphi}\|_{L_M^1(D)}} d\underline{q} \leq \frac{2}{\kappa} \int_D M(\underline{q}) \left| \nabla_{\underline{q}} \sqrt{\widehat{\varphi}(\underline{q})} \right|^2 d\underline{q}, \quad (5.4)$$

for all $\widehat{\varphi}$ such that $\widehat{\varphi} \geq 0$ on D and $\sqrt{\widehat{\varphi}} \in H_M^1(D)$. Taking $\widehat{\varphi} = \varphi/M$ where φ is a probability density function on D , we have that $\|\widehat{\varphi}\|_{L_M^1(D)} = \|\varphi\|_{L^1(D)} = 1$; thus, on denoting by μ the Gibbs measure defined by $d\mu = M(\underline{q}) d\underline{q}$, the left-hand side of (5.4) becomes

$$S(\varphi|M) := \int_D \frac{\varphi}{M} \left(\log \frac{\varphi}{M} \right) d\mu,$$

referred to as the *relative entropy* of φ with respect to M . The expression appearing on the right-hand side of (5.4) is $1/(2\kappa)$ times the *Fisher information*, $I(\widehat{\varphi})$, of $\widehat{\varphi}$:

$$I(\widehat{\varphi}) := \mathbb{E} \left[\left| \nabla_{\underline{q}} \log \widehat{\varphi}(\underline{q}) \right|^2 \right] = \int_D \left| \nabla_{\underline{q}} \log \widehat{\varphi}(\underline{q}) \right|^2 \widehat{\varphi}(\underline{q}) d\mu = 4 \int_D \left| \nabla_{\underline{q}} \sqrt{\widehat{\varphi}(\underline{q})} \right|^2 d\mu,$$

where, $\mathbb{E}$ is the expectation with respect to the Gibbs measure μ defined above. $\diamond$

Lemma 5.1. *Suppose that a sequence $\{\widehat{\varphi}_n\}_{n=1}^\infty$ converges in $L^1(0, T; L_M^1(\Omega \times D))$ to a function $\widehat{\varphi} \in L^1(0, T; L_M^1(\Omega \times D))$, and is bounded in $L^\infty(0, T; L_M^1(\Omega \times D))$, i.e. there exists $K_0 > 0$ such that $\|\varphi_n\|_{L^\infty(0, T; L_M^1(\Omega \times D))} \leq K_0$ for all $n \geq 1$. Then, $\widehat{\varphi} \in L^p(0, T; L_M^1(\Omega \times D))$ for all $p \in [1, \infty)$, and the sequence $\{\widehat{\varphi}_n\}_{n \geq 1}$ converges to $\widehat{\varphi}$ in $L^p(0, T; L_M^1(\Omega \times D))$ for all $p \in [1, \infty)$.*

The proof is easy and is therefore omitted. We refer to the paper⁹ for details.

6. Passage to the limit $L \rightarrow \infty$: existence of weak solutions to the FENE chain model with centre-of-mass diffusion

The bounds (4.28), (4.42) and (4.51) imply the existence of a constant $C_\star > 0$, depending only on $B(u_0, f, \hat{\psi}_0)$ and the constants $C_\star$ and C_{**} , which in turn depend only on $\varepsilon, \nu, C_P(\Omega), T, |A|, a_0, k, K, \lambda, \Omega, d$ and b , but *not* on L or Δt , such that:

$$\begin{aligned} & \text{ess.sup}_{t \in [0, T]} \|u_{\varepsilon, L}^{\Delta t, +}(t)\|^2 + \frac{1}{\Delta t} \int_0^T \|u_{\varepsilon, L}^{\Delta t, +} - u_{\varepsilon, L}^{\Delta t, -}\|^2 ds + \int_0^T \|\nabla_x u_{\varepsilon, L}^{\Delta t, +}(s)\|^2 ds \\ & + \text{ess.sup}_{t \in [0, T]} \int_{\Omega \times D} M \mathcal{F}(\hat{\psi}_{\varepsilon, L}^{\Delta t, +}(t)) dq d\tilde{x} \\ & + \frac{1}{\Delta t L} \int_0^T \int_{\Omega \times D} M (\hat{\psi}_{\varepsilon, L}^{\Delta t, +} - \hat{\psi}_{\varepsilon, L}^{\Delta t, -})^2 dq d\tilde{x} ds \\ & + \int_0^T \int_{\Omega \times D} M |\nabla_x \sqrt{\hat{\psi}_{\varepsilon, L}^{\Delta t, +}}|^2 dq d\tilde{x} ds + \int_0^T \int_{\Omega \times D} M |\nabla_q \sqrt{\hat{\psi}_{\varepsilon, L}^{\Delta t, +}}|^2 dq d\tilde{x} ds \\ & + \int_0^T \left\| \frac{\partial u_{\varepsilon, L}^{\Delta t}}{\partial t} \right\|_{V'_\sigma}^2 dt + \int_0^T \left\| M \frac{\partial \hat{\psi}_{\varepsilon, L}^{\Delta t}}{\partial t} \right\|_{(H^s(\Omega \times D))'}^2 dt \leq C_\star, \end{aligned} \quad (6.1)$$

where $\|\cdot\|_{V'_\sigma}$ denotes the norm of the dual space V'_σ of V_σ with $\sigma \geq \frac{1}{2}d, \sigma > 1$ (cf. the paragraph following (4.44)); and $\|\cdot\|_{(H^s(\Omega \times D))'}$ is the norm of the dual space $(H^s(\Omega \times D))'$ of $H^s(\Omega \times D)$, with $s > 1 + \frac{1}{2}(K+1)d$. The bounds in (6.1) on the time-derivatives follow from (4.51), and from (4.42) using the Sobolev embedding theorem.

By virtue of (4.38), (4.35), the definitions (4.1a,b), and with an argument completely analogous to (4.35) on noting (3.5) in the case of the fourth term in (6.1), and using (4.10), (3.20) and recalling that $L > 1$ we have (with a possible adjustment of the constant $C_\star$, if necessary,) that

$$\begin{aligned} & \text{ess.sup}_{t \in [0, T]} \|u_{\varepsilon, L}^{\Delta t, (\pm)}(t)\|^2 + \frac{1}{\Delta t} \int_0^T \|u_{\varepsilon, L}^{\Delta t, +} - u_{\varepsilon, L}^{\Delta t, -}\|^2 ds \\ & + \int_0^T \|\nabla_x u_{\varepsilon, L}^{\Delta t, (\pm)}(s)\|^2 ds + \text{ess.sup}_{t \in [0, T]} \int_{\Omega \times D} M \mathcal{F}(\hat{\psi}_{\varepsilon, L}^{\Delta t, (\pm)}(t)) dq d\tilde{x} \\ & + \frac{1}{\Delta t L} \int_0^T \int_{\Omega \times D} M (\hat{\psi}_{\varepsilon, L}^{\Delta t, +} - \hat{\psi}_{\varepsilon, L}^{\Delta t, -})^2 dq d\tilde{x} ds \\ & + \int_0^T \int_{\Omega \times D} M |\nabla_x \sqrt{\hat{\psi}_{\varepsilon, L}^{\Delta t, +}}|^2 dq d\tilde{x} ds + \int_0^T \int_{\Omega \times D} M |\nabla_q \sqrt{\hat{\psi}_{\varepsilon, L}^{\Delta t, +}}|^2 dq d\tilde{x} ds \\ & + \int_0^T \left\| \frac{\partial u_{\varepsilon, L}^{\Delta t}}{\partial t} \right\|_{V'_\sigma}^2 dt + \int_0^T \left\| M \frac{\partial \hat{\psi}_{\varepsilon, L}^{\Delta t}}{\partial t} \right\|_{(H^s(\Omega \times D))'}^2 dt \leq C_\star. \end{aligned} \quad (6.2)$$

On noting (4.31a,b), (4.1a,b), (3.20) and (3.5), we also have that

$$\hat{\psi}_{\varepsilon, L}^{\Delta t, (\pm)} \geq 0 \quad \text{a.e. on } \Omega \times D \times [0, T] \quad (6.3)$$

and

$$\int_D M(\underline{q}) \widehat{\psi}_{\varepsilon,L}^{\Delta t(\pm)}(\underline{x}, \underline{q}, t) d\underline{q} \leq 1 \quad \text{for a.e. } (\underline{x}, t) \in \Omega \times [0, T]. \quad (6.4)$$

Henceforth, we shall assume that

$$\Delta t = o(L^{-1}) \quad \text{as } L \rightarrow \infty. \quad (6.5)$$

Requiring, for example, that $0 < \Delta t \leq C_0/(L \log L)$, $L > 1$, with an arbitrary (but fixed) constant C_0 will suffice to ensure that (6.5) holds. The sequences $\{u_{\varepsilon,L}^{\Delta t(\pm)}\}_{L>1}$ and $\{\widehat{\psi}_{\varepsilon,L}^{\Delta t(\pm)}\}_{L>1}$ as well as all sequences of spatial and temporal derivatives of the entries of these two sequences will thus be, indirectly, indexed by L alone, although for reasons of consistency with our previous notation we shall not introduce new, compressed, notation with Δt omitted from the superscripts. Instead, whenever $L \rightarrow \infty$ in the rest of this section, it will be understood that Δt tends to 0 according to (6.5). We are now almost ready to pass to the limit with $L \rightarrow \infty$. Before doing so, however, we first need to state the definition of the function $\widehat{\psi}^0$ that obeys (3.20), for a given $\widehat{\psi}_0$ satisfying (3.5).

6.1. The definition of $\widehat{\psi}^0$

Given $\widehat{\psi}_0$ satisfying the conditions in (3.5) and $\Lambda > 1$, we consider the following discrete-in-time problem in weak form: find $\widehat{\zeta}^{\Lambda,1} \in H_M^1(\Omega \times D)$ such that

$$\int_{\Omega \times D} M \frac{\widehat{\zeta}^{\Lambda,1} - \widehat{\zeta}^{\Lambda,0}}{\Delta t} \widehat{\varphi} d\underline{q} d\underline{x} + \int_{\Omega \times D} M \left[\nabla_x \widehat{\zeta}^{\Lambda,1} \cdot \nabla_x \widehat{\varphi} + \nabla_q \widehat{\zeta}^{\Lambda,1} \cdot \nabla_q \widehat{\varphi} \right] d\underline{q} d\underline{x} = 0 \quad (6.6)$$

for all $\widehat{\varphi} \in H_M^1(\Omega \times D)$, with $\widehat{\zeta}^{\Lambda,0} := \beta^\Lambda(\widehat{\psi}_0) \in L_M^2(\Omega \times D)$. Here β^Λ is defined by (1.12), with L replaced by Λ . The function $\mathcal{F}^\Lambda$, which we shall encounter below, is defined by (4.7), with L replaced by Λ .

The existence of a unique solution $\widehat{\zeta}^{\Lambda,1} \in H_M^1(\Omega \times D)$ to (6.6), for each $\Delta t > 0$ and $\Lambda > 1$, follows immediately by applying the Lax–Milgram theorem.

Lemma 6.1. *Let $\widehat{\zeta}^{\Lambda,1}$ be defined by (6.6), and consider $\gamma^{\Lambda,n}$ defined by*

$$\gamma^{\Lambda,n}(\underline{x}) := \int_D M(\underline{q}) \widehat{\zeta}^{\Lambda,n}(\underline{x}, \underline{q}) d\underline{q}, \quad n = 0, 1. \quad (6.7)$$

Then, $\widehat{\zeta}^{\Lambda,1}$ is nonnegative a.e. on $\Omega \times D$, and $0 \leq \gamma^{\Lambda,1} \leq 1$ a.e. on Ω .

Proof. The proof of nonnegativity of $\widehat{\zeta}^{\Lambda,1}$ is straightforward (cf. the discussion following (3.66)). Indeed, we have that $[\widehat{\zeta}^{\Lambda,0}]_- = 0$ a.e. on $\Omega \times D$, thanks to (3.5) and the definition of β^Λ ; we then take $\varphi = [\widehat{\zeta}^{\Lambda,1}]_-$ as a test function in (6.6), noting that this is a legitimate choice since $\widehat{\zeta}^{\Lambda,1} \in H_M^1(\Omega \times D)$ and therefore $[\widehat{\zeta}^{\Lambda,1}]_- \in H_M^1(\Omega \times D)$ also (cf. Lemma 3.3 in Barrett, Schwab & Süli⁶). On decomposing

$\widehat{\zeta}^{\Lambda,1} = [\widehat{\zeta}^{\Lambda,1}]_+ + [\widehat{\zeta}^{\Lambda,1}]_-$, and using that $[\widehat{\zeta}^{\Lambda,1}]_+ [\widehat{\zeta}^{\Lambda,1}]_- = 0$, $\nabla_x [\widehat{\zeta}^{\Lambda,1}]_+ \cdot \nabla_x [\widehat{\zeta}^{\Lambda,1}]_- = 0$ and $\nabla_q [\widehat{\zeta}^{\Lambda,1}]_+ \cdot \nabla_q [\widehat{\zeta}^{\Lambda,1}]_- = 0$ a.e. on $\Omega \times D$, we deduce that

$$\begin{aligned} & \frac{1}{\Delta t} \|M^{\frac{1}{2}} [\widehat{\zeta}^{\Lambda,1}]_-\|^2 + \|M^{\frac{1}{2}} \nabla_x [\widehat{\zeta}^{\Lambda,1}]_-\|^2 + \|M^{\frac{1}{2}} \nabla_q [\widehat{\zeta}^{\Lambda,1}]_-\|^2 \\ &= \frac{1}{\Delta t} \int_{\Omega \times D} M \widehat{\zeta}^{\Lambda,0} [\widehat{\zeta}^{\Lambda,1}]_- \, dq \, dx = \frac{1}{\Delta t} \int_{\Omega \times D} M [\widehat{\zeta}^{\Lambda,0}]_+ [\widehat{\zeta}^{\Lambda,1}]_- \, dq \, dx \leq 0, \end{aligned}$$

where $\|\cdot\|$ denotes the $L^2(\Omega \times D)$ norm. This then implies that $\|M^{\frac{1}{2}} [\widehat{\zeta}^{\Lambda,1}]_-\|^2 \leq 0$. Hence, $[\widehat{\zeta}^{\Lambda,1}]_- = 0$ a.e. on $\Omega \times D$. In other words, $\widehat{\zeta}^{\Lambda,1} \geq 0$ a.e. on $\Omega \times D$, as claimed.

In order to prove the upper bound in the statement of the lemma, we proceed as follows. With $\gamma^{\Lambda,n}$ as defined in (6.7), we deduce from the definition of $\widehat{\zeta}^{\Lambda,1}$ and Fubini's theorem that $\gamma^{\Lambda,1} \in H^1(\Omega)$. Furthermore, on selecting $\widehat{\varphi} = \varphi \in H^1(\Omega) \otimes 1(D)$ in (6.6), recall (3.63), we have that

$$\int_{\Omega} \frac{\gamma^{\Lambda,1} - \gamma^{\Lambda,0}}{\Delta t} \varphi \, dx + \int_{\Omega} \nabla_x \gamma^{\Lambda,1} \cdot \nabla_x \varphi \, dx = 0 \quad \forall \varphi \in H^1(\Omega). \quad (6.8)$$

As $\widehat{\zeta}^{\Lambda,0} = \beta^{\Lambda}(\widehat{\psi}_0)$, and $0 \leq \beta^{\Lambda}(s) \leq s$ for all $s \in \mathbb{R}_{\geq 0}$, we also have by (3.5) that

$$0 \leq \gamma^{\Lambda,0} = \int_D M \beta^{\Lambda}(\widehat{\psi}_0) \, dq \leq \int_D M \widehat{\psi}_0 \, dq = 1 \quad \text{on } \Omega. \quad (6.9)$$

Consider $z^{\Lambda,n} := 1 - \gamma^{\Lambda,n}$, $n = 0, 1$. On substituting $\gamma^{\Lambda,n} = 1 - z^{\Lambda,n}$, $n = 0, 1$, into (6.8), we have that

$$\int_{\Omega} \frac{z^{\Lambda,1} - z^{\Lambda,0}}{\Delta t} \varphi \, dx + \int_{\Omega} \nabla_x z^{\Lambda,1} \cdot \nabla_x \varphi \, dx = 0 \quad \forall \varphi \in H^1(\Omega). \quad (6.10)$$

Also, by (6.9), we have that $0 \leq z^{\Lambda,0} \leq 1$. By using an identical procedure to the one in the first part of the proof, we then deduce that $[z^{\Lambda,1}]_- = 0$ a.e. on Ω . Thus, $z^{\Lambda,1} \geq 0$ a.e. on Ω , which then implies that $\gamma^{\Lambda,1} \leq 1$ a.e. on Ω , as claimed. $\square$

Next, we shall pass to the limit $\Lambda \rightarrow \infty$. To this end, we need to derive Λ -independent bounds on norms of $\widehat{\zeta}^{\Lambda,1}$, very similar to the L -independent bounds discussed in Section 4. Since the argument is almost identical to (but simpler than) the one there (viz. (6.6) can be viewed as a special case of (3.21b), with $\widehat{f}^n$, $\widehat{u}_{\varepsilon,L}^{n-1}$ and $\widehat{u}_{\varepsilon,L}^n$ taken to be identically zero, $\lambda = \frac{1}{2}$, $\varepsilon = 1$, $N = 1$, and A chosen as the $K \times K$ identity matrix), we shall not include the details here. It suffices to say that, on testing (6.6) with $\mathcal{F}'(\widehat{\zeta}^{\Lambda,1} + \alpha)$ and passing to the limit $\alpha \rightarrow 0_+$, analogously as in the proof of (4.27) in Section 4, we obtain that

$$\begin{aligned} & \int_{\Omega \times D} M \mathcal{F}(\widehat{\zeta}^{\Lambda,1}) \, dq \, dx + 4 \Delta t \int_{\Omega \times D} M |\nabla_x \sqrt{\widehat{\zeta}^{\Lambda,1}}|^2 \, dq \, dx \\ &+ 4 \Delta t \int_{\Omega \times D} M |\nabla_q \sqrt{\widehat{\zeta}^{\Lambda,1}}|^2 \, dq \, dx \leq \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_0) \, dq \, dx. \end{aligned} \quad (6.11)$$

Our passage to the limit $\Lambda \rightarrow \infty$ in (6.6) is based on a weak compactness argument, using (6.11), and is discussed below.

We have from Lemma 6.1 that $\{[\widehat{\zeta}^{\Lambda,1}]^{\frac{1}{2}}\}_{\Lambda>1}$ is a bounded sequence in $L_M^2(\Omega \times D)$. Using this in conjunction with the second and third bound in (6.11) we deduce that, for $\Delta t > 0$ fixed, $\{[\widehat{\zeta}^{\Lambda,1}]^{\frac{1}{2}}\}_{\Lambda>1}$ is a bounded sequence in $H_M^1(\Omega \times D)$. Thanks to the compact embedding of $H_M^1(\Omega \times D)$ into $L_M^2(\Omega \times D)$ (cf. Appendix D in the extended version of this paper⁹), we deduce that $\{[\widehat{\zeta}^{\Lambda,1}]^{\frac{1}{2}}\}_{\Lambda>1}$ has a strongly convergent subsequence in $L_M^2(\Omega \times D)$, whose limit we label by Z , and we then let $\widehat{\zeta}^1 := Z^2$. For future reference we note that, upon extraction of a subsequence (not indicated), $\widehat{\zeta}^{\Lambda,1}$ then converges to $\widehat{\zeta}^1$ a.e. on $\Omega \times D$; and $\widehat{\zeta}^{\Lambda,1}(\underline{x}, \cdot)$ converges to $\widehat{\zeta}^1(\underline{x}, \cdot)$ a.e. on D , for a.e. $\underline{x} \in \Omega$.

By definition, we have that $\widehat{\zeta}^1 \geq 0$; furthermore, thanks to the upper bound on $\gamma^{\Lambda,1}$ stated in Lemma 6.1, the remark in the last sentence of the previous paragraph, and Fatou's lemma, we also have that

$$\int_D M(\underline{q}) \widehat{\zeta}^1(\underline{x}, \underline{q}) d\underline{q} \leq 1 \quad \text{for a.e. } \underline{x} \in \Omega. \quad (6.12)$$

Further, again as a direct consequence of the definition of $\widehat{\zeta}^1$, we have that

$$\sqrt{\widehat{\zeta}^{\Lambda,1}} \rightarrow \sqrt{\widehat{\zeta}^1} \quad \text{strongly in } L_M^2(\Omega \times D). \quad (6.13)$$

Application of the factorization $c_1 - c_2 = (\sqrt{c_1} - \sqrt{c_2})(\sqrt{c_1} + \sqrt{c_2})$ with $c_1, c_2 \in \mathbb{R}_{\geq 0}$, the Cauchy–Schwarz inequality and (6.13) then yields that

$$\widehat{\zeta}^{\Lambda,1} \rightarrow \widehat{\zeta}^1 \quad \text{strongly in } L_M^1(\Omega \times D). \quad (6.14)$$

Finally, we define

$$\widehat{\psi}^0 := \widehat{\zeta}^1. \quad (6.15)$$

It follows from the nonnegativity of $\widehat{\zeta}^1$ and (6.12) that

$$\widehat{\psi}^0 \geq 0 \quad \text{a.e. on } \Omega \times D \quad \text{and} \quad 0 \leq \int_D M(\underline{q}) \widehat{\psi}^0(\underline{x}, \underline{q}) d\underline{q} \leq 1 \quad \text{for a.e. } \underline{x} \in \Omega. \quad (6.16)$$

Further, from the bound on the first term in (6.11) and Fatou's lemma, together with the fact that, thanks to the continuity of $\mathcal{F}$, (a subsequence, not indicated, of) $\{\mathcal{F}(\widehat{\zeta}^{\Lambda,1})\}_{\Lambda>0}$ converges to $\mathcal{F}(\widehat{\zeta}^1) = \mathcal{F}(\widehat{\psi}^0)$ a.e. on $\Omega \times D$, we also have that

$$\int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}^0) d\underline{q} d\underline{x} \leq \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_0) d\underline{q} d\underline{x}. \quad (6.17)$$

Next, we note that from (6.13) we have that, as $\Lambda \rightarrow \infty$,

$$M^{\frac{1}{2}} \sqrt{\widehat{\zeta}^{\Lambda,1}} \rightarrow M^{\frac{1}{2}} \sqrt{\widehat{\zeta}^1} \quad \text{strongly in } L^2(\Omega \times D). \quad (6.18)$$

We shall use (6.18) to deduce weak convergence of the sequences of $\underline{x}$ and $\underline{q}$ gradients of $\widehat{\zeta}^{\Lambda,1}$. We proceed as in the proof of Lemma 3.3. The bound on the third term on the left-hand side of (6.11) implies the existence of a subsequence (not indicated) and an element $\underline{g} \in \underline{L}^2(\Omega \times D)$, such that

$$M^{\frac{1}{2}} \nabla_{\underline{q}} \sqrt{\widehat{\zeta}^{\Lambda,1}} \rightarrow \underline{g} \quad \text{weakly in } \underline{L}^2(\Omega \times D). \quad (6.19)$$

Proceeding as in (3.60)–(3.61) in the proof of Lemma 3.3 with $\widehat{\psi}_{\varepsilon,L,\delta}^n$, $\widehat{\psi}_{\varepsilon,L}^n$ and $\delta \rightarrow 0_+$ replaced by $\sqrt{\widehat{\zeta}^{\Lambda,1}}$, $\sqrt{\widehat{\zeta}^1}$ and $\Lambda \rightarrow \infty$, respectively, we obtain the weak convergence result:

$$M^{\frac{1}{2}} \nabla_q \sqrt{\widehat{\zeta}^{\Lambda,1}} \rightarrow M^{\frac{1}{2}} \nabla_q \sqrt{\widehat{\zeta}^1} \quad \text{weakly in } L^2(\Omega \times D), \quad (6.20a)$$

and similarly for the $\mathcal{X}$ gradient

$$M^{\frac{1}{2}} \nabla_x \sqrt{\widehat{\zeta}^{\Lambda,1}} \rightarrow M^{\frac{1}{2}} \nabla_x \sqrt{\widehat{\zeta}^1} \quad \text{weakly in } L^2(\Omega \times D), \quad (6.20b)$$

as $\Lambda \rightarrow \infty$. Then, inequality (6.11), (6.20a,b) and the weak lower-semicontinuity of the $L^2(\Omega \times D)$ norm imply that

$$4 \Delta t \int_{\Omega \times D} M \left[|\nabla_x \sqrt{\widehat{\zeta}^1}|^2 + |\nabla_q \sqrt{\widehat{\zeta}^1}|^2 \right] dq dx \leq \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_0) dq dx. \quad (6.21)$$

After these preparations, we are now ready to state the central result of this subsection. Before we do so, a comment is in order. Strictly speaking, we should have written $\widehat{\psi}_{\Delta t}^0$ instead of $\widehat{\psi}^0$ in our definition (6.15), as $\widehat{\psi}^0$ depends on the choice of Δt . For notational simplicity, we prefer the more compact notation, $\widehat{\psi}^0$, with the dependence of $\widehat{\psi}^0$ on Δt implicitly understood; we shall only write $\widehat{\psi}_{\Delta t}^0$, when it is necessary to emphasize the dependence of Δt . Of course, $\widehat{\psi}_0$ is independent of Δt .

We shall show that, with our definition of $\widehat{\psi}^0$, the properties under (3.20) hold, together with some additional properties that we extract from (6.15).

Lemma 6.2. *The function $\widehat{\psi}^0 = \widehat{\psi}_{\Delta t}^0$ defined by (6.15) has the following properties:*

- ❶ $\widehat{\psi}^0 \in \widehat{Z}_1$;
- ❷ $\int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}^0) dq dx \leq \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_0) dq dx$;
- ❸ $4 \Delta t \int_{\Omega \times D} M \left[|\nabla_x \sqrt{\widehat{\psi}^0}|^2 + |\nabla_q \sqrt{\widehat{\psi}^0}|^2 \right] dq dx \leq \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_0) dq dx$;
- ❹ $\lim_{\Delta t \rightarrow 0_+} \widehat{\psi}^0 = \widehat{\psi}_0$, weakly in $L_M^1(\Omega \times D)$;
- ❺ $\lim_{\Delta t \rightarrow 0_+} \beta^L(\widehat{\psi}^0) = \widehat{\psi}_0$, weakly in $L_M^1(\Omega \times D)$.

Proof.

- ❶ This is an immediate consequence of (6.16) and the definition (3.19) of $\widehat{Z}_1$.
- ❷ This property was established in (6.17) above.
- ❸ The inequality follows by using (6.15) in the left-hand side of (6.21).
- ❹ We begin by noting that an argument, completely analogous to (but simpler than) the one in Section 4.2.1 that resulted in (4.42), applied to (6.6) now,

yields

$$\left| \int_{\Omega \times D} M \frac{\widehat{\zeta}^{\Lambda,1} - \widehat{\zeta}^{\Lambda,0}}{\Delta t} \widehat{\varphi} \, dq \, dx \right| \leq 2 \left(\int_{\Omega \times D} M \left[|\nabla_x \sqrt{\widehat{\zeta}^{\Lambda,1}}|^2 + |\nabla_q \sqrt{\widehat{\zeta}^{\Lambda,1}}|^2 \right] dq \, dx \right)^{\frac{1}{2}} \times \left(\int_{\Omega} \left[\|\nabla_x \widehat{\varphi}\|_{L^\infty(D)}^2 + \|\nabla_q \widehat{\varphi}\|_{L^\infty(D)}^2 \right] dx \right)^{\frac{1}{2}},$$

for all $\widehat{\varphi} \in H^1(\Omega; L^\infty(D)) \cap L^2(\Omega; W^{1,\infty}(D))$. By noting (6.11) we deduce that

$$\left| \int_{\Omega \times D} M (\widehat{\zeta}^{\Lambda,1} - \widehat{\zeta}^{\Lambda,0}) \widehat{\varphi} \, dq \, dx \right| \leq (\Delta t)^{\frac{1}{2}} \left(\int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_0) \, dq \, dx \right)^{\frac{1}{2}} \times \left(\int_{\Omega} \left[\|\nabla_x \widehat{\varphi}\|_{L^\infty(D)}^2 + \|\nabla_q \widehat{\varphi}\|_{L^\infty(D)}^2 \right] dx \right)^{\frac{1}{2}} \quad (6.22)$$

for all $\widehat{\varphi} \in H^1(\Omega; L^\infty(D)) \cap L^2(\Omega; W^{1,\infty}(D))$. As the right-hand side of (6.22) is independent of Λ , we can pass to the limit $\Lambda \rightarrow \infty$ on both sides of (6.22), using the strong convergence of $\widehat{\zeta}^{\Lambda,1}$ to $\widehat{\psi}^0$ in $L_M^1(\Omega \times D)$ as $\Lambda \rightarrow \infty$ (see (6.14) and the definition of (6.15)) together with the strong convergence of $\widehat{\zeta}^{\Lambda,0} = \beta^\Lambda(\widehat{\psi}_0)$ to $\widehat{\psi}_0$ in $L_M^1(\Omega \times D)$ as $\Lambda \rightarrow \infty$, with Δt kept fixed. We deduce that

$$\left| \int_{\Omega \times D} M (\widehat{\psi}^0 - \widehat{\psi}_0) \widehat{\varphi} \, dq \, dx \right| \leq (\Delta t)^{\frac{1}{2}} \left(\int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_0) \, dq \, dx \right)^{\frac{1}{2}} \times \left(\int_{\Omega} \left[\|\nabla_x \widehat{\varphi}\|_{L^\infty(D)}^2 + \|\nabla_q \widehat{\varphi}\|_{L^\infty(D)}^2 \right] dx \right)^{\frac{1}{2}} \quad (6.23)$$

for all $\widehat{\varphi} \in H^1(\Omega; L^\infty(D)) \cap L^2(\Omega; W^{1,\infty}(D))$ and therefore in particular for all $\widehat{\varphi} \in H^s(\Omega \times D)$ with $s > 1 + \frac{1}{2}(K+1)d$.

As the last two factors on the right-hand side of (6.23) are independent of Δt , we can pass to the limit $\Delta t \rightarrow 0_+$ on both sides of (6.23) to deduce that $\widehat{\psi}^0 = \widehat{\psi}_{\Delta t}^0$ converges to $\widehat{\psi}_0$ weakly in $M^{-1}(H^s(\Omega \times D))'$, $s > 1 + \frac{1}{2}(K+1)d$, as $\Delta t \rightarrow 0_+$.

Noting (6.17) and the fact that $\mathcal{F}(r)/r \rightarrow \infty$ as $r \rightarrow \infty$, we deduce from de la Vallée-Poussin's theorem that the family $\{\widehat{\psi}_{\Delta t}^0\}_{\Delta t > 0}$ is uniformly integrable in $L_M^1(\Omega \times D)$. Hence, by the Dunford–Pettis theorem, the family $\{\widehat{\psi}_{\Delta t}^0\}_{\Delta t > 0}$ is weakly relatively compact in $L_M^1(\Omega \times D)$. Consequently, one can extract a subsequence $\{\widehat{\psi}_{\Delta t_k}^0\}_{k=1}^\infty$ that converges weakly in $L_M^1(\Omega \times D)$; however the uniqueness of the weak limit together with the weak convergence of the (entire) sequence $\widehat{\psi}^0 = \widehat{\psi}_{\Delta t}^0$ to $\widehat{\psi}_0$ in $M^{-1}(H^s(\Omega \times D))'$, $s > 1 + \frac{1}{2}(K+1)d$, as $\Delta t \rightarrow 0_+$, established in the previous paragraph, then implies that the (entire) sequence $\widehat{\psi}^0 = \widehat{\psi}_{\Delta t}^0$ converges to $\widehat{\psi}_0$ weakly in $L_M^1(\Omega \times D)$, as $\Delta t \rightarrow 0_+$, on noting that $L_M^1(\Omega \times D)$ is (continuously) embedded in $M^{-1}(H^s(\Omega \times D))'$ for $s > 1 + \frac{1}{2}(K+1)d$ (cf. the discussion following Theorem 5.1).

⑤ It follows from $\widehat{\psi}^0 \in \widehat{Z}_1$ and (1.12) that

$$0 \leq \int_{\widehat{\psi}^0 \geq L} M L \, d\widetilde{q} \, d\widetilde{x} \leq \int_{\Omega \times D} M \beta^L(\widehat{\psi}^0) \, d\widetilde{q} \, d\widetilde{x} \leq \int_{\Omega \times D} M \widehat{\psi}^0 \, d\widetilde{q} \, d\widetilde{x} \leq |\Omega|. \quad (6.24)$$

On noting that $\mathcal{F}$ is nonnegative and monotonically increasing on $[1, \infty)$, and that $\mathcal{F}(s) \in [0, 1]$ for $s \in [0, 1]$, we deduce that

$$\begin{aligned} & \int_{\Omega \times D} M \mathcal{F}([\widehat{\psi}^0 - L]_+) \, d\widetilde{q} \, d\widetilde{x} \\ &= \int_{\widehat{\psi}^0 \in [0, L+1)} M \mathcal{F}([\widehat{\psi}^0 - L]_+) \, d\widetilde{q} \, d\widetilde{x} + \int_{\widehat{\psi}^0 \geq L+1} M \mathcal{F}([\widehat{\psi}^0 - L]_+) \, d\widetilde{q} \, d\widetilde{x} \\ &\leq \int_{\Omega \times D} M \, d\widetilde{q} \, d\widetilde{x} + \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}^0) \, d\widetilde{q} \, d\widetilde{x} \leq C. \end{aligned} \quad (6.25)$$

Let us recall the logarithmic Young's inequality

$$r s \leq r \log r - r + e^s \quad \text{for all } r, s \in \mathbb{R}_{\geq 0}. \quad (6.26)$$

This follows from the Fenchel–Young inequality:

$$r s \leq g^*(r) + g(s) \quad \text{for all } r, s \in \mathbb{R},$$

involving the convex function $g : s \in \mathbb{R} \mapsto g(s) \in (-\infty, +\infty]$ and its convex conjugate g^* , with $g(s) = e^s$ and

$$g^*(r) = \begin{cases} +\infty & \text{if } r < 0, \\ 0 & \text{if } r = 0, \\ r(\log r - 1) & \text{if } r > 0; \end{cases}$$

with the resulting inequality then restricted to $\mathbb{R}_{\geq 0}$. It immediately follows from (6.26) that $r s \leq \mathcal{F}(r) + e^s$ for all $r, s \in \mathbb{R}_{\geq 0}$.

Applying the last inequality with $r = [\widehat{\psi}^0 - L]_+$ and $s = \log L$, we have that

$$[\widehat{\psi}^0 - L]_+ (\log L) \leq \mathcal{F}([\widehat{\psi}^0 - L]_+) + L. \quad (6.27)$$

The bounds (6.24), (6.25) (noting that the integrand of the left-most integral in (6.25) is nonnegative) and (6.27) then imply

$$\begin{aligned} & \int_{\Omega \times D} M [\widehat{\psi}^0 - L]_+ \, d\widetilde{q} \, d\widetilde{x} = \int_{\widehat{\psi}^0 \geq L} M [\widehat{\psi}^0 - L]_+ \, d\widetilde{q} \, d\widetilde{x} \\ &\leq \frac{1}{\log L} \left[\int_{\widehat{\psi}^0 \geq L} M \mathcal{F}([\widehat{\psi}^0 - L]_+) \, d\widetilde{q} \, d\widetilde{x} + \int_{\widehat{\psi}^0 \geq L} M L \, d\widetilde{q} \, d\widetilde{x} \right] \leq \frac{C}{\log L}. \end{aligned} \quad (6.28)$$

Hence, for any $\widehat{\varphi} \in L^\infty(\Omega \times D)$ we have from (6.28) on recalling the relationship $\Delta t = o(L^{-1})$ that $\widehat{\psi}^0 = \widehat{\psi}_{\Delta t}^0$ satisfies

$$\begin{aligned} & \lim_{\Delta t \rightarrow 0+} \left| \int_{\Omega \times D} M (\widehat{\psi}^0 - \beta^L(\widehat{\psi}^0)) \widehat{\varphi} \, d\widetilde{q} \, d\widetilde{x} \right| = \lim_{\Delta t \rightarrow 0+} \left| \int_{\Omega \times D} M [\widehat{\psi}^0 - L]_+ \widehat{\varphi} \, d\widetilde{q} \, d\widetilde{x} \right| \\ &\leq \left(\lim_{\Delta t \rightarrow 0+} \int_{\Omega \times D} M [\widehat{\psi}^0 - L]_+ \, d\widetilde{q} \, d\widetilde{x} \right) \|\widehat{\varphi}\|_{L^\infty(\Omega \times D)} = 0. \end{aligned} \quad (6.29)$$

Therefore, similarly to (6.23), we have that the sequence $\{\widehat{\psi}_{\Delta t}^0 - \beta^L(\widehat{\psi}_{\Delta t}^0)\}_{\Delta t > 0}$ converges to zero weakly in $M^{-1}(H^s(\Omega \times D))'$ for $s > \frac{1}{2}(K+1)d$, as $\Delta t \rightarrow 0_+$. Noting (6.25) and the fact that $\mathcal{F}(r)/r \rightarrow \infty$ as $r \rightarrow \infty$, we deduce from de le Vallée Poussin's theorem that the family

$$\{\widehat{\psi}_{\Delta t}^0 - \beta^L(\widehat{\psi}_{\Delta t}^0)\}_{\Delta t > 0} \equiv \{[\widehat{\psi}_{\Delta t}^0 - L]_+\}_{\Delta t > 0}$$

is uniformly integrable in $L_M^1(\Omega \times D)$. Hence, we can proceed as for the sequence $\{\widehat{\psi}_{\Delta t}^0\}_{\Delta t > 0}$ in the proof of $\clubsuit$ to show that the (entire) sequence

$$\widehat{\psi}^0 - \beta^L(\widehat{\psi}^0) = \widehat{\psi}_{\Delta t}^0 - \beta^L(\widehat{\psi}_{\Delta t}^0) \rightarrow 0 \quad \text{weakly in } L_M^1(\Omega \times D), \text{ as } \Delta t \rightarrow 0_+,$$

on noting that $L_M^1(\Omega \times D)$ is (continuously) embedded in $M^{-1}(H^s(\Omega \times D))'$ for $s > \frac{1}{2}(K+1)d$ (cf. the discussion following Theorem 5.1). Hence, we have proved the desired result. $\square$

Noting item 3 in Lemma 6.2, we can now return to the inequality (6.2), and supplement it with additional bounds, in the sixth and seventh term on the left-hand side. The first additional bound can be seen as the analogue of (4.35):

$$\begin{aligned} & 4 \int_0^T \int_{\Omega \times D} M \left[|\nabla_x \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,-}}|^2 + |\nabla_q \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,-}}|^2 \right] dq d\tilde{x} dt \\ &= 4\Delta t \int_{\Omega \times D} M \left[|\nabla_x \sqrt{\beta^L(\widehat{\psi}^0)}|^2 + |\nabla_q \sqrt{\beta^L(\widehat{\psi}^0)}|^2 \right] dq d\tilde{x} \\ &\quad + 4 \int_0^{T-\Delta t} \int_{\Omega \times D} M \left[|\nabla_x \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}}|^2 + |\nabla_q \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}}|^2 \right] dq d\tilde{x} dt \\ &\leq 4\Delta t \int_{\Omega \times D} M \left[|\nabla_x \sqrt{\widehat{\psi}^0}|^2 + |\nabla_q \sqrt{\widehat{\psi}^0}|^2 \right] dq d\tilde{x} \\ &\quad + 4 \int_0^{T-\Delta t} \int_{\Omega \times D} M \left[|\nabla_x \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}}|^2 + |\nabla_q \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}}|^2 \right] dq d\tilde{x} dt \\ &\leq \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}^0) dq d\tilde{x} \\ &\quad + 4 \int_0^T \int_{\Omega \times D} M \left[|\nabla_x \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}}|^2 + |\nabla_q \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}}|^2 \right] dq d\tilde{x} dt \leq C_*, \quad (6.30) \end{aligned}$$

where in the last inequality we used (3.20) and the bounds on the sixth and seventh term in (6.2); here and henceforth C_* signifies a generic positive constant, independent of L and Δt . On combining (6.30) with our previous bounds on the sixth and seventh term in (6.2), we deduce that

$$4 \int_0^T \int_{\Omega \times D} M \left[|\nabla_x \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,\pm}}|^2 + |\nabla_q \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,\pm}}|^2 \right] dq d\tilde{x} dt \leq C_*. \quad (6.31)$$

A simple calculation⁹ then shows that these imply an analogous inequality for $\widehat{\psi}_{\varepsilon,L}^{\Delta t}$:

$$4 \int_0^T \int_{\Omega \times D} M \left[|\nabla_x \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t}}|^2 + |\nabla_q \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t}}|^2 \right] dq d\tilde{x} dt \leq C_*, \quad (6.32)$$

where, again, C_* denotes a generic positive constant independent of L and Δt . Finally, on combining (6.31) and (6.32) with (6.2) we arrive at the following bound, which represents the starting point for the convergence analysis that will be developed in the next subsection.

With $\sigma \geq \frac{1}{2}d$, $\sigma > 1$ and $s > 1 + \frac{1}{2}(K+1)d$, we have that:

$$\begin{aligned} & \text{ess.sup}_{t \in [0, T]} \|u_{\varepsilon, L}^{\Delta t, (\pm)}(t)\|^2 + \frac{1}{\Delta t} \int_0^T \|u_{\varepsilon, L}^{\Delta t, +}(t) - u_{\varepsilon, L}^{\Delta t, -}(t)\|^2 dt \\ & + \int_0^T \|\nabla_x u_{\varepsilon, L}^{\Delta t, (\pm)}(t)\|^2 dt + \text{ess.sup}_{t \in [0, T]} \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_{\varepsilon, L}^{\Delta t, (\pm)}(t)) dq d\tilde{x} \\ & + \frac{1}{\Delta t L} \int_0^T \int_{\Omega \times D} M (\widehat{\psi}_{\varepsilon, L}^{\Delta t, +} - \widehat{\psi}_{\varepsilon, L}^{\Delta t, -})^2 dq d\tilde{x} dt \\ & + \int_0^T \int_{\Omega \times D} M |\nabla_x \sqrt{\widehat{\psi}_{\varepsilon, L}^{\Delta t, (\pm)}}|^2 dq d\tilde{x} dt + \int_0^T \int_{\Omega \times D} M |\nabla_q \sqrt{\widehat{\psi}_{\varepsilon, L}^{\Delta t, (\pm)}}|^2 dq d\tilde{x} dt \\ & + \int_0^T \left\| \frac{\partial u_{\varepsilon, L}^{\Delta t}}{\partial t}(t) \right\|_{V'_\sigma}^2 dt + \int_0^T \left\| M \frac{\partial \widehat{\psi}_{\varepsilon, L}^{\Delta t}}{\partial t}(t) \right\|_{(H^s(\Omega \times D))'}^2 dt \leq C_*. \quad (6.33) \end{aligned}$$

Similarly,

$$\begin{aligned} & \text{ess.sup}_{t \in [0, T]} \|\rho_{\varepsilon, L}^{\Delta t, (\pm)}(t)\|_{L^\infty(\Omega)}^2 + \frac{1}{\Delta t} \int_0^T \|\rho_{\varepsilon, L}^{\Delta t, +}(t) - \rho_{\varepsilon, L}^{\Delta t, -}(t)\|^2 dt \\ & + \int_0^T \|\nabla_x \rho_{\varepsilon, L}^{\Delta t, (\pm)}(t)\|^2 dt + \int_0^T \left\| \frac{\partial \rho_{\varepsilon, L}^{\Delta t}}{\partial t}(t) \right\|_{(H^1(\Omega))'}^2 dt \leq C_*. \quad (6.34) \end{aligned}$$

Here, the bound on the first term on the left-hand side follows from (4.6), (6.3) and (6.4); the bound on the second term comes from (4.29), and the bound on the last term from (4.43). The bound on the third term on the left-hand side of (6.34) is obtained by applying ∇_x to both sides of (4.6) with the integrand

$$M(q) \widehat{\psi}_{\varepsilon, L}^{\Delta t, (\pm)}$$

rewritten as

$$M(q) \left[\sqrt{\widehat{\psi}_{\varepsilon, L}^{\Delta t, (\pm)}} \right]^2,$$

then exchanging the order of ∇_x and the integral over D on the right-hand side of the resulting identity, applying ∇_x to the integrand using the chain rule, followed by taking the modulus on both sides and applying the Cauchy–Schwarz inequality to the integral over D on the right, integrating the square of the resulting inequality over $[0, T] \times \Omega$ and, finally, recalling again the definition (4.6) and using the bound on the first term in (6.34) and the bound on the sixth term in (6.33). In fact, in the case of $\rho_{\varepsilon, L}^{\Delta t, +}$ the stated bound on the third term on the left-hand side of (6.34) follows directly from (4.29).

6.2. Passage to the limit $L \rightarrow \infty$

We are now ready to prove the central result of the paper.

Theorem 6.1. *Suppose that the assumptions (3.5) and the condition (6.5), relating Δt to L , hold. Then, there exists a subsequence of $\{(u_{\varepsilon,L}^{\Delta t}, \widehat{\psi}_{\varepsilon,L}^{\Delta t})\}_{L>1}$ (not indicated) with $\Delta t = o(L^{-1})$, and a pair of functions $(u_\varepsilon, \widehat{\psi}_\varepsilon)$ such that*

$$u_\varepsilon \in L^\infty(0, T; \widetilde{L}^2(\Omega)) \cap L^2(0, T; \widetilde{V}) \cap H^1(0, T; \widetilde{V}'_\sigma), \quad \sigma \geq \frac{1}{2}d, \quad \sigma > 1,$$

and

$$\widehat{\psi}_\varepsilon \in L^1(0, T; L_M^1(\Omega \times D)) \cap H^1(0, T; M^{-1}(H^s(\Omega \times D))'), \quad s > 1 + \frac{1}{2}(K+1)d,$$

with $\widehat{\psi}_\varepsilon \geq 0$ a.e. on $\Omega \times D \times [0, T]$,

$$\rho_\varepsilon(\underline{x}, t) := \int_D M(q) \widehat{\psi}_\varepsilon(\underline{x}, q, t) d\widetilde{q} = 1 \quad \text{for a.e. } (x, t) \in \Omega \times [0, T], \quad (6.35)$$

whereby $\widehat{\psi}_\varepsilon \in L^\infty(0, T; L_M^1(\Omega \times D))$; and finite relative entropy and Fisher information, with

$$\mathcal{F}(\widehat{\psi}_\varepsilon) \in L^\infty(0, T; L_M^1(\Omega \times D)) \quad \text{and} \quad \sqrt{\widehat{\psi}_\varepsilon} \in L^2(0, T; H_M^1(\Omega \times D)), \quad (6.36)$$

such that, as $L \rightarrow \infty$ (and thereby $\Delta t \rightarrow 0_+$),

$$u_{\varepsilon,L}^{\Delta t,(\pm)} \rightarrow u_\varepsilon \quad \text{weak}^* \text{ in } L^\infty(0, T; \widetilde{L}^2(\Omega)), \quad (6.37a)$$

$$u_{\varepsilon,L}^{\Delta t,(\pm)} \rightarrow u_\varepsilon \quad \text{weakly in } L^2(0, T; \widetilde{V}), \quad (6.37b)$$

$$u_{\varepsilon,L}^{\Delta t,(\pm)} \rightarrow u_\varepsilon \quad \text{strongly in } L^2(0, T; \widetilde{L}^r(\Omega)), \quad (6.37c)$$

$$\frac{\partial u_{\varepsilon,L}^{\Delta t}}{\partial t} \rightarrow \frac{\partial u_\varepsilon}{\partial t} \quad \text{weakly in } L^2(0, T; \widetilde{V}'_\sigma), \quad (6.37d)$$

where $r \in [1, \infty)$ if $d = 2$ and $r \in [1, 6)$ if $d = 3$; and

$$M^{\frac{1}{2}} \widetilde{\nabla}_x \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,(\pm)}} \rightarrow M^{\frac{1}{2}} \widetilde{\nabla}_x \sqrt{\widehat{\psi}_\varepsilon} \quad \text{weakly in } L^2(0, T; \widetilde{L}^2(\Omega \times D)), \quad (6.38a)$$

$$M^{\frac{1}{2}} \widetilde{\nabla}_q \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,(\pm)}} \rightarrow M^{\frac{1}{2}} \widetilde{\nabla}_q \sqrt{\widehat{\psi}_\varepsilon} \quad \text{weakly in } L^2(0, T; \widetilde{L}^2(\Omega \times D)), \quad (6.38b)$$

$$M \frac{\partial \widehat{\psi}_{\varepsilon,L}^{\Delta t}}{\partial t} \rightarrow M \frac{\partial \widehat{\psi}_\varepsilon}{\partial t} \quad \text{weakly in } L^2(0, T; (H^s(\Omega \times D))'), \quad (6.38c)$$

$$\widehat{\psi}_{\varepsilon,L}^{\Delta t,(\pm)} \rightarrow \widehat{\psi}_\varepsilon \quad \text{strongly in } L^p(0, T; L_M^1(\Omega \times D)), \quad (6.38d)$$

for all $p \in [1, \infty)$; and,

$$\widetilde{\nabla}_x \cdot \sum_{i=1}^K \widetilde{C}_i(M \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}) \rightarrow \widetilde{\nabla}_x \cdot \sum_{i=1}^K \widetilde{C}_i(M \widehat{\psi}_\varepsilon) \quad \text{weakly in } L^2(0, T; \widetilde{V}'_\sigma). \quad (6.38e)$$

The pair $(\underline{u}_\varepsilon, \widehat{\psi}_\varepsilon)$ is a global weak solution to problem (P_ε) , in the sense that

$$\begin{aligned} & - \int_0^T \int_\Omega \underline{u}_\varepsilon \cdot \frac{\partial \underline{w}}{\partial t} \, dx \, dt + \int_0^T \int_\Omega \left[\left[(\underline{u}_\varepsilon \cdot \nabla_x) \underline{u}_\varepsilon \right] \cdot \underline{w} + \nu \nabla_x \underline{u}_\varepsilon : \nabla_x \underline{w} \right] \, dx \, dt \\ & = \int_\Omega \underline{u}_0(x) \cdot \underline{w}(x, 0) \, dx + \int_0^T \left[\langle f, \underline{w} \rangle_{H_0^1(\Omega)} - k \sum_{i=1}^K \int_\Omega C_i(M \widehat{\psi}_\varepsilon) : \nabla_x \underline{w} \, dx \right] \, dt \\ & \quad \forall \underline{w} \in W^{1,1}(0, T; V_\sigma) \text{ s.t. } \underline{w}(\cdot, T) = 0, \end{aligned} \quad (6.39)$$

and

$$\begin{aligned} & - \int_0^T \int_{\Omega \times D} M \widehat{\psi}_\varepsilon \frac{\partial \widehat{\varphi}}{\partial t} \, dq \, dx \, dt + \int_0^T \int_{\Omega \times D} M \left[\varepsilon \nabla_x \widehat{\psi}_\varepsilon - \underline{u}_\varepsilon \widehat{\psi}_\varepsilon \right] \cdot \nabla_x \widehat{\varphi} \, dq \, dx \, dt \\ & + \frac{1}{2\lambda} \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \sum_{j=1}^K A_{ij} \nabla_{q_j} \widehat{\psi}_\varepsilon \cdot \nabla_{q_i} \widehat{\varphi} \, dq \, dx \, dt \\ & - \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \left[\sigma(\underline{u}_\varepsilon) q_i \right] \widehat{\psi}_\varepsilon \cdot \nabla_{q_i} \widehat{\varphi} \, dq \, dx \, dt = \int_{\Omega \times D} \widehat{\psi}_0(x, q) \widehat{\varphi}(x, q, 0) \, dq \, dx \\ & \quad \forall \widehat{\varphi} \in W^{1,1}(0, T; H^s(\Omega \times D)) \text{ s.t. } \widehat{\varphi}(\cdot, \cdot, T) = 0. \end{aligned} \quad (6.40)$$

In addition, the function $\underline{u}_\varepsilon$ is weakly continuous as a mapping from $[0, T]$ to $\underline{H}$, and $\widehat{\psi}_\varepsilon$ is weakly continuous as a mapping from $[0, T]$ to $L_M^1(\Omega \times D)$. The weak solution $(\underline{u}_\varepsilon, \widehat{\psi}_\varepsilon)$ satisfies the following energy inequality for a.e. $t \in [0, T]$:

$$\begin{aligned} & \|\underline{u}_\varepsilon(t)\|^2 + \nu \int_0^t \|\nabla_x \underline{u}_\varepsilon(s)\|^2 \, ds + k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_\varepsilon(t)) \, dq \, dx \\ & + 4k\varepsilon \int_0^t \int_{\Omega \times D} M |\nabla_x \sqrt{\widehat{\psi}_\varepsilon}|^2 \, dq \, dx \, ds + \frac{a_0 k}{\lambda} \int_0^t \int_{\Omega \times D} M |\nabla_q \sqrt{\widehat{\psi}_\varepsilon}|^2 \, dq \, dx \, ds \\ & \leq \|\underline{u}_0\|^2 + \frac{1}{\nu} \int_0^t \|f(s)\|_{(H_0^1(\Omega))'}^2 \, ds + k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_0) \, dq \, dx \leq [\mathbf{B}(\underline{u}_0, f, \widehat{\psi}_0)]^2, \end{aligned} \quad (6.41)$$

with $\mathcal{F}(s) = s(\log s - 1) + 1$, $s \geq 0$, and $[\mathbf{B}(\underline{u}_0, f, \widehat{\psi}_0)]^2$ as defined in (4.28).

Proof. Since the proof is long, we have broken it up into a number of steps.

Step 1. On recalling the weak* compactness of bounded balls in the Banach space $L^\infty(0, T; \underline{L}^2(\Omega))$ and noting the bound on the first term on the left-hand side of (6.33), upon three successive extractions of subsequences we deduce the existence of an unbounded index set $\mathcal{L} \subset (1, \infty)$ such that each of the three sequences $\{\underline{u}_{\varepsilon, L}^{\Delta t(\pm)}\}_{L \in \mathcal{L}}$ converges to its respective weak* limit in $L^\infty(0, T; \underline{L}^2(\Omega))$ as $L \rightarrow \infty$ with $L \in \mathcal{L}$. Thanks to (4.1a,b),

$$\int_0^T \|\underline{u}_{\varepsilon, L}^{\Delta t}(s) - \underline{u}_{\varepsilon, L}^{\Delta t, +}(s)\|^2 \, ds = \frac{1}{3} \int_0^T \|\underline{u}_{\varepsilon, L}^{\Delta t, +}(s) - \underline{u}_{\varepsilon, L}^{\Delta t, -}(s)\|^2 \, ds \leq \frac{1}{3} C_\star \Delta t, \quad (6.42)$$

where the last inequality is a consequence of the second bound in (6.33). On passing to the limit $L \rightarrow \infty$ with $L \in \mathcal{L}$ and using (6.5) we thus deduce that the weak*

limits of the sequences $\{u_{\varepsilon,L}^{\Delta t(\pm)}\}_{L \in \mathcal{L}}$ coincide. We label this common limit by u_ε ; by construction then, $u_\varepsilon \in L^\infty(0, T; \tilde{L}^2(\Omega))$. Thus we have shown (6.37a).

Upon further successive extraction of subsequences from $\{u_{\varepsilon,L}^{\Delta t(\pm)}\}_{L \in \mathcal{L}}$ and noting the bounds on the third and eighth term on the left-hand side of (6.33) the limits (6.37b,d) follow directly from the weak compactness of bounded balls in the Hilbert spaces $L^2(0, T; \tilde{V})$ and $L^2(0, T; \tilde{V}'_\sigma)$ and (6.37a) thanks to the uniqueness of limits of sequences in the weak topology of $L^2(0, T; \tilde{V})$ and $L^2(0, T; \tilde{V}'_\sigma)$, respectively.

By the Aubin–Lions–Simon compactness theorem (cf. (3.14)), we then deduce (6.37c) in the case of $u_{\varepsilon,L}^{\Delta t}$ on noting the compact embedding of $\tilde{V}$ into $\tilde{L}^r(\Omega) \cap \tilde{H}$, with the values of r as in the statement of the theorem. In particular, with $r = 2$, $\{u_{\varepsilon,L}^{\Delta t}\}_{L \in \mathcal{L}}$ converges to u_ε , strongly in $L^2(0, T; \tilde{L}^2(\Omega))$ as $L \rightarrow \infty$. Then, by the bound on the left-most term in (6.42), we deduce that $\{u_{\varepsilon,L}^{\Delta t,+}\}_{L \in \mathcal{L}}$ also converges to u_ε , strongly in $L^2(0, T; \tilde{L}^2(\Omega))$ as $L \rightarrow \infty$ (and thereby $\Delta t \rightarrow 0_+$). Further, by the bound on the middle term in (6.42) we have that the same is true of $\{u_{\varepsilon,L}^{\Delta t,-}\}_{L \in \mathcal{L}}$. Thus we have shown that the three sequences $\{u_{\varepsilon,L}^{\Delta t(\pm)}\}_{L \in \mathcal{L}}$ all converge to u_ε , strongly in $L^2(0, T; \tilde{L}^2(\Omega))$. Since the sequences $\{u_{\varepsilon,L}^{\Delta t(\pm)}\}_{L \in \mathcal{L}}$ are bounded in $L^2(0, T; \tilde{H}^1(\Omega))$ (cf. the bound on the third term in (6.33)) and strongly convergent in $L^2(0, T; \tilde{L}^2(\Omega))$, we deduce from (3.4) that (6.37c) holds, with the values of r as in the statement of the theorem. Thus we have proved (6.37a–d).

Step 2. Dubinskii’s theorem, with $\mathcal{A}_0$, $\mathcal{A}_1$ and $\mathcal{M}$ as in the discussion following the statement of Theorem 5.1 and selecting $p = 1$ and $p_1 = 2$, implies that

$$\left\{ \varphi : [0, T] \rightarrow \mathcal{M} : [\varphi]_{L^1(0, T; \mathcal{M})} + \left\| \frac{d\varphi}{dt} \right\|_{L^2(0, T; \mathcal{A}_1)} < \infty \right\} \\ \hookrightarrow L^1(0, T; \mathcal{A}_0) = L^1(0, T; L^1_M(\Omega \times D)).$$

Using this compact embedding, together with the bounds on the sixth, the seventh and the last term on the left-hand side of (6.33), in conjunction with (6.3) and (6.4), we deduce (upon extraction of a subsequence) strong convergence of $\{\hat{\psi}_{\varepsilon,L}^{\Delta t}\}_{L > 1}$ in $L^1(0, T; L^1_M(\Omega \times D))$ to an element $\hat{\psi}_\varepsilon \in L^1(0, T; L^1_M(\Omega \times D))$, as $L \rightarrow \infty$.

Thanks to the bound on the fifth term in (6.33), by the Cauchy–Schwarz inequality and an argument identical to the one in (6.42), we have that

$$\left(\int_0^T \int_{\Omega \times D} M |\hat{\psi}_{\varepsilon,L}^{\Delta t} - \hat{\psi}_{\varepsilon,L}^{\Delta t,+}| \, dq \, dx \, dt \right)^2 \leq \frac{T|\Omega|}{3} \int_0^T \int_{\Omega \times D} M (\hat{\psi}_{\varepsilon,L}^{\Delta t,+} - \hat{\psi}_{\varepsilon,L}^{\Delta t,-})^2 \, dq \, dx \, dt \\ \leq \frac{1}{3} C_* T |\Omega| \Delta t L. \quad (6.43)$$

On recalling (6.5), and using the triangle inequality in the $L^1(0, T; L^1_M(\Omega \times D))$ norm, together with (6.43) and the strong convergence of $\{\hat{\psi}_{\varepsilon,L}^{\Delta t}\}_{L > 1}$ to $\hat{\psi}_\varepsilon$ in $L^1(0, T; L^1_M(\Omega \times D))$, we deduce, as $L \rightarrow \infty$, strong convergence of $\{\hat{\psi}_{\varepsilon,L}^{\Delta t,+}\}_{L > 1}$ in $L^1(0, T; L^1_M(\Omega \times D))$ to the same element $\hat{\psi}_\varepsilon \in L^1(0, T; L^1_M(\Omega \times D))$. The inequality (6.43) then implies strong convergence of $\{\hat{\psi}_{\varepsilon,L}^{\Delta t,-}\}_{L > 1}$ to $\hat{\psi}_\varepsilon$, also. This completes the proof of (6.38d) for $p = 1$.

From (6.3) and (6.4) we have that

$$\|\widehat{\psi}_{\varepsilon,L}^{\Delta t(\pm)}(t)\|_{L_M^1(\Omega \times D)} \leq |\Omega| \quad (6.44)$$

for a.e. t in $[0, T]$ and all $L > 1$. In other words, the sequences $\{\widehat{\psi}_{\varepsilon,L}^{\Delta t(\pm)}\}_{L>1}$ are bounded in $L^\infty(0, T; L_M^1(\Omega \times D))$. By Lemma 5.1, the strong convergence of these to $\widehat{\psi}_\varepsilon$ in $L^1(0, T; L_M^1(\Omega \times D))$, shown above, then implies strong convergence in $L^p(0, T; L_M^1(\Omega \times D))$ to the same limit for all values of $p \in [1, \infty)$. That now completes the proof of (6.38d).

Since strong convergence in $L^p(0, T; L_M^1(\Omega \times D))$, $p \geq 1$, implies convergence almost everywhere on $\Omega \times D \times [0, T]$ of a subsequence, it follows from (6.3) that $\widehat{\psi}_\varepsilon \geq 0$ on $\Omega \times D \times [0, T]$. Furthermore, by Fubini's theorem, strong convergence of $\{\widehat{\psi}_{\varepsilon,L}^{\Delta t(\pm)}\}_{L>1}$ to $\widehat{\psi}_\varepsilon$ in $L^1(0, T; L_M^1(\Omega \times D))$ implies that

$$\int_D M(\underline{q}) |\widehat{\psi}_{\varepsilon,L}^{\Delta t(\pm)}(\underline{x}, \underline{q}, t) - \widehat{\psi}_\varepsilon(\underline{x}, \underline{q}, t)| d\underline{q} \rightarrow 0 \quad \text{as } L \rightarrow \infty$$

for a.e. $(\underline{x}, t) \in \Omega \times [0, T]$. Hence we have that $\int_D M(\underline{q}) \widehat{\psi}_{\varepsilon,L}^{\Delta t(\pm)}(\underline{x}, \underline{q}, t) d\underline{q}$ converges to $\int_D M(\underline{q}) \widehat{\psi}_\varepsilon(\underline{x}, \underline{q}, t) d\underline{q}$, as $L \rightarrow \infty$, for a.e. $(\underline{x}, t) \in \Omega \times [0, T]$, and then (6.4) implies that

$$\int_D M(\underline{q}) \widehat{\psi}_\varepsilon(\underline{x}, \underline{q}, t) d\underline{q} \leq 1 \quad \text{for a.e. } (x, t) \in \Omega \times [0, T]. \quad (6.45)$$

We will show later that the inequality here can in fact be sharpened to an equality.

As the sequences $\{\widehat{\psi}_{\varepsilon,L}^{\Delta t(\pm)}\}_{L>1}$ converge to $\widehat{\psi}_\varepsilon$ strongly in $L^1(0, T; L_M^1(\Omega \times D))$, it follows (upon extraction of suitable subsequences) that they converge to $\widehat{\psi}_\varepsilon$ a.e. on $\Omega \times D \times [0, T]$. This then, in turn, implies that the sequences $\{\mathcal{F}(\widehat{\psi}_{\varepsilon,L}^{\Delta t(\pm)})\}_{L>1}$ converge to $\mathcal{F}(\widehat{\psi}_\varepsilon)$ a.e. on $\Omega \times D \times [0, T]$; in particular, for a.e. $t \in [0, T]$, the sequences $\{\mathcal{F}(\widehat{\psi}_{\varepsilon,L}^{\Delta t(\pm)}(\cdot, \cdot, t))\}_{L>1}$ converge to $\mathcal{F}(\widehat{\psi}_\varepsilon(\cdot, \cdot, t))$ a.e. on $\Omega \times D$. Since $\mathcal{F}$ is nonnegative, Fatou's lemma then implies that, for a.e. $t \in [0, T]$,

$$\begin{aligned} & \int_{\Omega \times D} M(\underline{q}) \mathcal{F}(\widehat{\psi}_\varepsilon(\underline{x}, \underline{q}, t)) d\underline{x} d\underline{q} \\ & \leq \liminf_{L \rightarrow \infty} \int_{\Omega \times D} M(\underline{q}) \mathcal{F}(\widehat{\psi}_{\varepsilon,L}^{\Delta t(\pm)}(\underline{x}, \underline{q}, t)) d\underline{x} d\underline{q} \leq C_*, \end{aligned} \quad (6.46)$$

where the second inequality in (6.46) stems from the bound on the fourth term on the left-hand side of (6.33). As the integrand in the expression on the left-hand side of (6.46) is nonnegative, we deduce that $\mathcal{F}(\widehat{\psi}_\varepsilon)$ belongs to $L^\infty(0, T; L_M^1(\Omega \times D))$, as asserted in the statement of the theorem.

We observe in passing that since $|\sqrt{c_1} - \sqrt{c_2}| \leq \sqrt{|c_1 - c_2|}$ for any two nonnegative real numbers c_1 and c_2 , the strong convergence (6.38d) directly implies that, as $L \rightarrow \infty$ (and thereby $\Delta t \rightarrow 0_+$),

$$\sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t(\pm)}} \rightarrow \sqrt{\widehat{\psi}_\varepsilon} \quad \text{strongly in } L^p(0, T; L_M^2(\Omega \times D)) \quad \forall p \in [1, \infty), \quad (6.47)$$

and therefore, as $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$),

$$M^{\frac{1}{2}} \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t(\pm)}} \rightarrow M^{\frac{1}{2}} \sqrt{\widehat{\psi}_{\varepsilon}} \quad \text{strongly in } L^p(0, T; L^2(\Omega \times D)) \quad \forall p \in [1, \infty). \quad (6.48)$$

By proceeding in exactly the same way as in the previous subsection, between equations (6.18) and (6.20b), with $\widehat{\zeta}^{\Lambda,1}$ and $\widehat{\zeta}^1 = \widehat{\psi}^0$ replaced by $\widehat{\psi}_{\varepsilon,L}^{\Delta t(\pm)}$ and $\widehat{\psi}_{\varepsilon}$, respectively, but now using the sixth and the seventh bound in (6.33), and (6.4), we deduce that (6.38a,b) hold.

The convergence result (6.38c) follows from the bound on the last term on the left-hand side of (6.33) and the weak compactness of bounded balls in the Hilbert space $L^2(0, T; (H^s(\Omega \times D))')$, $s > 1 + \frac{1}{2}(K+1)d$.

The proof of (6.38e) is considerably more complicated, and will be given below.

After all these technical preparations we are now ready to return to (4.3) and (4.4) and pass to the limit $L \rightarrow \infty$ (and thereby also $\Delta t \rightarrow 0_+$); we shall also prove (6.38e). Since there are quite a few terms to deal with, we shall discuss them one at a time, starting with equation (4.4), and followed by equation (4.3).

Step 3. We begin by passing to the limit $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$) on equation (4.4). In what follows, we shall take test functions $\widehat{\varphi} \in C^1([0, T]; C^\infty(\overline{\Omega \times D}))$ such that $\widehat{\varphi}(\cdot, \cdot, T) = 0$. Note that, for any $s \geq 0$, the set of all such test functions $\widehat{\varphi}$ is a dense linear subspace of the linear space of functions in $W^{1,1}(0, T; H^s(\Omega \times D))$ vanishing at $t = T$. As each of the terms in (4.4) has been shown to be a continuous linear functional with respect to $\widehat{\varphi}$ on $L^2(0, T; H^s(\Omega \times D))$ for $s > 1 + \frac{1}{2}(K+1)d$, and therefore also on $W^{1,1}(0, T; H^s(\Omega \times D))$ for $s > 1 + \frac{1}{2}(K+1)d$, which is (continuously) embedded in $L^2(0, T; H^s(\Omega \times D))$ for $s > 1 + \frac{1}{2}(K+1)d$, the use of such test functions for the purposes of the argument below is fully justified.

Step 3.1. Integration by parts with respect to t in the first term in (4.4) gives

$$\begin{aligned} \int_0^T \int_{\Omega \times D} M \frac{\partial \widehat{\psi}_{\varepsilon,L}^{\Delta t}}{\partial t} \widehat{\varphi} \, dq \, dx \, dt &= - \int_0^T \int_{\Omega \times D} M \widehat{\psi}_{\varepsilon,L}^{\Delta t} \frac{\partial \widehat{\varphi}}{\partial t} \, dq \, dx \, dt \\ &\quad - \int_{\Omega \times D} M(q) \beta^L(\widehat{\psi}^0(x, q)) \widehat{\varphi}(x, q, 0) \, dq \, dx \end{aligned} \quad (6.49)$$

for all $\widehat{\varphi} \in C^1([0, T]; C^\infty(\overline{\Omega \times D}))$ such that $\widehat{\varphi}(\cdot, \cdot, T) = 0$. Using (6.38d) and noting point ⑤ of Lemma 6.2, we immediately have that, as $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$), the first term on the right-hand side of (6.49) converges to the first term on the left-hand side of (6.40) and the second term on the right-hand side of (6.49) converges to $-\int_{\Omega \times D} \widehat{\psi}_0(x, q) \widehat{\varphi}(x, q, 0) \, dq \, dx$, resulting in the first term on the right-hand side of (6.40). That completes Step 3.1.

Step 3.2. The second term in (4.4) will be dealt with by decomposing it into two further terms, the first of which tends to 0, while the second converges to the

expected limiting value. We proceed as follows:

$$\begin{aligned}
& \varepsilon \int_0^T \int_{\Omega \times D} M \nabla_x \widehat{\psi}_{\varepsilon,L}^{\Delta t,+} \cdot \nabla_x \widehat{\varphi} \, dq \, dx \, dt \\
&= 2\varepsilon \int_0^T \int_{\Omega \times D} M \left(\sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}} - \sqrt{\widehat{\psi}_\varepsilon} \right) \nabla_x \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}} \cdot \nabla_x \widehat{\varphi} \, dq \, dx \, dt \\
&\quad + 2\varepsilon \int_0^T \int_{\Omega \times D} M \sqrt{\widehat{\psi}_\varepsilon} \nabla_x \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}} \cdot \nabla_x \widehat{\varphi} \, dq \, dx \, dt \\
&=: V_1 + V_2.
\end{aligned}$$

We shall show that V_1 converges to 0 and that V_2 converges to the expected limit.

$$\begin{aligned}
|V_1| &\leq 2\varepsilon \int_0^T \int_{\Omega} \left(\int_D M |\sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}} - \sqrt{\widehat{\psi}_\varepsilon}|^2 \, dq \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_0^T \|\sqrt{M \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}} - \sqrt{M \widehat{\psi}_\varepsilon}\|_{L^2(\Omega \times D)}^r \, dt \right)^{\frac{1}{r}} \left(\int_0^T \|\nabla_x \widehat{\varphi}\|_{L^\infty(\Omega \times D)}^{\frac{2r}{r-2}} \, dt \right)^{\frac{r-2}{2r}},
\end{aligned}$$

where $r \in (2, \infty)$. Using the bound on the sixth term in (6.2) together with the Sobolev embedding theorem, we then have (with C_* now denoting a possibly different constant than in (6.2), but one that is still independent of L and Δt) that

$$\begin{aligned}
|V_1| &\leq 2C_*^{\frac{1}{2}} \varepsilon \|\sqrt{M \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}} - \sqrt{M \widehat{\psi}_\varepsilon}\|_{L^r(0,T;L^2(\Omega \times D))} \|\nabla_x \widehat{\varphi}\|_{L^{\frac{2r}{r-2}}(0,T;L^\infty(\Omega \times D))} \\
&\leq 2C_*^{\frac{1}{2}} T^{\frac{1}{2}} \varepsilon \|\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} - \widehat{\psi}_\varepsilon\|_{L^r(0,T;L_M^1(\Omega \times D))}^{\frac{1}{2}} \|\nabla_x \widehat{\varphi}\|_{L^{\frac{2r}{r-2}}(0,T;L^\infty(\Omega \times D))},
\end{aligned}$$

where we also used the elementary inequality $|\sqrt{c_1} - \sqrt{c_2}| \leq \sqrt{|c_1 - c_2|}$ with $c_1, c_2 \in \mathbb{R}_{\geq 0}$. The norm of the difference in the last displayed line is known to converge to 0 as $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$) by (6.38d). This then implies that the term V_1 converges to 0 as $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$).

Concerning the term V_2 , we have that

$$V_2 = 2\varepsilon \int_0^T \int_{\Omega \times D} M^{\frac{1}{2}} \nabla_x \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}} \cdot \sqrt{M \widehat{\psi}_\varepsilon} \nabla_x \widehat{\varphi} \, dq \, dx \, dt.$$

Once we have verified that $\sqrt{M \widehat{\psi}_\varepsilon} \nabla_x \widehat{\varphi}$ belongs to $L^2(0, T; L^2(\Omega \times D))$, the weak convergence result (6.38a) will imply that

$$\begin{aligned}
V_2 &\rightarrow 2\varepsilon \int_0^T \int_{\Omega \times D} M^{\frac{1}{2}} \nabla_x \sqrt{\widehat{\psi}_\varepsilon} \cdot \sqrt{M \widehat{\psi}_\varepsilon} \nabla_x \widehat{\varphi} \, dq \, dx \, dt \\
&= \varepsilon \int_0^T \int_{\Omega \times D} M \nabla_x \widehat{\psi}_\varepsilon \cdot \nabla_x \widehat{\varphi} \, dq \, dx \, dt
\end{aligned}$$

as $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$), and we will have completed Step 3.2. Let us therefore show that $\sqrt{M \widehat{\psi}_\varepsilon} \nabla_x \widehat{\varphi}$ belongs to $L^2(0, T; L^2(\Omega \times D))$; the justification is quite

straightforward: using (6.45) we have that

$$\int_0^T \int_{\Omega \times D} |\sqrt{M} \widehat{\psi}_\varepsilon \nabla_x \widehat{\varphi}|^2 dx dt \leq \|\nabla_x \widehat{\varphi}\|_{L^2(0,T;L^2(\Omega;L^\infty(D)))}^2 < \infty.$$

That now completes Step 3.2.

Step 3.3. The third term in (4.4) is dealt with as follows:

$$\begin{aligned} - \int_0^T \int_{\Omega \times D} M u_{\varepsilon,L}^{\Delta t,-} \widehat{\psi}_{\varepsilon,L}^{\Delta t,+} \cdot \nabla_x \widehat{\varphi} dq dx dt &= - \int_0^T \int_{\Omega \times D} M u_\varepsilon \widehat{\psi}_\varepsilon \cdot \nabla_x \widehat{\varphi} dq dx dt \\ &\quad + \int_0^T \int_{\Omega \times D} M (u_\varepsilon - u_{\varepsilon,L}^{\Delta t,-}) \widehat{\psi}_{\varepsilon,L}^{\Delta t,+} \cdot \nabla_x \widehat{\varphi} dq dx dt \\ &\quad + \int_0^T \int_{\Omega \times D} M u_\varepsilon (\widehat{\psi}_\varepsilon - \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}) \cdot \nabla_x \widehat{\varphi} dq dx dt. \end{aligned}$$

We label the last two terms by V_3 and V_4 and we show that each of them converges to 0 as $L \rightarrow 0$ (and $\Delta t \rightarrow 0_+$). We start with term V_3 ; below, we apply Hölder's inequality with $r \in (1, \infty)$ in the case of $d = 2$ and with $r \in (1, 6)$ when $d = 3$:

$$|V_3| \leq \|u_\varepsilon - u_{\varepsilon,L}^{\Delta t,-}\|_{L^2(0,T;L^r(\Omega))} \|\nabla_x \widehat{\varphi}\|_{L^2(0,T;L^{\frac{r}{r-1}}(\Omega;L^\infty(D)))},$$

where we made use of (4.31b). Thanks to (6.37c) the first factor converges to 0, and hence V_3 converges to 0 also, as $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$).

For V_4 , we have, by using Fubini's theorem, together with the factorization

$$M (\widehat{\psi}_\varepsilon - \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}) = M^{\frac{1}{2}} \left(\sqrt{\widehat{\psi}_\varepsilon} - \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}} \right) M^{\frac{1}{2}} \left(\sqrt{\widehat{\psi}_\varepsilon} + \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}} \right), \quad (6.50)$$

and in conjunction with the Cauchy-Schwarz inequality, (4.31b), (6.45) and the elementary inequality $|\sqrt{c_1} - \sqrt{c_2}| \leq \sqrt{|c_1 - c_2|}$ with $c_1, c_2 \in \mathbb{R}_{\geq 0}$, that

$$|V_4| \leq 2 \|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \|\widehat{\psi}_\varepsilon - \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}\|_{L^1(0,T;L_M^1(\Omega \times D))}^{\frac{1}{2}} \|\nabla_x \widehat{\varphi}\|_{L^2(0,T;L^\infty(\Omega \times D))}.$$

By (6.37a) the first factor is finite while, according to (6.38d) (with $p = 1$), the middle factor converges to 0 as $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$). This proves that V_4 converges to 0 as $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$), also. That completes Step 3.3.

Step 3.4. Thanks to (6.38b), as $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$),

$$M^{\frac{1}{2}} \nabla_q \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,(\pm)}} \rightarrow M^{\frac{1}{2}} \nabla_q \sqrt{\widehat{\psi}_\varepsilon} \quad \text{weakly in } L^2(0,T;L^2(\Omega \times D)).$$

This, in turn, implies that, componentwise, as $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$),

$$M^{\frac{1}{2}} \nabla_{q_j} \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,(\pm)}} \rightarrow M^{\frac{1}{2}} \nabla_{q_j} \sqrt{\widehat{\psi}_\varepsilon} \quad \text{weakly in } L^2(0,T;L^2(\Omega \times D)),$$

for each $j = 1, \dots, K$, whereby also,

$$M^{\frac{1}{2}} \sum_{j=1}^K A_{ij} \nabla_{q_j} \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,(\pm)}} \rightarrow M^{\frac{1}{2}} \sum_{j=1}^K A_{ij} \nabla_{q_j} \sqrt{\widehat{\psi}_\varepsilon} \quad \text{weakly in } L^2(0,T;L^2(\Omega \times D)),$$

for each $i = 1, \dots, K$. That places us in a very similar position as in the case of Step 3.2, and we can argue in an identical manner as there to show that

$$\begin{aligned} & \frac{1}{2\lambda} \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \sum_{j=1}^K A_{ij} \nabla_{q_j} \widehat{\psi}_{\varepsilon,L}^{\Delta t,+} \cdot \nabla_{q_i} \widehat{\varphi} \, dq \, dx \, dt \\ & \rightarrow \frac{1}{2\lambda} \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \sum_{j=1}^K A_{ij} \nabla_{q_j} \widehat{\psi}_{\varepsilon} \cdot \nabla_{q_i} \widehat{\varphi} \, dq \, dx \, dt \end{aligned}$$

as $L \rightarrow \infty$ and $\Delta t \rightarrow 0_+$, for all $\widehat{\varphi} \in L^{\frac{2r}{r-2}}(0, T; W^{1,\infty}(\Omega \times D))$, $r \in (2, \infty)$, and in particular for all $\widehat{\varphi} \in C^1([0, T]; C^\infty(\overline{\Omega \times D}))$. That completes Step 3.4.

Step 3.5. The final term in (4.4), the drag term, is the one in the equation that is the most difficult to deal with. We shall break it up into four subterms, three of which will be shown to converge to 0 in the limit of $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$), leaving the fourth term as the (expected) limiting value:

$$\begin{aligned} & - \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \left[\sigma(\widehat{u}_{\varepsilon,L}^{\Delta t,+}) q_i \right] \beta^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}) \cdot \nabla_{q_i} \widehat{\varphi} \, dq \, dx \, dt \\ & = - \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \left[\left(\nabla_x \widehat{u}_{\varepsilon,L}^{\Delta t,+} \right) q_i \right] \left(\beta^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}) - \beta^L(\widehat{\psi}_{\varepsilon}) \right) \cdot \nabla_{q_i} \widehat{\varphi} \, dq \, dx \, dt \\ & \quad - \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \left[\left(\nabla_x \widehat{u}_{\varepsilon,L}^{\Delta t,+} \right) q_i \right] \left(\beta^L(\widehat{\psi}_{\varepsilon}) - \widehat{\psi}_{\varepsilon} \right) \cdot \nabla_{q_i} \widehat{\varphi} \, dq \, dx \, dt \\ & \quad - \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \left[\left(\nabla_x \widehat{u}_{\varepsilon,L}^{\Delta t,+} - \nabla_x \widehat{u}_{\varepsilon} \right) q_i \right] \widehat{\psi}_{\varepsilon} \cdot \nabla_{q_i} \widehat{\varphi} \, dq \, dx \, dt \\ & \quad - \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \left[\left(\nabla_x \widehat{u}_{\varepsilon} \right) q_i \right] \widehat{\psi}_{\varepsilon} \cdot \nabla_{q_i} \widehat{\varphi} \, dq \, dx \, dt. \end{aligned} \tag{6.51}$$

Strictly speaking, we should have written “ $L \rightarrow \infty$, with $L \in \mathcal{L}$,” as in Step 1 above, instead of “ $L \rightarrow \infty$ ”; for the sake of brevity we chose to use the latter, compressed notation. The same notational convention applies below.

We label the first three terms on the right-hand side by V_5, V_6, V_7 , respectively. We shall show that each of the three terms converges to 0, leaving the fourth term as the limit of the left-most expression in the chain, as $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$).

We begin by bounding V_5 , noting that β^L is Lipschitz continuous with Lipschitz constant 1, writing, as before, $b := |\underline{b}|_1$ and using the factorization (6.50) together with (4.31b) and (6.45), and then proceeding as in the case of term V_4 in Step 3.3:

$$\begin{aligned} |V_5| & \leq 2\sqrt{b} \|\nabla_x \widehat{u}_{\varepsilon,L}^{\Delta t,+}\|_{L^2(0,T;L^2(\Omega))} \|\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} - \widehat{\psi}_{\varepsilon}\|_{L^1(0,T;L_M^1(\Omega \times D))}^{\frac{1}{2}} \\ & \quad \times \|\nabla_q \widehat{\varphi}\|_{L^\infty((0,T) \times \Omega \times D)}. \end{aligned}$$

By noting the bound on the third term on the left-hand side of (6.33) and the convergence result (6.38d) that was proved in Step 2, we deduce that term V_5

converges to 0 as $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$).

We move on to term V_6 , using an identical argument as in the case of term V_5 :

$$|V_6| \leq 2\sqrt{b} \|\nabla_{\tilde{x}} u_{\tilde{\varepsilon}, L}^{\Delta t, +}\|_{L^2(0, T; L^2(\Omega))} \|\beta^L(\hat{\psi}_{\tilde{\varepsilon}}) - \hat{\psi}_{\tilde{\varepsilon}}\|_{L^{\frac{1}{2}}(0, T; L_M^1(\Omega \times D))}^{\frac{1}{2}} \\ \times \|\nabla_q \hat{\varphi}\|_{L^\infty((0, T) \times \Omega \times D)}.$$

Observe that $0 \leq \widehat{\psi}_\varepsilon - \beta^L(\widehat{\psi}_\varepsilon) \leq \widehat{\psi}_\varepsilon$ and that $\widehat{\psi}_\varepsilon - \beta^L(\widehat{\psi}_\varepsilon)$ converges to 0 almost everywhere on $\Omega \times D \times (0, T)$ as $L \rightarrow \infty$. Note further that, thanks to (6.38d) with $p = 1$, $\widehat{\psi}_\varepsilon \in L^1(0, T; L_M^1(\Omega \times D))$. Thus, Lebesgue's dominated convergence theorem implies that, as $L \rightarrow \infty$, the middle factor in the last displayed line converges to 0. Hence, recalling the bound on the third term on the left-hand side of (6.33), we thus deduce that V_6 converges to 0 as $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$).

Finally, we consider the term V_7 :

$$V_7 := - \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \left[\left(\nabla_{\widetilde{x}} u_{\widetilde{\varepsilon}, L}^{\Delta t, +} - \nabla_{\widetilde{x}} u_{\widetilde{\varepsilon}} \right) q_i \right] \widehat{\psi}_{\widetilde{\varepsilon}} \cdot \nabla_{q_i} \widehat{\varphi} \, dq \, d\widetilde{x} \, dt.$$

We observe that, before starting to bound V_7 , we should perform an integration by parts in order to transfer the x -gradients from the difference $\nabla_x u_{\varepsilon, L}^{\Delta t, +} - \nabla_x u_{\varepsilon}$ onto the other factors under the integral sign, as we only have weak, but not strong, convergence of $\nabla_x u_{\varepsilon, L}^{\Delta t, +} - \nabla_x u_{\varepsilon}$ to 0, (cf. (6.37b)) whereas the difference $u_{\varepsilon, L}^{\Delta t, +} - u_{\varepsilon}$ converges to 0 strongly by virtue of (6.37c).

We note in this respect that the function $\underline{x} \in \Omega \mapsto \hat{\psi}_\varepsilon(\underline{x}, q, t) \in \mathbb{R}_{\geq 0}$ has a well-defined trace on $\partial\Omega$ for a.e. $(q, t) \in D \times (0, T)$, since, thanks to (6.38a),

$$\sqrt{\widehat{\psi}_\varepsilon(\cdot, \underline{q}, t)} \in H^1(\Omega), \quad \text{and therefore} \quad \sqrt{\widehat{\psi}_\varepsilon(\cdot, \underline{q}, t)} \Big|_{\partial\Omega} \in H^{1/2}(\partial\Omega),$$

for a.e. $(q, t) \in D \times (0, T)$, implying that $\sqrt{\widehat{\psi}_\varepsilon(\cdot, q, t)}|_{\partial\Omega} \in L^{2p}(\partial\Omega)$ for a.e. $(q, t) \in D \times (0, T)$, with $2p \in [1, \infty)$ when $d = 2$ and $2p \in [1, 4]$ when $d = 3$, whereby $\widehat{\psi}_\varepsilon|_{\partial\Omega} \in L^p(\partial\Omega)$ for a.e. $(q, t) \in D \times (0, T)$, with $p \in [1, \infty)$ when $d = 2$ and $p \in [1, 2]$ when $d = 3$. As the functions $\underline{u}_\varepsilon$ and $\underline{u}_{\varepsilon, L}^{\Delta t, +}$ have zero trace on $\partial\Omega$, the boundary integral that arises in the course of integration by parts is correctly defined and, in fact, vanishes. With these preliminary remarks in mind, we first write

$$V_7 = - \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \sum_{m,n=1}^d \frac{\partial}{\partial x_m} \left[\left((\underset{\sim}{u}_{\varepsilon,L}^{\Delta t,+})_n - (\underset{\sim}{u}_{\varepsilon})_n \right) (\underset{\sim}{q}_i)_m \right] \hat{\psi}_{\varepsilon} (\underset{\sim}{\nabla}_{q_i} \hat{\varphi})_n \underset{\sim}{d} \underset{\sim}{q} \underset{\sim}{d} \underset{\sim}{x} \underset{\sim}{d} t.$$

Here, $(\underline{u}_{\varepsilon,L}^{\Delta t,+})_n$ and $(\underline{u}_{\varepsilon})_n$ denote the n th among the d components of the vectors $\underline{u}_{\varepsilon,L}^{\Delta t,+}$ and $\underline{u}_{\varepsilon}$, $1 \leq n \leq d$, respectively, and $(\nabla_{q_i} \widehat{\varphi})_n$ denotes the n th among the d components of the vector $\nabla_{q_i} \widehat{\varphi}$, $1 \leq n \leq d$, for each $i \in \{1, \dots, K\}$. Similarly, $(\underline{q}_i)_m$ denotes the m th component, $1 \leq m \leq d$, of the d -component vector $\underline{q}_i$ for $i \in \{1, \dots, K\}$. Now, on integrating by parts w.r.t. x_m and cancelling the boundary integral terms, with the justification given above, we have that

$$\begin{aligned}
V_7 &= \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \sum_{m,n=1}^d \left[\left((u_{\varepsilon,L}^{\Delta t,+})_n - (u_\varepsilon)_n \right) (q_i)_m \right] \frac{\partial}{\partial x_m} \left(\widehat{\psi}_\varepsilon (\nabla_{q_i} \widehat{\varphi})_n \right) dq dx dt \\
&= \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \sum_{m,n=1}^d \left[\left((u_{\varepsilon,L}^{\Delta t,+})_n - (u_\varepsilon)_n \right) (q_i)_m \right] \frac{\partial \widehat{\psi}_\varepsilon}{\partial x_m} (\nabla_{q_i} \widehat{\varphi})_n dq dx dt \\
&\quad + \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \sum_{m,n=1}^d \left[\left((u_{\varepsilon,L}^{\Delta t,+})_n - (u_\varepsilon)_n \right) (q_i)_m \right] \left(\widehat{\psi}_\varepsilon \frac{\partial}{\partial x_m} (\nabla_{q_i} \widehat{\varphi})_n \right) dq dx dt \\
&=: V_{7,1} + V_{7,2}.
\end{aligned}$$

For the term $V_{7,1}$ we have, by the Cauchy–Schwarz inequality and (6.45), that

$$|V_{7,1}| \leq 2\sqrt{b} \int_0^T \left[\int_\Omega |u_{\varepsilon,L}^{\Delta t,+} - u_\varepsilon| \left(\int_D M |\nabla_x \sqrt{\widehat{\psi}_\varepsilon}|^2 dq \right)^{\frac{1}{2}} dx \right] \|\nabla_q \widehat{\varphi}\|_{L^\infty(\Omega \times D)} dt.$$

Hence,

$$\begin{aligned}
|V_{7,1}| &\leq 2\sqrt{b} \|u_{\varepsilon,L}^{\Delta t,+} - u_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \|\nabla_x \sqrt{\widehat{\psi}_\varepsilon}\|_{L^2(0,T;L_M^2(\Omega \times D))} \\
&\quad \times \|\nabla_q \widehat{\varphi}\|_{L^\infty(0,T;L^\infty(\Omega \times D))}.
\end{aligned}$$

Thanks to (6.37c) with $r = 2$ and (6.38a), $V_{7,1}$ tends to 0 as $L \rightarrow 0$ (and $\Delta t \rightarrow 0_+$).

Let us now consider the term $V_{7,2}$. Proceeding similarly as in the case of the term $V_{7,1}$, using (6.45), yields

$$|V_{7,2}| \leq \sqrt{b} \|u_{\varepsilon,L}^{\Delta t,+} - u_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \|\nabla_x \nabla_q \widehat{\varphi}\|_{L^2(0,T;L^2(\Omega;L^\infty(D)))}.$$

Noting (6.37c) with $r = 2$, we deduce that $V_{7,2}$ converges to 0 as $L \rightarrow 0$ (and $\Delta t \rightarrow 0_+$). Having shown that both $V_{7,1}$ and $V_{7,2}$ converge to 0 as $L \rightarrow 0$ (and $\Delta t \rightarrow 0_+$), it follows that the same is true of $V_7 = V_{7,1} + V_{7,2}$. We have already shown that V_5 and V_6 converge to 0 as $L \rightarrow 0$ (and $\Delta t \rightarrow 0_+$). Since the sum of the first three terms on the left-hand side of (6.51) converges to 0, it follows that the left-most expression in the chain (6.51) converges to the right-most term, in the limit of $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$). That completes Step 3.5.

Having dealt with (4.4), we now turn to (4.3), with the aim to pass to the limit with L (and Δt). In Steps 3.6 and 3.7 below we shall choose as our test function

$$w \in C^1([0, T]; \mathcal{C}_0^\infty(\Omega)) \quad \text{with } w(\cdot, T) = 0, \text{ and } \nabla_x \cdot w = 0 \text{ on } \Omega \text{ for all } t \in [0, T].$$

Clearly, any such w belongs to $L^1(0, T; \mathcal{V})$ and is therefore a legitimate choice of test function in (4.3). Furthermore, for any $\sigma \geq 1$, the set of such smooth test functions w is dense in the space of all functions in $W^{1,1}(0, T; \mathcal{V}_\sigma)$ that vanish at $t = T$. As each term in (4.3) has been shown before to be a continuous linear functional on $L^2(0, T; \mathcal{V}_\sigma)$, $\sigma \geq \frac{1}{2}d$, $\sigma > 1$ and $W^{1,1}(0, T; \mathcal{V}_\sigma)$ is (continuously) embedded in $L^2(0, T; \mathcal{V}_\sigma)$, $\sigma \geq \frac{1}{2}d$, $\sigma > 1$, the use of such smooth test functions for the purposes of the argument below is fully justified.

Step 3.6. The terms on the left-hand side of (4.3) are handled routinely, using (6.33) and, respectively, integration by parts in time in conjunction with (6.37c) with $r = 2$, (6.37b) and recalling that $u^0 \rightarrow u_0$ weakly in H . In particular, the second (nonlinear) term on the left-hand side of (4.3) is quite simple to deal with on rewriting it as $-\int_0^T (u_{\varepsilon,L}^{\Delta t,+} \otimes u_{\varepsilon,L}^{\Delta t,-}, \nabla_x w) dt$, and then considering the difference $\int_0^T (u_\varepsilon \otimes u_\varepsilon - u_{\varepsilon,L}^{\Delta t,+} \otimes u_{\varepsilon,L}^{\Delta t,-}, \nabla_x w) dt$, which is bounded by

$$\left(\int_0^T \|u_\varepsilon \otimes u_\varepsilon - u_{\varepsilon,L}^{\Delta t,+} \otimes u_{\varepsilon,L}^{\Delta t,-}\|_{L^1(\Omega)} dt \right) \|\nabla_x w\|_{L^\infty(0,T;L^\infty(\Omega))}.$$

By adding and subtracting $u_\varepsilon \otimes u_{\varepsilon,L}^{\Delta t,-}$ inside the first norm sign, using the triangle inequality, followed by the Cauchy–Schwarz inequality in each of the resulting terms, and then applying the first bound in (6.33), and (6.37c) with $r = 2$, we deduce that the above expression converges to 0 as $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$). The convergence of the first term on the right-hand side of (4.3) to the correct limit, as $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$), is an immediate consequence of (3.23). We refer the reader for a similar argument to Ch. 3, Sec. 4 of Temam³⁷. That completes Step 3.6.

Step 3.7. The extra-stress tensor appearing on the right-hand side of (4.3) is dealt with as follows. First, by using (3.16) and noting that w is, by assumption, divergence-free, and proceeding in exactly the same manner as in (4.48), but with $\hat{\psi}_{\varepsilon,L}^{\Delta t,+}$ now replaced by $\hat{\psi}_{\varepsilon,L}^{\Delta t,+} - \hat{\psi}_\varepsilon$, we have that

$$\begin{aligned} V_8 &:= \left| k \int_0^T \sum_{i=1}^K \int_{\Omega} C_i(M \hat{\psi}_{\varepsilon,L}^{\Delta t,+}) : \nabla_x w \, dx \, dt - k \int_0^T \sum_{i=1}^K \int_{\Omega} C_i(M \hat{\psi}_\varepsilon) : \nabla_x w \, dx \, dt \right| \\ &= k \left| \int_0^T \int_{\Omega} \left[\int_D M \sum_{i=1}^K (\nabla_x w) q_i \cdot \nabla_{q_i} (\hat{\psi}_{\varepsilon,L}^{\Delta t,+} - \hat{\psi}_\varepsilon) \, dq \right] dx \, dt \right|. \end{aligned}$$

We rewrite the second factor in the integrand of the last integral as follows:

$$\begin{aligned} M \left(\nabla_{q_i} \hat{\psi}_{\varepsilon,L}^{\Delta t,+} - \nabla_{q_i} \hat{\psi}_\varepsilon \right) &= 2 \left(\sqrt{M \hat{\psi}_{\varepsilon,L}^{\Delta t,+}} - \sqrt{M \hat{\psi}_\varepsilon} \right) \left[M^{\frac{1}{2}} \nabla_{q_i} \sqrt{\hat{\psi}_{\varepsilon,L}^{\Delta t,+}} \right] \\ &\quad + 2 \sqrt{M \hat{\psi}_\varepsilon} \left(M^{\frac{1}{2}} \nabla_{q_i} \sqrt{\hat{\psi}_{\varepsilon,L}^{\Delta t,+}} - M^{\frac{1}{2}} \nabla_{q_i} \sqrt{\hat{\psi}_\varepsilon} \right). \end{aligned}$$

Hence we obtain the following inequality:

$$\begin{aligned} V_8 &\leq 2k \sqrt{b} \int_0^T \int_{\Omega} |\nabla_x w| \left[\int_D \left| \sqrt{M \hat{\psi}_{\varepsilon,L}^{\Delta t,+}} - \sqrt{M \hat{\psi}_\varepsilon} \right| \left| M^{\frac{1}{2}} \nabla_{q_i} \sqrt{\hat{\psi}_{\varepsilon,L}^{\Delta t,+}} \right| dq \right] dx \, dt \\ &\quad + 2k \left| \int_0^T \int_{\Omega} \left[\int_D \sum_{i=1}^K (\nabla_x w) q_i \cdot \sqrt{M \hat{\psi}_\varepsilon} \left(M^{\frac{1}{2}} \nabla_{q_i} \sqrt{\hat{\psi}_{\varepsilon,L}^{\Delta t,+}} - M^{\frac{1}{2}} \nabla_{q_i} \sqrt{\hat{\psi}_\varepsilon} \right) dq \right] dx \, dt \right| \\ &=: V_{8,1} + V_{8,2}. \end{aligned}$$

We emphasize at this point that in the term $V_{8,2}$ we intentionally did not move the modulus sign under the integral: we shall be relying on the weak convergence result

(6.38b) to drive the term $V_{8,2}$ to 0 in the limit of $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$), so it is essential that the modulus sign is kept outside the integral.

For $V_{8,1}$, we have, by using that $|\sqrt{c_1} - \sqrt{c_2}| \leq \sqrt{|c_1 - c_2|}$ for any $c_1, c_2 \in \mathbb{R}_{\geq 0}$:

$$\begin{aligned} V_{8,1} &\leq 2k \sqrt{b} \int_0^T \|\nabla_x w\|_{L^\infty(\Omega)} \|\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} - \widehat{\psi}_\varepsilon\|_{L_M^1(\Omega \times D)}^{\frac{1}{2}} \|\nabla_q \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}}\|_{L_M^2(\Omega \times D)} dt \\ &\leq 2k \sqrt{b} \|\nabla_x w\|_{L^\infty(0,T;L^\infty(\Omega))} \\ &\quad \times \|\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} - \widehat{\psi}_\varepsilon\|_{L^1(0,T;L_M^1(\Omega \times D))}^{\frac{1}{2}} \|\nabla_q \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}}\|_{L^2(0,T;L_M^2(\Omega \times D))}. \end{aligned}$$

By noting (6.38d) with $p = 1$ and the bound on the seventh term in (6.2) we deduce that the term $V_{8,1}$ converges to 0 in the limit of $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$).

Finally, for $V_{8,2}$, we first define the (Kd) -component column-vector function $\Xi := [\Xi_1^T, \dots, \Xi_K^T]^T$, where $\Xi_i := \sqrt{M \widehat{\psi}_\varepsilon} (\nabla_x w)_{q_i}$, $i = 1, \dots, K$, and note that

$$\begin{aligned} V_{8,2} &= 2k \left| \int_0^T \int_\Omega \left[\int_D \sum_{i=1}^K \Xi_i \cdot \left(M^{\frac{1}{2}} \nabla_{q_i} \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}} - M^{\frac{1}{2}} \nabla_{q_i} \sqrt{\widehat{\psi}_\varepsilon} \right) dq \right] dx dt \right| \\ &= 2k \left| \int_0^T \int_{\Omega \times D} \left(M^{\frac{1}{2}} \nabla_q \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}} - M^{\frac{1}{2}} \nabla_q \sqrt{\widehat{\psi}_\varepsilon} \right) \cdot \Xi dq dx dt \right|. \end{aligned}$$

The convergence of $V_{8,2}$ to 0 will directly follow from (6.38b) once we have shown that $\Xi \in L^2(0, T; L^2(\Omega \times D))$. The latter is straightforward to verify, using (6.45):

$$\begin{aligned} \int_0^T \int_{\Omega \times D} |\Xi|^2 dq dx dt &= \sum_{i=1}^K \int_0^T \int_{\Omega \times D} \left| \sqrt{M \widehat{\psi}_\varepsilon} (\nabla_x w)_{q_i} \right|^2 dq dx dt \\ &\leq b \|\nabla_x w\|_{L^2(0,T;L^2(\Omega))}^2 < \infty. \end{aligned}$$

Thus we deduce from (6.38b) that $V_{8,2}$ converges to 0 as $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$). As both $V_{8,1}$ and $V_{8,2}$ converge to 0, the same is true of V_8 , which then implies (6.38e), thanks to the denseness of the set of divergence-free functions contained in $C^1([0, T]; \mathcal{C}_0^\infty(\Omega))$ and vanishing at $t = T$ in the function space $L^2(0, T; \mathcal{V}_\sigma)$, $\sigma \geq \frac{1}{2}d$, $\sigma > 1$. That completes Step 3.7, and the proof of (6.38e).

Step 3.8. Steps 3.1–3.7 enable us to pass to the limits $L \rightarrow \infty$, $\Delta t \rightarrow 0_+$, with $\Delta t = o(L^{-1})$ as $L \rightarrow \infty$, to deduce the existence of a pair $(u_\varepsilon, \widehat{\psi}_\varepsilon)$ satisfying (6.39), (6.40) for smooth test functions $\widehat{\varphi}$ and w , as above. The denseness of the set of divergence-free functions contained in $C^1([0, T]; \mathcal{C}_0^\infty(\Omega))$ and vanishing at $t = T$ in the set of all functions in $W^{1,1}(0, T; \mathcal{V}_\sigma)$ and vanishing at $t = T$, $\sigma \geq \frac{1}{2}d$, $\sigma > 1$, and the denseness of the set of functions contained in $C^1([0, T]; C^\infty(\overline{\Omega \times D}))$ and vanishing at $t = T$ in the set of all functions in $W^{1,1}(0, T; H^s(\Omega \times D))$ and vanishing at $t = T$, $s > 1 + \frac{1}{2}(K+1)d$, yield (6.39) and (6.40). That completes Step 3.8.

Step 3.9. Let X be a Banach space. We shall denote by $C_w([0, T]; X)$ the set of all functions $x : [0, T] \rightarrow X$ such that $t \in [0, T] \mapsto \langle x', x(t) \rangle_X \in \mathbb{R}$ is continuous on $[0, T]$ for all $x' \in X'$, the dual space of X . Whenever X has a predual, E , say, (viz.

$E' = X$), we shall denote by $C_{w*}([0, T]; X)$ the set of all functions x from $[0, T]$ into X such that $t \in [0, T] \mapsto \langle x(t), e \rangle_E \in \mathbb{R}$ is continuous on $[0, T]$ for all $e \in E$.

Lemma 6.3. *Let X and Y be Banach spaces.*

- (a) *If the space X is reflexive and is continuously embedded in the space Y , then $L^\infty(0, T; X) \cap C_w([0, T]; Y) = C_w([0, T]; X)$.*
- (b) *If X has separable predual E and Y has predual F such that F is continuously embedded in E , then $L^\infty(0, T; X) \cap C_{w*}([0, T]; Y) = C_{w*}([0, T]; X)$.*

Part (a) is due to Strauss³⁶ (cf. Lions & Magenes²⁷, Lemma 8.1, Ch. 3, Sec. 8.4); part (b) is proved analogously, *via* the sequential Banach–Alaoglu theorem. That $u_\varepsilon \in C_w([0, T]; \tilde{H})$ then follows from $u_\varepsilon \in L^\infty(0, T; \tilde{H}) \cap H^1(0, T; \tilde{V}'_\sigma)$ by Lemma 6.3(a), with $X := \tilde{H}$, $Y := \tilde{V}'_\sigma$, $\sigma \geq \frac{1}{2}d$, $\sigma > 1$. That $\hat{\psi}_\varepsilon \in C_w([0, T]; L^1_M(\Omega \times D))$ follows from $\mathcal{F}(\hat{\psi}_\varepsilon) \in L^\infty(0, T; L^1_M(\Omega \times D))$ and $\hat{\psi}_\varepsilon \in H^1(0, T; M^{-1}(H^s(\Omega \times D))')$ by Lemma 6.3(b) on taking $X := L^\Phi_M(\Omega \times D)$, the Maxwellian weighted Orlicz space with Young’s function $\Phi(r) = \mathcal{F}(1 + |r|)$ (cf. Kufner, John & Fučík²⁴, Sec. 3.18.2) whose separable predual $E := E^\Psi_M(\Omega \times D)$ has Young’s function $\Psi(r) = \exp|r| - |r| - 1$, and $Y := M^{-1}(H^s(\Omega \times D))'$ whose predual w.r.t. the duality pairing $\langle M \cdot, \cdot \rangle_{H^s(\Omega \times D)}$ is $F := H^s(\Omega \times D)$, $s > 1 + \frac{1}{2}(K + 1)d$, and noting that $C_{w*}([0, T]; L^\Phi_M(\Omega \times D)) \hookrightarrow C_w([0, T]; L^1_M(\Omega \times D))$. The last embedding and that $F \hookrightarrow E$ are proved by adapting Def. 3.6.1. and Thm. 3.2.3 in Kufner, John & Fučík²⁴ to the measure $M(q) dq d\tilde{x}$ to show that $L^\infty(\Omega \times D) \hookrightarrow L^\Xi_M(\Omega \times D)$ for any Young’s function Ξ , and then adapting Theorem 3.17.7 *ibid.* to deduce that $F \hookrightarrow L^\infty(\Omega \times D) \hookrightarrow E^\Psi_M(\Omega \times D) = E$. [The abstract framework in Temam³⁷, Ch. 3, Sec. 4 then implies that u_ε and $\hat{\psi}_\varepsilon$ satisfy $u_\varepsilon(\cdot, 0) = u_0(\cdot)$ and $\hat{\psi}_\varepsilon(\cdot, 0) = \hat{\psi}(\cdot, 0)$ in the sense of $C_w([0, T]; \tilde{H})$ and $C_w([0, T]; L^1_M(\Omega \times D))$, respectively.]

Step 3.10. The energy inequality (6.41) is a direct consequence of (6.37a–c) and (6.38a,b,d), on noting the (weak) lower-semicontinuity of the terms on the left-hand side of (4.28) and (6.46). That completes Step 3.10.

Step 3.11. It remains to prove (6.35). The bounds on the first and third term on the left-hand side of (6.34) imply that the sequences $\{\rho_{\varepsilon,L}^{\Delta t(\pm)}\}_{L>1}$ are bounded in $L^\infty(0, T; L^\infty(\Omega)) \cap L^2(0, T; H^1(\Omega))$; the bound on the fourth term in (6.34) yields that $\{\rho_{\varepsilon,L}^{\Delta t}\}_{L>1}$ is bounded in $H^1((0, T); (H^1(\Omega))')$. In fact, by noting (6.3) and (6.4), we have that $\{\rho_{\varepsilon,L}^{\Delta t(\pm)}\}_{L>1}$ is a bounded in $L^2(0, T; \mathcal{K})$. Thus, there exist subsequences of $\{\rho_{\varepsilon,L}^{\Delta t(\pm)}\}_{L>1}$ (not indicated) with $\Delta t = o(L^{-1})$ and, thanks to the uniform bound on the second term on the left-hand side of (6.34), a common limiting function $\rho_\varepsilon \in L^\infty(0, T; L^\infty(\Omega)) \cap L^2(0, T; \mathcal{K}) \cap H^1(0, T; (H^1(\Omega))')$ such that

$$\rho_{\varepsilon,L}^{\Delta t(\pm)} \rightarrow \rho_\varepsilon \quad \text{weak}^* \text{ in } L^\infty(0, T; L^\infty(\Omega)), \quad (6.52a)$$

$$\rho_{\varepsilon,L}^{\Delta t(\pm)} \rightarrow \rho_\varepsilon \quad \text{weakly in } L^2(0, T; H^1(\Omega)), \quad (6.52b)$$

$$\frac{\partial \rho_{\varepsilon,L}^{\Delta t}}{\partial t} \rightarrow \frac{\partial \rho_\varepsilon}{\partial t} \quad \text{weakly in } L^2(0, T; (H^1(\Omega))'), \quad (6.52c)$$

as $L \rightarrow \infty$ (and thereby $\Delta t \rightarrow 0_+$), It follows from (6.52b,c) and Lemma 1.2 in

Ch. 3 of Temam³⁷ that ρ_ε belongs to $C([0, T]; L^2(\Omega))$, in fact. Using (6.52a–c) and (6.37c), we can pass to the limit as $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$) in (4.5) to obtain that, for all $\varphi \in L^2(0, T; H^1(\Omega))$,

$$\int_0^T \left\langle \frac{\partial \rho_\varepsilon}{\partial t}, \varphi \right\rangle_{H^1(\Omega)} dt + \int_0^T \int_\Omega \left[\varepsilon \nabla_x \rho_\varepsilon - u_\varepsilon \rho_\varepsilon \right] \cdot \nabla_x \varphi dx dt = 0. \quad (6.53)$$

We note also that Fubini's theorem, (4.6) and (6.38d) yield that

$$\begin{aligned} \int_0^T \int_\Omega \left| \rho_{\varepsilon, L}^{\Delta t} - \int_D M \widehat{\psi}_\varepsilon dq \right| dx dt &= \int_0^T \int_\Omega \left| \int_D M (\widehat{\psi}_{\varepsilon, L}^{\Delta t} - \widehat{\psi}_\varepsilon) dq \right| dx dt \\ &\leq \int_0^T \int_{\Omega \times D} M |\widehat{\psi}_{\varepsilon, L}^{\Delta t} - \widehat{\psi}_\varepsilon| dq dx dt \rightarrow 0 \quad \text{as } L \rightarrow \infty \text{ (and } \Delta t \rightarrow 0_+). \end{aligned} \quad (6.54)$$

Thus, $\rho_{\varepsilon, L}^{\Delta t} \rightarrow \int_D M \widehat{\psi}_\varepsilon dq$ strongly in $L^1(0, T; L^1(\Omega))$ as $L \rightarrow \infty$ (and $\Delta t \rightarrow 0_+$). Comparing this with (6.52a) implies that

$$\rho_\varepsilon(x, t) = \int_D M(q) \widehat{\psi}_\varepsilon(x, q, t) dq \quad \text{for a.e. } (x, t) \in \Omega \times (0, T). \quad (6.55)$$

It follows from Step 3.9 that, for $s > 1 + \frac{1}{2}(K+1)d$, we have

$$\lim_{t \rightarrow 0_+} \int_{\Omega \times D} M(q) (\widehat{\psi}_\varepsilon(x, q, t) - \widehat{\psi}_0(x, q)) \widehat{\varphi}(x, q) dq dx = 0 \quad \forall \widehat{\varphi} \in H^s(\Omega \times D).$$

Consequently, using (6.55) and (3.5) we then deduce by selecting any $\widehat{\varphi} = \varphi \in C^\infty(\overline{\Omega}) \cong C^\infty(\overline{\Omega}) \otimes 1(D) \subset H^s(\Omega \times D)$ that

$$\begin{aligned} \lim_{t \rightarrow 0_+} \int_\Omega \rho_\varepsilon(x, t) \varphi(x) dx &= \lim_{t \rightarrow 0_+} \int_\Omega \left(\int_D M(q) \widehat{\psi}_\varepsilon(x, q, t) dq \right) \varphi(x) dx \\ &= \lim_{t \rightarrow 0_+} \int_{\Omega \times D} M(q) \widehat{\psi}_\varepsilon(x, q, t) \varphi(x) dq dx = \int_{\Omega \times D} M(q) \widehat{\psi}_0(x, q) \varphi(x) dq dx \\ &= \int_\Omega \left(\int_D M(q) \widehat{\psi}_0(x, q) dq \right) \varphi(x) dx = \int_\Omega \varphi(x) dx. \end{aligned} \quad (6.56)$$

As $\rho_\varepsilon \in C([0, T]; L^2(\Omega))$, it follows from (6.56) that $\rho_\varepsilon(x, 0) = 1$ for a.e. $x \in \Omega$.

Clearly the linear parabolic problem (6.53) with initial datum $\rho_\varepsilon(x, 0) = 1$ for a.e. $x \in \Omega$ has the unique solution $\rho_\varepsilon \equiv 1$ on $\Omega \times [0, T]$. Using this in (6.55) implies (6.35), and completes Step 3.11 and the proof. $\square$

7. Exponential decay to the equilibrium solution

We shall show that, in the absence of a body force (i.e. with $f \equiv 0$), weak solutions $(u_\varepsilon, \widehat{\psi}_\varepsilon)$ to (P_ε) , whose existence we have proved *via* our limiting procedure in the previous section, decay exponentially in time to the trivial solution of the steady counterpart of problem (P_ε) at a rate that is independent of the specific choice of the initial data for the Navier–Stokes and Fokker–Planck equations. Our result is similar to the one derived by Jourdain, Lelièvre, Le Bris & Otto²², except that the

arguments there were partially formal in the sense that the existence of a unique global-in-time solution, which was required to be regular enough, was assumed; in fact, the probability density function was supposed there to be a classical solution to the Fokker–Planck equation; $\widehat{\psi}_0$ was required to belong to $L^\infty(\Omega \times D)$ and to be strictly positive, and $\underline{u}$ was assumed to be in $L^\infty(0, \infty; W^{1,\infty}(\Omega))$ (cf. p.105, (B.128), (B.129) therein; as well as the recent paper of Arnold, Carrillo and Manzini³ for refinements and extensions). In contrast, we require no additional regularity hypotheses here.

Theorem 7.1. *Suppose the assumptions of Theorem 6.1 hold and M satisfies the Bakry–Émery condition (cf. Remark 5.1) with $\kappa > 0$; then, for all $T > 0$,*

$$\begin{aligned} & \|u_\varepsilon(T)\|^2 + \frac{k}{|\Omega|} \|\widehat{\psi}_\varepsilon(T) - 1\|_{L^1_M(\Omega \times D)}^2 \\ & \leq e^{-\gamma_0 T} \left[\|u_0\|^2 + 2k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_0) d\tilde{q} d\tilde{x} \right] + \frac{1}{\nu} \int_0^T \|f\|_{(H_0^1(\Omega))'}^2 ds, \end{aligned} \quad (7.1)$$

where $\gamma_0 := \min\left(\frac{\nu}{C_P^2}, \frac{\kappa a_0}{2\lambda}\right)$. In particular if $f \equiv 0$, the following inequality holds:

$$\|u_\varepsilon(T)\|^2 + \frac{k}{|\Omega|} \|\widehat{\psi}_\varepsilon(T) - 1\|_{L^1_M(\Omega \times D)}^2 \leq e^{-\gamma_0 T} \left[\|u_0\|^2 + 2k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_0) d\tilde{q} d\tilde{x} \right]. \quad (7.2)$$

Proof. We take $t = t_1 = \Delta t$ and write $0 = t_0$ in (4.23), and we replace the function $\mathcal{F}$ on the left-hand side of (4.23) by $\mathcal{F}^L$, noting that, prior to (4.23), in (4.19) we in fact had $\mathcal{F}^L$ on the left-hand side of the inequality, and $\mathcal{F}^L$ was subsequently bounded below by $\mathcal{F}$; thus we reinstate the $\mathcal{F}^L$ we previously had. We recall that $\underline{u}^0 = u_{\varepsilon,L}^{\Delta t,-}(t_1)$ and $\beta^L(\widehat{\psi}^0) = \widehat{\psi}_{\varepsilon,L}^{\Delta t,-}(t_1)$ and adopt the notational convention $t_{-1} := -\infty$ (say), which allows us to write $\underline{u}_{\varepsilon,L}^{\Delta t,+}(t_0)$ instead of $\underline{u}_{\varepsilon,L}^{\Delta t,-}(t_1)$ and $\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(t_0)$ instead of $\widehat{\psi}_{\varepsilon,L}^{\Delta t,-}(t_1)$. Hence we have that

$$\begin{aligned} & \|\underline{u}_{\varepsilon,L}^{\Delta t,+}(t_1)\|^2 + \frac{1}{\Delta t} \int_{t_0}^{t_1} \|\underline{u}_{\varepsilon,L}^{\Delta t,+} - \underline{u}_{\varepsilon,L}^{\Delta t,-}\|^2 ds + \left(\nu - \alpha \frac{2\lambda b k}{a_0} \right) \int_{t_0}^{t_1} \|\nabla_x \underline{u}_{\varepsilon,L}^{\Delta t,+}(s)\|^2 ds \\ & + 2k \int_{\Omega \times D} M \mathcal{F}^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(t_1) + \alpha) d\tilde{q} d\tilde{x} + \frac{k}{\Delta t L} \int_{t_0}^{t_1} \int_{\Omega \times D} M (\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} - \widehat{\psi}_{\varepsilon,L}^{\Delta t,-})^2 d\tilde{q} d\tilde{x} ds \\ & + 2k \varepsilon \int_{t_0}^{t_1} \int_{\Omega \times D} M \frac{|\nabla_x \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}|^2}{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha} d\tilde{q} d\tilde{x} ds + \frac{a_0 k}{2\lambda} \int_{t_0}^{t_1} \int_{\Omega \times D} M \frac{|\nabla_q \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}|^2}{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha} d\tilde{q} d\tilde{x} ds \\ & \leq \|\underline{u}_{\varepsilon,L}^{\Delta t,+}(t_0)\|^2 + \frac{1}{\nu} \int_{t_0}^{t_1} \|f^{\Delta t,+}(s)\|_{(H_0^1(\Omega))'}^2 ds \\ & + 2k \int_{\Omega \times D} M \mathcal{F}^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(t_0) + \alpha) d\tilde{q} d\tilde{x}. \end{aligned} \quad (7.3)$$

Closer inspection of the procedure that resulted in inequality (4.23) reveals that (4.23) could have been equivalently arrived at by repeating the argument that gave

us (7.3) on each time interval $[t_{n-1}, t_n]$, $n = 1, \dots, N$; viz.,

$$\begin{aligned}
 & \|u_{\varepsilon,L}^{\Delta t,+}(t_n)\|^2 + \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \|u_{\varepsilon,L}^{\Delta t,+} - u_{\varepsilon,L}^{\Delta t,-}\|^2 ds \\
 & + \left(\nu - \alpha \frac{2\lambda b k}{a_0} \right) \int_{t_{n-1}}^{t_n} \|\nabla_x u_{\varepsilon,L}^{\Delta t,+}(s)\|^2 ds + 2k \int_{\Omega \times D} M \mathcal{F}^L(\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(t_n) + \alpha) dq d\tilde{x} \\
 & + \frac{k}{\Delta t L} \int_{t_{n-1}}^{t_n} \int_{\Omega \times D} M (\hat{\psi}_{\varepsilon,L}^{\Delta t,+} - \hat{\psi}_{\varepsilon,L}^{\Delta t,-})^2 dq d\tilde{x} ds \\
 & + 2k \varepsilon \int_{t_{n-1}}^{t_n} \int_{\Omega \times D} M \frac{|\nabla_x \hat{\psi}_{\varepsilon,L}^{\Delta t,+}|^2}{\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha} dq d\tilde{x} ds + \frac{a_0 k}{2\lambda} \int_{t_{n-1}}^{t_n} \int_{\Omega \times D} M \frac{|\nabla_q \hat{\psi}_{\varepsilon,L}^{\Delta t,+}|^2}{\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha} dq d\tilde{x} ds \\
 & \leq \|u_{\varepsilon,L}^{\Delta t,+}(t_{n-1})\|^2 + \frac{1}{\nu} \int_{t_{n-1}}^{t_n} \|f^{\Delta t,+}(s)\|_{(H_0^1(\Omega))'}^2 ds \\
 & + 2k \int_{\Omega \times D} M \mathcal{F}^L(\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(t_{n-1}) + \alpha) dq d\tilde{x}, \quad n = 1, \dots, N, \quad (7.4)
 \end{aligned}$$

summing these through n and then bounding $\mathcal{F}^L$ on the left-hand side below by $\mathcal{F}$.

Here we proceed differently: we shall retain $\mathcal{F}^L$ on both sides of (7.4), and omit the second, fifth and sixth term from the left-hand side of (7.4). Thus we have that

$$\begin{aligned}
 & \|u_{\varepsilon,L}^{\Delta t,+}(t_n)\|^2 + \left(\nu - \alpha \frac{2\lambda b k}{a_0} \right) \int_{t_{n-1}}^{t_n} \|\nabla_x u_{\varepsilon,L}^{\Delta t,+}(s)\|^2 ds \\
 & + 2k \int_{\Omega \times D} M \mathcal{F}^L(\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(t_n) + \alpha) dq d\tilde{x} \\
 & + \frac{2a_0 k}{\lambda} \int_{t_{n-1}}^{t_n} \int_{\Omega \times D} M |\nabla_q \sqrt{\hat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha}|^2 dq d\tilde{x} ds \\
 & \leq \|u_{\varepsilon,L}^{\Delta t,+}(t_{n-1})\|^2 + \frac{1}{\nu} \int_{t_{n-1}}^{t_n} \|f^{\Delta t,+}(s)\|_{(H_0^1(\Omega))'}^2 ds \\
 & + 2k \int_{\Omega \times D} M \mathcal{F}^L(\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(t_{n-1}) + \alpha) dq d\tilde{x}, \quad n = 1, \dots, N. \quad (7.5)
 \end{aligned}$$

Thanks to Poincaré's inequality, recall (4.34), there exists a positive constant $C_P = C_P(\Omega)$, such that

$$\|u_{\varepsilon,L}^{\Delta t,+}(\cdot, s)\| \leq C_P(\Omega) \|\nabla_x u_{\varepsilon,L}^{\Delta t,+}(\cdot, s)\| \quad (7.6)$$

for $s \in (t_{n-1}, t_n]$; $n = 1, \dots, N$. Also, by the logarithmic Sobolev inequality (5.4), we have for a.e. $\tilde{x} \in \Omega$ that

$$\begin{aligned}
 & \int_D M(q) [\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(x, q, s) + \alpha] \log \frac{\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(x, q, s) + \alpha}{\int_D M(q) [\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(x, q, s) + \alpha] dq} dq \\
 & \leq \frac{2}{\kappa} \int_D M(q) \left| \nabla_q \sqrt{\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(x, q, s) + \alpha} \right|^2 dq,
 \end{aligned}$$

for $s \in (t_{n-1}, t_n]$; $n = 1, \dots, N$. Hence, for a.e. $\underline{x} \in \Omega$,

$$\begin{aligned} & \int_D M(\underline{q}) [\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(\underline{x}, \underline{q}, s) + \alpha] \log[\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(\underline{x}, \underline{q}, s) + \alpha] d\underline{q} \\ & \leq \frac{2}{\kappa} \int_D M(\underline{q}) \left| \nabla_{\underline{q}} \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(\underline{x}, \underline{q}, s) + \alpha} \right|^2 d\underline{q} \\ & + \left(\int_D M(\underline{q}) [\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(\underline{x}, \underline{q}, s) + \alpha] d\underline{q} \right) \log \left(\int_D M(\underline{q}) [\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(\underline{x}, \underline{q}, s) + \alpha] d\underline{q} \right), \quad (7.7) \end{aligned}$$

for $s \in (t_{n-1}, t_n]$, $n = 1, \dots, N$. Note that, thanks to (4.31b) and the monotonicity of the mapping $s \in \mathbb{R}_{>0} \mapsto \log s \in \mathbb{R}$, the second factor in the second term on the right-hand side of (7.7) is $\leq \log(1+\alpha)$. Since $\alpha \in (0, 1)$, we have $\log(1+\alpha) > 0$; also, the first factor in the second term on the right-hand side of (7.7) is positive thanks to (4.31a) and by (4.31b) it is bounded above by $(1+\alpha)$. Hence the second term on the right-hand side of (7.7) is bounded above by the product $(1+\alpha) \log(1+\alpha)$. We integrate the resulting inequality over Ω to deduce that

$$\begin{aligned} & \int_{\Omega \times D} M(\underline{q}) [\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(\underline{x}, \underline{q}, s) + \alpha] \log[\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(\underline{x}, \underline{q}, s) + \alpha] d\underline{q} d\underline{x} \\ & \leq \frac{2}{\kappa} \int_{\Omega \times D} M(\underline{q}) \left| \nabla_{\underline{q}} \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(\underline{x}, \underline{q}, s) + \alpha} \right|^2 d\underline{q} d\underline{x} + |\Omega| (1+\alpha) \log(1+\alpha), \end{aligned}$$

for $s \in (t_{n-1}, t_n]$, $n = 1, \dots, N$. Equivalently, on noting that $s \log s = \mathcal{F}(s) - (1-s)$, we can rewrite the last inequality in the following form:

$$\begin{aligned} & \int_{\Omega \times D} M(\underline{q}) \mathcal{F}(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(\underline{x}, \underline{q}, s) + \alpha) d\underline{q} d\underline{x} \\ & \leq \frac{2}{\kappa} \int_{\Omega \times D} M(\underline{q}) \left| \nabla_{\underline{q}} \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(\underline{x}, \underline{q}, s) + \alpha} \right|^2 d\underline{q} d\underline{x} \\ & + \int_{\Omega \times D} M(\underline{q}) (1 - \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(\underline{x}, \underline{q}, s) - \alpha) d\underline{q} d\underline{x} + |\Omega| (1+\alpha) \log(1+\alpha), \quad (7.8) \end{aligned}$$

for $s \in (t_{n-1}, t_n]$, $n = 1, \dots, N$. This then in turn implies, thanks to the fact that $\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(\underline{x}, \underline{q}, \cdot)$ is constant on the interval $(t_{n-1}, t_n]$ for all $(\underline{x}, \underline{q}) \in \Omega \times D$, that

$$\begin{aligned} & \frac{\kappa a_0 k}{\lambda} \Delta t \int_{\Omega \times D} M(\underline{q}) \mathcal{F}(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(t_n) + \alpha) d\underline{q} d\underline{x} \\ & \leq \frac{2a_0 k}{\lambda} \int_{t_{n-1}}^{t_n} \int_{\Omega \times D} M(\underline{q}) \left| \nabla_{\underline{q}} \sqrt{\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} + \alpha} \right|^2 d\underline{q} d\underline{x} ds \\ & + \frac{\kappa a_0 k}{\lambda} \left[\int_{t_{n-1}}^{t_n} \int_{\Omega \times D} M(1 - \widehat{\psi}_{\varepsilon,L}^{\Delta t,+} - \alpha) d\underline{q} d\underline{x} ds + \Delta t |\Omega| (1+\alpha) \log(1+\alpha) \right], \end{aligned}$$

for $n = 1, \dots, N$. Using this and (7.6) in (7.5) then yields

$$\begin{aligned}
 & \left(1 + \frac{\Delta t}{C_p^2} \left(\nu - \alpha \frac{2\lambda b k}{a_0}\right)\right) \|u_{\varepsilon,L}^{\Delta t,+}(t_n)\|^2 \\
 & + \left(1 + \frac{\kappa a_0}{2\lambda} \Delta t\right) 2k \int_{\Omega \times D} M \mathcal{F}(\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(t_n) + \alpha) \, dq \, d\tilde{x} \\
 & + 2k \int_{\Omega \times D} M [\mathcal{F}^L(\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(t_n) + \alpha) - \mathcal{F}(\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(t_n) + \alpha)] \, dq \, d\tilde{x} \\
 & \leq \|u_{\varepsilon,L}^{\Delta t,+}(t_{n-1})\|^2 + 2k \int_{\Omega \times D} M \mathcal{F}(\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(t_{n-1}) + \alpha) \, dq \, d\tilde{x} \\
 & + 2k \int_{\Omega \times D} M [\mathcal{F}^L(\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(t_{n-1}) + \alpha) - \mathcal{F}(\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(t_{n-1}) + \alpha)] \, dq \, d\tilde{x} \\
 & + \frac{\kappa a_0 k}{\lambda} \int_{t_{n-1}}^{t_n} \int_{\Omega \times D} M (1 - \hat{\psi}_{\varepsilon,L}^{\Delta t,+} - \alpha) \, dq \, d\tilde{x} \, dt \\
 & + \frac{\kappa a_0 k}{\lambda} \Delta t |\Omega| (1 + \alpha) \log(1 + \alpha) + \frac{1}{\nu} \int_{t_{n-1}}^{t_n} \|f_{\sim}^{\Delta t,+}\|_{(H_0^1(\Omega))'}^2 \, ds, \quad (7.9)
 \end{aligned}$$

for $n = 1, \dots, N$. We now introduce, for $n = 1, \dots, N$, the following notation:

$$\begin{aligned}
 \gamma(\alpha) &:= \min \left(\frac{1}{C_p^2} \left(\nu - \alpha \frac{2\lambda b k}{a_0} \right), \frac{\kappa a_0}{2\lambda} \right), \\
 A_n(\alpha) &:= \|u_{\varepsilon,L}^{\Delta t,+}(t_n)\|^2 + 2k \int_{\Omega \times D} M \mathcal{F}(\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(t_n) + \alpha) \, dq \, d\tilde{x}, \\
 B_n(\alpha) &:= 2k \int_{\Omega \times D} M [\mathcal{F}^L(\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(t_n) + \alpha) - \mathcal{F}(\hat{\psi}_{\varepsilon,L}^{\Delta t,+}(t_n) + \alpha)] \, dq \, d\tilde{x}, \\
 C_n(\alpha) &:= \frac{\kappa a_0 k}{\lambda} \int_{t_{n-1}}^{t_n} \int_{\Omega \times D} M (1 - \hat{\psi}_{\varepsilon,L}^{\Delta t,+}(x, q, s)) \, dq \, d\tilde{x} \, ds \\
 & + \frac{\kappa a_0 k}{\lambda} \Delta t |\Omega| [(1 + \alpha) \log(1 + \alpha) - \alpha] + \frac{1}{\nu} \int_{t_{n-1}}^{t_n} \|f_{\sim}^{\Delta t,+}\|_{(H_0^1(\Omega))'}^2 \, ds.
 \end{aligned}$$

We shall assume henceforth that α is sufficiently small in the sense that (4.22) holds. For all such α , $\gamma(\alpha) > 0$; further, trivially, $A_n(\alpha)$ is nonnegative; by (4.12), we have that $B_n(\alpha)$ is nonnegative, and by (6.4) and since $\mathcal{F}(1 + \alpha) \geq 0$ for all $\alpha \geq 0$, $C_n(\alpha)$ is also nonnegative. In terms of this notation (7.9) can be rewritten as follows:

$$(1 + \gamma(\alpha) \Delta t) A_n(\alpha) + B_n(\alpha) \leq A_{n-1}(\alpha) + B_{n-1}(\alpha) + C_n(\alpha), \quad n = 1, \dots, N.$$

Hence a straightforward argument based on induction⁹ implies that:

$$A_n(\alpha) + B_n(\alpha) \leq (1 + \gamma(\alpha) \Delta t)^{-n} A_0(\alpha) + B_0(\alpha) + \sum_{j=1}^n C_j(\alpha), \quad n = 1, \dots, N.$$

In particular, with $n = N$, by omitting the nonnegative term $B_N(\alpha)$ from the

left-hand side of the resulting inequality, and recalling that $T = t_N = N\Delta t$, we get

$$\begin{aligned}
& \|u_{\varepsilon,L}^{\Delta t,+}(T)\|^2 + 2k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(T) + \alpha) \, dq \, dx \\
& \leq \left(1 + \frac{\gamma(\alpha)T}{N}\right)^{-N} \left[\|u_{\varepsilon,L}^{\Delta t,+}(0)\|^2 + 2k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(0) + \alpha) \, dq \, dx \right] \\
& \quad + 2k \int_{\Omega \times D} M \left[\mathcal{F}^L(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(0) + \alpha) - \mathcal{F}(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(0) + \alpha) \right] \, dq \, dx \\
& \quad + \frac{\kappa a_0 k}{\lambda} \int_0^T \int_{\Omega \times D} M (1 - \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(x, q, s)) \, dq \, dx \, ds \\
& \quad + \frac{\kappa a_0 k}{\lambda} T |\Omega| \mathcal{F}(1 + \alpha) + \frac{1}{\nu} \int_0^T \|f^{\Delta t,+}\|_{(H_0^1(\Omega))'}^2 \, ds. \tag{7.10}
\end{aligned}$$

Using that $u_{\varepsilon,L}^{\Delta t,+}(0) = u^0$ and $\widehat{\psi}_{\varepsilon,L}^{\Delta t,+} = \beta^L(\widehat{\psi}^0)$, we then obtain from (7.10) that

$$\begin{aligned}
& \|u_{\varepsilon,L}^{\Delta t,+}(T)\|^2 + 2k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(T) + \alpha) \, dq \, dx \\
& \leq \left(1 + \frac{\gamma(\alpha)T}{N}\right)^{-N} \left[\|u^0\|^2 + 2k \int_{\Omega \times D} M \mathcal{F}(\beta^L(\widehat{\psi}^0) + \alpha) \, dq \, dx \right] \\
& \quad + 2k \int_{\Omega \times D} M \left[\mathcal{F}^L(\beta^L(\widehat{\psi}^0) + \alpha) - \mathcal{F}(\beta^L(\widehat{\psi}^0) + \alpha) \right] \, dq \, dx \\
& \quad + \frac{\kappa a_0 k}{\lambda} \int_0^T \int_{\Omega \times D} M (1 - \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(x, q, s)) \, dq \, dx \, ds \\
& \quad + \frac{\kappa a_0 k}{\lambda} T |\Omega| \mathcal{F}(1 + \alpha) + \frac{1}{\nu} \int_0^T \|f^{\Delta t,+}\|_{(H_0^1(\Omega))'}^2 \, ds. \tag{7.11}
\end{aligned}$$

Applying (4.12) and (4.25) in the second factor in the first term on the right-hand side of (7.11) and using (4.24) in the square brackets in the second term on the right-hand side, we have that

$$\begin{aligned}
& \|u_{\varepsilon,L}^{\Delta t,+}(T)\|^2 + 2k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(T) + \alpha) \, dq \, dx \\
& \leq \left(1 + \frac{\gamma(\alpha)T}{N}\right)^{-N} \left[\|u^0\|^2 + 3\alpha k |\Omega| + 2k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}^0 + \alpha) \, dq \, dx \right] \\
& \quad + 3\alpha k |\Omega| + \frac{\kappa a_0 k}{\lambda} \int_0^T \int_{\Omega \times D} M (1 - \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(x, q, s)) \, dq \, dx \, ds \\
& \quad + \frac{\kappa a_0 k}{\lambda} T |\Omega| \mathcal{F}(1 + \alpha) + \frac{1}{\nu} \int_0^T \|f^{\Delta t,+}\|_{(H_0^1(\Omega))'}^2 \, ds. \tag{7.12}
\end{aligned}$$

We now pass to the limit $\alpha \rightarrow 0_+$, with L and Δt fixed, in much the same way as in Section 4.1. Noting that $\lim_{\alpha \rightarrow 0_+} \gamma(\alpha) = \gamma_0$, we thus obtain from (7.12), (3.18)

and (3.20), that

$$\begin{aligned} & \|u_{\varepsilon,L}^{\Delta t,+}(T)\|^2 + 2k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(T)) \, d\tilde{q} \, d\tilde{x} \\ & \leq \left(1 + \frac{\gamma_0 T}{N}\right)^{-N} \left[\|u_0\|^2 + 2k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_0) \, d\tilde{q} \, d\tilde{x} \right] \\ & \quad + \frac{\kappa a_0 k}{\lambda} \int_0^T \int_{\Omega \times D} M (1 - \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}) \, d\tilde{q} \, d\tilde{x} \, ds + \frac{1}{\nu} \int_0^T \|f^{\Delta t,+}\|_{(H_0^1(\Omega))'}^2 \, ds. \end{aligned} \quad (7.13)$$

In order to pass to the limits $L \rightarrow \infty$ and $\Delta t \rightarrow 0_+$ (with $\Delta t = o(L^{-1})$) in the first two terms on the left-hand side of (7.13) we require additional considerations.

Noting (6.37c) for the sequence $\{u_{\varepsilon,L}^{\Delta t}\}_{L>1}$ of Theorem 6.1, and passing to a subsequence (not indicated), as $L \rightarrow \infty$ and $\Delta t \rightarrow 0_+$ (with $\Delta t = o(L^{-1})$) we have that $\|u_{\varepsilon,L}^{\Delta t}(t) - u_\varepsilon(t)\|$ converges to 0 for a.e. $t \in (0, T)$; let t_* be one such t in $(0, T)$. It then follows from (6.37d) that, for any $v \in V_\sigma \subset \underline{H}$,

$$\begin{aligned} |\langle u_\varepsilon(T) - u_{\varepsilon,L}^{\Delta t}(T), v \rangle_{V_\sigma}| & \leq \left| \int_{t_*}^T \left\langle \frac{\partial(u_\varepsilon - u_{\varepsilon,L}^{\Delta t})}{\partial t}(t), v \right\rangle_{V_\sigma} \, dt \right| + \|u_{\varepsilon,L}^{\Delta t}(t_*) - u_\varepsilon(t_*)\| \|v\| \\ & \rightarrow 0 \text{ as } L \rightarrow \infty \text{ and } \Delta t \rightarrow 0_+ \text{ with } \Delta t = o(L^{-1}). \end{aligned} \quad (7.14)$$

Since $u_\varepsilon : [0, T] \rightarrow \underline{H}$ is weakly continuous, we have that $u_\varepsilon(T) \in \underline{H}$. It follows from the bound on the first term in (6.33), as $t \in [0, T] \mapsto \|u_{\varepsilon,L}^{\Delta t}(t)\| \in \mathbb{R}_{\geq 0}$ is a *continuous* (piecewise linear) function, that, for any $v_0 \in \underline{H}$ and any $v \in V_\sigma$,

$$|(u_\varepsilon(T) - u_{\varepsilon,L}^{\Delta t}(T), v_0)| \leq |\langle u_\varepsilon(T) - u_{\varepsilon,L}^{\Delta t}(T), v \rangle_{V_\sigma}| + [\|u_\varepsilon(T)\| + C_*^{\frac{1}{2}}] \|v_0 - v\|.$$

Recalling (7.14), it follows from the last inequality that

$$\limsup_{L \rightarrow \infty} |(u_\varepsilon(T) - u_{\varepsilon,L}^{\Delta t}(T), v_0)| \leq C \|v_0 - v\| \quad \forall v_0 \in \underline{H}, \quad \forall v \in V_\sigma.$$

As V_σ is dense in $\underline{H}$, we thus deduce that $\{u_{\varepsilon,L}^{\Delta t}(T)\}_{L>1}$ converges to $u_\varepsilon(T)$ weakly in $\underline{H}$ as $L \rightarrow \infty$ and $\Delta t \rightarrow 0_+$, with $\Delta t = o(L^{-1})$. Hence, by the weak lower-semicontinuity of the $L^2(\Omega)$ norm and noting that $u_{\varepsilon,L}^{\Delta t}(T) = u_{\varepsilon,L}^{\Delta t,+}(T)$, we have

$$\|u_\varepsilon(T)\| \leq \liminf_{L \rightarrow \infty} \|u_{\varepsilon,L}^{\Delta t,+}(T)\|. \quad (7.15)$$

Analogously to (7.14), noting (6.38d) for the sequence $\{\widehat{\psi}_{\varepsilon,L}^{\Delta t}\}_{L>1}$ of Theorem 6.1, we have (on passing to a subsequence, not indicated,) that $\widehat{\psi}_{\varepsilon,L}^{\Delta t}(T)$ converges weakly to $\widehat{\psi}_\varepsilon(T)$ in $M^{-1}(H^s(\Omega \times D))'$ as $L \rightarrow \infty$ and $\Delta t \rightarrow 0_+$, with $\Delta t = o(L^{-1})$. Thanks to Theorem 6.1, $\widehat{\psi}_\varepsilon : [0, T] \rightarrow L_M^1(\Omega \times D)$ is weakly continuous; hence we have that $\widehat{\psi}_\varepsilon(T) \in L_M^1(\Omega \times D)$. Similarly to the argument in the proof of 4 of Lemma 6.2, it follows from the bound on the fourth term in (6.33), on noting that $\mathcal{F}(r)/r \rightarrow \infty$ as $r \rightarrow \infty$, together with the de la Vallée-Poussin and Dunford–Pettis theorems, that, upon subtraction of a further subsequence (not indicated), $\widehat{\psi}_{\varepsilon,L}^{\Delta t}(T)$ converges weakly in $L_M^1(\Omega \times D)$ to some limit A , as $L \rightarrow \infty$ and $\Delta t \rightarrow 0_+$, with $\Delta t = o(L^{-1})$. The fact that $A = \widehat{\psi}_\varepsilon(T)$ follows from the weak convergence of $\widehat{\psi}_{\varepsilon,L}^{\Delta t}(T)$ to $\widehat{\psi}_\varepsilon(T)$.

in $M^{-1}(H^s(\Omega \times D))'$ ($\leftrightarrow L_M^1(\Omega \times D)$). Finally, since $r \in [0, \infty) \mapsto \mathcal{F}(r) \in \mathbb{R}_{\geq 0}$ is continuous and convex, on applying Tonelli's weak lower semicontinuity theorem in $L_M^1(\Omega \times D)$ (cf. Theorem 3.20 in Dacorogna¹³),

$$\int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_\varepsilon(T)) \, dq \, dx \leq \liminf_{L \rightarrow \infty} \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(T)) \, dq \, dx, \quad (7.16)$$

where we have noted that $\widehat{\psi}_{\varepsilon,L}^{\Delta t}(T) = \widehat{\psi}_{\varepsilon,L}^{\Delta t,+}(T)$.

We are now ready to pass to the limit in (7.13). Using (7.15) and (7.16), (6.38d), (6.35) and (3.23), and letting $L \rightarrow \infty$ (whereby $\Delta t \rightarrow 0_+$ according to $\Delta t = o(L^{-1})$ and therefore $N = T/\Delta t \rightarrow \infty$), we deduce from (7.13) that

$$\begin{aligned} \|u_\varepsilon(T)\|_{\sim}^2 + 2k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_\varepsilon(T)) \, dq \, dx \\ \leq e^{-\gamma_0 T} \left[\|u_0\|_{\sim}^2 + 2k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_0) \, dq \, dx \right] + \frac{1}{\nu} \int_0^T \|f(s)\|_{(H_0^1(\Omega))'}^2 \, ds. \end{aligned} \quad (7.17)$$

The Csiszár–Kullback inequality (cf., for example, (1.1) and (1.2) in the work of Unterreiter et al.³⁸) with respect to the Gibbs measure μ defined by $d\mu = M(q) \, dq$ yields, on noting (6.35), for a.e. $x \in \Omega$, that

$$\|\widehat{\psi}_\varepsilon(x, \cdot, T) - 1\|_{L_M^1(D)} \leq \left[2 \int_D M \mathcal{F}(\widehat{\psi}_\varepsilon(x, q, T)) \, dq \right]^{\frac{1}{2}},$$

which, after integration over Ω implies, by the Cauchy–Schwarz inequality, that

$$\|\widehat{\psi}_\varepsilon(T) - 1\|_{L_M^1(\Omega \times D)}^2 \leq 2|\Omega| \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_\varepsilon(T)) \, dq \, dx.$$

Combining this with (7.17) yields (7.1). Taking $f \equiv 0$, the stated exponential decay in time of $(u_\varepsilon, \widehat{\psi}_\varepsilon)$ to $(0, 1)$ in the $L^2(\Omega) \times L_M^1(\Omega \times D)$ norm follows from (7.1). $\square$

Remark 7.1. By introducing the “free energy” as the sum of the kinetic energy and the relative entropy:

$$\mathfrak{E}(t) := \frac{1}{2} \|u_\varepsilon(t)\|^2 + k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_\varepsilon(t)) \, dq \, dx,$$

we deduce from (7.17) that, for any $T > 0$,

$$\mathfrak{E}(T) \leq e^{-\gamma_0 T} \mathfrak{E}(0) + \frac{1}{2\nu} \int_0^T \|f(s)\|_{(H_0^1(\Omega))'}^2 \, ds.$$

Thus in particular when $f = 0$, the free energy decays to 0 as a function of time from any initial datum (u_0, ψ_0) with initial velocity $u_0 \in \underline{H}$ and initial probability density function ψ_0 that has finite relative entropy with respect to the log-concave Maxwellian M .

It is interesting to note the dependence of $\gamma_0 = \min \left(\frac{\nu}{C_P^2}, \frac{\kappa a_0}{2\lambda} \right)$, the rate at which the fluid relaxes to equilibrium, on the dimensionless viscosity coefficient ν of the solvent, the minimum eigenvalue a_0 of the Rouse matrix A , the geometry of the flow domain encoded in the Poincaré constant $C_P(\Omega)$, the Weissenberg number λ , and

the Bakry–Émery constant κ for the Maxwellian M of the model. We also observe that the right-hand side of the energy inequality (6.41) and γ_0 are independent of the centre-of-mass diffusion coefficient ε appearing in the equation (1.9). $\diamond$

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