Well-posedness for Hall-magnetohydrodynamics

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Abstract

We prove local existence of smooth solutions for large data and global smooth solutions for small data to the incompressible, resitive, viscous or inviscid Hall-MHD model. We also show a Liouville theorem for the stationary solutions.

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1 Introduction

In this paper, we study the existence of smooth solutions for the incompressible resistive Hall-MagnetoHydroDynamics system (in short, Hall-MHD). While usual incompressible resistive MHD equations are well understood for quite long time (see e.g. [6]), Hall-MHD has received little attention from mathematicians. However, in many current physics problems, Hall-MHD is required. The first systematic study of Hall-MHD is due to Lighthill [10] followed by Campos [3]. The Hall-MHD is indeed needed for such problems as magnetic reconnection in space plasmas [7, 9], star formation [17, 2], neutron stars [14] and geo-dynamo [12]. A physical review on these questions can be found in [13]. Mathematical derivations of Hall-MHD equations from either two-fluids or kinetic models can be found in [1] and in this paper, the first existence result of global weak solutions is given. In [4], a stability analysis of a Vlasov equation modeling the Hall effect in plasmas is carried over.

Hall-MHD is believed to be an essential feature in the problem of magnetic reconnection. Magnetic reconnection corresponds to changes in the topology of magnetic field lines which are ubiquitously observed in space. However, in ideal MHD, due to ideal Ohm's law, the magnetic field undergoes a passive transport by the fluid velocity and its topology is preserved. The Hall term restores the influence of the electric current in the Lorenz force occuring in Ohm's law, which was neglected in conventional MHD. This term is quadratic in the magnetic field and involves second order derivatives. So its influence becomes dominant in the cases where the magnetic shear is large, which precisely occurs during reconnection events. In laminar situations, this term is usually small and can be neglected, which is why conventional MHD models ignore it.

In this paper, we focus on the mathematical analysis of this model and investigate the existence and uniqueness of smooth solutions. We also prove a Liouville theorem for stationary solutions. The main results are stated in section 2. Theorem 2.1 provides the global existence of weak solutions for any data. Compared to [1] which dealt with a periodic setting, the present result concerns the whole space case. However, the proof is identical and is omitted. Theorem 2.2 shows the local existence of smooth solutions for large data and provides a blow-up criterion. Theorem 2.3 proves the global existence of smooth solutions for small data. Theorem 2.4 gives the uniqueness of the solution. Finally, a Liouville theorem for stationary solutions is provided in Theorem 2.5. The main technical point is to control the second order derivatives in the Hall term by the diffusion term induced by the resistivity. This can be done thanks to the special antisymmetric structure of the Hall term. The proofs are carried over in section 3.

2 Statement of the main results

We consider the following viscous or invisicd, resistivity incompressible MHD-Hall equations.

$$\partial_t u + u \cdot \nabla u + \nabla p - \nu \Delta u = (\nabla \times B) \times B, \tag{2.1}$$

$$\nabla \cdot u = 0, \tag{2.2}$$

$$\partial_t B - \nabla \times (u \times B) - \Delta B = -\nabla \times ((\nabla \times B) \times B), \tag{2.3}$$

Eq. (2.1) represents the momentum conservation equation for the plasma fluid while (2.3) is the Maxwell-Faraday equation for the magnetic field. The incompressibility condition (2.2) is what is left from the continuity equation when the fluid density is a constant. The left-hand side of (2.1) is the standard Navier-Stokes equation while the right-hand side describes the Lorentz force acting on a charged fluid. For the simplicity of the presentation, the density is assumed equal to 1, along with all the physical parameters except for the viscosity ν. Keeping ν will allow us to distinguish between inviscid flow ($\nu = 0$) and viscous flow ($\nu > 0$). The left-hand side of (2.3) is the standard Maxwell-Faraday equation for standard MHD, while the right-hand side is the Hall term. The collection of (2.1), (2.2) and the left-hand side of (2.3) makes the standard incompressible viscous resistive MHD. The addition of the Hall term at the right-hand side of (2.3) gives rise to the incompressible viscous resistive Hall-MHD. The second term at the left-hand side of (2.3) describes the passive transport of the magnetic field by the plasma velocity and arises from ideal Ohm's law. The third term at the left-hand side of (2.3) is added to Ohm's law when finite resistivity effects are present. Here, the resistivity term is essential for the well-posedness. Consequently, we set the resistivity equal to 1. Finally, the Hall term restores the influence of the discrepancy between electron and ion velocities in Ohm's law. This discrepancy is usually neglected in conventional MHD models but it may become significant in some situations such as magnetic reconnection and dynamo effects.

The following theorem is the first step of the present paper. It shows the existence of global weak solutions in the whole space case.

Theorem 2.1 (existence of global weak solutions) *Let $\nu > 0$ and $u_0, B_0 \in L^2(\mathbb{R}^3)$, with $\nabla \cdot u_0 = 0$. Then, there exists a global weak solution $u, B \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+; H^1(\mathbb{R}^3))$ satisfying energy inequality*

$$\frac{1}{2}\left(\|u(\cdot,t)\|_{L^2}^2 + \|B(\cdot,t)\|_{L^2}^2\right) + \nu \int_0^t \|\nabla u(\cdot s)\|_{L^2}^2 ds + \int_0^t \|\nabla B(\cdot,s)\|_{L^2}^2 ds \leq \frac{1}{2}\left(\|u_0\|_{L^2}^2 + \|B_0\|_{L^2}^2\right)$$

for almost every $t \in [0,\infty)$. Furthermore, if $\nabla \cdot B_0 = 0$, then we have $\nabla \cdot B(\cdot,t) = 0$ for all $t > 0$.

A previous version of this theorem in the case of a periodic domain has been proved in [1] using a Galerkin approximation. Here, in the whole space case, the proof is based on mollifiers (see eqs. (3.1)-(3.3)) and the main estimates will be given at Proposition 3.1 below. We note that this theorem does not require that $\nabla \cdot B_0 = 0$. However, if $\nabla \cdot B_0 = 0$, the divergence free condition is propagated.

The main results of this paper are the establishment of short-time existence of smooth solutions and a blow-up criterion (Theorem 2.2). We also establish the existence of global smooth solutions for small data (Theorem 2.3). Additionally, we show the uniqueness of solutions (Theorem 2.4). Finally, we state a Liouville theorem for smooth stationary solutions.

For the sharp blow-up criterion, we need to introduce the following functional setting. We recall the homogeneous Besov space $\dot{B}^0_{\infty,\infty}$, which is defined as follows. Let $\{\psi_k\}_{k\in\mathbb{Z}}$ be the Littlewood-Paley partition of unity, where the Fourier transform $\hat{\psi}_k(\xi)$ is supported on the annulus $\{\xi \in \mathbb{R}^N \,|\, 2^{k-1} \le |\xi| < 2^k\}$(see e.g. [5, 16]). Then,

$$f \in \dot{B}^0_{\infty,\infty} \quad \text{if and only if} \quad \sup_{k\in\mathbb{Z}} \|\psi_k * f\|_{L^\infty} =: \|f\|_{\dot{B}^0_{\infty,\infty}} < \infty.$$

The following is a well-known embedding result, c.f. [16, pp. 244]

$$L^\infty(\mathbb{R}^N) \hookrightarrow \mathrm{BMO}(\mathbb{R}^N) \hookrightarrow \dot{B}^0_{\infty,\infty}(\mathbb{R}^N), \tag{2.4}$$

where BMO denotes the Bounded Mean Oscillation space [16]. Now, we state the theorems. The first one is the local existence theorem for smooth solutions and the blow-up criterion:

Theorem 2.2 (local existence and uniqueness of smooth solutions and blow-up criterion) *Let $m > 5/2$ be an integer, $\nu \ge 0$ and $u_0, B_0 \in H^m(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. Then:*

(i) There exists $T = T(\|u_0\|_{H^m}, \|B_0\|_{H^m})$ such that there exists a unique solution $u, B \in L^\infty([0,T); H^m(\mathbb{R}^3)) \cap Lip(0,T; H^{m-2}(\mathbb{R}^3))$.

(ii) Define

$$X(t) := 1 + \|B(\cdot,t)\|^2_{H^m} + \|u(\cdot,t)\|^2_{H^m}, \tag{2.5}$$

and

$$A(t) := \|\omega(t)\|_{\dot{B}^0_{\infty,\infty}} + \frac{1 + \|u(t)\|^2_{L^\infty} + \|B(t)\|^2_{L^\infty} + \|\nabla B(t)\|^2_{L^\infty}}{1 + \log(1 + \|u(t)\|_{L^\infty} + \|B(t)\|_{L^\infty} + \|\nabla B(t)\|_{L^\infty})} \tag{2.6}$$

where we denoted $\omega = \nabla \times u$ the vorticity. For $T^* < \infty$ then the following two statements are equivalent:

$$(i) \qquad X(t) < \infty, \quad \forall t < T^* \quad \text{and} \quad \limsup_{t\to T^*} X(t) = \infty, \tag{2.7}$$

$$(ii) \qquad \int_0^t A(s)\,ds < \infty, \quad \forall t < T^* \quad \text{and} \quad \int_0^{T^*} A(s)\,ds = \infty. \tag{2.8}$$

If such T^* exists, then, T^* is called the first-time blow up and (2.8) is a blow-up criterion.

Note that this theorem is valid in both the viscous ($\nu > 0$) and inviscid ($\nu = 0$) cases. Next, we state the global existence theorem for smooth solutions with small data:

Theorem 2.3 (global existence and uniqueness of smooth solutions for small data) *Let $m > 5/2$ be an integer, $\nu > 0$, and $u_0, B_0 \in H^m(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. There exists a universal constant $K = K(m, \nu)$ such that if $\|u_0\|_{H^m} + \|B_0\|_{H^m} < K$, then, there exists a unique solution $u, B \in L^\infty(\mathbb{R}_+; H^m(\mathbb{R}^3)) \cap Lip(\mathbb{R}_+; H^{m-2}(\mathbb{R}^3))$.*

This theorem is only valid in the viscous case ($\nu > 0$). The next theorem states the uniqueness of the solution:

Theorem 2.4 (weak-strong uniqueness) *Let (u_1, B_1) and (u_2, B_2) are two weak solutions.*

(i) Assume $\nu \geq 0$, and (u_2, B_2) satisfies

$$\int_0^T \left(\|\nabla u_2\|_{L^\infty} + \|u_2\|_{L^\infty}^2 + \|B_2\|_{L^\infty}^2 + \|\nabla B_2\|_{L^\infty}^2 \right) dt < \infty,$$

then we have $u_1 \equiv u_2$, and $B_1 \equiv B_2$ a.e., in $(0, T) \times \mathbb{R}^3$.

(ii) Assume $\nu > 0$, $u_2 \in L^\infty\big(0, T; H^1(\mathbb{R}^3)\big) \cap L^2\big(0, T; H^2(\mathbb{R}^3)\big)$, $B_2 \in L^2\big(0, T; W^{1,\infty}(\mathbb{R}^3)\big)$, Then we have $u_1 \equiv u_2$, and $B_1 \equiv B_2$ a.e., in $(0, T) \times \mathbb{R}^3$.

More precisely, this theorem states that, if two solutions exist with the same data and if one of them is smooth, then they must coincide. The first statement is valid in both the viscous ($\nu > 0$) and inviscid ($\nu = 0$) cases but requires stronger regularity on the smooth solution. The second result is only valid in the viscous case ($\nu > 0$) and the regularity for the u-component of the smooth solution reduces to that of the strong solution.

Finally, we state a Liouville type theorem for the smooth solutions of the following stationary Hall-MHD system.

$$u \cdot \nabla u + \nabla p = (\nabla \times B) \times B + \nu \Delta u, \tag{2.9}$$

$$\nabla \cdot u = 0, \tag{2.10}$$

$$-\nabla \times (u \times B) + \nabla \times ((\nabla \times B) \times B) = \Delta B, \tag{2.11}$$

$$\nabla \cdot B = 0. \tag{2.12}$$

The Liouville theorem reads as follows:

Theorem 2.5 (Liouville theorem for steady smooth solutions) *Let (u, B) be a $C^2(\mathbb{R}^3)$ solution to (2.9)-(2.12) satisfying*

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx + \int_{\mathbb{R}^3} |\nabla B|^2 dx < \infty, \tag{2.13}$$

and

$$u, B \in L^\infty(\mathbb{R}^3) \cap L^{\frac{9}{2}}(\mathbb{R}^3). \tag{2.14}$$

We assume $\nu > 0$. Then, we have $u = B = 0$.

Remark 2.1 *If we set $B = 0$ in the Hall-MHD system, the above theorem reduces to the well-known Galdi result [8] for the Navier-Stokes equations.*

3 Proofs of the main results

We use the mollifier technique as described in [11]. We consider the following mollifier operator:

$$\mathcal{J}_\varepsilon v = \rho_\varepsilon * v, \qquad \rho_\varepsilon = \varepsilon^{-3}\rho(x/\varepsilon),$$

where ρ is a non-negative C_0^∞ function, with unit integral. We introduce the regularized system as follows:

$$\partial_t u_\varepsilon + \mathcal{J}_\varepsilon\big((\mathcal{J}_\varepsilon u_\varepsilon \cdot \nabla)\mathcal{J}_\varepsilon u_\varepsilon\big) + \nabla p_\varepsilon = \mathcal{J}_\varepsilon\big((\nabla \times \mathcal{J}_\varepsilon B_\varepsilon) \times \mathcal{J}_\varepsilon B_\varepsilon\big) + \nu\Delta\mathcal{J}_\varepsilon^2 u_\varepsilon, \tag{3.1}$$

$$\nabla \cdot u_\varepsilon = 0, \tag{3.2}$$

$$\partial_t B_\varepsilon - \nabla \times \Big(\mathcal{J}_\varepsilon\big(\mathcal{J}_\varepsilon u_\varepsilon \times \mathcal{J}_\varepsilon B_\varepsilon\big)\Big) + \nabla \times \Big(\mathcal{J}_\varepsilon\big((\nabla \times \mathcal{J}_\varepsilon B_\varepsilon) \times \mathcal{J}_\varepsilon B_\varepsilon\big)\Big) = \Delta\mathcal{J}_\varepsilon^2 B_\varepsilon, \tag{3.3}$$

with initial condition

$$(u_\varepsilon, B_\varepsilon)\big|_{t=0} = \mathcal{J}_\varepsilon(u_0, B_0).$$

First, we have the

Proposition 3.1 *Let m be an integer such that $m > 5/2$, let $\varepsilon > 0$, $\nu \geq 0$ and $u_0, B_0 \in H^m(\mathbb{R}^3)$, with $\nabla\cdot u_0 = 0$. Then, there exists a unique global solution $u_\varepsilon, B_\varepsilon \in C^\infty\big(\mathbb{R}_+; C^\infty \cap H^m(\mathbb{R}^3)\big)$ which satisfies:*

(i) Energy inequality:

$$\frac{1}{2}\left(\|u_\varepsilon(\cdot,t)\|_{L^2}^2 + \|B_\varepsilon(\cdot,t)\|_{L^2}^2\right) + \nu\int_0^t \|\nabla\mathcal{J}_\varepsilon u_\varepsilon(\cdot,s)\|_{L^2}^2 ds + \int_0^t \|\nabla\mathcal{J}_\varepsilon B_\varepsilon(\cdot,s)\|_{L^2}^2 ds$$
$$\leq \frac{1}{2}\left(\|u_0\|_{L^2}^2 + \|B_0\|_{L^2}^2\right) \quad \forall t \in (0,\infty). \tag{3.4}$$

(ii) There are positive constants C depending only on m, and constant T depending only on m, $\|u_0\|_{H^m}$, and $\|B_0\|_{H^m}$ such that:

$$\|(u_\varepsilon, B_\varepsilon)\|_{L^\infty(0,T;H^m(\mathbb{R}^3))\cap Lip(0,T;H^{m-2}(\mathbb{R}^3))} \leq C(\|u_0\|_{H^m}^2 + \|B_0\|_{H^m}^2). \tag{3.5}$$

Sketch of proof. The existence and uniqueness comes directly from the abstract Picard iteration theorem in H^m (see details in [11]). The energy estimate comes directly from (3.1), (3.3). The proof of estimate (3.5) is almost identical as the proof of the a priori estimate (3.20) of Theorem 2.2 below and is skipped. ∎

We can now prove Theorem 2.1.

Sketch of proof of Theorem 2.1. Thanks to Proposition 3.1, we can construct a sequence of regularized solutions to problem (2.1), (2.3) and the energy estimate (3.4) provides the required compactness allowing to pass to the limit $\varepsilon \to 0$ in the weak formulation and to get a global weak solution. The details of the functional analysis are the same as in [1]. ∎

Property (ii) of Proposition 3.1 will be used as an a priori estimate for the construction of regularized solutions in the proof of Theorem 2.2 below.

Proposition 3.2 *Let $m > 5/2$ be an integer. Let (u, B) be a smooth solution to (2.1)-(2.3). Then, there are two positive universal constants C_1 and C_2 such that the following a priori estimates hold:*

$$\frac{d}{dt}(\|B\|_{H^m}^2 + \|u\|_{H^m}^2) + \|\nabla B\|_{H^m}^2 + 2\nu\|Du\|_{H^m}^2 \leq$$
$$C_1(1 + \|B\|_{L^\infty}^2 + \|\nabla B\|_{L^\infty}^2 + \|u\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty})(\|B\|_{H^m}^2 + \|u\|_{H^m}^2 + 1). \quad (3.6)$$
$$\frac{d}{dt}(\|B\|_{H^m}^2 + \|u\|_{H^m}^2) + 2\|\nabla B\|_{H^m}^2 + 2\nu\|Du\|_{H^m}^2 \leq$$
$$\leq C_2\big(\|\nabla B\|_{H^m}^2 + \|\nabla u\|_{H^m}^2\big)(\|B\|_{H^m}^2 + \|u\|_{H^m}^2 + \|u\|_{H^m} + \|B\|_{H^m}). \quad (3.7)$$

Proof. We first concentrate ourselves on (3.6). Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ be a multi-index. We operate $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \dots \partial x_3^{\alpha_3}$ (where $|\alpha| = \alpha_1 + \ldots + \alpha_3$) on (2.1) and (2.3) respectively and take the scalar product of them with $D^\alpha B$ and $D^\alpha u$ respectively, add them together and then sum the result over $|\alpha| \leq m$. We obtain

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{H^m}^2 + \|B\|_{H^m}^2) + \nu\|Du\|_{H^m}^2 + \|\nabla B\|_{H^m}^2 =$$
$$- \sum_{0<|\alpha|\leq m} \int_{\mathbb{R}^3} D^\alpha\big((\nabla \times B) \times B\big) \cdot D^\alpha(\nabla \times B)\, dx$$
$$+ \sum_{0<|\alpha|\leq m} \int_{\mathbb{R}^3} D^\alpha(u \times B) \cdot D^\alpha(\nabla \times B)\, dx$$
$$- \sum_{0<|\alpha|\leq m} \int_{\mathbb{R}^3} D^\alpha(u \cdot \nabla u) \cdot D^\alpha u\, dx$$
$$+ \sum_{0<|\alpha|\leq m} \int_{\mathbb{R}^3} D^\alpha\{(\nabla \times B) \times B\} \cdot D^\alpha u\, dx$$
$$=: I_1 + I_2 + I_3 + I_4. \quad (3.8)$$

Notice that the $|\alpha| = 0$ terms on the right hand side above have exactly cancelled each other by energy conservation. The cancellation is crucially important for the existence of

global smooth solution for small data. Then, we estimate successively each of the I_1 to I_4 terms. We have:

$$I_1 \;=\; -\sum_{0<|\alpha|\leq m}\int_{\mathbb{R}^3}\big[D^\alpha\big((\nabla\times B)\times B\big)-(D^\alpha(\nabla\times B))\times B\big]\cdot D^\alpha(\nabla\times B)\,dx$$

where the second term of the right-hand side is simply zero. Using the well-known calculus inequality,

$$\sum_{|\alpha|\leq m}\|D^\alpha(fg)-(D^\alpha f)g\|_{L^2}\leq C(\|f\|_{H^{m-1}}\|\nabla g\|_{L^\infty}+\|f\|_{L^\infty}\|g\|_{H^m}),\tag{3.9}$$

we get:

$$\begin{aligned} I_1 &\leq C(\|B\|_{H^m}\|\nabla B\|_{L^\infty}+\|\nabla B\|_{L^\infty}\|B\|_{H^m})\|\nabla B\|_{H^m} &(3.10)\\ &\leq \frac{1}{4}\|\nabla B\|^2_{H^m}+C\|B\|^2_{H^m}\|\nabla B\|^2_{L^\infty}, &(3.11)\end{aligned}$$

On the other hand, using Leibnitz formula and the Sobolev inequality, we obtain

$$\begin{aligned} I_2 &\leq \sum_{0<|\alpha|\leq m}\|D^\alpha(u\times B)\|_{L^2}\|\nabla B\|_{H^m}\\ &\leq C(\|u\|_{L^\infty}\|B\|_{H^m}+\|u\|_{H^m}\|B\|_{L^\infty})\|\nabla B\|_{H^m}\\ &\leq \frac{1}{4}\|\nabla B\|^2_{H^m}+C\|u\|^2_{L^\infty}\|B\|^2_{H^m}+C\|u\|^2_{H^m}\|B\|^2_{L^\infty}. &(3.12)\end{aligned}$$

Then, we remark that

$$I_3 \;=\; -\sum_{0<|\alpha|\leq m}\int_{\mathbb{R}^3}[D^\alpha(u\cdot\nabla u)-u\cdot\nabla D^\alpha u]\cdot D^\alpha u\,dx.$$

Indeed, the second term is zero by the fact that u is divergence free. Then, similarly to the above calculation, using the calculus inequality (3.9), we obtain

$$I_3 \;\leq\; \sum_{0<|\alpha|\leq m}\|D^\alpha(u\cdot\nabla u)-u\cdot\nabla D^\alpha u\|_{L^2}\|\nabla u\|_{H^{m-1}}\leq C\|\nabla u\|_{L^\infty}\|\nabla u\|^2_{H^{m-1}}.\tag{3.13}$$

From (3.8) we get

$$I_4 \;\leq\; \sum_{0<|\alpha|\leq m}\|(\nabla\times B)\times B\|_{H^m}\|\nabla u\|_{H^{m-1}}.$$

Note that we can take $\|\nabla u\|_{H^{m-1}}$ instead of $\|u\|_{H^m}$ because $|\alpha|>0$. This remark is important for the proof of the next theorem about the global existence of smooth solutions for small data. Using Leibnitz formula, we derive

$$\begin{aligned} I_4 &\leq C(\|\nabla B\|_{L^\infty}\|B\|_{H^m}+\|\nabla B\|_{H^m}\|B\|_{L^\infty})\|\nabla u\|_{H^{m-1}} &(3.14)\\ &\leq C\|\nabla B\|_{L^\infty}\|B\|_{H^m}\|\nabla u\|_{H^{m-1}}+\frac{1}{2}\|\nabla B\|^2_{H^m}+C\|B\|^2_{L^\infty}\|u\|^2_{H^m}. &(3.15)\end{aligned}$$

From estimates (3.11), (3.12), (3.13) and (3.15), we obtain

$$\begin{aligned}\frac{d}{dt}(\|B\|_{H^m}^2+\|u\|_{H^m}^2)+\|\nabla B\|_{H^m}^2+2\nu\|Du\|_{H^m}^2\\ \leq C(\|B\|_{L^\infty}^2+\|\nabla B\|_{L^\infty}^2+\|u\|_{L^\infty}^2)(\|B\|_{H^m}^2+\|u\|_{H^m}^2)\\ +C\|\nabla u\|_{L^\infty}\|\nabla u\|_{H^{m-1}}^2+C\|\nabla B\|_{L^\infty}\|B\|_{H^m}\|\nabla u\|_{H^{m-1}},\end{aligned}\tag{3.16}$$

from which we easily deduce (3.6).

We now turn towards estimate (3.7). It is deduced through a small change in the estimate (3.12) for I_2. From Leibnitz formula, we have:

$$\begin{aligned}I_2 &\leq \sum_{0<|\alpha|\leq m}\|D^\alpha(u\times B)\|_{L^2}\|\nabla B\|_{H^m}\\ &\leq \sum_{0<|\alpha|\leq m}\sum_{j=1}^3\|D^{\alpha-e_j}(\partial_{x_j}u\times B+u\times\partial_{x_j}B)\|_{L^2}\|\nabla B\|_{H^m}\\ &\leq C(\|u\|_{L^\infty}\|\nabla B\|_{H^{m-1}}+\|\nabla u\|_{L^\infty}\|B\|_{H^{m-1}}\\ &\qquad +\|B\|_{L^\infty}\|\nabla u\|_{H^{m-1}}+\|\nabla B\|_{L^\infty}\|u\|_{H^{m-1}})\,\|\nabla B\|_{H^m}\\ &\leq C(\|u\|_{H^m}\|\nabla B\|_{H^{m-1}}+\|B\|_{H^m}\|\nabla u\|_{H^{m-1}})\,\|\nabla B\|_{H^m}\end{aligned}\tag{3.17}$$

From estimates (3.10), (3.17), (3.13) and (3.14), we easily deduce (3.7). This completes the proof of Proposition 3.2. ∎

In order to prove Theorem 2.3, we need to use the following

Lemma 3.3 *Assume that a is a positive constant, $x(t)$, $y(t)$ are two nonnegative $C^1(\mathbb{R}_+)$ functions, and $D(t)$ is a nonnegative function, satisfying*

$$\frac{d}{dt}\left(x^2+y^2\right)+D\leq a(x^2+y^2+x+y)D.$$

If additionally, the initial data satisfy

$$x^2(0)+y^2(0)+\sqrt{2(x^2(0)+y^2(0))}<\frac{1}{a},\tag{3.18}$$

then, for any $t>0$, one has

$$x^2(t)+y^2(t)+x(t)+y(t)<x^2(0)+y^2(0)+\sqrt{2(x^2(0)+y^2(0))}<\frac{1}{a}$$

Proof: Notice that

$$\frac{d}{dt}\left(x^2+y^2\right)+D\leq a(\,x^2+y^2+\sqrt{2(x^2+y^2)}\,)D$$

Since (3.18) is true initially, it is still true for short time, so that one has

$$x^2(t) + y^2(t) + \sqrt{2(x^2(t) + y^2(t))} < \frac{1}{a}$$

Then $x^2(t) + y^2(t)$ is a decreasing function in this short period. Hence

$$x^2(t) + y^2(t) + \sqrt{2(x^2(t) + y^2(t))} \leq x^2(0) + y^2(0) + \sqrt{2(x^2(0) + y^2(0))}$$

Then by an extension argument, it holds true for all time. ∎

Proof of Theorem 2.2. We construct a sequence of weak solutions of the regularized system (3.1)-(3.3) and remark that such solutions do actually satisfy the a priori estimate (3.6). This allows us to pass to the limit in a subsequence and show the existence of smooth solutions on short times.

Below we set

$$X(t) := \|B(\cdot, t)\|^2_{H^m} + \|u(\cdot, t)\|^2_{H^m} + 1.$$

Then, from (3.6), and using the Sobolev inequality, we have

$$\begin{aligned} \frac{d}{dt}X &\leq C(1 + \|B\|^2_{L^\infty} + \|\nabla B\|^2_{L^\infty} + \|u\|^2_{L^\infty} + \|\nabla u\|_{L^\infty})X \\ &\leq C(1 + \|B\|^2_{H^m} + \|u\|^2_{H^m} + \|u\|_{H^m})X \\ &\leq CX^2. \end{aligned}$$

Therefore, thanks to nonlinear Gronwall's inequality, we have:

$$X(t) \leq \frac{X(0)}{1 - C_0 X(0)t}.$$

Now, choose $T = \frac{1}{2C_0X(0)}$. Then:

$$X(t) \leq 2X(0) \quad \forall t \in [0, T). \tag{3.19}$$

This implies the following a priori estimate:

$$\|(u, B)\|_{L^\infty(0,T;H^m(\mathbb{R}^3))} \leq C(\|u_0\|^2_m + \|B_0\|^2_m).$$

Now, thanks to a direct estimate on the time derivatives using eqs. (2.1), (2.3), we have also the a priori estimate:

$$\|(u, B)\|_{Lip(0,T;H^{m-2}(\mathbb{R}^3))} \leq C(\|u_0\|^2_m + \|B_0\|^2_m).$$

Adding these two inequalities together, we get the a priori estimate:

$$\|(u, B)\|_{L^\infty(0,T;H^m(\mathbb{R}^3))\cap Lip(0,T;H^{m-2}(\mathbb{R}^3))} \leq C(\|u_0\|^2_m + \|B_0\|^2_m). \tag{3.20}$$

Inequality (3.20) also holds true for the modified equations (3.1)-(3.3). This is exactly estimate (3.5) of Proposition 3.1. As in the proof in Theorem 2.1, there exists a subsequence, still denoted by $(u_\varepsilon, B_\varepsilon)$ whose limit gives rise to a global weak solution (u, B). Using the lower-semicontinuity of the norm, one has that this weak solution satisfies the inequality

$$\|(u,B)\|_{L^\infty(0,T;H^m(\mathbb{R}^3))\cap Lip(0,T;H^{m-2}(\mathbb{R}^3))} \le C(\|u_0\|_m^2 + \|B_0\|_m^2),$$

which proves the local existence of a smooth solution on $[0,T)$. Uniqueness follows from the same kind of proof as Theorem 2.4.

Next, in order to prove the blow-up criterion (2.8) we recall the following version of Beale-Kato-Majda type logarithmic Sobolev inequality in $\mathbb{R}^3$(see [15, formula 14.2] or [5]),

$$\|f\|_{L^\infty} \le C(1+\|f\|_{\dot{B}^0_{\infty,\infty}})\{\log(1+\|f\|_{H^{m-1}})\}, \quad m > 5/2. \tag{3.21}$$

Substituting f for ∇u in (3.21) and inserting the result into (3.16), we have

$$\begin{aligned}\frac{d}{dt}X \le C(1+\|B\|_{L^\infty}^2 + \|\nabla B\|_{L^\infty}^2 + \|u\|_{L^\infty}^2)X \\ +C(1+\|\nabla u\|_{\dot{B}^0_{\infty,\infty}})X\log(1+X).\end{aligned} \tag{3.22}$$

We use the fact that the Calderon-Zygmund operator is bounded from the homogeneous Besov space $\dot{B}^0_{\infty,\infty}$ into itself [16], namely we have

$$\|\nabla u\|_{\dot{B}^0_{\infty,\infty}} \le C\|\omega\|_{\dot{B}^0_{\infty,\infty}}. \tag{3.23}$$

We also use the Sobolev inequality to obtain

$$\begin{aligned}1+\|B\|_{L^\infty}^2 + \|\nabla B\|_{L^\infty}^2 + \|u\|_{L^\infty}^2 \\ \le C\frac{1+\|B\|_{L^\infty}^2 + \|\nabla B\|_{L^\infty}^2 + \|u\|_{L^\infty}^2}{1+\log(1+\|u\|_{L^\infty}+\|B\|_{L^\infty}+\|\nabla B\|_{L^\infty})}\log(1+X).\end{aligned} \tag{3.24}$$

Inserting (3.23) and (3.24) into (3.22), we get:

$$\frac{d}{dt}X \;\le\; C\,A(t)\,X\ln(1+X),$$

where $A(t)$ is given by (2.6). We now recall that by Sobolev imbedding, there exists $C>0$ such that we have $A(t) \le CX(t)$. Then, by this inequality and Gronwall's lemma, we obtain the equivalence of (2.7) and (2.8). ∎

Remark 3.1 *By applying the same argument, we can show the local well-posedness for the simple Hall problem without coupling to the fluid velocity u, which is written as follows:*

$$\partial_t B + \nabla\times((\nabla\times B)\times B) = \Delta B,$$

with a initial data B_0. We can also extend the result to the following generalized Hall problem

$$\partial_t B + \nabla \times ((\Lambda^\alpha \nabla \times B) \times B) = -\Lambda^\beta B,$$

where Λ^α is the fractional power of the Laplacian $\Lambda^\alpha = (-\Delta)^{\alpha/2}$, supplemented with a initial data B_0. With the same argument, we can show that if $\beta \geq \alpha + 2$, $\alpha \geq 0$, then this generalized Hall problem is also locally well posed. When $\beta < \alpha + 2$ it is an open problem to determine if this problem is well-posed or not.

Proof of Theorem 2.3. We use the inequality (3.7), and estimate, using the Sobolev inequality,

$$\begin{aligned}
&\frac{d}{dt}(\|B\|_{H^m}^2 + \|u\|_{H^m}^2) + 2\|\nabla B\|_{H^m}^2 + 2\nu\|\nabla u\|_{H^m}^2 \\
&\quad \leq C(\|B\|_{L^\infty}^2 + \|\nabla B\|_{L^\infty}^2 + \|u\|_{L^\infty}^2)(\|B\|_{H^m}^2 + \|u\|_{H^m}^2) \\
&\qquad + C\|\nabla u\|_{L^\infty}\|\nabla u\|_{H^{m-1}}^2 + C\|\nabla B\|_{L^\infty}\|B\|_{H^m}\|\nabla u\|_{H^{m-1}} \\
&\quad \leq C_1\big(\|\nabla B\|_{H^m}^2 + \|\nabla u\|_{H^m}^2\big)(\|B\|_{H^m}^2 + \|u\|_{H^m}^2 + \|u\|_{H^m} + \|B\|_{H^m}). \qquad (3.25)
\end{aligned}$$

Therefore if

$$\|B_0\|_{H^m}^2 + \|u_0\|_{H^m}^2 + \sqrt{\|B_0\|_{H^m}^2 + \|u_0\|_{H^m}^2} < \frac{\min\{1, 2\nu\}}{C_1},$$

then, by Lemma 3.3, we have for any $t > 0$

$$\frac{d}{dt}(\|B(\cdot, t)\|_{H^m}^2 + \|u(\cdot, t)\|_{H^m}^2) \leq 0, \qquad (3.26)$$

and

$$\|B(t)\|_{H^m}^2 + \|u(t)\|_{H^m}^2 \leq \|B_0\|_{H^m}^2 + \|u_0\|_{H^m}^2 \leq \frac{\min\{1, 2\nu\}}{C_1},$$

for all $t > 0$. Hence the so obtained solution is global in time, which ends the proof. ∎

Proof of Theorem 2.4. The proof of this theorem uses the same estimates as for the proof of Theorem 2.2 (i) applied to $u = u_1 - u_2$ and $B = B_1 - B_2$. The details are omitted. ∎

Proof of Theorem 2.5 We first estimate the pressure in (2.9). Taking the divergence of (2.9), and using the identity, $(\nabla \times B) \times B = -\nabla \frac{|B|^2}{2} + (B \cdot \nabla)B$, we obtain,

$$\Delta\left(p + \frac{|B|^2}{2}\right) = -\sum_{j,k=1}^{3} \partial_j \partial_k (u_j u_k - B_j B_k),$$

from which we have the representation formula of the pressure, using the Riesz transforms in $\mathbb{R}^3$,

$$p = \sum_{j,k=1}^{3} R_j R_j (u_j u_k - B_j B_k) - \frac{|B|^2}{2}. \tag{3.27}$$

By the Calderon-Zygmund inequality, one has

$$\|p\|_{L^q} \le C\|u\|_{L^{2q}}^2 + C\|B\|_{L^{2q}}^2, \quad 1 < q < \infty, \tag{3.28}$$

if $u, B \in L^{2q}(\mathbb{R}^3)$. Let σ_R be the standard cut-off function defined as follows. Consider $\sigma \in C_0^\infty(\mathbb{R}^N)$ such that

$$\sigma(|x|) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 2, \end{cases}$$

and $0 \le \sigma(x) \le 1$ for $1 < |x| < 2$. Then, for each $R > 0$, let us define

$$\sigma\left(\frac{|x|}{R}\right) := \sigma_R(|x|) \in C_0^\infty(\mathbb{R}^N).$$

We take the inner product of Eq. (2.9) with $u\sigma_R$ and the inner product of Eq. (2.11) with $B\sigma_R$, add the result together and integrate over $\mathbb{R}^3$. After integration by parts, we have

$$\begin{aligned} \nu \int_{\mathbb{R}^3} |\nabla u|^2 \sigma_R dx + \int_{\mathbb{R}^3} |\nabla B|^2 \sigma_R dx &= \frac{1}{2}\int_{\mathbb{R}^3} |u|^2 (u\cdot\nabla)\sigma_R dx + \int_{\mathbb{R}^3} p(u\cdot\nabla)\sigma_R dx \\ &- \int_{\mathbb{R}^3} u \times B \cdot \nabla\sigma_R \times B dx + \int_{\mathbb{R}^3} (\nabla \times B) \times B \cdot \nabla\sigma_R \times B dx \\ &+ \frac{\nu}{2}\int_{\mathbb{R}^3} |u|^2 \Delta\sigma_R dx + \frac{1}{2}\int_{\mathbb{R}^3} |B|^2 \Delta\sigma_R dx \\ &:= I_1 + \cdots + I_6. \end{aligned} \tag{3.29}$$

We have the following estimates,

$$\begin{aligned} |I_1| &\le \int_{\{R\le|x|\le 2R\}} |u|^3 |\nabla\sigma_R| dx \\ &\le \frac{1}{2R}\|\nabla\sigma\|_{L^\infty} \left(\int_{\{R\le|x|\le 2R\}} |u|^{\frac{9}{2}} dx\right)^{\frac{2}{3}} \left(\int_{\{R\le|x|\le 2R\}} dx\right)^{\frac{1}{3}} \\ &\le C\|u\|^3_{L^{\frac{9}{2}}(R\le|x|\le 2R)} \to 0 \quad \text{as } R \to \infty. \end{aligned}$$

Using the estimate (3.28), one has

$$\begin{aligned} |I_2| &\le \int_{\{R\le|x|\le 2R\}} |p||u||\nabla\sigma_R| dx \\ &\le \frac{1}{R}\|\nabla\sigma\|_{L^\infty} \left(\int_{\mathbb{R}^3} |p|^{\frac{9}{4}} dx\right)^{\frac{4}{9}} \left(\int_{\{R\le|x|\le 2R\}} |u|^{\frac{9}{2}} dx\right)^{\frac{2}{9}} \left(\int_{\{R\le|x|\le 2R\}} dx\right)^{\frac{1}{3}} \\ &\le C(\|u\|^2_{L^{\frac{9}{2}}} + \|B\|^2_{L^{\frac{9}{2}}})\|u\|_{L^{\frac{9}{2}}(R\le|x|\le 2R)} \to 0 \quad \text{as } R \to \infty. \end{aligned}$$

$$
\begin{aligned}
|I_3| &\leq \int_{\{R\leq|x|\leq 2R\}} |u||B|^2|\nabla\sigma|dx \\
&\leq \frac{1}{R}\|\nabla\sigma\|_{L^\infty}\left(\int_{\mathbb{R}^3}|u|^{\frac{9}{2}}dx\right)^{\frac{2}{9}}\left(\int_{\mathbb{R}^3}|B|^{\frac{9}{2}}dx\right)^{\frac{4}{9}}\left(\int_{\{R\leq|x|\leq 2R\}}dx\right)^{\frac{1}{3}} \\
&\leq C\|B\|^2_{L^{\frac{9}{2}}}\|u\|_{L^{\frac{9}{2}}(R\leq|x|\leq 2R)} \to 0 \quad \text{as } R\to\infty.
\end{aligned}
$$

$$
\begin{aligned}
|I_4| &\leq \int_{\{R\leq|x|\leq 2R\}} |\nabla B||B|^2|\nabla\sigma|dx \\
&\leq \frac{1}{R}\|\nabla\sigma\|_{L^\infty}\|B\|_{L^\infty}\|\nabla B\|_{L^2}\left(\int_{\{R\leq|x|\leq 2R\}}|B|^6dx\right)^{\frac{1}{6}}\left(\int_{\{R\leq|x|\leq 2R\}}dx\right)^{\frac{1}{3}} \\
&\leq C\|B\|_{L^\infty}\|\nabla B\|_{L^2}\|B\|_{L^6(R\leq|x|\leq 2R)} \to 0 \quad \text{as } R\to\infty,
\end{aligned}
$$

since $\|B\|_{L^6}\leq C\|\nabla B\|_{L^2}<\infty$ by the Sobolev embedding. Then,

$$
\begin{aligned}
|I_5|+|I_6| &\leq C\int_{\{R\leq|x|\leq 2R\}}(|u|^2+|B|^2)|\Delta\sigma|dx \\
&\leq \frac{C}{R^2}\|D^2\sigma\|_{L^\infty}\left(\int_{\{R\leq|x|\leq 2R\}}(|u|^2+|B|^2)^3dx\right)^{\frac{1}{3}}\left(\int_{\{R\leq|x|\leq 2R\}}dx\right)^{\frac{2}{3}} \\
&\leq C(\|u\|^2_{L^6(R\leq|x|\leq 2R)}+\|B\|^2_{L^6(R\leq|x|\leq 2R)}) \to 0 \quad \text{as } R\to\infty.
\end{aligned}
$$

Therefore, passing to the limit $R\to\infty$ in (3.29), and using the dominated convergence theorem, one has

$$
\int_{\mathbb{R}^3}|\nabla u|^2dx+\int_{\mathbb{R}^3}|\nabla B|^2dx=0,
$$

which implies the conclusion of the theorem. ∎

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