

Nikolskii inequality and functional classes on compact Lie groups

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ABSTRACT. In this note we study Besov, Triebel–Lizorkin, Wiener, and Beurling function spaces on compact Lie groups. A major role in the analysis is played by the Nikolskii inequality.

1. Introduction. The classical Nikolskii inequality for trigonometric polynomials T_L of degree up to L can be written as ([**Nik51**]):

$$\|T_L\|_{L^q(\mathbb{T})} \leq 2L^{1/p-1/q} \|T_L\|_{L^p(\mathbb{T})},$$

where $1 \leq p < q \leq \infty$. The Nikolskii inequality plays an important role in the analysis of different function spaces (for example, see [**Tri83**]) and in the approximation theory (for example, see [**DT05**]).

In the Euclidean case, for functions $f \in L^p(\mathbb{R}^n)$ such that $\text{supp}(\hat{f})$ is compact (cf. [**NW78**]) we have

$$\|f\|_{L^q(\mathbb{R}^n)} \leq \left(C(p) \mu(\text{conv}[\text{supp}(\hat{f})]) \right)^{1/p-1/q} \|f\|_{L^p(\mathbb{R}^n)}, \quad (1)$$

where $1 \leq p \leq q \leq \infty$, $\mu(E)$ is the Lebesgue measure of E , and $\text{conv}[E]$ is the convex hull of E . Inequalities of the form (1) are often called Plancherel–Polya–Nikolskii inequalities.

Recently, in [**Pes09**] Pesenson obtained the Bernstein–Nikolskii inequality on symmetric spaces of noncompact type, and in [**Pes08**] on compact homogeneous spaces.

Let G be a compact Lie group of dimension $\dim G$ and let $\hat{G}$ be its unitary dual. If we fix bases in representation spaces we can work with matrix representations $\xi : G \rightarrow \mathbb{C}^{d_\xi \times d_\xi}$ of dimensions d_ξ . By the Peter–Weyl theorem the system $\{\sqrt{d_\xi} \xi_{ij} : [\xi] \in \hat{G}, 1 \leq i, j \leq d_\xi\}$ is an orthonormal basis in $L^2(G)$ with respect to the normalized Haar measure on G . All the integrals below and the spaces $L^p(G)$ will be always considered with respect to this normalized bi-invariant Haar measure on G .

For $f \in C^\infty(G)$ we define its Fourier coefficient at $\xi \in [\xi] \in \hat{G}$ by

$$\hat{f}(\xi) = \int_G f(x) \xi(x)^* dx.$$

Thus, we have $\hat{f}(\xi) \in \mathbb{C}^{d_\xi \times d_\xi}$. The Fourier series of a function f takes the form

$$f(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}(\hat{f}(\xi) \xi(x)). \quad (2)$$

E. Nursultanov was supported by grants MOH PK 4080 GF-4, 3311 GF-4. M. Ruzhansky was supported by EPSRC Leadership Fellowship, EPSRC EP/K039407/1 and Leverhulme Grant RPG-2014-02. S. Tikhonov was supported by grants 2014-SGR-289 from AGAUR (Generalitat de Catalunya), MTM 2014-59174-P, and RFFI-13-01-00043.

For $[\xi] \in \widehat{G}$ by $\langle \xi \rangle$ we denote the eigenvalue of the operator $(I - \mathcal{L}_G)^{1/2}$ corresponding to the representation class $[\xi] \in \widehat{G}$, where $\mathcal{L}_G$ is the Laplacian on G , see, for example [Ste70, Chapter 1.7].

In [RT10] the following Lebesgue spaces $\ell^p(\widehat{G})$ on $\widehat{G}$ were defined as follows: using the Fourier coefficients of f , we set

$$\|\widehat{f}\|_{\ell^p(\widehat{G})} = \left(\sum_{[\xi] \in \widehat{G}} d_\xi^{p(\frac{2}{p}-\frac{1}{2})} \|\widehat{f}(\xi)\|_{\text{HS}}^p \right)^{1/p}, \quad 1 \leq p < \infty, \quad (3)$$

and

$$\|\widehat{f}\|_{\ell^\infty(\widehat{G})} = \sup_{[\xi] \in \widehat{G}} d_\xi^{-\frac{1}{2}} \|\widehat{f}(\xi)\|_{\text{HS}}, \quad (4)$$

where $\|\widehat{f}(\xi)\|_{\text{HS}} = \text{Tr}(\widehat{f}(\xi)\widehat{f}(\xi)^*)^{1/2}$. For these spaces the following Hausdorff–Young inequalities are valid:

$$\|\widehat{f}\|_{\ell^{p'}(\widehat{G})} \leq \|f\|_{L^p(G)}, \quad \|f\|_{L^{p'}(G)} \leq \|\widehat{f}\|_{\ell^p(\widehat{G})}, \quad 1 \leq p \leq 2, \quad \frac{1}{p'} + \frac{1}{p} = 1.$$

Let $N(L)$ be the Weyl eigenvalue counting function for the elliptic pseudo-differential operator $(1 - \mathcal{L}_G)^{1/2}$, denoting the number of its eigenvalues $\leq L$ counted with multiplicities. Then

$$N(L) = \sum_{\substack{\langle \xi \rangle \leq L \\ [\xi] \in \widehat{G}}} d_\xi^2.$$

For sufficiently large L the Weyl asymptotic formula says that

$$N(L) \sim C_0 L^n, \quad C_0 = (2\pi)^{-n} \int_{\sigma_1(x, \omega) < 1} dx d\omega,$$

where $n = \dim G$, and the integral is taken with respect to the canonical measure on the cotangent bundle $T^*(G)$ induced by the canonical symplectic form. Here σ_1 is the principal symbol of the operator $(1 - \mathcal{L}_G)^{1/2}$, see e.g. [Shu01].

Full proofs of our results below will appear in [NRT15].

2. Nikolskii inequality. Let T be a trigonometric polynomial on a compact Lie group G , i. e. a function with only finitely many non-zero Fourier coefficients. Let D be the Dirichlet kernel, i.e. the function $D \in C^\infty(G)$ such that

$$\widehat{D}(\xi) := I_{d_\xi} \quad \text{for } \langle \xi \rangle \leq L,$$

and zero otherwise. Here $I_{d_\xi} \in \mathbb{C}^{d_\xi \times d_\xi}$ denote the identity matrix.

THEOREM 1. *Let $0 < p < q \leq \infty$. For $0 < p \leq 2$ set $\rho := 1$, and for $2 < p < \infty$ let ρ be the smallest integer $\geq p/2$. Then*

$$\|T\|_{L^q(G)} \leq \left(\sum_{\widehat{T}^\rho(\xi) \neq 0} d_\xi^2 \right)^{\frac{1}{p} - \frac{1}{q}} \|T\|_{L^p(G)}.$$

Moreover, this inequality is sharp for $p = 2$ and $q = \infty$, and the equality is attained at $T = D$.

We note that for the classical trigonometric polynomials of several variables the Nikolskii inequality is well known ([Nik51]).

REMARK 2. Note that if $T = T_L$, i.e. if $\widehat{T}(\xi) = 0$ for $\langle \xi \rangle > L$, then $\sum_{\widehat{T}(\xi) \neq 0} d_\xi^2 \leq N(L)$ and, therefore,

$$\|T_L\|_{L^q(G)} \leq N(\rho L)^{\frac{1}{p} - \frac{1}{q}} \|T_L\|_{L^p(G)} \asymp (\rho L)^{n(\frac{1}{p} - \frac{1}{q})} \|T_L\|_{L^p(G)}.$$

For a partial sum of the Fourier series of f :

$$S_L f(x) = \sum_{\langle \xi \rangle \leq L} d_\xi \operatorname{Tr}(\widehat{f}(\xi) \xi(x))$$

one can prove the following result.

COROLLARY 3. Let G be a compact Lie group and let $1 \leq p < q \leq \infty$ be such that $\frac{1}{p} > \frac{1}{q} + \frac{1}{2}$. Then we have

$$\left(\sum_{k=1}^{\infty} \frac{\left(k^{1-1/p+1/q} \sup_{N(L) \geq k} \frac{1}{N(L)} \|S_L f\|_{L^q(G)} \right)^p}{k} \right)^{1/p} \leq C \|f\|_{L^p(G)}$$

for all $f \in L^p(G)$. In particular, we have $N(L)^{\frac{1}{q} - \frac{1}{p}} \|S_L f\|_{L^q(G)} = o(1)$ as $L \rightarrow \infty$.

3. Embeddings of functional classes. Here we investigate embedding theorems and interpolation properties of several classes of functions on a compact Lie group G . Using the definition (2) of the Fourier series, we can defined Sobolev, Besov, and Triebel–Lizorkin spaces, respectively, as follows:

$$\begin{aligned} H_p^r &= H_p^r(G) = \left\{ f \in \mathcal{D}'(G) : \|f\|_{H_p^r} := \left\| (1 - \mathcal{L}_G)^{r/2} f \right\|_p < \infty \right\}, \\ B_{p,q}^r &= B_{p,q}^r(G) = \left\{ f \in \mathcal{D}'(G) : \|f\|_{B_{p,q}^r} := \left(\sum_{s=0}^{\infty} 2^{srq} \left\| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \operatorname{Tr}(\widehat{f}(\xi) \xi(x)) \right\|_p^q \right)^{1/q} < \infty \right\}, \\ F_{p,q}^r &= F_{p,q}^r(G) = \left\{ f \in \mathcal{D}'(G) : \|f\|_{F_{p,q}^r} := \left\| \left(\sum_{s=0}^{\infty} 2^{srq} \left| \sum_{2^s \leq \langle \xi \rangle < 2^{s+1}} d_\xi \operatorname{Tr}(\widehat{f}(\xi) \xi(x)) \right|^q \right)^{1/q} \right\|_p < \infty \right\}. \end{aligned}$$

Then we have the following result:

THEOREM 4. Let G be a compact Lie group of dimension n . Then

- (1) $B_{p_1,q}^{r_1} \hookrightarrow B_{p_2,q}^{r_2}$, $0 < p_1 \leq p_2 \leq \infty$, $0 < q \leq \infty$, $r_2 = r_1 - n(\frac{1}{p_1} - \frac{1}{p_2})$;
- (2) $B_{p,\min\{p,2\}}^r \hookrightarrow H_p^r \hookrightarrow B_{p,\max\{p,2\}}^r$, $r \in \mathbb{R}$, $1 < p < \infty$;
- (3) $B_{p,q}^r \hookrightarrow L_q$, $1 < p < q < \infty$, $r = n(\frac{1}{p} - \frac{1}{q})$;
- (4) $B_{p,1}^r \hookrightarrow L_\infty$, $0 < p \leq \infty$, $r = \frac{n}{p}$;
- (5) $B_{p,\min\{p,q\}}^r \hookrightarrow F_{p,q}^r \hookrightarrow B_{p,\max\{p,q\}}^r$, $1 < p < \infty$, $0 < q \leq \infty$, $0 < r < \infty$;
- (6) $(B_{p,\beta_0}^{r_0}, B_{p,\beta_1}^{r_1})_{\theta,q} = (H_p^{r_0}, H_p^{r_1})_{\theta,q} = (F_{p,\beta_0}^{r_0}, F_{p,\beta_1}^{r_1})_{\theta,q} = B_{p,q}^r$, $0 < r_1 < r_0 < \infty$, $0 < \beta_0, \beta_1, q \leq \infty$, $1 < p < \infty$, $r = (1 - \theta)r_0 + \theta r_1$, $0 < \theta < 1$.

For functions on the torus the corresponding results can be found, for example, in the book [Tri83].

Consequently, using norms (3) and (4), we can investigate the embeddings between Wiener and Beurling classes defined as follows:

$$A^\beta(\widehat{G}) = \left\{ f \in \mathcal{D}'(G) : \|f\|_{A^\beta} := \|\widehat{f}\|_{\ell^\beta(\widehat{G})} = \left(\sum_{[\xi] \in \widehat{G}} d_\xi^{\beta(\frac{2}{\beta} - \frac{1}{2})} \|\widehat{f}(\xi)\|_{\text{HS}}^\beta \right)^{1/\beta} < \infty \right\}$$

and

$$A^{*,\beta}(\widehat{G}) = \left\{ f : \|f\|_{A^{*,\beta}(\widehat{G})} := \left(\sum_{s=0}^{\infty} 2^{ns} \left(\sup_{2^s \leq \langle \xi \rangle} d_{\xi}^{-1/2} \|\widehat{f}(\xi)\|_{\text{HS}} \right)^{\beta} \right)^{1/\beta} < \infty \right\},$$

where $0 < \beta < \infty$. For periodic functions, i.e. for $G = \mathbb{T}^n$, we have $d_{\xi} \equiv 1$, $\widehat{G} \simeq \mathbb{Z}^n$, and $\|\widehat{f}(\xi)\|_{\text{HS}} = |\widehat{f}(\xi)|$, and such spaces have been investigated, for example, in [Beu48, BLT97] and [TB04, Ch. 6].

THEOREM 5. *Let G be a compact Lie group of dimension n .*

(A). *Let $\alpha > 0$ and $\frac{1}{\beta} = \frac{n}{\alpha} + \frac{1}{p}$. Then*

$$\|f\|_{A^{\beta}} \leq C \|f\|_{B_{p,\beta}^{\alpha}}, \quad 1 < p \leq 2; \quad (5)$$

$$\|f\|_{B_{p,\beta}^{\alpha}} \leq C \|f\|_{A^{\beta}}, \quad 2 \leq p < \infty.$$

(B). *Let $0 < \beta < \infty$ $p \geq 2$. Then*

$$C_1 \|f\|_{B_{p,\beta}^{n(\frac{1}{\beta} - \frac{1}{p})}} \leq \|f\|_{A^{*,\beta}} \leq C_2 \|f\|_{B_{1,\beta}^{n/\beta}}$$

As a consequence, we obtain

$$C_1 \|f\|_{A^{\beta}} \leq \|f\|_{B_{2,\beta}^{n(1/\beta - 1/2)}} \leq C_2 \|f\|_{A^{*,\beta}}.$$

The left inequality is an analogue of Bernstein theorem on the absolute convergence of Fourier series. It strengthens the following inequality proved by Faraut in [Far08] for groups G of unitary matrices: if $f \in C^k(G)$ for an even $k > \frac{\dim G}{2}$ then $\widehat{f} \in \ell^1(\widehat{G})$, i.e. $f \in A(G)$. For periodic functions of several variables inequality (5) follows from the results in [MS47].

Finally, we look at the following Beurling-type spaces:

$$A_r^{*,\beta} = \left\{ f : \|f\|_{A_r^{*,\beta}} := \left(\sum_{s=0}^{\infty} \left(2^{rns} \sup_{2^s \leq \langle \xi \rangle} d_{\xi}^{-1/2} \|\widehat{f}(\xi)\|_{\text{HS}} \right)^{\beta} \right)^{\frac{1}{\beta}} < \infty \right\}.$$

Note that $A_{1/\beta}^{*,\beta} = A^{*,\beta}$. These spaces are interpolation spaces in the following sense:

THEOREM 6. *Let $0 < r_1 < r_0 < \infty$, $0 < \beta_0, \beta_1, q \leq \infty$, $r = (1 - \theta)r_0 + \theta r_1$ $0 < \theta < 1$. Then*

$$(A_{r_0}^{*,\beta_0}, A_{r_1}^{*,\beta_1})_{\theta,q} = A_r^{*,q}.$$

In particular,

$$(A^{*,1/r_0}, A^{*,1/r_1})_{\theta,1/r} = A^{*,1/r}.$$

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