

BOUNDARY INTEGRAL EQUATION ANALYSIS
OF THE BENDING OF THIN PLATES

by

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ABSTRACT

This thesis describes the application of the boundary integral equation (BIE) method to the bending of thin plates under lateral loading. The plate theory used is that of Kirchhoff & Love. A brief outline of this classical plate theory is given, followed by the derivation of the governing differential equation of the bending of thin plates under lateral loading in rectangular Cartesian coordinates. This is followed by a detailed presentation of the BIE formulation for plate bending problems.

Two implementations of the BIE method are described. The first one employs constant elements and the second one incorporates quadratic isoparametric elements. The constant element implementation contains numerical tests using six possible pairs of integral equations applicable to plate bending problems. The relative accuracies of the solutions obtained using each pair are compared. It is also shown that the constant element formulations published to date do not provide an accurate solution for certain boundary variables, namely, the effective shear force, for problems in which the Kirchhoff corner forces are non-zero. This is due to the fact that the effect of these corner forces has been ignored in the formulations. It is also shown that certain pairs of boundary integral equations yield relatively accurate solutions only if the Kirchhoff corner forces are included in the formulation.

The quadratic implementation employs the integral equations for deflection and normal slope at the boundary. The computer program

developed using this formulation is applied to a wide range of problems to assess its effectiveness. These include rectangular and circular plates under various loading and boundary conditions. Further numerical tests have also been carried out on plates with various geometrical shapes. These include triangular plates, rhombic and skew plates, plates in the shape of a sector of a circle, and corner plates. The formulation has also been extended to include plates with interior supports having various edge conditions.

To my parents, my wife and children

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NOMENCLATURE

The notation used in this thesis is briefly described in alphabetical order in the following list. Where there are two or more definitions for the same symbol, the relevant chapter or section is indicated in parentheses. When no such indication is provided, the definition of the symbol applies everywhere other than for the alternative definitions. Where appropriate, the number of the equation in which the symbol first appears is given.

<u>Symbol</u>	<u>Definition</u>	<u>Equation</u>
a	Width of rectangular plates (Chapter 5)	
a	Radius of circular or partly circular plates (Chapter 6)	
A	A two-dimensional region	
A_ϵ	A small circular region of radius ϵ	
$[A]$	Coefficient matrix for nodal unknowns	(3.9)
$[\bar{A}]$	Coefficient matrix for nodal boundary conditions	(3.9)
b	Base of an equilateral triangular plate or the length of rectangular plates (Chapter 5)	
$\{\bar{b}\}$	Column vector containing prescribed boundary conditions	(3.9)
C_n	Free term coefficient for normal slope at a boundary point	(2.107)
C_t	Free term coefficient for tangential derivative of deflection	(2.107)
C_1^I	Coefficient for deflection at the self node in the equation for deflection at a boundary point	(3.5)
C_1^{II}	Coefficient for deflection at the self node in the equation for normal slope at a boundary point	(3.6)
C_2^{II}	Coefficient for normal slope at the self node in the equation for normal slope at a boundary point	(3.6)

<u>Symbol</u>	<u>Definition</u>	<u>Equation</u>
C_1^{III}	Coefficient for deflection at the self node in the equation for normal moment at a boundary point	(3.7)
C_2^{III}	Coefficient for normal slope at the self node in the equation for normal moment at a boundary point	(3.7)
C_3^{III}	Coefficient for normal moment at the self node in the equation for normal moment at a boundary point	(3.7)
C_1^{IV}	Coefficient for deflection at the self node in the equation for effective shear force at a boundary point	(3.8)
C_2^{IV}	Coefficient for normal slope at the self node in the equation for effective shear force at a boundary point	(3.8)
C_3^{IV}	Coefficient for normal moment at the self node in the equation for effective shear force at a boundary point	(3.8)
C_4^{IV}	Coefficient for effective shear force at the self node in the equation for effective shear force at a boundary point	(3.8)
C^I	Coefficient for deflection at the self node in the equation for deflection at a boundary point	(4.10)
C_n^{II}	Coefficient for normal slope at the self node in the equation for normal slope at a boundary point	(4.11)
C_t^{II}	Coefficient for tangential derivative of deflection at the self node in the equation for normal slope at a corner point on the boundary	(4.11)
D	Flexural rigidity of the plate	(2.36)
E	Young's modulus	(2.37)
δ	Intensity of a uniformly distributed load or the maximum intensity for other types of distributed loadings (Chapters 5 and 6)	
F	Lateral loading on a differential element of the plate	(2.42)
F	Concentrated load (Chapters 3, 5 and 6)	
F_1, F_2	Concentrated loads	(3.46)

<u>Symbol</u>	<u>Definition</u>	<u>Equation</u>
F_i	Intensity of loading at the i th node of area elements	(4.106)
F_z	Any lateral load acting on the plates surface	(2.3)
$\{\bar{F}\}$	Column vector containing the contribution from the external loading on the plate to the right hand member of the set of linear algebraic equations for the nodal unknowns	(3.45)
G	Shear modulus	(2.15)
J	Jacobian	(3.53)
K_x	Curvature in the x direction	(2.26)
K_y	Curvature in the y direction	(2.27)
K_{xy}	Warping	(2.28)
K	Included angle at a corner	(2.89)
L	Element length	(3.20)
L	A side of a square or rhombic plate or the height of an equilateral triangular plate (Section 3.7 and Chapter 5)	
m_x	Moment about the y axis	(2.1)
m_y	Moment about the x axis	(2.2)
m_{xy}, m_{yx}	Twisting moments	(2.4)
m_n	Normal moment at the boundary	(2.55)
m_t	Twisting moment at the boundary	(2.56)
m_p	Moment at point p	(2.124)
$[m_t]$	Corner jump in the twisting moment	(2.70)
M_n	Normal moment kernel in the integral equation for deflection at a point	(2.71)
M_2	Normal moment kernel in the integral equation for deflection gradient at a point	(2.96)
M_3	Normal moment kernel in the integral equation for moment at a point	(2.125)
M_4	Normal moment kernel in the integral equation for effective shear force at a point	(2.132)
M_5	Normal moment kernel in the integral equation for twisting moment at a point	(2.140)

<u>Symbol</u>	<u>Definition</u>	<u>Equation</u>
M_6	Normal moment kernel in the integral equation for shear force at a point	(2.144)
n	Normal direction at a boundary point	(2.55)
n_p	An arbitrary direction at an interior point p	(2.95)
n_P	Normal to the boundary at the source point P	
n_Q	Normal to the boundary at a field point Q	(2.77)
n_P^+	Normal direction on the positive side of a corner point P	(2.102)
n_P^-	Normal direction on the negative side of a corner point P	
$n^+(p^a)$	Normal direction on the positive side of the a th node; the a th node is a corner node	(4.97)
$n(p^{d(b,c)})$	Normal direction at the c th node of the b th element	(4.97)
N_1 to N_8	Shape functions	
p	Source point in the interior of the domain	
P	Source point on the boundary of the domain	
$p^{d(b,c)}$	The c th node of the b th element	(4.10)
p^a	The a th boundary node	(4.10)
$p^e(\lambda)$	The λ th corner node	(4.10)
q	Field point in the interior of the domain	
Q	Field point on the boundary of the domain	
Q_x	Shear force along lines parallel to the y axis	(2.4)
Q_y	Shear force along lines parallel to the x axis	(2.5)
Q_p	Shear force at point p	(2.144)
r	Distance between source point and field point	(2.75)
r	Polar coordinate	(2.62)
R	Radius of curvature of the boundary curve	(2.78d)
R^- or R_-	Radius of curvature of the boundary curve on the negative side of a corner	(2.90a)
R^+ or R_+	Radius of curvature of the boundary curve on the positive side of a corner	(2.90b)

<u>Symbol</u>	<u>Definition</u>	<u>Equation</u>
R_P	Radius of curvature at the source point P on the boundary	(2.135a)
R_Q	Radius of curvature at a field point Q on the boundary	(2.135d)
S	Arc length along the boundary curve	(2.57)
S^-	Boundary curve on the negative side of a corner	(2.90a)
S^+	Boundary curve on the positive side of a corner	(2.90b)
S_ϵ	Boundary of domain A_ϵ	(2.77)
S_b	b th boundary element	(4.10)
t	Plate thickness (Section 2.2.5)	(2.29)
$\hat{t}$	Tangential direction at a boundary point	(2.55)
$\hat{t}_P^+$	Tangential direction on the positive side of a corner point P	(2.102)
$\hat{t}_P^-$	Tangential direction on the negative side of a corner point P	(2.112)
$[t]$	Kirchhoff corner forces	(2.71)
T	Twisting moment kernel in the integral equation for deflection at a point	(2.78e)
$[T]$	Corner jump in the twisting moment kernel in the integral equation for deflection at a point	
T_2	Twisting moment kernel in the integral equation for deflection gradient at a point	(2.97e)
$[T_2]$	Corner jump in the twisting moment kernel in the integral equation for deflection gradient at a point	(2.96)
T_3	Twisting moment kernel in the integral equation for moment at a point	(2.126e)
$[T_3]$	Corner jump in the twisting moment kernel in the integral equation for moment at a point	(2.125)
T_4	Twisting moment kernel in the integral equation for effective shear force at a point	(2.133e)
$[T_4]$	Corner jump in the twisting moment kernel in the integral equation for effective shear force at a point	(2.132)
T_5	Twisting moment kernel in the integral equation for twisting moment at a point	(2.141e)

<u>Symbol</u>	<u>Definition</u>	<u>Equation</u>
$[T_5]$	Corner jump in the twisting moment kernel in the integral equation for twisting moment at a point	(2.140)
T_6	Twisting moment kernel in the integral equation for shear force at a point	(2.145e)
$[T_6]$	Corner jump in the twisting moment kernel in the integral equation for shear force at a point	(2.144)
u	Deflection in the x direction	(2.21)
$\{\bar{u}\}$	Column vector of nodal unknowns	(3.9)
v	Deflection in the y direction	(2.22)
v_x	Effective shear force along lines parallel to the y axis	(2.52)
v_y	Effective shear force along lines parallel to the x axis	(2.54)
v_n	Effective shear force at a boundary point	(2.57)
v_p	Effective shear force at point p	(2.131)
V_n	Effective shear force kernel in the integral equation for deflection at a point	(2.70)
V_2	Effective shear force kernel in the integral equation for deflection gradient at a point	(2.96)
V_3	Effective shear force kernel in the integral equation for moment at a point	(2.125)
V_4	Effective shear force kernel in the integral equation for effective shear force at a point	(2.132)
V_5	Effective shear force kernel in the integral equation for twisting moment at a point	(2.140)
V_6	Effective shear force kernel in the integral equation for shear force at a point	(2.144)
w	Deflection in the direction normal to the plate surface	(2.16)
w_Q	Deflection at the field point Q	(2.84)
w_P or w_p	Deflection at the source point P or p	(2.84)
w_i	Weighting coefficients for Gaussian quadrature	(3.10)
$\partial w / \partial n_p$	Deflection derivative at point p in the n_p direction	(2.96)

<u>Symbol</u>	<u>Definition</u>	<u>Equation</u>
$\partial w / \partial n_p^+$	Normal slope at a corner point P in the n_p^+ direction	(2.102)
$\partial w / \partial t_p^+$	Tangential slope at a corner point P in the t_p^+ direction	(2.102)
$\partial w / \partial n_Q$	Normal slope at a boundary point Q	(2.98)
$\partial w / \partial h_{P(Q)}$	Deflection derivative with respect to h at point $P(Q)$	(2.103)
$\frac{\partial w}{\partial n}(p^a)$	Normal slope at node p^a	(4.11)
$\frac{\partial w}{\partial t}(p^a)$	Tangential slope at node p^a	(4.11)
w	Deflection kernel in the integral equation for deflection at a point	(2.68)
w_2	Deflection kernel in the integral equation for deflection gradient at a point	(2.96)
w_3	Deflection kernel in the integral equation for moment at a point	(2.125)
w_4	Deflection kernel in the integral equation for effective shear force at a point	(2.132)
w_5	Deflection kernel in the integral equation for twisting moment at a point	(2.140)
w_6	Deflection kernel in the integral equation for shear force at a point	
x	Cartesian coordinate in the plane of the plate	(2.1)
y	Cartesian coordinate in the plane of the plate	(2.2)
z	Cartesian coordinate perpendicular to the plane of the plate	(2.3)
α	Angle between the x axis and the normal direction at a field point Q on the boundary	(2.58)
α	Angle between the normals on either side of a corner (Section 4.5)	(4.12a)
α	Angle of skew for a rhombic plate (Section 5.17)	
β	Angle between the position vector and the normal direction at a field point Q on the boundary	(2.63)
γ_1	Angular distortion due to in-plane shear	(3.21)
γ_2	Angular distortion due to in-plane shear	(3.22)
γ_{xy}	Shear strain	

<u>Symbol</u>	<u>Definition</u>	<u>Equation</u>
$\delta(p,q)$	Dirac delta function	(2.72)
ϵ	Radius of the circular domain A_ϵ	(2.81)
ζ	Intrinsic coordinate along a boundary element varying from zero to one	(3.22)
ζ	Intrinsic coordinate for triangular area elements (Section 4.7.2.2)	(4.110)
ζ_i	Coordinates for logarithmic quadrature	(3.22)
η	Intrinsic coordinate for area elements	(3.52)
η_j	Coordinates for Gaussian quadrature	(3.49)
θ	Angle in polar coordinates	(2.62)
ν	Poisson's ratio	(2.11)
ξ	Intrinsic coordinate of a boundary or area element	(3.1)
ξ_i	Coordinates for Gaussian quadrature	(3.10)
σ_x	Direct stress in the x direction	(2.10)
σ_y	Direct stress in the y direction	(2.11)
τ_{xy}, τ_{yx}	Shear stress	(2.15)
ϕ	Angle of rotation of lines normal to the middle surface of the plate (Section 2.2.4)	(2.15)
ψ	Angle between the position vector of a field point Q on the boundary and an arbitrary direction at the source point p or P	(2.97a)
ψ^-	ψ on the negative side of a corner point P	(2.110a)
ψ^+	ψ on the positive side of a corner point P	(2.110b)
∇^2	Laplacian operator	(2.43)
∇^4	Biharmonic operator	(2.43)
$\in$	Belongs to	(4.93)
$\notin$	Does not belong to	(4.88)

CHAPTER 1

INTRODUCTION

The analysis of many engineering and scientific problems requires the solution of the appropriate governing differential equations with a prescribed set of boundary conditions. In elasticity, the problem is usually that of evaluating the stresses and displacements in an elastic body subjected to some form of loading. For many practical problems, the complex nature of the geometries and boundary conditions involved make an analytical solution to the differential equation impossible. Numerical solutions are frequently the best alternatives available.

The advent of the high speed digital computer has been a major factor in the emergence of many numerical techniques for engineering analysis. The finite difference method was one of the first methods which was widely used for the solution of differential equations encountered in many engineering and scientific disciplines. However, the difficulties involved in its application to problems with complex geometries and boundary conditions led to the more recently developed finite element method being preferred in many cases. The common feature of these two techniques is that they both require some form of discretisation of the problem domain. The final product of either method is a set of linear algebraic equations in terms of the values of the unknown function and, possibly, its derivatives at nodal or collocation points inside and on the boundary of the domain. In general, the accuracy of the solution obtained using these techniques increases as the domain discretisation is refined. This leads to a very large set of linear algebraic equations.

The finite element method has been by far the most popular numerical tool used by engineers in stress analysis problems in the last two decades. Its superiority lies in its ability to treat, with relative ease, problems

involving complex geometries and varying material properties. However, the method has some disadvantages. The preparation of a finite element mesh and other data can be a tedious and expensive process. As already mentioned, the discretisation involves the whole of the problem domain, resulting in a large system of linear algebraic equations, although the symmetric and banded nature of the associated coefficient matrix does help to reduce computational costs. In the context of plate bending problems, which is the subject of this thesis, many of the "finite plate elements" developed violate the continuity conditions in the plate domain along inter-element boundaries.

An alternative numerical technique developed more recently is the boundary integral equation (BIE) or boundary element method (BEM). This technique uses Green's integral identities to transform the differential equation of the particular problem, which is valid throughout the problem domain, into an integral equation over the boundary of the domain, thus reducing the dimensionality of the problem by one. This integral equation is a statement of Betti's reciprocal theorem related to the specific problem under consideration. In the context of plate bending problems, the combination of this and a suitable singular solution to the governing differential equation of plate bending leads to integral equations over the plate boundary connecting the boundary variables. These boundary integral equations are then solved numerically by discretising the boundary of the domain only, and setting up a system of linear algebraic equations in terms of the nodal variables on the boundary. The solution of these equations gives all the boundary variables, from which the value of any required variable can be obtained at selected points in the interior of the domain.

The BIE method has a number of advantages over domain type methods such as finite elements. The reduction in the dimensionality of the

problem leads to an equation over the boundary of the plate only. This means that any discretisation of the problem is carried out over the boundary and involves boundary variables only. This results in a smaller coefficient matrix associated with the final system of linear algebraic equations. Another important feature of the BIE method is that the solution throughout the domain is continuous since there is no interior discretisation of the domain involved. This enables the analyst to obtain values for required variables at points inside the domain with a high resolution. In most engineering problems, and in the specific case of plate bending problems, the analyst is usually interested in the value of certain variables at selected points in the problem domain, for example, deflection and moments at the centre of a square plate under a uniformly distributed load. In the finite element method, the solution process involves the computation of the unknown variables at all the interior nodes, thus providing a lot of information which is of no particular interest. However, in the BIE method, the required variables at interior points need only be computed for predetermined points in the interior of the domain.

The BIE method is not without its drawbacks. Although the coefficient matrix associated with the system of linear algebraic equations to be solved is usually smaller than those obtained in domain type techniques, it is not symmetric and is fully populated. Its application to problems involving varying material properties throughout the domain may be difficult and would require additional analytical and numerical work, whereas in the finite element method these types of problems are easily treated with little additional effort. Despite these disadvantages, there is still a very wide field of numerical stress analysis where the BIE method can be used efficiently, thus providing the analyst with an attractive alternative to other techniques.

The early applications of the BIE method were in the fields of potential theory and two- and three-dimensional elastostatics. The applicability of the method in these fields seems to be firmly established and numerous texts are available in these areas (see, for example, Banerjee & Butterfield [1979,1981], and Brebbia [1978a,1978b,1981]). However, there are relatively few publications on the application of the BIE method to problems of bending of thin plates under lateral loading.

This thesis presents a detailed description of the application of the boundary integral equation method to the solution of the problem of the bending of thin plates under lateral loading. The plate model is that attributed to Kirchhoff & Love. The governing differential equation of the Kirchhoff-Love plate theory is derived in the next chapter, followed by a detailed derivation of the integral equation formulation applied to plate bending problems. Chapter 3 presents a review of some current literature on the application of the BIE method to plate bending problems. This is followed by a constant element implementation aimed at assessing the accuracy of the numerical solution obtained using six possible pairs of integral equations. Chapter 4 presents a higher order formulation of the BIE method employing curved isoparametric quadratic elements. Chapters 5 and 6 assess the accuracy and versatility of the computer program based on the quadratic formulation by applying it to a variety of problems involving various geometric, loading and boundary conditions. The formulation is extended to include plates involving support conditions inside the domain, as well as the usual boundary conditions prescribed along the boundaries.

CHAPTER 2

ANALYTICAL FORMULATION

2.1 INTRODUCTORY REMARKS

In engineering applications, the term "flat plate" is applied to a three-dimensional body whose dimensions in one direction are small compared to its dimensions in the other two directions. A plate can be thought of as a compressed right prism whose depth or height, which is referred to as its thickness, t , is small compared to the dimensions of its base. The surface equidistant from the top and bottom surfaces is called the middle surface. When t is constant, the plate is said to be of uniform thickness. The plate geometry is, therefore, fully defined by the shape of its middle surface and its thickness at any point. The equations derived in this chapter are applicable to flat plates of uniform thickness and of homogeneous linearly elastic material under lateral loading.

2.2 DIFFERENTIAL EQUATION FOR THE BENDING OF PLATES UNDER LATERAL LOADING

2.2.1 Basic Assumptions

In the derivation of the differential equation of bending of plates under the action of lateral loads, certain simplifying assumptions are made. This leads to the "small deflection plate theory", generally attributed to Kirchhoff & Love, on which numerous texts exist (see, for example, Timoshenko & Woinowsky-Krieger [1959], and Szilard [1974]).

These assumptions are as follows:

- (i) The material of the plate is elastic, homogeneous and isotropic.
- (ii) The plate is initially flat.
- (iii) The thickness of the plate is small compared to its other

dimensions. The smallest lateral dimension of the plate is at least ten times larger than its thickness.

(iv) The deflections are small compared to the plate thickness.

A maximum deflection from one tenth to one fifth of the thickness is considered as the limit of small deflection theory (see Szilard [1974]).

(v) The slopes of the deflected middle surface are small compared to unity.

(vi) The deformations are such that straight lines, initially normal to the middle surface, remain straight and normal to the middle surface. This implies that the deformation due to transverse shear forces (shear forces across the thickness) is neglected.

(vii) The deflection of the plate is produced by the deflection of points of the middle surface normal to its initial plane.

(viii) The stresses normal to the middle surface are of a negligible order of magnitude.

(ix) The strains in the middle surface produced by in-plane forces can usually be neglected in comparison to strains due to bending.

2.2.2 Sign Convention for Generalised Forces and Deflections

The governing differential equation for the bending of plates under the action of lateral loading can conveniently be derived with respect to a Cartesian coordinate system. In the context of plate bending formulations, the term "generalised deflections" refers not only to deflections in coordinate directions but includes slopes as well. Similarly, moments and vertical forces are referred to as "generalised

forces". Furthermore, these forces are given as moments, or shear forces, per unit length along a particular direction. The external and internal forces and deflection components are considered positive when they point towards the positive directions of the coordinate axes. Moments are considered positive if they produce tension in the fibres located at the top surface of the plate. This sign convention is shown in Figure 2.1.

2.2.3 Equilibrium of a Differential Element of the Plate

Consider an elemental parallelepiped cut out of the plate, as shown in Figure 2.2, under an equilibrating set of internal and external forces. It must be noted that where double subscripts are used, the first subscript indicates the direction of the surface normal pertinent to the section on which the force acts. The quantities m_{xy} and m_{yx} are twisting moments.

For the plate element to be in equilibrium, the following equations must be satisfied:

$$\sum m_x = 0 \quad (2.1)$$

$$\sum m_y = 0 \quad (2.2)$$

$$\sum F_z = 0 \quad (2.3)$$

where $\sum m_x$ is the sum of moments about the y axis, $\sum m_y$ is the sum of the moments about the x axis, and $\sum F_z$ is the sum of all vertical forces. As mentioned in Section 2.2.2, in the theory of plates it is customary to deal with internal forces and moments per unit length of the middle surface. With this in mind, equations (2.1), (2.2) and (2.3) can be expanded as follows:

$$\begin{aligned}
 (m_x + \frac{\partial m_x}{\partial x} dx) dy - m_x dy + (m_{yx} + \frac{\partial m_{yx}}{\partial y} dy) dx - m_{yx} dx \\
 - Q_x dy \frac{dx}{2} - (Q_x + \frac{\partial Q_x}{\partial x} dx) dy \frac{dx}{2} = 0
 \end{aligned} \tag{2.4}$$

$$\begin{aligned}
 (m_y + \frac{\partial m_y}{\partial y} dy) dx - m_y dx + (m_{xy} + \frac{\partial m_{xy}}{\partial x} dx) dy - m_{xy} dy \\
 - Q_y dx \frac{dy}{2} - (Q_y + \frac{\partial Q_y}{\partial y} dy) dx \frac{dy}{2} = 0
 \end{aligned} \tag{2.5}$$

$$\begin{aligned}
 (Q_y + \frac{\partial Q_y}{\partial y} dy) dx - Q_y dx + (Q_x + \frac{\partial Q_x}{\partial x} dx) dy \\
 - Q_x dy + F_z dx dy = 0
 \end{aligned} \tag{2.6}$$

where Q_x and Q_y represent shear forces per unit length along lines parallel to the y and x directions, respectively, F_z is the intensity of the lateral loading, and m_{xy} and m_{yx} are twisting moments. After simplification and neglecting the terms containing $(dx)^2 dy$ and $(dy)^2 dx$, the following equations are obtained for the equilibrium of the differential plate element:

$$\frac{\partial m_x}{\partial x} + \frac{\partial m_{yx}}{\partial y} = Q_x \tag{2.7}$$

$$\frac{\partial m_y}{\partial y} + \frac{\partial m_{xy}}{\partial x} = Q_y \tag{2.8}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = - F_z \tag{2.9}$$

Substitution of equations (2.7) and (2.8) into equation (2.9), and observing that $m_{xy} = m_{yx}$ (see equations (2.31) and (2.32)), gives:

$$\frac{\partial^2 m_x}{\partial x^2} + \frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_y}{\partial y^2} = - F_z \quad (2.10)$$

2.2.4 Relation Between Stresses, Strains and Deflections

The assumption that the material of the plate is linearly elastic permits the use of the two-dimensional form of Hooke's law in the absence of thermal strains:

$$\sigma_x = E \epsilon_x + \nu \sigma_y \quad (2.11)$$

$$\sigma_y = E \epsilon_y + \nu \sigma_x \quad (2.12)$$

where E is Young's modulus, ν is Poisson's ratio, and σ_x , σ_y , ϵ_x and ϵ_y are direct stresses and strains in the x and y directions, respectively. Elimination of σ_y from equations (2.11) and (2.12) gives:

$$\sigma_x = \frac{E}{(1 - \nu^2)} (\epsilon_x + \nu \epsilon_y) \quad (2.13)$$

Similarly:
$$\sigma_y = \frac{E}{(1 - \nu^2)} (\epsilon_y + \nu \epsilon_x) \quad (2.14)$$

The twisting moments m_{xy} and m_{yx} produce in-plane shear stresses, τ_{xy} and τ_{yx} (see Figure 2.3), which are again related to the shear strain, γ_{xy} , by the pertinent Hookean relationship, giving:

$$\tau_{xy} = G \gamma_{xy} = \frac{E}{2(1 + \nu)} \gamma_{xy} = \tau_{yx} \quad (2.15)$$

where G is the shear modulus.

The strain deflection relationships are derived by

considering the geometry of the deflected plate before and after deformation. Taking a section at a constant x , as shown in Figure 2.4, and considering it before and after deformation, the angle of rotation of lines I-I and II-II can be expressed by:

$$\phi = - \frac{\partial w}{\partial y} \quad \text{for I-I} \quad (2.16)$$

and:
$$\phi + \frac{\partial \phi}{\partial y} dy \quad \text{for II-II} \quad (2.17)$$

respectively. After the deformation, the length AB of a fibre located at a distance z from the middle surface becomes $A'B'$ (Figure 2.4). The direct strain, ϵ_y , is given, by definition, as:

$$\epsilon_y = \frac{\Delta dy}{dy} = \frac{A'B' - AB}{AB} = \frac{[dy + z (\partial \phi / \partial y) dy] - dy}{dy} = z \frac{\partial \phi}{\partial y} \quad (2.18)$$

Substitution of equation (2.16) into equation (2.18) gives the direct strain as:

$$\epsilon_y = - z \frac{\partial^2 w}{\partial y^2} \quad (2.19)$$

In a similar manner, the direct strain in the x direction, ϵ_x , is obtained as:

$$\epsilon_x = - z \frac{\partial^2 w}{\partial x^2} \quad (2.20)$$

The angular distortion, γ_{xy} , can be determined by comparing a rectangular parallelogram, $ABCD$ (Figure 2.5), located at a constant distance z from the middle surface, with its deformed shape, $A'B'C'D'$, on the deflected plate surface. From Figure 2.5, it can be deduced that:

$$\gamma_{xy} = \frac{\partial u}{\partial y} \quad (2.21)$$

and:

$$\gamma_2 = \frac{\partial v}{\partial x} \quad (2.22)$$

where u and v are deflection components in the x and y directions, respectively, and γ_1 and γ_2 are angles, shown in Figure 2.5. From Figure 2.4:

$$v = z \phi = -z \frac{\partial w}{\partial y} \quad (2.23)$$

Similarly:
$$u = -z \frac{\partial w}{\partial x} \quad (2.24)$$

Consequently:

$$\gamma_{xy} = \gamma_1 + \gamma_2 = -z \frac{\partial^2 w}{\partial x \partial y} - z \frac{\partial^2 w}{\partial x \partial y} = -2z \frac{\partial^2 w}{\partial x \partial y} \quad (2.25)$$

The curvature changes of the deflected middle surface are now defined as:

$$K_x = - \frac{\partial^2 w}{\partial x^2} \quad (2.26)$$

$$K_y = - \frac{\partial^2 w}{\partial y^2} \quad (2.27)$$

$$K_{xy} = - \frac{\partial^2 w}{\partial x \partial y} \quad (2.28)$$

where K_{xy} represents the warping of the plate.

2.2.5 Internal Forces Expressed in Terms of the Lateral Deflections

The bending moments acting on the plate element can be obtained by integrating the moments of the direct stress components, σ_x and σ_y , about the middle surface, over the thickness, giving:

$$m_x = \int_{-t/2}^{t/2} \sigma_x z \, dz \quad (2.29)$$

$$m_y = \int_{-t/2}^{t/2} \sigma_y z \, dz \quad (2.30)$$

Similarly, the twisting moments produced by the shear stresses, τ_{xy} and τ_{yx} , can be written as:

$$m_{xy} = \int_{-t/2}^{t/2} \tau_{xy} z \, dz \quad (2.31)$$

$$m_{yx} = \int_{-t/2}^{t/2} \tau_{yx} z \, dz \quad (2.32)$$

and since $\tau_{xy} = \tau_{yx}$, $m_{xy} = m_{yx}$. Substitution of equations (2.19) and (2.20) into equations (2.11) and (2.12) gives the direct stresses, σ_x and σ_y , in terms of the lateral deflection as:

$$\sigma_x = - \frac{E z}{1 - \nu^2} \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \quad (2.33)$$

and:
$$\sigma_y = - \frac{E z}{1 - \nu^2} \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \quad (2.34)$$

Integration of equations (2.29) and (2.30) after the above substitutions for σ_x and σ_y yields:

$$m_x = - D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \quad (2.35)$$

and:
$$m_y = - D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \quad (2.36)$$

where:
$$D = \frac{E t^3}{12(1 - \nu^2)} \quad (2.37)$$

which is called the flexural rigidity of the plate. In a similar manner, the expression for twisting moment in terms of the lateral deflections is obtained:

$$m_{xy} = m_{yx} = \int_{-t/2}^{t/2} \tau_{xy} z dz = - 2G \int_{-t/2}^{t/2} \frac{\partial^2 w}{\partial x \partial y} z^2 dz \quad (2.38)$$

After integration:

$$m_{xy} = - (1 - \nu) D \frac{\partial^2 w}{\partial x \partial y} \quad (2.39)$$

Expressions for the transverse shear forces, Q_x and Q_y , in terms of the lateral deflections can be obtained by substituting equations (2.35), (2.36) and (2.38) into equations (2.7) and (2.8), which yields:

$$Q_x = - D \frac{\partial}{\partial x} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = - D \frac{\partial}{\partial x} \nabla^2 w \quad (2.40)$$

$$Q_y = - D \frac{\partial}{\partial y} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = - D \frac{\partial}{\partial y} \nabla^2 w \quad (2.41)$$

where $\nabla^2 = ((\partial/\partial x^2) + (\partial/\partial y^2))$ is the Laplacian operator in two dimensions.

2.2.6 The Differential Equation of the Plate

The substitution of equations (2.35), (2.36) and (2.39) into equation (2.10) yields the governing differential equation of the bending of plates subjected to lateral loading:

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{F}{D} \quad (2.42)$$

Using the two-dimensional Laplacian operator, ∇^2 , equation (2.41) can be written as:

$$\nabla^2 \nabla^2 w = \nabla^4 w = \frac{F}{D} \quad (2.43)$$

where w is the lateral deflection, D is the flexural rigidity as given by equation (2.37), and F is the intensity of the lateral loading. This equation is a fourth order, non-homogeneous, linear partial differential equation of the elliptic type with constant coefficients, often called a non-homogeneous biharmonic equation.

2.2.7 Boundary Conditions for the Bending Theory

An exact solution of the governing differential equation of the plate, equation (2.43), must simultaneously satisfy the differential equation and the boundary conditions of any given plate problem. The most common type of boundary conditions encountered in practice are clamped, simply-supported and free.

Referring to Figure 2.6, considering plate edges having normals in the x and y directions, respectively, the boundary conditions for a clamped edge can be written as:

$$w = \frac{\partial w}{\partial x} = 0 \quad \text{along } x = 0 \text{ and } x = a \quad (2.44)$$

$$\text{and: } w = \frac{\partial w}{\partial y} = 0 \quad \text{along } y = 0 \text{ and } y = b \quad (2.45)$$

The boundary conditions for a simply-supported edge can be expressed as:

$$w = m_x = 0 \quad \text{along } x = 0 \text{ and } x = a \quad (2.46)$$

$$w = m_y = 0 \quad \text{along } y = 0 \text{ and } y = b \quad (2.47)$$

Along a free edge, all edge forces must vanish. For instance, if the boundary $x = a$ is a free boundary, the pertinent boundary conditions would be:

$$m_x = 0 \quad (2.48)$$

$$m_{xy} = 0 \quad (2.49)$$

$$Q_x = 0 \quad (2.50)$$

However, the governing differential equation of the plate is a fourth order one and, therefore, requires the prescription of two boundary conditions only along each edge. This difficulty occurs as a result of the simplifying assumptions of Kirchhoff's plate theory. It can be overcome by defining an effective shear force, v_x , which is assumed to be equivalent to the combined action of the shear force, Q_x , and the twisting moment, m_{xy} (see Lloyd, Hamilton & Darnell [1976]). Therefore, the two boundary conditions represented by equations (2.49) and (2.50) are replaced by a single one which is assumed to involve the sum of all vertical forces along the edge. The free edge boundary conditions can now be expressed as:

$$m_x = -D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) = 0 \quad (2.51)$$

$$v_x = Q_x + \frac{\partial m_{xy}}{\partial y} = -D \left[\frac{\partial^3 w}{\partial x^3} + (2 - \nu) \frac{\partial^3 w}{\partial x \partial y^2} \right] = 0 \quad (2.52)$$

The term $\partial m_{xy} / \partial y$ represents additional shearing forces along the edge $x = a$ produced by the twisting moments, m_{xy} . Similarly, for a free edge parallel to the x axis, the pertinent boundary conditions would be:

$$m_y = -D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) = 0 \quad (2.53)$$

$$v_y = Q_y + \frac{\partial m_{yx}}{\partial x} = -D \left[\frac{\partial^3 w}{\partial y^3} + (2 - \nu) \frac{\partial^3 w}{\partial x^2 \partial y} \right] = 0 \quad (2.54)$$

2.2.8 Expressions for Moments and Shear Force at a General Boundary Point

Figure 2.7 depicts a portion of the plate. The plate is bounded by a plane region A with total boundary S . The boundary curve is assumed to be piecewise twice continuously differentiable. That is, the curvature is bounded and continuous, except at corner points. Along the boundary curve, the positive direction is chosen such that if the boundary curve is traversed in the positive direction, the plate domain, A , would always be on the left. The curvature at a smooth point on the boundary is denoted by $1/R$, where R is the radius of curvature at the boundary point; a negative curvature is one where the centre of curvature lies on the outward normal to the boundary.

At a regular point on the boundary, the normal moment, m_n , the twisting moment, m_t , and the effective shear force, v_n , can be written in the local n, t coordinates as:

$$m_n = -D \left(\frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial t^2} \right) \quad (2.55)$$

$$m_t = -D (1 - \nu) \frac{\partial^2 w}{\partial n \partial t} \quad (2.56)$$

$$v_n = -D \frac{\partial}{\partial n} \nabla^2 w + \frac{\partial}{\partial S} m_t \quad (2.57)$$

Using the coordinate transformation relationship:

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial x} \cos \alpha + \frac{\partial}{\partial y} \sin \alpha \quad (2.58)$$

$$\frac{\partial}{\partial t} = - \frac{\partial}{\partial x} \sin \alpha + \frac{\partial}{\partial y} \cos \alpha \quad (2.59)$$

the expressions for moments can be written as:

$$m_n = D \left\{ -\nabla^2 w + (1-\nu) \left[\frac{\partial^2 w}{\partial x^2} \sin^2 \alpha + \frac{\partial^2 w}{\partial y^2} \cos^2 \alpha - 2 \frac{\partial^2 w}{\partial x \partial y} \sin \alpha \cos \alpha \right] \right\} \quad (2.60)$$

$$m_t = -D (1-\nu) \left\{ \left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2} \right) \sin \alpha \cos \alpha + \frac{\partial^2 w}{\partial x \partial y} (\cos^2 \alpha - \sin^2 \alpha) \right\} \quad (2.61)$$

where α is the angle between the x axis and the normal, n , as indicated in Figure 2.7. It is seen that when α is such that n and t are parallel to the x and y axes, equations (2.60) and (2.61) reduce to the usual expressions for moments along lines parallel to the coordinate axes (equations (2.35), (2.36) and (2.39)).

It is also helpful for later use to obtain expressions for the governing differential equation of the plate, the moments and the effective shear force in plane polar coordinates (see Appendix A). In Figure 2.7, let P be the origin of polar coordinates (r, θ) . The governing differential equation of the plate, equation (2.43), can now be transformed and written in polar coordinates as:

$$\begin{aligned} & \frac{\partial^4 w}{\partial r^4} + \frac{2}{r} \frac{\partial^3 w}{\partial r^3} - \frac{1}{r^2} \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^3} \frac{\partial w}{\partial r} + \frac{4}{r^4} \frac{\partial^2 w}{\partial \theta^2} \\ & - \frac{2}{r^3} \frac{\partial^3 w}{\partial r \partial \theta^2} + \frac{2}{r^2} \frac{\partial^4 w}{\partial r^2 \partial \theta^2} + \frac{1}{r^4} \frac{\partial^4 w}{\partial \theta^4} = \frac{F}{D} \end{aligned} \quad (2.62)$$

The normal and twisting moments are given by:

$$m_n = \frac{\mathcal{D}}{2} \left\{ - (1 + \nu) \nabla^2 w + (1 - \nu) \left[\left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} - \frac{\partial^2 w}{\partial r^2} \right) \cos 2\beta \right. \right. \\ \left. \left. - 2 \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial w}{\partial \theta} \right) \sin 2\beta \right] \right\} \quad (2.63)$$

$$m_t = - \mathcal{D} \frac{(1 - \nu)}{2} \left\{ \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} - \frac{\partial^2 w}{\partial r^2} \right) \sin 2\beta + 2 \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial w}{\partial \theta} \right) \cos 2\beta \right\} \quad (2.64)$$

where β is the angle between the radial direction and the outer normal (Figure 2.7).

The effective shear force contains the term $\partial m_t / \partial S$. It can be shown (see Appendix B) that:

$$\frac{\partial \beta}{\partial S} = \frac{1}{R} - \frac{\cos \beta}{r} \quad (2.65)$$

Consequently, using equation (2.65), the expression for effective shear force (equation (2.57)) becomes:

$$v_n = \left[- \mathcal{D} \frac{\partial}{\partial r} \nabla^2 w + \frac{1}{r} \left(\frac{\partial}{\partial \theta} m_t - \frac{\partial}{\partial \beta} m_t \right) \right] \cos \beta \\ - \left[\frac{\mathcal{D}}{r} \frac{\partial}{\partial \theta} \nabla^2 w + \frac{\partial}{\partial r} m_t \right] \sin \beta + \frac{1}{R} \frac{\partial m_t}{\partial \beta} \quad (2.66)$$

where R is the curvature at the particular point on the boundary. Note that when differentiating m_t with respect to the arc length, S , it must be considered as a function of three variables, r , θ and β .

2.3 FORMULATION OF THE BOUNDARY INTEGRAL EQUATION METHOD

Basic to the formulation of the integral equation method presented in this work is the existence of:

- (i) A reciprocal relationship between two functions which are continuous and possess continuous derivatives up to the fourth order.
- (ii) A fundamental solution to the differential equation of the plate. This is a singular solution of the biharmonic equation with the Dirac delta function for a discrete inhomogeneity. This is equivalent to obtaining a solution to equation (2.43) for an infinitely large plate with F , a concentrated load.

2.3.1 Derivation of the Reciprocal Theorem

Consider a plate loaded by a transverse load of intensity $F = F(x, y)$ defined over the plate domain, A , giving rise to a deflection $w = w(x, y)$, as in Figure 2.7. The deflection, w , must satisfy the differential equation:

$$\nabla^4 w = \frac{F}{D} \quad \text{in } A \quad (2.67)$$

with given boundary conditions on the plate boundary, S . Let w be the deflection associated with some other particular loading on the plate with suitable boundary conditions on S . Consider the bilinear expression:

$$E(w, w) = \iint_A \left\{ \nabla^2 w \nabla^2 w - (1-\nu) \left(\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} \right) \right\} \quad (2.68)$$

Note that in equation (2.68), if W is replaced by w , $E(w, w)$ will represent twice the strain energy of the plate subject to the deflection field, w .

Integrating equation (2.68) by parts twice in W , and integrating the resulting tangential derivations on the boundary between corners, gives (see Stern [1979], and Bergman & Schiffer [1953]):

$$E(w, W) = \mathcal{D} \iint_A W \nabla^4 w \, dA + \int_S (v_n w - m_n \frac{\partial w}{\partial n}) \, dS + \sum_{\lambda=1}^k \{ [m_{\mathcal{L}}] w \}_{\lambda} \quad (2.69)$$

where m_n , $m_{\mathcal{L}}$ and v_n are defined by equations (2.60), (2.61) and (2.57), respectively; $[m_{\mathcal{L}}]$ denotes the discontinuity jump of the twisting moment at a corner in the direction of increasing arc length, referred to usually as the Kirchhoff corner reaction or corner force, and the summation is extended over all corners. A corner point is defined as a point on the boundary where the normal to the boundary is not continuous (see Figure 2.7). For a derivation of equation (2.69), see Dym & Sham [1973]. Interchanging w and W in equation (2.69) and noting the symmetry of $E(\cdot, \cdot)$ gives:

$$\begin{aligned} E(W, w) = E(w, W) = \mathcal{D} \iint_A w \nabla^4 W \, dA + \int_S (V_n w - M_n \frac{\partial w}{\partial n}) \, dS \\ + \sum_{\lambda=1}^k \{ [M_{\mathcal{L}}] w \}_{\lambda} \end{aligned} \quad (2.70)$$

where M_n , $M_{\mathcal{L}}$ and V_n are defined by equations (2.60), (2.61) and (2.57), respectively, except that w is replaced by W . Subtracting equation (2.69) from equation (2.70) and rearranging gives:

$$\begin{aligned} \int_S (V_n w - M_n \frac{\partial w}{\partial n} + \frac{\partial W}{\partial n} m_n - W v_n) \, dS = \mathcal{D} \iint_A (W \nabla^4 w - w \nabla^4 W) \, dA \\ + \sum_{\lambda=1}^k \{ [t] w - [\tau] W \}_{\lambda} \end{aligned} \quad (2.71)$$

where the change of notation, $m_x = x$ and $M_x = T$, has been made. Equation (2.71) relates the forces and deflections of two different states of loading on the plate. Close inspection of equation (2.71) reveals that it is, in fact, a statement of Betti's reciprocal theorem (see Stern [1979], Bezine & Gamby [1978], and Tottenham [1979]).

2.3.2 Fundamental Solution for the Biharmonic Equation

Let point p denote the origin of polar coordinates (r, θ) and q any other point in the plane. The fundamental solution for the biharmonic equation is defined as a function, $w(p, q)$, which satisfies the differential equation:

$$\nabla^4 w(p, q) = \delta(p, q) \quad (2.72)$$

where $\delta(p, q)$ is the Dirac delta function defined as:

$$\delta(p, q) = 0 \quad \text{when } p \neq q \quad (2.73a)$$

$$\delta(p, q) = \infty \quad \text{when } p = q \quad (2.73b)$$

$$\iint_A \delta(p, q) \, dA = 1 \quad (2.73c)$$

Physically, the fundamental solution can be interpreted as the solution to the differential equation of the plate under the action of a concentrated load at point p . The point p is referred to as the source point and the points q are called field points.

An immediate consequence of the definition of the Dirac delta function is that if $\sigma(q)$ is a continuous and bounded function in a two-dimensional Euclidean space, then:

$$\iint_A \sigma(q) \delta(p, q) dA(q) = \sigma_p \quad (2.74)$$

The notation $dA(q)$ indicates that the integration is with respect to the field point q , while p is kept constant; σ_p is the value of the function $\sigma(q)$ when p and q coincide. The fundamental solution for the governing differential equation of the plate is given by:

$$W(p, q) = \frac{1}{8\pi D} \kappa^2 \ln \kappa \quad (2.75)$$

where κ is the distance between p and q (see, for example, Forbes & Robinson [1969], and Hansen [1976]). It can be verified that equation (2.75) satisfies the differential equation (2.72) (see Appendix C).

2.3.3 Integral Equation for Deflection at Interior Points

Figure 2.8 depicts the plate as a domain A bounded by a piecewise smooth curve, S . Place the origin of polar coordinates (κ, θ) at a point p inside A . Let points on S be denoted by Q and points inside A , other than p , by q .

In equation (2.71), consider W to be the fundamental solution as defined by equation (2.74). It can be verified that $\nabla^4 W$ exhibits a singularity at point p . In view of this, it is necessary to exclude the singularity by deleting from A a circular region, A_ϵ , of radius ϵ , assumed to be small, and centre at p (see Figure 2.8). The limiting values of the integrals, in equation (2.71), which contain the singularity can then be examined as $\epsilon \rightarrow 0$.

Adopting the procedure outlined above, equation (2.71) can now be written as:

$$\begin{aligned}
 & \int_S \left\{ V_n w - M_n \frac{\partial w}{\partial n} + \frac{\partial w}{\partial n} m_n - w v_n \right\} dS(Q) \\
 & + \int_{S_\epsilon} \left\{ V_n w - M_n \frac{\partial w}{\partial n} + \frac{\partial w}{\partial n} m_n - w v_n \right\} dS(Q) \\
 & = \mathcal{D} \iint_{A-A_\epsilon} w \nabla^4 w \, dA(q) + \sum_{\lambda=1}^k \left\{ [t] w - [T] w \right\}_\lambda
 \end{aligned} \tag{2.76}$$

where S_ϵ is the boundary of the circular region A_ϵ . In equation (2.76), the functions w , $\partial w/\partial n$, M_n and V_n are referred to as the kernels of the integrals, and are functions of the positions of points p and q (or Q). However, w , $\partial w/\partial n$, m_n and v_n are functions of the positions of points q (or Q) only; $dS(Q)$ and $dA(q)$ denote integration with respect to the field points Q or q .

The kernels in equation (2.76) are obtained by substituting $w = 1/(\delta\pi\mathcal{D}) \kappa^2 \ln \kappa$ in the expressions for moments and effective shear force, noting that at a boundary point Q (see Appendix B):

$$\frac{\partial}{\partial n_Q} = \frac{\partial}{\partial \kappa} \cos \beta + \frac{1}{\kappa} \frac{\partial}{\partial \theta} \sin \beta \tag{2.77}$$

where $\partial/\partial n_Q$ is the normal derivative at point Q , and β is the angle between the radial direction and the outer normal at point Q , as shown in Figure 2.8. Using equation (2.77), the kernels can now be written as:

$$w = \frac{1}{\delta\pi\mathcal{D}} \kappa^2 \ln \kappa \tag{2.78a}$$

$$\frac{\partial w}{\partial n_Q} = \frac{\kappa}{\delta\pi\mathcal{D}} (2 \ln \kappa + 1) \cos \beta \tag{2.78b}$$

$$M_n = - \frac{1}{\delta\pi} \left\{ 2(1+\nu) (1 + \ln \kappa) + (1-\nu) \cos 2\beta \right\} \tag{2.78c}$$

$$V_n = - \frac{1}{8\pi \kappa} \{ (5 - \nu) \cos \beta + (1 - \nu) \cos 3\beta \} + \frac{1 - \nu}{4\pi R} \cos 2\beta \quad (2.78d)$$

The twisting moment is given by:

$$T = \frac{1 - \nu}{8\pi} \sin 2\beta \quad (2.78e)$$

Equations (2.78) have been presented by Stern [1979]; alternatively, Tottenham [1979] has suggested $W = 1/(16\pi D) (\kappa^2 \ln \kappa^2 - \kappa^2)$ as the fundamental solution for the biharmonic equations. Both are equally valid and satisfy equation (2.75).

The integrals over S and the area integral over $A-A_\epsilon$ do not contain any singularities and therefore need no special attention. The effective shear force kernel has an κ^{-1} singularity, and the integral over S_ϵ must be examined as $\epsilon \rightarrow 0$. Noting that along S_ϵ , $\beta = \pi$ and the curvature is given by $-\kappa^{-1}$, the kernels given by equations (2.78) become:

$$W = \frac{1}{8\pi D} \kappa^2 \ln \kappa \quad (2.79a)$$

$$\frac{\partial W}{\partial n_Q} = - \frac{\kappa}{8\pi D} (2 \ln \kappa + 1) \quad (2.79b)$$

$$M_n = - \frac{1}{8\pi} \{ 2(1 + \nu) (1 + \ln \kappa) + (1 - \nu) \} \quad (2.79c)$$

$$V_n = \frac{1}{2\pi \kappa} \quad (2.79d)$$

Substitution of equations (2.79) into the integral over S_ϵ in equation (2.76) and writing each integral separately gives:

$$\begin{aligned}
 \int_{S_\epsilon} (\cdot) dS(Q) &= \int_{S_\epsilon} \frac{1}{2\pi \kappa} w dS(Q) \\
 &- \int_{S_\epsilon} - \frac{1}{8\pi} \{2(1+\nu) (1 + \ln \kappa) + (1-\nu)\} \frac{\partial w}{\partial n_Q} dS(Q) \\
 &+ \int_{S_\epsilon} - \frac{\kappa}{8\pi \mathcal{D}} (2 \ln \kappa + 1) m_n dS(Q) \\
 &+ \int_{S_\epsilon} \frac{\kappa^2 \ln \kappa}{8\pi \mathcal{D}} \nu_n dS(Q)
 \end{aligned} \tag{2.80}$$

The last three integrals in equation (2.80) vanish as $\epsilon \rightarrow 0$ because, observing that $dS(Q) = \kappa d\theta$ and $\kappa = \epsilon$:

$$\lim_{\epsilon \rightarrow 0} \int_{S_\epsilon} - \frac{1}{8\pi} \{2(1+\nu) (1 + \ln \kappa) + (1-\nu)\} \frac{\partial w}{\partial n_Q} \kappa d\theta = 0 \tag{2.81}$$

$$\lim_{\epsilon \rightarrow 0} \int_{S_\epsilon} - \frac{\kappa}{8\pi \mathcal{D}} (2 \ln \kappa + 1) m_n \kappa d\theta = 0 \tag{2.82}$$

$$\lim_{\epsilon \rightarrow 0} \int_{S_\epsilon} \frac{\kappa^2 \ln \kappa}{8\pi \mathcal{D}} \nu_n \kappa d\theta = 0 \tag{2.83}$$

The first integral can be written as follows by noting that $w \equiv w(Q)$:

$$\begin{aligned}
 \int_{S_\epsilon} \frac{1}{2\pi \kappa} w_Q dS(Q) &= \int_{S_\epsilon} \frac{1}{2\pi \kappa} (w_Q - w_p + w_p) dS(Q) \\
 &= \int_{S_\epsilon} \frac{1}{2\pi \kappa} (w_Q - w_p) \kappa d\theta + \int_{S_\epsilon} \frac{1}{2\pi \kappa} w_p \kappa d\theta
 \end{aligned} \tag{2.84}$$

where w_p is the displacement at point p . Using the continuity condition imposed on $w = w(Q)$ (continuous and four times differentiable in A) gives:

$$\lim_{\epsilon \rightarrow 0} \int_{S_\epsilon} \frac{1}{2\pi \kappa} (w_Q - w_p) \kappa d\theta = 0 \tag{2.85}$$

In view of the above, equation (2.80) can now be written finally as:

$$\lim_{\epsilon \rightarrow 0} \int_{S_\epsilon} (\cdot) dS(Q) = \lim_{\epsilon \rightarrow 0} \int_0^{2\pi} \frac{1}{2\pi} \frac{1}{r} w_p r d\theta = w_p \quad (2.86)$$

Using equations (2.81) to (2.86) and rearranging gives equation (2.76) as:

$$\begin{aligned} w_p = & - \int_S (V_n w - M_n \frac{\partial w}{\partial n} + \frac{\partial w}{\partial n} m_n - w v_n) dS(Q) \\ & + \mathcal{D} \iint_A w \nabla^4 w dA(q) + \sum_{\lambda=1}^k \{ [\mathcal{L}] w - [\mathcal{T}] w \}_\lambda \end{aligned} \quad (2.87)$$

Equation (2.87) is an expression for the deflection w_p at a point p inside the plate domain A in terms of the boundary values of deflection w , normal slope $\partial w / \partial n$, normal moments m_n , effective shear force v_n , and twisting moment corner jumps $[\mathcal{L}]$, if the boundary has any corner points. For a smooth boundary, the term under the $\sum$ sign is omitted.

Directional derivatives, at point p , of equation (2.87) can be obtained for any direction making an angle ϕ with the x axis. Therefore, by obtaining appropriate directional derivatives of this equation, expressions for slope, moments and effective shear force at an interior point of the plate can be obtained, providing the integral involving $\nabla^4 w$ continues to converge (see Sections 2.3.5, 2.3.7, 2.3.9 and 2.3.11).

2.3.4 Integral Equation for Deflection at a Boundary Point

Consider now the point p described in the preceding section to be a boundary point P and, for generality, let P be a corner point with interior angle $K\pi$, as shown in Figure 2.9. Proceeding in a manner similar to that described in the previous section, a small area A_ϵ in the shape of a sector of a circle of radius ϵ and centre at P is excluded from A , as shown.

Equation (2.71), with w taken to be the fundamental solution for the biharmonic equation, can be written as:

$$\begin{aligned} & \int_S (V_n w - M_n \frac{\partial w}{\partial n} + \frac{\partial w}{\partial n} m_n - w v_n) dS(Q) \\ & + \int_{S_\epsilon} (V_n w - M_n \frac{\partial w}{\partial n} + \frac{\partial w}{\partial n} m_n - w v_n) dS(Q) \\ & = \iint_{A-A_\epsilon} w \nabla^4 w dA(Q) + \sum_{\lambda=1} \{ [t] w - [T] w \}_\lambda \end{aligned} \quad (2.88)$$

where all the symbols have the same definition as in equation (2.76). It must also be noted that in equation (2.88), the summation over the corners excludes the point P . However, two additional corners, C_1 and C_2 , have now been created and their contribution must be included. Due to the singular nature of some of the kernels in equation (2.88), the behaviour of the boundary integrals, as $\epsilon \rightarrow 0$, needs closer examination as was done in the preceding section. It is important to note that the principal aim here is not to evaluate these integrals but to show that, as $\epsilon \rightarrow 0$, they converge to finite limiting values. Their evaluation is carried out numerically and will be explained in Chapters 3 and 4.

Along the arc S_ϵ , the angle $\beta = \pi$ and the curvature is given by κ^{-1} . The kernels take the form given by equations (2.79). Using equations (2.81) to (2.86), and noting that in equation (2.86) the limits of the integration must now be changed to 0 and $K\pi$, gives:

$$\int_{S_\epsilon} (\cdot) dS(Q) = \int_0^{K\pi} \frac{1}{2\pi} w_p d\theta = \frac{1}{2\pi} w_p [\theta]_0^{K\pi} = \frac{1}{2} K w_p \quad (2.89)$$

To examine the convergence of the integral over S , the following observations are made along the arcs S^- and S^+ (see Figure 2.9):

$$\beta \approx \frac{\pi}{2} - \frac{\kappa}{2R^-} \quad \text{along } S^- \quad (2.90a)$$

$$\beta \approx \frac{3\pi}{2} + \frac{\kappa}{2R^+} \quad \text{along } S^+ \quad (2.90b)$$

$$\cos \beta \approx \frac{\kappa}{2R^-} \quad \text{along } S^- \quad (2.90c)$$

$$\cos \beta \approx \frac{\kappa}{2R^+} \quad \text{along } S^+ \quad (2.90d)$$

$$\sin \beta \approx 1 \quad \text{along } S^- \quad (2.90e)$$

$$\sin \beta \approx -1 \quad \text{along } S^+ \quad (2.90f)$$

$$\cos 2\beta \approx -1 \quad \text{along } S^- \text{ and } S^+ \quad (2.90g)$$

$$\sin 2\beta \approx \frac{\kappa}{R^-} \quad \text{along } S^- \quad (2.90h)$$

$$\sin 2\beta \approx \frac{-\kappa}{R^+} \quad \text{along } S^+ \quad (2.90i)$$

Referring to Figure 2.9(b) for a fixed ρ away from $\mathcal{P}$ and with $0 < \varepsilon < \kappa < \rho$, the kernels along S^- and S^+ in the integral over S become:

$$W = \frac{1}{8\pi \mathcal{D}} \kappa^2 \ln \kappa \quad (2.91a)$$

$$\frac{\partial W}{\partial n_Q} = \frac{\kappa}{8\pi \mathcal{D}} (2 \ln \kappa + 1) \frac{\kappa}{2R^\pm} \quad (2.91b)$$

$$M_n = -\frac{1}{8\pi} \{ 2(1+\nu) (1 + \ln \kappa) + (1-\nu) (-1) \} \quad (2.91c)$$

$$V_n = -\frac{1}{8\pi} \left\{ (5-\nu) \left(\frac{1}{2R^+} \right) + (1-\nu) \left(-\frac{1}{R^+} \right) \right\} + \frac{1-\nu}{4\pi R^\pm} (-1) \quad (2.91d)$$

The part of the boundary integral over S^- and S^+ can now be written as:

$$\lim_{\epsilon \rightarrow 0} \left\{ \int_{\epsilon}^{\rho} (V_n w - M_n \frac{\partial w}{\partial n} + \frac{\partial w}{\partial n} m_n - w v_n) dS^-(Q) \right. \\ \left. + \int_{\epsilon}^{\rho} (V_n w - M_n \frac{\partial w}{\partial n} + \frac{\partial w}{\partial n} m_n - w v_n) dS^+(Q) \right\} \quad (2.92)$$

Using equations (2.91), it can be verified that as $\epsilon \rightarrow 0$, the integrals in equation (2.92) converge to finite limiting values. The evaluation of these integrals, however, is carried out numerically and is described in the chapters on numerical implementation of the Boundary Integral Equation method. Finally, at the two corners C_1 and C_2 , the twisting moment jumps are given by:

$$[T] = (0 - \frac{(1-\nu)}{8\pi} \frac{\rho}{2R^-}) \quad \text{at } C_1 \quad (2.93a)$$

$$\text{and:} \quad [T] = (\frac{(1-\nu)}{8\pi} \frac{\rho}{2R^+} - 0) \quad \text{at } C_2 \quad (2.93b)$$

Therefore, as $\rho \rightarrow 0$, $[T] = 0$. Equation (2.88) can now be written as:

$$\frac{1}{2}K w_P + \int_S (V_n w - M_n \frac{\partial w}{\partial n} + \frac{\partial w}{\partial n} m_n - w v_n) dS(Q) \\ - \sum_{\lambda=1}^k \{ [t] w - [T] w \}_{\lambda} = \mathcal{D} \iint_A w \nabla^4 w dA \quad (2.94)$$

If point P is on a smooth part of the boundary, $K = 1$ in equation (2.94) and the coefficient for w_P will be $\frac{1}{2}$. Equation (2.94) provides an expression for the deflection at the boundary in terms of all the pertinent boundary variables.

2.3.5 Integral Equation for Deflection Derivatives at an Interior Point

As was mentioned in Section 2.3.3, directional derivatives of equation (2.87) can be obtained in any direction making an angle ϕ with the x axis. Referring to Figure 2.10, directional derivatives of equation (2.87) at point p in the direction n_p can be obtained on noting that:

- (i) Only the functions V_n , M_n , $\partial W/\partial n$, W and M_x are dependent on point p ; and
- (ii) If ψ is the angle between the position vector of point Q on the boundary and the direction n_p , as shown in Figure 2.10, then (see Appendix B):

$$\frac{\partial}{\partial n_p} = - \frac{\partial}{\partial \kappa} \cos \psi - \frac{1}{\kappa} \frac{\partial}{\partial \theta} \sin \psi \quad (2.95)$$

where $\partial/\partial n_p$ indicates directional differentiation at point p .

The directional derivative of w at point p inside A in the direction n_p can now be written as:

$$\begin{aligned} \frac{\partial w}{\partial n_p} = & - \int_S (V_2 w - M_2 \frac{\partial w}{\partial n_Q} + \frac{\partial w_2}{\partial n_Q} m_n - w_2 v_n) dS(Q) \\ & + \mathcal{D} \iint_A w_2 \nabla^4 w dA(q) + \sum_{\lambda=1}^k \{ [m_x] w_2 - [T_2] w \}_\lambda \end{aligned} \quad (2.96)$$

$$\text{where: } w_2 = \frac{\partial w}{\partial n_p} = - \frac{1}{8\pi \mathcal{D}} \{ 2\kappa \ln \kappa + \kappa \} \cos \psi \quad (2.97a)$$

$$\frac{\partial w_2}{\partial n_Q} = \frac{\partial^2 w}{\partial n_p \partial n_Q} = - \frac{1}{8\pi \mathcal{D}} \{ 2(\ln \kappa + 1) \cos (\psi - \beta) + \cos (\psi + \beta) \} \quad (2.97b)$$

$$M_2 = \frac{\partial M_n}{\partial n_p} = \frac{1}{8\pi \kappa} \{2(1+\nu) \cos \psi + (1-\nu) [\cos (\psi - 2\beta) - \cos (\psi + 2\beta)]\} \quad (2.97c)$$

$$\begin{aligned} \nu_2 = \frac{\partial \nu_n}{\partial n_p} = & - \frac{1}{8\pi \kappa^2} \{ (5-\nu) \cos (\psi + \beta) + 2(1-\nu) \cos (\psi + 3\beta) \\ & - (1-\nu) \cos (\psi - 3\beta) \} - \frac{1}{4\pi \kappa R} [\cos (\psi - 2\beta) - \cos (\psi + 2\beta)] \end{aligned} \quad (2.97d)$$

$$\tau_2 = \frac{\partial \tau}{\partial n_p} = \frac{1}{4\pi \kappa} (1 - \nu) \sin \psi \cos 2\beta \quad (2.97e)$$

where $\partial/\partial n_p$ denotes differentiation at point p . Equation (2.96) gives an expression for the derivative of the deflection, w , at an interior point of the plate domain, in terms of the pertinent boundary variables.

2.3.6 Integral Equation for the Normal Derivative of Deflection at the Boundary

In Section 2.3.5, the directional derivative of equation (2.87) was obtained by carrying out the differentiation under the integral sign. This is permissible because of the fact that for p inside A , the integrals occurring in equation (2.87) are all bounded and none of the kernels exhibit any singularity. However, the same procedure cannot be adopted for obtaining deflection derivatives at the boundary by differentiating equation (2.94). This is because of the singular behaviour of some of the kernels at point P and, in general, the derivative of singular integrals should not be obtained by carrying out the differentiation under the integral sign (see Bui [1978]). However, if the singularity is excluded from the integration domain, then differentiation can be carried out under the integral sign and the behaviour of the integral examined as the exclusion area shrinks to zero. A similar procedure is adopted here.

Consider, as in Section 2.3.4, the point P to be on the boundary, and for generality let P be a corner point with interior angle $K\pi$ (see Figure 2.11). In a manner similar to that described in Section 2.3.4, a small region A_ϵ is excluded from the plate domain, A . Since in the remaining domain, $A-A_\epsilon$, all the kernels are non-singular, directional derivatives of equation (2.76) in the n_p^+ or n_p^- directions can readily be obtained by carrying out the differentiation under the integral sign.

This gives:

$$\begin{aligned} & \int_S (V_2 w - M_2 \frac{\partial w}{\partial n_Q} + \frac{\partial W_2}{\partial n_Q} m_n - W_2 v_n) dS(Q) \\ & + \int_{S_\epsilon} (V_2 w - M_2 \frac{\partial w}{\partial n} + \frac{\partial W_2}{\partial n} m - W v_n) dS(Q) \\ & = \mathcal{D} \iint_{A-A_\epsilon} W_2 \nabla^4 w dA(Q) + \sum_{\lambda=1}^k \{ [t] W_2 - [T_2] w \}_\lambda \end{aligned} \quad (2.98)$$

where the kernels W_2 , V_2 , M_2 , $\partial W_2 / \partial n_Q$ and T_2 are as given by equations (2.97). Note that in equation (2.98) the summation over the corners does not include the self point P . However, the contributions from the corner points at C_1 and C_2 (see Figure 2.11) must be considered. Equation (2.98) contains integrals with singular kernels and the convergence of these integrals must be examined as $\epsilon \rightarrow 0$. Once again, as was mentioned in Section 2.3.4, it must be noted that the principal aim here is not to evaluate these integrals but to show that as $\epsilon \rightarrow 0$, they converge to finite limiting values.

As before, along the arc S_ϵ , the angle $\beta = \pi$ and the curvature is $-\kappa^{-1}$. The kernels along S_ϵ , therefore, become:

$$W_2 = - \frac{1}{8\pi \mathcal{D}} (2\kappa \ln \kappa + \kappa) \cos \psi \quad (2.99a)$$

$$\frac{\partial w_2}{\partial n_Q} = \frac{1}{8\pi \mathcal{D}} \{2(\ln \kappa + 1) \cos \psi + \cos \psi\} \quad (2.99b)$$

$$M_2 = \frac{1+\nu}{4\pi \kappa} \cos \psi \quad (2.99c)$$

$$V_2 = \frac{3-\nu}{4\pi \kappa^2} \cos \psi \quad (2.99d)$$

On noting that along S_ϵ , $dS(Q) = \kappa d\theta$:

$$\lim_{\epsilon \rightarrow 0} \int_{S_\epsilon} \frac{1}{8\pi \mathcal{D}} \{2(\ln \kappa + 1) \cos \psi + \cos \psi\} \kappa d\theta = 0 \quad (2.100)$$

$$\text{and: } \lim_{\epsilon \rightarrow 0} \int_{S_\epsilon} -\frac{1}{8\pi \mathcal{D}} (2\kappa \ln \kappa + \kappa) \cos \psi \kappa d\theta = 0 \quad (2.101)$$

The shear kernel, V_2 , contains an κ^{-2} singularity and the integral containing this kernel is divergent. To remedy the situation, $w \equiv w(Q)$ is replaced by $(w_Q - w_P)$ (see Forbes & Robinson [1969], and Stern [1979]). This is equivalent to subtracting a rigid body translation, w_P , from the deflection field, $w \equiv w(Q)$, and does not affect the slopes and stresses in the plate. Expanding $(w_Q - w_P)$ using a Taylor series gives:

$$w_Q - w_P = \kappa \left(\frac{\partial w}{\partial n_P^+} \cos \psi - \frac{\partial w}{\partial t_P^+} \sin \psi \right) + O(\kappa^2) \quad (2.102)$$

Noting that along the arc S_ϵ , $\partial/\partial n_Q = -\partial/\partial \kappa$, the first two integrals along S_ϵ can now be written as:

$$\begin{aligned}
 \int_{S_\epsilon} (\cdot) dS(Q) &= \int_{S_\epsilon} \left(\frac{3-\nu}{4\pi h^2} \cos \psi (w_Q - w_P) + \frac{1+\nu}{4\pi h} \cos \psi \frac{\partial w}{\partial r_Q} \right) dS(Q) \\
 &= \int_{S_\epsilon} \left\{ \frac{3-\nu}{4\pi h} \cos \psi \left(-\frac{\partial w}{\partial n_P^+} \cos \psi - \frac{\partial w}{\partial t_P^+} \sin \psi \right) \right. \\
 &\quad \left. + \frac{1+\nu}{4\pi h} \cos \psi \left(-\frac{\partial w}{\partial r_Q} - \frac{\partial w}{\partial r_P} + \frac{\partial w}{\partial r_P} \right) \right\} dS(Q)
 \end{aligned} \tag{2.103}$$

Due to the continuity conditions imposed on the first derivatives of the deflection, the integral containing $(\partial w / \partial r_Q) - (\partial w / \partial r_P)$ goes to zero as $\epsilon \rightarrow 0$. Equation (2.103), therefore, reduces to:

$$\begin{aligned}
 \int_{S_\epsilon} (\cdot) dS(Q) &= \int_{S_\epsilon} \left\{ \frac{3-\nu}{4\pi h} \cos \psi \left(-\frac{\partial w}{\partial n_P^+} \cos \psi - \frac{\partial w}{\partial t_P^+} \sin \psi \right) \right. \\
 &\quad \left. + \frac{1+\nu}{4\pi h} \cos \psi \left(-\frac{\partial w}{\partial n_P^+} \cos \psi - \frac{\partial w}{\partial t_P^+} \sin \psi \right) \right\} dS(Q)
 \end{aligned} \tag{2.104}$$

which reduces to:

$$\begin{aligned}
 \int_{S_\epsilon} (\cdot) dS(Q) &= \int_{S_\epsilon} \left\{ \frac{1}{4\pi h} \frac{\partial w}{\partial n_P^+} (4 \cos^2 \psi) \right. \\
 &\quad \left. - \frac{1}{4\pi h} \frac{\partial w}{\partial t_P^+} (-4 \sin \psi \cos \psi) \right\} dS(Q)
 \end{aligned} \tag{2.105}$$

Finally, substituting $h d\theta$ for $dS(Q)$ in equation (2.105) gives:

$$\int_{S_\epsilon} (\cdot) dS(Q) = \int_{S_\epsilon} \left\{ \frac{1}{\pi} \frac{\partial w}{\partial n_P^+} \left(\frac{1 + \cos^2 \psi}{2} \right) + \frac{1}{\pi} \frac{\partial w}{\partial t_P^+} \frac{\sin 2\psi}{2} \right\} d\theta \tag{2.106}$$

As can be seen, the integral over S_ϵ is no longer singular and can be written as:

$$\int_{S_\epsilon} (\cdot) dS(Q) = C_n \frac{\partial w}{\partial n_P^+} + C_t \frac{\partial w}{\partial t_P^+} \tag{2.107}$$

The evaluation of the coefficients C_n and C_t is carried out numerically and is described in subsequent chapters. For the case when P is a point on a smooth part of the boundary, equation (2.106) gives (see Figure 2.12):

$$\begin{aligned} \int_{S_\epsilon} (\cdot) dS(Q) &= \frac{1}{2\pi} \frac{\partial w}{\partial n_P^+} \int_0^\pi d\theta + \frac{1}{2\pi} \frac{\partial w}{\partial n_P^+} \int_0^\pi \cos 2\psi d\theta \\ &+ \frac{1}{2\pi} \frac{\partial w}{\partial t_P^+} \int \sin 2\psi d\theta \end{aligned} \quad (2.108)$$

The second and third integrals in equation (2.108) vanish, leaving:

$$\int_{S_\epsilon} (\cdot) dS(Q) = \frac{1}{2} \frac{\partial w}{\partial n_P^+} \quad (2.109)$$

To examine the convergence of the integrals over S^- and S^+ , an approach similar to that adopted in Section 2.3.4 is employed. The only integrals which need to be examined are the ones containing the moment and the effective shear force kernels. The deflection and slope kernels behave like $\kappa \ln \kappa$ and $\ln \kappa$, respectively, and their integrals are convergent. Therefore, using equations (2.90) for a fixed ρ away from P , with $0 < \epsilon < \kappa < \rho$, the moment and effective shear force kernels along S^- and S^+ become:

$$M_2 = \frac{1+\nu}{4\pi \kappa} \cos \psi^\pm \mp \frac{1-\nu}{8\pi} \frac{\sin \psi^\pm}{R^\pm} \quad (2.110a)$$

$$V_2 = -\frac{1+\nu}{4R_\pm^2 \kappa} \cos \psi^\pm \mp \frac{1+\nu}{4\pi \kappa^2} \sin \psi^\pm \pm \frac{2(1-\nu)}{16\pi R_\pm^2} \sin \psi^\pm \pm \frac{1-\nu}{4\pi R_\pm^2} \quad (2.110b)$$

By replacing $w = w(Q)$ by $(w_Q - w_P)$, as shown before, and using equations (2.102), it can be seen that the only remaining singular terms are $((1+\nu)/(4\pi \kappa^2)) \sin \psi^\pm$ in the effective shear force kernel and $((1+\nu)/(4\pi \kappa)) \cos \psi^\pm$ in the moment kernel. Therefore, the integral to

be considered is:

$$\begin{aligned}
 I = & \int_{S^-} + \frac{1+\nu}{4\pi \kappa^2} \sin \psi^- (w_Q - w_P) dS(Q) - \int_{S^-} \frac{1+\nu}{4\pi \kappa} \cos \psi^- \frac{\partial w}{\partial n_Q} dS(Q) \\
 & + \int_{S^+} - \frac{1+\nu}{4\pi \kappa^2} \sin \psi^+ (w_Q - w_P) dS(Q) - \int_{S^+} \frac{1+\nu}{4\pi \kappa} \cos \psi^+ \frac{\partial w}{\partial n_Q} dS(Q) \quad (2.111)
 \end{aligned}$$

Using equation (2.102), this gives:

$$\begin{aligned}
 I = & \int_{S^-} \frac{1+\nu}{4\pi \kappa} \sin \psi^- \left(-\frac{\partial w}{\partial t_P^-}\right) dS(Q) - \int_{S^-} \frac{1+\nu}{4\pi \kappa} \cos \psi^- \frac{\partial w}{\partial n_Q} dS(Q) \\
 & + \int_{S^+} - \frac{1+\nu}{4\pi \kappa} \sin \psi^+ \frac{\partial w}{\partial t_P^+} dS(Q) - \int_{S^+} \frac{1+\nu}{4\pi \kappa} \cos \psi^+ \frac{\partial w}{\partial n_Q} dS(Q) \quad (2.112)
 \end{aligned}$$

where $\partial w / \partial t_P^-$ is the directional derivative of the deflection at point P in the tangential direction t_P^- on S^- . Along S^- and S^+ , the following observations can be made:

$$\psi^+ \approx \frac{3\pi}{2} - \frac{\kappa}{2R_+} \quad (2.113a)$$

$$\psi^- \approx \psi^+ - K\pi \quad (2.113b)$$

$$\sin \psi^+ \approx -1 \quad (2.113c)$$

$$\cos \psi^+ \approx \frac{\kappa}{2R_+} \quad (2.113d)$$

$$\sin \psi^- \approx -\cos K\pi \quad (2.113e)$$

$$\cos \psi^- \approx -\sin K\pi \quad (2.113f)$$

Equation (2.112) can now be written as:

$$\begin{aligned}
 I = & \int_{S^-} -\frac{1+\nu}{4\pi h} \cos K\pi \left(-\frac{\partial w}{\partial x_p^-}\right) dS(Q) - \int_{S^-} -\frac{1+\nu}{4\pi h} \sin K\pi \frac{\partial w}{\partial n_Q} dS(Q) \\
 & + \int_{S^+} \frac{1+\nu}{4\pi h} \frac{\partial w}{\partial x_p^+} dS(Q) - \int_{S^+} \frac{1+\nu}{4\pi h} \frac{h}{2R_+} \frac{\partial w}{\partial n_Q} dS(Q)
 \end{aligned} \quad (2.114)$$

The last integral in equation (2.114) is no longer singular. Thus, considering the first three integrals only gives:

$$\begin{aligned}
 I = & \int_{S^-} -\frac{1+\nu}{4\pi h} \cos K\pi \left(-\frac{\partial w}{\partial x_p^-}\right) dS(Q) - \int_{S^-} \frac{1+\nu}{4\pi h} \sin K\pi \left(\frac{\partial w}{\partial n_Q} - \frac{\partial w}{\partial n_p^-}\right) dS(Q) \\
 & - \int_{S^-} -\frac{1+\nu}{4\pi h} \sin K\pi \frac{\partial w}{\partial n_p^-} dS(Q) + \int_{S^+} \frac{1+\nu}{4\pi h} \frac{\partial w}{\partial x_p^+} dS(Q)
 \end{aligned} \quad (2.115)$$

Due to the continuity conditions imposed on the first derivatives of

deflections, the integral containing $(\partial w/\partial n_Q) - (\partial w/\partial n_p^-)$ is finite.

Expressing $\partial w/\partial x_p^-$ and $\partial w/\partial n_p^-$ in terms of $\partial w/\partial x_p^+$ and $\partial w/\partial n_p^+$, the remaining integrals in equation (2.115) can be written as:

$$\begin{aligned}
 I = & \int_{S^-} \frac{1+\nu}{4\pi h} \cos K\pi \left(\frac{\partial w}{\partial n_p^+} \sin K\pi - \frac{\partial w}{\partial x_p^+} \cos K\pi\right) dS(Q) \\
 & + \int_{S^-} \frac{1+\nu}{4\pi h} \sin K\pi \left(-\frac{\partial w}{\partial n_p^+} \cos K\pi - \frac{\partial w}{\partial x_p^+} \sin K\pi\right) dS(Q) \\
 & + \int_{S^+} \frac{1+\nu}{4\pi h} \frac{\partial w}{\partial x_p^+} dS(Q)
 \end{aligned} \quad (2.116)$$

Simplifying:

$$\begin{aligned}
 I = & \int_{S^-} \frac{1+\nu}{4\pi h} \left[\frac{\partial w}{\partial n_p^+} (\sin K\pi \cos K\pi - \sin K\pi \cos K\pi) \right. \\
 & \left. - \frac{\partial w}{\partial x_p^+} (\cos^2 K\pi + \sin^2 K\pi) \right] dS(Q) + \int_{S^+} \frac{1+\nu}{4\pi h} \frac{\partial w}{\partial x_p^+} dS(Q)
 \end{aligned} \quad (2.117)$$

Therefore:

$$I = \int_{S^-} - \frac{1+\nu}{4\pi\kappa} \frac{\partial w}{\partial t_p^+} dS(Q) + \int_{S^+} \frac{1+\nu}{4\pi\kappa} \frac{\partial w}{\partial t_p^+} \quad (2.118)$$

which is a Cauchy principal value integral and, hence, the integral over S converges as $\varepsilon \rightarrow 0$. Finally, using equations (2.113), the twisting moment jumps at C_1 and C_2 are given by:

$$[T_2]_{C_1} = \frac{1}{4\pi\kappa} (1-\nu) \sin \psi^- [(1) - (-1)] = - \frac{2(1-\nu)}{4\pi\kappa} \cos K\pi \quad (2.119)$$

$$\text{and: } [T_2]_{C_2} = \frac{1-\nu}{4\pi\kappa} \sin \psi^+ [(-1) - (1)] = + \frac{2(1-\nu)}{4\pi\kappa} \quad (2.120)$$

Thus, replacing $w = w(Q)$ by $(w_Q - w_P)$ and using equations (2.102) and (2.113), the contributions from the twisting moment jumps at C_1 and C_2 are given by:

$$[T_2]_{C_1} (w_Q - w_P) = - \frac{1-\nu}{2\pi} \cos K\pi \left(- \frac{\partial w}{\partial n_p^+} \sin K\pi + \frac{\partial w}{\partial t_p^+} \cos K\pi \right) \quad (2.121)$$

$$[T_2]_{C_2} (w_Q - w_P) = \frac{1-\nu}{2\pi\kappa} \left(\kappa \frac{\partial w}{\partial t_p^+} \right) = \frac{1-\nu}{2\pi} \left(\frac{\partial w}{\partial t_p^+} \right) \quad (2.122)$$

Therefore, in general, non-zero contributions from the twisting moment corner jumps are computed which appear as coefficients for $\partial w / \partial t_p^+$ and $\partial w / \partial n_p^+$ and must be added to C_n and C_t in equation (2.107). In subsequent chapters, it is shown how the final coefficients for $\partial w / \partial n_p^+$ and $\partial w / \partial t_p^+$ are obtained numerically. For a point on a smooth part of the boundary, the sum of the contributions from C_1 and C_2 vanish. Equation (2.98) in its final form can be written as:

$$\begin{aligned}
 C_n \frac{\partial w}{\partial n_p} + C_t \frac{\partial w}{\partial t_p} + \int_S [V_2 (w - w_p) - M_2 \frac{\partial w}{\partial n} + \frac{\partial w_2}{\partial n} m - w_2 v_n] dS(Q) \\
 + \sum_{\lambda=1}^k \{ [T_2] (w - w_p) - [t] w_2 \}_\lambda = \mathcal{D} \iint_A w_2 \nabla^4 w dA(Q) \quad (2.123)
 \end{aligned}$$

A similar equation is obtained by forming the directional derivative of equation (2.76) in the n^- direction.

2.3.7 Integral Equation for Normal Moment at an Interior Point

Defining any two perpendicular directions, n_p and t_p , at an interior point, p , of the domain, A , of the plate, the normal moment, m_p , is given by:

$$m_p = - \mathcal{D} \left(\frac{\partial^2 w}{\partial n_p^2} + \nu \frac{\partial^2 w}{\partial t_p^2} \right) \quad (2.124)$$

Therefore, differentiating equation (2.87) accordingly gives:

$$\begin{aligned}
 m_p = - \int_S (V_3 w - M_3 \frac{\partial w}{\partial n} + \frac{\partial w_3}{\partial n} m - w_3 v_n) dS(Q) \\
 - \sum_{\lambda=1}^k \{ [T_3] w - [t] w_3 \}_\lambda + \mathcal{D} \iint_A w_3 \nabla^4 w dA \quad (2.125)
 \end{aligned}$$

where the kernels are given by:

$$w_3 = - \frac{1}{8\pi} \{ (1 + \nu) (1 + \ln \kappa^2) + (1 - \nu) \cos 2\psi \} \quad (2.126a)$$

$$\frac{\partial w_3}{\partial n} = - \frac{1}{8\pi \kappa} \{ 2(1 + \nu) \cos \beta + (1 - \nu) [\cos (2\psi - \beta) - \cos (2\psi + \beta)] \} \quad (2.126b)$$

$$M_3 = - \frac{\mathcal{D}}{4\pi \kappa^2} \{ (1 - \nu^2) (\cos 2\psi + \cos 2\beta) - (1 - \nu)^2 \cos (2\psi + 2\beta) \} \quad (2.126c)$$

$$V_3 = \frac{\mathcal{D}(1-\nu)}{4\pi \kappa^3} \{-2(1+\nu) \cos 3\beta + (5-\nu) \cos (2\psi+\beta) + 3(1-\nu) \cos (2\psi+3\beta)\} \\ + \frac{\mathcal{D}(1-\nu)}{2\pi R \kappa^2} \{(1+\nu) \cos 2\beta - (1-\nu) \cos (2\psi+2\beta)\} \quad (2.126d)$$

$$T_3 = -\frac{\mathcal{D}(1-\nu)}{4\pi \kappa^2} \{(1-\nu) \sin (2\psi+2\beta) - (1+\nu) \sin 2\beta\} \quad (2.126e)$$

where the angles ψ and β have their previous meaning.

2.3.8 Integral Equation for Normal Moment at a Boundary Point

An integral equation for the normal moment at a boundary point can be obtained by adopting a similar limiting process as that described in Section 2.3.6. Referring to Figure 2.11 and differentiating equation (2.76) according to equation (2.124) gives:

$$\int_{S_\epsilon} (V_3 w - M_3 \frac{\partial w}{\partial n} + \frac{\partial w_3}{\partial n} m_n - w_3 v_n) dS(Q) \\ + \int_{S-S_\epsilon} (V_3 w - M_3 \frac{\partial w}{\partial n} + \frac{\partial w_3}{\partial n} m_n - w_3 v_n) dS(Q) \\ = \iint_{A-A_\epsilon} w_3 \nabla^4 w dA(Q) + \sum \{ [t] w_3 - [T_3] w_Q \} \quad (2.127)$$

where the kernels w_3 , $\partial w_3/\partial n$, M_3 , V_3 and T_3 are given by equations (2.126). Once again, note that if P is a corner point, its contribution is excluded from the summation over the corners. However, the contribution from the corner points C_1 and C_2 must be considered. Equation (2.127) contains singular integrals and the convergence of these integrals as $\epsilon \rightarrow 0$ needs closer examination. This can be done by expanding $w = w(Q)$, about the point P using Taylor series, as:

$$w_Q = w_P + n^+ \frac{\partial w}{\partial n_P^+} + t^+ \frac{\partial w}{\partial t_P^+} + \frac{1}{2} (n^{+2} \frac{\partial^2 w}{\partial n_P^{+2}} + 2n^+ t^+ \frac{\partial^2 w}{\partial n_P^+ \partial t_P^+} + t^{+2} \frac{\partial^2 w}{\partial t_P^{+2}}) + O(\kappa^3) \quad (2.128)$$

Proceeding in a manner similar to that described in Section 2.3.6, $w = w(Q)$ is now replaced by:

$$w = w_Q - (w_P + n^+ \frac{\partial w}{\partial n_P^+} + t^+ \frac{\partial w}{\partial t_P^+}) = \frac{1}{2} (n^{+2} \frac{\partial^2 w}{\partial n_P^{+2}} + 2n^+ t^+ \frac{\partial^2 w}{\partial n_P^+ \partial t_P^+} + t^{+2} \frac{\partial^2 w}{\partial t_P^{+2}}) \quad (2.129)$$

and upon noting that $n^+ = \kappa \cos \psi$ and $t^+ = -\kappa \sin \psi$, it can be verified that all the integrals in equation (2.127) become κ^{-1} singular and, therefore, can be evaluated as Cauchy principal value integrals. However, the explicit evaluation of these integrals is very tedious and, for the purpose of obtaining numerical solutions using equation (2.127), unnecessary. In the next chapter, it is shown how the contribution from these integrals is computed when numerical solutions, using equation (2.127), are obtained. Equation (2.127) for a point P on the boundary is, therefore, written as:

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \left\{ \int_{S_\epsilon} (V_3 w - M_3 \frac{\partial w}{\partial n} + \frac{\partial w_3}{\partial n} m_n - w_3 v_n) dS(Q) \right. \\ & \quad \left. + \int_{S-S_\epsilon} (V_3 w - M_3 \frac{\partial w}{\partial n} + \frac{\partial w_3}{\partial n} m_n - w_3 v_n) dS(Q) \right\} \\ & = D \iint_A w_3 v^4 w dA(Q) + \sum_{\lambda=1}^k \{ [t] w_3 - [T_3] w \} \end{aligned} \quad (2.130)$$

Note that, at a corner point, two such equations can be written, since there are two sets of normals and tangents at a corner.

2.3.9 Integral Equation for Effective Shear Force at an Interior Point

The effective shear force at a point, p , inside A is given by:

$$v_p = -D \left(\frac{\partial^3 w}{\partial n_p^3} + (2-\nu) \frac{\partial^3 w}{\partial n_p \partial t_p^2} \right) \quad (2.131)$$

Once again, differentiating equation (2.87) accordingly gives:

$$\begin{aligned} v_p = & - \int_S \left(V_4 w - M_4 \frac{\partial w}{\partial n} + \frac{\partial w_4}{\partial n} m_n - w_4 v_n \right) dS(Q) \\ & - \sum_{\lambda=1}^k \{ [T_4] w - [t] w_4 \}_\lambda + D \iint w_4 \nabla^4 w dA(Q) \end{aligned} \quad (2.132)$$

where:

$$w_4 = \frac{1}{8\pi \kappa} \{ (5-\nu) \cos \psi + (1-\nu) \cos 3\psi \} - \frac{1-\nu}{4\pi R_p} \cos 2\psi \quad (2.133a)$$

$$\begin{aligned} \frac{\partial w_4}{\partial n} = & - \frac{1}{8\pi \kappa^2} \{ (5-\nu) \cos (\psi+\beta) + 2(1-\nu) \cos (3\psi+\beta) - (1-\nu) \cos (3\psi-\beta) \} \\ & - \frac{1-\nu}{4\pi \kappa R_p} \{ \cos (2\psi-\beta) - \cos (2\psi+\beta) \} \end{aligned} \quad (2.133b)$$

$$\begin{aligned} M_4 = & \frac{D(1-\nu)}{4\pi \kappa^3} \{ 2(1+\nu) \cos 3\psi - (5-\nu) \cos (\psi+2\beta) - 3(1-\nu) \cos (3\psi+2\beta) \} \\ & - \frac{D(1-\nu)}{4\pi \kappa^2 R_p} \{ 2(1+\nu) \cos 2\psi - 2(1-\nu) \cos (2\psi+2\beta) \} \end{aligned} \quad (2.133c)$$

$$\begin{aligned} V_4 = & - \frac{3D(1-\nu)}{4\pi \kappa^4} \{ (5-\nu) [\cos (\psi+3\beta) + \cos (3\psi+\beta)] + 4(1-\nu) \cos (3\psi+3\beta) \} \\ & + \frac{D(1-\nu)}{2\pi R_Q \kappa^3} \{ (5-\nu) \cos (\psi+2\beta) + 3(1-\nu) \cos (3\psi+2\beta) \} \end{aligned} \quad (2.133d)$$

$$T_4 = - \frac{D(1-\nu)}{4\pi \kappa^3} \{ -3(1-\nu) \sin (3\psi+2\beta) - (5-\nu) \sin (\psi+2\beta) \} \quad (2.133e)$$

2.3.10 Integral Equation for Effective Shear Force at a Boundary Point

An integral equation for the effective shear force at a point on the boundary can also be obtained by differentiating equation (2.76), giving:

$$\begin{aligned} & \int_{S_E} (V_4 w - M_4 \frac{\partial w}{\partial n} + \frac{\partial w_4}{\partial n} m_n - w_4 v_n) dS(Q) \\ & + \int_{S-S_E} (V_4 w - M_4 \frac{\partial w}{\partial n} + \frac{\partial w_4}{\partial n} m_n - w_4 v_n) dS(Q) \\ & = D \iint_{A-A_E} w_4 \nabla^4 w dA(q) + \sum_{\lambda=1}^k \{ [t] w_4 - [T_4] w \}_\lambda \end{aligned} \quad (2.134)$$

Referring to equations (2.57), it is seen that the effective shear expression involves differentiation with respect to the arc length, S , along the boundary. Therefore, by noting equation (2.65), it is seen that the kernels in equation (2.134) are slightly different from those given by equations (2.133), due to the presence of the curvature term, $1/R_p$, at point P on the boundary; in this case:

$$w_4 = \frac{1}{8\pi \kappa} \{ (5-\nu) \cos \psi + (1-\nu) \cos 3\psi \} - \frac{1-\nu}{4\pi R_p} \cos 2\psi \quad (2.135a)$$

$$\begin{aligned} \frac{\partial w_4}{\partial n} &= -\frac{1}{8\pi \kappa} \{ (5-\nu) \cos (\psi+\beta) + 2(1-\nu) \cos (3\psi+\beta) - (1-\nu) \cos (3\psi-\beta) \} \\ &\quad - \frac{1-\nu}{4\pi \kappa R_p} \{ \cos (2\psi-\beta) - \cos (2\psi+\beta) \} \end{aligned} \quad (2.135b)$$

$$\begin{aligned} M_4 &= \frac{D(1-\nu)}{4\pi \kappa^3} \{ 2(1+\nu) \cos 3\psi - (5-\nu) \cos (\psi+2\beta) - 3(1-\nu) \cos (3\psi+2\beta) \} \\ &\quad - \frac{D(1-\nu)}{4\pi \kappa^2 R_p} \{ 2(1+\nu) \cos 2\psi - 2(1-\nu) \cos (2\psi+2\beta) \} \end{aligned} \quad (2.135c)$$

$$\begin{aligned}
 V_4 = & -\frac{3D(1-\nu)}{4\pi \kappa^4} \{ (5-\nu) [\cos(\psi+3\beta) + \cos(3\psi+\beta)] + 4(1-\nu) \cos(3\psi+3\beta) \} \\
 & + \frac{D(1-\nu)}{2\pi R_Q \kappa^3} \{ (5-\nu) \cos(\psi+2\beta) + 3(1-\nu) \cos(3\psi+2\beta) \} \\
 & + \frac{D(1-\nu)^2}{2\pi R_Q R_P \kappa^2} \{ \cos(2\psi+2\beta) \} \quad (2.135d)
 \end{aligned}$$

$$\begin{aligned}
 T_4 = & + \frac{D(1-\nu)}{4\pi \kappa^3} \{ 3(1-\nu) \sin(3\psi+2\beta) + (5-\nu) \sin(\psi+2\beta) \} \\
 & + \frac{D(1-\nu)^2}{2\pi R_P \kappa^2} \{ \sin 2\psi - \sin(2\psi+2\beta) \} \quad (2.135e)
 \end{aligned}$$

The kernels given by equations (2.78), (2.97), (2.126) and (2.135) reduce to those given by Tottenham [1979] for plates having only straight boundaries, where the curvature terms are zero. Effectively, identical equations have also been presented by Stern [1979] and Bezine & Gamby [1978] for equations (2.78) and (2.97).

The convergence of the singular integrals in equation (2.134), for vanishingly small ϵ , can be examined by expanding $w = w(Q)$ about the point P using a Taylor series as:

$$\begin{aligned}
 w_Q = & w_P + n^+ \frac{\partial w}{\partial n_P^+} + t^+ \frac{\partial w}{\partial t_P^+} + \frac{1}{2} (n^{+2} \frac{\partial^2 w}{\partial n_P^{+2}} + 2n^+ t^+ \frac{\partial^2 w}{\partial n_P^+ \partial t_P^+} + t^{+2} \frac{\partial^2 w}{\partial t_P^{+2}}) \\
 & + \frac{1}{6} (n^{+3} \frac{\partial^3 w}{\partial n_P^{+3}} + 3n^{+2} t^+ \frac{\partial^3 w}{\partial n_P^{+2} \partial t_P^+} + 3n^+ t^{+2} \frac{\partial^3 w}{\partial n_P^+ \partial t_P^{+2}} + t^{+3} \frac{\partial^3 w}{\partial t_P^{+3}}) \quad (2.136)
 \end{aligned}$$

and subsequently replacing $w \equiv w(Q)$ by:

$$\begin{aligned}
 w = & w_Q - \{ w_P + n^+ \frac{\partial w}{\partial n_P^+} + t^+ \frac{\partial w}{\partial t_P^+} + \frac{1}{2} (n^{+2} \frac{\partial^2 w}{\partial n_P^{+2}} + 2n^+ t^+ \frac{\partial^2 w}{\partial n_P^+ \partial t_P^+} + t^{+2} \frac{\partial^2 w}{\partial t_P^{+2}}) \} \\
 = & \frac{1}{6} (n^{+3} \frac{\partial^3 w}{\partial n_P^{+3}} + 3n^{+2} t^+ \frac{\partial^3 w}{\partial n_P^{+2} \partial t_P^+} + 3n^+ t^{+2} \frac{\partial^3 w}{\partial n_P^+ \partial t_P^{+2}} + t^{+3} \frac{\partial^3 w}{\partial t_P^{+3}}) \quad (2.137)
 \end{aligned}$$

However, as already explained, this equation will also be written in a limiting form and the evaluation of the integrals is carried out numerically as shown in Chapter 3. Equation (2.134) for a point P on the boundary is, therefore, written as:

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \left\{ \int_{S_\epsilon} (V_4 w - M_4 \frac{\partial w}{\partial n} + \frac{\partial w_4}{\partial n} m_n - w_4 v_n) dS(Q) \right. \\ & \quad \left. + \int_{S-S_\epsilon} (V_4 w - M_4 \frac{\partial w}{\partial n} + \frac{\partial w_4}{\partial n} m_n - w_4 v_n) dS(Q) \right\} \\ & = \mathcal{D} \iint_A w_4 \nabla^4 w dA(q) + \sum_{\lambda=1}^k \{ [t] w_4 - [T_4] w_Q \}_\lambda \end{aligned} \quad (2.138)$$

where w_4 , $\partial w_4 / \partial n$, M_4 , V_4 and T_4 are given by equations (2.133). Note that if P is a corner point, two such equations can be written, one either side of the corner.

2.3.11 Integral Equation for Twisting Moment at an Interior Point

The twisting moment at a point p inside the plate domain is given by:

$$m_t = - \mathcal{D} (1 - \nu) \frac{\partial^2 w}{\partial n_p \partial t_p} \quad (2.139)$$

Therefore, differentiating equations (2.87) accordingly gives:

$$\begin{aligned} m_t & = - \int_S (V_5 w - M_5 \frac{\partial w}{\partial n} + \frac{\partial w_5}{\partial n} m_n - w_5 v_n) dS(Q) \\ & \quad - \sum_{\lambda=1}^k \{ [T_5] w - [t] w_5 \}_\lambda + \mathcal{D} \iint_A w_5 \nabla^4 w dA(q) \end{aligned} \quad (2.140)$$

$$\text{where: } w_5 = \frac{1 - \nu}{8\pi} \sin 2\psi \quad (2.141a)$$

$$\frac{\partial w_5}{\partial n} = - \frac{1 - \nu}{8\pi h} \{ \sin (2\psi + \beta) - \sin (2\psi - \beta) \} \quad (2.141b)$$

$$M_5 = \frac{\mathcal{D} (1 - \nu)}{8\pi \kappa^2} \{ (1 + \nu) \sin 2\psi - (1 - \nu) \sin (2\psi + 2\beta) \} \quad (2.141c)$$

$$\begin{aligned} V_5 = & - \frac{\mathcal{D} (1 - \nu)}{4\pi \kappa^3} \{ 3(1 - \nu) \sin (2\psi + 3\beta) - (5 - \nu) \sin (2\psi + \beta) \} \\ & + \frac{\mathcal{D} (1 - \nu)^2}{2\pi R \kappa^2} \sin (2\psi + 2\beta) \end{aligned} \quad (2.141d)$$

$$T_5 = - \frac{\mathcal{D} (1 - \nu)^2}{4\pi \kappa^2} \cos (2\psi + 2\beta) \quad (2.141e)$$

2.3.12 Integral Equation for Twisting Moment at a Boundary Point

An integral equation for the twisting moment at a point on the boundary can be obtained by adopting the same limiting process as the one described in deriving equation (2.130), thus giving:

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \left\{ \int_{S_\epsilon} (V_5 \omega - M_5 \frac{\partial \omega}{\partial n} + \frac{\partial \omega_5}{\partial n} m_n - \omega_5 \nu_n) dS(Q) \right. \\ & \quad \left. + \int_{S-S_\epsilon} (V_5 \omega - M_5 \frac{\partial \omega}{\partial n} + \frac{\partial \omega_5}{\partial n} m_n - \omega_5 \nu_n) dS(Q) \right\} \\ & = \mathcal{D} \iint_A \omega_5 \nabla^4 \omega dA(q) + \sum_{\lambda=1}^k \{ [t] \omega_5 - [T_5] \omega_Q \}_\lambda \end{aligned} \quad (2.142)$$

where the kernels ω_5 , $\partial \omega_5 / \partial n$, M_5 , V_5 and T_5 are given by equations (2.141).

2.3.13 Integral Equations for Shear Force at an Interior Point

The shear force at an interior point p is given by:

$$Q_p = - \mathcal{D} \frac{\partial}{\partial n_p} \nabla^2 \omega = - \mathcal{D} \frac{\partial}{\partial n_p} \left(\frac{\partial^2 \omega}{\partial n_p^2} + \frac{\partial^2 \omega}{\partial t_p^2} \right) \quad (2.143)$$

Hence, differentiating equation (2.87) gives:

$$Q_p = - \int_S (V_6 w - M_6 \frac{\partial w}{\partial n} + \frac{\partial w_6}{\partial n} m_n - w_6 v_n) dS(Q) \\ - \sum_{\lambda=1}^k \{ [T_6] w - [t] w_6 \}_\lambda + D \iint_A w_6 \nabla^4 w_Q dA(q) \quad (2.144)$$

where:

$$w_6 = \frac{\cos \psi}{2\pi \kappa} \quad (2.145a)$$

$$\frac{\partial w_6}{\partial n} = - \frac{1}{2\pi \kappa^2} \cos (\psi + \beta) \quad (2.145b)$$

$$M_6 = - \frac{D (1 - \nu)}{\pi \kappa^3} \cos (\psi + 2\beta) \quad (2.145c)$$

$$V_6 = - \frac{3D (1 - \nu)}{\pi \kappa^4} \cos (\psi + 3\beta) + \frac{2D (1 - \nu)}{\pi \kappa^4} \cos (\psi + 2\beta) \quad (2.145d)$$

$$T_6 = \frac{D (1 - \nu)}{\pi \kappa^3} \sin (\psi + 2\beta) \quad (2.145e)$$

2.4 FORMULATION STRATEGY AND CHOICE OF BOUNDARY INTEGRAL EQUATIONS

In Section 2.3, expressions were obtained for the boundary values of deflection, normal slope, normal moment and effective shear force in terms of the same variables around the boundary and the transverse loading, $F = D \nabla^4 w$, on the plate. Equation (2.142) gives the twisting moment at a boundary point in terms of the boundary variables. An equation can also be obtained for the jump in the twisting moment at a corner point by writing equation (2.142) on either side of a corner and finding the difference between them.

At a point on a smooth part of the boundary, there are, in general, four variables: deflection, normal derivative of deflection, normal

moment and effective shear force. The four equations expressing these variables at a boundary point are given by equations (2.94), (2.123), (2.130) and (2.138). In equations (2.94) and (2.123), the first terms before the integrals are referred to as the "free terms" and relate to the boundary variables for which the equations have been written. These represent the limiting values of the integrals over the arc S_ϵ as $\epsilon \rightarrow 0$ (see Figure 2.11). For example, in equation (2.94), the free term is a deflection term, whereas the free terms for equation (2.123) are deflection derivatives. Furthermore, in Section 2.3, it was also shown that for a point on a smooth part of the boundary, the free term coefficients are each equal to $\frac{1}{2}$ (see equations (2.94) and (2.109)). The explicit evaluation of the "free terms" in equations (2.130) and (2.138) has not been presented here. However, following the procedure outlined in Sections 2.3.8 and 2.3.10, it can be verified that these free term coefficients are also $\frac{1}{2}$ on a smooth part of the boundary (see Tottenham [1979]). For ease of reference, the four boundary integral equations referred to above are rewritten below with the pertinent free terms stated explicitly. Note that the coefficient of $\frac{1}{2}$ is only applicable if the boundary point lies on a smooth part of the boundary curve:

$$\begin{aligned} \frac{1}{2} w_p = & - \int_S (V w - M \frac{\partial w}{\partial n} + \frac{\partial w}{\partial n} m_n - w v_n) dS(Q) \\ & + \sum_{\lambda=1}^k \{ [T] w - w [t] \}_\lambda + \iint_A w F dA(q) \end{aligned} \quad (2.146/I)$$

$$\begin{aligned} \frac{1}{2} \frac{\partial w}{\partial n_p} = & - \int_S (V_2 w - M_2 \frac{\partial w}{\partial n_Q} + \frac{\partial w}{\partial n} m_n - w_2 v_n) dS(Q) \\ & + \sum_{\lambda=1}^k \{ [T_2] w - [t] w_2 \}_\lambda + \iint_A w_2 F dA(q) \end{aligned} \quad (2.146/II)$$

$$\begin{aligned} \frac{1}{2} m_n &= - \int_S (V_3 w - M_3 \frac{\partial w}{\partial n} + \frac{\partial w_3}{\partial n} m_n - w_3 v_n) dS(Q) \\ &+ \sum_{\lambda=1}^k \{ [T_3] w - [t] w_3 \}_{\lambda} + \iint w_3 F dA(q) \end{aligned} \quad (2.146/III)$$

$$\begin{aligned} \frac{1}{2} v_n &= - \int_S (V_4 w - M_4 \frac{\partial w}{\partial n} + \frac{\partial w_4}{\partial n} m_n - w_4 v_n) dS(Q) \\ &+ \sum_{\lambda=1}^k \{ [T_4] w - [t] w_4 \}_{\lambda} + \iint w_4 F dA(q) \end{aligned} \quad (2.146/IV)$$

Furthermore, it must be noted that for the convergence of the integrals in these equations:

$$w = w_Q - w_P \quad \text{in equation (2.146/II)}$$

$$w = w_Q - w_P - n_P \frac{\partial w}{\partial n_P} - t_P \frac{\partial w}{\partial t_P} \quad \text{in equation (2.146/III)}$$

and:

$$w = w_Q - w_P - n_P \frac{\partial w}{\partial n_P} - t_P \frac{\partial w}{\partial t_P} - \frac{1}{2} n_P^2 \frac{\partial^2 w}{\partial n_P^2} - n_P t_P \frac{\partial^2 w}{\partial n_P \partial t_P} - \frac{1}{2} t_P^2 \frac{\partial^2 w}{\partial t_P^2}$$

$$\text{in equation (2.146/IV)}$$

If the boundary point is at a corner, the free term coefficients will, in general, not be equal to $\frac{1}{2}$. There will also be additional free terms for other deflection derivatives of the same order. For example, if the boundary point is at a corner, equation (2.123) would have two free terms, one for $\partial w / \partial n_P$ and one for $\partial w / \partial t_P$.

Equations (2.146/I) to (2.146/IV) provide four equations expressing the boundary variables w , $\partial w / \partial n$, m_n and v_n , respectively, in terms of the

same variables around the boundary and the twisting moment jumps at corners. At a point on a smooth part of the boundary, two of the four variables are given as boundary conditions. The other two solutions required must be chosen from equations (2.146/I) to (2.146/IV). In most of the published work to date, the first two equations have been used (see, for example, Stern [1979], and Bezine [1978]). Tottenham [1979] has adopted an intuitive approach, choosing one from the pair of equations giving w and v_n and one from those giving $\partial w/\partial n$ and m_n . There are six different ways that a pair of integral equations can be chosen from the given four. The kernels in the first two equations are simpler than the kernels in the other two. However, to date, there seems to be no investigation available regarding the accuracy of each pair, when these equations are solved numerically for a given degree of mesh discretisation. This particular aspect is examined in more detail in Chapter 3.

2.5 CONCLUDING REMARKS

Equations (2.146/I) to (2.146/IV), presented in the last section, provide four equations expressing the boundary variables w , $\partial w/\partial n$, m_n and v_n , respectively, in terms of the same variables along the boundary and the twisting moment jumps at the corners. The Kirchhoff thin plate theory, described in the earlier parts of this chapter, requires the prescription of two boundary conditions at a general boundary point. This means that there are two remaining unknowns to be determined, for which two equations are required. In principle, any two of the four equations (2.146) can be used to provide the extra relations required. However, as yet, no investigation has been reported as to the degree of accuracy obtained using each pair. In the next chapter, the results of a series of numerical tests are presented which attempt to assess the accuracy of each pair.

The two boundary conditions plus a pair of integral equations chosen from equations (2.146) provide four relations for the four boundary variables at a boundary point. (At a corner, there are more than four variables and their treatment is explained later.) Once a solution has been obtained, all the boundary variables will be known. Equations (2.87), (2.96), (2.125), (2.132), (2.140) and (2.144) can then be used to obtain any desired quantity at interior points of the plate. Note that this is in contrast with the so-called domain approaches where interior values are part of the solution process. In most practical applications, internal values are required at only a few points of interest and, therefore, it could be said that the domain methods provide a great deal of information that is not required.

The Boundary Integral formulation described in this chapter is known as the direct formulation. A specific feature of this approach is that the integral equations are in terms of the physical variables, with boundary conditions involving these same variables.

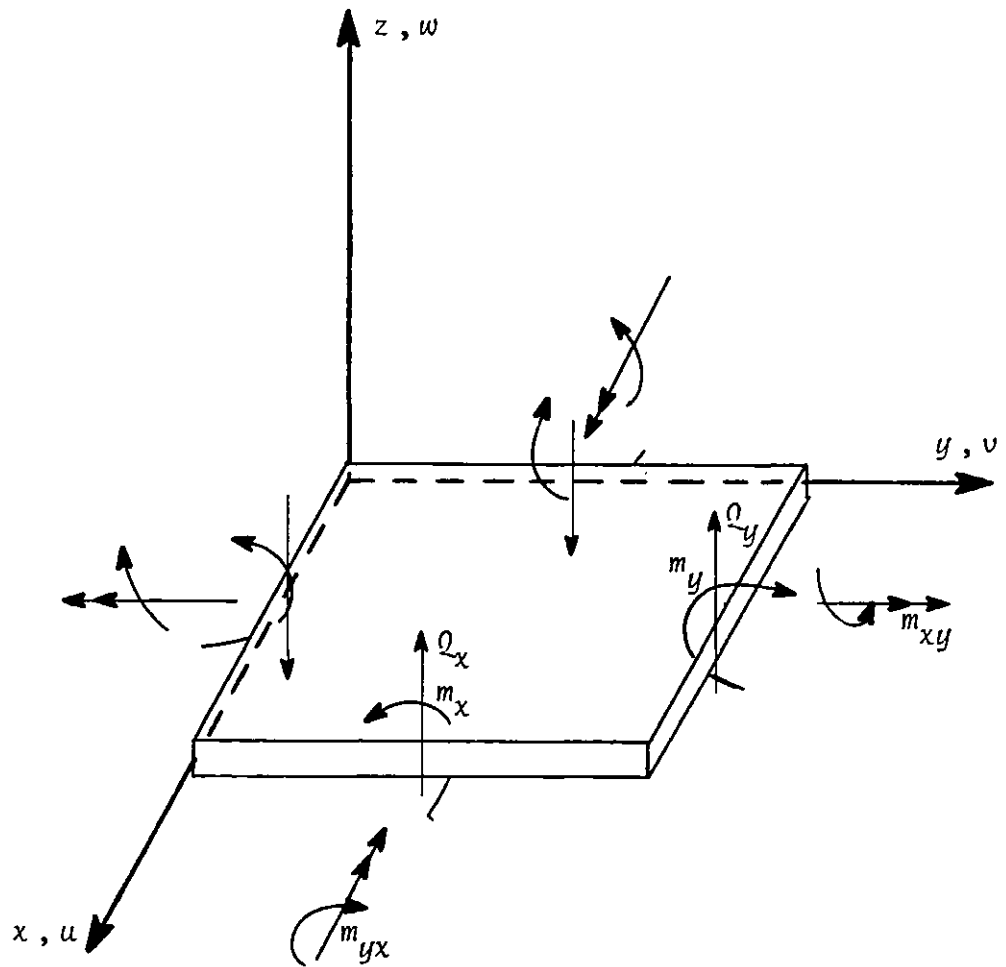


Figure 2.1: Sign convention showing positive direction for displacements, stress and couple resultants

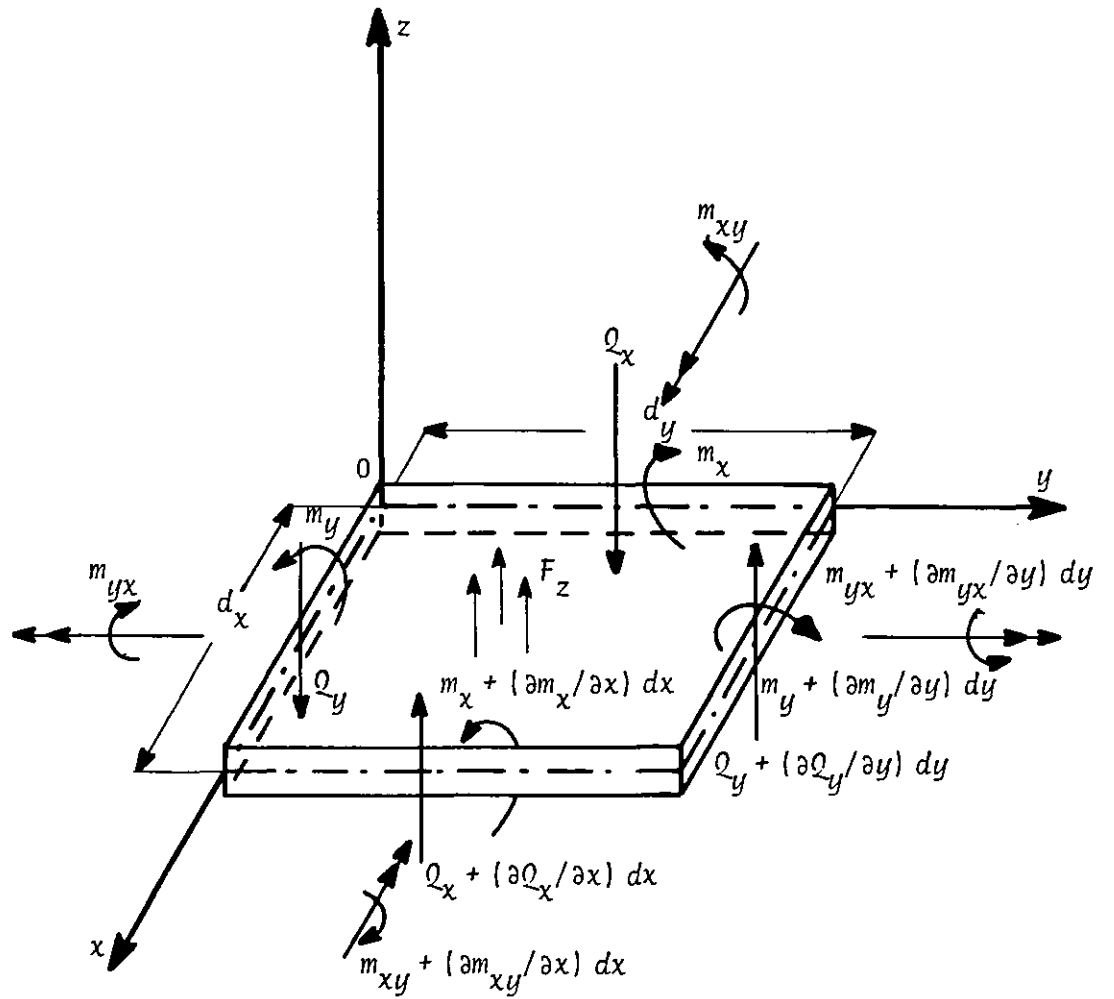


Figure 2.2: Differential plate element under an equilibrating system of internal and external forces

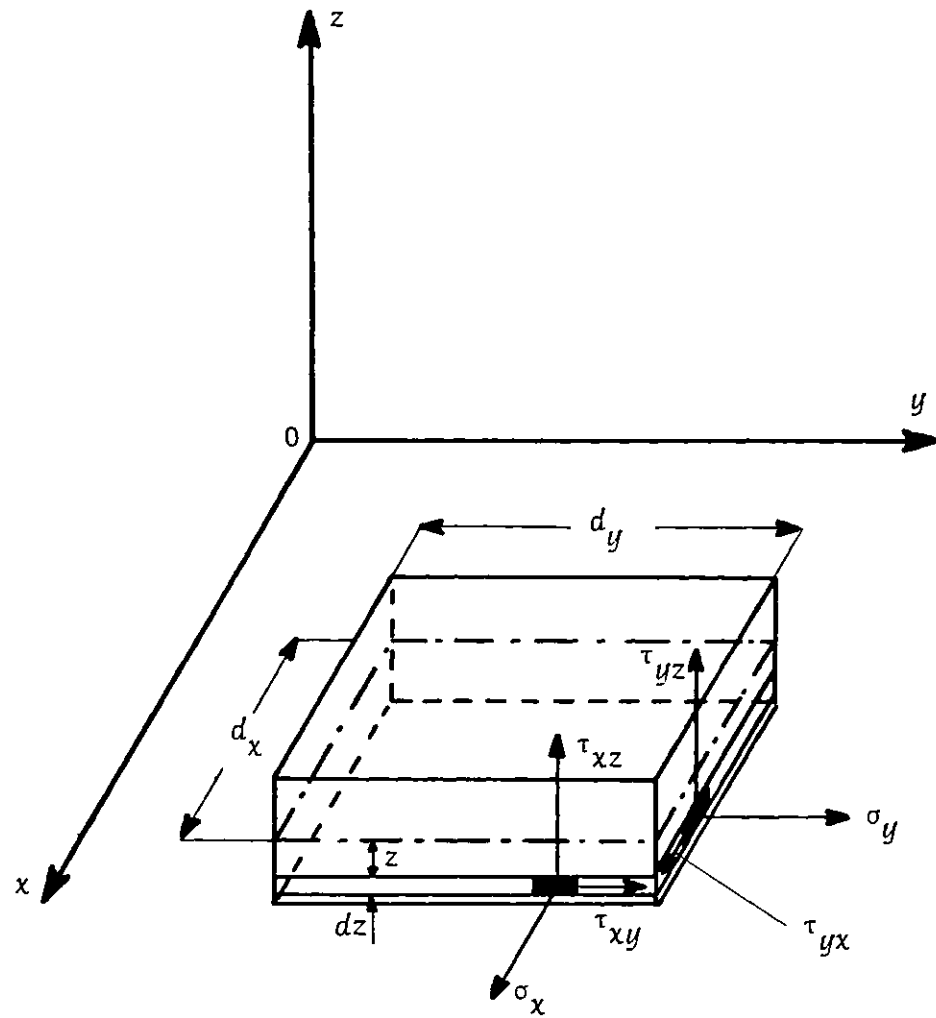


Figure 2.3: Direct and shearing stresses on a differential plate element

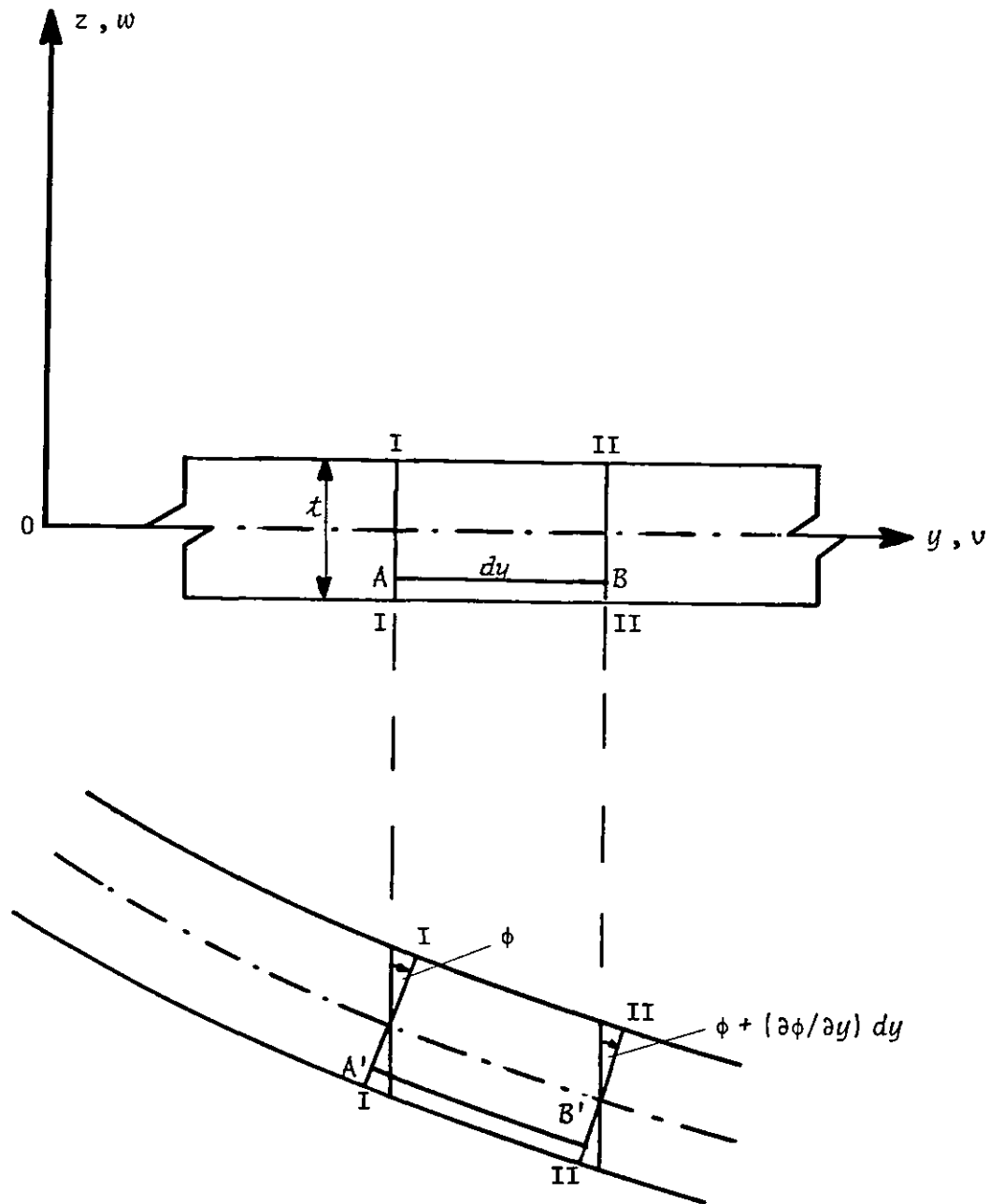


Figure 2.4: Section of a plate before and after deflection

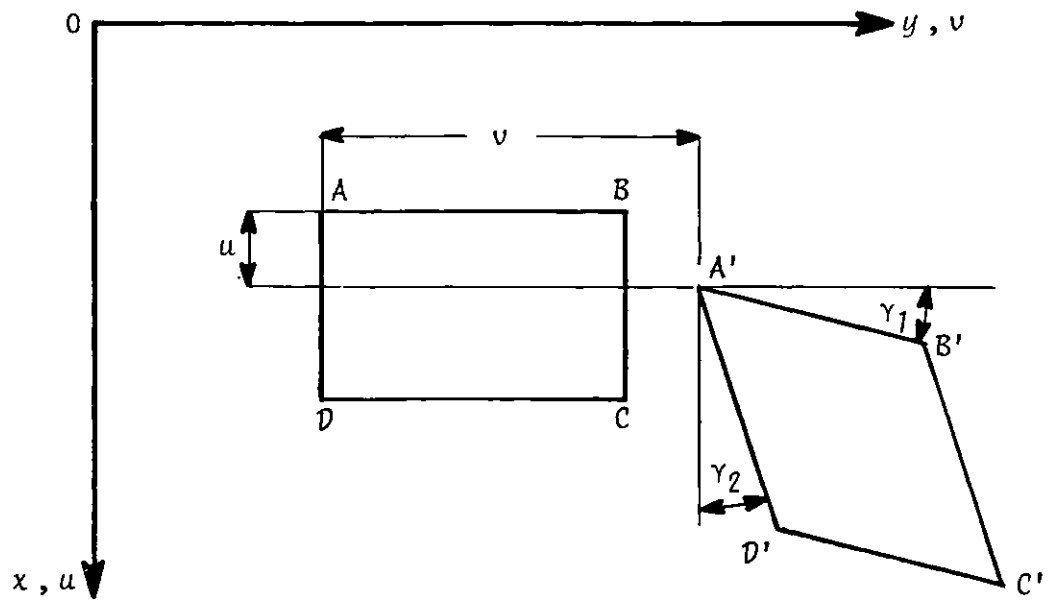


Figure 2.5: Angular distortion of a rectangular parallelogram

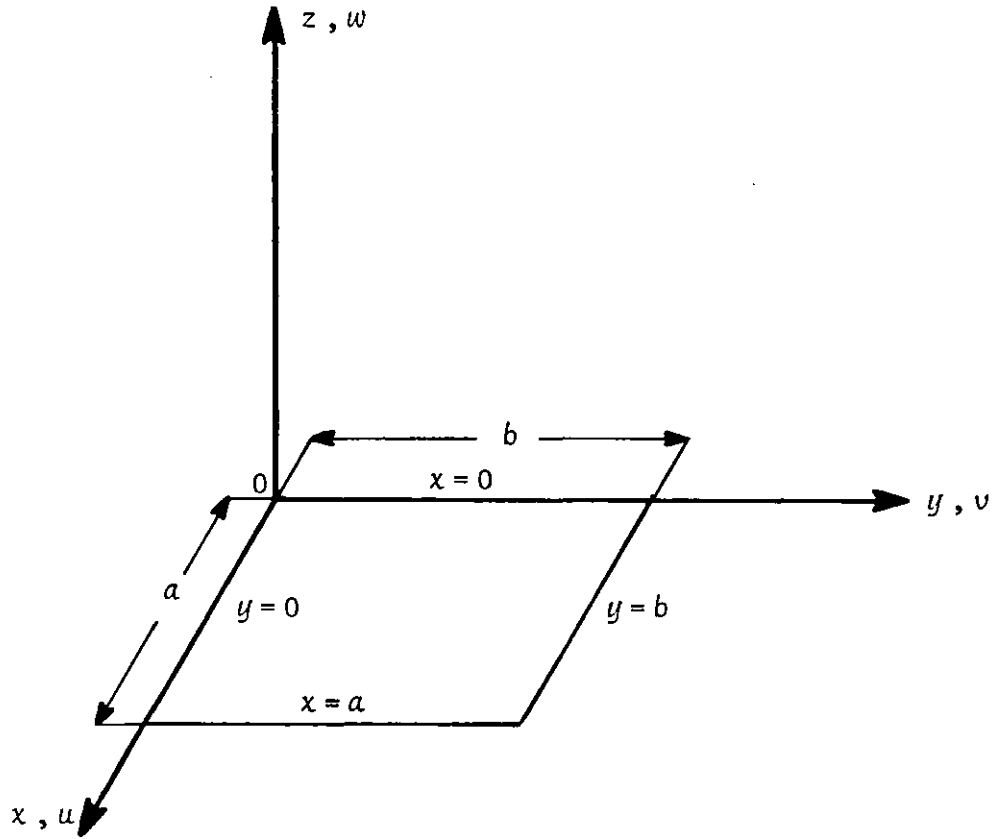


Figure 2.6: Rectangular plate in a rectangular Cartesian coordinate

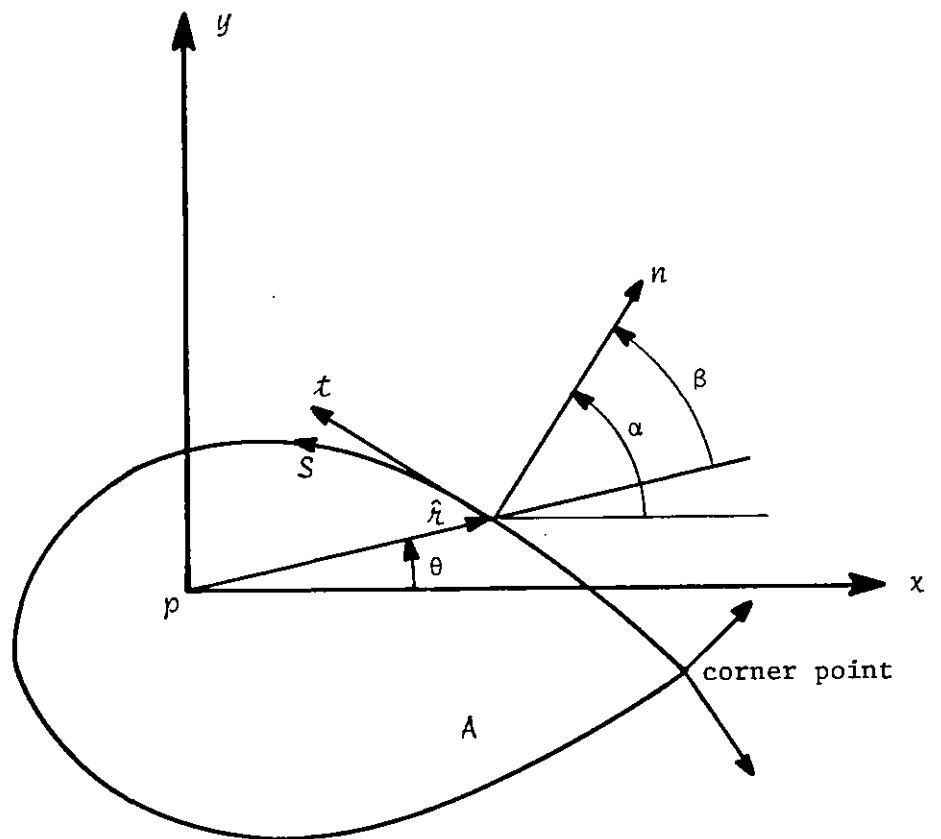


Figure 2.7: Normal and tangential directions at a boundary point of the plate

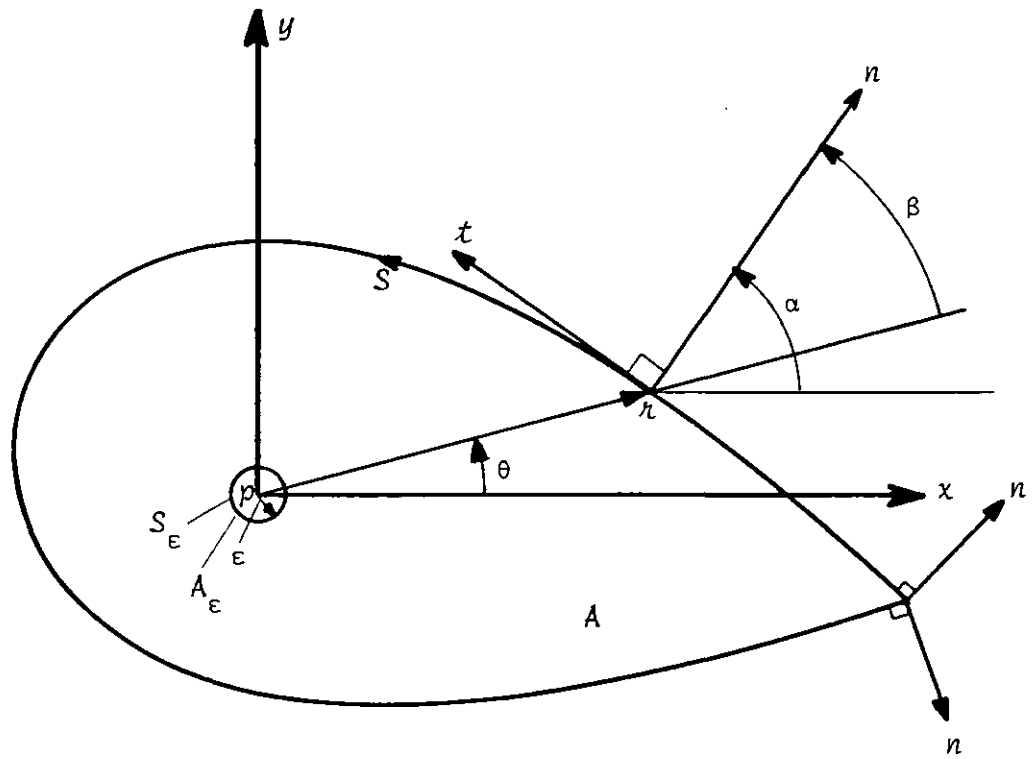
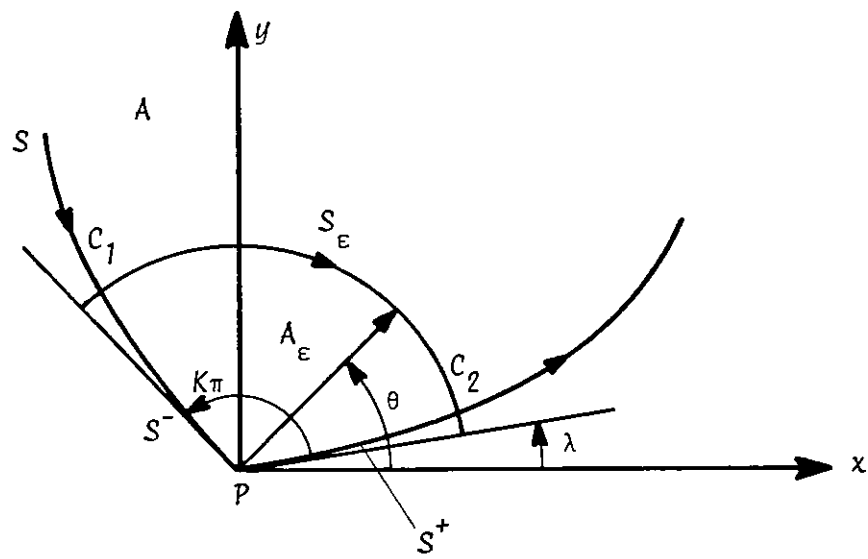
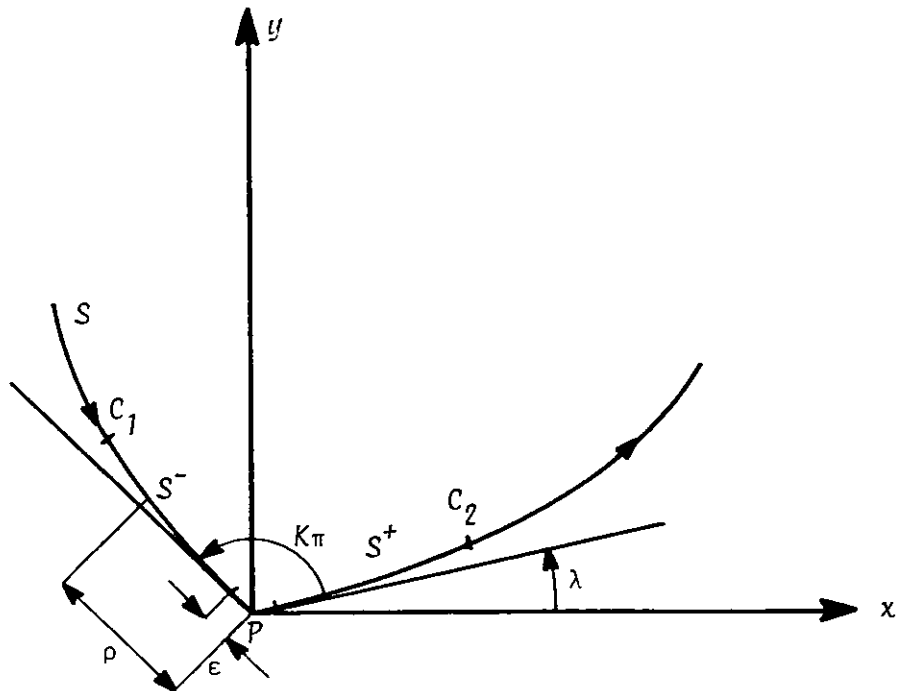


Figure 2.8: Plate domain, A , with a small region, A_ϵ , excluded from it



(a)



(b)

Figure 2.9: Plate boundary showing region of exclusion

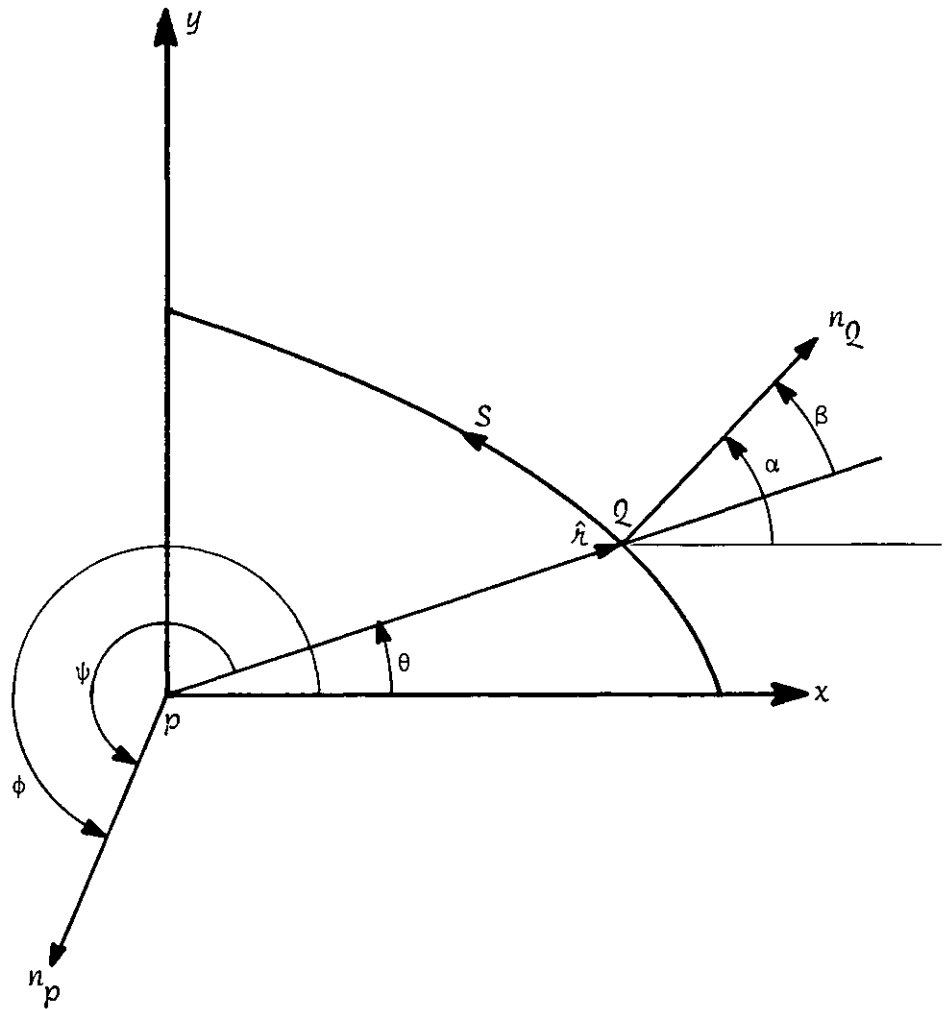


Figure 2.10: Notation for directional derivatives at an interior point

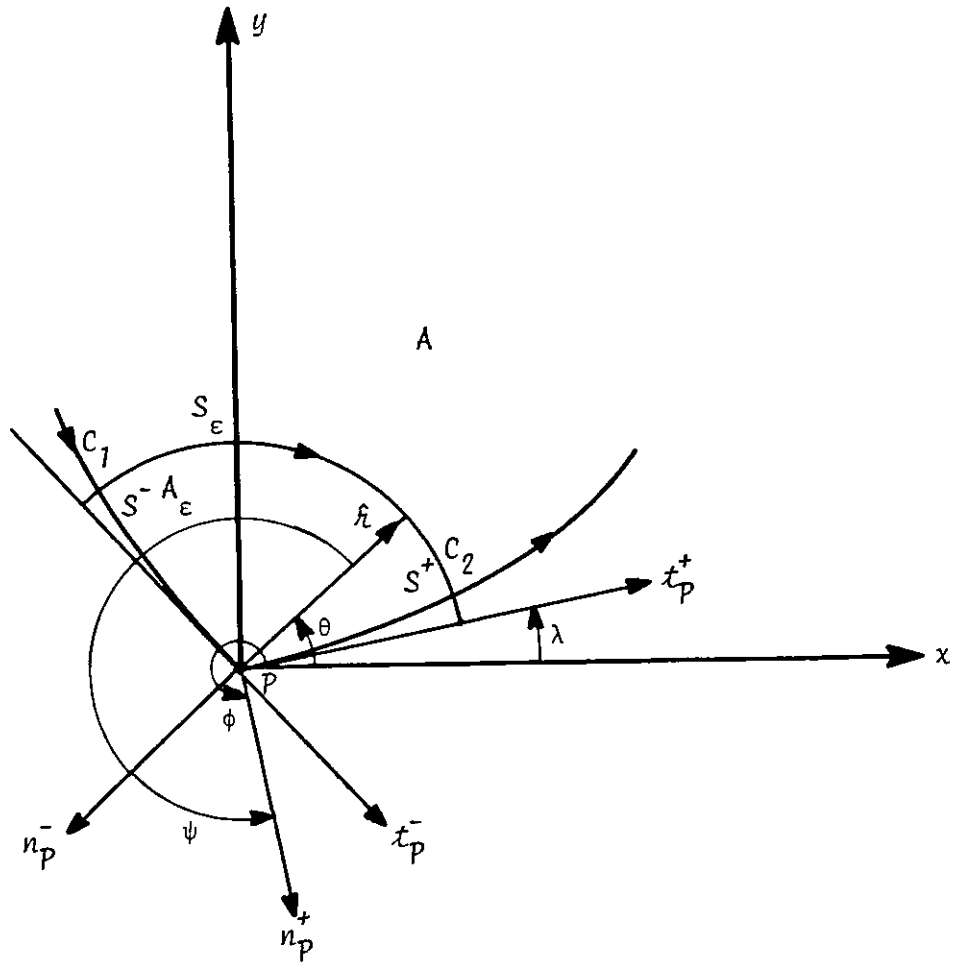


Figure 2.11: Notation for directional derivatives at a boundary point

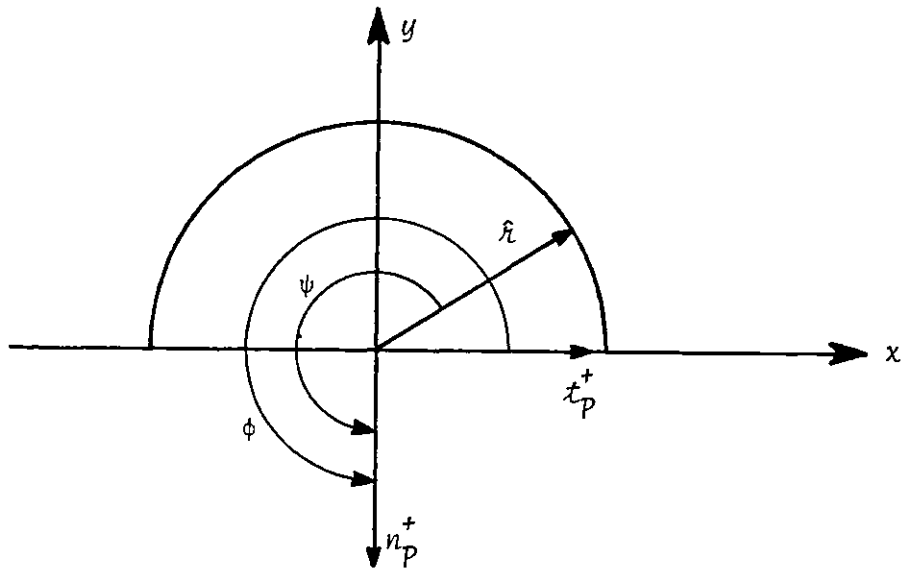


Figure 2.12: Point P on a regular part of the boundary

CHAPTER 3

NUMERICAL SOLUTION OF THE BOUNDARY INTEGRAL EQUATIONS

USING CONSTANT ELEMENTS

3.1 INTRODUCTION

The analytical solution of the integral equations derived in the previous chapter is, in general, extremely difficult if not impossible. However, the advent of the digital computer has made it possible to obtain numerical solutions to these, otherwise intractable, problems in a very elegant way. The basic steps involved in the numerical solution procedure are as follows:

- (i) The boundary curve is divided into a finite number of segments or elements with each of which is associated a certain number of nodal points at which the unknowns are to be determined.
- (ii) The number of nodes for each element depends on the order of interpolation functions employed in approximating the variation of the variables, along the element length, in terms of their nodal values. For example, if the value of a variable is assumed to be constant throughout the length of the element, only one node is associated with such an element and it is situated at the mid-point of the element. This type of element is referred to as a constant element. For a linear element, two nodes are required, one at either end, and the variation of the variables is assumed to be linear along the element length. A quadratic element has three nodes, and so on.
- (iii) A pair of integral equations from equations (2.146) is now set

up at each nodal point in turn, and after invoking the boundary conditions, a set of linear algebraic equations in terms of the nodal unknowns is obtained.

- (iv) After solving this system of equations, all the boundary values will be known.
- (v) Interior values of any required variable can then be computed by using the appropriate integral equation for interior points.

Jaswon & Maiti [1968] have applied an integral equation formulation to the problem of simply supported and clamped rectangular plates. In their formulation, the lateral deflection of the plate is expressed in terms of harmonic functions which are, in turn, expressed as potentials generated by continuous simple source distributions on the boundary, with densities to be determined. The boundary integral equations thus obtained are numerically solved for these unknown densities. The authors discretise the plate boundary using constant elements. Due to difficulties encountered near corners (see Jaswon, Maiti & Symm [1967]), the corners of the plate are rounded off by a circular arc. Altiero & Sikarskie [1978] have presented a formulation which involves the embedding of the plate domain under consideration in one for which the Green's functions for a concentrated load and moment are known. An unspecified line load of concentrated force and moment is then applied along the boundary of the original plate region and the solution throughout the plate is represented in terms of a boundary integral involving these unknown densities and the Green's functions for the fictitious plate. The prescribed boundary conditions for the original plate are then used to generate a pair of coupled integral equations for the force and moment densities. The authors, however, report difficulties with boundary conditions involving high derivatives of deflections and only treat clamped boundary conditions.

Furthermore, they report appreciable inaccuracy for the clamping moments along the boundary. Maiti & Chakrabarty [1974] have applied an integral equation method to the solution of simply supported polygonal plates under uniformly distributed loading conditions. Their discretisation is, in principle, similar to that presented by Jaswon & Maiti [1968], mentioned above. The authors report difficulties in obtaining accurate solutions for problems involving built-in edges. A similar formulation has been adopted by Segedin & Brickell [1968] for the solution of simply supported corner plates. Their results seem to be in close agreement with the finite difference solution presented by Salvadori & Regini [1968] (see also Brickell [1970]). Tottenham [1979] has applied the so-called indirect method to the problem of a uniformly loaded square plate with a square central opening. The boundary conditions chosen for this problem were simply supported on the external boundary and free along the internal boundary. In these types of formulations, the unknown variables in the integral equations do not usually have a direct physical meaning.

As mentioned in the previous chapter, the integral equation formulation presented in this work is the so-called direct formulation. An attractive feature of the direct method is the fact that the variables present in the integral equations have a physical meaning in terms of the deflections and forces along the plate boundary. One of the early contributions of this type is by Forbes & Robinson [1969]. Their discretisation assumes constant values of the nodal variables along each segment. The boundary integral equations used are those for deflection and normal slope at boundary points (similar to equations (2.94) and (2.123)). They apply their formulation to a variety of plate and shallow shell problems but restrict themselves to domains with smooth boundary curves. Bazine & Gamby [1978] also use these same equations in a constant element boundary integral equation formulation for plate bending problems.

They present results for square plates under centrally applied loads, with various boundary conditions. Other publications including constant element formulations include Niwa, Kobayashi & Fukui [1974] and Dawson [1980]. Stern [1979] uses linear interpolation of the nodal variables and presents results for square plates under uniformly distributed load, with simply supported and clamped edges. In addition to integral equations for deflection and slope at the boundary, Tottenham [1979] also writes equations for normal moment and effective shear force at the boundary. Of these four equations, only two need be used at any boundary point. This author chooses these equations such that at any nodal point, the equations used are those relating to the boundary conditions at that node. For example, at a node having clamped boundary conditions, $w = \partial w / \partial n = 0$, the equations used are those for deflection and normal slope at the boundary. Linear variation of the variables between nodal values is assumed. Hansen [1976] applies what is basically the direct method to the solution of infinite plates containing one or more holes. Free boundary conditions are considered along the edges of the holes. Bazine [1981a] has extended the direct method to the analysis of plate flexure problems with conditions inside the domain. The formulation assumes constant variation of the variables along each boundary segment. The plate domain is also divided into square panels on which the variation of lateral force and deflection is assumed to be constant. This author presents results for square plates supported on four interior supports, having free, clamped and simply supported boundaries. In another paper, Bazine [1981b] presents an alternative formulation using the method of similarities. More recently, the boundary integral equation method has been applied to plate bending problems based on Reissner's plate model (see, for example, van der Weeën [1982a,1982b]). Recent publications include the application of the BIE method to the non-linear bending of

thin elastic plates (see, for example, Tanaka [1982], and Kamiya, Sawaki & Nakamura [1982]), computation of stress intensity factors (see, for example, Stern & Kim [1980], and Kim [1982]), BIE solutions to the bending of thin orthotropic plates (see, for example, Kamiya & Sawaki [1982]), and dynamic and stability analysis of plates (see Gospodinov & Ljutskanov [1982]). For a further list of references and bibliography, the reader is referred to Hartmann [1981].

This chapter describes a numerical implementation of the boundary integral equation formulation presented in Chapter 2. The discretisation assumes the variables to remain constant along each boundary segment. Numerical tests are presented using all possible pairs of integral equations from equations (2.146) for square plates with clamped and simply supported edges under a centrally applied load.

3.2 PARAMETRIC REPRESENTATION OF GEOMETRY

Figure 3.1 depicts a general plate domain in a cartesian coordinate system, with the x,y plane taken to be the middle surface of the plate. The plate boundary is approximated by a series of straight line elements, and a node is placed at the mid-point of each element. It can be seen that if sufficient numbers of elements are used, even curved boundaries can be approximated with good accuracy. The geometry of each element is described in terms of the coordinates of its end points and its intrinsic coordinate, ξ . This can be understood more clearly by referring to Figure 3.2, which shows a typical element AB of length L . A local coordinate, ξ , is defined along the length of the element, and its direction with respect to the outward normal is indicated. As shown, ξ varies between -1 and $+1$ along the element.

The geometry of the element is now expressed as:

$$x = N_1(\xi) x_1 + N_2(\xi) x_2 \quad (3.1)$$

$$y = N_1(\xi) y_1 + N_2(\xi) y_2 \quad (3.2)$$

where: $N_1 = \frac{1}{2}(1 - \xi)$ (3.3)

and: $N_2 = \frac{1}{2}(1 + \xi)$ (3.4)

The functions N_1 and N_2 are called shape functions. Note that an important characteristic of the shape functions is that at each end of the element, the shape function pertaining to that end is equal to unity and the shape function for the other end vanishes. Furthermore, the variation of these functions along the element is linear, indicating that straight boundaries can be represented exactly using only the end node coordinates.

3.3 INTEGRAL EQUATIONS IN DISCRETISED FORM

An important step in the numerical solution of the system of equations formed by using a pair of integral equations from equations (2.146), with suitable boundary conditions, is to approximate the variation of the boundary variables along an element. The approximation adopted here is the simplest, and the value of each variable is assumed to be constant along each element. In view of this, for a general point P on the boundary, the integral equations (2.146) of Chapter 2 can now be written as:

$$\begin{aligned}
 C_1^I w_p + \sum_{\substack{i=1 \\ i \neq p}}^{\ell} w_i \int_i v_n dS - \sum_{i=1}^{\ell} \frac{\partial w_i}{\partial n} \int_i M_n dS + \sum_{i=1}^{\ell} m_i \int_i \frac{\partial w}{\partial n} dS \\
 - \sum_{i=1}^{\ell} v_i \int_i w dS + \sum_{\lambda=1}^k \{ [T] w - w [t] \}_{\lambda} = \iint w F dA
 \end{aligned} \quad (3.5)$$

$$\begin{aligned}
 C_1^{II} w_p + C_2^{II} \frac{\partial w}{\partial n_p} + \sum_{\substack{i=1 \\ i \neq p}}^{\ell} w_i \int_i v_2 dS - \sum_{\substack{i=1 \\ i \neq p}}^{\ell} \frac{\partial w_i}{\partial n} \int_i M_2 dS + \sum_{i=1}^{\ell} m_i \int_i \frac{\partial w_2}{\partial n} \\
 - \sum_{i=1}^{\ell} v_i \int_i w_2 dS + \sum_{\lambda=1}^k \{ [T_2] w - w_2 [t] \}_{\lambda} = \iint w_2 F dA
 \end{aligned} \quad (3.6)$$

$$\begin{aligned}
 C_1^{III} w_p + C_2^{III} \frac{\partial w}{\partial n_p} + C_3^{III} m_p + \sum_{\substack{i=1 \\ i \neq p}}^{\ell} w_i \int_i v_3 dS \\
 - \sum_{\substack{i=1 \\ i \neq p}}^{\ell} \frac{\partial w_i}{\partial n} \int_i M_3 dS + \sum_{\substack{i=1 \\ i \neq p}}^{\ell} m_i \int_i \frac{\partial w_3}{\partial n} dS - \sum_{i=1}^{\ell} v_i \int_i w_3 dS \\
 + \sum_{\lambda=1}^k \{ [T_3] w - w_3 [t] \}_{\lambda} = \iint w_3 F dA
 \end{aligned} \quad (3.7)$$

$$\begin{aligned}
 C_1^{IV} w_p + C_2^{IV} \frac{\partial w}{\partial n_p} + C_3^{IV} m_p + C_4^{IV} v_p + \sum_{\substack{i=1 \\ i \neq p}}^{\ell} w_i \int_i v_4 dS \\
 - \sum_{\substack{i=1 \\ i \neq p}}^{\ell} \frac{\partial w_i}{\partial n} \int_i M_4 dS + \sum_{\substack{i=1 \\ i \neq p}}^{\ell} m_i \int_i \frac{\partial w_4}{\partial n} dS - \sum_{\substack{i=1 \\ i \neq p}}^{\ell} v_i \int_i w_4 dS \\
 + \sum_{\lambda=1}^k \{ [T_3] w - w_2 [t] \}_{\lambda} = \iint w_4 F dA
 \end{aligned} \quad (3.8)$$

where ℓ denotes the number of elements. The following points should be noted in relation to equations (3.5) to (3.8):

- (i) The integration over the boundary has now been replaced by the sum of the integrals over each element.
- (ii) Since the value of each variable is assumed to be constant along each element, it can be placed outside the integral sign when integrating over an element.

- (iii) It was shown in Chapter 2 that some of the kernels in equations (3.5) to (3.8) are singular. The integrals containing singularities of the order of κ^{-1} or higher cannot be evaluated over the self element (the element containing the point P) by using the same quadrature rule that is used to evaluate the integrals over the other elements. This is indicated in equations (3.5) to (3.8) by excluding the contribution from these integrals in the summation signs and replacing them by the coefficients C_1^I, C_2^{II} , etc. Their evaluation is carried out by using certain known deflection fields and will be fully described in Section 3.5.2.2.
- (iv) $\mathcal{D} \nabla^4 w$ has been replaced by F , which is the intensity of the lateral loading (see equation (2.67)).

An important point needs to be emphasised in equations (3.5) to (3.8) regarding the terms relating to deflections and forces at corner points. Figure 3.3 shows a square plate with its boundary discretised using twelve constant elements. As already mentioned, there are four variables at each node. Two of these are given as boundary conditions, thus leaving two unknowns at each nodal point. For ℓ nodes, therefore, the total number of unknowns will be 2ℓ . If the boundary has corners, there will, in general, be eight variables associated with each corner point. This is due to the fact that at each corner point there are two normals to the boundary, one on either side of the corner, therefore providing two values for each of the variables of slope, moment and effective shear force. These six, plus the deflection and the Kirchhoff force, result in eight variables at a corner point. However, a constant element discretisation does not provide for a node at the corner and the proper consideration of all corner variables is not possible.

Specifically, the values of slope, moment and effective shear force for each side of the corner are the same as that for the node nearest to that corner. So, the only extra variables to be considered at the corner are the deflection and the Kirchhoff corner force. Assuming that one of these is given as a boundary condition, this leaves one unknown for each corner. Therefore, the total number of unknowns would be $(2\ell + k)$ where k is the number of corners.

Now, for every node on the boundary, two integral equations can be written, providing a total of 2ℓ equations for the $(2\ell + k)$ unknowns, which is not sufficient for a solution to be obtained. There are two ways of overcoming this problem. The first is to ignore the summation over the corners in equations (3.5) to (3.8), and solve only for the nodal unknowns. In this case, the total number of unknowns will be 2ℓ and, therefore, a solution can be obtained using the 2ℓ integral equations written at the nodal points. However, as will be shown later, this method suffers from the disadvantage that the computed values of the nodal variables in the vicinity of a corner are not very accurate. Furthermore, and perhaps more important, it will be shown that certain pairs of equations chosen from the set (3.5) to (3.8) do not provide any reasonable solution if the corner unknowns are ignored. This seems to be the method adopted in most constant element formulations presented to date.

An alternative approach presented here is to provide an additional integral equation at each corner and thus obtain $(2\ell + k)$ equations for the $(2\ell + k)$ unknowns. In this way, the solution would include the corner forces when they are present, whereas the former approach does not allow for their computation. The additional integral equation required is chosen either as equation (3.5) written at the corner, or the discretised version of the corner force equation. This latter equation is obtained by writing equation (2.142) for positive and negative sides of a corner

and subtracting one equation from the other (see Figure 3.10). Figure 3.10 shows what is meant by a positive and negative side of a corner. The side of a corner on which the positive direction of travel is away from the corner is taken as the positive side and the negative side is defined as that side of the corner on which the positive direction of travel is towards the corner.

3.4 MATRIX REPRESENTATION OF THE SYSTEM OF EQUATIONS

Consider a square plate with its boundary discretised using, for example, ℓ constant elements, as in Figure 3.3. With each element is associated a node at the element mid-point. A pair of integral equations from (3.5) to (3.8) is then written at every node in turn, providing 2ℓ equations involving the boundary variables at the nodes. Depending on whether corner forces are considered or not, extra equations may or may not be written at corner points. The final system of linear simultaneous equations can be written in matrix form as:

$$[A] \{\bar{u}\} + [\bar{A}] \{\bar{b}\} = \{\bar{F}\} \quad (3.9)$$

In this equation, $\{\bar{u}\}$ contains the nodal unknowns and $\{\bar{b}\}$ contains the boundary conditions. $[A]$ and $[\bar{A}]$ are $(2\ell \times 2\ell)$ or $(2\ell + k) \times (2\ell + k)$ matrices and their coefficients are evaluated by integrating the appropriate kernels along the boundary elements. The right hand side of equation (3.9) is obtained by integrating, over the plate domain, the right hand side of equations (3.5) to (3.8).

3.5 EVALUATION OF THE EQUATION COEFFICIENTS

The analytical evaluation of the integrals in equations (3.5) to (3.8) is extremely cumbersome and its extension to general boundary curves may not be possible. Numerical integration coupled with interpolation of the boundary curve provides an attractive alternative. Various numerical integration schemes have been employed by different authors. Bezzine & Gamby [1978] use a trapezoidal formula with 20 sampling points. Some authors have used Simpson's rule (see, for example, Brickell [1970], and Segedin & Brickell [1968]) and others have employed Gaussian quadrature (see, for example, Stern [1979], and Tottenham [1979]). Tottenham [1979] has employed both analytical and numerical integration with linear elements.

The evaluation of the integrals presented in this work is carried out using Gaussian quadrature formulae. However, care must be taken when the integration interval contains the self point, P . In these cases, an alternative treatment is necessary.

3.5.1 Integration Over an Element Not Containing the Self Point, P

For the evaluation of the boundary integrals over elements which do not contain the self point, P , a one-dimensional Gaussian quadrature formula can be used. This is given by:

$$\int_{-1}^{+1} f(\xi) d\xi = \sum_{i=1}^n w_i f(\xi_i) + E_n \quad (3.10)$$

where the w_i are weighting coefficients, ξ_i are the coordinates of the i th integration point, $f(\xi_i)$ is the value of the function at this integration point, and n is the total number of integration points (see, for example, Stroud & Secrest [1966]).

Tables of Gauss point coordinates and weighting coefficients

can be found in most texts on numerical analysis (see, for example, Scheid [1968]). The integration in equations (3.5) to (3.8) is with respect to the arc length, S . Since the coordinates and the coefficients for the numerical integration are referred to the ξ coordinate, a transformation relationship between S and ξ is required. This can be obtained using the chain rule, giving:

$$dS = \frac{dS}{d\xi} d\xi \quad (3.11)$$

$dS/d\xi$ is called the Jacobian of the transformation and is derived as follows:

$$dS = (dx^2 + dy^2)^{\frac{1}{2}} \quad (3.12)$$

Since x and y are written as functions of ξ by the use of the shape functions, the derivative in equation (3.9) with respect to ξ can be written as:

$$dx = \frac{dx}{d\xi} d\xi \quad (3.13)$$

$$dy = \frac{dy}{d\xi} d\xi \quad (3.14)$$

Substitution of equations (3.13) and (3.14) into equation (3.12) gives:

$$dS = \left\{ \left[\left(\frac{dx}{d\xi} \right)^2 + \left(\frac{dy}{d\xi} \right)^2 \right] d\xi^2 \right\}^{\frac{1}{2}} \quad (3.15)$$

$$\text{which reduces to: } \frac{dS}{d\xi} = \left[\left(\frac{dx}{d\xi} \right)^2 + \left(\frac{dy}{d\xi} \right)^2 \right]^{\frac{1}{2}} \quad (3.16)$$

For the geometrical representation described by equations (3.1) to (3.4), the Jacobian is computed as follows:

$$\frac{dx}{d\xi} = \frac{dN_1}{d\xi} x_1 + \frac{dN_2}{d\xi} x_2 = -\frac{1}{2}x_1 + \frac{1}{2}x_2 \quad (3.17)$$

$$\frac{dy}{d\xi} = \frac{dN_1}{d\xi} y_1 + \frac{dN_2}{d\xi} y_2 = -\frac{1}{2}y_1 + \frac{1}{2}y_2 \quad (3.18)$$

From equation (3.16):

$$\frac{dS}{d\xi} = \left\{ \frac{1}{4}(x_2 - x_1)^2 + \frac{1}{4}(y_2 - y_1)^2 \right\}^{\frac{1}{2}} \quad (3.19)$$

$$\frac{dS}{d\xi} = \frac{1}{2}L \quad (3.20)$$

where L is the length of the element. Therefore, a typical integral with respect to dS can now be written as:

$$\int (\cdot) dS = \int (\cdot) \frac{L}{2} d\xi \quad (3.21)$$

The computer program which was written based on the implementation described in this chapter incorporates a six point Gaussian quadrature rule. This allows for a very accurate evaluation of the integrals over each element.

3.5.2 Integration Over an Element Containing the Self Point, P

3.5.2.1 Logarithmic singularities

The Gaussian quadrature described in Section 3.5.1 is not suitable when the integrand is singular. For logarithmically singular kernels, an alternative numerical integration scheme is employed which is given by the formula:

$$\int_0^1 \ln \left(\frac{1}{\zeta} \right) \delta(\zeta) d\zeta = \sum_{i=1}^n w_i \delta(\zeta_i) \quad (3.22)$$

where the w_i are weighting coefficients, ζ_i is the coordinate of the

i th sampling point, and n is the total number of such points in the range $0 < \zeta < 1$. Tables of coordinates and weights for logarithmic quadrature are given by Stroud & Secrest [1966]. Note that the limits of integration vary between zero and one. This means that the integration has to be carried out over the two halves of the element separately (see Figure 3.4). Furthermore, the kernels in equations (3.5) and (3.6) must be altered to take the form shown in equation (3.22). This is done by noting that, along the element, the distance r from P is given by:

$$r = \frac{1}{2} L \zeta \quad (3.23)$$

where L is the length of the element. The kernels in equation (3.5) which contain logarithmic terms can now be written as (see equations (2.78)):

$$W = \frac{1}{8\pi D} r^2 \ln \frac{1}{2} L \zeta = \frac{1}{8\pi D} r^2 \ln \frac{1}{2} L - \frac{1}{8\pi D} r^2 \ln \frac{1}{\zeta} \quad (3.24)$$

$$\begin{aligned} \frac{\partial W}{\partial n_Q} &= \frac{1}{8\pi D} 2r \ln \frac{1}{2} L \zeta + \frac{r \cos \beta}{8\pi D} = \frac{1}{4\pi D} r \ln \frac{1}{2} L - \frac{1}{4\pi D} r \ln \frac{1}{\zeta} + \frac{r \cos \beta}{8\pi D} \\ &= \frac{1}{8\pi D} (2r \ln \frac{1}{2} L + r \cos \beta) - \frac{1}{4\pi D} r \ln \frac{1}{\zeta} \end{aligned} \quad (3.25)$$

$$\begin{aligned} M_n &= - \frac{1-\nu}{8\pi} \cos 2\beta - \frac{2(1+\nu)}{8\pi} - \frac{2(1+\nu)}{8\pi} \ln \frac{1}{2} L \zeta \\ &= - \frac{1-\nu}{8\pi} \cos 2\beta - \frac{2(1+\nu)}{8\pi} - \frac{2(1+\nu)}{8\pi} \ln \frac{1}{2} L + \frac{2(1+\nu)}{8\pi} \ln \frac{1}{\zeta} \end{aligned} \quad (3.26)$$

However, the shear kernel in equation (3.5) is r^{-1} singular, and cannot be integrated numerically. The evaluation of coefficients which involve singularities of r^{-1} or higher order will be discussed in Section 3.5.2.2.

Proceeding as above, the kernels in equation (3.6)

can now be written as (see equations (2.97)):

$$w_2 = - \frac{1}{8\pi D} \kappa \cos \psi - \frac{2\kappa}{8\pi D} \cos \psi \ln \frac{1}{2} L + \frac{2\kappa \cos \psi}{8\pi D} \ln \frac{1}{\zeta} \quad (3.27)$$

$$\begin{aligned} \frac{\partial w_2}{\partial n_Q} = & - \frac{1}{8\pi D} \{ 2 \cos (\psi - \beta) + \cos (\psi + \beta) \} - \frac{2 \cos (\psi - \beta)}{8\pi D} \ln \frac{L}{2} \\ & + \frac{2 \cos (\psi - \beta)}{8\pi D} \ln \frac{1}{\zeta} \end{aligned} \quad (3.28)$$

The moment and effective shear kernels in equation (3.6) behave like κ^{-1} and κ^{-2} , respectively, and their evaluation will be discussed in Section 3.5.2.2. Equation (3.7) has only one logarithmically singular kernel, and can be written as:

$$\begin{aligned} w_3 = & - \frac{1}{8\pi} \{ (1 + \nu) + 2(1 + \nu) \ln \frac{1}{2} L \zeta + (1 - \nu) \cos 2\psi \} \\ w_3 = & - \frac{1}{8\pi} \{ (1 + \nu) + (1 - \nu) \cos 2\psi + 2(1 + \nu) \ln \frac{L}{2} \} \\ & + \frac{1}{8\pi} \{ 2(1 + \nu) \ln \frac{1}{\zeta} \} \end{aligned} \quad (3.29)$$

Equation (3.8), however, does not contain any logarithmic singular kernels.

The transformed kernels expressed by equations (3.24) to (3.29) can now be numerically integrated upon noting that only the terms involving $\ln 1/\zeta$ must be integrated using the logarithmic quadrature. The remaining terms are integrated using ordinary Gaussian quadrature. Along the self element, these kernels can be further simplified upon noting that for $\xi < 0$, $\psi = \beta = \pi/2$, and for $\xi > 0$, $\psi = \beta = 3\pi/2$; therefore, equations (3.24) to (3.29) simplify to:

$$w = \frac{1}{8\pi D} \kappa^2 \ln \frac{1}{2} L - \frac{1}{8\pi D} \kappa^2 \ln \frac{1}{\zeta} \quad (3.30)$$

$$\frac{\partial W}{\partial n_Q} = \frac{\kappa}{4\pi D} \ln \frac{1}{2} L - \frac{\kappa}{4\pi D} \ln \frac{1}{\zeta} \quad (3.31)$$

$$M = - \frac{1+3\nu}{8\pi} - \frac{2(1+\nu)}{8\pi} \ln \frac{1}{2} L + \frac{1+\nu}{4\pi} \ln \frac{1}{\zeta} \quad (3.32)$$

$$W_2 = 0 \quad (3.33)$$

$$\frac{\partial W_2}{\partial n_Q} = - \frac{1}{8\pi D} (1 + 2 \ln \frac{L}{2}) + \frac{1}{4\pi D} \ln \frac{1}{\zeta} \quad (3.34)$$

$$W_3 = - \frac{1}{8\pi} \{2\nu + 2(1+\nu) \ln \frac{L}{2}\} + \frac{1}{4\pi} (1+\nu) \ln \frac{1}{\zeta} \quad (3.35)$$

As can be seen, some of the above terms are constant along the element and their integration can be carried out simply by multiplying the pertinent constant by the element length.

3.5.2.2 Singularities of order κ^{-1} and higher

The numerical integration schemes described above cannot be used for the evaluation of integrals containing kernels which have singularities of the order κ^{-1} or higher over the self element. These integrals can either be evaluated analytically, as outlined in Chapter 2, or, alternatively, their effective values can be evaluated by using certain known deflection fields. It must be noted that, for example, the coefficient C_1^I in equation (3.5) is made up of two terms: (a) the integral of the effective shear kernel, V , over the self element, and (b) a free term which represents the discontinuity jump in deflection as the point p , inside the domain, approaches the boundary. The procedure described here for determining these coefficients allows for any such compound terms to be computed directly and no distinction need be made between free terms or terms involving integrals.

The coefficient for w_p in equations (3.5) to (3.8) can be computed by noting that for a constant deflection field, $w = 1$, these equations yield (see Tottenham [1979]):

$$C_1^I + \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_i V \, dS + \sum_{k=1}^k [T] = 0 \quad (3.36a)$$

$$C_1^{II} + \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_i V_2 \, dS + \sum_{k=1}^k [T_2] = 0 \quad (3.36b)$$

$$C_1^{III} + \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_i V_3 \, dS + \sum_{k=1}^k [T_3] = 0 \quad (3.36c)$$

$$C_1^{IV} + \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_i V_4 \, dS + \sum_{k=1}^k [T_4]_k = 0 \quad (3.36d)$$

where ℓ is the total number of elements, the first summation excludes the self element and the second summation is over all the corners. Equations (3.36), therefore, provide expressions for the coefficient of w_p in equations (3.5) to (3.8), which hold irrespective of the order of singularity of the kernels. The integrals are evaluated numerically using ordinary Gaussian quadrature formulae. The coefficient for $\partial w / \partial n_p$ in equations (3.6) to (3.8) can be computed by imposing a deflection field on the plate, giving $\partial w / \partial n_p = 1$. Such a deflection field can be written as:

$$w = n_p \quad (3.37)$$

where n_p is the unit outward normal direction at point P . As can be seen from Figure 3.5, this is a linearly varying deflection field. Substituting equation (3.37) into equations (3.6) to (3.8), and noting that $w_p = 0$ (see Figure 3.5) for this deflection field, gives:

$$C_2^{II} = - \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_i V_2 \, w \, dS + \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_i M_2 \frac{\partial w}{\partial n} \, dS - \sum_{\lambda=1}^k [T_2]_{\lambda} w \quad (3.38a)$$

$$C_2^{III} = - \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_i V_3 w \, dS + \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_i M_3 \frac{\partial w}{\partial n} \, dS - \sum_{k=1}^k [\Gamma_3] w_k \quad (3.38b)$$

$$C_2^{IV} = - \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_i V_4 w \, dS + \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_i M_4 \frac{\partial w}{\partial n} \, dS - \sum_{k=1}^k [\Gamma_4] w_k \quad (3.38c)$$

In equations (3.38), all integrations are computed numerically using Gaussian quadrature. Proceeding in the same manner, the coefficient for moment in equations (3.7) and (3.8) is obtained by imposing a quadratic deflection field, giving unit moment at point P . This is given by:

$$w = - \frac{1}{2D} n_p^2 \quad (3.39)$$

It can be verified that:

$$w_P = 0 \quad (3.40a)$$

$$\frac{\partial w}{\partial n_P} = - \frac{1}{D} n_P = 0 \quad (3.40b)$$

$$m_P = - D \left(\frac{\partial^2 w}{\partial n_P^2} + \nu \frac{\partial^2 w}{\partial t_P^2} \right) = - D \times \left(- \frac{1}{D} \right) = 1 \quad (3.40c)$$

Substituting equation (3.39) into equations (3.7) and (3.8) gives:

$$\begin{aligned} C_3^{III} = & - \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_i V_3 w \, dS + \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_i M_3 \frac{\partial w}{\partial n} \, dS - \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_i \frac{\partial w_3}{\partial n} m \, dS \\ & + \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_i w_3 v_n \, dS - \sum_{\lambda=1}^k \{ [\Gamma_3] w - [t] w_3 \} \end{aligned} \quad (3.41a)$$

$$\begin{aligned} C_3^{IV} = & - \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_i V_4 w \, dS + \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_i M_4 \frac{\partial w}{\partial n} \, dS - \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_i \frac{\partial w_4}{\partial n} m \, dS \\ & + \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_i w_4 v_n \, dS - \sum_{\lambda=1}^k \{ [\Gamma_4] w - [t] w_4 \} \end{aligned} \quad (3.41b)$$

It is important to note that for the quadratic deflection field given by equations (3.39), the twisting moment is equal to zero only along directions parallel and perpendicular to n_P . Along an arbitrary direction, there is, in general, a non-zero value for twisting moment, hence resulting in a non-zero value of effective shear force. Finally, the coefficient for effective shear force in equation (3.8) is obtained by imposing a cubic deflection field on the plate, giving unit effective shear force at point P . This is given by:

$$w = - \frac{1}{6D} n_P^3 \quad (3.42)$$

From equation (3.42), it can be seen that:

$$w_P = - \frac{1}{6D} n_P^3 = 0 \quad (3.43a)$$

$$\frac{\partial w}{\partial n_P} = - \frac{1}{2D} n_P^2 = 0 \quad (3.43b)$$

$$m_P = - D \left(\frac{\partial^2 w}{\partial n_P^2} + \nu \frac{\partial^2 w}{\partial t_P^2} \right) = - D \times \left(- \frac{n_P}{D} \right) = 0 \quad (3.43c)$$

$$\nu_P = - D \left(\frac{\partial^3 w}{\partial n_P^3} + (2-\nu) \frac{\partial^3 w}{\partial n_P \partial t_P^2} \right) = - D \times \left(- \frac{1}{D} \right) = 1 \quad (3.43d)$$

Substitution of equation (3.38) into equation (3.8) now gives:

$$\begin{aligned} C_4^{IV} = & - \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_{\lambda} V_4 w \, dS + \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_{\lambda} M_4 \frac{\partial w}{\partial n} \, dS - \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_{\lambda} \frac{\partial w_4}{\partial n} m \, dS \\ & + \sum_{\substack{i=1 \\ i \neq P}}^{\ell} \int_{\lambda} w_4 \nu_n \, dS - \sum_{\lambda=1}^k \{ [T_4] w - [t] w_4 \}_{\lambda} \end{aligned} \quad (3.44)$$

3.6 EVALUATION OF THE SECOND MEMBER

Having computed the equation coefficients, as described in the previous section, they are then placed in the $[A]$ and $[\bar{A}]$ matrices in equation (3.7) according to the following rules:

- (i) The coefficients for all unknown variables are placed in the $[A]$ matrix.
- (ii) The coefficients for all variables given as boundary conditions are placed in the $[\bar{A}]$ matrix.

Equation (3.7) can now be written as:

$$[A] \{\bar{u}\} = \{\bar{F}\} - [\bar{A}] \{\bar{b}\} \quad (3.45)$$

The second member of this equation contains two parts, $\{\bar{F}\}$ and $[\bar{A}]\{\bar{b}\}$. The latter term can be computed simply by multiplying the $[\bar{A}]$ matrix by the boundary conditions $\{\bar{b}\}$. The first term, $\{\bar{F}\}$, is obtained by evaluating the integral, over the domain of the plate, of the product of the transverse load intensity, F , and the deflection kernel for equations (3.5) and (3.6) at each point P . In the chapter relating to the quadratic implementation, a procedure is described for evaluating the domain integral referred to above for a general type of loading. However, only concentrated and uniformly distributed loads are considered here.

3.6.1 Concentrated Loads

If the loading on the plate consists of concentrated loads, F_i , the value of $\iint F W da$ is simply the work done by the concentrated loads, F_i , through the deflections of the fundamental solution at the point of application of the load. For example, considering Figure 3.6,

the contribution to the boundary integral equations (3.5) and (3.6) from the concentrated loads F_1 and F_2 are given by:

$$\iint W F dA = W(\kappa_1) F_1 + W(\kappa_2) F_2 \quad (3.46)$$

$$\iint W_2 dA = W_2(\kappa_1, \psi_1) F_1 + W_2(\kappa_2, \psi_2) F_2 \quad (3.47)$$

3.6.2 Uniformly Distributed Load

For a uniformly distributed load, the evaluation of the contributions from the transverse loading involves the integration of the deflection kernel over the plate domain. The load intensity, F , is a constant and can remain outside the integral sign, i.e.

$$\iint W F dA = F \iint W dA \quad (3.48)$$

This integral is also evaluated numerically using a two-dimensional Gaussian quadrature. This is given by (see, for example, Stroud & Secrest [1966]):

$$\int_{-1}^{+1} \int_{-1}^{+1} F(\xi, n) d\xi dn = \sum_{i=1}^n \sum_{j=1}^n w_i w_j F(\xi_i, n_j) \quad (3.49)$$

where w_i and w_j are weighting coefficients, ξ_i and n_j are the coordinates of the Gaussian quadrature points, and n is the number of quadrature points.

To evaluate the integral in equation (3.48) using equation (3.49), the plate domain is discretised into a number of rectangular elements, as shown in Figure 3.7. On each element, a set of coordinates ξ, n are defined as depicted by Figure 3.8. The element has associated with it four nodes, as shown. The element geometry is then defined using

the coordinates of the four nodes and appropriate interpolation or shape functions as follows (see, for example, Zienkiewicz [1977]):

$$x = \sum_{i=1}^4 N_i x_i \quad (3.50)$$

$$y = \sum_{i=1}^4 N_i y_i \quad (3.51)$$

where: $N_1 = \frac{1}{4}(1 - \xi)(1 - \eta) \quad (3.52a)$

$$N_2 = \frac{1}{4}(1 + \xi)(1 - \eta) \quad (3.52b)$$

$$N_3 = \frac{1}{4}(1 + \xi)(1 + \eta) \quad (3.52c)$$

$$N_4 = \frac{1}{4}(1 - \xi)(1 + \eta) \quad (3.52d)$$

As mentioned in Section 3.5.1, a transformation rule is required in changing from $dx dy$ to $d\xi d\eta$. This is given by (see Chirgwin & Plumpton [1972]):

$$\iint f(x, y) dx dy = \iint g(\xi, \eta) J d\xi d\eta \quad (3.53)$$

where J is the Jacobian of the transformation and is given by:

$$J = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix} = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \eta} \quad (3.54)$$

The integrals in equations (3.5) to (3.8) which contain the transverse load intensity, F , are therefore written as:

$$\iint w F dA = F \iint w(\xi, n) J d\xi dn = F \sum_{i=1}^n \sum_{j=1}^n w_i w_j w(\xi_i, n_j) J \quad (3.55)$$

$$\iint w_2 F dA = F \iint w_2(\xi, n) J d\xi dn = F \sum_{i=1}^n \sum_{j=1}^n w_i w_j w_2(\xi_i, n_j) J \quad (3.56)$$

$$\iint w_3 F dA = F \iint w_3(\xi, n) J d\xi dn = F \sum_{i=1}^n \sum_{j=1}^n w_i w_j w_3(\xi_i, n_j) J \quad (3.57)$$

$$\iint w_4 F dA = F \iint w_4(\xi, n) J d\xi dn = F \sum_{i=1}^n \sum_{j=1}^n w_i w_j w_4(\xi_i, n_j) J \quad (3.58)$$

In the present study, these integrals have been evaluated using a 3×3 Gaussian quadrature rule. Tables for the relevant weights and abscissae are given by Stroud & Secrest [1966].

3.7 NUMERICAL RESULTS

The necessary steps involved in obtaining a numerical solution to the problem of the bending of plates under lateral loading by the use of the integral equation formulation presented in Chapter 2 have been described in Section 3.1. As mentioned before, there does not appear to be any investigation available as to which pair of integral equations from (3.5) to (3.8) should be used at each boundary point. Furthermore, in the constant element implementation published to date, the effect of the Kirchhoff corner forces seem to have been ignored. In this section, numerical tests are presented with two objectives in view:

- (1) to ascertain the accuracy of each possible pair of integral equations from (3.5) to (3.8) by applying them to some standard plate bending problems;
- (2) to investigate the effect of the inclusion of the Kirchhoff corner forces in the constant element formulation.

The following numerical tests were carried out using a computer program written by the author incorporating constant elements. In the examples that follow, the numerical integration over non-self elements have been carried out using six Gauss points. Over the self element, the logarithmic numerical integrations have been carried out using seven sampling points.

In most of what follows, the discussion will be mainly focused on the equation pairs combining [(3.5) and (3.6)] and [(3.7) and (3.8)]. As will be seen later, in most of the cases studied, those two pairs provide the most diverse results, and solutions obtained using other equation pairs usually lie between the results obtained using these two.

An important question to consider at this stage is whether a solution to the integral equation pairs chosen from equations (3.5) to (3.8) exists, and if so, is the solution unique? The rigorous treatment of this problem is outside the scope of the present work; however, some authors have investigated this problem in the case of integral equations applied to harmonic (see Christiansen [1975]) and biharmonic (see Christiansen & Hougard [1977]) problems. They have shown that for certain exceptional boundary curves, the solution may be non-unique. As an example, consider a circular plate of unit radius under a centrally applied load. Suppose the pair of integral equations chosen is equations (3.5) and (3.6). The deflection kernels in these equations contain $\ln r$ terms. The contribution to the right hand member of the system of equations, due to the external loading, computed from equations (3.46) and (3.47) would be zero, since $\ln 1.0 = 0.0$. It is, therefore, important to scale the problem domain when such equation pairs are used. The above authors suggest, however, additional integral constraints in order to overcome this difficulty. In the examples presented in this chapter, and later on in Chapters 5 and 6, the problem domain has been scaled appropriately and

no difficulties were encountered.

3.7.1 Clamped and Simply Supported Square Plates

Figure 3.9 shows a square plate of side l with its boundary discretised by a number of constant elements. The plate is assumed to carry a concentrated load applied at its centre. In the present context, two types of boundary conditions are considered: (i) all four edges clamped, and (ii) all four edges simply supported. Furthermore, the corner forces have not been included. The effect of including the corner forces is shown in Section 3.7.2.

Table 3.1 gives the computed values of the deflection at the centre, for each possible pair of equations from (3.5) to (3.8) and various degrees of discretisation, when all four boundaries of the plate are clamped. The figures in brackets indicate percentage errors with respect to the solution given by Timoshenko & Woinowsky-Krieger [1959]. The number of elements in each case has been chosen such that a node is provided at the mid-point of each side. Table 3.2 gives the values of the normal moment along half an edge of the plate and compares them with the finite element solution given by Sanders [1969] as quoted by Bezine [1978]. The mid-point figure of -0.126 is given by Timoshenko & Woinowsky-Krieger [1959]. The rest of the figures due to Sanders have been measured from the graphical representation as given by Bezine & Gamby [1978]. Table 3.3 gives computed values of effective shear force along a half edge of the plate and compares them with the finite element solution given in graphical form in the same reference.

Table 3.4 gives the computed values of deflection at the centre of the plate with simply supported boundary conditions. The result for each pair of equations and each degree of discretisation is compared against the solution given by Timoshenko & Woinowsky-Krieger

[1959]. Computed values of slope along a half edge are given in Table 3.5, and are compared against the series solution given by Timoshenko & Woinowsky-Krieger [1959]. Finally, Table 3.6 compares the computed values of effective shear force acting along the half edge of the plate with the values measured from the graphical representation given by Timoshenko & Woinowsky-Krieger [1959].

From Table 3.1, it can be seen that when the chosen pair of equations does not include equation (3.5), the accuracy for the computed values of the central deflection deteriorates. This effect seems much more pronounced for the simply supported case, as is evident from Table 3.4. In fact the figures suggest that the inaccuracy increases with respect to the order of singularity of the kernels in the chosen pair of equations. The most inaccurate values are obtained using equations (3.7) and (3.8). A more important point to note is that in this case, the results given for simply supported boundary conditions is totally unreliable compared with that given for clamped boundary conditions. In fact, by increasing the number of elements to 76, the error in the computation of the central deflection was reduced to 17% for the clamped boundary condition case, whereas no significant improvement was obtained for the simply supported one. This would suggest that for clamped boundary conditions, using equations (3.7) and (3.8), convergence to the correct solution is possible; however, a very large number of elements must be used in discretising the boundary. Equations (3.7) and (3.8), however, do not seem to provide any reliable solution for the central deflection when the plate boundaries are simply supported. The most accurate values seem to be given by equations (3.5) and (3.6), even for relatively coarse meshes.

In order to understand the reason for the above observation, it is necessary to examine the boundary solutions obtained using each

equation pair. From Table 3.2, it can be seen that for points remote from the corners, the computed values for moments along a half edge differ by an average of 2.5% between the solution given by equations [(3.5) and (3.6)] and [(3.7) and (3.8)]. However, for points near corners, this discrepancy is larger. A similar observation can be made for the effective shear force along a half edge of the plate given in Table 3.3. In fact, for the more singular equations, the effective shear force near the corners seems to be slightly high. The inaccuracies observed in the computation of the central deflections can, therefore, be attributed to the discrepancies between the computed values along the boundary. For the plate with simply supported boundary conditions, the situation is somewhat different. Taking equations [(3.5) and (3.6)] and [(3.7) and (3.8)] as the two extreme conditions, it can be seen that the computed values along the edge for slope and effective shear force using these equations vary considerably. Table 3.5 shows that the computed values for slope using equations (3.5) and (3.6) agree very well with the series solution given by Timoshenko & Woinowsky-Krieger [1959]. However, the slopes computed using equations (3.7) and (3.8) are inaccurate, even for points remote from corners. In both cases, however, the computed values of effective shear force for points remote from the mid-point deviate from those given by Timoshenko & Woinowsky-Krieger to such an extent that as the corner is approached, the computed values become unreliable. In short, numerical results obtained by using equations (3.5) and (3.6) are satisfactory, except for the computation of effective shear force along the boundary for the simply supported case. The most inaccurate results are given by equations (3.7) and (3.8). If the plate boundaries are clamped, the tests show that convergence to the correct solution using equations (3.7) and (3.8) is possible, only if a very large number of elements is used. However, if the plate boundaries are simply supported,

these equations provide totally unreliable results. As can be seen from the tables, the results obtained by other equation pairs seem to lie between those given by equations [(3.5) and (3.6)] and [(3.7) and (3.8)].

The exact explanation as to why the above equations behave in this way is probably in the realm of mathematicians; however, an intuitive argument based on the numerical results obtained here is given as follows: As mentioned at the beginning of this section, the numerical solutions obtained above using equation pairs from (3.5) to (3.8) do not include the terms containing the corner reactions. If the two edges either side of a rectangular corner are clamped, the corner reaction at that corner is zero. However, if the two edges either side of a right-angled corner are simply supported, there will, in general, be a non-zero value for the corner reaction at that point. The author believes that it is the exclusion of these corner reactions that results in the poor performance of some of the equation pairs. This is qualitatively supported by the fact that the results for the simply supported case are worse than those for the clamped case for equation pairs which do not provide accurate results, for example, (3.7) and (3.8). It is believed that this is because, in the clamped case, the value for the corner reaction is zero and ignoring it, therefore, has a much lesser effect than in the simply supported case, where there is a non-zero value for the corner reaction. For example, the results obtained using equations (3.7) and (3.8) for the simply supported case are totally unreliable, whereas in the clamped case convergence to the correct solution can be detected, albeit slow. Furthermore, the results show that, in the simply supported case, for all equation pairs, the computed values of effective shear force along the boundary deteriorates as the corner is approached. In fact, even for equations (3.5) and (3.6), the computed values near the corner are highly inaccurate. It appears as if the effect of all vertical forces have been combined together and that the

computed values of the effective shear force near the corners have tried to replace the corner reactions. Although in this case the effective shear force along the boundary near the corners is computed incorrectly, the results show that the computed values for slope and internal deflection are still accurate for equation pairs containing the least singular kernels. However, for equation pairs containing the more strongly singular kernels, this situation does not seem to hold.

In Section 3.7.2, numerical results for test cases similar to those presented above are given, but this time the corner forces are accounted for, so that the validity of the above argument can be verified.

3.7.2 Clamped and Simply Supported Square Plates: Corner Reactions Included in BIE Formulations

In this section, numerical results are presented for simply supported square plates under a centrally applied point load. However, in this case, the effect of the corner reactions present in equations (3.5) to (3.8) is incorporated in the computer program. The inclusion of the corner reactions introduces an extra unknown at each corner point. Therefore, an additional equation is required at each corner to solve for this unknown. As mentioned in Section 3.3, this additional equation is provided either by equation (3.5) or by the discretised version of the corner force equation. This latter equation is obtained by writing equation (2.141) on either side of a corner and subtracting the resulting two equations from one another. Note that since, by definition, the twisting moment jump at a corner is in the direction of increasing arc length, equation (2.141) on the negative side of a corner has to be subtracted from equation (2.141) on the positive side (see Figure 3.10).

Tables 3.7 to 3.10 indicate clearly that the inclusion of the corner reactions dramatically affect the computed value of the central

deflection. A comparison with Tables 3.1 and 3.2 shows that accurate values for the central deflection can now be obtained for almost any pair of integral equations chosen from (3.5) to (3.8). It is seen that by including the corner reactions in the computations, accurate results can be obtained, even for equation pairs containing highly singular kernels, particularly equations (3.7) and (3.8), where previous numerical tests provided unreliable results. Nevertheless, it appears that the choice of the corner equation affects the accuracy of the solution obtained. It is seen that the choice of the corner equation has little effect on the computation of the central deflection, when the equation pairs used do not contain highly singular kernels, for example, equations (3.5) and (3.6). However, if the equation pair used contains highly singular kernels, the difference in the solutions obtained can be large. Furthermore, this discrepancy is much more pronounced when the plate boundaries are simply supported. Tables 3.9 and 3.10 indicate that the most favourable choice of corner equation is dependent on the choice of the boundary integral equations used. If the equation pair used contains highly singular kernels, equation (3.5) provides a better choice for the corner equation. However, if the equation pair used contains kernels which have singularities of a lower order, both corner equations provide a similar solution.

The above discussion is based on the accuracy of the computed values of the central deflection. However, in order to decide which combination of equations provides better overall results, it is also necessary to examine the solution obtained along the boundary. Tables 3.11 and 3.12 show that for a square plate with clamped boundaries, the computed value of normal moments along an edge agrees very favourably with the results given by Sanders, as quoted by Bezine & Gamby [1978], for all pairs of integral equations and both choices of corner equation.

Similarly, Tables 3.13 and 3.14 show that the computed values of slope along an edge are also satisfactory for all equation pairs and both choices of corner equations. However, once again, it is seen that the solution obtained using equation (3.5) at the corner is more accurate when the boundary integral equations used contain highly singular kernels. Furthermore, comparing these results with Table 3.5 shows that the inclusion of the corner reactions in the computations has markedly increased the accuracy of the solution obtained using these equation pairs. Tables 3.15 and 3.16 show that, for all combinations of equations, the computed value of effective shear force along the boundary is in close agreement with the numerical solution given by Sanders [1969], as quoted by Bezine & Gamby [1978], for boundary points remote from corners. However, in the vicinity of a corner point, the differences between the various BIE solutions and the Sanders solution are magnified. Nevertheless, it can be seen that there is overall agreement, at least qualitatively, on the variation of the effective shear force along the boundary. For the case of a simply supported square plate, a somewhat similar situation holds. As can be seen from Tables 3.17 and 3.18, the computed value of the effective shear force, for points remote from corners, is in close agreement with those obtained from the graphical representation given by Timoshenko & Woinowsky-Krieger [1959]. However, in the vicinity of a corner, the solution becomes less reliable. Nevertheless, a comparison with Table 3.6 shows that, for some equation pairs, a significant increase in accuracy has been obtained. This is demonstrated by the small numerical value computed for the effective shear force, in the vicinity of a corner, and its gradual increase to a maximum value at the mid-point. The equation combinations which roughly provide such a solution profile are:

- (i) equations (3.5) and (3.6) with the equation for twisting moment jump at corners;
- (ii) equations (3.5) and (3.8) with the equation for twisting moment jump at corners;
- (iii) equations (3.6) and (3.7) with the equation for twisting moment jump at corners.

Finally, Tables 3.19 to 3.22 indicate that, for the clamped boundary condition case, all equation combinations provide a satisfactory value for the corner reaction. However, for the plate with simply supported boundary conditions, more accurate results are obtained if the corner equation used is that for twisting moment jump at a corner.

3.8 CONCLUDING REMARKS

The numerical test cases presented in the last section were aimed at assessing the accuracy of the various possible pairs of the boundary integral equations that can be used in obtaining a numerical solution to the problem of bending of plates under lateral loading. In the constant element formulations presented to date, the effects of the Kirchhoff corner forces have not been included in the numerical treatment. The numerical results show that, if the Kirchhoff corner forces are ignored, only those equation pairs containing less singular kernels provide an accurate solution. Equation pairs containing the more singular kernels provide unreliable results, especially if the plate boundaries are simply supported. For example, equations (3.5) and (3.6) usually provide more accurate results than equations (3.7) and (3.8). However, even when equations (3.5) and (3.6) are used, the computed values of the effective shear force along the boundary become unreliable in the vicinity of a simply supported corner. This effect is not so strong if the boundaries

are clamped either side of a corner (see Figures 3.11 and 3.12).

The Kirchhoff corner forces can be included in the analysis by writing an equation at each corner of the plate and associating an unknown corner reaction with each corner. This extra equation can be chosen either as equation (3.5) or the equation for twisting moment jump at a corner. It is seen from Tables 3.9 to 3.14 that, by including the Kirchhoff forces in the numerical treatment, accurate results can be obtained for the computed value of the central deflection and slopes and moments along the boundary, for almost all pairs of boundary integral equation. The accuracy of the results, however, is to a certain extent dependent on the choice of the corner equations. For example, the computed value of the central deflection, using equations (3.7) and (3.8), is more accurate if the corner equation chosen is equation (3.5). Slopes and moments along the boundary, however, do not seem to be significantly affected by the choice of the corner equation.

Tables 3.15 to 3.18 show that for points remote from the corners, the computed values of effective shear force along the boundary agree favourably with the finite element results given by Sanders [1969]. However, for points on the boundary which are near corners, the various solutions do not agree. It is important to note that, once again, the simply supported case proves to be the more difficult of the two types of boundary conditions considered. Figures 3.11 and 3.12 show the variation of the effective shear force along a half edge for both type of boundary conditions considered. The boundary integral solutions shown are those obtained using equations (3.5) and (3.6). This seems to be an appropriate choice since this pair of equations usually provides as good as, or better, accuracy than the other equation pairs, and requires less computational effort due to its simpler kernels. It is seen from Figure 3.11 that for clamped boundary conditions, the computed values of

the effective shear force along the boundary agrees reasonably well with the solution given by Sanders. However, it seems that the more accurate values are obtained if corner reactions are ignored. This is to be expected since ignoring the corner reactions has the same effect as including them, but setting their values to be zero, which is the correct value for the corner reaction at a clamped corner. Tables 3.19 to 3.22 show that when corner reactions are included, the computed values for corner reactions are small but not zero for clamped boundary conditions. It appears that these non-zero corner forces affect the computed values of effective shear force along the boundary in the vicinity of a clamped corner.

Figure 3.12 shows that in order to compute the effective shear force along the boundary, for a simply supported plate, to a tolerable degree of accuracy, it is essential to include the corner reactions in the analysis. Furthermore, it is seen that the more acceptable results are obtained by choosing, as the corner equation, the equation for twisting moment jump at a corner. This is also true for the computation of the corner reactions as seen in Tables 3.19 to 3.22.

TABLE 3.1

Deflection, w , at the Centroid of a Clamped Square Plate Under a Centrally Applied Point Load, F (Figure 3.9)

Poisson's ratio, $\nu = 0.3$

$w \times \frac{D}{F L^2}$	Integral Equation Solutions						Exact
	I & II	I & III	I & IV	II & III	II & IV	III & IV	
4 elements	0.00525 (-6.25%)	0.00565 (0.89%)	0.03055 (>500%)	0.00854 (52.5%)	0.16299 (>2000%)	0.54563 (>9000%)	0.00560
12 elements	0.00567 (1.25%)	0.00571 (1.96%)	0.00538 (-3.93%)	0.00573 (2.32%)	0.00397 (-29.11%)	0.00351 (-37.32%)	
28 elements	0.00562 (0.36%)	0.00571 (1.96%)	0.00559 (-0.18%)	0.00617 (10.18%)	0.00281 (-49.82%)	-0.00188 (-66%)	
60 elements	0.00561 (0.18%)	0.00557 (-0.54%)	0.00560 (exact)	0.00574 (2.5%)	0.00508 (-9.29%)	0.00385 (-31.25%)	

- I : equation (3.5)
 II : equation (3.6)
 III : equation (3.7)
 IV : equation (3.8)

TABLE 3.2

Normal Moment, m_n/F , Along a Half Edge for a Clamped Square Plate with a Load, F , at its Centre

60 Constant Elements Used (Figure 3.9)

Poisson's ratio, $\nu = 0.3$

x/L	Sanders [1969]	I & II	I & III	I & IV	II & III	II & IV	III & IV
0.0	-0.1257 (TWK)	-0.126	-0.126	-0.128	-0.126	-0.126	-0.123
0.07	-0.1194	-0.120	-0.120	-0.122	-0.120	-0.120	-0.117
0.13	-0.101	-0.103	-0.103	-0.104	-0.103	-0.102	-0.100
0.2	-0.0796	-0.079	-0.079	-0.079	-0.078	-0.078	-0.076
0.27	-0.049	-0.051	-0.051	-0.051	-0.050	-0.050	-0.049
0.33	-0.0245	-0.025	-0.025	-0.023	-0.023	-0.023	-0.023
0.4	-0.0061	-0.0044	-0.004	-0.006	-0.004	-0.003	-0.003
0.47	-0.0012	+0.002	-0.008	+0.012	+0.002	+0.005	+0.004

TWK : Timoshenko & Woinowsky-Krieger [1959]

I : equation (3.5)

II : equation (3.6)

III : equation (3.7)

IV : equation (3.8)

TABLE 3.3

Effective Shear Force, $-v_n \times (L/F)$, Along a Half Edge for a Clamped Square Plate
with a Load, F , at its Centre. 60 Constant Elements Used (Figure 3.9)

Poisson's ratio, $\nu = 0.3$

x/L	Sanders [1969]	I & II	I & III	I & IV	II & III	II & IV	III & IV
0.0	0.800	0.798	0.795	0.798	0.802	0.795	0.79
0.07	0.740	0.745	0.742	0.745	0.748	0.742	0.738
0.13	0.6	0.598	0.593	0.599	0.6	0.597	0.594
0.2	0.4	0.390	0.388	0.388	0.391	0.39	0.390
0.27	0.16	0.154	0.135	0.160	0.155	0.158	0.158
0.33	-0.084	-0.063	-0.033	-0.083	-0.071	-0.064	-0.059
0.4	-0.228	-0.251	-0.434	-0.153	-0.240	-0.213	-0.208
0.47	-0.116	-0.097	0.086	-0.181	0.110	-0.136	-0.133

I : equation (3.5)
 II : equation (3.6)
 III : equation (3.7)
 IV : equation (3.8)

TABLE 3.4

Deflection, w , at the Centroid of a Simply Supported Square Plate Under a
Centrally Applied Point Load, F (Figure 3.9)

Poisson's ratio, $\nu = 0.3$

$w \times \frac{D}{F L^2}$	Integral Equation Solutions						Exact
	I & II	I & III	I & IV	II & III	II & IV	III & IV	
4 elements	0.01221 (5.26%)	0.01276 (10%)	0.10652 (>800%)	0.01629 (40.43%)	-0.54188 (>-4000%)	-0.25476 (>-2000%)	0.01160
12 elements	0.01204 (3.79%)	0.01186 (2.24%)	0.02701 (>1000%)	0.00259 (-77.67%)	-0.06118 (>-600%)	-0.05581 (>-500%)	
28 elements	0.01161 (0.09%)	0.01149 (-0.95%)	0.01372 (18.28%)	0.00679 (-41.47%)	-0.1952 (>-1000%)	-0.03044 (>-300%)	
60 elements	0.01160 (exact)	0.01165 (0.43%)	0.01224 (5.52%)	0.00860 (-25.86%)	-0.00527 (>-140%)	-0.01944 (>-200%)	

- I : equation (3.5)
 II : equation (3.6)
 III : equation (3.7)
 IV : equation (3.8)

TABLE 3.5

Normal Slope, $-(\partial w / \partial n) \times (L / F D)$, Along Half an Edge of a Simply Supported Square Plate
with a Load, F , at its Centre. 60 Constant Elements Used (Figure 3.9)

Poisson's ratio, $\nu = 0.3$

x/L	TWK	I & II	I & III	I & IV	II & III	II & IV	III & IV
0.0	0.030	0.030	0.028	0.033	0.030	0.022	0.017
0.07	0.029	0.029	0.028	0.032	0.029	0.022	0.016
0.13	0.027	0.027	0.025	0.03	0.027	0.020	0.014
0.2	0.023	0.023	0.022	0.026	0.024	0.016	0.012
0.27	0.019	0.019	0.018	0.021	0.019	0.012	0.009
0.33	0.014	0.014	0.013	0.016	0.014	0.008	0.005
0.4	0.009	0.009	0.007	0.009	0.009	0.003	0.002
0.47	0.003	0.003	0.000	-0.001	0.003	-0.001	-0.001

TWK : Timoshenko & Woinowsky-Krieger [1959]

I : equation (3.5)
 II : equation (3.6)
 III : equation (3.7)
 IV : equation (3.8)

TABLE 3.6

Effective Shear Force, $-\nu_n \times (L/F)$, Along a Half Edge for a Simply Supported Square Plate

Under a Centrally Applied Point Load, F . 60 Constant Elements Used (Figure 3.9)

Poisson's ratio, $\nu = 0.3$

x/L	TWK	I & II	I & III	I & IV	II & III	II & IV	III & IV
0.0	0.621	0.654	0.670	0.661	0.642	0.642	0.623
0.07	0.6	0.629	0.642	0.637	0.618	0.617	0.598
0.13	0.557	0.566	0.584	0.572	0.552	0.55	0.529
0.2	0.471	0.455	0.488	0.480	0.458	0.452	0.430
0.27	0.351	0.412	0.427	0.376	0.35	0.337	0.314
0.33	0.278	0.083	0.173	0.265	0.234	0.209	0.185
0.4	0.163	0.833	1.048	0.110	0.202	0.044	0.026
0.47	0.043	-1.428	-1.812	-0.886	-0.876	-0.706	-0.478

TWK : Timoshenko & Woinowsky-Krieger [1959]

I : equation (3.5)

II : equation (3.6)

III : equation (3.7)

IV : equation (3.8)

TABLE 3.7

Deflection, w , at the Centroid of a Square Plate Under a Centrally Applied Point Load.
Corner Reactions Included Using Equation (3.5). 44 Constant Elements Used (Figure 3.9)

Poisson's ratio, $\nu = 0.3$

$w \times \frac{D}{F L^2}$	Integral Equation Solutions						Exact
	I & II	I & III	I & IV	II & III	II & IV	III & IV	
Clamped	0.00561	0.00562	0.00560	0.00560	0.00563	0.00562	0.0056
Simply Supported	0.01143	0.01158	0.01126	0.01171	0.01156	0.01192	0.0116

I : equation (3.5)
 II : equation (3.6)
 III : equation (3.7)
 IV : equation (3.8)

TABLE 3.8

Deflection, w , at the Centroid of a Square Plate Under a Centrally Applied Point Load.

Corner Reactions Included Using Equation for Twisting Moment Jump at Corners.

44 Constant Elements Used (Figure 3.9)

Poisson's ratio, $\nu = 0.3$

$w \times \frac{D}{F L^2}$	Integral Equation Solutions						Exact
	I & II	I & III	I & IV	II & III	II & IV	III & IV	
Clamped	0.00561	0.00561	0.00560	0.00625	0.00592	0.00615	0.0056
Simply Supported	0.01153	0.01140	0.01170	0.01283	0.00801	0.00271	0.0116

I : equation (3.5)

II : equation (3.6)

III : equation (3.7)

IV : equation (3.8)

TABLE 3.9

Deflection, w , at the Centroid of a Square Plate Under a Centrally Applied Point Load.

Corner Reactions Included Using Equation (3.5). 60 Constant Elements Used (Figure 3.9)

Poisson's ratio, $\nu = 0.3$

$w / (\frac{F a^2}{D})$	Integral Equation Solutions						Exact
	I & II	I & III	I & IV	II & III	II & IV	III & IV	
Clamped	0.00561	0.00562	0.00560	0.00561	0.00562	0.00562	0.0056
Simply Supported	0.0115	0.0116	0.0114	0.0117	0.01156	0.01183	0.0116

I : equation (3.5)

II : equation (3.6)

III : equation (3.7)

IV : equation (3.8)

TABLE 3.10

Deflection, w , at the Centroid of a Square Plate Under a Centrally Applied Point Load.

Corner Reactions Included Using Equation for Twisting Moment Jump at Corners.

60 Constant Elements Used (Figure 3.9)

Poisson's ratio, $\nu = 0.3$

$w/(\frac{F a^2}{D})$	Integral Equation Solutions						Exact
	I & II	I & III	I & IV	II & III	II & IV	III & IV	
Clamped	0.00561	0.00561	0.00560	0.00603	0.00582	0.00616	0.0056
Simply Supported	0.0116	0.0116	0.0117	0.01267	0.00879	0.0033	0.0116

I : equation (3.5)

II : equation (3.6)

III : equation (3.7)

IV : equation (3.8)

TABLE 3.11

Normal Moment, m_n/F , Along a Half Edge for a Clamped Square Plate with a Load, F , at its Centre.

Corner Reactions Included Using Equation (3.5). 60 Constant Elements Used (Figure 3.9)

Poisson's ratio, $\nu = 0.3$

x/L	Sanders [1969]	I & II	I & III	I & IV	II & III	II & IV	III & IV
0.0	-0.126 (TWK)	-0.126	-0.126	-0.127	-0.126	-0.126	-0.125
0.07	-0.119	-0.120	-0.120	-0.121	-0.120	-0.120	-0.119
0.13	-0.101	-0.103	-0.103	-0.104	-0.103	-0.103	-0.103
0.2	-0.080	-0.078	-0.078	-0.079	-0.078	-0.079	-0.078
0.27	-0.049	-0.050	-0.050	-0.050	-0.050	-0.051	-0.051
0.33	-0.025	-0.024	-0.024	-0.023	-0.024	-0.024	-0.025
0.4	-0.006	-0.004	-0.005	-0.003	-0.004	-0.005	-0.006
0.47	-0.001	0.001	0.002	0.002	0.001	0.001	0.001

TWK : Timoshenko & Woinowsky-Krieger [1959]

I : equation (3.5)

II : equation (3.6)

III : equation (3.7)

IV : equation (3.8)

TABLE 3.12

Normal Moment, m_n/F , Along a Half Edge for a Clamped Square Plate with a Load, F , at its Centre.

Corner Reactions Included Using Equation for Twisting Moment Jump at Corners.

60 Constant Elements Used (Figure 3.9)

Poisson's ratio, $\nu = 0.3$

x/L	Sanders [1969]	I & II	I & III	I & IV	II & III	II & IV	III & IV
0.0	-0.126 (TWK)	-0.126	-0.126	-0.127	-0.126	-0.127	-0.126
0.07	-0.119	-0.120	-0.120	-0.121	-0.120	-0.120	-0.120
0.13	-0.101	-0.103	-0.103	-0.104	-0.103	-0.103	-0.103
0.2	-0.080	-0.078	-0.078	-0.079	-0.078	-0.079	-0.079
0.27	-0.049	-0.050	-0.050	-0.050	-0.050	-0.051	-0.052
0.33	-0.025	-0.024	-0.024	-0.024	-0.024	-0.024	-0.026
0.4	-0.006	-0.004	-0.004	-0.002	-0.004	-0.005	-0.007
0.47	-0.001	0.003	0.000	0.000	-0.003	0.000	0.000

TWK : Timoshenko & Woinowsky-Krieger [1959]

I : equation (3.5)

II : equation (3.6)

III : equation (3.7)

IV : equation (3.8)

TABLE 3.13

Normal Slope, $-\{\partial w/\partial n\} \times (D/FL)$, Along Half an Edge of a Simply Supported Square Plate with a Concentrated Load at its Centre. Corner Reactions Included Using Equation (3.5).

60 Constant Elements Used (Figure 3.9)

Poisson's ratio, $\nu = 0.3$

x/L	TWK	I & II	I & III	I & IV	II & III	II & IV	III & IV
0.0	0.030	0.029	0.030	0.028	0.030	0.029	0.029
0.07	0.029	0.028	0.030	0.028	0.029	0.029	0.028
0.13	0.027	0.026	0.028	0.026	0.027	0.027	0.026
0.2	0.023	0.023	0.024	0.022	0.023	0.023	0.023
0.27	0.019	0.019	0.020	0.018	0.019	0.019	0.019
0.33	0.014	0.014	0.015	0.013	0.014	0.014	0.014
0.4	0.009	0.008	0.010	0.008	0.009	0.008	0.009
0.47	0.003	0.002	0.004	0.004	0.003	0.003	0.003

TWK : Timoshenko & Woinowsky-Krieger [1959]

I : equation (3.5)

II : equation (3.6)

III : equation (3.7)

IV : equation (3.8)

TABLE 3.14

Normal Slope, $-(\partial w / \partial n) \times (D / FL)$, Along Half an Edge of a Simply Supported Square Plate with
a Concentrated Load, F , at its Centre. Corner Reactions Included Using Equation for
Twisting Moment Jump at Corners. 60 Constant Elements Used (Figure 3.9)

Poisson's ratio, $\nu = 0.3$

x/L	TWK	I & II	I & III	I & IV	II & III	II & IV	III & IV
0.0	0.030	0.029	0.031	0.030	0.030	0.028	0.026
0.07	0.029	0.029	0.030	0.029	0.029	0.028	0.025
0.13	0.027	0.027	0.028	0.027	0.027	0.025	0.023
0.2	0.023	0.023	0.025	0.024	0.023	0.022	0.020
0.27	0.019	0.019	0.021	0.019	0.019	0.018	0.016
0.33	0.014	0.014	0.016	0.014	0.014	0.013	0.012
0.4	0.009	0.008	0.010	0.009	0.009	0.008	0.007
0.47	0.003	0.003	0.005	0.002	0.003	0.002	0.002

TWK : Timoshenko & Woinowsky-Krieger [1959]

I : equation (3.5)

II : equation (3.6)

III : equation (3.7)

IV : equation (3.8)

TABLE 3.15

Effective Shear Force, $-v_n \times \{L/F\}$, Along Half an Edge of a Clamped Square Plate with
a Load, F , at its Centre. Corner Reactions Included Using Equation (3.5).

60 Constant Elements Used (Figure 3.9)

Poisson's ratio, $\nu = 0.3$

x/L	Sanders [1969]	I & II	I & III	I & IV	II & III	II & IV	III & IV
0.0	0.80	0.798	0.799	0.798	0.802	0.794	0.789
0.07	0.740	0.745	0.745	0.745	0.748	0.741	0.737
0.13	0.600	0.599	0.599	0.598	0.600	0.596	0.593
0.2	0.400	0.390	0.390	0.389	0.390	0.389	0.388
0.27	0.160	0.155	0.155	0.154	0.154	0.155	0.157
0.33	-0.084	-0.066	-0.065	-0.065	-0.072	-0.067	-0.062
0.4	-0.228	-0.238	-0.239	-0.236	-0.243	-0.223	-0.213
0.47	-0.116	-0.142	-0.128	-0.122	-0.134	-0.133	-0.143

I : equation (3.5)

II : equation (3.6)

III : equation (3.7)

IV : equation (3.8)

TABLE 3.16

Effective Shear Force, $-\nu_n \times (L/F)$, Along Half an Edge of a Clamped Square Plate with
a Load, F , at its Centre. Corner Reactions Included Using Equation for
Twisting Moment Jump at Corners. 60 Constant Elements Used (Figure 3.9)

Poisson's ratio, $\nu = 0.3$

x/L	Sanders [1969]	I & II	I & III	I & IV	II & III	II & IV	III & IV
0.0	0.80	0.798	0.798	0.798	0.803	0.793	0.788
0.07	0.74	0.745	0.745	0.744	0.750	0.740	0.736
0.13	0.600	0.598	0.598	0.597	0.602	0.595	0.593
0.2	0.400	0.390	0.390	0.389	0.392	0.388	0.388
0.27	0.150	0.151	0.151	0.153	0.157	0.155	0.157
0.33	-0.080	-0.054	-0.060	-0.062	-0.068	-0.68	-0.063
0.4	-0.220	-0.283	-0.271	-0.252	-0.233	-0.226	-0.214
0.47	-0.10	0.004	-0.093	-0.111	-0.056	-0.132	-0.146

I : equation (3.5)

II : equation (3.6)

III : equation (3.7)

IV : equation (3.8)

TABLE 3.17

Effective Shear Force, $-v_n \times (L/F)$, Along Half an Edge of a Simply Supported Square Plate with a Load, F , at its Centre. Corner Reactions Included Using Equation (3.5).

60 Constant Elements Used (Figure 3.9)

Poisson's ratio, $\nu = 0.3$

x/L	TWK	I & II	I & III	I & IV	II & III	II & IV	III & IV
0.0	0.621	0.647	0.642	0.646	0.651	0.649	0.648
0.07	0.6	0.624	0.617	0.622	0.627	0.625	0.624
0.13	0.557	0.552	0.550	0.556	0.563	0.560	0.558
0.2	0.471	0.469	0.454	0.462	0.470	0.466	0.464
0.27	0.351	0.305	0.340	0.355	0.366	0.359	0.358
0.33	0.278	0.376	0.236	0.245	0.261	0.249	0.249
0.4	0.163	-0.470	0.014	0.144	0.214	0.136	0.145
0.47	0.043	1.908	0.088	0.564	-0.269	0.260	0.464

TWK : Timoshenko & Woinowsky-Krieger [1959]

I : equation (3.5)

II : equation (3.6)

III : equation (3.7)

IV : equation (3.8)

TABLE 3.18

Effective Shear Force, $-v_n \times (L/F)$, Along Half an Edge of a Simply Supported Square Plate with
a Load, F , at its Centre. Corner reactions Included Using Equation for
Twisting Moment Jump at Corners. 60 Constant Elements Used (Figure 3.9)

Poisson's ratio, $\nu = 0.3$

x/L	TWK	I & II	I & III	I & IV	II & III	II & IV	III & IV
0.0	0.621	0.652	0.635	0.651	0.654	0.648	0.640
0.07	0.6	0.626	0.610	0.627	0.631	0.624	0.617
0.13	0.557	0.560	0.542	0.562	0.566	0.558	0.550
0.2	0.471	0.461	0.446	0.468	0.475	0.463	0.455
0.27	0.351	0.368	0.319	0.362	0.372	0.355	0.346
0.33	0.278	0.204	0.252	0.252	0.270	0.242	0.232
0.4	0.163	0.295	-0.236	0.132	0.218	0.119	0.113
0.47	0.043	-0.052	0.547	0.062	-0.072	0.101	0.181

TWK : Timoshenko & Woinowsky-Krieger [1959]

I : equation (3.5)

II : equation (3.6)

III : equation (3.7)

IV : equation (3.8)

TABLE 3.19

Corner Reactions, $[t]/F$, for a Square Plate Under a Centrally Applied Point Load.

Corner Reactions Included Using Equation (3.5). 44 Constant Elements Used (Figure 3.9)

$[t]/F$	Integral Equation Solutions						Exact
	I & II	I & III	I & IV	II & III	II & IV	III & IV	
Clamped	-0.008	-0.004	-0.004	-0.007	-0.005	-0.005	0.000
Simply Supported	0.291	0.076	0.182	0.092	0.148	0.167	0.122

I : equation (3.5)

II : equation (3.6)

III : equation (3.7)

IV : equation (3.8)

TABLE 3.20

Corner Reactions, $[t]/F$, for a Square Plate Under a Centrally Applied Point Load.

Corner Reactions Included Using Equation (3.5). 60 Constant Elements Used (Figure 3.9)

$[t]/F$	Integral Equation Solutions						Exact
	I & II	I & III	I & IV	II & III	II & IV	III & IV	
Clamped	0.004	0.003	0.002	0.004	0.003	0.003	0.000
Simply Supported	0.295	0.100	0.186	0.091	0.147	0.169	0.122

I : equation (3.5)

II : equation (3.6)

III : equation (3.7)

IV : equation (3.8)

TABLE 3.21

Corner Reactions, $[t]/F$, for a Square Plate Under a Centrally Applied Point Load.

Corner Reactions Included Using Equation for Twisting Moment Jump at Corners.

44 Constant Elements Used (Figure 3.9)

$[t]/F$	Integral Equation Solutions						Exact
	I & II	I & III	I & IV	II & III	II & IV	III & IV	
Clamped	0.013	-0.005	-0.004	0.010	-0.006	-0.006	0.000
Simply Supported	0.122	0.136	0.122	0.121	0.123	0.123	0.122

I : equation (3.5)
 II : equation (3.6)
 III : equation (3.7)
 IV : equation (3.8)

TABLE 3.22

Corner Reactions, $[t]/F$, for a Square Plate Under a Centrally Applied Point Load.

Corner Reactions Included Using Equation for Twisting Moment Jump at Corners.

60 Constant Elements Used (Figure 3.9)

$[t]/F$	Integral Equation Solutions						Exact
	I & II	I & III	I & IV	II & III	II & IV	III & IV	
Clamped	0.010	0.002	0.002	0.009	0.004	0.004	0.000
Simply Supported	0.122	0.125	0.122	0.121	0.123	0.123	0.122

I : equation (3.5)

II : equation (3.6)

III : equation (3.7)

IV : equation (3.8)

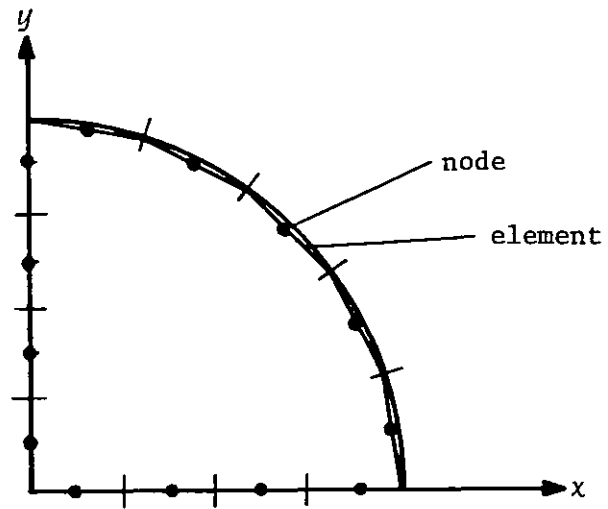


Figure 3.1: Typical boundary discretisation using constant elements

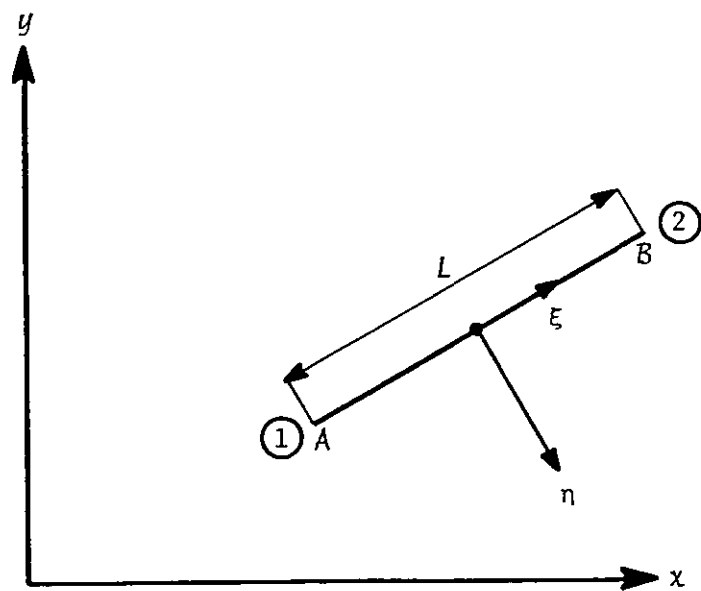


Figure 3.2: A constant element and its local coordinates

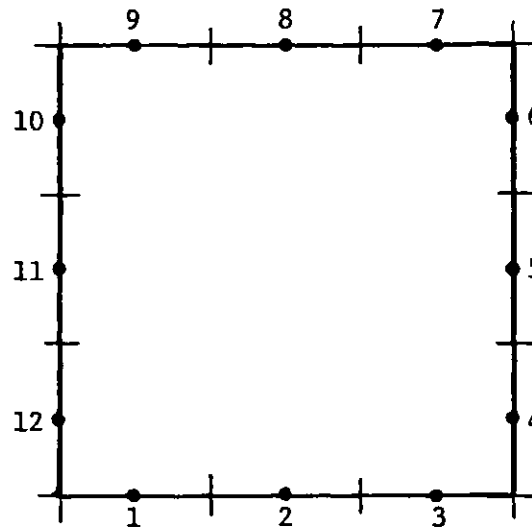
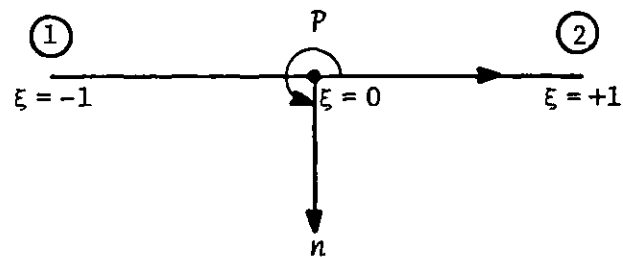
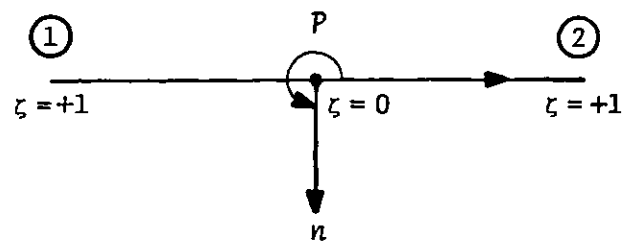


Figure 3.3: Square plate discretised using three constant elements along each side



(a) ξ coordinate system



(b) ζ coordinate system

Figure 3.4: Intrinsic coordinates along a constant element

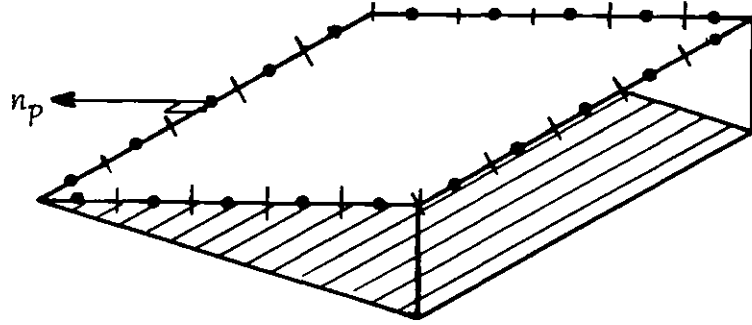


Figure 3.5: A linearly varying displacement field, $w = n_p$

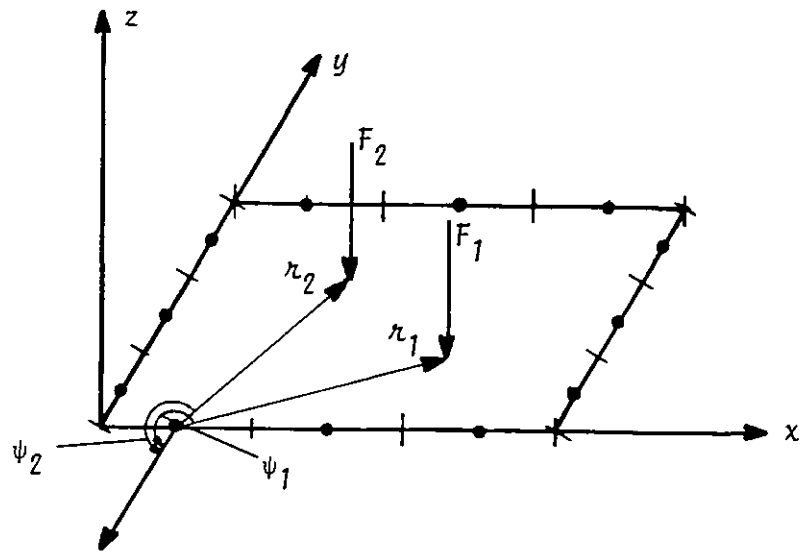


Figure 3.6: Plate under concentrated loads, F_i

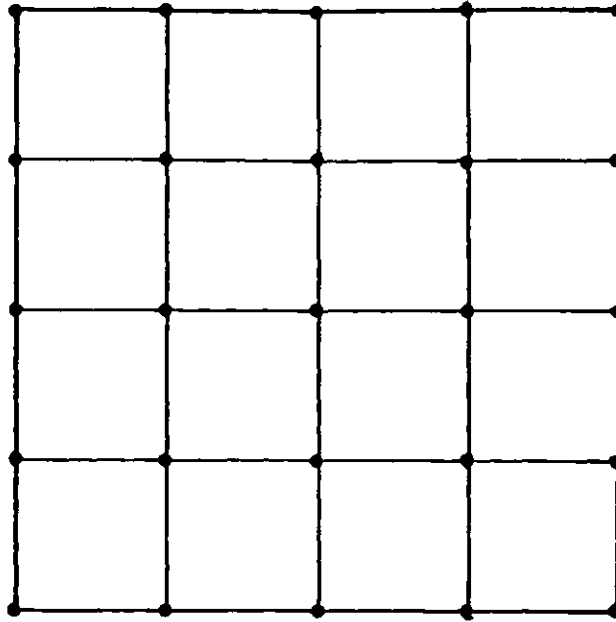


Figure 3.7: Interior discretisation using four noded elements

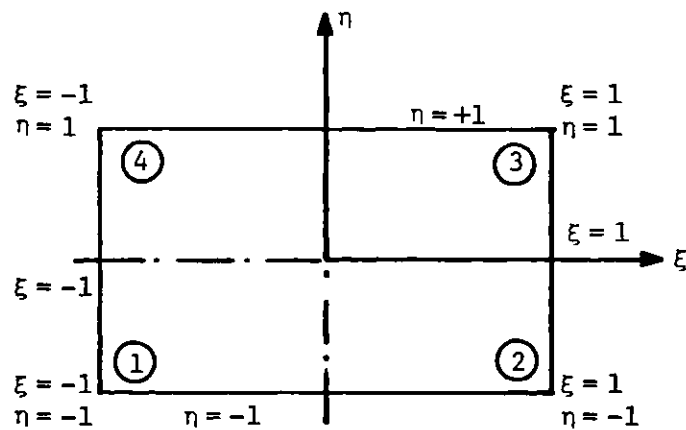


Figure 3.8: An area element in its local ξ, η coordinate system

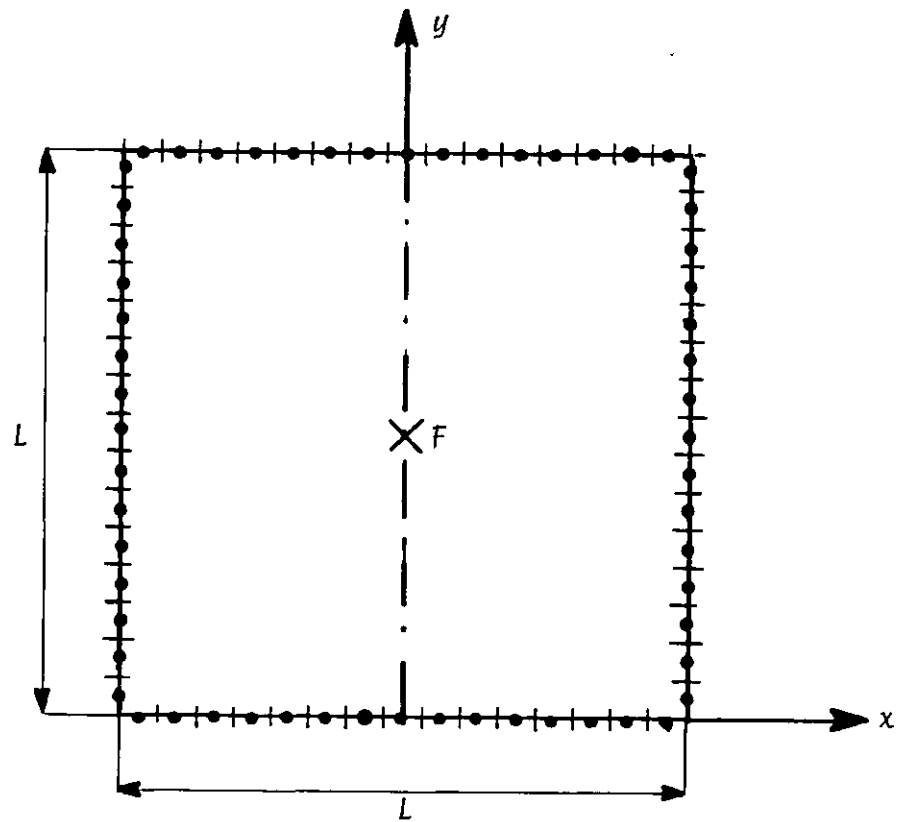


Figure 3.9: Boundary discretisation for a square plate under a centrally applied load, F . 60 constant elements.

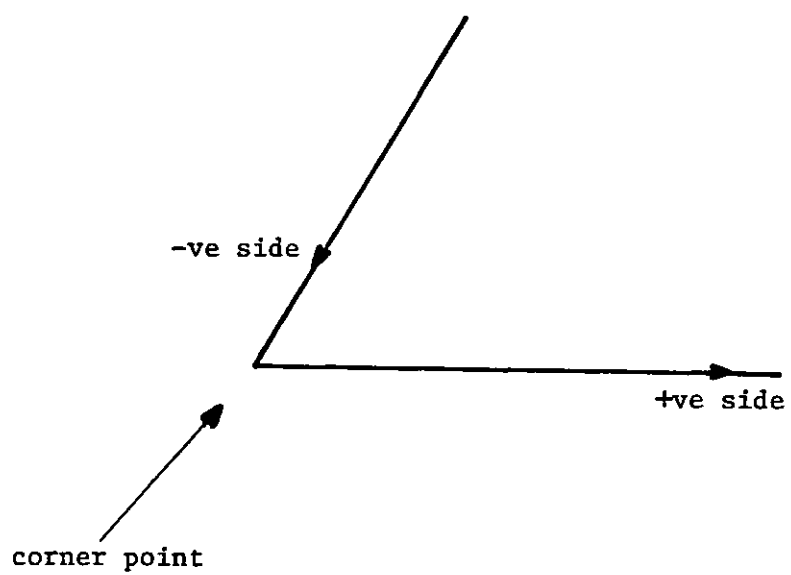


Figure 3.10: Sign convention for positive and negative sides of a corner

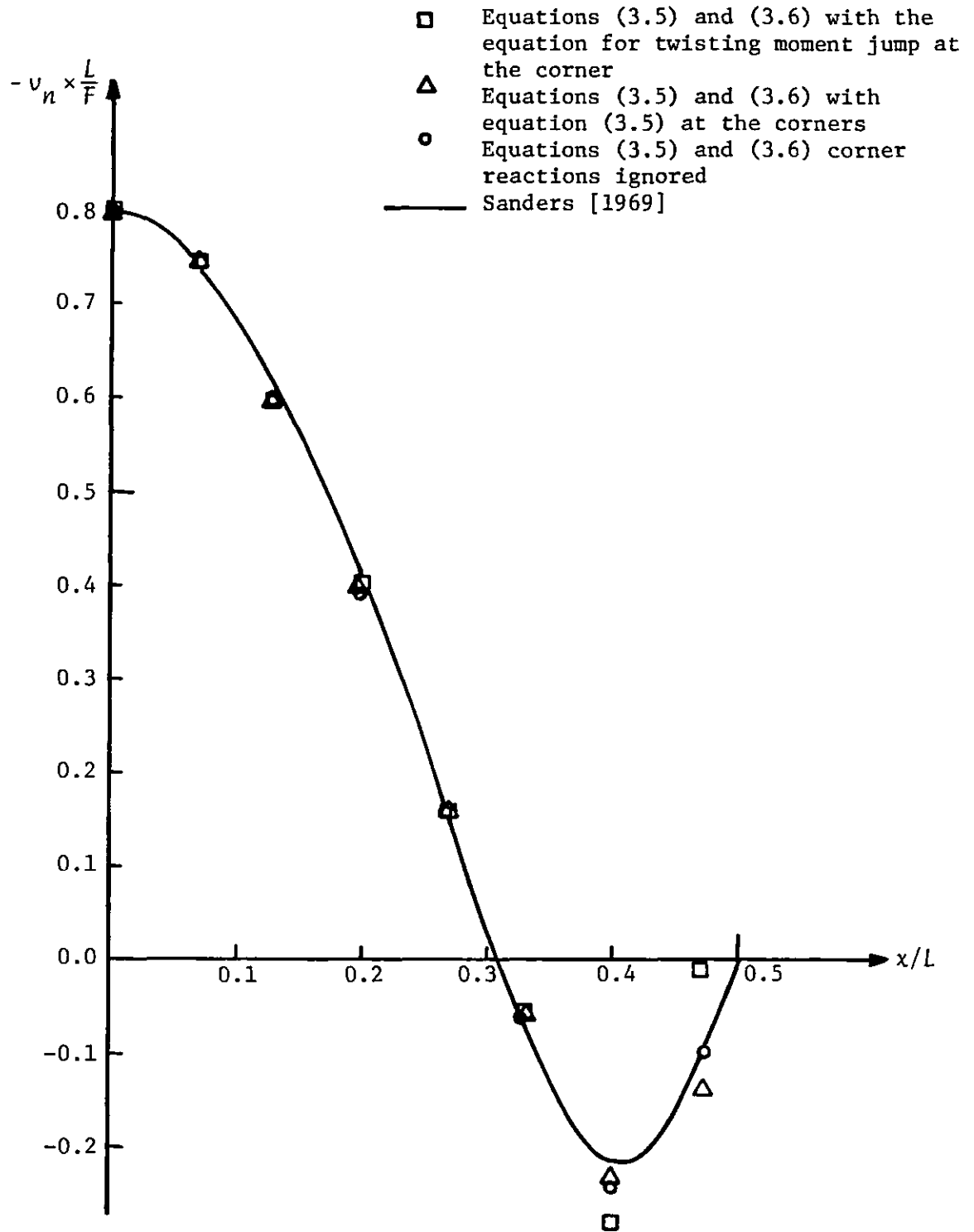


Figure 3.11: Effective shear force along a half edge for a clamped square plate under a centrally applied load, F (see Figure 3.9)

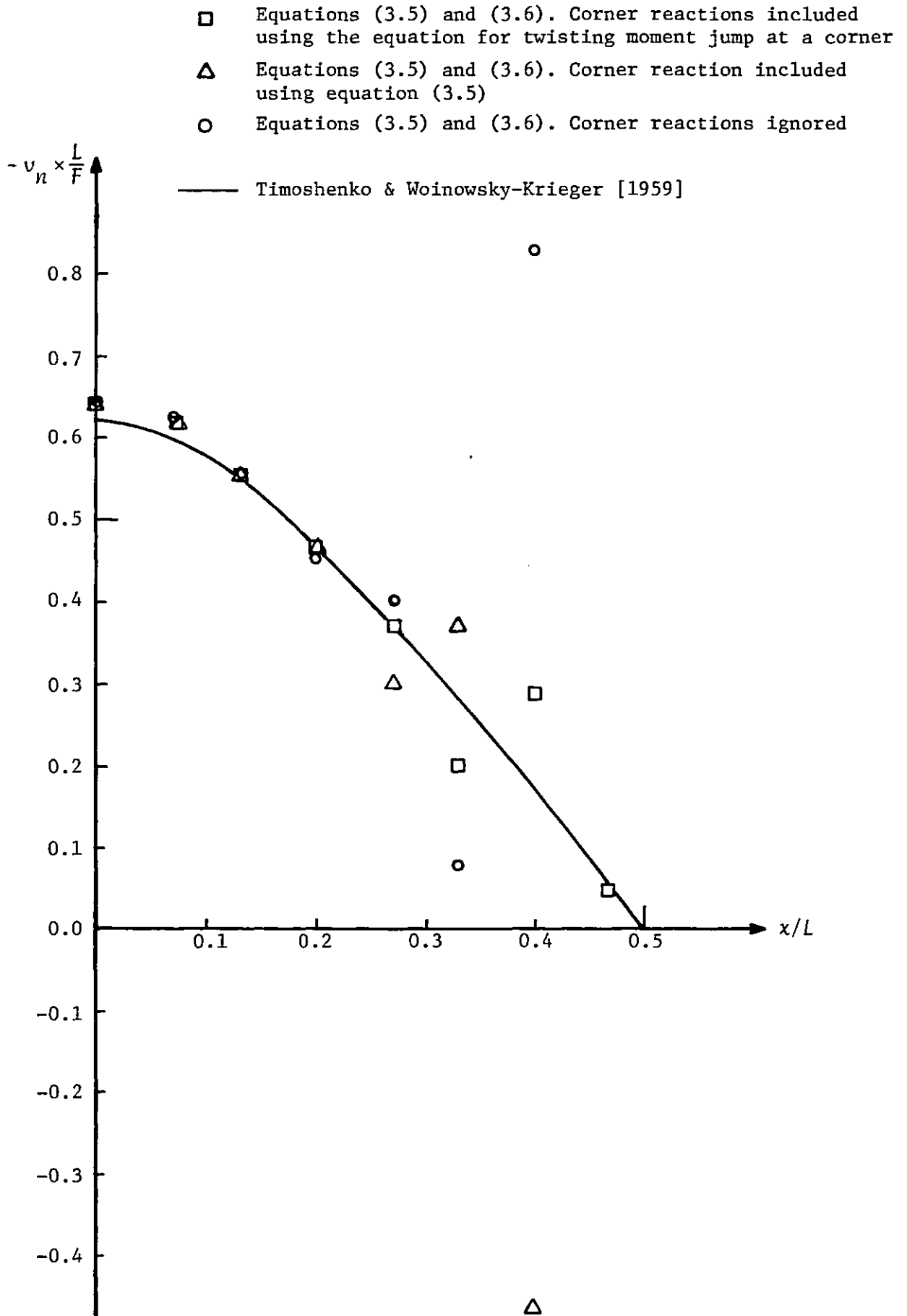


Figure 3.12: Effective shear force along a half edge for a simply supported square plate under a centrally applied load, F (see Figure 3.9)

CHAPTER 4

NUMERICAL FORMULATION OF THE BIE METHOD

USING ISOPARAMETRIC QUADRATIC ELEMENTS

4.1 INTRODUCTION

In the constant element formulation described in Chapter 3, the numerical value of the boundary values are assumed to remain constant along each element. It was shown that accurate solutions can be obtained using this type of formulation applied to some simple problems. However, to obtain a satisfactory degree of accuracy, for some boundary variables, a large number of elements must be used. This is particularly true for variables which vary rapidly along the boundary. The accurate modelling of curved boundaries would also require a fine degree of discretisation. The consequence of using a large number of constant elements is that the final system of linear simultaneous equations will be relatively large and this increases the execution time of the computer program. Furthermore, it was shown in Section 3.1 that at a corner of a plate, there are, in general, eight variables and a constant element formulation does not provide for their computation. This could be one reason why the accuracy of the computed values of the effective shear force along the boundary deteriorates in the vicinity of a corner point.

In this chapter, a numerical solution procedure is described which employs higher order interpolation functions in the approximation of the boundary variables and boundary geometry. The plate boundary is discretised using three-noded boundary elements. The variation of each boundary variable along an element is given in terms of its nodal values and quadratic interpolation functions. The elements are isoparametric (see Zienkiewicz [1977]) in that the geometry of the element is described using the same interpolation functions that are used for the approximation

of the nodal variables. This allows for a more accurate approximation of curved boundaries. Finally, the formulation provides for a choice of eight-noded quadrilaterals or six-noded triangles, or a mixture of the two, to be used in the discretisation of the interior region of the plate. These elements also incorporate quadratic interpolation functions for the description of nodal variables and element geometry. This allows for a more accurate evaluation of the contribution of the transverse loading on the plate for general loading conditions.

The boundary integral equation pairs used in this chapter are the equations for deflection and normal slope at the boundary, i.e. equations (2.146/I) and (2.146/II). This appears to be an appropriate choice based on the numerical results presented in Chapter 3, and the fact that the expressions for the kernels in these equations are simpler than those in the other possible equation pairs.

4.2 PARAMETRIC REPRESENTATION OF GEOMETRY

Figure 4.1 depicts a typical plate domain in a Cartesian coordinate system, with the x,y plane taken to be the middle surface of the plate. The plate boundary is approximated by a number of three-noded elements (nine in this case). Each element has a node at its mid-point and a node at either end. Figure 4.2 shows a typical element in the x,y plane. The geometry of the element is described in terms of its nodal coordinates and its intrinsic coordinate, ξ . The direction of ξ with respect to the outward normal is as shown, and it takes the values -1 to $+1$ at nodes 1 and 3 of the element, respectively. The middle node, node number 2, is situated at $\xi = 0$.

The geometry of the element is now expressed as:

$$x = N_1(\xi) x_1 + N_2(\xi) x_2 + N_3(\xi) x_3 \quad (4.1)$$

$$y = N_1(\xi) y_1 + N_2(\xi) y_2 + N_3(\xi) y_3 \quad (4.2)$$

where the suffixes refer to the element node numbers, and the "shape functions" $N_1(\xi)$, $N_2(\xi)$ and $N_3(\xi)$ are given by:

$$N_1 = -\frac{1}{2}\xi(1-\xi) \quad (4.3)$$

$$N_2 = (1-\xi^2) \quad (4.4)$$

$$N_3 = \frac{1}{2}\xi(1+\xi) \quad (4.5)$$

Note that at each node of the element, the shape function pertaining to that node is equal to unity and the shape functions for the other two nodes vanish. Furthermore, the variation of these functions along an element is quadratic, therefore, allowing a better representation of curved boundaries than is possible with constant straight line elements.

4.3 APPROXIMATION OF THE NODAL VARIABLES ALONG AN ELEMENT

In the constant element formulation described in Chapter 3, the value of each boundary variable was assumed to remain constant along each element. To obtain an acceptable degree of accuracy when using this type of formulation, a relatively large number of elements must be employed in discretising the boundary. This leads to a large system of linear simultaneous equations, requiring lengthy execution times.

In the formulation presented in this chapter, the variation of each boundary variable along an element is given in terms of its nodal values and quadratic shape functions as:

$$w = N_1(\xi) w_1 + N_2(\xi) w_2 + N_3(\xi) w_3 \quad (4.6)$$

$$\frac{\partial w}{\partial n} = N_1(\xi) \frac{\partial w_1}{\partial n} + N_2(\xi) \frac{\partial w_2}{\partial n} + N_3(\xi) \frac{\partial w_3}{\partial n} \quad (4.7)$$

$$m = N_1(\xi) m_1 + N_2(\xi) m_2 + N_3(\xi) m_3 \quad (4.8)$$

$$v = N_1(\xi) v_1 + N_2(\xi) v_2 + N_3(\xi) v_3 \quad (4.9)$$

where $N_1(\xi)$, $N_2(\xi)$ and $N_3(\xi)$ are given by equations (4.3) to (4.5), and w , $\partial w/\partial n$, m and v are the deflection, normal slope, normal moment and effective shear force, respectively, along the element.

4.4 INTEGRAL EQUATIONS IN DISCRETISED FORM

The integral equation pair used in this chapter consists of the equation for deflection at the boundary, equation (2.94), and the equation for slope at the boundary, equation (2.123). Figure 4.3 depicts a rectangular plate with its boundary discretised into m quadratic elements giving rise to a total of $n = 2m$ nodal points. Substitution of equations (4.6) to (4.9) into the integral equation (2.94) gives:

$$\begin{aligned} c^I w(p^a) + \sum_{b=1}^m \sum_{c=1}^3 \{ w(p^{d(b,c)}) \int_{S_b} v_n N_c dS - \frac{\partial w}{\partial n}(p^{d(b,c)}) \int_{S_b} M_n N_c dS \\ + m(p^{d(b,c)}) \int_{S_b} \frac{\partial w}{\partial n} N_c dS - v_n(p^{d(b,c)}) \int_{S_b} w N_c dS \} \\ + \sum_{\lambda=1}^k \{ [T] w(p^e(\lambda)) - [t(p^e(\lambda))] w \}_{\lambda} = \mathcal{D} \iint_A w \nabla^4 w dA \end{aligned} \quad (4.10)$$

where p^a is the a th boundary node, $p^{d(b,c)}$ is the c th node of the b th element (for example, $p^{d(5,3)}$ is the 3rd node of the 5th element, which is

node number 11 in Figure 4.3), $p^{e(\lambda)}$ is the λ th corner node (for example, $p^{e(3)}$ is node number 13 in Figure 4.3), and S_b refers to element b . The coefficient C^I is the free term, which is given by $\frac{1}{2}k$ in equation (2.94). The right hand side of this equation has not been written in the discretised form. The discretisation of the integral over the domain, and its evaluation, will be described in Section 4.7. Similarly, substituting equations (4.6) to (4.9) into equation (2.123) gives:

$$\begin{aligned}
 C_n^{II} \frac{\partial w}{\partial n}(p^a) + C_t^{II} \frac{\partial w}{\partial t}(p^a) + \sum_{b=1}^m \sum_{c=1}^3 \{ [w(p^{d(b,c)}) - w(p^a)] \int_{S_b} v_2 N_c dS \\
 - \frac{\partial w}{\partial n}(p^{d(b,c)}) \int_{S_b} M_2 N_c dS + m_n(p^{d(b,c)}) \int_{S_b} \frac{\partial w_2}{\partial n} N_c dS \\
 - v_n(p^{d(b,c)}) \int_{S_b} w_2 N_c dS \} \\
 + \sum_{\lambda=1}^k \{ [T_2] [w(p^{e(\lambda)}) - w(p^a)] - [t(p^{e(\lambda)})] w_2 \} = D \iint_A w_2 \nabla^4 w dA \quad (4.11)
 \end{aligned}$$

where p^a , $p^{d(b,c)}$, $p^{e(\lambda)}$ and S_b have the same meaning as in equation (4.10). Note that this equation is written at a corner node and n and t refer to the normal and tangential directions on one side of a corner. A similar equation can be written in terms of the normal and tangential directions on the other side of a corner. If p^a is a point on a smooth part of the boundary, then the coefficient C_t^{II} will vanish, and the coefficient C_n^{II} will be equal to $\frac{1}{2}$.

4.5 MATRIX REPRESENTATION OF THE SYSTEM OF EQUATIONS

Figure 4.3 shows a rectangular plate with its boundary discretised into a number of three-noded elements. The matrix representation of the system of linear simultaneous equations, obtained by using equations (4.10) and (4.11), follows basically the procedure described in Section 3.4. However, since the quadratic formulation involves nodes at the corners of the plate, certain complications arise which require more detailed description.

At each node on a smooth part of the boundary, there are four variables. These are the deflection, normal slope, normal moment and effective shear force. Therefore, four relations are required at each of these nodes in order to make a solution possible. These four relations are provided by the two boundary conditions required by the Kirchhoff plate theory and equations (4.10) and (4.11) written at the node.

The situation at a corner is somewhat less simple. At a corner node, there are, in general, eight variables. This is because there are two normal directions at a corner, n^+ and n^- , as shown in Figure 4.4. The eight variables are the corner deflection, the slope, moment and effective shear force on either side of the corner, and the Kirchhoff corner force. Although equation (4.11) contains the tangential derivative at a corner, this can be transformed in terms of the normal derivatives either side of a corner using the following transformation:

$$\frac{\partial}{\partial t^+} = \cot \alpha \frac{\partial}{\partial n^+} - \frac{1}{\sin \alpha} \frac{\partial}{\partial n^-} \quad (4.12a)$$

$$\frac{\partial}{\partial t^-} = \frac{1}{\sin \alpha} \frac{\partial}{\partial n^+} - \cot \alpha \frac{\partial}{\partial n^-} \quad (4.12b)$$

where α is the angle between the two normal directions, as shown in Figure 4.4. Equation (4.11) now contains only normal derivatives either side of

a corner. Now, at a corner, eight relations are required to solve for the eight variables present. One of these is given by equation (4.10), and two more are provided by equation (4.11) written for either side of a corner. The other five relations required can usually be given as boundary conditions and they depend on the nature of support conditions either side of a corner. In some cases, however, only four boundary conditions are available. The treatment of these cases is discussed in the next chapters in the context of the particular problem encountered.

Summarising the above, there are two unknowns at a node on a smooth part of the boundary and three unknowns at each corner node. This leads to $(2n + k)$ number of simultaneous equations, where n is the total number of nodes around the boundary and k is the total number of corners. The resulting system of equations can be written in matrix form as:

$$[A] \{\bar{u}\} + [\bar{A}] \{\bar{b}\} = \{\bar{F}\} \quad (4.13)$$

In this equation, $[A]$ is a $(2n + k) \times (2n + k)$ square matrix, $\{\bar{u}\}$ is a column vector containing all the nodal unknowns, $[\bar{A}]$ is a $(2n + k) \times (2n + 3k)$ rectangular matrix, and $\{\bar{b}\}$ is a $(2n + 3k)$ column vector containing the boundary conditions. $\{\bar{F}\}$ is a $(2n + k)$ column vector containing the contributions from the transverse loading on the plate. The coefficients in $[A]$ and $[\bar{A}]$ are evaluated by integrating the appropriate products of kernels and shape functions along the elements. The right hand side of equation (4.13) is evaluated by integrating, over the plate domain, the right hand sides of equations (4.10) and (4.11).

4.6 EVALUATION OF THE EQUATION COEFFICIENTS

The explicit evaluation of the coefficients arising from the integrals in equations (4.10) and (4.11) is cumbersome and, as mentioned before in Section 3.5, a numerical integration scheme is employed here as an attractive alternative. The method of evaluation of the integrals depends on whether the element, over which the integration is being carried out, contains the node p^a or not. Each case needs to be considered separately.

4.6.1 Integration Over an Element Which Does Not Contain Node p^a

If the integration interval does not contain node p^a , the distance r , between p^a and any point in the integration interval, is always greater than zero (see Figure 4.5). Therefore, all the kernels in equations (4.10) and (4.11) remain bounded. In this case, ordinary Gaussian quadrature formulae can be employed in the evaluation of the integrals. The one-dimensional Gaussian quadrature formula is given by:

$$\int_{-1}^{+1} f(\xi) d\xi = \sum_{i=1}^n w_i f(\xi_i) + E_n \quad (4.14)$$

where the w_i are weighting coefficients, ξ_i is the coordinate of the i th integration point, $f(\xi_i)$ is the value of the function $f(\xi)$ at this point, and n is the total number of integration points (see, for example, Stroud & Secrest [1966]).

As mentioned in Section 3.5.1, the integration in equations (4.10) and (4.11) are with respect to the arc length, S , whereas the Gaussian quadrature formula (equation (4.14)) is given in terms of ξ . A transformation relationship is, therefore, required between S and ξ . This can be obtained using the chain rule, thus giving:

$$dS = \frac{dS}{d\xi} d\xi \quad (4.15)$$

The quantity $dS/d\xi$ is called the Jacobian of the transformation and is given as follows (see equations (3.12) to (3.16)):

$$\frac{dS}{d\xi} = \left[\left(\frac{dx}{d\xi} \right)^2 + \left(\frac{dy}{d\xi} \right)^2 \right]^{\frac{1}{2}} \quad (4.16)$$

For the geometrical representation given by equations (4.1) to (4.5), the Jacobian is derived by straightforward differentiation of equations (4.1) and (4.2). This gives:

$$J = \frac{dS}{d\xi} = \sqrt{\left\{ \left(\xi - \frac{1}{2} \right) x_1 + \left(\xi + \frac{1}{2} \right) x_3 - 2\xi x_2 \right\}^2 + \left\{ \left(\xi - \frac{1}{2} \right) y_1 + \left(\xi + \frac{1}{2} \right) y_3 - 2\xi y_2 \right\}^2} \quad (4.17)$$

It can be verified that if the element is straight, the Jacobian reduces to $\frac{1}{2}L$ as in equation (3.20), where L is the length of the element. A typical integral from equations (4.10) or (4.11) can now be written as:

$$\int_{S_b} (\cdot) dS = \int_{-1}^{+1} (\cdot) J d\xi \quad (4.18)$$

where $(\cdot)$ indicates the product of a kernel and a shape function. As an example, the first integral in equation (4.10) over element 5 in Figure 4.3 is written as:

$$\begin{aligned} & w^9 \int_{S_5} v_n N_1 dS + w^{10} \int_{S_5} v_n N_2 dS + w^{11} \int_{S_5} v_n N_3 dS \\ &= w^9 \int_{-1}^{+1} v_n N_1 J d\xi + w^{10} \int_{-1}^{+1} v_n N_2 J d\xi + w^{11} \int_{-1}^{+1} v_n N_3 J d\xi \end{aligned} \quad (4.19)$$

The numerical integration scheme incorporated in the computer program

written based on the quadratic implementation described in this chapter employs a six point Gaussian quadrature. As mentioned in Section 3.5.1, this allows for a very accurate evaluation of the integrals over each element.

4.6.2 Integration Over Element(s) Containing Node p^a

The Gaussian quadrature scheme described in Section 4.6.1 does not provide an accurate evaluation of the integrals if the integrand becomes unbounded in the integration interval. Such is the case when the node p^a belongs to an element over which integration is being carried out. The integration interval contains the point $\kappa = 0$, where $\ln \kappa$, κ^{-1} and κ^{-2} become infinite, i.e. the integrands are singular. These types of integrals are divided into two categories. Integrals containing logarithmic singularities and integrals containing singularities of the order κ^{-1} and higher.

4.6.2.1 Integrals containing logarithmic singularities

For logarithmically singular kernels, the numerical integration scheme employed is given by:

$$\int_0^1 \ln \frac{1}{\zeta} f(\zeta) d\zeta \approx \sum_{i=1}^n w_i f(\zeta_i) \quad (4.20)$$

where the w_i are the Gaussian weight coefficients, ζ_i is the coordinate of the i th sampling point, and n is the total number of integration points. Tables of coordinates and weights for logarithmic quadrature are given by Stroud & Secrest [1966].

The logarithmic kernels appearing in equations (4.10) and (4.11) do not have the form shown in equation (4.20) and must, therefore, be modified accordingly. Furthermore, since the limits of

integration, in equation (4.20), are between zero and one, different transformation rules are required from the S to the ζ coordinate system, depending on whether P^a is node one, two or three of the element.

(a) P^a is node one of the element: By referring to Figure 4.6(a), it can be verified that the relation between the ξ and ζ coordinates is given by:

$$\xi = 2\zeta - 1 \quad (4.21)$$

Substitution of equation (4.21) into equations (4.3) to (4.5) gives:

$$N_1 = (2\zeta - 1)(\zeta - 1) \quad (4.22)$$

$$N_2 = 4\zeta(1 - \zeta) \quad (4.23)$$

$$N_3 = \zeta(2\zeta - 1) \quad (4.24)$$

Referring to Figure 4.6(a), the distance r of point P^a from another point with coordinates x and y is given as:

$$\begin{aligned} r &= [(x - x_1)^2 + (y - y_1)^2]^{\frac{1}{2}} \\ &= [(N_1 x_1 + N_2 x_2 + N_3 x_3 - x_1)^2 + (N_1 y_1 + N_2 y_2 + N_3 y_3 - y_1)^2]^{\frac{1}{2}} \end{aligned} \quad (4.25)$$

Substituting equations (4.22) to (4.24) into equation (4.25) gives:

$$\begin{aligned} \kappa = & \zeta \{ [(2\zeta - 3) x_1 + 4(1 - \zeta) x_2 + (2\zeta - 1) x_3]^2 \\ & + [(2\zeta - 3) y_1 + 4(1 - \zeta) y_2 + (2\zeta - 1) y_3]^2 \}^{\frac{1}{2}} \end{aligned} \quad (4.26)$$

The term $\ln \kappa$ can now be written as:

$$\ln \kappa = \ln \zeta U_1(\zeta) = - \ln \frac{1}{\zeta} + \ln U_1(\zeta) \quad (4.27)$$

where $U_1(\zeta)$ is defined as:

$$\begin{aligned} U_1(\zeta) = & \{ [(2\zeta - 3) x_1 + 4(1 - \zeta) x_2 + (2\zeta - 1) x_3]^2 \\ & + [(2\zeta - 3) y_1 + 4(1 - \zeta) y_2 + (2\zeta - 1) y_3]^2 \}^{\frac{1}{2}} \end{aligned}$$

The kernels in equations (4.10) and (4.11) containing logarithmic terms can now be written as (see equations (3.24) to (3.28)):

$$\omega = \frac{1}{8\pi \mathcal{D}} \kappa^2 \ln \zeta U_1(\zeta) = - \frac{1}{8\pi \mathcal{D}} \kappa^2 \ln \frac{1}{\zeta} + \frac{1}{8\pi \mathcal{D}} \kappa^2 \ln U_1(\zeta) \quad (4.28)$$

$$\frac{\partial \omega}{\partial n_Q} = - \frac{1}{4\pi \mathcal{D}} \kappa \ln \frac{1}{\zeta} + \frac{1}{8\pi \mathcal{D}} (2\kappa \ln U_1(\zeta) + \kappa \cos \beta) \quad (4.29)$$

$$M_n = \frac{(1 + \nu)}{4\pi} \ln \frac{1}{\zeta} - \frac{(1 - \nu)}{8\pi} \cos 2\beta - \frac{(1 + \nu)}{4\pi} - \frac{(1 + \nu)}{4\pi} \ln U_1(\zeta) \quad (4.30)$$

$$\omega_2 = \frac{\kappa \cos \psi}{4\pi \mathcal{D}} \ln \frac{1}{\zeta} - \frac{1}{8\pi \mathcal{D}} \kappa \cos \psi - \frac{\kappa}{4\pi \mathcal{D}} \cos \psi \ln U_1(\zeta) \quad (4.31)$$

$$\begin{aligned} \frac{\partial \omega_2}{\partial n_Q} = & \frac{\cos (\psi - \beta)}{8\pi \mathcal{D}} \ln \frac{1}{\zeta} - \frac{1}{8\pi \mathcal{D}} \{ 2 \cos (\psi - \beta) + \cos (\psi + \beta) \} \\ & - \frac{\cos (\psi - \beta)}{4\pi \mathcal{D}} \ln U_1(\zeta) \end{aligned} \quad (4.32)$$

The modified kernels expressed by equations (4.28) to (4.32) can now be integrated upon noting that only the terms involving $\ln 1/\zeta$ must be integrated using logarithmic quadrature. The remaining terms can be integrated using ordinary Gaussian quadrature since the expression $U_1(\zeta)$ is always greater than zero. The expression $U_1(\zeta)$ can, therefore, be written in terms of ξ using equation (4.21), thus giving:

$$U_1(\xi) = \{[(\xi - 2) x_1 + 2(1 - \xi) x_2 + \xi x_3]^2 + [(\xi - 2) y_1 + 2(1 - \xi) y_2 + \xi y_3]^2\}^{\frac{1}{2}} \quad (4.33)$$

Finally, since the logarithmic quadrature formula is with respect to the ζ coordinates, an expression is required for the Jacobian of the transformation. This is, once again, obtained by using the chain rule, giving:

$$dS = \frac{dS}{d\xi} \frac{d\xi}{d\zeta} d\zeta \quad (4.34)$$

Substitution of equations (4.17) and (4.21) into equation (4.34) yields:

$$dS = 2J d\zeta \quad (4.35)$$

As an example, the integral containing the term $(-1/(\delta\pi D) \kappa^2 \ln 1/\zeta)$ of the kernel W in equation (4.10), over element L_4 in Figure 4.3 with $P^a = 7$, is written as:

$$\begin{aligned} v_n^7 \int_0^1 \left(-\frac{1}{\delta\pi D} \kappa^2 \ln \frac{1}{\zeta}\right) N_1 \times 2J d\zeta + v_n^8 \int_0^1 \left(-\frac{1}{\delta\pi D} \kappa^2 \ln \frac{1}{\zeta}\right) N_2 \times 2J d\zeta \\ + v_n^9 \int_0^1 \left(-\frac{1}{\delta\pi D} \kappa^2 \ln \frac{1}{\zeta}\right) N_3 \times 2J d\zeta \end{aligned} \quad (4.36)$$

(b) p^a is node two of the element: Figure 4.6(b) shows a quadratic element with p^a as its mid-node. The relation between the ξ and ζ coordinates is now given by:

$$\xi = \zeta \quad \text{between nodes 2 and 3} \quad (4.37a)$$

$$\xi = -\zeta \quad \text{between nodes 1 and 2} \quad (4.37b)$$

Between nodes 2 and 3, the shape functions are given as (see equations (4.3) to (4.5)):

$$N_1 = -\frac{\zeta}{2} (1 - \zeta) \quad (4.38)$$

$$N_2 = (1 - \zeta^2) \quad (4.39)$$

$$N_3 = \frac{\zeta}{2} (1 + \zeta) \quad (4.40)$$

The distance r between p^a and any other point along the arc (2,3) is given by:

$$\begin{aligned} r &= [(x - x_1)^2 + (y - y_1)^2]^{\frac{1}{2}} \\ &= [(N_1 x_1 + N_2 x_2 + N_3 x_3 - x_1)^2 + (N_1 y_1 + N_2 y_2 + N_3 y_3 - y_1)^2]^{\frac{1}{2}} \quad (4.41) \end{aligned}$$

Substitution of equations (4.38) to (4.40) into equation (4.41) gives:

$$\begin{aligned} r &= \zeta \left\{ \left[-\frac{1}{2}(1 - \zeta) x_1 - \zeta x_2 + \frac{1}{2}(1 + \zeta) x_3 \right]^2 \right. \\ &\quad \left. + \left[-\frac{1}{2}(1 - \zeta) y_1 - \zeta y_2 + \frac{1}{2}(1 + \zeta) y_3 \right]^2 \right\}^{\frac{1}{2}} \quad (4.42) \end{aligned}$$

This gives:

$$\ln r = \ln \zeta u_{2,3}(\zeta) = -\ln \frac{1}{\zeta} + \ln u_{2,3}(\zeta) \quad (4.43)$$

where $u_{2,3}(\zeta)$ is given by:

$$\begin{aligned} u_{2,3}(\zeta) = & \left\{ \left[-\frac{1}{2}(1-\zeta) x_1 - \zeta x_2 + \frac{1}{2}(1+\zeta) x_3 \right]^2 \right. \\ & \left. + \left[-\frac{1}{2}(1-\zeta) y_1 - \zeta y_2 + \frac{1}{2}(1+\zeta) y_3 \right]^2 \right\}^{\frac{1}{2}} \end{aligned} \quad (4.44)$$

The kernels in equations (4.10) and (4.11) containing logarithmic terms are written in exactly the same manner as equations (4.28) to (4.32), with $u_1(\zeta)$ replaced by $u_{2,3}(\zeta)$. The modified kernels can now be integrated upon noting that only the terms involving $\ln 1/\zeta$ must be integrated using logarithmic quadrature. The remaining terms can be integrated using ordinary Gaussian quadrature since $u_2(\zeta)$ is always greater than zero. Finally, since $\xi = \zeta$ for this case, the Jacobian is simply given by J in equation (4.17). Between nodes 1 and 2, the shape functions are obtained by substituting equation (4.37b) into equations (4.3) to (4.5), thus giving:

$$N_1 = \frac{\zeta}{2} (1 + \zeta) \quad (4.45)$$

$$N_2 = (1 - \zeta^2) \quad (4.46)$$

$$N_3 = -\frac{\zeta}{2} (1 - \zeta) \quad (4.47)$$

The distance r between P^a and any other point along the arc (1,2) is given by equation (4.41). Using equations (4.45) to (4.47) gives:

$$\begin{aligned} h = & \zeta \left\{ \left[\frac{1}{2}(1+\zeta) x_1 - \zeta x_2 - \frac{1}{2}(1-\zeta) x_3 \right]^2 \right. \\ & \left. + \left[\frac{1}{2}(1+\zeta) y_1 - \zeta y_2 - \frac{1}{2}(1-\zeta) y_3 \right]^2 \right\}^{\frac{1}{2}} \end{aligned} \quad (4.48)$$

This gives:

$$\ln h = \ln \zeta u_{2,1}(\zeta) = -\ln \frac{1}{\zeta} + \ln u_{2,1}(\zeta) \quad (4.49)$$

where $u_{2,1}(\zeta)$ is given by:

$$\begin{aligned} u_{2,1}(\zeta) = & \left\{ \left[\frac{1}{2}(1+\zeta) x_1 - \zeta x_2 - \frac{1}{2}(1-\zeta) x_3 \right]^2 \right. \\ & \left. + \left[\frac{1}{2}(1+\zeta) y_1 - \zeta y_2 - \frac{1}{2}(1-\zeta) y_3 \right]^2 \right\}^{\frac{1}{2}} \end{aligned} \quad (4.50)$$

The kernels in equations (4.10) and (4.11) containing logarithmic terms are the same as those given by equations (4.28) to (4.32) with $u_1(\zeta)$ replaced by $u_{2,1}(\zeta)$. As mentioned already, only the terms containing $\ln 1/\zeta$ must be integrated using logarithmic quadrature. Finally, since $\xi = -\zeta$ in this case, the Jacobian is given by $-J$. However, it must be noted that, in this case, the integration with respect to ζ is in the opposite sense to that with respect to the arc length S , i.e.

$$\int_{12} (\cdot) dS = - \int_0^1 (\cdot) d\zeta \quad (4.51)$$

This is because, along the arc (1,2), the direction of increasing ζ is opposite to that of increasing S . Therefore, equation (4.51) can now be written as:

$$\int_{12} (\cdot) dS = - \int_0^1 (\cdot) \frac{dS}{d\xi} \frac{d\xi}{d\zeta} d\zeta = \int_0^1 (\cdot) J d\zeta \quad (4.52)$$

where, once again, J is given by equation (4.17).

(c) P^a is node three of the element: Figure 4.6(c) shows that, in this case, the relation between the ξ and ζ coordinates is given by:

$$\xi = 1 - 2\zeta \quad (4.53)$$

Substitution of this equation into equations (4.3) to (4.5) gives:

$$N_1 = \zeta (2\zeta - 1) \quad (4.54)$$

$$N_2 = 4\zeta (1 - \zeta) \quad (4.55)$$

$$N_3 = (2\zeta - 1)(\zeta - 1) \quad (4.56)$$

The distance r of point P^a from another point with coordinates x and y along the element is now given by:

$$\begin{aligned} r = \zeta \{ [(2\zeta - 1) x_1 + 4(1 - \zeta) x_2 + (2\zeta - 3) x_3]^2 \\ + [(2\zeta - 1) y_1 + 4(1 - \zeta) y_2 + (2\zeta - 3) y_3]^2 \}^{\frac{1}{2}} \end{aligned} \quad (4.57)$$

Using equation (4.57) gives:

$$\ln r = \ln \zeta U_3(\zeta) = - \ln \frac{1}{\zeta} + \ln U_3(\zeta) \quad (4.58)$$

where $U_3(\zeta)$ is defined as:

$$U_3(\zeta) = \{[(2\zeta - 1) x_1 + 4(1 - \zeta) x_2 + (2\zeta - 3) x_3]^2 + [(2\zeta - 1) y_1 + 4(1 - \zeta) y_2 + (2\zeta - 3) y_3]^2\}^{\frac{1}{2}} \quad (4.59)$$

The kernels in equations (4.10) and (4.11) which contain logarithmic singularities are the same as those given by equations (4.28) to (4.32), except that $U_1(\zeta)$ must be replaced by $U_3(\zeta)$. Note that, as in (a) above, only the terms involving $\ln 1/\zeta$ need be integrated using the logarithmic quadrature rule. The remaining terms are integrated using ordinary Gaussian quadrature, since $U_3(\zeta) > 0$ for $0 < \zeta < 1$. The expression for $U_3(\zeta)$ is, therefore, transformed back to a function of ξ , giving:

$$U_3(\xi) = \{[-\xi x_1 + 2(1 + \xi) x_2 - (2 + \xi) x_3]^2 + [-\xi y_1 + 2(1 + \xi) y_2 - (2 + \xi) y_3]^2\}^{\frac{1}{2}} \quad (4.60)$$

Finally, referring to equations (4.34) and (4.53) gives:

$$dS = \frac{dS}{d\xi} \frac{d\xi}{d\zeta} d\zeta = -2J d\zeta \quad (4.61)$$

However, since in this case the integration with respect to ζ is in the opposite sense to that with respect to the arc length S , a typical integral over the element would be written as:

$$\int(\cdot) dS = - \int_0^1 (\cdot) \frac{dS}{d\xi} \frac{d\xi}{d\zeta} d\zeta = \int_0^1 (\cdot) 2J d\zeta \quad (4.62)$$

It is seen that this is identical to the expression obtained in (a) above.

4.6.2.2 Integrals containing singularities of the order of κ^{-1} and higher

The only kernel in equation (4.10) which has a singularity of κ^{-1} is the shear kernel. Referring to equations (2.90), it can be seen that the shear kernel along the elements containing the node p^a can be adequately expressed by equation (2.91d), which is repeated below:

$$v_n = - \frac{1}{8\pi} \frac{(3-\nu)}{2R} \quad (4.63)$$

It is seen that the singularity is effectively removed and that the shear kernel vanishes if the element is along a straight boundary, that is having zero curvature. Therefore, the contributions from this kernel to nodes other than p^a can be computed using ordinary Gaussian quadrature. Over the self element(s), therefore, this gives integrals of the form:

$$\sum_{\substack{c=1 \\ a \neq d(b,c)}}^3 w(p^{d(b,c)}) \int_{S_b} v_n N_c dS \quad (4.64)$$

Note that if p^a is a mid-side node, there is only one self element to be considered. However, if p^a is an inter-element node, it is connected to two elements and both of these elements should be considered as self elements. Finally, the coefficient for $w(p^a)$ is obtained, as explained in Section 3.5.2.2, by considering a constant deflection field, $w = 1$. This gives:

$$c^I = - \sum_{b=1}^m \sum_{\substack{c=1 \\ a \neq d(b,c)}}^3 \left\{ \int_{S_b} v_n N_c dS \right\} - \sum_{\substack{\lambda=1 \\ a \neq e(\lambda)}}^k \{ [T] \}_\lambda \quad (4.65)$$

In equation (4.11), there are two singular kernels. The shear kernel has a singularity of the order κ^{-2} and the moment kernel has a singularity of the order κ^{-1} . It may be recalled from Chapter 2, Section 2.3.6, that in

order to remove the κ^{-2} singularity in equation (2.98), the deflection field, w , is expanded in the neighbourhood of the singularity using a Taylor series (see equation (2.102)). In the context of the present quadratic formulation, a contribution to the nodes on the self element is obtained by considering the term $O(\kappa^2)$ in equation (2.103).

The deflection field $w \equiv w(Q)$ can be expanded in the neighbourhood of the self point, P , by using a Taylor series:

$$w_Q = w_P + \kappa \frac{\partial w}{\partial \kappa_P} + \frac{\kappa^2}{2} \frac{\partial^2 w}{\partial \kappa_P^2} \quad (4.66)$$

giving:

$$w_Q - w_P = \kappa \frac{\partial w}{\partial \kappa_P} + \frac{\kappa^2}{2} \frac{\partial^2 w}{\partial \kappa_P^2} \quad (4.67)$$

It was shown in Section 2.3.6 that the first term on the right hand side of equation (4.67), combined with the integral containing $\partial w / \partial n_Q$ in equation (2.98), yields the coefficients for normal and tangential derivatives given in equation (2.107). In the numerical implementation discussed in this chapter, the deflection field is assumed to vary quadratically along each element. Therefore, a contribution to the coefficients for nodal deflections along the self element is obtained from the second term on the right hand side of equation (4.67) as:

$$\int V_2 \frac{\kappa^2}{2} \frac{\partial^2 w}{\partial \kappa_P^2} dS \quad (4.68)$$

Along each element, the deflection field is given in terms of its nodal values and quadratic interpolation functions as:

$$w = -\frac{\xi}{2} (1 - \xi) w_1 + (1 - \xi^2) w_2 + \frac{\xi}{2} (1 + \xi) w_3 \quad (4.69)$$

(see also equation (4.6)). Upon noting that:

$$\frac{\partial w}{\partial \xi} = \frac{\partial w}{\partial \kappa} \frac{\partial \kappa}{\partial \xi} \quad (4.70)$$

and, therefore:

$$\frac{\partial w}{\partial \kappa} = \frac{1}{\partial \kappa / \partial \xi} \frac{\partial w}{\partial \xi} \quad (4.71)$$

the required second derivative in equation (4.67) can be written as:

$$\frac{\partial^2 w}{\partial \kappa^2} = \frac{1}{(\partial \kappa / \partial \xi)^3} \left\{ \frac{\partial \kappa}{\partial \xi} \frac{\partial^2 w}{\partial \xi^2} - \frac{\partial^2 \kappa}{\partial \xi^2} \frac{\partial w}{\partial \xi} \right\} \quad (4.72)$$

Since the x and y coordinates of the points along each element are given as functions of the intrinsic coordinate, ξ , only (see equations (4.1) and (4.2)), the partial derivatives in equation (4.72) can be replaced by ordinary derivatives, giving:

$$\frac{d^2 w}{d\kappa^2} = \frac{1}{(d\kappa/d\xi)^3} \left\{ \frac{d\kappa}{d\xi} \frac{d^2 w}{d\xi^2} - \frac{d^2 \kappa}{d\xi^2} \frac{dw}{d\xi} \right\} \quad (4.73)$$

Now, the distance κ between any two points with coordinates x and y on the self element and the self node, P , is given by:

$$\kappa = [(x - x_p)^2 + (y - y_p)^2]^{\frac{1}{2}} \quad (4.74)$$

Therefore, differentiating equation (4.74) with respect to ξ gives:

$$\frac{d\kappa}{d\xi} = \frac{(x - x_p) dx/d\xi + (y - y_p) dy/d\xi}{\kappa} \quad (4.75)$$

Further differentiation gives:

$$\begin{aligned} \frac{d^2 h}{d\xi^2} = & \frac{1}{h} \left[\left(\frac{dx}{d\xi} \right)^2 + \left(\frac{dy}{d\xi} \right)^2 + (x - x_p) \frac{d^2 x}{d\xi^2} + (y - y_p) \frac{d^2 y}{d\xi^2} \right] \\ & - \frac{1}{h^2} \frac{dh}{d\xi} \left[(x - x_p) \frac{dx}{d\xi} + (y - y_p) \frac{dy}{d\xi} \right] \end{aligned} \quad (4.76)$$

The derivatives $dx/d\xi$, $dy/d\xi$, $d^2x/d\xi^2$ and $d^2y/d\xi^2$ can all be computed using equations (4.1) and (4.2), giving:

$$\frac{dx}{d\xi} = \left(-\frac{1}{2} + \xi\right) x_1 - 2\xi x_2 + \left(\frac{1}{2} + \xi\right) x_3 \quad (4.77)$$

$$\frac{dy}{d\xi} = \left(-\frac{1}{2} + \xi\right) y_1 - 2\xi y_2 + \left(\frac{1}{2} + \xi\right) y_3 \quad (4.78)$$

$$\frac{d^2 x}{d\xi^2} = x_1 - 2x_2 + x_3 \quad (4.79)$$

$$\frac{d^2 y}{d\xi^2} = y_1 - 2y_2 + y_3 \quad (4.80)$$

Similarly, $dw/d\xi$ and $d^2w/d\xi^2$ can be obtained using equation (4.6):

$$\frac{dw}{d\xi} = \left(-\frac{1}{2} + \xi\right) w_1 - 2\xi w_2 + \left(\frac{1}{2} + \xi\right) w_3 \quad (4.81)$$

and:
$$\frac{d^2 w}{d\xi^2} = w_1 - 2w_2 + w_3 \quad (4.82)$$

Equation (4.68) can now be written as:

$$\int V_2 \frac{h^2}{2} \frac{\partial^2 w}{\partial h_p^2} dS = \int V_2 \frac{h^2}{2} \left[\frac{d^2 w/d\xi^2}{(dh/d\xi)^2} - \frac{d^2 h/d\xi^2}{(dh/d\xi)^3} \frac{dw}{d\xi} \right] dS \quad (4.83)$$

where V_2 is the effective shear kernel in equation (2.98). Using equations (4.77) to (4.82) gives this equation as:

$$\int V_2 \frac{\kappa^2}{2} \left\{ \frac{1}{(dr/d\xi)^2} (w_1 - 2w_2 + w_3) - \frac{d^2\kappa/d\xi^2}{(dr/d\xi)^3} \left[\left(-\frac{1}{2} + \xi\right) w_1 - 2\xi w_2 + \left(\frac{1}{2} + \xi\right) w_3 \right] \right\} dS \quad (4.84)$$

The coefficient for each node will, therefore, be given by taking the nodal values outside the integral sign as:

$$C^{II}(w_1) = \int V_2 \frac{\kappa^2}{2} \left\{ \frac{1}{(dr/d\xi)^2} - \frac{d^2\kappa/d\xi^2}{(dr/d\xi)^3} \left(\xi - \frac{1}{2}\right) \right\} dS \quad (4.85)$$

$$C^{II}(w_2) = \int V_2 \frac{\kappa^2}{2} \left\{ \frac{-2}{(dr/d\xi)^2} + \frac{d^2\kappa/d\xi^2}{(dr/d\xi)^3} (2\xi) \right\} dS \quad (4.86)$$

$$C^{II}(w_3) = \int V_2 \frac{\kappa^2}{2} \left\{ \frac{1}{(dr/d\xi)^2} - \frac{d^2\kappa/d\xi^2}{(dr/d\xi)^3} \left(\xi + \frac{1}{2}\right) \right\} dS \quad (4.87)$$

The integrals in equations (4.85) to (4.87) are no longer singular and their evaluation can be carried out using Gaussian quadrature.

Equations (4.85) to (4.87) give the contribution to the coefficients for nodal deflections, including the self node, along the self element resulting from the quadratic term in the Taylor series expansion of the deflection, w , in the neighbourhood of the self point, P . For the self node, P , a further contribution is obtained from the integration of the terms containing $(w_Q - w_P)$ over the elements not containing the self node, P . This contribution is obtained from equation (4.11) as:

$$C^{II} = - \sum_{b=1}^m \sum_{\substack{c=1 \\ p^a \notin b}}^3 \left\{ \int_{S_b} V_2 N_c dS \right\} - \sum_{\substack{\lambda=1 \\ a \notin e(\lambda)}}^k \{ [\Pi_2] \}_\lambda \quad (4.88)$$

where $p^a \notin b$ means p^a is not a node of element b .

The full contribution for the self node, P , is therefore given by the sum of equation (4.88) and one of the equations (4.85) to (4.87), depending on

whether the self node is node 1, 2 or 3 of the self element, respectively, as:

$$C_w^{II} = - \sum_{b=1}^m \sum_{c=1}^3 \left\{ \int_{S_b} v_2 N_c dS \right\} - \sum_{\substack{\lambda=1 \\ a \neq e(\lambda)}}^k \{ [T_2] \}_\lambda + \sum_{b=1}^m \sum_{\substack{c=1 \\ a=d(b,c)}}^3 C_w^{II}(w_c) \quad (4.89)$$

Note that the second term involving summation signs in equation (4.89) represents either $C_w^{II}(w_1)$, $C_w^{II}(w_2)$ or $C_w^{II}(w_3)$, depending on whether the self node is node 1, 2 or 3 of the self element. As before, if the self node is a mid-side node, there is only one self element to be considered; however, if it is an inter-element node, the number of self elements to be considered is two.

The moment kernel, M_2 , in equation (4.11) has a singularity of the order κ^{-1} . To establish whether the contribution, to nodes on the self element other than p^a , of the product of the moment kernel and appropriate shape functions can be integrated using Gaussian quadrature, the behaviour of the integrand along the self element must be examined. This is shown below for the case when p^a is the first node of the element. Referring to Figure 4.6(a), the distance κ from p^a of any point on the element is given by:

$$\kappa = \frac{1+\xi}{2} u_1(\xi) \quad (4.90)$$

where $u_1(\xi)$ is given by equation (4.33). This is obtained by substituting, in equation (4.26), for ζ in terms of ξ as obtained from equation (4.21). Therefore, the contributions to nodes 2 and 3 of the element can be written as:

$$\begin{aligned} & \frac{\partial w}{\partial n}(2) \int M_2 N_2 dS + \frac{\partial w}{\partial n}(3) \int M_2 N_3 dS \\ &= \frac{\partial w}{\partial n}(2) \int \frac{1}{8\pi\kappa} g(\xi, \beta) N_2 dS + \frac{\partial w}{\partial n}(3) \int \frac{1}{8\pi\kappa} g(\xi, \beta) N_3 dS \end{aligned} \quad (4.91)$$

where $g(\xi, \beta)$ is defined by (see equation (2.97c)):

$$M_2 = \frac{1}{8\pi\kappa} g(\xi, \beta) \quad (4.92)$$

Substituting equation (4.90) and the expressions for the shape functions given by equations (4.4) and (4.5) into equation (4.91) gives:

$$\begin{aligned} & \frac{\partial w}{\partial n}(2) \int_{S_b} \frac{g(\xi, \beta)}{4\pi} \frac{(1-\xi^2)}{(1+\xi) u_1(\xi)} dS + \frac{\partial w}{\partial n}(3) \int_{S_b} \frac{g(\xi, \beta)}{4\pi} \frac{\xi/2 (1+\xi)}{(1+\xi) u_1(\xi)} dS \\ & \quad p^a \in b \quad \quad \quad p^a \in b \\ & = \frac{\partial w}{\partial n}(2) \int_{S_b} \frac{g(\xi, \beta)}{4\pi} \frac{(1-\xi)}{u_1(\xi)} dS + \frac{\partial w}{\partial n}(3) \int_{S_b} \frac{g(\xi, \beta)}{4\pi} \frac{\xi/2}{u_1(\xi)} dS \quad (4.93) \\ & \quad \quad \quad p^a \in b \quad \quad \quad p^a \in b \end{aligned}$$

where $p^a \in b$ means p^a is a node of element b .

It is seen that the integrals are not singular, and that these coefficients can be evaluated using ordinary Gaussian quadrature formula. For cases when p^a is node 2 or 3 of the element, a similar approach can be adopted, and in both cases a similar result is obtained.

Finally, the coefficients for $\partial w / \partial n(p^a)$ and $\partial w / \partial t(p^a)$ can be obtained by considering equation (4.11) for deflection fields giving rise to unit $\partial w / \partial n(p^a)$ and $\partial w / \partial t(p^a)$, respectively. Referring to Figure 4.7, and assuming for generality that p^a is a corner node, the coefficient for $\partial w / \partial n^+(p^a)$ and $\partial w / \partial t^+(p^a)$ can be obtained by considering equation (4.11) for the deflection fields:

$$w = n^+ \quad (4.94)$$

$$\text{and:} \quad w = t^+ \quad (4.95)$$

respectively. Equation (4.94) gives rise to unit normal derivative, and

equation (4.95) gives rise to unit tangential derivative. A similar approach can be adopted for the case when equation (4.11) is written on the negative side of a corner. The transformation of the tangential derivatives in terms of the normal derivatives on either side of a corner is then carried out using equations (4.12). If point P^a is not a corner node, the coefficient for $\partial W / \partial t(P^a)$ is zero at mid-side nodes, and at inter-element nodes, provided the elements either side of that node are of equal length. If the elements on either side are of unequal length, there is a non-zero contribution for $\partial W / \partial t(P^a)$ (see Appendix D). Assuming equal length elements, therefore, the only coefficient required is that of $\partial W / \partial n(P^a)$, which can be obtained by considering the deflection field:

$$w = n_{P^a} \quad (4.96)$$

(see Figure 4.8). As an example, the coefficient for $\partial W / \partial n^+(P^a)$ is given as:

$$\begin{aligned} C_n^{II} = & - \sum_{b=1}^m \sum_{c=1}^3 \left\{ \int_{S_b} V_2 N_c (r \cos \psi) dS - \int M_2 N_c \cos (n^+(P^a), n) dS \right\} \\ & P^a \notin b \\ & - \sum_{b=1}^m \sum_{c=1}^3 \cos (n^+(P^a), n(P^{d(b,c)})) \int M_2 N_c dS \\ & P^a \in b \quad a \neq d(b,c) \\ & - \sum_{\substack{\lambda=1 \\ e(\lambda) \neq a}}^k \{ [T_2] r \cos \psi \}_\lambda \end{aligned} \quad (4.97)$$

noting that: $w = n^+ = r \cos \psi \quad (4.98)$

$$\frac{\partial w}{\partial n} = \frac{\partial w}{\partial n^+} \cos (n^+(P^a), n) = \cos (n^+(P^a), n) \quad (4.99)$$

where n is the normal at some other point on the boundary, and $\cos (n^+(P^a), n)$ is the cosine of the angle between $n^+(P^a)$ and n (see Figure 4.7).

4.7 EVALUATION OF THE SECOND MEMBER

Having computed the coefficients arising from the boundary integrals, as described in the previous section, they are then placed in the $[A]$ and $[\bar{A}]$ matrices in equation (4.13), noting that:

- (i) The coefficients for all unknown variables are placed in the $[A]$ matrix; and
- (ii) The coefficients for all variables given as boundary conditions are placed in the $[\bar{A}]$ matrix. Equation (4.13) is now written as:

$$[A] \{\bar{u}\} = \{\bar{F}\} - [\bar{A}] \{\bar{b}\} \quad (4.100)$$

The right hand member of this set of equations consists of two parts. The first part, $\{\bar{F}\}$, is obtained by integrating, over the domain of the plate, the product of the transverse load intensity, F , and the deflection kernel for each of the equations (4.10) and (4.11). The second part is merely the product of the $[\bar{A}]$ matrix and the vector $\{\bar{b}\}$ containing the prescribed boundary conditions. This section describes the procedure adopted for the evaluation of the contribution, to the right hand side of equation (4.100), of the transverse loading on the plate.

4.7.1 Concentrated Loads

If the transverse loading on the plate consists of concentrated loads, F_i , their contribution to the integral equations (4.10) and (4.11) is simply the work done by the concentrated loads, F_i , through the deflections given by the fundamental solutions w_1 and w_2 , respectively, at the points of application of the loads. For example, if the loading on the plate consists of the concentrated forces F_1 and F_2 , as shown in Figure 3.6, the contribution from these forces to equations (4.10) and (4.11) is given by:

$$\iint w F dA = w(r_1) F_1 + w(r_2) F_2 \quad (4.101)$$

and:
$$\iint w_2 F dA = w_2(r_1, \psi_1) F_1 + w_2(r_2, \psi_2) F_2 \quad (4.102)$$

respectively.

4.7.2 Distributed Loads

If the lateral loading on the plate is a distributed one, the area integrals in equations (4.10) and (4.11) must be evaluated numerically. This involves the discretisation of the plate domain into a number of two-dimensional elements, over which numerical integration is carried out. Two types of elements are incorporated in the formulation presented in this section. These are an eight-noded quadrilateral element and a six-noded triangular element.

4.7.2.1 Numerical integration over quadrilateral elements

Figure 4.9(a) depicts a quadrilateral element and its natural or intrinsic coordinates, (ξ, η) . The element geometry is defined in terms of its nodal coordinates and quadratic shape functions of the

intrinsic coordinates, as given by Zienkiewicz [1977], as follows:

$$x = \sum_{i=1}^8 N_i x_i \quad (4.103)$$

$$y = \sum_{i=1}^8 N_i y_i \quad (4.104)$$

where: $N_1 = \frac{1}{4}(1 - \xi)(1 - \eta)(-\xi - \eta - 1)$ (4.105a)

$$N_2 = \frac{1}{4}(1 + \xi)(1 - \eta)(\xi - \eta - 1) \quad (4.105b)$$

$$N_3 = \frac{1}{4}(1 + \xi)(1 + \eta)(\xi + \eta + 1) \quad (4.105c)$$

$$N_4 = \frac{1}{4}(1 - \xi)(1 + \eta)(-\xi + \eta - 1) \quad (4.105d)$$

$$N_5 = (1 - \xi^2)(1 - \eta) \quad (4.105e)$$

$$N_6 = (1 - \eta^2)(1 + \xi) \quad (4.105f)$$

$$N_7 = (1 - \xi^2)(1 + \eta) \quad (4.105g)$$

$$N_8 = (1 - \eta^2)(1 - \xi) \quad (4.105h)$$

The load intensity, F , is also defined in terms of its nodal values and the above shape functions as:

$$F = \sum_{i=1}^8 N_i F_i \quad (4.106)$$

The domain integrals in equations (4.10) and (4.11) can now be written as:

$$\iint_A w F dA = \sum_{b=1}^m \sum_{c=1}^8 \int_{A_b} N_c F(p^{d(b,c)}) w dA_b \quad (4.107a)$$

$$\text{and: } \iint_A w_2 F dA = \sum_{b=1}^m \sum_{c=1}^8 \int_{A_b} N_c F(p^{d(b,c)}) w_2 dA_b \quad (4.107b)$$

respectively, where m is the total number of area elements having area A_b , and $p^{d(b,c)}$ is the c th node of element b . Recalling equations (3.53) and (3.54) for the transformation from the Cartesian, $(dx dy)$, to the intrinsic, $(d\xi dn)$, coordinate system, gives:

$$\iint_A w F dA = \sum_{b=1}^m \sum_{c=1}^8 \int_{-1}^{+1} \int_{-1}^{+1} N_c F(p^{d(b,c)}) w J d\xi dn \quad (4.108a)$$

$$\text{and: } \iint_A w_2 F dA = \sum_{b=1}^m \sum_{c=1}^8 \int_{-1}^{+1} \int_{-1}^{+1} N_c F(p^{d(b,c)}) w_2 J d\xi dn \quad (4.108b)$$

where J is the Jacobian of the transformation, as given by equation (3.54).

The integrals in equations (4.108) can be numerically integrated using the quadrature formula given by equation (3.49), thus giving:

$$\begin{aligned} & \int_{-1}^{+1} \int_{-1}^{+1} N_c F(p^{d(b,c)}) w J d\xi dn \\ &= \sum_{i=1}^n \sum_{j=1}^n w_i w_j N_c(\xi_i, n_j) F_c(\xi_i, n_j) w(\xi_i, n_j) J(\xi_i, n_j) \end{aligned} \quad (4.109a)$$

$$\begin{aligned} \text{and: } & \int_{-1}^{+1} \int_{-1}^{+1} N_c F(p^{d(b,c)}) w_2 J d\xi dn \\ &= \sum_{i=1}^n \sum_{j=1}^n w_i w_j N_c(\xi_i, n_j) F_c(\xi_i, n_j) w_2(\xi_i, n_j) J(\xi_i, n_j) \end{aligned} \quad (4.109b)$$

where n is the total number of quadrature points in each direction. A choice of $n = 2$ or $n = 3$ is incorporated in the computer program developed, based on the above formulations, giving a total of 4 or 9 quadrature

points, respectively.

4.7.2.2 Numerical integration over triangular elements

Figure 4.9(b) shows a six-noded triangular element with its intrinsic coordinates. These coordinates have the property that they vary between zero, along a side of the triangle, and unity at the vertex opposite that side. Furthermore, at any given point in the triangle, the sum of the intrinsic coordinates is equal to unity, i.e.

$$\xi + \eta + \zeta = 1 \quad (4.110)$$

The numerical integration scheme described here is the one given by Cowper [1973] and is written as:

$$\iint \phi(\xi, \eta, \zeta) dA = A \sum_{i=1}^n w_i \phi(\xi_i, \eta_i, \zeta_i) J(\xi_i, \eta_i, \zeta_i) \quad (4.111)$$

where A is the area of the triangle, ξ_i , η_i and ζ_i are the coordinates of the i th quadrature point, w_i is its weight, n is the total number of quadrature points, and J is the Jacobian.

Now, the geometry of each triangular element is, once again, defined in terms of its nodal coordinates and quadratic interpolation functions of the intrinsic coordinates as (see Zienkiewicz [1977]):

$$x = \sum_{i=1}^6 N_i x_i \quad (4.112)$$

$$y = \sum_{i=1}^6 N_i y_i \quad (4.113)$$

where: $N_1 = (2\xi - 1) \xi \quad (4.114a)$

$$N_2 = (2n - 1) n \quad (4.114b)$$

$$N_3 = (2\zeta - 1) \zeta \quad (4.114c)$$

$$N_4 = 4\xi n \quad (4.114d)$$

$$N_5 = 4n \zeta \quad (4.114e)$$

$$N_6 = 4\xi \zeta \quad (4.114f)$$

The intensity of the transverse loading, F , is also interpolated in terms of its nodal values and the above shape functions as:

$$F = \sum_{i=1}^6 N_i F_i \quad (4.115)$$

In a similar manner to that used in arriving at equations (4.108), the domain integrals in equations (4.10) and (4.11) can be written as:

$$\iint_A w F dA = \sum_{b=1}^m \sum_{c=1}^6 \iint N_c F(p^{d(b,c)}) w dA_b \quad (4.116a)$$

$$\text{and: } \iint_A w_2 F dA = \sum_{b=1}^m \sum_{c=1}^6 \iint N_c F(p^{d(b,c)}) w_2 dA_b \quad (4.116b)$$

The integrals appearing in equations (4.116) can now be integrated using equation (4.111). However, there is one remaining point to be clarified and that concerns the Jacobian in equation (4.111). From equation (4.110), it is clear that:

$$\zeta = 1 - \xi - n \quad (4.117)$$

Therefore, ξ and η can be treated as independent variables in equations (4.112) and (4.113), and the Jacobian can be computed using equation (3.54).

The triangular quadrature scheme incorporated in the computer program developed based on the above formulation employs seven sampling points. Tables for coordinates and weighting coefficients for integration over a triangle are given by Cowper [1973].

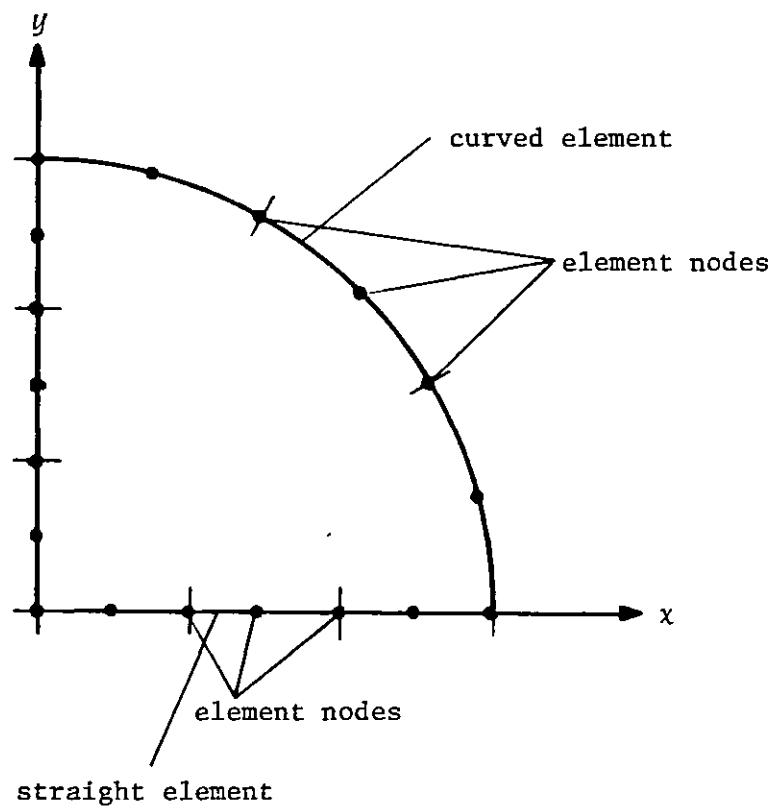


Figure 4.1: Typical boundary discretisation using quadratic elements

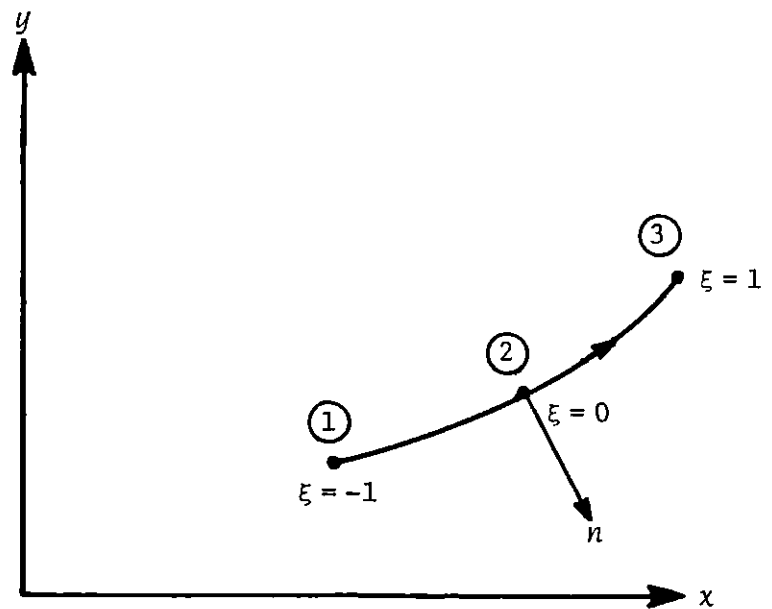


Figure 4.2: A quadratic element and its intrinsic coordinates

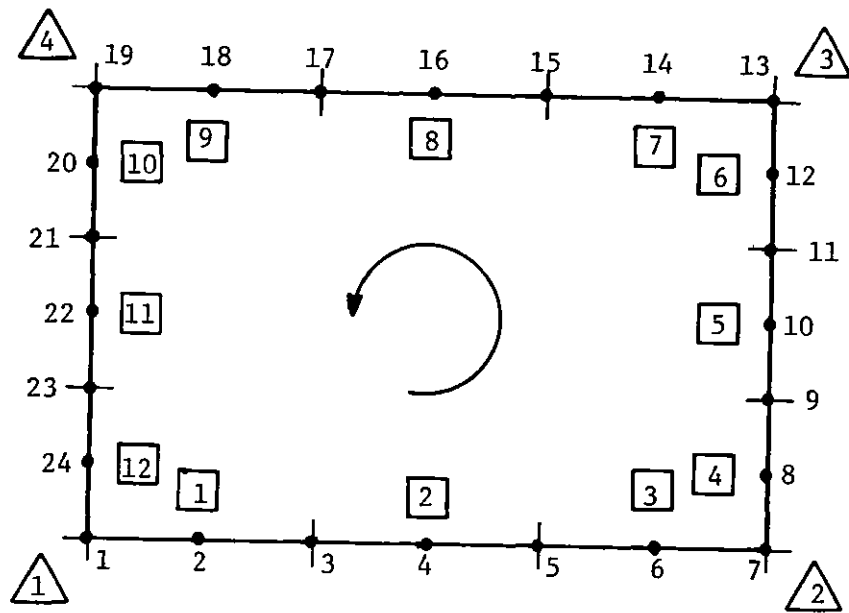


Figure 4.3: A typical discretisation employing 12 quadratic elements

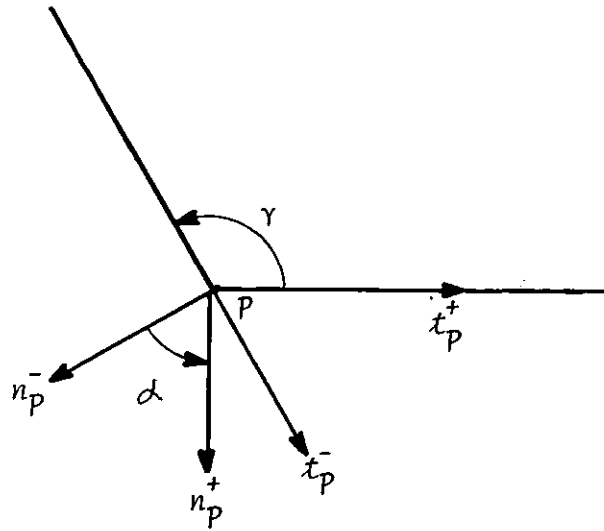


Figure 4.4: Normal and tangential directions at a corner point

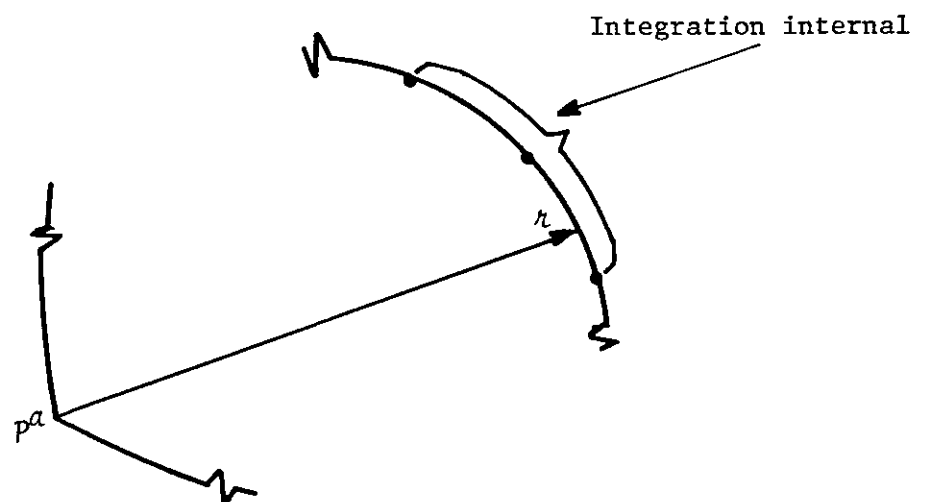
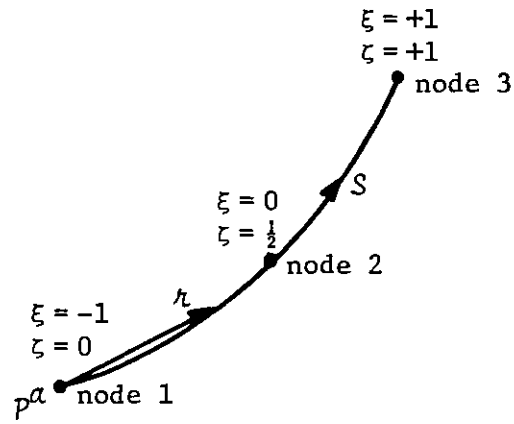
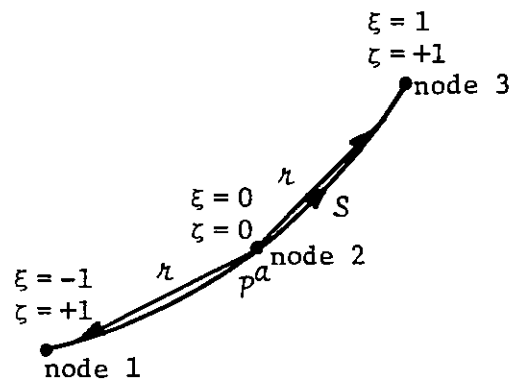


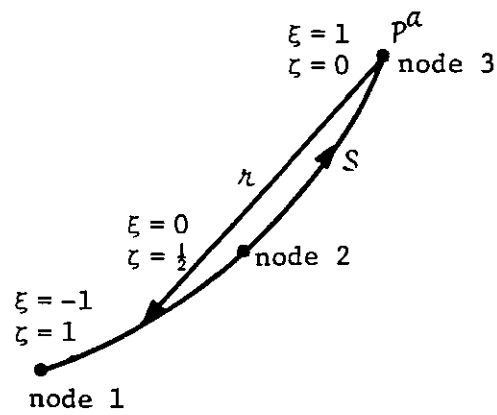
Figure 4.5: Element not containing the self point, P



(a) Self node is node 1 of the element



(b) Self node is node 2 of the element



(c) Self node is node 3 of the element

Figure 4.6: Element containing the self point, $\mathcal{P}$

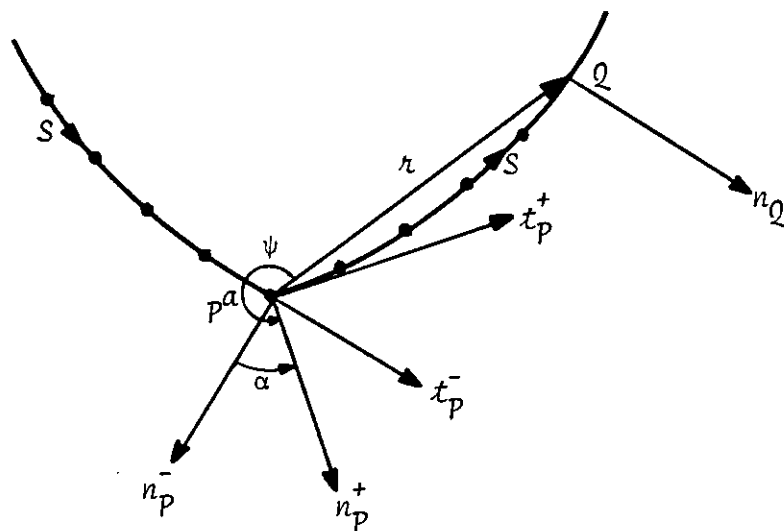


Figure 4.7: Notation for normal and tangential directions at a corner point

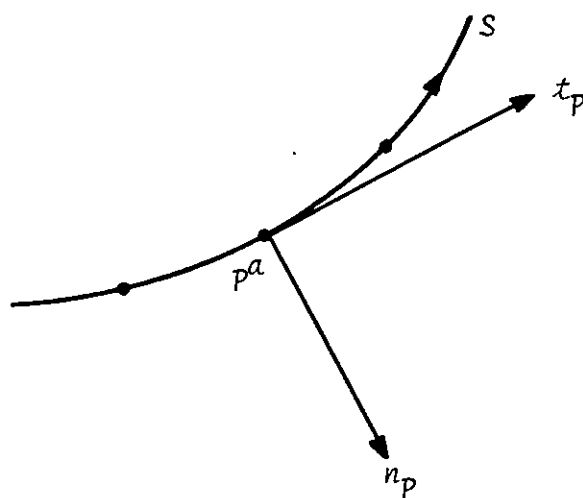
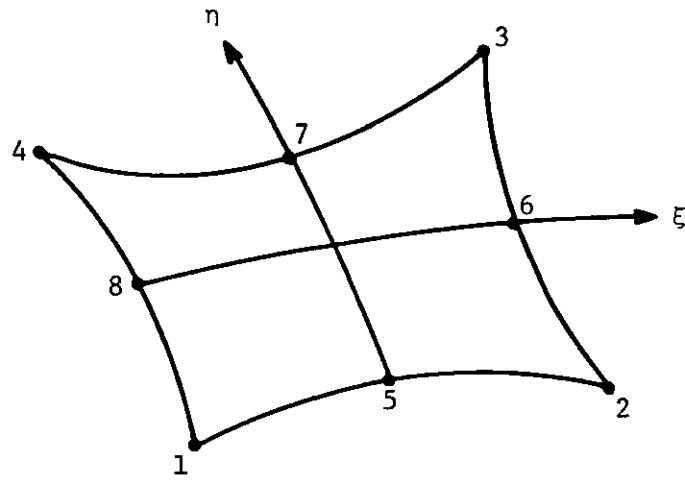
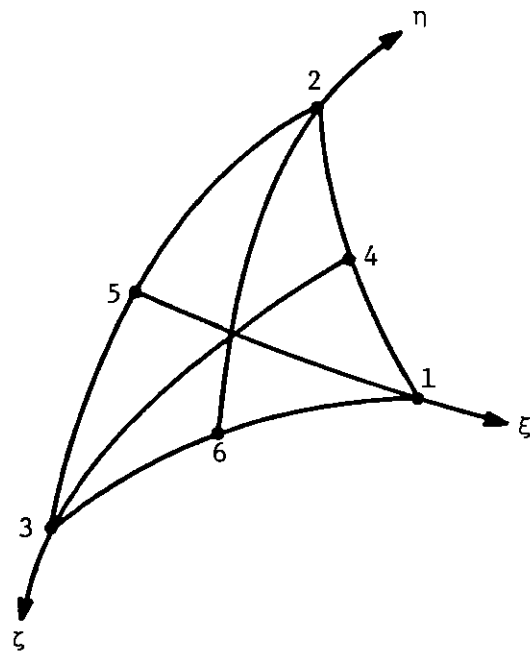


Figure 4.8: Notation for normal and tangential directions at a regular point on the boundary



(a) Eight noded quadrilaterals



(b) Six noded triangles

Figure 4.9: Quadratic area elements and their intrinsic coordinate systems

CHAPTER 5
NUMERICAL RESULTS FOR
PLATES WITH STRAIGHT BOUNDARIES

5.1 INTRODUCTION

In this chapter, the accuracy of the quadratic formulation described in Chapter 4 is assessed by applying it to a variety of plate bending problems. For this purpose, a computer program was written based on this formulation. Fortunately, there is a wealth of analytical solutions available (see Timoshenko & Woinowsky-Krieger [1959]) for a variety of plate bending problems involving various loading and boundary conditions. These provide a valuable check on the accuracy of any numerical solution algorithm. In choosing the examples described in the following pages, an effort has been made to cover a wide range of boundary conditions and a variety of loading types. Most of the cases studied have been compared with analytical or other numerical solutions. These examples begin with some standard test cases involving square plates loaded at the centre with their boundaries clamped or simply supported, and are followed by other examples involving various loading and boundary conditions.

It may be recalled from Section 4.5 that at a corner point, there are usually eight variables. Therefore, eight relations are required in order to make a solution possible. Of these, three are provided by using equations (4.10) and (4.11), as already discussed. The other five relations would depend on the nature of the support conditions either side of a corner. These will be discussed for each of the test cases in turn. In most cases, these corner boundary conditions may be deduced from the edge conditions on either side of the corner and coordinate transformation rules between the normal and tangential directions on these sides.

In all the test cases discussed in this chapter, the numerical

integration over elements along the boundary employs a six-point Gaussian quadrature. Over the self elements, the logarithmically singular kernels are integrated using seven sampling points. The numerical integration over the eight-noded quadrilateral elements employs 3×3 Gaussian quadrature and the numerical integration over the six-noded triangular elements uses seven sampling points, as given by Cowper [1973].

An important point should be mentioned at this stage regarding the relation between the interior mesh design and the computation of moments at interior points. Since the deflection kernel for the equation for moments at interior points is logarithmically singular, it is preferable to have the internal points as far as possible from the integration sampling points. It was found that in most cases, if the interior points are on the element boundaries, the computed values of the interior moments are satisfactory. This aspect has been considered in all the cases where a domain integration was required.

5.2 SQUARE PLATE UNDER A CONCENTRATED LOAD AT THE CENTRE

The problem of a square plate under a concentrated load at the centre is considered here with the plate boundaries being either clamped or simply supported. The boundary conditions used at a clamped to clamped corner were that deflection must vanish as well as the slope and normal moment either side of a corner. The same boundary conditions can also be used for a simply supported to simply supported corner. Stern [1979] suggests that at a simply supported rectangular corner, the deflection, normal moment either side and effective shear force either side of a corner must vanish. This, however, seems to hold for rectangular corners only, whereas the former set of boundary conditions is more generally applicable. In any case, as will be seen later, the effect of choosing either boundary condition is very localised and does not affect the overall

deflected shape of the plate.

To assess the rate of convergence to the exact series solution given by Timoshenko & Woinowsky-Krieger [1959], this problem was solved using different degrees of mesh discretisation. Figure 5.1 shows a typical mesh design using 24 quadratic elements. The computed values of the central deflections for different meshes are given in Table 5.1, and compared with the finite element results given by Martin [1976]. It can be seen that, even for a very coarse mesh of 4 elements, that is one quadratic element along each side of the plate, the computed values of the central deflection are in close agreement with the series solution. Table 5.2 and Figure 5.3 compare the computed values of the deflections along the centreline OA, in Figure 5.1, with those computed from the series solution given by Timoshenko & Woinowsky-Krieger [1959]. Table 5.3 and Figure 5.4 give slopes along half an edge of the simply supported plate and compare them with the series solution given in the same reference. In both cases, agreement between the BIE solution and the analytical solution is seen to be very good. The above results do not seem to be affected to any significant degree by the choice of the corner boundary conditions used. However, the computed values of the Kirchhoff corner forces are affected, as can be seen in Table 5.4. Another quantity affected is the computed value of effective shear force along the boundary, at the corner point, as is seen in Figure 5.2. Timoshenko & Woinowsky-Krieger [1959] give the value of the effective shear force at the corner for this problem as zero. Therefore, by using the boundary conditions which set effective shear force to zero at the corner, a better representation of the vertical forces in the vicinity of a corner can be achieved, which is indicated by the given results. Furthermore, Figure 5.2 shows that this effect is very localised and does not affect the computed values of effective shear force away from the corner.

Figure 5.5 shows the deflection along OA, in Figure 5.1, when the plate boundaries are clamped, and Figure 5.6 shows the normal moment along BC, in Figure 5.1, for the same problem. The results given by Sanders [1969] have been taken from Bezine & Gamby [1978]. Figure 5.7 gives the variation of effective shear force along BC, in Figure 5.1, and compares it with the finite element results attributed to Sanders [1969], as given by Bezine & Gamby [1978]. It is seen that, apart from the corner, the two solutions are in close agreement.

5.3 SIMPLY SUPPORTED RECTANGULAR PLATES UNDER UNIFORMLY DISTRIBUTED LOADS

Figure 5.8(a) shows a rectangular plate in the plane of Cartesian coordinates x and y . The discretisation of the boundary using 24 elements is shown in Figure 5.8(b). The area discretisation is shown in Figure 5.8(c). The boundary conditions prescribed at the simply supported corners were that the deflections, normal moment either side and effective shear force either side of a corner must vanish. As was seen in the last section, this yields more accurate computed values of vertical forces at the corners.

Table 5.5 gives the computed values of the central deflections for different degrees of boundary discretisation, for a square plate, and compares them with the exact series solution given by Timoshenko & Woinowsky-Krieger [1959] and the finite element solution presented by Martin [1976]. The finite element solution uses the Semiloof shell element (see Irons [1976]). Table 5.6 gives the computed values of deflection, moment and effective shear force, at selected points in the plate domain and on its boundary, for rectangular plates of differing aspect ratios. The computed values of the Kirchhoff corner forces are also included. It can be seen that the results obtained using the present BIE implementation agree very well with the series solution given by

Timoshenko & Woinowsky-Krieger [1959].

The deflections along the centreline and the slopes along half a side of a simply supported square plate are given in Tables 5.7 and 5.8, respectively. Figures 5.9 and 5.10 show the variations of slope along the longer and shorter sides, respectively, for rectangular plates with different aspect ratios. In Figure 5.10, the results for $b/a = 1.8$ and $b/a = 1.4$ are omitted for the sake of clarity. The results for $b/a = 1.8$ are practically coincident with those for $b/a = 2.0$. The variations of the effective shear force along the longer and shorter sides are shown in Figures 5.11 and 5.12, respectively. The results plotted are for a square plate and a rectangular plate with $b/a = 2.0$. The computed values of effective shear force along the edges of rectangular plates with aspect ratios within these limits lie in between the results shown. Finally, Figure 5.13 shows the variation of the bending moments in the x and y directions along the centreline for a simply supported square plate.

5.4 CLAMPED RECTANGULAR PLATES UNDER UNIFORMLY DISTRIBUTED LOADS

These problems are similar to those discussed in the last section, except that the plate boundaries are built-in and not simply supported. The plate geometry and loading are shown in Figure 5.14(a). The discretisation of the boundary using 24 elements is shown in Figure 5.14(b). The interior discretisation is shown in Figure 5.14(c). The boundary conditions prescribed at a clamped to clamped corner are as given in Section 5.2.

Table 5.9 gives the computed values for the central deflection of a square plate for different degrees of boundary discretisation and compares them with the exact series solution given by Timoshenko & Woinowsky-Krieger [1959] and the finite element solution presented by Martin [1976] using the Semiloof shell element (see Irons [1976]). Table 5.10 gives the

computed values of the deflection and moments at selected points inside and on the boundary of the plate, for rectangular plates of different aspect ratios. As can be seen, these are in good agreement with the series solution. The variations of the normal moments along the longer and shorter sides are shown in Figures 5.15 and 5.16, respectively. It is seen, in both cases, that with increased aspect ratio, the change in the normal moments becomes smaller and smaller. In fact, the changes in moments along the shorter side were so small that, in the interest of clarity, the results for $b/a = 1.6, 1.8$ and 2.0 have been omitted.

Figures 5.17 and 5.18 show the variations of the effective shear force along the longer and shorter sides of the plate. These are given for Poisson's ratio of 0.2 and are compared with the finite difference solution given by Moody [1960] for the same value of Poisson's ratio. It is seen that there is little change in the value of effective shear force along the shorter side for the two aspect ratios shown. Table 5.11 gives values for deflections and bending moments along AC in Figure 5.14(a). The figures given by Moody [1960] have been obtained with a Poisson's ratio of $\nu = 0.2$. The BIE results are given for $\nu = 0.3$ and $\nu = 0.2$. It is interesting to note that the effect of Poisson's ratio on m_x decreases as the boundary is approached. This is to be expected because, as the $x = 0$ boundary is approached, $\partial^2 w / \partial y^2$ reduces to zero and therefore its contribution, $\nu (\partial^2 w / \partial y^2)$, to m_x becomes smaller and smaller. A similar observation can be made for m_y when the $y = 0$ or $y = b$ boundaries are approached.

5.5 SIMPLY SUPPORTED RECTANGULAR STRIP UNDER A UNIFORMLY DISTRIBUTED LOAD

The problem considered here is that of the bending of a simply supported infinite strip under a uniformly distributed load. To model the problem, a rectangular plate is chosen with a 10:1 aspect ratio, as shown in Figure 5.19(a). The plate boundary is discretised, as shown in Figure 5.19(b), using 12 equal length elements along the length and 3 equal length elements along the width. Figure 5.19(c) shows the interior discretisation.

Along the length of the plate, the boundary conditions are those of a simply supported edge. Along the width of the plate at the two ends, two types of boundary conditions are considered. In the first case, this boundary is assumed to be free. In the second case, the boundary conditions are chosen so as to simulate an infinitely long strip bent into a cylindrical surface. This requires the vanishing of the normal slope and the effective shear force along the width of the plate. For convenience, this boundary condition will be referred to as "cylindrical". The boundary conditions used at a corner point are shown in Figures 5.20(a) and 5.20(b) for each of the above two types of end conditions.

An important point to mention at this stage is that the boundary conditions given in Figure 5.20(a) for a free to simply supported corner is not true for an angle other than 90° . In such cases, the condition that $\partial W / \partial n_f = 0$ is not correct because n_f and t_s in Figure 5.20(a) are at an angle to each other. It may be recalled from Section 4.5 that, in this case, four boundary conditions are available and not five as required. Therefore, an alternative relation is needed in order to solve for the eight variables present at a corner. This extra equation may be obtained by writing the equation for twisting moment jump at the corner node. Alternatively, the slope on one side of the corner can be related to the slope on the other side by use of coordinate transformation relationships.

The numerical results obtained using the present implementation are given in Table 5.12. It can be seen that for points away from the far ends, the results agree with the analytical solution given by Timoshenko & Woinowsky-Krieger [1959] to a high degree of accuracy. Furthermore, they are, as expected, insensitive to the choice of boundary conditions at the ends. However, the computed values of the deflections at the mid-points of the ends of the plate are affected by the choice of the end boundary conditions. In both cases, the normal slopes along the sides of the strip are largely unaffected by the end conditions and are in good agreement with the analytical solution given by Timoshenko & Woinowsky-Krieger [1959]. The figures in brackets in the column for normal slope in Table 5.12 refer to the plate with its end boundary conditions taken to be free.

5.6 SIMPLY SUPPORTED RECTANGULAR PLATES BENT BY MOMENTS ALONG TWO OPPOSITE EDGES

Figure 5.21 shows a rectangular plate acted upon by uniformly distributed moments along two opposite edges. This problem is different from those discussed so far in that there is no lateral load on the plate. The moments applied along the boundary are incorporated in the boundary conditions. Along the sides $x = \pm a/2$, the simply supported boundary conditions require that deflection and normal moment be zero. Along the sides $y = \pm b/2$, the boundary conditions are deflection equal to zero and normal moment equal to M_0 , which is the applied moment. The boundary conditions used at a corner are shown in Figure 5.22. The results obtained using the present formulation are given in Table 5.13. It is seen that they are in very good agreement with the exact series solution given by Timoshenko & Woinowsky-Krieger [1959].

5.7 CANTILEVER PLATE UNDER A CONCENTRATED LOAD

This problem is considered for two different rectangular geometries with widely different aspect ratios. In both cases, one edge of the plate is assumed to be clamped and the other three free. The boundary conditions used at a clamped to free corner are the vanishing of the deflection, normal slope either side of a corner and normal moment and effective shear force on the free side. At a free to free corner, the boundary conditions used are that normal moment and effective shear force must vanish either side of a corner, plus the vanishing of the Kirchhoff corner force.

The problem considered first is that of a long and narrow rectangular strip, built in along one of its longer edges, as shown in Figure 5.23(a). The aspect ratio of the plate analysed is chosen as 10:1. The plate boundary is discretised using 10 elements along the length and 2 elements across the width of the plate. The loading considered on the plate is that of a concentrated load at the centroid. The computed distributions of normal moments and effective shear force along the clamped side are shown in Figures 5.24 and 5.25, respectively. It is seen that, in both cases, the maximum value of the stress resultant occurs at the mid-point along the clamped side and rapidly reduces to practically zero, with increased distance from this point.

The second problem considered is that of a square cantilever plate under a concentrated load at the centre. Williams [1951] has shown that, within the framework of Kirchhoff's thin plate theory, for certain types of boundary conditions the moment and/or the effective shear force become unbounded near a corner. It would, therefore, be interesting to examine the solution provided by the present formulation for such problems. The clamped to free corner provides an example of this type.

The plate geometry and boundary discretisations used are shown in

Figure 5.26, there being three different types of boundary discretisation. Figure 5.27 shows the computed variation of normal moment along the length of the clamped side. The results obtained using the two different meshes are in good agreement with each other for all the points along the boundary, except those very near a corner. The steady drop in the magnitude of the normal moment as the corner is approached would seem to suggest that this quantity is not unbounded at the corner.

The results obtained for the effective shear force, however, suggest that this quantity is unbounded near a corner. This is exhibited by the oscillatory behaviour of the solution near a corner, as is shown in Figures 5.28, 5.29 and 5.30. The results shown in Figure 5.30 are obtained using a very refined mesh near the corner, as shown in Figure 5.26(c). The computed values of the effective shear force for the last few nodes near the corner continued to oscillate, and were so large that they could not be plotted to scale in Figure 5.30. Williams [1951] has shown that for the particular case of a clamped to free corner, the dominant solution to the eigen equation is complex, which implies an oscillatory behaviour. It is seen, therefore, that the BIE solution obtained is consistent with the analytical work of Williams [1951]. The computation of the intensity of the singularities of the type seen above is outside the scope of the present work. However, the boundary integral equation method has been used successfully by some authors in the calculation of these so-called stress intensity factors (see, for example, Stern [1975], Stern & Kim [1980], Kim [1982], and Stern [1983]). Figures 5.31 to 5.33 give the deflections of the cantilever along the boundary and the centreline OC shown in Figure 5.26. They are in good agreement with the BIE solution given by Bezzine & Gamby [1978]. The BIE solutions presented by these authors employ constant elements. It is important to note that the effect of the singularity at the corner is

very localised and does not affect the overall deflected shape of the plate. Figure 5.34 gives the values of slopes along AB and BC in Figure 5.26.

5.8 SIMPLY SUPPORTED SQUARE PLATE UNDER A LINEARLY VARYING LOAD

This type of loading occurs in engineering practice in, for example, retaining wall structures and liquid containers. The plate geometry and boundary discretisation are shown in Figures 5.35(a) and 5.35(b). Figure 5.35(c) shows the interior discretisation. The intensity of the lateral loading is shown in the shaded triangle on top of the figure. The boundary conditions employed at the simply supported corners are those of vanishing deflection, normal slope and normal moment on either side.

Table 5.14 gives the computed values of deflection and moments along the centreline AB, shown in Figure 5.35(a), and compares them with the series solution given by Timoshenko & Woinowsky-Krieger [1959]. It can be seen that for both moments and deflections, there is very good agreement between the two solutions. Figure 5.36 shows the variation of slope along the edges $x = 0$ and $x = L$. Figure 5.37 shows the variation of slope along the sides $y = 0$ or $y = L$. The distributions of the effective shear force along the boundary are shown in Figures 5.38 and 5.39.

5.9 SIMPLY SUPPORTED SQUARE PLATE UNDER A LOAD IN THE SHAPE OF A TRIANGULAR PRISM

This problem has been solved by Timoshenko & Woinowsky-Krieger [1959]. Because of the unusual type of loading, it is an interesting test case for the present computer program. The geometry and loading on the plate are shown in Figure 5.40(a). Figure 5.40(b) shows the boundary discretisation and Figure 5.40(c) shows the interior discretisation. The

boundary conditions used at the simply supported corners are those given by Stern [1979] and described earlier in Section 5.2.

Table 5.15 gives the results for deflection, moments and vertical forces at selected points inside and on the boundary of the plate. It is seen that the solution obtained by using the present formulation agrees very well with the series solution given by Timoshenko & Woinowsky-Krieger [1959]. The deflections along the two lines of symmetry shown in Figure 5.40 are shown in Figure 5.41. These are symmetric about the mid-point. The variations of the moments along the same lines are also symmetric about the mid-point and are shown in Figures 5.42 and 5.43. It is seen that the magnitude of the deflections along AO is larger than those along CO, except, of course, at O. This is expected since the intensity of the loading along AO is greater than that along CO. This means that at O, the curvature in the x direction is larger than that in the y direction, giving a higher value of moment in the x direction.

5.10 RECTANGULAR PLATES HAVING ONE EDGE CLAMPED AND THE OTHER THREE SIMPLY SUPPORTED, UNDER A UNIFORMLY DISTRIBUTED LOAD

This problem introduces another type of boundary condition, that of a clamped to simply supported corner, which has not been considered so far. The geometry and boundary and interior discretisations employed are shown in Figure 5.44. The boundary conditions used at a clamped to simply supported corner are that the deflection, normal slope either side and normal moment either side of the corner must vanish. In fact, these are the same boundary conditions used at the simply supported corners.

The results obtained using the present formulation are given in Table 5.16. It is seen that these are in very good agreement with the series solution given by Timoshenko & Woinowsky-Krieger [1959].

5.11 RECTANGULAR PLATES WITH TWO OPPOSITE EDGES CLAMPED AND THE OTHER TWO
SIMPLY SUPPORTED, UNDER A UNIFORMLY DISTRIBUTED LOAD

Figure 5.45(a) shows the geometry of the plate under consideration. The discretisation of the boundary is shown in Figure 5.45(b) and that of the interior is shown in Figure 5.45(c). The clamped to simply supported boundary conditions used were described in the last section. The results obtained using the present formulation are given in Table 5.17 for a range of aspect ratios. They are found to be in very good agreement with those recorded by Timoshenko & Woinowsky-Krieger [1959].

5.12 RECTANGULAR PLATES WITH TWO OPPOSITE EDGES SIMPLY SUPPORTED AND THE
OTHER TWO FREE, UNDER A UNIFORMLY DISTRIBUTED LOAD

The corner boundary conditions for this problem have already been discussed in Section 5.5 and are shown in Figure 5.20(a). The geometry and mesh designs are shown in Figure 5.46. The results obtained using the present formulation are given in Table 5.18 for a range of aspect ratios. The high degree of accuracy of the solution obtained is shown by comparing them with the exact series solution given by Timoshenko & Woinowsky-Krieger [1959]. However, for $b/a = 0.5$, the value of m_y at the centre of the plate is significantly different from that given in this reference. Figure 5.46 shows the computed variation of m_y along the $y = L/2$ centreline. The figure quoted for the series solution is plotted on the same graph. By comparing the various moment profiles, it would appear that the numerical value obtained by the present formulation is the correct one and that the figure published by Timoshenko & Woinowsky-Krieger [1959] may be a printing error.

5.13 RECTANGULAR PLATES WITH TWO OPPOSITE EDGES SIMPLY SUPPORTED, A THIRD
EDGE FREE AND THE LAST ONE CLAMPED, UNDER A UNIFORMLY DISTRIBUTED
LOAD

The corner boundary conditions encountered in this problem have already been discussed in Sections 5.5 and 5.10. However, this problem is interesting in that it mixes all three types of boundary conditions encountered so far, that is, simply supported, clamped and free. The geometry and mesh designs are shown in Figure 5.48.

Table 5.19 gives the computed values of deflection and bending moments at selected points inside and on the boundary of the plate. Where possible, these have been compared with the exact series solution given by Timoshenko & Woinowsky-Krieger [1959] and the agreement between them is seen to be very good. The variation of the deflection along the line of symmetry, DE, for a square plate is shown in Figure 5.49. It is seen that the variation is almost linear for points away from the clamped boundary. Figure 5.50 shows the deflections along the $y = b/2$ axis for the case of a square plate. The deflection profile is symmetric about the centroid of the plate and, therefore, half of it is shown only. Figure 5.51 shows the variations of the moments along the line of symmetry, DE, for a square plate, and Figure 5.52 shows the variation of moments along the line $y = b/2$. For the latter case, the profile is shown over half the length of the plate since the results are symmetric about the centroid of the plate. Finally, the distribution of moments along the clamped side is shown in Figure 5.53. Due to symmetry, the results are shown for only half the length of the clamped side. It may be recalled from Table 5.19 that the computed value for the maximum normal moment along the clamped side is in very close agreement with that given by Timoshenko & Woinowsky-Krieger [1959].

5.14 SQUARE PLATE WITH THREE EDGES CLAMPED AND ONE EDGE FREE

This type of boundary condition is encountered in practice in, for example, retaining walls and rectangular tanks. The boundary conditions at the clamped to clamped and clamped to free corners have already been discussed in Sections 5.2 and 5.7, respectively. Three types of loading are considered. These are uniformly distributed load, linearly varying load and a load uniformly distributed over two thirds of the plate area. The plate geometry and the loading types are shown in Figure 5.54(a). The boundary discretisation is shown in Figure 5.54(b). This mesh design was used so that the sampling points became coincident with those presented in the finite difference solution given by Moody [1960]. This allows a direct comparison between the two solutions. The interior discretisation is shown in Figure 5.54(c). This conveniently accommodates the third type of loading considered. For this case, the plate interior is, theoretically, divided into seven segments along the y axis. However, at $y/L = 2/3$, the coordinates are chosen so as to give a row of elements with zero areas. The intensity of loading along one side of these elements, parallel to the x direction, is set to $\frac{1}{2}$ and along the other side to zero. So, although across these elements the intensity of the lateral loading varies, in the y direction, from $\frac{1}{2}$ to zero, their net contribution to the area integration is zero because the elements have zero areas. This effectively produces a jump in the lateral loading from $\frac{1}{2}$ to zero, bearing in mind that loading data are entered for nodes rather than elements.

The variations of deflection along the line of symmetry AC, for the three different types of loading, are shown in Figure 5.55. As expected, the largest deflection occurs under the uniformly distributed load. Figure 5.56 shows the variation of the effective shear force along the edge OB for the uniformly loaded case. It is seen that, once again, as in Section 5.7, the computed values of the effective shear force near the

clamped to free corner behave in an oscillatory manner, suggesting a possible singularity (see Section 5.7, and also Williams [1951]). The finite difference procedure used by Moody [1960] solves first for deflections. The moments and effective shear forces at each node are then computed by using the appropriate finite difference molecule at that point, averaging out the singularity. Figures 5.57 to 5.60 present further results for the uniformly loaded case. It is seen that for all cases, the results obtained using the present formulation are in good agreement with those given by Moody [1960], except for points near or at the corners. Figure 5.60 suggests a possible singularity for the normal moment on the clamped side in the vicinity of a clamped to free corner.

Figures 5.61 to 5.65 give the results for moments and effective shear force for the plate carrying a linearly varying load. It is seen that, once again, the effective shear force near the clamped to free corner behaves in a singular manner. However, for points away from corners, the integral equation results are in good agreement with those given by Moody [1960]. The results for moments and effective shear forces in a plate carrying a uniformly distributed load on two thirds of its area are shown in Figures 5.66 to 5.70. It is worth noting that the effective shear force near the clamped to free corner is, once again, singular. Furthermore, although there is, in general, good agreement between the BIE and the finite difference solutions, the agreement between the two solutions for normal moments along OB, in Figure 5.54, is not as good as the previous two cases considered. The author believes that this may be due to the fact that the jump in the lateral loading is accommodated more effectively in the present formulation than in the finite difference method.

5.15 SQUARE PLATE UNDER A UNIFORMLY DISTRIBUTED LOAD SUPPORTED ON FOUR COLUMNS AT ITS CORNERS

The geometry and mesh discretisation for this problem are shown in Figure 5.71. This problem introduces a new boundary condition, that of a column support at the plate's corners. The rest of the plate boundary is assumed to be free. Therefore, the boundary condition used at each corner is that deflection, normal moment either side and effective shear force either side of the corner must vanish.

Table 5.20 gives the values of deflections and moments along the centreline AC, in Figure 5.71. The values for the central deflection and moments are compared with those given by Timoshenko & Woinowsky-Krieger [1959]. From symmetry, it is expected that the value obtained for the corner reactions would be equal to a quarter of the total load. This is shown also in Table 5.20. Figures 5.72 and 5.73 show the variations of deflection and normal slope, respectively, along an edge of the plate. As these are symmetrical about the mid-point, only half of each distribution is shown.

5.16 SQUARE PLATES WITH SUPPORTS INSIDE THE DOMAIN

In all the examples considered so far, the plate has been supported along its boundary. However, in practice, there are situations where additional supports are provided for the plate inside the plate domain; for example, columns under a floor slab. This section considers square plates under a concentrated load at the centre, having simply supported, clamped or free boundaries plus additional internal supports. The plate geometry and the positions of interior supports are shown in Figure 5.74(a). Figure 5.74(b) shows the boundary discretisation.

The incorporation of internal supports in the present implementation is achieved by treating them as unknown concentrated loads on the plate.

The deflection under this load is assumed to be zero. Therefore, by writing the equation for deflection at interior points in terms of the boundary variables and the transverse loading, that is equation (2.87), at every interior support point and equating it to zero, a sufficient number of equations is available for the unknown support reactions. Although Bezzine [1981a] has already applied the BIE method to such problems using a constant element formulation, the above approach was formulated by the author independently.

In the examples presented in this section, three types of boundary conditions are considered; simply supported, clamped and free. The treatment of the corner boundary conditions in each case has already been discussed in Sections 5.2 and 5.7.

Table 5.21 gives values for the Kirchhoff corner forces and the support reactions for a square plate under a centrally applied load. For the plate with free boundaries, it is seen that the support reaction is computed very accurately as a quarter of the total load. The magnitudes of these reactions are larger than 0.25 for the clamped and simply supported cases. This is to be expected because, looking at Figure 5.78, it is seen that when the plate boundary is free, it deflects in the opposite direction to that of the applied load. Therefore, the vertical forces required along the boundary to bring these deflections back to zero (as is the case for clamped and simply supported boundaries) would be primarily in the same direction as the applied load (see Figure 5.81). The support reactions must balance these, therefore their magnitude is larger than when the plate boundary is free. When the plate boundary is clamped, additional moments would be required to reduce the boundary slopes to zero. These moments have the effect of reducing the magnitude of the deflections inside the plate. This, in turn, reduces the magnitude of the support reactions, giving a smaller value relative to the simply

supported case. The results given in Table 5.21 seem to confirm this. Figures 5.75, 5.76 and 5.77 show the variation of deflection along the centreline CF, in Figure 5.74, when the plate boundaries are simply supported, clamped and free, respectively. It is seen that the results obtained using the present formulation are in good agreement with the BIE solution given by Bezine [1981a]. This author employed constant elements around the boundary. Figure 5.78 shows the deflection along the boundary when the plate boundary is free. The slopes along the boundary are shown in Figure 5.79. As the solution is symmetrical about the mid-point of each side, the variations of deflection and slope are shown along half a side only. The variation of normal moment along half an edge for the plate with clamped boundaries is shown in Figure 5.80, and the variation of effective shear force along the boundary for the simply supported and clamped cases is shown in Figure 5.81. Where possible, these have been compared with the solution given by Bezine [1981a].

5.17 RHOMBIC PLATES UNDER A UNIFORMLY DISTRIBUTED LOAD

The preceding sections contain examples in which the plate geometry is either square or rectangular. In this and the following sections, plates with polygonal geometries other than square or rectangular are considered. Simply supported rhombic plates under uniformly distributed load are treated first.

Figure 5.82(a) shows the plate geometry. The boundary and interior discretisations are shown in Figures 5.82(b) and 5.82(c). Table 5.22 gives the computed values of the central deflection for rhomboids of different angles of skew, α . It is seen that the results obtained using the present formulation agree very well with those given by Morley [1963], using finite differences, and Maiti & Chakrabarty [1974], using BIE, for values of the skew angle, α , near 90° . However, as α decreases from 90° ,

the agreement between the solutions gradually deteriorates. The greatest discrepancy occurs at the smallest angle considered, $\alpha = 30^\circ$. The important point to note is that for the smaller angles of skew, the deflections obtained using the present formulation are always smaller than the ones given by Morley [1963] and Maiti & Chakrabarty [1974]. The author believes that the reason for this "stiffening" effect is the different ways in which the boundary conditions are treated in the present formulation and the other methods. In all cases, the boundary conditions along a simply supported edge are that deflection and normal moment must vanish. However, at corner points, the situation is somewhat different. The BIE solution presented by Maiti & Chakrabarty [1974] employs constant elements (but their constant element formulation is different to the one mentioned in Section 5.3). In this type of formulation, there are not, in general, any corner nodes present. Therefore, the boundary conditions prescribed on the elements either side of a corner are the same as those given along the boundary, that is, zero deflection and normal moment. This would mean that the elements either side of a corner are free to rotate. A similar argument may be used in the case of finite difference solutions and the series solution given by Morley [1963]. The important point is that in these cases, no specific boundary conditions are prescribed at corner points. However, in the quadratic formulation presented in this work, the normal slopes either side of a corner are set to zero. The author believes that this satisfies more correctly the condition of the continuity of the first derivatives, that is, if the gradient in two different directions is zero, they should be zero in all directions in the plane of the two original ones. The alternatives mentioned above would allow the discontinuity of the first derivatives, in the direction along the boundary, at the corners which would not be acceptable. The vanishing of normal slope either side of a corner would

seem to have an effectively clamping effect at the corners. This, therefore, results in a smaller deflection profile. This effect seems to be more severe for an obtuse corner angle because, in these cases, the plate is narrower and the centroid closer to the corner. This is probably why the maximum discrepancy between the above solutions occurs at $\alpha = 30^\circ$.

Figure 5.83 shows the slopes along OA in Figure 5.82 for various angles of skew. For $\alpha = 30^\circ$, the slopes are compared with those obtained using the constant element formulation presented in Chapter 3. This is shown in Figure 5.84. It is seen that, although near the acute corner the two solutions agree, in the vicinity of the obtuse corner the slopes obtained using the constant element formulation are larger than those obtained by using the quadratic elements. In fact, the non-dimensionalised value of the central deflection for this case, obtained using the constant element formulation, was found to be 0.000440. This result appears to give some support to the preceding argument concerning the differences in central deflection seen in Table 5.22. Lyons [1977] has solved the problem of a 30° skew rhombic plate using the compatible element proposed by Veubeck [1968]. This author gives a value of 0.00033 for the non-dimensionalised value of the central deflection. Referring to Table 5.22, it can be seen that this is in good agreement with the result obtained using the present quadratic elements.

Table 5.23 gives values for the moments at the centroid of the plate for various angles of skew. These results seem to be consistent with those given in Table 5.22, that is, for angles of skew at which the deflections obtained using the present formulation are smaller than those given by the other authors, the moments are also smaller.

5.18 TRIANGULAR PLATES

This section considers the bending of a plate in the shape of an equilateral triangle. The plate is assumed to be simply supported. Three kinds of loading are considered; a concentrated load at the centroid, a uniformly distributed load and moments along the edges.

5.18.1 Simply Supported Equilateral Triangular Plate Under a Concentrated Load at Its Centroid

Figure 5.85(a) shows an equilateral triangular plate lying in the x,y plane. The boundary and interior discretisation are shown in Figures 5.85(b) and 5.85(c). The plate is assumed to be simply supported along its boundaries and carrying a concentrated load at its centroid. The simply supported boundary conditions used at the corner are as discussed in Section 5.2.

Table 5.24 gives the computed values of deflections along a median. The deflection at the centroid, $\mathcal{D}$, is seen to agree very well with the analytical solution given by Timoshenko & Woinowsky-Krieger [1959]. Table 5.25 gives the computed values of slopes along half a side. These are symmetric about the mid-point of each side.

5.18.2 Simply Supported Equilateral Triangular Plate Under a Uniformly Distributed Load

Timoshenko & Woinowsky-Krieger [1959] present an exact analytical expression for the deflection along the median for this problem. The results obtained by the present formulation are given in Table 5.26. The deflection at the centroid given by the integral equation solution of Maiti & Chakrabarty [1974] is also shown. These authors use constant elements, but their formulation is different to that presented in Chapter 3. It is seen that the accuracy of the results obtained using

the present formulation is very good when compared with the analytical solution. Table 5.27 compares the computed values of slope along the boundary again with the analytical solution. The computed values of moments at interior points are given in Table 5.28.

5.18.3 Simply Supported Equilateral Triangular Plate Bent by Constant Moments Applied Along Its Boundary

Table 5.29 gives the computed values of deflections along the median for this problem and compares them with the analytical solution given by Timoshenko & Woinowsky-Krieger [1959]. It is seen that the agreement between the two solutions is remarkably good. The computed values of the slopes along the boundary are shown in Table 5.30 and those for effective shear force along the boundary and the Kirchhoff corner forces in Table 5.31. In all cases, the present BIE solution is in very close agreement with the analytical solution. Finally, Table 5.32 gives the computed values of moments along the median and compares them again with those obtained using the analytical solution.

5.19 SQUARE PLATE WITH A SQUARE CENTRAL OPENING

The geometry of the plate under consideration is shown in Figure 5.86(a). The outer boundary is assumed to be simply supported and the inner one free. The plate is assumed to carry a uniformly distributed load. Taking advantage of symmetry, only quarter of the plate is modelled, as shown in Figure 5.86(b). The interior discretisation is shown in Figure 5.86(c). The boundary conditions used along the boundaries DB and FG are the cylindrical boundary conditions described in Section 5.5. The corner boundary conditions at the simply supported to "cylindrical" corner are given in Figure 5.20(b). The boundary conditions used at the "cylindrical" to free corners are that the normal

slope and effective shear force on the "cylindrical" sides, the normal moment and effective shear force on the free sides and the Kirchhoff corner forces must vanish.

Table 5.33 gives the computed values of deflections at selected points on the boundary and in the interior of the plate, and compares them with other solutions. The tabulated values for finite difference and finite element solutions were presented by Tottenham [1979]. The BIE results derived from the same reference employ linear elements. The BIE results due to Du, et al. [1984] employ Hermitian interpolation functions. It is seen that the results obtained using the present formulation agree best with those presented by Du, et al. Figure 5.87 shows the variation of slopes along the boundaries, and Figure 5.88 shows the variation of normal moments along the "cylindrical" end.

5.20 SIMPLY SUPPORTED CORNER PLATE UNDER A UNIFORMLY DISTRIBUTED LOAD

Figure 5.89 shows the geometry of the plate under consideration and the boundary discretisation. As can be seen, the discretisation is refined in the vicinities of the two corners. The reason for this choice of discretisation is that near the corners, especially the re-entrant corner, B, very large moments are expected, as discussed by Williams [1951]. The interior discretisation is shown in Figure 5.90. Each limb of the plate is assumed to be very long, and, therefore, the plate is assumed to bend to a cylindrical surface away from the corner (see Salvadori & Reggini [1960]). The boundary conditions used along the two sides OA and DE are, therefore, taken as the "cylindrical" boundary conditions discussed in Section 5.5.

Table 5.34 gives the computed values of deflections along the boundaries OA and DE, and compares them with the finite difference solution given by Salvadori & Reggini [1960]. It is seen that the finite

difference results are larger than the exact values by around 5%.

However, the difference between the present BIE solution and the exact one is less than 1%. The exact solution is that of the bending of a simply supported rectangular strip under a uniformly distributed load (see Section 5.5).

The variations of deflections along the diagonal CB are shown in Figure 5.91. It is seen that the deflections obtained using the simply supported boundary conditions discussed in Section 5.2 at the corners B and C are significantly less than those given by the finite difference solution of Salvadori & Reggini [1960] and the integral equation method of Segedin & Brickell [1968]. The author believes that the reason for this discrepancy is similar to that discussed for the case of the rhombic plates in Section 5.17. It is believed that the condition of vanishing normal slopes either side of a corner produces a clamping effect at these corners, which in turn contributes to a smaller deflection than would be obtained if the corner was free to rotate. This effect is more pronounced in the case of the re-entrant corner. Figure 5.91 shows also results obtained when the re-entrant corner is left free. This was done by setting the boundary conditions at the corner to those of a free to free corner with vanishing normal moment and effective shear force either side of the corner plus vanishing Kirchhoff corner force. The freeing of the outer corner, C, did not have any significant effect on the solution. It is seen that by freeing the re-entrant corner, the computed values of the deflections along the diagonal increase. Finally, results were also obtained for this problem using the constant element formulation given in Chapter 3. It is seen that this case gives larger deflections along the diagonal compared with those obtained by the quadratic formulation. These results may be explained by looking at Figure 5.29 showing the slopes along the boundary AB, in Figure 5.89, obtained using the quadratic and constant

element formulations presented in this work. It is seen that the largest computed values of normal slope along the boundary are given by the constant element formulation and the smallest ones by the quadratic formulation with the re-entrant corner, B, simply supported. The variation of moments along the diagonal CB is shown in Figure 5.93 and is compared with the finite difference solution given by Salvadori & Reggini [1960]. It is seen that as the re-entrant corner is approached, the two solutions differ widely. The BIE solution seems to confirm the singularity present at such corners (see Williams [1951]). The computation of the intensity of such singularities needs special attention, as was discussed in Section 5.7.

The question as to whether the BIE solution presented here is more accurate than the other solutions given above is not a simple one. It is seen that, although the constant element and quadratic formulations presented in this thesis, which were used to solve this problem, are derived from the same integral equations, there is a large discrepancy between the solutions obtained using them for the deflections along the diagonal CB. The results shown in Figure 5.92 seem to support the argument that the reason for this is the larger slopes obtained by the constant element formulation along the boundaries either side of the re-entrant corner, B, in Figure 5.89. This, in turn, is due to the fact that in the quadratic formulation, the normal slopes either side of a simply supported corner are set to zero, as was discussed in Section 5.17.

Table 5.34 shows that along the width of the plate at its ends, the finite difference solution over-estimates the deflections by about 6%. It may be that this error is repeated throughout the plate domain and perhaps amplified near the corner. This would be one reason why the deflections obtained using finite differences are, in general, larger than the BIE solutions presented in this work. A further contribution could

also arise from the lack of any specific restraints on the normal slopes either side of the re-entrant corner in the finite difference solution. This has also been discussed in Section 5.17.

TABLE 5.1

Deflection, w , at the Centre of a Square Plate With a Load, F , at Its Centre (Figure 5.1)

Poisson's ratio, $\nu = 0.3$

$w/(\frac{F L^2}{D})$	4 Elements	8 Elements	12 Elements	24 Elements	48 Elements	Semiloof	Exact TWK
Square plate clamped along all edges	0.00549	0.00563	0.00561	0.00561	0.00561	0.00577	0.00560
Square plate simply supported along all edges	0.01175	0.01154	0.01159	0.01160	0.01160	0.01780	0.01160

TWK : Timoshenko & Woinowsky-Krieger [1959]

Results quoted for the Semiloof element are given by Martin [1976]

TABLE 5.2

Deflection Along the Centre Line OA, in Figure 5.1, for a
Simply Supported Square Plate Under a Central Load, F

Poisson's ratio, $\nu = 0.3$

x/L	$w \times (D/F L^2)$	
	TWK	QUDPLT
0.0	0.011600	0.011601
0.1	0.010531	0.010531
0.2	0.008404	0.008405
0.3	0.005791	0.005792
0.4	0.002943	0.002943
0.5	0.000000	0.000000

TWK : Timoshenko & Woinowsky-Krieger [1959]
QUDPLT : Present quadratic formulation

TABLE 5.3

Slope, $-(\partial W / \partial n) \times (D / F L)$, Along BC, in Figure 5.1, for a
Simply Supported Square Plate Under a Centrally Applied Load, F
Poisson's ratio, $\nu = 0.3$

x/L	TWK	QUDPLT
0.0	0.02958	0.02957
1/12	0.02844	0.02843
2/12	0.02518	0.02518
3/12	0.02022	0.02022
4/12	0.01407	0.01407
5/12	0.00720	0.00720
0.5	0.00000	0.00000

TWK : Timoshenko & Woinowsky-Krieger [1959]
QUDPLT : Present quadratic formulation

TABLE 5.4

Corner Reactions for a Square Plate Under a Central Load

Poisson's ratio, $\nu = 0.3$

t/F	4 Elements	8 Elements	12 Elements	24 Elements	48 Elements	Exact TWK
Clamped $w = 0$ $(\partial w / \partial n)_{\pm} = 0$ $(m)_{\pm} \equiv 0$	0.236	0.077	0.022	0.002	0.003	0.000
Simply supported $w = 0$ $(\partial w / \partial n)_{\pm} = 0$ $(m)_{\pm} \equiv 0$	0.021	0.066	0.099	0.117	0.121	0.122
Simply supported $w = 0$ $(m)_{\pm} = 0$ $(v)_{\pm} = 0$	0.132	0.120	0.121	0.122	-	0.122

TWK : Timoshenko & Woinowsky-Krieger [1959]

TABLE 5.5

Deflection, $w \times (D/\delta L^4)$, at the Centre of a Simply Supported Square Plate
of Side L Under a Uniformly Distributed Load of Intensity, δ (Figure 5.8)

Poisson's ratio, $\nu = 0.3$

Number of Elements	QUDPLT	Martin [1976]	TWK
4	0.00409	0.00406	0.00406
8	0.00405		
12	0.00406		
24	0.00406		
48	0.00406		

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 5.6

Maximum Values of Deflection, Moment and Effective Shear Force Plus the Corner Forces for
Simply Supported Rectangular Plates Under a Uniformly Distributed Load, q (Figure 5.8)

Poisson's ratio, $\nu = 0.3$

b/a	$w/(q a^4/D)$ at $x = a/2, y = b/2$		$M_x/(q a^2)$ at $x = a/2, y = b/2$		$M_y/(q a^2)$ at $x = a/2, y = b/2$		$-V_x/(q a)$ at $x = 0, y = b/2$		$-V_y/(q a)$ at $x = a/2, y = 0$		$t/(q a^2)$	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
1.0	0.00406	0.00406	0.0479	0.0479	0.0479	0.0479	0.420	0.421	0.420	0.421	0.065	0.063
1.2	0.00564	0.00565	0.0627	0.0628	0.0501	0.0501	0.455	0.456	0.453	0.454	0.074	0.075
1.4	0.00705	0.00708	0.0755	0.0757	0.0502	0.0502	0.478	0.474	0.471	0.477	0.083	0.082
1.6	0.00830	0.00831	0.0862	0.0864	0.0492	0.0493	0.491	0.490	0.485	0.486	0.088	0.087
1.8	0.00931	0.00932	0.0948	0.0950	0.0479	0.0479	0.499	0.497	0.491	0.493	0.090	0.089
2.0	0.01013	0.01013	0.1017	0.1018	0.0464	0.0463	0.503	0.500	0.496	0.497	0.092	0.091

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 5.7

Deflection Along the Centreline AC, in Figure 5.8, for a Simply Supported

Square Plate of Side a Under a Uniformly Distributed Load, $\frac{q}{D}$

Poisson's ratio, $\nu = 0.3$

x/a	$w \times (D/\frac{q}{D} a^4)$	
	TWK	QUDPLT
0.0	0.000000	0.000000
0.1	0.001316	0.001315
0.2	0.002463	0.002462
0.3	0.003336	0.003336
0.4	0.003879	0.003876
0.5	0.004062	0.004062

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 5.8

Slope Along Half an Edge, OE in Figure 5.8, of a Simply Supported Square

Plate Subjected to a Uniformly Distributed Load, q

Poisson's ratio, $\nu = 0.3$

x/a	Slope, $-(\partial w / \partial n) \times (D / q a^3)$	
	TWK	QUDPLT
0.0	0.00000	0.00000
0.1	0.00448	0.00449
0.2	0.00829	0.00829
0.3	0.01114	0.01115
0.4	0.01289	0.01290
0.5	0.01348	0.01349

TWK : Timoshenko & Woinowsky-Krieger [1959]
QUDPLT : Present quadratic formulation

TABLE 5.9

Deflection, $w \times (D/\delta L^4)$, at the Centre of a Clamped Square Plate
Under a Uniformly Distributed Load of Intensity, δ (Figure 5.14)

Poisson's ratio, $\nu = 0.3$

Number of Elements	QUDPLT	Martin [1976]	TWK
4	0.00125	0.00126	0.00126
8	0.00127		
12	0.00127		
24	0.00127		
48	0.00127		

TWK : Timoshenko & Woinowsky-Krieger [1959]
QUDPLT : Present quadratic formulation

$a = b = L$ in Figure 5.14

TABLE 5.10

Deflection and Bending Moments in Uniformly Loaded Rectangular Plates With All Edges Clamped (Figure 5.14)

Poisson's ratio, $\nu = 0.3$

b/a	$w \times (D/F a^4)$ at $x = a/2, y = b/2$		$M_x/(F a^2)$ at $x = 0, y = b/2$		$M_y/(F a^2)$ at $x = a/2, y = 0$		$M_x/(F a^2)$ at $x = a/2, y = b/2$		$M_y/(F a^2)$ at $x = a/2, y = b/2$	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
1.0	0.00126	0.00127	-0.0513	-0.0514	-0.0513	-0.0514	0.0230	0.0230	0.0230	0.0230
1.2	0.00172	0.00173	-0.0639	-0.0642	-0.0554	-0.0557	0.0299	0.0301	0.0222	0.0229
1.4	0.00207	0.00207	-0.0726	-0.0729	-0.0568	-0.0571	0.0349	0.0351	0.0212	0.0213
1.6	0.00230	0.00230	-0.0780	-0.0784	-0.0571	-0.0574	0.0381	0.0383	0.0193	0.0193
1.8	0.00245	0.00246	-0.0812	-0.0816	-0.0571	-0.0574	0.401	0.402	0.0174	0.0173
2.0	0.00254	0.00254	-0.0829	-0.0834	-0.0571	-0.0573	0.0417	0.0413	0.0158	0.0158

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 5.11

Deflections and Moments Along the Centreline AC, in Figure 5.14, for a Square Plate Under a
Uniformly Distributed Load and Clamped Along the Boundaries

Poisson's ratio, $\nu = 0.3$; figures in brackets () for $\nu = 0.2$

x/a	$m_x / (q a^2)$			$m_y / (q a^2)$			$w \times (D / q a^4)$	
	TWK	Moody [1960]	QUDPLT	TWK	Moody [1960]	QUDPLT	TWK	QUDPLT
0.0	-0.0513	-0.0500	-0.0516 (-0.0513)	-	-0.0100	-	0.00000	0.00000
0.1	-	-0.0156	-0.0163 (-0.0166)	-	-0.0007	-0.0026 (-0.0009)	-	0.00019
0.2	-	0.0040	0.0041 (0.0034)	-	0.0082	0.0081 (0.0080)	-	0.00057
0.3	-	0.0147	0.0155 (0.0143)	-	0.0153	0.0162 (0.0151)	-	0.00093
0.4	-	0.0198	0.0213 (0.0197)	-	0.0197	0.0213 (0.0196)	-	0.00118
0.5	0.0230	0.0213	0.0230 (0.0212)	0.0230	0.0213	0.0230 (0.0212)	0.00126	0.00127

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 5.12

Deflections, Bending Moments and Normal Slope for a Simply Supported Long Rectangular Strip Under a Uniformly Distributed Load (see Figure 5.19(a))

Poisson's ratio, $\nu = 0.3$; figures for normal slope given in brackets are for free end conditions

x/L	TWK		QUDPLT End conditions cylindrical		QUDPLT End conditions free		QUDPLT	TWK
	$w \times (D/\delta b^4)$ along $y = 0$	$m_y/(\delta b^2)$ along $y = 0$	$w \times (D/\delta b^4)$ along $y = 0$	$m_y/(\delta b^2)$ along $y = 0$	$w \times (D/\delta b^4)$ along $y = 0$	$m_y/(\delta b^2)$ along $y = 0$	$-(\partial w/\partial n) \times (D/\delta b^3)$ along $y = \pm b/2$	$-(\partial w/\partial n) \times (D/\delta b^3)$ along $y = \pm b$
- 1/6	0.01302	0.125	0.01302	0.125	0.01303	0.125	0.04161 (0.04165)	0.04167
0.0	0.01302	0.125	0.01302	0.125	0.01302	0.125	0.04161 (0.04162)	0.04167
1/6	0.01302	0.125	0.01302	0.125	0.01303	0.125	0.04161 (0.04165)	0.04167
$\pm 1/2$	0.01302	-	0.01301	-	0.01428	-	0.04169 (0.04119)	0.04167

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 5.13

Deflections and Bending Moments at the Centre of Rectangular Plates Simply Supported and
Subjected to Uniformly Distributed Moments Along Two Opposite Edges (see Figure 5.21)

Poisson's ratio, $\nu = 0.3$

b/a	w		m_x		m_y	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
0.5	$0.0964 \frac{M_o b^2}{D}$	$0.0966 \frac{M_o b^2}{D}$	$0.387 M_o$	$0.387 M_o$	$0.770 M_o$	$0.771 M_o$
1.0	$0.0368 \frac{M_o a^2}{D}$	$0.0369 \frac{M_o a^2}{D}$	$0.394 M_o$	$0.394 M_o$	$0.256 M_o$	$0.256 M_o$
1.5	$0.0280 \frac{M_o a^2}{D}$	$0.0281 \frac{M_o a^2}{D}$	$0.264 M_o$	$0.264 M_o$	$0.046 M_o$	$0.047 M_o$

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 5.14

Deflection and Bending Moments Along AB, in Figure 5.35, for a
Simply Supported Square Plate Subjected to a Linearly Varying Load

Poisson's ratio, $\nu = 0.3$

x/L	$w \times (D/\delta \quad L^4)$ along $y = L/2$		$m_x/(\delta \quad L^2)$ along $y = L/2$		$m_y/(\delta \quad L^2)$ along $y = L/2$	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
0.25	0.00131	0.00131	0.0132	0.0131	0.0149	0.0149
0.5	0.00203	0.00203	0.0239	0.0240	0.0239	0.0240
0.6	0.00201	0.00203	0.0264	0.0265	0.0245	0.0245
0.75	0.00162	0.00163	0.0259	0.0258	0.0207	0.0208

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 5.15

Deflection, Moments, Effective Shear Force and the Kirchhoff Corner Force for a Simply Supported Square Plate Under a Load in the Shape of a Triangular Prism (see Figure 5.40)

Poisson's ratio, $\nu = 0.3$

$w \times (D/\delta \quad L^4)$ at $x = y = L/2$		$m_x/(\delta \quad L^2)$ at $x = y = L/2$		$m_y/(\delta \quad L^2)$ at $x = y = L/2$		$V_x/(\delta \quad L)$ at $x = 0, y = L/2$		$V_y/(\delta \quad L)$ at $x = L/2, y = 0$		$t/(\delta \quad L^2)$ corner reactions	
TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
0.00263	0.00263	0.0340	0.0341	0.0317	0.0317	0.199	0.200	0.315	0.319	0.038	0.038

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 5.16

Deflection and Bending Moments in a Rectangular Plate With One Edge Clamped, the Other Three Simply Supported, Under a Uniformly Distributed Load (Figure 5.44)

Poisson's ratio, $\nu = 0.3$

b/a	Deflection at $x = y = 0$		m_x at $x = y = 0$		m_y at $x = y = 0$		m_y at $x = 0, y = -b/2$	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
0.5	$0.0049 \frac{b^4}{D}$	$0.0049 \frac{b^4}{D}$	$0.023 f b^2$	$0.023 f b^2$	$0.060 f b^2$	$0.060 f b^2$	$-0.122 f b^2$	$-0.122 f b^2$
1.0	$0.0028 \frac{a^4}{D}$	$0.0028 \frac{a^4}{D}$	$0.034 f a^2$	$0.034 f a^2$	$0.039 f a^2$	$0.039 f a^2$	$-0.084 f a^2$	$-0.084 f a^2$
2.0	$0.0093 \frac{a^4}{D}$	$0.0093 \frac{a^4}{D}$	$0.094 f a^2$	$0.094 f a^2$	$0.047 f a^2$	$0.047 f a^2$	$-0.122 f a^2$	$-0.122 f a^2$

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 5.17

Deflection, w , and Moments, m_x, m_y , for Rectangular Plates Under a Uniformly Distributed Load

With Two Opposite Edges Clamped and the Other Two Simply Supported (Figure 5.45)

Poisson's ratio, $\nu = 0.3$

b/a	$w \times (D/\delta a^4)$ at $x = a/2, y = b/2$		$m_x/(\delta b^2)$ at $x = a/2, y = b/2$		$m_y/(\delta b^2)$ at $x = a/2, y = b/2$		$m_y/(\delta b^2)$ at $x = a/2, y = 0$	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
1.0	0.00192	0.00192	0.0244	0.0244	0.0332	0.0333	-0.0697	-0.0700
1.4	0.00460	0.00461	0.0514	0.0518	0.0448	0.0446	-0.0998	-0.1000
1.6	0.00603	0.00602	0.0650	0.0651	0.0469	0.0469	-0.1090	-0.1092
1.8	0.00732	0.00731	0.0768	0.0769	0.0477	0.0476	-0.1152	-0.1153
2.0	0.00844	0.00844	0.0869	0.0869	0.0474	0.0473	-0.1191	-0.1191

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 5.18

Deflection and Moments for a Rectangular Plate With Two Opposite Edges Free and the Other Two Simply Supported, Under a Uniformly Distributed Load, q (see Figure 5.46)

Poisson's ratio, $\nu = 0.3$

b/a	$w \times (D/q a^4)$ at $x = a/2, y = b/2$		$m_x/(q a^2)$ at $x = a/2, y = b/2$		$m_y/(q a^2)$ at $x = a/2, y = b/2$		$w \times (D/q a^4)$ at $x = a/2, y = 0, b$	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
0.5	0.01377	0.01375	0.1235	0.1241	0.0102	0.0126	0.01443	0.01467
1.0	0.01309	0.01311	0.1225	0.1228	0.0271	0.0276	0.01509	0.01497
1.2	-	0.0130	-	0.1228	-	0.0310	-	0.01503
1.6	-	0.01291	-	0.1232	-	0.0348	-	0.01507
2.0	0.01289	0.01291	0.1235	0.1237	0.0364	0.0365	0.01521	0.01506

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 5.19

Deflections and Bending Moments for a Uniformly Loaded Rectangular Plate With Two Opposite Edges

Simply Supported, the Third Edge Free and the Fourth Clamped (Figure 5.48)

Poisson's ratio, $\nu = 0.3$

b/a	$w \times (D/f b^4)$ at $x = a/2, y = b$		$w \times (D/f b^4)$ at $x = a/2, y = b/2$		$m_x/(f b^2)$ at $x = a/2, y = b/2$		$m_y/(f b^2)$ at $x = a/2, y = b/2$		$m_y/(f b^2)$ at $x = a/2, y = 0$	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
1.0	0.0113	0.0112	-	0.0057	-	0.0563	-	0.0282	-0.119	-0.119
0.5	0.0582	0.0580	-	0.0231	-	0.0105	-	-0.0069	-0.319	-0.319

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 5.20

Deflection and Moments Along the Centreline AC, in Figure 5.69, Plus the Support Reactions
at the Corners for a Uniformly Loaded Square Plate Supported at its Corners

Poisson's ratio, $\nu = 0.25$

x/L	$w \times (D/\delta L^4)$ along $x = L/2$		$m_x/(\delta L^2)$ along $x = L/2$		$m_y/(\delta L^2)$ along $x = L/2$		$t/(\delta L^2)$ at corners	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT	Exact	QUDPLT
0.1	-	0.01960	-	0.1389	-	0.0374	-0.2500	-0.2502
0.2	-	0.02191	-	0.1268	-	0.0683		
0.3	-	0.02383	-	0.1182	-	0.0916		
0.4	-	0.02509	-	0.1130	-	0.1063		
0.5	0.0257	0.02553	0.1109	0.1112	0.1109	0.1112		

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 5.21

Kirchhoff Corner Force, t , and Support Reaction, R ,
for a Square Plate Under a Centrally Applied Point
Load and on Four Interior Supports (Figure 5.72)

Poisson's ratio, $\nu = 0.3$

Edge Support Conditions	t/F	R/F
Clamped	0.0102	-0.4041
Simply Supported	-0.0190	-0.4145
Free	0.0000	-0.2500

TABLE 5.22

Deflections, $w \times (D/4 L^4)$, at the Centre of a Simply Supported Rhomboid

Plate Carrying a Uniformly Distributed Load (Figure 5.82)

Poisson's ratio, $\nu = 0.3$

Angle, α	Morley (Finite Difference)	Morley (Series)	Maiti & Chakrabarty	Present Quadratic
$\alpha = 90^\circ$	0.00406	-	0.00408	0.00406
$\alpha = 85^\circ$	-	0.00401	0.00403	0.00401
$\alpha = 80^\circ$	0.00386	0.00387	0.00388	0.00387
$\alpha = 60^\circ$	0.00254	0.00256	0.00256	0.00251
$\alpha = 50^\circ$	0.00167	0.00172	0.00171	0.00163
$\alpha = 45^\circ$	-	-	-	0.00122
$\alpha = 40^\circ$	0.00092	0.00096	0.00094	0.00085
$\alpha = 30^\circ$	0.00038	0.00041	0.00038	0.00033

TABLE 5.23

Moments, $m_x/(l^2 L^2)$ and $m_y/(l^2 L^2)$, at the Centre of a Simply Supported Rhombic Plate Carrying
a Uniformly Distributed Load (Figure 5.82)

Poisson's ratio, $\nu = 0.3$

Angle, α	Morley (Finite Difference)		Morley (Series)		Maiti & Chakrabarty		Present Quadratic	
	m_x	m_y	m_x	m_y	m_x	m_y	m_x	m_y
90°	0.0477	0.0477	-	-	0.0480	0.0480	0.0479	0.0479
85°	-	-	0.0486	0.0466	0.0486	0.0468	0.0485	0.0467
80°	0.0485	0.0444	0.0486	0.0448	0.0486	0.0450	0.0486	0.0449
60°	0.0427	0.0325	0.0425	0.0333	0.0426	0.0333	0.0421	0.0327
50°	0.0364	0.0245	0.0362	0.0258	0.0363	0.0257	0.0352	0.0243
45°	-	-	-	-	-	-	0.0309	0.0199
40°	0.0283	0.0168	0.0281	0.0180	0.0284	0.0177	0.0264	0.0156
30°	0.0192	0.0098	0.0191	0.0108	0.0200	0.0098	0.0171	0.0078

TABLE 5.24

Deflection, $w \times (D/\delta L^2)$, for a Simply Supported Equilateral Triangle

With a Concentrated Load, F , at Its Centroid (Figure 5.85)

Poisson's ratio, $\nu = 0.3$

Point	TWK	QUDPLT
A	-	0.00020
B	-	0.00143
C	-	0.00380
D	0.00575	0.00571
E	-	0.00360

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 5.25

Normal Slope, $(\partial w / \partial n) \times (D / F L)$, Along Half an Edge of a Simply Supported
Equilateral Triangular Plate Under a Concentrated Load, F , at Its Centre

(see Figure 5.85)

Poisson's ratio, $\nu = 0.3$

x/b	$-\frac{\partial w}{\partial n} \times \frac{D}{F L}$
8/16	0.02274
7/16	0.02162
6/16	0.01861
5/16	0.01447
4/16	0.01003
3/16	0.00597
2/16	0.00275
1/16	0.00070
0.0	0.00000

TABLE 5.26

Deflection, $w \times (D/\delta L^4)$, for a Simply Supported Equilateral Triangle

Under a Uniformly Distributed Load, F (see Figure 5.85)

Poisson's ratio, $\nu = 0.3$

Point	TWK	MCH	QUDPLT
A	0.00007033	-	0.00007027
B	0.00038580	-	0.00038579
C	0.00081380	-	0.00081383
D	0.00102881	0.001034	0.00102882
E	0.00075352	-	0.00075352

TWK : Timoshenko & Woinowsky-Krieger [1959]

MCH : Maiti & Chakrabarty [1974]

QUDPLT : Present quadratic formulation

TABLE 5.27

Normal Slope, $-(\partial w / \partial n) \times (D / F L^4)$, for Half a Side of a Simply Supported

Equilateral Triangular Plate Carrying a Uniformly Distributed Load

(see Figure 5.85)

Poisson's ratio, $\nu = 0.3$

x/b	TWK	QUDPLT
8/16	0.00521	0.00521
7/16	0.00505	0.00505
6/16	0.00458	0.00456
5/16	0.00385	0.00386
4/16	0.00293	0.00295
3/16	0.00193	0.00192
2/16	0.00100	0.00099
1/16	0.00029	0.00029
0.0	0.00000	0.00000

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 5.28

Bending Moments, $m_x/(\frac{1}{6} L^2)$ and $m_y/(\frac{1}{6} L^2)$, Along a Median for an Equilateral Triangular Plate
Simply Supported Along the Boundary Carrying a Uniformly Distributed Load (see Figure 5.85)

Poisson's ratio, $\nu = 0.3$

Point	TWK		MCH		QUDPLT	
	m_x	m_y	m_x	m_y	m_x	m_y
A, $y/L = 5/6$	0.01288	-0.00535	-	-	0.01287	-0.00536
B, $y/L = 2/3$	0.02176	0.00231	-	-	0.02177	0.00237
C, $y/L = 1/2$	0.02578	0.01484	-	-	0.02576	0.01483
D, $y/L = 1/3$	0.02407	0.02407	0.02410	0.02410	0.02414	0.02414
E, $y/L = 1/6$	0.01577	0.02185	-	-	0.01584	0.02189

TWK : Timoshenko & Woinowsky-Krieger [1959]

MCH : Maiti & Chakrabarty [1974]

QUDPLT : Present quadratic formulation

TABLE 5.29

Deflection, $w \times (D/m_0 L^2)$, for a Simply Supported Equilateral Triangular Plate With a Constant Moment, m_0 , Applied Along Its Edges (Figure 5.85)

Poisson's ratio, $\nu = 0.3$

Point	TWK	QUDPLT
A	0.005787	0.005787
B	0.018519	0.018519
C	0.031250	0.031251
D	0.037037	0.037038
E	0.028935	0.028936

TWK : Timoshenko & Woinowsky-Krieger [1959]
QUDPLT : Present quadratic formulation

TABLE 5.30

Normal Slope, $(\partial w / \partial n) \times (D / m_o L)$, for Half a Side of a Simply Supported
Equilateral Triangular Plate, With a Moment, m_o , Applied Along Its Edges

(see Figure 5.85)

Poisson's ratio, $\nu = 0.3$

x/b	TWK	QUDPLT
8/16	0.25000	0.25001
7/16	0.24609	0.24610
6/16	0.23438	0.23438
5/16	0.21484	0.21485
4/16	0.18750	0.18751
3/16	0.15234	0.15235
2/16	0.10938	0.10938
1/16	0.05859	0.05854
0.0	0.00000	0.00000

TWK : Timoshenko & Woinowsky-Krieger [1959]
QUDPLT : Present quadratic formulation

TABLE 5.31

Effective Shear Force, $v_n \times (L/m_o)$, and Kirchhoff Corner Force, $[t]/m_o$, for
a Simply Supported Equilateral Triangular Plate With Moment, m_o , Applied

Along the Edges

Poisson's ratio, $\nu = 0.3$

Variable	TWK	QUDPLT
Effective shear force, $v_n \times (a/m_n)$	1.0500	1.0501
Corner force, $[t]/m_o$	1.2124	1.2207

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 5.32

Bending Moments, m_x/m_o and m_y/m_o , for a Simply Supported Equilateral
Triangular Plate With a Constant Moment, m_o , Applied Along Its Edges

(Figure 5.85)

Poisson's ratio, $\nu = 0.3$

Point	TWK		QUDPLT	
	m_x/m_o	m_y/m_o	m_x/m_o	m_y/m_o
A	1.17500	0.12500	1.17495	0.12504
B	1.00000	0.30000	1.00004	0.30002
C	0.82500	0.47500	0.82503	0.47501
D	0.65000	0.65000	0.65002	0.65002
E	0.47500	0.82500	0.47501	0.82502

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 5.33

Deflection, $w \times (100D/\delta L^4)$, for a Square Plate With a Square Central Opening With a Uniformly Distributed Load.

Outer Boundary Simply Supported, Inner Boundary Free (Figure 5.86)

Poisson's ratio, $\nu = 0.3$

Point	Direct BIE (Tottenham [1979])	Finite Element	Finite Difference	Present Quadratic	Du, et al. [1984]
A	0.2188	0.2185	0.2174	0.2178	0.2172
B	0.3107	0.3156	0.3006	0.3017	0.3011
C	0.1558	-	0.1541	0.1545	0.1542
H	-	-	-	0.0633	-

TABLE 5.34

Deflection Along OA for a Simply Supported Corner Plate Under a

Uniformly Distributed Load (Figure 5.89)

Poisson's ratio, $\nu = 0.3$

x/b	SRG	QUDPLT	Exact
0.0	0.0	-	0.000
0.25	2.5	2.360	2.375
0.5	3.5	3.310	3.333
0.75	2.5	2.360	2.375
1.0	0.0	0.000	0.000

SRG : Salvadori & Reggini [1960]

QUDPLT : Present quadratic formulation

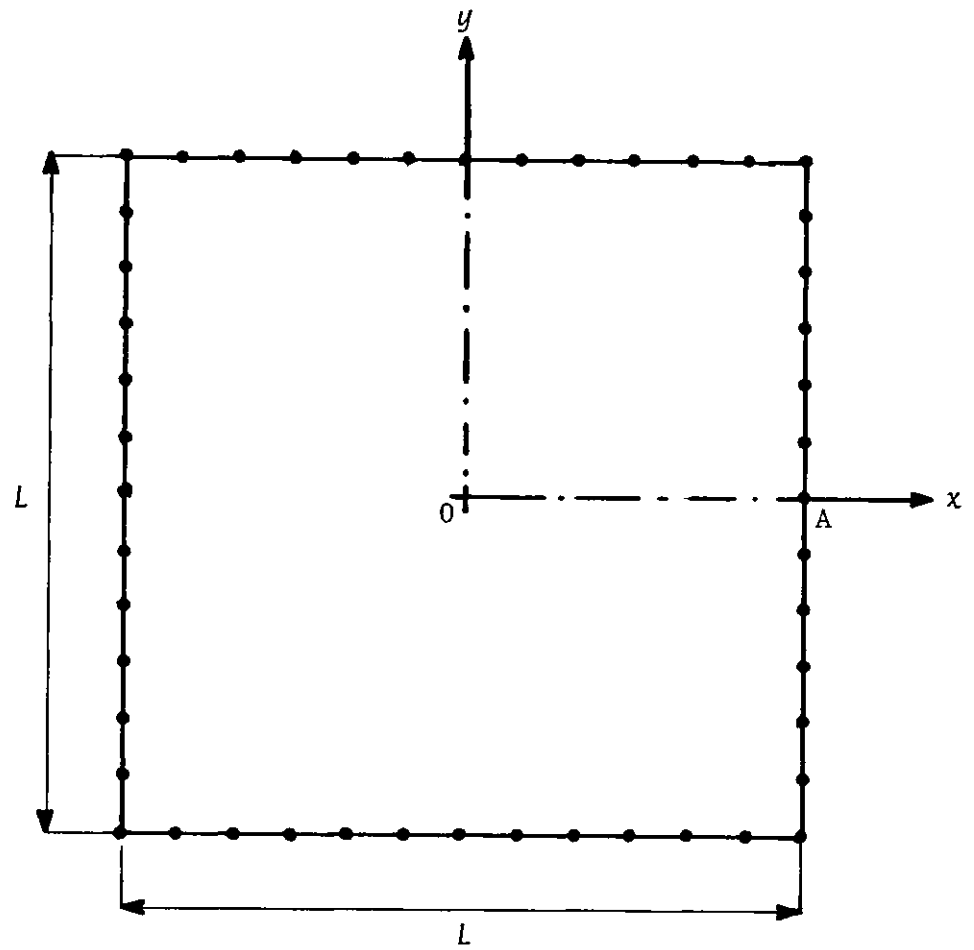


Figure 5.1: Boundary discretisation for a square plate under a central load, F . 24 quadratic elements.

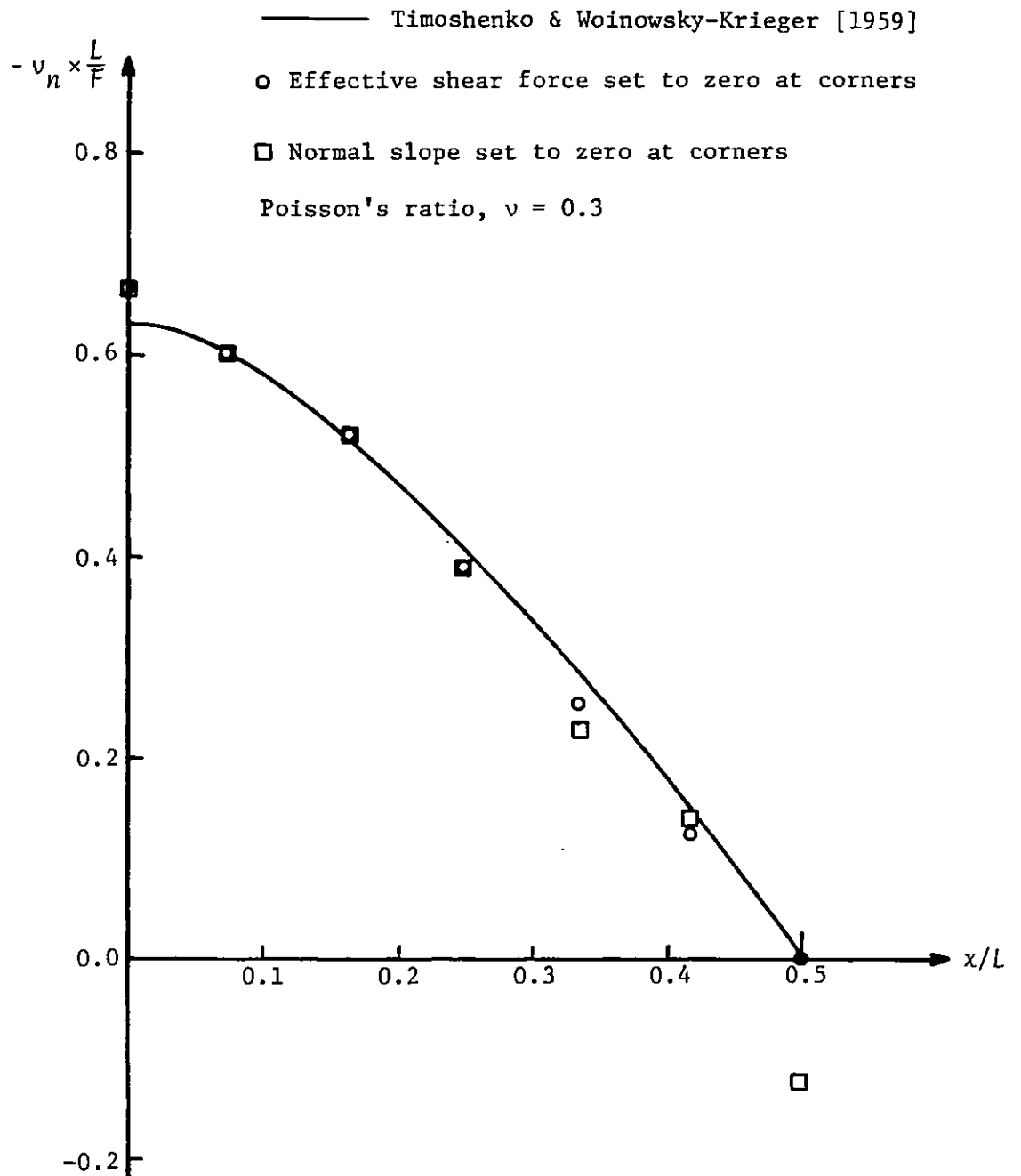


Figure 5.2: Effective shear force along a half edge of a simply supported square plate under a centrally applied load, F (see Figure 5.1)

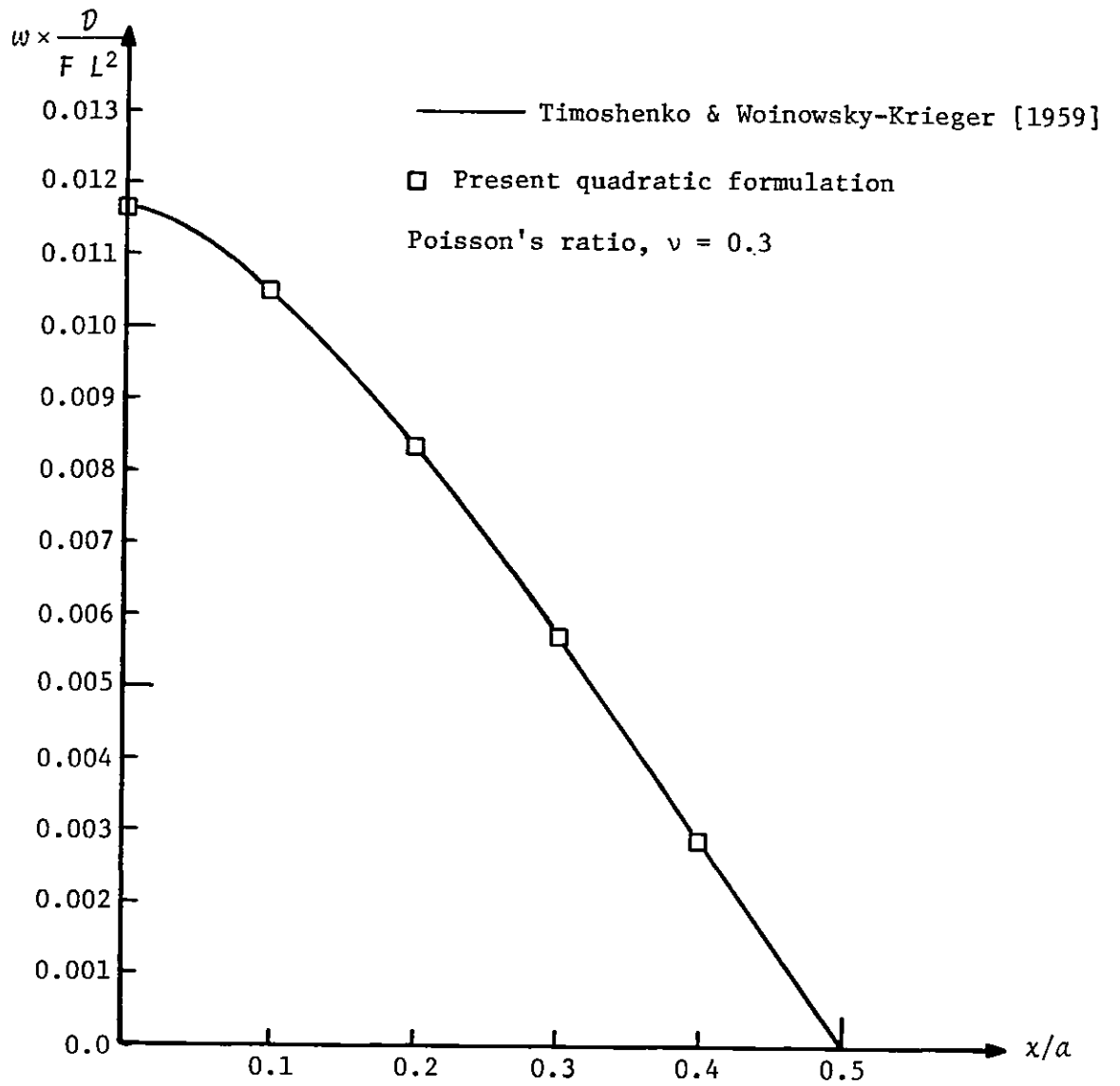


Figure 5.3: Deflection, w , along a half axis of symmetry (OA in Figure 5.1) for a simply supported square plate under a centrally applied point load

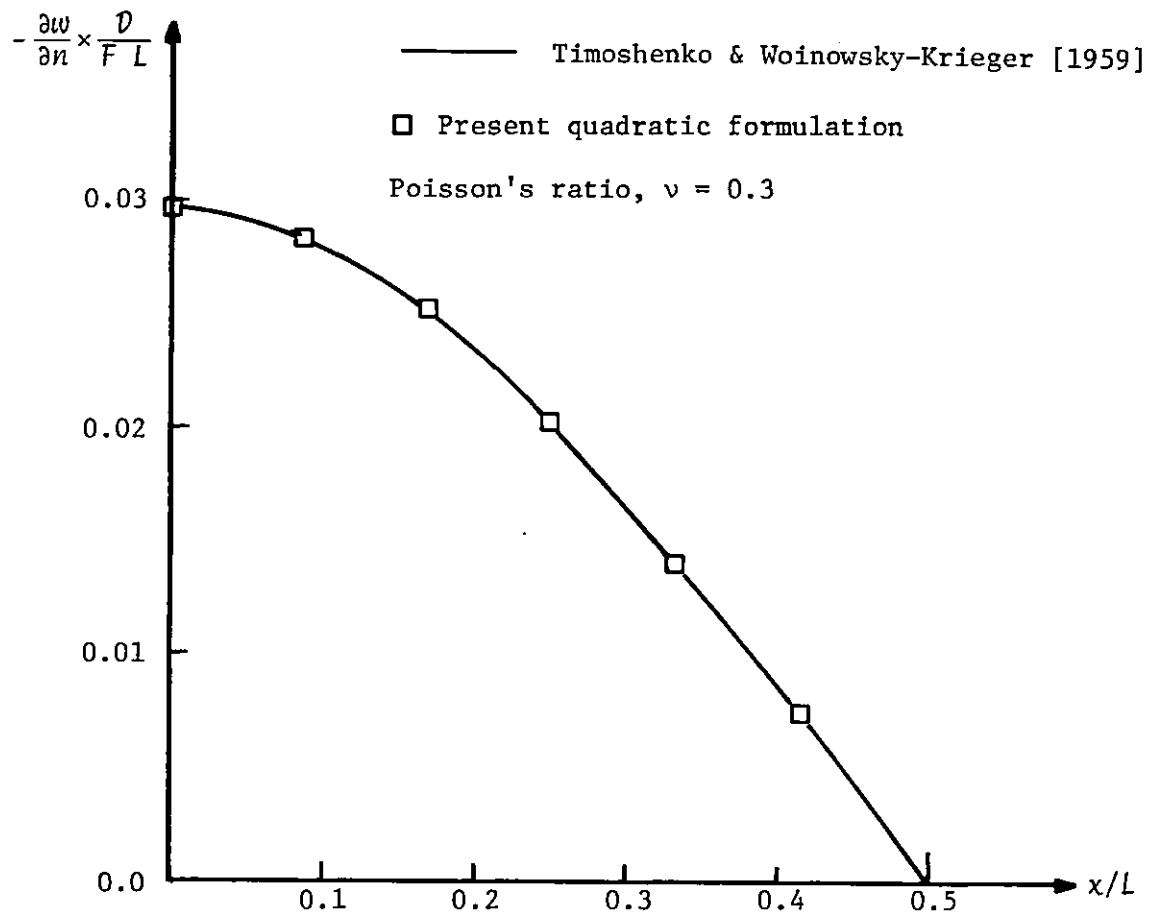


Figure 5.4: Slope along a half edge of a simply supported square plate under a central load, F (see Figure 5.1)

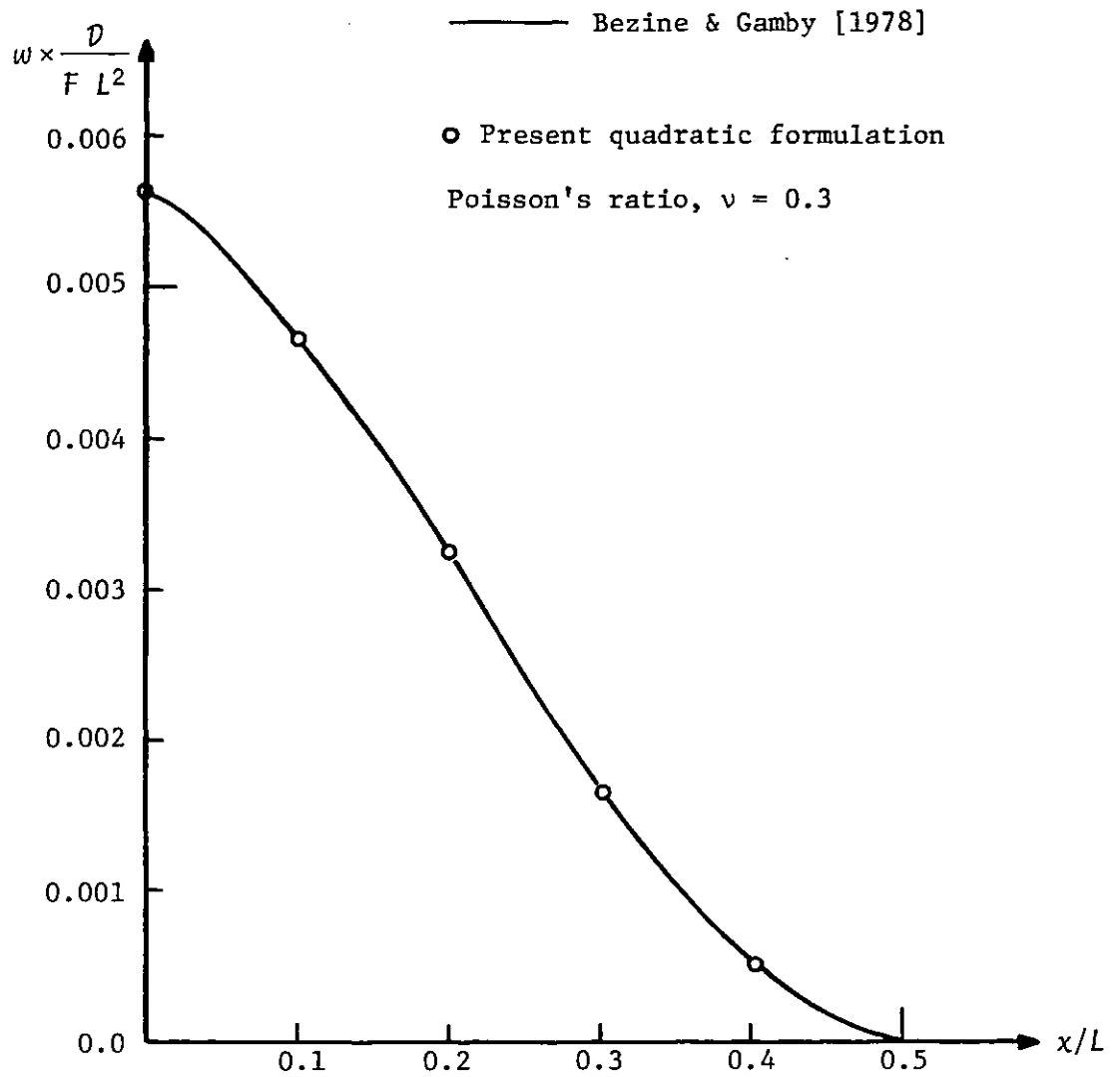


Figure 5.5: Deflection, w , along a half axis of symmetry (OA in Figure 5.1) for a clamped square plate under a centrally applied load, F

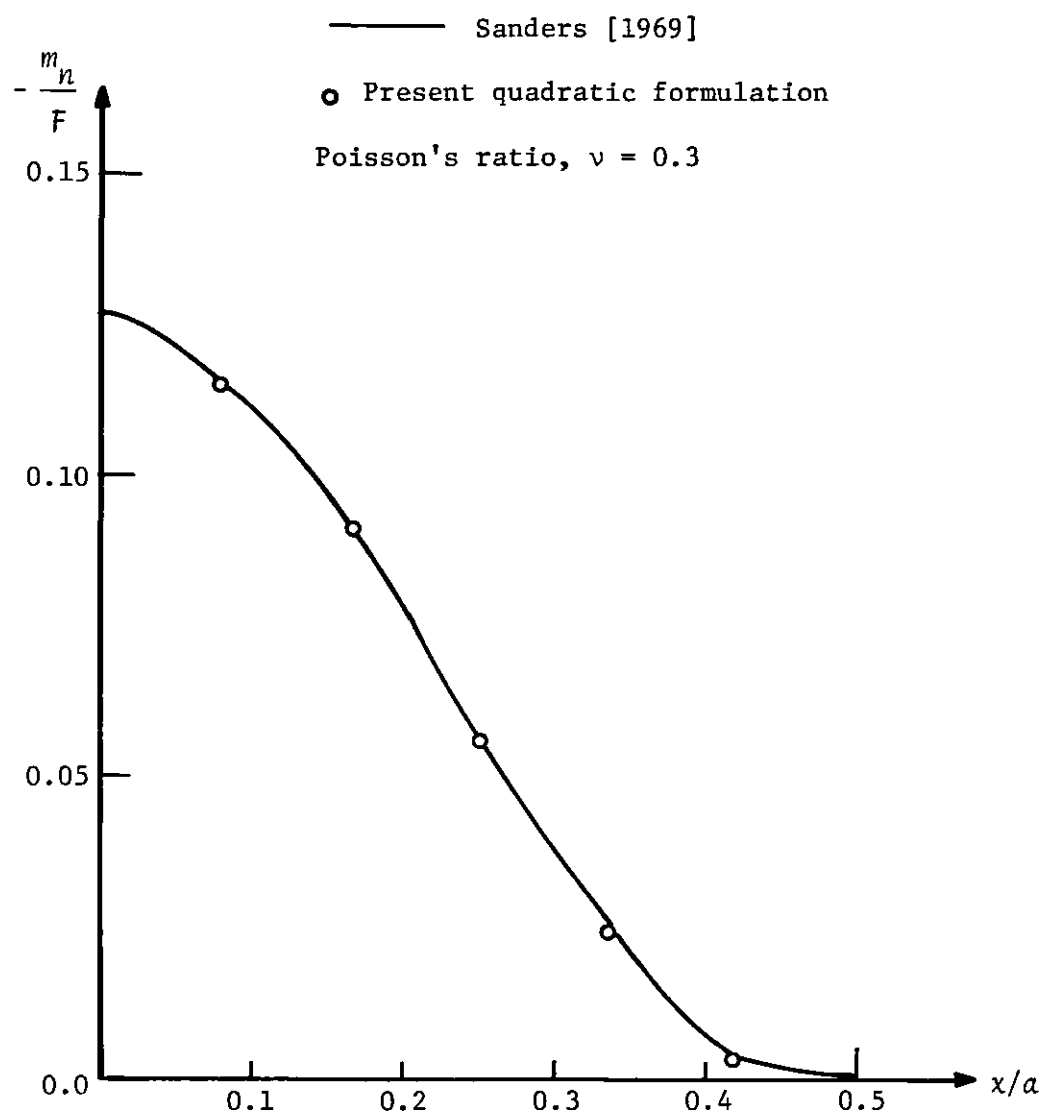


Figure 5.6: Normal moment, m_n , along a half edge of a clamped square plate under a centrally applied load (see Figure 5.1)

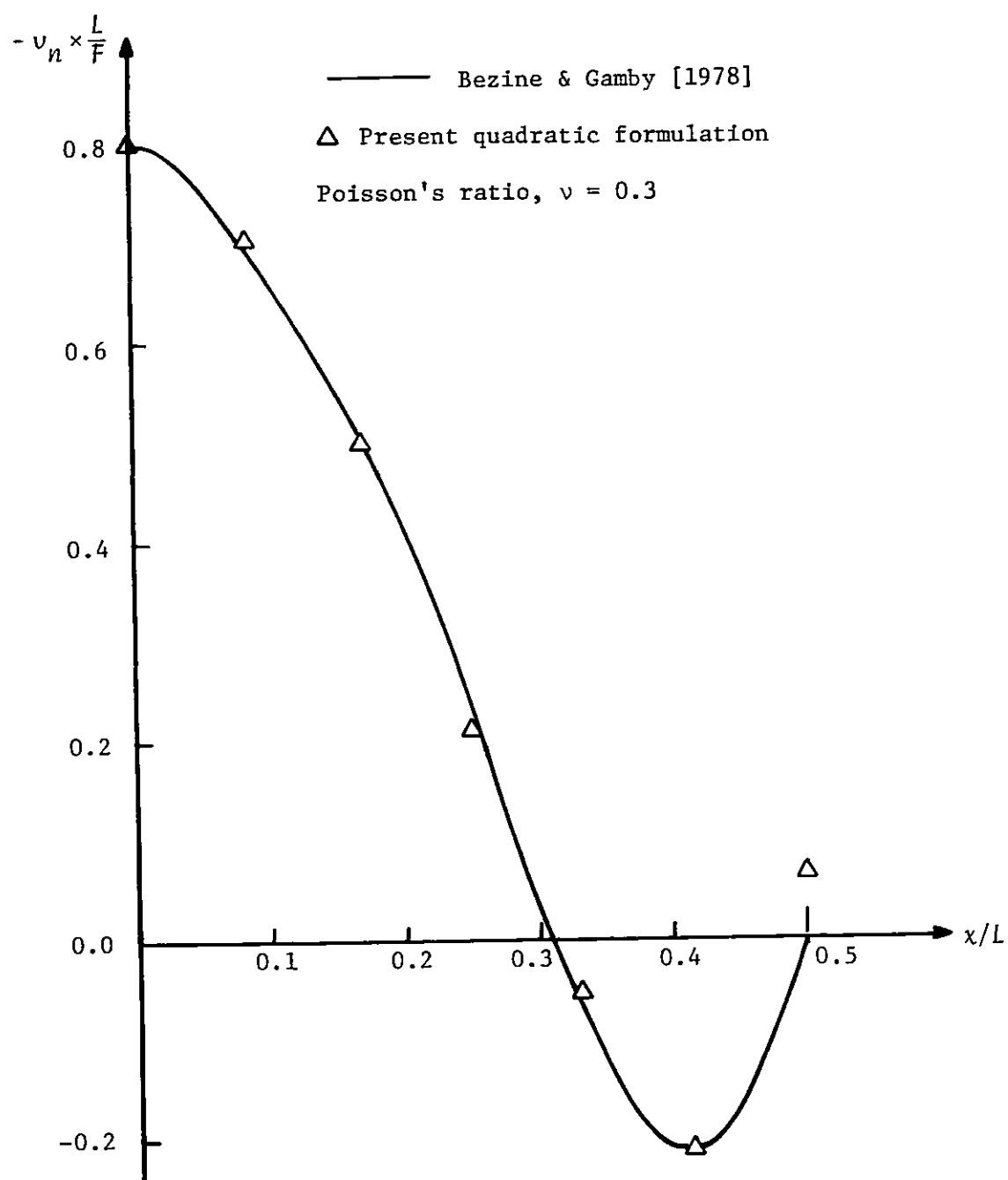
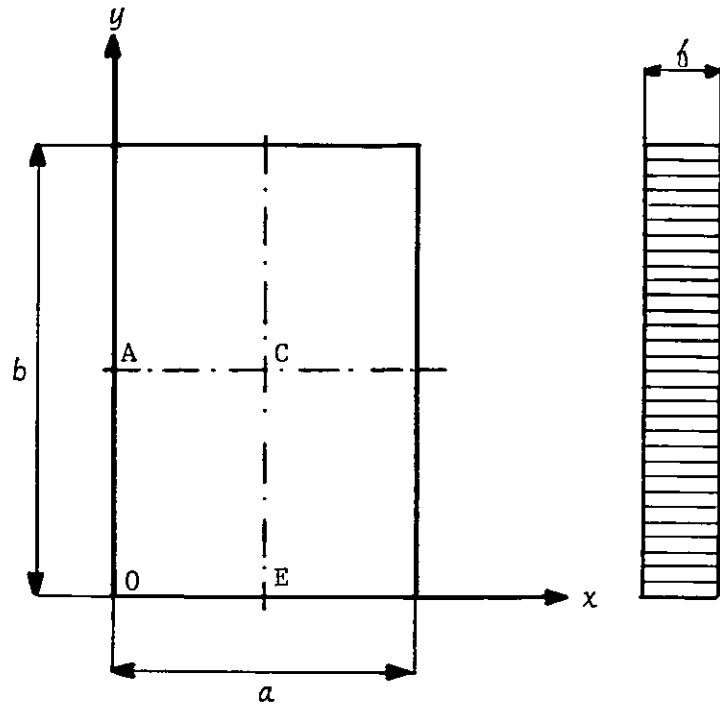
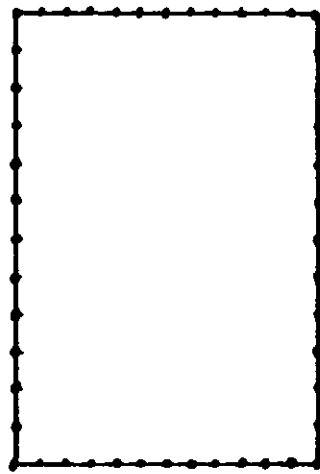


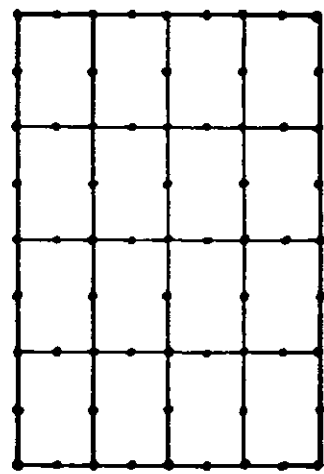
Figure 5.7: Effective shear force along a half edge of a clamped square plate under a centrally applied load (see Figure 5.1)



(a) Plate geometry and loading



(b) Boundary discretisation.
6 elements along each
side



(c) Interior discretisation

Figure 5.8: Simply supported rectangular plate under a uniformly distributed load

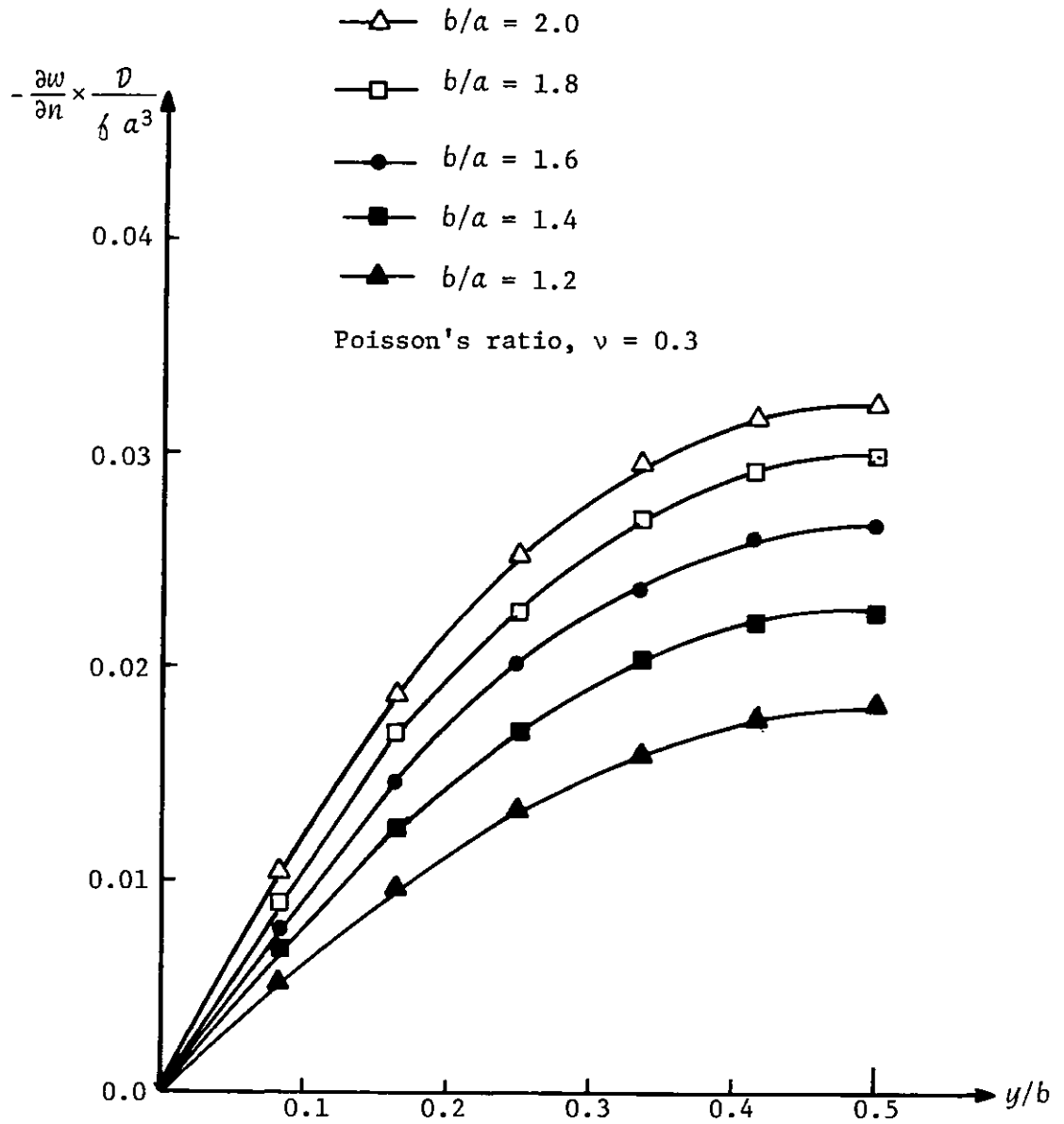


Figure 5.9: Normal slope along half the length of the longer side (OA in Figure 5.8) of a simply supported rectangular plate under a uniformly distributed load

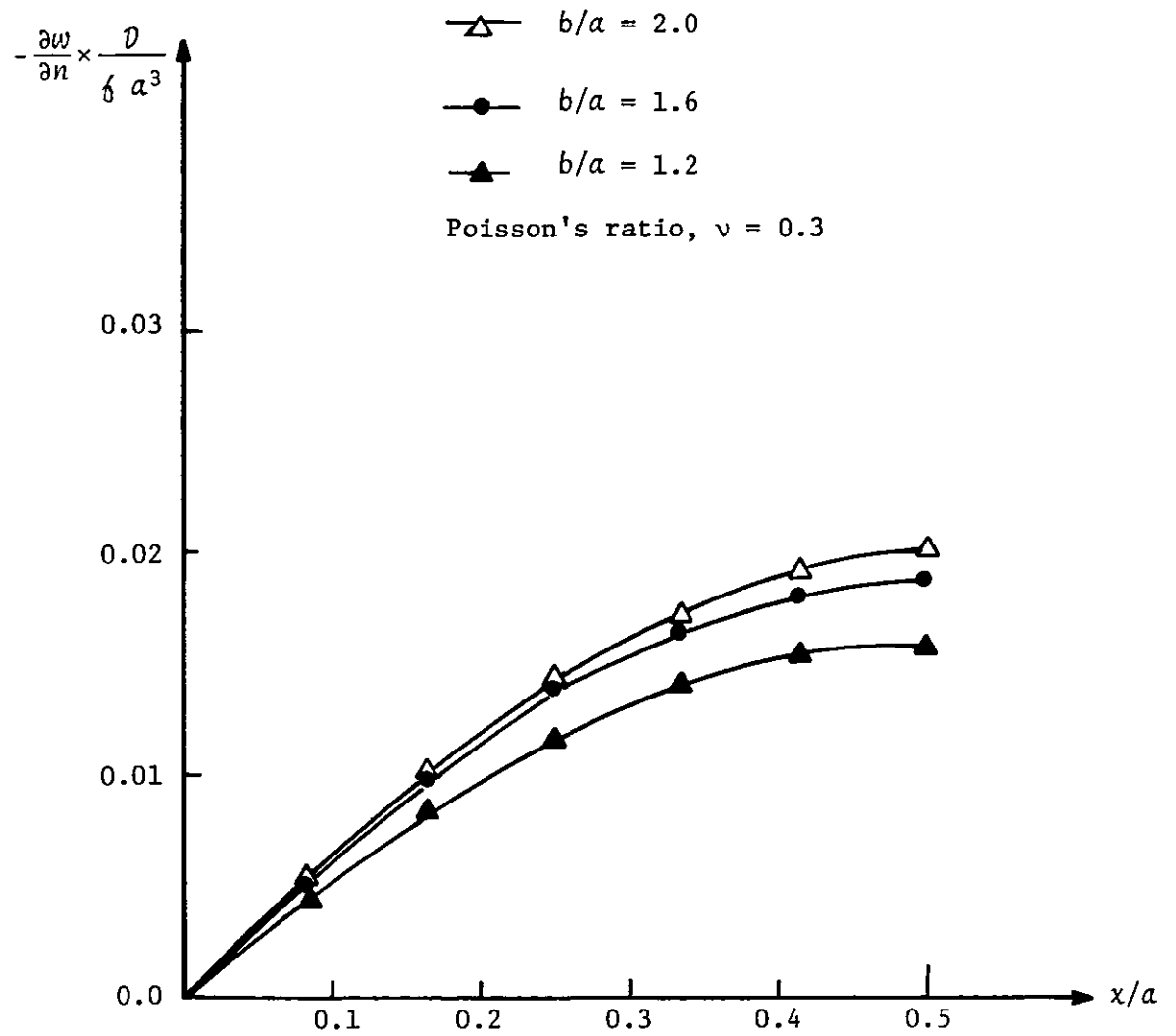


Figure 5.10: Normal slope along half the length of the shorter side (OE in Figure 5.8) of a simply supported rectangular plate under a uniformly distributed load

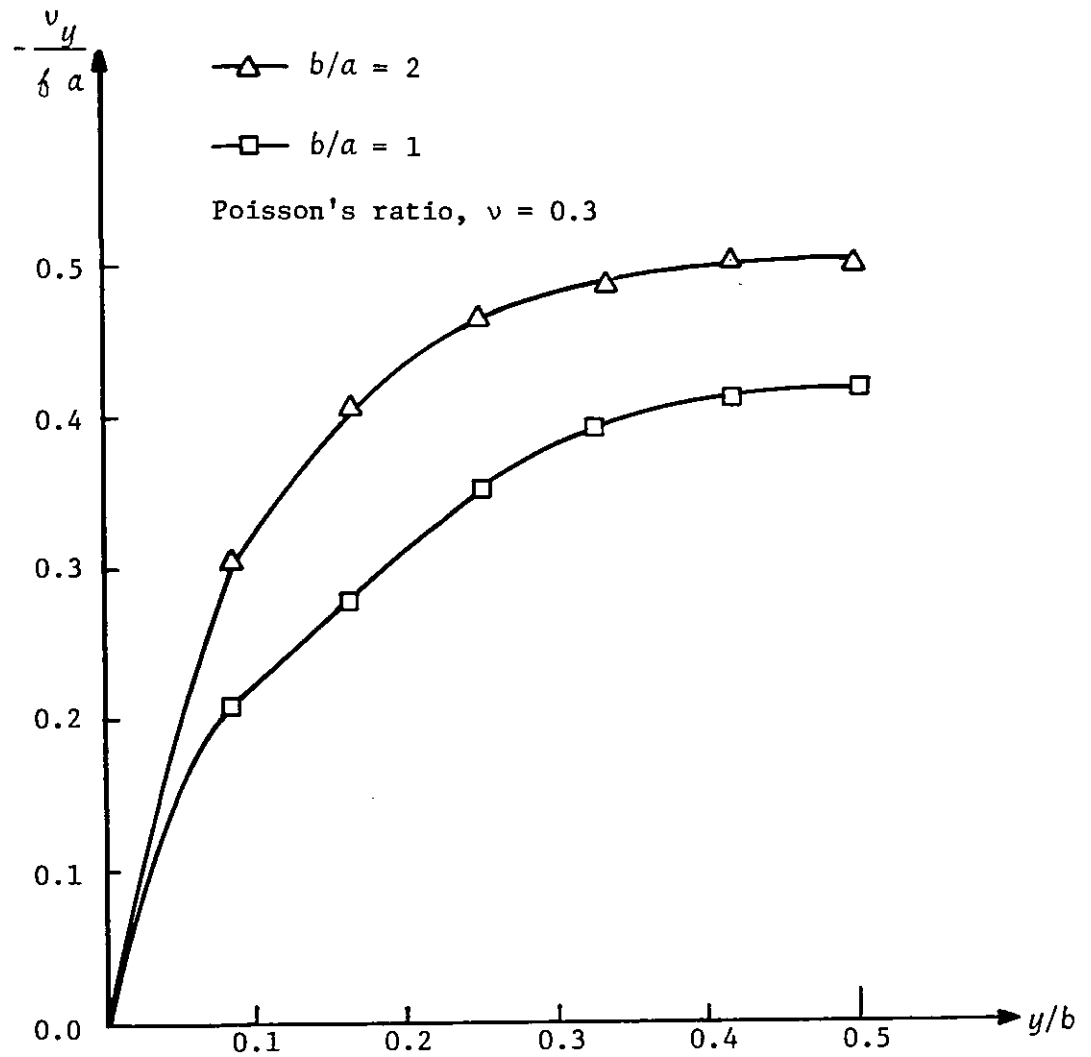


Figure 5.11: Effective shear force along half the length of the longer side (OA in Figure 5.8) of a simply supported rectangular plate under a uniformly distributed load

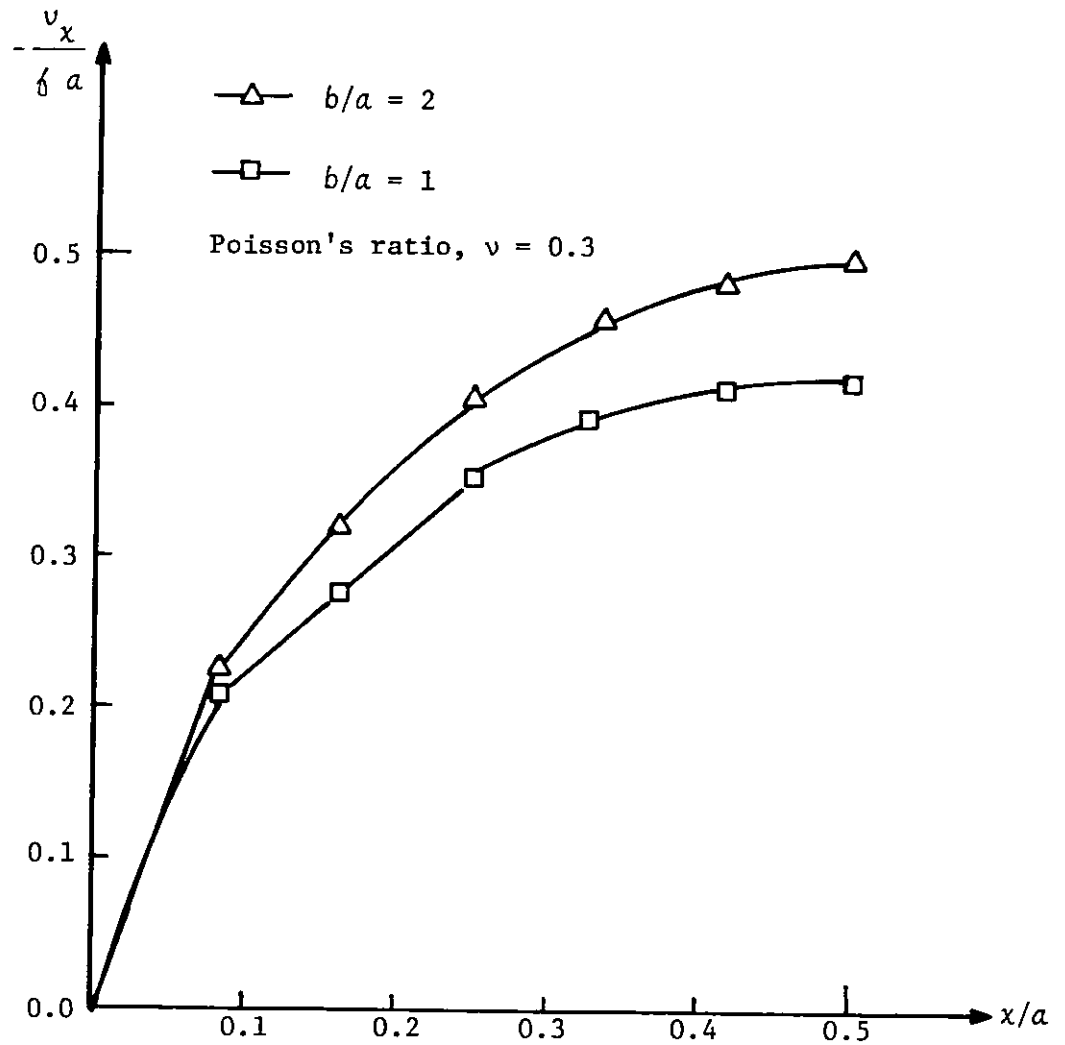


Figure 5.12: Effective shear force along half the length of the shorter side (OE in Figure 5.8) of a simply supported rectangular plate under a uniformly distributed load

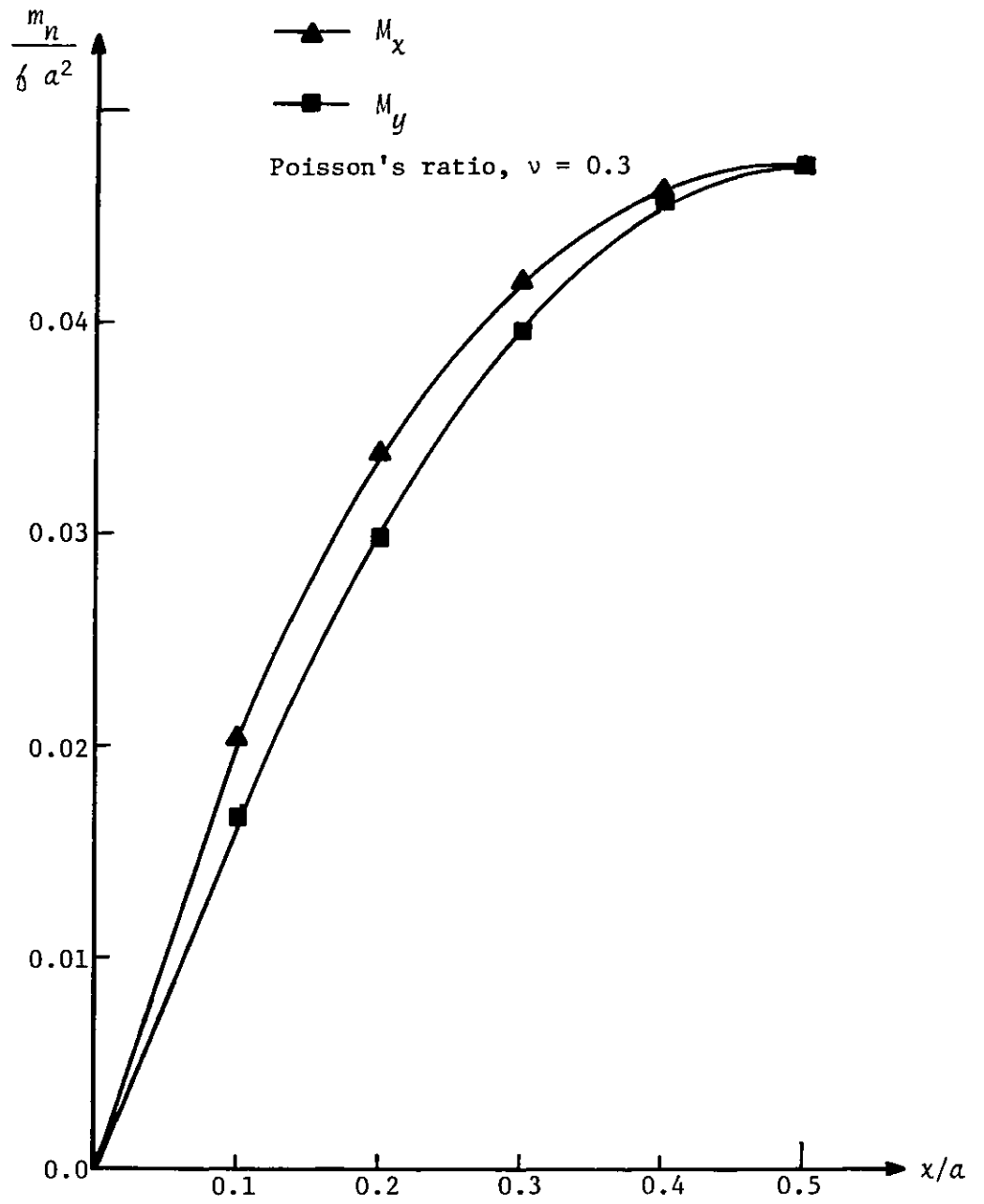
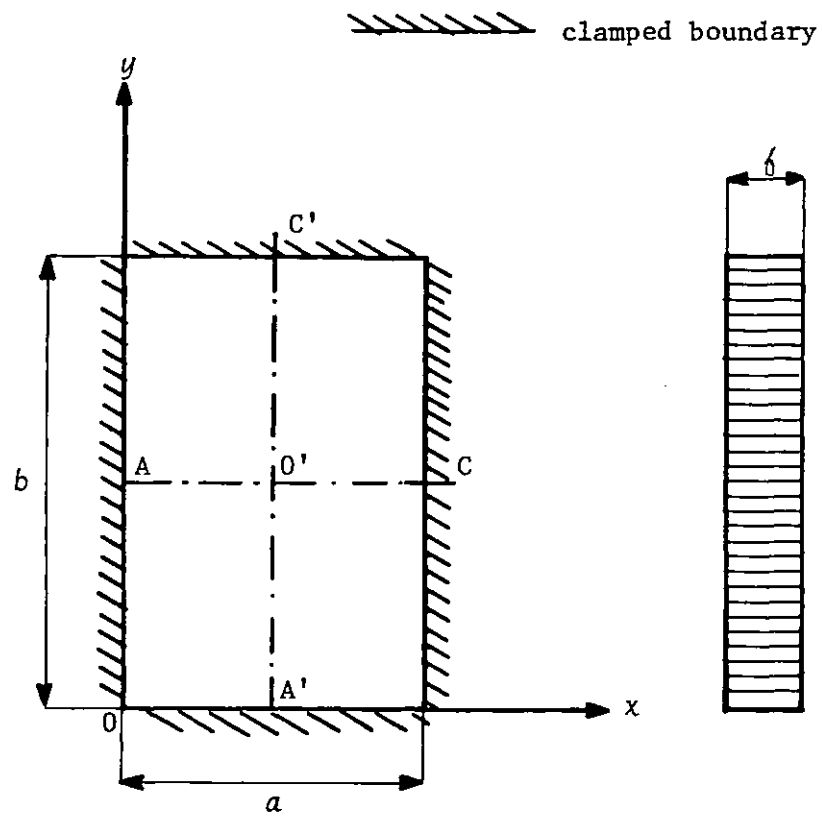
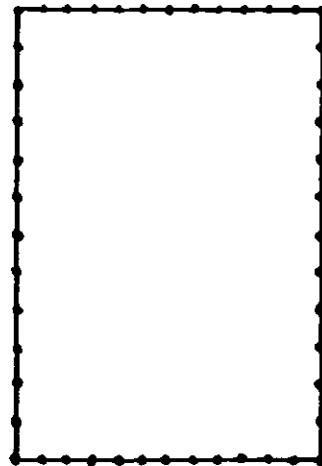


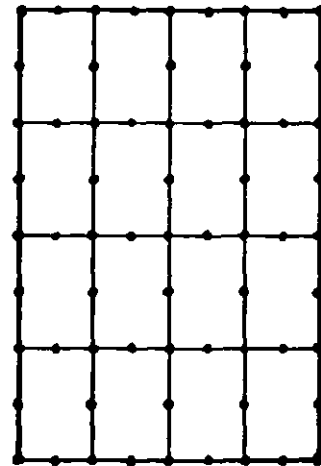
Figure 5.13: Moments along the centreline (AC in Figure 5.8) for a simply supported square plate under a uniformly distributed load. $a = b$ in Figure 5.8.



(a) Plate geometry, boundary conditions and loading



(b) Boundary discretisation.
6 quadratic elements along
each side



(c) Interior discretisation

Figure 5.14: Clamped rectangular plate under a uniformly distributed load

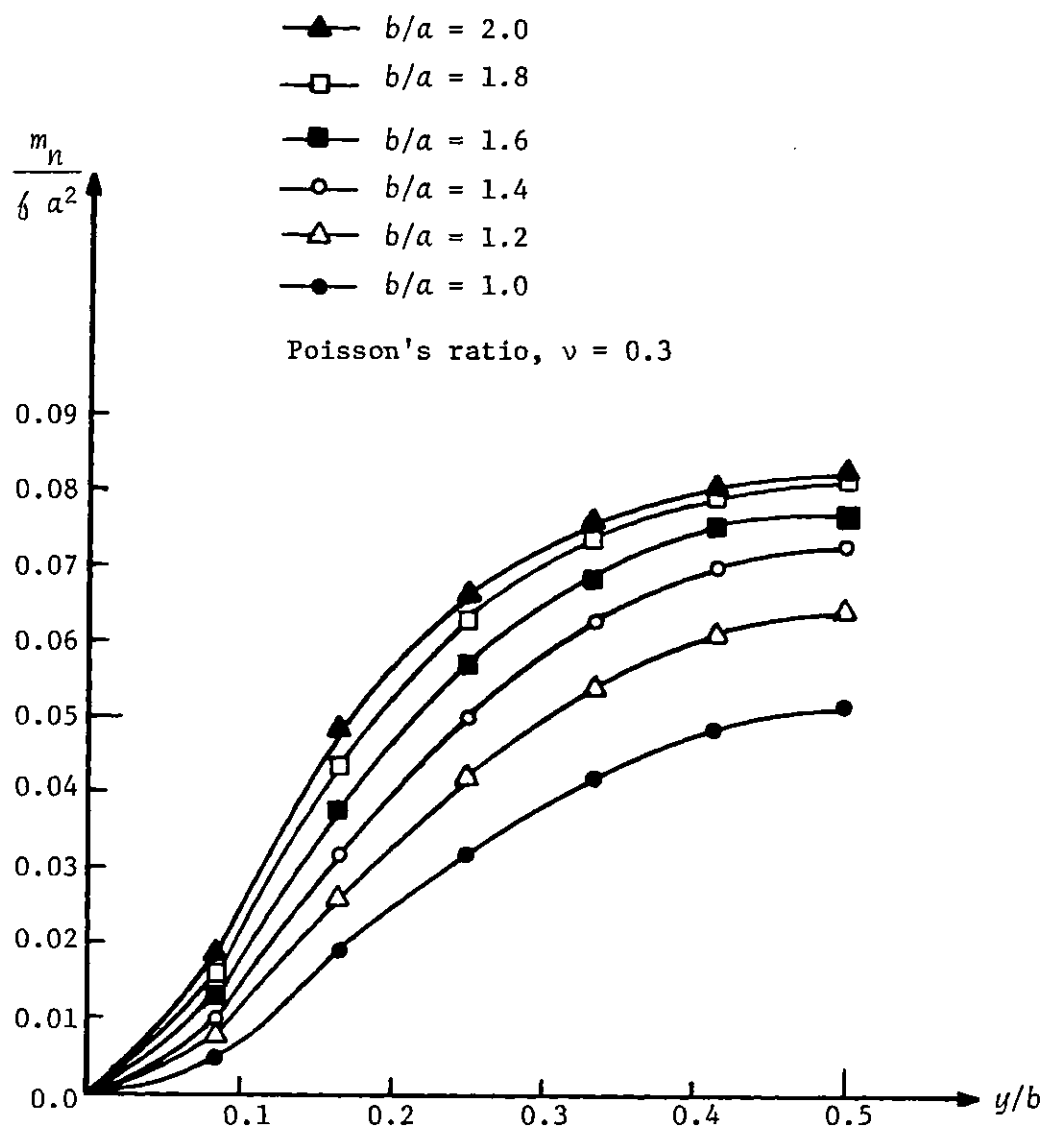


Figure 5.15: Normal moment along half the length of the longer side (OA in Figure 5.14) of a clamped rectangular plate under a uniformly distributed load

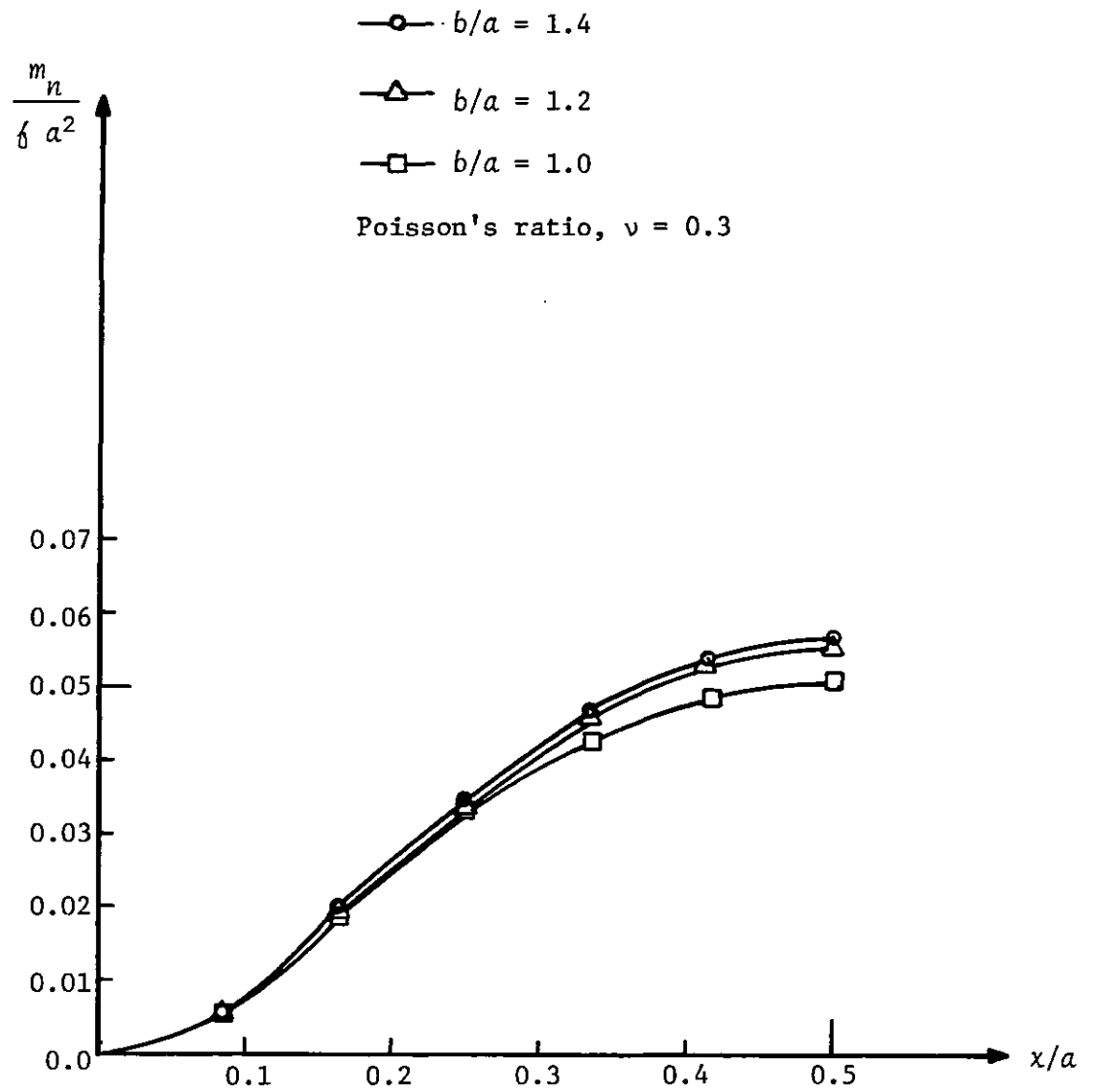


Figure 5.16: Normal moment along half the length of the shorter side (OA' in Figure 5.16) of a clamped rectangular plate under a uniformly distributed load

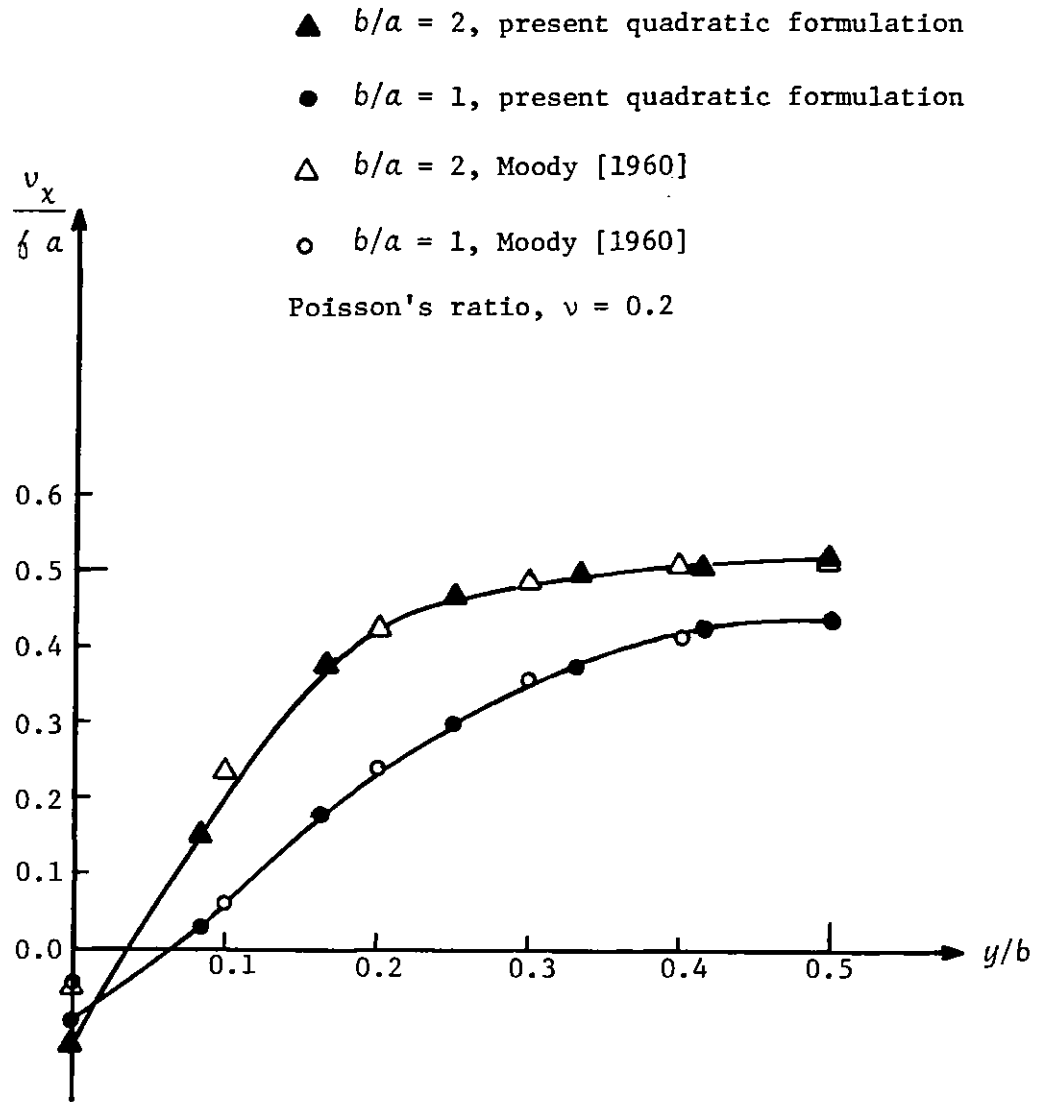


Figure 5.17: Effective shear force along half the length of the longer side (OA in Figure 5.14) of a clamped rectangular plate under a uniformly distributed load

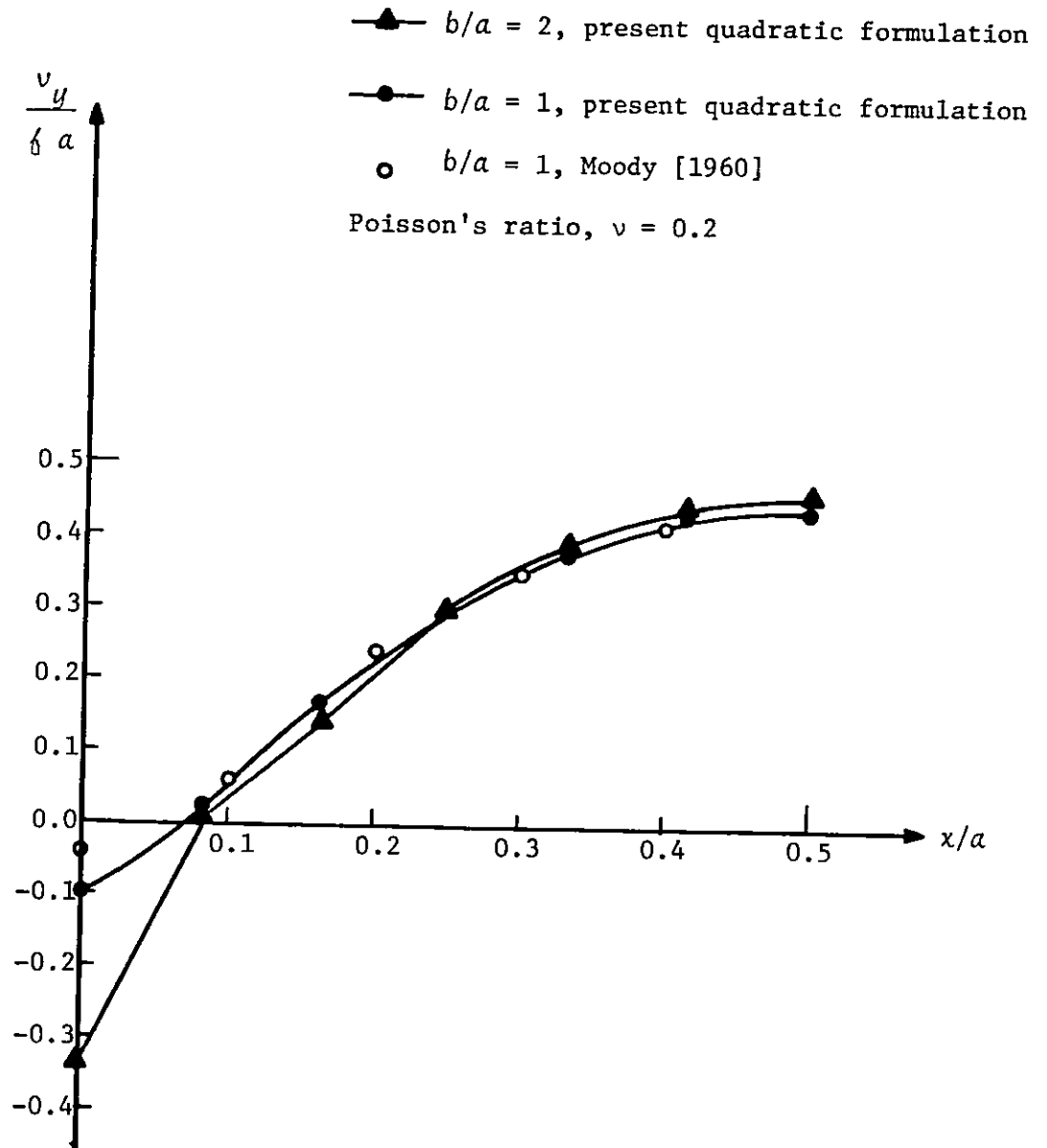
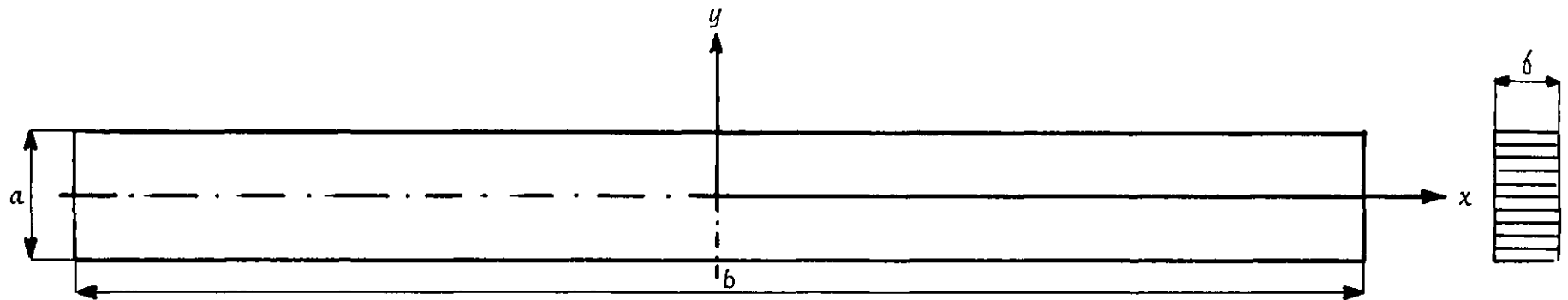
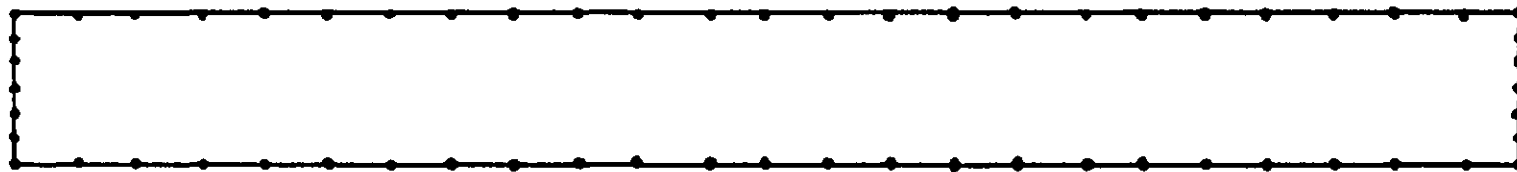


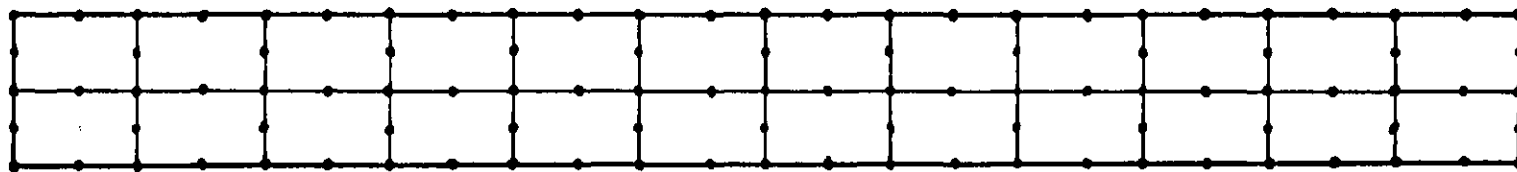
Figure 5.18: Effective shear force along half the length of the shorter side (OA' in Figure 5.14) of a clamped rectangular plate under a uniformly distributed load



(a) Plate geometry and loading



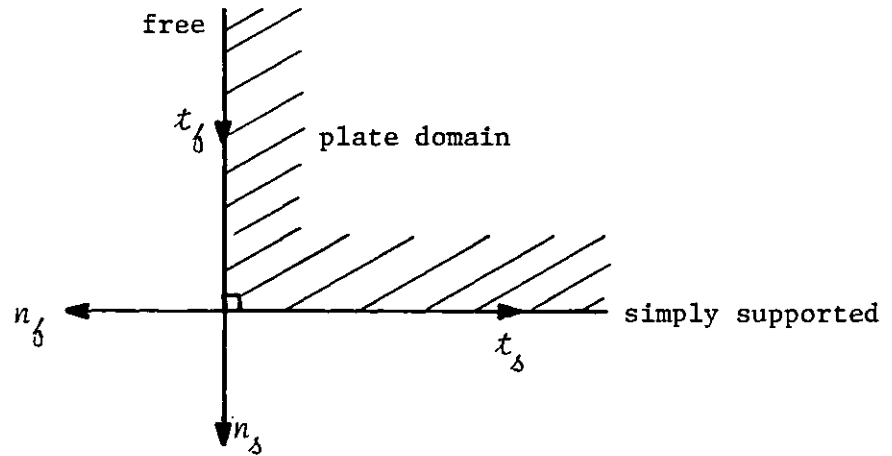
(b) Boundary discretisation. 12 elements along the length, 3 elements across the width.



(c) Interior discretisation

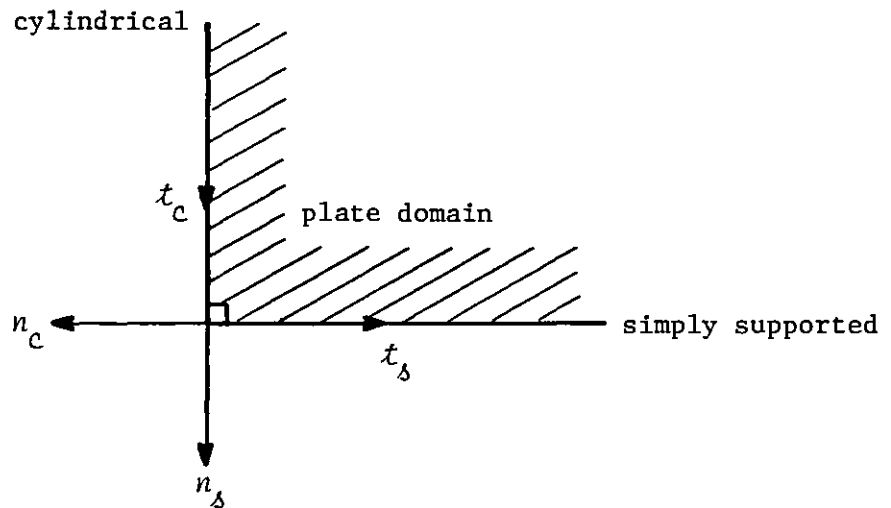
Figure 5.19: Simply supported rectangular strip under a uniformly distributed load

n_f, t_f : normal and tangent on the free side
 n_s, t_s : normal and tangent on the simply supported side



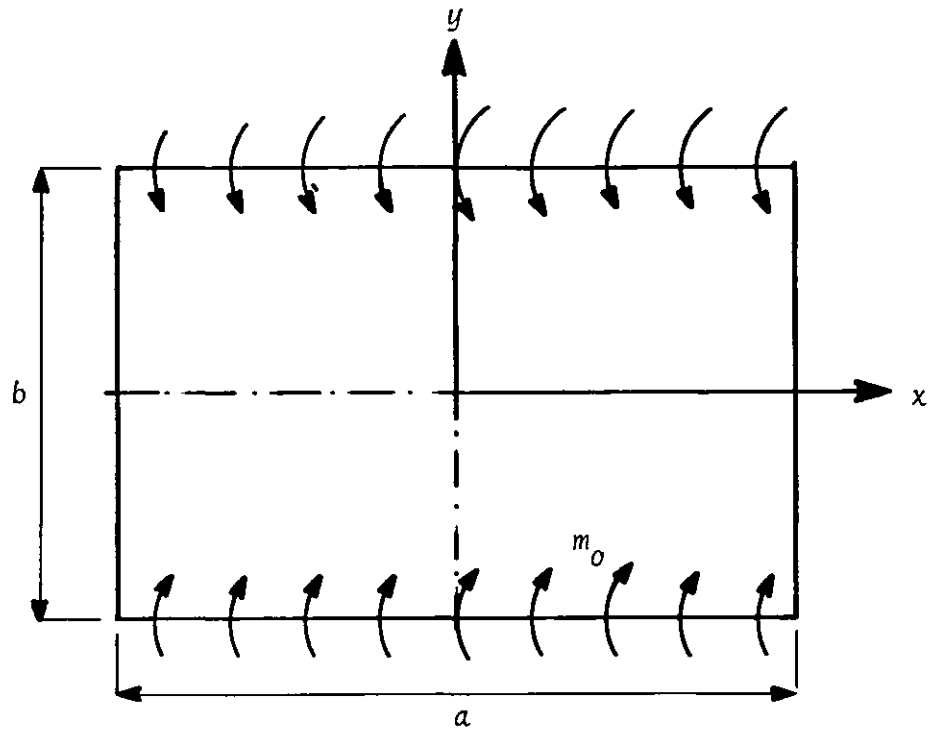
$w = 0$: corner deflection
 $m_{n_s} = 0$: normal moment on simply supported side
 $m_{n_f} = 0$: normal moment on free side
 $v_{n_f} = 0$: effective shear force on the free side
 $\partial w / \partial n_f = 0$: normal slope on the free side

Figure 5.20(a): Boundary conditions at a simply supported to free rectangular corner

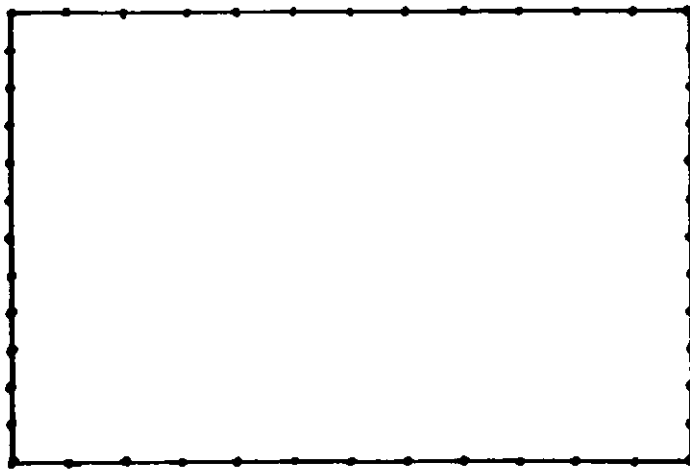


n_c, t_c : normal and tangent on the cylindrical side
 $w = 0$: corner deflection
 $m_{n_c} = 0$: normal moment on the simply supported side
 $v_{n_c} = 0$: effective shear force on the cylindrical side
 $\partial w / \partial n_c = 0$: normal slope on the cylindrical side
 $[\tau] = 0$: Kirchhoff corner forces

Figure 5.20(b): Boundary conditions at a simply supported to cylindrical corner



(a) Plate geometry and loading



(b) Boundary discretisation. 6 quadratic elements along each side.

Figure 5.21: Simply supported rectangular plate bent by uniformly distributed moments along two opposite edges

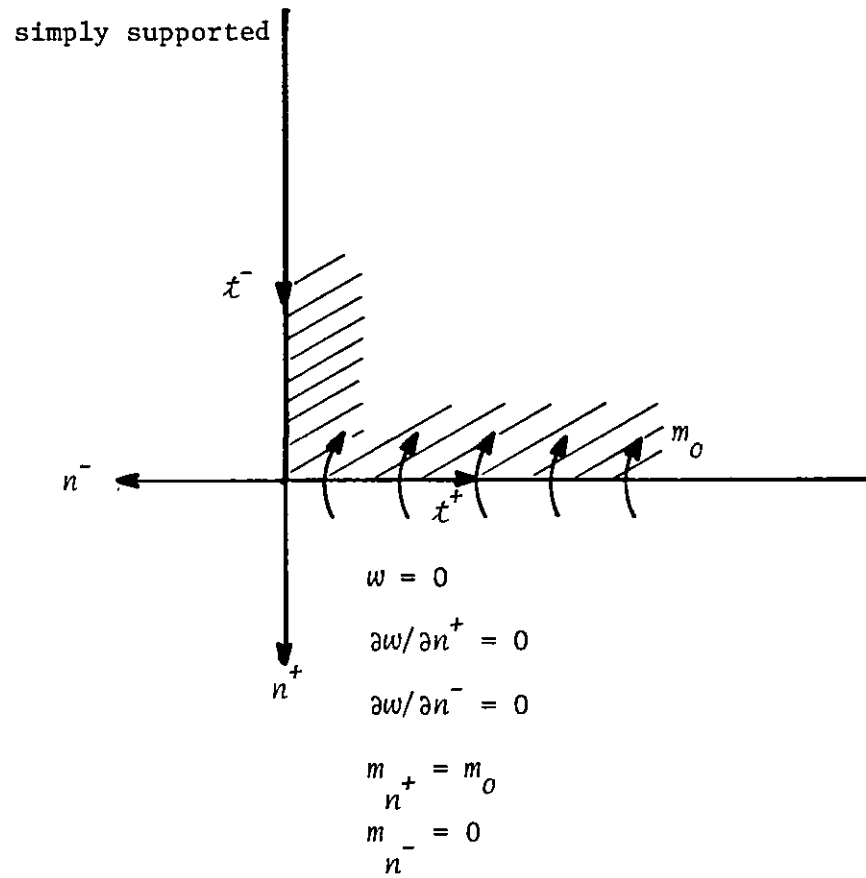
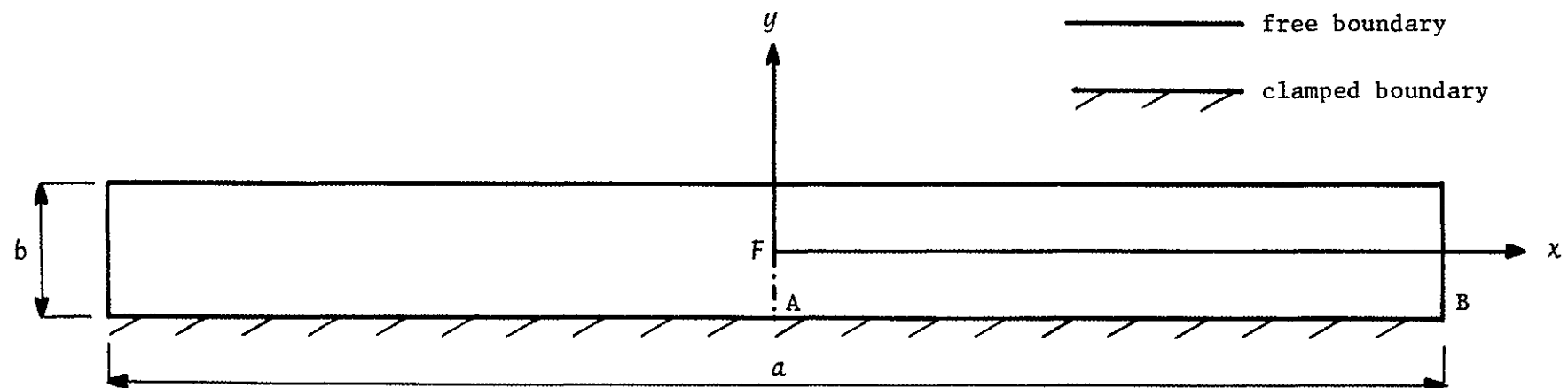


Figure 5.22: Boundary conditions at a corner of a rectangular plate bent by moments applied to two opposite edges



(a) Plate geometry, loading and boundary conditions



(b) Boundary discretisation. 10 quadratic elements along the length and 2 across the width

Figure 5.23: Long rectangular cantilever under a centrally applied load, F

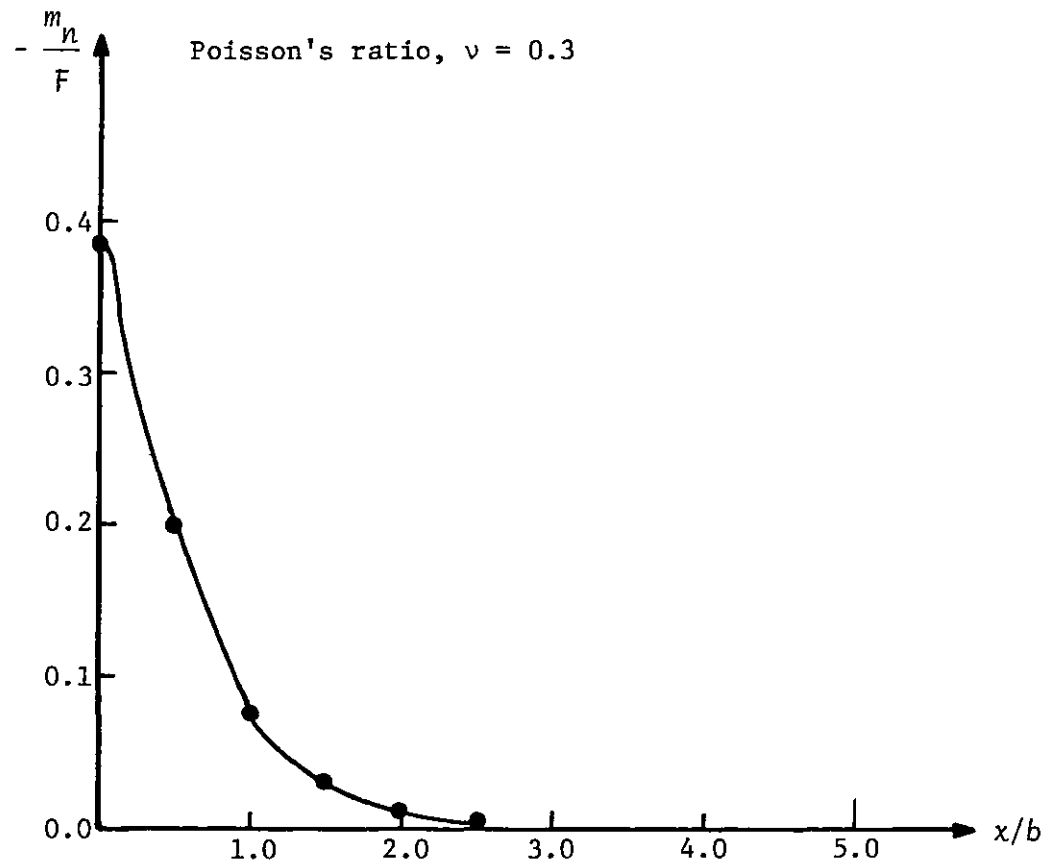


Figure 5.24: Normal moment along half the length of the clamped edge (AB in Figure 5.23) of a long cantilever plate under a centrally applied load, F

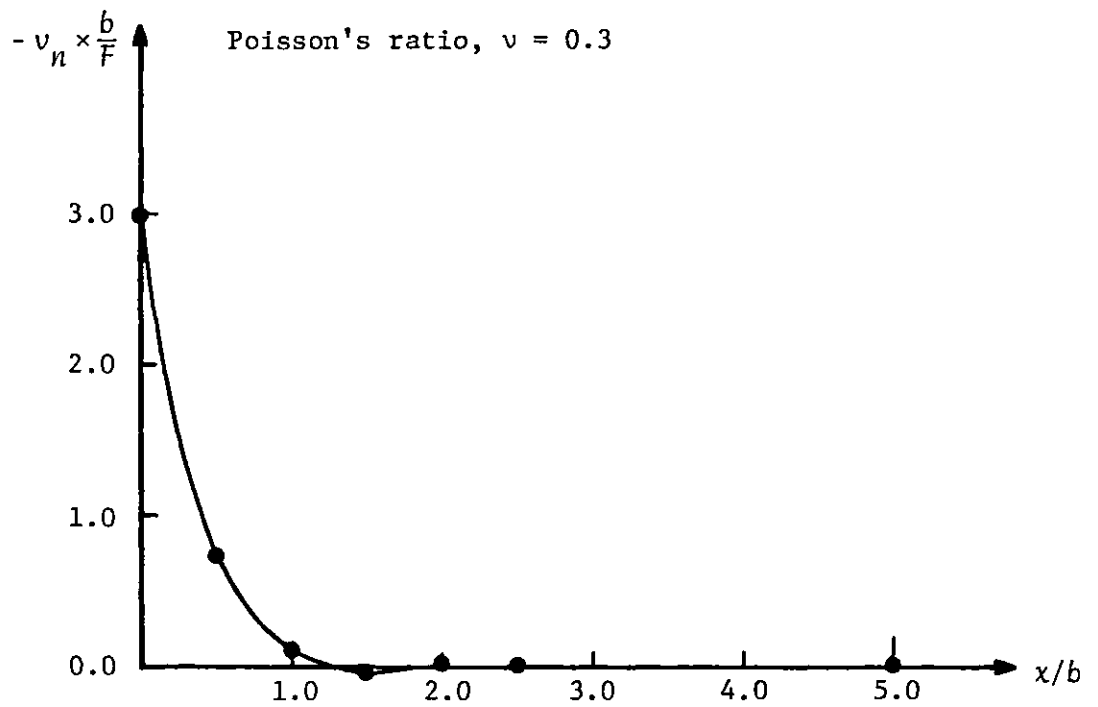
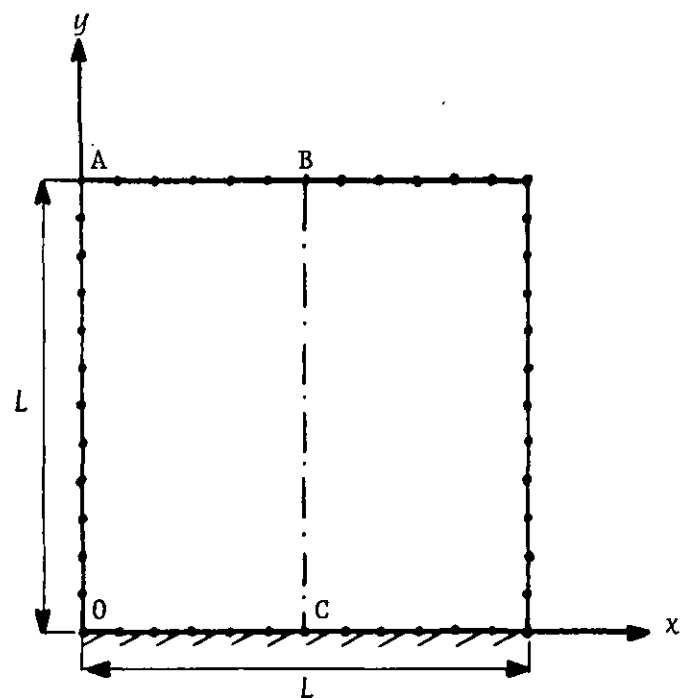
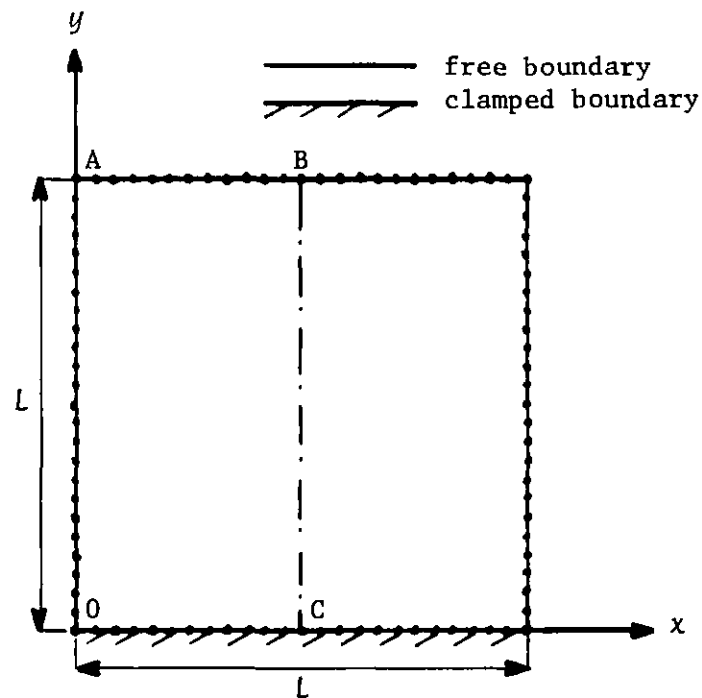


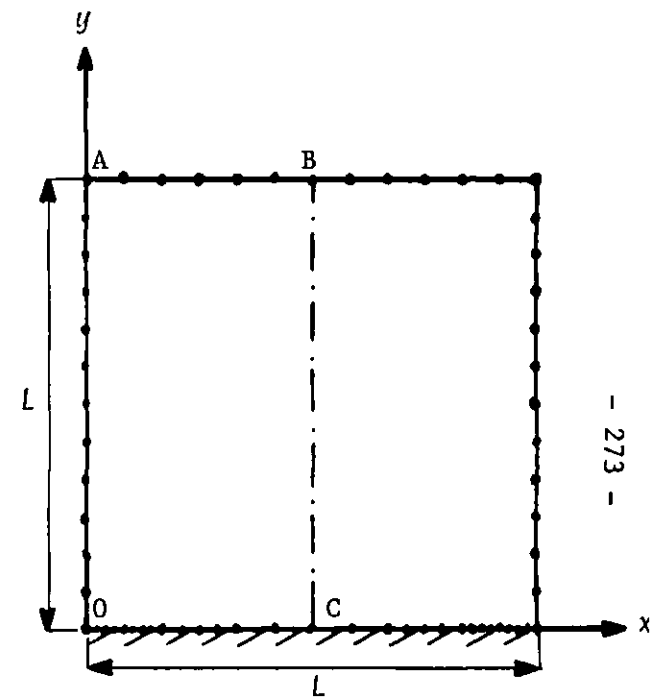
Figure 5.25: Effective shear force along half the length of the clamped side (AB in Figure 5.23) of a long cantilever plate under a centrally applied load, F



(a) Boundary discretisation.
6 quadratic elements
along each side.



(b) Boundary discretisation.
12 quadratic elements
along each side.



(c) Refined discretisation
near the clamped to free
corner. 32 elements in
total.

Figure 5.26: Square cantilever plate under a centrally applied load, F

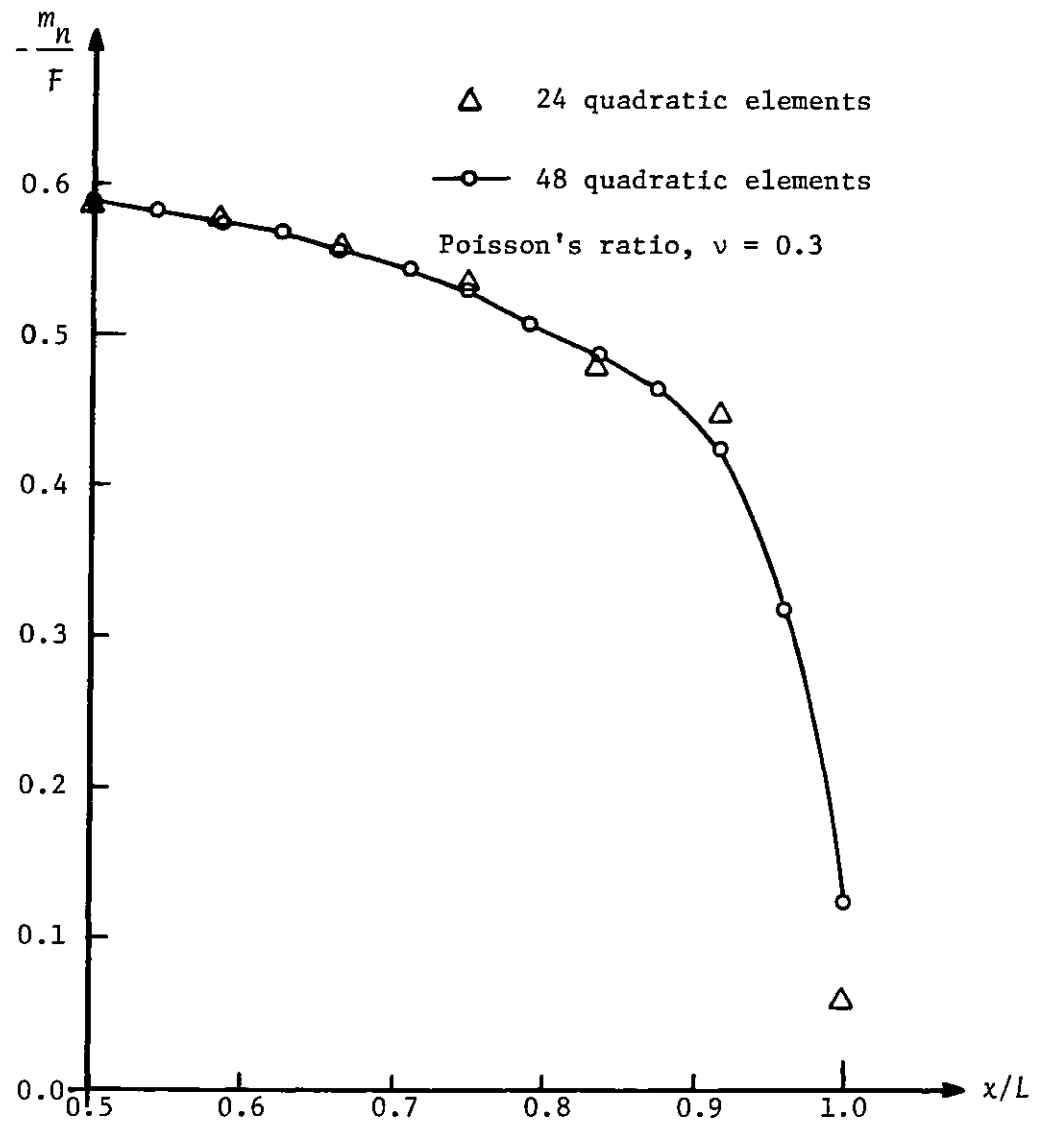


Figure 5.27: Normal moment along half the length of the clamped side (OC in Figure 5.26) of a square cantilever plate under a centrally applied load, F

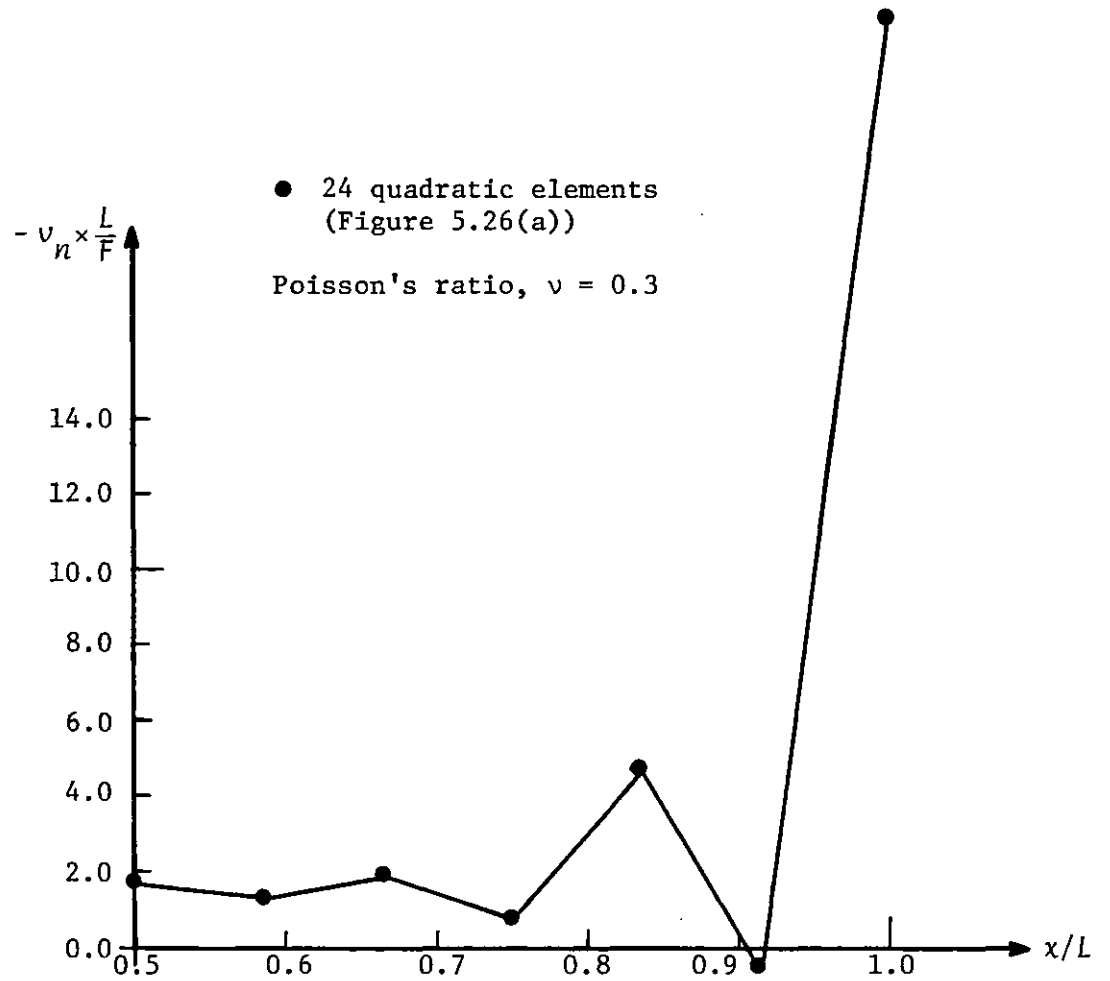


Figure 5.28: Effective shear force along half the length of the clamped side (OC in Figure 5.26) of a square cantilever plate under a centrally applied load, F

● 48 quadratic elements (Figure 5.26(b))

Poisson's ratio, $\nu = 0.3$

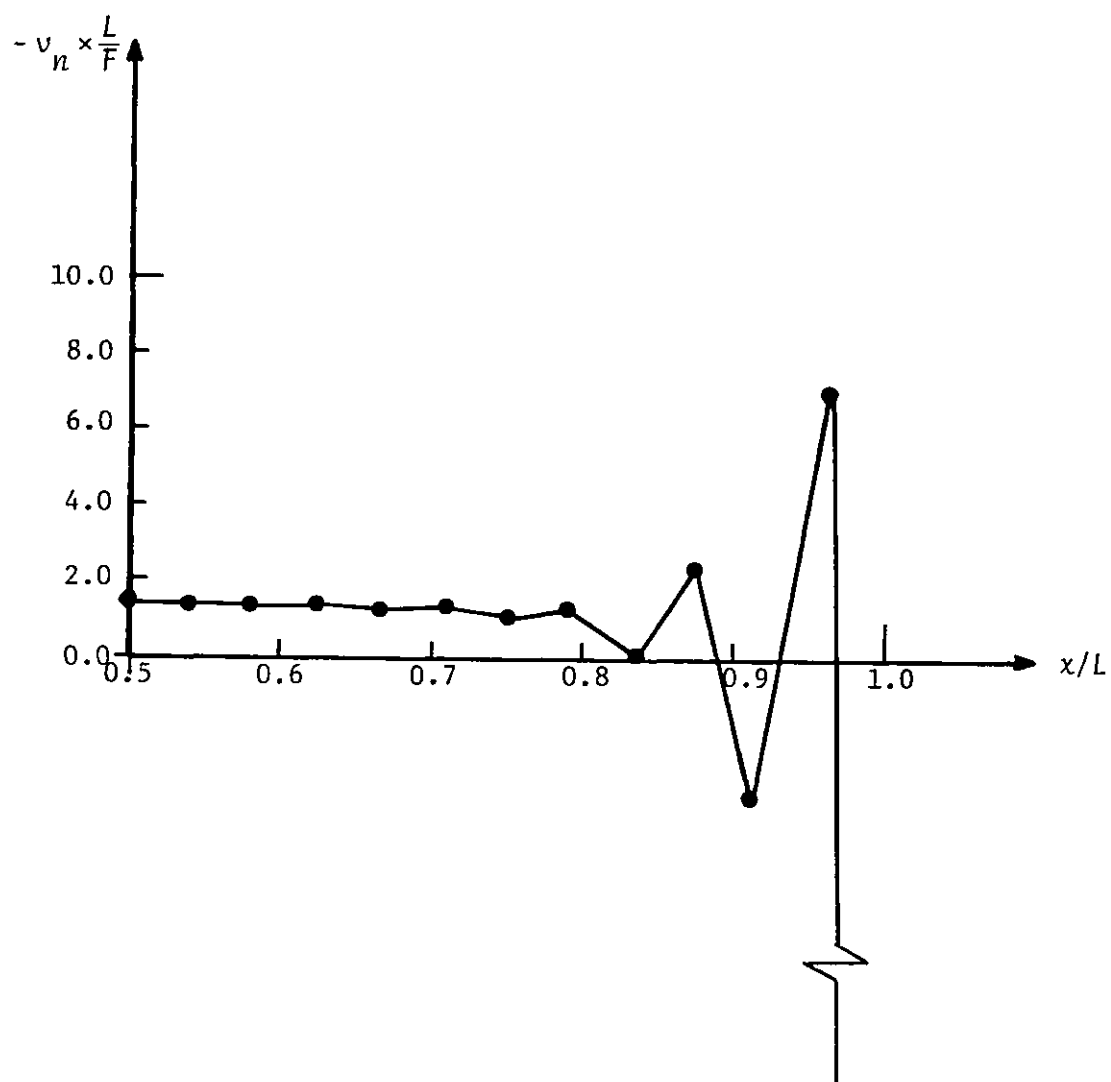


Figure 5.29: Effective shear force along half the length of the clamped side (OC in Figure 5.26) of a clamped square cantilever under a centrally applied load, $\bar{F}$

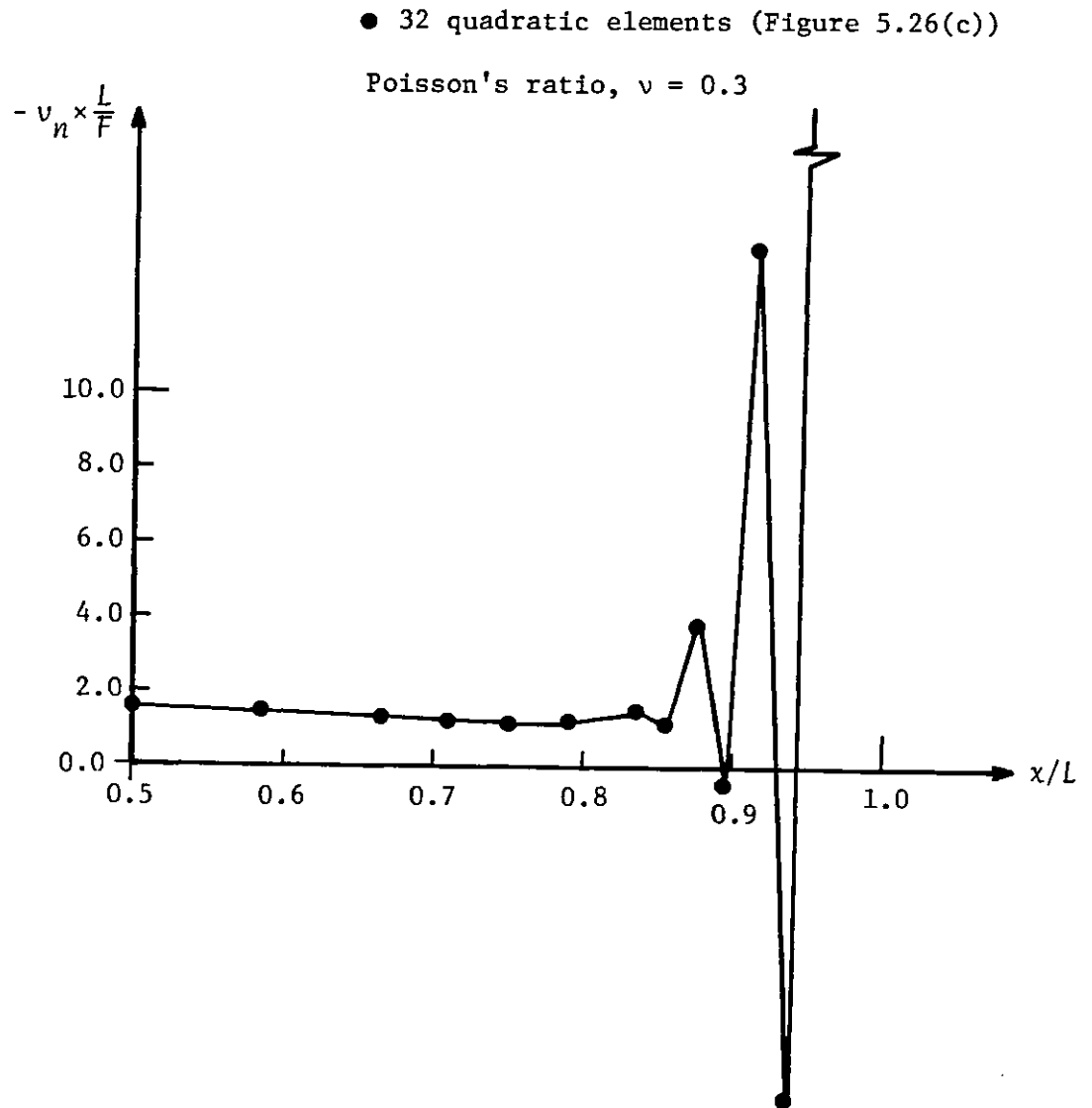


Figure 5.30: Effective shear force along half the length of the clamped side (OC in Figure 5.26) of a square cantilever plate under a centrally applied load, F

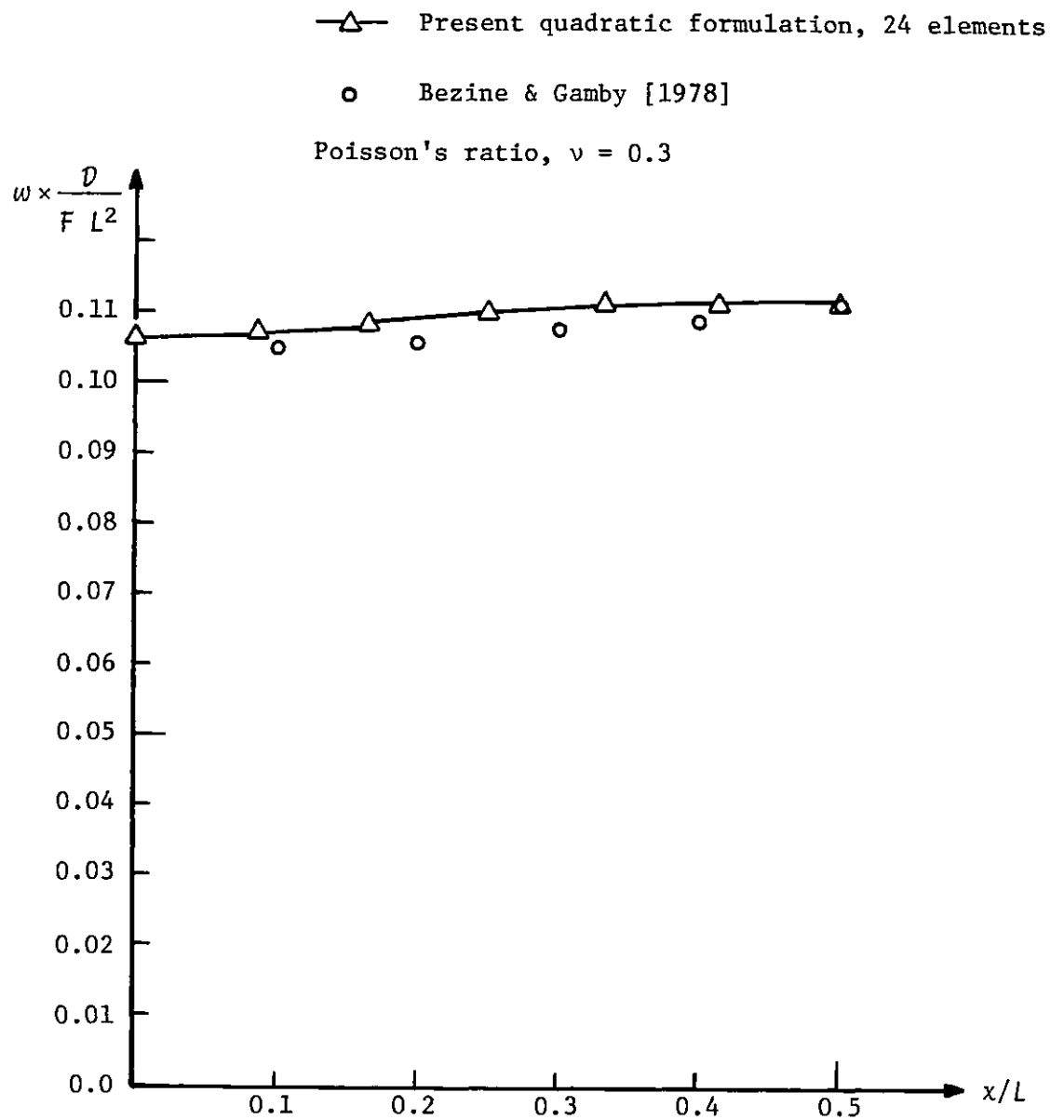


Figure 5.31: Deflection along half the free edge (AB in Figure 5.26) of a square cantilever plate under a centrally applied load, F

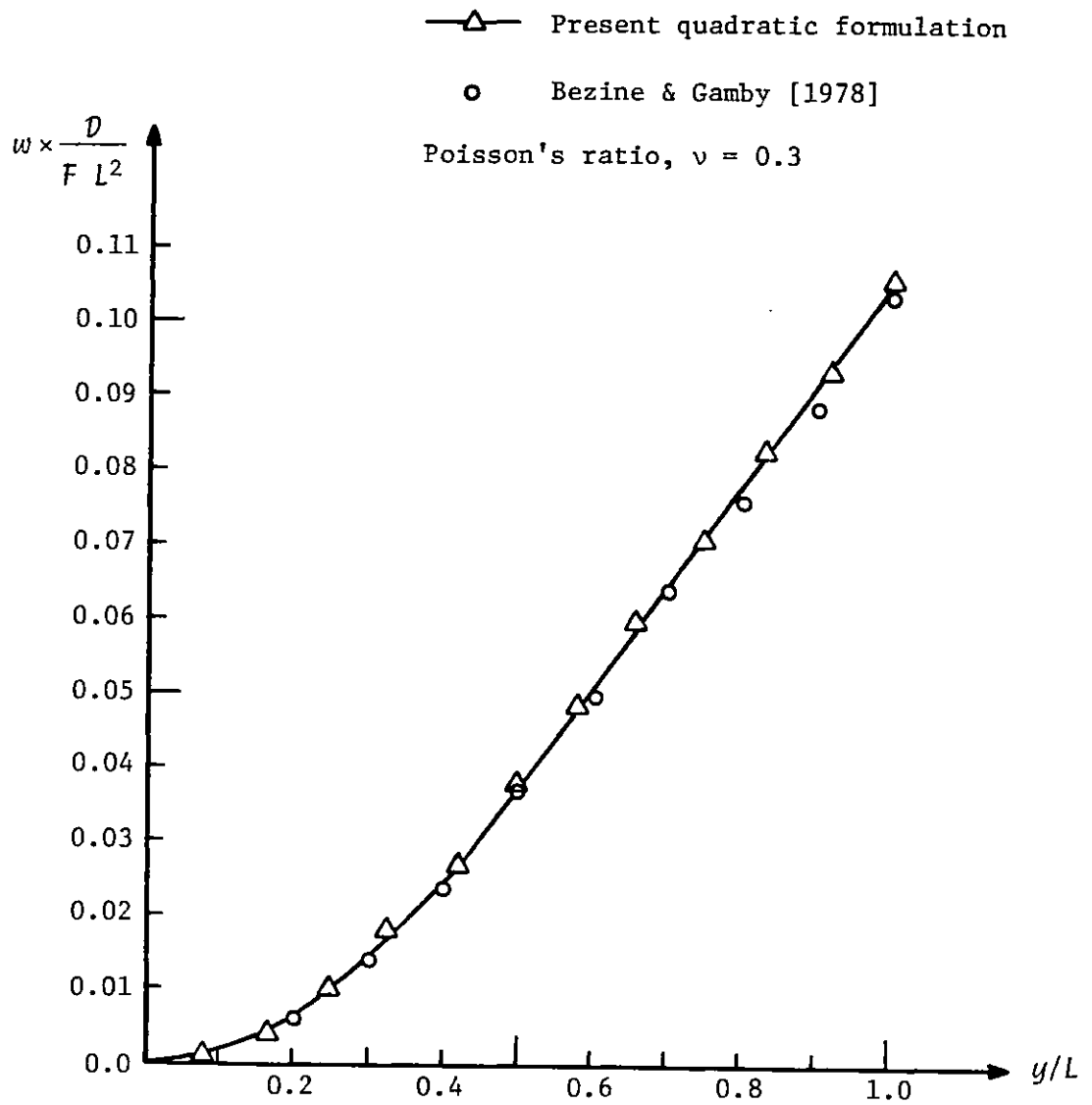


Figure 5.32: Deflection along the free edge (OA in Figure 5.26) of a square cantilever plate under a centrally applied load, F

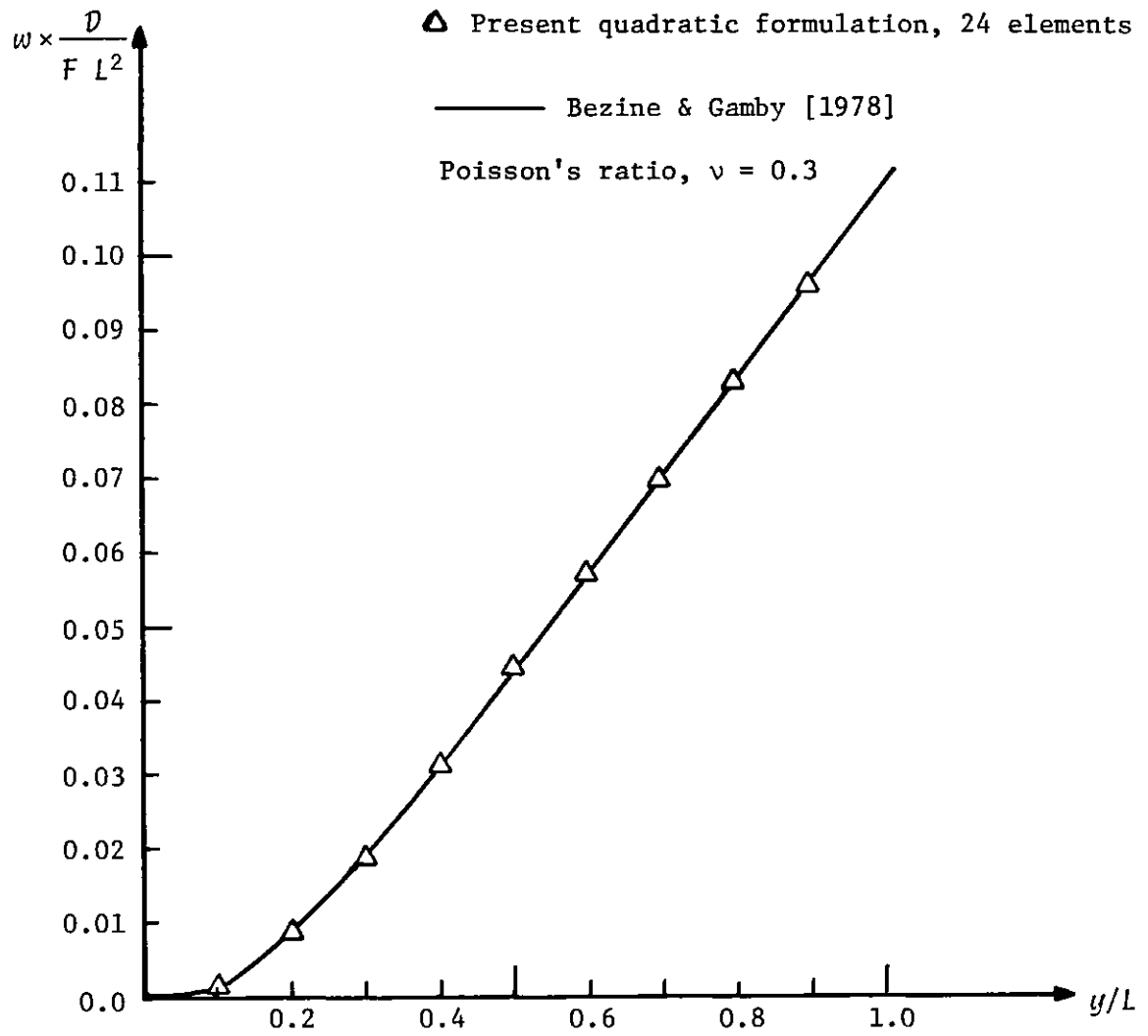


Figure 5.33: Deflection along the axis of symmetry (CB in Figure 5.26) of a square cantilever plate under a centrally applied load, F

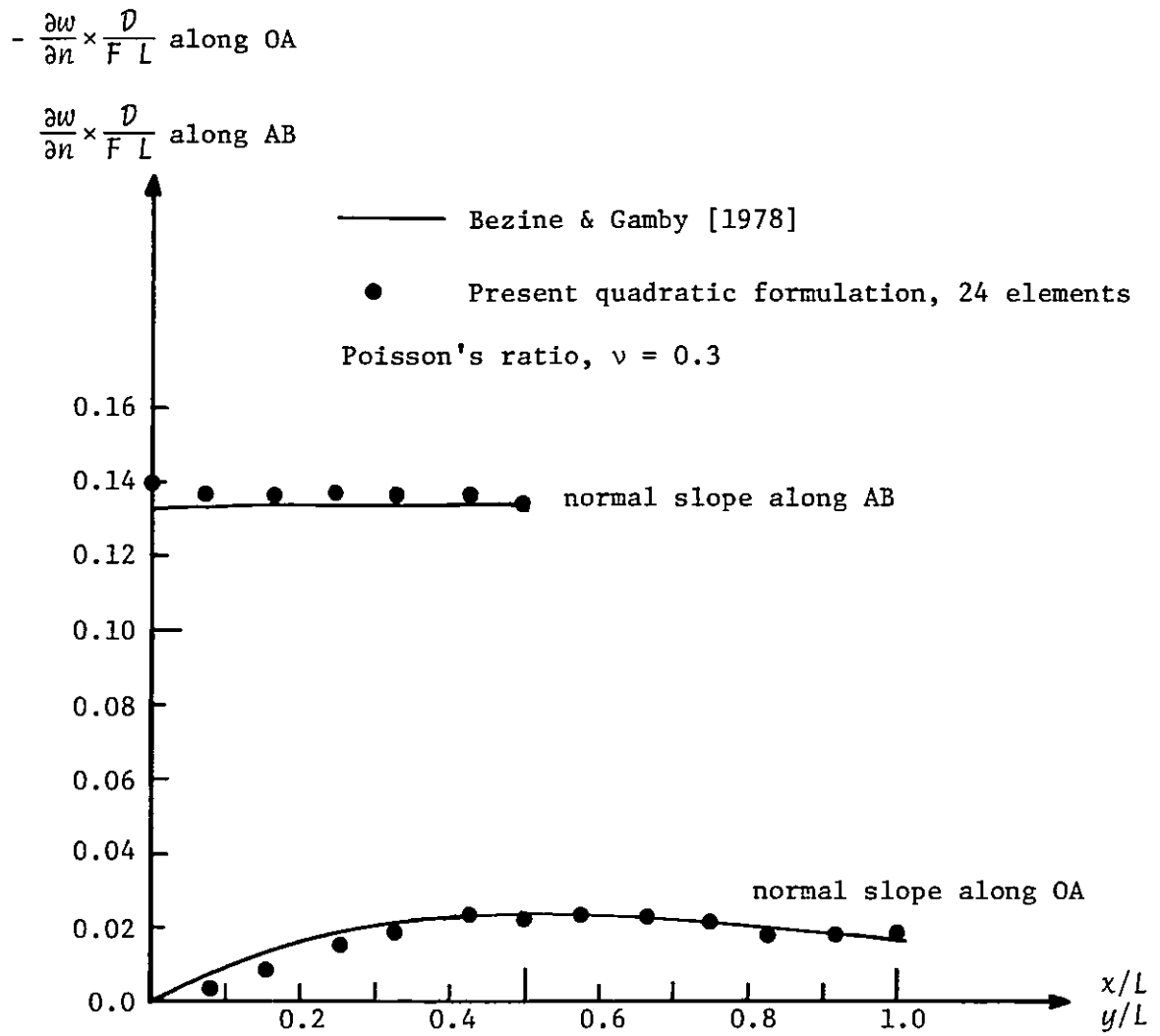
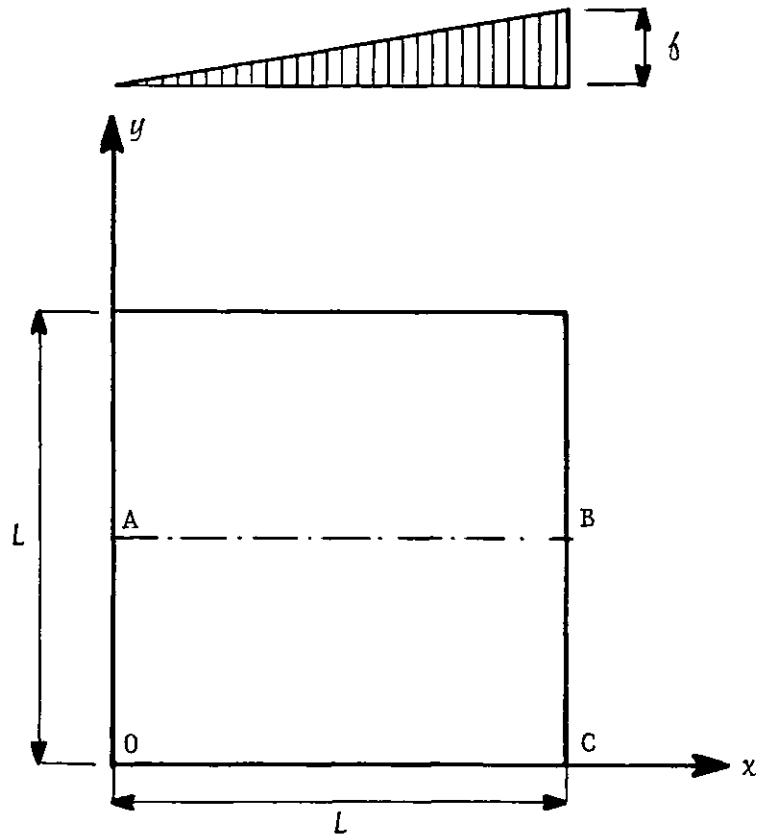
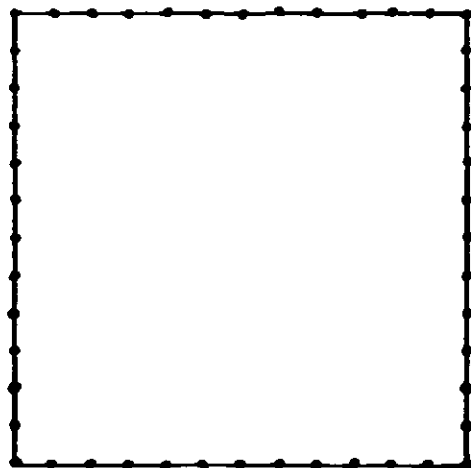


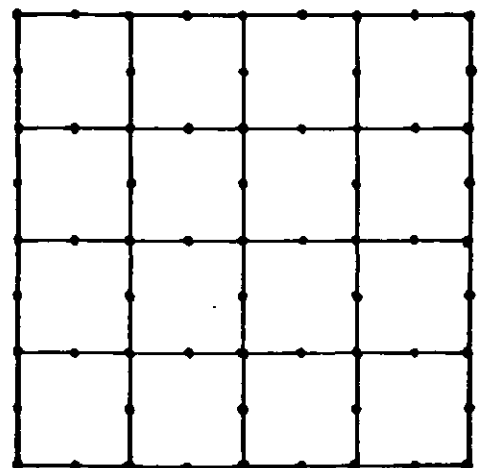
Figure 5.34: Normal slope along the free edges (OA and AB in Figure 5.26) of a square cantilever plate under a centrally applied load, F



(a) Plate geometry and loading



(b) Boundary discretisation.
6 elements along each
side.



(c) Interior discretisation

Figure 5.35: Simply supported square plate under a linearly varying load

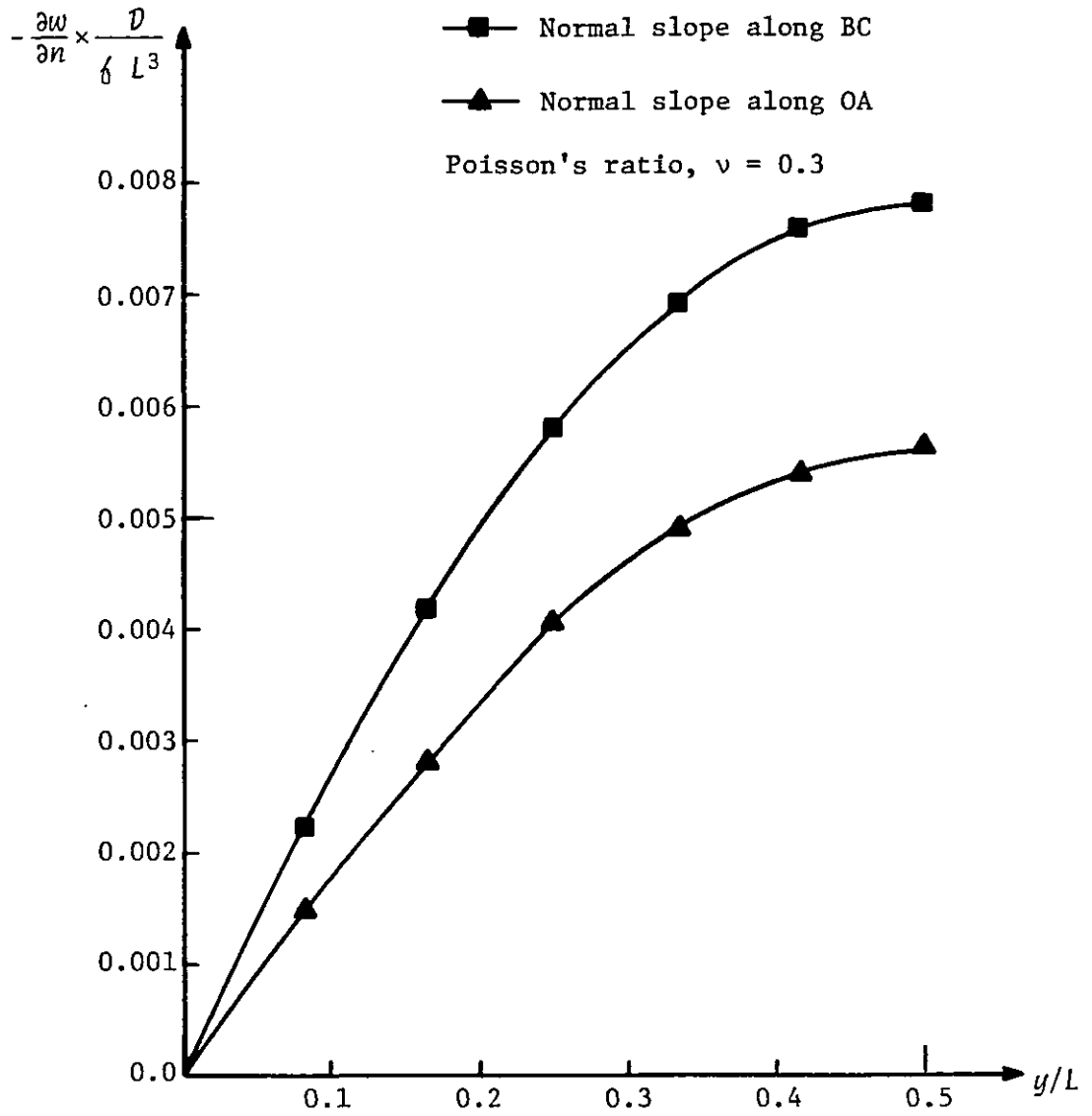


Figure 5.36: Normal slope along OA and BC in Figure 5.35 for a simply supported square plate under a linearly varying load

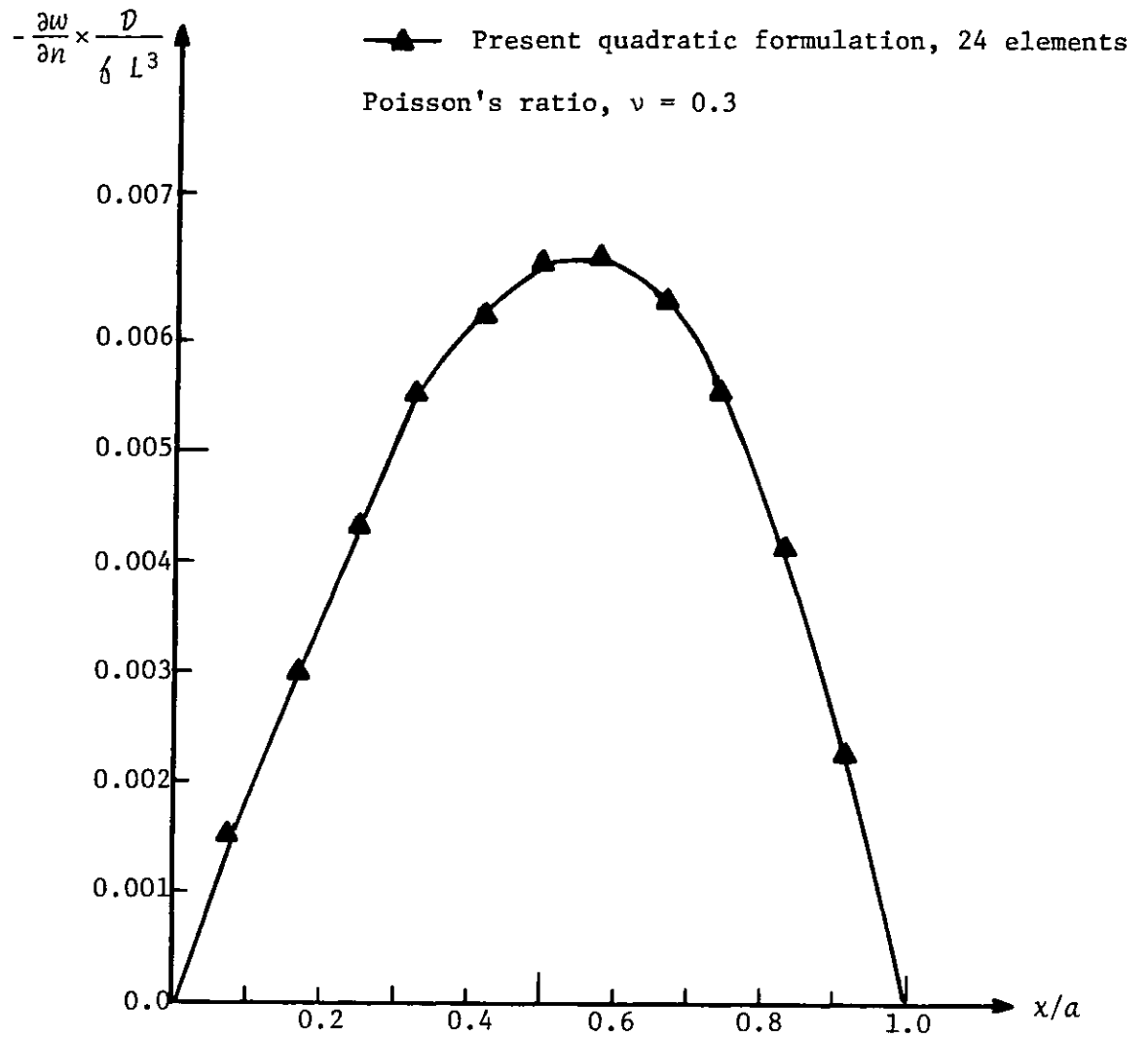


Figure 5.37: Normal slope along OC in Figure 5.35 for a simply supported square plate under a linearly varying load

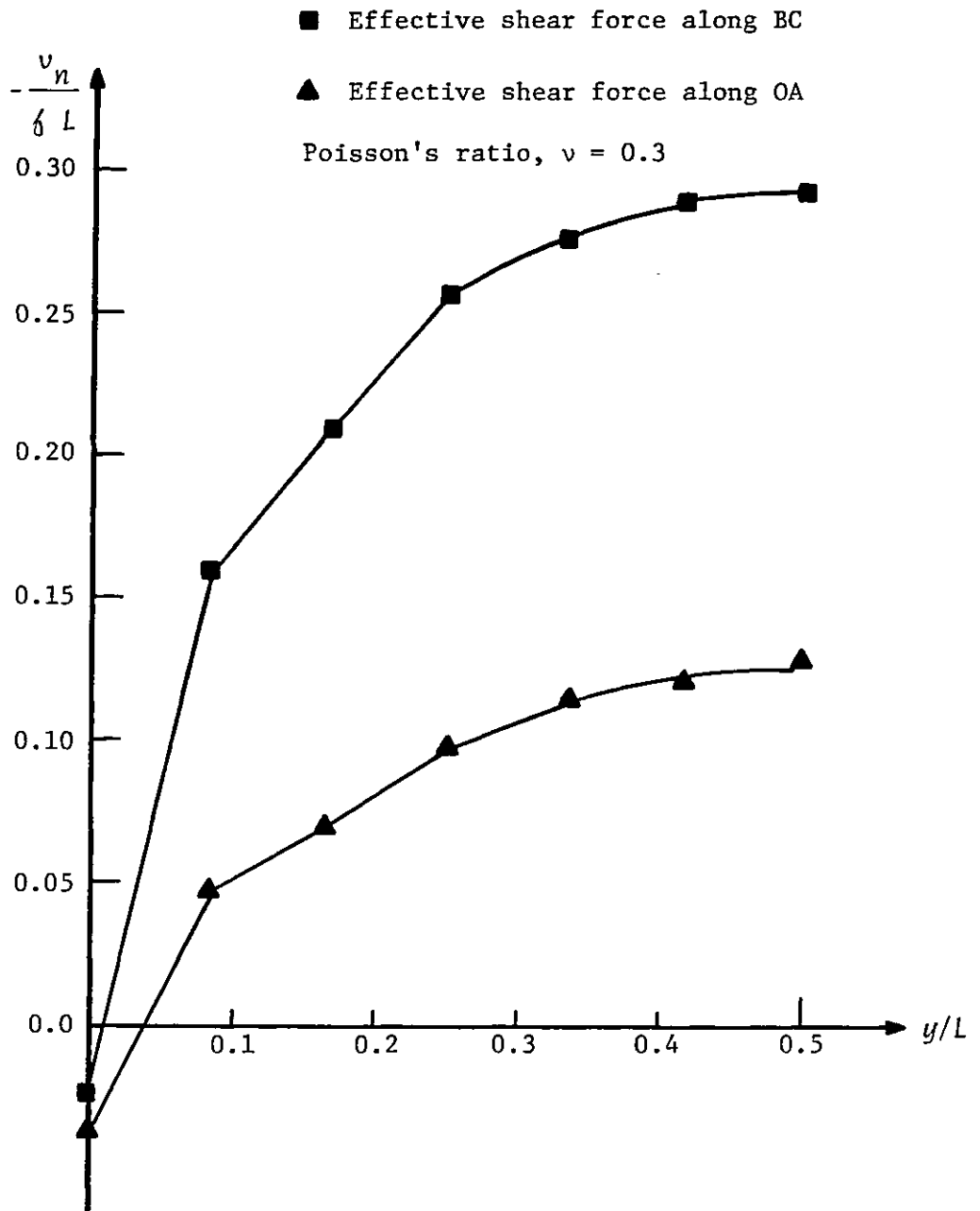


Figure 5.38: Effective shear force along OA and BC in Figure 5.35 for a simply supported square plate under a linearly varying load

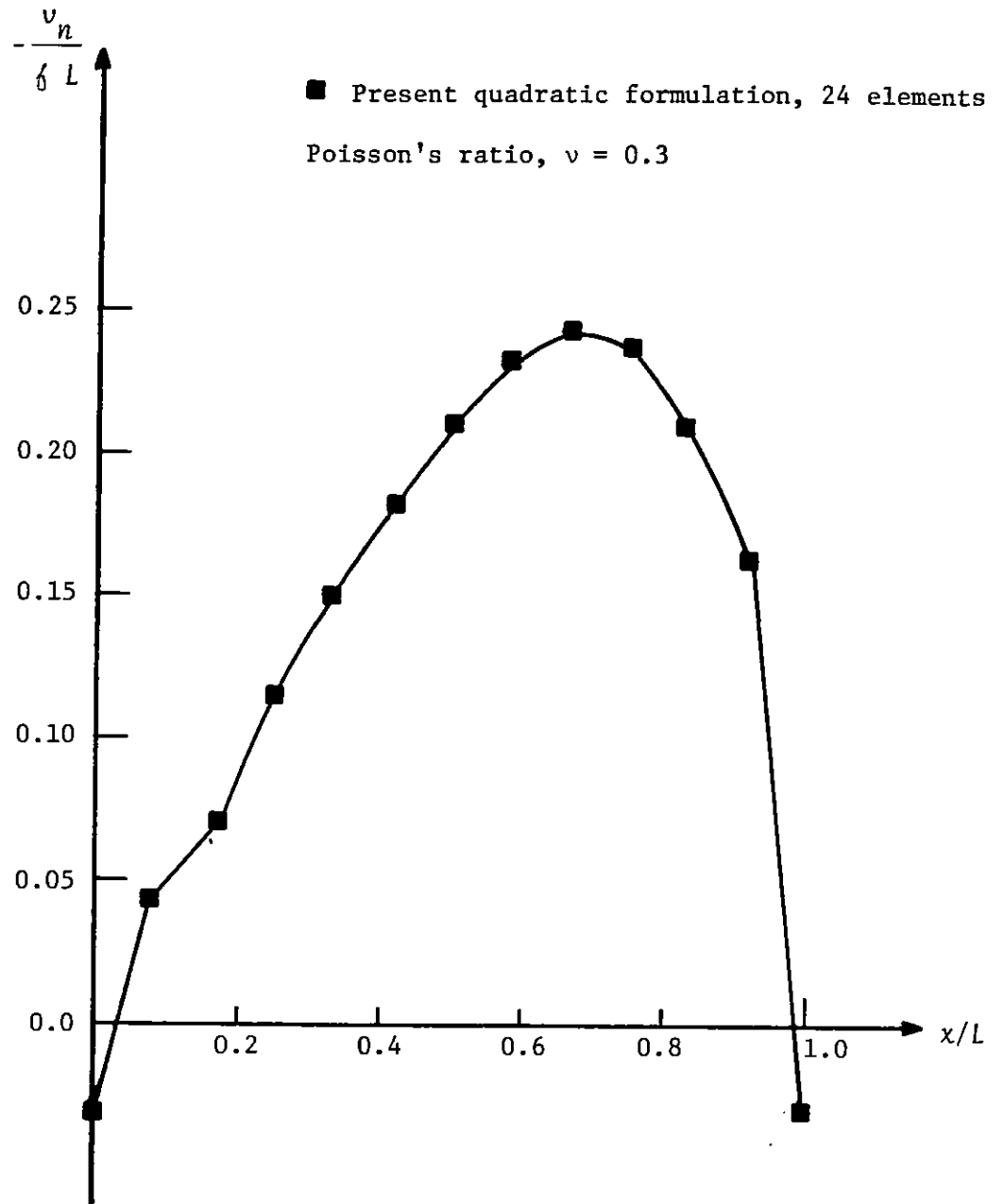
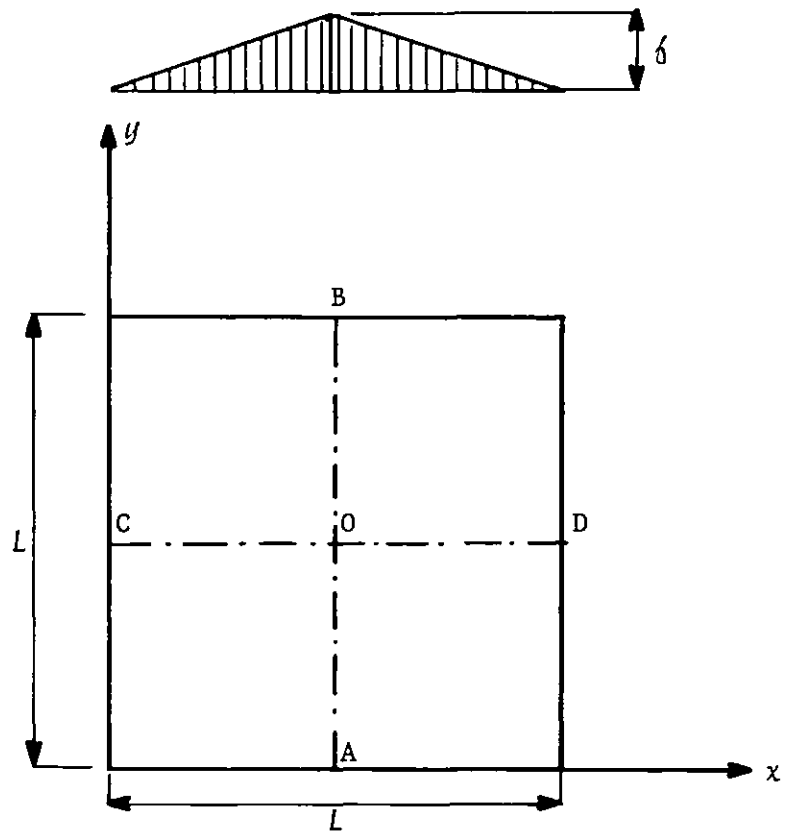
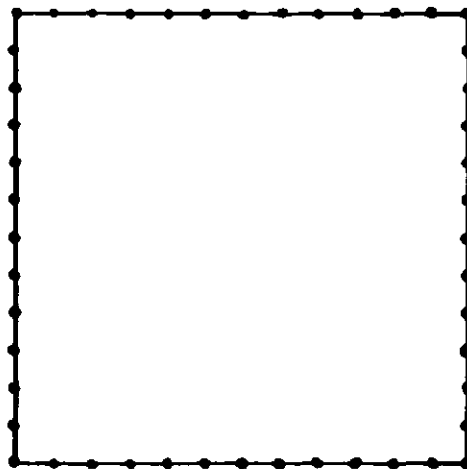


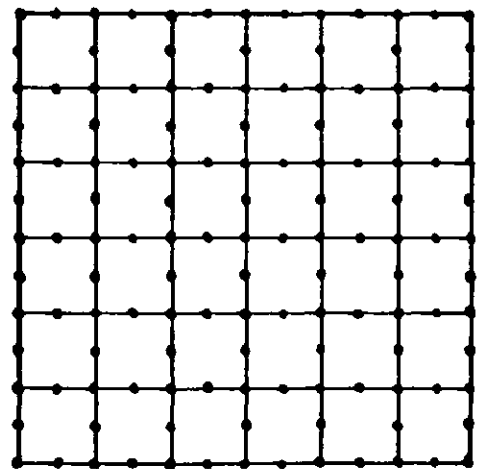
Figure 5.39: Effective shear force along OC in Figure 5.35 for a simply supported square plate under a linearly varying load



(a) Plate geometry and loading



(b) Boundary discretisation.
6 elements along each
side.



(c) Interior discretisation

Figure 5.40: Simply supported square plate under a load in the shape of a triangular prism

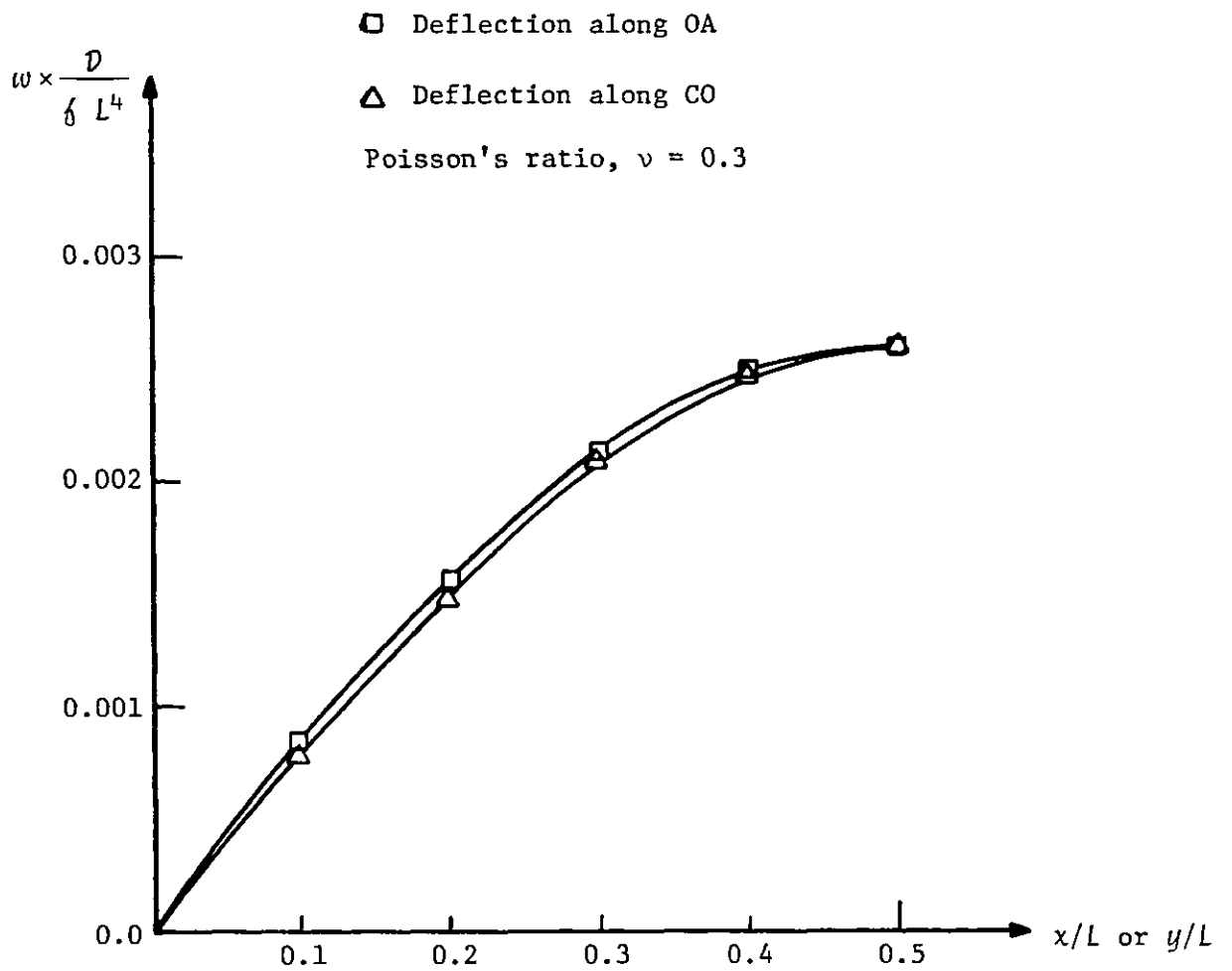


Figure 5.41: Deflection along the two centrelines OA and CO in Figure 5.40 for a simply supported square plate under a load in the shape of a triangular prism

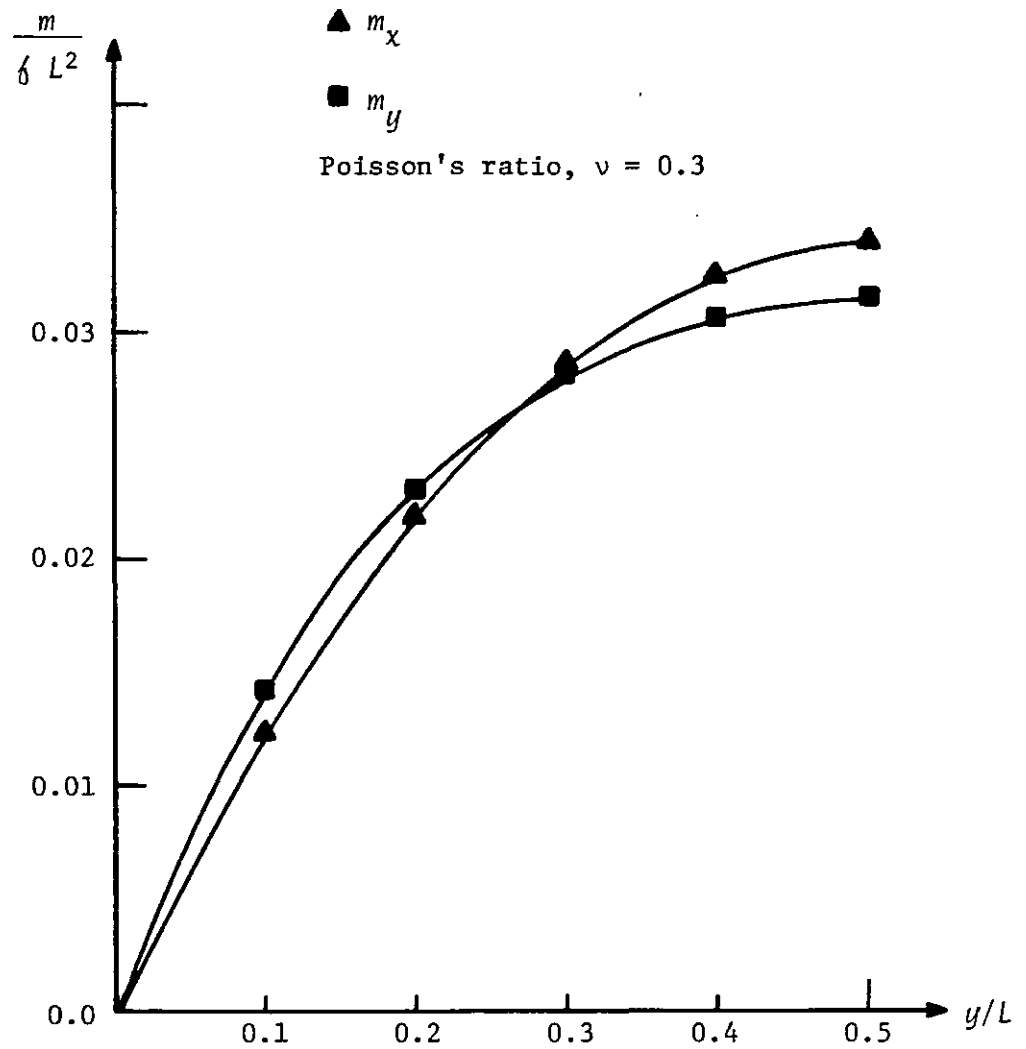


Figure 5.42: Moments along OA in Figure 5.40 for a simply supported square plate under a load in the shape of a triangular prism

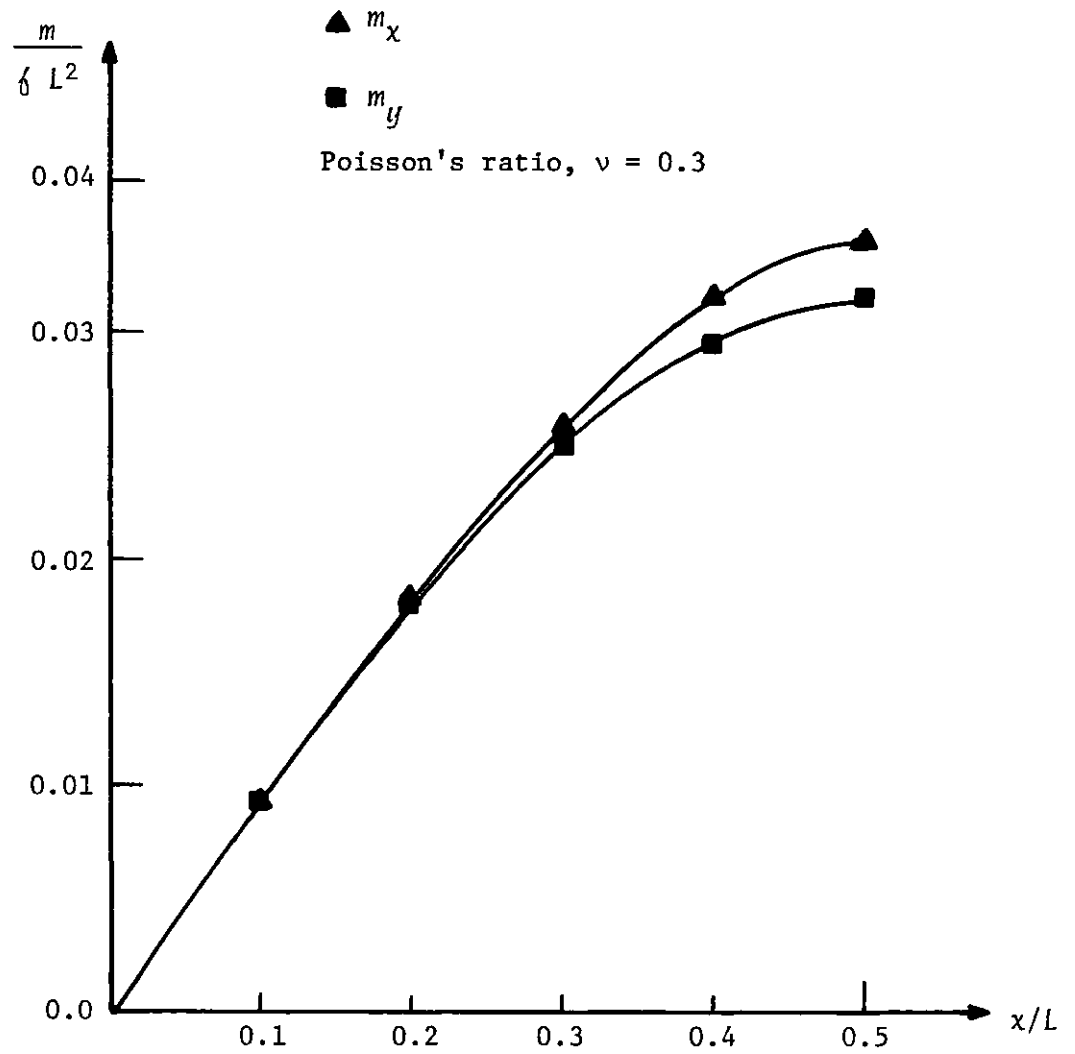
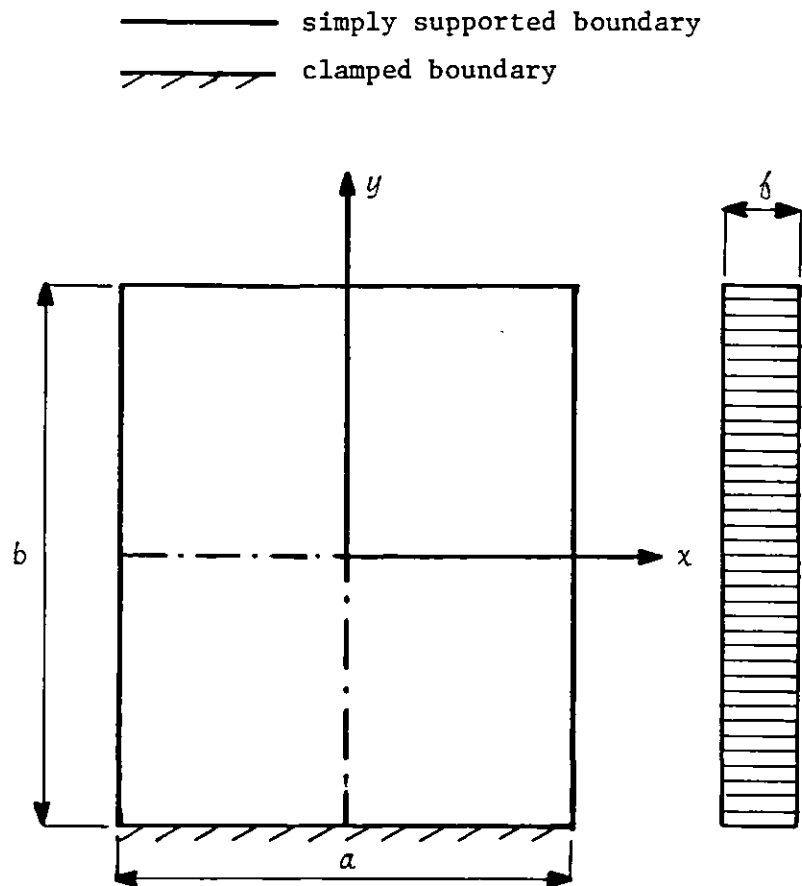
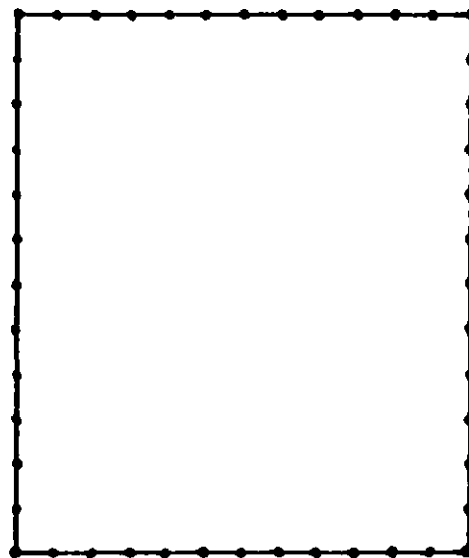


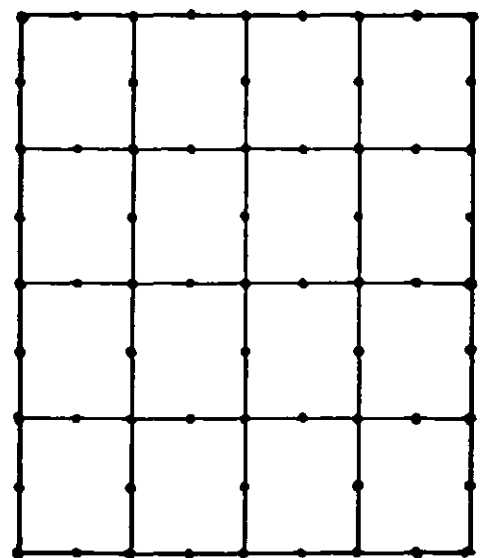
Figure 5.43: Moments along CO in Figure 5.40 for a simply supported square plate under a load in the shape of a triangular prism



(a) Plate geometry, loading and boundary conditions

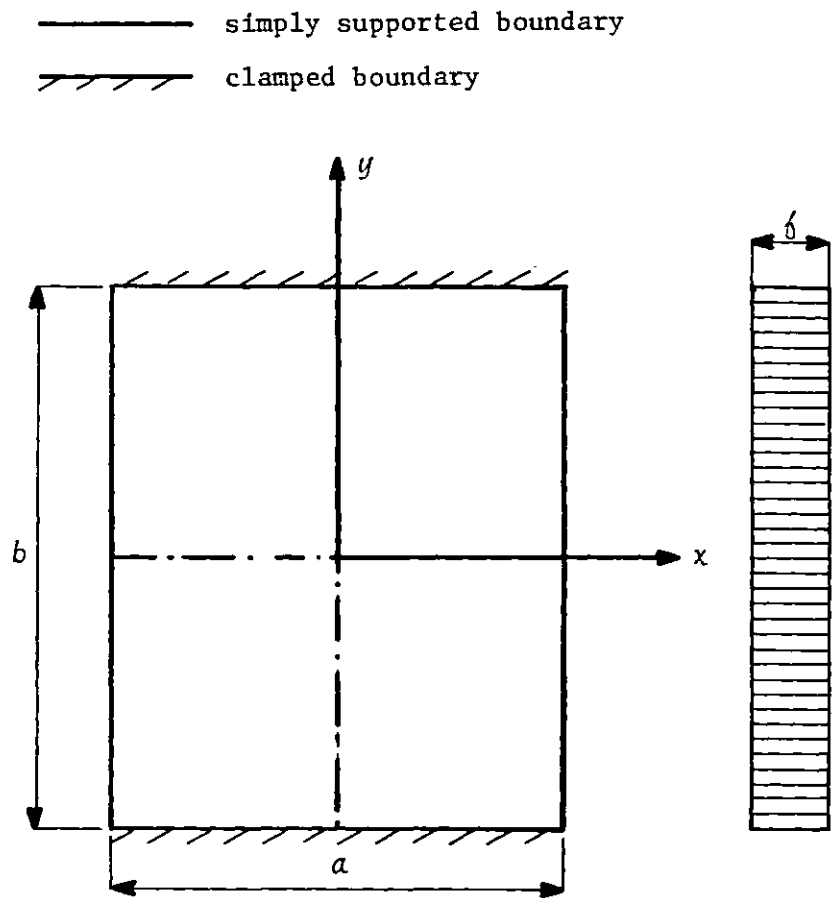


(b) Boundary discretisation.
6 elements along each side.

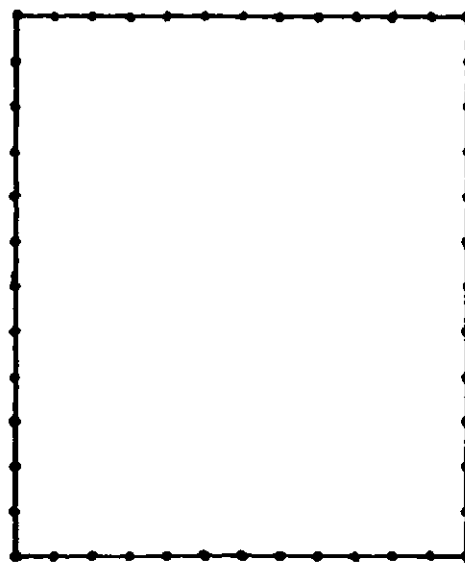


(c) Interior discretisation

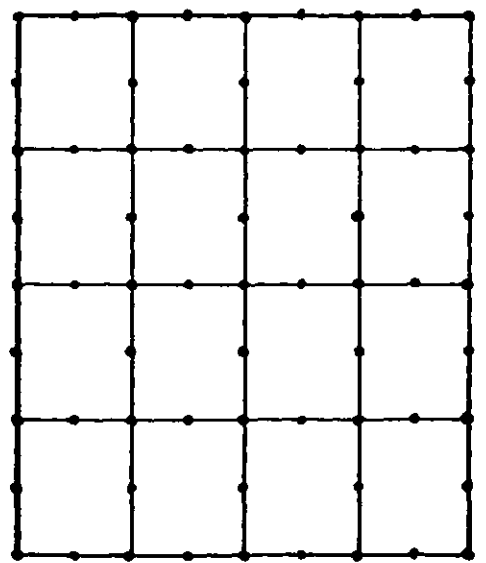
Figure 5.44: Rectangular plate having one clamped edge and the other three simply supported under a uniformly distributed load



(a) Plate geometry, loading and boundary conditions

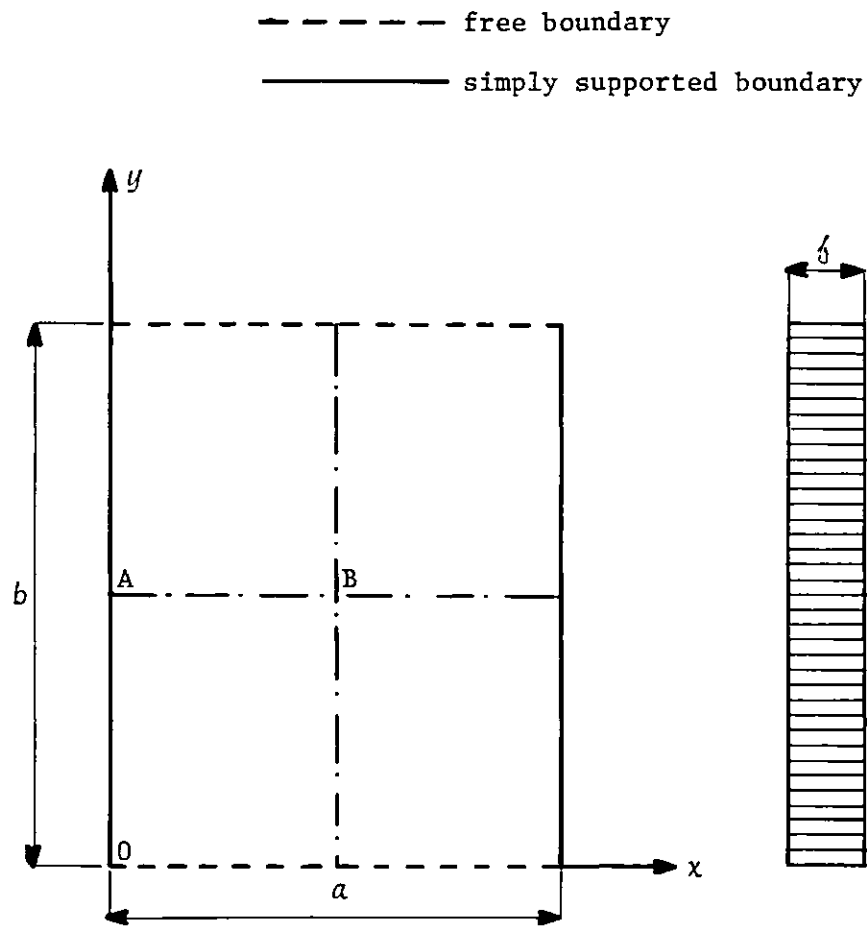


(b) Boundary discretisation.
6 elements along each
side.

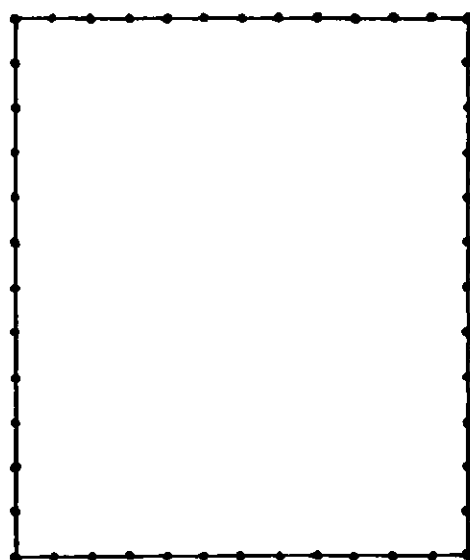


(c) Interior discretisation

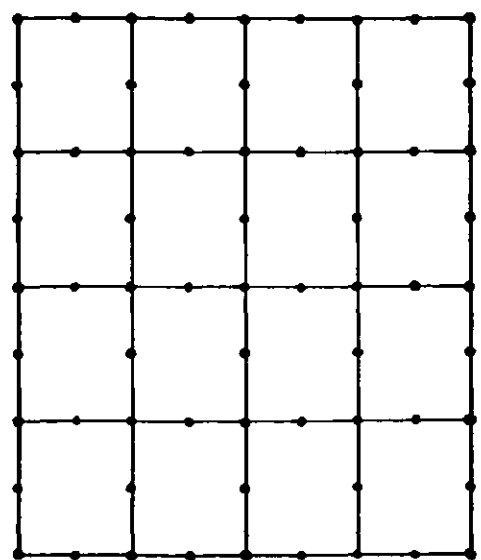
Figure 5.45: Rectangular plate with two opposite edges clamped and the other two simply supported under a uniformly distributed load



(a) Plate geometry, loading and boundary conditions



(b) Boundary discretisation.
6 elements along each
side.



(c) Interior discretisation

Figure 5.46: Rectangular plate with two opposite edges simply supported and the other two free under a uniformly distributed load

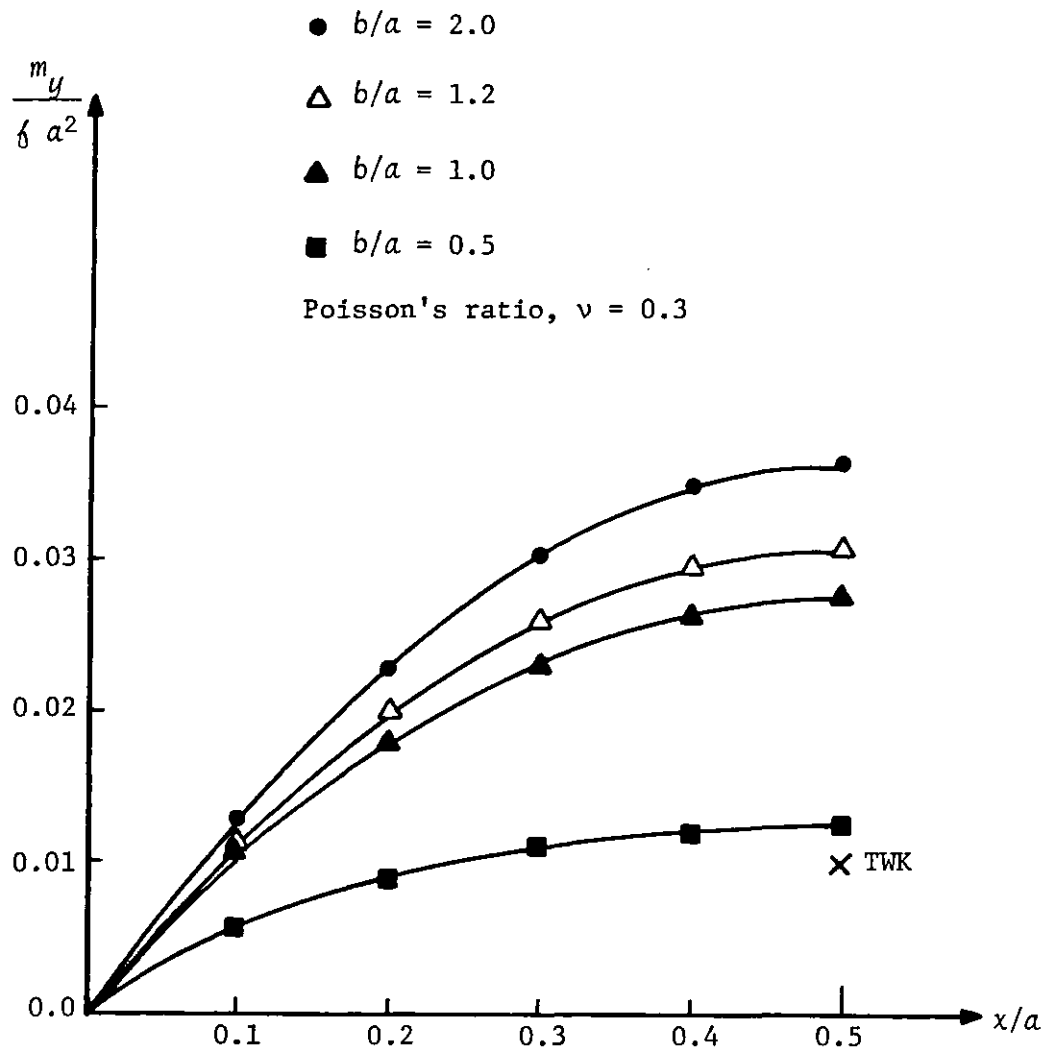


Figure 5.47: Moment, m_y , along the centreline parallel to the free boundaries (AB in Figure 5.46) for a rectangular plate with two opposite edges free and the other two simply supported under a uniformly distributed load

TWK : Timoshenko & Woinowsky-Krieger [1959]

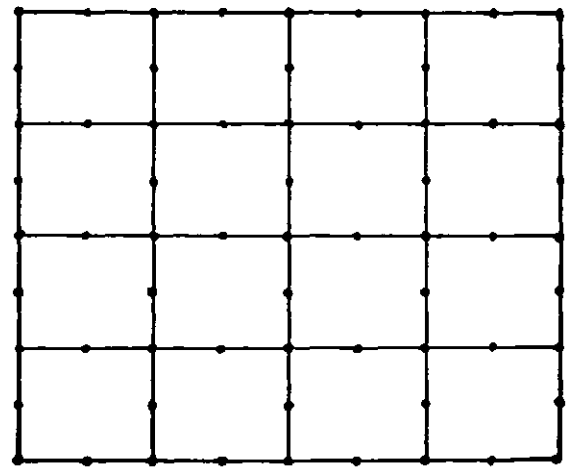
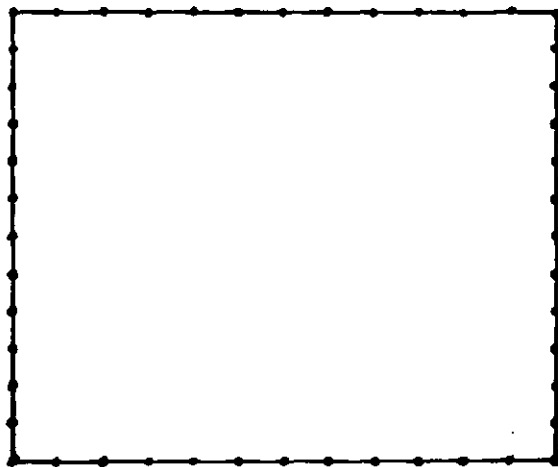
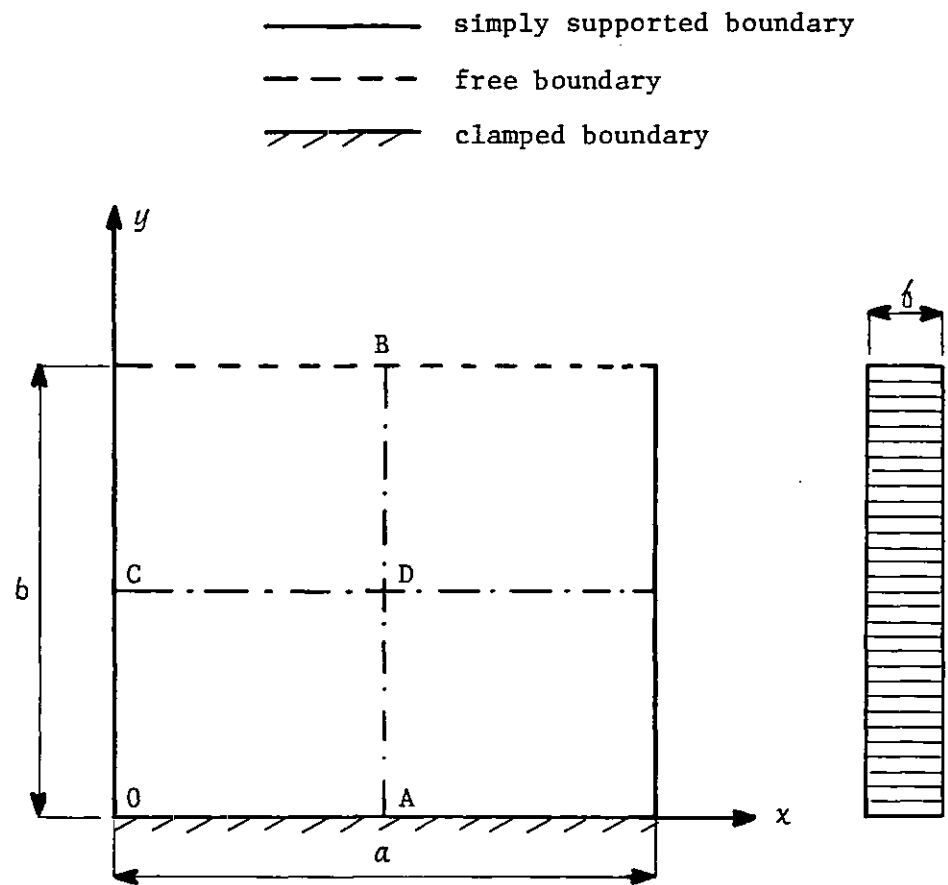


Figure 5.48: Rectangular plate with two opposite edges simply supported, the third edge free and the fourth clamped, under a uniformly distributed load

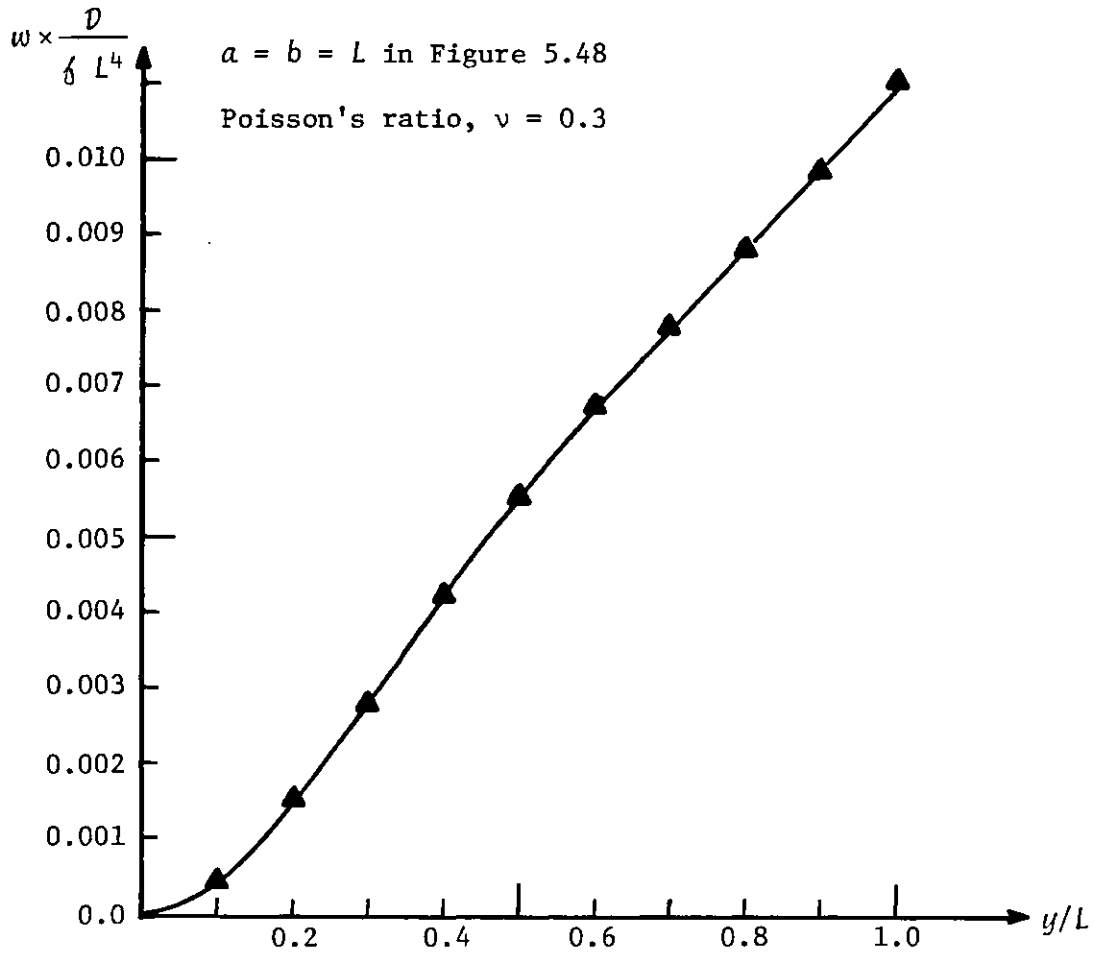


Figure 5.49: Deflection along the centreline (AB in Figure 5.48) for a square plate with two opposite edges simply supported, the third edge clamped and the fourth edge free, under a uniformly distributed load

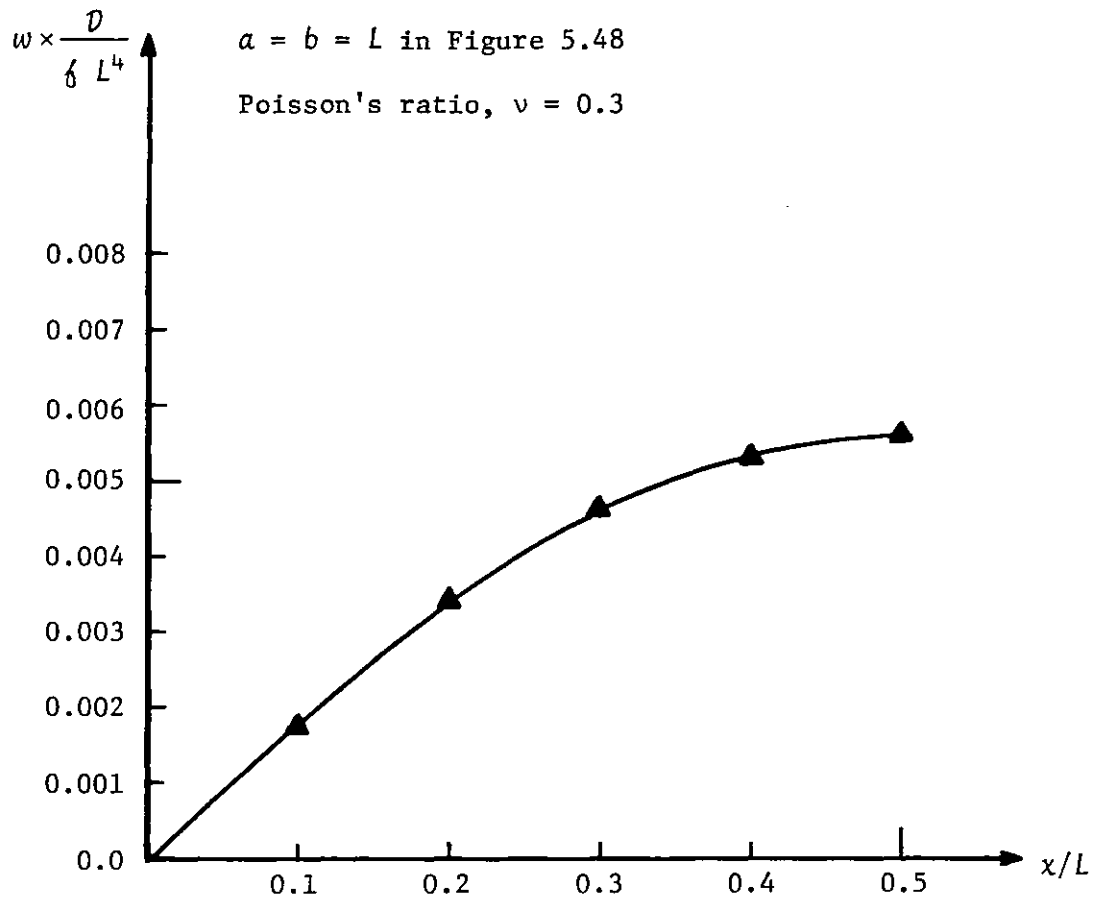


Figure 5.50: Deflection along CD in Figure 5.48 for a square plate with two opposite edges simply supported, the third edge clamped and the fourth edge free, under a uniformly distributed load

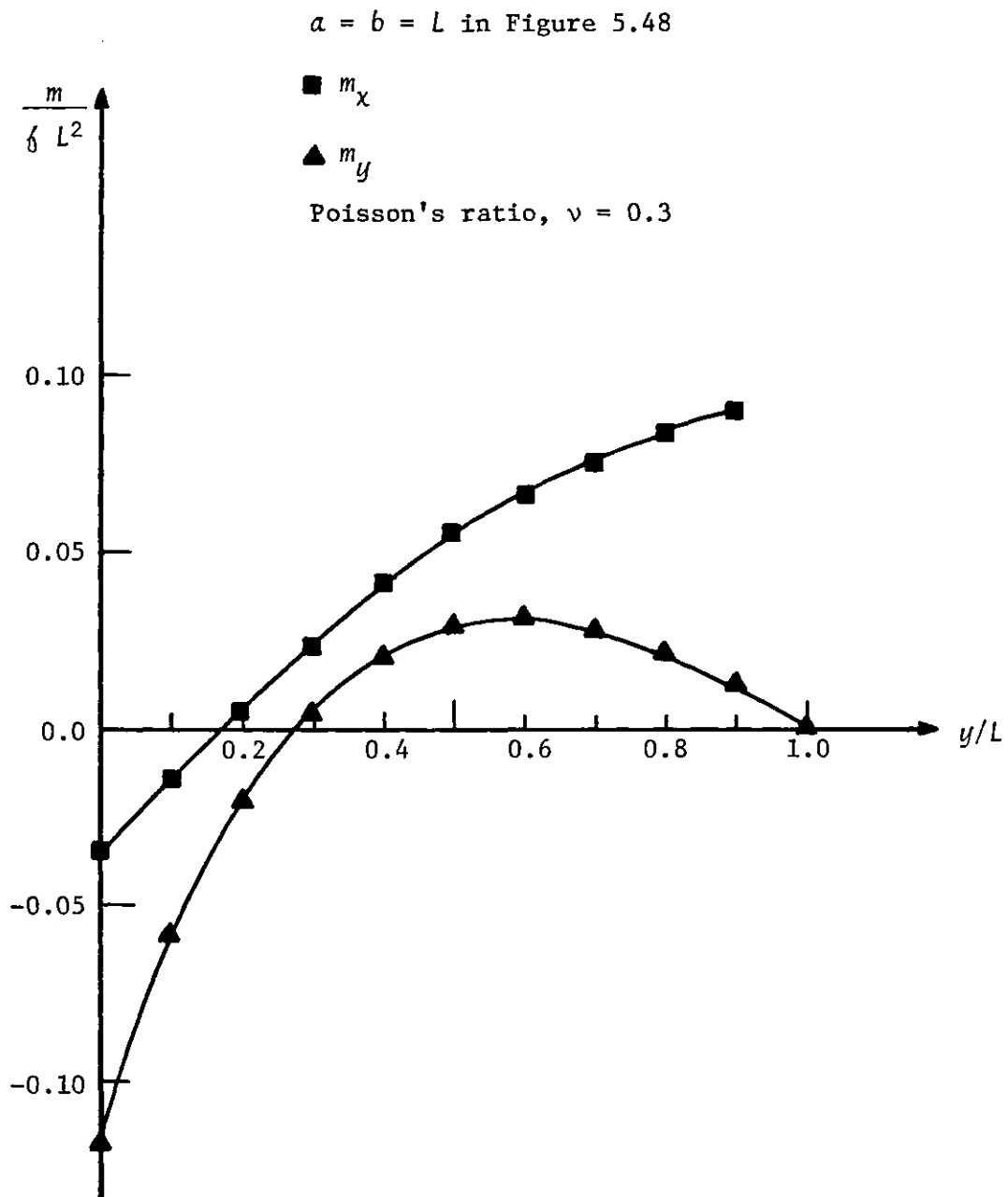


Figure 5.51: Moments along the line of symmetry (AB in Figure 5.48) for a square plate with two opposite edges simply supported, the third edge free and the fourth edge clamped, under a uniformly distributed load

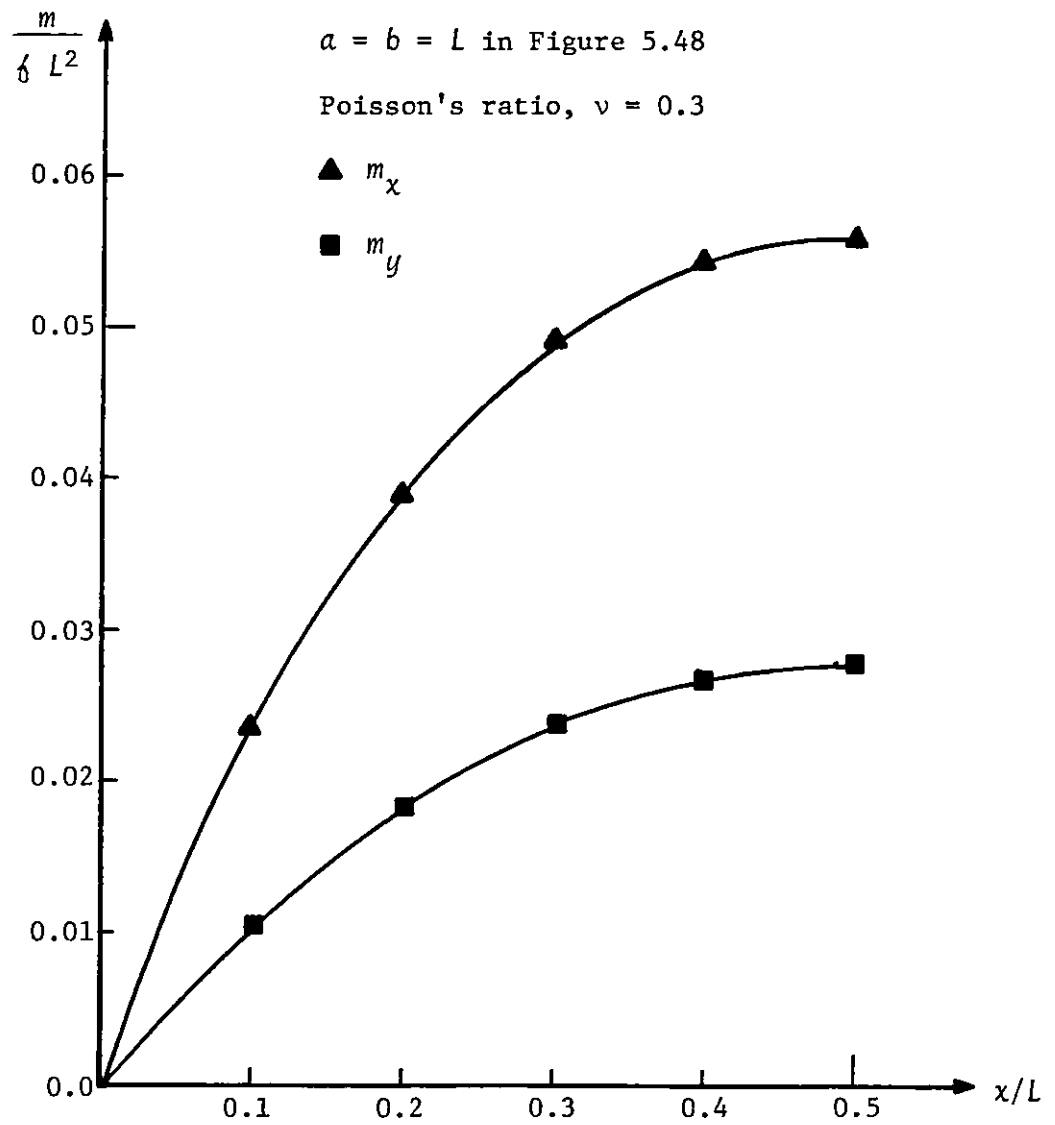


Figure 5.52: Moments along CD in Figure 5.48 for a square plate with two opposite edges simply supported, the third edge free and the fourth edge clamped, under a uniformly distributed load

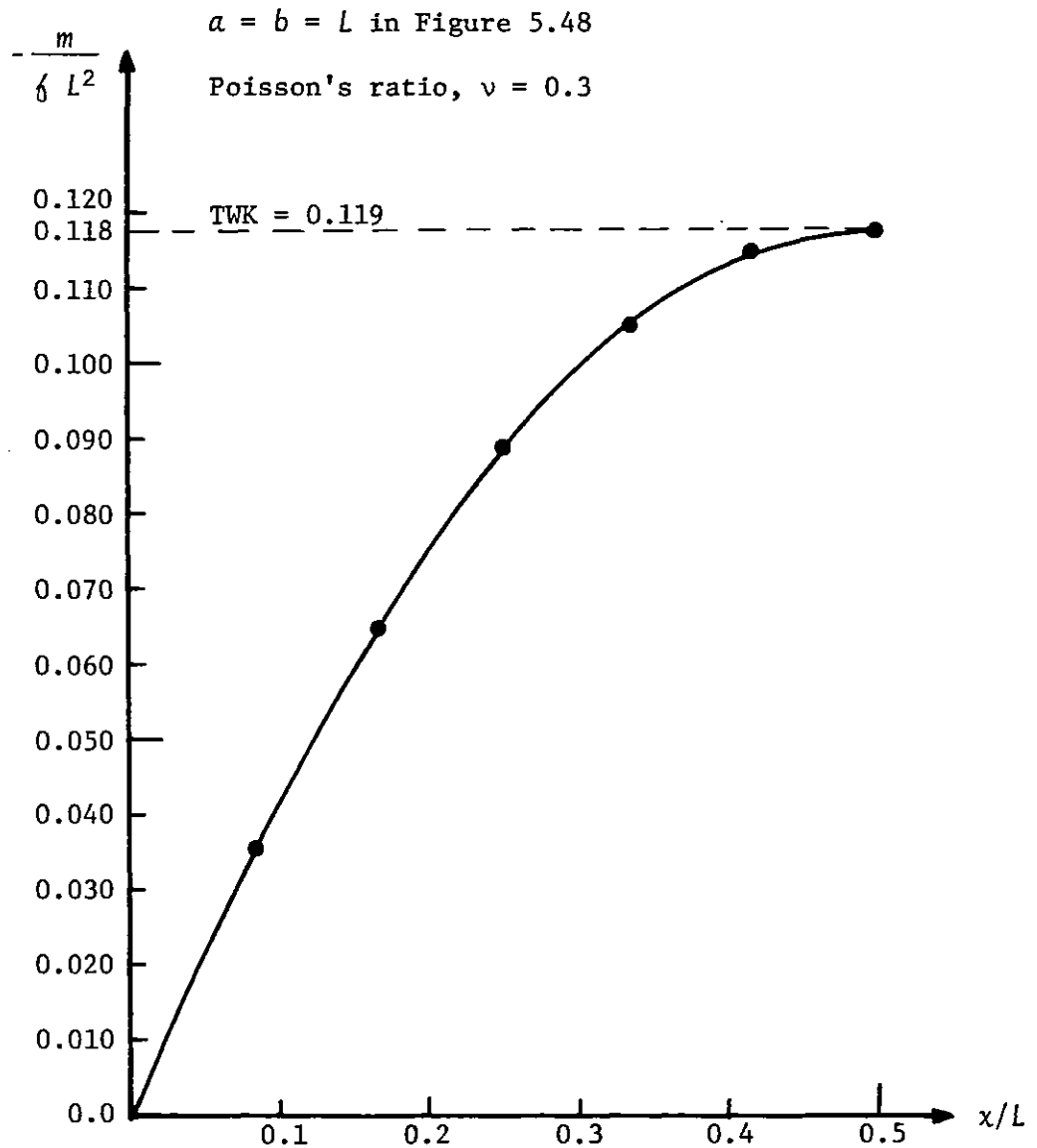
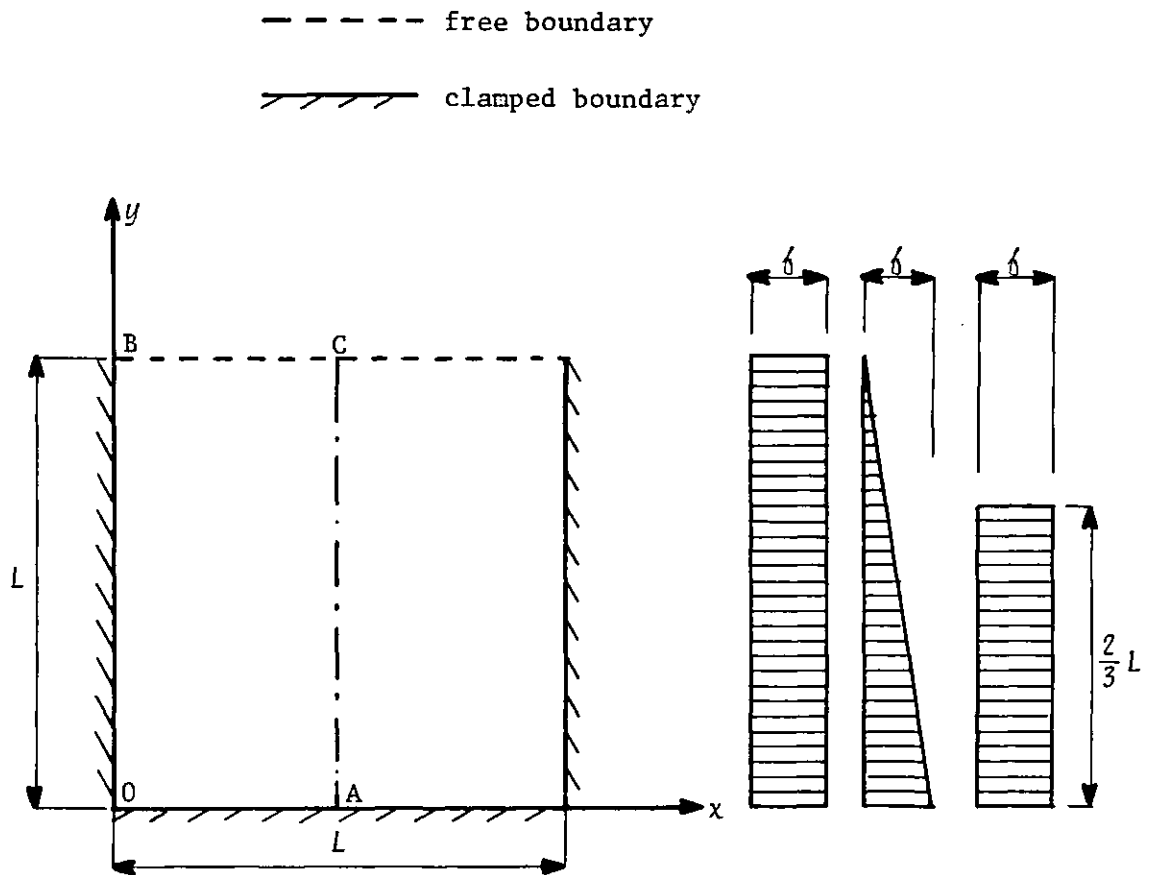
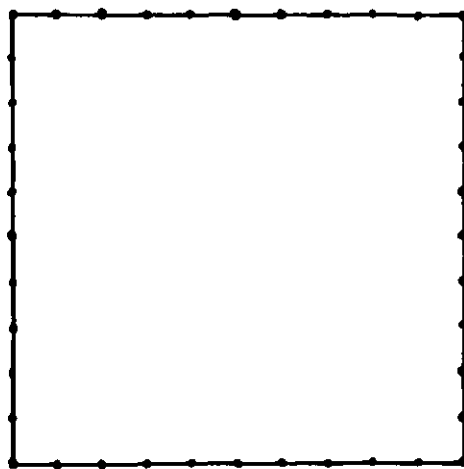


Figure 5.53: Normal moment along half the length of the clamped side (OA in Figure 5.48) for a square plate with two opposite edges simply supported, the third edge free and the fourth edge clamped, under a uniformly distributed load

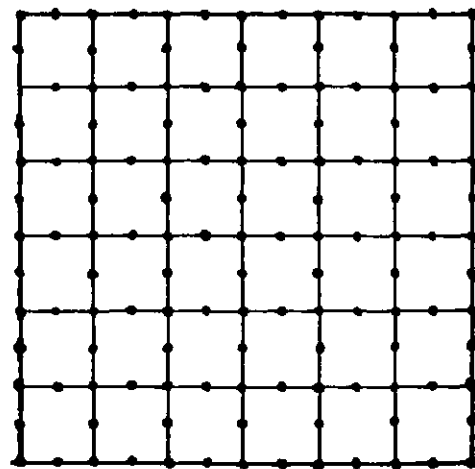
TWK : Timoshenko & Woinowsky-Krieger [1959]



(a) Plate geometry, loading and boundary conditions



(b) Boundary discretisation.
5 elements along each
side.



(c) Interior discretisation

Figure 5.54: Square plate clamped along three edges and free along the fourth edge under:
 i) a uniformly distributed load
 ii) a linearly varying load
 iii) a uniformly distributed load on two thirds of the plate

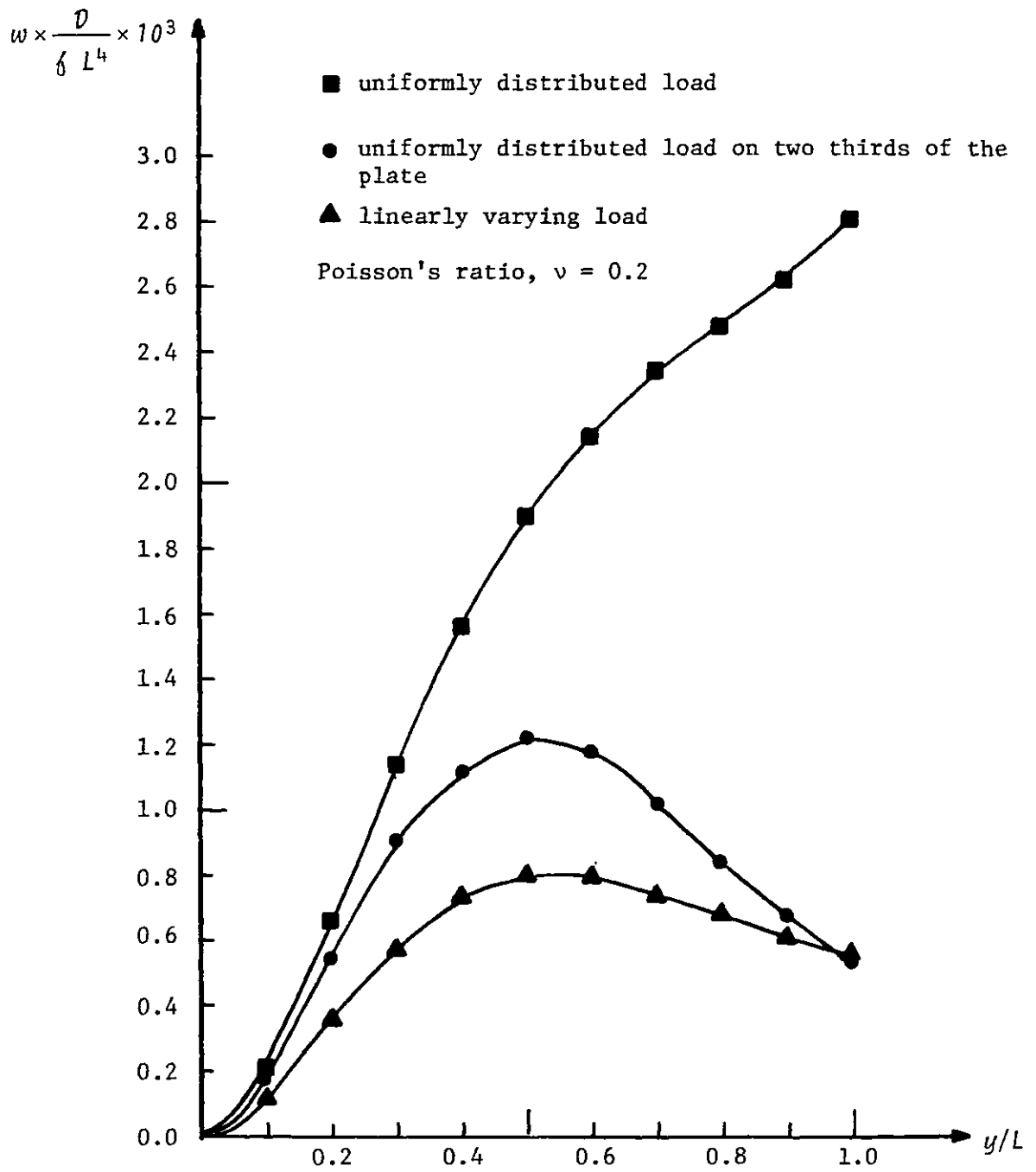


Figure 5.55: Deflections along the centreline (AC in Figure 5.54) for a square plate with three edges clamped and one edge free

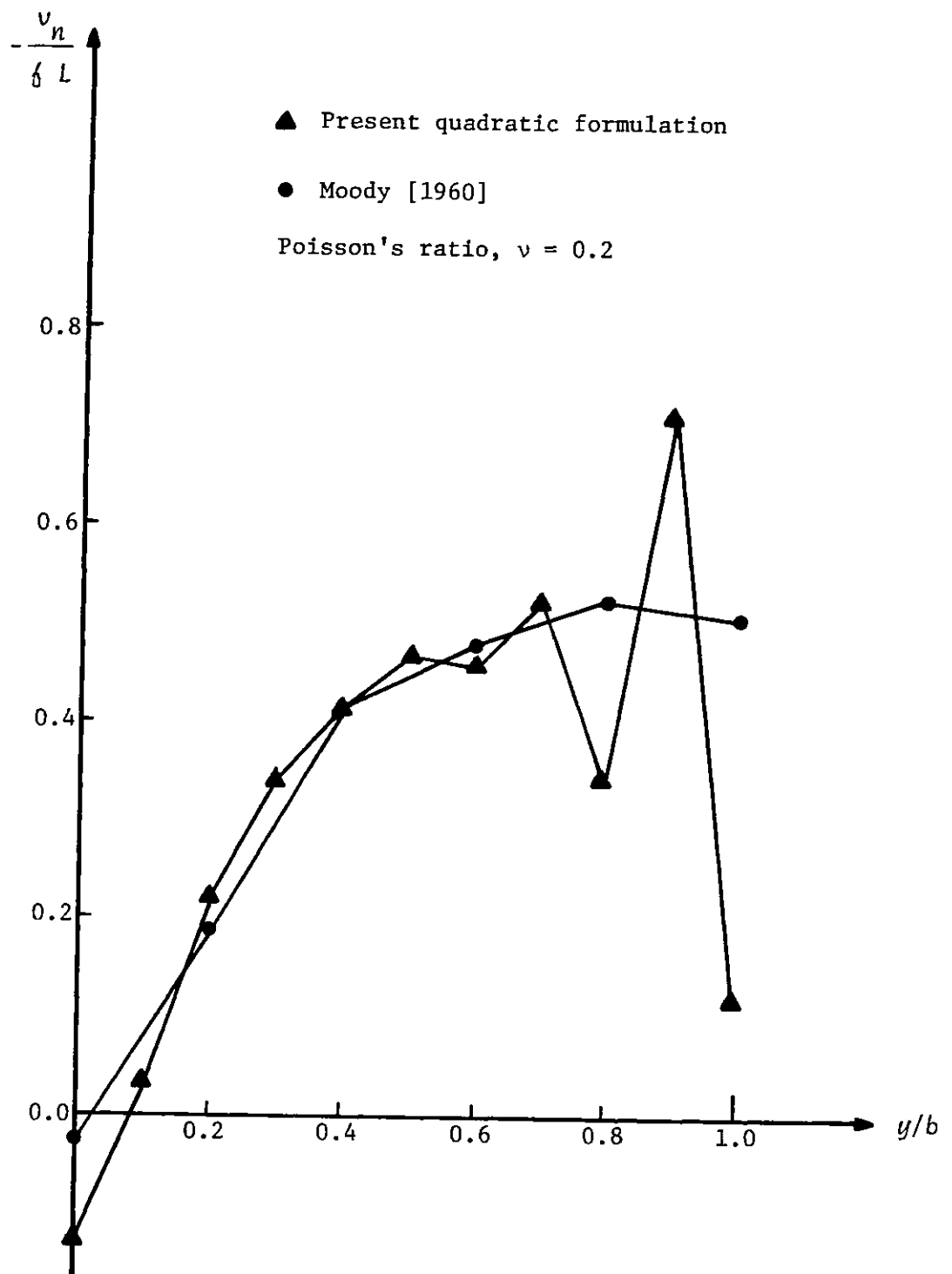


Figure 5.56: Effective shear force along OB in Figure 5.54 for a square plate with three edges clamped and the fourth edge free under a uniformly distributed load

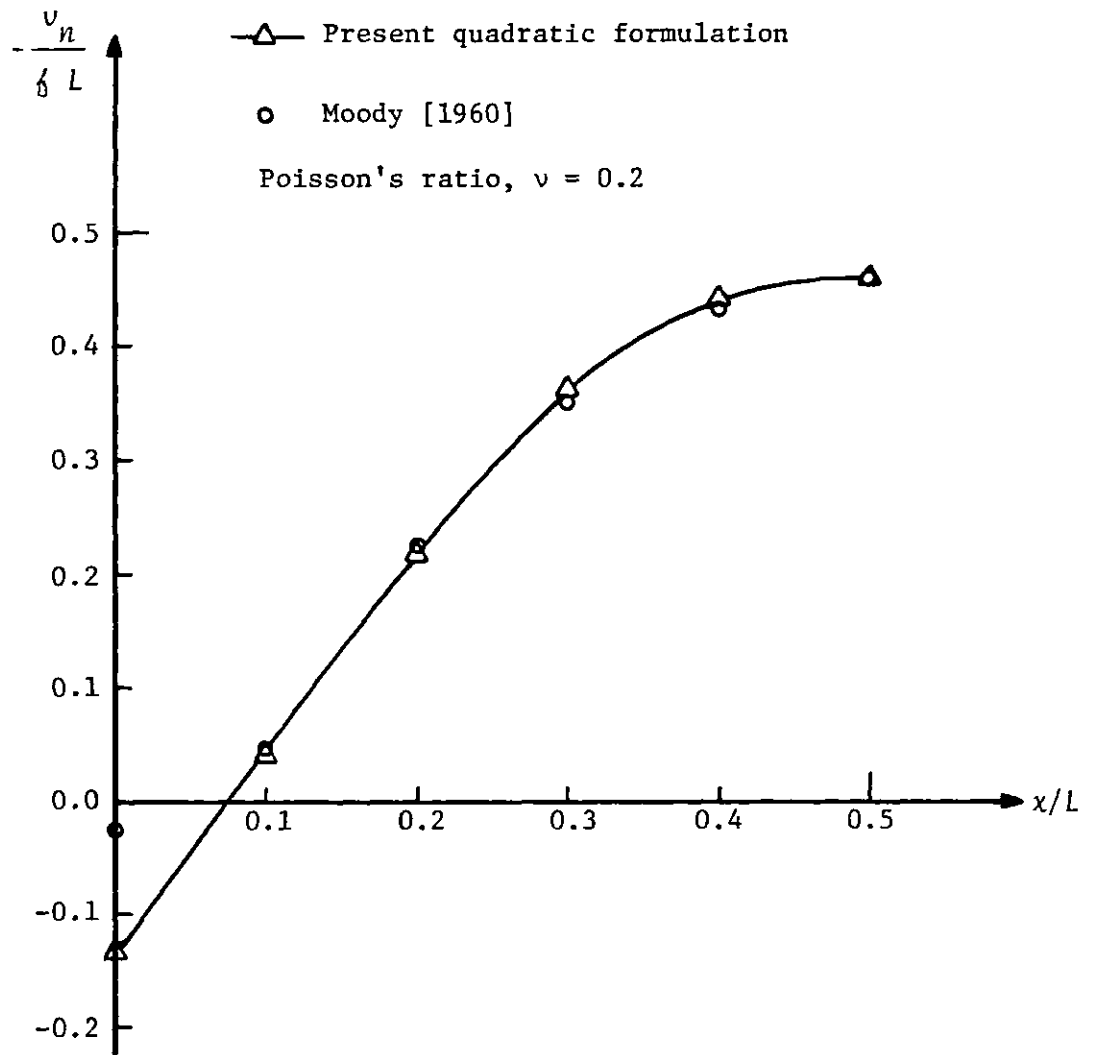


Figure 5.57: Effective shear force along OA in Figure 5.54 for a square plate with three edges clamped and the fourth edge free, under a uniformly distributed load

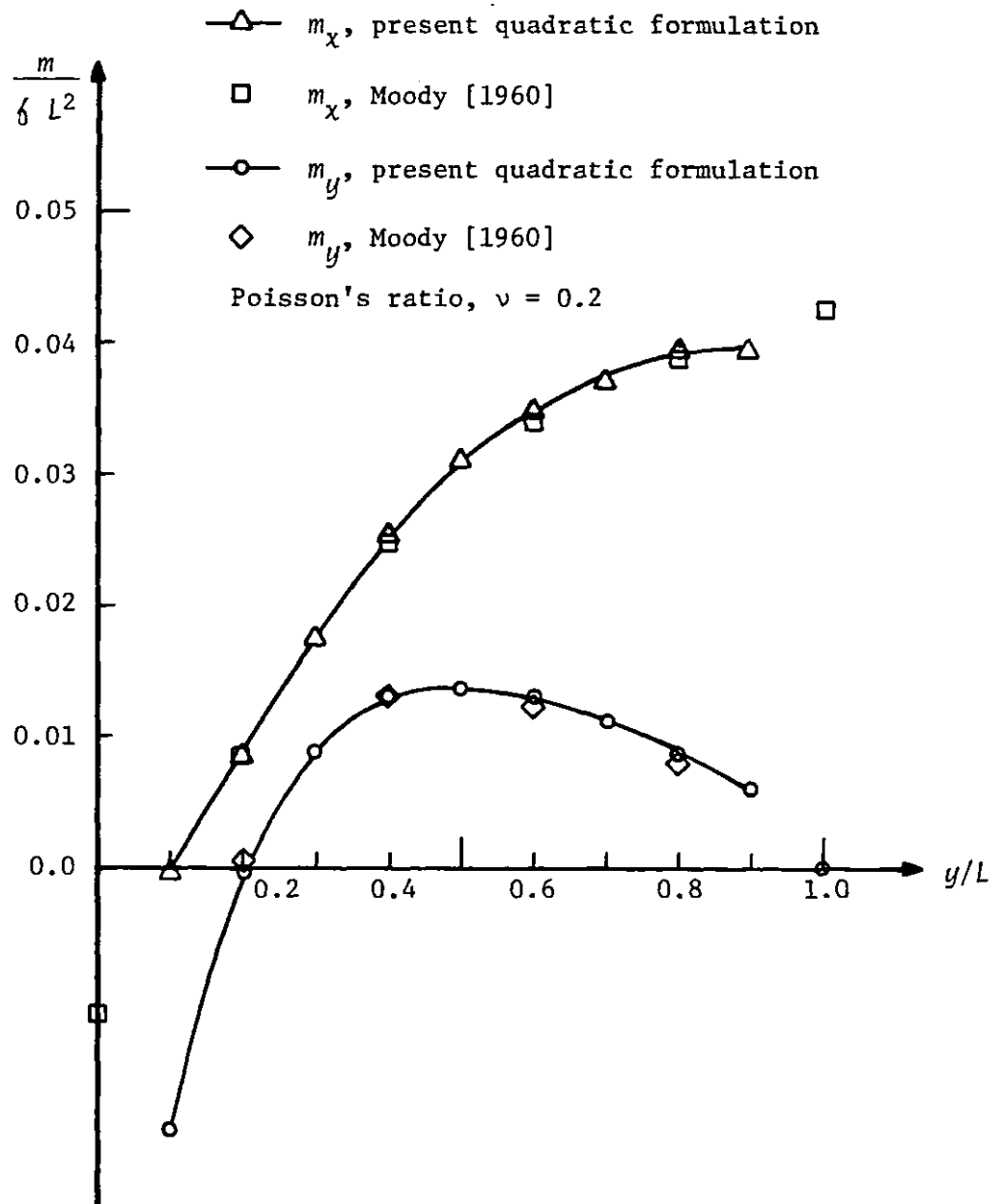


Figure 5.58: Moments, m_x and m_y , along the line of symmetry (AC in Figure 5.54) for a square plate with three edges clamped and the fourth edge free, under a uniformly distributed load

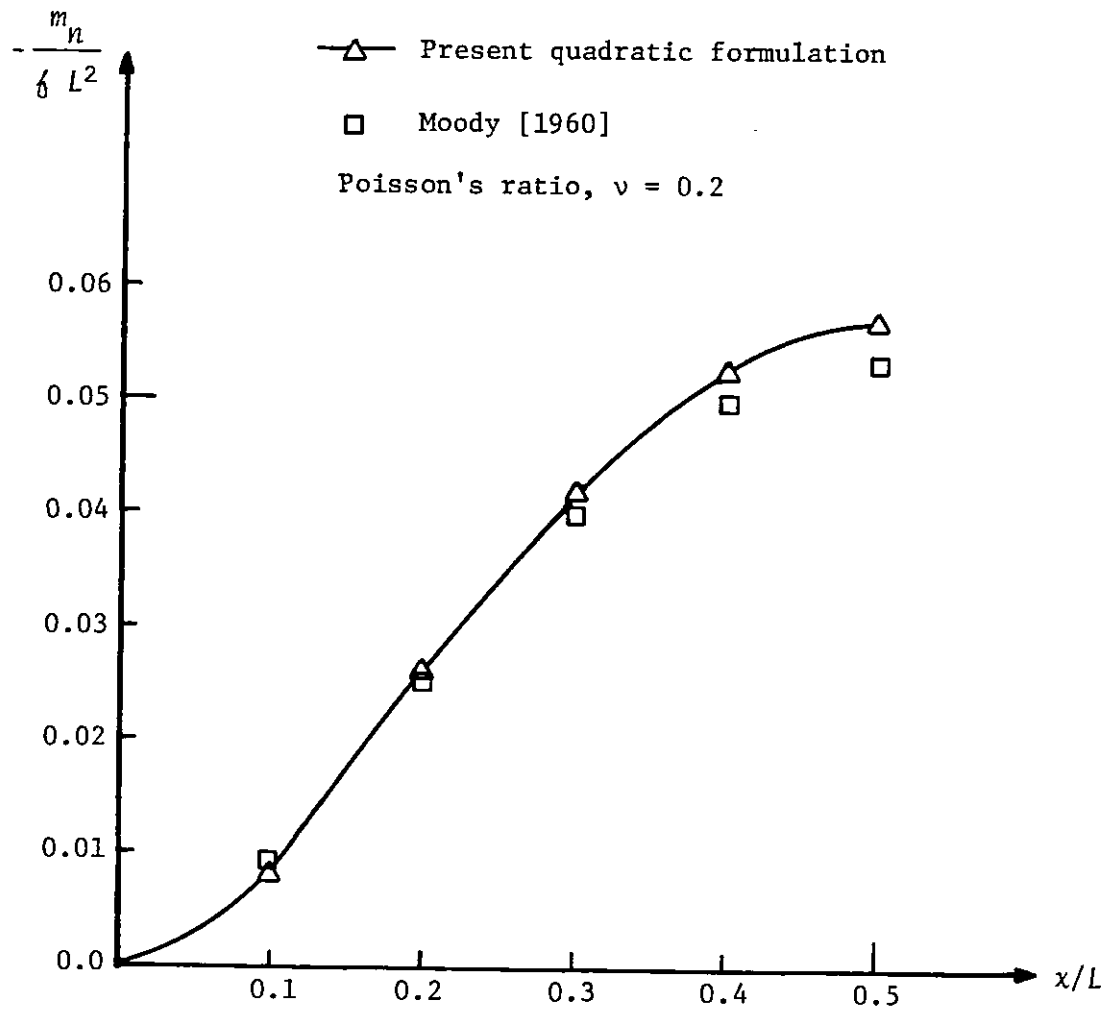


Figure 5.59: Normal moment along OA in Figure 5.54 for a square plate with three edges clamped and one edge free under a uniformly distributed load

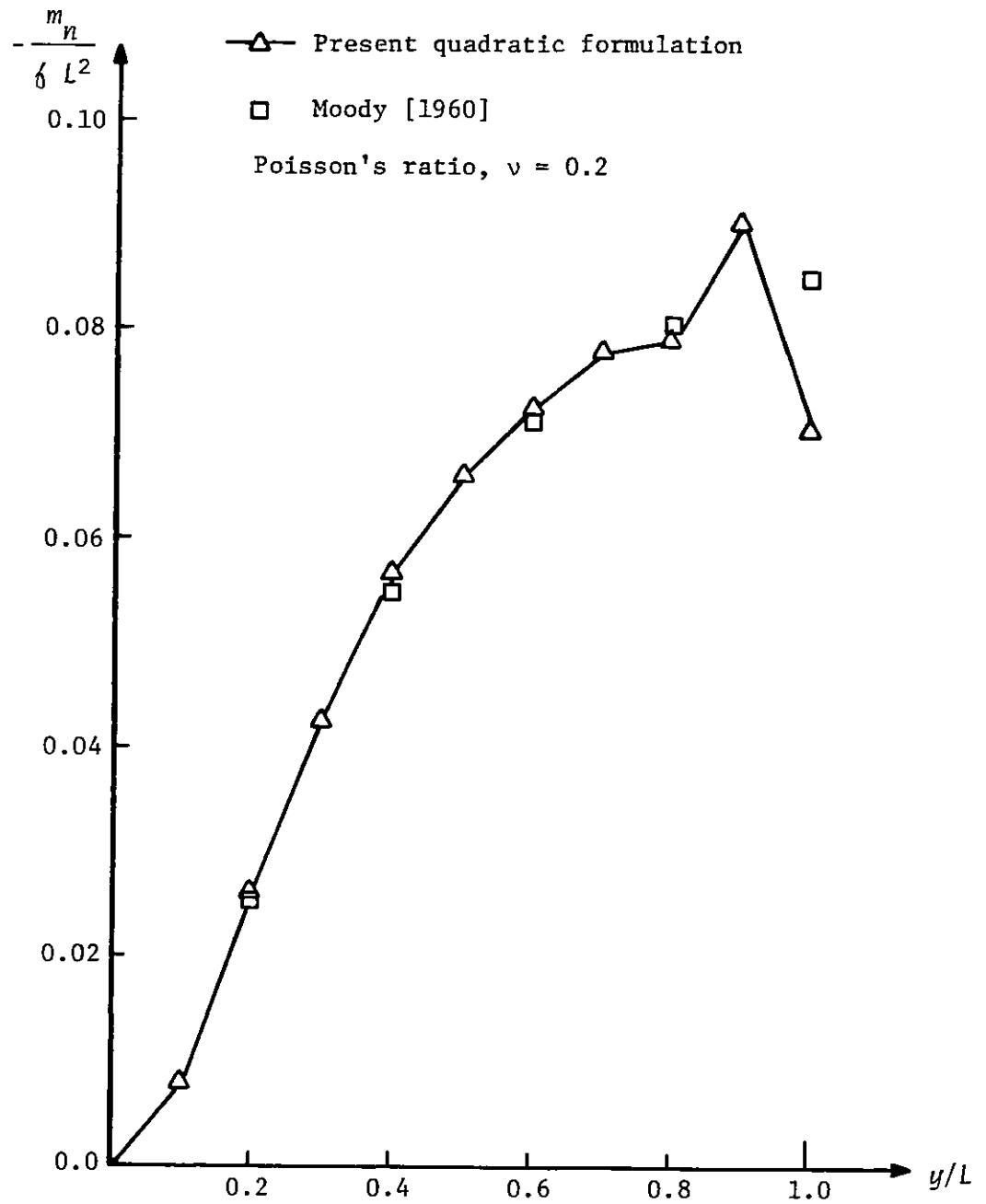


Figure 5.60: Normal moment along OB in Figure 5.54 for a square plate with three edges clamped and the fourth edge free under a uniformly distributed load

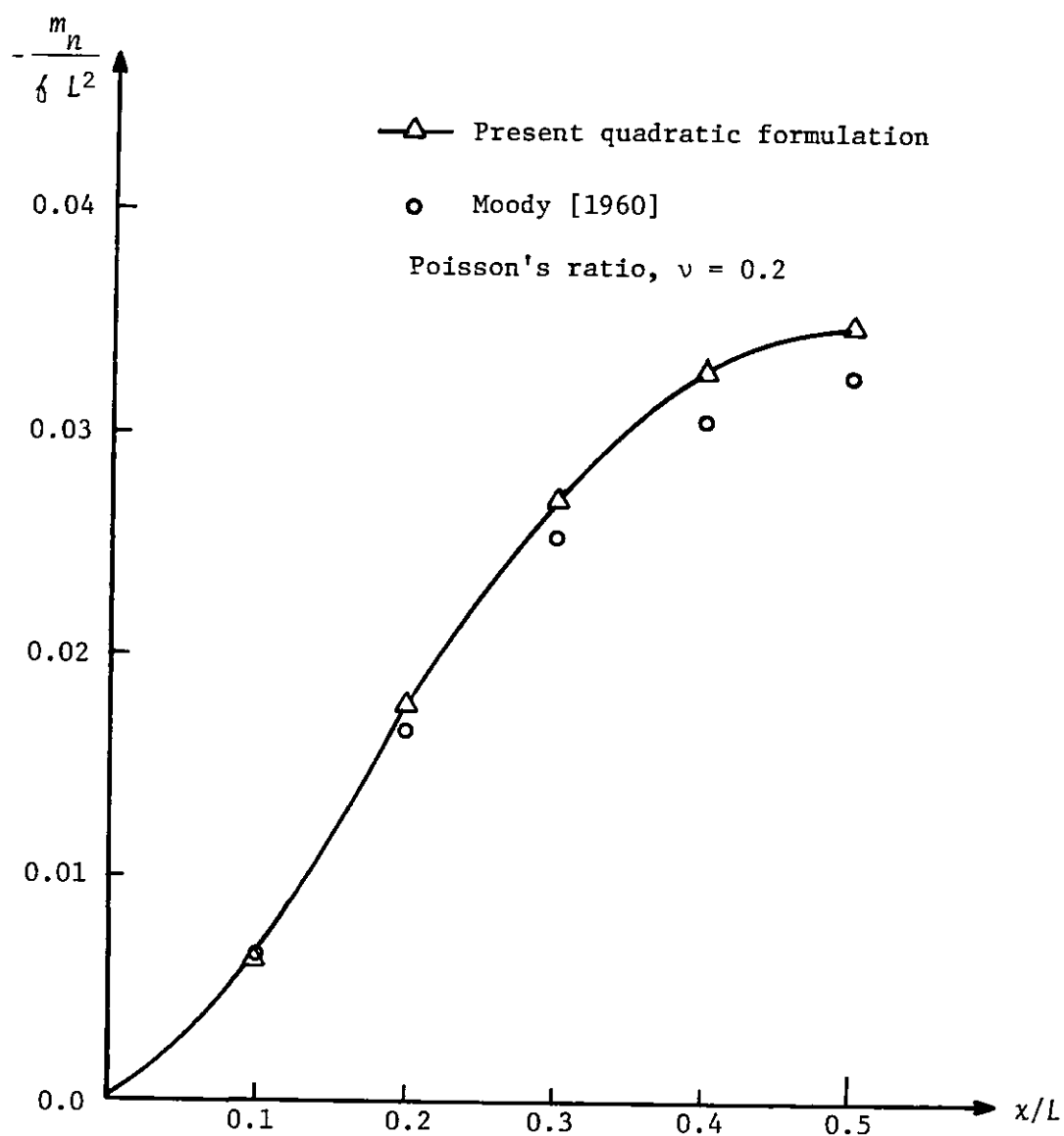


Figure 5.61: Normal moment along OA in Figure 5.54 for a square plate with three edges clamped and the fourth edge free under a linearly varying load

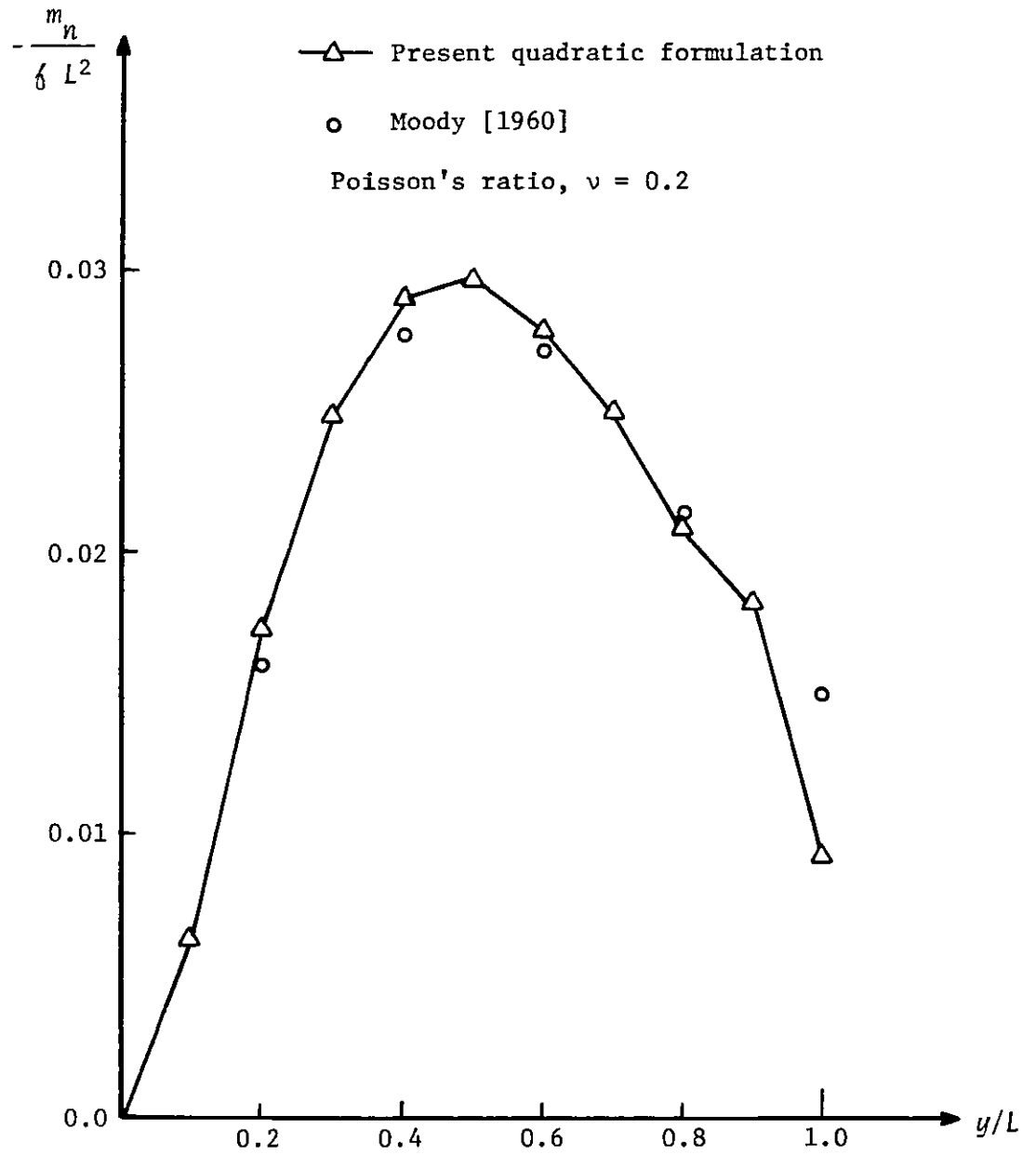


Figure 5.62: Normal moment along OB in Figure 5.54 for a square plate with three edges clamped and the fourth edge free under a linearly varying load

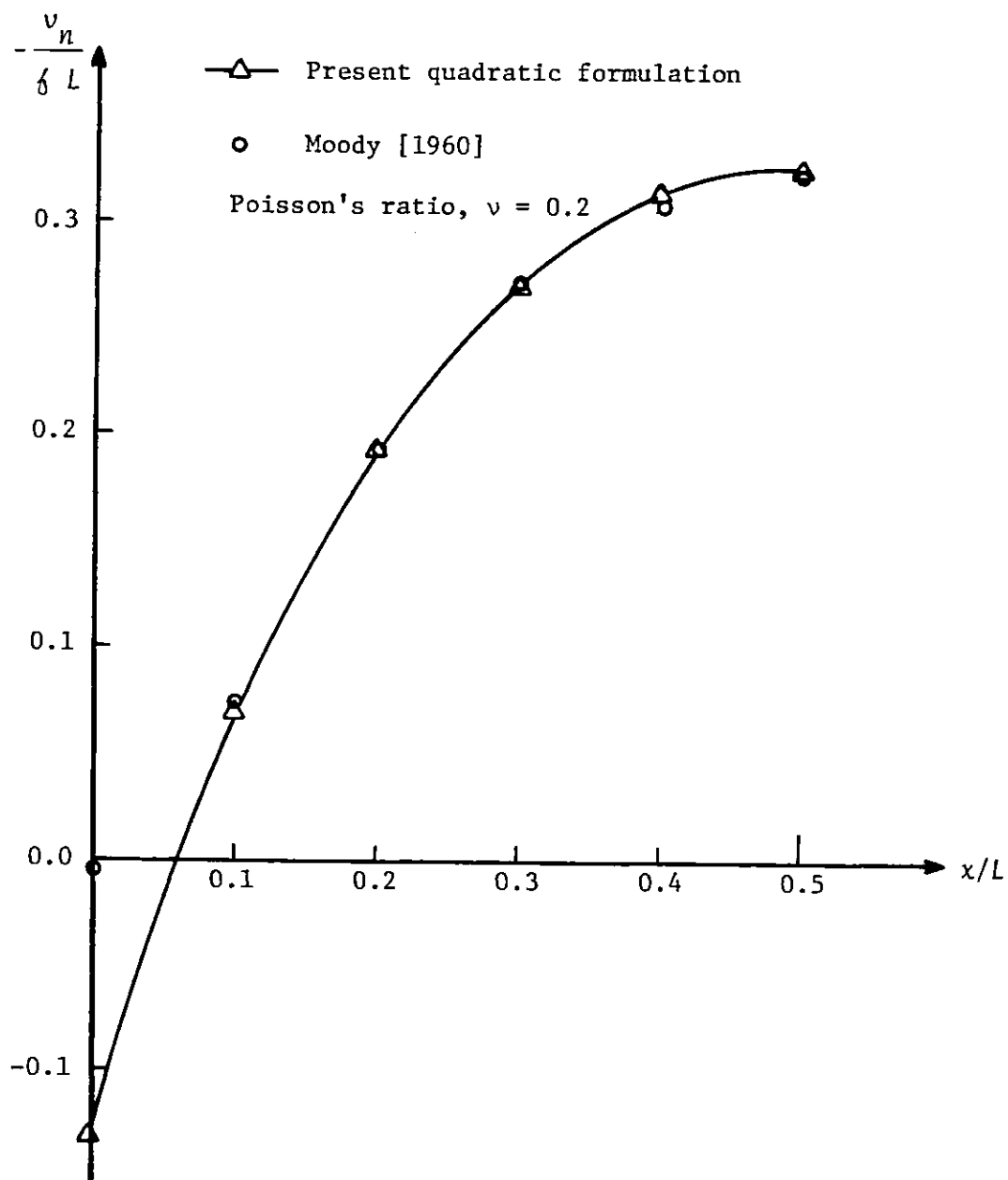


Figure 5.63: Effective shear force along OA in Figure 5.54 for a square plate with three edges clamped and the fourth edge free under a linearly varying load

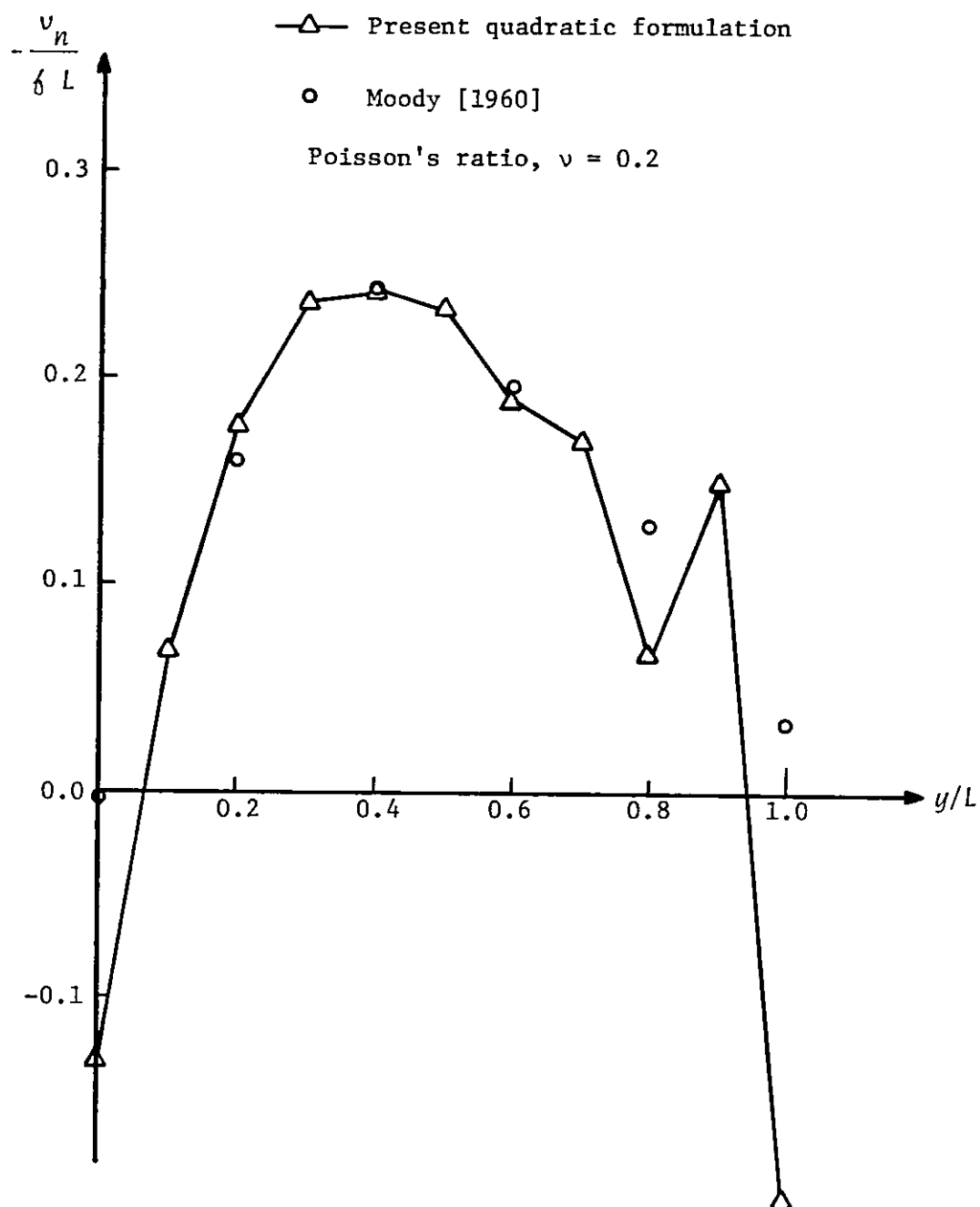


Figure 5.64: Effective shear force along OB in Figure 5.54 for a square plate with three edges clamped and the fourth edge free under a linearly varying load

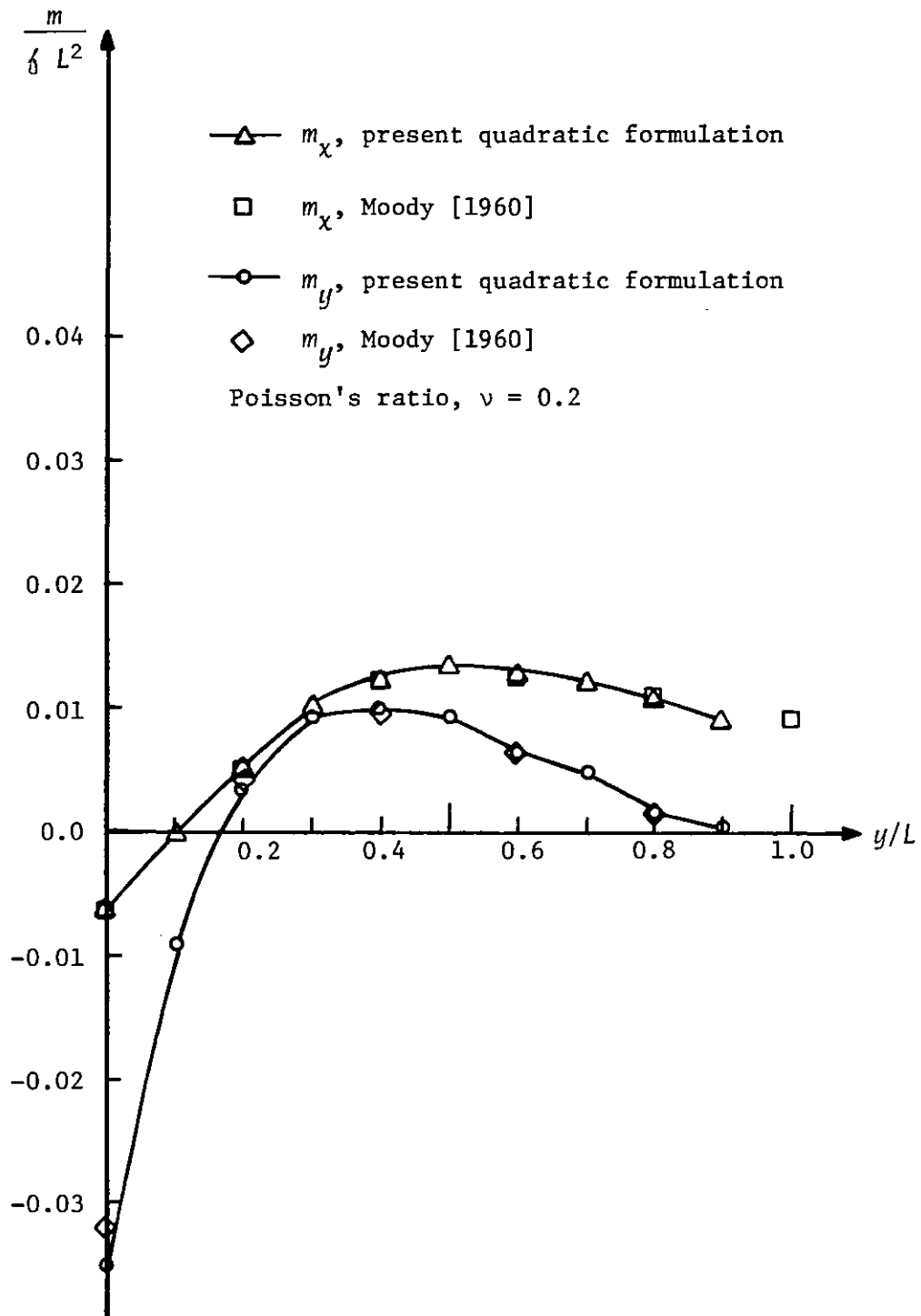


Figure 5.65: Moments, m_x and m_y , along the centreline AC in Figure 5.54 for a square plate with three edges clamped and the fourth edge free under a linearly varying load

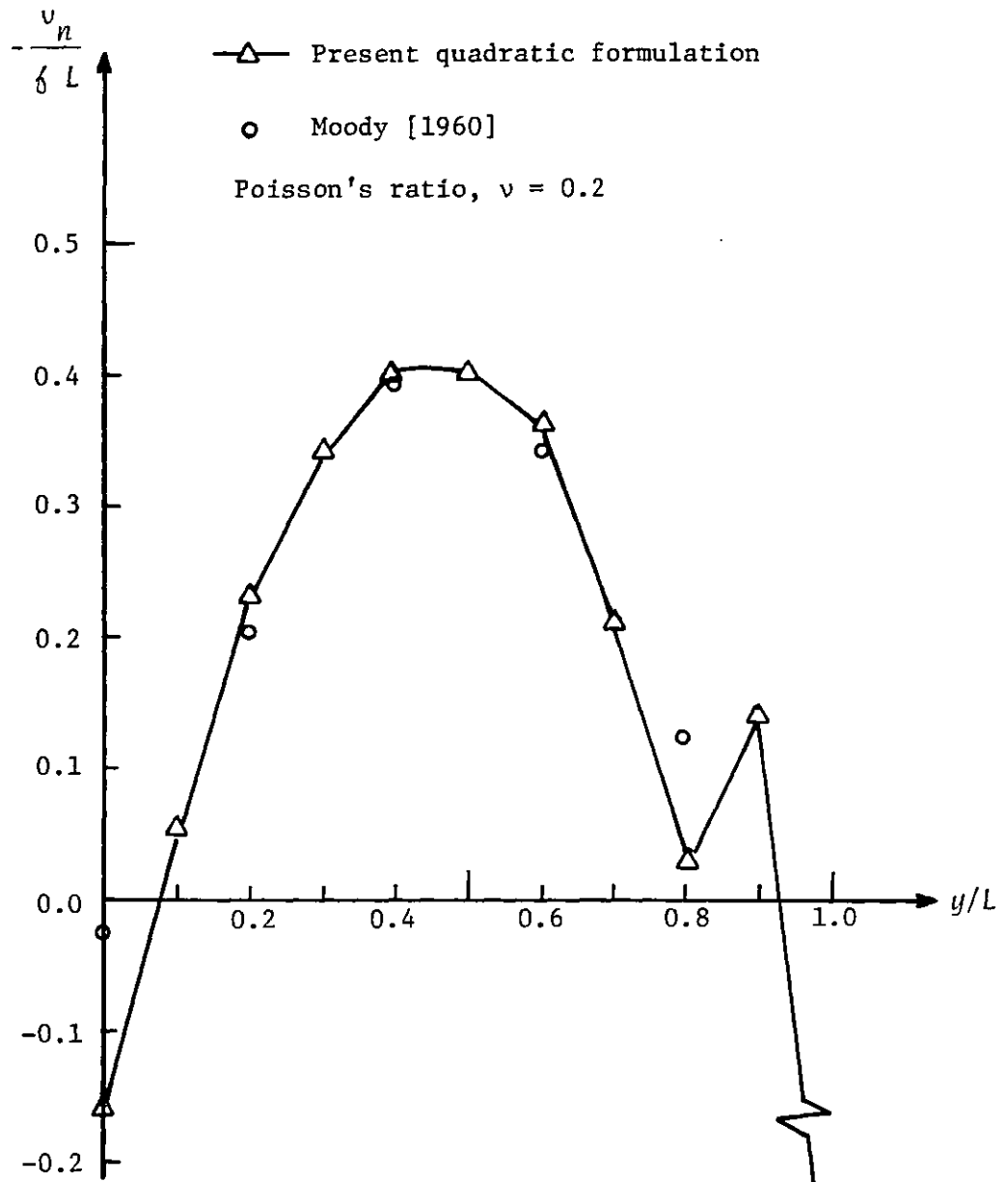


Figure 5.66: Effective shear force along OB in Figure 5.54 for a square plate with three edges clamped and the fourth edge free under a uniformly distributed load on two thirds of the plate

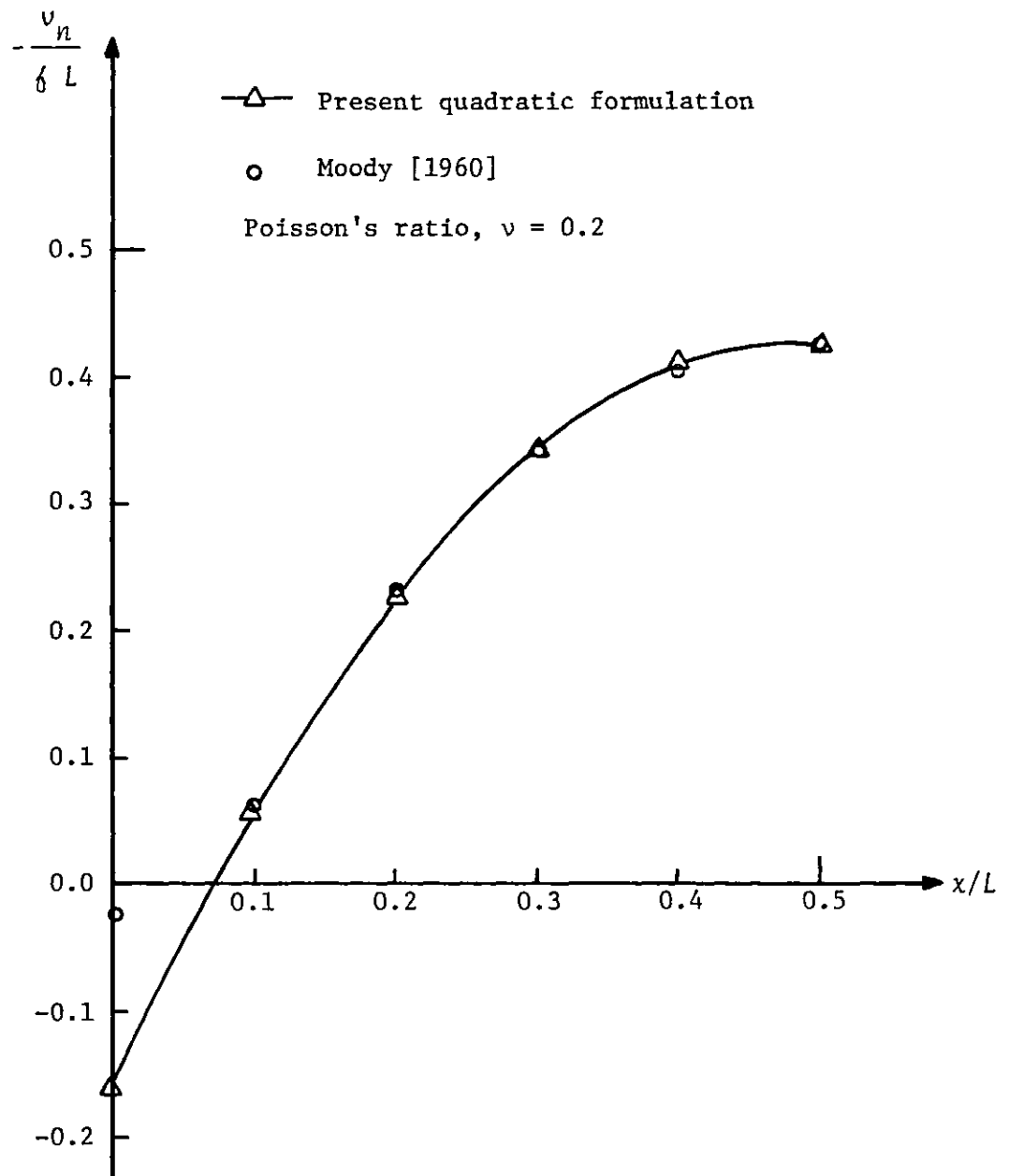


Figure 5.67: Effective shear force along OA in Figure 5.54 for a square plate with three edges clamped and the fourth edge free under a uniformly distributed load on two thirds of the plate

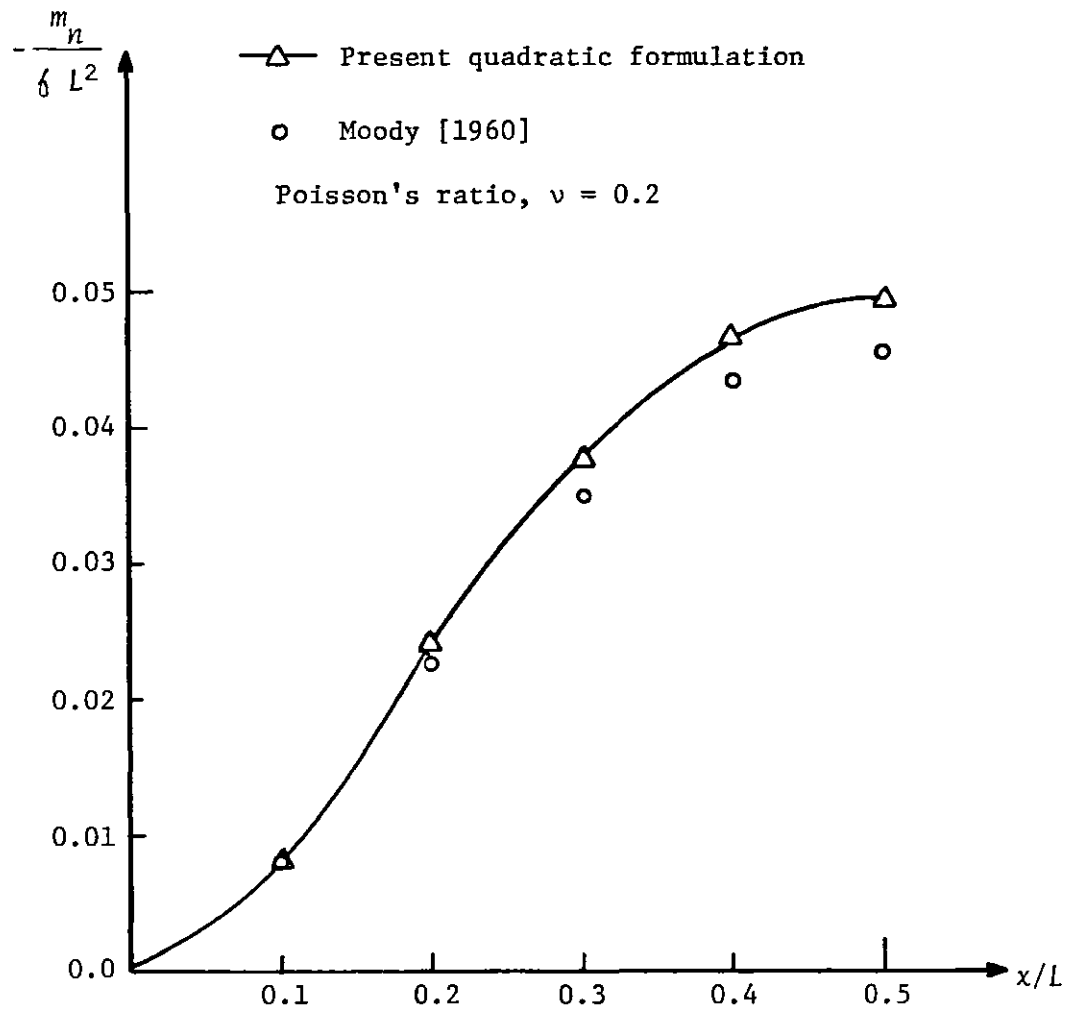


Figure 5.68: Normal moment along OA in Figure 5.54 for a square plate with three edges clamped and the fourth edge free with a uniformly distributed load on two thirds of the plate

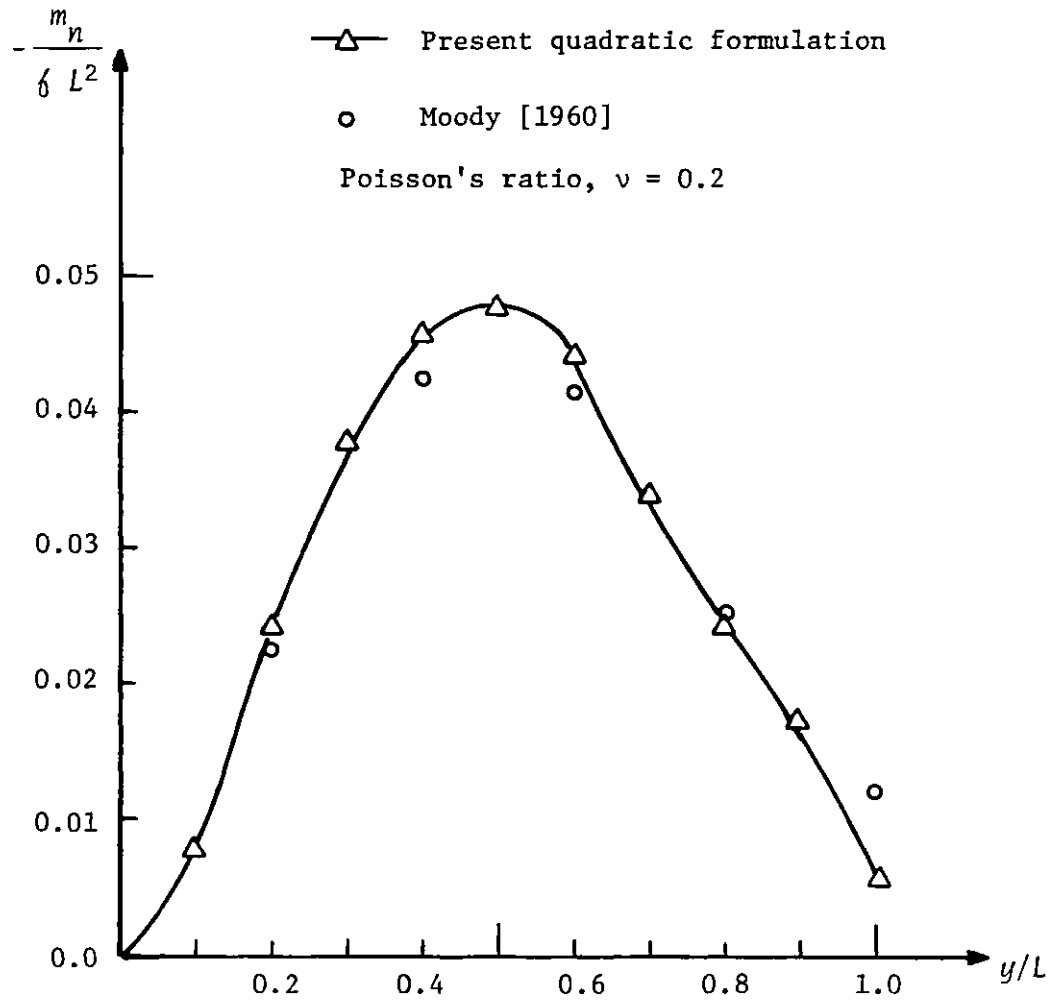


Figure 5.69: Normal moment along OB in Figure 5.54 for a square plate with three edges clamped and the fourth edge free under a uniformly distributed load on two thirds of the plate

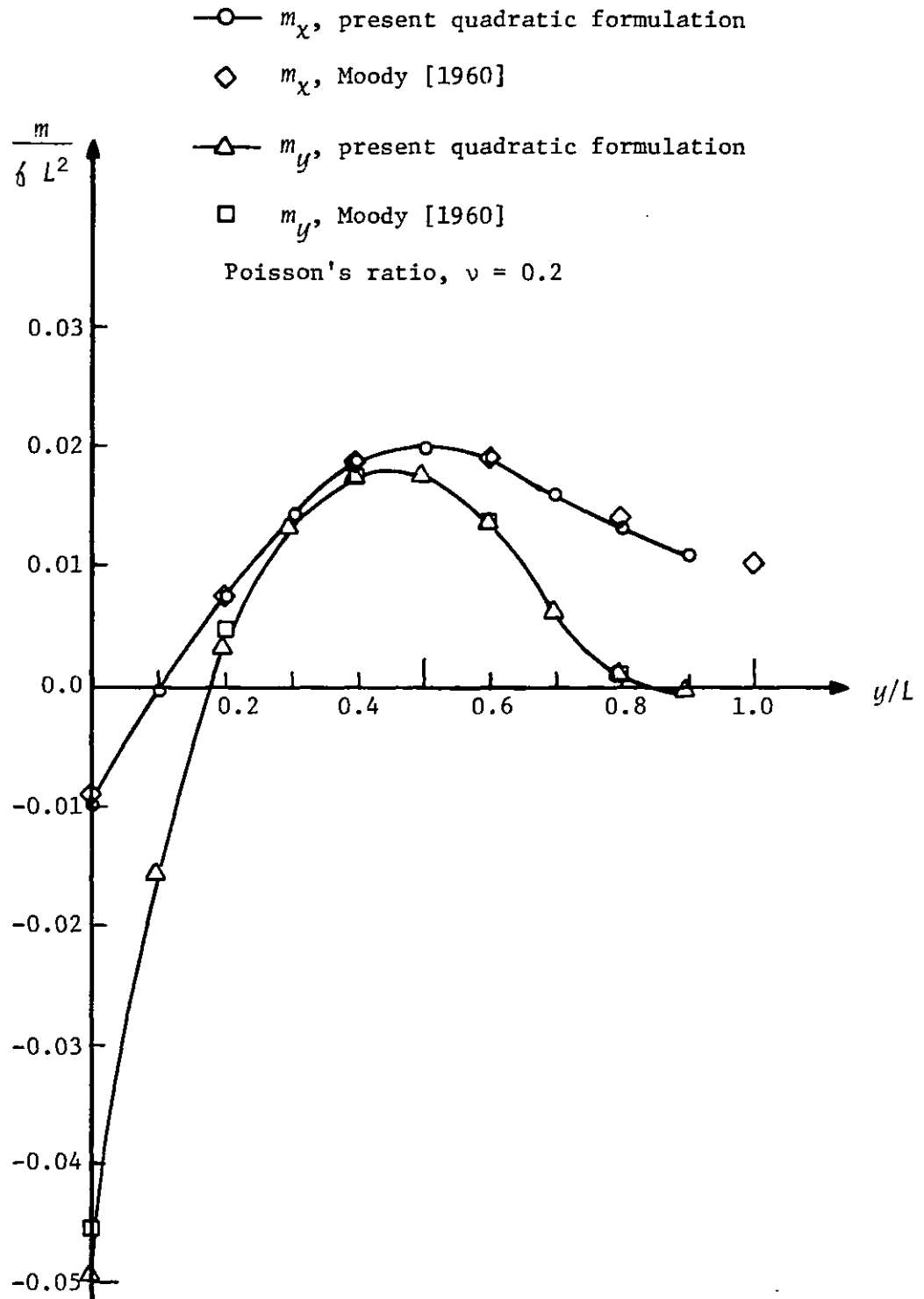
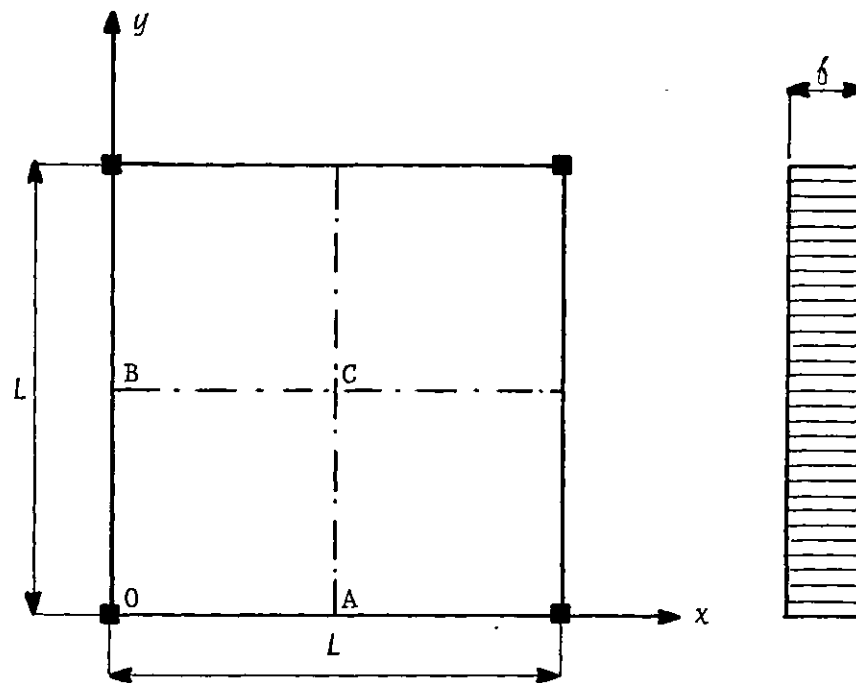
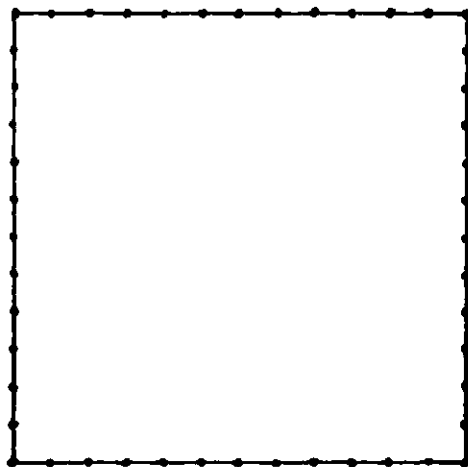


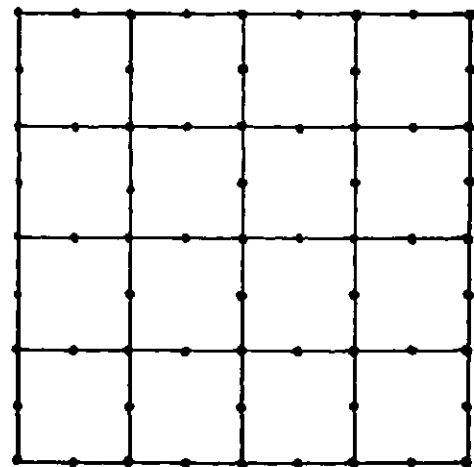
Figure 5.70: Moments, m_x and m_y , along the centreline (AC in Figure 5.54) for a square plate with three edges clamped and the fourth edge free under a uniformly distributed load on two thirds of the plate



(a) Plate geometry and loading



(b) Boundary discretisation.
6 elements along each
side.



(c) Interior discretisation

Figure 5.71: Square plate supported only at the corners under a uniformly distributed load

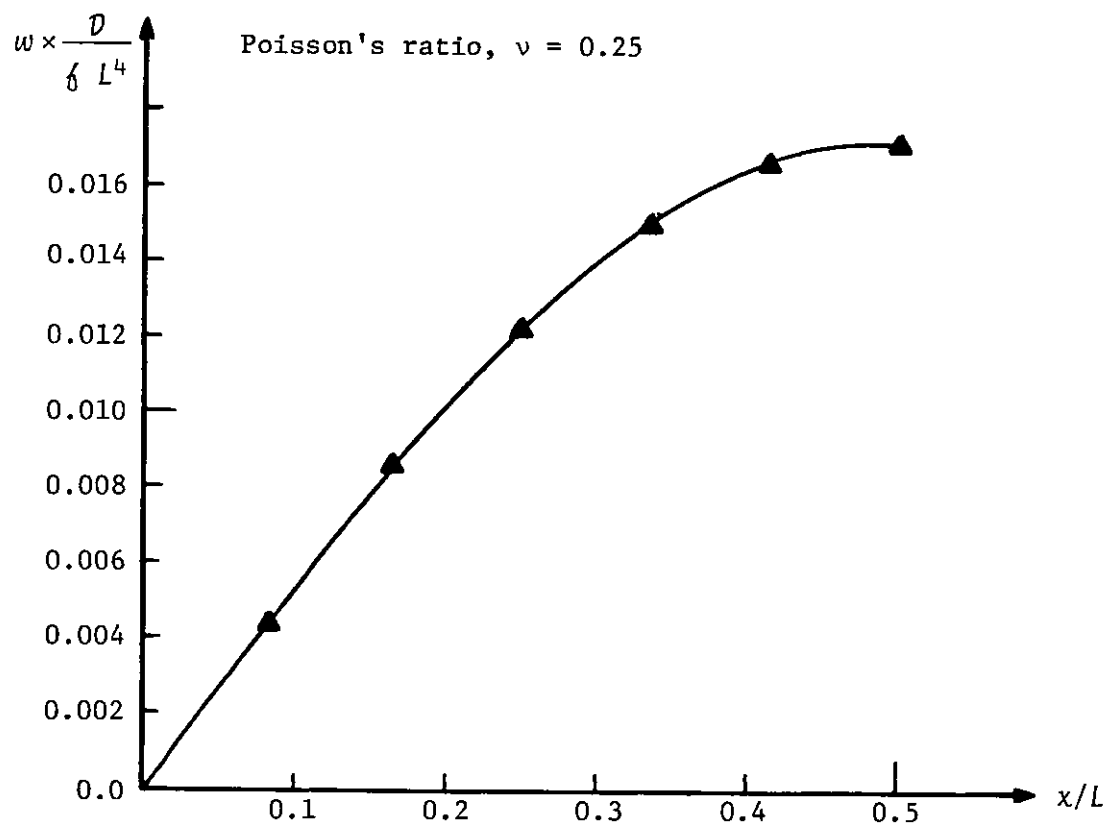


Figure 5.72: Deflection along half a side (OA in Figure 5.71) of a square plate supported only at its corners under a uniformly distributed load

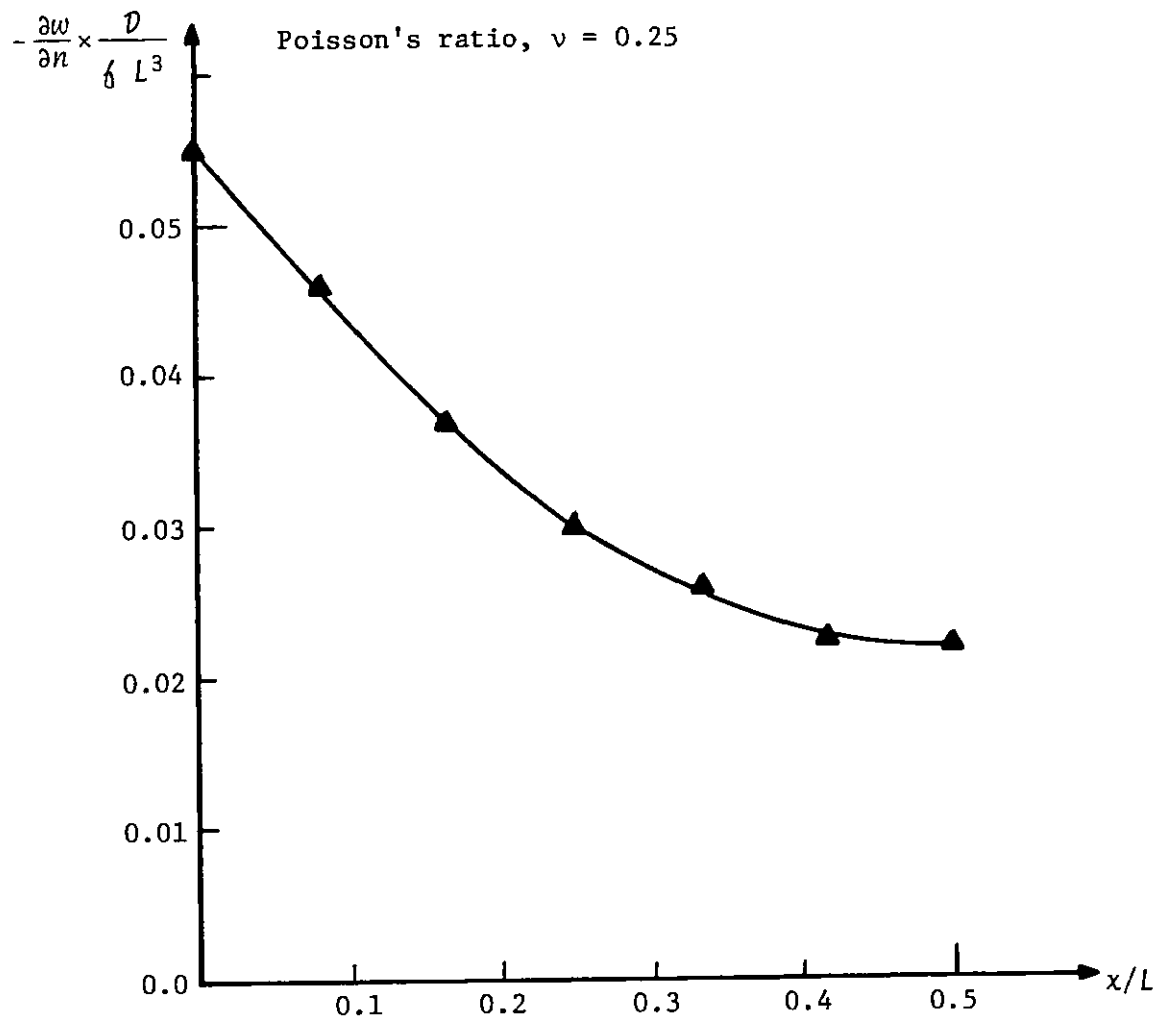
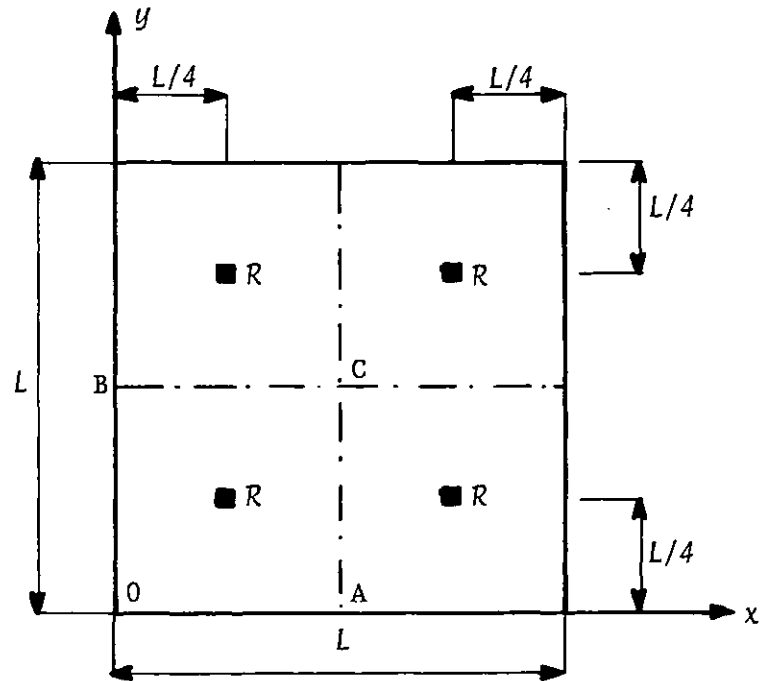
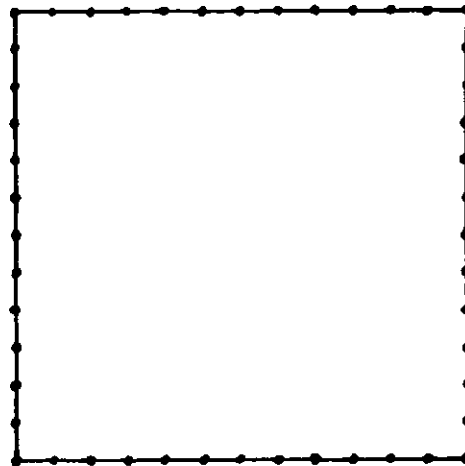


Figure 5.73: Normal slope along half a side (OA in Figure 5.71) of a square plate supported only at its corners under a uniformly distributed load



(a) Plate geometry and interior support positions



(b) Boundary discretisation.
6 elements along each
side.

Figure 5.74: Square plate on four interior supports under a centrally applied load, F

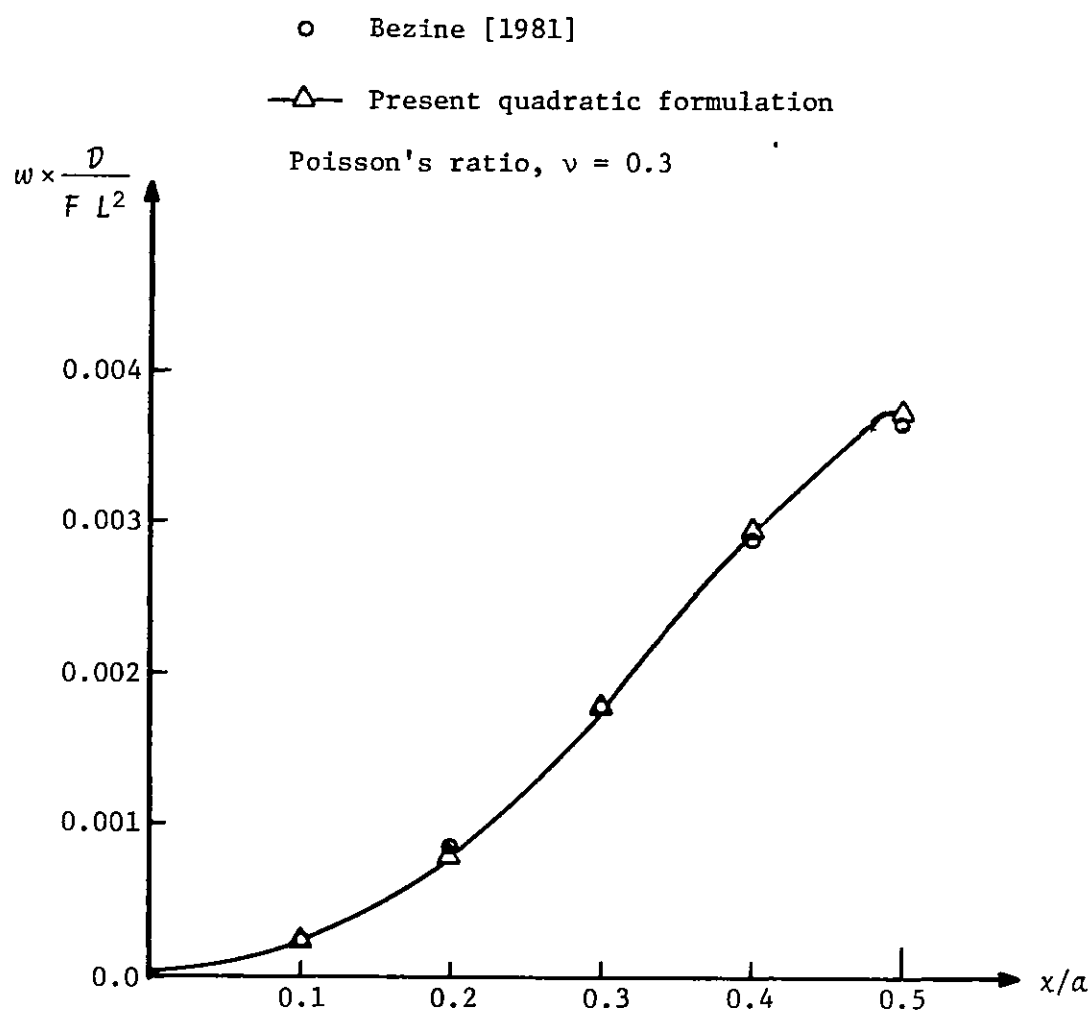


Figure 5.75: Deflection along a half axis of symmetry (BC in Figure 5.74) of a simply supported square plate on four interior supports under a centrally applied load, $\bar{F}$

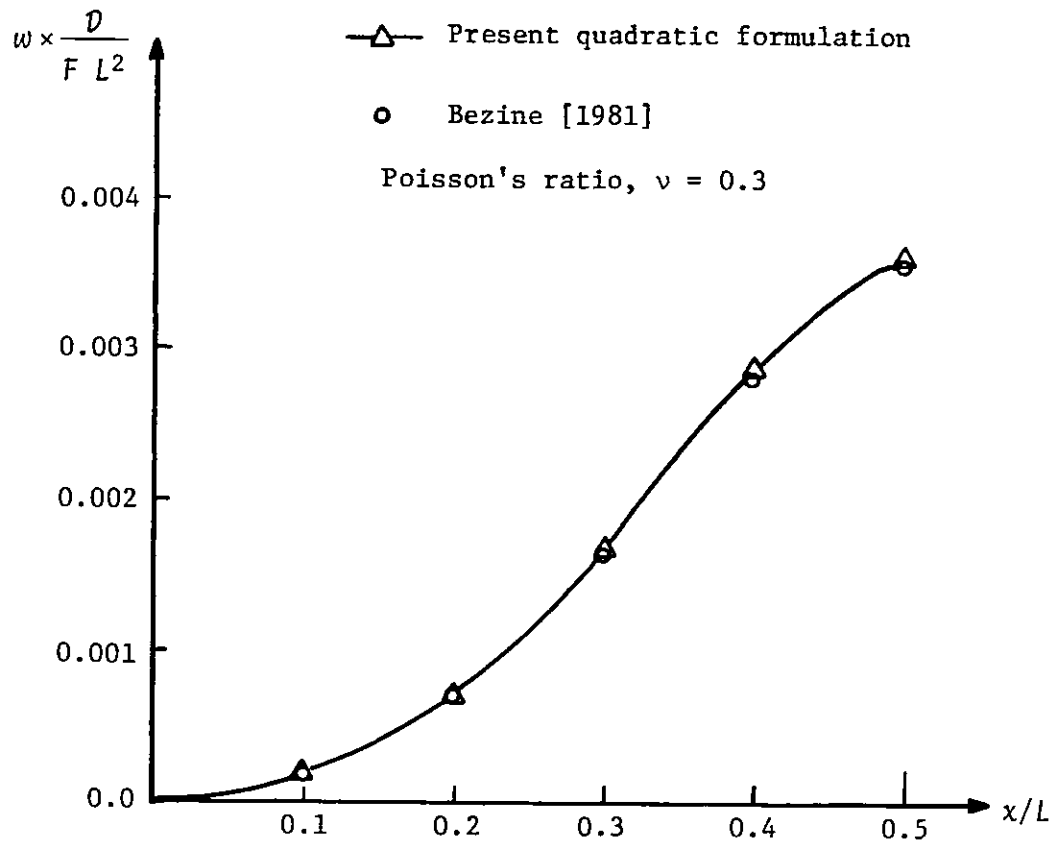


Figure 5.76: Deflection along a half axis of symmetry (BC in Figure 5.74) of a clamped square plate on four interior supports under a centrally applied load, F

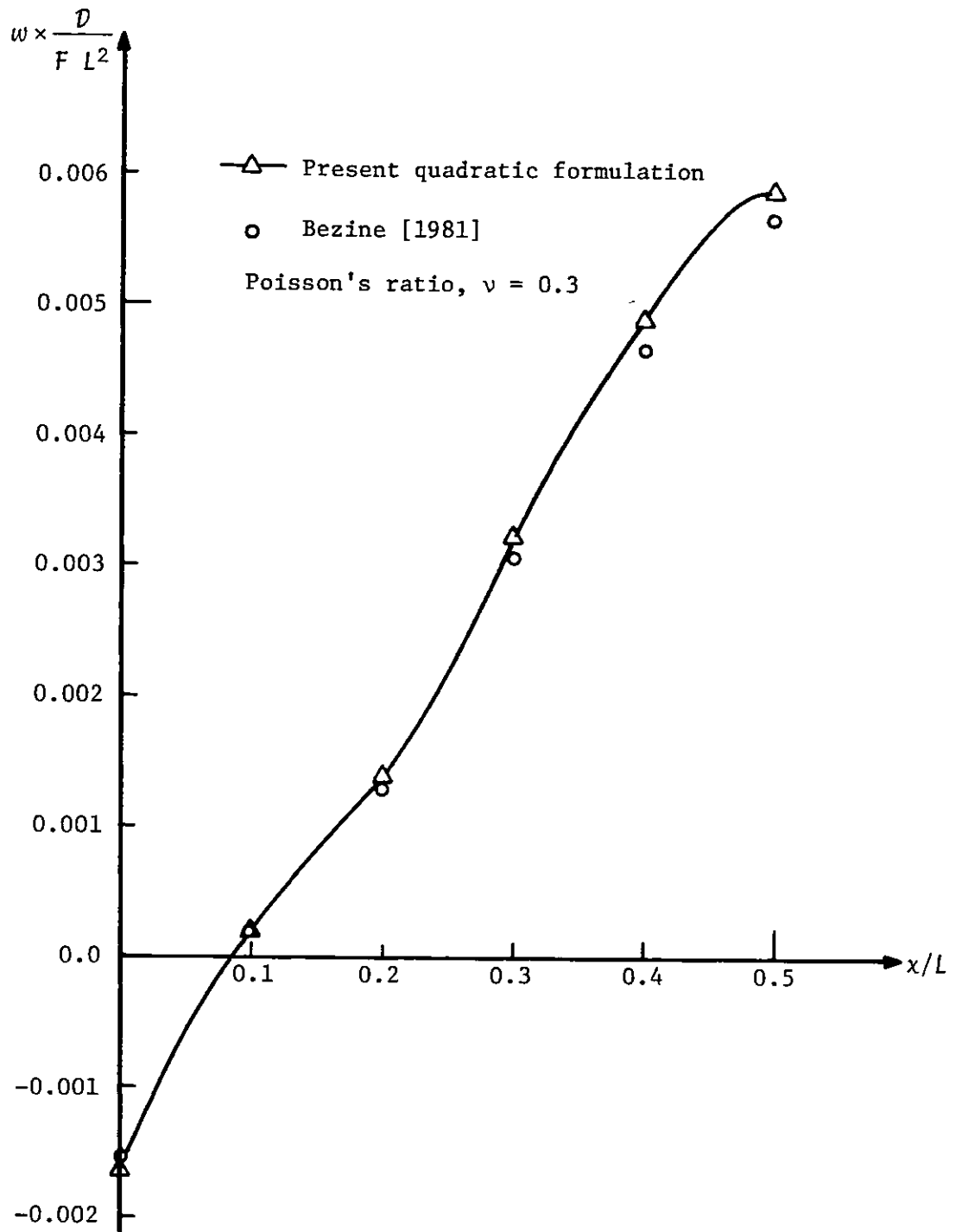


Figure 5.77: Deflection along a half axis of symmetry (BC in Figure 5.74) for a square plate on four interior supports under a centrally applied load, F . Plate boundaries free.

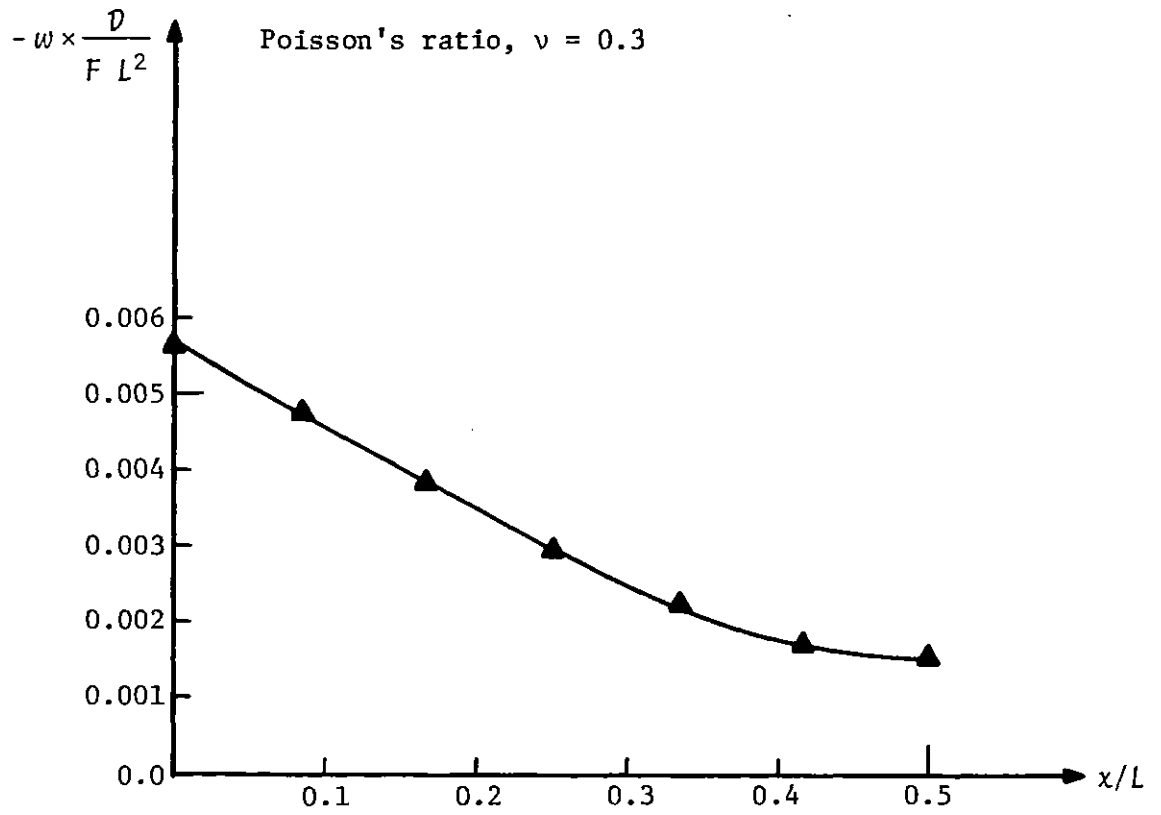


Figure 5.78: Deflection along half an edge (OA in Figure 5.74) of a square plate supported on four interior supports under a centrally applied load, F . All plate boundaries free.

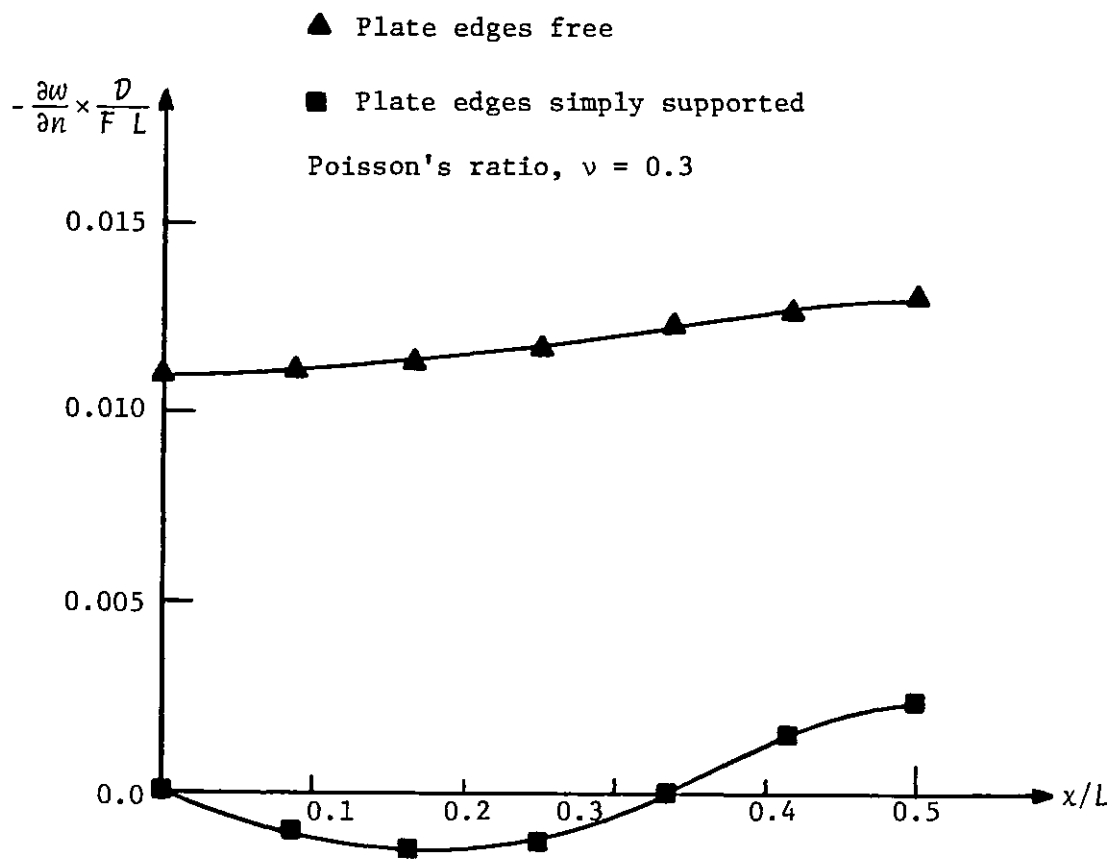


Figure 5.79: Normal slope along a half edge (OA in Figure 5.74) of a square plate supported on four interior supports under a centrally applied load, F

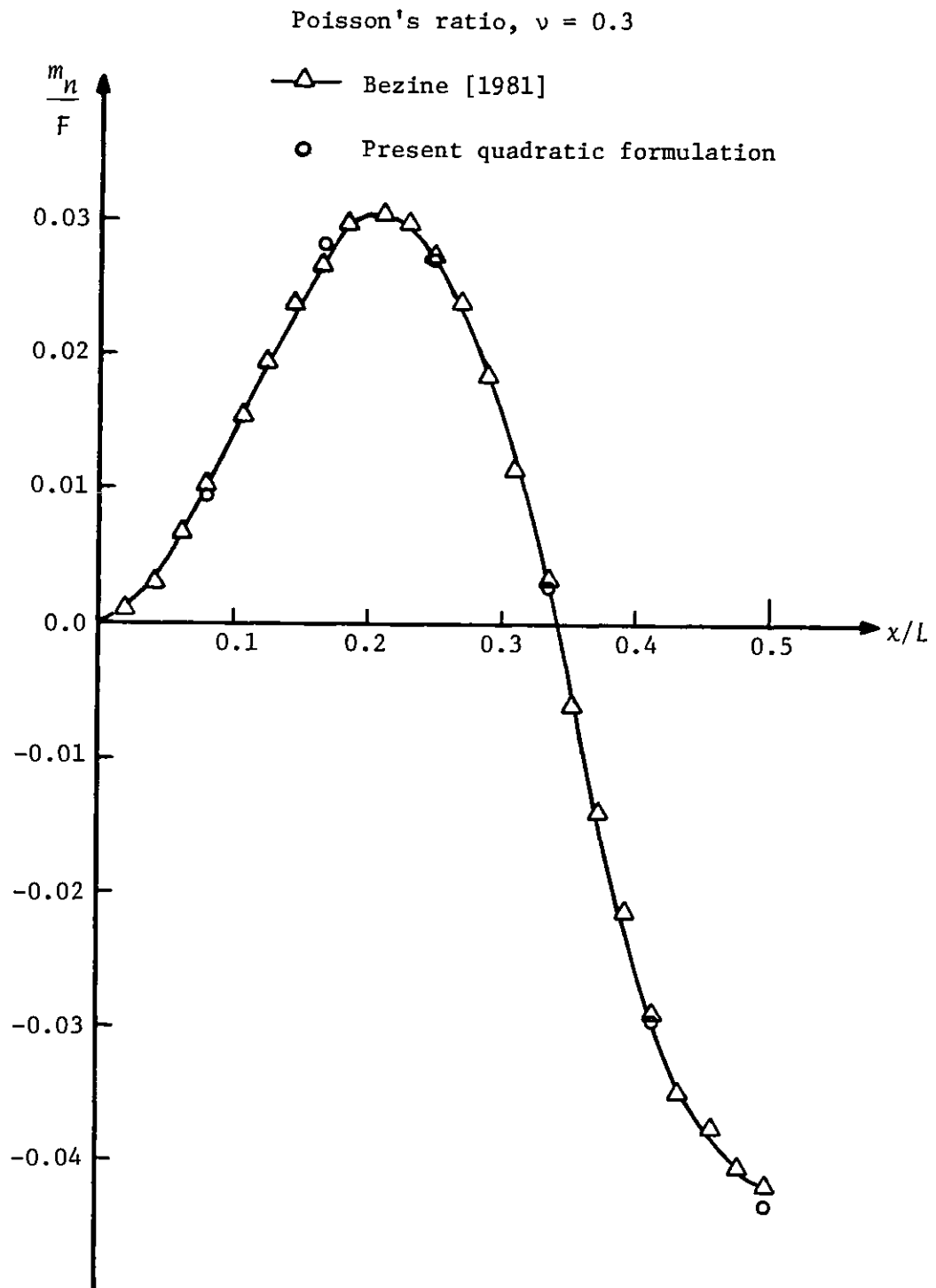


Figure 5.80: Normal moment along half an edge (OA in Figure 5.74) for a clamped square plate on four interior supports under a centrally applied load, F

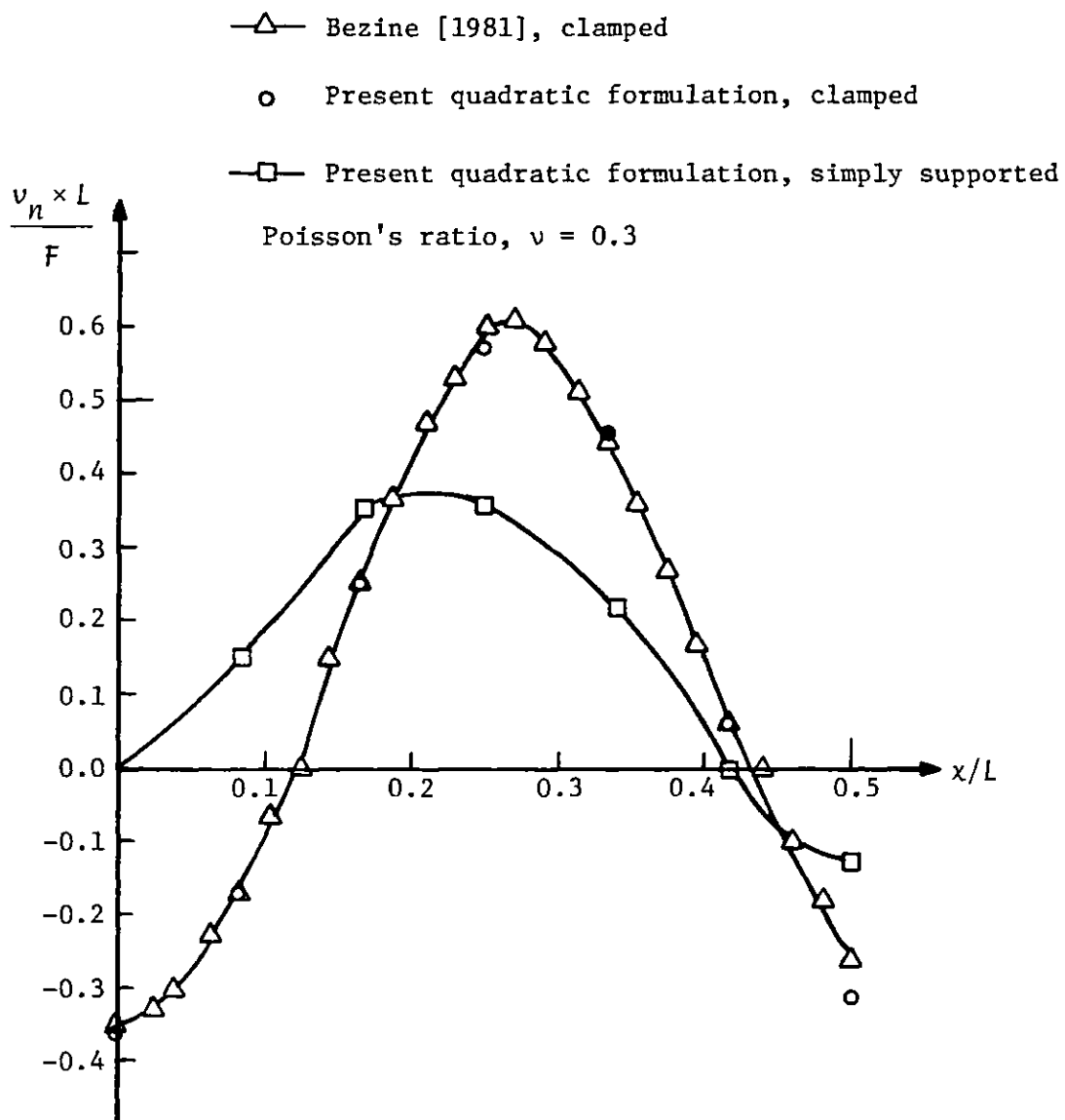
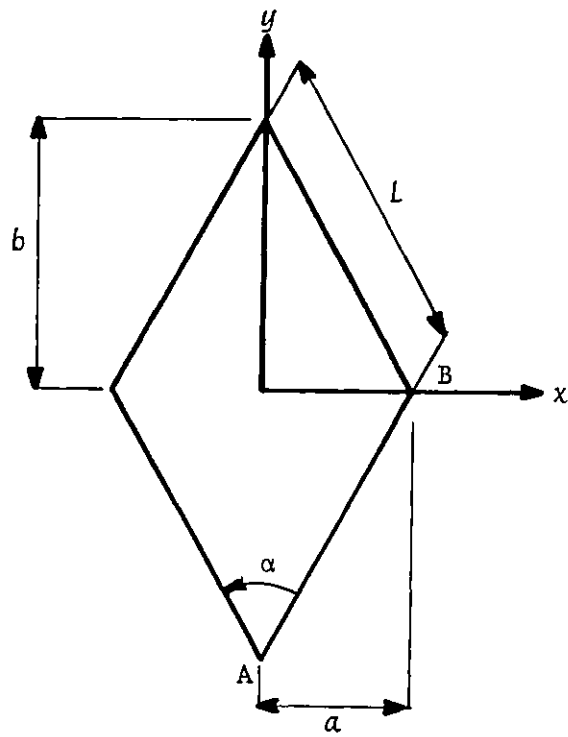
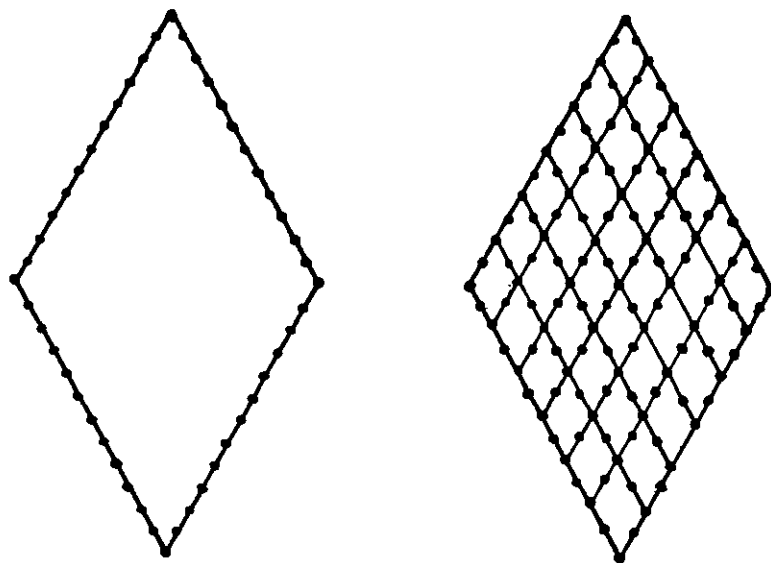


Figure 5.81: Effective shear force along half an edge (OA in Figure 5.74) of a square plate on four interior supports under a centrally applied load, F



(a) Plate geometry



(b) Boundary discretisation. (c) Interior discretisation
6 elements along each side.

Figure 5.82: Simply supported rhombic plate under a uniformly distributed load

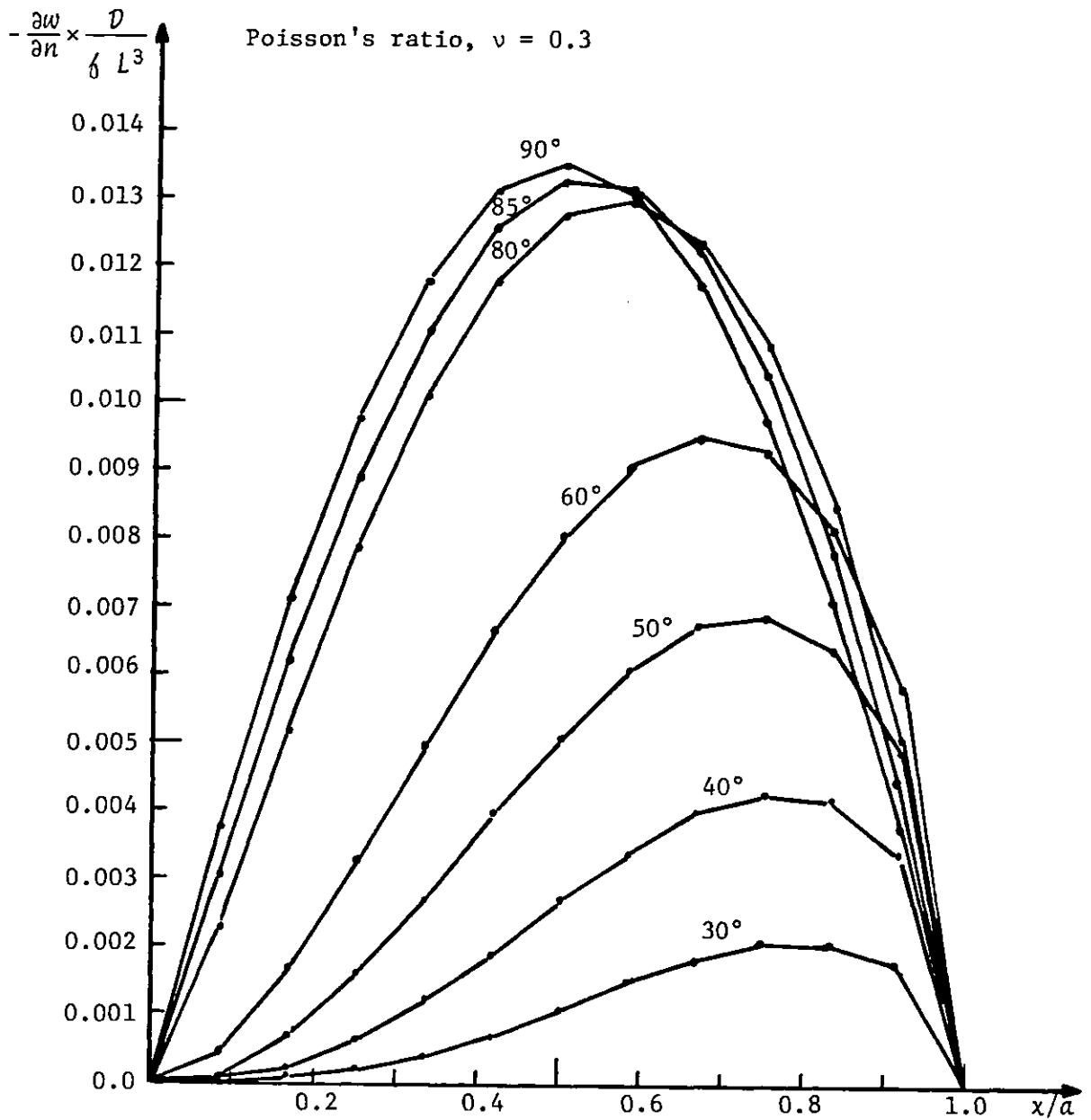


Figure 5.83: Normal slope along AB in Figure 5.82 for a simply supported rhombic plate under a uniformly distributed load

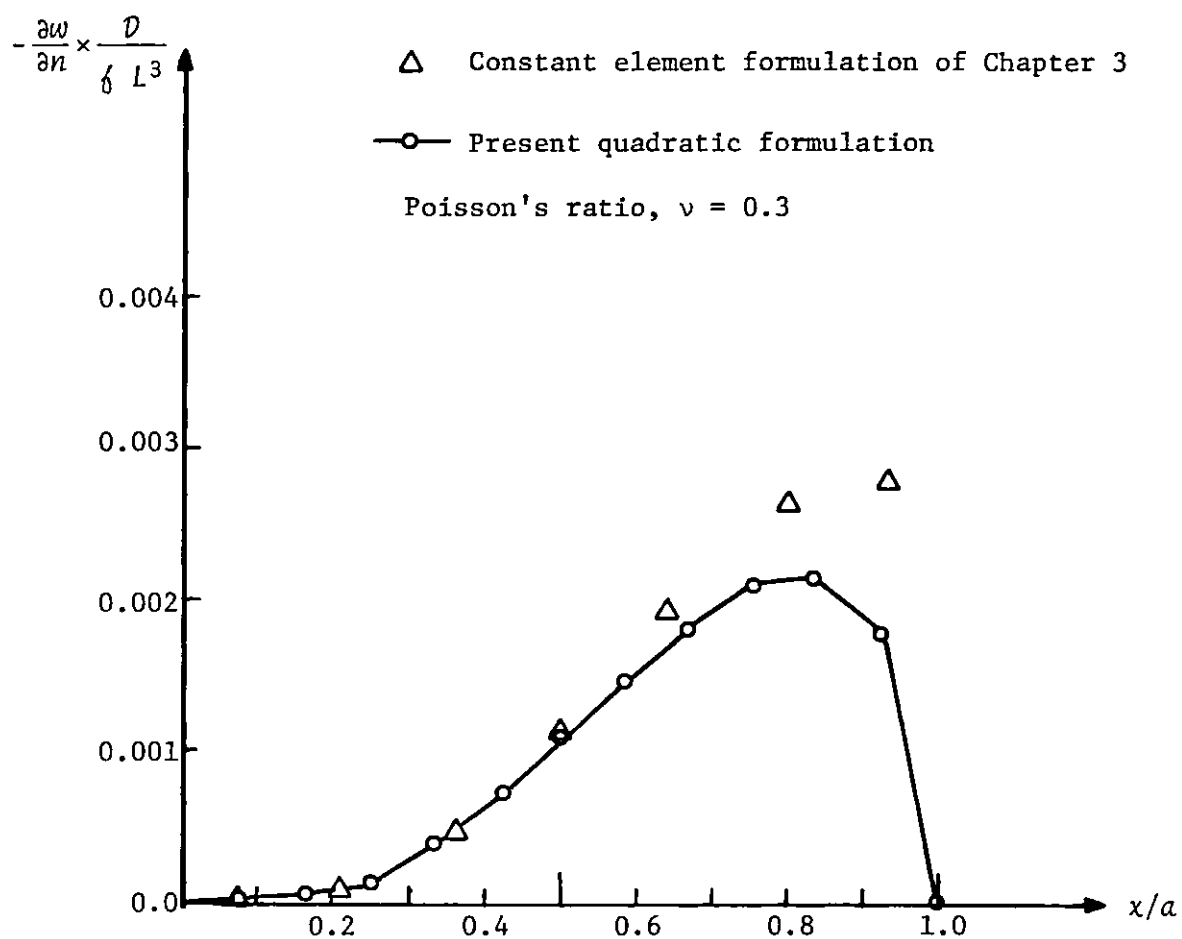
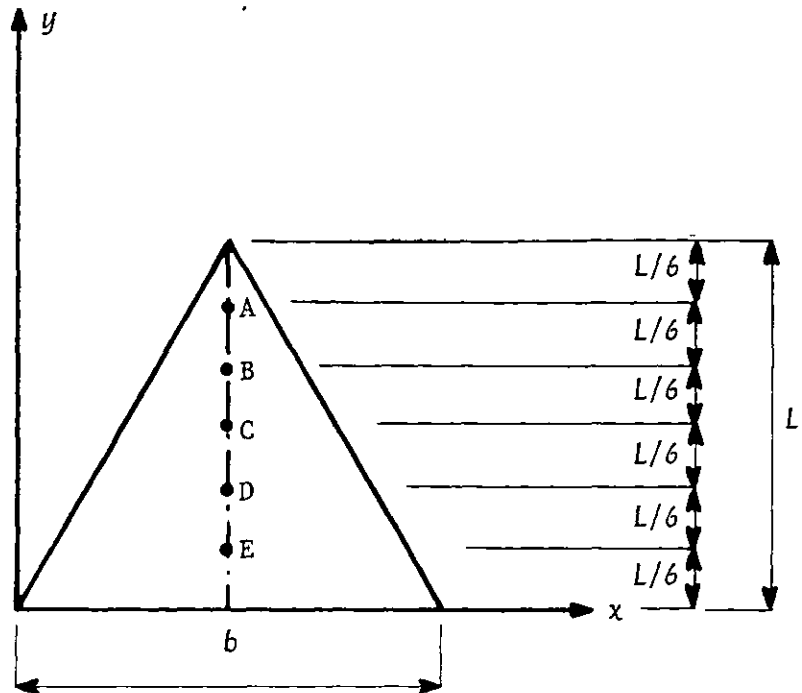
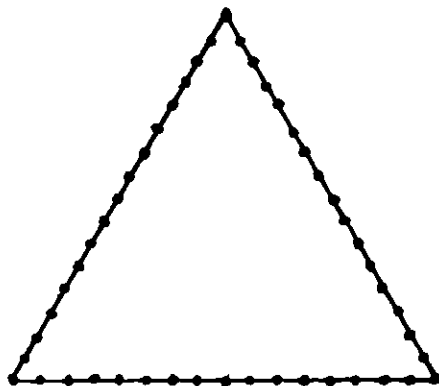


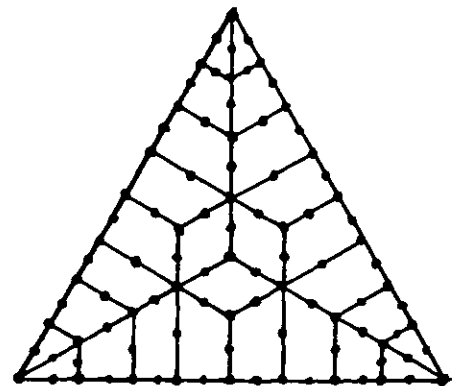
Figure 5.84: Normal slope along AB in Figure 5.82 for a simply supported rhombic plate with 30° angle of skew under a uniformly distributed load



(a) Plate geometry

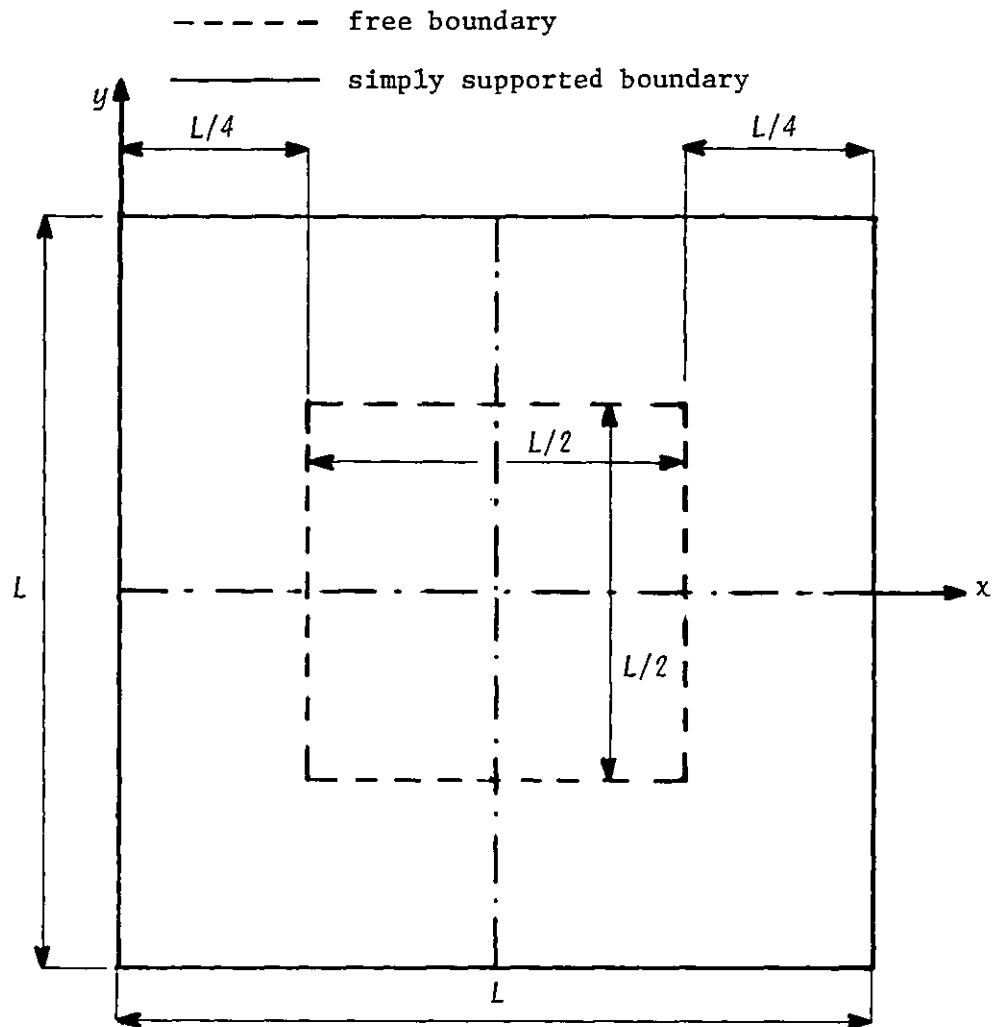


(b) Boundary discretisation.
8 elements along each
side.

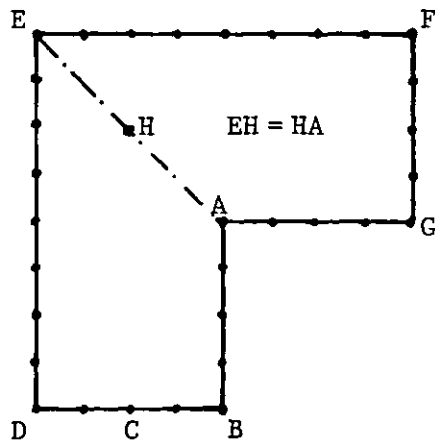


(c) Interior discretisation

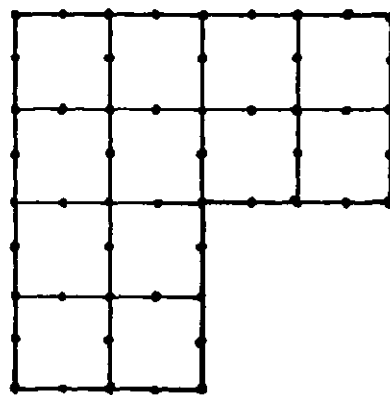
Figure 5.85: Simply supported plate in the shape of an equilateral triangle



(a) Plate geometry and boundary conditions



(b) Boundary discretisation for a quarter of the plate



(c) Interior discretisation for a quarter of the plate

Figure 5.86: Square plate with a square central opening under a uniformly distributed load. Outer boundary simply supported and inner boundary free.

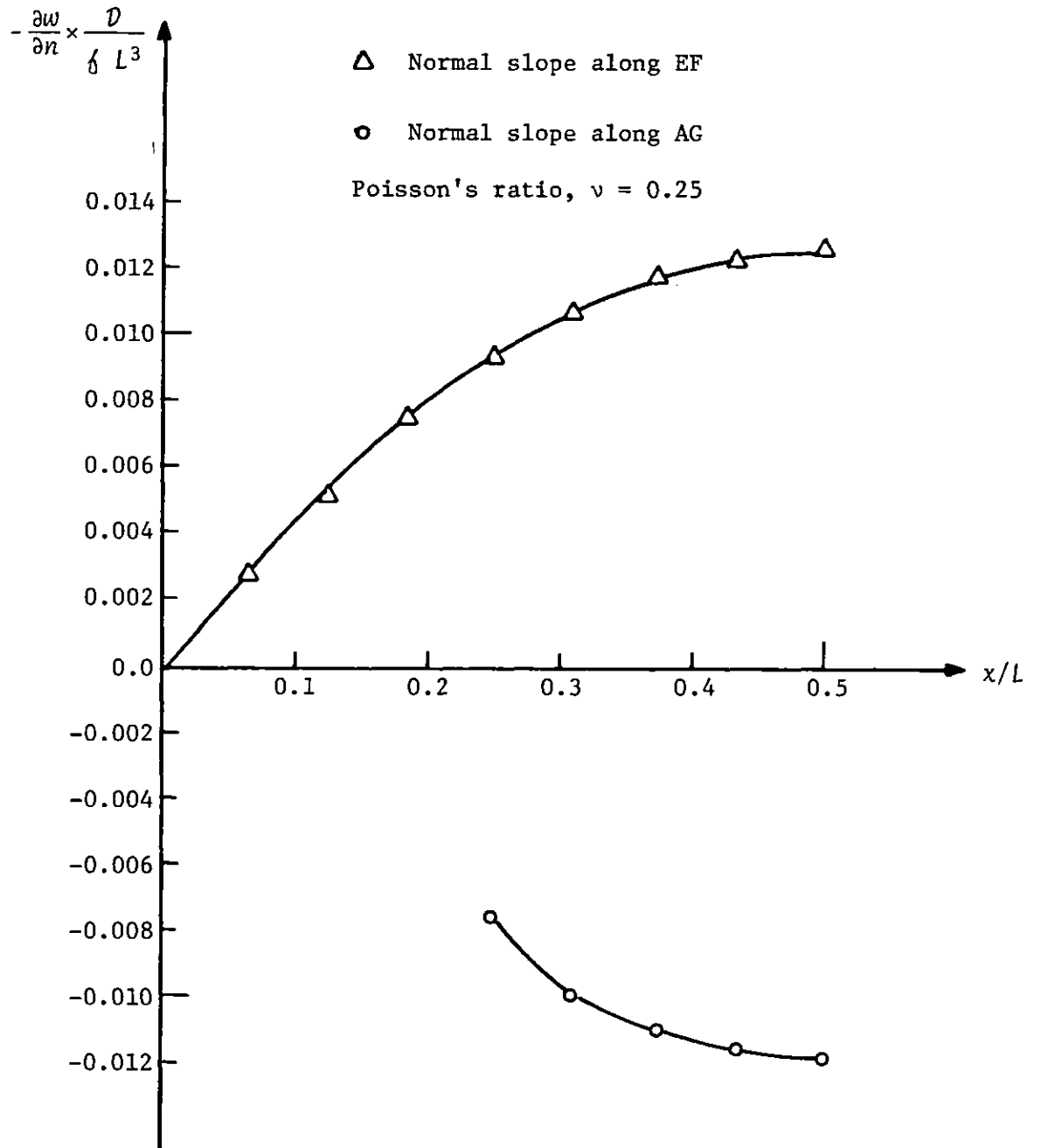


Figure 5.87: Normal slope along EF and AG in Figure 5.86(b) for a square plate with a square central opening under a uniformly distributed load. Simply supported on the outer boundary and free along the inner boundary.

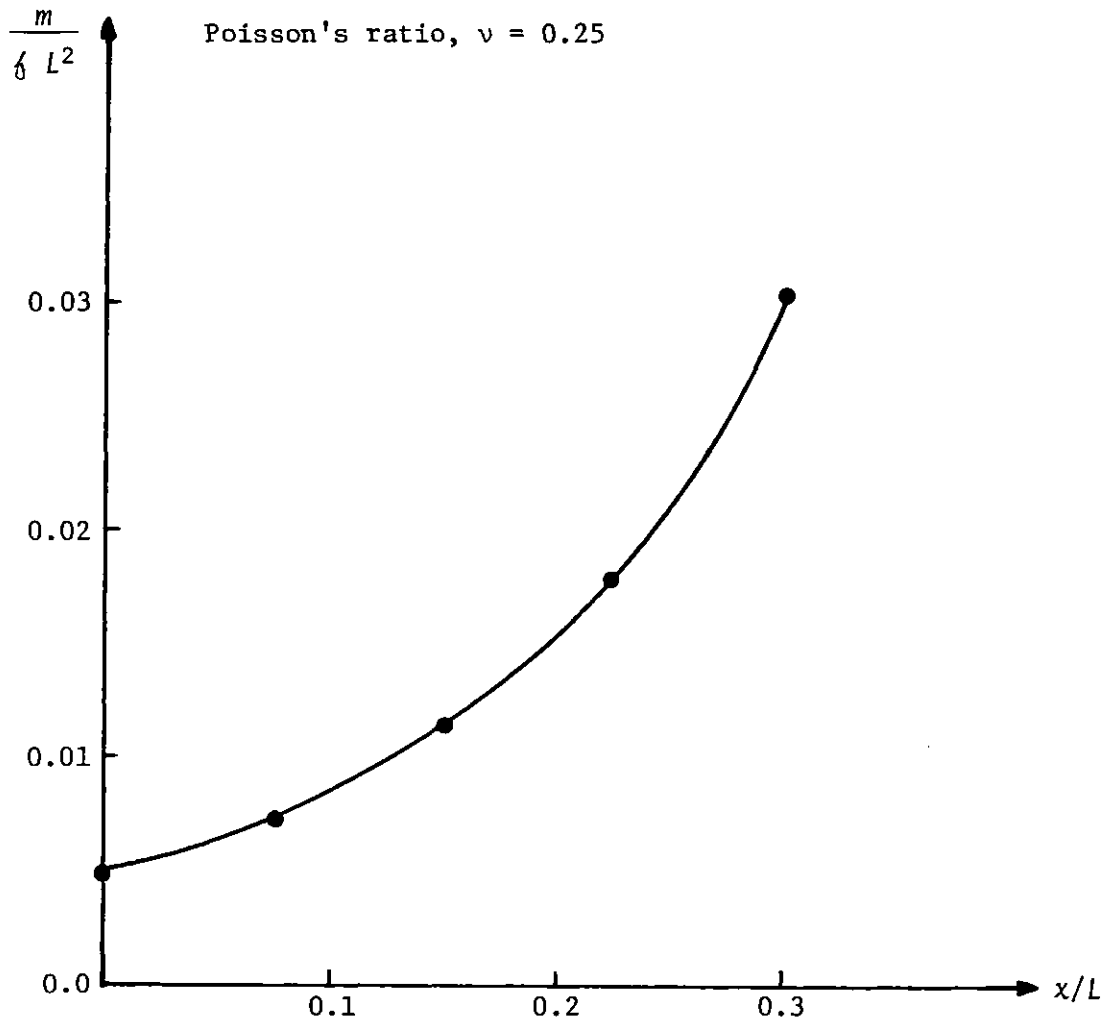


Figure 5.88: Normal moment along DB in Figure 5.86(b) for a square plate with a square central opening under a uniformly distributed load. Simply supported on the outer boundary and free along the inner boundary.

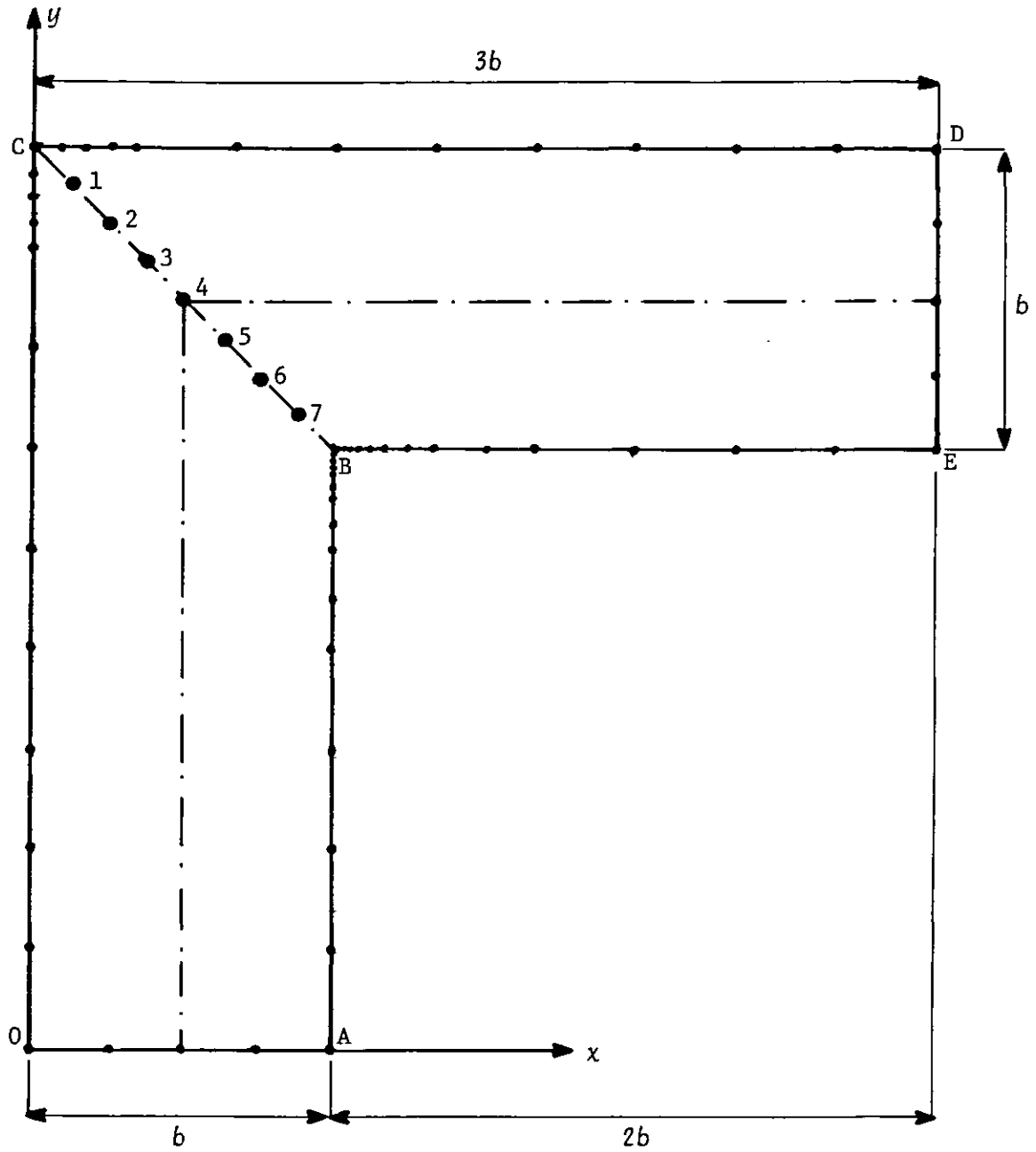


Figure 5.89: Geometry and boundary discretisation for a simply supported right angled corner plate under a uniformly distributed load. 30 elements used in total.

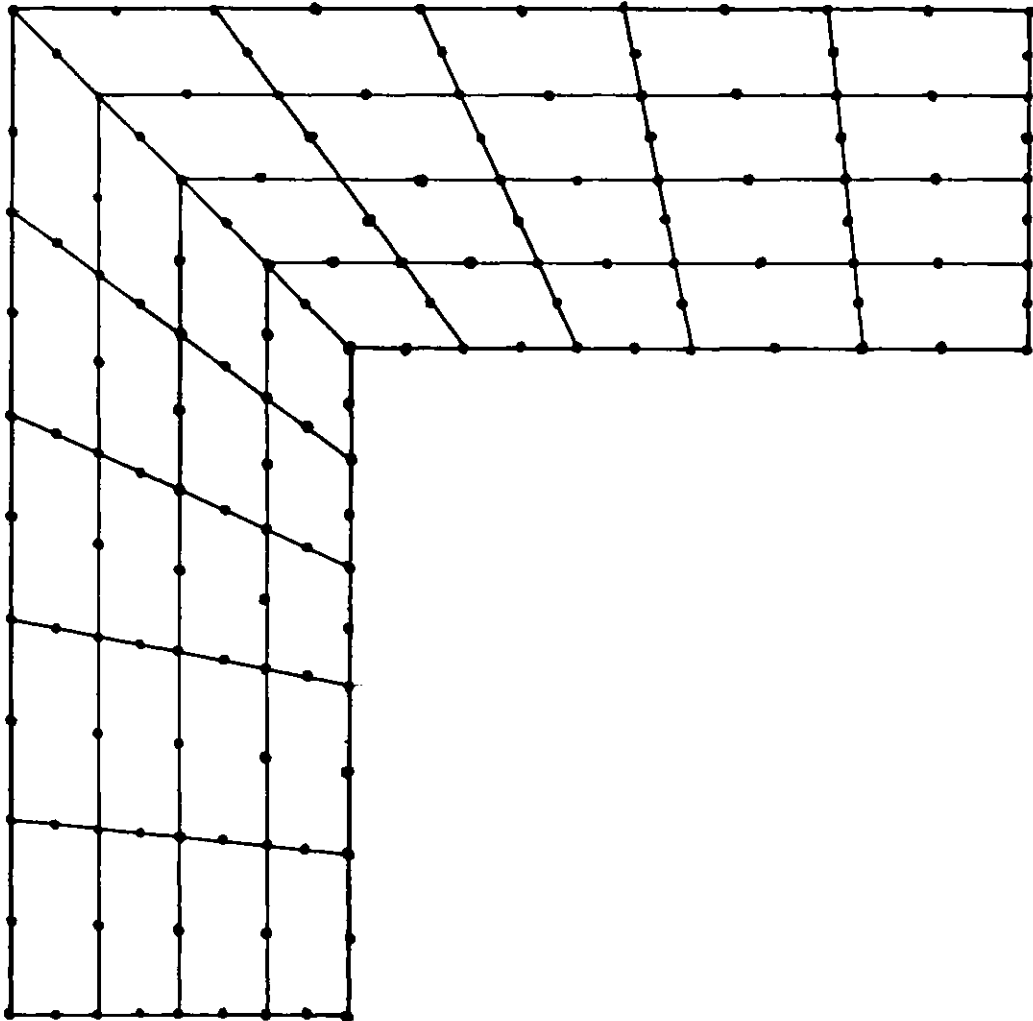


Figure 5.90: Interior discretisation of a right angled corner plate (see Figure 5.89)

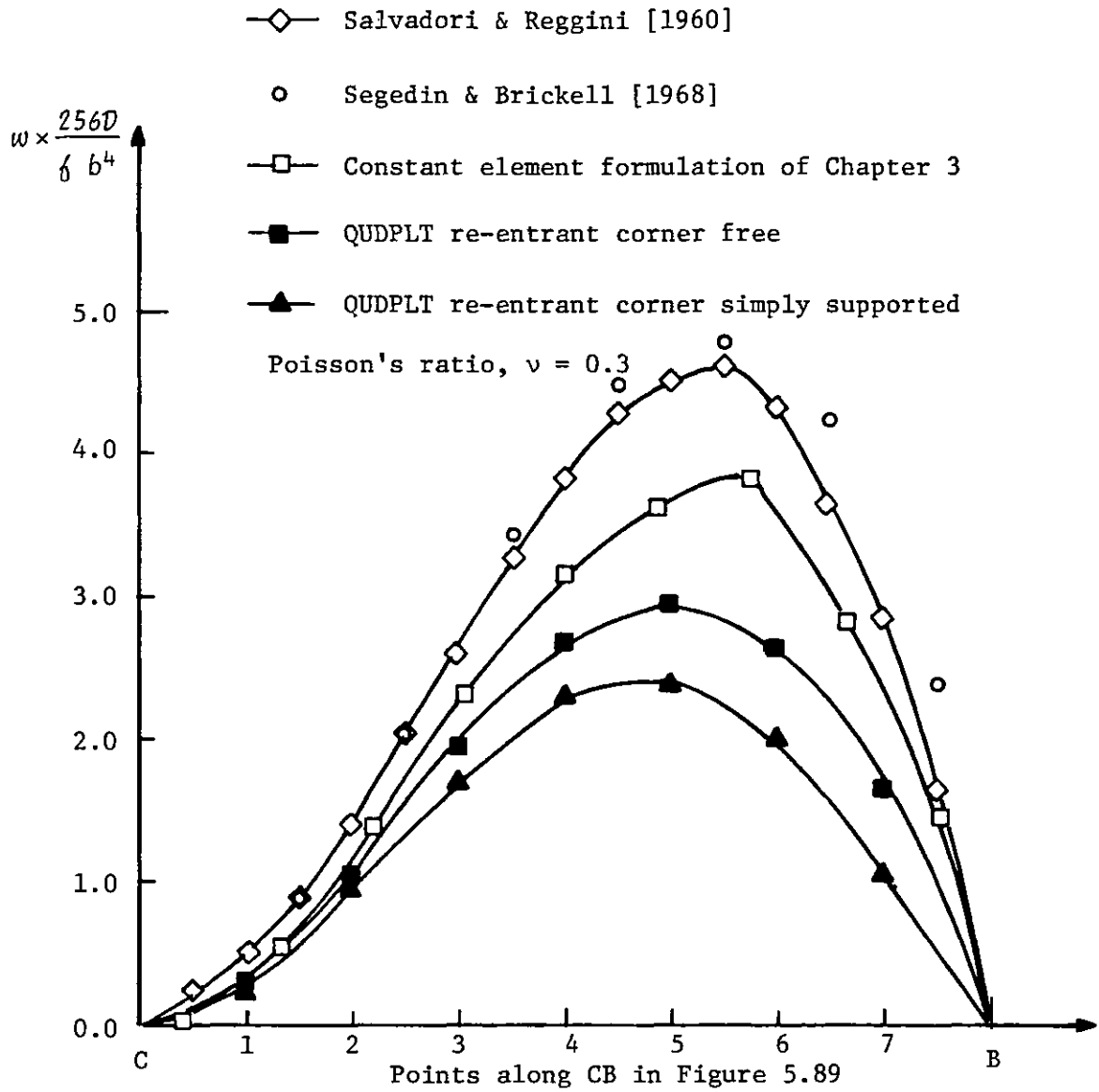


Figure 5.91: Deflection along the diagonal CB in Figure 5.89 for a simply supported right angled corner plate under a uniformly distributed load

QUDPLT : Present quadratic formulation

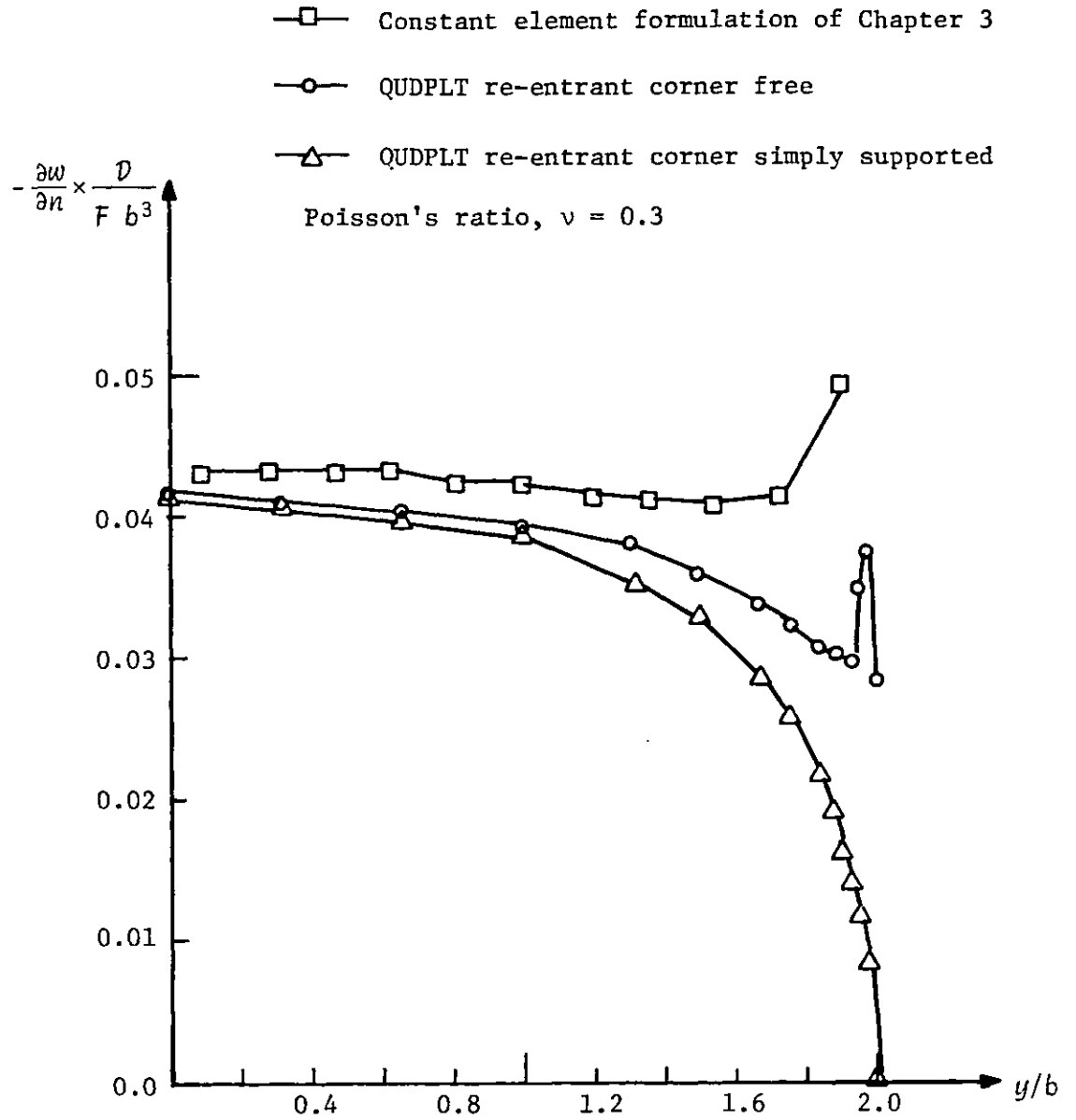


Figure 5.92: Normal slope along AB in Figure 5.89 for a simply supported corner plate under a uniformly distributed load

QUDPLT : Present quadratic formulation

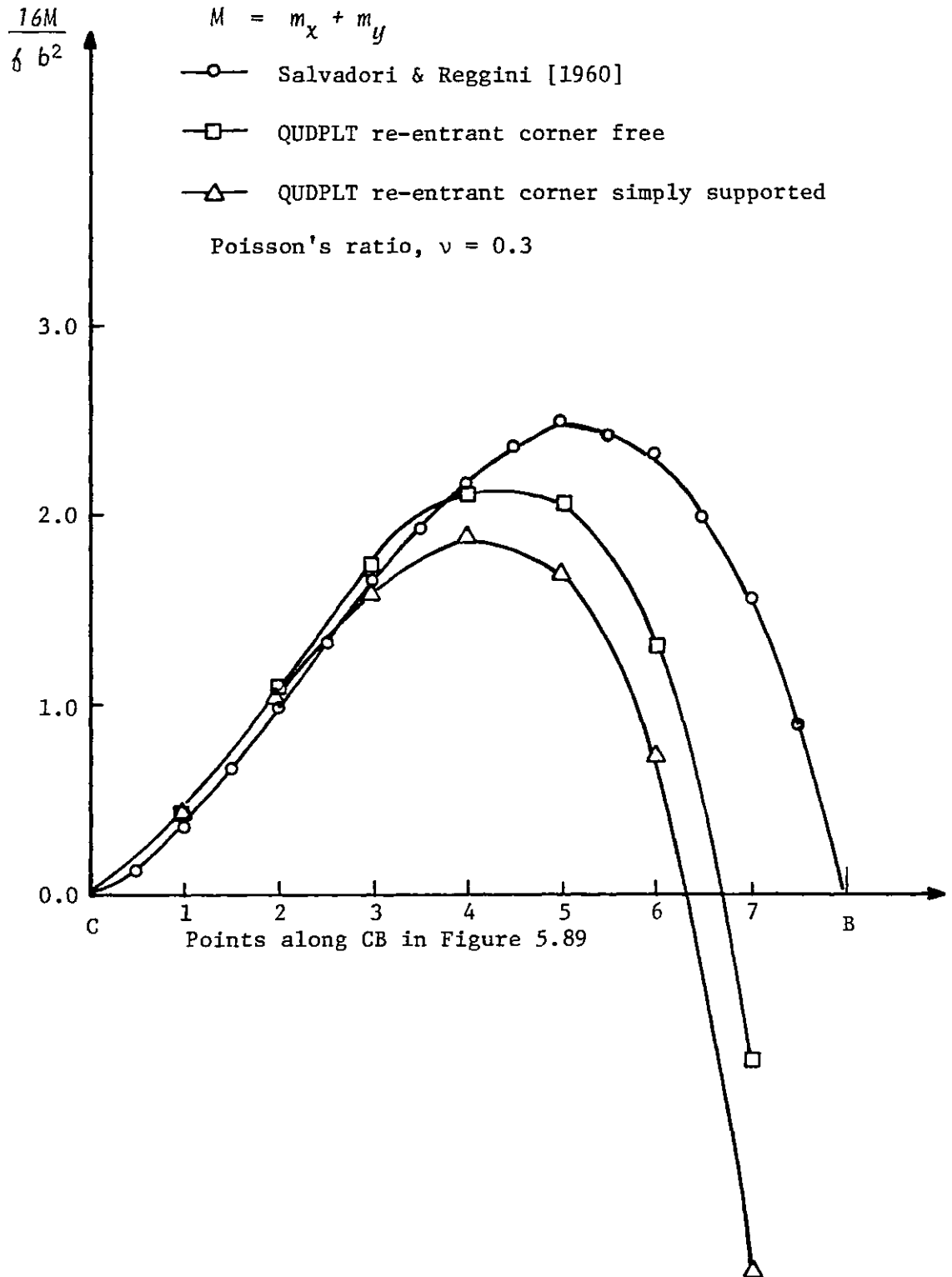


Figure 5.93: Moment sum, $(m_x + m_y) = M$, along the diagonal CB in Figure 5.89 for a simply supported right angled corner plate under a uniformly distributed load

QUDPLT : Present quadratic formulation

CHAPTER 6
NUMERICAL RESULTS FOR PLATES
WITH CURVED BOUNDARIES

6.1 INTRODUCTION

The plate examples presented in the previous chapter consisted of plates with geometries which involved straight boundaries only. The isoparametric elements described in Chapter 4 can model such boundaries exactly. However, when the plate geometry involves curved boundaries, the geometrical interpolations, in general, approximate the geometry. It would be expected, however, that the approximation would improve with finer discretisation. In this chapter, an attempt has been made to assess the accuracy of the present quadratic formulation when it is used to solve plate problems involving curved boundaries.

6.2 CIRCULAR PLATE UNDER A CONCENTRATED LOAD AT THE CENTRE

Figure 6.1(a) shows a circular plate which is assumed to be carrying a concentrated load, F , at its centre. The plate boundary is assumed to be either simply supported or clamped. A typical boundary discretisation is shown in Figure 6.1(b).

Table 6.1 gives the computed values of the central deflections for the two boundary conditions considered. The exact solutions are taken from the analytical expressions given by Timoshenko & Woinowsky-Krieger [1959]. It is seen that the agreement between the BIE solution and the analytical one is very good, even for relatively coarse meshes. Tables 6.2 and 6.3 give values of deflections and moments at interior points, along a radial direction, for simply supported and clamped plates, respectively. It is seen that the agreement between the BIE solution and the analytical solution is excellent.

It must be noted that a quadratic interpolation of the boundary geometry does not model a circle exactly. Specifically, the normal to the boundary is not continuous at inter-element nodes. This is a direct result of the isoparametric formulation described in Chapter 4. However, refining the mesh does help to decrease the discontinuity. For example, if the boundary of the circle is discretised using 8 elements, there is an angle of 1.612° between the normals to the elements either side of an inter-element node. However, this angle is reduced to 0.0637° if the boundary is discretised using 24 elements. Although in the numerical implementation employed, the normal direction at an inter-element node is taken to be the average between the normals on the elements either side of that node, nevertheless this incompatibility creates artificial corners in the boundary geometry. The effect of this incompatibility on the computed boundary values is shown in Tables 6.4 and 6.5. It is seen that the computed values of the boundary variables at the mid-side nodes are different from those obtained at inter-element nodes, whereas, for this particular problem, all boundary variables are expected to be constant along the boundary. Nevertheless, it is seen that refining the mesh reduces the difference between the computed nodal values. The effective shear force, for the plate with simply supported boundaries, is most adversely affected. The author believes that this is due to the fact that the artificial corners require the existence of a Kirchhoff corner force. It was shown in Chapter 3 that ignoring the corner forces in the constant element formulation would result in incorrect values of the equivalent shear force near corners for problems involving simply supported boundaries. For a simply supported circular plate, there are no corner forces to be considered. Therefore, the corner forces which might be present at the artificial corners are not computed as independent boundary variables. It is believed that the effect of these corner forces is

reflected in the value of the equivalent shear force at the inter-element nodes, leading to the discrepancy seen between the equivalent shear force at mid-side and end nodes of the elements. It would appear that a more accurate figure for the effective shear force along the boundary can be obtained as the average between two mid-side node values and an inter-element nodal value. This is equivalent to averaging the effective shear force along an element by using the appropriate quadratic shape functions. The effective shear force along the boundary of the clamped circular plate seems to be much less affected by the incompatibility between adjacent elements. It is believed that this is due to the fact that at a clamped corner, the Kirchhoff corner force is zero and, therefore, the vertical forces along the boundary would not be adversely affected by the existence of the artificial corners.

6.3 CIRCULAR PLATE UNDER A UNIFORMLY DISTRIBUTED LOAD

Figure 6.3 shows the geometry, loading and the boundary and interior discretisations employed for this particular problem. Two different boundary conditions are considered; simply supported and clamped. The convergence of the computed value of the central deflection to the analytical solution given by Timoshenko & Woinowsky-Krieger [1959] is shown in Table 6.6. It is seen that there is very good agreement between the BIE and analytical solutions. Tables 6.7 and 6.8 compare the computed values of deflection and moments, at interior points along a radius, with those obtained by the analytical solution.

The solutions obtained for the boundary variables are shown in Tables 6.9 and 6.10. Once again, as in Section 6.2, it is seen that there are discrepancies between the computed values of the variables at mid-nodes and end nodes of the elements. However, this difference is reduced with mesh refinement. The effective shear force, for the simply supported

case, is once again the worst affected. As mentioned in the previous section, this is due to the way in which the circular boundary is modelled by the quadratic elements. This problem has already been discussed in Section 6.1, to which the reader is referred.

6.4 SIMPLY SUPPORTED CIRCULAR PLATE BENT BY A UNIFORMLY DISTRIBUTED MOMENT ALONG ITS BOUNDARY

The geometry and boundary discretisation for this problem are shown in Figure 6.3. Timoshenko & Woinowsky-Krieger [1959] give an analytical solution for this problem. The results obtained using the present quadratic formulation are compared with this solution in Table 6.11. It is seen that the agreement is very good. The figure given for normal slope at the boundary is the average between the normal slope at a mid-node and an end node of an element, which differed from each other by only 0.11%.

6.5 SIMPLY SUPPORTED CIRCULAR PLATE UNDER A LINEARLY VARYING LOAD

Figure 6.4(a) shows the geometry and loading for this problem. The boundary and interior discretisations are shown in Figures 6.4(b) and 6.4(c), respectively. An analytical solution is given by Timoshenko & Woinowsky-Krieger [1959]. Since the values of the computed variables are symmetric about the x axis and anti-symmetric about the y axis, the results presented are shown for a quarter of the plate only. Tables 6.12 to 6.18 present the computed values of deflections and moments along radial lines, making an angle θ with the x axis. Results are given for different values of θ . It is seen that there is a very good agreement between the BIE and the analytical solutions. Table 6.19 gives the variation of the normal slope along a quarter of the length of the boundary, showing again very good agreement.

6.6 PLATE IN THE SHAPE OF A SECTOR OF A CIRCLE UNDER A UNIFORMLY DISTRIBUTED LOAD

This type of geometry involves both curved and straight boundaries, which makes it an interesting test case for the computer program. The plate geometry and mesh designs are shown in Figures 6.5(a), 6.5(b) and 6.5(c). Two types of boundary conditions are considered. In the first instance, the whole of the plate boundary is considered to be simply supported. The second type of boundary condition assumes the plate to be simply supported along the two radial edges and clamped along the curved boundary. Timoshenko & Woinowsky-Krieger [1959] provide tabulated values for deflections and moments along the axis of symmetry of the plate for various sector angles.

Tables 6.20 to 6.23 compare the computed values of deflections and moments, along the axis of symmetry of the sector, with those given in this reference for the simply supported sector plates. It is seen that there is good agreement between the two solutions. Tables 6.24 to 6.27 give values of deflections and moments in the radial direction, along the axis of symmetry, when the circular boundary is clamped.

TABLE 6.1

Deflection at the Centre of a Circular Plate Loaded at the Centre

Poisson's ratio, $\nu = 0.3$

$w \times \left(\frac{D}{F a^2} \right)$	4 Elements	8 Elements	12 Elements	24 Elements	Exact
Simply Supported	0.05062	0.05062	0.05058	0.05054	0.05050
Clamped	0.01980	0.01989	0.01989	0.01989	0.01989

TABLE 6.2

Deflection and Bending Moments Along a Radial Direction for a
Simply Supported Circular Plate Loaded at the Centre (see Figure 6.1)

Poisson's ratio, $\nu = 0.3$

r/a	$w \times (D/F a^2)$		m_r/F		m_t/F	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
0.0	0.05050	0.05054	-	-	-	-
0.2	0.04592	0.04596	0.16650	0.16659	0.22220	0.22229
0.4	0.03659	0.03662	0.09479	0.09488	0.15050	0.15059
0.6	0.02500	0.02503	0.05285	0.05294	0.10855	0.10864
0.8	0.01250	0.01251	0.02308	0.02317	0.07879	0.07889

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 6.3

Deflections and Bending Moments Along a Radial Direction for a Clamped Circular Plate

Loaded at the Centre (see Figure 6.1)

Poisson's ratio, $\nu = 0.3$

r/a	$w \times (D/F a^2)$		m_r/F		m_θ/F	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
0.0	0.0198944	0.0198943	-	-	-	-
0.2	0.0165371	0.0165370	0.0869200	0.0869198	0.1426243	0.1426240
0.4	0.0108780	0.0108779	0.0152135	0.0152132	0.0709177	0.0709177
0.6	0.0054154	0.0054153	-0.0267322	-0.0267325	0.0289720	0.0289718
0.8	0.0014797	0.0014796	-0.0564931	-0.0564933	-0.0007889	-0.0007892
1.0	0.0000000	0.0000000	-0.0795775	-0.0795776	-0.0238732	-0.0238733

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 6.4

Normal Slope and Effective Shear Force at the Boundary of a
Simply Supported Circular Plate Under a
Concentrated Load, F , at the Centre
 Poisson's ratio, $\nu = 0.3$

Number of Elements	$-\frac{\partial w}{\partial n} \times \frac{D}{F a}$		$-\nu \times \frac{a}{F}$	
	Mid-Node	End Node	Mid-Node	End Node
4	0.06180	0.05993	0.14351	0.19074
8	0.06131	0.06119	0.15508	0.16757
12	0.06123	0.06129	0.16222	0.15333
24	0.06121	0.06128	0.17702	0.12362

Exact value for $-\frac{\partial w}{\partial n} \times \frac{D}{F a} = 0.06121$

Exact value for $-\nu \times \frac{a}{F} = 0.15915$

TABLE 6.5

Normal Moment and Effective Shear Force at the Boundary of a Clamped

Circular Plate Under a Concentrated Load, F , at the Centre

Poisson's ratio, $\nu = 0.3$

Number of Elements	m_h/F		$-V \times \frac{a}{F}$	
	Mid-Node	End Node	Mid-Node	End Node
4	-0.07960	-0.07982	0.15384	0.17140
8	-0.07958	-0.07959	0.15756	0.16246
12	-0.07958	-0.07958	0.15842	0.16065
24	-0.07958	-0.07958	0.15897	0.15953

Exact value for $\frac{m_h}{F} = -0.07958$

Exact value for $-V \times \frac{a}{F} = 0.15915$

TABLE 6.6

Deflection at the Centre of a Uniformly Loaded Circular Plate (Figure 6.2)

Poisson's ratio, $\nu = 0.3$

$w \times \left(\frac{D}{\delta a^4} \right)$	4 Elements	8 Elements	12 Elements	24 Elements	Exact
Simply Supported	0.06327	0.06385	0.06382	0.06376	0.06370
Clamped	0.01546	0.01562	0.01563	0.01562	0.01563

TABLE 6.7

Deflection and Moments at Interior Points for a Simply Supported
Circular Plate Under a Uniformly Distributed Load (see Figure 6.2)

Poisson's ratio, $\nu = 0.3$

r/a	$w \times (D/\delta a^4)$		$m_r/(\delta a^2)$		$m_t/(\delta a^2)$	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
0.0	0.06370	0.06376	0.2063	0.2063	0.2063	0.2063
0.2	0.06055	0.06061	0.1980	0.1981	0.2015	0.2016
0.4	0.05141	0.05146	0.1733	0.1734	0.1873	0.1874
0.6	0.03717	0.03721	0.1320	0.1321	0.1635	0.1637
0.8	0.01933	0.01936	0.0743	0.0744	0.1303	0.1305

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 6.8

Deflection and Moments Along a Radial Direction for a Clamped Circular
Plate Under a Uniformly Distributed Load (Figure 6.2)

Poisson's ratio, $\nu = 0.3$

r/a	$w \times (D/\delta a^4)$		$m_r/(\delta a^2)$		$m_t/(\delta a^2)$	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
0.0	0.01563	0.01562	0.0813	0.0812	0.0813	0.0812
0.2	0.01440	0.01440	0.0730	0.0730	0.0765	0.0765
0.4	0.01102	0.01103	0.0483	0.0482	0.0623	0.0622
0.6	0.00640	0.00640	0.0070	0.0070	0.0385	0.0385
0.8	0.00202	0.00203	-0.0508	-0.0507	-0.0053	-0.0053
1.0	0.00000	0.00000	-0.1250	-0.1249	-0.0375	-0.0375

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 6.9

Normal Slope and Effective Shear Force at the Boundary of a
Simply Supported Circular Plate Under a Uniformly Distributed

Load, f (see Figure 6.2)

Poisson's ratio, $\nu = 0.3$

Number of Elements	$-\frac{\partial w}{\partial n} \times \frac{D}{F a^3}$		$-\frac{\nu}{f a}$	
	Mid-Node	End Node	Mid-Node	End Node
4	0.09595	0.09290	0.47464	0.055465
8	0.09623	0.09602	0.49370	0.51323
12	0.096171	0.09625	0.50474	0.49100
24	0.09615	0.09624	0.52967	0.44099

$$\text{Exact value for } -\frac{\partial w}{\partial n} \times \frac{D}{f a^3} = 0.09615$$

$$\text{Exact value for } -\frac{\nu}{f a} = 0.5$$

TABLE 6.10

Normal Moment and Effective Shear Force at the Boundary of a Clamped Circular Plate Under a Uniformly Distributed Load, δ (see Figure 6.2)

Poisson's ratio, $\nu = 0.3$

Number of Elements	$m_h / (\delta a^2)$		$- V / (\delta a)$	
	Mid-Node	End Node	Mid-Node	End Node
4	-0.12408	-0.12289	0.49228	0.52176
8	-0.12504	-0.12465	0.49832	0.50381
12	-0.12497	-0.12497	0.49884	0.50237
24	-0.12516	-0.12467	0.49884	0.50237

Exact value for $\frac{m_h}{\delta a^2} = -0.12500$

Exact value for $-\frac{V}{\delta a} = 0.5$

TABLE 6.11

Deflections and Moments at Interior Points Plus Normal Slope
at the Boundary for a Simply Supported Circular Plate Bent by a Constant
Moment Applied Along Its Circumference (see Figure 6.3)

Poisson's ratio, $\nu = 0.3$

h/a	$w \times (D/m_o a^2)$		m_h/m_o		$-\frac{\partial w}{\partial n} \times \frac{D}{m_o h}$	
	Exact	QUDPLT	Exact	QUDPLT	Exact	QUDPLT
0.0	0.38462	0.38510	1.0	1.001	0.7692	0.7700
0.1	0.36920	0.36970	1.0	1.001		
0.2	0.32308	0.32348	1.0	1.001		
0.3	0.24615	0.24646	1.0	1.001		
0.4	0.13846	0.13863	1.0	1.001		

QUDPLT : Present quadratic formulation

TABLE 6.12

Deflection and Bending Moments Along $\theta = 0^\circ$ Radial Direction for a
Simply Supported Circular Plate Under a Linearly Varying Load (see Figure 6.4)

Poisson's ratio, $\nu = 0.3$

r/a	$w \times (D/\delta a^4)$		$m_r/(\delta a^2)$		$m_t/(\delta a^2)$	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
0.0	0.00000	1.5×10^{-11}	0.00000	2.8×10^{-11}	0.00000	4.8×10^{-11}
1/3	0.00324	0.00324	0.03272	0.03271	0.01926	0.01926
2/3	0.00341	0.00341	0.04090	0.04088	0.02695	0.02699

TWK : Timoshenko & Woinowsky-Krieger [1959]
 QUDPLT : Present quadratic formulation

TABLE 6.13

Deflection and Bending Moments Along $\theta = 15^\circ$ Radial Direction for a
Simply Supported Circular Plate Under a Linearly Varying Load (see Figure 6.4)

Poisson's ratio, $\nu = 0.3$

r/a	$w \times (D/\delta a^4)$		$m_r/(\delta a^2)$		$m_t/(\delta a^2)$	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
0.0	0.00000	1.5×10^{-11}	0.00000	3.1×10^{-11}	0.00000	4.7×10^{-11}
1/3	0.00313	0.00313	0.03160	0.03159	0.01861	0.01860
2/3	0.00329	0.00329	0.03950	0.03949	0.02603	0.02607

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 6.14

Deflection and Bending Moments Along $\theta = 30^\circ$ Radial Direction for a
Simply Supported Circular Plate Under a Linearly Varying Load (see Figure 6.4)

Poisson's ratio, $\nu = 0.3$

r/a	$w \times (D/\delta a^4)$		$m_r/(\delta a^2)$		$m_t/(\delta a^2)$	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
0.0	0.00000	1.5×10^{-11}	0.00000	3.4×10^{-11}	0.00000	4.3×10^{-11}
1/3	0.00281	0.00281	0.02833	0.02833	0.01668	0.01668
2/3	0.00295	0.00295	0.03542	0.03541	0.02334	0.02337

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 6.15

Deflection and Bending Moments Along $\theta = 45^\circ$ Radial Direction for a
Simply Supported Circular Plate Under a Linearly Varying Load (see Figure 6.4)

Poisson's ratio, $\nu = 0.3$

r/a	$w \times (D/\delta a^4)$		$m_r/(\delta a^2)$		$m_t/(\delta a^2)$	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
0.0	0.00000	1.5×10^{-11}	0.00000	3.9×10^{-11}	0.00000	3.9×10^{-11}
1/3	0.00229	0.00229	0.02313	0.02313	0.01362	0.01362
2/3	0.00241	0.00241	0.02892	0.02891	0.01906	0.01908

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 6.16

Deflection and Bending Moments Along $\theta = 60^\circ$ Radial Direction for a
Simply Supported Circular Plate Under a Linearly Varying Load (see Figure 6.4)

Poisson's ratio, $\nu = 0.3$

r/a	$w \times (D/\delta a^4)$		$m_r/(\delta a^2)$		$m_t/(\delta a^2)$	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
0.0	0.00000	1.5×10^{-11}	0.00000	4.3×10^{-11}	0.00000	3.4×10^{-11}
1/3	0.00162	0.00162	0.01636	0.01635	0.00963	0.00963
2/3	0.00170	0.00170	0.02045	0.02044	0.01348	0.01349

TWK : Timoshenko & Woinowsky-Krieger [1959]
 QUDPLT : Present quadratic formulation

TABLE 6.17

Deflection and Bending Moments Along $\theta = 75^\circ$ Radial Direction for a
Simply Supported Circular Plate Under a Linearly Varying Load (see Figure 6.4)

Poisson's ratio, $\nu = 0.3$

r/a	$w \times (D/\delta a^4)$		$m_r/(\delta a^2)$		$m_\theta/(\delta a^2)$	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
0.0	0.00000	1.5×10^{-11}	0.00000	4.7×10^{-11}	0.00000	3.0×10^{-11}
1/3	0.00084	0.00084	0.00847	0.00847	0.00499	0.00498
2/3	0.00088	0.00088	0.01058	0.01058	0.00698	0.00699

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 6.18

Deflection and Bending Moments Along $\theta = 90^\circ$ Radial Direction for a
Simply Supported Circular Plate Under a Linearly Varying Load

(see Figure 6.4)

Poisson's ratio, $\nu = 0.3$

r/a	$w \times (D/\delta a^4)$		$m_r/(\delta a^2)$		$m_t/(\delta a^2)$	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
0.0	0.0	1.5×10^{-11}	0.0	4.8×10^{-11}	0.0	2.9×10^{-11}
1/3	0.0	1.2×10^{-11}	0.0	3.7×10^{-11}	0.0	2.9×10^{-11}
2/3	0.0	6.7×10^{-12}	0.0	1.6×10^{-11}	0.0	2.5×10^{-11}

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 6.19

Normal Slope, $-(\partial w / \partial n) \times (D / \delta a^3)$, Along a Quarter of the Boundary
of a Simply Supported Circular Plate Under a Linearly Varying Load

(see Figure 6.4)

Poisson's ratio, $\nu = 0.3$

θ	Exact	QUDPLT
0.0	0.01263	0.01262
$\pi/24$	0.01252	0.01251
$2\pi/24$	0.01220	0.01219
$3\pi/24$	0.01167	0.01166
$4\pi/24$	0.01093	0.01093
$5\pi/24$	0.01002	0.01001
$6\pi/24$	0.00893	0.00892
$7\pi/24$	0.00769	0.00768
$8\pi/24$	0.00631	0.00631
$9\pi/24$	0.00483	0.00483
$10\pi/24$	0.00327	0.00327
$11\pi/24$	0.00165	0.00165
$12\pi/24$	0.00000	0.3×10^{-9}

QUDPLT : Present quadratic formulation

TABLE 6.20

Deflection and Moments Along the Line of Symmetry of a Simply Supported 45°

Sector Plate Under a Uniformly Distributed Load (Figure 6.5)

Poisson's ratio, $\nu = 0.3$

r/a	$w \times (D/\delta a^4)$		$m_r/(\delta a^2)$		$m_t/(\delta a^2)$	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
0.25	0.00006	0.00005	-0.0015	-0.0015	0.0093	0.0093
0.5	0.00033	0.00033	0.0069	0.0069	0.0183	0.0183
0.75	0.00049	0.00048	0.0161	0.0161	0.0169	0.0170

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 6.21

Deflection and Moments Along the Line of Symmetry of a Simply Supported 60°

Sector Plate Under a Uniformly Distributed Load (Figure 6.5)

Poisson's ratio, $\nu = 0.3$

r/a	$w \times (D/\delta a^4)$		$m_r/(\delta a^2)$		$m_t/(\delta a^2)$	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
0.25	0.00019	0.00020	-0.0025	-0.0025	0.0177	0.0177
0.5	0.00080	0.00080	0.0149	0.0149	0.0255	0.0256
0.75	0.00092	0.00093	0.0243	0.0243	0.0213	0.0214

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 6.22

Deflection and Moments Along the Line of Symmetry of a Simply Supported 90°

Sector Plate Under a Uniformly Distributed Load (Figure 6.5)

Poisson's ratio, $\nu = 0.3$

r/a	$w \times (D/\delta a^4)$		$m_r/(\delta a^2)$		$m_t/(\delta a^2)$	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
0.25	0.00092	0.00093	0.0036	0.0036	0.0319	0.0319
0.5	0.00225	0.00226	0.0353	0.0354	0.0352	0.0354
0.75	0.00203	0.00204	0.0381	0.0383	0.0286	0.0288

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 6.23

Deflection and Moments Along the Line of Symmetry of a Simply Supported Semi-Circular Plate Under a Uniformly Distributed Load (see Figure 6.5)

Poisson's ratio, $\nu = 0.3$

r/a	$w \times (D/\delta a^4)$		$m_r/(\delta a^2)$		$m_t/(\delta a^2)$	
	TWK	QUDPLT	TWK	QUDPLT	TWK	QUDPLT
0.25	0.00594	0.00594	0.0692	0.0693	0.0357	0.0359
0.5	0.00811	0.00811	0.0868	0.0869	0.00515	0.00515
0.75	0.00560	0.00561	0.0617	0.0616	0.0468	0.0472

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 6.24

Deflection and Moments Along the Line of Symmetry of a 45° Sector Plate
Under a Uniformly Distributed Load, Having the Two Radial Edges Simply
Supported and the Circular Boundary Clamped (Figure 6.5)

Poisson's ratio, $\nu = 0.3$

r/a	$w \times (D/\delta a^4)$		$m_r/(\delta a^2)$	
	TWK	QUDPLT	TWK	QUDPLT
0.25	0.00005	0.00005	-0.0008	-0.0008
0.5	0.00026	0.00026	0.0087	0.0075
0.75	0.00028	0.00028	0.0107	0.0106
1.0	0.00000	0.00000	-0.0250	-0.0251

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 6.25

Deflection and Moments Along the Line of Symmetry of a 60° Sector Plate
Under a Uniformly Distributed Load, Having the Two Radial Edges Simply
Supported and the Circular Boundary Clamped (Figure 6.5)

Poisson's ratio, $\nu = 0.3$

r/a	$w \times (D/\delta a^4)$		$m_r/(\delta a^2)$	
	TWK	QUDPLT	TWK	QUDPLT
0.25	0.00017	0.00016	-0.0006	-0.0006
0.5	0.00057	0.00056	0.0143	0.0143
0.75	0.00047	0.00047	0.0123	0.0124
1.0	0.00000	0.00000	-0.0340	-0.0342

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

TABLE 6.26

Deflection and Moment Along the Line of Symmetry of a 90° Sector Plate
Under a Uniformly Distributed Load, Having the Two Radial Edges Simply
Supported and the Circular Boundary Clamped (Figure 6.5)

Poisson's ratio, $\nu = 0.3$

r/a	$w \times (D/l^3 a^4)$		$m_r/(l^3 a^2)$	
	TWK	QUDPLT	TWK	QUDPLT
0.25	0.00063	0.00063	0.0068	0.0068
0.5	0.00132	0.00132	0.0272	0.0273
0.75	0.00082	0.00081	0.0113	0.0118
1.0	0.00000	0.00000	-0.0488	-0.0492

TWK : Timoshenko & Woinowsky-Krieger [1959]

QUDPLT : Present quadratic formulation

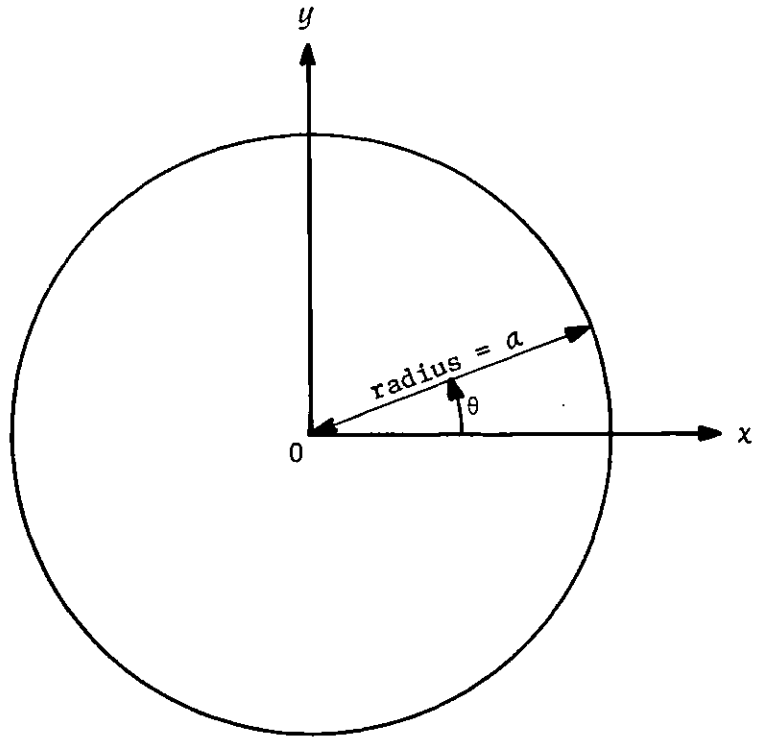
TABLE 6.27

Deflection and Moment Along the Line of Symmetry of a Semi-Circular Disc
Under a Uniformly Distributed Load With Its Straight Edge Simply Supported
and Its Circular Boundary Clamped (Figure 6.5)

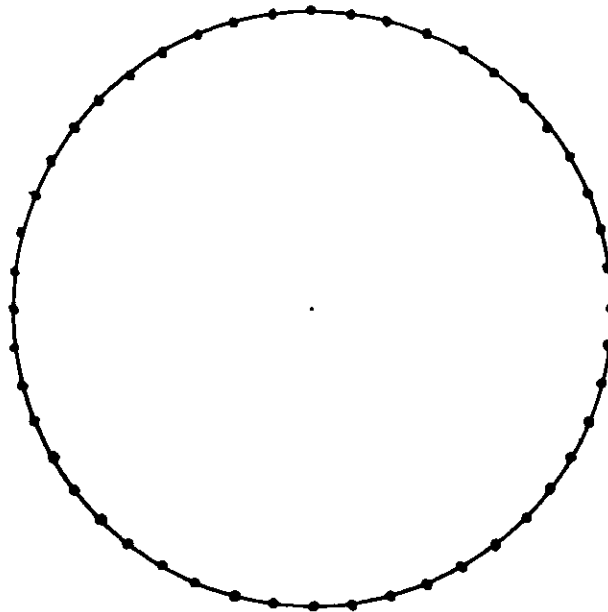
Poisson's ratio, $\nu = 0.3$

r/a	$w \times (D/\delta a^4)$		$m_r/(\delta a^2)$	
	TWK	QUDPLT	TWK	QUDPLT
0.25	0.00293	0.00294	0.0473	0.0473
0.5	0.00337	0.00337	0.0446	0.0448
0.75	0.00153	0.00153	0.0016	0.0017
1.0	0.00000	0.00000	-0.0756	-0.0768

TWK : Timoshenko & Woinowsky-Krieger [1959]
QUDPLT : Present quadratic formulation

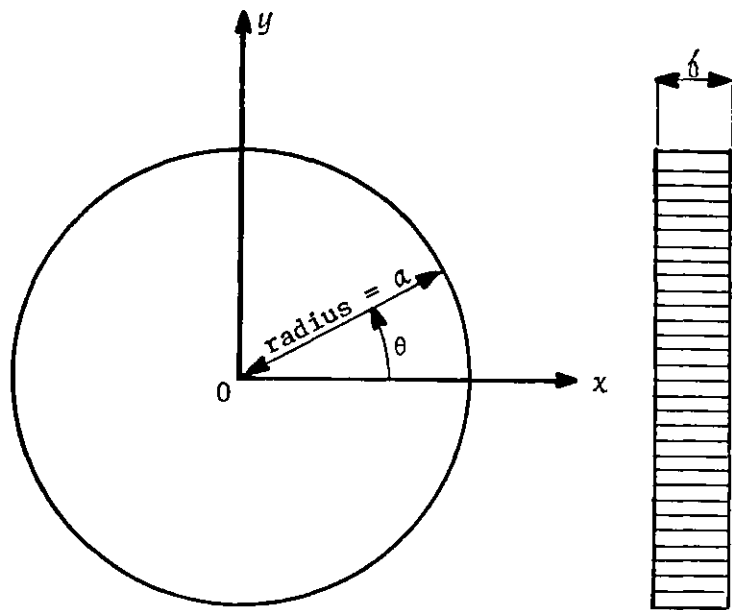


(a) Plate geometry

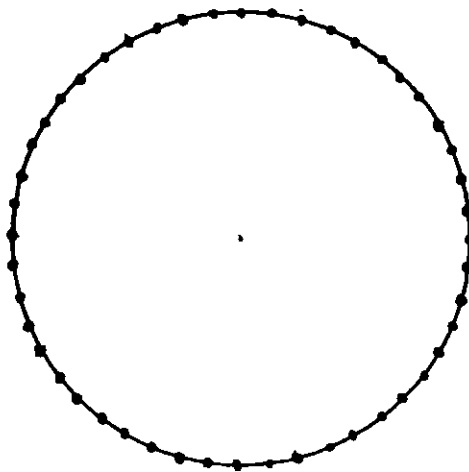


(b) Boundary discretisation.
24 elements.

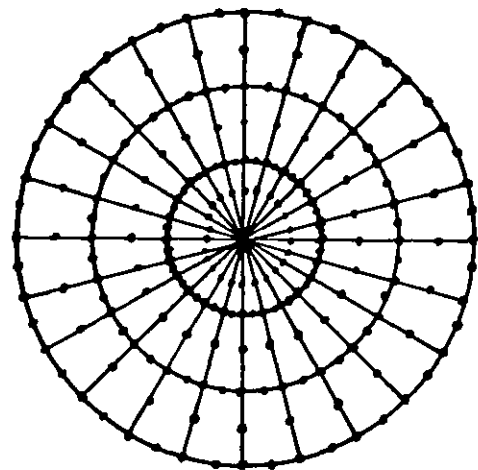
Figure 6.1: Circular plate under a concentrated load at the centre



(a) Plate geometry and loading

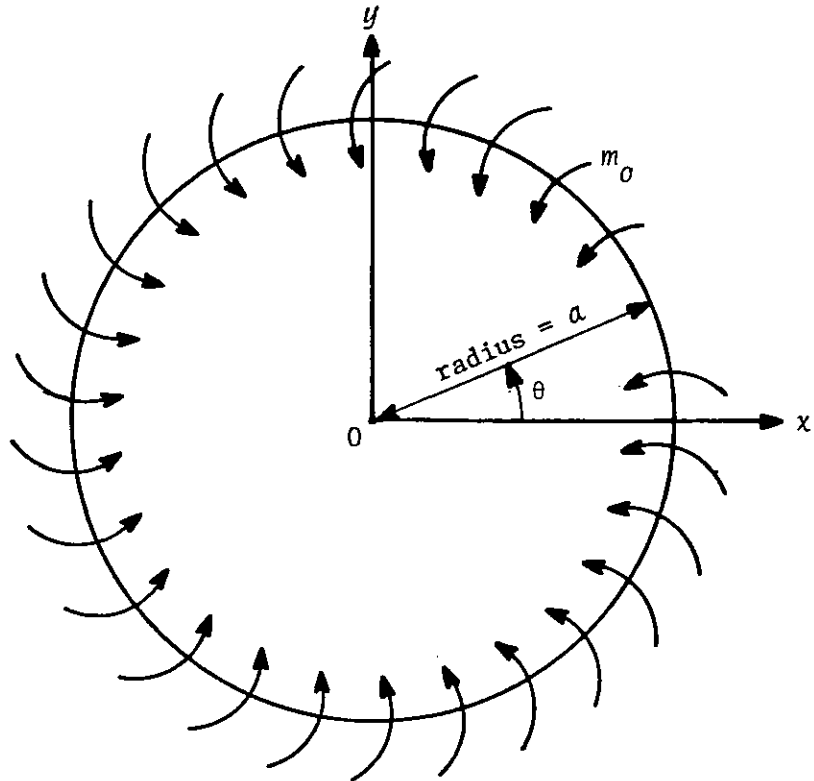


(b) Boundary discretisation.
24 elements.

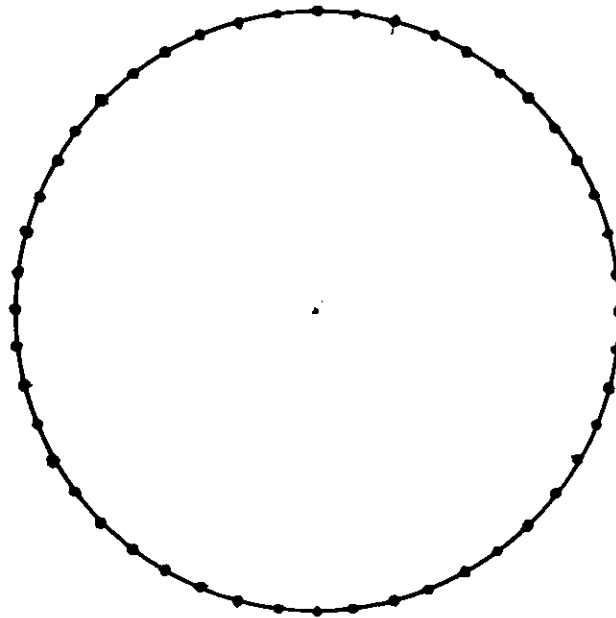


(c) Interior discretisation

Figure 6.2: Circular plate under a uniformly distributed load

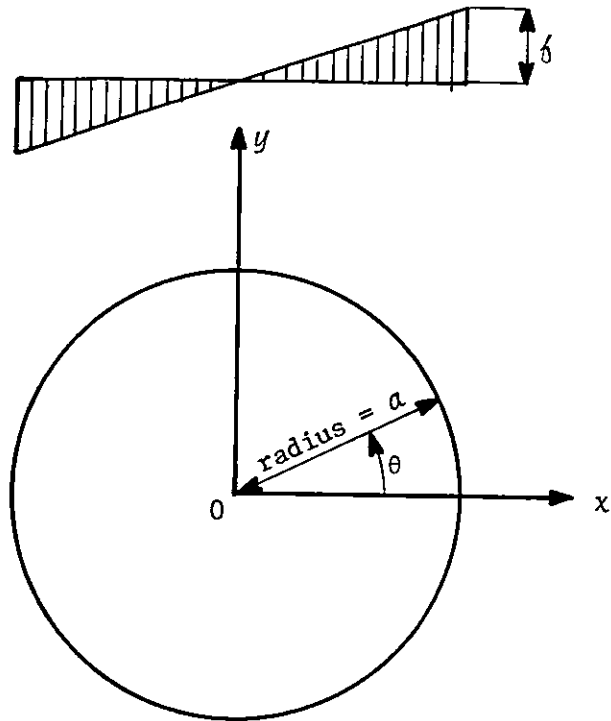


(a) Plate geometry and loading

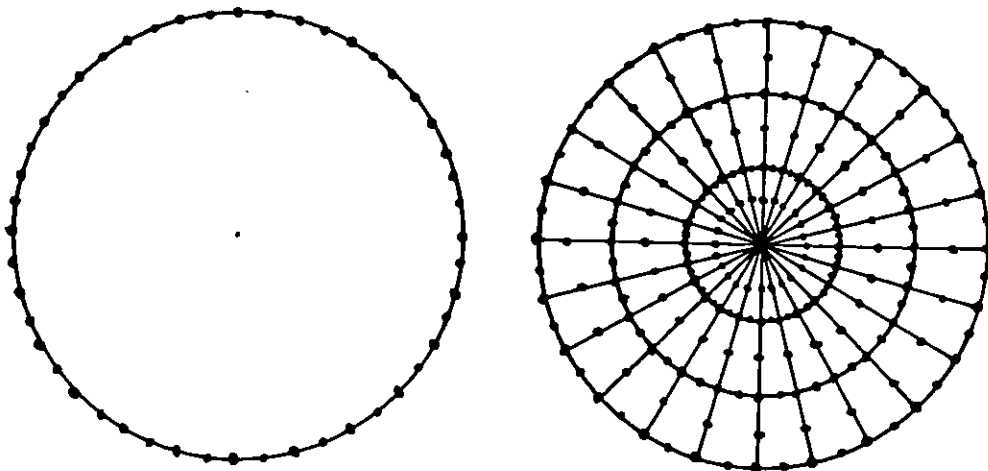


(b) Boundary discretisation.
24 elements.

Figure 6.3: Simply supported circular plate bent by a uniformly distributed moment, m_0 , along its boundary

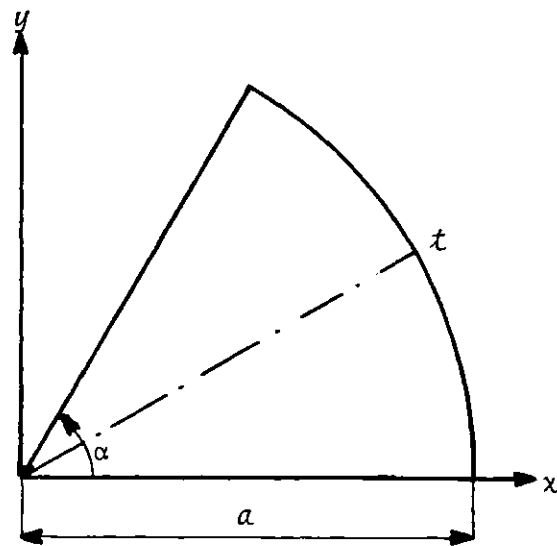


(a) Plate geometry and loading

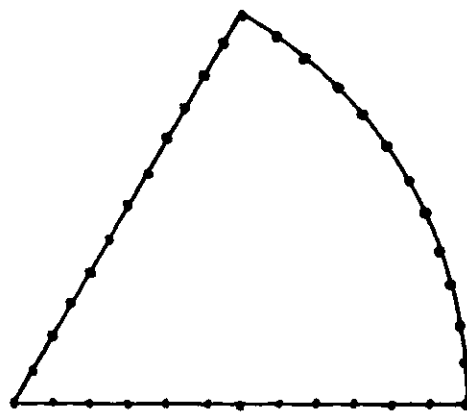


(b) Boundary discretisation. 24 elements. (c) Interior discretisation

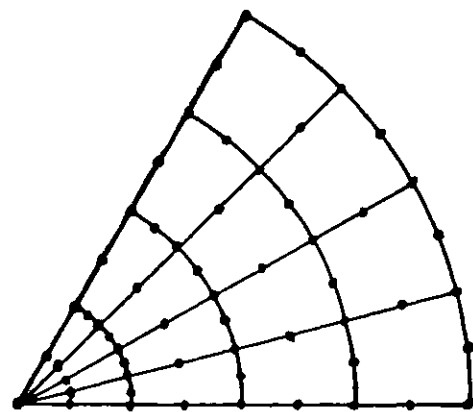
Figure 6.4: Simply supported circular plate under a linearly varying load



(a) Plate geometry



(b) Boundary discretisation.
6 elements along each
side.



(c) Interior discretisation

Figure 6.5: Plates in the shape of a sector of a circle under a uniformly distributed load

CHAPTER 7

DISCUSSION AND CONCLUSION

The boundary integral equations for the bending of thin plates under lateral loading have been derived from the Kirchhoff plate theory. Integral equations have been obtained for deflection, deflection gradients, moments, and effective shear forces at interior points of the plate. The limiting processes involved in taking the interior point to the boundary have also been described, resulting in boundary integral equations for deflection, normal slope, normal moment and effective shear force at points along the boundary. These equations provide four relations connecting the boundary variables and the lateral loading on the plate with each other. At a regular point on the plate boundary, there are four variables present. Two of these are effectively given as boundary conditions. To obtain a solution for the other two variables, two more relations are required. These are provided by choosing a pair of boundary integral equations from the four available.

A constant element computer program was written and applied to problems of simply supported and clamped square plates loaded at the centre. These were solved by using, in turn, all possible pairs out of the four boundary integral equations. It was found that by adopting the approach usual in the current literature, that is, ignoring the Kirchhoff corner forces in constant element formulations, a reliable solution could not be obtained when the integral equation pair employed involved the more singular kernels, namely, the equations for normal moment and effective shear force at a boundary point. Furthermore, in these situations, the solution obtained for simply supported plates had a much greater inaccuracy than those obtained for a plate with clamped edges. It was also seen that, for a simply supported plate, the computed values of the

effective shear force near the corners were very unreliable, even for equation pairs which provided a very accurate overall solution, namely, the equations for deflection and normal slope at a boundary point.

It was shown that the above difficulties can be overcome if the Kirchhoff corner reactions are included in the numerical implementation as additional unknowns. To solve for these unknowns, extra equations are required at the corners. These were chosen as either the equation for deflection at the corner or the equation for twisting moment jump at the corner. The results showed that, in most cases, the equation for twisting moment jump at the corners provided a better choice for the corner equation. However, for equation pairs containing the more singular kernels, a more accurate value for the central deflection was obtained when the corner equation was chosen as that of the deflection at the boundary. This was found to be the case when the equation pairs employed did not include the integral equation for deflection at the boundary. The computed values of the slopes and moments along the boundary seemed to be affected much less by the choice of the corner equations. However, the computed values of the effective shear force were found to be more accurate, when the equation for twisting moment jump at the corner was used in conjunction with an integral equation pair containing the less singular kernels, namely, the equations for deflection and normal slope at a boundary point. For all equation pairs, however, the equation for twisting moment jump at a corner provided a more accurate value for the Kirchhoff corner reactions. In view of these findings, it would appear that the equations for deflection and normal slope, coupled with the equation for twisting moment jump at corners, provide the best choice for a constant element direct BIE formulation for plate bending problems. In the test cases considered, the solutions obtained using this combination were usually as good as, or better than, the solution

obtained using the other combinations of equations. Furthermore, the evaluation of the matrix coefficients for these equations is less time-consuming due to their simpler kernel expressions.

The quadratic element implementation described employs the equations for deflection and normal slope at the boundary. The computer program written incorporates three-noded isoparametric elements along the boundary, along which the geometry, as well as the boundary variables, is interpolated using quadratic shape functions. The evaluation of the integral coefficients by using various quadrature formulae is fully described. In order not to restrict the type of loading on the plate, eight-noded quadrilateral and six-noded triangular elements have been incorporated in the computer program for the numerical quadrature over the plate domain.

This program was applied to a variety of plate problems involving various geometries, loadings and boundary conditions. Where for a particular problem an analytical or an exact series solution was available, it was found that the results obtained using the quadratic BIE formulation were in very close agreement with it. In practically all such cases, the discrepancy between the exact solution and the BIE solutions was of the order of a few tenths of a percent. However, for simply supported circular plates, the computed value of the effective shear force at nodes along the boundary differed from the exact solution by between 4% to 23%. It is believed that this is due to the deficiency of the quadratic interpolation of the geometry in exactly modelling a circular boundary. However, by averaging the effective shear force along the boundary using the appropriate shape functions, a very accurate value can be obtained. Where exact solutions to problems were not available, comparisons were made with other numerical solutions, when available. Some of the more interesting results obtained are discussed below.

It was found that, near a clamped to free boundary, the behaviour of the effective shear force is oscillatory, with the magnitude becoming very large as the corner is approached. This type of behaviour was seen in the problem of a square cantilever under a central load, and also for square plates with three edges clamped and one edge free. For the latter problem, existing finite difference results were available, but did not exhibit such singular behaviour for the effective shear force near the clamped to free corner. However, Williams [1951] has shown that, within the framework of Kirchhoff's thin plate theory, the effective shear force at a clamped to free right-angled corner is singular. Furthermore, for the particular case of clamped to free corners, the dominant solution to the eigen equation was shown to be complex, which implies an oscillatory singularity. Although the detailed treatment of these types of singularities is outside the scope of the present work, nevertheless it is interesting to note that the present BIE formulation at least detects their presence and confirms the limitations of the plate theory from which it is derived.

Simply supported rhombic plates under a uniformly distributed load were analysed for various angles of skew. The present BIE solutions were compared with a finite difference solution, an approximate series solution and a constant element BIE solution. The finite difference and series solutions are due to Morley [1963]. The constant element BIE solution is given by Maiti & Chakrabarty [1974], but their formulation employs a different set of equations from those presented in this work. It was found that, for small angles of skew, the computed values of deflections and moments at the centre of the plate were smaller than those obtained by the other authors. The author believes that the reason for this discrepancy is the zero slope boundary condition prescribed at either side of a simply supported corner in the quadratic BIE formulation. In the

other solutions, no specific restriction is placed on the normal slopes either side of a corner. This allows the slope either side of a simply supported corner to be non-zero. The problem of a 30° skew rhombic plate has been solved using the constant element BIE program described in this work. It was found that, near the obtuse corner, the magnitudes of the normal slopes obtained by the constant element formulation were much larger than those given by the quadratic one. As a result, the computed value of the central deflection obtained by the constant element formulation was also larger.

A difficulty similar to that discussed above was also found in the case of a simply supported corner plate under a uniformly distributed load. For this problem, existing finite difference and BIE solutions using constant elements were available. Once again, the constant element BIE solutions employed different equations to those given in this work. Along the limbs of the plate at a sufficient distance away from the corner (about twice the width of each limb), the plate is expected to behave as a simply supported infinite strip. For this simpler problem, an analytical solution is available, and it was found that the solution obtained, for the deflections along the width of the plate in this region, by the quadratic BIE formulation was more accurate than the finite difference one. The discrepancy between the quadratic BIE and the exact solutions was less than 1%, whereas the finite difference results over-estimated the exact ones by around 5%. The discrepancy between the quadratic BIE and the finite difference solutions, for the deflections along the diagonal connecting the outer corner of the plate to the re-entrant corner, was found to be much greater. Along this diagonal, the quadratic BIE solution gave smaller deflections than the finite difference solution and the constant element BIE formulations. Once again, as in the case of the rhombic plates, it is believed that this

discrepancy is mainly due to the zero normal slope restriction made at the simply supported re-entrant corner in the quadratic BIE formulation. It was found that, by freeing this corner node and allowing for non-zero normal slopes either side of it, larger deflections were obtained along the diagonal. Even larger deflections were obtained when the same problem was solved using the constant element formulation presented in this work. An examination of the variation of the normal slope, along the boundary adjacent to the re-entrant corner, showed that the solutions giving the larger deflections along the diagonal also gave large normal slopes either side of a corner. The finite difference solution over-estimated the value of deflections away from the corner by about 5%. It may be that this error is repeated throughout the plate and probably amplified in the region near the corner. This would, therefore, be an additional contribution to the discrepancy between the finite difference and the quadratic BIE formulation for this particular problem. An examination of the variation of the moment sum ($m_x + m_y$) along the diagonal showed that the quadratic BIE solution exhibits a singularity near the re-entrant corner. This seems to be in agreement with the solution given by Williams [1951], which showed that the moments are singular at a simply supported corner with a corner angle of greater than 90° .

The present quadratic formulation was also extended to the analysis of plates having interior supports. Although a constant element formulation incorporating interior support conditions is already published by other authors, the work carried out here was done independently and was in close agreement with the existing published work.

CONCLUSIONS

The principal conclusions that may be drawn from the work presented in this thesis can be summarised as follows:

- i) Provided the Kirchhoff corner forces are included in a constant element formulation, all six possible pairs of integral equations applicable to plate bending problems may be used successfully to obtain numerical solutions for problems of thin plates under lateral loading. The equation pairs consisting of the equations for deflection and normal slope at the boundary appear to be the best choice, in that they generally provide results which are as good as, or better than, those obtained using the other equation combinations, with less computational effort.
- ii) Ignoring the corner forces does not affect the equation pairs involving the less singular kernels. However, it results in unreliable values for effective shear force in the vicinity of the corners, even for equation pairs providing an accurate overall solution. This effect is more severe for simply supported plate problems, because by ignoring the Kirchhoff corner forces they are effectively taken to be zero, which is not correct.
- iii) The quadratic formulation, employing the equations for deflection and normal slope at the boundary, has been shown to provide very accurate solutions for a range of plate bending problems, involving varying geometry, loading and boundary conditions, for which existing exact solutions are available for comparison. In the case of plates with boundaries in the form of circular arcs, the nodal values of effective shear force along simply supported boundaries were not

computed to a high degree of accuracy. This is attributed to the discontinuity in the normal direction created at inter-element nodes, due to the approximation of the boundary geometry using quadratic interpolation functions. This effectively creates artificial corners, which generate Kirchhoff corner forces. Since, for a circular boundary, the computer program does not solve for any corner variables, the effect of ignoring these "Kirchhoff corner forces" affects the value of the effective shear force. It is shown that an accurate solution for the effective shear force in this type of situation can be obtained by averaging out the nodal values by using the shape functions.

- iv) For certain types of boundary conditions at the corners, where the Kirchhoff plate theory predicts a singularity, the present quadratic formulation was shown to be able to detect these singularities, whereas the existing finite difference solutions for the problems considered appeared to be insensitive to these singularities.
- v) For geometries involving simply supported corners, with corner angles larger than 90° , the quadratic BIE formulation provides a stiffer solution than those obtained by the finite difference method or the constant element BIE method. This was shown to be due to the fact that in the quadratic formulation, the normal slopes either side of a simply supported corner are set to zero, whereas in the finite difference and constant element BIE methods, the boundaries meeting at such corners are allowed to rotate, resulting in larger deflections inside the plate domain.

- vi) The quadratic formulation was also successfully employed in the solution of plate bending problems involving support conditions inside the domain.

APPENDIX A
TRANSFORMATION OF THE GOVERNING DIFFERENTIAL EQUATION OF
THE PLATE AND STRESS RESULTANTS ALONG THE BOUNDARY
TO PLANE POLAR COORDINATES

The governing differential equation of the bending of thin plates under lateral loading is given by equation (2.42) as:

$$\nabla^4 w = \nabla^2 \nabla^2 w = \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{F}{D} \quad (\text{A.1})$$

where ∇^2 is the Laplacian operator:

$$\nabla^2 = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} \quad (\text{A.2})$$

The relation between the Cartesian coordinates x, y and the polar coordinates r, θ is given by:

$$x = r \cos \theta \quad (\text{A.3a})$$

$$y = r \sin \theta \quad (\text{A.3b})$$

where r is the modulus of the position vector of a point with coordinates x, y , and θ is the angle between the x axis and the position vector $\hat{r}$, as shown in Figure A.1.

The derivatives with respect to x and y can be written using the chain rule for differentiation as:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta} \frac{\partial \theta}{\partial x} \quad (\text{A.4a})$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta} \frac{\partial \theta}{\partial y} \quad (\text{A.4b})$$

The modulus r of the position vector $\hat{r}$ is given by:

$$r = (x^2 + y^2)^{\frac{1}{2}} \quad (\text{A.5})$$

Therefore:
$$\frac{\partial r}{\partial x} = \frac{x}{(x^2 + y^2)^{\frac{1}{2}}} = \frac{x}{r} = \cos \theta \quad (\text{A.6})$$

and:
$$\frac{\partial r}{\partial y} = \frac{y}{(x^2 + y^2)^{\frac{1}{2}}} = \frac{y}{r} = \sin \theta \quad (\text{A.7})$$

Differentiating equation (A.3a) with respect to x and using equation (A.6) gives:

$$\begin{aligned} 1 &= \frac{\partial r}{\partial x} \cos \theta - r \frac{\partial \theta}{\partial x} \sin \theta \\ &= \cos^2 \theta - r \frac{\partial \theta}{\partial x} \sin \theta \end{aligned} \quad (\text{A.8})$$

Taking $\cos^2 \theta$ to the left hand side and noting that $1 - \cos^2 \theta = \sin^2 \theta$ gives:

$$\frac{\partial \theta}{\partial x} = - \frac{1}{r} \sin \theta \quad (\text{A.9})$$

In a similar manner, by differentiating equation (A.3b) with respect to y and using equation (A.7):

$$\frac{\partial \theta}{\partial y} = \frac{1}{r} \cos \theta \quad (\text{A.10})$$

Substituting equations (A.6), (A.7), (A.9) and (A.10) into equations (A.4a) and (A.4b) gives:

$$\frac{\partial}{\partial x} = \cos \theta \frac{\partial}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta} \quad (\text{A.11})$$

and:
$$\frac{\partial}{\partial y} = \sin \theta \frac{\partial}{\partial r} + \frac{1}{r} \cos \theta \frac{\partial}{\partial \theta} \quad (\text{A.12})$$

Having obtained expressions for $\partial/\partial x$ and $\partial/\partial y$, the Laplacian operator given by equation (A.2) is obtained as follows:

$$\begin{aligned}
 \frac{\partial^2}{\partial x^2} &= \left(\cos \theta \frac{\partial}{\partial \rho} - \frac{1}{\rho} \sin \theta \frac{\partial}{\partial \theta} \right) \left(\cos \theta \frac{\partial}{\partial \rho} - \frac{1}{\rho} \sin \theta \frac{\partial}{\partial \theta} \right) \\
 &= \cos^2 \theta \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho^2} \sin \theta \cos \theta \frac{\partial}{\partial \theta} - \frac{1}{\rho} \sin \theta \cos \theta \frac{\partial^2}{\partial \rho \partial \theta} \\
 &\quad + \frac{1}{\rho} \sin^2 \theta \frac{\partial}{\partial \rho} - \frac{1}{\rho} \sin \theta \cos \theta \frac{\partial^2}{\partial \rho \partial \theta} + \frac{1}{\rho^2} \sin \theta \cos \theta \frac{\partial}{\partial \theta} \\
 &\quad + \frac{1}{\rho^2} \sin^2 \theta \frac{\partial^2}{\partial \theta^2}
 \end{aligned} \tag{A.13}$$

and:

$$\begin{aligned}
 \frac{\partial^2}{\partial y^2} &= \left(\sin \theta \frac{\partial}{\partial \rho} + \frac{1}{\rho} \cos \theta \frac{\partial}{\partial \theta} \right) \left(\sin \theta \frac{\partial}{\partial \rho} + \frac{1}{\rho} \cos \theta \frac{\partial}{\partial \theta} \right) \\
 &= \sin^2 \theta \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho^2} \sin \theta \cos \theta \frac{\partial}{\partial \theta} + \frac{1}{\rho} \sin \theta \cos \theta \frac{\partial^2}{\partial \rho \partial \theta} \\
 &\quad + \frac{1}{\rho} \cos^2 \theta \frac{\partial}{\partial \rho} + \frac{1}{\rho} \sin \theta \cos \theta \frac{\partial^2}{\partial \rho \partial \theta} - \frac{1}{\rho^2} \sin \theta \cos \theta \frac{\partial}{\partial \theta} \\
 &\quad + \frac{1}{\rho^2} \cos^2 \theta \frac{\partial^2}{\partial \theta^2}
 \end{aligned} \tag{A.14}$$

Adding equations (A.13) and (A.14) and noting that $\cos^2 \theta + \sin^2 \theta = 1$ gives the Laplacian operator as:

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} \tag{A.15}$$

The biharmonic operator, ∇^4 , can now be obtained as:

$$\begin{aligned}
 \nabla^4 &= \nabla^2 \nabla^2 = \left(\frac{\partial^2}{\partial \kappa^2} + \frac{1}{\kappa} \frac{\partial}{\partial \kappa} + \frac{1}{\kappa^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2}{\partial \kappa^2} + \frac{1}{\kappa} \frac{\partial}{\partial \kappa} + \frac{1}{\kappa^2} \frac{\partial^2}{\partial \theta^2} \right) \\
 &= \frac{\partial^4}{\partial \kappa^4} + \frac{2}{\kappa} \frac{\partial^3}{\partial \kappa^3} - \frac{1}{\kappa^2} \frac{\partial^2}{\partial \kappa^2} + \frac{1}{\kappa^3} \frac{\partial}{\partial \kappa} + \frac{4}{\kappa^4} \frac{\partial^2}{\partial \theta^2} \\
 &\quad - \frac{2}{\kappa^3} \frac{\partial^3}{\partial \kappa \partial \theta^2} + \frac{2}{\kappa^2} \frac{\partial^4}{\partial \kappa^2 \partial \theta^2} + \frac{1}{\kappa^4} \frac{\partial^4}{\partial \theta^4}
 \end{aligned} \tag{A.16}$$

The expressions for stress resultants on the boundary are given by equations (2.55), (2.56) and (2.57), which, for convenience, are repeated below as:

$$m_n = - \mathcal{D} \left(\frac{\partial^2 \omega}{\partial x^2} + \nu \frac{\partial^2 \omega}{\partial t^2} \right) \tag{A.17}$$

$$m_t = - \mathcal{D} (1 - \nu) \frac{\partial^2 \omega}{\partial n \partial t} \tag{A.18}$$

$$v_n = - \mathcal{D} \frac{\partial}{\partial n} \nabla^2 \omega + \frac{\partial}{\partial S} m_t \tag{A.19}$$

where n and t are the normal and tangent to the boundary at a point Q on the boundary (see Figure A.2). It can be shown (see Appendix B) that:

$$\frac{\partial}{\partial n_Q} = \cos \beta \frac{\partial}{\partial \kappa} + \frac{\sin \beta}{\kappa} \frac{\partial}{\partial \theta} \tag{A.20}$$

where $\partial/\partial n_Q$ denotes the normal derivative at a boundary point Q , and β is the angle between the position vector $\hat{\kappa}$ and the normal n , as shown in Figure A.2. Substituting β by $\beta + \pi/2$ gives the tangential derivative as:

$$\frac{\partial}{\partial t_Q} = - \sin \beta \frac{\partial}{\partial \kappa} + \frac{\cos \beta}{\kappa} \frac{\partial}{\partial \theta} \tag{A.21}$$

Using equations (A.20) and (A.21) and noting that (see Figure A.2):

$$\beta = \alpha - \theta \quad (\text{A.22})$$

and therefore:
$$\frac{\partial \beta}{\partial \theta} = -1 \quad (\text{A.23})$$

the following second derivatives can be obtained:

$$\begin{aligned} \frac{\partial^2}{\partial n_Q^2} &= \left(\cos \beta \frac{\partial}{\partial \kappa} + \frac{\sin \beta}{\kappa} \frac{\partial}{\partial \theta} \right) \left(\cos \beta \frac{\partial}{\partial \kappa} + \frac{\sin \beta}{\kappa} \frac{\partial}{\partial \theta} \right) \\ &= \left(\frac{1 + \cos 2\beta}{2} \right) \frac{\partial^2}{\partial \kappa^2} - \frac{\sin 2\beta}{\kappa^2} \frac{\partial}{\partial \theta} + \frac{\sin 2\beta}{\kappa} \frac{\partial^2}{\partial \kappa \partial \theta} \\ &\quad + \left(\frac{1 - \cos 2\beta}{\kappa} \right) \frac{\partial}{\partial \kappa} + \left(\frac{1 - \cos 2\beta}{\kappa^2} \right) \frac{\partial^2}{\partial \theta^2} \end{aligned} \quad (\text{A.24})$$

$$\begin{aligned} \text{and: } \frac{\partial^2}{\partial t_Q^2} &= \left(-\sin \beta \frac{\partial}{\partial \kappa} + \frac{\cos \beta}{\kappa} \frac{\partial}{\partial \theta} \right) \left(-\sin \beta \frac{\partial}{\partial \kappa} + \frac{\cos \beta}{\kappa} \frac{\partial}{\partial \theta} \right) \\ &= \left(\frac{1 - \cos 2\beta}{2} \right) \frac{\partial^2}{\partial \kappa^2} + \frac{\sin 2\beta}{\kappa^2} \frac{\partial}{\partial \theta} - \frac{\sin 2\beta}{\kappa} \frac{\partial^2}{\partial \kappa \partial \theta} \\ &\quad + \left(\frac{1 + \cos 2\beta}{2\kappa} \right) \frac{\partial}{\partial \kappa} + \left(\frac{1 + \cos 2\beta}{2\kappa^2} \right) \frac{\partial^2}{\partial \theta^2} \end{aligned} \quad (\text{A.25})$$

In deriving equations (A.24) and (A.25), the following trigonometric identities have been used:

$$\sin 2\beta = 2 \sin \beta \cos \beta \quad (\text{A.26})$$

$$\cos^2 \beta = \frac{1 + \cos 2\beta}{2} \quad (\text{A.27})$$

$$\sin^2 \beta = \frac{1 - \cos 2\beta}{2} \quad (\text{A.28})$$

Substitution of equations (A.24) and (A.25) into equation (A.17) gives:

$$m_n = \frac{D}{2} \left\{ - (1+\nu) \nabla^2 w + (1-\nu) \left[\left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} - \frac{\partial^2 w}{\partial r^2} \right) \cos 2\beta \right. \right. \\ \left. \left. - 2 \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial w}{\partial \theta} \right) \sin 2\beta \right] \right\} \quad (A.29)$$

In a similar manner, the mixed derivative $\partial^2 w / (\partial n_Q \partial t_Q)$ is obtained as:

$$\begin{aligned} \frac{\partial^2 w}{\partial n_Q \partial t_Q} &= \left(\cos \beta \frac{\partial}{\partial r} + \frac{\sin \beta}{r} \frac{\partial}{\partial \theta} \right) \left(- \sin \beta \frac{\partial}{\partial r} + \frac{\cos \beta}{r} \frac{\partial}{\partial \theta} \right) \\ &= - \frac{\sin 2\beta}{2} \frac{\partial^2}{\partial r^2} + \frac{1 + \cos 2\beta}{2r} \frac{\partial^2}{\partial r \partial \theta} - \frac{1 + \cos 2\beta}{r^2} \frac{\partial}{\partial \theta} \\ &\quad - \frac{1 - \cos 2\beta}{2r} \frac{\partial^2}{\partial r \partial \theta} + \frac{\sin 2\beta}{r} \frac{\partial}{\partial r} + \frac{\sin 2\beta}{r^2} \frac{\partial^2}{\partial \theta^2} \\ &\quad + \frac{1 - \cos 2\beta}{2r^2} \frac{\partial}{\partial \theta} \end{aligned} \quad (A.30)$$

Substitution of equation (A.30) into equation (A.18) gives:

$$m_t = - \frac{D (1-\nu)}{2} \left\{ \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} - \frac{\partial^2 w}{\partial r^2} \right) \sin 2\beta + 2 \cos 2\beta \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial w}{\partial \theta} \right) \right\} \quad (A.31)$$

The derivation of the expression for the effective shear force, equation (A.19), requires some care due to the presence of a curvature term. It can be shown (see Appendix B) that:

$$\frac{\partial \beta}{\partial S} = \frac{1}{R} - \frac{\cos \beta}{r} \quad (A.32)$$

$$\frac{\partial r}{\partial S} = - \sin \beta \quad (A.33)$$

$$\frac{\partial \theta}{\partial S} = \frac{\cos \beta}{r} \quad (A.34)$$

where R^{-1} is the curvature of the boundary at point Q (see Figure A.2).

Furthermore, the expression for m_t is a function of β , as well as κ and θ .

Therefore, using the chain rule:

$$\frac{\partial m_t}{\partial S} = \frac{\partial m_t}{\partial \kappa} \frac{\partial \kappa}{\partial S} + \frac{\partial m_t}{\partial \theta} \frac{\partial \theta}{\partial S} + \frac{\partial m_t}{\partial \beta} \frac{\partial \beta}{\partial S} \quad (\text{A.35})$$

Using equations (A.32) to (A.34), this gives:

$$\frac{\partial m_t}{\partial S} = - \frac{\partial m_t}{\partial \kappa} \sin \beta + \frac{\cos \beta}{\kappa} \left(\frac{\partial m_t}{\partial \theta} - \frac{\partial m_t}{\partial \beta} \right) + \frac{1}{R} \frac{\partial m_t}{\partial \beta} \quad (\text{A.36})$$

Finally, $(\partial/\partial n) \nabla^2 \omega$ can be written using equation (A.20) as:

$$\frac{\partial}{\partial n_Q} \nabla^2 \omega = \cos \beta \frac{\partial}{\partial \kappa} \nabla^2 \omega + \frac{\sin \beta}{\kappa} \frac{\partial}{\partial \theta} \nabla^2 \omega \quad (\text{A.37})$$

Substitution of equations (A.36) and (A.37) into equation (A.19) gives:

$$\begin{aligned} v_n = & \left[- \mathcal{D} \frac{\partial}{\partial \kappa} \nabla^2 \omega + \frac{1}{\kappa} \left(\frac{\partial m_t}{\partial \theta} - \frac{\partial m_t}{\partial \beta} \right) \right] \cos \beta \\ & - \left[\frac{\mathcal{D}}{\kappa} \frac{\partial}{\partial \theta} \nabla^2 \omega + \frac{\partial m_t}{\partial \kappa} \right] \sin \beta + \frac{1}{R} \frac{\partial m_t}{\partial \beta} \end{aligned} \quad (\text{A.38})$$

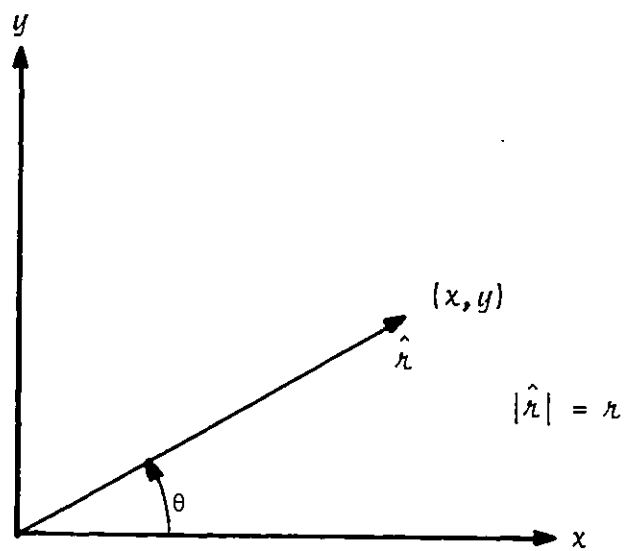


Figure A.1: Position vector $\hat{h}$ in two-dimensional rectangular Cartesian coordinates

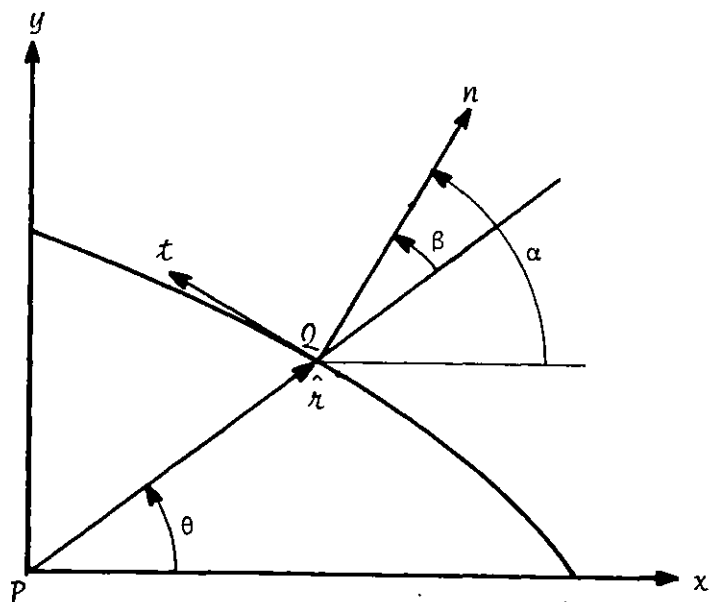


Figure A.2: Notation for normal and tangent at a boundary point

APPENDIX B

PROOF OF SOME EQUATIONS USED IN THIS WORK

(a) To Show That $\frac{\partial}{\partial n_Q} = \cos \beta \left(\frac{\partial}{\partial r} \right) + \left(\sin \beta / r \right) \left(\frac{\partial}{\partial \theta} \right)$

A directional derivative of a function, ϕ , in any particular direction, n , is defined as:

$$\frac{\partial \phi}{\partial n} = \nabla \phi \cdot \hat{n} \quad (B.1)$$

where $\nabla = \left[\left(\frac{\partial}{\partial x} \right) i + \left(\frac{\partial}{\partial y} \right) j \right]$ is the gradient operator, and $\hat{n}$ denotes unit vector in the direction n (see, for example, Kreyszig [1979]). Referring to Figure B.1, equations (A.3) can be written as:

$$x_Q - x_p = r \cos \theta \quad (B.2a)$$

$$y_Q - y_p = r \sin \theta \quad (B.2b)$$

Therefore, equations (A.11) and (A.12) take the form:

$$\frac{\partial}{\partial x_Q} = \cos \theta \frac{\partial}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta} \quad (B.3)$$

$$\frac{\partial}{\partial y_Q} = \sin \theta \frac{\partial}{\partial r} + \frac{1}{r} \cos \theta \frac{\partial}{\partial \theta} \quad (B.4)$$

Upon noting that a unit vector in the direction n is given by (see Figure B.2):

$$\hat{n} = \cos \alpha i + \sin \alpha j \quad (B.5)$$

equation (B.1) can be written as:

$$\frac{\partial \phi}{\partial n_Q} = \left(\frac{\partial}{\partial x_Q} i + \frac{\partial}{\partial y_Q} j \right) (\cos \alpha i + \sin \alpha j) \quad (B.6)$$

Substituting equations (B.3) and (B.4) for $\partial/\partial x_Q$ and $\partial/\partial y_Q$, respectively, in equation (B.6) gives:

$$\begin{aligned} \frac{\partial \phi}{\partial n_Q} = & \cos \theta \cos \alpha \frac{\partial}{\partial \kappa} - \frac{1}{\kappa} \sin \theta \cos \alpha \frac{\partial}{\partial \theta} \\ & + \sin \alpha \sin \theta \frac{\partial}{\partial \kappa} + \frac{1}{\kappa} \sin \alpha \cos \theta \frac{\partial}{\partial \theta} \end{aligned} \quad (B.7)$$

which reduces to:

$$\frac{\partial \phi}{\partial n_Q} = \cos \beta \frac{\partial}{\partial \kappa} + \frac{1}{\kappa} \sin \beta \frac{\partial}{\partial \theta} \quad (B.8)$$

where $\beta = \alpha - \theta$. The directional derivative in the tangential direction can readily be obtained by substituting $\beta + \pi/2$ in place of β in equation (B.8). This gives:

$$\frac{\partial \phi}{\partial \tau_Q} = -\sin \beta \frac{\partial}{\partial \kappa} + \frac{1}{\kappa} \cos \beta \frac{\partial}{\partial \theta} \quad (B.9)$$

Referring to equations (B.2), it can be seen that $\partial/\partial x_Q = -\partial/\partial x_p$ and $\partial/\partial y_Q = -\partial/\partial y_p$. The directional derivative at point p in a direction making an angle ψ with the vector $\hat{\kappa}$ is therefore given by:

$$\frac{\partial \phi}{\partial n_p} = -\cos \psi \frac{\partial}{\partial \kappa} - \frac{1}{\kappa} \sin \psi \frac{\partial}{\partial \theta} \quad (B.10)$$

and for a direction τ_p perpendicular to n_p (see Figure B.1):

$$\frac{\partial \phi}{\partial \tau_p} = \sin \psi \frac{\partial}{\partial \kappa} - \frac{1}{\kappa} \cos \psi \frac{\partial}{\partial \theta} \quad (B.11)$$

(b) To Show That $\partial\theta/\partial S = (\cos \beta/\kappa)$ and $\partial\kappa/\partial S = -\sin \beta$

This can be shown by observing that $\partial/\partial\tau = \partial/\partial S$ and using equation (A.21) of Appendix A. This gives:

$$\frac{\partial\theta}{\partial S} = \frac{\partial\theta}{\partial\tau_Q} = \frac{\cos \beta}{\kappa} \quad (\text{B.12})$$

where the symbols have their usual meaning (see Figure B.1). Similarly, $\partial\kappa/\partial S$ is given as:

$$\frac{\partial\kappa}{\partial S} = \frac{\partial\kappa}{\partial\tau_Q} = -\sin \beta \quad (\text{B.13})$$

(c) To Show That $\partial\beta/\partial S = 1/R - (\cos \beta/\kappa)$

Referring to Figure B.1:

$$\sin \beta = -\frac{\hat{\kappa}}{\kappa} \cdot \frac{\partial\hat{\kappa}}{\partial S} \quad (\text{B.14})$$

where κ is the modulus of the position vector $\hat{\kappa}$. Equation (B.14) is simply the dot product of a unit vector along $\hat{\kappa}$ and the unit tangent vector given by $\partial\hat{\kappa}/\partial S$. Differentiation of equation (B.14) with respect to S gives:

$$\frac{\partial\beta}{\partial S} \cos \beta = -\frac{\partial}{\partial S} \left(\frac{\hat{\kappa}}{\kappa} \right) \cdot \frac{\partial\hat{\kappa}}{\partial S} - \frac{\hat{\kappa}}{\kappa} \cdot \frac{\partial^2\hat{\kappa}}{\partial S^2} \quad (\text{B.15})$$

Now, a standard result from differential geometry gives (see Lipschutz [1969]):

$$\frac{\partial^2\hat{\kappa}}{\partial S^2} = \left(-\frac{1}{R} \right) \cdot \hat{n} \quad (\text{B.16})$$

where $\hat{n}$ is the unit normal vector at Q , and R is the radius of curvature of the boundary at Q . Substituting equation (B.16) into equation (B.15)

gives:

$$\frac{\partial \beta}{\partial S} \cos \beta = - \frac{\partial}{\partial S} \left(\frac{\hat{h}}{h} \right) \cdot \frac{\partial \hat{h}}{\partial S} + \frac{1}{R} \frac{\hat{h}}{h} \cdot \hat{n} \quad (\text{B.17})$$

$(\hat{h}/h) \cdot \hat{n} = \cos \beta$ because it is the dot product of the unit normals in the $\hat{h}$ and $\hat{n}$ directions. Equation (B.17) can, therefore, be written as:

$$\frac{\partial \beta}{\partial S} \cos \beta = \frac{1}{R} \cos \beta - \left\{ \frac{h \frac{\partial \hat{h}}{\partial S} \cdot \frac{\partial \hat{h}}{\partial S} - \frac{\partial h}{\partial S} \hat{h} \cdot \frac{\partial \hat{h}}{\partial S}}{h^2} \right\} \quad (\text{B.18})$$

Noting that $\hat{h} \cdot (\partial \hat{h} / \partial S) = h (\partial h / \partial S)$ (see Spiegel [1974]) and $(\partial \hat{h} / \partial S) \cdot (\partial \hat{h} / \partial S) = 1$:

$$\frac{\partial \beta}{\partial S} = \frac{1}{R} - \frac{1}{\cos \beta} \left[\frac{h - h \left(\frac{\partial h}{\partial S} \right)^2}{h^2} \right] \quad (\text{B.19})$$

Using equation (B.13) to replace $\partial h / \partial S$ gives:

$$\frac{\partial \beta}{\partial S} = \frac{1}{R} - \frac{\cos \beta}{h} \quad (\text{B.20})$$

which is the required result.

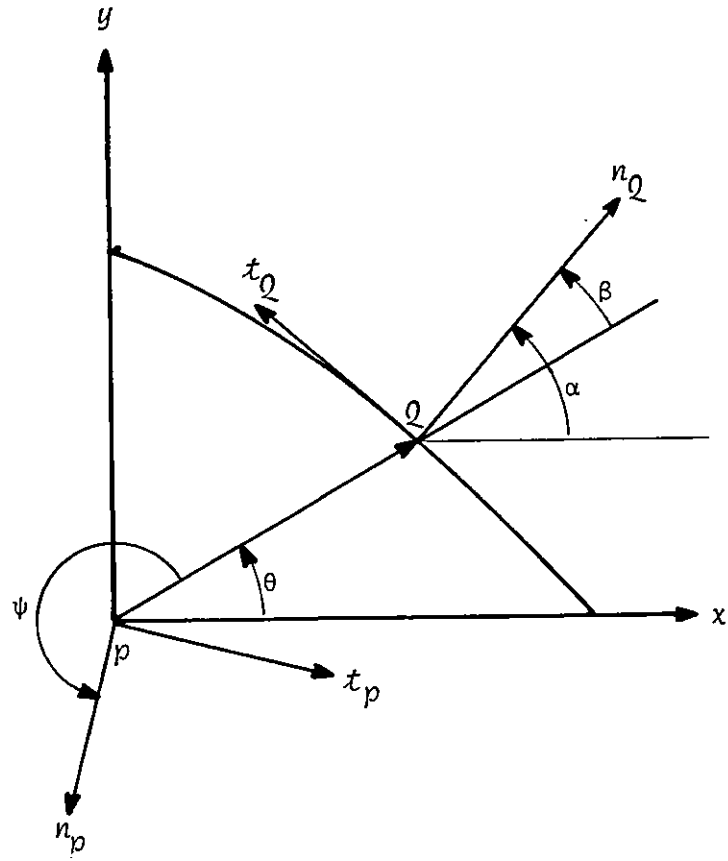


Figure B.1: Notation for obtaining directional derivatives at p and Q

APPENDIX C

VERIFICATION OF THE FUNDAMENTAL SOLUTION

$$\underline{w = (1/8\pi) \kappa^2 \ln \kappa}$$

The fundamental solution for the biharmonic equation is given in Chapter 2 as:

$$w(p,q) = \frac{1}{8\pi} \kappa^2 \ln \kappa \quad (C.1)$$

where κ is the distance between the source point p and the field point q . The constant, $\mathcal{D}$, in equation (2.75) is the flexure rigidity of the plate, and for the purpose of the present analysis, need not be included. The function to be considered is, therefore, given by $w = 1/8\pi \kappa^2 \ln \kappa$ and it must be shown that:

$$\nabla^4 w = \nabla^4 \left(\frac{1}{8\pi} \kappa^2 \ln \kappa \right) = \begin{matrix} 0 & \text{for } p \neq q \\ \infty & \text{for } p = q \end{matrix} \quad (C.2)$$

$$\text{and:} \quad \iint_A \nabla^4 w \, dA = 1 \quad (C.3)$$

where A is the area of the plate domain. Using the biharmonic operator given by equation (A.16) of Appendix A and noting that equation (C.1) is a function of κ only gives:

$$\begin{aligned} \nabla^4 w &= \frac{1}{8\pi} \left(-\frac{2}{\kappa^2} + \frac{4}{\kappa^2} - \frac{2 \ln \kappa}{\kappa^2} - \frac{3}{\kappa^2} + \frac{2 \ln \kappa}{\kappa^2} + \frac{1}{\kappa^2} \right) \\ &= \frac{1}{8\pi} \left[\frac{1}{\kappa^2} (-2 + 4 - 3 + 1) - \frac{2 \ln \kappa}{\kappa^2} + \frac{2 \ln \kappa}{\kappa^2} \right] \end{aligned} \quad (C.4)$$

Equation (C.4) is equal to zero for all values of $\kappa \neq 0$. However, for $\kappa = 0$, it is undefined, i.e. it has a singularity.

In order to verify equation (C.3), Green's second identity is used

(see Spiegel [1974]). This is given as:

$$\iint_A (u \nabla^2 v - v \nabla^2 u) dA = \int_S (u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}) dS \quad (C.5)$$

Now, choosing $u = 1$ and $v = \nabla^2 w$ in equation (C.5) gives:

$$\iint_A \nabla^4 w dA = \int_S \frac{\partial}{\partial n} \nabla^2 w dS \quad (C.6)$$

Figure C.1 shows a plate domain A with its boundary curve S . The point P is isolated from A by excluding a small disc, A_ϵ , of radius ϵ from the domain A . The domain integral given by equation (C.3) is now written as:

$$\iint_A \nabla^4 w dA = \iint_{A-A_\epsilon} \nabla^4 w dA + \iint_{A_\epsilon} \nabla^4 w dA \quad (C.7)$$

The first integral on the right hand side of equation (C.7) is equal to zero because $\nabla^4 w = 0$ in $A - A_\epsilon$. The second integral on the right hand side of equation (C.7) is now written with the help of equation (C.6) as:

$$\iint_{A_\epsilon} \nabla^4 w dA = \int_{S_\epsilon} \frac{\partial}{\partial n} \nabla^2 w dS \quad (C.8)$$

Since, along the contour S_ϵ , the direction of the normal is the same as the radius r , equation (C.8) can be written as:

$$\iint_{A_\epsilon} \nabla^4 w dA = \int_{S_\epsilon} \frac{\partial}{\partial r} \nabla^2 w dS \quad (C.9)$$

Using equation (A.15) of Appendix A and equation (C.1) gives:

$$\frac{\partial}{\partial r} \nabla^2 w = \frac{1}{8\pi} \frac{4}{r} \quad (C.10)$$

Substituting equation (C.10) into equation (C.9) and noting that along S_ϵ , $dS = r d\theta$, gives:

$$\iint_{A_\epsilon} \nabla^4 w \, dA = \int_0^{2\pi} \frac{1}{8\pi} \left(\frac{4}{r}\right) r \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} d\theta = 1 \quad (\text{C.11})$$

which completes the proof.

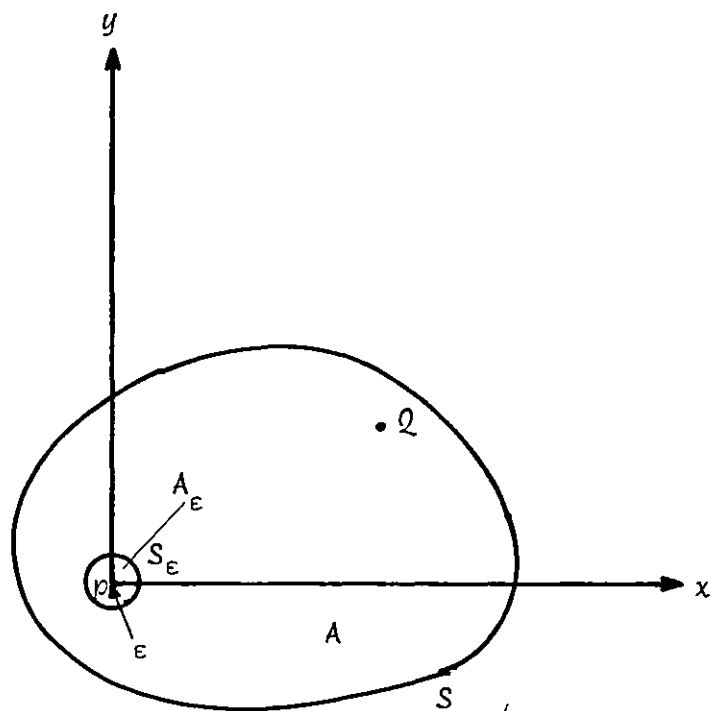


Figure C.1: Domain A_ϵ excluded from domain A

APPENDIX D

CONTRIBUTION TO TANGENTIAL DERIVATIVE OF DEFLECTION AT

A REGULAR POINT ON THE BOUNDARY IF THE

ELEMENTS EITHER SIDE ARE OF UNEQUAL LENGTH

Consider the point P^a to be an inter-element node at a regular point on the boundary. For simplicity, a straight line boundary is considered here, as shown in Figure D.1. The integration of the term containing $(w_Q - w_P)$ in equation (4.11) over these two elements can be written as:

$$I = \lim_{\epsilon \rightarrow 0} \left\{ \int_{-L_1}^{-\epsilon} \frac{\phi}{h^2} (w_Q - w_P) dS + \int_{\epsilon}^{L_2} \frac{\phi}{h^2} (w_Q - w_P) dS \right\} \quad (D.1)$$

where $\phi = V_2 h^2$, and 1 and 2 refer to elements 1 and 2 in Figure D.1. A Taylor series expansion of $(w_Q - w_P)$ about P^a gives:

$$\frac{w_Q - w_P}{h} = - \frac{\partial w}{\partial \bar{x}}(P^a) \quad \text{on 1} \quad (D.2)$$

$$\text{and:} \quad \frac{w_Q - w_P}{h} = \frac{\partial w}{\partial \bar{x}}(P^a) \quad \text{on 2} \quad (D.3)$$

where the terms of the order of h^2 are ignored, and $\partial w / \partial \bar{x}(P^a)$ refers to the tangential derivative of deflection at point P^a . Substitution of equations (D.2) and (D.3) into equation (D.1) gives:

$$I = \lim_{\epsilon \rightarrow 0} \left\{ \int_{-L_1}^{-\epsilon} - \frac{\phi}{h} \frac{\partial w}{\partial \bar{x}}(P^a) dS + \int_{\epsilon}^{L_2} \frac{\phi}{h} \frac{\partial w}{\partial \bar{x}}(P^a) dS \right\} \quad (D.4)$$

Noting that:

$$\left. \begin{aligned} S &= -h \\ dS &= -dr \end{aligned} \right\} \quad \text{on 1} \quad (D.5)$$

$$\text{and:} \quad \left. \begin{aligned} S &= h \\ dS &= dr \end{aligned} \right\} \quad \text{on 2} \quad (D.6)$$

gives:

$$I = \frac{\partial \omega}{\partial \mathcal{E}}(p^a) \lim_{\epsilon \rightarrow 0} \left\{ \int_{L_1}^{\epsilon} \frac{\phi}{\kappa} dr + \int_{\epsilon}^{L_2} \frac{\phi}{\kappa} dr \right\} \quad (D.7)$$

Recalling the expression for the effective shear kernel, V_2 (equation (2.97d)), it is seen that in this case $\phi = \kappa^2 V_2$ is a constant and can be taken out of the integral sign. Equation (D.7), therefore, becomes:

$$I = \phi \frac{\partial \omega}{\partial \mathcal{E}}(p^a) \lim_{\epsilon \rightarrow 0} \left\{ [\ln \kappa]_{L_1}^{\epsilon} + [\ln \kappa]_{\epsilon}^{L_2} \right\} \quad (D.8)$$

$$I = \phi \frac{\partial \omega}{\partial \mathcal{E}}(p^a) \lim_{\epsilon \rightarrow 0} \left\{ -\ln L_1 + \ln L_2 \right\} \quad (D.9)$$

$$I = \phi \frac{\partial \omega}{\partial \mathcal{E}}(p^a) \ln \frac{L_2}{L_1} \quad (D.10)$$

It is clear that for $L_1 = L_2$, this contribution is zero. Since the formulation presented in this work does not allow for a tangential derivative as a nodal variable, the incorporation of this contribution for each element can be obtained by differentiating the element shape functions in a similar manner to the evaluation of the contribution from $\partial^2 \omega / \partial \kappa^2$ in equation (4.67).

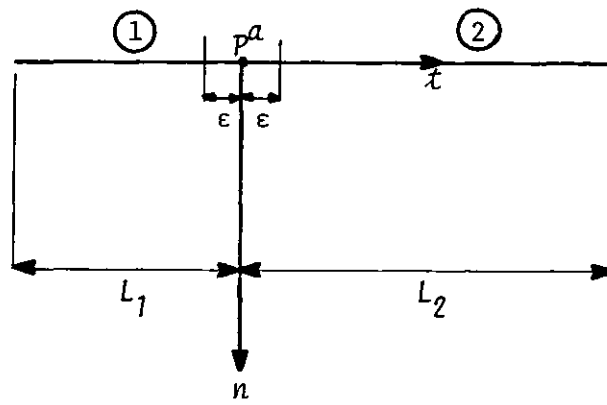


Figure D.1: Two self elements of unequal length

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