Comment on "Anomalous Discontinuity at the Percolation Critical Point of Active Gels"

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(Dated: January 29, 2016)

In their recent work Sheinman *et al.* introduce a variation of percolation which they call no-enclaves percolation (NEP). The main claims are 1) the salient physics captured in NEP is closer to what happens experimentally; 2) The Fisher exponent of NEP, is $\tau = 1.82(1)$; 3) Due to the different Fisher exponent, NEP constitutes a universality class distinct from random percolation (RP). While we fully agree with 1) and found NEP to be a very interesting variation of random percolation, we disagree with 2) and 3). We will demonstrate that τ is exactly 2, directly derivable from RP, and thus there is no foundation of a new universality class.

In [1] NEP-clusters are introduced as the union of all RP clusters with their largest enclosing RP cluster. It is argued that their histogram n_s, that is the average number of NEP-clusters of size s (number of sites) per realisation of a d-dimensional system with M sites, is given by $n_s \sim M^{\tau-1}s^{-\tau}$ with exponent $\tau < 2$. Strictly, this cannot be the scaling form of a finite system, because it lacks a cutoff or scaling function. Assuming standard finite size scaling [2], one may redefine $n_s = M^{\tau-1}s^{-\tau}\mathcal{G}(s/s_c)$, with scaling function $\mathcal{G}(x)$ and cutoff s_c, which requires further qualification, because it leaves the exponent τ undefined, unless the choice of $\mathcal{G}(x)$ is constrained. One may demand $\lim_{x\to 0}\mathcal{G}(x) = \mathcal{G}_0$ positive and finite, which renders τ the *apparent* exponent [3], that is visible as the slope of the intermediate asymptote in double logarithmic plots, provided the scaling function is essentially constant. This is extremely difficult to ascertain and to test [4, 5].

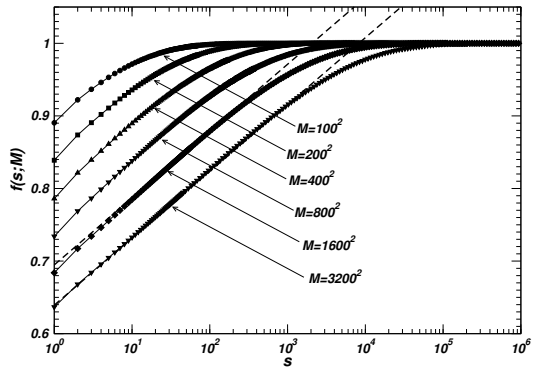


FIG. 1: Log-linear plot of the fraction $h(s;M)/\tilde{h}(s;M) = f(s;M)$ shown (binned) for $M = 100^2, \ldots, 3200^2$, which is merely a logarithmic correction; Full lines to guide the eye. The dashed lines are linear approximations, $1.285 + 0.04\ln(s/M)$.

An unambiguous and commonly used definition of the scaling exponent τ is $h(s;M) = s^{-\tau}\mathcal{G}(s/s_c(M))$ for $s \gg s_0$, the lower cutoff, demanding that all dependence on the cutoff $s_c(M)$ (which includes finite size) is contained inside the scaling function. With this definition, the classical Fisher exponent is recovered in finite size scaling.

We will now use that equation to characterise the scaling of the *site-normalised* histograms, as normally used in percolation [6], of the following models: $h(s;M) = n_s/M$ (NEP), $\tilde{h}(\tilde{s};M)$ (nested NEP or nNEP to be introduced below) and $h'(s';M)$ (RP), having exponents τ, $\tilde{\tau}$ and τ' respectively.

The RP site-normalised histogram $h'(s';M)$ follows $s'^{-\tau'}\mathcal{G}'(s'/M^{d_f/d})$ where $\tau' = 187/91$ is the Fisher exponent in $d = 2$ dimensions and the clusters' fractal dimension d_f is 91/48 [6].

Next, we introduce nNEP, which consists of clusters from RP but with all of the interior filled up, *i.e.* the histogram $\tilde{h}(\tilde{s};M)$ counts cluster sizes $\tilde{s}$, which are the total areas inside the outermost perimeter of every RP cluster [7]. Because nNEP clusters are compact, their size $\tilde{s}$ is related to that of their "hosting" RP cluster of size s' by $\tilde{s} = s'^{d/d_f}$, so that their histogram $\tilde{h}(\tilde{s};M)$ follows $(d_f/d)\tilde{s}^{-2}\mathcal{G}'((\tilde{s}/M)^{d_f/d})$, using $-\tau' d_f/d - 1 + d_f/d = -2 = -\tilde{\tau}$.

By construction, nNEP differs from NEP only by the latter discounting all but the largest cluster of a set of nested clusters. To construct $h(s;M)$ of NEP from $\tilde{h}(s;M)$ of nNEP, we consider the fraction of those not discounted, $f(s;M) = h(s;M)/\tilde{h}(s;M)$, *i.e.* the fraction of those clusters which feature in both histograms. In the presence of scale invariance $f(s;M)$ is necessarily a function only of s/M, so that $f(s;M) = F(s/M)$, and can therefore be absorbed into the scaling function, $h(s;M) = \tilde{h}(s;M)F(s/M) = (d_f/d)s^{-2}\mathcal{G}'((s/M)^{d_f/d})F(s/M)$. This *proves* that $\tau = \tilde{\tau} = -2$ exactly. To determine the value of the *apparent* exponent, we study $f(s;M)$ directly and find that it is merely a very small, logarithmic correction, as shown in Fig. 1, which is a log-linear plot with a remarkably small range at the ordinate. For small arguments, $f(s;M)$ follows $a + b\ln(s/M)$, before reaching unity and staying there for $s/M \gtrsim 0.04$; It is clearly not a power law.

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In summary, NEP is RP with a different observable, which is not the order parameter, thus allowing for a "discontinuity" in the approach of the transition. It is an important model to understand recent experimental work [8], but it does not represent a novel universality class. The exponent τ was strongly underestimated in [1] due to an unsuitable definition, and if measured correctly is exactly 2.

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