

~~THE~~ INTERACTION OF
FREE AND FORCED CONVECTIVE
HEAT OR MASS TRANSFER FROM
BODIES OF REVOLUTION

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ABSTRACT

The interaction of free and forced convective heat or mass transfer from bodies of revolution and the vertical flat plate has been investigated for both aiding and opposing flows.

The momentum, continuity and transfer equations have been expressed in terms of an orthogonal curvilinear coordinate system. These equations were reduced to a pair of coupled partial differential equations by the introduction of the boundary layer simplifications.

An integral method was used to reduce the two coupled partial differential equations to a pair of coupled ordinary differential equations.

These equations were solved in order to predict local transfer rates from spheres and oblate spheroids with ratios of the minor to the major axis of 13/16, 10/16, 7/16 and 4/16. Solutions have been calculated for a range of values of the parameter Grashof number divided by the square of the Reynolds number, $0.0 \leq Gr/Re^2 \leq 100.0$, and for Schmidt numbers, $0.72 \leq Sc \leq 1000.0$. The former parameter indicated the relative predominance of either free or forced convection.

The integral method was found to fail over part of the range of values of the parameter Gr/Re^2 . The extent of the failure was a function of the Schmidt number. For a Schmidt number of 2.4 the range was $0.2 < Gr/Re^2 < 2.0$, whereas for a Schmidt number of 100.0 the range was $0.2 < Gr/Re^2 < 80.0$.

A number of modifications were investigated in an attempt to improve the range of values of the parameter Gr/Re^2 over which the method failed. The order of the assumed polynomials for the velocity and concentration profiles was found to have

no influence on the failure. The range was extended by the use of a scaling of the hydrodynamic boundary layer equation over the region of the transfer boundary layer. By the use of this modification, the integral method could be used up to a value of the parameter Gr/Re^2 of 50.0 for a Schmidt number of 100.0.

An exact series method was also developed to reduce the coupled partial differential equations to two infinite sets of coupled ordinary differential equations.

The first equation of each set was independent of shape and the solution of the first pair of equations represented the stagnation point solution. The two point boundary value problem associated with the solution of these two equations was solved by a quasilinearization technique.

An exact solution for the vertical flat plate in a uniform free stream was also developed.

The approximate solutions for the stagnation point and the vertical flat plate were compared with the exact solutions. When the approximate solutions could be used for the interaction region, the predicted transfer rates were shown to be in excellent agreement with the exact solutions. The approximate solutions for the local transfer rates for the limiting cases of free convection and forced convection were shown to be in very good agreement with other workers.

The extent of the interaction region was found to be dependent upon the ~~upon the~~ Schmidt number. However, in general, the interaction region was found to lie between values of the parameter Gr/Re^2 of 0.1 and 30.0.

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<u>Symbol</u>	<u>Meaning</u>
$\underline{a}$	Arbitrary vector.
a_C, b_C, c_C, d_C, e_C	Coefficients in the polynomial which represents the concentration profile for the approximate method.
a_H, b_H, c_H, d_H, e_H	Coefficients in the polynomial which represents the velocity profile for the approximate method.
A	Arbitrary constant.
$\underline{b}$	Arbitrary vector.
B	Arbitrary constant.
c	The radius of the trajectory of the focus of the oblate spheroid.
$\bar{c}$	$(g\beta\Delta C/4\nu^2)^{1/4}$ for mass transfer, $(g\beta\Delta T/4\nu^2)^{1/4}$ for heat transfer.
C_f	Friction factor.
C_p	Specific heat at constant pressure.
C	Concentration.
ΔC	Defined as $C_0 - C_\infty$
C_1, C_2, C_3	Arbitrary constants.
C^*	Dimensionless concentration defined as $\left(\frac{C - C_\infty}{C_0 - C_\infty} \right)$
$C^{****}, C^{***}, C^{**}$	Dimensionless concentrations, all of which are equal to C^* .
$\bar{d}, d'$	Semi - major axis of oblate spheroid, $c \cosh \xi_0$
D	Arbitrary constant.
D_C	Characteristic dimension.
$\mathcal{D}$	Coefficient of diffusivity.
$\underline{e}_1, \underline{e}_2, \underline{e}_3$	Unit tangent vectors associated with the orthogonal curvilinear coordinate system, (x_1, x_2, x_3)

f	Arbitrary function, variously defined.
$\underline{f}$	n - dimensional vector.
$f_1(\eta), f_3(\eta) \dots$	Functions in power series expansion for the stream function $\mathcal{F}$.
$\bar{f}$	The semi - minor axis of oblate spheroid, $c \sinh \xi_0$.
$f(\lambda)$	Function of λ defined by equation (4-66).
$\underline{F}$	Vector representing the body forces.
$F_0(\eta), F_2(\eta) \dots$	Functions in the power series for the concentration distribution.
g	Acceleration due to gravity.
$g(\eta_c)$	Function which represents the concentration profile.
g_1, g_2, g_3	Components of the gravity force in the three orthogonal curvilinear coordinate directions.
$\underline{g}$	Gravity vector.
g_1^*, g_2^*, g_3^*	Dimensionless gravitational components, $g_1^* = g_1 / \underline{g}$, etc.
$g(\lambda)$	Function defined by equation (4-67).
Gr	Grashof number ($g \beta D_C^3 \Delta T / \nu^2$; heat transfer $g \beta D_C^3 \Delta C / \nu^2$; mass transfer).
$\overline{Gr}$	Grashof number $4g \beta \bar{I}_0^3 \Delta C / \nu^2$.
h, H	Integration step length.
$\bar{h}$	Overall heat transfer coefficient.
h_1, h_2, h_3	Scale factors in the coordinate directions (x_1, x_2, x_3)
h_1^*, h_2^*, h_3^*	Dimensionless scale factors : $h_1^* = h_1 / D_C$, etc.
h_ξ, h_η, h_ϕ	Scale factors in oblate spheroidal coordinates (ξ, η, ϕ).

$h(\lambda)$	Function defined by equation (4-68).
$j(\lambda)$	Function defined by equation (4-69).
J	Jacobi matrix.
k	Thermal conductivity.
k_c, K'	Mass transfer coefficient.
$K'(x_1)$	Local mass transfer coefficient.
$k_1, k_3, k_5 \dots$	Dimensionless coefficients in the power series equation (5-15).
$\bar{k}_1, \bar{k}_3, \bar{k}_5 \dots$	Coefficients in power series expansion equation (5-11).
$k(\lambda)$	Function defined by equation (4-70).
K	$\cosh^2 \zeta_0 \tan^{-1} (1/\sinh \zeta_0) - \sinh \zeta_0$.
$l(\lambda)$	Function defined by equation (4-71).
$\bar{l}_0, \bar{l}_2 \dots$	Coefficients in power series expansion, equation (5-10).
$l_0, l_2 \dots$	Dimensionless coefficients in power series expansion equation (5-14).
L	Characteristic length.
L_Z	The right hand side of equation (4-89).
L_X	The right hand side of equation (4-90).
m	Arbitrary constant.
$m(X)$	Function of X defined by equation (4-76).
M_L	Mass transfer number $\overline{Sh} / \overline{Re}^2 Sc^{\frac{1}{3}}$
n	Arbitrary constant.
$n(X)$	Function of X defined by equation (4-77).
$\underline{n}$	Outward normal vector at body surface.
N^*	Dimensionless geometric normal at the body surface.
Nu	Nusselt number $\bar{h} D_C / k$
$\overline{Nu}$	Nusselt number $\bar{h} \bar{l}_0 / k$

Nu_{forced}	Nusselt number for forced convection.
Nu_{free}	Nusselt number for free convection.
Nu_0	Nusselt number for pure conduction.
∇P	Pressure gradient.
P^*	Dimensionless pressure $P / \rho U_\infty^2$
Pr	Prandtl number $C_p \mu / k$, ν / α
q	The mass flux at the body surface.
$q(x_1)$	The local mass flux at the body surface.
r	Radius of transverse cross section.
$r(x)$	Radius of transverse cross section at position x .
r^*	Dimensionless radius of transverse cross section r / D_C .
r^{****} , r^{***} , r^{**}	Dimensionless radii of transverse cross section, all of which are equal to r^* .
$\underline{r}$	Position vector.
R	Radius of sphere.
Re	Reynolds number $U_\infty D_C / \nu$
$\overline{Re}$	Reynolds number $2\bar{u}_1 \bar{l}_0 / \nu$
Re_I	Reynolds number used by Ingram ³⁷ based upon the characteristic dimension of Pasternak and Gauvin ⁸⁶ .
Re_{forced}	Reynolds number due to forced convection only.
Re_{equiv}	Equivalent Reynolds number defined by equation (2-6).
Sc	Schmidt number $\nu / \mathcal{D}$, ν / α
Sh	Sherwood number $k_c D_C / \mathcal{D}$.
$\overline{Sh}$	Sherwood number $K' \bar{l}_0 / \mathcal{D}$.

Sh_o	Sherwood number for forced convection transfer.
Sh_{x_1}	Point Sherwood number.
t	Time.
t_f	The final value of t .
T	Temperature.
ΔT	Defined as $T_o - T_\infty$
u	Velocity component parallel to the body surface.
$\bar{u}_1, \bar{u}_3, \bar{u}_5$	Coefficients in power series expansion in equation (5-9).
u_1, u_3, u_5	Dimensionless coefficients in power series expansion in equation (5-13).
u^*	Dimensionless velocity $u^* = u / U_\infty$
u^{***}, u^{**}, u_o	Dimensionless velocities $= u^*$.
u^{****}, u_i	Dimensionless velocities $= u^{**} Sc^{\frac{1}{2}}$.
U	Potential flow velocity calculated at the body surface.
U^*	Dimensionless velocity $U^* = U / U_\infty$
U_∞	Free stream velocity.
$U^{****}, U^{***}, U^{**}$	Dimensionless velocities $= U^*$.
U_L	Limiting value of U^* defined by equation (4-91).
U_L'	Limiting value of $dU^* / d\Theta$ defined by equation (4-91).
v	Velocity component normal to the body surface.
$\underline{v}$	Velocity vector.
v^*	Dimensionless velocity $v^* = v / U_\infty$
v^{**}	Dimensionless velocity $= v^* Re^{\frac{1}{2}}$.

v^{***}	Dimensionless velocity = $v^{**} (Re^2/Gr)^{\frac{1}{2}}$
v^{****}	Dimensionless velocity = $v^{**} Sc^{\frac{2}{3}}$
V_x	x - velocity component in boundary layer.
V_y	y - velocity component in boundary layer.
V_1, V_2, V_3	Velocity components associated with (x_1, x_2, x_3) coordinate system.
V_1^*, V_2^*, V_3^*	Dimensionless velocities $V_1^* = V_1/U_\infty$, etc.
V_N	Normal velocity at the body surface.
V_N^*	Dimensionless normal velocity at the body surface.
$\bar{w}$	The maximum velocity of free convection motion.
$W_1, W_2, \dots, W_n$	Arbitrary variables.
$\underline{W}$	n - dimensional vector representing $W_1, \dots, W_n$.
$W_{j,0}$	The value of the initial condition for W_j .
$W_{q,f}$	The value of the final condition for W_q .
x	The coordinate parallel to the body surface.
x^*, x^{**}, x^{****}	Dimensionless coordinate x/D_C .
x^{***}	Dimensionless variable $x^{**} (Gr/Re^2)$.
x_1, x_2, x_3	General orthogonal curvilinear coordinates.
y	The coordinate normal to the body surface.
y^*	Dimensionless variable y/D_C .
y^{**}	Dimensionless variable = $y^* Re^{\frac{1}{2}}$.
y^{***}	Dimensionless variable = $y^{**} (Gr/Re^2)^{\frac{1}{2}}$.
y^{****}, y_i	Dimensionless variable = $y^{**} Sc^{\frac{1}{3}}$.
$Y(1) \dots Y(5)$	Variables defined by equation (5-71).
Z	$\sum^2$

α	Mass or heat diffusivity.
β	Volume coefficient of expansion due to either concentration or temperature differences.
β_1	Variable defined by equation (2-8).
δ, δ_H	Hydrodynamic boundary layer thickness.
$\Delta, \delta_T, \delta_C$	Transfer boundary layer thickness.
$\delta_H^*, \delta^*, \delta_T^*$	Dimensionless boundary layer thickness $\delta^* = \delta/D_C$.
$\bar{e}$	Defined by Table 4.1 for the sphere and oblate spheroid.
$\bar{e}'$	Defined by Table 4.1 for the sphere and oblate spheroid.
ζ	Function defined by equation (5-53).
η	Dimensionless normal coordinate, variously defined.
η_H	$\eta \times Sc^{1/3}$ η defined for the hydrodynamic boundary layer.
η_T, η_C	η defined for the transfer boundary layer.
η_{MAX}	Effective infinity for the integration.
Θ	Parametric angle measured from the front stagnation point.
χ	The ratio of the transfer to the hydrodynamic boundary layer thicknesses.
λ	Defined as $\delta^2 \frac{dU}{dx}$
λ_L	Limit of λ defined by equation (4-92).
μ	Fluid viscosity.
ν	Kinematic viscosity (μ/ρ).
ξ	Defined by Szewczyk ⁵⁰ as $(Gr/Re^2)^{1/2}$.
(ξ, η, ϕ)	Oblate spheroidal coordinates.

ξ_0	The value of ξ , the oblate spheroidal coordinate, at the surface of the spheroid.
ρ	Fluid density.
σ	The value of $Y(2)_\infty$
$\tau_i, i = 1, n$	Superposition constants.
ϕ	Arbitrary scalar.
ψ	The angle between the normal at the body surface and the axis of rotation.
$\bar{\psi}_1, \bar{\psi}_3$	Coordinates in power series expansion equation (5-12).
$\psi_1, \psi_3 \dots$	Dimensionless coefficients in power series expansion equation (5-16).
Γ_2	Mass transfer number Sh (or Nu) / $Re^{\frac{1}{2}}(Gr/Re^2)^{\frac{1}{2}}$.
$\bar{\Psi}$	Stream function.
Ω	Function defined by equation (5-56).
∇	Vector operator "del".
<u>Subscripts</u>	
p	Particular solution.
h, i	The i^{th} homogeneous solution.
o, s, w	The value of the variable at the body surface.
∞	The value of the variable at infinity.
<u>Superscripts</u>	
$'$	Denotes differentiation w.r.t.
$k, k+1$	Denotes the iteration number.
$0, 1, 2$	Denotes the iteration number.
$*$	Dimensionless variable.

CHAPTER 1INTRODUCTION.

In the field of convective heat and mass transfer, a great deal of work has been carried out to predict heat and mass transfer rates for many physical problems associated with dispersed systems.

Most of the fundamental work has been in the study of the transport processes of heat, mass and momentum as applied to a single drop or a single solid particle in a continuous surrounding fluid. This situation is a first approximation to multi - drop or multi - particle systems which are frequently encountered in chemical engineering heat or mass transfer operations such as: absorption, distillation, drying, evaporation, extraction, fluidization and humidification. The idealized picture ignores many phenomena that may be encountered in the physical situation. For example in the case of drops, drop oscillation, internal circulation and the presence of surface - active agents and for both drops and particles, the influence of surrounding bodies.

Many workers have employed a spherical shape in their theoretical treatments of transfer from drops. However, it is well known from experimental studies, that in many cases drop shapes may be greatly distorted from a spherical shape and frequently may be approximated by an oblate spheroidal shape.

Convective heat or mass transfer occurs by two different modes which are termed free convection and forced convection. Although generally both of these modes are acting at the same time, one will normally be predominant and transfer by the other may be neglected. Frequently, however, situations arise in which both modes of transfer are important.

There are two particular cases which often arise. First, the free convective forces act in the same direction as the forced convective forces. Secondly, the free convective forces oppose the forced convective forces. These two regimes are called aiding flows and opposing flows respectively.

The two limiting cases of free convection only and forced convection only have been treated in the literature for many years, but the interaction regime has only recently become the subject of study. Interest was first shown in the interaction as a consequence of experimental work on mass transfer at low Reynolds numbers. The data obtained could not be correlated by existing forced convection correlations, and this was explained as being due to the influence of free convection.

At this time, suggestions were put forward that free convective transfer rates and forced convective transfer rates were directly additive.

Since the original experimental studies, there have been a number of both experimental and theoretical investigations of the interaction regime. The theoretical study requires the solution of the momentum equation, the continuity equation and the energy or diffusion equation. These equations form a set of interdependent partial differential equations which are rendered more easily soluble by the introduction of a number of simplifying assumptions.

In the case of pure forced convection it is possible to solve the momentum and continuity equations independently of the transfer equation, and then to employ this solution to solve the transfer equation. This uncoupling of the equations is not possible for either pure free convection or for interaction and hence all the equations must be solved simultaneously.

Analytical solutions have been obtained for only a few

problems in which the equations of motion could be simplified. Numerical solutions, in the form of finite difference equations have been obtained, and a large number of problems have been solved for situations when the boundary layer assumptions have been applied.

Solutions of the boundary layer equations are termed to be either exact or approximate.

Exact solutions are those for which the boundary layer equations are satisfied at every point within the boundary layer, whereas approximate solutions satisfy the boundary layer equations only at a number of points within or at the edge of the boundary layers. For these latter solutions, each of the boundary layer equations is satisfied on average over the respective boundary layer.

Exact solutions may be conveniently dealt with under three main categories: first, those for which similarity transforms are used to reduce the partial differential equations to ordinary differential equations; secondly, for non - similarity flow bodies, series expansions are frequently used to reduce the partial differential equations to infinite sets of ordinary differential equations; and finally, numerical solutions.

A considerable amount of work is required to obtain solutions for flow around bodies when similarity transforms cannot be obtained, and either series or numerical methods have to be used. For this reason, approximate methods, based on integral equations, were developed to produce solutions more readily.

The approximate methods considered here employ integral forms of the momentum and transfer equations. Assumed velocity and temperature or concentration distributions are introduced

and the integral equations simplified to give two ordinary differential equations in terms of two boundary layer parameters.

In most previous work in which comparisons between approximate and exact methods have been made, the approximate solutions have been shown to be in excellent agreement with the exact solutions, whilst requiring much less computer time.

Similarity transforms have been frequently employed in problems associated with the two regimes of free convection and forced convection. However, in the case of interaction the number of "similar" solutions is greatly reduced since the free stream velocity and the surface temperature or surface concentration variations cannot both be selected arbitrarily. One consequence of this is that the isothermal, vertical flat plate can only be treated as a similarity body when the free stream velocity varies as the square root of the distance in the direction of flow.

Many cases of interaction have been treated as perturbations on either free or forced convection. However, very little consideration has been given to matching these solutions in the interaction region.

A number of solutions have been obtained for similarity bodies and for the vertical flat plate in a uniform free stream, but although a large amount of experimental work has been carried out on drops and solid particles, virtually no theoretical studies have been made in the interaction regime for bluff bodies.

For the vertical flat plate and similarity flow bodies, a number of solutions have been obtained for the asymptotic cases of the Prandtl or Schmidt number going to either zero or infinity. Also, many solutions have been presented when this parameter has a value of unity, i.e. when the momentum boundary layer thickness is approximately the same as that of the transfer boundary layer.

Treatments with a Schmidt number of 2.4 and a Prandtl number of 0.7 are frequently found, but very few solutions have been reported for higher values of either the Schmidt or Prandtl number.

The present work was initiated to investigate the interaction of free and forced convection heat or mass transfer from bodies of revolution, with special reference being made to spheres and a number of oblate spheroidal shapes.

The boundary layer approach is employed and an integral technique is used to reduce the two partial differential equations to ordinary differential equations. This approximate method is then used to predict local transfer rates from a given body of revolution.

During the course of the work, some doubts have been cast upon the general applicability of the approximate technique. A number of attempts to overcome some of the short comings of the method are described.

An exact method is also developed in order to calculate solutions in the region for which the integral methods fail, and also to assess the accuracy of the approximate solutions which are obtained.

LITERATURE SURVEY.2.1. Introduction.

The treatment of heat and mass transfer from single solid bodies has been the subject of many theoretical and experimental studies. Since the present analysis is confined to a solid body, only transfer from the surface into the bulk of the continuous fluid is considered.

The flow regimes of pure free convection and pure forced convection have received much attention and a large amount of work on these two subjects has been published. The present study is concerned with situations in which free convection and forced convection are both of importance. This subject has received much less attention than either of the limiting cases, and, as will be seen from the subsequent review, much further work is needed before a satisfactory understanding of the subject is achieved.

The present literature survey will be primarily confined to the interaction of free and forced convective transfer, but wherever appropriate the limiting solutions of pure free convection and pure forced convection will be mentioned.

In the present work, the mass transfer process is considered to be analogous to that of heat transfer. This implies that only small rates of mass transfer are to be considered such that there is only a negligibly small normal velocity at the surface of the body.

The second section of the survey, 2.2, reviews the boundary layer simplifications of the Navier Stokes, energy and diffusion equations. Section 2.3 describes the literature for the interaction

of free and forced convection heat or mass transfer. The section is sub-divided into a review of experimental work, 2.3.1 and theoretical work, 2.3.2.

2.2. Boundary Layer Simplifications.

The concept of the boundary layer is due to Prandtl,¹ who suggested that viscosity effects are significant and comparable in magnitude with inertial forces in layers adjoining solid boundaries and are small outside these layers. The hydrodynamic boundary layer is effectively the layer in which the velocity changes from the value of zero at the surface of the body to that of the free stream.

A consequence of the boundary layer being thin in comparison with the linear dimensions of the body makes possible certain assumptions which in many cases enable the flow in the boundary layer to be determined.

The thickness of the boundary layer, however it is defined, is supposed to be small everywhere in comparison with distances over which there is appreciable change in the external flow velocity. Within the boundary layer the derivative of each of the flow quantities with respect to the coordinate distance normal to the surface is in general much larger than that with respect to the coordinate distance parallel to the surface. These assumptions enable the Navier Stokes equations to be simplified and the resulting boundary layer equations are in a more readily soluble form.

The major disadvantage with boundary layer theory is that it is no longer valid when flow separation occurs. Thus in

forced convective transfer from a bluff body of revolution, boundary layer theory may be used to predict transfer rates over only approximately one half of the body.

Prandtl presented the boundary layer approach for two dimensional flows and Boltze² extended the treatment to include axisymmetric bodies of revolution. Boundary layer assumptions may be applied to simplify the transfer equation when the transfer boundary layer is also thin, i.e. when the Prandtl or Schmidt number is of order unity or larger.

The solution of the transfer equation for free convection and also forced convection over a vertical flat plate was first given by Pohlhausen³. Since this time, boundary layer theory has been applied to a large number of problems.

2.3. Interaction of Free and Forced Convection.

Interest shown in the interaction of free and forced convection heat or mass transfer can be traced from experimental studies carried out in the early 1950's. Since then the interaction regime has been the subject of a number of experimental and theoretical investigations. However, as will be seen from the subsequent review, the interaction regime presents quite different problems from those of free convection alone or forced convection alone.

2.3.1. Experimental Investigations.

An early reference to the influence of free convection on experimental results was made by Ranz and Marshall⁴. They were investigating the factors which influence the rate of evaporation of drops of a pure liquid and of drops of water containing

dissolved and suspended solids. Their study was confined to a Reynolds number range of 0-200. Although they treated their results under the categories of pure free convection and pure forced convection, they commented that little consideration had been given in the past to experimental errors caused by free convection. They stated that in their study and also in that of Frössling⁵, the free convective forces had been in opposition to those of forced convection, whereas in Kramer's⁶ work they had been mutually aiding. They further stated that in the work of Maisel and Sherwood⁷ the forces had been partially aiding. They commented that none of these other workers had commented upon the presence of free convection.

Chronologically the next relevant experimental study was by Dryden et al⁸, who measured mass transfer rates from solid pellets in packed beds to water flowing at low velocities through the voids. They noticed that their transfer results differed for up-flow and downflow of the water. Attributing this to the existence of free convection currents, which differed for upflow and downflow, they stated that much more data were needed to investigate the influence of the Grashof number on the transfer rate. They did not, therefore, feel justified in presenting any correlations to take account of the free convective effects.

Garner and Grafton⁹ investigated local and overall mass transfer rates from benzoic acid spheres to water in a horizontal water tunnel at Reynolds numbers of 20 - 1,000. They first correlated their data by the semi-empirical relationship:

$$Sh = 44.0 + 0.48 Re^{0.5} Sc^{0.33} \quad (2-1)$$

On comparing their additive constant in the expression (2-1) with that of 2.0, the theoretical value associated with

diffusion into an infinite stagnant medium, used by Frössling⁵ and by Ranz and Marshall⁴, they suggested that the reason that theirs was much larger was due to the influence of free convection. Although this may, in part, have been true, they did not seem to realize that the value of 2.0 had been chosen a priori by both of the other workers.

In order to account for the free convective effect they proposed the following correlation:

$$Sh = 0.5(GrSc)^{\frac{1}{4}} + 0.48 Re^{0.5} Sc^{0.33} \quad (2-2)$$

Equation (2-2) suggests that the contributions of free convection and forced convection on the total transfer are directly additive. Although a number of other reviewers have reported that Garner and Grafton claim direct additivity they do not mention an important fact. Garner and Grafton in their paper acknowledged that their data were not sufficient for such a claim and suggested further work to ascertain whether the relationship was as simple as they had suggested. Their work was probably the first that set out specifically to include both the effects of free convection and forced convection in an experimental correlation. They observed flow patterns using a dye which also showed that as the Reynolds number was increased the separation angle moved forward around the body, towards the separation angle expected for forced flows.

Garner and Keey¹⁰ followed the work of Garner and Grafton with the expressed intention of investigating some aspects of mass transfer when free and forced convection interact. They supported pelleted benzoic acid spheres, $\frac{3}{4}$ inch in diameter, in a vertical water tunnel which allowed for either upflow or downflow and worked in the range of Reynolds numbers 2.3 - 255. They observed a minimum value of the Sherwood number for aiding

(i.e. down) flow at a Reynolds number of about fifty and attributed this to the interaction of free and forced convective forces.

Contrary to their expectations the plots of Sherwood number against Reynolds number for overall transfer showed that the Sherwood numbers for aiding flow were lower than those for opposing flow. An interesting point to note is that their curves for local transfer rates showed that in the vicinity of the forward stagnation point the transfer in aiding flow was in fact higher than for opposing flow. However, over the rear of the sphere the transfer rate in aiding flow was markedly reduced.

They extrapolated their curves for aiding and opposing flows and found that they would have met at a Reynolds number of approximately 750. Thus suggesting that above this value the effect of natural convection on the overall transfer rate would have been negligible. They acknowledged that their data were insufficient to draw up a general correlation for overall transfer in the interaction regime, but postulated the following relationship to predict the value of the Reynolds number when the effect of natural convection was negligible.

$$Re > 0.4 Gr^{\frac{1}{2}} Sc^{-\frac{1}{6}} \quad (2-3)$$

The relationship, which was obtained by equating the Sherwood number for free convection and forced convection, does not appear to have any logical justification.

Garner and Hoffman¹¹ using a similar experimental system to Garner and Keey¹⁰ measured local and overall mass transfer rates from 3/8 inch in diameter benzoic acid spheres to water. The minimum mass transfer rate occurred at a Reynolds number of 25 for aiding flows and the extrapolation of their overall mass transfer curves for aiding and opposing flows would have met at

a Reynolds number of approximately 250. The effect of the smaller sphere, and hence of a smaller Grashof number, compared with that used by Garner and Keey was clearly to make the influence of natural convection negligible at a smaller Reynolds number. It is interesting to note that the value of the parameter Grashof number divided by the square of the Reynolds number, Gr/Re^2 , at which the extrapolated curves met, was approximately the same for both systems. As will be shown in the theoretical section later, this parameter, Gr/Re^2 , is used to describe the predominance of free or forced convection in the interaction regime.

Garner and Hoffman noted that as the Reynolds number approached zero, the Sherwood number became asymptotic to a value that was almost twice that expected for natural convection alone. This phenomenon, which was similar to that found by Garner and Keey, could not be ascribed to experimental scatter and none of the authors could explain the behaviour.

The complex flow patterns which were observed were very interesting. Garner and Hoffman noted that for opposing flows the system was very unstable and that the wake region behind the body was much larger than for aiding flows. At low Reynolds numbers in opposing flows, the separation angle changed very rapidly when there were small changes in the flow conditions and oscillatory behaviour of the wake was observed. Later workers^{33,34,35} have suggested that these wake oscillations could explain the higher mass transfer rates at low Reynolds numbers.

Garner and Suckling¹² presented mass transfer data for the dissolution of adipic and benzoic acid spheres to water in a horizontal water tunnel operating at Reynolds numbers of

100 - 700. They correlated their data, and that of Garner and Grafton, with recalculated diffusivities, by the relationship-

$$Sh = 2 + 0.95 Re^{0.5} Sc^{0.33} \quad (2-4)$$

in which they fixed the values of the additive constant and both exponents on theoretical grounds. This possibly explains why their correlation differs from that of Garner and Grafton, equation (2-1), who used a similar experimental system but only fixed the two exponents. Based on their results they stated that for the system studied, natural convective effects were of importance when the parameter $Re^{1/2} Sc^{1/3}$ was less than 100. This corresponded to a Reynolds number of approximately 100, which was much lower than the values suggested previously by Garner and his co-workers. Garner and Suckling expected that at a higher Grashof number this criterion might not be valid.

Since the number of experimental studies of particles and drops for the interaction regime has been small, it is of interest to consider briefly studies in which other body shapes have been used.

Van der Hegge Zijnen¹³ proposed that the transfer due to both natural and forced convective effects could be added vectorially. He studied heat transfer rates from horizontal cylinders to air blowing over them and proposed an empirical correlation, based on his own results and those of other workers, to predict when the heat transfer due to natural convection was only five per cent of the total heat transfer rate.

Krischer and Loos¹⁴ obtained experimental data for mass transfer from a number of different bodies to downward flowing air. In order to correlate their data in the interaction regime they suggested the use of an equivalent Reynolds number which had been previously introduced by Krischer¹⁵. The

equivalent Reynolds number was derived as follows. The group $D_c \rho (T - T_\infty)$ in the Grashof number represents the buoyancy energy which produces a maximum kinetic energy of $\bar{w}^2/2g$ in the case of frictionless flow. A Reynolds number for free convection, Re_{free} , was then defined as follows.

$$Gr = \frac{1}{2} \frac{D_c^2 \bar{w}^2}{\nu^2} = \frac{1}{2} Re_{free}^2 \quad (2-5)$$

The equivalent Reynolds number, Re_{equiv} , was then defined on the assumption that the contributions of free and forced convection were directly additive:

$$Re_{equiv} = Re_{forced} + Re_{free} \quad (2-6)$$

This method of correlation appeared to be most successful for the vertical plate and for the cylinder.

A number of early experimentalists assumed that the transfer rates due to free and forced convection were directly additive, and presented correlations on this basis.

Kudryashev and Ipatenko¹⁶ investigated the effect of free stream motion on heat transfer from steel spheres to air up to a Reynolds number of 300. Above a Reynolds number of 200 they found that the effect of the Grashof number was negligible. They correlated their experimental data by the following relationship for $Re < 55Gr^{0.138}$:

$$Nu = 2 + 0.181 Re^{0.67} \left(1 + P_1 \frac{Gr}{Re} \right)^{0.67} \quad (2-7)$$

$$\text{where} \quad P_1 = \frac{1 - \frac{Re}{55Gr^{0.138}}}{\left(\frac{Gr}{7.68} \right)^{0.49}} \quad (2-8)$$

They stated that when the Reynolds number was greater than $55Gr^{0.138}$, free convective effects were negligible. They also presented a theoretical analysis for both aiding and opposing

flows, based on the assumption that the combined velocity could be represented as the sum of the induced and forced motion velocities. Their analysis was criticised by Spalding¹⁷ because of the inconsistency in the definitions presented for the thermal boundary layer thickness. Although they claimed reasonable agreement between their theoretical and experimental correlations, the published relationships showed that this was clearly not so.

Narasimhan¹⁸ reported some experimental work by Kudryashev and Berezanski¹⁹. These workers also assumed direct additivity of transfer rates resulting from the two convective modes. Their observations are of interest since they found that in their experimental study of heat transfer, from stationary copper spheres to an air stream, the Nusselt number for aiding flows was higher than that for opposing flows up to a Reynolds number of 50. However, beyond this value, up to a Reynolds number of 120, the Nusselt number for opposing flows was higher. They explained this behaviour on the basis of the differences of the separation angle which they observed for aiding and opposing flows.

Two other authors have proposed direct additivity of transfer by natural and forced convection. Steinberger and Treybal²⁰ studied mass transfer from cast benzoic acid spheres to water and aqueous glycol which was flowing vertically upwards. Their range of Reynolds numbers was 90 - 16,900 and Schmidt numbers 987 - 69,680. They proposed a correlation of the form:

$$Sh = A + B Re^n \quad (2-9)$$

This relation predicted $Sh_0 = A$ at zero Reynolds number where A was assumed to be a function of the Grashof and Schmidt

numbers. Two formulae for A were presented:

$$\text{for } GrSc < 10^8 \text{ (laminar free convection),}$$

$$A = Sh_o = 2 + 0.569 (Gr Sc)^{0.250} \quad (2-10)$$

$$\text{and for } GrSc > 10^8 \text{ (turbulent free convection),}$$

$$A = Sh_o = 2 + 0.0254 (Gr Sc)^{\frac{1}{3}} Sc^{0.244} \quad (2-11)$$

The parameter B was given as:

$$B = 0.347 Sc^{0.312} \quad (2-12)$$

and their final correlation became:

$$Sh = Sh_o + 0.347 (ReSc^{\frac{1}{2}})^{0.62} \quad (2-13)$$

Another correlation presented on the assumption of additivity was that of Jenson et al²¹, who studied overall heat transfer coefficients from a $\frac{3}{4}$ inch diameter sphere to flowing oil. They correlated their data by the expression:

$$Nu = 4.5 + 0.48 Re^{\frac{1}{2}} Pr^{\frac{1}{3}} \quad (2-14)$$

in which the two exponents were chosen a priori. The constant, 4.5, was split up to take account of molecular conduction and natural convection.

The authors did not discuss the interaction regime in detail, but simply presented the following correlation for their data:

$$Nu = 2.0 + 0.75 Pr^{\frac{1}{4}} + 0.48 Re^{\frac{1}{2}} Pr^{\frac{1}{3}} \quad (2-15)$$

Yuge²² presented a very complicated graphical procedure to calculate heat transfer rates in the interaction flow regime. The method was based on his own experimental results for heat transfer from spheres in aiding, opposing and cross flows for a range of Reynolds numbers $3.5 - 1.4 \times 10^5$ and Grashof number $1 - 10^5$. The practical usefulness of the method is questionable because of its complicated nature.

Yuge's results were analysed independently by two

other workers. Klyachko²³ criticised Yuge's results because the heat transfer predictions for free convection were not in agreement with a previous relation proposed by Mikheev²⁴. He suggested that the errors may be due to the influence of the natural movement of the air in the experimental room. However, he made no mention of similar discrepancies found by other workers. In order to test his hypothesis he introduced his own, very complicated correlations for the interaction region without mentioning their derivation, and used these to assess the influence, if any, of air movement. He concluded from his analysis that Yuge's transfer results had been influenced by external air movements.

The second analysis of Yuge's data was by Narasimhan and Gauvin²⁵, who made no mention of Klyachko. They "picked off" Yuge's data from the published graphs and treated them conventionally. They found that the transfer rates for opposing flows were depressed in the interaction region compared with those for aiding flows, and that the magnitude of the Grashof number affected the extent of the interaction region. The results showed that for the range of Grashof numbers considered, natural convection was negligible when the value of the parameter $Gr/Re^2 < 0.8$ and that forced convection appeared to have little influence when the value of the parameter $Gr/Re^2 > 10.0$. One convective mode was assumed to be negligible when the predicted transfer rate was within five per cent of the value due solely to the other mode.

A number of other workers have commented upon the influence of free and forced convection as an aside from their main work. Tsubouchi and Sato²⁶ measured heat transfer rates from thermistors to air, in both natural and forced convection, and attempted to

describe transfer rates for the interaction region by vectorial addition of the transfer caused by each mode.

Vliet and Leppert²⁷ measured heat transfer rates from an isothermal plate to water in aiding forced convection flows for a Reynolds number range of 50 - 300,000 . They did not use any data, in the determination of their correlations, for which the dominant flow could not be stated a priori to be forced. They suggested that in the interaction region the overall heat transfer coefficient should be calculated from both a natural convection and a forced convection correlation, and that the higher of the two should be used. They stated that any error introduced should not be greater than twenty per cent.

Most previous workers who simply added the effects of transfer by free and forced convection either did so because they did not realize the complicated nature of the problem, or because they wished to obtain a simple correlation for their own data, after realizing that both modes of transfer should be considered for their experimental system. The work of Kinard et al²⁸ differs from these in that they mentioned that the theoretical justification for additivity of transfer due to free and forced convection was questionable, but, since they wished to introduce a novel correlation for previously published experimental data, they proposed:

$$Sh = 2 + Sh_N + 0.45 Re^{\frac{1}{4}} Sc^{\frac{1}{3}} + C_3 Re Sc^{\frac{1}{3}} \quad (2-16)$$

where Sh_N was the Sherwood number for pure natural convection, and the last term was an attempt to take account of forced convective transfer in the wake region.

A very misleading statement was made by Pritchard and Biswas²⁹ in a review paper of mass transfer from drops.

They stated that Skelland and Cornish³⁶ have shown that the effects of free convection were negligible above a Reynolds number of 15 and that in their own review paper, since they were not concerned with Reynolds numbers less than 15, they were justified in ignoring the effects of free convection. They clearly overlooked the fact that the relationship used by Skelland and Cornish suggested that the Reynolds number, at which free convection may be neglected, was a function of both the Grashof and the Schmidt number of the system. A general statement such as the one made by Pritchard and Biswas is, therefore, very misleading and completely false.

Most subsequent studies in which free and forced convection interact have been extremely detailed in their descriptions of flow phenomena and associated transfer rates.

Pei and a number of co-workers^{30,31} have studied the evaporation of water, benzene and methanol from drops and particles in high temperature surroundings. They also studied heat transfer in both aiding and opposing flows over a Reynolds number range of 5.5 - 510 . They found that the heat transfer rates for opposing flows were lower than those for aiding flows, and noted an alternating predominance between natural and forced convective effects on the flow in the interaction zone. They observed the flows using a Schlieren technique.

They commented on the contradictory evidence regarding transfer rates in aiding and opposing flows for heat or mass transfer and suggested that the oscillatory behaviour which they, and a number of other workers had observed, could explain this phenomenon.

At low Prandtl numbers Torobin and Gauvin³² had previously

reported an almost instantaneous reformation of the boundary layer after a disturbance, and thus the periodic oscillations observed in Yuge's²² experiments were probably relatively short and the overall rates of transfer were not greatly affected. In Garner and Hoffman's system, however, the Schmidt number was 788 and a considerable time was required to re-establish the boundary layer. This resulted in a larger effect on the mass transfer due to the oscillatory behaviour than for the previous system.

As a consequence of their work, Pei and his co-workers concluded that the effects of free and forced convection were non-additive, and they found by dimensional analysis, that the parameter Gr/Re^2 was of importance in characterizing the dominant force in the interaction regime. Their results showed that the transfer due to free convection was less than five per cent of the total transfer when the value of the parameter $Gr/Re^2 < 0.2$, and that the contribution to the total transfer by forced convection was less than five per cent when the value of the parameter $Gr/Re^2 > 10.0$. They added that these same values may be found not to govern systems for which the Prandtl number was different.

Another comprehensive study by Pei³³ on heat transfer confirmed the results of his earlier work. He investigated experimentally the effects of combined free and forced convective transfer from brass spheres, supported at their equators in a 6 inch diameter tube which allowed for either aiding or opposing flows. In his experiments, the transfer rates were higher for aiding than for opposing flows, thus agreeing with the heat transfer results of previous workers, but in contrast with the overall mass transfer rates of Garner et al^{10,11}.

Pei used a Schlieren photography technique to observe the flow behaviour, and he found that for opposing flows the boundary layer was unstable when the two effects were of the same order of magnitude. He observed the same sweeping action, characteristic of the alternating predominance of free and forced convection as in his previous work.

The interaction regime was found to lie between values of the parameter Gr/Re^2 of 0.05 - 100 . Here, the extent of the range was defined by the criterion that one transfer mode was negligible in comparison with the other, and not the 'five per cent' criterion used in Pei's other works. Pei once again emphasized that he expected that the interaction region might be a function of the Prandtl number.

Narasimhan³⁴ and Narasimhan and Gauvin³⁵ studied simultaneous heat and mass transfer from spheres in high temperature surroundings. Their experimental investigation in the combined flow regime involved the measurement of the rate of evaporation of water from stationary, porous spheres under steady conditions in an electrically heated stainless steel chamber. The spheres were suspended in a stream of superheated steam at temperatures ranging from 500- 1,000 °K, and flowing at Reynolds numbers of 0 - 1,489 for aiding and opposing flows. The main conclusions of the work were very similar to those of Pei and his co-workers and will not be repeated. However, the relative magnitudes of the transfer rates found for aiding and opposing flows were of interest since the transfer rates for opposing flows were greater than for aiding flows. This agreed with some previous transfer results for mass transfer, but was in direct contrast with previous results for heat transfer. They suggested that the contribution of transfer from the wake region when mass

is transferred is considerably different than when heat is transferred. This finding may not be due solely to the difference between heat transfer and mass transfer, but could result from the magnitude of the Prandtl or Schmidt number, irrespective of whether heat or mass is transferred. In order to clarify this, further work on heat transfer where the Prandtl number is large, and mass transfer for which the Schmidt number is small is required.

Ingram³⁷ studied, experimentally, mass transfer from benzoic acid spheres and spheroids to water in opposing free and forced convection. His experiments were performed using a low speed water tunnel with benzoic acid bodies of five different oblate spheroidal shapes. The overall mass transfer rates were obtained by direct weighing, and the flow patterns were studied using a Schlieren technique.

The transfer data for the Reynolds number range of 45 - 195 were statistically analysed, and the equation which gave the minimum variance, within the 90% confidence range, was presented as:

$$Sh = 23.7 + 0.196 Re_I^{0.78} Sc^{\frac{1}{3}} \quad (2-17)$$

Ingram suggested that the increase of the exponent of the Reynolds number from what he called the theoretical value of 0.5 may be due to the free convective influence, but he added that for any exponent between 0.2 and 1.4, an expression could be obtained for which the estimate of variance fell within the 90% confidence range of the minimum estimate of variance. He presented a simpler correlation than equation (2-17) which was still within the 90% confidence limits:

$$Sh = 0.632 Re_I^{0.59} Sc^{\frac{1}{3}} \quad (2-18)$$

Ingram made some extremely interesting observations in the interaction regime, similar to those of Pei for heat transfer, and discussed his observations in great detail. He noted four different flow regimes over the Reynolds number range studied of 0 - 195 .

Regime I was that of pure free convection, in which there was downward motion of the saturated solution, forming a tail flowing away from the body. The tail was much thicker for the flatter oblate spheroids than for the narrow, well defined tail for the sphere. Once the flow of water upwards began, the second regime was observed. This consisted of basically downward motion, as above, but with occasional scattering and break up of the tail. The range of Reynolds number over which this regime was observed differed for each body shape considered.

As the water flow rate was further increased the upflow caused a build-up of the dense solution in the wake region, but the flow was not sufficient to carry the solution away. Once the gravity forces overcame the bulk flow forces which were acting on this mass of solution, the solution flowed downwards causing a highly disturbed wake. Once this solution had moved away from the body, upflow once again became predominant. Ingram reported that the frequency of the cyclic behaviour was very irregular.

The final flow regime was upflow with a well defined separation angle. The dense solution caused the separation angle to be pushed towards the front of the body. As the flow rate was further increased, the separation angle moved towards the rear of the body and approached the separation angle associated with forced convection. Ingram found that above a value of the parameter Gr/Re^2 of 160, the mass transfer

rates were independent of the parameter Gr/Re^2 . He noted that the mass transfer rate for the sphere was that due to free convection alone, but for the flatter oblate spheroids the mass transfer rate was about five per cent higher than expected. This result is in contrast with some other workers who found markedly higher transfer rates as the free convection asymptote was approached. Below a value of the parameter Gr/Re^2 of 16, the experimental data for each shape fell onto a straight line. When the transfer rates for the sphere were compared with other authors, they were found to be higher than the rates predicted for forced convection alone. This indicated that there was still an appreciable effect due to free convection.

It is of interest to note that whilst the flow patterns, observed by Ingram, changed from those associated with free convection to those associated with forced convection, the overall mass transfer rates remained approximately constant. However, since the flow patterns changed appreciably, the local mass transfer rates probably also changed. Ingram did not obtain any data for local transfer rates, and therefore, these variations cannot be discussed.

Bantin and Wilson³⁸, using Ingram's equipment and technique, studied aiding free and forced convection mass transfer from oblate spheroids. They used thymol bodies and obtained overall transfer rates by direct weighing and observed the flow patterns using a Schlieren technique. Their mass transfer results will be discussed together with the observed flow patterns. The transfer rates gradually increased with increasing flow rates, up to a Reynolds number of about 10, and then decreased to a minimum at a Reynolds number of about 50. With a further increase of the Reynolds number beyond 50, the transfer

rate then continued to increase. The explanation of these phenomena was given on a qualitative basis. The flow patterns showed that below a Reynolds number of approximately 10, no separation was observed. The increase in the Sherwood number here was assumed to be purely the result of the bulk motion of the surrounding fluid. Above a Reynolds number of 10, separation was observed and the separation angle, measured from the front stagnation point, decreased with increasing Reynolds number. In this flow regime, the wake was observed to oscillate from side to side. The reason for the decrease in the transfer was presumed to be due to the increase in the wake area, in which it was assumed that the transfer rate was reduced. As the Reynolds number increased further, the effect of the reduction in the effective area for transfer also decreased and resulted in a lower transfer rate. Finally, at a Reynolds number of approximately 50 the separation angle had reached its final value and had become independent of the Reynolds number. The transfer was now affected solely by the increase in the external flow, and continued to increase.

From their graph of Sh vs Gr/Re^2 they concluded that natural convective effects have little influence when the value of the parameter $Gr/Re^2 < 1.0$, but that for the range of Reynolds numbers considered, the effects of forced convective transfer could not be ignored. The maximum value of the parameter Gr/Re^2 in their work was 10^3 . This value is much larger than has been previously reported, and requires further confirmation.

Another very interesting study of both aiding and opposing flows was presented by Pearson and Dickson³⁹ who studied free convective effects on forced convection mass transfer in the Stokes flow regime. They used single drops of methyl acetate

and 2-ethoxyethyl acetate, supported at their rear pole, with distilled water as the continuous medium. Their experiments were over the ranges: Re , 0.1 - 30 ; Sc , 578 - 1,149 ; Gr , 12.3 - 75.7 . The small flow rates were maintained using a constant head tank and a needle valve. The transfer was recorded by measurement of the times, at various water flow rates, for drops to change from one diameter to a chosen smaller diameter. To aid measurement, an enlarged image of the drop was projected onto a screen. The results for aiding flows showed consistently higher mass transfer rates than those for opposing flows in the interaction regime. Pearson and Dickson stated that in the opposing flow regime the organic material was convected away from the vicinity of the drop much more slowly than in the aiding case. This, therefore, resulted in a markedly reduced concentration gradient. As a consequence of their visual observations, they were able to describe the physical flow conditions associated with the phenomenon of the minimum transfer rate in the opposing flow case. They observed that as the continuous flow was increased from a pure free convective situation, opposition to the free convection increased until a situation was reached when forced convection was on the point of becoming dominant. At this point the minimum rate of mass transfer occurred. On further increase of the external flow, the mass transfer rate approaches that for pure forced convection.

2.3.2. Theoretical Studies.

It is clear from the experimental studies described above that the transfer due to the interaction of free and forced convection is very complicated, especially for opposing flows for which very complex flow phenomena have been observed. A number of theoretical studies have been presented in the literature, most of which have been confined to studies of the vertical flat plate in a uniformly flowing free stream, and similarity flow bodies.

An early theoretical investigation was that of Tanaev⁴⁰ who considered the flow of a compressible gas around a flat plate for small values of the Froude number, Fr. He employed an approximate method to solve the equations of motion and obtained a relationship from which the effect of the gravitational forces on the total coefficient of resistance could be studied:

$$\frac{C_f}{C_{f(Gr/Re^2=0)}} = 1 - 0.498 \frac{Gr}{Re^2} \quad (2-19)$$

From this relationship, he concluded that when the parameter Gr/Re^2 had a value of less than 0.1, the resistance due to free convection was less than five per cent of the total resistance.

A general theoretical analysis of combined laminar free and forced convection heat transfer from an arbitrary 2-dimensional surface was presented by Acrivos⁴¹. He presented the dimensionless forms of the boundary layer equations which he then simplified for the vertical flat plate. The Pohlhausen⁴² - Von Karman⁴³ integral method was used to reduce these equations to two ordinary differential equations. The first equation represented the rate of change of the hydrodynamic boundary layer thickness with respect to the dimensionless distance parallel to the

body surface, and the second equation represents the rate of change of the ratio of the transfer to the hydrodynamic boundary layer thickness with respect to the same variable. Acrivos assumed a quartic polynomial to represent the velocity distribution within the boundary layer, and a cubic to represent the temperature. The velocity profile was represented as the sum of a free convection and a forced convection profile. This followed directly from one of the boundary conditions at the body surface which the assumed profile had to satisfy. Acrivos solved the two ordinary differential equations for values of the Prandtl number of 0.73, 10, and 100. The heat transfer results were presented as plots of the heat transfer group, Γ_2 , against the parameter Gr/Re^2 , with the Prandtl number as a parameter.

Acrivos commented that over part of the integration of the two equations he had had to completely ignore the first differential equation, which resulted from the momentum integral equation. This was necessary because the denominator of this equation became zero and meaningless results were obtained. In an attempt to overcome this difficulty, he continued to integrate the second differential equation on its own. He thus allowed the ratio of the two boundary layer thicknesses to vary whilst keeping the hydrodynamic boundary layer thickness constant. When the numerator of the first equation became zero, he once again integrated both equations simultaneously.

Acrivos made no mention of the range of integration over which he had only integrated one equation and did not report the consequent effect on the transfer rates which were predicted for the three different Prandtl numbers.

He compared his asymptotic solutions as the value of

the parameter Gr/Re^2 approached zero and infinity with some solutions of other workers and found them to be in good agreement. The interaction region was defined for each of the Prandtl numbers studied, and the results were summarized as follows:

Table 2.1 Results of Acrivos⁴¹, describing interaction region.

Prandtl Number	Forced convection negligible Gr/Re^2	Free convection negligible Gr/Re^2
0.73	> 2	< 0.02
10.	> 5	< 0.02
100.	> 30	< 0.02

Acrivos concluded that the mechanism of interaction was non-additive, and stated that whereas aiding flows would tend to stabilize the boundary layer, opposing flows would hasten the appearance of separation. These results of Acrivos have been widely reported, but no investigation of the effect of disregarding one equation, or the range of values of the parameter Gr/Re^2 over which this occurred has been published.

Oosthuizen⁴⁴, seemingly unaware of Acrivos' paper, presented an approximate solution of the boundary layer equations for the vertical flat plate, in which he introduced quartic profiles for both velocity and temperature. His work was unnecessarily restricted by the assumption that the ratio of the thermal to the hydrodynamic boundary layer thickness was constant for a particular Prandtl number.

He followed the same procedure as Acrivos in order to obtain two ordinary differential equations which he then simplified in an attempt to obtain expressions for pure free convection and pure forced convection. The transfer rates predicted using these expressions were found to be in good agreement with those of previous workers. He considered the combined flow regime,

but found that because of his earlier assumption that the ratio of the two boundary layer thicknesses was constant, he could not solve the problem. He stated that the reason for this was obvious, because of the form of the basic equations, but he did not explain this statement.

Hauptman⁴⁵ has also presented an approximate method similar to that of Acrivos. He investigated small buoyancy effects on heat transfer from both horizontal and vertical plates. He did not mention any difficulties in the application of the method, but stated that the applicability of the method was limited to values of the parameter Gr/Re^3 less than 10^{-5} for the horizontal plate, and less than 10^{-7} for the vertical flat plate over a range of Prandtl numbers 0.01 - 100 . This suggests that he probably found similar difficulties in the application of the method as Acrivos.

Several workers have used perturbation methods to study the interaction regime. One of the first papers was by Sparrow and Gregg^{46,47} who investigated buoyancy effects on forced convection heat transfer from a vertical flat plate in a uniform free stream. They introduced a stream function into the momentum and transfer boundary layer equations for interaction, and obtained two coupled partial differential equations. In the absence of the buoyancy term, the equations were identical to the equations for pure forced convection. Sparrow and Gregg's intention was to find deviations from pure forced convection due to buoyancy effects. They therefore introduced series expansions, in terms of ascending powers of the parameter Gr/Re^2 , for the stream function and the dimensionless temperature, in which the first term of each series was that for pure forced

convection, and successive terms represented the deviations due to buoyancy. The series were then substituted into the two partial differential equations, and powers of the parameter Gr/Re^2 were equated. Two infinite sets of ordinary differential equations were obtained in which the first equation of each series were those associated with pure forced convection and have been previously solved by Blasius⁴⁸ and Pohlhausen³.

Sparrow and Gregg solved the equations for Prandtl numbers of 0.01, 1.0, and 10. The results of their investigation showed that the local transfer rates could be predicted from forced convection heat transfer theory, with a maximum error of five per cent, when the value of the parameter $Gr/Re^2 \leq 0.075$, and overall rates when $Gr/Re^2 \leq 0.225$. They commented that the overall transfer rates were less sensitive to the effects of buoyancy than the local rates, which explained the difference between their criteria for local and overall transfer rates. This finding was similar to that of Ingram³⁷ discussed earlier.

In contrast with the previous perturbation solution, Eshig⁴⁹ considered an analytical model of a vertical flat plate, with aiding or opposing flows, and investigated the influence of forced convection upon a free flow situation. He assumed series for both velocity and temperature in which the first term of each series corresponded to that for pure free convection. He found that the series used to calculate the overall and local transfer rates were convergent only if the parameter Gr/Re^2 was greater than a value of approximately unity for all Prandtl numbers studied.

Further perturbation solutions were presented by Szewczyk⁵⁰ for deviations from both pure free convection and from pure forced convection. The latter solution was the

same as that presented by Sparrow and Gregg. The series used in the free convection perturbation problem, for the stream function and dimensionless temperature, were expressed in terms of the independent distance variable. The first term of each series was that for pure free convection and successive terms represented the deviations due to forced convection. Upon substitution of these series into the free convection boundary layer equations and equating powers of the parameter $(Re/Gr^{1/2})$, he obtained two infinite sets of ordinary differential equations which had to be solved simultaneously. The first equation of each set was that for free convection alone, and subsequent equations represented perturbations on this solution. He found that the perturbation solution for free convection was only convergent when the value of the parameter $Re/Gr^{1/2}$ was less than or equal to 0.1. He also commented on the difficulty, which is frequently encountered when perturbation techniques are used, of matching two solutions in the interaction zone. As a consequence of this, Szewczyk stated that for his problem the interaction zone remained essentially unsolved.

Merkin⁵¹ mathematically investigated the effects of buoyancy forces on the boundary layer flow over a semi-infinite vertical flat plate in a uniform free stream for both aiding and opposing flows. The Prandtl number of the system studied was unity, although all Prandtl numbers could be studied in a similar manner. He commented on the solution presented by Szewczyk for the effects of forced convection on the free convection solution, and pointed out that although Szewczyk expanded in terms of parameters he actually obtained a coordinate expansion. Stewartson⁵² has discussed the difficulties associated with coordinate

expansions for obtaining asymptotic solutions, and Merkin stated that in Szewczyk's expansion a term of $O(\log \xi / \xi)$ must be included in order for a solution to be obtained for the term of $O(\xi^{-1})$.

Merkin discussed both perturbation solutions and adapted the step - by - step method of Terrill⁵³ to match the two in the intermediate region. His solution for predominantly forced convective flows was the same as that of Sparrow and Gregg, and that for predominantly free convection was similar to that of Szewczyk with the addition of the term of $O(\log \xi / \xi)$. His results, for a Prandtl number of unity, showed that the transfer rates were within five per cent of those for pure free convection and pure forced convection when the value of the parameter Gr/Re^2 was approximately 6.0 and 0.1 respectively. Merkin also studied the opposing flow regime and found that separation was predicted when the value of the parameter Gr/Re^2 was 0.192357.

Lloyd and Sparrow⁵⁴ introduced the method of local similarity to solve the boundary layer equations for heat transfer from a vertical flat plate in the interaction flow regime. Their treatment is in effect identical with that of Merkin, for the predominantly forced convective situation, but in order to simplify the analysis they assumed that all derivatives with respect to the independent distance variable were zero. They obtained solutions for a range of Prandtl numbers 0.003 - 100, and for a range of values of the parameter Gr/Re^2 0.0 - 4.0.

Gunness and Gebhart⁵⁵ have presented a perturbation solution of the forced convection problem for wedge flows in which the dimensionless velocity was expanded as a double power series so that the variations in temperature gradients in

both coordinate directions could be investigated. They obtained three infinite sets of ordinary differential equations which they solved in order to obtain the heat transfer rates and drags. They were mainly interested in the predictions of the drags and studied the occurrence of separation for opposing flows.

The theoretical studies which have been described above have been mainly confined to the isothermal, vertical flat plate in a uniform stream. The work of Sparrow, Eichhorn and Gregg⁵⁵ was an interesting study which sought to obtain "similar" solutions for heat transfer in the interaction regime. The general problem of boundary layer flow with a free stream pressure gradient and heat transfer from a non-isothermal surface was considered. Previous forced convection "similar" solutions have been found when the free stream velocity varied as:

$$U_{\infty} = A x^m \quad (2-20)$$

and, for the temperature, "similar" solutions have been obtained when the wall temperature varies as:

$$T_w - T_{\infty} = B x^n \quad (2-21)$$

These expressions were therefore assumed in an attempt to find conditions for similar solutions in the interaction regime. In their analysis they considered steady laminar flow, disregarded viscous dissipation and assumed that all fluid properties were constant other than essential density variations. They found that in order to obtain similar solutions for the interaction region one very important condition which related the two exponents in equations (2-20) and (2-21), must be satisfied, namely:

$$2m - 1 = n \quad (2-22)$$

This very interesting result meant that the free stream velocity and surface temperature variations could not both be selected arbitrarily if a similar solution was to be obtained. In the case of an isothermal vertical flat plate, the exponent n of equation (2-21) is zero, then from equation (2-22), the free stream velocity must vary as the square root of the distance in the direction of flow. This means that the isothermal vertical flat plate in a uniform free stream is not a similarity body for the interaction regime.

Sparrow et al found that there was very great latitude in the selection of the independent variable and the dimensionless stream function. They showed that either of the forms normally associated with pure free or forced convection could be used for the interaction regime. They investigated two problems, the case of an isothermal wall, and a wall with a constant heat flux. Both of these problems were solved using the forced convection transformations for the independent variable and the dimensionless stream function. They did not solve any problem using the free convection transformations, and therefore, did not obtain any matching of solutions using both methods. Had they done so, they would have found excellent agreement, as has been recently shown by Harris⁵⁷ for his own investigation of similarity solutions in the interaction regime.

Sparrow et al reported both the predictions of heat transfer rates and the values for the shear stress for each of the problems investigated. Their main comment was that the heat transfer rates in the mixed flow regime showed surprisingly small deviations from the envelope formed by

the two limiting curves. The maximum error anywhere was found to be 23%.

The criteria, in the interaction region, for which the heat transfer rate was found to be within five per cent of that for either pure free or forced convection were:

for aiding flows, Prandtl number 0.7 :

$$\begin{array}{lll} 0.0 < Gr/Re^2 < 0.3 & \text{forced convection} \\ 0.3 < Gr/Re^2 < 16.0 & \text{mixed convection} \\ 16.0 < Gr/Re^2 & \text{free convection} \end{array}$$

for opposing flows, Prandtl number 0.7 :

$$\begin{array}{lll} 0.0 < Gr/Re^2 < 0.3 & \text{forced convection} \\ 0.3 < Gr/Re^2 & \text{mixed convection} \end{array}$$

From the form of their equations, Sparrow et al pointed out that for opposing flows the buoyancy term plays the same role as an adverse pressure gradient in pure forced convection. Thus, if the parameter Gr/Re^2 were increased, a situation of flow separation would be reached. For this reason, free convective situations were never achieved in opposing flows. The phenomenon of flow separation was investigated for opposing flows, and, for the constant temperature wall separation was found to occur at a value of the parameter Gr/Re^2 slightly greater than 0.95 when the Prandtl number was 0.7. For the constant flux case, separation did not occur until a value of the parameter Gr/Re^2 of 1.15. The reason for this difference was suggested to be caused by the fact that the free stream flow exerted a stronger pressure gradient, $U_\infty \propto x^{0.6}$, in the latter case as opposed to $U_\infty \propto x^{0.5}$ in the former.

Brindley⁵⁸ extended Meksyn's⁵⁹ approximate method to solve the free convection boundary layer problem at a uniformly heated vertical wall. The method, which may only

be used for similarity flows, was then used by Brindley to solve the boundary layer problem in the interaction regime. He used two different transforms, one when free convection was dominant, and the other when forced convection was dominant. He did not mention how the appropriate transform was chosen in the region in which the dominant force was unknown. The transfer rates predicted by Brindley's approximate method were found to be seven per cent higher than those obtained by Sparrow et al⁵⁶ who used a more exact method. The purpose of Brindley's paper seemed to be to test the applicability of Meksyn's method for problems in which the momentum and transfer equations are coupled, rather than to study the transfer results which he obtained.

Another solution which employed "similar" solutions was that of Barton and Trass⁶⁰, who investigated simultaneous heat and mass transfer in the combined flow regime from a vertical flat plate both experimentally and theoretically. They treated the problem in the same way as Sparrow et al⁵⁶, with the addition of the diffusion equation. They found that their theoretical predictions were in very good agreement with their own experimental data. They commented that in the interaction zone their results showed that the heat and mass transfer rates differed only slightly from those for the pure flows. They defined the interaction zone as lying between values of the parameter Gr/Re^2 from 0.3 to 16.0, where the usual five per cent criterion was used.

Hellums and Churchill⁶¹ commented that in the interaction regime, the range of "similar" solutions is very much reduced than in either of the two limiting cases. Their work presented an efficient method which may be used to remove parameters

from a set of equations and they showed that, in many cases, this led directly to similarity transforms.

Harris⁵⁷ solved the boundary layer equations in order to predict heat transfer rates from a range of similarity flow bodies in the interaction regime, over a very wide range of Prandtl numbers, up to 10,000. He solved the two simultaneous ordinary differential equations using the same quasilinearization technique as has been used in the present work, and this will be described in Chapter 5.

He found, as did Sparrow et al.⁵⁶, that he could use transformations for the independent variable and the stream function which were either those normally associated with pure free convection or with pure forced convection. He solved a number of problems using both transformations and found that over the whole interaction region the results were the same, within the accuracy of the numerical method used.

Harris is probably the only worker to have made a detailed study of the direct matching of two solutions in the interaction region.

Kitaura and Tanaka⁶² investigated the effects of combined free and forced flows on mass transfer rates from the region of the front stagnation point of a sphere. They obtained an approximate solution of the integral forms of the boundary layer equations. In their method they defined the combined velocity profile as the sum of a free convection velocity distribution and the forced convection profile of Tomotika⁶³. The free convection profile was chosen on the assumption that the hydrodynamic and concentration boundary layers were of the same thickness. They assumed a cubic

concentration profile which was applicable to both flow regimes. Since the analysis was restricted to the forward stagnation point, they were able to considerably simplify the integral equations.

They attempted to obtain a relationship between the ratio of the two boundary layer thicknesses as a function of the Schmidt number, but this seems to be rather artificial, since they chose the free convection profile on the assumption that the hydrodynamic and transfer boundary layers were of the same thickness. They presented a relationship to predict the Sherwood number in the combined flow regime, based on the assumption of additivity of the Sherwood number for free flows alone and forced flows alone:

$$\frac{Sh}{Sh_0} = 1 + \left[0.002381 \frac{Gr}{Re^2} \left\{ A (0.1875 + Sc) \right\}^{-1} \right]^{\frac{1}{4}} \quad (2-23)$$

Where A is a constant for the system under investigation and Sh_0 is the Sherwood number for forced convection transfer and is effectively Tomotika's solution. They commented that their predictions of transfer rates were consistently higher than those of other workers.

They discussed the concept of the equivalent Reynolds number, but stated that the introduction of a proportionality constant, in the definition of this Reynolds number, made the method unsuitable for general application. They attempted to obtain the value of the constant for low values of the parameter Gr/Re^2 by perturbing the forced convection equations, and found that the method involving the equivalent Reynolds number predicted reasonable transfer rates.

Acrivos⁶⁴, commenting on the disadvantages of other

approaches which had been used to obtain solutions of the boundary layer equations in the interaction regime, treated the heat transfer problem in laminar boundary layer flows using an approximate method. He investigated the asymptotic behaviour as the Prandtl number approached zero or infinity, and developed the approximate method to predict transfer rates in the interaction regime. In the asymptotic approach for large Prandtl numbers, he considered the thin thermal boundary layer separately from the bulk of the hydrodynamic boundary layer. He characterized the flow in this regime by the dimensionless group $Gr/Re^2Pr^{1/3}$. The inertial terms of the momentum boundary layer equation were neglected, and the parameter $Gr/Re^2Pr^{1/3}$ was removed from the equations by a scaling. The equations were then solved by the use of a previously derived similarity transform⁶⁵. The approximate method was used by Acrivos to predict transfer rates from a horizontal cylinder. The results were compared with previously obtained exact solutions and were found to agree with an error of approximately ten per cent. On this basis, Acrivos stated that in the intermediate region it was "highly probable" that the heat transfer rates would be described with little error. A modified expression to calculate intermediate Nusselt numbers was presented as:

$$Nu = (Nu_{\text{forced}}^4 + Nu_{\text{free}}^4)^{1/4} \quad (2-24)$$

The asymptotic solution for very small Prandtl numbers was calculated by an approximate procedure which used the asymptotic forms of the velocities and the previously mentioned similarity transform. The resulting formula for the calculation of the heat transfer rates in the

intermediate zone was:

$$\frac{d}{dx} \left[\int_0^x Nu \, dx \right]^2 = \left\{ \left[\frac{d}{dx} \left(\int_0^x Nu_{\text{forced}} \, dx \right)^2 \right]^2 + \left[\frac{d}{dx} \left(\int_0^x Nu_{\text{free}} \, dx \right)^2 \right]^2 \right\}^{\frac{1}{2}} \quad (2-25)$$

Upon comparison with an exact solution for heat transfer from a horizontal cylinder, the transfer rates predicted by the approximate method were found to agree to within a 'few' per cent.

Acrivos mentioned, at the end of his paper, that the results at high Prandtl numbers in his previous paper on the interaction regime⁴¹ were substantially in error, but he did not elaborate on this statement. The present author has investigated the problem, and the results are discussed in Chapter 6.

The literature survey has shown that for systems in which the effects of both free and forced convection are of importance, most of the experimental investigations have been concerned with the prediction of overall heat or mass transfer rates from single drops or single solid bodies.

In contrast with this, the majority of theoretical investigations have been confined to the prediction of local transfer rates, by the solution of the boundary layer equations, from similarity flow bodies and the vertical flat plate in a uniform free stream. In the case of the vertical flat plate in a uniform free stream, which is not a similarity body for the interaction regime, a number of approximate methods have had to be used in order to solve the relevant boundary layer equations.

Very few theoretical studies in the interaction regime have been made to predict either local or overall transfer rates from axisymmetric bluff bodies of revolution.

The importance of the parameter Gr/Re^2 in the interaction regime has been noted in both the experimental and the theoretical studies which have been reported. The magnitude of the parameter Gr/Re^2 has been shown to provide an indication of the relative importance of free and forced convection for a particular system.

The majority of studies have shown that the total transfer rate is within five per cent of that for either pure free or pure forced convection when the value of the parameter Gr/Re^2 is approximately 10 and 0.1 respectively. These values appear to depend to some extent upon the shape of the body and upon the value of the Schmidt or Prandtl number.

On the basis of the literature survey, the present project was initiated to investigate local heat or mass transfer rates from bodies of revolution when the transfer is due to the effects of both free and forced convection.

The general theory which is developed is used to predict local transfer rates from the sphere and oblate spheroid in particular.

CHAPTER 3

THEORETICAL ANALYSIS

3.1. Introduction.

The present theoretical analysis presents the Navier Stokes, Continuity and Transfer equations, for an axisymmetric body, in orthogonal curvilinear coordinates. In order to predict transfer rates from bodies of revolution when transfer is due to both free and forced convection, the boundary layer simplifications are introduced to reduce the equations to a more readily soluble form.

The boundary layer equations which are obtained are shown to simplify to the familiar ones for two dimensional and axially symmetrical flows.

3.2. The Navier Stokes, Continuity and Transfer Equations for an Axisymmetric Body in Orthogonal Curvilinear Coordinates.

The basic differential equations from which the prediction of heat or mass transfer rates may be calculated express the physical concept of conservation. These equations are the momentum equation, the continuity equation and the transfer equation (either the energy equation for heat transfer or the diffusion equation for mass transfer). In the current analysis the mass transfer case will be considered, but the same arguments apply for heat transfer.

For detailed derivations of these equations, reference may be made to a number of standard texts^{66,67}.

The equations governing transfer at a point in an incompressible Newtonian fluid, where the heat generated by viscous dissipation is negligible, and for which the only body forces which will be considered are those due to gravity, are:

$$\frac{D\mathbf{v}}{Dt} = + \mathbf{g} \beta (C - C_\infty) - \frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{v}$$

Navier Stokes equation (3-1)

$$\nabla \cdot \mathbf{v} = 0$$

Continuity equation (3-2)

$$\frac{DC}{Dt} = \alpha \nabla^2 C$$

Transfer equation (3-3)

where $\mathbf{v}$ is the velocity vector, $\mathbf{g}$ the gravitational vector, β the coefficient of volume expansion, ρ the density, ν the kinematic viscosity, P the pressure, and C the concentration (for mass transfer) or temperature (for heat transfer), and α is the diffusivity.

$\frac{D}{Dt}$ is the substantial derivative and is defined as:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \quad (3-4)$$

For steady state conditions, $\frac{\partial}{\partial t} = 0$ and equations (3-1) to (3-3) become:

$$(\mathbf{v} \cdot \nabla) \mathbf{v} = + \mathbf{g} \beta (C - C_\infty) - \frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{v} \quad (3-5)$$

$$\nabla \cdot \mathbf{v} = 0 \quad (3-6)$$

$$(\mathbf{v} \cdot \nabla) C = \alpha \nabla^2 C \quad (3-7)$$

The boundary conditions, relating to equations (3-5) to (3-7) for the flow of a viscous fluid past a stationary body in the regime of the interaction of free and forced convective transfer are:-

- (i) There is no slip of the fluid at the solid wall.
- (ii) There is assumed to be a negligibly small normal velocity at the wall in the mass transfer case.
- (iii) The value of the concentration or the temperature is assumed to be constant at all points on the surface.
- (iv) As the distance from the wall increases, the velocity and concentration become asymptotic to their values in the undisturbed free stream.

These conditions may be written as:

$$\begin{aligned} \text{At } \underline{n} = 0 : \underline{v} = 0 ; C = C_0 \\ \underline{n} \rightarrow \infty : \underline{v} \rightarrow \underline{v}_\infty ; C \rightarrow C_\infty \end{aligned} \quad (3-8)$$

where:

- $\underline{n}$ is the outward normal vector from the surface.
- $\underline{v}_\infty$ is the free stream velocity vector.
- C_0 is the surface concentration.
- C_∞ is the concentration in the free stream.

The orthogonal curvilinear coordinate system and the necessary standard vector relationships which are now required have been presented in a number of previous works^{68,69}, but are repeated here in Appendix A for ease of reference.

Upon substitution of the expressions for $\nabla^2 \underline{v}$, $(\underline{v} \cdot \nabla) \underline{v}$ and $(\underline{v} \cdot \nabla) C$, equations (3-5) to (3-7) become:

$$\frac{1}{2} \nabla (\underline{v} \cdot \underline{v}) - \underline{v} \times (\nabla \times \underline{v}) = + \frac{\rho}{\mu} \beta (C - C_\infty) - \frac{1}{\rho} \nabla P - \nabla \nabla \times (\nabla \times \underline{v}) \quad (3-9)$$

$$\nabla \cdot \underline{v} = 0 \quad (3-10)$$

$$\underline{v} \cdot \nabla C = \alpha \nabla^2 C \quad (3-11)$$

Equations (3-9) to (3-11) can now be expressed in orthogonal curvilinear coordinates (see Appendix A).

For an axisymmetric body in a flow which has axial symmetry:

$$\frac{\partial}{\partial x_3} = 0 \quad ; \quad v_3 = 0 \quad (3-12)$$

The momentum equation:

$$\begin{aligned} & \frac{v_1}{h_1} \frac{\partial v_1}{\partial x_1} + \frac{v_2}{h_1} \frac{\partial v_2}{\partial x_1} + \frac{v_2}{h_1 h_2} \left\{ \frac{\partial h_1 v_1}{\partial x_2} - \frac{\partial h_2 v_2}{\partial x_1} \right\} \\ = & + \beta \varepsilon_1 (C - C_\infty) - \frac{1}{\rho h_1} \frac{\partial P}{\partial x_1} + \frac{\gamma}{h_2 h_3} \frac{\partial}{\partial x_2} \left\{ \frac{h_3}{h_1 h_2} \left(\frac{\partial h_1 v_1}{\partial x_2} - \frac{\partial h_2 v_2}{\partial x_1} \right) \right\} \end{aligned} \quad (3-13)$$

$$\begin{aligned} & \frac{v_1}{h_2} \frac{\partial v_1}{\partial x_2} + \frac{v_2}{h_2} \frac{\partial v_2}{\partial x_2} - \frac{v_1}{h_1 h_2} \left\{ \frac{\partial h_1 v_1}{\partial x_2} - \frac{\partial h_2 v_2}{\partial x_1} \right\} \\ = & + \beta \varepsilon_2 (C - C_\infty) - \frac{1}{\rho h_2} \frac{\partial P}{\partial x_2} - \frac{\gamma}{h_1 h_3} \frac{\partial}{\partial x_1} \left\{ \frac{h_3}{h_1 h_2} \left(\frac{\partial h_1 v_1}{\partial x_2} - \frac{\partial h_2 v_2}{\partial x_1} \right) \right\} \end{aligned} \quad (3-14)$$

The continuity equation:

$$\frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial x_1} (h_2 h_3 V_1) + \frac{\partial}{\partial x_2} (h_1 h_3 V_2) \right\} = 0 \quad (3-15)$$

The transfer equation:

$$\begin{aligned} \frac{V_1}{h_1} \frac{\partial C}{\partial x_1} + \frac{V_2}{h_2} \frac{\partial C}{\partial x_2} = \frac{\alpha}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial x_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial C}{\partial x_1} \right) \right. \\ \left. + \frac{\partial}{\partial x_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial C}{\partial x_2} \right) \right\} \quad (3-16) \end{aligned}$$

The boundary conditions are:

$$\begin{aligned} \text{At } x_2 = 0 : \quad V_1 = V_2 = 0 ; \quad C = C_0 \\ \text{as } x_2 \rightarrow \infty : \quad V_1 \rightarrow V_{1\infty} ; \quad C \rightarrow C_\infty \\ V_2 \rightarrow V_{2\infty} . \quad (3-17) \end{aligned}$$

3.3. Boundary Layer Simplifications.

The Navier Stokes, continuity and transfer equations are extremely difficult to solve as they stand. When there is interaction between free and forced convection the equations are coupled, and hence, must be solved simultaneously. In order to simplify the equations, the boundary layer assumptions are introduced.

These assumptions are that there is a very thin layer next to the body in which viscous effects are important and in which the velocity gradients are generally large. A similar, thin layer is assumed to exist in which the concentration gradients are confined and outside which the concentration is that of the free stream.

The boundary layer equations are derived by making an estimate of the order of magnitude of each term in a dimensionless form of the momentum, continuity and transfer equations.

The Navier Stokes, continuity and transfer equations are now re-written in a dimensionless form. A characteristic length, D_c , is defined and the free stream velocity, U_∞ is defined as a characteristic velocity.

The dimensionless velocities are:

$$V_1^* = \frac{V_1}{U_\infty} ; \quad V_2^* = \frac{V_2}{U_\infty} \quad (3-18)$$

$$h_1^* = \frac{h_1}{D_c} ; \quad h_2^* = \frac{h_2}{D_c} ; \quad h_3^* = \frac{h_3}{D_c} \quad (3-19)$$

The dimensionless components of the gravitational vector are:

$$g_1^* = \frac{g_1}{g} ; \quad g_2^* = \frac{g_2}{g} \quad (3-20)$$

The dimensionless pressure is defined as :

$$P^* = \frac{P}{\rho U_\infty^2} \quad (3-21)$$

and the dimensionless concentration or temperature as:

$$C^* = \frac{C - C_\infty}{C_0 - C_\infty} \quad (3-22)$$

The dimensionless variables are substituted into equations (3-13) to (3-16) :

momentum equations:

$$\begin{aligned} & \frac{v_1^*}{h_1^*} \frac{\partial v_1^*}{\partial x_1} + \frac{v_2^*}{h_1^*} \frac{\partial v_2^*}{\partial x_1} + \frac{v_2^*}{h_1^* h_2^*} \left\{ \frac{\partial h_1^* v_1^*}{\partial x_2} - \frac{\partial h_2^* v_2^*}{\partial x_1} \right\} \\ & = + \frac{Gr}{Re^2} C^* g_1^* - \frac{1}{\rho h_1^*} \frac{\partial P^*}{\partial x_1} + \frac{1}{Re} \frac{1}{h_2^* h_3^*} \frac{\partial}{\partial x_2} \left\{ \frac{h_3^*}{h_1^* h_2^*} \left(\frac{\partial h_1^* v_1^*}{\partial x_2} - \frac{\partial h_2^* v_2^*}{\partial x_1} \right) \right\} \end{aligned} \quad (3-23)$$

$$\begin{aligned} & \frac{v_1^*}{h_2^*} \frac{\partial v_1^*}{\partial x_2} + \frac{v_2^*}{h_2^*} \frac{\partial v_2^*}{\partial x_2} - \frac{v_1^*}{h_1^* h_2^*} \left\{ \frac{\partial h_1^* v_1^*}{\partial x_2} - \frac{\partial h_2^* v_2^*}{\partial x_1} \right\} \\ & = + \frac{Gr}{Re^2} C^* g_2^* - \frac{1}{\rho h_2^*} \frac{\partial P^*}{\partial x_2} - \frac{1}{Re} \frac{1}{h_1^* h_3^*} \frac{\partial}{\partial x_2} \left\{ \frac{h_3^*}{h_1^* h_2^*} \left(\frac{\partial h_1^* v_1^*}{\partial x_2} - \frac{\partial h_2^* v_2^*}{\partial x_1} \right) \right\} \end{aligned} \quad (3-24)$$

continuity equation:

$$\frac{1}{h_1^* h_2^* h_3^*} \left\{ \frac{\partial}{\partial x_1} (h_2^* h_3^* v_1^*) + \frac{\partial}{\partial x_2} (h_1^* h_3^* v_2^*) \right\} = 0 \quad (3-25)$$

transfer equation:

$$\begin{aligned} & \frac{v_1^*}{h_1^*} \frac{\partial c^*}{\partial x_1} + \frac{v_2^*}{h_2^*} \frac{\partial c^*}{\partial x_2} = \frac{1}{ReSc} \frac{1}{h_1^* h_2^* h_3^*} \left\{ \frac{\partial}{\partial x_1} \left(\frac{h_2^* h_3^*}{h_1^*} \frac{\partial c^*}{\partial x_1} \right) \right. \\ & \quad \left. + \frac{\partial}{\partial x_2} \left(\frac{h_1^* h_3^*}{h_2^*} \frac{\partial c^*}{\partial x_2} \right) \right\} \end{aligned} \quad (3-26)$$

The boundary conditions are:

$$\begin{aligned}
 \text{At } x_2 = 0 : \quad V_1^* &= V_2^* = 0 ; \quad C^* = 1 \\
 x_2 \rightarrow \infty : \quad V_1^* &\rightarrow V_{1\infty}^* ; \quad C^* \rightarrow 0 \\
 &V_2^* \rightarrow V_{2\infty}^* ; \quad (3-27)
 \end{aligned}$$

where Gr is the Grashof Number, $\frac{g \beta D_c^3 (C - C_\infty)}{\nu^2}$

Re is the Reynolds Number, $\frac{U_\infty D_c}{\nu}$

Sc is the Schmidt Number, $\frac{\nu}{\alpha}$
(mass transfer)

or the Prandtl number for heat transfer.

The hydrodynamic boundary layer thickness, δ_H , is assumed to be very small compared with the characteristic dimension of the body, and thus the dimensionless boundary layer thickness $\delta_H/D_c = \delta_H^*$ is very small with respect to unity, i.e. $\delta_H^* \ll 1$.

When the transfer boundary layer thickness, δ_T , is also very small compared with the characteristic dimension of the body, it follows that the dimensionless transfer boundary layer thickness is also very small with respect to unity, $\delta_T^* \ll 1$.

The order of magnitude of each term in equations (3-23) to (3-26) can now be estimated in order that small terms may be neglected, and the desired simplifications may be achieved. The desired simplification of the Navier Stokes and transfer equations can also be achieved by other methods which have been presented in a number of other places^{70,71}. However, for the present study of the interaction region, the order of magnitude analysis can be seen to be more readily applicable.

The body is assumed to be smoothly curved and to have no sharp points. Thus, all scale factors and their derivatives can be assumed to be of order unity.

The velocity parallel to the surface, V_1 , increases from zero at the surface, to the potential flow velocity, at the edge of the boundary layer, and is thus of order unity. It now follows that $\frac{\partial V_1}{\partial x_1}$ is also of order unity, and from the continuity equation (3-25), $\frac{\partial V_2}{\partial x_2}$ must also be of order unity.

Since the distance x_2 is of the order of the boundary layer thickness, δ_H^* , V_2 must also be of order δ_H^* .

Following directly from this:

$$\frac{\partial V_1}{\partial x_2} = O\left(\frac{1}{\delta_H^*}\right)$$

and the second derivative:

$$\frac{\partial^2 V_1}{\partial x_2^2} = O\left(\frac{1}{\delta_H^{*2}}\right)$$

This latter derivative can become very large near the body surface, whereas the other second derivative:

$$\frac{\partial^2 V_1}{\partial x_1^2} = O(1).$$

Over the region for which the boundary layer assumptions are valid, Rosenhead⁷⁰ has shown that the dimensionless hydrodynamic boundary layer thickness is of the same order as the reciprocal of the square root of the Reynolds number, i.e. $Re = O(1/\delta_H^{*2})$.

The pressure term and the body force term are assumed to be of the same order of magnitude as all of the terms retained.

The continuity equation (3-25) remains unaltered since both terms are of order unity.

All of the terms of the second momentum equation (3-24) can be seen, by inspection, to be negligible. The total pressure change across the boundary layer, for a smooth body without sharp corners, is therefore of order δ_H^* and can be neglected, as can be body force in this direction. The pressure in the normal direction can be taken to be constant and may be assumed to be that at the outer edge of the boundary layer. This pressure distribution may now be obtained from the momentum equation for inviscid flow.

The momentum equation for the inviscid flow:

$$\begin{aligned} \frac{v_1^*}{h_1^*} \frac{\partial v_1^*}{\partial x_1} + \frac{v_2^*}{h_1^*} \frac{\partial v_2^*}{\partial x_1} + \frac{v_2^*}{h_1^* h_2^*} \left[\frac{\partial h_1^* v_1^*}{\partial x_2} - \frac{\partial h_2^* v_2^*}{\partial x_1} \right] \\ = - \frac{1}{h_1^*} \frac{\partial P^*}{\partial x_1} \end{aligned} \quad (3-28)$$

for a thin boundary layer, by an order of magnitude analysis, equation (3-28) reduces to:

$$v_1^* \frac{\partial v_1^*}{\partial x_1} = - \frac{\partial P^*}{\partial x_1} \quad (3-29)$$

replacing v_1^* by U^* , the dimensionless potential flow velocity, calculated at the body surface, equation (3-29) becomes:

$$U^* \frac{dU^*}{dx_1} = - \frac{\partial P^*}{\partial x_1} \quad (3-30)$$

When the Schmidt number is of at least order unity, the transfer boundary layer thickness is of the same order as, or smaller than, the hydrodynamic boundary layer, then the transfer equation (3-26) can also be simplified by an order of magnitude analysis.

The dimensionless transfer boundary layer thickness is δ_T^* . The tangential gradients will be small in relation to the normal gradients and the orders of magnitude of the various terms can be seen to be:

$$C^* = O(1) ; \quad \frac{\partial C^*}{\partial x_1} = O(1) ; \quad \frac{\partial^2 C^*}{\partial x_1^2} = O(1)$$

$$\frac{\partial C^*}{\partial x_2} = O\left(\frac{1}{\delta_T^*}\right) ; \quad \frac{\partial^2 C^*}{\partial x_2^2} = O\left(\frac{1}{\delta_T^{*2}}\right)$$

When the Schmidt number is of order unity or greater, the order of magnitude of the Schmidt number is $O(\delta_H^*/\delta_T^*)^2$ or greater.

Equation (3-26) reduces to:

$$\frac{v_1^*}{h_1^*} \frac{\partial C^*}{\partial x_1} + \frac{v_2^*}{h_2^*} \frac{\partial C^*}{\partial x_2} = \frac{1}{ReSc} \frac{1}{h_2^*} \frac{\partial^2 C^*}{\partial x_2^2} \quad (3-31)$$

Equations (3-23) to (3-26) have now been reduced to:

$$\frac{v_1^*}{h_1^*} \frac{\partial v_1^*}{\partial x_1} + \frac{v_2^*}{h_2^*} \frac{\partial v_1^*}{\partial x_2} = \frac{Gr}{Re^2} C^* \varepsilon_1^* + \frac{U^*}{h_1^*} \frac{dU^*}{dx_1} + \frac{1}{Re} \frac{1}{h_2^{*2}} \frac{\partial^2 v_1^*}{\partial x_2^2} \quad (3-32)$$

The second momentum equation (3-24) has been neglected.

$$\frac{1}{h_1^* h_2^* h_3^*} \left[\frac{\partial}{\partial x_1} (h_2^* h_3^* V_1^*) + \frac{\partial}{\partial x_2} (h_1^* h_3^* V_2^*) \right] = 0 \quad (3-33)$$

$$\frac{V_1^*}{h_1^*} \frac{\partial C^*}{\partial x_1} + \frac{V_2^*}{h_2^*} \frac{\partial C^*}{\partial x_2} = \frac{1}{\text{ReSc}} \frac{1}{h_2^*} \frac{\partial^2 C^*}{\partial x_2^2} \quad (3-34)$$

where the boundary conditions are:

$$\begin{aligned} \text{at } x_2 = 0 : V_1^* = V_2^* = 0 ; C^* = 1 \\ x_2 \rightarrow \infty : V_1^* \rightarrow U^* ; C^* \rightarrow 0 \end{aligned} \quad (3-35)$$

The assumption of a thin boundary layer enables a simplification of the coordinate system to be introduced. The normal coordinate is, in fact curved, but in the vicinity of the body this may be approximated by a straight line.

This coordinate is therefore replaced by the geometric normal at the surface. Thus $h_2^* \partial x_2$ is replaced by ∂N^* . Also, by virtue of the boundary layer being thin, the general scale factors h_1 and h_3 are replaced by their values at the body surface which will be denoted by $h_{1,s}$ and $h_{3,s}$ respectively.

These simplifications are introduced into equations (3-32) to (3-35), which become:

$$\frac{V_1^*}{h_{1,s}} \frac{\partial V_1^*}{\partial x_1} + \frac{V_N^*}{\partial N^*} \frac{\partial V_1^*}{\partial N^*} = + \frac{\text{Gr}}{\text{Re}^2} C^* g_1^* + \frac{U^*}{h_{1,s}} \frac{dU^*}{dx_1} + \frac{1}{\text{Re}} \frac{\partial^2 V_1^*}{\partial N^{*2}} \quad (3-36)$$

$$\frac{\partial}{\partial x_1} (h_{3,s}^* V_1^*) + h_{1,s}^* h_{3,s}^* \frac{\partial V_N^*}{\partial N^*} = 0 \quad (3-37)$$

$$\frac{V_1^*}{h_{1,s}} \frac{\partial C^*}{\partial x_1} + V_N^* \frac{\partial C^*}{\partial N^*} = \frac{1}{ReSc} \frac{\partial^2 C^*}{\partial N^{*2}} \quad (3-38)$$

with the boundary conditions:

$$\begin{aligned} \text{at } N^* = 0 & : V_1^* = V_N^* = 0 ; C^* = 1 \\ \text{as } N^* \rightarrow \infty & : V_1^* \rightarrow U^* ; C^* \rightarrow 0 \end{aligned} \quad (3-39)$$

These equations can be expressed in terms of Boltze's boundary layer coordinates by putting:

$$\begin{aligned} h_{1,s}^* \frac{\partial x_1}{\partial x} &= \frac{\partial x^*}{\partial x} \\ \frac{\partial N^*}{\partial y} &= \frac{\partial y^*}{\partial y} \\ h_{3,s}^* &= r^* \\ \xi_1^* &= \sin \phi \end{aligned} \quad (3-40)$$

Equations (3-36) to (3-39) become:

$$\frac{u^*}{Re} \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{Gr}{Re^2} C^* \sin \phi + U^* \frac{dU^*}{dx^*} + \frac{1}{Re} \frac{\partial^2 u^*}{\partial y^{*2}} \quad (3-41)$$

$$\frac{\partial}{\partial x^*} (u^* r^*) + r^* \frac{\partial v^*}{\partial y^*} = 0 \quad (3-42)$$

$$u^* \frac{\partial C^*}{\partial x^*} + v^* \frac{\partial C^*}{\partial y^*} = \frac{1}{ReSc} \frac{\partial^2 C^*}{\partial y^{*2}} \quad (3-43)$$

With the boundary layer conditions:

$$\begin{aligned} \text{At } y^* = 0 & : u^* = v^* = 0 ; C^* = 1 \\ y^* \rightarrow \infty & : u^* \rightarrow U^* ; C^* \rightarrow 0 \end{aligned} \quad (3-44)$$

The dimensionless radius of transverse cross section, r^* , is assumed to be independent of the distance normal to the body surface, y^* , and is therefore a function only of the distance parallel to the body surface, x^* .

The equations for two dimensional flows follow immediately by the removal of the radius of transverse cross section, r^* . The boundary layer equations for a vertical flat plate in a uniform free stream are obtained by putting $\sin \psi$ equal to unity and dU^*/dx^* equal to zero.

The above boundary layer equations clearly show the importance of the parameter Gr/Re^2 , in the interaction regime. The importance of this parameter has been discussed previously in connection with the literature review.

When the parameter is of order unity and increasing, the effects of free convection are becoming predominant, whereas for values of the parameter less than order unity and decreasing, the free convective effects are becoming less significant.

The boundary layer equations may be solved using either exact or approximate methods. Even for those cases for which similarity transforms may be used to reduce the partial differential equations to ordinary differential equations, the exact methods have considerable mathematical difficulties associated with them. The problem involving flow around bodies, for which similarity transforms may not be used, necessitates the use of numerical methods which frequently require a large amount of both work and computer time, especially when a large number of problems have to be solved. Exact series solutions often have convergence problems associated with them and, therefore, a large number of ordinary differential equations

have to be solved sequentially. Any numerical errors which arise in the solution of one equation then affect the solution of all subsequent equations. Numerical step by step methods have been used to solve boundary layer problems, but these also involve a considerable amount of work and computer time, especially when a very fine mesh is required to cover the field.

In order to obtain reliable solutions quickly, even if they were slightly less accurate than the exact solutions, approximate methods were developed which did not involve such difficulties as the exact methods.

The integral method originally proposed by von Karman⁴³ and Pohlhausen⁴² has been used to solve a large number of two dimensional and axially symmetric boundary layer problems. The velocity and concentration profiles within the boundary layers are represented by polynomials, the order of which may be varied by considering various boundary conditions at the body surface and in the free stream.

Very good agreement has been found between approximate and exact solutions and a number of comparisons have been reported by Schlichting⁶⁷ and Spalding⁷². With particular relevance to the present study, the approximate method has been found to be equally successful in the prediction of heat or mass transfer rates for both free convection and forced convection. Green⁷³ used an exact series method to predict forced convective transfer rates from spheres and oblate spheroids. He found excellent agreement between his solutions and those of Rojey⁷⁴, who used an integral method.

Merk and Prins⁷⁵ used an approximate method to solve the integral equations to predict transfer rates from spheres in pure free convection and found excellent agreement between

their results and those of some previously reported exact solutions^{76,77}. Pandya⁷⁸ also used both exact and approximate methods to predict transfer rates in pure free convection from a number of different body shapes and found very good agreement between the solutions.

The approximate method has been reported to fail by Schlichting⁷⁹, when applied to solve the boundary layer equations for a rotating body. Failure of the method occurred once a certain spin parameter was reached. Sanders⁸⁰ also reported this break - down of the approximate method and found that he had to resort to an exact method in order to obtain solutions of the boundary layer equations above the spin parameter at which the approximate method failed.

For the majority of problems studied, the approximate method has been shown to predict transfer rates which have been in very good agreement with those predicted by exact methods, and to require much less computer time to obtain the solution. Thus, it was decided to apply the von Karman - Pohlhausen integral method to the present problem of the prediction of transfer rates from spheres and oblate spheroids in the interaction regime.

CHAPTER 4.

THEORETICAL ANALYSIS.

THE APPROXIMATE METHOD.

4.1. The Simplification of the Boundary Layer Equations by the Removal of Dimensionless Parameters.

It has been shown, in the previous chapter, that the boundary layer equations derived in orthogonal curvilinear coordinates reduce to the form presented below when Boltze's coordinates for an axisymmetric body of revolution are used. These will be used in the derivation of the equations for the approximate method.

The equations are:

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = + \frac{Gr}{Re^2} C^* \sin \psi + \frac{U^*}{dx^*} + \frac{1}{Re} \frac{\partial^2 u^*}{\partial y^{*2}} \quad (4-1)$$

$$\frac{\partial(u^* r^*)}{\partial x^*} + r^* \frac{\partial v^*}{\partial y^*} = 0 \quad (4-2)$$

$$u^* \frac{\partial C^*}{\partial x^*} + v^* \frac{\partial C^*}{\partial y^*} = \frac{1}{ReSc} \frac{\partial^2 C^*}{\partial y^{*2}} \quad (4-3)$$

with the boundary conditions:

$$\text{At } y^* = 0 : u^* = v^* = 0 ; - \frac{1}{Re} \frac{\partial^2 u^*}{\partial y^{*2}} = + \frac{Gr}{Re^2} \sin \psi + \frac{U^*}{dx^*}$$

$$C^* = 1 \quad (4-4)$$

$$\text{As } y^* \rightarrow \infty: u^* \rightarrow U^*; \quad \frac{\partial u^*}{\partial y^*} = \frac{\partial^2 u^*}{\partial y^{*2}} \rightarrow 0; \quad C^* \rightarrow 0 \quad (4-4)$$

The parameters Re and Gr/Re^2 may be removed immediately from the equations (4-1) to (4-3) and the boundary conditions (4-4) by a dimensionless substitution. However, they will be removed individually since this enables some interesting observations to be made.

The Reynolds number, the Schmidt number, and the group Grashof number divided by the square of the Reynolds number are all present, either in the equations themselves or in the boundary conditions or in both. Ideally, if all the parameters were removed from both of the equations, and the boundary conditions, the problem could be solved once, and only once, for all possible values of the parameters.

The Reynolds number is simply removed, in the same manner as in pure forced convection problems, by the introduction of a second set of dimensionless variables.

The method proposed by Hellums and Churchill⁶¹ was used to obtain the dimensionless variables, which are as follows:

$$\begin{aligned} u^{**} &= u^* & ; & & v^{**} &= v^* Re^{\frac{1}{2}} \\ x^{**} &= x^* & ; & & y^{**} &= y^* Re^{\frac{1}{2}} \\ C^{**} &= C^* & ; & & U^{**} &= U^* \\ r^{**} &= r^* & ; & & & \end{aligned} \quad (4-5)$$

Where the double asterisk represents the second dimensionless set.

Upon substitution of these new variables, equations (4-1) to (4-3) become:

$$u^* \frac{\partial u^*}{\partial x^*} + v^{**} \frac{\partial u^*}{\partial y^{**}} = + \frac{Gr}{Re^2} C^* \sin \phi + \frac{U^*}{dx^*} + \frac{\partial^2 u^*}{\partial y^{**2}} \quad (4-6)$$

$$\frac{\partial(u^* r^*)}{\partial x^*} + r^* \frac{\partial v^{**}}{\partial y^{**}} = 0 \quad (4-7)$$

$$u^* \frac{\partial C^*}{\partial x^*} + v^{**} \frac{\partial C^*}{\partial y^{**}} = \frac{1}{Sc} \frac{\partial^2 C^*}{\partial y^{**2}} \quad (4-8)$$

With the boundary conditions:

$$\text{At } y^{**} = 0 : u^* = v^{**} = 0; \quad \frac{\partial^2 u^*}{\partial y^{**2}} = - \frac{Gr}{Re^2} \sin \phi - \frac{U^*}{dx^*}$$

$$C^* = 1 \quad (4-9)$$

$$\text{as } y^{**} \rightarrow \infty : u^* \rightarrow U^* ; \quad \frac{\partial u^*}{\partial y^{**}} = \frac{\partial^2 u^*}{\partial y^{**2}} \rightarrow 0 ; \quad C^* \rightarrow 0$$

The equations (4-6) to (4-9) are very similar to those for pure forced convection. In fact they differ only by the presence of the buoyancy term $\frac{Gr}{Re^2} C^* \sin \phi$ in equations (4-6) and (4-9).

The parameter Gr/Re^2 is readily removed by a third dimensionless substitution.

The dimensionless variables are:

$$\begin{aligned}
 u^{***} &= u^{**} = u^* \\
 v^{***} &= v^{**} \frac{Gr}{Re^2}^{-\frac{1}{2}} = v^* Re^{\frac{1}{2}} \left(\frac{Gr}{Re^2} \right)^{-\frac{1}{2}} \\
 x^{***} &= x^{**} \frac{Gr}{Re^2} = x^* \frac{Gr}{Re^2} \\
 y^{***} &= y^{**} \frac{Gr}{Re^2}^{\frac{1}{2}} = y^* Re^{\frac{1}{2}} \left(\frac{Gr}{Re^2} \right)^{\frac{1}{2}} \\
 c^{***} &= c^{**} = c^* \\
 U^{***} &= U^{**} = U^* \\
 r^{***} &= r^{**} = r^*
 \end{aligned} \tag{4-10}$$

Substitution of these variables into equations (4-6) to (4-8) and the boundary conditions (4-9) gives:

$$u^* \frac{\partial u^*}{\partial x^{***}} + v^{***} \frac{\partial u^*}{\partial y^{***}} = + C^* \sin \gamma + \frac{U^* dU^*}{dx^{***}} + \frac{\partial^2 u^*}{\partial y^{***2}} \tag{4-11}$$

$$\frac{\partial(u^* r^*)}{\partial x^{***}} + r^* \frac{\partial v^{***}}{\partial y^{***}} = 0 \tag{4-12}$$

$$u^* \frac{\partial c^*}{\partial x^{***}} + v^{***} \frac{\partial c^*}{\partial y^{***}} = \frac{1}{Sc} \frac{\partial^2 c^*}{\partial y^{***2}} \tag{4-13}$$

With the boundary conditions:

$$\text{At } y^{***} = 0 : u^* = v^{***} = 0 ;$$

$$\frac{\partial^2 u^*}{\partial y^{***2}} = + \sin \gamma - \frac{U^* dU^*}{dx^{***}} ; \quad C^* = 1 \tag{4-14}$$

$$\begin{aligned} \text{As } y^{***} \rightarrow \infty : \quad u^* - U^* & ; \quad \frac{\partial u^*}{\partial y^{***}} = \frac{\partial^2 u^*}{\partial y^{***2}} \rightarrow 0 \\ c^* & \rightarrow 0 \end{aligned} \quad \begin{matrix} 86 \\ (4-14) \end{matrix}$$

In general, r^* and $\sin \psi$ are both functions of x^{***} . Therefore, any substitution of an expression for r^* and $\sin \psi$ in terms of x^* reintroduces the parameter Gr/Re^2 into the equations. Except for body shapes for which both r^* and $\sin \psi$ are independent of x^* , equations (4-11) to (4-14) are not quite as convenient to solve as they appear. Either equations (4-6) to (4-10) or equations (4-11) to (4-14) can be used to investigate the removal of the Schmidt number.

Any attempt to remove the Schmidt number completely, ends in failure. The parameter may be removed from the equations, but it then reappears in the boundary conditions. Upon consideration of the two limiting cases of pure free convection and pure forced convection, the Schmidt number can only be removed in the limit as it approaches zero or infinity. It is not surprising, therefore, that the Schmidt number cannot be completely removed from the interaction equations.

Since the Schmidt number can be removed in the two limiting cases for both free and forced convection, it is of interest to investigate whether the parameter may also be removed in the interaction regime. Only the limiting case as the Schmidt number approaches infinity is considered since the present study is confined to systems, in which the Schmidt number is of order unity or greater, for which the boundary layer simplifications may be applied.

It is found that the Schmidt number cannot be completely removed from the interaction equations in the limit as the Schmidt number $\rightarrow \infty$. This is not really surprising since for

free convection as the $Sc \rightarrow \infty$ the Sherwood number is proportional to the fourth root of the Schmidt number, whereas for forced convection the Sherwood number is proportional to the cube root of the Schmidt number.

It is of interest to introduce a scaling which removes the Schmidt number from the transfer equation and introduces it into the momentum equation. This scaling has been found to be very useful by a number of other workers^{64,81} for systems where the Schmidt number is large, i.e. the transfer boundary layer is very much thinner than the hydrodynamic boundary layer.

The scaling is introduced into equations (4-6) to (4-9).

The dimensionless variables are:

$$\begin{aligned} u^{****} &= u^{**} Sc^{\frac{1}{3}} & ; & & v^{****} &= v^{**} Sc^{\frac{2}{3}} \\ x^{****} &= x^{*} & ; & & y^{****} &= y^{**} Sc^{\frac{1}{3}} \\ C^{****} &= C^{*} & ; & & U^{****} &= U^{*} \\ r^{****} &= r^{*} & ; & & & \end{aligned} \quad (4-15)$$

Substitution of these variables into equations (4-6) to (4-8) with the boundary conditions (4-9) gives:

$$\begin{aligned} \frac{1}{Sc} \left[u^{****} \frac{\partial u^{****}}{\partial x^{*}} + v^{****} \frac{\partial u^{****}}{\partial y^{****}} \right] &= + \frac{Gr}{Re^2 Sc^{\frac{1}{3}}} C^{*} \sin \psi \\ &+ \frac{\partial^2 u^{****}}{\partial y^{****2}} + \frac{1}{Sc^{\frac{1}{3}}} U^{*} \frac{dU^{*}}{dx^{*}} \end{aligned} \quad (4-16)$$

$$\frac{\partial u^{****} r^{*}}{\partial x^{*}} + r^{*} \frac{\partial v^{****}}{\partial y^{****}} = 0 \quad (4-17)$$

$$u^{****} \frac{\partial C^{*}}{\partial x^{*}} + v^{****} \frac{\partial C^{*}}{\partial y^{****}} = \frac{\partial^2 C^{*}}{\partial y^{****2}} \quad (4-18)$$

with the boundary conditions :

$$\text{At } y^{****} = 0 : u^{****} = v^{****} = 0$$

$$\frac{\partial^2 u^{****}}{\partial y^{****2}} = + \frac{Gr}{Re^2 Sc^{\frac{1}{3}}} \sin \phi - \frac{1}{Sc^{\frac{1}{3}}} U^* \frac{dU^*}{dx^*}$$

$$C^* = 1 \quad (4-19)$$

$$\text{as } y^{****} \rightarrow \infty : u^{****} \rightarrow U^* Sc^{\frac{1}{3}} ; \frac{\partial u^{****}}{\partial y^{****}} = \frac{\partial^2 u^{****}}{\partial y^{****2}} \rightarrow 0$$

$$C^* \rightarrow 0$$

4.2. The Development of the Integral Equations.

The method of solution of the boundary layer equations requires that the momentum , continuity and transfer equations are integrated over some arbitrarily chosen hydrodynamic and transfer boundary layer thicknesses.

Two integral methods have been used. The first method is developed from equations (4-11) to (4-14) which are re-written without the asterisks in order to simplify the presentation as:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = + C \sin \phi + \frac{U}{dx} \frac{dU}{dx} + \frac{\partial^2 u}{\partial y^2} \quad (4-20)$$

$$\frac{\partial (ur)}{\partial x} + r \frac{\partial v}{\partial y} = 0 \quad (4-21)$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = \frac{1}{Sc} \frac{\partial^2 C}{\partial y^2} \quad (4-22)$$

With the boundary conditions:

$$\text{At } y = 0 : u = v = 0 ; \quad \frac{\partial^2 u}{\partial y^2} = \frac{U}{r} \sin \psi - \frac{U}{r} \frac{dU}{dx}$$

$$C = 1 \quad (4-23)$$

$$\text{as } y \rightarrow \infty : u \rightarrow U ; \quad \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} \rightarrow 0 ; \quad C \rightarrow 0$$

The normal velocity, v , is obtained from the continuity equation (4-21) and the resulting expression is substituted into the momentum and transfer equations (4-20) and (4-22).

From (4-21) :

$$\begin{aligned} v &= - \int_0^y \frac{1}{r} \frac{\partial(ur)}{\partial x} dy \\ &= - \int_0^y \frac{1}{r} \left\{ r \frac{\partial u}{\partial x} + u \frac{\partial r}{\partial x} \right\} dy \end{aligned} \quad (4-24)$$

$$\text{Since } r \text{ is independent of } y, \quad \frac{\partial r}{\partial x} \equiv \frac{dr}{dx} \quad (4-25)$$

The momentum equation is now integrated over the hydrodynamic boundary layer thickness, δ :

Considering the left hand side of equation (4-20) :

$$\int_0^\delta u \frac{\partial u}{\partial x} dy + \int_0^\delta v \frac{\partial u}{\partial y} dy \quad (4-26)$$

Substituting for v , from (4-24), in the second term gives:

$$\int_0^\delta v \frac{\partial u}{\partial y} dy = - \int_0^\delta \frac{\partial u}{\partial y} \left\{ \int_0^y \left(\frac{\partial u}{\partial x} + \frac{u}{r} \frac{dr}{dx} \right) dy \right\} dy \quad (4-27)$$

Upon integration by parts, the integral becomes:

$$\begin{aligned} \int_0^{\delta} v \frac{\partial u}{\partial y} dy &= - U \int_0^{\delta} \frac{u}{r} \frac{dr}{dx} dy - U \int_0^{\delta} \frac{\partial u}{\partial x} dy \\ &+ \int_0^{\delta} \frac{u^2}{r} \frac{dr}{dx} dy + \int_0^{\delta} u \frac{\partial u}{\partial x} dy \end{aligned} \quad (4-28)$$

Then upon substitution back into equation (4-26):

$$\begin{aligned} \int_0^{\delta} u \frac{\partial u}{\partial x} dy + \int_0^{\delta} v \frac{\partial u}{\partial y} dy &= \int_0^{\delta} u \frac{\partial u}{\partial x} dy - U \int_0^{\delta} \frac{u}{r} \frac{dr}{dx} dy \\ &- U \int_0^{\delta} \frac{\partial u}{\partial x} dy + \int_0^{\delta} \frac{u^2}{r} \frac{dr}{dx} dy + \int_0^{\delta} u \frac{\partial u}{\partial x} dy \end{aligned} \quad (4-29)$$

$$\begin{aligned} &= \frac{d}{dx} \int_0^{\delta} u^2 dy - U \frac{d}{dx} \int_0^{\delta} u dy \\ &+ \frac{1}{r} \frac{dr}{dx} \left[\int_0^{\delta} u^2 dy - U \int_0^{\delta} u dy \right] \end{aligned} \quad (4-30)$$

The right hand side of equation (4-20) is integrated straight away to give:

$$\int_0^{\delta} U \frac{dU}{dx} dy + \int_0^{\Delta} C \sin f dy + \int_0^{\delta} \frac{\partial^2 u}{\partial y^2} dy \quad (4-31)$$

Since at the edge of the boundary layer $\frac{\partial u}{\partial y} = 0$, equation (4-31) can be written:

$$U \frac{dU}{dx} \delta + \int_0^{\Delta} C \sin f dy - \left. \frac{\partial u}{\partial y} \right|_{y=0} \quad (4-32)$$

The complete momentum integral equation is:

$$\begin{aligned} \frac{d}{dx} \int_0^{\delta} u^2 dy - U \frac{d}{dx} \int_0^{\delta} u dy + \frac{1}{r} \frac{dr}{dx} \left[\int_0^{\delta} u^2 dy - U \int_0^{\delta} u dy \right] \\ = U \frac{dU}{dx} \delta + \int_0^{\Delta} C \sin \psi dy - \left. \frac{\partial u}{\partial y} \right|_y = 0 \end{aligned} \quad (4-33)$$

which may be written in a more compact form as:

$$\frac{1}{r} \frac{d}{dx} \left(r \int_0^{\delta} u(u-U) dy \right) = U \frac{dU}{dx} \delta + \int_0^{\Delta} C \sin \psi dy - \left. \frac{\partial u}{\partial y} \right|_y = 0 \quad (4-34)$$

The transfer integral equation is derived by integrating equation (4-22) over the transfer boundary layer thickness, Δ .

Thus equation (4-22) becomes:

$$\int_0^{\Delta} u \frac{\partial C}{\partial x} + \int_0^{\Delta} v \frac{\partial C}{\partial y} = \int_0^{\Delta} \frac{1}{Sc} \frac{\partial^2 C}{\partial y^2} \quad (4-35)$$

The right hand side of equation (4-35) integrates readily to give:

$$\frac{1}{Sc} \left[\left. \frac{\partial C}{\partial y} \right|_{y=\Delta} - \left. \frac{\partial C}{\partial y} \right|_{y=0} \right] \quad (4-36)$$

at the edge of the boundary layer $\frac{\partial C}{\partial y} = 0$, therefore equation (4-36) becomes:

$$- \frac{1}{Sc} \left. \frac{\partial C}{\partial y} \right|_{y=0} \quad (4-37)$$

The left hand side of equation (4-35) is treated in a similar manner to the momentum equation. Upon substitution for v in the second term:

$$\int_0^{\Delta} v \frac{\partial C}{\partial y} dy = - \int_0^{\Delta} \frac{\partial C}{\partial y} \left(\int_0^y \left\{ \frac{1}{r} \frac{\partial(ur)}{\partial x} \right\} dy \right) dy \quad (4-38)$$

Again integrating by parts:

$$\int_0^{\Delta} v \frac{\partial C}{\partial y} dy = \frac{1}{r} \int_0^{\Delta} C \frac{\partial(ur)}{\partial x} dy \quad (4-39)$$

Therefore equation (4-35) can now be written as:

$$\int_0^{\Delta} u \frac{\partial C}{\partial x} dy + \frac{1}{r} \int_0^{\Delta} C \frac{\partial(ur)}{\partial x} dy = - \frac{1}{Sc} \frac{\partial C}{\partial y} \Big|_{y=0} \quad (4-40)$$

which simplifies to:

$$\frac{1}{r} \frac{d}{dx} \left\{ r \int_0^{\Delta} uC dy \right\} = - \frac{1}{Sc} \frac{\partial C}{\partial y} \Big|_{y=0} \quad (4-41)$$

The integral method is based on the assumption of suitable profiles for the velocity and concentration distributions. The profiles are chosen such that they contain one free parameter, ideally the particular boundary layer thickness to be determined. Each function in the profile is dependent upon only one of the independent variables and the boundary conditions at the body surface and in the free stream are satisfied exactly. The profiles are introduced into the integral equations, which are then simplified to give two ordinary differential equations for the two free parameters.

The order of the polynomials chosen has been the subject of much discussion. Some workers have suggested that higher

order polynomials increase the accuracy of the solution^{67,82}, whilst others have suggested that high order profiles are no better, and may be worse, than cubic or quartic profiles.⁸³

Initially, quartic profiles for both velocity and concentration are assumed in the present study.

The velocity function is chosen in terms of the dimensionless distance from the wall:

$$\eta_H = \frac{y}{\delta} \quad (4-42)$$

and the concentration function in terms of another dimensionless distance:

$$\eta_C = \frac{y}{\Delta} \quad (4-43)$$

where δ is defined as the hydrodynamic boundary layer thickness, and Δ is defined as the transfer boundary layer thickness.

The assumed velocity profile is:

$$\frac{u}{U_\infty} = u^* = f(\eta_H) = a_H + b_H \eta_H + c_H \eta_H^2 + d_H \eta_H^3 + e_H \eta_H^4$$

$$0 \leq \eta_H \leq 1 \quad (4-44)$$

and for $\eta \geq 1$, the potential flow condition is satisfied, $\frac{u}{U_\infty} = 1$ (4-45)

$$\frac{C - C_\infty}{C_0 - C_\infty} = C^* = g(\eta_C) = a_C + b_C \eta_C + c_C \eta_C^2 + d_C \eta_C^3 + e_C \eta_C^4$$

$$0 \leq \eta_C \leq 1 \quad (4-46)$$

$$C^* = 0 \quad \eta_C \geq 1 \quad (4-47)$$

The coefficients a_H, b_H, c_H, d_H, e_H and a_C, b_C, c_C, d_C, e_C are determined from the boundary conditions:

For the velocity profile, (dimensionless variables):

$$\begin{aligned} y = 0 : u = 0 ; \quad \frac{\partial^2 u}{\partial y^2} &= - \frac{U}{dx} \frac{dU}{dx} + \sin \phi \\ y = \delta : u = U ; \quad \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} &= 0 \end{aligned} \quad (4-48)$$

For the concentration profile, (dimensionless variables):

$$\begin{aligned} y = 0 : C = 1 ; \quad \frac{\partial^2 C}{\partial y^2} &= 0 \\ y = \Delta : C = \frac{\partial C}{\partial y} = \frac{\partial^2 C}{\partial y^2} &= 0 \end{aligned} \quad (4-49)$$

These ten boundary conditions enable the ten coefficients to be calculated:

For the velocity profile:

$$a_H = 0 \quad (4-50)$$

$$b_H = (\pm \delta^2 \sin \phi + U \lambda + 12 U) / 6 \quad (4-51)$$

$$c_H = -(\pm \delta^2 \sin \phi + U \lambda) / 2 \quad (4-52)$$

$$d_H = (\pm \delta^2 \sin \phi + U \lambda - 4 U) / 2 \quad (4-53)$$

$$e_H = (6 U - (\pm \delta^2 \sin \phi + U \lambda)) / 6 \quad (4-54)$$

where $\lambda = \delta^2 \frac{dU}{dx}$

For the concentration profile :

$$a_C = 1 ; \quad b_C = -2 ; \quad c_C = 0 \quad (4-55)$$

$$d_C = 2 ; \quad e_C = -1$$

Any increase or decrease in the order of the polynomials for the velocity or the concentration profiles means that either more or less boundary conditions have to be satisfied

at the wall and at the external boundary of the respective boundary layer. The coefficients of the new profiles are then calculated in the same way as above.

The velocity and concentration profiles are now substituted into the two integral equations (4-32) and (4-41). Each term will be presented separately, since the final expression is rather complicated and the origin of each term is not clear.

$$\int_0^{\delta} u \, dy = \frac{U\delta}{120} (84 + \lambda) + \frac{\delta^3 \sin \psi}{120} \quad (4-56)$$

$$\begin{aligned} \int_0^{\delta} u^2 \, dy = & \frac{U^2 \delta}{120} \left(734 + \frac{71\lambda}{6} + \frac{5\lambda^2}{36} \right) + U\delta^3 \sin \psi \left(\frac{71}{7520} + \frac{\lambda}{4536} \right) \\ & + \frac{\delta^5 \sin^2 \psi}{9072} \end{aligned} \quad (4-57)$$

$$\begin{aligned} \frac{1}{r} \frac{dr}{dx} \left[\int_0^{\delta} u^2 \, dy - U \int_0^{\delta} u \, dy \right] = & \frac{1}{r} \frac{dr}{dx} \left[\frac{U^2 \delta}{1260} \left(734 + \frac{71\lambda}{6} + \frac{5\lambda^2}{36} \right) \right. \\ & + U\delta^3 \sin \psi \left(\frac{71}{7520} + \frac{\lambda}{4536} \right) + \frac{\delta^5 \sin^2 \psi}{9072} \\ & \left. - \left\{ \frac{U^2 \delta}{120} (84 + \lambda) + \frac{U\delta^3 \sin \psi}{120} \right\} \right] \end{aligned} \quad (4-58)$$

$$\int_0^{\delta} U \frac{dU}{dx} \, dy = \delta U \frac{dU}{dx} \quad (4-59)$$

$$-\left. \frac{\partial u}{\partial y} \right|_{y=0} = - \left(+ \delta^2 \sin \psi + U \lambda + 12 U \right) / 6\delta \quad (4-60)$$

$$+ \int_0^{\Delta} C \sin \psi \, dy = + \frac{3\Delta \sin \psi}{10} \quad (4-61)$$

This term is only integrated over the transfer boundary layer since $C^* = 0$ for the remainder of the hydrodynamic boundary layer.

Substitution of equations (4-56) to (4-61) into the full integral equation (4-33) gives :

$$\begin{aligned}
 & \frac{d}{dx} \left[\frac{U^2 \delta}{1260} \left(734 + \frac{71\lambda}{6} + \frac{5\lambda^2}{36} \right) + U \delta^3 \sin \psi \left(\frac{71}{7520} + \frac{\lambda}{4536} \right) \right. \\
 & \quad \left. + \frac{\delta^5 \sin^2 \psi}{9072} \right] - U \frac{d}{dx} \left[\frac{U \delta}{120} (84 + \lambda) + \frac{\delta^3 \sin \psi}{120} \right] \\
 & \quad + \frac{1}{r} \frac{dr}{dx} \left[\frac{U^2 \delta}{120} \left(734 + \frac{71\lambda}{6} + \frac{5\lambda^2}{36} \right) + U \delta^3 \sin \psi \left(\frac{71}{7520} + \frac{\lambda}{4536} \right) \right. \\
 & \quad \left. + \frac{\delta^5 \sin^2 \psi}{9072} - \left\{ \frac{U^2 \delta}{120} (84 + \lambda) + \frac{U \delta^3 \sin \psi}{120} \right\} \right] \\
 & = \delta U \frac{dU}{dx} - \left(\delta^2 \sin \psi + U \lambda + 12 U \right) / 6 \delta + \frac{3 \Delta \sin \psi}{10}
 \end{aligned}
 \tag{4-62}$$

After performing the prescribed differentiations and simplifying, the following ordinary differential equation is obtained :

$$\begin{aligned}
 & \frac{dZ}{dx} \left\{ U k(\lambda) + Z \sin \psi l(\lambda) - \frac{5Z^2 \sin^2 \psi}{9072 U} \right\} \\
 & = f(\lambda) + U \frac{d^2 U}{dx^2} Z^2 g(\lambda) + \frac{Z \sin \psi}{U} j(\lambda) + Z^3 \frac{d^2 U}{dx^2} \frac{\sin \psi}{2268} \\
 & \quad + Z^2 \cos \psi \frac{d\psi}{dx} g(\lambda) + \frac{Z^3 \sin \psi \cos \psi}{2268 U} \frac{d\psi}{dx} + \frac{3 Z \lambda \sin \psi}{5 U} \\
 & \quad + \frac{1}{r} \frac{dr}{dx} \left[\frac{U h(\lambda)}{dU/dx} + Z^2 \sin \psi g(\lambda) + \frac{Z^3 \sin^2 \psi}{4536 U} \right]
 \end{aligned}
 \tag{4-63}$$

where :

$$Z = \delta^2 \tag{4-64}$$

$$\lambda = \Delta / \delta \quad \text{-- the ratio of the transfer} \tag{4-65}$$

to the hydrodynamic boundary layer thickness.

and where the functions $f(\lambda)$, etc are polynomials in λ and are defined as follows :

$$f(\lambda) = 4 - \frac{232\lambda}{315} + \frac{79\lambda^2}{3780} + \frac{\lambda^3}{2268} \quad (4-66)$$

$$g(\lambda) = \frac{2}{945} + \frac{\lambda}{2268} \quad (4-67)$$

$$h(\lambda) = -\frac{74\lambda}{315} + \frac{2\lambda^2}{945} + \frac{\lambda^3}{4536} \quad (4-68)$$

$$j(\lambda) = \frac{1}{3} + \frac{71\lambda}{3780} + \frac{2\lambda^2}{4536} \quad (4-69)$$

$$k(\lambda) = \frac{37}{315} - \frac{\lambda}{315} - \frac{5\lambda^2}{9072} \quad (4-70)$$

$$l(\lambda) = \frac{1}{315} - \frac{5\lambda}{4536} \quad (4-71)$$

The transfer equation is treated in the same way, where the individual terms are as follows :

$$\begin{aligned} \int_0^{\Delta} u C dy &= \left(+\delta^2 \sin \psi + U \lambda + 12 U \right) \frac{\Delta^2}{90 \delta} \\ &- \left(+ \sin \psi + U \frac{dU}{dx} \right) \frac{\Delta^3}{84} \\ &+ \left(+\delta^2 \sin \psi + U \lambda - 4 U \right) \frac{3 \Delta^4}{560 \delta^3} \\ &+ \left(6 U + \delta^2 \sin \psi - U \lambda \right) \frac{\Delta^5}{1080 \delta^4} \end{aligned} \quad (4-72)$$

and ,

$$-\frac{1}{Sc} \frac{\partial C}{\partial y} \bigg|_{y=0} = \frac{2}{Sc \Delta} \quad (4-73)$$

Upon substitution of equations (4-72) and (4-73), equation (4-41) becomes :

$$\begin{aligned}
 & \frac{d}{dx} \left[\left(+\delta^2 \sin \psi + U \lambda + 12 U \right) \frac{\Delta^2}{90 \delta} - \left(+\sin \psi + U \frac{dU}{dx} \right) \frac{\Delta^3}{84} \right. \\
 & + \left(+\delta^2 \sin \psi + U \lambda - 4 U \right) \frac{3\Delta^4}{560 \delta^3} \\
 & \left. + \left(6 U + \delta^2 \sin \psi - U \lambda \right) \frac{\Delta^5}{1080 \delta^4} \right] \\
 & + \frac{1}{r} \frac{dr}{dx} \left[\left(+\delta^2 \sin \psi + U \lambda + 12 U \right) \frac{\Delta^2}{90 \delta} \right. \\
 & - \left(+\sin \psi + U \frac{dU}{dx} \right) \frac{\Delta^3}{84} + \left(+\delta^2 \sin \psi + U \lambda - 4 U \right) \frac{3\Delta^4}{560 \delta^3} \\
 & \left. + \left(6 U + \delta^2 \sin \psi - U \lambda \right) \frac{\Delta^5}{1080 \delta^4} \right] \\
 & = \frac{2}{Sc \Delta} \tag{4-74}
 \end{aligned}$$

which, upon differentiation and further simplification reduces to an ordinary differential equation for χ :

$$\begin{aligned}
 & \frac{d\chi}{dx} \left[+ Z^2 \sin \psi \frac{d(m(\chi))}{d\chi} + U Z \frac{d(n(\chi))}{d\chi} + U \lambda Z \frac{d(m(\chi))}{d\chi} \right] \\
 & = \frac{2}{Sc \chi} - \left\{ \frac{1}{r} \frac{dr}{dx} \left[+ Z^2 \sin \psi m(\chi) + U Z n(\chi) + U \lambda Z m(\chi) \right] \right. \\
 & \quad + \frac{dZ}{dx} \left[+ \frac{3 Z \sin \psi}{2} m(\chi) + \frac{U}{2} n(\chi) + \frac{3 U \lambda}{2} m(\chi) \right] \\
 & \quad \left. + Z^2 \cos \psi \frac{d\psi}{dx} m(\chi) + U \frac{d^2 U}{dx^2} Z^2 m(\chi) + \lambda n(\chi) + \lambda^2 m(\chi) \right\} \tag{4-75}
 \end{aligned}$$

where $m(\chi)$ and $n(\chi)$ are polynomials in χ and are defined as follows :

$$m(\chi) = \frac{\chi^2}{90} - \frac{\chi^3}{84} + \frac{3\chi^4}{560} - \frac{\chi^5}{1080} \quad (4-76)$$

$$n(\chi) = \frac{2\chi^2}{15} - \frac{3\chi^4}{140} + \frac{\chi^5}{180} \quad (4-77)$$

The starting conditions which are required for the numerical integration of equations (4-63) and (4-75) are the two boundary layer thicknesses at the forward stagnation point. These two values are obtained by solving the limiting forms of the two differential equations at the stagnation point.

For a bluff body in an axisymmetric flow, the gradients $\frac{dz}{dx}$ and $\frac{dx}{dx}$ are zero at the front stagnation point. Therefore, the limits of the right hand sides of equations (4-63) and (4-75) are also zero at the stagnation point. Further discussion of the form and solution of the limiting equations will be left until the application of the method to specific body shapes is discussed.

4.2.1. Point Transfer Rates.

The point transfer rate may be readily calculated once the two differential equations, for the two boundary layer thicknesses, have been solved.

The mass transfer from any point on the body surface is given by Fick's Law :

$$q(x_1) = -D \left. \frac{\partial c}{\partial N} \right|_{\text{surface}} \quad (4-78)$$

where $q(x_1)$ is the mass flux from the surface at the

point (x_1, N) and D is the diffusivity.

The mass transfer may also be expressed in terms of the point mass transfer coefficient $K'(x_1)$ by the equation :

$$q(x_1) = K'(x_1) (C_o - C_\infty) \quad (4-79)$$

Equations (4-78) and (4-79) are equated to give :

$$\frac{K'(x_1)}{D} = \frac{1}{(C_o - C_\infty)} \left. \frac{\partial C}{\partial N} \right|_{\text{surface}} \quad (4-80)$$

Upon the introduction of the following dimensionless variables :

$$C^* = \frac{(C - C_\infty)}{(C_o - C_\infty)} ; \quad N^* = \frac{N}{D_c} \quad (4-81)$$

equation (4-80) becomes :

$$\frac{K'(x_1) D_c}{D} = - \left. \frac{\partial C^*}{\partial N^*} \right|_{\text{surface}} \quad (4-82)$$

A local Sherwood number is defined as follows:

$$Sh_{x_1} = \frac{K'(x_1) D_c}{D}$$

Thus, equation (4-82) becomes:

$$Sh_{x_1} = - \left. \frac{\partial C^*}{\partial N^*} \right|_{\text{surface}} \quad (4-83)$$

Upon the introduction of the second set of dimensionless variables, equation (4-5), equation (4-83) becomes:

$$Sh_{x_1} = - Re^{\frac{1}{2}} \left. \frac{\partial C^*}{\partial N^{**}} \right|_{\text{surface}} \quad (4-84)$$

Upon the introduction of the third set of dimensionless variables, equation (4-10) , equation (4-84) becomes :

$$Sh_{x_1} = - Re^{\frac{1}{2}} \left(\frac{Gr}{Re^2} \right)^{\frac{1}{2}} \left. \frac{\partial C^*}{\partial N^{***}} \right|_{\text{surface}} \quad (4-85)$$

From equation (4-46), the dimensionless concentration gradient at the surface is equal to b_C .

For the present assumed polynomial, $b_C = - 2$, therefore :

$$Sh_{x_1} = + Re^{\frac{1}{2}} \left(\frac{Gr}{Re^2} \right)^{\frac{1}{2}} \frac{2}{\Delta} \quad (4-86)$$

4.3 The Solution of the Differential Equations for a Specific Body.

In their present form, equations (4-63) and (4-75) are completely general. To solve them for a particular body two expressions are required. One expression is necessary to describe the body shape, and another for the potential flow velocity distribution. The equations for a bluff body were integrated with the angle from the front stagnation point as the independent variable instead of x , the coordinate parallel to the body surface. Therefore, an expression for $d\Theta/dx$ is required.

The expressions for the potential flow and the description of the body shape are presented in Table 4.1 for three shapes which were studied in the present work. The derivations of the expressions for the potential flow around an oblate spheroid may be found in a number of other works^{84,85}.

Table 4.1. Expressions for the sphere, oblate spheroid and the vertical flat plate to describe the body shape and the potential flow velocity distribution.

	<u>SPHERE</u>	<u>OBLATE</u> <u>SPHEROID</u>	<u>VERTICAL</u> <u>FLAT PLATE</u>
U	$\frac{3}{2} \sin \Theta$	$\frac{\sin \Theta}{K(\sinh^2 \xi_0 + \cos^2 \Theta)^{1/2}}$	1
$\frac{dU}{d\Theta}$	$\frac{3}{2} \cos \Theta$	$\frac{\cosh^2 \xi_0 \cos \Theta}{K(\sinh^2 \xi_0 + \cos^2 \Theta)^{3/2}}$	0
$\frac{d^2U}{d\Theta^2}$	$-\frac{3}{2} \sin \Theta$	$\frac{\sin \Theta \cosh^2 \xi_0 (2\cos^2 \Theta - \sinh^2 \xi_0)}{K(\sinh^2 \xi_0 + \cos^2 \Theta)^{5/2}}$	-
$\frac{d\Theta}{dx}$	$\frac{2Re^2}{Gr}$	$\frac{D_c}{\bar{d}} \frac{1}{\left(1 - \frac{\sin^2 \Theta}{\cosh^2 \xi_0}\right)^{1/2}} \frac{Re^2}{Gr}$	-
$\frac{df}{dx}$	$\frac{d\Theta}{dx}$	$\frac{\bar{f} \cos^2 \psi}{\bar{d} \cos^2 \Theta} \frac{d\Theta}{dx}$	-
$\frac{1}{r} \frac{dr}{dx}$	$\frac{1}{\tan \Theta} \frac{d\Theta}{dx}$	$\frac{1}{\tan \Theta} \frac{d\Theta}{dx}$	0

In the calculation of the above expressions, the characteristic dimension for the sphere has been taken to be the diameter.

For the oblate spheroid, the characteristic linear dimension which was used was that proposed by Pasternak and Gauvin⁸⁶.

This is defined as the ratio of the total surface area of the body to the maximum perimeter normal to the flow. This

characteristic dimension was chosen because it has been previously shown to correlate data for oblate spheroids^{37,85}.

In order to maintain generality in the equations, a variable $\bar{\epsilon}$ was defined which has the value 2 for the sphere, and the value $\frac{D_c}{\bar{d}} \frac{1}{\left(1 - \frac{\sin^2 \Theta}{\cosh^2 \xi_0}\right)^{\frac{1}{2}}}$ for the oblate spheroid.

Also, for the spheroid $\bar{\epsilon}' = \bar{\epsilon} \frac{\bar{f} \cos^2 \psi}{\bar{d} \cos^2 \Theta}$; for the sphere $\bar{\epsilon}' = \bar{\epsilon}$.

Upon substitution of the expressions for the body shape and the potential flow distribution, equations (4-63) and (4-75) become :

$$\begin{aligned} \frac{dZ}{d\Theta} \left\{ U k(\lambda) + Z \sin \psi l(\lambda) - \frac{5 Z^2 \sin^2 \psi}{9072 U} \right\} & \bar{\epsilon} \frac{Re^2}{Gr} \\ = f(\lambda) + \bar{\epsilon}^2 Z^2 g(\lambda) U \frac{d^2 U}{d\Theta^2} \left(\frac{Re^2}{Gr} \right)^2 + Z j(\lambda) \frac{\sin \psi}{U} \\ + \frac{Z^3 \bar{\epsilon}^2 \sin \psi}{2268} \frac{d^2 U}{d\Theta^2} \left(\frac{Re^2}{Gr} \right)^2 + Z^2 g(\lambda) \bar{\epsilon}' \cos \psi \frac{Re^2}{Gr} \\ + \frac{Z^3 \bar{\epsilon}' \sin \psi \cos \psi}{2268 U} \frac{Re^2}{Gr} + \frac{3 Z \chi \sin \psi}{5 U} \\ + \bar{\epsilon} \frac{Re^2}{Gr} \frac{\cos \Theta}{\sin \Theta} \left[\frac{U h(\lambda)}{dU/d\Theta} \frac{Gr}{Re^2} + Z^2 g(\lambda) \sin \psi + \frac{Z^3 \sin^2 \psi}{4536 U} \right] \end{aligned}$$

(4-87)

and :

$$\begin{aligned}
 & \frac{d\chi}{d\Theta} \left\{ + Z^2 \sin \psi \frac{d(m(\chi))}{d\chi} + U Z \frac{d(n(\chi))}{d\chi} + U \lambda Z \frac{d(m(\chi))}{d\chi} \right\} \bar{\epsilon} \frac{Re^2}{Gr} \\
 &= \frac{2}{Sc \chi} - \left\{ \bar{\epsilon} \frac{Re^2}{Gr} \frac{\cos \Theta}{\sin \Theta} \left[+ Z^2 \sin \psi m(\chi) + U Z n(\chi) + U \lambda Z n(\chi) \right] \right. \\
 & \quad + \frac{dZ}{d\Theta} \bar{\epsilon} \frac{Re^2}{Gr} \left[+ \frac{3}{2} Z \sin \psi m(\chi) + \frac{U}{2} n(\chi) + \frac{3}{2} U \lambda m(\chi) \right] \\
 & \quad + Z^2 \cos \psi m(\chi) \bar{\epsilon}' \frac{Re^2}{Gr} + Z^2 m(\chi) \bar{\epsilon}^2 U \frac{d^2 U}{d\Theta^2} \left(\frac{Re^2}{Gr} \right)^2 \\
 & \quad \left. + \lambda n(\chi) + \lambda^2 m(\chi) \right\}
 \end{aligned} \tag{4-88}$$

The boundary layer thicknesses at the front stagnation point are required to start the integration. As has been stated earlier, these are calculated from the limiting forms of the two differential equations at the front stagnation point. The limiting forms of equations (4-87) and (4-88) are :

$$\begin{aligned}
 0 &= f(\lambda_L) + \frac{Z j(\lambda_L)}{U_L} + Z^2 g(\lambda_L) \bar{\epsilon}' \frac{Re^2}{Gr} + \frac{Z^3 \bar{\epsilon}'}{2268 U_L} \frac{Re^2}{Gr} \\
 & \quad + \frac{3 Z \chi}{5 U_L} + \bar{\epsilon} \frac{Re^2}{Gr} \left[\frac{h(\lambda_L)}{\bar{\epsilon} (Re^2/Gr)} + Z^2 g(\lambda_L) + \frac{Z^3}{4536 U_L} \right]
 \end{aligned} \tag{4-89}$$

$$\begin{aligned}
 0 &= \frac{2}{Sc \chi} - \left\{ \bar{\epsilon} \frac{Re^2}{Gr} \left[+ Z^2 m(\chi) + U_L Z n(\chi) \right. \right. \\
 & \quad \left. + U_L U_L' \bar{\epsilon} Z^2 m(\chi) \frac{Re^2}{Gr} \right] + Z^2 \bar{\epsilon}' m(\chi) \frac{Re^2}{Gr} \\
 & \quad \left. + Z U_L' \bar{\epsilon} n(\chi) \frac{Re^2}{Gr} + Z^2 U_L'^2 \bar{\epsilon}^2 m(\chi) \left(\frac{Re^2}{Gr} \right)^2 \right\}
 \end{aligned} \tag{4-90}$$

where the limiting forms U_L , U'_L , and λ_L are :

SPHERE

OBLATE
SPHEROID

$$U_L \quad \frac{3}{2} \quad \frac{1}{K (\sinh^2 \zeta_0 + 1)^{1/2}} \quad (4-91)$$

$$U'_L \quad \frac{3}{2} \quad \frac{\cosh^2 \zeta_0}{K (\sinh^2 \zeta_0 + 1)^{3/2}}$$

$$\text{and } \lambda_L = Z \bar{\epsilon} \frac{Re^2}{Gr} U'_L \quad (4-92)$$

Equations (4-89) and (4-90) were solved simultaneously for Z and χ by the Newton-Raphson procedure⁸⁷.

If the right hand side of equation (4-89) is written as L_Z , and the right hand side of equation (4-90) is written as L_χ , the solution procedure may be written in the form of a recurrence relationship from which the two initial conditions are generated.

$$Z^{k+1} = Z^k - \left[\frac{L_Z \frac{dL}{d\chi} - L_\chi \frac{dL_Z}{d\chi}}{\frac{dL_Z}{dZ} \frac{dL_\chi}{d\chi} - \frac{dL_Z}{d\chi} \frac{dL_\chi}{dZ}} \right] \quad (4-93)$$

$$\chi^{k+1} = \chi^k + \left[\frac{L_Z \frac{dL_\chi}{dZ} - L_\chi \frac{dL_Z}{dZ}}{\frac{dL_Z}{dZ} \frac{dL_\chi}{d\chi} - \frac{dL_Z}{d\chi} \frac{dL_\chi}{dZ}} \right] \quad (4-94)$$

For small values of the parameter Gr/Re^2 the initial guessed values which were used for the two boundary layer thicknesses, Z^0 and χ^0 , were those for pure forced convective transfer

from a sphere. These values are well known and have been discussed in detail elsewhere^{63,88}.

Once the roots had been found for small values of the parameter Gr/Re^2 , they were used as first approximations to the solutions for larger values of the parameter Gr/Re^2 . This method was found to work equally well for all of the body shapes studied.

By the use of this method, convergence to the correct solution was very rapid. However, if extreme care was not taken to provide initial guesses close to the correct values, convergence to another pair of roots was often obtained. The error was usually immediately obvious upon inspection for one of a number of reasons; because the hydrodynamic boundary layer thickness was negative, or the ratio of the transfer to the hydrodynamic boundary layer thickness was very much greater than unity for a Schmidt number greater than unity. Another pair of incorrect roots, which could be detected upon inspection, predicted the ratio of the two boundary layers approximately one half of the value which was expected for a particular Schmidt number. This pair of incorrect roots predicted a value of the velocity gradient at the body surface of zero or less. Therefore, any test incorporated in the computer programme to detect the point of separation would immediately detect the error.

4.3.1 Numerical Integration Method.

The numerical integration technique used for both the approximate and exact solutions in the present work was the Runge-Kutta method.

There are two main types of method which could have been used; these are predictor-corrector type methods and single step methods. Single step methods are self starting, and for this reason require little storage space when proceeding from step to step since the information from previous steps may be overwritten. Changes of integration step length, during the computation, are readily achieved with single step methods.

Predictor-corrector methods require values of the function at a number of previous steps in order to proceed to the next point. The methods are therefore not self starting, and a single step method must be used to calculate the initial points before a switch can be made to the predictor-corrector method. Changes of integration step length are more difficult to cope with when predictor-corrector methods are used than with single step methods.

The single step methods either approximate the curve by a series of straight lines or use higher order profiles. If a low order method is used, smaller step lengths are usually necessary to obtain the solution than with higher order methods.

The Runge-Kutta method⁸⁹ which is used in the present work is a fourth order single step method in which the truncation error is of order h^5 , where h is the integration step length. The method requires four function evaluations for each step compared with either one or two for the simplest predictor-corrector methods. With a complicated function for which each function evaluation is very time consuming, a switch to a predictor-corrector method is likely to save computer time. Both methods were used initially, but the Runge-Kutta method was finally chosen.

The reasons for the choice were : that after running a number of programmes, it was clear that for a given accuracy the Runge-Kutta method would be computationally faster than the predictor-corrector method which was used. Smaller integration steps were found to be necessary for the predictor-corrector method, and therefore the extra function evaluations which were necessary for the Runge-Kutta method were not significant regarding computer time.

The ease with which the step length may be varied with the single step method was also very important, as will become clear when the results are discussed later. Virtually no extra programming is required if the integration step length is to be increased or decreased with a single step method, whereas extra programming and storage are both necessary if there are any changes of step when a predictor-corrector method is used.

The computer which was used in the latter part of the present study, the University of London CDC 6600, enabled single precision arithmetic to be used, and the round-off error introduced was very small. The CDC 6600 works with fifteen significant figures on single precision. The Imperial College IBM 7094, which was used in the early part of the work, only worked with eight significant figures on single precision. Double precision arithmetic was used on the CDC 6600, thus working with twenty nine significant figures, but no increase in accuracy was achieved and therefore, single precision was used throughout the present work.

4.4 Modifications to the Integral Method.

The approximate method presented above has been used to predict transfer rates from spheres and oblate spheroids for forced convection and for the forced convection end of the interaction region at Schmidt numbers of 0.7 to 1,000. Over part of the interaction range solutions could not be obtained. However the method was again successfully used for predominantly free convective situations. The region over which solutions could not be obtained differed according to the value of the Schmidt number. For small Schmidt numbers 0.72, 1.0, and 2.4, the method could not be used to predict transfer rates over the relatively small range of values of the parameter Gr/Re^2 : 5 to 10. However, for a Schmidt number of 1,000 the range extended from approximately 3 to 100 .

The failure of the method became apparent when, for a particular value of the parameter Gr/Re^2 , no initial conditions could be found for the two boundary layer thicknesses. To test whether only the initial condition routine was failing, the integration was started by using the boundary layer thicknesses which had been previously calculated for the next smallest value of the parameter Gr/Re^2 . In cases for which this was attempted, the method immediately became unstable.

A number of modifications to the method were investigated in an attempt to obtain results in the region where the original method failed. At high Schmidt numbers the velocity profile was simplified by the assumption of a linear distribution over the relatively thin transfer boundary layer. The modified profiles were introduced into the integral equations. The two integral equations were simplified, to give

the ordinary differential equations, in the same way as has been described above. This modification had no effect on the breakdown of the method.

The basic equations (4-6) to (4-8) were reconsidered. A set of dimensionless variables was introduced such that the parameter Gr/Re^2 was removed from the equations by scaling only the dimensionless concentration. The two integral equations were derived, and quartic velocity and concentration profiles were used to reduce the equations to two ordinary differential equations. The region over which solutions could not be obtained remained unchanged.

A third modification which was investigated was to introduce the Schmidt number, by a suitable scaling, into the equation of motion, and to remove it completely from the transfer equation. There was no improvement in the range of solutions obtained, even in the limiting case when the Schmidt number approached infinity and the inertial terms disappeared from the equation of motion.

In order to investigate the breakdown of the integral method more easily, the case of a vertical flat plate in a uniform free stream was studied in more detail. The simpler case of the vertical flat plate was studied because the integral equations are not as complicated as those for axisymmetric bodies. Therefore the amount of work and time required to make modifications was considerably less.

As has been mentioned in the literature survey, Acrivos⁴¹ has presented a theoretical analysis of the problem.

Acrivos stated in his paper that he had found it necessary to integrate over three regions. In the first region, in which forced convection was dominant, he integrated both of the ordinary differential equations simultaneously until the

denominator of the differential equation for the hydrodynamic boundary layer thickness became zero. Over the second integration region, he allowed only the ratio of the two boundary layer thicknesses to vary, and kept the hydrodynamic boundary layer thickness constant, at the value which it had when the denominator became zero. He continued the integration in this way until the numerator of the first differential equation also became zero. Then he again allowed both of the boundary layer thicknesses to vary.

He did not state, for any Prandtl number, the range of values of the parameter Gr/Re^2 over which the hydrodynamic boundary layer thickness was kept constant. The present author has recalculated Acrivos' mass transfer predictions so that these regions could be found. The results are presented in Fig. 4.1. The regions, for each Prandtl number, over which the hydrodynamic boundary layer thickness was kept constant are shown between the arrows marked "A" and "B".

It is clear from Fig. 4.1. that at high Prandtl numbers the problem has not, in fact, been solved for values of the parameter Gr/Re^2 greater than approximately 0.5. It is not surprising, therefore, that Acrivos' results at high Prandtl numbers are, as Acrivos mentioned in a later paper⁶⁴, disappointing.

As has been discussed earlier, in a number of previous applications the integral method has been shown to predict transfer rates which are in very good agreement with exact solutions, and also to be computationally very efficient. Therefore, an investigation was carried out to see whether any improvements could be introduced to extend the range of values of the parameter Gr/Re^2 for which solutions could be obtained.

Sanders⁸⁰ mentioned that the approximate method had been

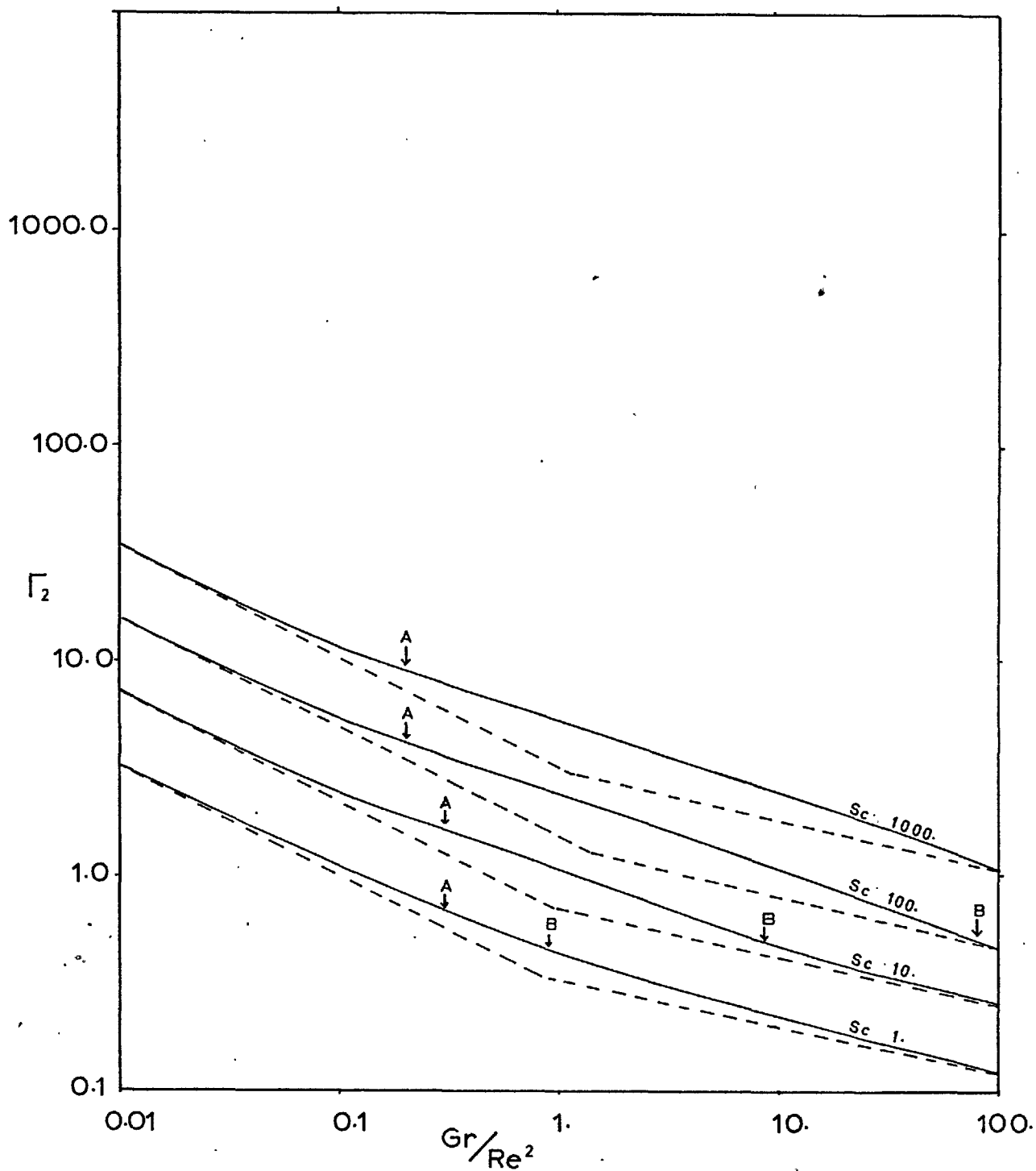


FIG. 4.1. The Plot of Γ_2 Against Gr/Re^2
 (Data of Acrivos⁴¹), for the Vertical Flat Plate.

found to fail⁷⁹ when applied to the solution of the boundary layer equations for a rotating body once a certain value of a dimensionless spin parameter had been reached. He suggested that a possible reason for the failure was that the assumed form of the velocity profile did not truly represent the physical behaviour when the rotation of the body was such that there was a velocity within the boundary layer larger than that of the free stream. It is doubtful whether this is the sole reason for the failure since, as has been mentioned previously, the approximate method has been used successfully by Merk and Prins⁷⁵ and Pandya⁷⁸ for free convective transfer. In this case, the velocity profile also exhibits a maximum, and the shape of the profiles is very similar for the two cases.

In an attempt to obtain improved predictions of the angle of boundary layer separation, Curle⁸², and Schlichting and Ulrich⁹⁰ used high order velocity profiles. By the use of fifth and sixth order polynomials, respectively, they solved the momentum integral equation and achieved improved predictions of separation.

In order to see whether the order of the polynomial for the velocity profile influenced the failure of the present method, three velocity profiles were chosen, a cubic, a quartic and a sextic. The procedure described in Section 4.2. was followed and the two ordinary differential equations which were obtained for each profile were solved for a range of Schmidt numbers: 1 to 5,000. The order of the polynomial was found to have no influence upon the region over which the method failed.

A number of velocity profiles were calculated for each of the three chosen polynomials. The shape of the profiles which were obtained for forced convective flows agreed with those of previous exact solutions. Those for predominantly free

convective flows at low Schmidt numbers predicted a maximum velocity within the boundary layer. Therefore, the suggestion that the failure of the method is due to the polynomial not being able to predict this maximum does not appear to be valid.

It is interesting to note that at high Schmidt numbers the integral method failed before a maximum velocity was predicted. The shape of the velocity profiles at high Schmidt numbers will be discussed in more detail in Chapter 6 in connection with the exact solutions which have been obtained.

Acrivos used a cubic polynomial to represent the concentration profile, whereas a quartic has also been used in the present work. The order of the concentration profile did not affect the failure of the method.

To ensure that the velocity over the thinner transfer boundary layer was being correctly represented, two separate velocity polynomials were chosen. One polynomial was chosen to represent the velocity within the transfer boundary layer. The other represents the velocity outside the transfer boundary layer but within the remainder of the hydrodynamic boundary layer. The polynomials were chosen so that in addition to satisfying the boundary conditions at the body surface and in the free stream, the velocities and their derivatives were matched at the edge of the transfer boundary layer. As in the case of the single profile, the order of either polynomial was varied by changing the number of boundary conditions which were satisfied.

When this method was used two momentum integral equations were required, one over each region of the hydrodynamic boundary layer. These were then added together in order to eliminate the pressure term at the edge of the transfer boundary layer. This procedure resulted in a single integral equation

for the momentum boundary layer. The labour involved to obtain the equation was very much greater than when a single velocity profile was used, and the computational effort required to solve the equation was correspondingly increased. The use of the double velocity profile did not affect the range of solutions which could be obtained.

As has been previously mentioned, when the Schmidt number is large, the transfer boundary layer is very much thinner than the hydrodynamic boundary layer. When finite difference techniques are employed, a very fine grid is frequently used over the region of most interest, compared with a coarse grid over the remainder of the field. An attempt was made to treat the present problem in an analagous manner.

The coordinates of the hydrodynamic boundary layer in the region of the transfer boundary layer were scaled, whilst the coordinates of the remainder of the hydrodynamic boundary layer were left unaltered. The number of boundary and matching conditions which were satisfied determined the order of the assumed velocity profile polynomials. At the edge of the transfer boundary layer, the scaled velocities and their derivatives were matched.

The introduction of the simple double velocity profile modification greatly increased the amount of work needed to investigate the vertical flat plate. As a result, very little extra work would be required to obtain stagnation point solutions. It was therefore decided to apply the modification outlined above directly to the stagnation point of a bluff body.

Two momentum integral equations were again required, one over the region of the scaled transfer boundary layer, and the other over the remainder of the unscaled hydrodynamic boundary layer. These were then added in order to obtain a single

ordinary differential equation for the hydrodynamic boundary layer thickness. The second ordinary differential equation was derived in the usual manner from the transfer integral equation.

Predictions of mass transfer rates, from the front stagnation point of a bluff body, were calculated with this method over a larger range of values of the parameter Gr/Re^2 than had been possible with the original method. It should be emphasized, however, that the "gap" over which the approximate method failed, although considerably reduced, was not completely closed. The results for both approximate methods are presented in Fig. 6.60 and are discussed in more detail in Chapter 6.

The integral equations for the latter modification are now presented, together with the boundary and matching conditions which the velocity profile must satisfy.

4.5. The Development of the Second Set of Integral Equations.

When the Schmidt number is large, the hydrodynamic boundary layer region can be considered in two distinct parts. These are a thin region, which corresponds to the diffusion boundary layer, within which all concentration changes are confined, and the remainder of the momentum boundary layer in which the velocity increases to that of the free stream.

Equations (4-16) to (4-18), in which the Schmidt number has been removed from the transfer equation, are used in the region of the diffusion boundary layer. The scaling is not carried to the outer region of the momentum boundary layer and equations (4-6) to (4-8) are used in this region. The velocity components, together with their derivatives, are matched at the edge of the diffusion boundary layer.

The boundary conditions are:

$$\text{At } y^{****} = 0 : u^{****} = v^{****} = 0$$

$$\begin{aligned} ; \frac{\partial^2 u^{****}}{\partial y^{****2}} &= + \frac{Gr}{Re^2 Sc^{\frac{1}{3}}} \sin \phi - \frac{1}{Sc^{\frac{1}{3}}} U^* \frac{dU^*}{dx^*} \\ ; C^* &= 1 \end{aligned} \quad (4-95)$$

At the edge of the diffusion boundary layer, $y^{****} = y^{**} Sc^{\frac{1}{3}} :$

$$\begin{aligned} u^{****} &= u^{**} Sc^{\frac{1}{3}} \\ \frac{\partial u^{****}}{\partial y^{****}} &= \frac{\partial u^{**}}{\partial y^{**}} \end{aligned} \quad (4-96)$$

$$\frac{\partial^2 u^{****}}{\partial y^{****2}} = \frac{1}{Sc^{\frac{1}{3}}} \frac{\partial^2 u^{**}}{\partial y^{**2}}$$

$$\begin{aligned} \text{As } y^{**} \rightarrow \infty : u^{**} \rightarrow U^* ; \frac{\partial u^{**}}{\partial y^{**}} &= \frac{\partial^2 u^{**}}{\partial y^{**2}} \rightarrow 0 \\ C^* &= 0 \end{aligned} \quad (4-97)$$

Within the outer region of the hydrodynamic boundary layer, equations (4-6) to (4-8) remain unchanged except that, since $C^* = 0$, the buoyancy term is absent.

Thus, when the Schmidt number is large, the equations become partially uncoupled and the buoyancy forces do not directly affect the velocity distribution in the outer region of the hydrodynamic boundary layer.

The integral equations will not be developed in detail since the procedure is identical with that previously described.

Two momentum integral equations are required, one over the diffusion boundary layer and the other over the remainder of the hydrodynamic boundary layer. Only one transfer integral equation is required since the dimensionless concentration is,

by definition, zero outside the diffusion boundary layer.

In order to simplify the presentation, asterisks will be dropped, and the dimensionless variables will be written as:

$$\begin{aligned} u^{****} &= u_i & u^{**} &= u_o \\ y^{****} &= y_i & y^{**} &= y_o \end{aligned}$$

all of the other variables are identical for both sets of equations. The edge of the diffusion boundary layer occurs when $y_o = \Delta$, thus $y_i = \Delta Sc^{\frac{1}{3}}$.

The momentum integral equation for the region of the diffusion boundary layer, which is obtained by integrating equation (4-16), is:

$$\begin{aligned} \frac{1}{Sc} \left\{ \frac{1}{r} \frac{d}{dx} \left[r \int_0^{\Delta Sc^{\frac{1}{3}}} u_i (u_i - U) dy_i \right] \right. \\ = + \frac{Gr}{Re^2 Sc^{\frac{1}{3}}} \int_0^{\Delta Sc^{\frac{1}{3}}} C \sin \psi dy_i + \frac{1}{Sc} \int_0^{\Delta Sc^{\frac{1}{3}}} U \frac{dU}{dx} dy_i \\ \left. + \int_0^{\Delta Sc^{\frac{1}{3}}} \frac{\partial^2 u_i}{\partial y_i^2} dy_i \right\} \end{aligned} \quad (4-98)$$

The momentum integral equation for the remainder of the boundary layer, which is obtained by integrating equation (4-6) without the buoyancy term, is:

$$\frac{1}{r} \frac{d}{dx} \left[r \int_{\Delta}^{\delta} u_o (u_o - U) dy_o \right] = \int_{\Delta}^{\delta} U \frac{dU}{dx} dy_o + \int_{\Delta}^{\delta} \frac{\partial^2 u_o}{\partial y_o^2} dy_o \quad (4-99)$$

These two integral equations may now be added to give:

$$\begin{aligned} & \frac{1}{Sc} \left\{ \frac{1}{r} \frac{d}{dx} \left[r \int_0^{\Delta Sc^{\frac{1}{3}}} u_i (u_i - U) dy_i \right] \right\} + \frac{1}{r} \frac{d}{dx} \left[r \int_{\Delta}^{\delta} u_o (u_o - U) dy_o \right] \\ &= \frac{Gr}{Re^2 Sc^{\frac{1}{3}}} \int_0^{\Delta Sc^{\frac{1}{3}}} C \sin \left\{ dy_i + U \frac{dU}{dx} \delta - \frac{\partial u_i}{\partial y_i} \right\} \bigg|_{y_i=0} \\ & \hspace{25em} (4-100) \end{aligned}$$

The diffusion integral equation is:

$$\frac{1}{r} \frac{d}{dx} \left[r \int_0^{\Delta Sc^{\frac{1}{3}}} u_i C dy_i \right] = - \frac{\partial C}{\partial y_i} \bigg|_{y_i=0} \quad (4-101)$$

Polynomials are assumed to represent the two velocity profiles and the concentration profile. Upon substitution of the polynomials into the integral equations, two ordinary differential equations are obtained after considerable simplification. The equations which have been used in the present work are presented in Appendix E.

As has been mentioned in the previous section, this method has been successfully employed to predict stagnation point solutions at high Schmidt numbers over a larger range of values of the parameter Gr/Re^2 than was possible with the original integral method. However, it was still not possible to use the method over the entire range of values of the parameter Gr/Re^2 .

The reliability of the local transfer rates which had been calculated in the interaction regime could not readily be assessed because of the lack of other results in the literature.

Therefore, for two reasons, it was thought necessary to attempt to obtain solutions by a more exact method. First, so

that the reliability of the solutions which had been found could be judged, and secondly to try to close the "gap" over which the approximate method still completely failed.

CHAPTER 5

EXACT SOLUTION

Either a series method or a finite difference step-by-step method could be used to calculate exact solutions for the bluff body.

The main disadvantage of the step-by-step method is the difficulty in finding the initial condition at the front stagnation point with which to start the marching procedure. Frequently, a guessed initial profile is chosen and an iteration procedure is used until convergence to the correct initial profile is achieved.

An exact solution which is applicable to the region of the front stagnation point may readily be calculated by the use of a series method. The solution of the first equation of each infinite set, of ordinary differential equations, is the solution for the region of the front stagnation point. Therefore, even if considerable convergence difficulties are encountered when a series method is used to predict local transfer rates around the body, the series method may still be used to calculate the stagnation point solution. Once this solution has been calculated, it could then be used to start a step-by-step method. Because of the problems associated with a step-by-step method at the front stagnation point, it was decided to use a conventional series method to calculate the solution for this region.

5.1. The Theoretical Development of the Series Solution.

The dimensional forms of the boundary layer equations which were developed in Chapter 3, are:

$$\frac{V_1}{h_{1,s}} \frac{\partial V_1}{\partial x_1} + V_N \frac{\partial V_1}{\partial N} = \frac{U}{h_{1,s}} \frac{dU}{dx_1} + g\beta(C-C_0) \sin \gamma + \nu \frac{\partial^2 V_1}{\partial N^2} \quad (5-1)$$

$$\frac{\partial(h_{3,s} V_1)}{\partial x_1} + h_{1,s} h_{3,s} \frac{\partial V_N}{\partial N} = 0 \quad (5-2)$$

$$\frac{V_1}{h_{1,s}} \frac{\partial C}{\partial x_1} + V_N \frac{\partial C}{\partial N} = \alpha \frac{\partial^2 C}{\partial N^2} \quad (5-3)$$

A stream function is introduced which is the same as that used by Green⁷³. This enables the present solutions to be compared with the forced convection solutions of Green.

$$V_1 = \frac{1}{h_{3,s}} \frac{\partial(\Psi h_{3,s})}{\partial N} \quad (5-4)$$

$$V_N = - \frac{1}{h_{1,s} h_{3,s}} \frac{\partial(\Psi h_{3,s})}{\partial x_1}$$

This stream function reduces to that used by Frössling⁹¹ when the usual boundary layer coordinates are substituted into equation (5-4):

$$u = \frac{1}{r} \frac{\partial(\Psi r)}{\partial x} ; \quad v = - \frac{1}{r} \frac{\partial(\Psi r)}{\partial x} \quad (5-5)$$

Upon substitution of the expressions for V_1 and V_N , the

momentum equation (5-1) becomes :

$$\begin{aligned} \frac{\partial \bar{\Psi}}{\partial N} \frac{\partial^2 \bar{\Psi}}{\partial x_1 \partial N} - \frac{\partial^2 \bar{\Psi}}{\partial N^2} \left[\frac{\partial \bar{\Psi}}{\partial x_1} + \frac{1}{h_{3,s}} \frac{\partial h_{3,s}}{\partial x_1} \bar{\Psi} \right] \\ = \frac{U}{dx} \frac{dU}{dx} + g \beta h_{1,s} (C - C_0) \sin \psi + \nu h_{1,s} \frac{\partial^3 \bar{\Psi}}{\partial N^3} \end{aligned} \quad (5-6)$$

the transfer equation (5-3) becomes :

$$\begin{aligned} h_{3,s} \frac{\partial \bar{\Psi}}{\partial N} \frac{\partial C}{\partial x_1} - \frac{\partial C}{\partial N} \left[h_{3,s} \frac{\partial \bar{\Psi}}{\partial x_1} + \frac{\partial h_{3,s}}{\partial x_1} \bar{\Psi} \right] \\ = \alpha h_{1,s} h_{3,s} \frac{\partial^2 C}{\partial N^2} \end{aligned} \quad (5-7)$$

with the boundary conditions :

$$\begin{aligned} \text{at } N = 0 : \quad \frac{\partial \bar{\Psi}}{\partial N} = \frac{1}{h_{1,s} h_{3,s}} \frac{\partial (\bar{\Psi} h_{3,s})}{\partial x_1} = 0 ; \\ C = C_s \\ \text{as } N \rightarrow \infty : \quad \frac{\partial \bar{\Psi}}{\partial N} \rightarrow U ; \quad C \rightarrow C_\infty \end{aligned} \quad (5-8)$$

The series method upon which the present method is based was first presented by Blasius⁴⁸ for two dimensional flows. The series method was extended to axially symmetrical flows by Frössling⁹¹ who used the usual boundary layer coordinates. Green⁷³ found that in general, Frössling's method could not be applied to all axisymmetric bodies. Therefore, in order to introduce greater generality, he further extended the series method by the use of orthogonal curvilinear coordinates. The present series method also uses the orthogonal curvilinear coordinate system.

The potential flow velocity, the two scale factors, and the term $\sin \psi$ are expanded as power series in terms of the coordinate parallel to the body surface.

The series are :

$$U = \bar{u}_1 x_1 + \bar{u}_3 x_1^3 + \dots + \bar{u}_{2n-1} x_1^{2n-1} + \dots \quad (5-9)$$

$$h_{1,s} = \bar{l}_0 + \bar{l}_2 x_1^2 + \dots + \bar{l}_{2(n-1)} x_1^{2(n-1)} + \dots \quad (5-10)$$

$$h_{3,s} = \bar{k}_1 x_1 + \bar{k}_3 x_1^3 + \dots + \bar{k}_{2n-1} x_1^{2n-1} + \dots \quad (5-11)$$

$$\sin \psi = \bar{\psi}_1 x_1 + \bar{\psi}_3 x_1^3 + \dots + \bar{\psi}_{2n-1} x_1^{2n-1} + \dots \quad (5-12)$$

The shape parameters are removed from the first term of each series by division throughout by the coefficient of the first term.

Equations (5-9) to (5-12) become :

$$U = \bar{u}_1 (u_1 x_1 + u_3 x_1^3 + \dots + u_{2n-1} x_1^{2n-1} + \dots) \quad (5-13)$$

$$h_{1,s} = \bar{l}_0 (l_0 + l_2 x_1^2 + \dots + l_{2(n-1)} x_1^{2(n-1)} + \dots) \quad (5-14)$$

$$h_{3,s} = \bar{k}_1 (k_1 x_1 + k_3 x_1^3 + \dots + k_{2n-1} x_1^{2n-1} + \dots) \quad (5-15)$$

$$\sin \psi = \bar{\psi}_1 (\psi_1 x_1 + \psi_3 x_1^3 + \dots + \psi_{2n-1} x_1^{2n-1} + \dots) \quad (5-16)$$

where

$$u_{2i-1} = \frac{\bar{u}_{2i-1}}{\bar{u}_1} \quad ; \quad l_{2i-2} = \frac{\bar{l}_{2i-2}}{\bar{l}_0} \quad (5-17)$$

$$k_{2i-1} = \frac{\bar{k}_{2i-1}}{\bar{k}_1} \quad ; \quad \psi_{2i-1} = \frac{\bar{\psi}_{2i-1}}{\bar{\psi}_1}$$

$$i = 1, n$$

$$\text{and } u_1 = l_0 = k_1 = \psi_1 = 1$$

The parameter ν is removed from equations (5-6) and (5-7) by the choice of a suitable series for the stream function.

A dimensionless normal distance is introduced :

$$\eta = \left(\frac{2\bar{u}_1}{\nu \bar{I}_0} \right)^{\frac{1}{2}} N \quad (5-18)$$

This differs from the Blasius⁴⁸ coordinate by a factor of $\sqrt{2}$. The present form is used to maintain consistency with the work of Frossling["] and Green in order to facilitate the comparison of results later.

The stream function is now expanded as a power series in x_1 in which the coefficients are functions of the dimensionless distance :

$$\begin{aligned} \Psi(x_1, N) = \left(\frac{\nu \bar{I}_0}{2\bar{u}_1} \right)^{\frac{1}{2}} \left\{ \bar{u}_1 (u_1 f_1(\eta) x_1 + 2 u_3 f_3(\eta) x_1^3 \right. \\ \left. + \dots + n u_{2n-1} f_{2n-1}(\eta) x_1^{2n-1} + \dots) \right\} \end{aligned} \quad (5-19)$$

The series must be chosen so that the derivative $\partial \Psi / \partial N$ matches the potential flow series at the edge of the hydrodynamic boundary layer.

A similar series is assumed for the dimensionless transfer distribution :

$$\frac{C - C_\infty}{C_s - C_\infty} = F_0(\eta) + F_2(\eta) x_1^2 + \dots + F_{2(n-1)}(\eta) x_1^{2(n-1)} + \dots \quad (5-20)$$

If the series presented above for U , $h_{1,s}$, $h_{3,s}$, $\sin \phi$, $\bar{\Psi}$, and $\frac{C - C_0}{C_\infty - C_0}$ are assumed to be absolutely convergent, the

derivatives required in equations (5-6) and (5-7) can be obtained by term by term differentiation. The necessary derivatives are :

$$\frac{\partial \bar{\Psi}}{\partial N} = \bar{u}_1 (u_1 f_1'(\gamma) x_1 + 2u_3 f_3'(\gamma) x_1^3 + \dots + nu_{2n-1} f_{2n-1}'(\gamma) x_1^{2n-1} + \dots) \quad (5-21)$$

$$\begin{aligned} \frac{\partial^2 \bar{\Psi}}{\partial N^2} &= \left(\frac{2\bar{u}_1}{\partial \bar{I}_0} \right)^{\frac{1}{2}} \bar{u}_1 (u_1 f_1''(\gamma) x_1 + 2u_3 f_3''(\gamma) x_1^3 + \dots \\ &\quad + nu_{2n-1} f_{2n-1}''(\gamma) x_1^{2n-1} + \dots) \end{aligned} \quad (5-22)$$

$$\begin{aligned} \frac{\partial^3 \bar{\Psi}}{\partial N^3} &= \left(\frac{2\bar{u}_1}{\partial \bar{I}_0} \right) \bar{u}_1 (u_1 f_1'''(\gamma) x_1 + 2u_3 f_3'''(\gamma) x_1^3 + \dots \\ &\quad + nu_{2n-1} f_{2n-1}'''(\gamma) x_1^{2n-1} + \dots) \end{aligned} \quad (5-23)$$

$$\begin{aligned} \frac{\partial \bar{\Psi}}{\partial x_1} &= \left(\frac{\partial \bar{I}_0}{2\bar{u}_1} \right)^{\frac{1}{2}} \bar{u}_1 (u_1 f_1(\gamma) + 6u_3 f_3(\gamma) x_1^2 + \dots \\ &\quad + n(2n-1)u_{2n-1} f_{2n-1}(\gamma) x_1^{2n-2} + \dots) \end{aligned} \quad (5-24)$$

$$\begin{aligned} \frac{\partial^2 \bar{\Psi}}{\partial x_1 \partial N} &= \bar{u}_1 (u_1 f_1'(\gamma) + 6u_3 f_3'(\gamma) x_1^2 + \dots + \\ &\quad + n(2n-1)u_{2n-1} f_{2n-1}'(\gamma) x_1^{2n-2} + \dots) \end{aligned} \quad (5-25)$$

$$\frac{dh_{3,s}}{dx_1} = k_1 (k_1 + 3k_3 x_1^2 + \dots + (2n-1)k_{2n-1} x_1^{2n-2} + \dots) \quad (5-26)$$

$$\frac{dU}{dx_1} = \bar{u}_1 (u_1 + 3u_3 x_1^2 + \dots + (2n-1)u_{2n-1} x_1^{2n-2} + \dots)$$

(5-27)

$$\frac{\partial C}{\partial x_1} = (C_s - C_\infty) (2F_2(\gamma)x_1 + 4F_4(\gamma)x_1^3 + \dots + 2(n-1)F_{2(n-1)}(\gamma)x_1^{2n-3} + \dots)$$

(5-28)

$$\frac{\partial C}{\partial N} = (C_s - C_\infty) \left(\frac{2\bar{u}_1}{\sqrt{I_0}} \right)^{\frac{1}{2}} (F'_0(\gamma) + F'_2(\gamma)x_1^2 + \dots + F'_{2(n-1)}(\gamma)x_1^{2(n-1)} + \dots)$$

(5-29)

$$\frac{\partial^2 C}{\partial N^2} = (C_s - C_\infty) \left(\frac{2\bar{u}_1}{\sqrt{I_0}} \right) (F''_0(\gamma) + F''_2(\gamma)x_1^2 + \dots + F''_{2(n-1)}(\gamma)x_1^{2(n-1)} + \dots)$$

(5-30)

Upon substitution of the above expressions, the momentum equation (5-6) becomes :

$$\begin{aligned}
 & \bar{u}_1^2 \bar{k}_1 (u_1 f_1'(\gamma) x_1 + 2u_3 f_3'(\gamma) x_1^3 + \dots + nu_{2n-1} f_{2n-1}'(\gamma) x_1^{2n-1} + \dots) \\
 & (u_1 f_1'(\gamma) + 6u_3 f_3'(\gamma) x_1^2 + \dots + n(2n-1)u_{2n-1} f_{2n-1}'(\gamma) x_1^{2n-1} + \dots) \\
 & (k_1 x_1 + k_3 x_1^3 + \dots + k_{2n-1} x_1^{2n-1} + \dots) - \bar{u}_1^2 \bar{k}_1 (u_1 f_1''(\gamma) x_1 \\
 & + 2u_3 f_3''(\gamma) x_1^3 + \dots + nu_{2n-1} f_{2n-1}''(\gamma) x_1^{2n-1} + \dots) (u_1 f_1(\gamma) \\
 & + 6u_3 f_3(\gamma) x_1^2 + \dots + n(2n-1)u_{2n-1} f_{2n-1}(\gamma) x_1^{2n-2} + \dots) \\
 & (k_1 x_1 + k_3 x_1^3 + \dots + k_{2n-1} x_1^{2n-1} + \dots) - \bar{u}_1^2 \bar{k}_1 (u_1 f_1'''(\gamma) x_1 \\
 & + 2u_3 f_3'''(\gamma) x_1^3 + \dots + nu_{2n-1} f_{2n-1}'''(\gamma) x_1^{2n-1} + \dots) (k_1 \\
 & + 3k_3 x_1^2 + \dots + (2n-1)k_{2n-1} x_1^{2n-2} + \dots) (u_1 f_1(\gamma) x_1 \\
 & + 2u_3 f_3(\gamma) x_1^3 + \dots + nu_{2n-1} f_{2n-1}(\gamma) x_1^{2n-1} + \dots) \\
 = & \bar{u}_1^2 \bar{k}_1 (u_1 x_1 + u_3 x_1^3 + \dots + u_{2n-1} x_1^{2n-1} + \dots) (u_1 + 3u_3 x_1^2 \\
 & + \dots + (2n-1)u_{2n-1} x_1^{2n-2} + \dots) (k_1 x_1 + k_3 x_1^3 + \dots \\
 & + k_{2n-1} x_1^{2n-1} + \dots) - g\beta(C_s - C_\infty) \bar{k}_1 \bar{l}_0 \bar{\gamma}_1 (F_0(\gamma) + F_2(\gamma) x_1^2 \\
 & + \dots + F_{2(n-1)}(\gamma) x_1^{2(n-1)} + \dots) (\psi_1 x_1 + \psi_3 x_1^3 + \dots \\
 & + \psi_{2n-1} x_1^{2n-1} + \dots) (l_0 + l_2 x_1^2 + \dots + l_{2(n-1)} x_1^{2(n-1)} + \dots) \\
 & (k_1 x_1 + k_3 x_1^3 + \dots + k_{2n-1} x_1^{2n-1} + \dots) + 2\bar{u}_1^2 \bar{k}_1 (l_0 \\
 & + l_2 x_1^2 + \dots + l_{2(n-1)} x_1^{2(n-1)} + \dots) (u_1 f_1'''(\gamma) x_1 \\
 & + 2u_3 f_3'''(\gamma) x_1^3 + \dots + nu_{2n-1} f_{2n-1}'''(\gamma) x_1^{2n-1} + \dots) (k_1 x_1 \\
 & + k_3 x_1^3 + \dots + k_{2n-1} x_1^{2n-1} + \dots)
 \end{aligned}$$

Similarly, the diffusion equation (5-7) becomes :

$$\begin{aligned}
 & \bar{k}_1 \bar{u}_1 (C_s - C_{\infty}) (k_1 x_1 + k_3 x_1^3 + \dots + k_{2n-1} x_1^{2n-1} + \dots) (u_1 f_1'(\eta) x_1 \\
 & + 2u_3 f_3'(\eta) x_1^3 + \dots + nu_{2n-1} f_{2n-1}'(\eta) x_1^{2n-1} + \dots) \\
 & (2F_2(\eta) x_1 + 4F_4(\eta) x_1^3 + \dots + 2(n-1)F_{2(n-1)}(\eta) x_1^{2n-3} + \dots) \\
 & - \bar{u}_1 \bar{k}_1 (C_s - C_{\infty}) (F_0'(\eta) + F_2'(\eta) x_1^2 + \dots + F_{2(n-1)}'(\eta) x_1^{2(n-1)} \\
 & + \dots) (k_1 x_1 + k_3 x_1^3 + \dots + k_{2n-1} x_1^{2n-1} + \dots) (u_1 f_1(\eta) + \\
 & 6u_3 f_3(\eta) x_1^2 + \dots + n(2n-1)u_{2n-1} f_{2n-1}(\eta) x_1^{2n-1} + \dots) \\
 & - \bar{u}_1 \bar{k}_1 (C_s - C_{\infty}) (F_0'(\eta) + F_2'(\eta) x_1^2 + \dots + F_{2(n-1)}'(\eta) x_1^{2(n-1)} \\
 & + \dots) (k_1 + 3k_3 x_1^2 + \dots + (2n-1)k_{2n-1} x_1^{2n-2} + \dots) \\
 & (u_1 f_1(\eta) x_1 + 2u_3 f_3(\eta) x_1^3 + \dots + nu_{2n-1} f_{2n-1}(\eta) x_1^{2n-1} + \dots) \\
 & = \propto \bar{k}_1 \bar{l}_0 \frac{2\bar{u}_1}{\eta \bar{l}_0} (C_s - C_{\infty}) (F_0''(\eta) + F_2''(\eta) x_1^2 + \dots + F_{2(n-1)}''(\eta) x_1^{2(n-1)} \\
 & + \dots) (k_1 x_1 + k_3 x_1^3 + \dots + k_{2n-1} x_1^{2n-1} + \dots) (l_0 \\
 & + l_2 x_1^2 + \dots + l_{2(n-1)} x_1^{2(n-1)} + \dots)
 \end{aligned}
 \tag{5-32}$$

The boundary conditions for equations (5-31) and (5-32) are :

$$\begin{aligned}
 \text{at } \eta = 0 : \quad & f_1 = f_3 = \dots = f_{2n-1} = \dots = 0 \\
 & f_1' = f_3' = \dots = f_{2n-1}' = \dots = 0 \\
 & F_0 = 1, F_2 = \dots = F_{2n} = \dots = 0 \\
 & \tag{5-33} \\
 \text{as } \eta \rightarrow \infty : \quad & f_1' \rightarrow 1, f_3' \rightarrow \frac{1}{2}, f_{2n-1}' \rightarrow \frac{1}{n} \\
 & F_0 \rightarrow 0, F_2 \rightarrow 0, F_{2n} \rightarrow 0
 \end{aligned}$$

After the series in equations (5-31) and (5-32) have been multiplied together, the equations may be rewritten in ascending powers of x_1 . Two sets of simultaneous ordinary differential equations are then obtained when the coefficients of each ascending power of x_1 are equated.

Equating coefficients of powers of x_1 in equation (5-31):

The first two ordinary differential equations are:

$$f_1'''(\eta) = \frac{1}{2} (f_1'^2(\eta) - 1 + \frac{\overline{Gr}}{Re^2} F_0(\eta)) - f_1(\eta) f_1''(\eta) \quad (5-34)$$

$$\begin{aligned} f_3'''(\eta) = & 2 f_1(\eta) f_3'(\eta) - 2 f_1''(\eta) f_3(\eta) - f_1(\eta) f_3''(\eta) - 1 \\ & + \frac{\overline{Gr}}{Re^2} \frac{1}{4u_3} \left[F_2 + F_0(\psi_3 + 1_2) \right] - \frac{k_3}{2u_3} f_1(\eta) f_1''(\eta) \\ & - \frac{1_2}{2u_3} f_1'''(\eta) \end{aligned} \quad (5-35)$$

Equating coefficients of powers of x_1 in equation (5-32):

The first two ordinary differential equations are:

$$F_0''(\eta) = -Sc F_0'(\eta) f_1(\eta) \quad (5-36)$$

$$\begin{aligned} F_2''(\eta) = & Sc (f_1'(\eta) F_2(\eta) - 4u_3 f_3(\eta) F_0'(\eta) \\ & - f_1(\eta) F_2'(\eta) - 2k_3 f_1(\eta) F_0'(\eta) - (1_2 + k_3) F_0''(\eta)) \end{aligned} \quad (5-37)$$

If required, the higher order equations may be found by equating higher order terms. Apart from the first equation of each set, all higher order equations are shape dependent. A considerable amount of extra work would be required to remove all the shape parameters from the equations.

The usefulness of the series method is restricted by the convergence properties of the power series. For very slender

body shapes a large number of terms are generally required to achieve convergence. Green⁷³, who studied the sphere and four oblate spheroids, found that two terms in each power series were necessary to achieve convergence at an angle of twenty degrees from the front stagnation point. Between twenty and fifty degrees four or five terms were required, and beyond an angle of sixty degrees the rates of convergence of the series were very slow. However, even where convergence problems are such that the point of separation cannot be reached, the series method can be used to calculate the solution in the region of the stagnation point with great accuracy.

A step by step method could then be used to proceed around the body to the separation point.

It is clear from equations (5-35) and (5-37) that the higher order equations become very complex. Since a large number of these equations would have to be solved if the usual convergence problems were encountered, it was decided to solve only the first equation of each of the two infinite sets. This solution, as mentioned above, is that for the region of the front stagnation point.

Equations (5-34) and (5-36) are coupled and must be solved simultaneously. The system represents a two point boundary value problem, and is solved by a quasilinearization technique which is described in Section 5.

5.1.1 Point Mass Transfer Rates.

In Chapter 4 the equation for the point mass transfer rate was presented as equation (4-80):

$$\frac{K'(x_1)}{\Delta D} = - \frac{1}{C_o - C_\infty} \left. \frac{\partial C}{\partial N} \right|_{\text{surface}} \quad (5-38)$$

The concentration gradient at the surface is substituted from equation (5-28) to give:

$$\frac{K'_1(x_1)}{\mathfrak{D}} = - \left(\frac{2\bar{u}_1}{\sqrt{\bar{l}_0}} \right)^{\frac{1}{2}} (F'_0(0) + F'_2(0) x_1^2 + \dots + F'_{2(n-1)}(0) x_1^{2(n-1)}) \quad (5-39)$$

Since only the first equations are solved equation (5-39) becomes:

$$\frac{K'_1(x_1)}{\mathfrak{D}} = \left(\frac{2\bar{u}_1}{\sqrt{\bar{l}_0}} \right)^{\frac{1}{2}} - F'_0(0) \quad (5-40)$$

A Reynolds number and a local Sherwood number are defined as:

$$\overline{Re} = \frac{2\bar{u}_1 \bar{l}_0}{\nu} \quad (5-41)$$

$$\overline{Sh} = \frac{K'_1(x_1) \bar{l}_0}{\mathfrak{D}} \quad (5-42)$$

Thus equation (5-40) becomes:

$$\frac{\overline{Sh}}{\overline{Re}^{\frac{1}{2}}} = - F'_0(0) \quad (5-43)$$

It has been shown that for the sphere $\bar{u}_1 = 3 U_{\infty} / 2$ and $\bar{l}_0$ is the radius.¹³ However, in the approximate method developed in Chapter 4, the characteristic velocity was U_{∞} and the characteristic dimension was the diameter. Therefore, care needs to be taken when the local transfer rates are compared for the approximate and the exact methods.

5.1.2 The Series Solution for the Vertical Flat Plate.

An exact method has been developed for the vertical flat plate in order to be able to assess the accuracy of the approximate solutions which have been calculated.

The boundary layer equations for the vertical flat plate which govern transfer in the regime of combined free and forced convection are:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + g\beta(C - C_0) + \nu \frac{\partial^2 u}{\partial y^2} \quad (5-44)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (5-45)$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = \alpha \frac{\partial^2 C}{\partial y^2} \quad (5-46)$$

These equations may be reduced to two infinite sets of ordinary differential equations, as described in the previous section, by the introduction of series. The development will not be discussed in detail since the steps are the same as those for the previous method.

The series for the stream function may be written as:

$$\Psi = (U_\infty \nu x)^{\frac{1}{2}} \sum_0 \left(\frac{x}{D_C}\right)^n f_n(\eta) \quad (5-47)$$

The series for the concentration is:

$$\frac{C - C_\infty}{C_s - C_\infty} = \sum_0 \left(\frac{x}{D_C}\right)^{2n} F_{2n}(\eta) \quad (5-48)$$

$$\text{where } \eta = \left(\frac{U_\infty}{\nu x}\right)^{\frac{1}{2}} y \quad (5-49)$$

when there is a uniform free stream velocity, U_∞ , past the body, $dU/dx = 0$, and the derivative of the stream function, $\partial \Psi / \partial y$, must match the free stream velocity at all values of x . This will be true when only one term in the series expansion for the stream function is considered.

When equations (5-47) to (5-49) and their derivatives are

substituted into equations (5-44) and (5-46) two infinite sets of ordinary differential equations are obtained when the coefficients of powers of x are equated. The first equation of each set is:

$$f_0 f_0'' + 2 f_0''' - \frac{Gr}{Re^2} F_0 = 0 \quad (5-50)$$

$$F_0'' + \frac{1}{2} Sc f_0 F_0' = 0 \quad (5-51)$$

with the boundary conditions:

$$\begin{aligned} \text{at } \eta = 0 : f_0 &= f_0'' = 0 ; F_0 = 1 \\ \text{as } \eta \rightarrow \infty : f_0' &\rightarrow 1 ; F_0 \rightarrow 0 \end{aligned} \quad (5-52)$$

As the parameter Gr/Re^2 approaches zero, the equations reduce to those for pure forced convection. These equations also present a two point boundary value problem and are solved by the quasilinearization technique. During the course of the work, Lloyd and Sparrow⁵⁴ published a communication which treated the problem of the vertical flat plate by a local similarity method. The equations which they obtained were the same as the first equation of each infinite set presented above.

In addition to employing equations (5-47) to (5-49), which are associated with forced convection, those for free convection have also been investigated.

The series for the stream function is written as:

$$\Psi = 4\sqrt{\bar{c}} x^{\frac{3}{4}} \sum_0 \left(\frac{x}{D_c} \right)^n \zeta_n(\eta) \quad (5-53)$$

$$\eta = \frac{\bar{c}y}{x^{\frac{1}{4}}} \quad (5-54)$$

$$\text{where } \bar{c} = \left[\frac{g\beta(C_s - C_\infty)}{4\nu^2} \right]^{\frac{1}{4}} \quad (5-55)$$

and the concentration series is written as:

$$\frac{C - C_{\infty}}{C_s - C_{\infty}} = \sum_0^{\infty} \left(\frac{x}{D_c} \right)^{2n} \Omega_{2n}(\eta) \quad (5-56)$$

Upon substitution of these variables into equations (5-44) and (5-46) and equating coefficients, the first equation of each infinite set is:

$$\zeta_0''' + 3\zeta_0''\zeta_0 - 2\zeta_0 + \Omega_0 = 0 \quad (5-57)$$

$$\Omega_0'' + 3Sc\zeta_0\Omega_0' = 0 \quad (5-58)$$

with the boundary conditions:

$$\begin{aligned} \text{at } \eta = 0 : \zeta_0 = \zeta_0' = 0 ; \Omega_0 = 1 \\ \text{as } \eta \rightarrow \infty : \zeta_0' \rightarrow \frac{1}{2} \left(\frac{Re^2}{Gr} \right)^{\frac{1}{2}} ; \Omega_0 \rightarrow 0 \end{aligned} \quad (5-59)$$

As the parameter Gr/Re^2 increases the boundary condition for ζ_0' as $\eta \rightarrow \infty$ tends to zero, and the equations are those for free convection.

5.2. Numerical Techniques.

In order to calculate the exact solution, a non-linear third order ordinary differential equation and a linear second order ordinary differential equation have to be solved simultaneously. The system is fully defined in that there are five boundary conditions. However, as only three of these boundary conditions are initial conditions, the system represents a two point boundary value problem. The two remaining boundary conditions are known at infinity. These are the dimensionless free stream velocity and the dimensionless free stream concentration.

A trial and error procedure is frequently employed to solve two point boundary value problems. Green⁷³ successfully used a trial and error method to predict the unknown conditions for forced convection. A trial and error procedure was used by Pandya⁷⁸ to calculate the two unknown initial conditions for pure free convection. He found that the initial guesses of the two unknown boundary conditions had to be fairly close to the correct values, often to within two or more significant figures, before the procedure would converge. Even when convergence was achieved, the rate of convergence was often very slow.

Sanders⁸⁰ encountered considerable difficulties with the two point boundary value problem associated with his work on rotating bodies. As in the case of free convection, Sanders had to determine two unknown initial conditions simultaneously. He successfully overcame the problem by the use of the quasilinearization technique of Bellman⁹² and Kalaba⁹³.

Basically, the technique quasi-linearizes the non-linear equations and solves the resulting equations by the superposition principle which is normally used to solve the linear equations. The quasilinearization technique has been used by Lee^{94,95} to

solve a number of two point boundary value problems. The technique has been recently used by Harris⁵⁷ to solve the two point boundary value problem associated with free convection.

Since two unknown initial conditions have to be determined simultaneously for the present system of equations, the quasilinearization technique was used to solve the two point boundary value problem.

The main advantage of the quasilinearization technique is that even with very poor initial guesses, rapid convergence is achieved to the solution of the original non-linear differential equations. In a recent paper Lee⁹⁴ presented a detailed mathematical treatment of the method and discussed the criteria which must be satisfied for monotonic convergence.

For a number of problems, the quasilinearization technique has been shown⁹⁵ to converge to a solution, with five digit accuracy, in three to six iterations.

The general method will be discussed here so that its application to the present problem may be easily understood.

Consider the general system of n first order simultaneous ordinary differential equations:

$$\frac{dW_i}{dt} = f_i (W_1, W_2, \dots W_n, t)$$

$$i = 1, 2, \dots, n \quad (5-60)$$

in which W_i are functions of the independent variable t and where the boundary conditions are split and are:

Initial conditions:

$$W_j(0) = W_{j,o} \quad ; \quad j = 1, \dots, m$$

Final conditions: (5-61)

$$W_q(t_f) = W_{q,f} \quad ; \quad q = m + 1, \dots, n$$

t_f is the final value of t .

The system of equations (5-60) and (5-61) represents a non-linear, two point boundary value problem since not all of the boundary conditions are given at the initial point.

The system may be written in vector notation:

$$\frac{d\underline{W}}{dt} = \underline{f}(\underline{W}, t) \quad (5-62)$$

with the same boundary conditions as shown in equation (5-61)
 $\underline{f}$ and $\underline{W}$ represent n-dimensional vectors.

Since any n^{th} order ordinary differential equation may be reduced to a set of n first order ordinary differential equations, the system represented by equations (5-61) and (5-62) is a fairly general non-linear system.

An approximate solution of (5-62) is assumed and will be represented by $\underline{W}^0$. This may be a pure guess, or a solution to a similar problem whose solution is known.

The Taylor series is used to expand the function about $\underline{W}^0$. Equation (5-62) becomes:

$$\frac{d\underline{W}}{dt} = \underline{f}(\underline{W}^0, t) + J(\underline{W}^0)(\underline{W} - \underline{W}^0) \quad (5-63)$$

where all second and higher order derivatives have been neglected.

$J(\underline{W}^0)$ is the Jacobi matrix and in expanded form is the matrix of first derivatives:

$$J(\underline{W}^0) = \begin{bmatrix} \frac{\partial f_1}{\partial W_1^0} & \frac{\partial f_1}{\partial W_2^0} & \cdots & \frac{\partial f_1}{\partial W_n^0} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial W_1^0} & \frac{\partial f_n}{\partial W_2^0} & \cdots & \frac{\partial f_n}{\partial W_n^0} \end{bmatrix} \quad (5-64)$$

The boundary conditions are still those given by equation (5-61).

The system (5-63) represents a set of linear differential equations whose coefficients are functions of the known solution $\underline{W}^0$. There is no analytical solution to the system, but this may now be solved by the superposition principle.

A recurrence relation is obtained which, according to Lee⁹⁴, represents a monotone sequence of functions which converge to the required solution $\underline{W}$ if the convergence criteria are satisfied.

The solution of equation (5-63) may be written as $\underline{W}^1$. This may now be expanded in Taylor series to give:

$$\frac{d\underline{W}}{dt} = \underline{f}(\underline{W}^1, t) + J(\underline{W}^1)(\underline{W} - \underline{W}^1) \quad (5-65)$$

The solution may be written as $\underline{W}^2$ and the same procedure repeated. The general recurrence relation may now be written as :

$$\frac{d\underline{W}^{k+1}}{dt} = \underline{f}(\underline{W}^k, t) + J(\underline{W}^k)(\underline{W}^{k+1} - \underline{W}^k) \quad (5-66)$$

The boundary conditions for this recurrence relation are :

$$\begin{aligned} \underline{W}_j^{k+1}(0) &= \underline{W}_{j,0} & : j &= 1, \dots, m \\ \underline{W}_q^{k+1}(t_f) &= \underline{W}_{q,t_f} & : q &= m+1, \dots, n \end{aligned} \quad (5-67)$$

The method may now be applied to the stagnation point equations (5-34) to (5-36) which were derived earlier. The problem is represented by a non-linear set of two coupled ordinary differential equations with split boundary conditions.

The equations are:

$$f_1''' = \frac{1}{2} (f_1'^2 - 1 + \frac{\overline{Gr}}{Re^2} F_0) - f_1 f_1'' \quad (5-68)$$

$$F_0'' = - Sc F_0' f_1 \quad (5-69)$$

with the boundary conditions:

$$\begin{aligned} f_1(0) &= 0 ; & f_1'(0) &= 0 ; & F_0(0) &= 1 \\ f_1'(\infty) &\rightarrow 1 ; & F_0(\infty) &\rightarrow 0 \end{aligned} \quad (5-70)$$

Five new variables are now defined:

$$\begin{aligned} Y(1) &= f_1 \\ Y(2) &= f_1' \\ Y(3) &= f_1'' \\ Y(4) &= F_0 \\ Y(5) &= F_0' \end{aligned} \quad (5-71)$$

The system of equations (5-68) and (5-69) may be written as a set of five first order ordinary differential equations:

$$\begin{aligned} \frac{dY(1)}{d\eta} &= Y(2) \\ \frac{dY(2)}{d\eta} &= Y(3) \\ \frac{dY(3)}{d\eta} &= -Y(1)Y(3) - 0.5 \left(1.0 + \frac{\overline{Gr}}{Re^2} Y(4) - Y(2)^2 \right) \\ \frac{dY(4)}{d\eta} &= Y(5) \\ \frac{dY(5)}{d\eta} &= -Sc Y(1)Y(5) \end{aligned} \quad (5-72)$$

with the boundary conditions rewritten as:

$$\begin{aligned}
 Y(1)_0 &= 0 \\
 Y(2)_0 &= 0 \\
 Y(2)_\infty &= 1 \\
 Y(4)_0 &= 1 \\
 Y(4)_\infty &= 0
 \end{aligned}
 \tag{5-73}$$

where the subscript represents the point at which the boundary condition is known.

These equations may now be linearized using the recurrence relation, equation (5-66), to give the following set of equations:

$$\begin{aligned}
 \frac{dY(1)^{k+1}}{d\eta} &= Y(2)^{k+1} \\
 \frac{dY(2)^{k+1}}{d\eta} &= Y(3)^{k+1} \\
 \frac{dY(3)^{k+1}}{d\eta} &= - \left[Y(3)^k Y(1)^{k+1} + Y(1)^k Y(3)^{k+1} - Y(1)^k Y(3)^k \right. \\
 &\quad \left. + 0.5 \left(1.0 \pm \frac{\overline{Gr}}{\overline{Re}^2} Y(4)^{k+1} + Y(2)^k Y(2)^k \right. \right. \\
 &\quad \left. \left. - 2Y(2)^k Y(2)^{k+1} \right) \right] \\
 \frac{dY(4)^{k+1}}{d\eta} &= Y(5)^{k+1} \\
 \frac{dY(5)^{k+1}}{d\eta} &= - Sc \left(Y(1)^{k+1} Y(5)^k + Y(1)^k Y(5)^{k+1} \right. \\
 &\quad \left. - Y(1)^k Y(5)^k \right)
 \end{aligned}
 \tag{5-74}$$

with the boundary conditions:

$$\begin{aligned}
 Y(1)_0^{k+1} &= 0 \\
 Y(2)_0^{k+1} &= 0 \\
 Y(2)^{k+1} &= 1 \\
 Y(4)_0^{k+1} &= 1 \\
 Y(4)^{k+1} &= 0
 \end{aligned} \tag{5-75}$$

This linearized set of equations (5-74) and (5-75) may now be solved by the superposition principle. The homogeneous equations are:

$$\begin{aligned}
 \frac{dY(1)^{k+1}}{d\eta} &= Y(2)^{k+1} \\
 \frac{dY(2)^{k+1}}{d\eta} &= Y(3)^{k+1} \\
 \frac{dY(3)^{k+1}}{d\eta} &= - \left[Y(3)^k Y(1)^{k+1} + Y(1)^k Y(3)^{k+1} + 0.5 \left(\frac{\overline{Gr}}{Re} \right)^2 Y(4)^{k+1} \right. \\
 &\quad \left. - 2Y(2)^k Y(2)^{k+1} \right] \\
 \frac{dY(4)^{k+1}}{d\eta} &= Y(5)^{k+1} \\
 \frac{dY(5)^{k+1}}{d\eta} &= - Sc \left(Y(1)^k Y(5)^{k+1} + Y(1)^{k+1} Y(5)^k \right)
 \end{aligned} \tag{5-76}$$

with arbitrarily chosen, non trivial boundary conditions.

The solution of the system may now be written as:

$$Y(I)^{k+1}(\eta) = Y(I)_p^{k+1}(\eta) + \sum_{j=1}^5 \tau_j^{k+1} Y(I)_{h,j}^{k+1}(\eta)$$

$$I = 1, 5 \tag{5-77}$$

where $Y(I)_p^{k+1}(\gamma)$, $I = 1, 5$ represents one complete solution of the system of equations (5-74) which will be called the particular solution and will be denoted by the subscript p. $Y(I)_{h,j}^{k+1}(\gamma)$, $I = 1, 5$, $j = 1, 5$ represent the five distinct non-trivial solutions of the homogeneous equations (5-76). These are designated by the subscript h and are called the homogeneous solutions. The subscript j refers to the j^{th} homogeneous set of equations, and I to the I^{th} variable.

Since the system is fifth order, τ_j^{k+1} represent the five superposition constants which are required to calculate the general solution.

Theoretically, these five constants τ_j^{k+1} could be calculated by obtaining one particular solution and five distinct, non-trivial homogeneous solutions.

However, since three of the five initial conditions are already known, these may be used to reduce the number of homogeneous solutions which are required. Thus only two superposition constants are needed and the set of homogeneous equations only has to be integrated twice. Equation (5-77) becomes :

$$Y(I)^{k+1}(\gamma) = Y(I)_p^{k+1}(\gamma) + \sum_{j=1}^2 \tau_j^{k+1} Y(I)_{h,j}^{k+1}(\gamma)$$

$$I = 1, 5 \quad (5-78)$$

The initial conditions for the particular and homogeneous solutions are:

$$\underline{Y}_p^{k+1}(0) = \begin{bmatrix} Y(1)_p^{k+1}(0) \\ Y(2)_p^{k+1}(0) \\ Y(3)_p^{k+1}(0) \\ Y(4)_p^{k+1}(0) \\ Y(5)_p^{k+1}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad (5-79)$$

$$\underline{Y}_{h,1}^{k+1}(0) = \begin{bmatrix} Y(1)_{h,1}^{k+1}(0) \\ Y(2)_{h,1}^{k+1}(0) \\ Y(3)_{h,1}^{k+1}(0) \\ Y(4)_{h,1}^{k+1}(0) \\ Y(5)_{h,1}^{k+1}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad (5-80)$$

$$\underline{Y}_{h,2}^{k+1}(0) = \begin{bmatrix} Y(1)_{h,2}^{k+1}(0) \\ Y(2)_{h,2}^{k+1}(0) \\ Y(3)_{h,2}^{k+1}(0) \\ Y(4)_{h,2}^{k+1}(0) \\ Y(5)_{h,2}^{k+1}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (5-81)$$

It is clear from equations (5-79) to (5-81) that the first, second and fourth boundary conditions of the original set (5-75) are immediately satisfied.

The following procedure is then used to calculate the two unknown initial boundary conditions.

The two superposition constants may be obtained by the use of equations (5-75) and (5-78):

For $I = 2$ and $\gamma = \infty$:

$$Y(2)_p^{k+1}(\infty) + \tilde{\gamma}_1^{k+1} Y(2)_{h,1}^{k+1}(\infty) + \tilde{\gamma}_2^{k+1} Y(2)_{h,2}^{k+1}(\infty) = 1 \quad (5-82)$$

For $I = 4$ and $\eta = \infty$:

$$Y(4)_p^{k+1}(\infty) + \tau_1^{k+1} Y(4)_{h,1}^{k+1}(\infty) + \tau_2^{k+1} Y(4)_{h,2}^{k+1}(\infty) = 0 \quad (5-83)$$

Once one particular and two homogeneous solutions have been calculated, equations (5-82) and (5-83) are used to calculate the two superposition constants.

$$\tau_1^{k+1} = \frac{Y(4)_{h,2}^{k+1} + Y(4)_p^{k+1} Y(2)_{h,2}^{k+1} - Y(2)_p^{k+1} Y(4)_{h,2}^{k+1}}{Y(2)_{h,1}^{k+1} Y(4)_{h,2}^{k+1} - Y(4)_{h,1}^{k+1} Y(2)_{h,2}^{k+1}} \quad (5-84)$$

$$\tau_2^{k+1} = \frac{Y(4)_{h,1}^{k+1} + Y(4)_p^{k+1} Y(2)_{h,1}^{k+1} - Y(2)_p^{k+1} Y(4)_{h,1}^{k+1}}{Y(2)_{h,2}^{k+1} Y(4)_{h,1}^{k+1} - Y(4)_{h,2}^{k+1} Y(2)_{h,1}^{k+1}} \quad (5-85)$$

When the values of the two constants have been calculated, equation (5-70) is used to calculate the updated solution $\underline{Y}^{k+1}(\eta)$ from $\underline{Y}^k(\eta)$. The procedure is then repeated until there is no further change in the solution from iteration k to iteration $k+1$.

The form of the initial boundary conditions, equations (5-79) to (5-81), means that the two superposition constants, τ_1^{k+1} and τ_2^{k+1} are in fact the two unknown initial conditions $Y(3)_o^{k+1}$ and $Y(5)_o^{k+1}$ respectively.

The steps in the numerical solution procedure are now straightforward. An initial guess of the solution is chosen. This may be arbitrary, or the solution of a similar problem which is readily available. The crude guessed approximation which is used throughout the present work is:

$$\begin{aligned}
 Y(1)^0(\eta) &= 3.0 \\
 Y(2)^0(\eta) &= 0.5 \\
 Y(3)^0(\eta) &= 0.5 \\
 Y(5)^0(\eta) &= -0.3
 \end{aligned}
 \tag{5-86}$$

$$0 \leq \eta \leq \eta_{\text{MAX}}$$

The value of $Y(4)^k(\eta)$ is not required in the calculation of the solution for the $k+1^{\text{th}}$ iteration, and therefore no value for $Y(4)^0(\eta)$ is required.

The particular solution is calculated by the integration of the set of equations (5-74) with the initial conditions (5-79). The integration is carried out from the body surface, $\eta = 0$, to the outer edge of the hydrodynamic boundary layer, $\eta = \eta_{\text{MAX}}$. This value, η_{MAX} , is arbitrarily chosen. The effect of the choice of different values for this variable will be discussed in Chapter 6.

The first homogeneous solution is then computed by integration of the set of equations (5-76) with the initial conditions (5-80) from $\eta = 0$ to $\eta = \eta_{\text{MAX}}$.

The second homogeneous solution is calculated in the same way. In this case, equations (5-76) are integrated with the initial conditions (5-81).

The two superposition constants are now calculated using equations (5-84) and (5-85). The new approximation to the solution is then computed by the use of equation (5-78).

This solution is then used to compute new particular and homogeneous solutions and new superposition constants. These constants are compared with those of the previous iteration. If they are within the required accuracy the computation is stopped. A number of convergence criteria from 1×10^{-2} to 1×10^{-10}

were investigated and the results are discussed in Chapter 6.

If the solution has not converged, the above procedure is repeated until convergence is achieved.

The effects of instability in the numerical solution procedure were found to be much greater for high Schmidt numbers than for low Schmidt numbers.

This occurred because any round off errors which appeared were first observed in the calculation of $YG(4)$ and $YG(5)$, where $YG(I) = dY(I)/d\eta$. Since these were multiplied by the Schmidt number, the instabilities built up very rapidly.

Stability was achieved by a scaling which removed the Schmidt number from the transfer equation and put it into the momentum equation. This scaling is similar to that introduced into the approximate method. Instead of using the unscaled equations over the remainder of the hydrodynamic boundary layer, the integration step length was varied to achieve the same effect.

Very small integration steps were used within the transfer boundary layer. When the effects of the concentration became negligible the step length was increased and the integration was allowed to continue over the remainder of the hydrodynamic boundary layer.

The relevant non-linearized, linearized, and homogeneous equations for all of the problems which have been studied are presented in Appendix B.

The computer programme which was written to solve the two point boundary value problem by the quasilinearization technique is presented in Appendix D. The programme listing is presented, together with a written description and the flow charts for all of the important subroutines.

5.2.1. Limitation Due to the Storage Capacity of the Computer.

In a number of situations the use of the quasilinearization technique may be limited by the storage capacity of the computer. This may be more important if, in addition to having to make a large number of steps, double precision arithmetic is necessary because of the computer word length.

The method requires four arrays, each with five variables with n steps from $\eta = 0$ to $\eta = \infty$. Thus for problems where small steps are necessary, and the effective infinity is large, the storage requirement will become very large and prohibitive if double precision arithmetic is necessary.

A method has been devised which halves the storage requirements, but slightly increases the computation time.

The procedure is to calculate once through the particular and homogeneous solutions without storing any intermediate values. The final values of $Y(2)$ and $Y(4)$ are stored so that the superposition constants may be determined. This calculation is repeated but the constants are now used to build up the new solution. Thus:

$$\text{Particular} \quad Y(I)^{k+1}(\eta) = Y(I)_p^{k+1}(\eta) \quad (5-87)$$

$$\text{1st Homogeneous} \quad Y(I)^{k+1}(\eta) = Y(I)^{k+1}(\eta) + r_1^{k+1} Y(I)_{h,1}^{k+1}(\eta) \quad (5-88)$$

$$\text{2nd Homogeneous} \quad Y(I)^{k+1}(\eta) = Y(I)^{k+1}(\eta) + r_2^{k+1} Y(I)_{h,2}^{k+1}(\eta) \quad (5-89)$$

The final $Y(I)^{k+1}(\eta)$ is the required new approximation to $Y(I)$. The procedure is repeated until the two sets of constants are within 30% of each other. This usually occurs in two iterations.

The constants at step k are now used to calculate the solution at step $k+1$, thus:

$$Y(I)^{k+1}(\gamma) = Y(I)_P^{k+1}(\gamma) \quad (5-90)$$

$$Y(I)^{k+1}(\gamma) = Y(I)^{k+1}(\gamma) + \tau_1^k Y(I)_{h,1}^{k+1}(\gamma) \quad (5-91)$$

$$Y(I)^{k+1}(\gamma) = Y(I)^{k+1}(\gamma) + \tau_2^k Y(I)_{h,2}^{k+1}(\gamma) \quad (5-92)$$

This procedure is repeated until convergence is achieved.

Tests have shown that two or three extra iterations are required for a solution to be obtained than the number of iterations needed in the original method.

Since the solution is built up as the computation progresses, only one array is required. This is as opposed to three, one particular and two homogeneous, for the original method.

Thus if the IBM 709⁴ were used with 32K storage, and there were 800 integration steps using double precision arithmetic, the storage for the four large arrays alone would be $4 \times 5 \times 800 \times 2$ (for double precision) = 32,000. With the rest of the programme this would exceed the total storage. But with the revised method only 16,000 storage would be required for the large arrays, and the programme would be able to be run, even though the computer time required would increase.

CHAPTER 6.

RESULTS AND DISCUSSION

In order to assess the reliability of the numerical results, a discussion of the accuracy of the numerical techniques precedes the detailed presentation of the results.

6.1. The Accuracy of the Numerical Techniques.

6.1.1. The Approximate Method.

The discussion of the accuracy of the numerical techniques which are used in the approximate method is concerned mainly with the numerical integration of the two ordinary differential equations.

The two initial boundary layer thicknesses for the body of revolution were calculated by the Newton - Raphson method to an accuracy of about four decimal places in four or five iterations. As discussed in Chapter 4, a considerable amount of care was necessary to ensure that the correct roots were calculated. However, as long as small integration steps were taken initially, the numerical integration was fairly insensitive to the accuracy with which the correct initial conditions were calculated. Even when the two initial conditions were only determined to an accuracy of two significant figures, stability was immediately achieved in the integration.

The numerical integration was carried out by the use of a fourth order Runge - Kutta method. Small integration steps of one hundredth of a degree were taken in the vicinity of the front stagnation point to ensure that stability was achieved.

At an angle of one degree from the front stagnation point it was found that the integration step length could be increased to one tenth of a degree for all body shapes. The step length was further increased to one half of a degree at an angle of ten degrees, and maintained at this value up to an angle of seventy five degrees. For the sphere and the 13/16 spheroid, a step length of one degree could be employed. It was then found that it was necessary to reduce the step length to one twentieth of a degree in order to integrate past the equator of the body. The reduction of the integration step length was found to be increasingly necessary as the body shapes become flatter. This was caused by the rapid variation of the two boundary layer thicknesses around the equator. If the separation point had not already been found, the integration step length was increased to one half of a degree at an angle of one hundred and five degrees. The integration was then continued with this step length until the separation point was reached.

The separation point was taken to be the point at which the velocity gradient at the body surface was zero. The reduction of the integration step length in the vicinity of the equator enabled improved predictions of the separation point to be made. This in turn led to some interesting results when aiding and opposing flows were investigated. These results are discussed in a later section.

No further improvements in the results were obtained by increasing the accuracy with which the initial conditions were calculated, or by any reduction in the size of the integration step lengths. When the integration step lengths were halved, the predicted mass transfer rates were unchanged to four decimal places.

By the use of the above technique a complete solution could be calculated in four to five seconds computer time on the CDC 6600.

At this point it is of interest to discuss some important findings which resulted from the modifications which were made to the original integral method for the vertical flat plate. As discussed in Chapter 4, the order of the polynomials, which were chosen to represent the velocity and concentration profiles for the vertical flat plate problem, had no effect upon the extent of the region over which the approximate method failed. In addition, over the range of values of the parameter Gr/Re^2 for which the method could be used, the predicted transfer rates were in very good agreement for each of the different profiles. Some mass transfer predictions are presented in Table C.12. which show the relatively small effect of the order of the polynomials. These results are, as will be shown later, in very good agreement with exact solutions.

Thus the use of high order polynomials increases both the work and computer time which are required to calculate a solution. There is no resulting increase in either the accuracy of the mass transfer predictions, or the range of values of the parameter Gr/Re^2 over which the method may be used. Therefore, if the approximate method may be used for a particular set of parameters, the use of cubic or quartic polynomials for the concentration and velocity will produce an accurate solution very quickly.

6.1.2. The Exact Solution.

The analysis of the numerical solutions involves the accuracy of both the Runge - Kutta numerical integration technique, and the quasilinearization technique which is used to predict the two unknown boundary conditions, $f_1''(0)$ and $F_0'(0)$.

The Runge - Kutta integration technique was tested in the usual manner by varying the integration step lengths. Table C.1 shows the effects of varying the step length for the integration of the stagnation point equations. The values of the two unknown initial conditions $f_1''(0)$ and $F_0'(0)$ are compared for a number of different step lengths between 0.005 and 0.5. The solutions were calculated for both large and small Schmidt numbers over a range of values of the parameter $\overline{Gr}/\overline{Re}^2$.

In Table C.1.(a) the solutions for a Schmidt number of 1.0 can be seen to be very different for step lengths of 0.5 and 0.25. The difference then becomes smaller for subsequent solutions as the step length is reduced. For step lengths of 0.025 and 0.01, the difference between the solutions for $f_1''(0)$ and $F_0'(0)$ is very small. A further reduction of the step length to 0.005 results in no further change in the predicted values to five decimal places. A step length of 0.01 was therefore employed in the remaining calculations. Tables C.1.(b), (c), and (d) all show similar behaviour for other ranges of values of the parameters.

A second test was made in which the integration step length was varied. In this case the two initial conditions, $f_1''(0)$ and $F_0'(0)$, which were predicted by the quasilinearization method, were used to integrate the appropriate non - linear equations with the same integration step length. The results

for a Schmidt number of 2.4, using four integration step lengths: 0.01, 0.05, and 0.1 and 0.2, are shown in Table C.2 for values of the parameter $\overline{Gr}/\overline{Re}^2$ of 0.5 and 50.0. Two different values of the parameter $\overline{Gr}/\overline{Re}^2$ were studied in order to investigate whether the same integration step length could be used for all values of that parameter.

The two unknown initial conditions, $f_1''(0)$ and $F_0'(0)$, which were predicted by the quasilinearization technique for each integration step length are presented. The results of the integration of the non - linear equations, using these initial conditions, are tabulated for f_1' and F_0 at dimensionless distances, η , from the body surface. Tables C.2.(a) and C.2.(b) show that for a Schmidt number of 2.4, a smaller step length is required for the larger value of the parameter $\overline{Gr}/\overline{Re}^2$ than is required for the smaller value. As the velocity profile exhibits a maximum within the boundary layer for predominantly free convective flows, it is not surprising that small integration steps are required to negotiate this maximum.

The second important variable in the integration is the value of the effective infinity, η_{MAX} . In practice, the asymptotic boundary condition will be satisfied more accurately as the effective infinity becomes progressively larger. However, once a certain value of η_{MAX} is reached, there is no improvement in the solution when the value of η_{MAX} is further increased.

In Table C.3., the initial conditions, $f_1''(0)$ and $F_0'(0)$, for the stagnation point solution are tabulated for various values of η_{MAX} . Table C.3.(a) presents the effect of η_{MAX} for a Schmidt number of 2.4 and a value of the parameter $\overline{Gr}/\overline{Re}^2$ of 1.0. The difference between the values of the two initial

conditions can be seen to decrease rapidly as the value of η_{MAX} is increased. Constant values are attained when the value of η_{MAX} is greater than 5.0. The corresponding results for a Schmidt number of 1000 are presented in Table C.3.(b). A similar situation is noted, and constant values of $f_1''(0)$ and $F_0'(0)$ are attained above a value of $\overline{\eta}_{MAX}$ of 50.0.

The parameter $\overline{Gr}/\overline{Re}^2$ does not have any influence on the value of η_{MAX} which is required for the integration. Therefore, throughout this work a value of η_{MAX} not smaller than 6.0 was used, so that a small safety margin was included.

For Schmidt numbers of 10.0 and smaller, all five first order equations were integrated simultaneously to the same η_{MAX} . For higher Schmidt numbers, at a pre-specified value of $\overline{\eta}$, the value of $F_0''(\overline{\eta})$ was arbitrarily set to zero for the rest of the integration. This was necessary because of the build - up of round - off errors which caused the solution to "blow - up". The solution "blew - up" very rapidly once the round - off errors were transmitted through to the hydrodynamic equations. The effects of varying the point at which the value of $F_0''(\overline{\eta})$ was set to zero are shown in Table C.4. for a Schmidt number of 1000.0. Once again, the value of the parameter $\overline{Gr}/\overline{Re}^2$ had no influence on the results.

The results for the initial conditions, $f_1''(0)$ and $F_0'(0)$ at various values of $\overline{\eta}_{MAX}$ are presented for three different values of $\overline{\eta}$ at which $F_0''(\overline{\eta})$ was arbitrarily set to zero. If $F_0''(\overline{\eta})$ was set to zero earlier than a value of $\overline{\eta} = 5.0$, no convergence was achieved. The solution "blew - up" if a value greater than 7.0 was chosen. As can be seen from the table, no differences between the values which were predicted for

$f_1''(0)$ and $F_0'(0)$ were found. Therefore, throughout the work, F_0'' was set to zero at $\bar{\eta} = 6.0$ for a Schmidt number of 1000.0. A corresponding value was chosen for other Schmidt numbers.

It is interesting to note that when $F_0''(\bar{\eta})$ is put to zero the assumption is made that the concentration is so small that it no longer affects the hydrodynamic equations. The value at which $F_0''(\bar{\eta})$ was set to zero is seen to be approximately $\eta_{\text{MAX}}/\text{Sc}^{\frac{1}{3}}$. This relationship was also used as a criterion for setting $F_0''(\bar{\eta})$ to zero by both Green⁷³ and Sanders⁸⁰.

Now that the accuracy of the numerical integration technique has been discussed, the method used to improve the guesses of the unknown initial conditions must be investigated to ensure that convergence to the correct solution has been achieved.

The use of the method of quasilinearization to solve non-linear two point boundary value problems has been discussed in varying detail, by a number of authors.^{57, 94, 95}

The present discussion will therefore be limited to a discussion of the applicability of the method to the present problem. One of the advantages of the method has been stated to be the very rapid, monotonic convergence to the final solution from very approximate initial guesses.

A series of graphs for convergence are shown in Figs. 6.1 to 6.5 for the front stagnation point problem at a Schmidt number of 2.4 and a value of the parameter $\overline{\text{Gr}}/\overline{\text{Re}}^2$ of 0.1. The set of curves which have been chosen show the results for a problem which almost became unstable. However, the fact that convergence was finally achieved indicates the power of the method.

The equations were integrated out to an effective infinity of 8.0 with an integration step length of 0.01. The

Fig. 6.1. The Successive Iterations in the Determination of f_1 by Quasilinearization. 157

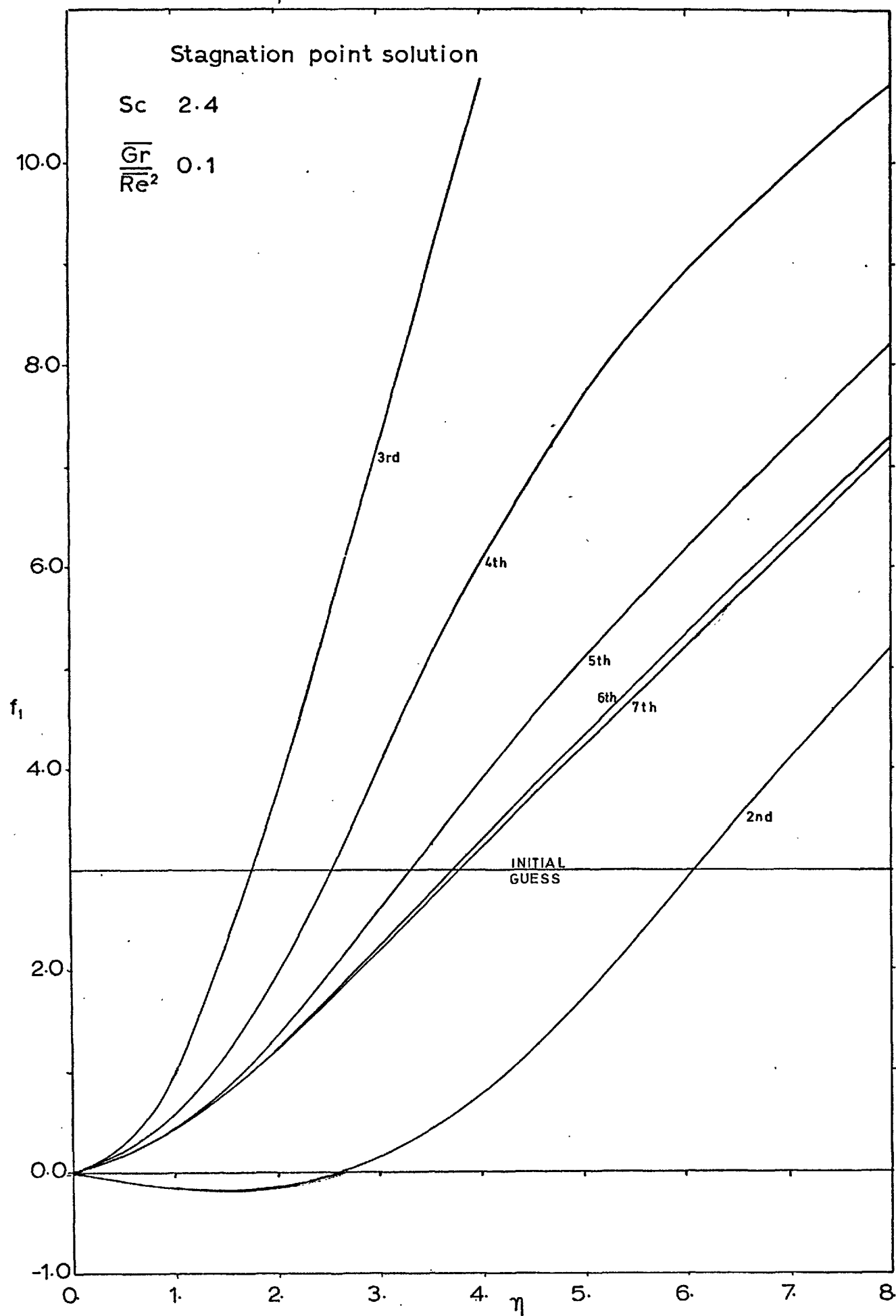
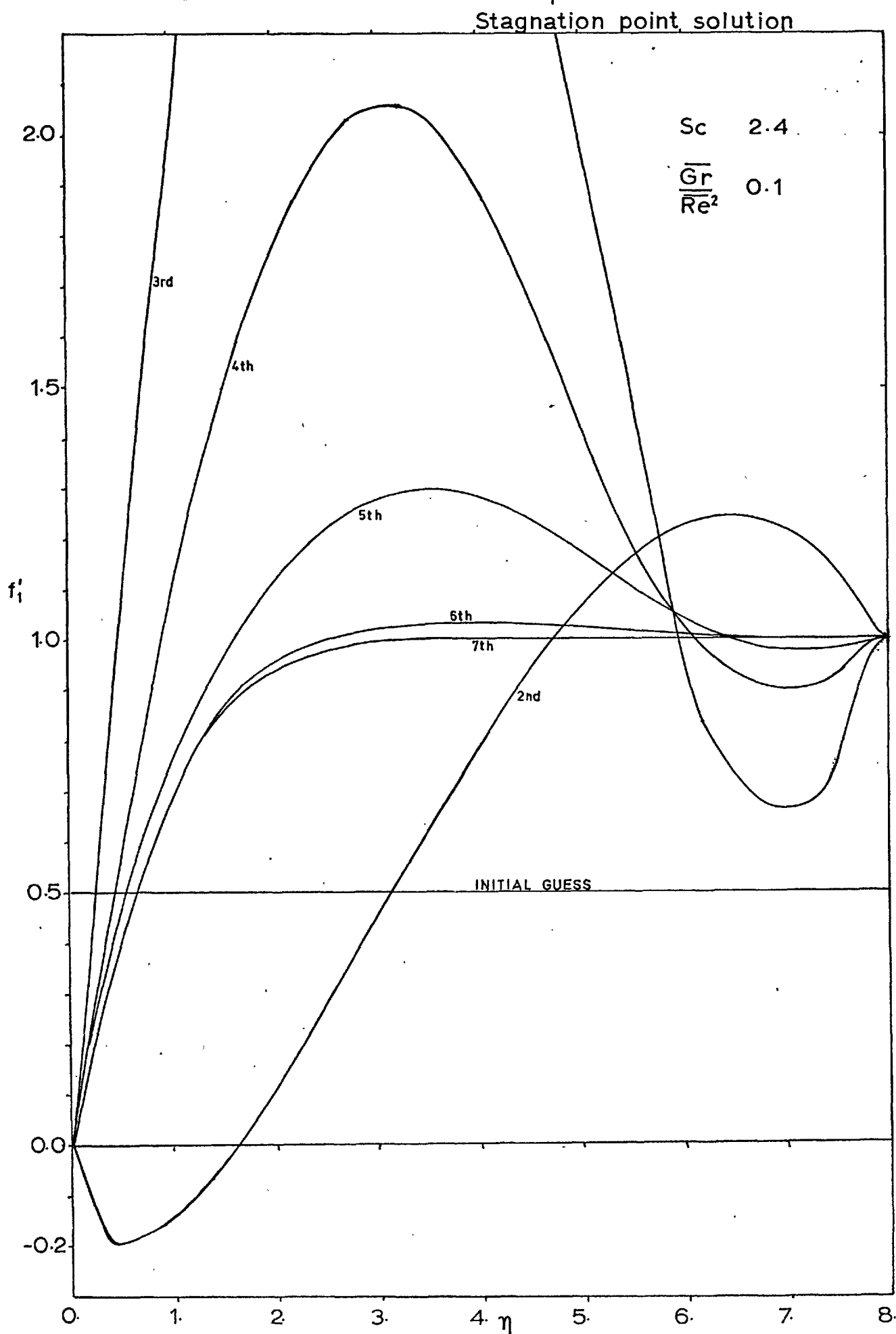
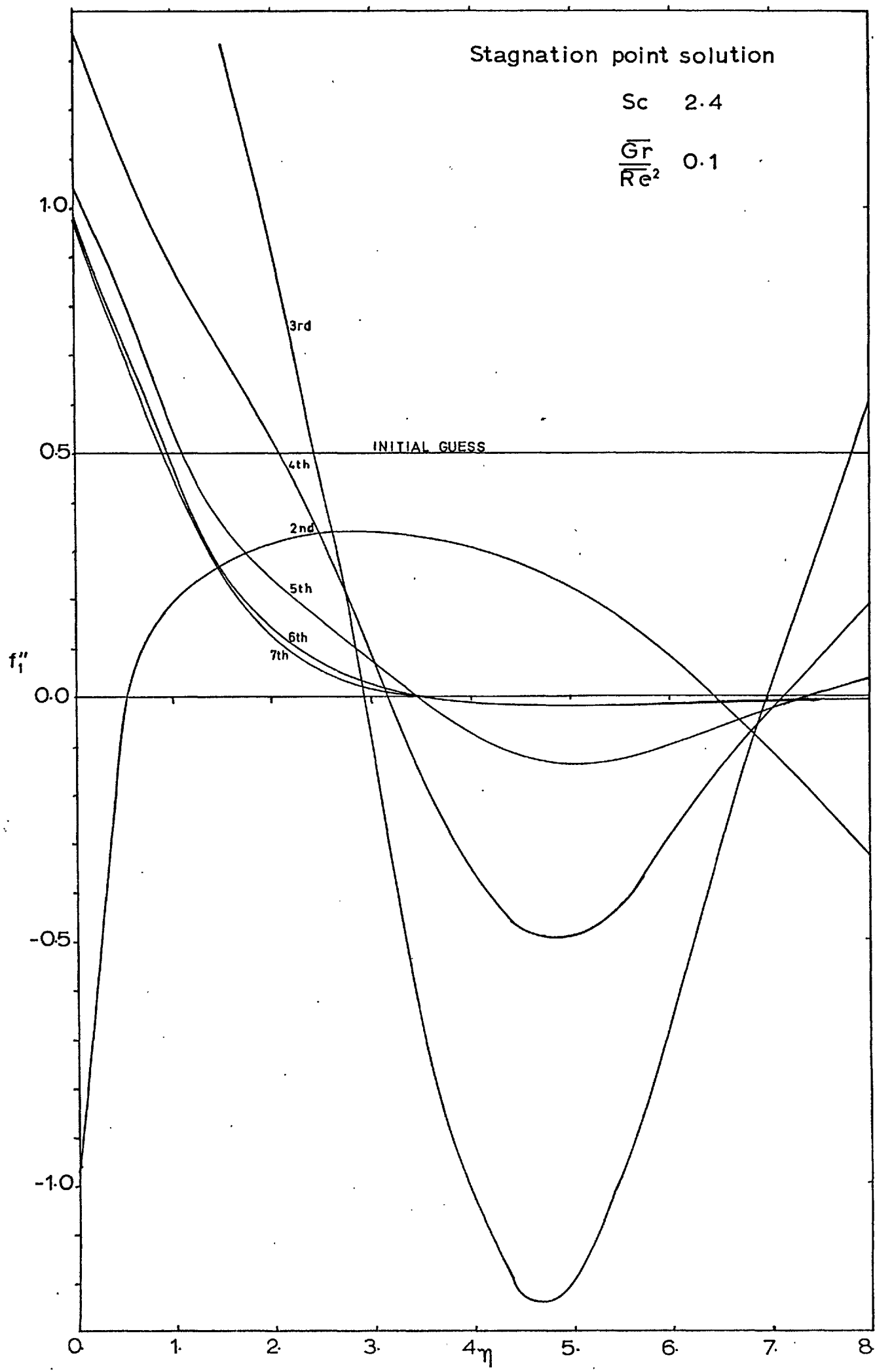
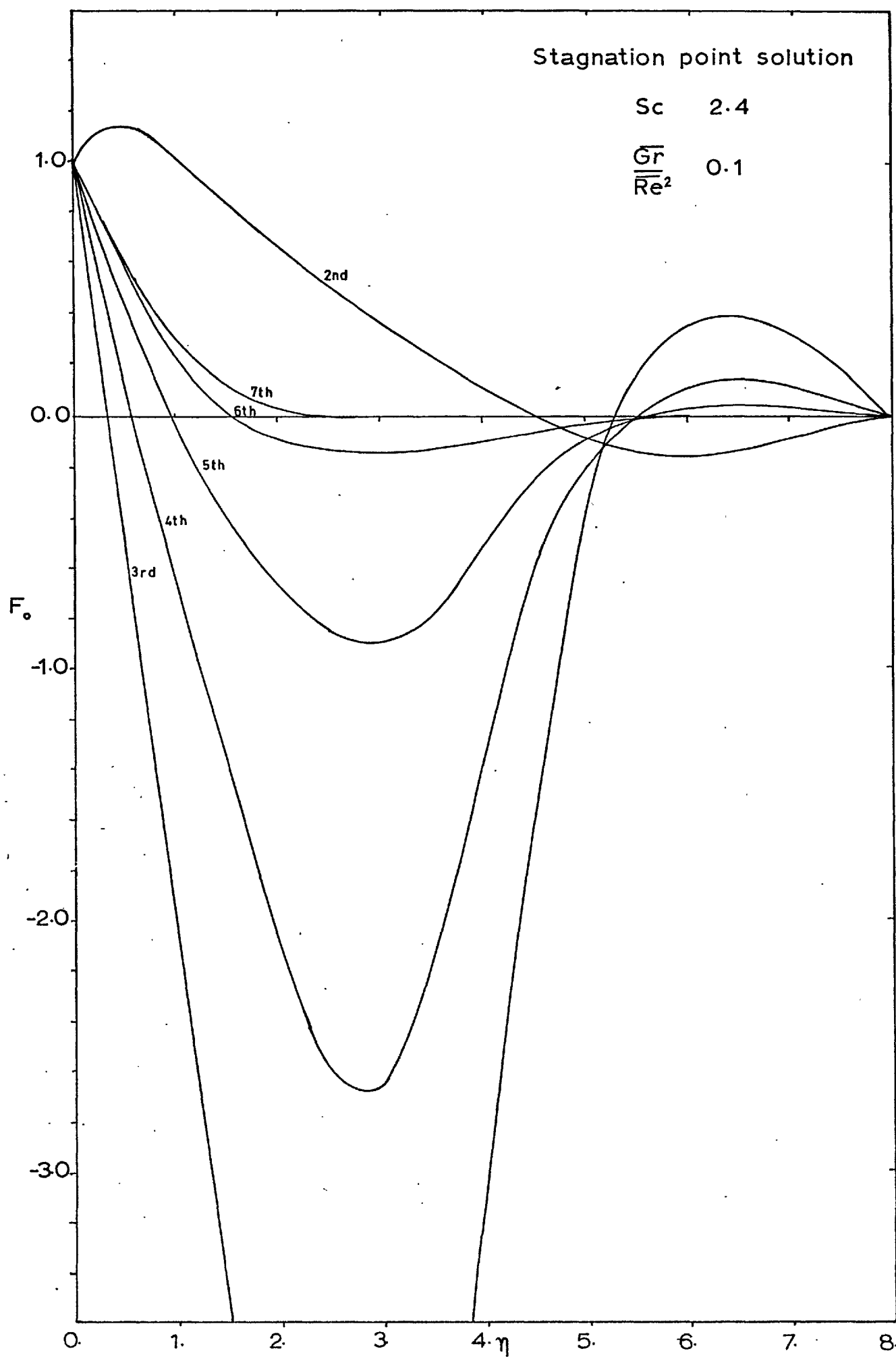
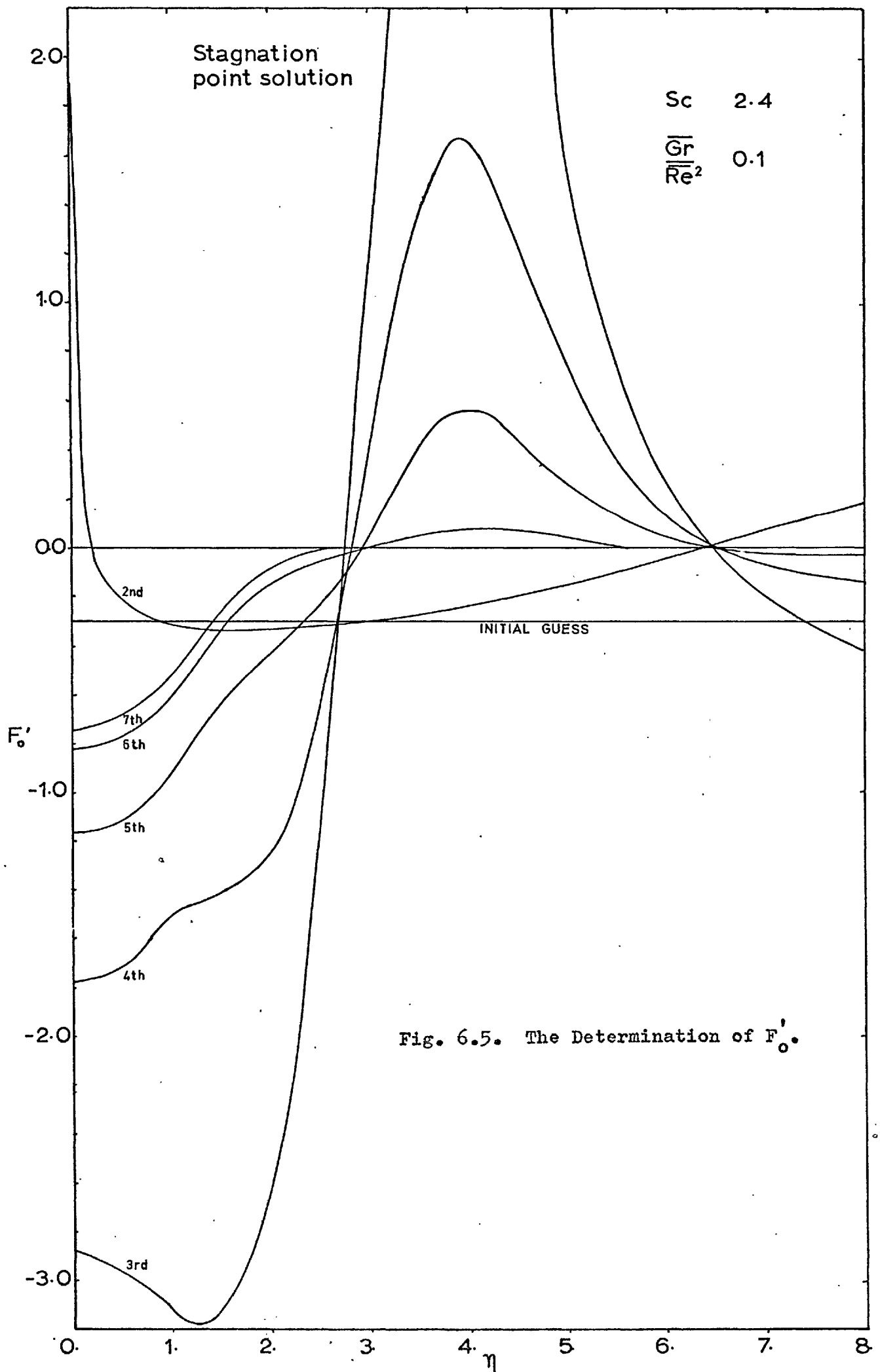


Fig. 6.2. The Determination of f_1' .









curves show that convergence is achieved after seven iterations. The initial guess is called the first iteration. The second iteration produces a very poor solution for all the variables, f_1 , f_1' , f_1'' , F_0 , and F_0' . The third and subsequent iterations can be seen to converge extremely rapidly to the final solution.

Convergence was usually achieved after four or five iterations with a convergence criterion of 1×10^{-6} . For unstable problems, stability could generally be achieved by initially integrating to a small value of η_{MAX} . Then, once convergence had been obtained, the effective infinity was increased and the quasilinearization method was used to update the extended solution.

Fig. 6.6 shows the successive approximations of f_1' with $\eta_{MAX} = 8.0$ for a Schmidt number of 2.4 and a value of the parameter $\overline{Gr}/\overline{Re}^2$ of 1.0.

Although a reasonable initial guess was introduced, the solution began to oscillate, and then after seven iterations it "blew - up". For the same problem, if η_{MAX} is initially set to 4.0, convergence is achieved in four iterations. If the effective infinity is then extended to 8.0 and the quasilinearization method is used again, convergence is immediately obtained. This is shown very clearly in Fig. 6.7 since all the curves after the fourth iteration are superimposed.

This method of solution was found to be necessary for all Schmidt numbers less than 20.0 for values of the parameter $\overline{Gr}/\overline{Re}^2$ greater than approximately unity.

When a large number of solutions were calculated during a single computer run, a second method was employed to gain rapid convergence. The converged solution for one problem was used as the initial guess for the subsequent problem. In

Stagnation point solution

Sc. 2.4

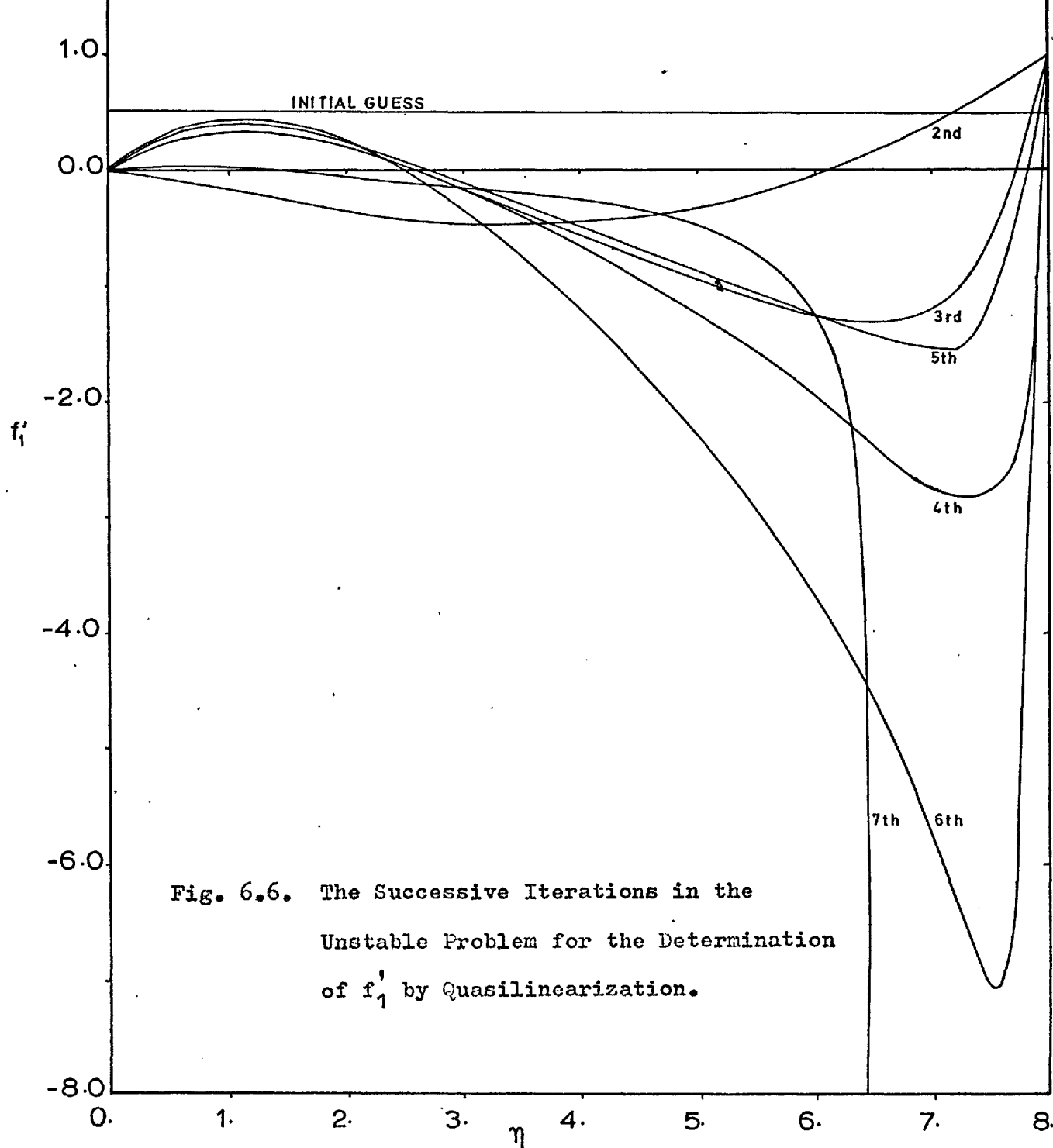
 $\frac{\overline{Gr}}{Re^2}$ 1.0

Fig. 6.6. The Successive Iterations in the Unstable Problem for the Determination of f_1' by Quasilinearization.

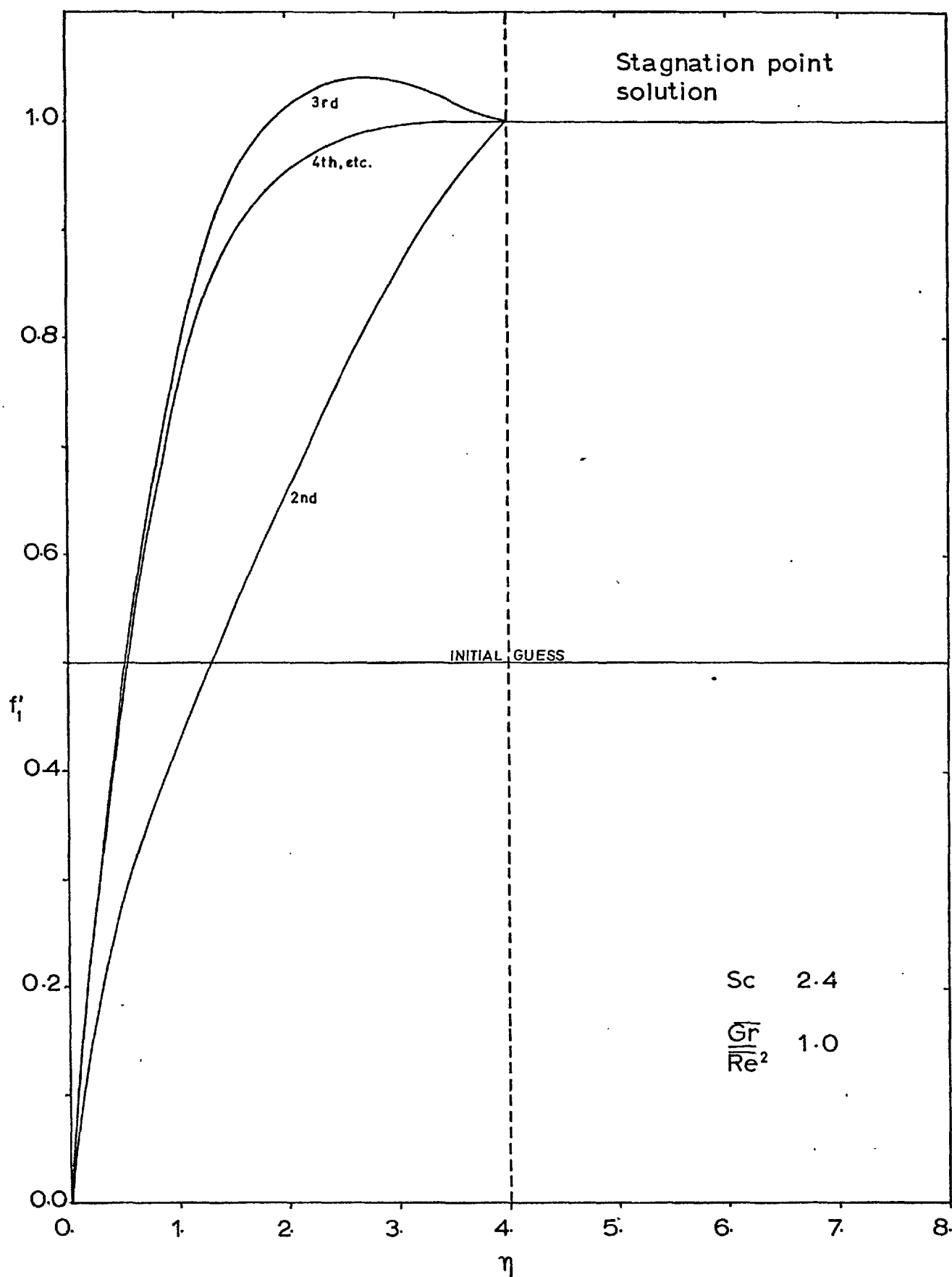


Fig. 6.7. The Successive Iterations in the Stabilized Problem for the Determination of f_1' .

this way, convergence was obtained in three or four iterations. This technique obviated the need to use intermediate values of η_{MAX} .

The solution was said to have converged when two subsequent iterations using the quasilinearization technique, produced initial conditions, $f_1''(0)$ and $F_0'(0)$ which were identical to within a pre-specified accuracy. The effects of altering this convergence criterion are shown in Table C.5 for Schmidt numbers of 1.0 and 1000.0 at values of the parameter $\overline{Gr}/\overline{Re}^2$ of 5.0 and 0.1 respectively.

These results probably show most clearly the rapid convergence of the quasilinearization technique to a solution. Five different convergence criteria were investigated, from 1×10^{-2} to 1×10^{-10} . The tabulated results show that in almost all of the cases studied, for both high and low Schmidt numbers, convergence of $f_1''(0)$ and $F_0'(0)$ to ten decimal places was achieved even when the stated convergence was only 1×10^{-6} or 1×10^{-8} . Even a specified accuracy of 1×10^{-2} produced extremely accurate predictions of the two unknown boundary conditions.

The accuracy of the quasilinearization technique was investigated in two ways. First by comparing the initial conditions which were predicted using the quasilinearization method with previously published results. Secondly, the non-linear equations were integrated using the initial conditions predicted from the quasilinearization technique, and the two solutions were compared.

In Table C.6.(a) the initial conditions for the forced convection regime are compared with those of Green⁷³, who used a trial and error method followed by linear interpolation. The agreement can be seen to be excellent. Similarly, the

results for the vertical flat plate are compared with those of Lloyd and Sparrow⁵⁴ in Table C.6.(b). The agreement between the two sets of results is again excellent.

The results presented in Table C.7.(a),(b),(c), and(d) show that for both large and small Schmidt numbers the solutions which are obtained by the use of the quasilinearization technique, and also by the integration of the non-linear equations are identical. These investigations show conclusively that the converged solution, which is obtained when the quasilinearization technique is used, is the solution of the non-linear problem. Any inaccuracies are introduced by virtue of the numerical techniques which are employed either for the numerical integration, or in the calculation of the updated solution.

6.2. The Velocity and Concentration Profiles.

The form of the velocity and concentration profiles showed exactly the same features for both the approximate and the exact methods of solution for the stagnation point solution and the vertical flat plate. Some comparisons between predicted velocity and concentration profiles for the stagnation point solution, which were calculated by the use of the approximate and exact methods are shown in Figs. 6.8 and 6.9. It is clear from these figures that the profiles predicted by the approximate method are in very good agreement with the exact profiles. Thus, although Figs. 6.10 to 6.30 represent the curves for the exact stagnation point solutions, the discussion which follows is applicable to all the other results.

Since the hydrodynamic and transfer equations are coupled, the velocity profile^{is} affected by the Schmidt number. However, the magnitude of the coupling is also directly influenced by

Profiles for the Stagnation Point Solution.

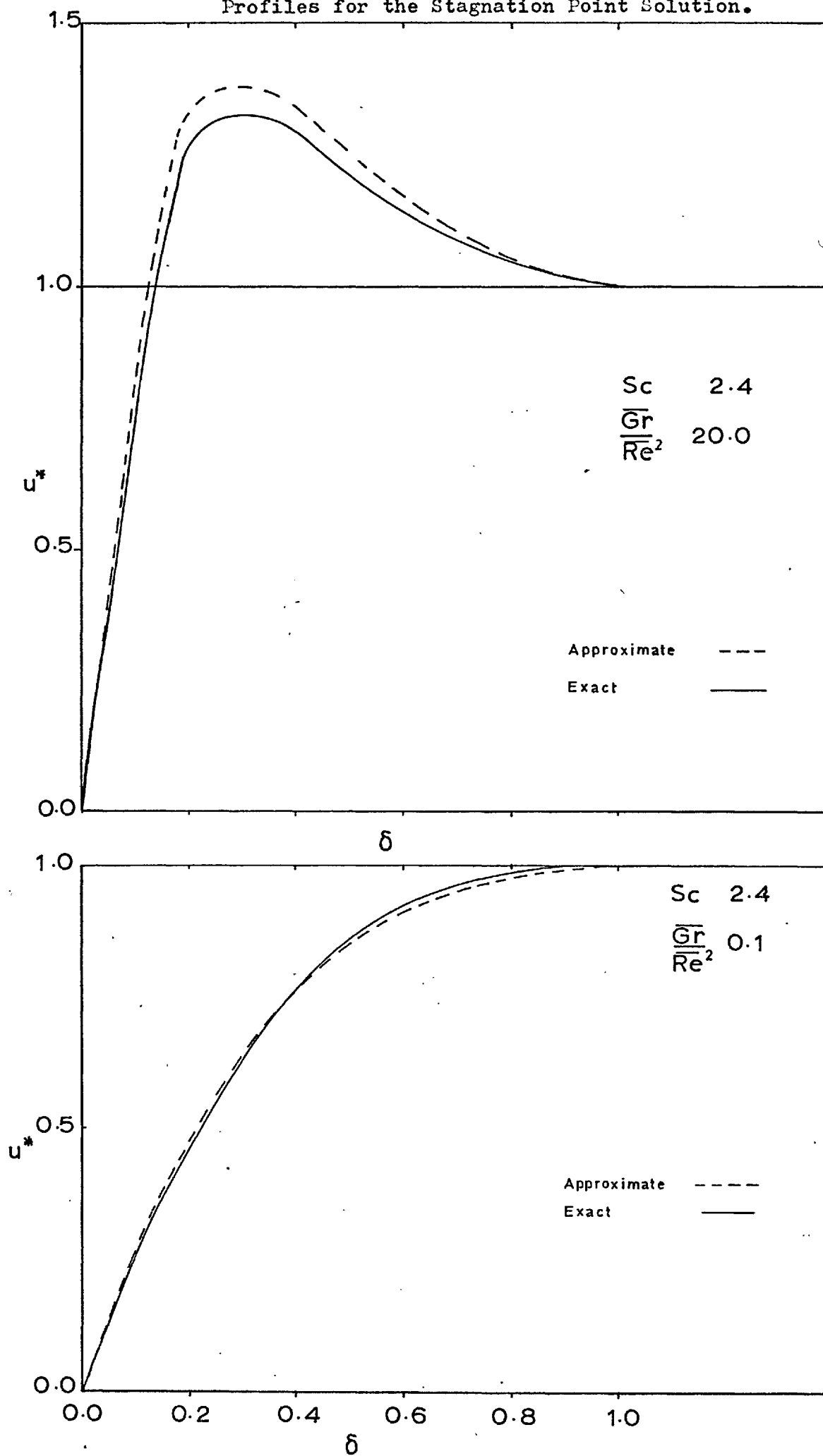
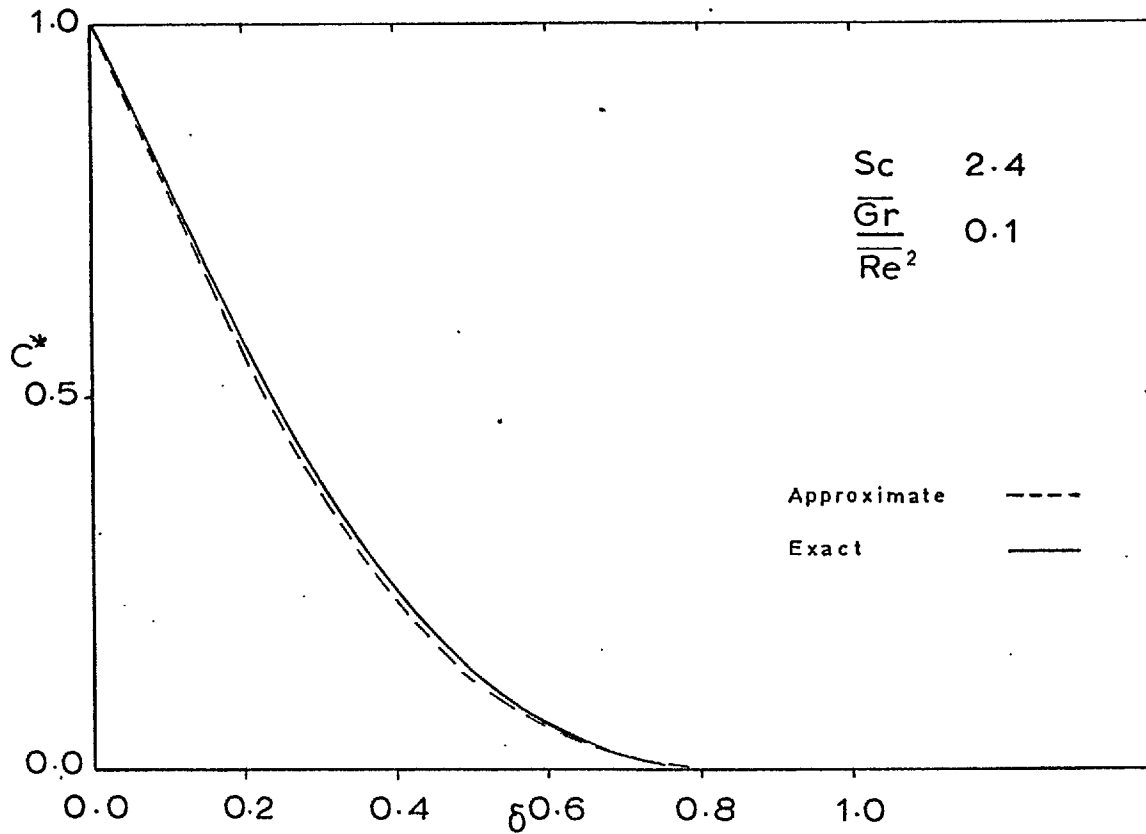
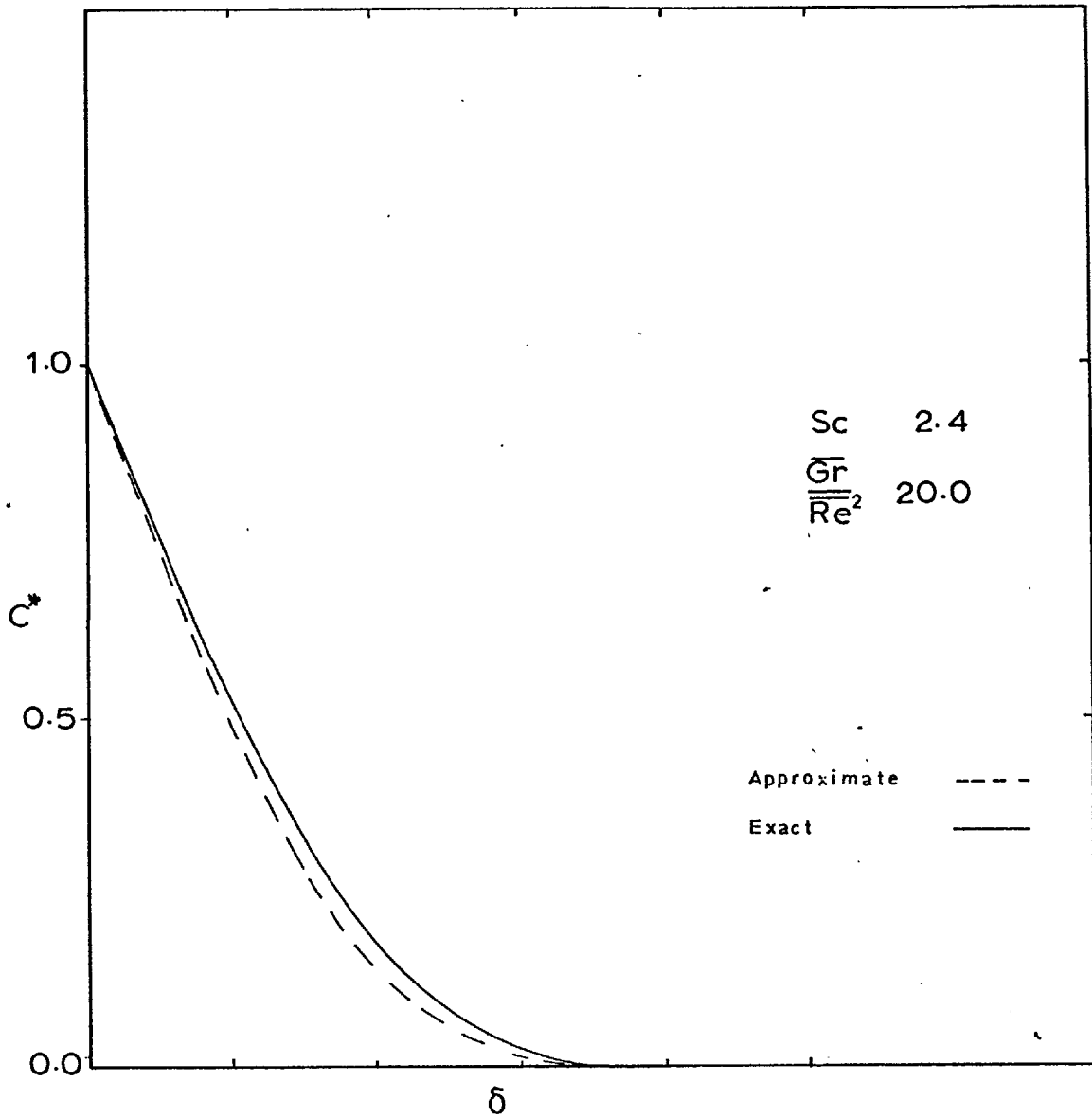


Fig. 6.9. A Comparison between Exact and Approximate Concentration Profiles for the Stagnation Point Solution. 168



the value of the parameter Gr/Re^2 . Therefore, it is difficult to discuss the influence of one of the parameters without the other. In the present section the influence of the parameter Gr/Re^2 will be discussed in detail, and will then be followed by a brief discussion of the influence of the Schmidt number. Because of the inter dependence of the effects of these two parameters, some of the latter discussion will draw on points noted in the more detailed discussion of the influence of the parameter Gr/Re^2 .

6.2.1. The Influence of the Parameter Gr/Re^2 , Aiding Flows.

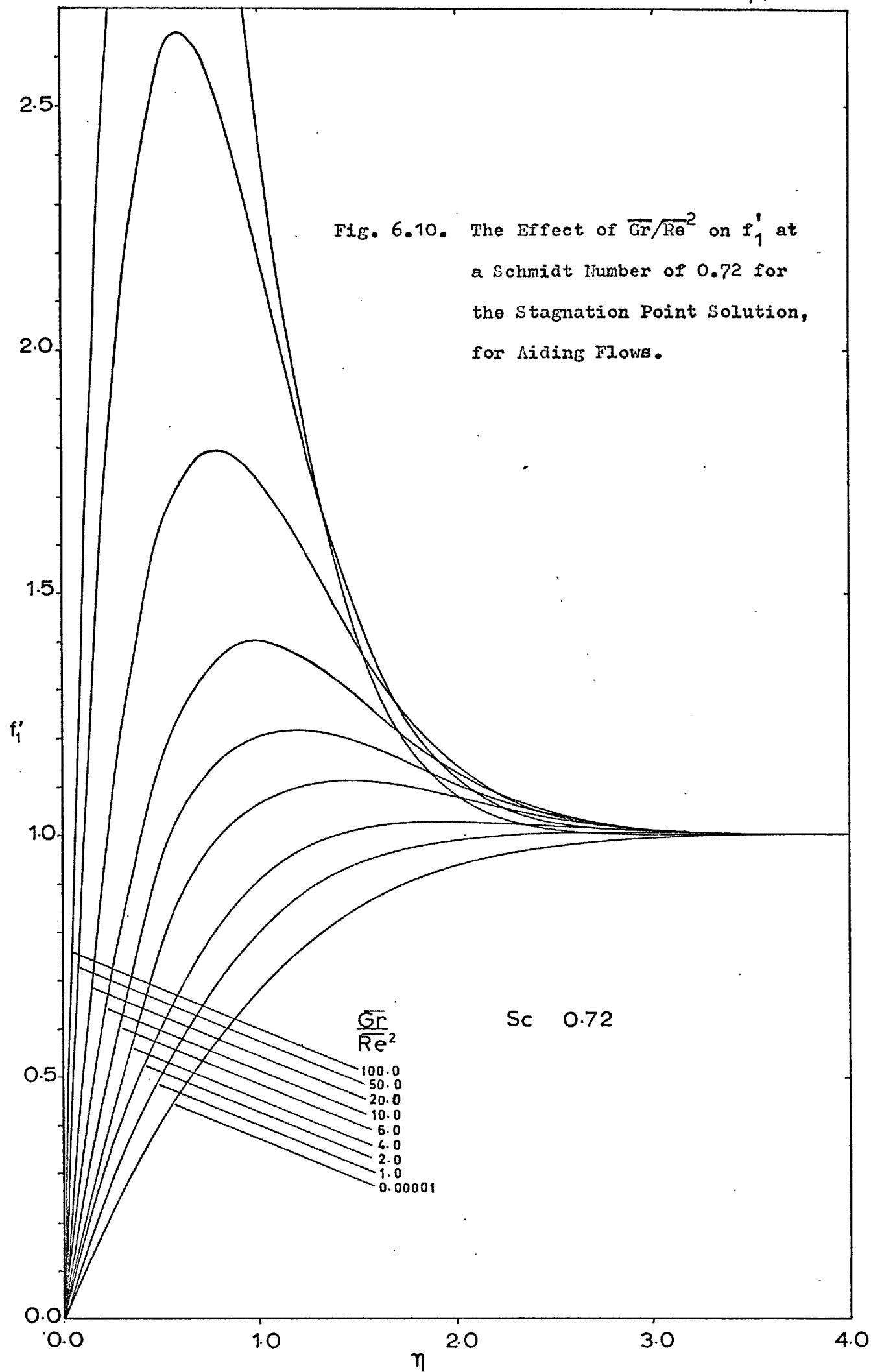
The values of $f_1'(\eta)$ and $F_0(\eta)$, which represent the dimensionless velocity and concentration respectively, were determined by the use of the quasilinearization technique. As discussed previously, for Schmidt numbers less than 50.0, a constant integration step length of 0.01 was employed. For Schmidt numbers of 100.0 and 1000.0, variable integration step lengths were used. The values of these step lengths were 0.01, 0.05, 0.1, and 0.5. A value of 8.0 was chosen for the effective infinity, η_{MAX} . The results are only plotted to $\eta = 4.0$ because there were no changes in any of the variables from $\eta = 4.0$ to $\eta = 8.0$.

The velocity and concentration profiles are presented for a range of values of the parameter $\overline{Gr}/\overline{Re}^2$ of 0.00001 to 100.0 at Schmidt numbers of 0.72, 2.4, 10.0, and 1000.0 in Figs. 6.10 to 6.17. The plots of the velocity distributions, $f_1'(\eta)$, for a Schmidt number of 0.72 are shown in Fig. 6.10. From these curves it can be seen that when the parameter $\overline{Gr}/\overline{Re}^2$ has a value of between 1.0 and 2.0, there is a velocity within the boundary layer greater than that of the free stream. The

corresponding plots of the concentration profile, $F_o(\eta)$, for a Schmidt number of 0.72 are shown in Fig. 6.11. These show that the transfer boundary layer becomes thinner as the parameter $\overline{Gr}/\overline{Re}^2$ increases. When the parameter has a value of 1.0, the value $F_o(\infty)$ is reached at a value of η of approximately 4.0. However, when the value of the parameter is 100.0, $F_o(\infty)$ is reached at a value of η of approximately 2.5. As the transfer boundary layer becomes thinner, there is a corresponding effect in the velocity profiles. As the parameter $\overline{Gr}/\overline{Re}^2$ increases, the maximum in the velocity profile occurs at a smaller value of η , i.e. nearer to the body surface. The velocity then approaches its asymptotic value more rapidly than at smaller values of the parameter $\overline{Gr}/\overline{Re}^2$. This occurs because as the parameter $\overline{Gr}/\overline{Re}^2$ increases the influence of free convection is becoming confined to a thinner layer adjacent to the surface.

The same behaviour as noted above is apparent for the higher Schmidt numbers. However, as the Schmidt number increases the value of the parameter $\overline{Gr}/\overline{Re}^2$ at which there is a maximum velocity within the boundary layer also increases. In Fig. 6.12, the plots of $f_1'(\eta)$ for a Schmidt number of 2.4 are presented. The maximum in the profile is first noted at a value of the parameter $\overline{Gr}/\overline{Re}^2$ of approximately 6.0. The reduction in the transfer boundary layer thickness can be seen in Fig. 6.13 as the influence of free convection increases. The reduction in the transfer boundary layer thickness is also reflected in the velocity plots. This has been discussed above for a Schmidt number of 0.72.

The velocity and concentration profiles for a Schmidt number of 10.0 are shown in Figs. 6.14 and 6.15. For this Schmidt number, the maximum velocity within the boundary layer



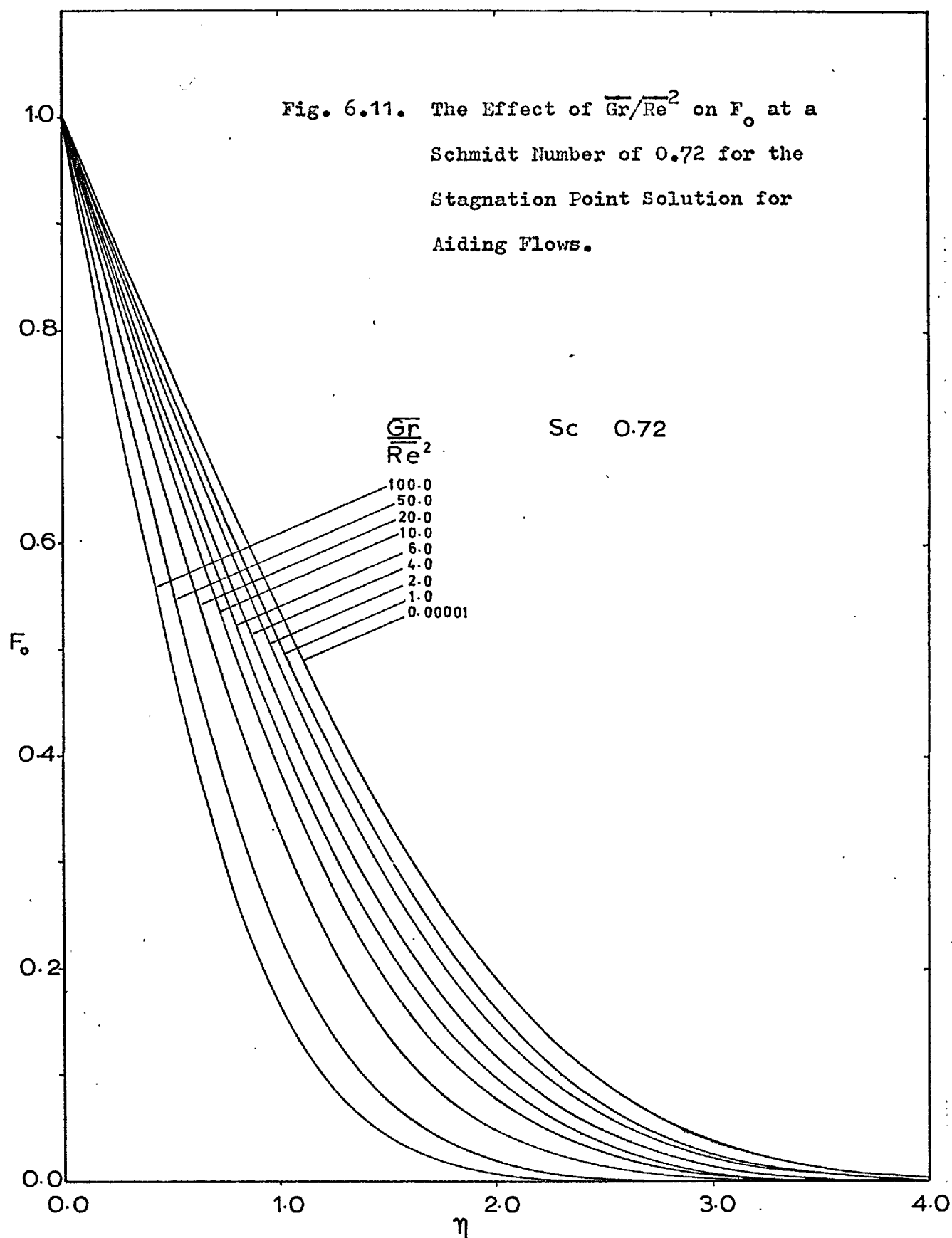
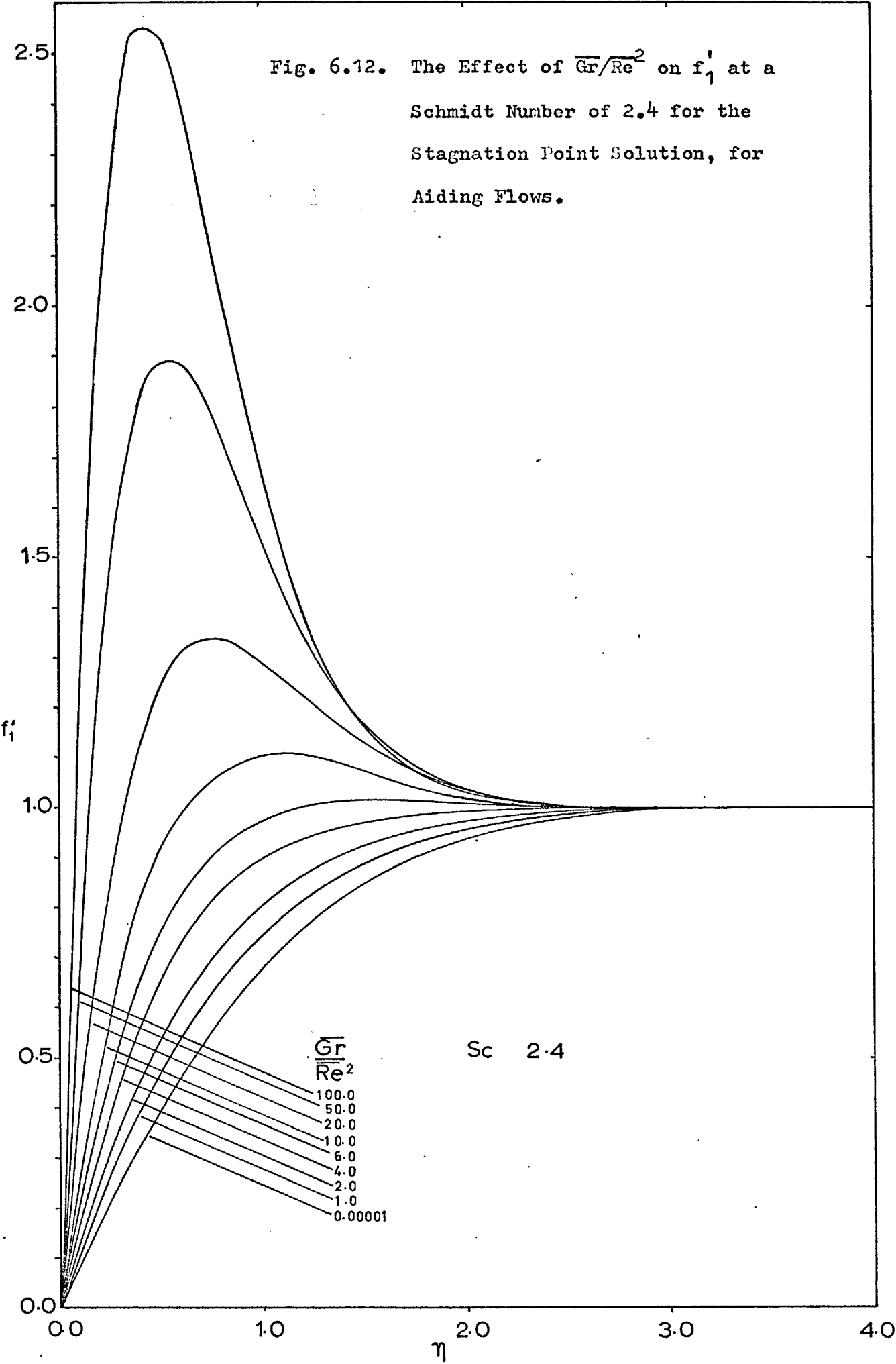


Fig. 6.12. The Effect of $\overline{Gr}/\overline{Re}^2$ on f'_η at a Schmidt Number of 2.4 for the Stagnation Point Solution, for Aiding Flows.



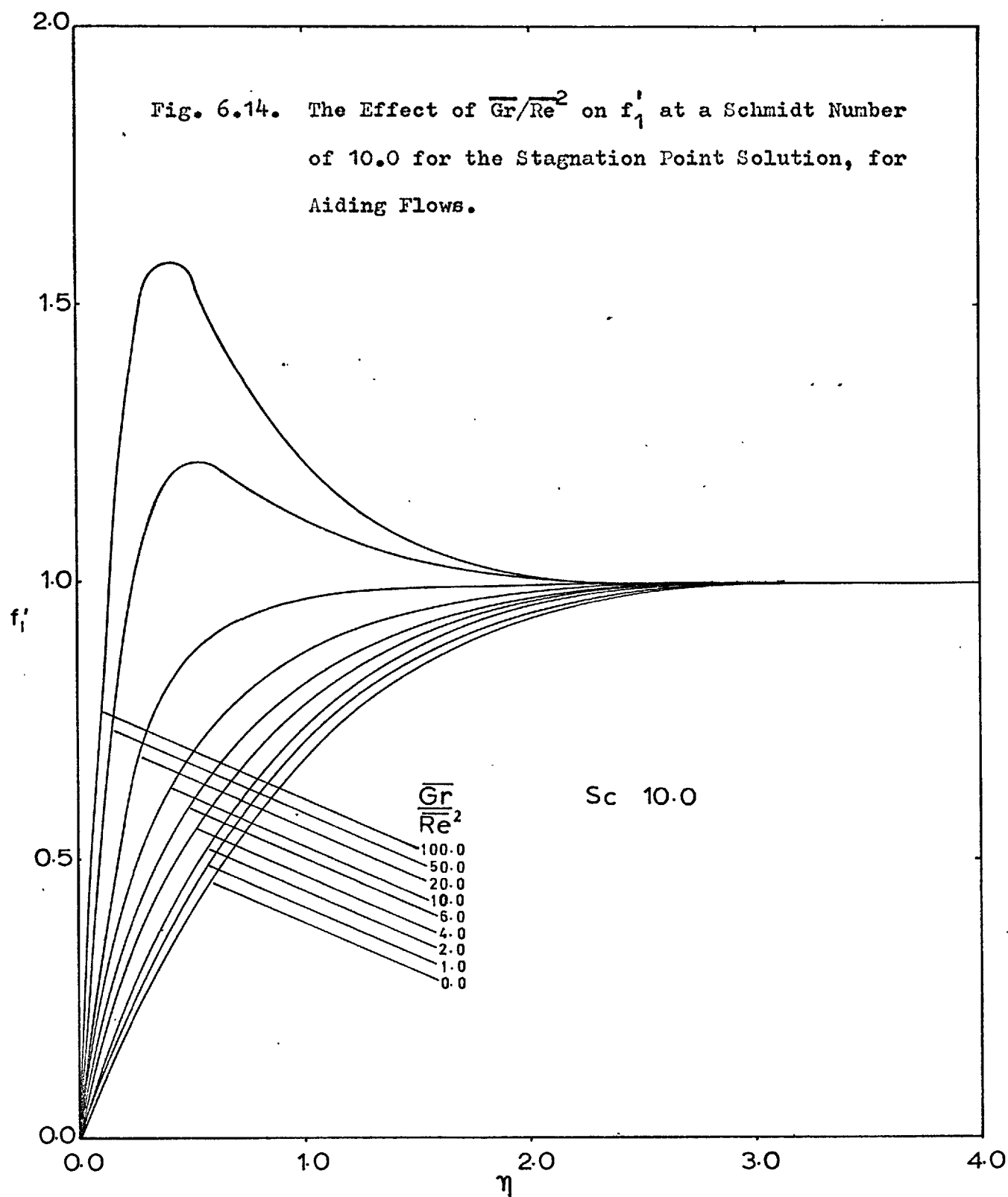
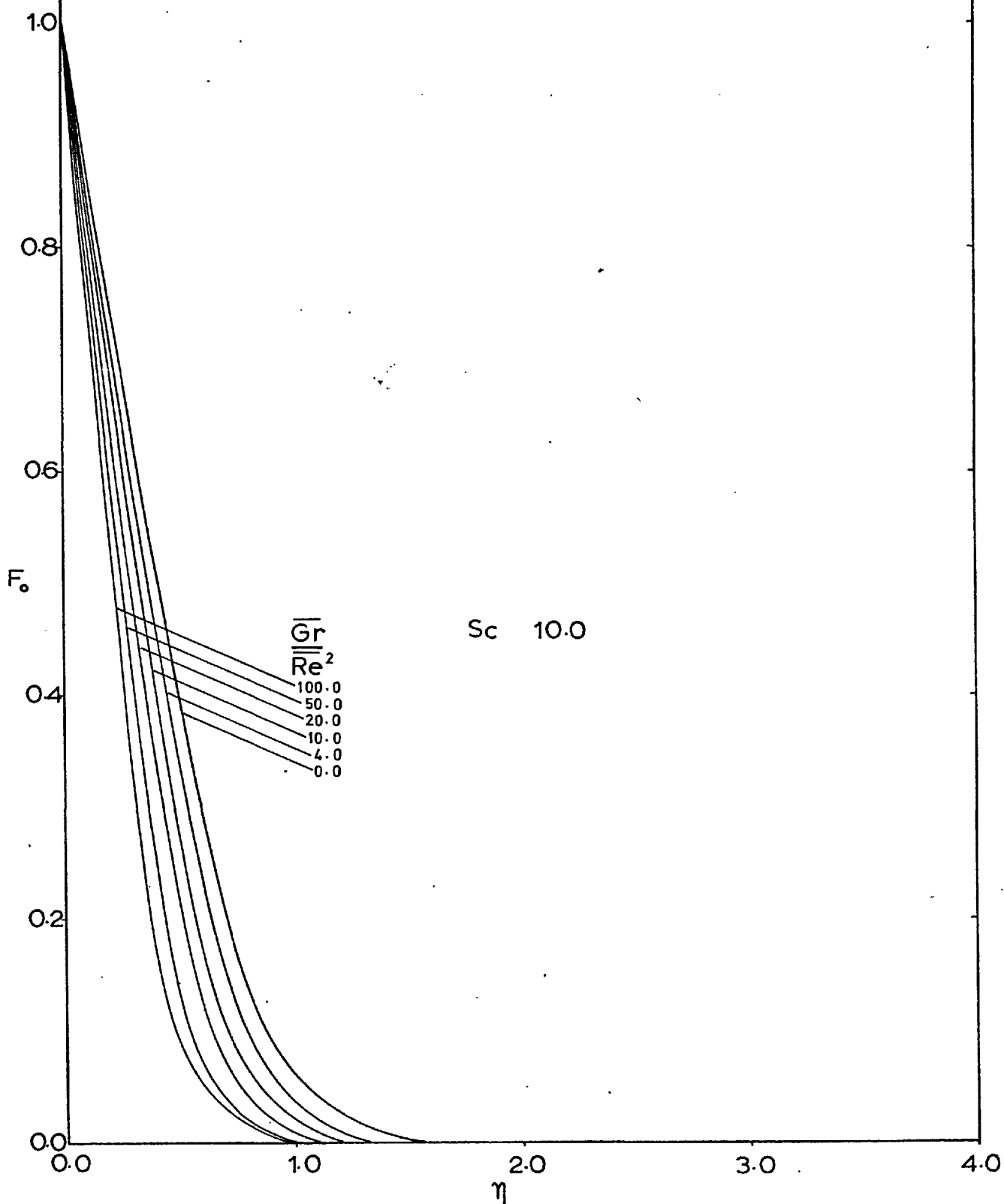
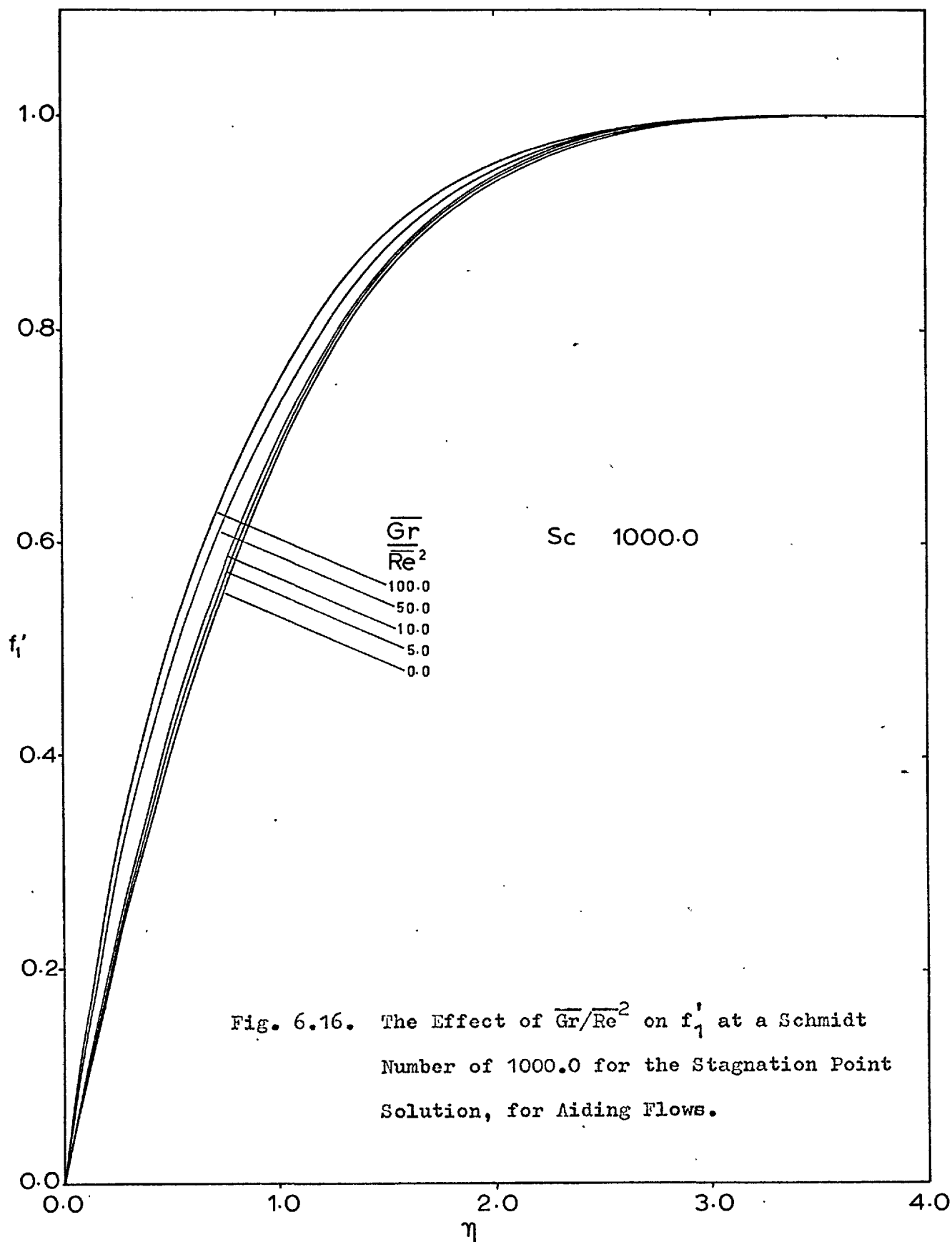
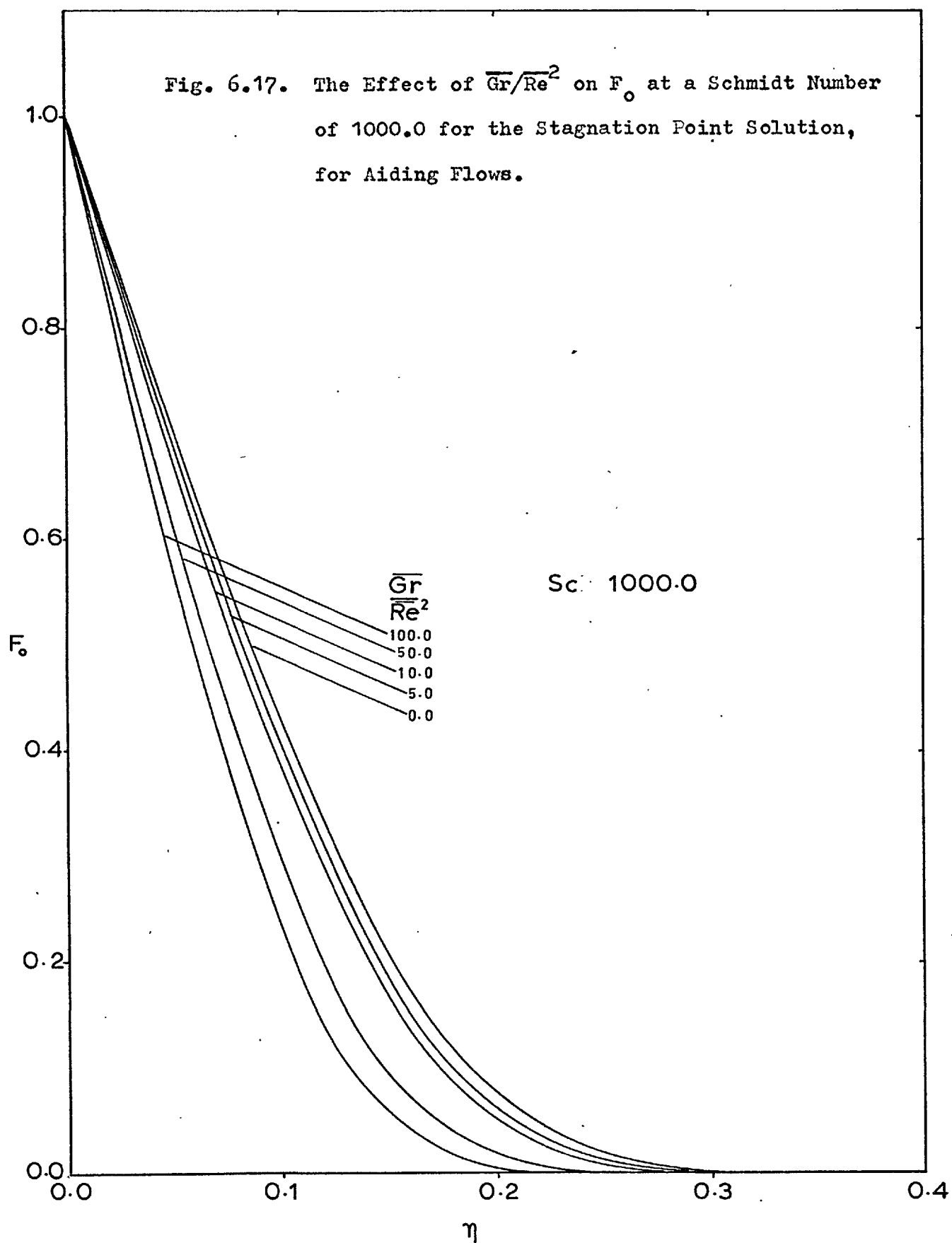


Fig. 6.15. The Effect of $\overline{Gr}/Re^2$ on F_0 at a Schmidt Number of 10.0 for the Stagnation Point Solution, for Aiding Flows.







is not attained until the parameter $\overline{Gr}/Re^2$ has a value of between 10.0 and 20.0. The reduction of the transfer boundary layer thickness is again noticed, together with the corresponding movement of the maxima in the curves of the velocity profiles towards the surface. However, since the concentration changes are becoming confined to a much thinner layer than for low Schmidt numbers, the behaviour, observed above, as the velocity becomes asymptotic to the free stream is not as noticeable.

The final curves in this set, Figs. 6.16 and 6.17, are for a Schmidt number of 1000.0. They show that a velocity greater than the free stream velocity is not present even at a value of the parameter $\overline{Gr}/Re^2$ of 100.0. Moreover, as can be seen from the curves in Fig. 6.16, it is probable that a very much larger value of the parameter would be required before such a maximum could be achieved. The reduction in the transfer boundary layer thickness is again observed in the plots for a Schmidt number of 1000.0, Fig. 6.17. The scale of the abscissa in this graph has been increased to present the curves at the various values of the parameter $\overline{Gr}/Re^2$.

6.2.2. The Influence of the Parameter $\overline{Gr}/Re^2$, Opposing Flows.

The opposing flow problem was studied until backflow was observed. Some solutions in which there is a reverse flow next to the body surface have been calculated. However, as the parameter $\overline{Gr}/Re^2$ was increased beyond the point at which backflow was first observed, considerable care was required to maintain stability in the quasilinearization solution procedure.

The velocity and concentration profiles, $f_1'(\eta)$ and $F_0(\eta)$

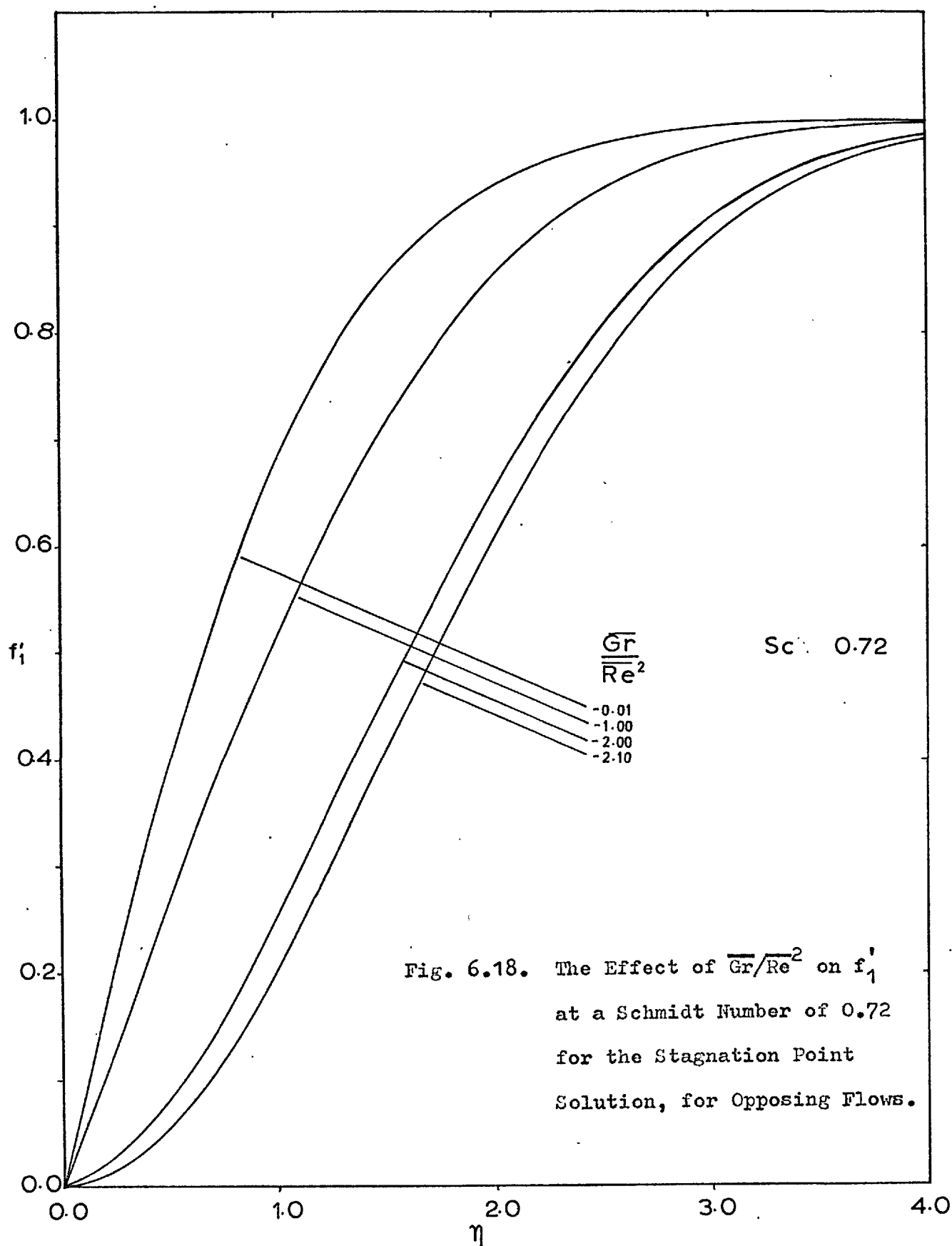
for opposing flows are shown in Figs. 6.18 to 6.25 for Schmidt numbers of 0.72, 2.4, 10.0 and 1000.0. When the value of the parameter $\overline{Gr}/\overline{Re}^2$ is very small the velocity and concentration profiles are those for pure forced convection. This was noted previously in the case of aiding flows. For larger values of the parameter $\overline{Gr}/\overline{Re}^2$ all the plots show that when the forces are in opposition, both the hydrodynamic and transfer boundary layers are thicker than when the forces are acting in the same direction.

As in the case of aiding flows, the effect of free convection is clearly more pronounced for the lower Schmidt numbers. Hence, as expected, the appearance of backflow occurs at lower values of the parameter $\overline{Gr}/\overline{Re}^2$ than for high Schmidt numbers.

Fig. 6.18 shows that for a Schmidt number of 0.72 backflow is first observed at a value of the parameter $\overline{Gr}/\overline{Re}^2$ of approximately 2.2. The figures 6.18 and 6.19 clearly show the increase in the two boundary layer thicknesses. For pure forced convective flows $f_1'(\eta)$ reaches 99% of the asymptotic value $f_1'(\infty)$ at a value of η of approximately 2.8. Whereas, at the point at which backflow appears, $f_1'(\eta)$ does not reach 99% of $f_1'(\infty)$ until η is approximately 5.0.

For a Schmidt number of 2.4 backflow is seen to occur at a value of the parameter $\overline{Gr}/\overline{Re}^2$ of approximately 2.7. Since, as the Schmidt number increases, the influence of free convection is becoming confined to a thinner layer next to the body, the thickening of both boundary layers is not as pronounced as for the lower Schmidt numbers. In this case, $f_1'(\eta)$ reaches 99% of $f_1'(\infty)$ at a value of η of approximately 3.5 when the parameter $\overline{Gr}/\overline{Re}^2$ has a value of 2.7.

When the Schmidt number is 10.0, backflow does not appear until the parameter $\overline{Gr}/\overline{Re}^2$ has reached a value of approximately



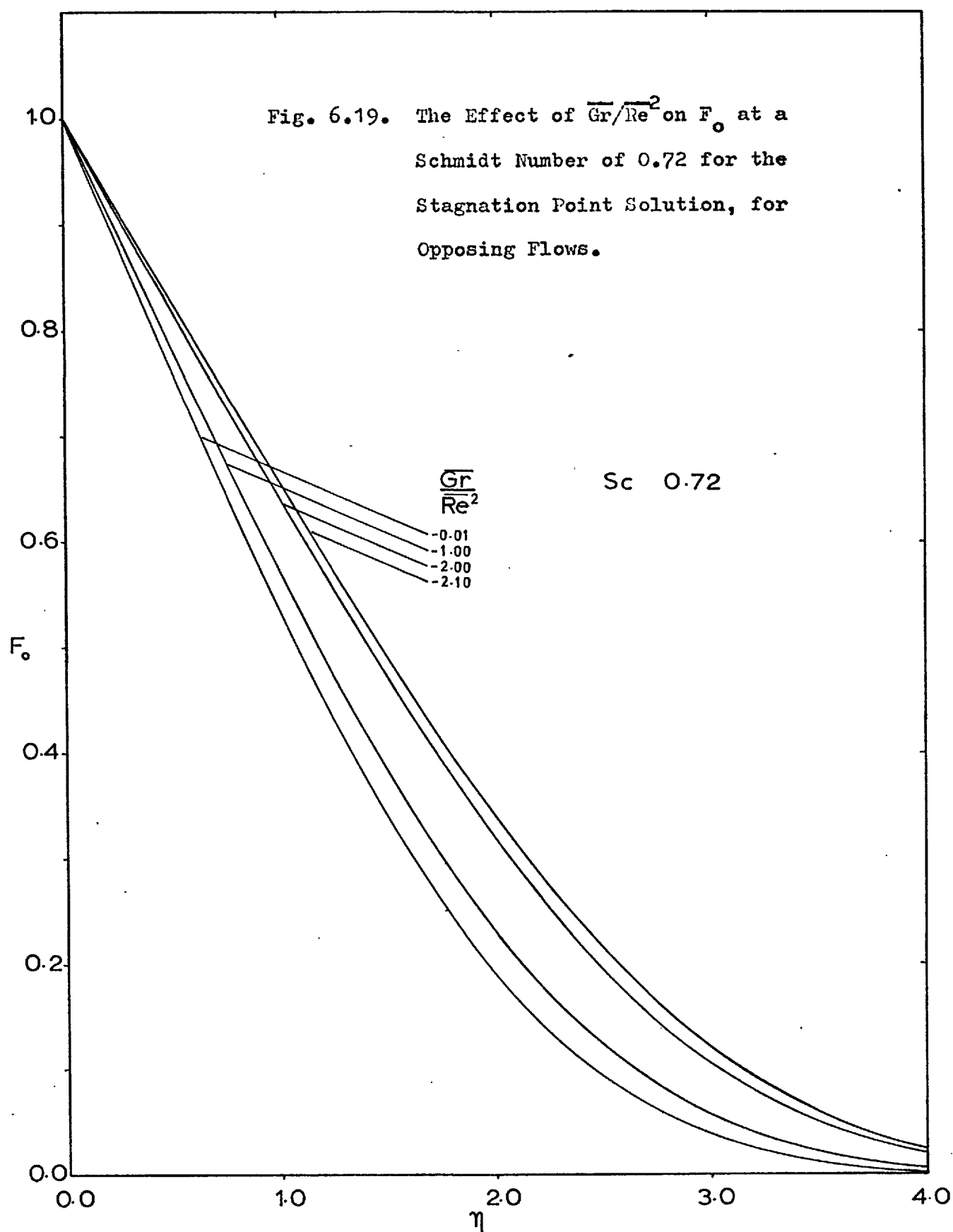
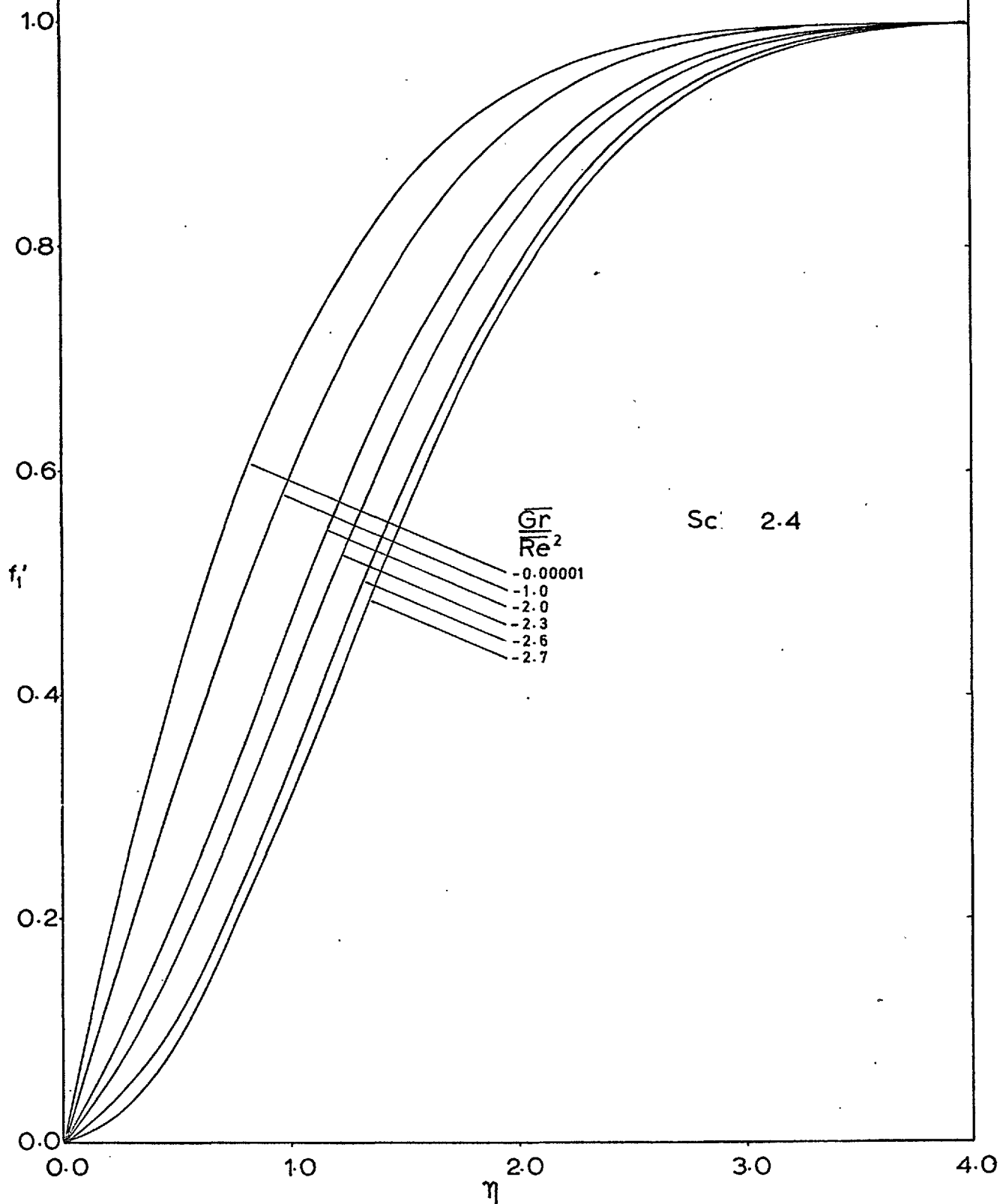
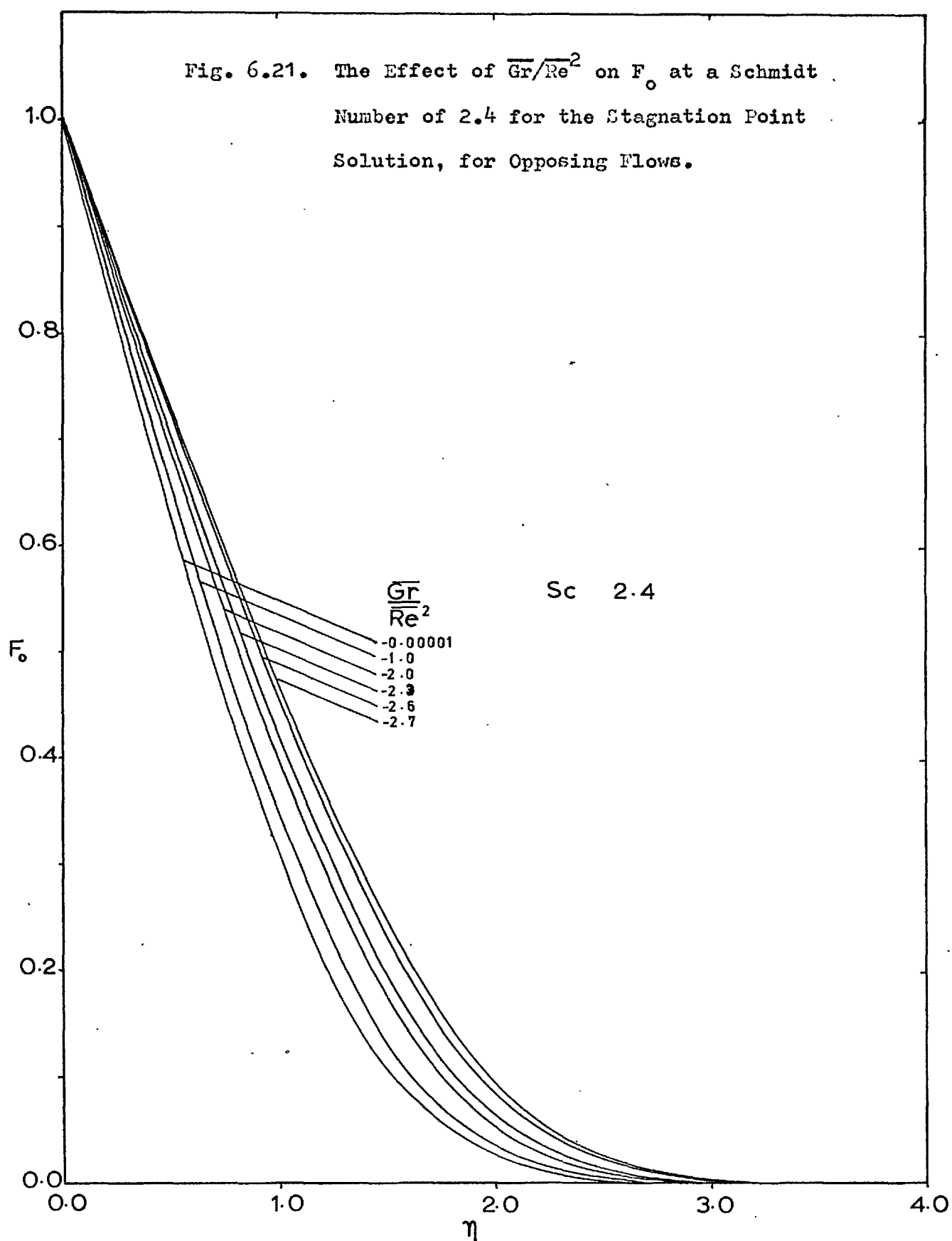
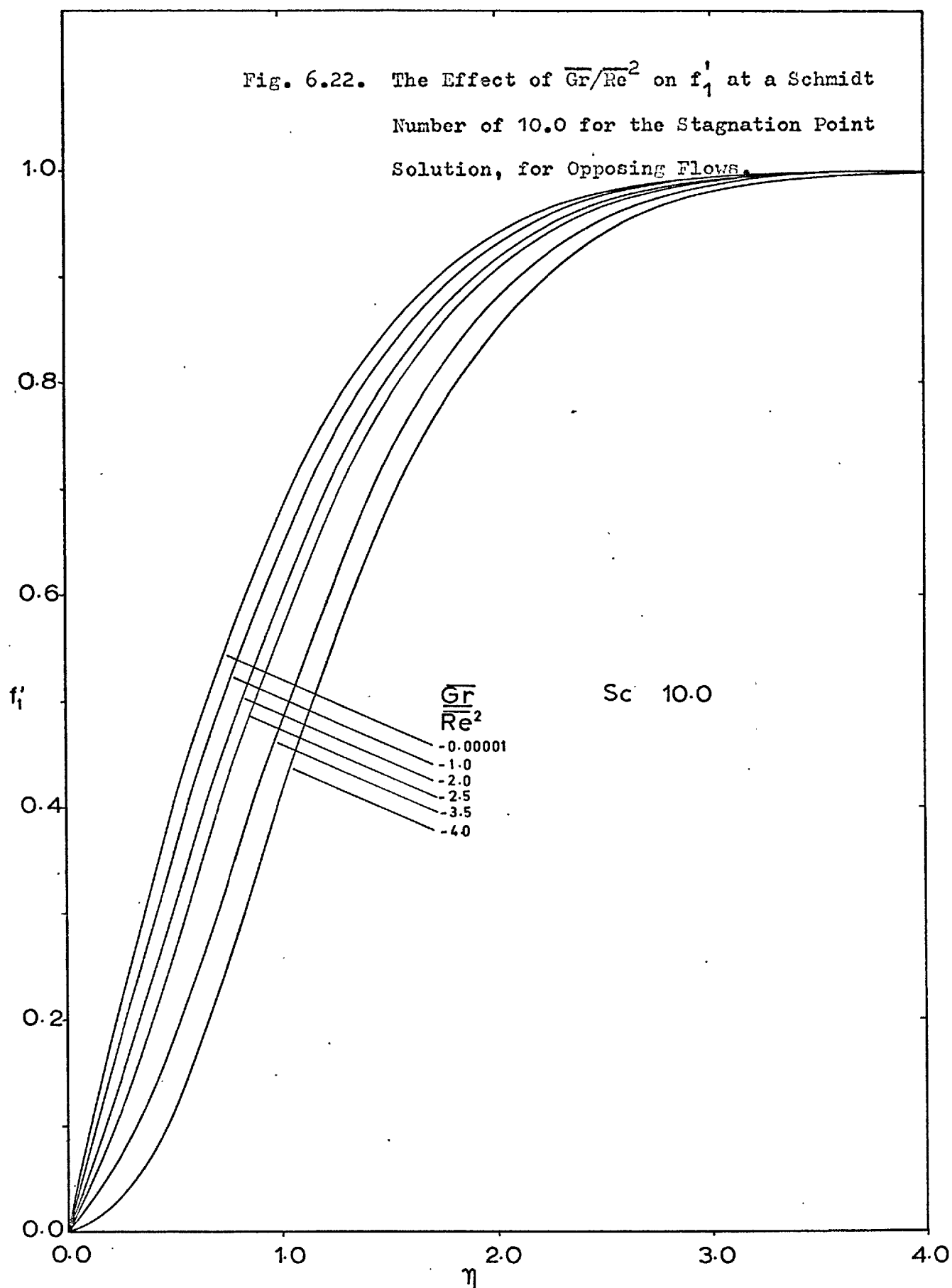
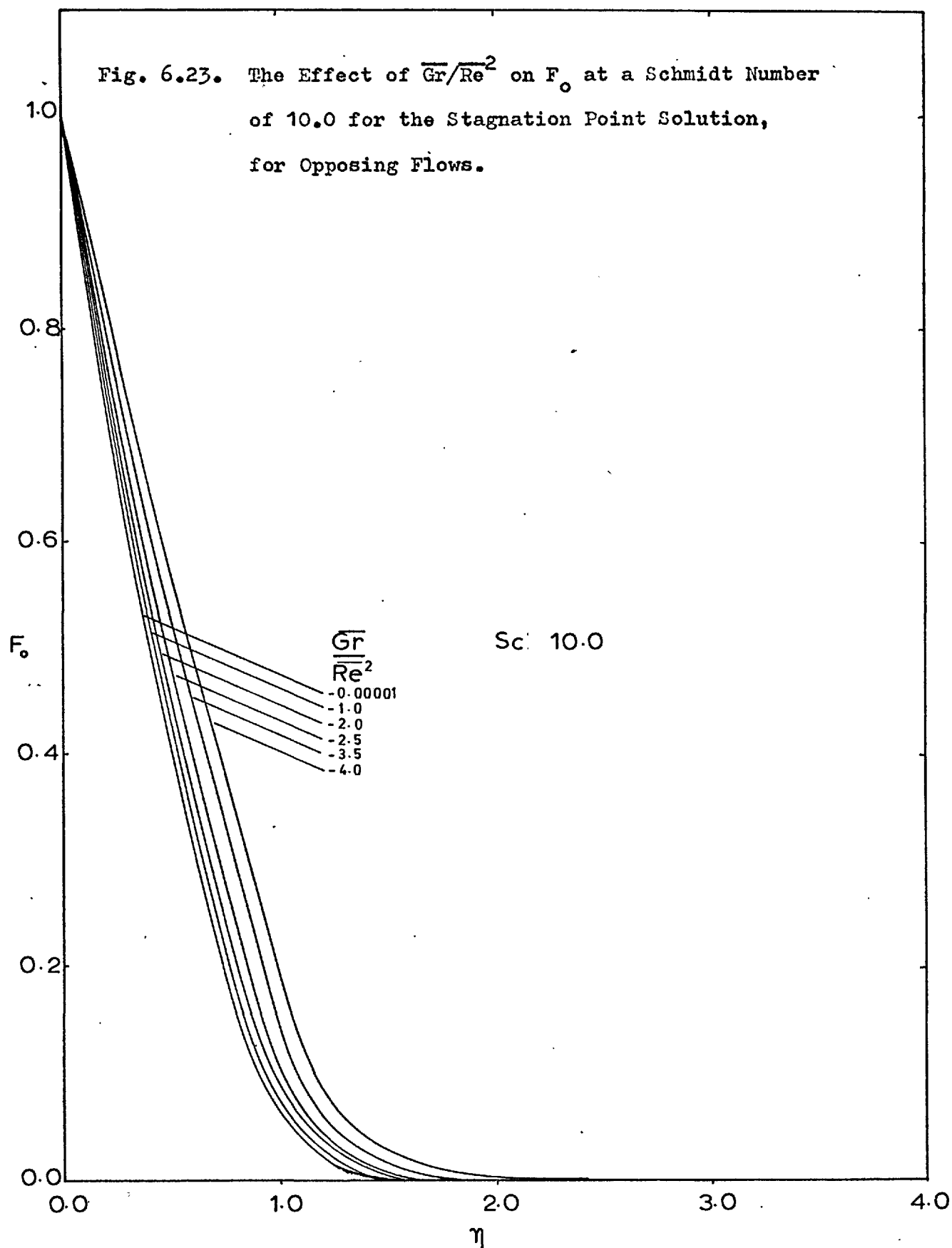


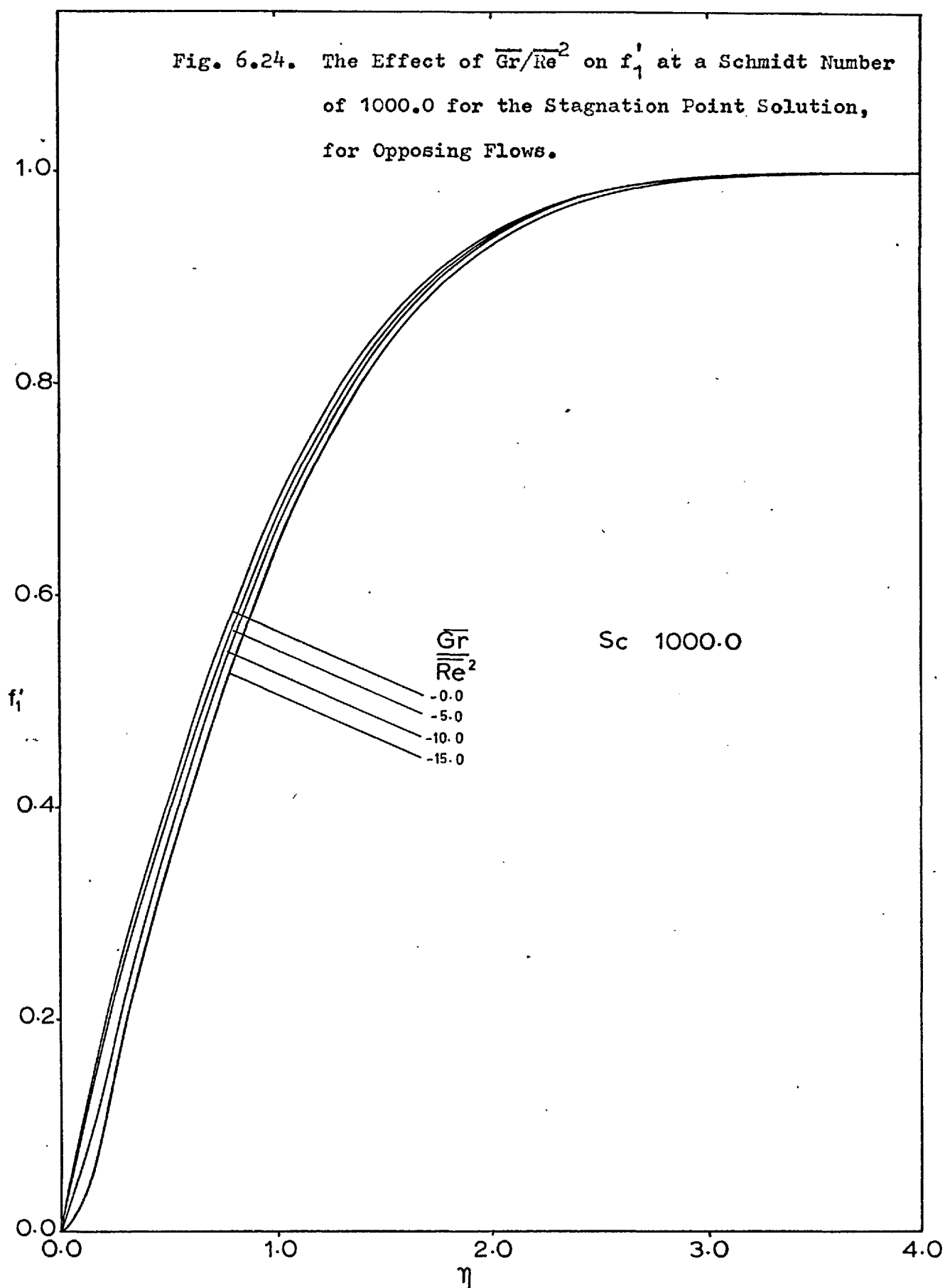
Fig. 6.20. The Effect of $\overline{Gr}/\overline{Re}^2$ on f_1' at a Schmidt Number of 2.4 for the Stagnation Point Solution, for Opposing Flows.

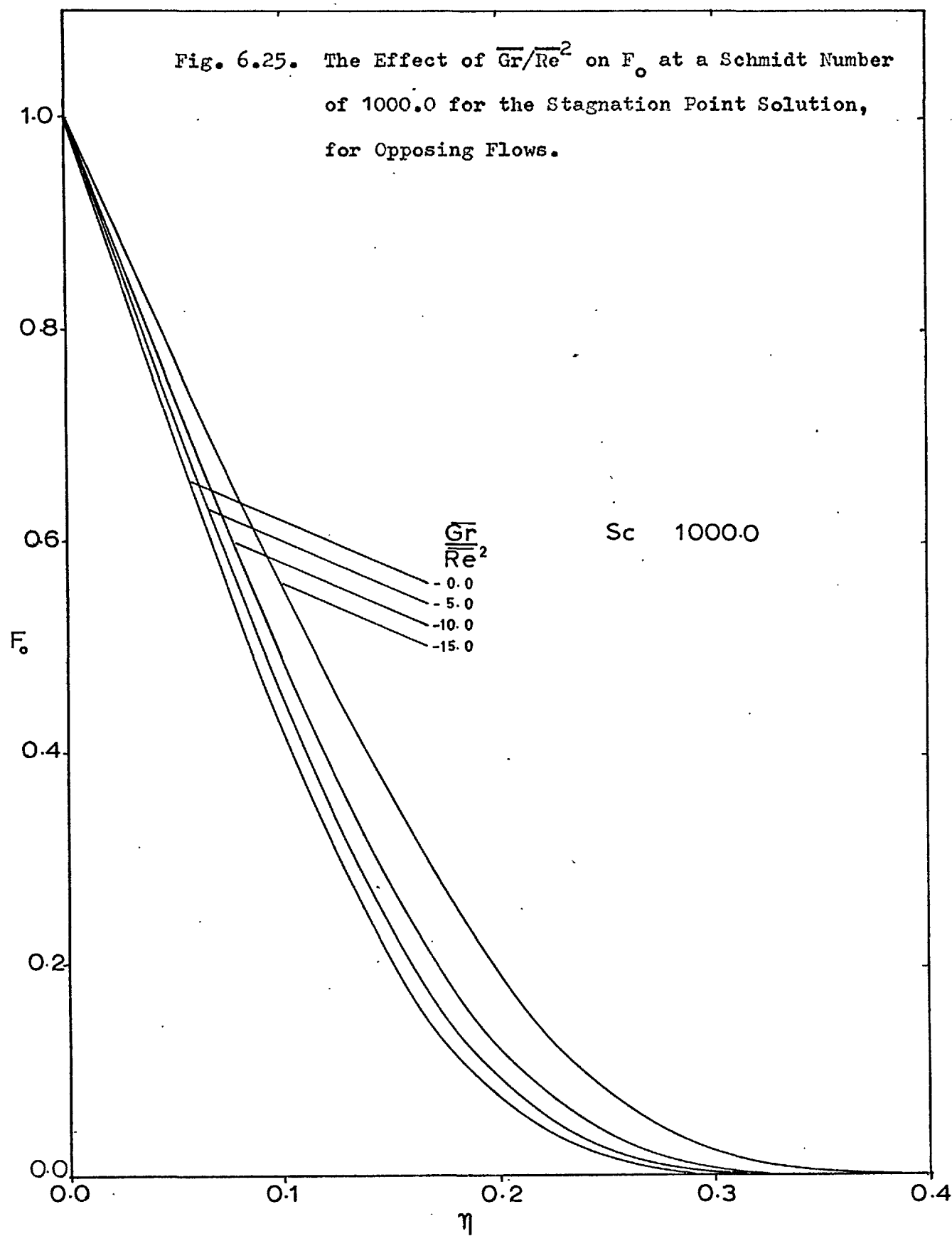












4.0, as shown in Fig. 6.22. The thickening of the boundary layers is smaller than for the lower Schmidt numbers. The value $f_1'(\eta)$ is within 1% of $f_1'(\infty)$ when the value of η is 3.3.

In the aiding flow situation it was observed that for a Schmidt number of 1000.0 the velocity profile only deviates very slightly from that for pure forced convection, even for relatively large values of the parameter $\overline{Gr}/Re^2$. As expected, the same behaviour is noted for opposing flows, Figs. 6.24 and 6.25. Backflow does not occur until a value of the parameter $\overline{Gr}/Re^2$ of greater than 15.0 is reached. There is only a very slight increase in the hydrodynamic boundary layer thickness. The value of $f_1'(\eta)$ becomes within 1% of $f_1'(\infty)$ at a value of η of approximately 29.0 compared with a value of 28.0 for pure forced convection.

6.2.3. The Influence of the Schmidt Number.

As has been discussed earlier, the coupling of the hydrodynamic and transfer equations for the interaction regime means that the influence of either the Schmidt number or the parameter $\overline{Gr}/Re^2$ cannot be treated in isolation.

In the previous two sections plots were presented for a number of values of the parameter $\overline{Gr}/Re^2$ at a constant Schmidt number. It is of interest to present these results in a different way by plotting profiles for a number of Schmidt numbers at a constant value of the parameter $\overline{Gr}/Re^2$.

For very small values of the parameter $\overline{Gr}/Re^2$, the velocity profiles for all Schmidt numbers are identical. Figure 6.26 shows the plot of $f_1'(\eta)$ against η for a value of the parameter $\overline{Gr}/Re^2$ of 0.00001. This situation is that for forced convection

with virtually no influence from free convection. The hydrodynamic equation reduces to that for pure forced convection. As a consequence, the two equations are virtually uncoupled and the Schmidt number does not affect the velocity distribution.

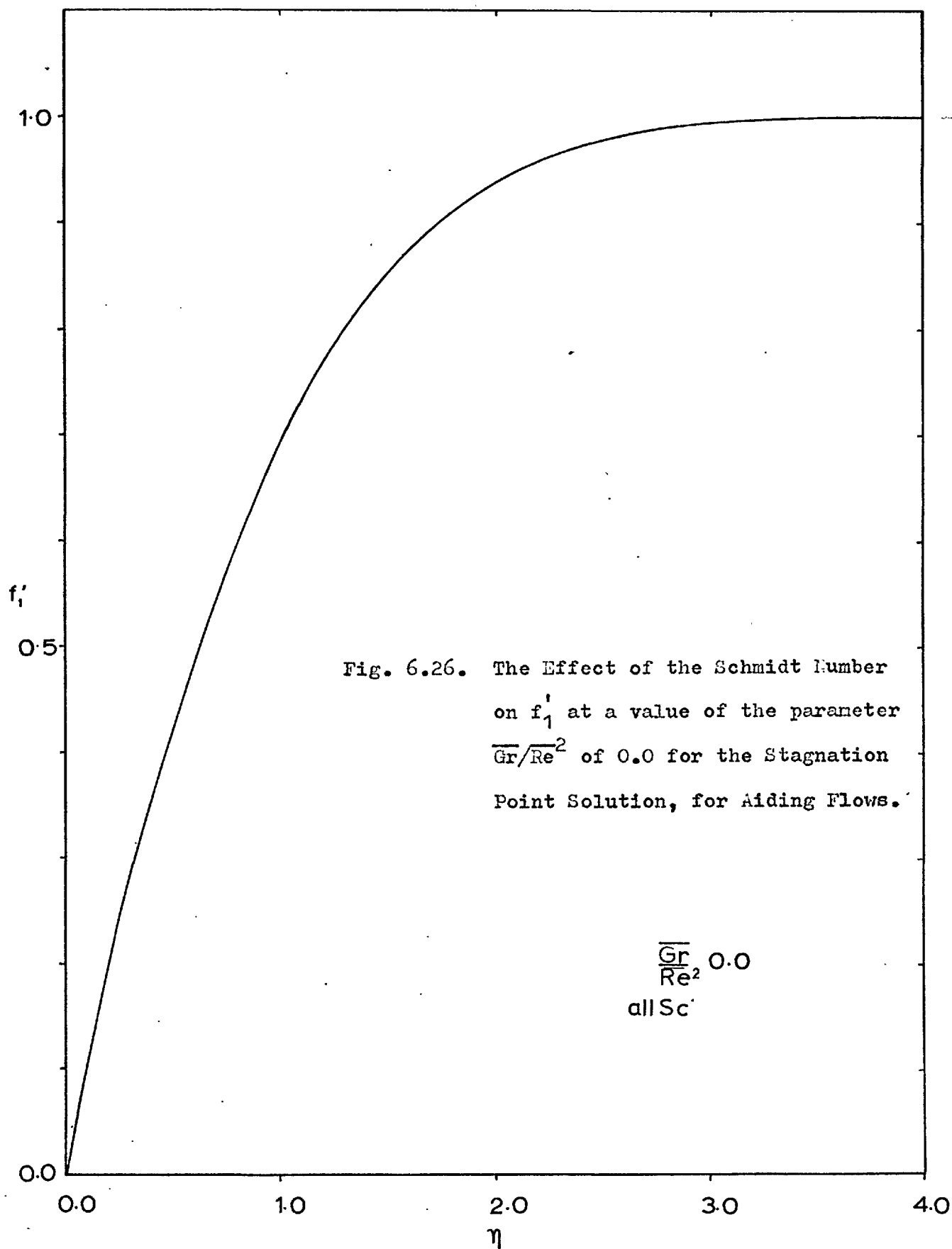
As the parameter $\overline{Gr}/\overline{Re}^2$ increases the equations become progressively more and more strongly coupled. The velocity and concentration profiles both show effects which are dependent upon the magnitude of the Schmidt number. This behaviour was clearly observed in the figures presented in the previous section.

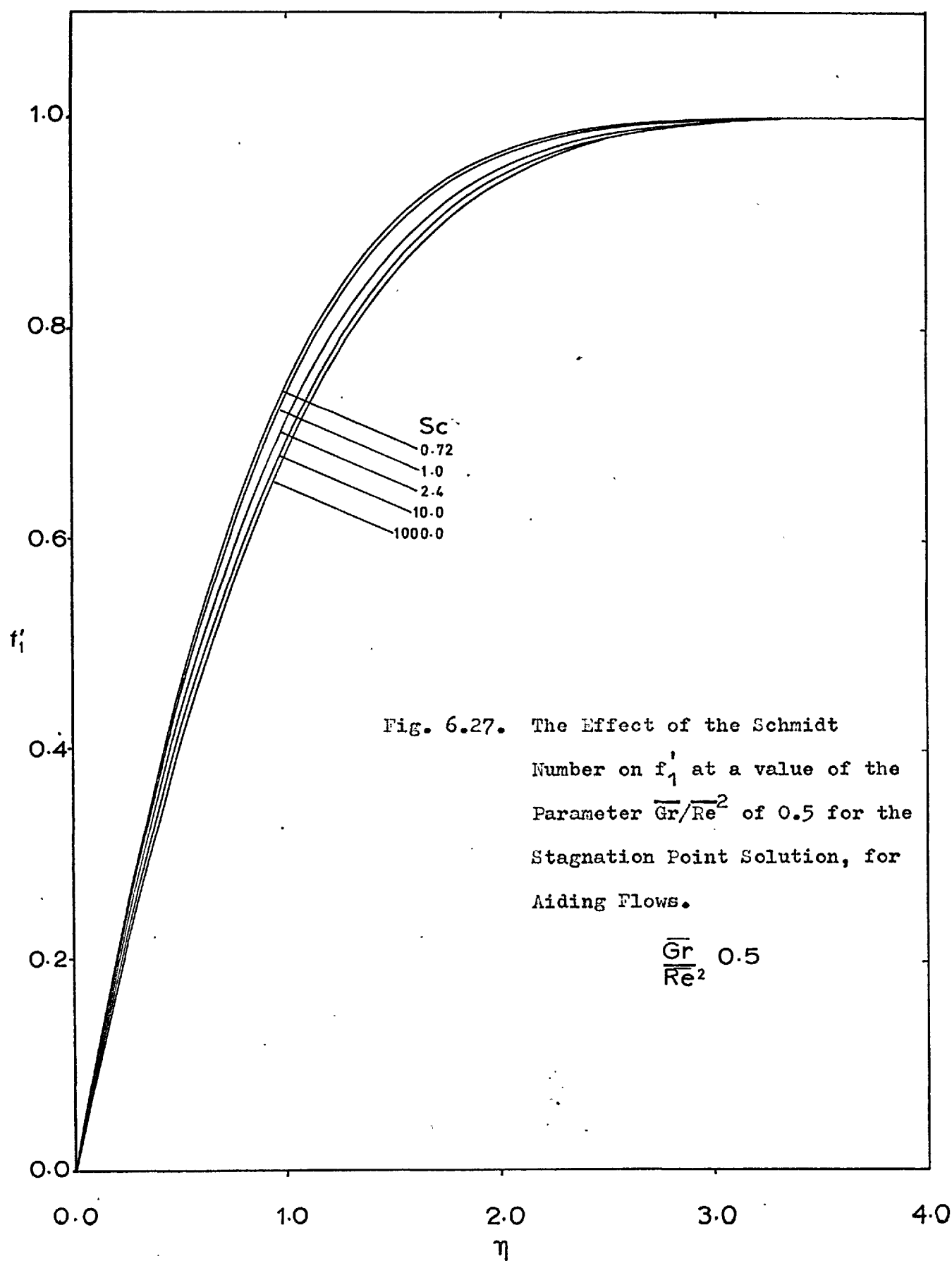
The velocity profiles for a value of the parameter $\overline{Gr}/\overline{Re}^2$ of 0.5 are shown in Fig. 6.27. For Schmidt numbers of 100.0 and 1000.0, the profile remains almost unchanged from that for pure forced convection. As the Schmidt number becomes progressively smaller, the profile deviates more from the forced convection profile. The velocities are larger than those for smaller values of the parameter $\overline{Gr}/\overline{Re}^2$ because of the aiding influence of free convection.

The velocity profiles for a value of the parameter $\overline{Gr}/\overline{Re}^2$ of 2.0, Fig. 6.28, show that there is a velocity within the boundary layer larger than that in the free stream for Schmidt numbers of 0.72 and 1.0. The profiles for the large Schmidt numbers are still very close to those for pure forced convection.

Figure 6.29 shows that when the parameter $\overline{Gr}/\overline{Re}^2$ has a value of 10.0, even more pronounced maxima in the velocity profiles exist. The velocity profile for a Schmidt number of 2.4 now also shows a maximum. The maximum velocity for the Schmidt number of 0.72 is now almost one and a half times that of the free stream velocity.

It is clear from Figs. 6.27 to 6.29 that the influence of free convection on the forced convection velocity profiles





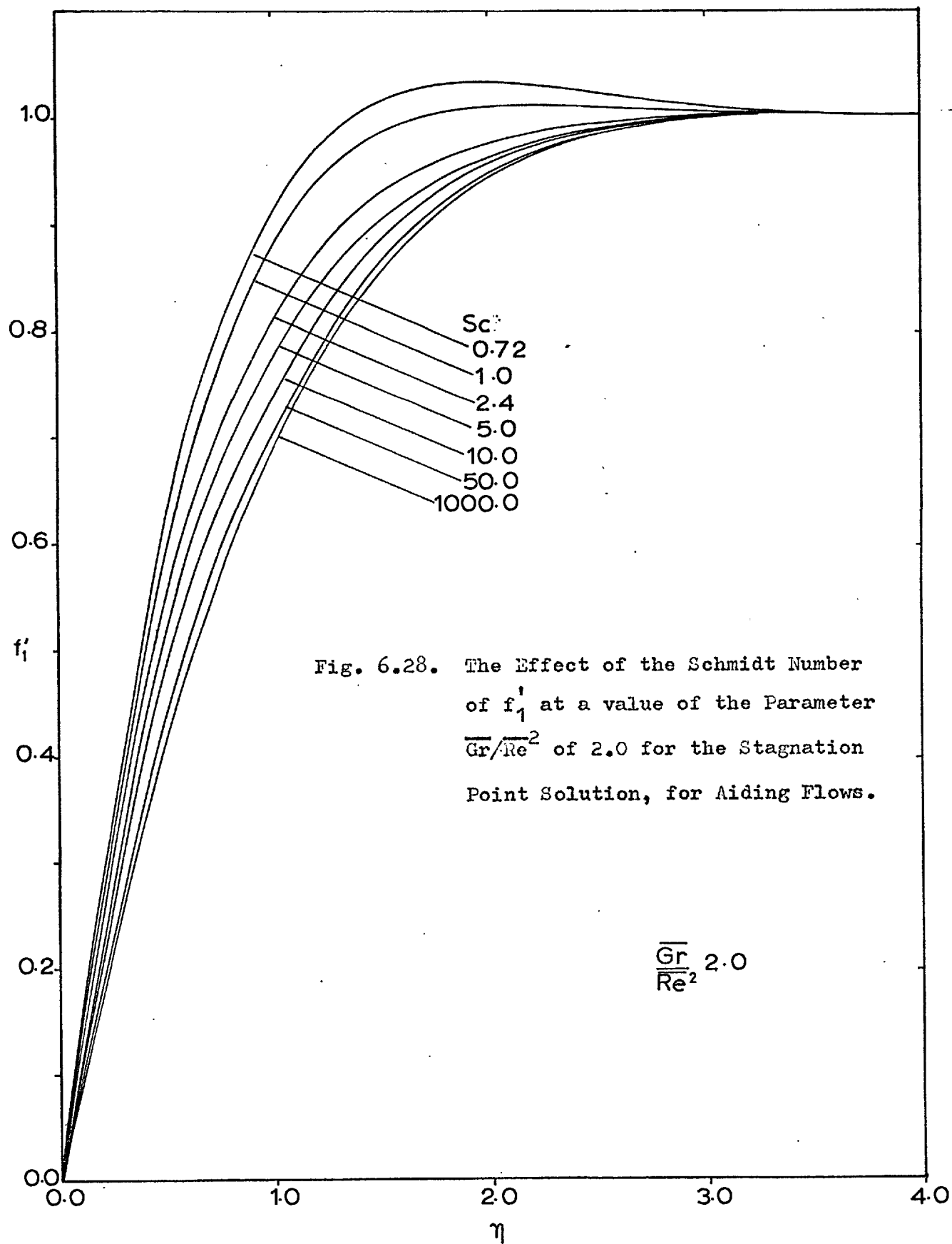


Fig. 6.29. The Effect of the Schmidt Number of f_1' at a value of the Parameter $\overline{Gr}/\overline{Re}^2$ of 10.0 for the Stagnation Point Solution, for Aiding Flows.

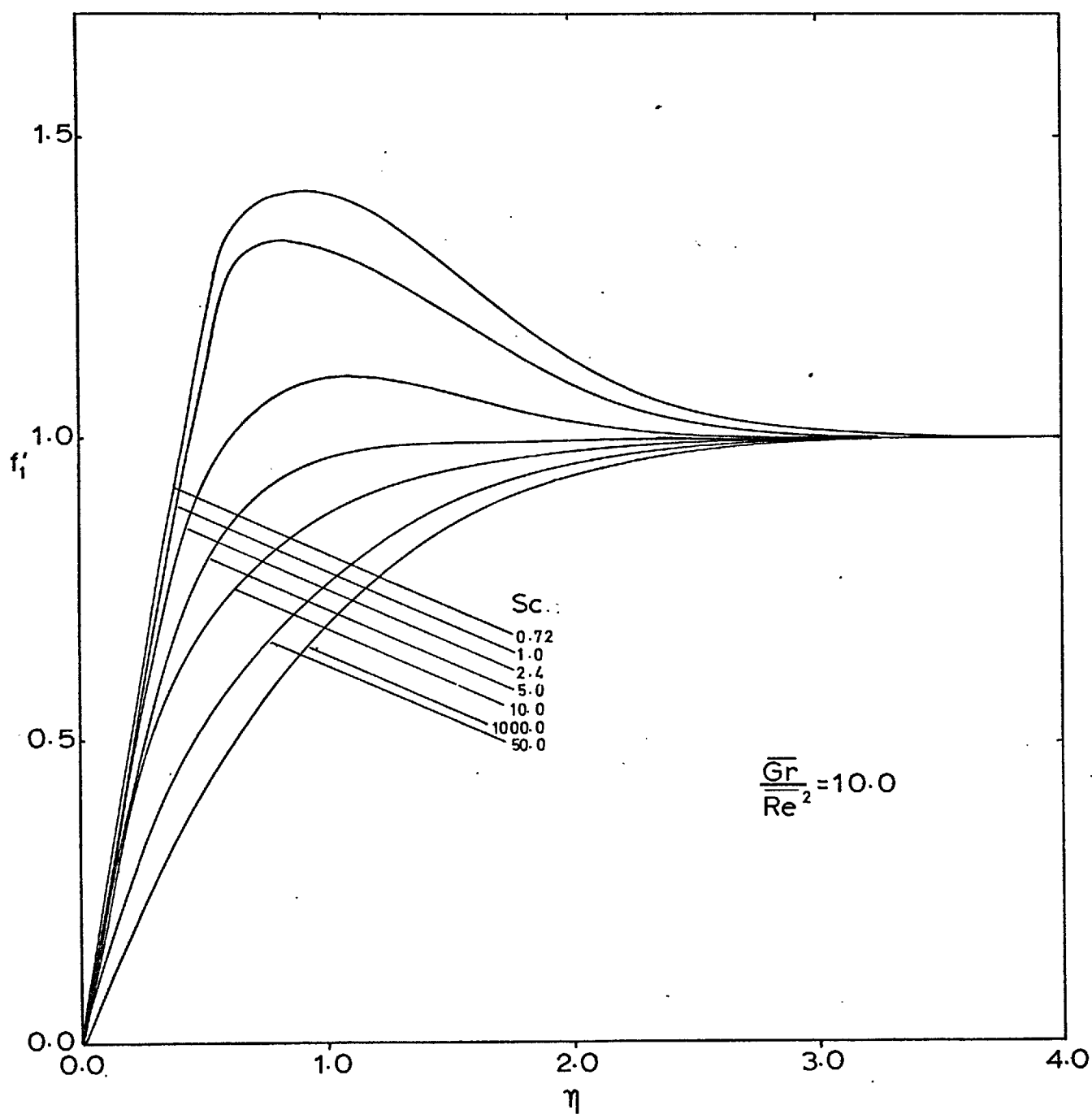
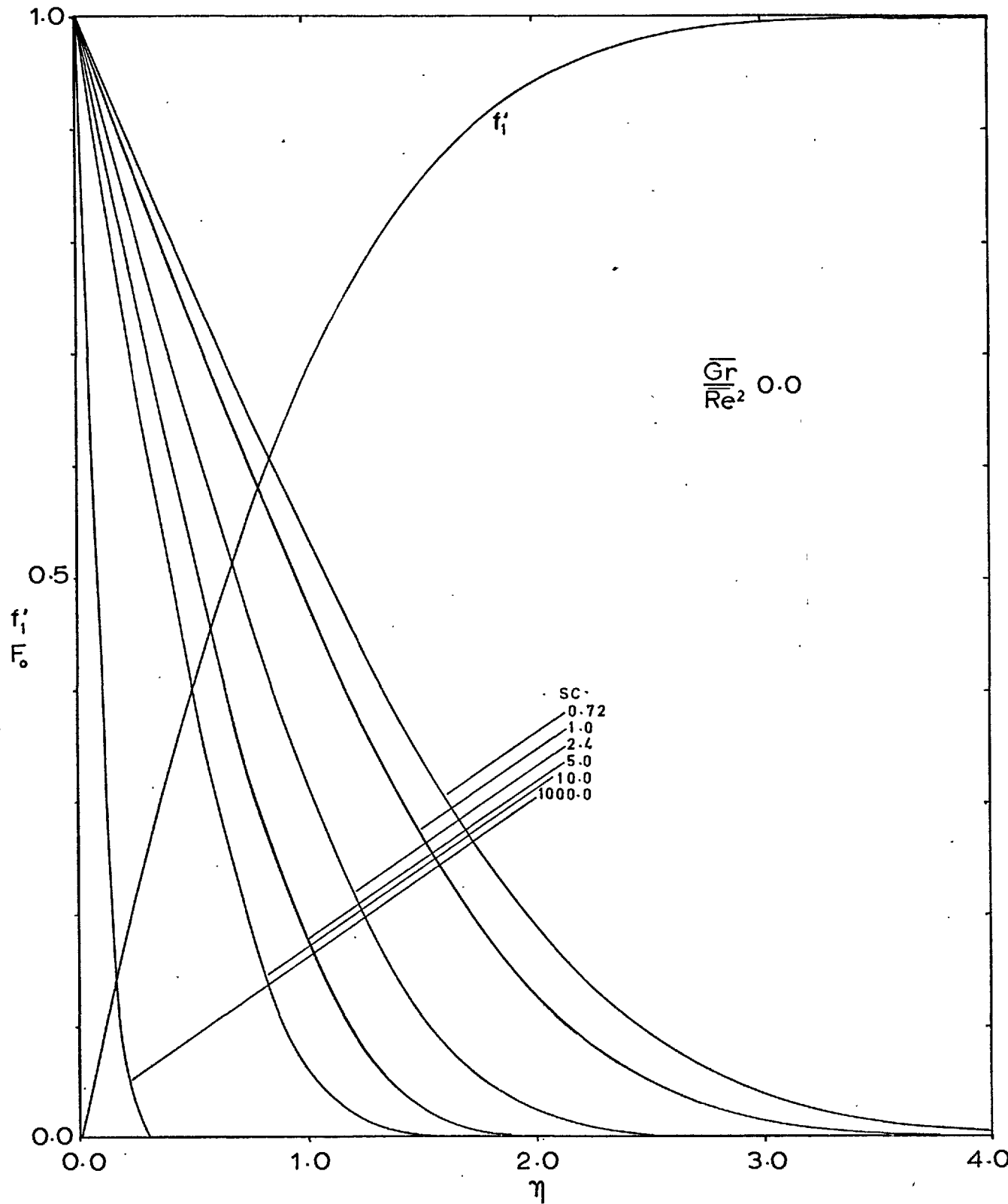


Fig. 6.30. The Effect of the Schmidt Number on F_0 at a value of the Parameter $\overline{Gr}/\overline{Re}^2$ of 0.0 for the Stagnation Point Solution, for Aiding Flows.



is more pronounced for small than for large Schmidt numbers. This is expected since in the previous discussion it was noted that for large Schmidt numbers the transfer boundary layer is very much thinner than the hydrodynamic boundary layer. Therefore, the effect of free convection is confined to a much smaller region than for low Schmidt numbers for which the free convective forces influence the velocity over almost all of the hydrodynamic boundary layer.

The relative thicknesses of the transfer to the hydrodynamic boundary layer thicknesses for Schmidt numbers of 0.72 to 1000.0 at a value of the parameter $\overline{Gr}/Re^2$ of 0.00001 are shown in Fig. 6.30.

For a Schmidt number of unity, the two boundary layer thicknesses are of the same order of magnitude. The transfer boundary layer is approximately one half of the thickness of the hydrodynamic boundary layer when the Schmidt number is 10.0, and only one tenth of the thickness when the Schmidt number is 1000.0.

6.3. Mass Transfer Results, Local Mass Rates for the Sphere and the Oblate Spheroids.

The integral method, which was developed in Chapter 4, has been used to predict local transfer rates from a sphere and four oblate spheroidal shapes, for ranges of both the Schmidt number and the parameter $\overline{Gr}/Re^2$. The four spheroids have ratios of the minor to the major axis of 13/16, 10/16, 7/16 and 4/16.

The effect of shape and the influence of the two other parameters will be discussed in subsequent sections, and a

comparison between the present results and those of other authors will be made.

6.3.1. The Influence of the Parameter Gr/Re^2 on the Local Transfer Rates.

Two sets of curves are plotted for the different body shapes so that the effects of the parameter Gr/Re^2 can be clearly seen.

The first series of curves represents the local mass transfer number, for values of the parameter Gr/Re^2 ranging from 0.0 to 100.0, plotted against the angle θ from the front stagnation point for a Schmidt number of 1.0.

The second set of curves represents the ratio of the local mass transfer number to the mass transfer number at the front stagnation point, plotted for a number of values of the parameter Gr/Re^2 for a Schmidt number of 2.4. Since the Reynolds number and the Schmidt number are both constant, this represents a plot of the local Sherwood number to the Sherwood number at the front stagnation point.

The two sets of curves will be discussed in detail for the sphere, and further discussion for the oblate spheroidal shapes will stem from this.

Figure 6.31. represents the effect of the parameter Gr/Re^2 on the local mass transfer number for the sphere. The forced convection solution is shown by the curve for a value of the parameter Gr/Re^2 of 0.0. Up to a value of the parameter Gr/Re^2 of approximately 0.2, only a very small influence of free convection is noticed. After this, free convection can be seen to play an increasingly dominant role. The local mass transfer number increases as the parameter

Gr/Re^2 increases. This results from the fact that the transfer boundary layer is observed to decrease in thickness, and the concentration gradient to increase, as the value of the parameter Gr/Re^2 becomes larger.

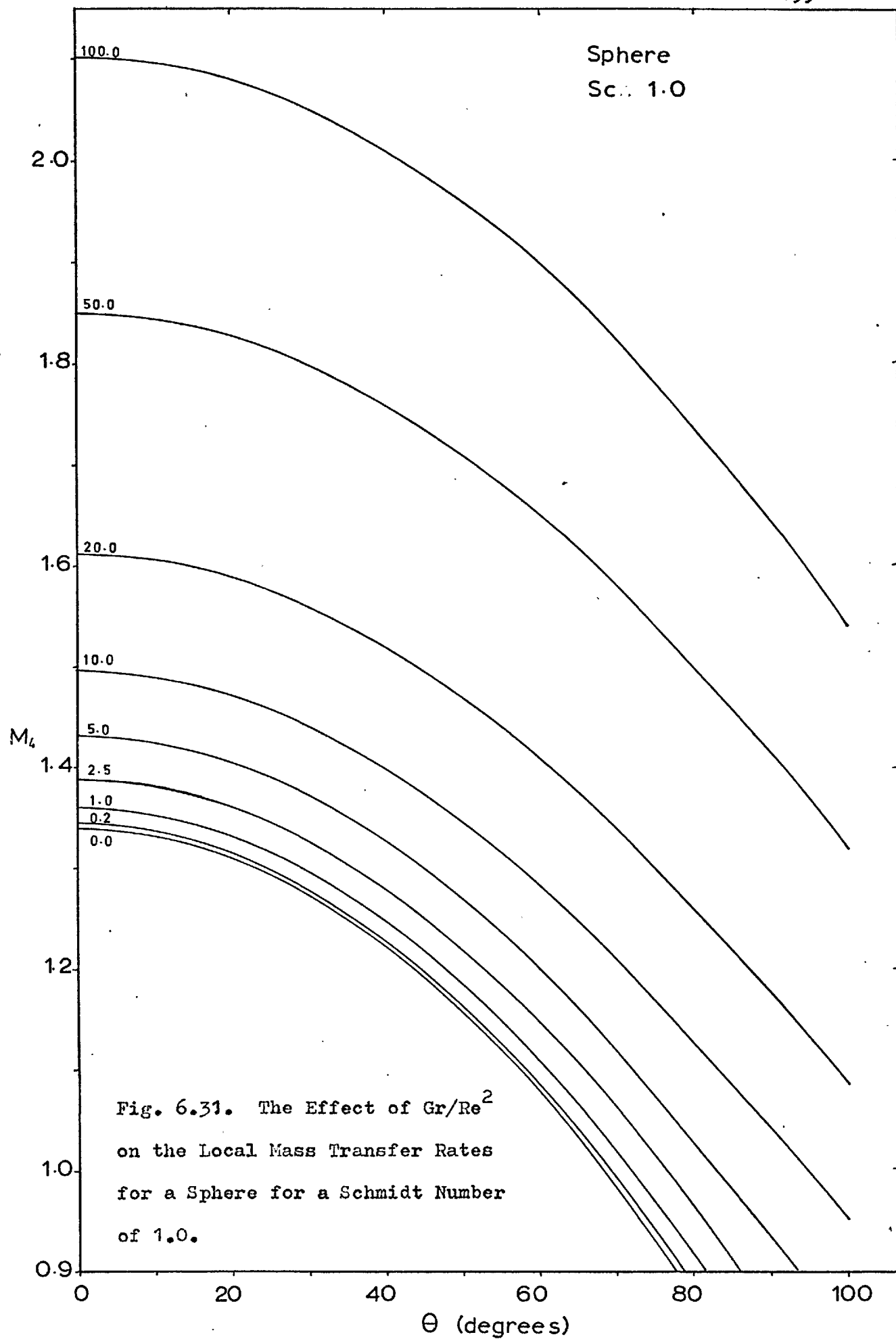
For the sphere, the hydrodynamic and transfer boundary layer thicknesses both increase around the body from the front stagnation point to the separation point. This thickening of the transfer boundary layer is reflected in the mass transfer plots, for all values of the parameter Gr/Re^2 , in which the local mass transfer number is observed to decrease, around the body, from a maximum at the front stagnation point.

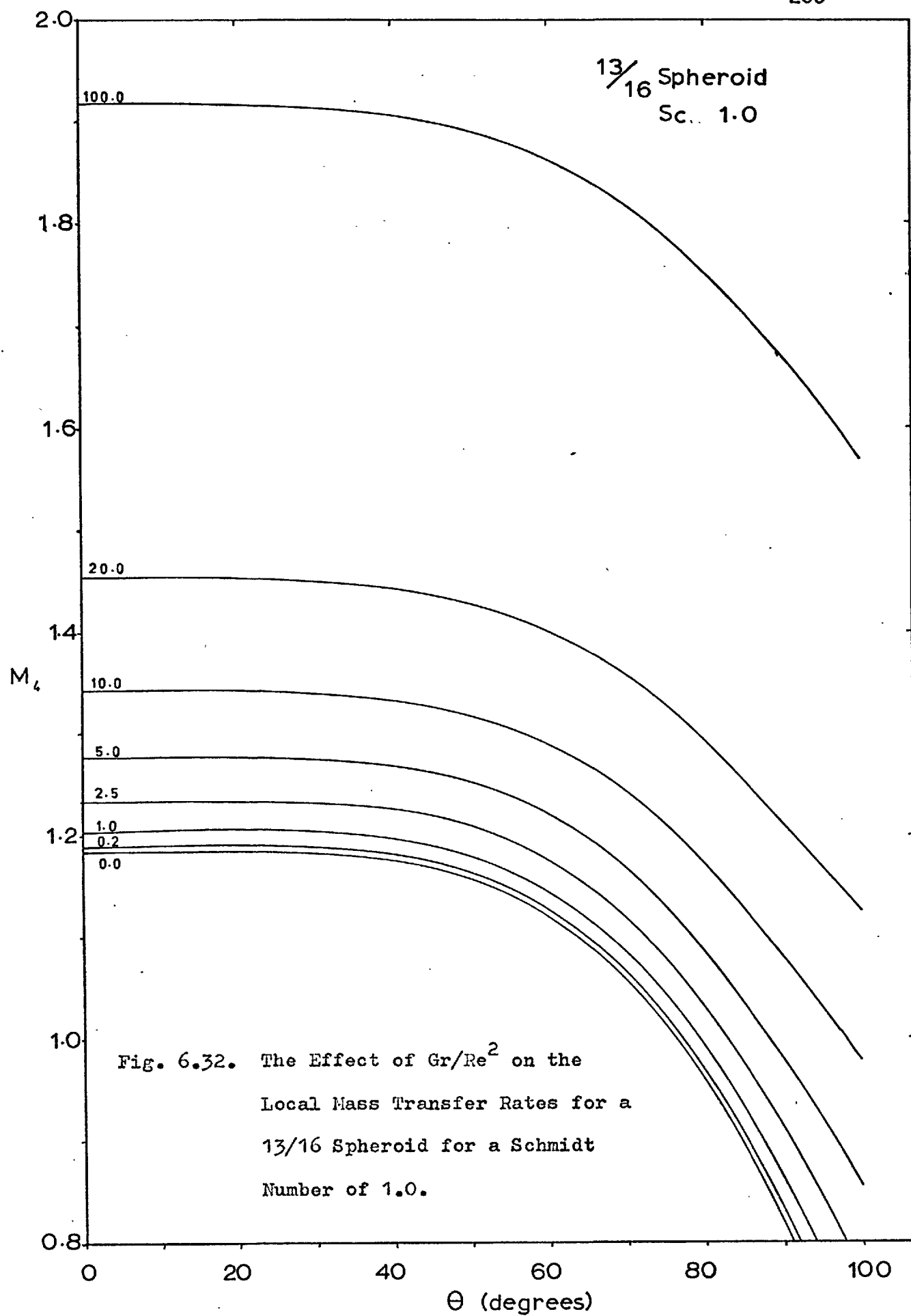
The curves in figure 6.31, are not parallel over the entire range of values of the parameter Gr/Re^2 . This suggests that the value of this parameter has a definite influence on the local transfer rates.

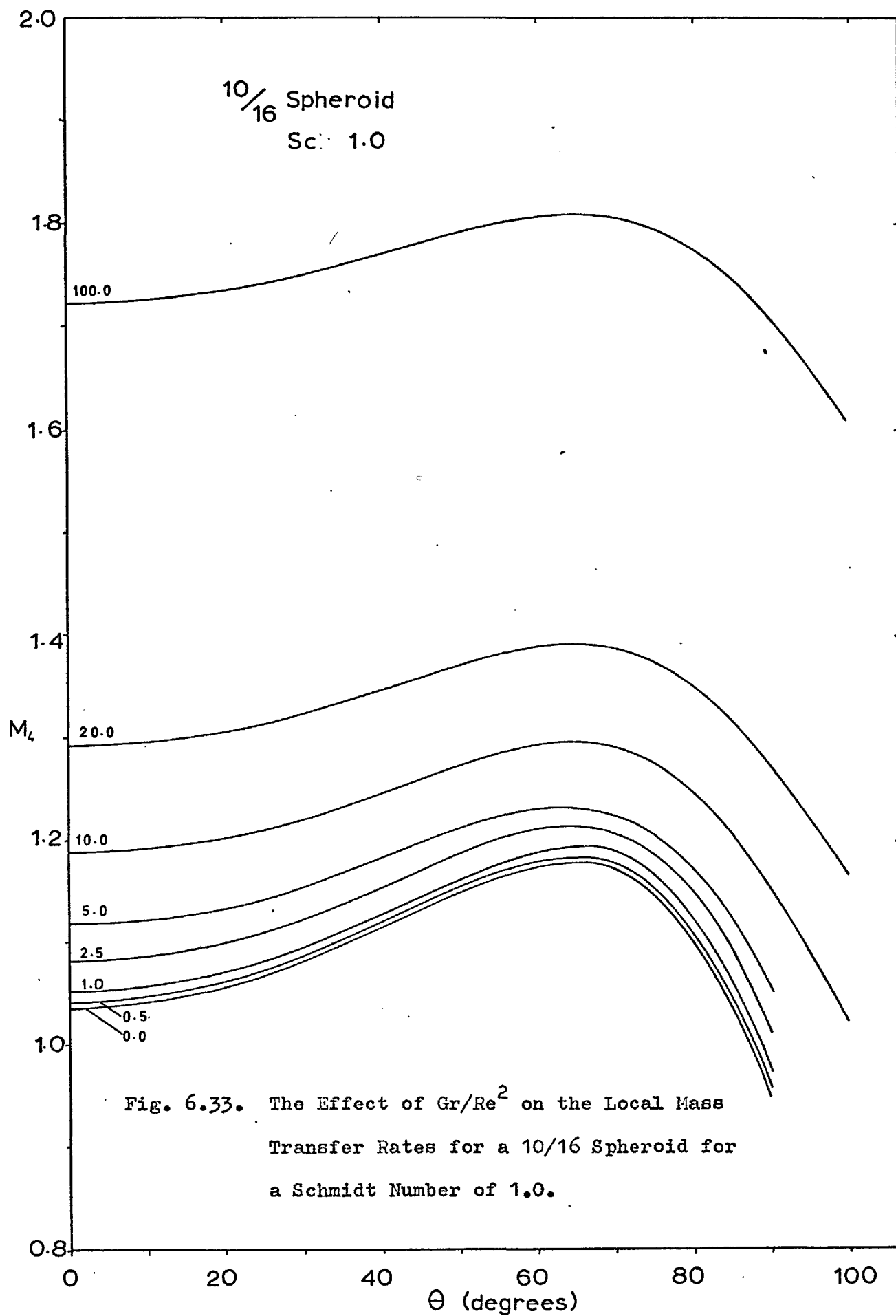
This effect is shown more clearly in the second plot for the sphere, Fig. 6.35. The local Sherwood number at the front stagnation point is plotted for a Schmidt number of 2.4 over a range of values of the parameter Gr/Re^2 . The effect of increasing the parameter can be seen to result in a decrease in the ratio of the two Sherwood numbers at a particular angular position.

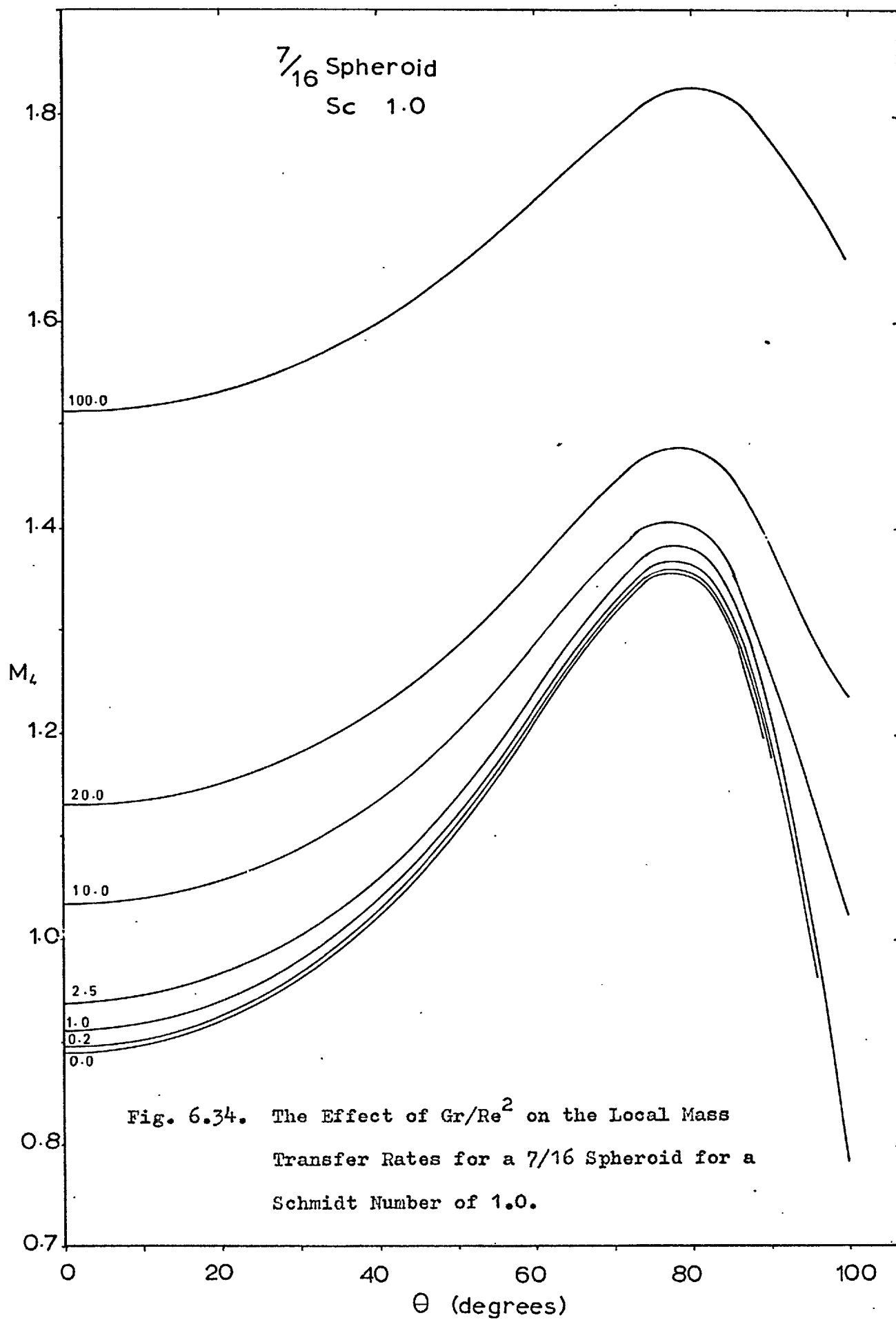
The curves for high values of the parameter Gr/Re^2 can be seen to approach a limit which represents the free convection transfer distribution.

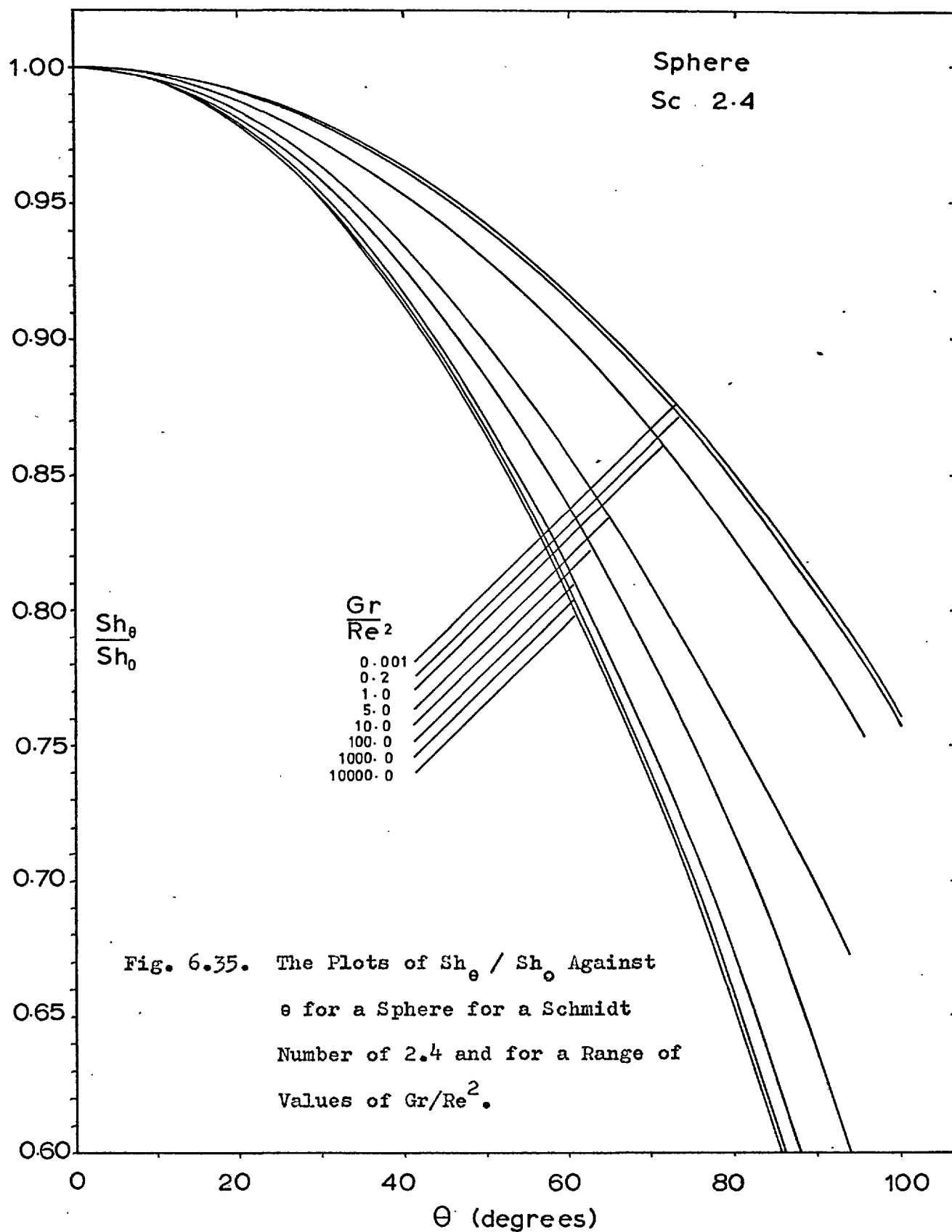
The curves for the oblate spheroidal bodies, Figs. 6.32 to 6.34 and Figs. 6.36 to 6.39, show similar results to those for the sphere. However, as the body shape becomes flatter, the effect of the parameter Gr/Re^2 is more pronounced.











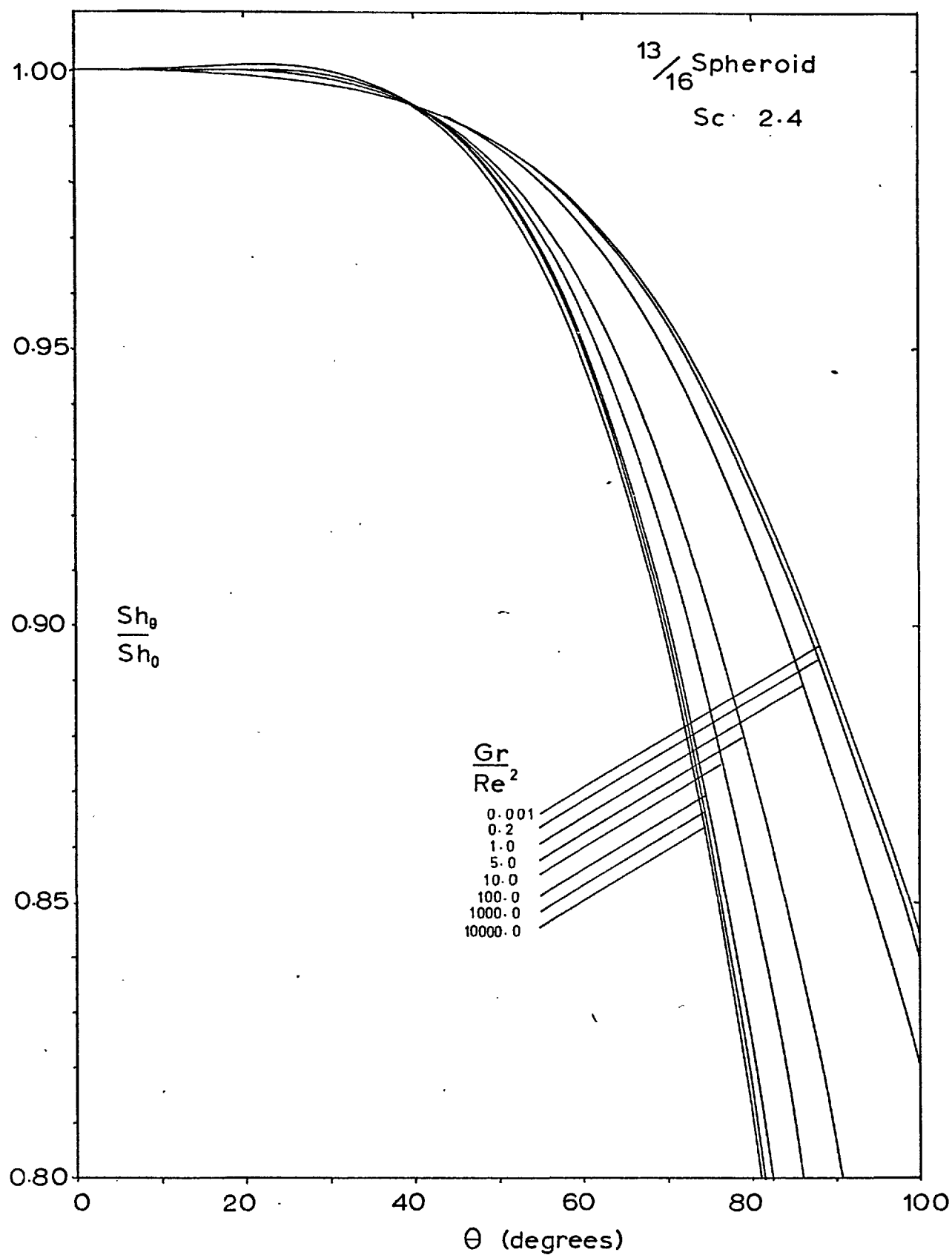
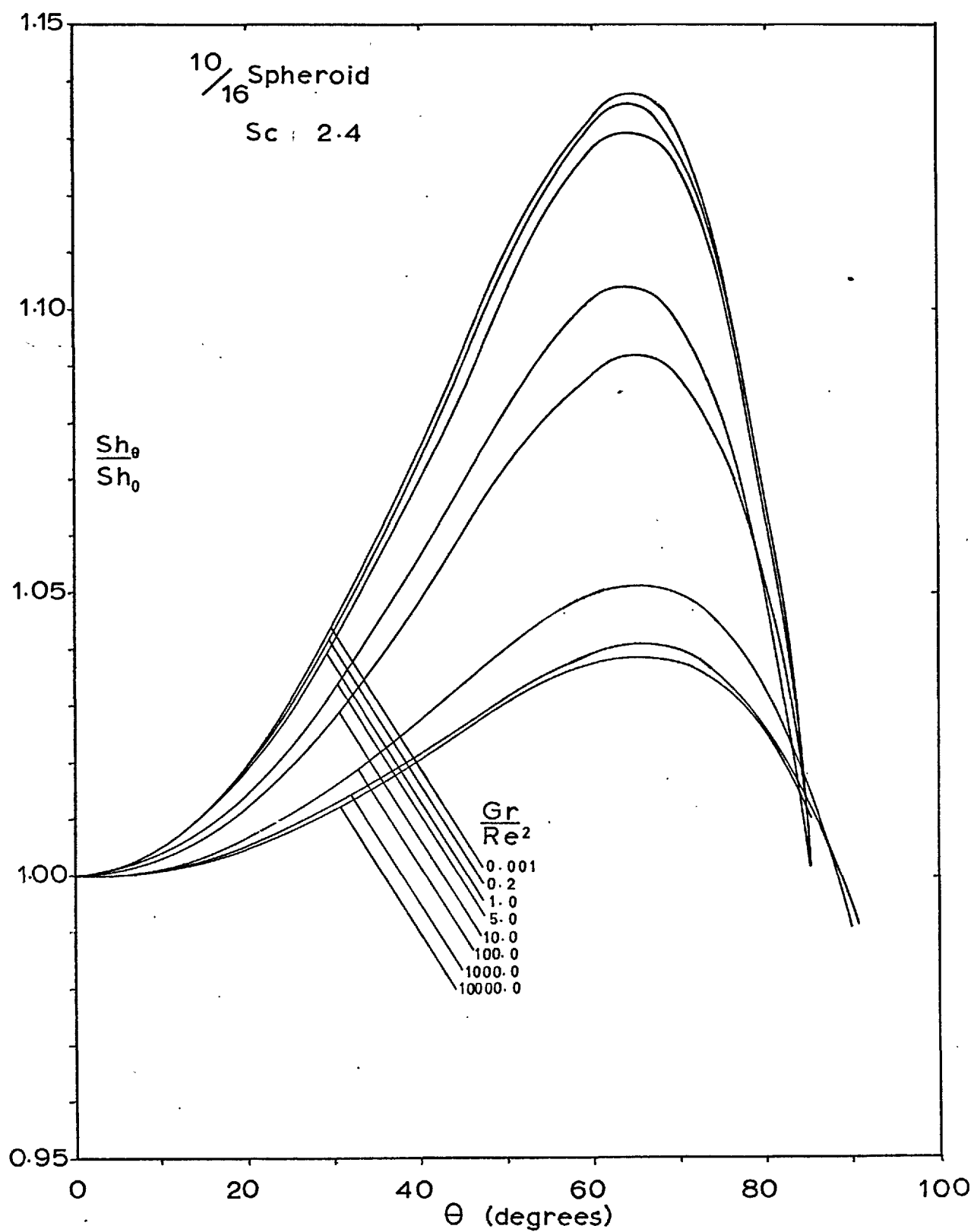


Fig. 6.36. The Plots of Sh_θ / Sh_0 Against θ for a $13/16$ Spheroid for a Schmidt Number of 2.4 and for a Range of Values of Gr/Re^2 .

Fig. 6.37 . The Plots of Sh_θ / Sh_0 Against θ for a 10/16 Spheroid for a Schmidt Number of 2.4 and for a Range of Values of Gr/Re^2 .



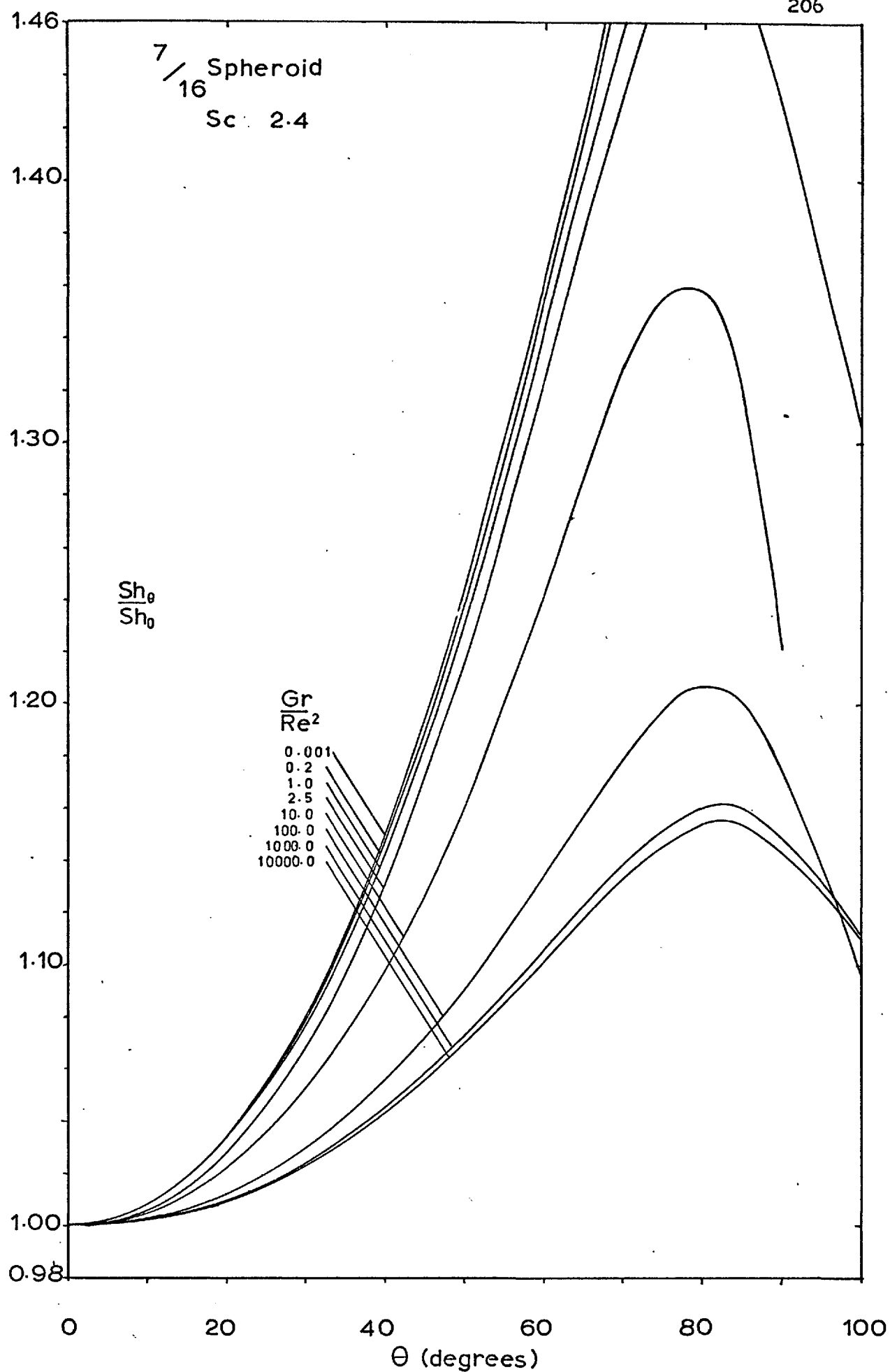
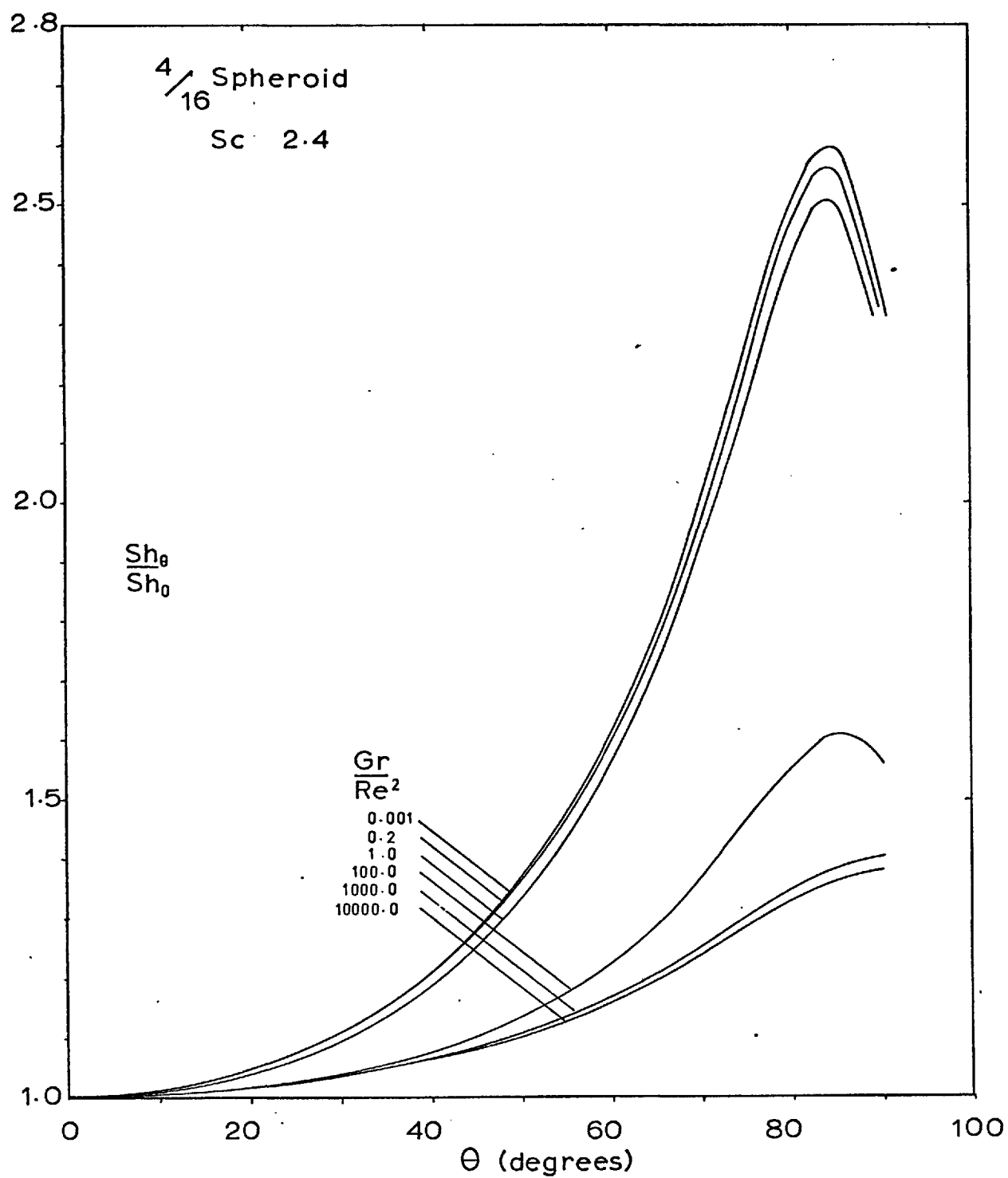


Fig. 6.38. The Plots of Sh_θ / Sh_0 Against θ for a 7/16 Spheroid for a Schmidt Number of 2.4 and for a Range of Values of Gr/Re^2 .

Fig. 6.39. The Plots of Sh_θ / Sh_0 Against θ for a $4/16$ Spheroid for a Schmidt Number of 2.4 and for a Range of Values of Gr/Re^2 .



At this stage it is of interest to consider the effect of shape upon the boundary layer thicknesses.

The oblate spheroid with a ratio of the minor to the major axis of $13/16$, produces an interesting result in that the thickness of each boundary layer remains approximately constant from the front stagnation point, up to an angle of between thirty and forty degrees. After this angle, both of the boundary layers become thicker round the body to the separation point.

The $10/16$ spheroid is of great interest since there is initially a thinning of both boundary layers. The boundary layers decrease in thickness from the front stagnation point around the body to an angle of approximately sixty five degrees, after which they both increase round the body to separation.

This decrease in the two boundary layer thicknesses becomes more pronounced as the body shapes become flatter. The minimum boundary layer thickness for the $7/16$ spheroid occurs at an angle of approximately seventy five degrees from the front stagnation point, and at an angle of approximately eighty five degrees for the $4/16$ spheroid.

The thinning of the boundary layers causes an increase of the local mass transfer rates. This can be clearly seen from the plots of the local mass transfer numbers for the oblate spheroids.

Figures 6.32 to 6.34 show that, as for the sphere, the free convective forces influence the transfer when the parameter Gr/Re^2 reaches a value of approximately 0.2. As the value of the parameter increases, free convection becomes more

important. As the body shape becomes flatter, the maxima in the mass transfer curves are more pronounced at low values of the parameter Gr/Re^2 , than when free convection is dominant. This effect is very clearly seen in the second set of curves, Figs. 6.36 to 6.39.

This means that for predominantly forced convective flows the ratio of the local boundary layer thickness divided by the thickness at the front stagnation point is smaller than that for free convective flows. This behaviour is caused by the potential flow around the body.

The increase of mass transfer rates for oblate spheroids was observed experimentally by Beg⁹⁶, who studied forced convective mass transfer from spheres and oblate spheroids. Comparisons between Beg's experimental predictions and those of the present work are presented later.

The influence of the Schmidt number was discussed in Section 6.2.3. in connection with the velocity and concentration profiles. It was shown that an increase of the value of the Schmidt number results in a decrease of the transfer boundary layer thickness. Therefore, the concentration gradient at the surface increases, and thus the local mass transfer number $Sh/Re^{1/2}$, also increases.

This effect is clear from the mass transfer plots, Figs. 6.40 to 6.44, for the five body shapes which were considered in the present work. The plots which are presented for a predominantly forced convective situation show the same features for each of the body shapes. An increase of the Schmidt number results in an increase of the local mass transfer number. In the plots the transfer number $Sh/Re^{1/2}$

Fig. 6.40. The Effect of the Schmidt Number on Local Transfer Rates for a Sphere.

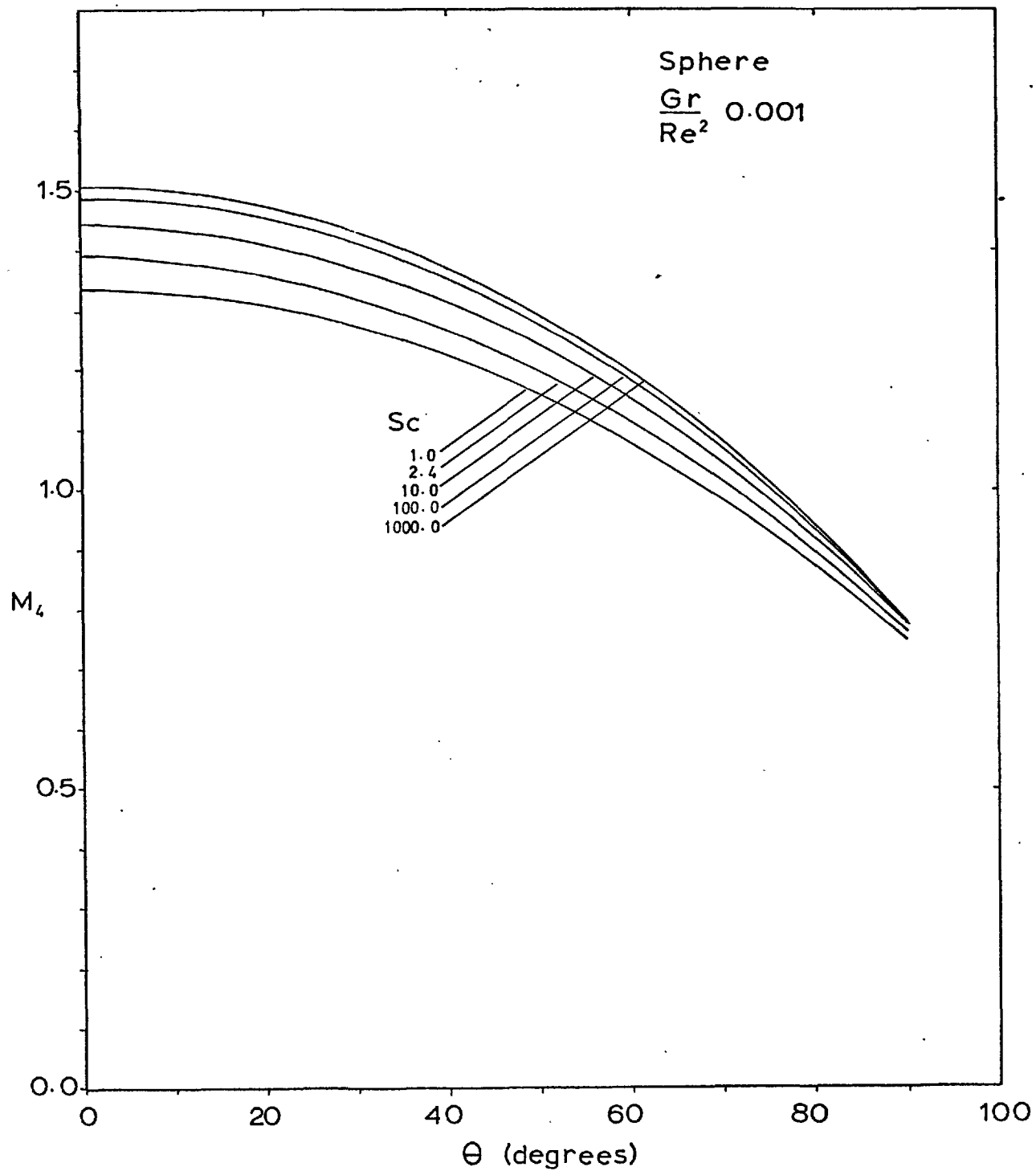


Fig. 6.41. The Effect of the Schmidt Number on Local Transfer Rates for a $13/16$ Spheroid.

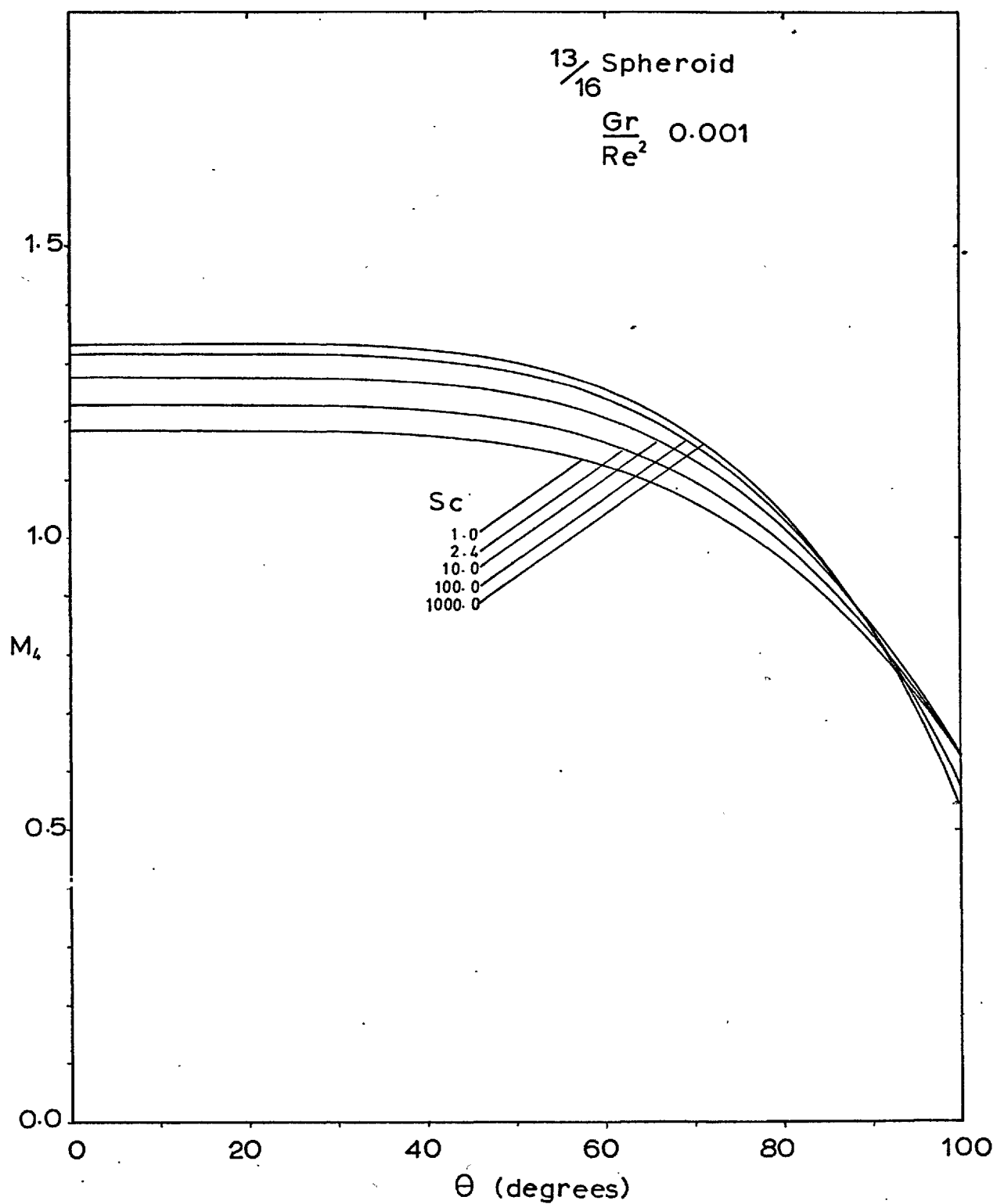


Fig. 6.42. The Effect of the Schmidt Number on Local
Transfer Rates for a 10/16 Spheroid.

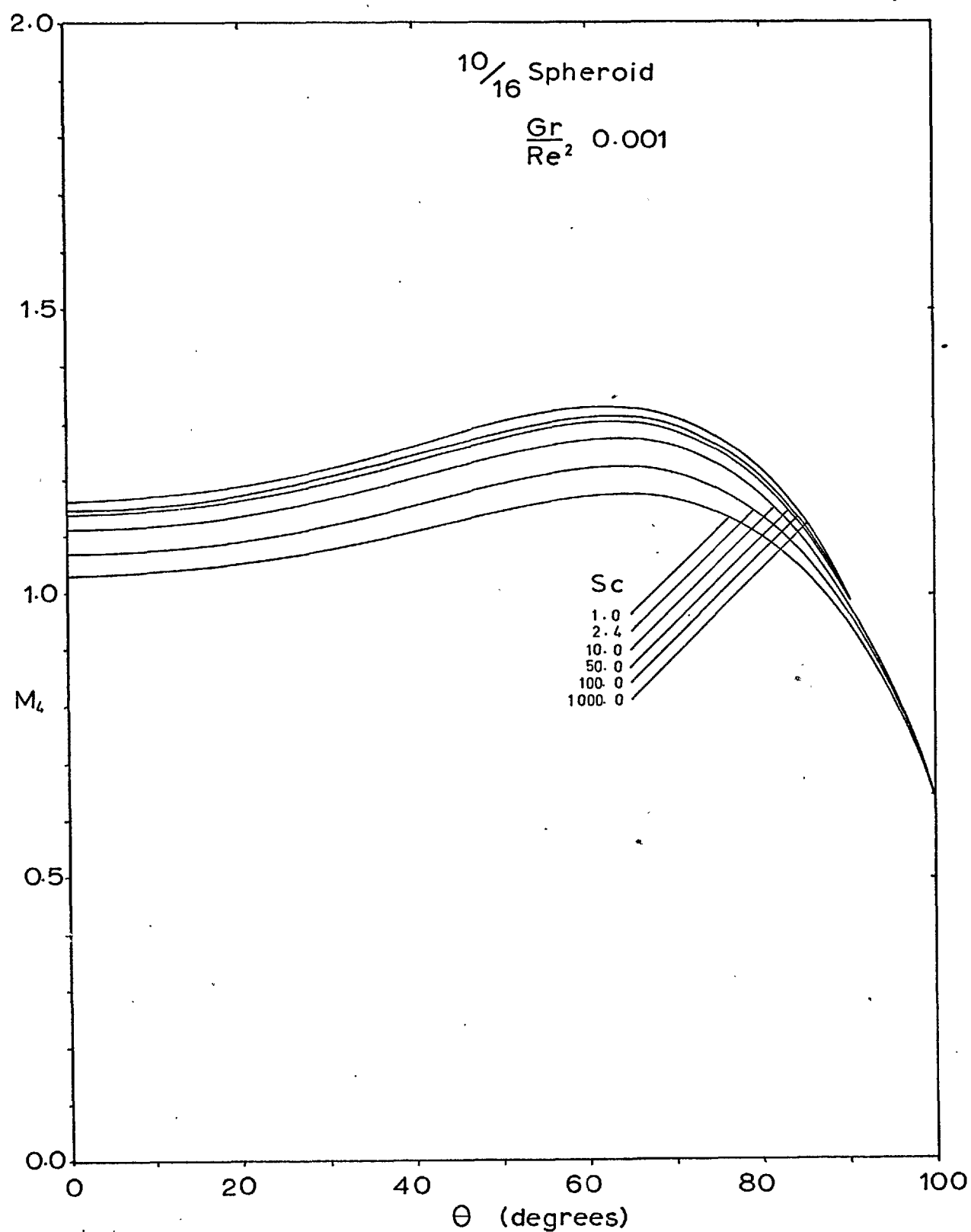
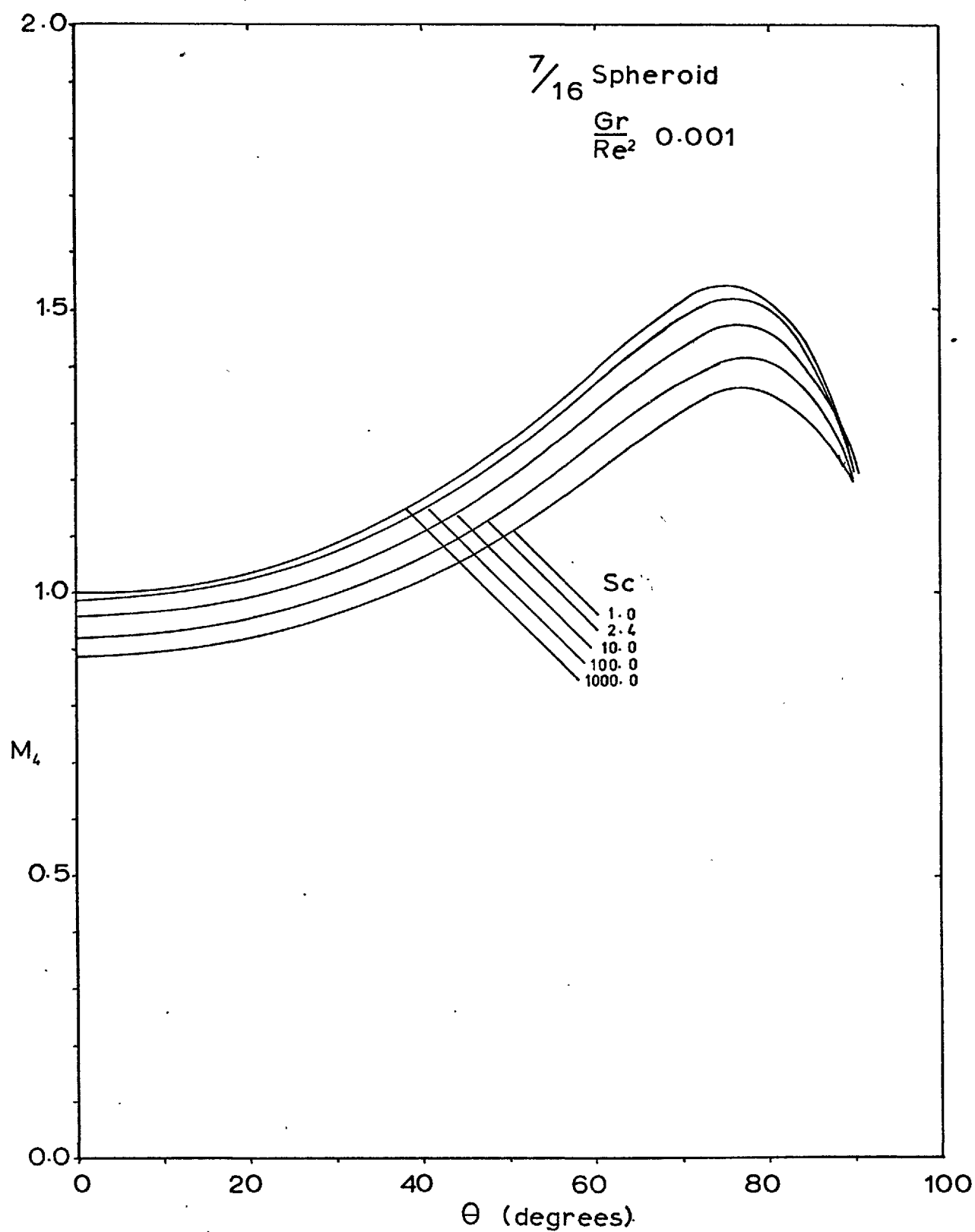


Fig. 6.43. The Effect of the Schmidt Number on Local
Transfer Rates for a $7/16$ Spheroid.



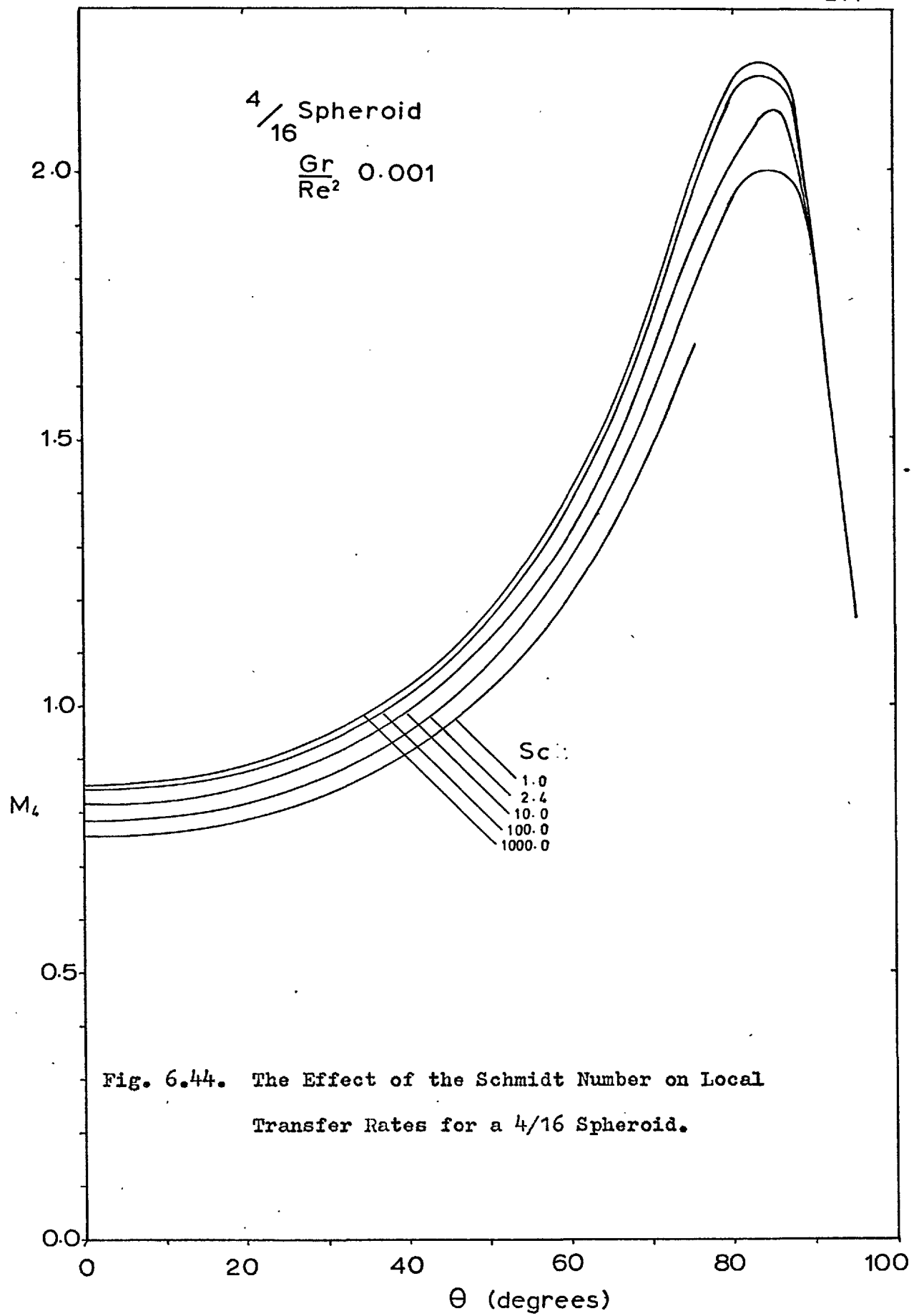


Fig. 6.44. The Effect of the Schmidt Number on Local Transfer Rates for a 4/16 Spheroid.

has been divided by the cube root of the Schmidt number to show that for predominantly forced convective flows, the local mass transfer number $Sh/Re^{1/2} Sc^{1/3}$ becomes independent of the Schmidt number.

A similar plot for which there is a strong interaction between free and forced convection cannot be made because of the breakdown of the integral method.

The influence of free convection on forced convective transfer may be observed either by comparison of the solutions at two different values of the parameter Gr/Re^2 , or by comparison of the aiding and the opposing solutions for the same value of the parameter.

The former analysis will be discussed in more detail when the transfer rates which have been predicted for the stagnation point solution are discussed in detail.

Figures 6.45 to 6.48 show the local transfer rates for a 10/16 spheroid for Schmidt numbers of 10.0 and 1000.0, at two different values of the parameter Gr/Re^2 for aiding and opposing flows. These figures show that when the value of the parameter Gr/Re^2 is 0.1, the difference predicted between the mass transfer rates for aiding and opposing flows is very small. This is expected since it was shown earlier that the effect of free convection is minimal when the value of the parameter Gr/Re^2 is less than 0.2. If the curves for aiding and opposing flows were superimposed, the situation would be that of forced convection with negligible effects due to free convection.

For each Schmidt number, the effect of free convection can be seen to be greater for the larger value of the

Fig. 6.45. The Effect of Aiding and Opposing Flows on Local Transfer Rates from a 10/16 Spheroid for a Schmidt Number of 10.0 for a Value Gr/Re^2 of 0.1.

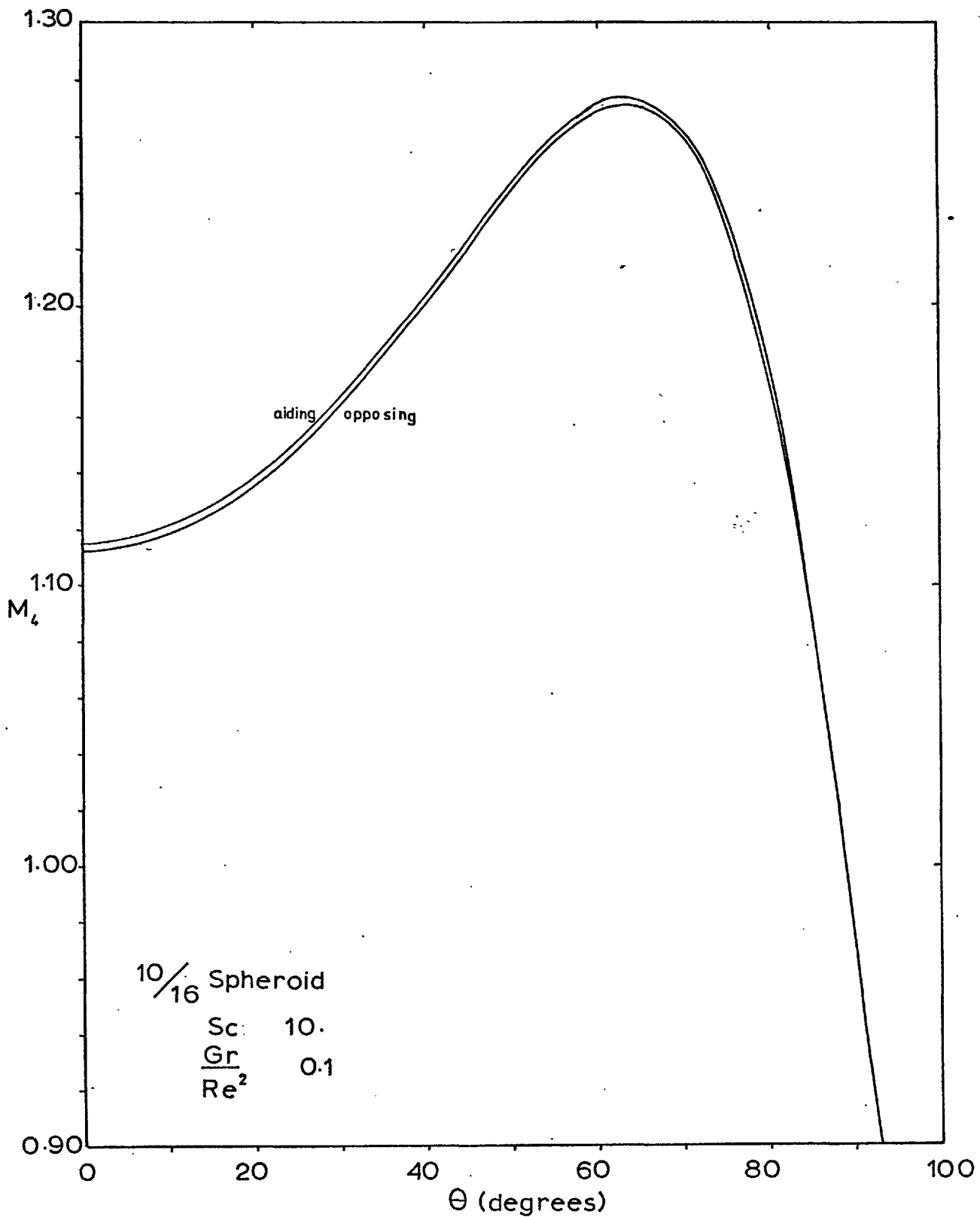


Fig.6.46. The Effect of Aiding and Opposing Flows on Local Transfer Rates from a 10/16 Spheroid for a Schmidt Number of 10.0 for a Value Gr/Re^2 of 1.0.

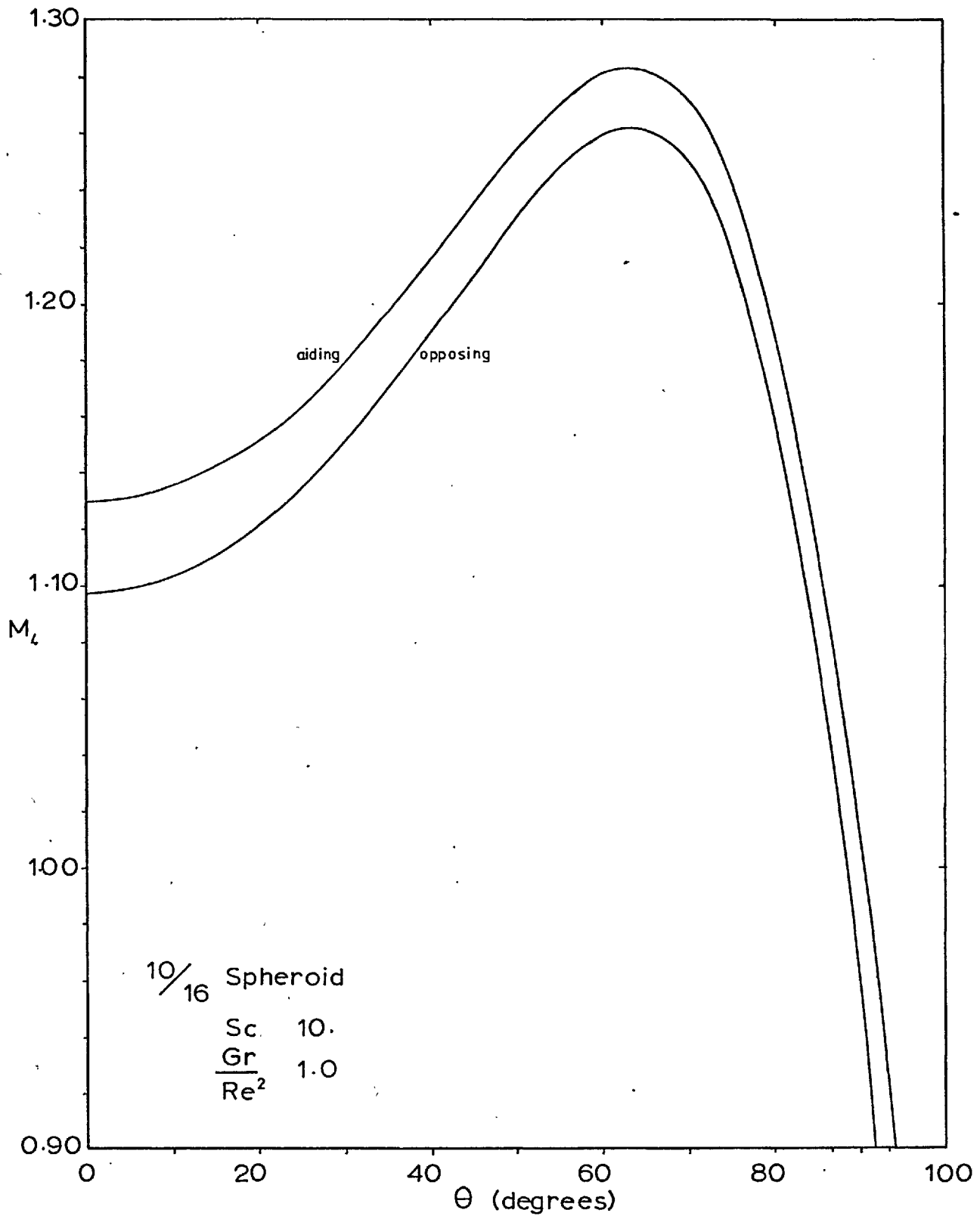


Fig.6.47. The Effect of Aiding and Opposing Flows on Local Transfer Rates from a 10/16 Spheroid for a Schmidt Number of 1000.0 for a Value Gr/Re^2 of 0.1.

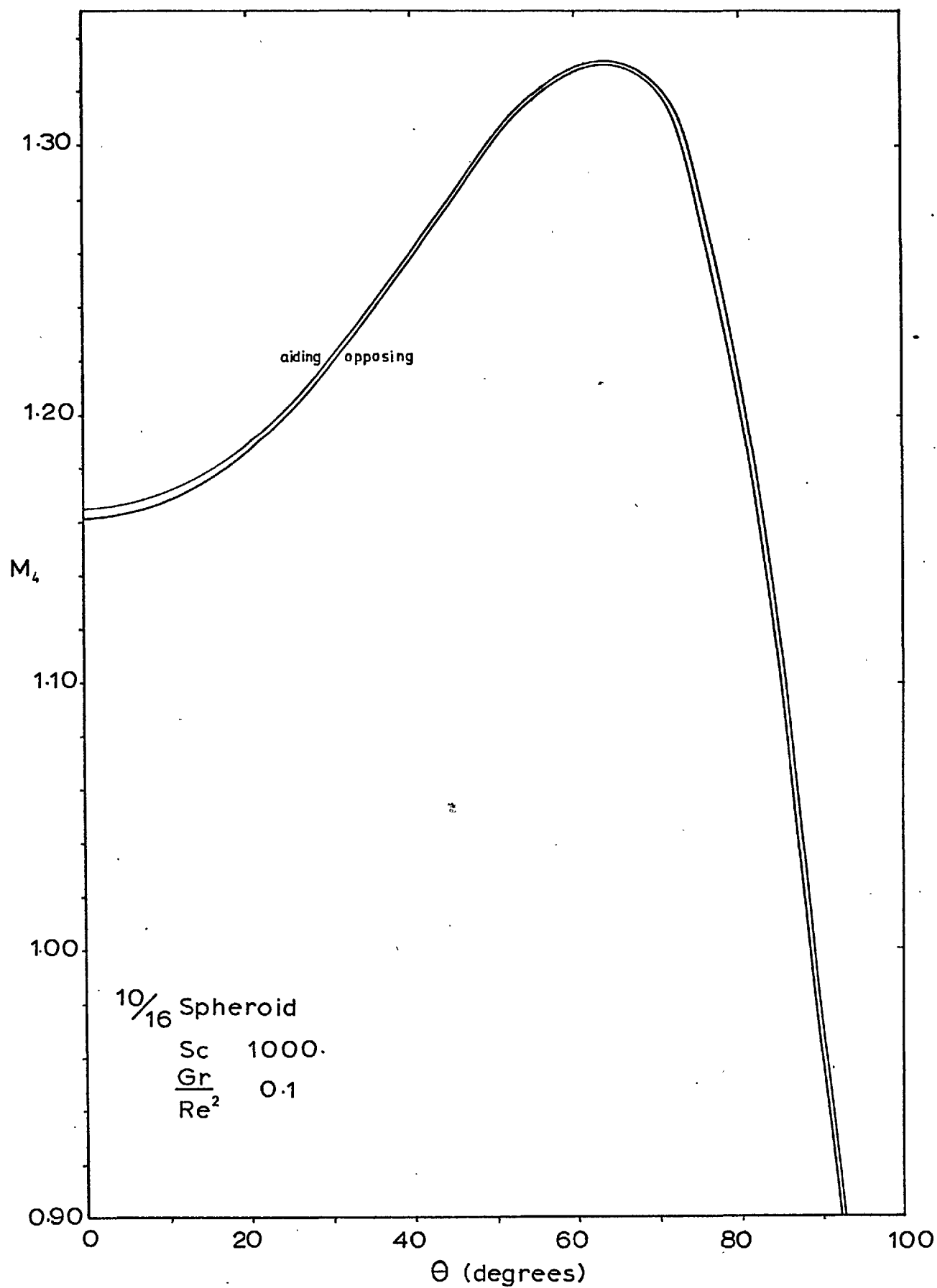
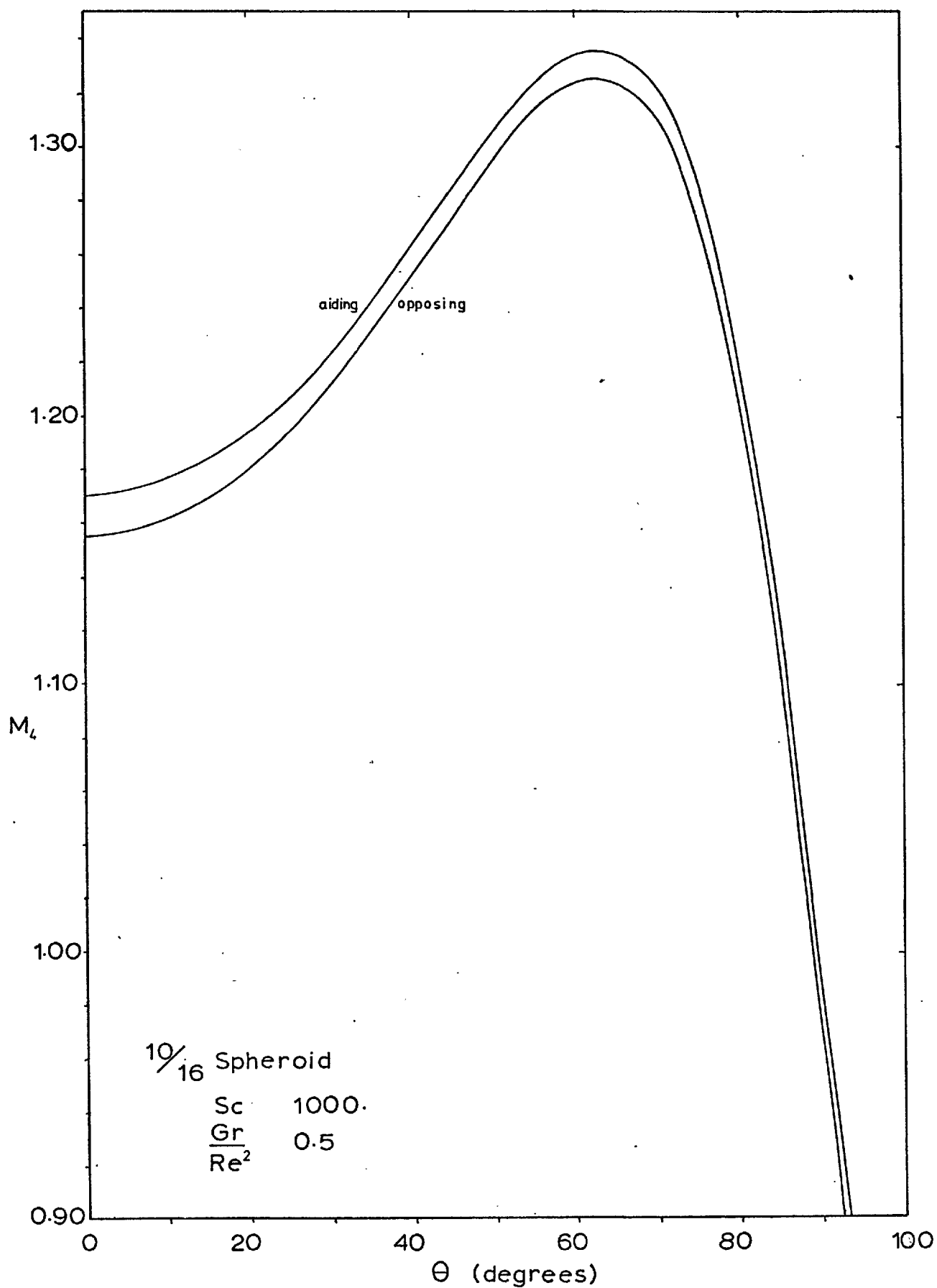


Fig.6.48. The Effect of Aiding and Opposing Flows on Local
Transfer Rates from a 10/16 Spheroid for a Schmidt
Number of 1000.0 for a Value Gr/Re^2 of 0.5.



parameter Gr/Re^2 . The transfer for the opposing flow regime can be seen to be depressed in comparison with the aiding flow.

It is very interesting that the transfer rates predicted for aiding flows are higher than those for opposing flows. As has already been noted in the literature survey, a number of workers have found that overall mass transfer rates for spheres in opposing flows were greater than for aiding flows. It was also noted that some workers have shown that local transfer rates for aiding flows around the body from the stagnation point to the separation point, were higher than those for opposing flows. However, in the wake region transfer in opposing flow regimes is greater than that for aiding flows.

These observations lead directly to a comparison of the separation angles for aiding and opposing flows at the same value of the parameter Gr/Re^2 . Although the approximate method can be relied upon to give only a rough prediction of separation, some very interesting conclusions may be drawn from the results for aiding and opposing flows.

The position of the separation point is determined by the calculation of the velocity gradient at the body surface. This has a value of zero at the separation point. For all of the body shapes considered, the separation point was seen to be nearer the front stagnation point of the body, when the free and forced convective forces were acting in opposition, than when they were acting in the same direction. This behaviour is expected because the buoyancy term in the hydrodynamic equation acts as an adverse pressure

gradient.

The prediction of wake transfer is not possible with boundary layer theory. Therefore no predictions of overall transfer rates can be made. However, it should be noted that the area of the body upstream of separation in the aiding flow regime is greater than for opposing flows. It is conceivable that if the wake transfer were larger for opposing flows than for aiding flows, the overall transfer rates for opposing flows could be larger than that for aiding flows.

In the present work the case of opposing flows was much more difficult to solve with boundary layer theory. Once the influence of free convection on a predominantly forced convective flow became appreciable, the solution procedure became unstable. This instability is not surprising when the visual flow pattern for opposing flows which were described in Chapter 2 are recalled.

Many of the difficulties associated with the interaction regime have arisen because neither the problem of the prediction of local transfer rates around bodies of revolution, nor the vertical flat plate in a uniform free stream, could be treated exactly by the method of similarity transforms. A number of plots have been made in order to show the deviation of the dimensionless velocity distributions at various angles around the body from similar profiles.

These plots, figures 6.49 to 6.53 are presented for a Schmidt number of 2.4 for different values of the parameter Gr/Re^2 . The extent to which the body shape influences the results can be seen from the curves which have been plotted

Fig. 6.49. The Variation in Local Velocity Profiles for a
13/16 Spheroid for a Schmidt Number of 2.4 and
a value Gr/Re^2 of 0.0 .

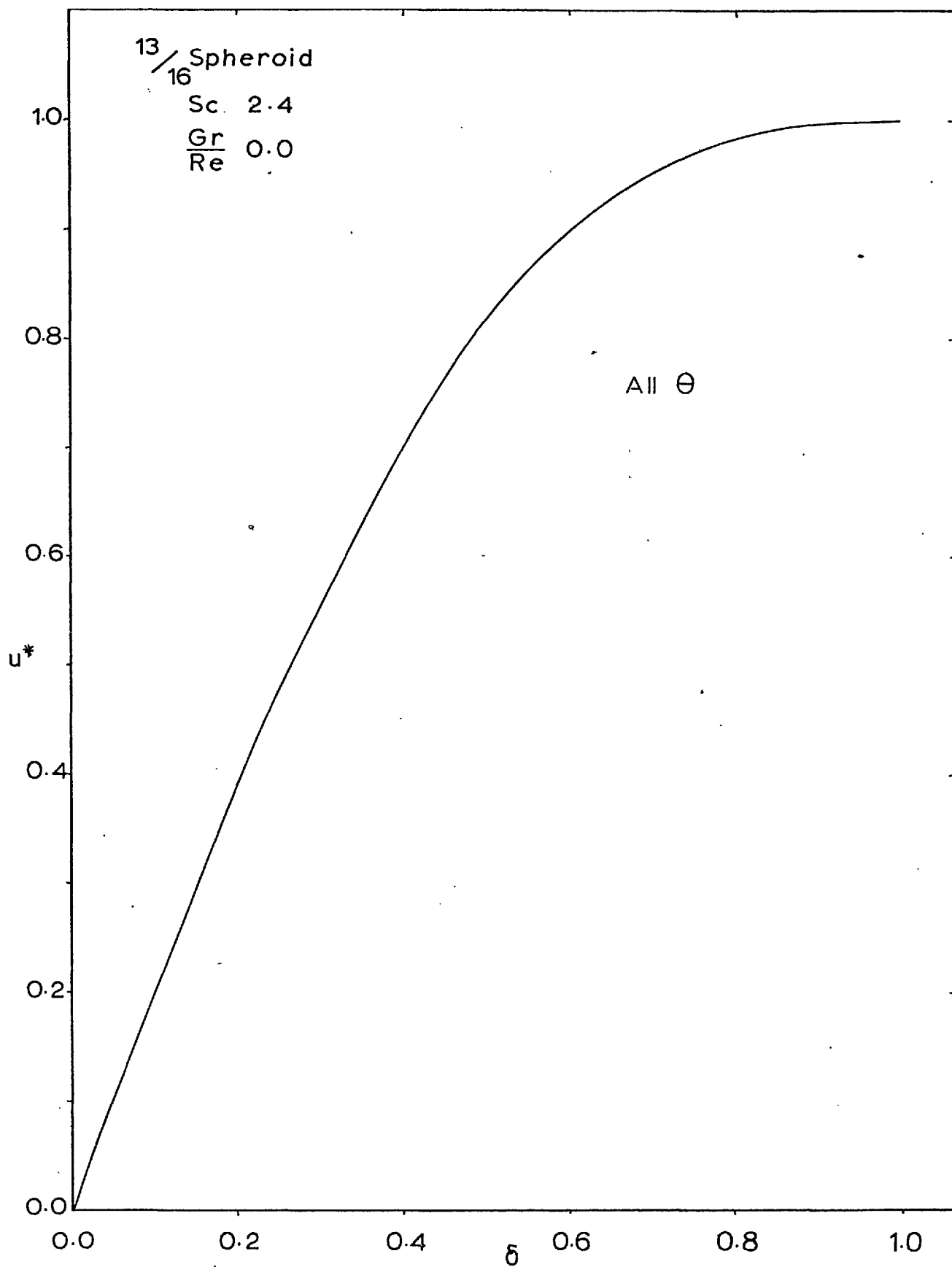


Fig. 6.50. The Variation in Local Velocity Profiles for a $13/16$ Spheroid for a Schmidt Number of 2.4 and a value Gr/Re^2 of 0.2 .

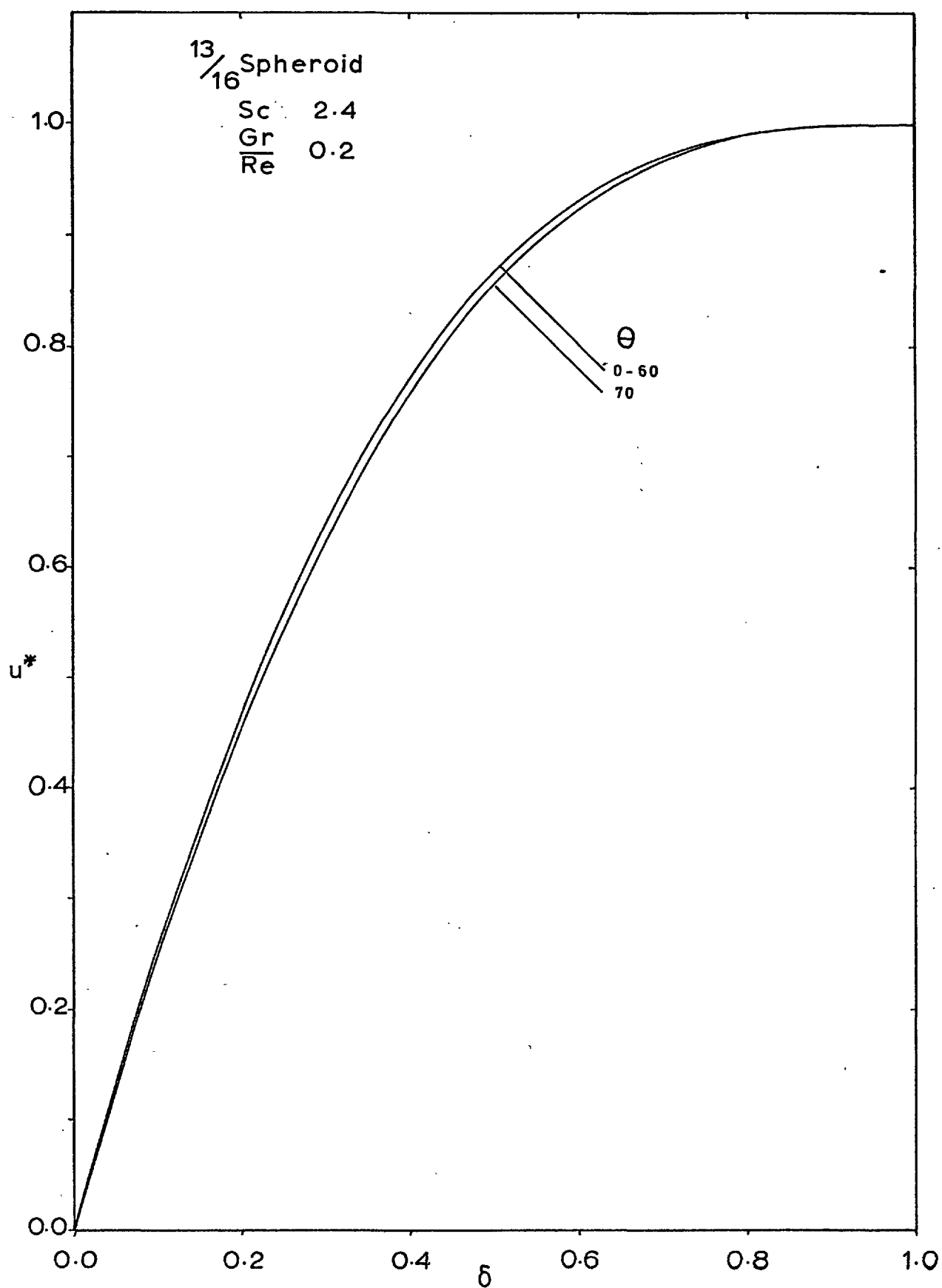


Fig. 6.51. The Variation in Local Velocity Profiles for a 13/16 Spheroid for a Schmidt Number of 2.4 and a value Gr/Re^2 of 3.3 .

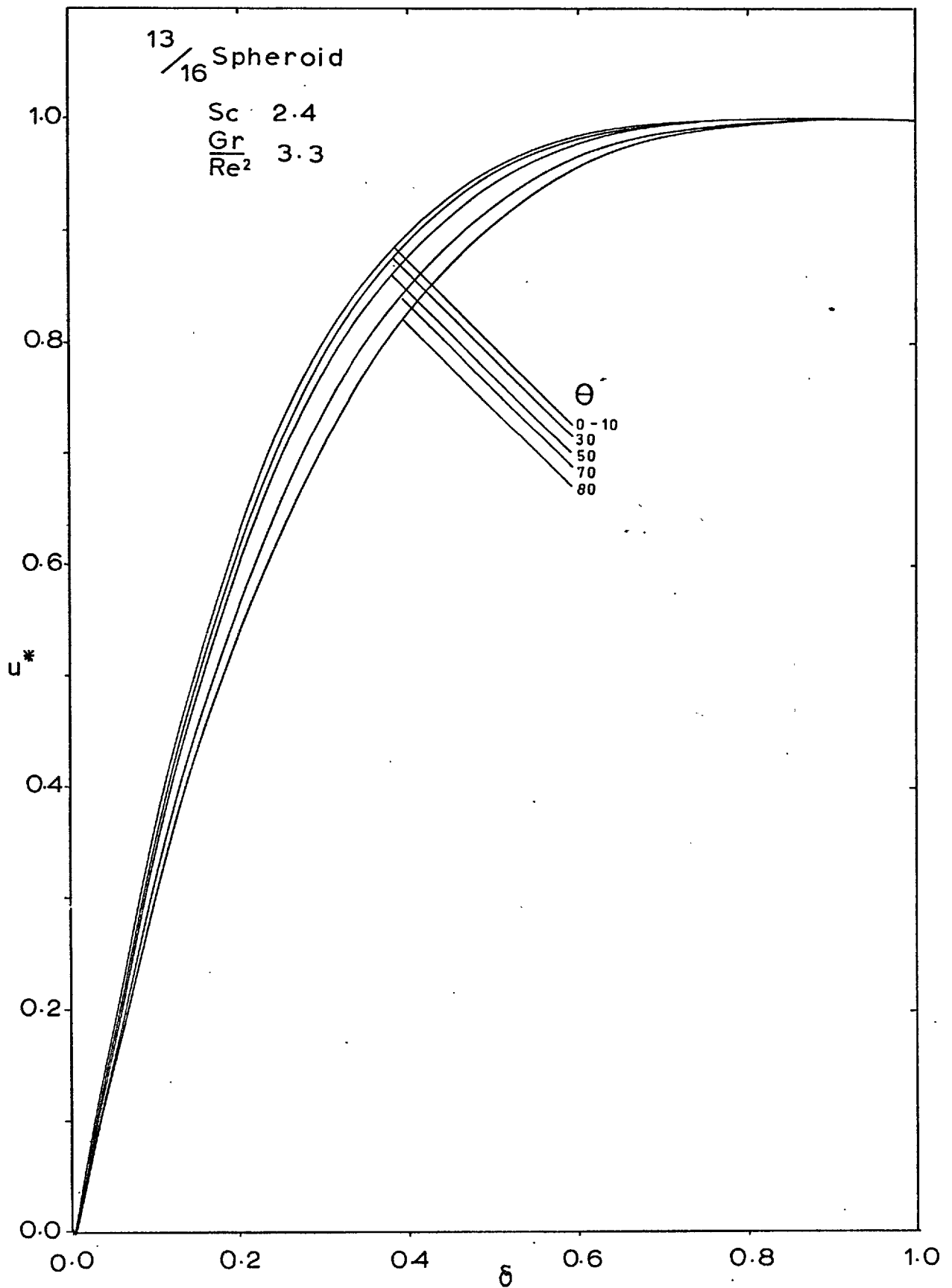


Fig. 6.52. The Variation in Local Velocity Profiles for a 7/16 Spheroid for a Schmidt Number of 2.4 and a value Gr/Re^2 of 0.2 .

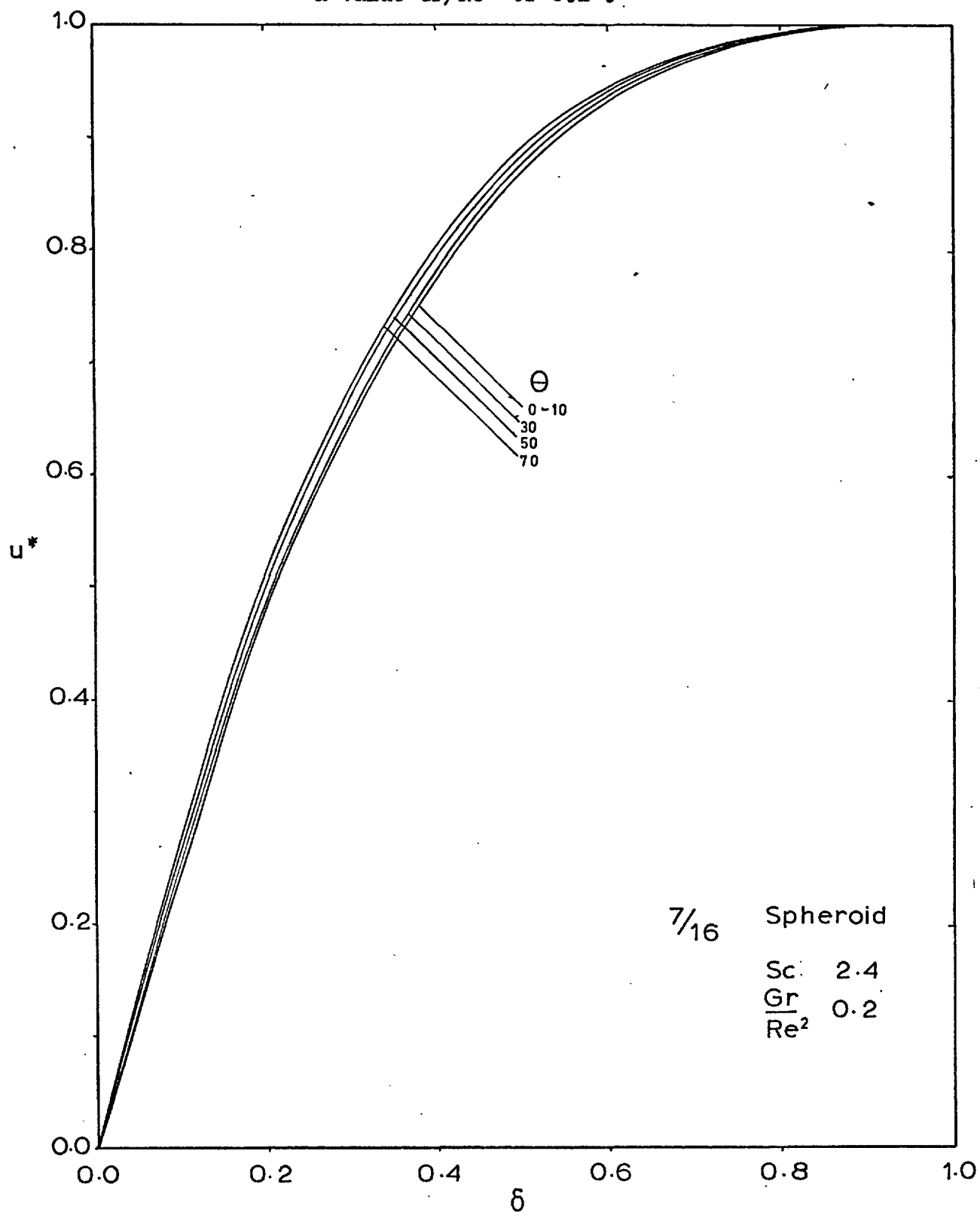
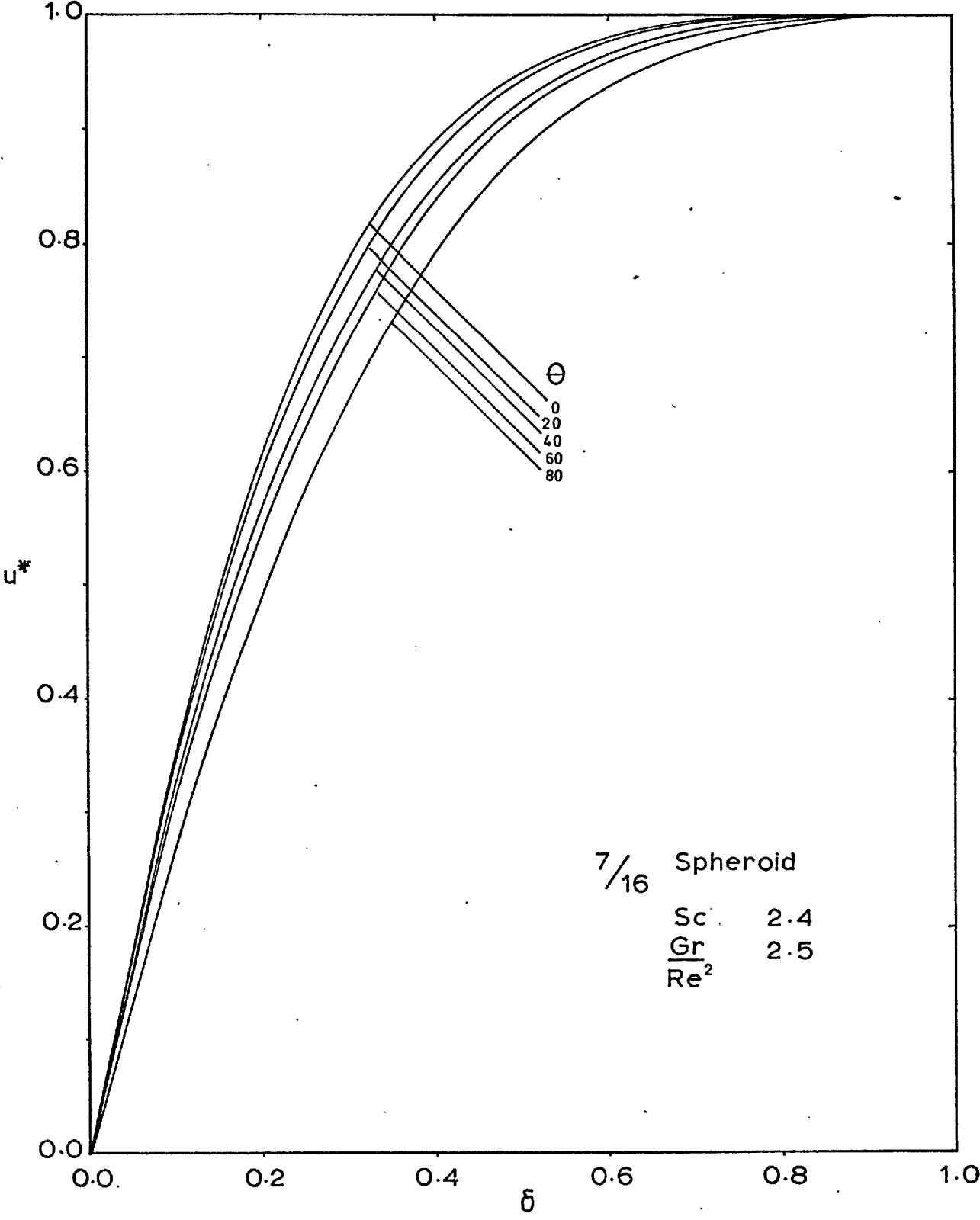


Fig. 6.53. The Variation of Local Velocity Profiles for a 7/16 Spheroid for a Schmidt number of 2.4 and a value Gr/Re^2 of 2.5 .



for the 13/16 and 7/16 spheroids.

For the 13/16 body, the dimensionless velocities at progressively larger angles from the front stagnation point are smaller than the velocities at smaller angles. However, for the 7/16 spheroid although this is true for the larger value of the parameter Gr/Re^2 , the opposite is seen to occur for the predominantly forced convective flow. This is probably caused by the thinning of the boundary layers which has already been mentioned.

6.3.2. Comparison of Transfer Rates for the Limiting Solutions of Free and Forced Convection.

In order to assess the reliability of the present method, the predicted transfer rates for the regimes of pure free convection and pure forced convection are compared with those of other workers. There have been no previous experimental or theoretical predictions of local mass transfer rates for the interaction regime with which the present results can be compared.

Garner and his co-workers^{10,11} reported some local transfer rates from spheres in the interaction regime for a system with a Schmidt number of greater than 1000.0. These results cannot be used since the present approximate method could not be used for high Schmidt numbers in the interaction regime.

In figures 6.54 and 6.55 the local transfer rates which were calculated by the use of the approximate method are presented for a sphere in a free convective flow. The results of other workers are also presented in these plots.

Figure 6.54 shows the exact solutions of Chiang et al.⁹⁷ and Pandya⁷⁸ together with the present approximate solution and that of Pandya. The agreement between the predicted transfer rates can be seen to be very good. Those of the present work are within one and a half per cent of the transfer rates predicted by the other workers.

In figure 6.55 the present approximate solution and that of Pandya are plotted with the experimental data of Kranse and Schenk⁹⁸. The agreement between both of the approximate solutions and the experimental data is very good around the body to an angle of 140° . Over the front half of the body, the agreement between the two approximate solutions is within five per cent.

In figures 6.56 and 6.57 the forced convective transfer rates calculated in the present work, for the sphere and the 7/16 spheroid, are compared with a number of other theoretical and experimental results.

The transfer rates for the sphere are within five per cent of the exact solutions of Green⁷³ and Frössling⁹¹, and the approximate solution of Rojey⁷⁴. The curve also lies within the experimental data of Beg⁹⁶ around the body to an angle of approximately 70° . The predictions of Akselrud's⁹⁹ approximate

Fig. 6.54. Comparison of Theoretical Free Convective Transfer Rates for the Sphere.

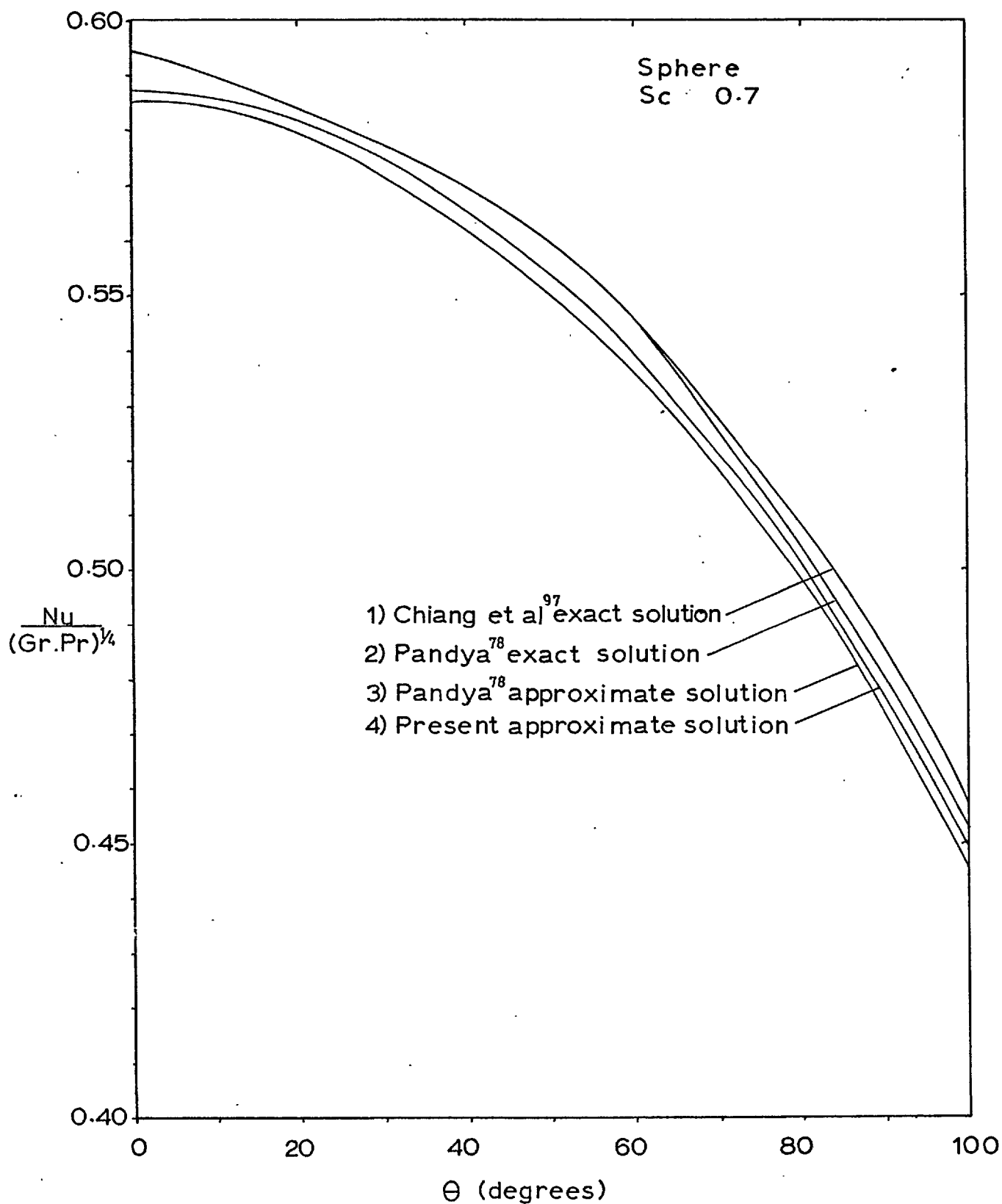
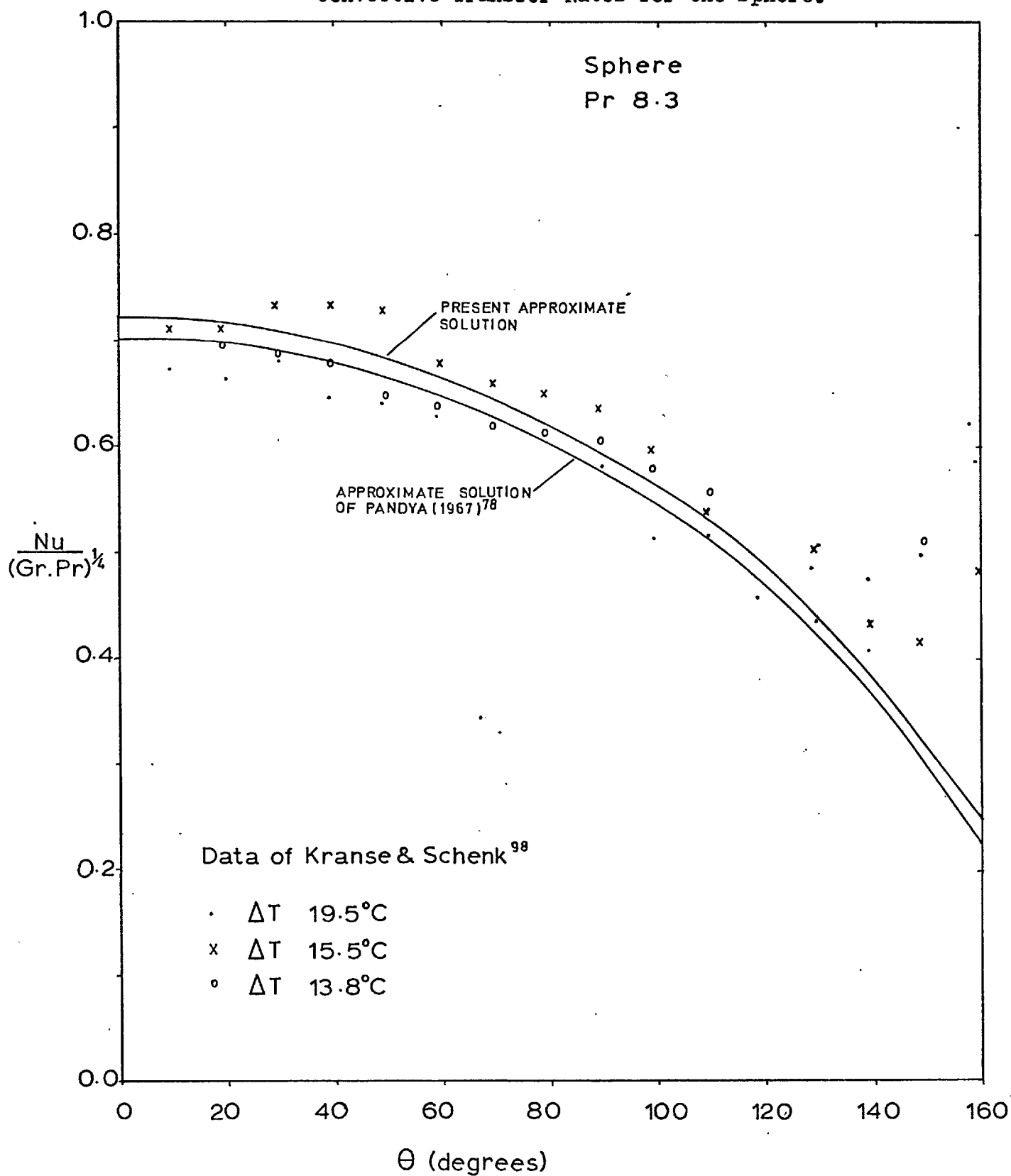
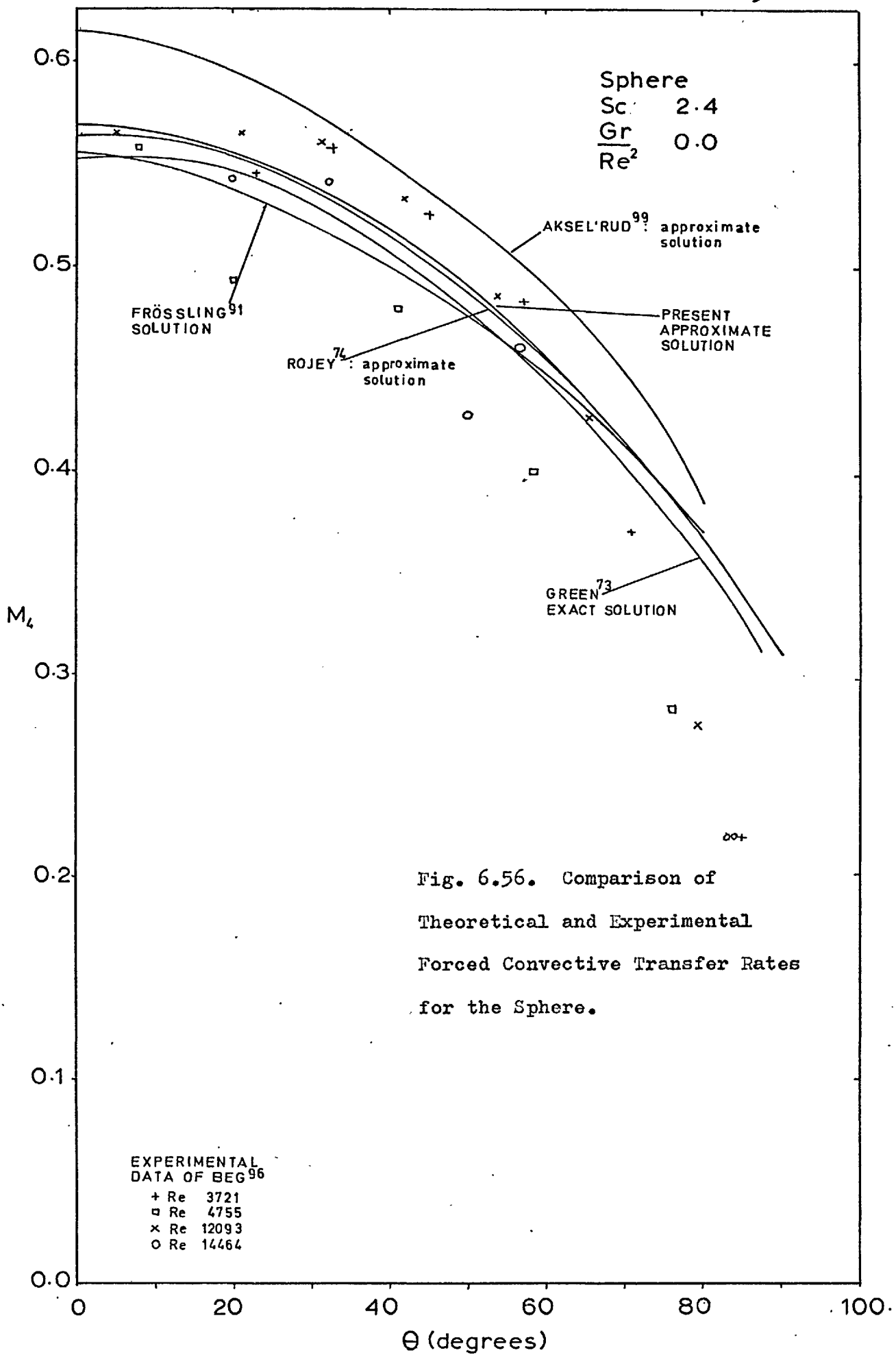
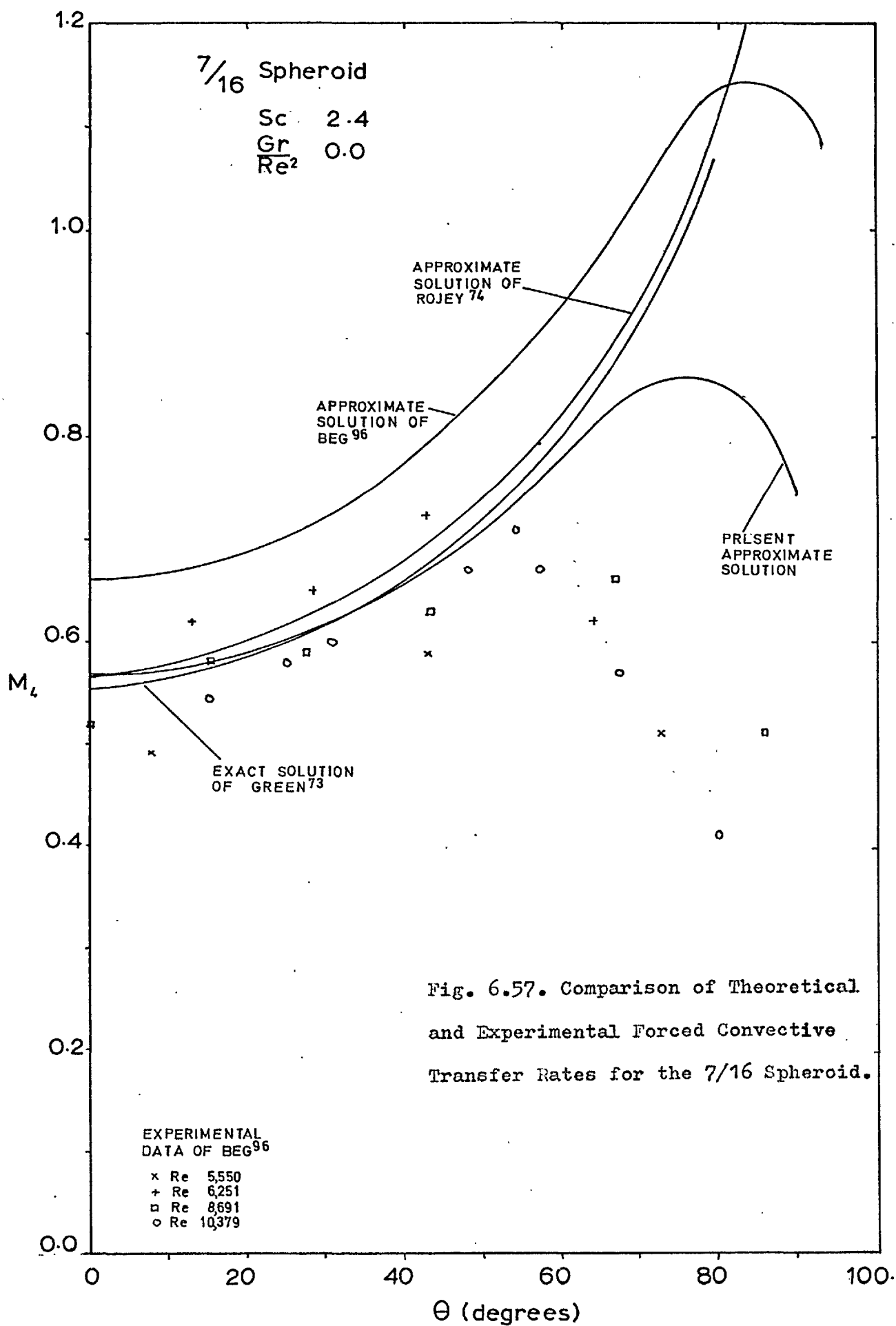


Fig. 6.55. Comparison of Theoretical and Experimental Free Convective Transfer Rates for the Sphere.







method are eight percent higher. This probably arises because his assumption, that the transfer boundary layer is thin in comparison with the hydrodynamic boundary layer, is not valid at a Schmidt number of 2.4.

The comparison of results for the 7/16 spheroid also shows very good agreement around the body to an angle of approximately 55 degrees. After this the experimental transfer rates of Beg fall off more rapidly than the theoretical predictions. Beg's theoretical predictions are higher than those of the other theoretical work because of the approximate method which he used to calculate the transfer boundary layer thickness.

6.4. Mass Transfer Results, Exact Stagnation Point Solutions.

The stagnation point solution is independent of body shape since the shape parameters were removed from the first term of each power series expansion which was used in Chapter 5.

As has been discussed in connection with the local transfer rates, the mass transfer rates for the stagnation point are influenced by the parameter $\overline{Gr}/\overline{Re}^2$, the Schmidt number, and whether the free and forced convective forces are acting in the same direction or whether they are in opposition.

The influence of these various factors on the velocity and concentration profiles has been discussed in detail earlier in this chapter. Since the

concentration gradient at the surface gives a direct measure of the transfer rate, further general discussion of the effect of these parameters is unnecessary. Therefore, the mass transfer rates will be presented and discussed on a quantitative basis.

The predicted values of the velocity and concentration gradients at the surface, $f_1''(0)$ and $F_0'(0)$, for aiding flows are presented in Table C.8 for a range of Schmidt numbers and values of the parameter $\overline{Gr}/\overline{Re}^2$. The mass transfer rates are plotted in Fig. 6.58 as the mass transfer number $Sh/\overline{Re}^{1/2}$ against the parameter $\overline{Gr}/\overline{Re}^2$.

The mass transfer number for each Schmidt number is approximately constant for very small values of the parameter $\overline{Gr}/\overline{Re}^2$. This represents the stagnation point solution for predominantly forced convection.

Free convection does not influence the mass transfer number until the parameter $\overline{Gr}/\overline{Re}^2$ has a value of approximately 0.1. As the parameter $\overline{Gr}/\overline{Re}^2$ is increased further, the mass transfer number then begins to increase for all Schmidt numbers. For smaller values of the Schmidt number the increase is more rapid. This would be expected from the earlier discussion of the effect of the Schmidt number on the concentration profiles.

Free convective forces play an increasingly dominant role as the value of the parameter $\overline{Gr}/\overline{Re}^2$ continues to increase. The mass transfer number increases and deviates more and more from its forced convection value. When free convection becomes more

Fig. 6.58. The Plot of $Sh/Re^{1/2}$ versus Gr/Re^2 for the Exact Stagnation Point Solution for Aiding Flows.

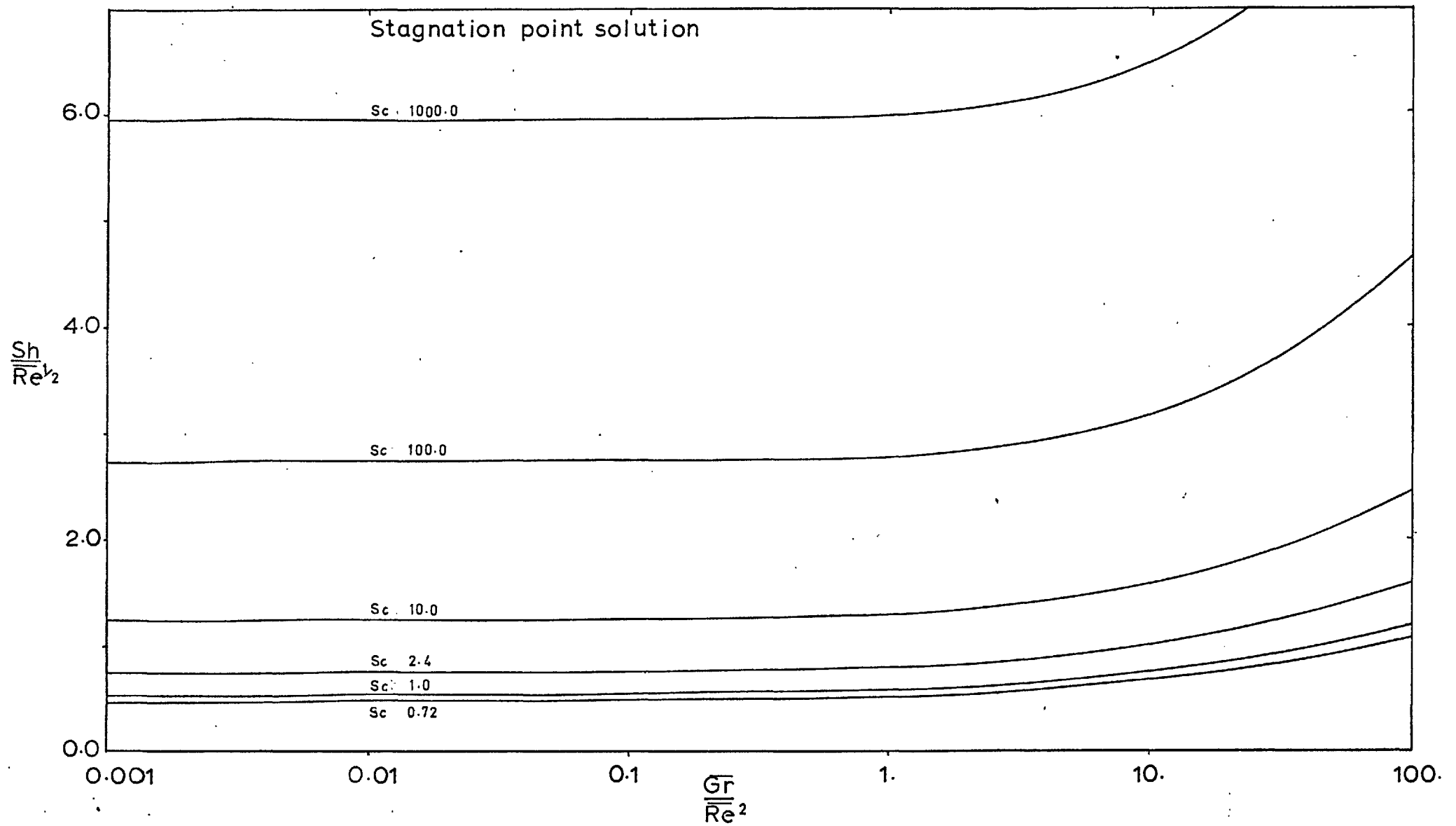
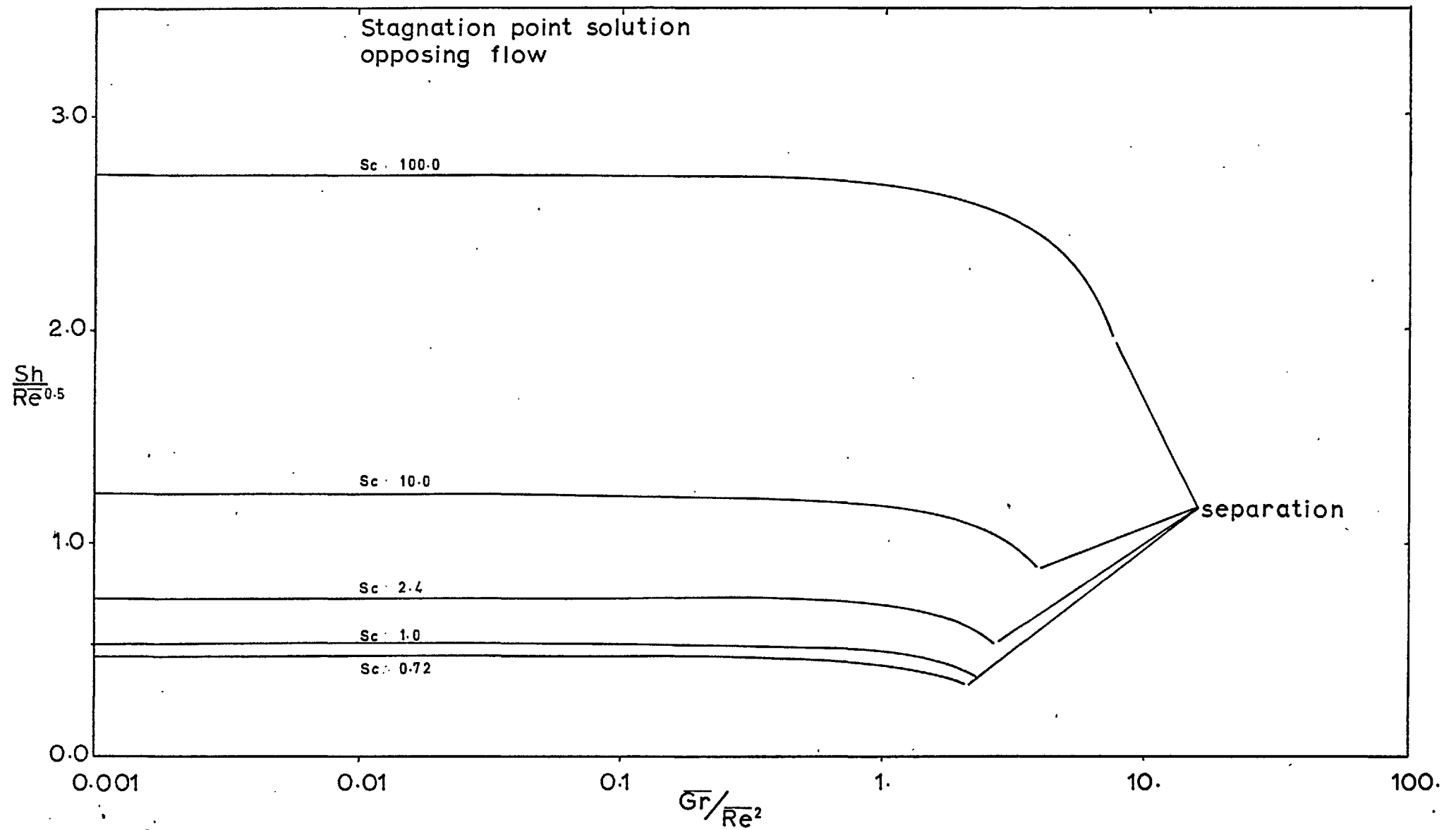


Fig. 6.59, The Plot of $\overline{Sh}/\overline{Re}^{0.5}$ versus $\overline{Gr}/\overline{Re}^2$ for the Exact Stagnation Point Solution for Opposing Flows.



dominant, the mass transfer becomes independent of the Reynolds number, i.e. of the forced convective forces.

The value of the parameter $\overline{Gr}/\overline{Re}^2$ at which there is a five per cent deviation from either of the limiting cases may be readily calculated. These two values show the range of values of the parameter $\overline{Gr}/\overline{Re}^2$ over which the interaction of free and forced convection is of importance.

It is found that for Schmidt numbers of 0.72, 2.4, 10.0 and 1000.0, the values of the parameter $\overline{Gr}/\overline{Re}^2$ at which the mass transfer number is five per cent are different from those for pure forced convection which are 0.5, 0.8, 1.1 and 5.0 respectively. As the effects of free convection further increase, forced convective transfer accounts for only 5% of the total transfer for Schmidt numbers of 0.72, 2.4, 10.0 and 100.0, when the parameter $\overline{Gr}/\overline{Re}^2$ has reached values of approximately 20, 25, 30 and 50 respectively.

For a Schmidt number of 0.72, the maximum deviation of the mass transfer number in the interaction region, from the mass transfer number of the dominant pure convective flow is 25%. For a Schmidt number of 1000.0, the corresponding maximum deviation is only 10%. Thus the interaction of free and forced convection can be seen to have a greater effect on the mass transfer for small Schmidt numbers.

When the opposing flow regime was considered earlier, it was noted that the boundary layer thickness increased as the opposing free convective forces

became more dominant. The concentration gradient, and hence the mass transfer number therefore decrease until the point at which backflow, (separation), is observed.

The surface velocity and concentration gradients for the stagnation point solution are tabulated in Table C.9 for Schmidt numbers from 0.72 to 1000.0, and the mass transfer numbers are plotted in figure 6.59.

6.5. Comparison of the Mass Transfer Predictions of the Approximate and Exact Methods for the Stagnation Point Solution.

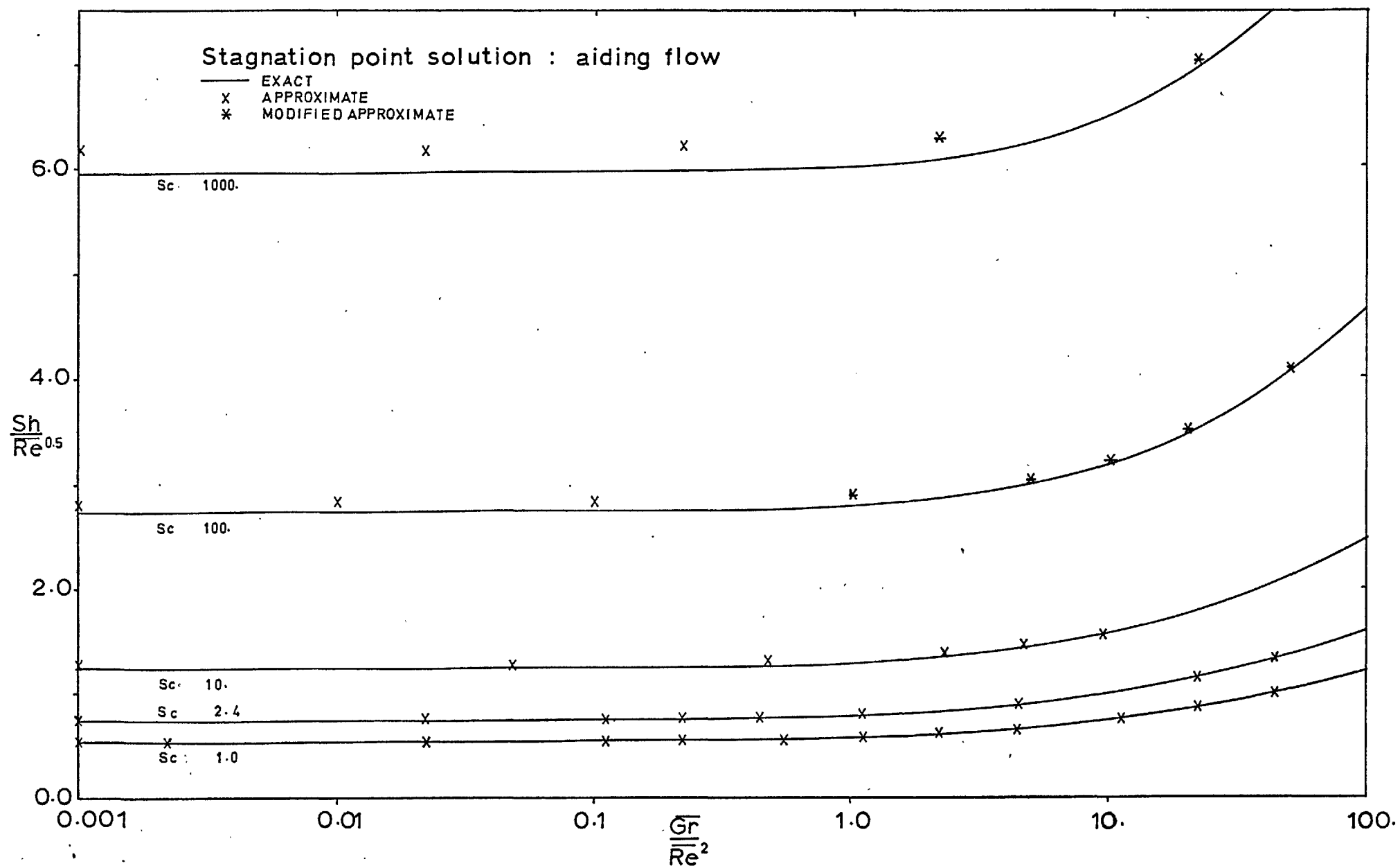
Figure 6.60 shows a comparison between the stagnation point solutions which have been calculated by the exact method and the two main approximate methods which have been developed in the present thesis.

The results for the two larger Schmidt numbers, 100.0 and 1000.0, for values of the parameter $\overline{Gr}/\overline{Re}^2$ greater than 2.0. were calculated by the use of the modified approximate method described in Section 4.5. The remaining results were calculated by the use of the approximate method which was developed in Section 4.2.

The results for the exact solution and the original approximate method show very good agreement, to within five per cent, for all Schmidt numbers at all values of the parameter $\overline{Gr}/\overline{Re}^2$ for which transfer rates have been predicted.

The predictions which were obtained by the use of the modified integral method show the same very good agreement with the exact solutions over the region

Fig. 6.60. The Comparison Between the Approximate and Exact Stagnation Point Solutions for Aiding Flows.



for which the original integral method failed. This integral method failed above a value of the parameter $\overline{Gr}/\overline{Re}^2$ of approximately 50.0 for Schmidt numbers of 100.0 and 1000.0.

The excellent agreement between the exact and approximate methods, enables the same criteria as described in Section 6.4. to be used to describe the interaction region.

This comparison suggests that when the simple approximate method works it can be relied on to predict accurate transfer rates in the interaction regime. The modified approximate method can be seen to predict accurate transfer rates over a larger range of values of the parameter $\overline{Gr}/\overline{Re}^2$ than was possible with the original method. However, because of the large amount of extra work involved with the modified method, and the fact that the method still fails, it is not thought that the method is of any great practical value.

The use of the quasilinearization technique has made the calculation of the exact stagnation point solution very straightforward, and computationally fast. This solution could be used as an initial condition to start a step by step finite difference method to predict local transfer rates. It is thought that over the region for which the approximate method fails, it would be more profitable to use such a step by step method rather than to use a very complicated integral method.

6.6. Mass Transfer Results, The Vertical Flat Plate.

The mass transfer solutions for the vertical flat plate, using the forced convection transforms, which correspond with those discussed above for the stagnation point solution, are presented in figures 6.61 to 6.65, and in Tables C.10 and C.11. As can be seen from the graphs and tables, the same behaviour which was observed for the stagnation point solution is again evident and further detailed discussion is unnecessary. The solutions which have been obtained for the vertical flat plate are now discussed in relation to the results of other workers who have used both approximate and exact methods.

Before discussing the mass transfer results which were predicted by the approximate and exact methods, the errors which were introduced by Acrivos⁴¹ when he kept the hydrodynamic boundary layer thickness constant, are discussed. In the region for which he only allowed the ratio of the two boundary layers to vary, the ratio was allowed to increase to a value of 8/9. The plots of the ratio for Schmidt numbers of 1.0, 2.4, 10.0 and 1000.0 are presented in figures 6.66 to 6.69.

The ratio which was calculated by the exact method has also been plotted. The edge of each boundary layer was assumed to have been reached when the velocity and concentration were within 1% of their asymptotic values. Therefore, the ratio is simply an approximation, based on this assumption, which indicates the trend of the variation of the ratio. At low Schmidt numbers, the range of values of the parameter Gr/Re^2 over which the hydrodynamic boundary layer thickness was kept constant can be seen to be small. Also the general

Fig. 6.61. The Comparison Between the Present Exact Solution for the Vertical Flat Plate and that of Merkin⁵¹ for Aiding Flows.

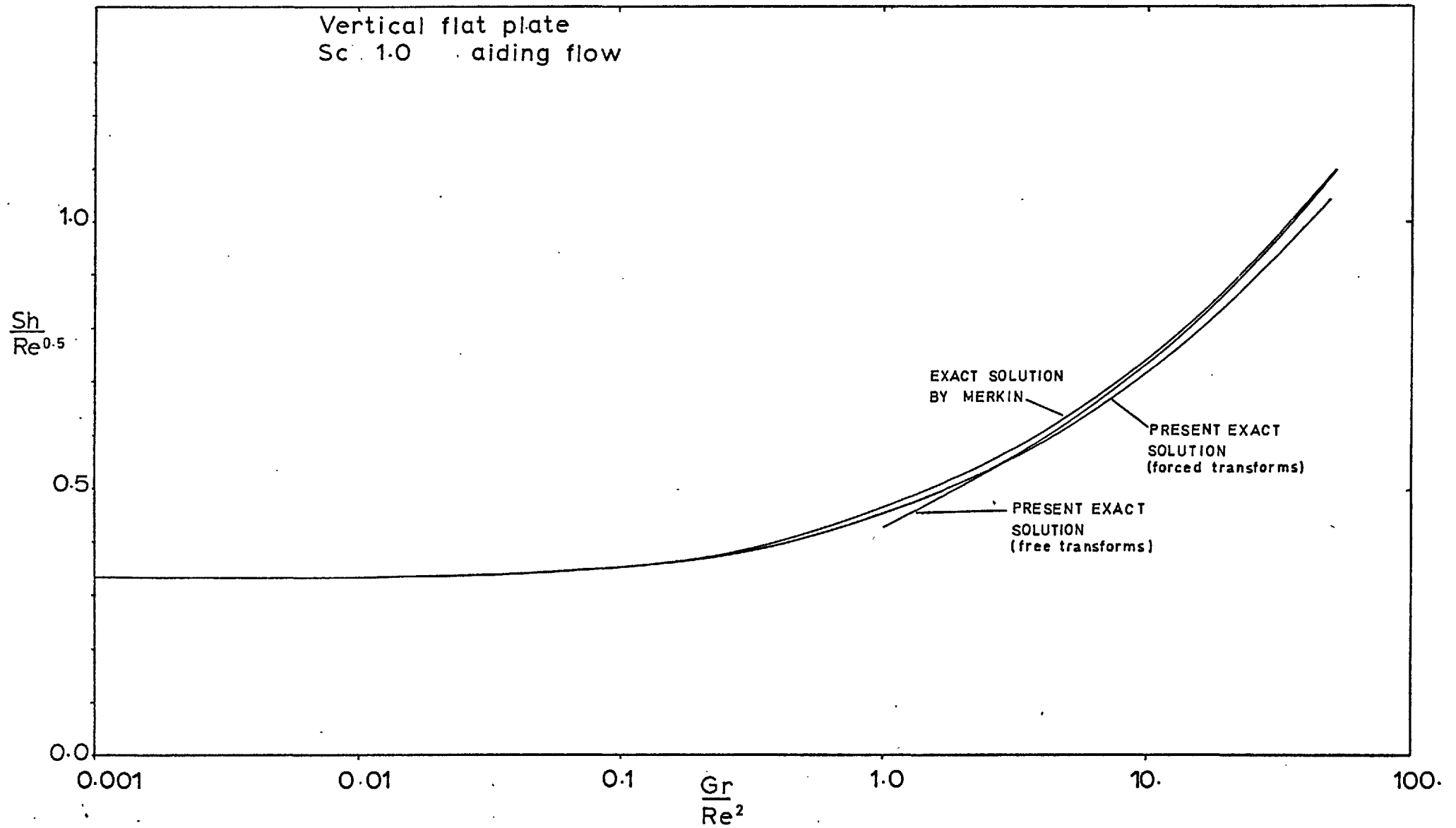


Fig. 6.62. Comparison Between the Present Exact Solution for the Vertical Flat Plate and that of Merkin⁵¹ for Opposing Flows.

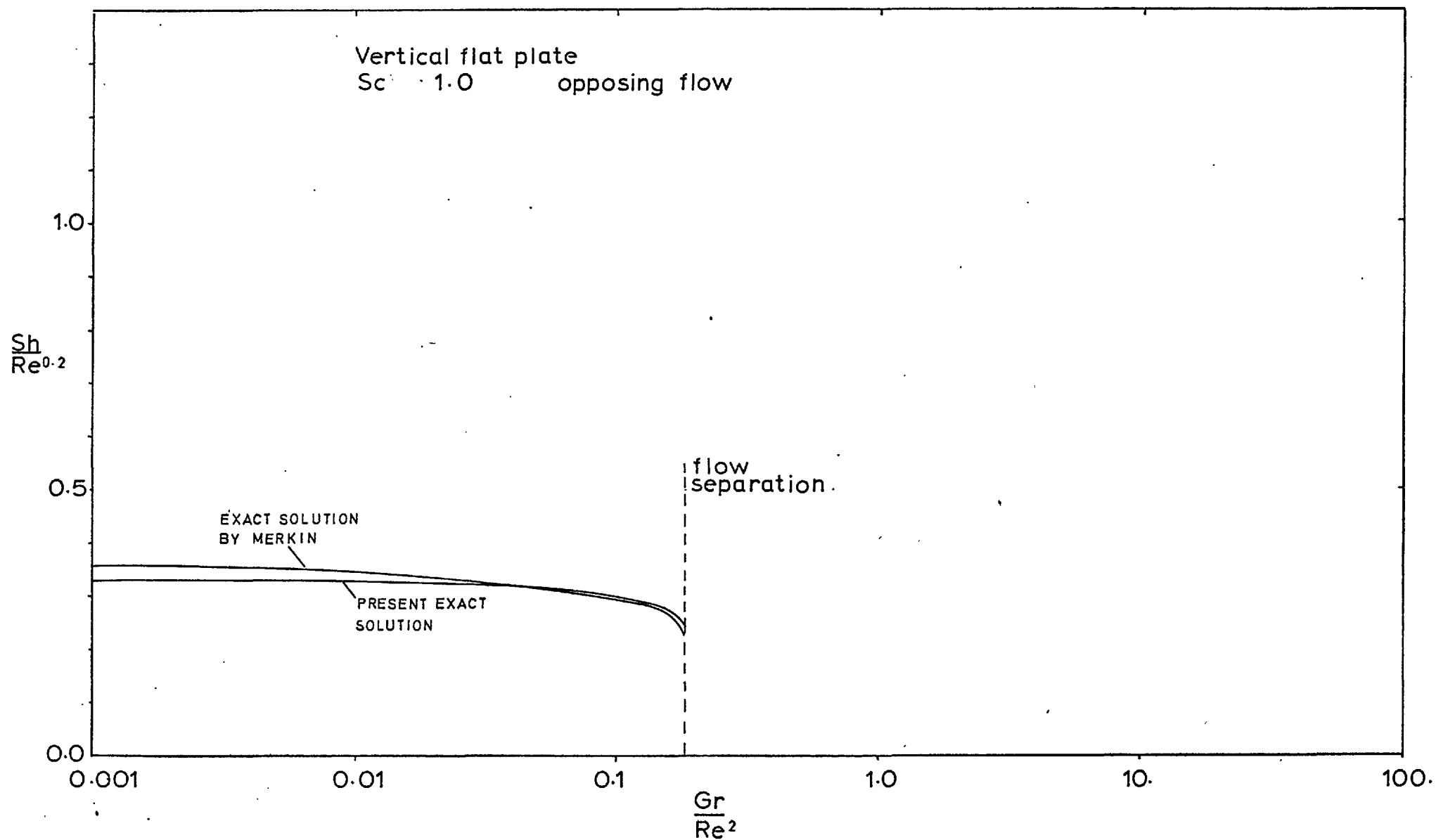


Fig. 6.63. Comparison Between the Present Exact Solution and the Approximate Solution of Acrivos⁴¹ for the Vertical Flat Plate for a Schmidt Number of 2.4.

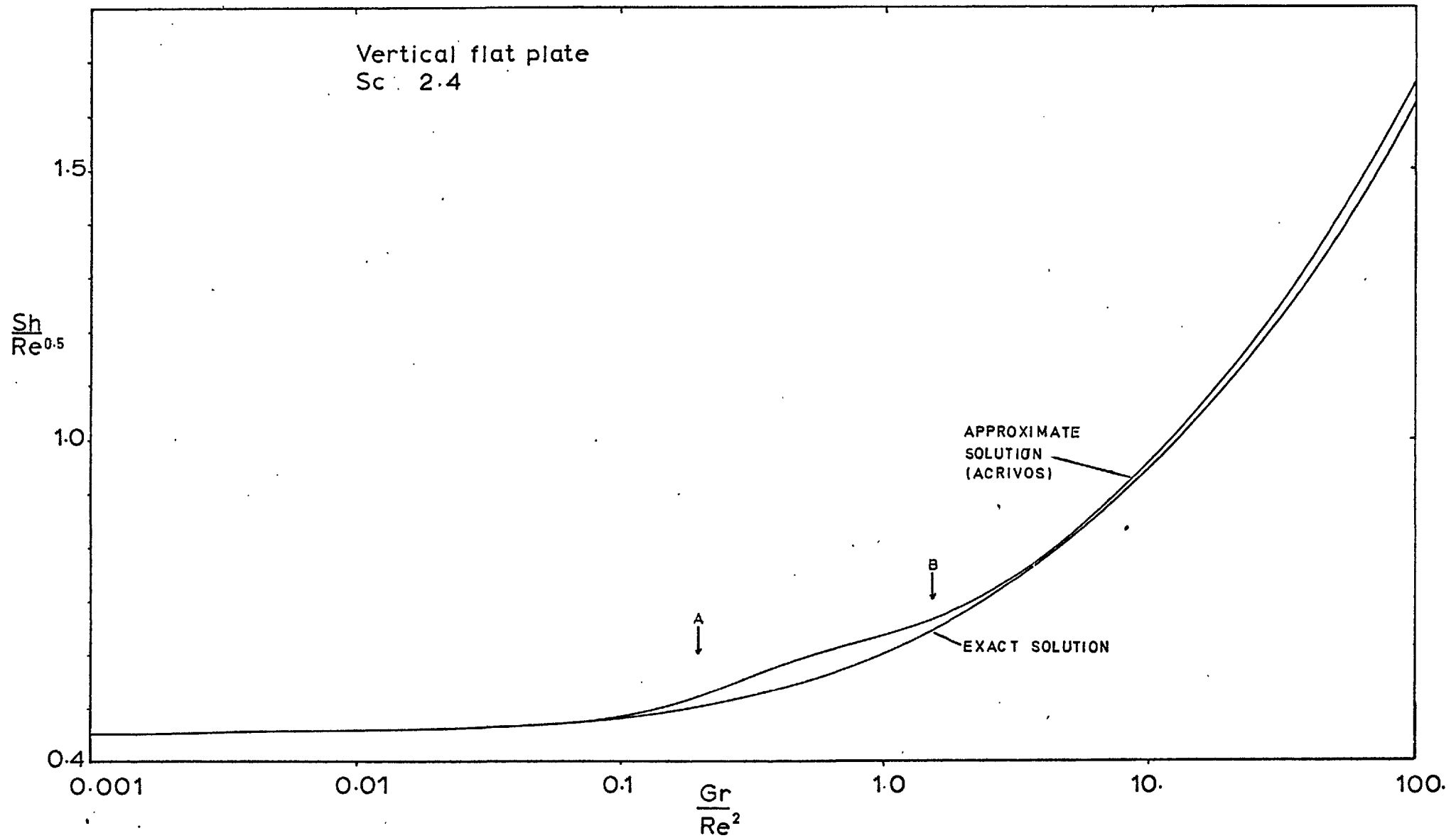


Fig. 6.64. Comparison Between the Present Exact Solution and the Approximate Solution of Acrivos⁴¹ for the Vertical Flat Plate for a Schmidt Number of 10.0.

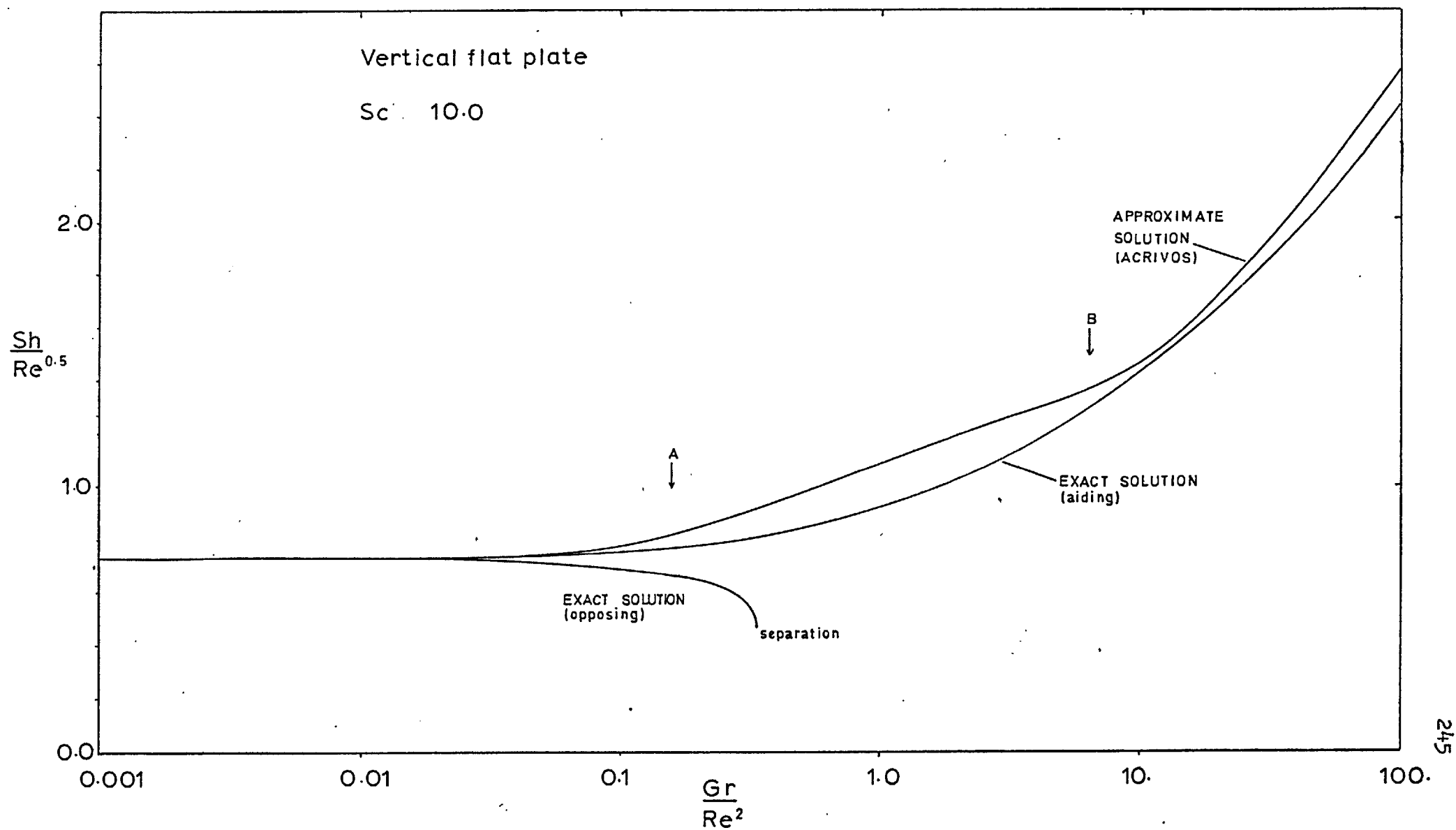
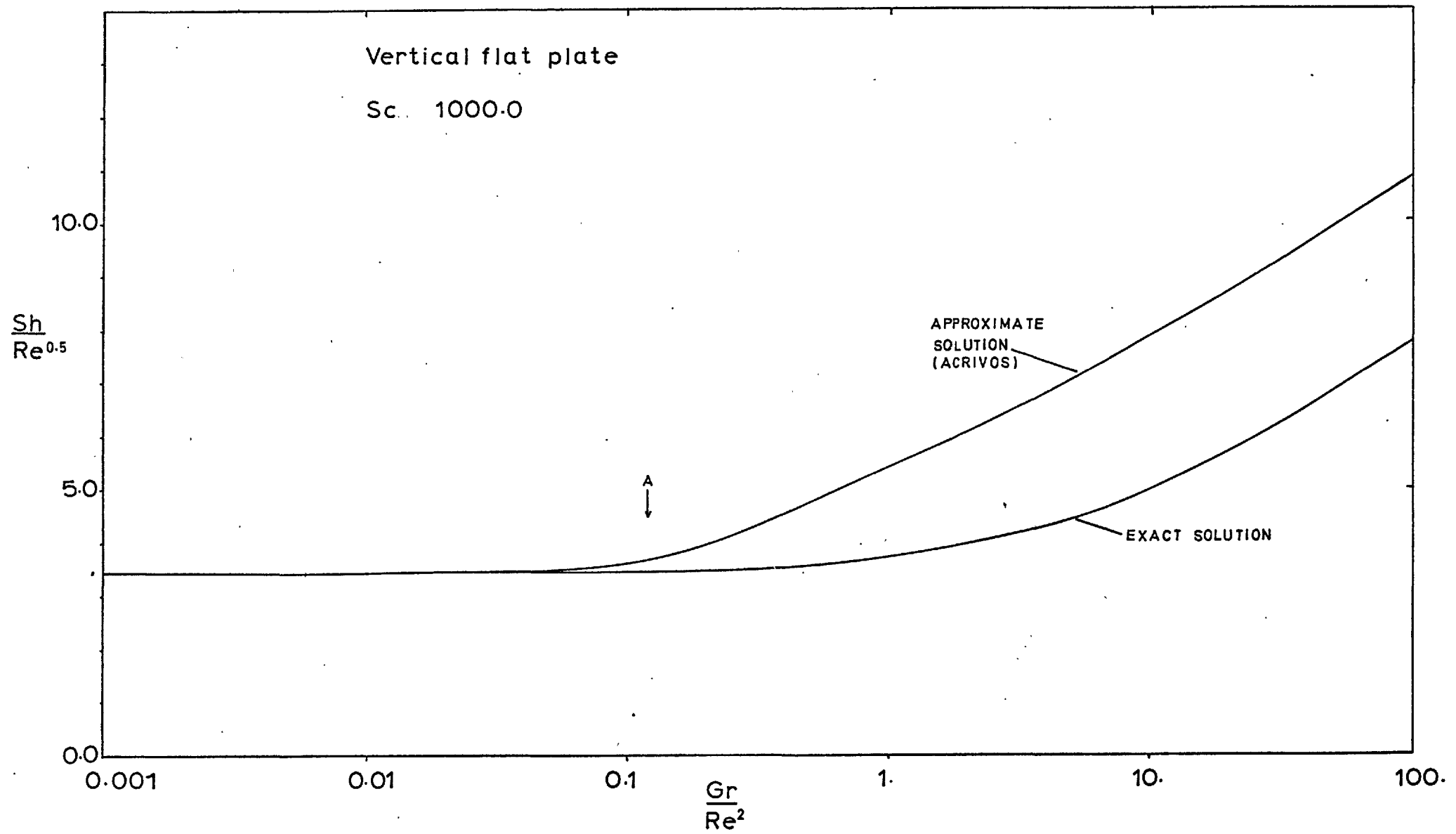


Fig. 6.65. Comparison Between the Present Exact Solution and the Approximate Solution of Acrivos for the Vertical Flat Plate for a Schmidt Number of 1000.0.



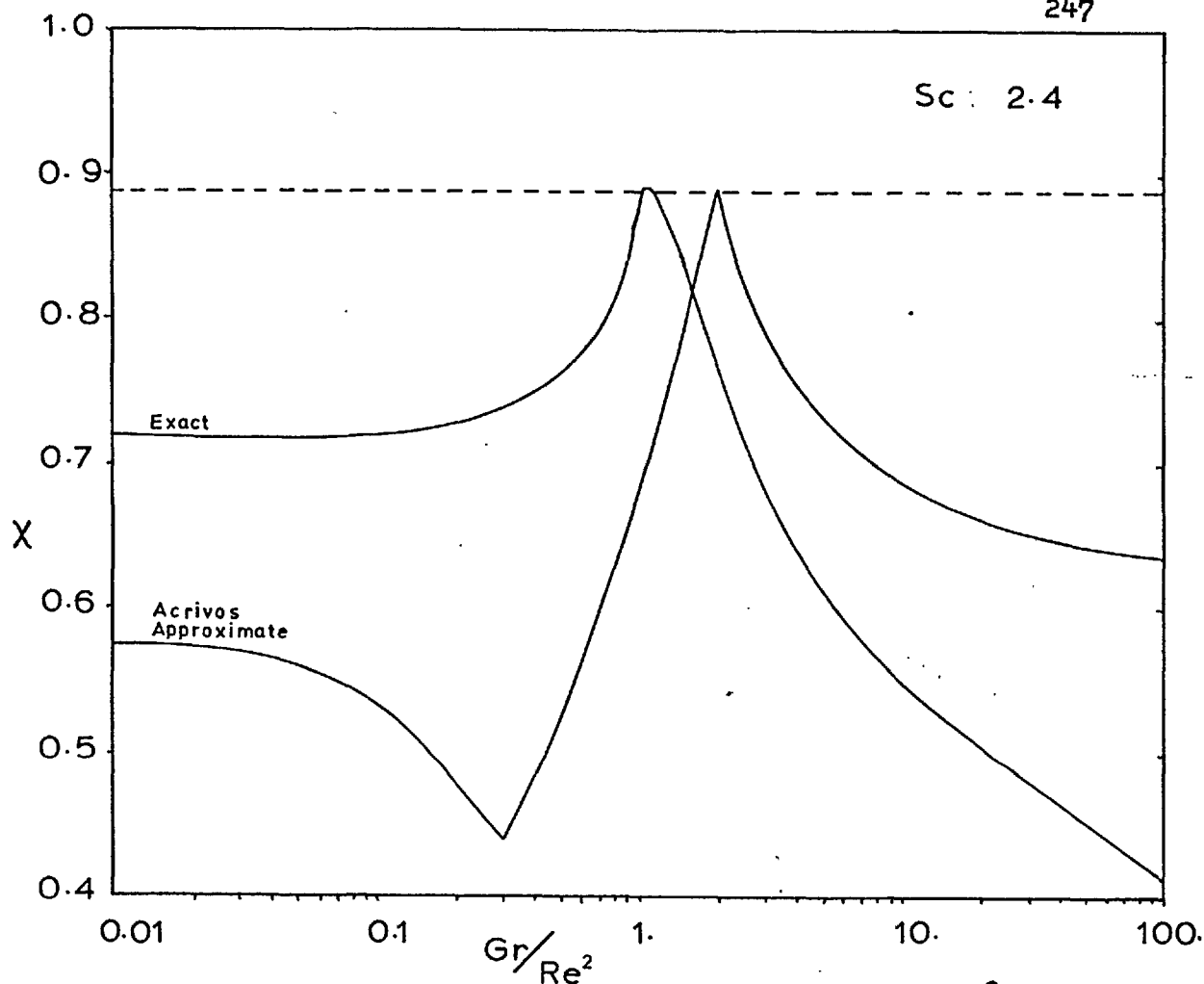


Fig. 6.67. The Variation of X with the value Gr/Re^2 for the Exact and Approximate Solutions for a Schmidt Number of 2.4 .

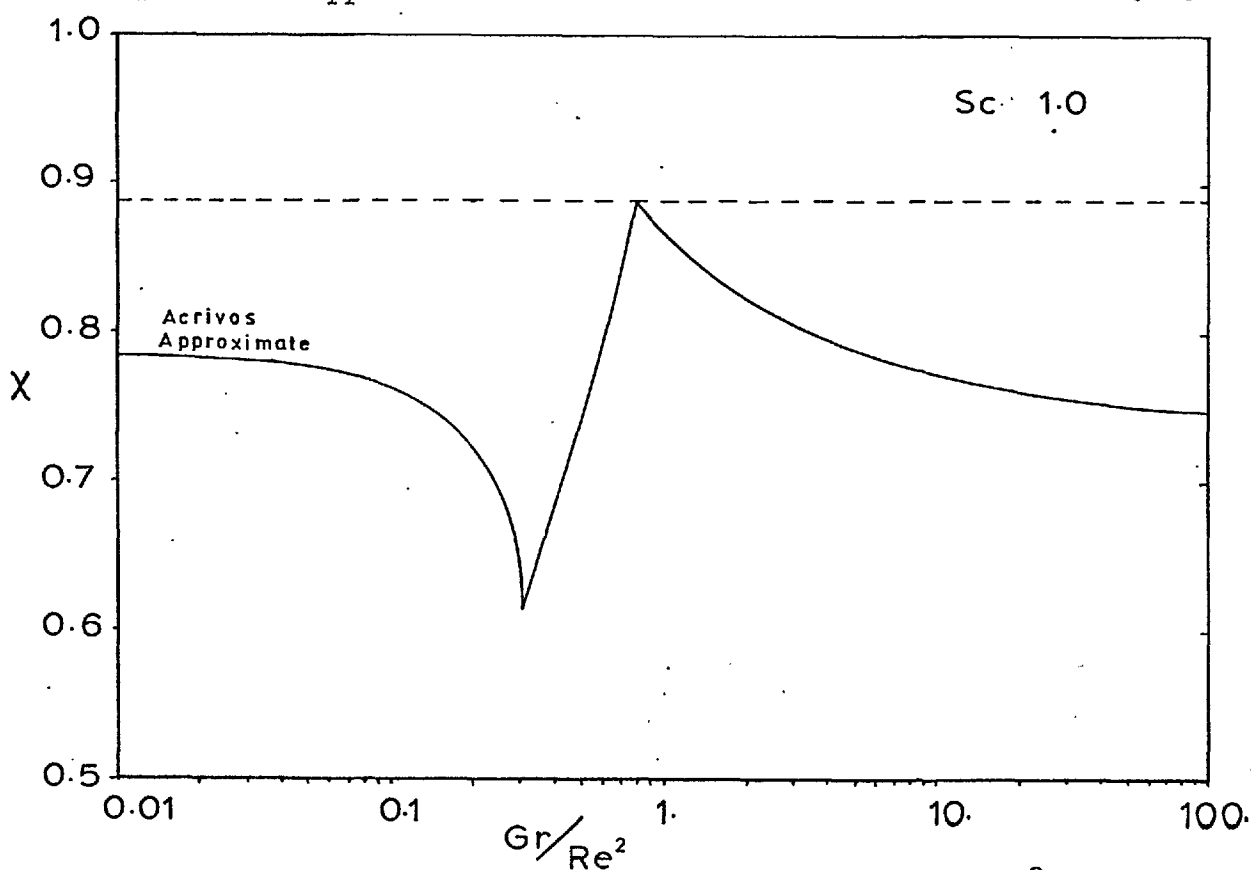


Fig. 6.66. The Variation of X with the value Gr/Re^2 for the Exact and Approximate Solutions for a Schmidt Number of 1.0 .

Fig. 6.68 The Variation of χ with the value Gr/Re^2 for the Exact and Approximate Solutions for a Schmidt Number of 10.0 .

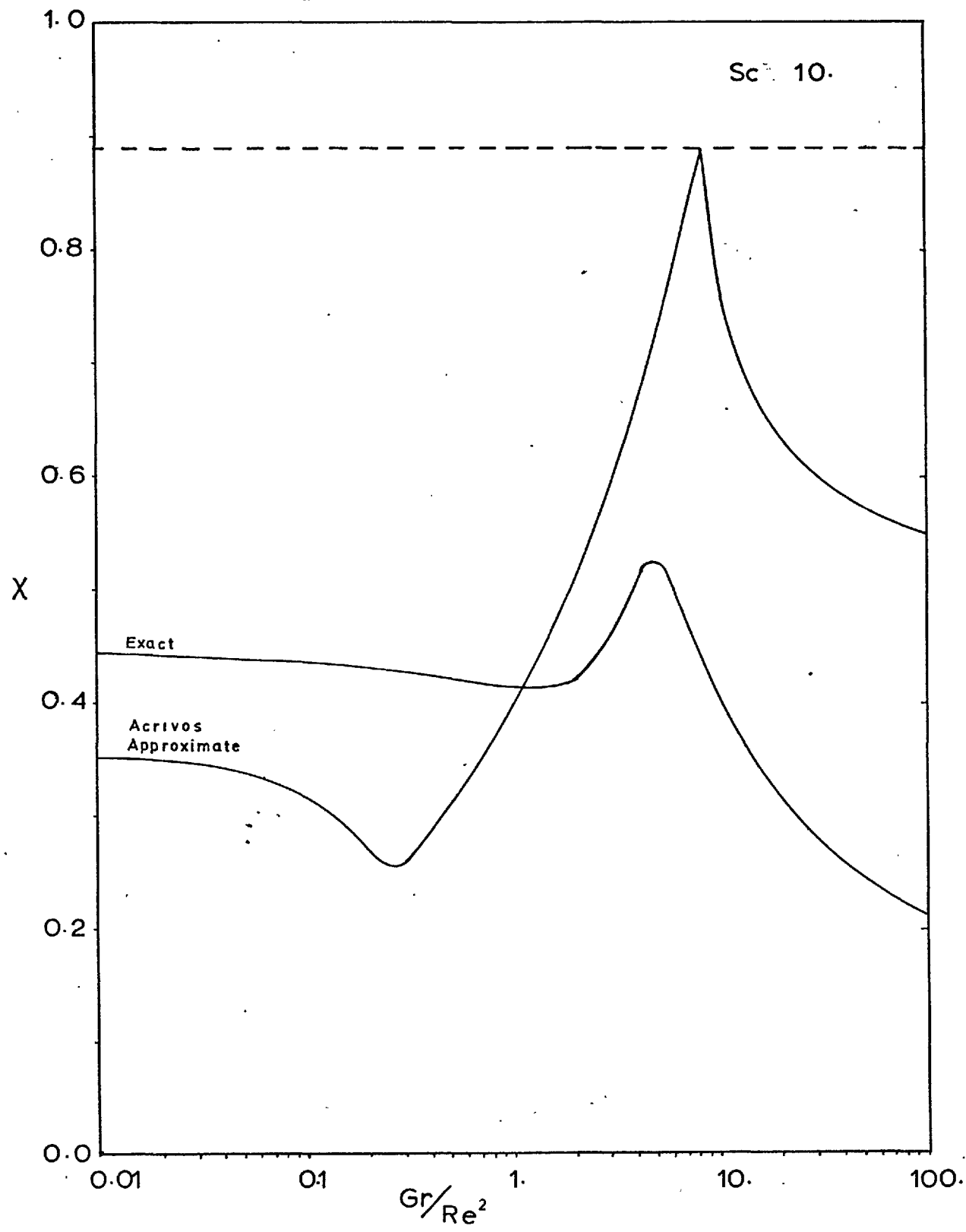
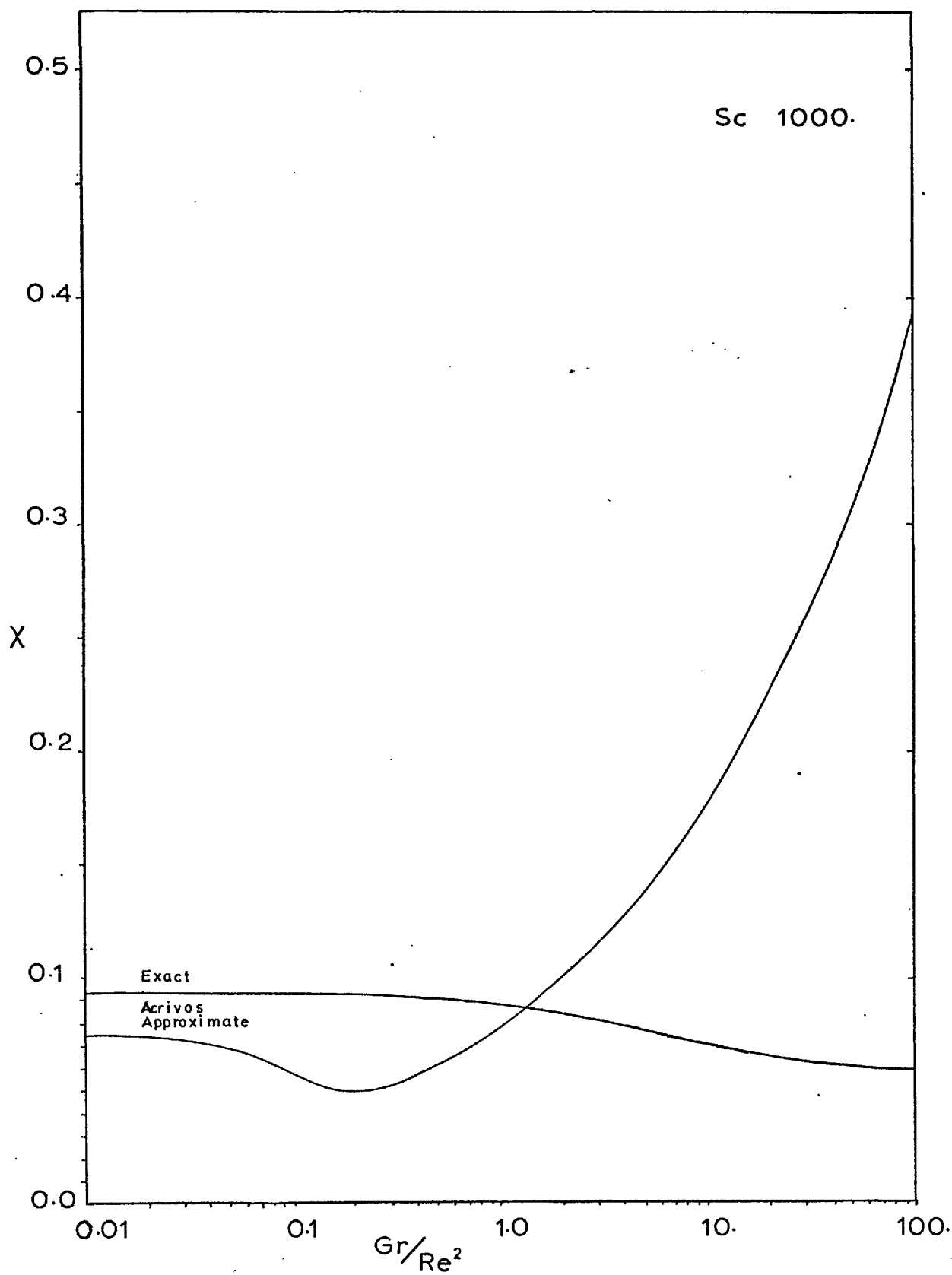


Fig. 6.69. The Variation of χ with the value Gr/Re^2
for the Exact and Approximate Solutions
for a Schmidt Number of 1000.0 .



trend for the ratio of the boundary layer thicknesses can be seen to be similar for both the exact and approximate solutions.

The maximum in the curves for the approximate solution occurred by virtue of the method, i.e. because the ratio of the two boundary layer thicknesses is allowed to increase to a value of $8/9$. However, the maximum in the exact curve can be thought of physically, by considering the form of the velocity profile at various values of the parameter Gr/Re^2 . When, for a particular value of the parameter Gr/Re^2 , the velocity in the boundary layer almost reaches a value greater than that in the free stream, the point at which the velocity within the boundary layer attains 99% of the free stream velocity appears to suddenly approach the body surface. Since a similar behaviour is not noticed with the concentration profile, the ratio of the two boundary layers attains a maximum.

The graphs show that this maximum is present for Schmidt numbers of 1.0, 2.4 and 10.0 but not for a Schmidt number of 1000.0. It has already been noted that no maximum velocity has been observed for a Schmidt number of 1000.0. Therefore the absence of a maximum in the curve of the ratio of the boundary layers is not surprising.

A comparison between the transfer results presented by Acrivos⁴¹ and those of the present exact solution are presented in figures 6.63 to 6.65. These plots show that up to the point at which the hydrodynamic boundary layer thickness was fixed, shown by the arrow A, the approximate results are within five per cent of the exact results. Over the region in which only the boundary layer thickness was

allowed to vary, the two sets of results do not show such good agreement. At the position where both of the boundary layers are again allowed to vary, shown by the arrow B, the results show close agreement to within approximately five per cent.

It is clear that because the intermediate region, in which the hydrodynamic boundary layer was maintained at a constant value, is very large for high Schmidt numbers, the transfer results do not show such good agreement as for low Schmidt numbers.

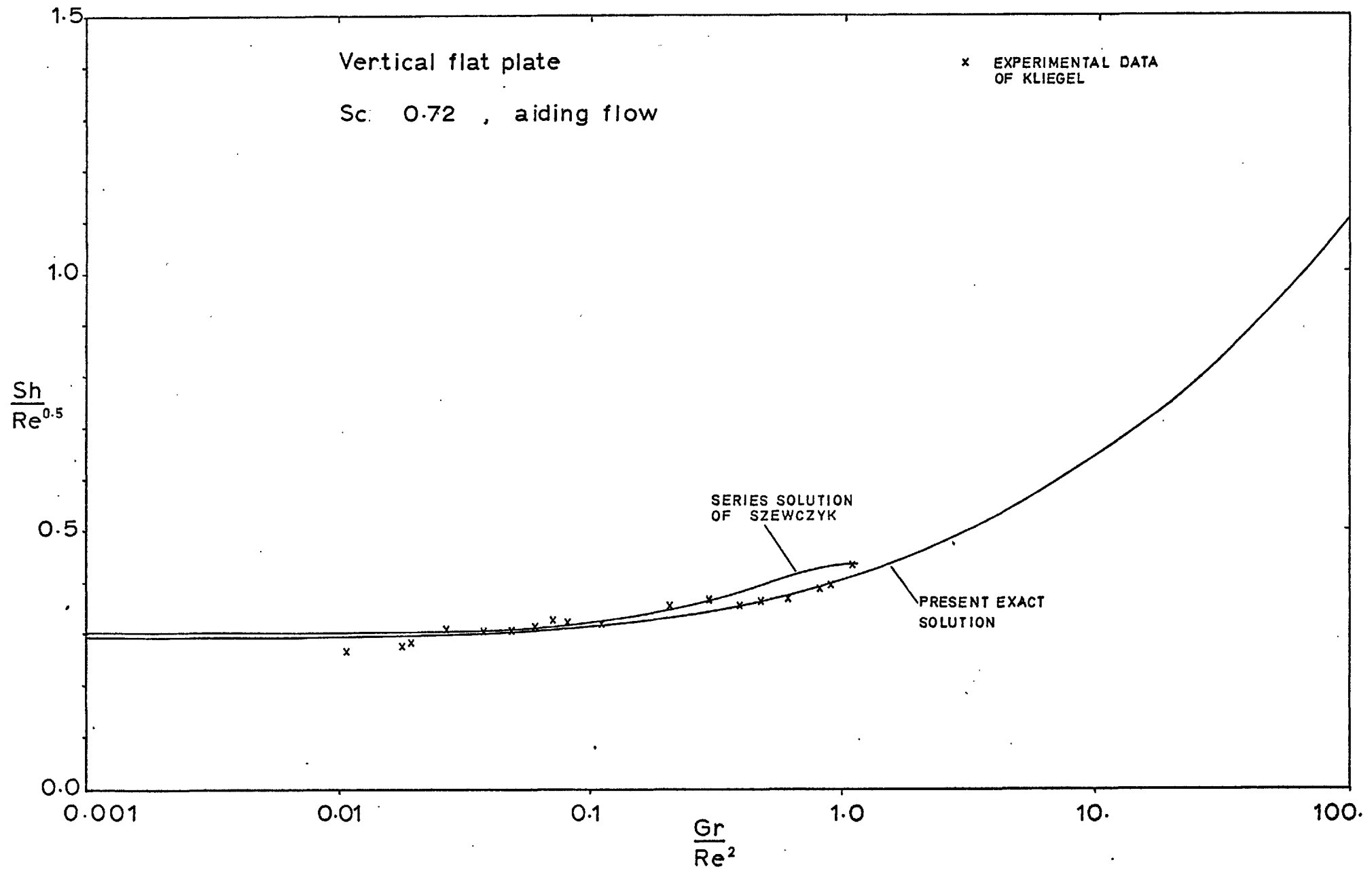
It should be emphasised that when the "complete" interaction problem is solved, the results which are predicted by the approximate method are in excellent agreement with the exact solution.

The results of the present exact solution for the vertical flat plate, which was developed in Chapter 5, have been compared with a number of other exact and experimental solutions.

The results of the present exact solution and those of Merkin⁵¹ for both aiding and opposing flows are presented in figures 6.61 and 6.62 for a Schmidt number of 1.0. The results which were calculated by the use of the free convection transforms are also shown in figure 6.61.

The predictions of transfer rates can be seen to be in excellent agreement. Those which are calculated from the equations which use the free convective transforms agree to within four per cent for values of the parameter Gr/Re^2 greater than 2.0. The mass transfer numbers which were calculated by the use of the forced convection transforms

Fig. 6.70. Comparison Between Theoretical and Experimental Transfer Rates for the Vertical Flat Plate
for a Schmidt Number of 0.72.



differ by approximately four per cent for values of the parameter Gr/Re^2 greater than 10.0. The two sets of results become identical as the parameter Gr/Re^2 approaches zero.

The results for the opposing flow regime show the same excellent agreement. The separation point which is predicted by each method can be seen to be at approximately the same value of the parameter Gr/Re^2 .

In figure 6.70, the data of Kliegel¹⁰⁰, and the results of Szewczyk's⁵⁰ theoretical analysis, are compared with the present exact results. The present curve lies within the scatter of Kliegel's experimental data. The predicted curve of Szewczyk deviates from the present curve at a value of the parameter Gr/Re^2 of approximately 0.2. This is probably due to the error in Szewczyk's paper which was reported by Merkin⁵¹.

As shown below, the predicted mass transfer numbers, $Sh/Re^{1/2}$, for a value of the parameter Gr/Re^2 of 100.0 are close to the exact solution of Ostrach¹⁰¹ for pure free convection, i.e. $Gr/Re^2 \rightarrow \infty$. The difference is 2.5% for a Schmidt number of 0.72, and 7.5% for a Schmidt number of 1000.0.

Sc	$\frac{Sh}{Re^{1/2}}(Ostrach)$	$\frac{Sh}{Re^{1/2}}(Present)$
0.72	1.1283	1.1000
1.00	1.2680	1.2282
10.0	2.6148	2.4454
1000.0	8.8682	8.1929

It is found that for Schmidt numbers of 0.72, 2.4, 10.0 and 1000.0, the mass transfer numbers are five per cent different from that for pure forced convection at values of the parameter Gr/Re^2 of 0.1, 0.1, 0.15 and 0.5 respectively.

The comparison of the mass transfer numbers with those

of Ostrach serve to indicate when the effects of forced convection are negligible.

It is of interest to check whether, for a predominantly free convective flow, the effect of forced convection results in a decrease of the Sherwood number. A value of the Grashof number of 10^6 was chosen and the data for a Schmidt number of 1000.0 for both the vertical flat plate and the stagnation point, were used to calculate the Sherwood number. There was no decrease of the Sherwood number observed, and the mass transfer continued to increase with increasing Reynolds number.

CHAPTER 7

CONCLUSIONS

The results of the present study may be summarized as follows:-

1) The boundary layer equations for the interaction of free and forced convective heat or mass transfer from a general axisymmetric body have been derived in terms of a general orthogonal curvilinear coordinate system.

The parameter Grashof number divided by the square of the Reynolds number, and the Schmidt number appeared in the equations.

2) The boundary layer equations were solved by two different techniques.

i) An approximate method was used in which polynomials were chosen to represent the velocity and concentration profiles within the boundary layer. These polynomials were substituted into the integral forms of the boundary layer equations.

ii) An exact series method was developed by the substitution of power series into the boundary layer equations. This resulted in two infinite sets of ordinary differential equations.

3) A general programme was written to solve the two ordinary differential equations which resulted from the integral method. This programme was written so that it can be used for any axisymmetric body for which the potential flow distribution and a description of the body shape are known.

4) The approximate method was used to predict local transfer rates from spheres, four oblate spheroids, with ratios of the minor to the major axis of 13/16, 10/16, 7/16 and 4/16, and the vertical flat plate in a uniform free stream.

Predictions of mass transfer rates for Schmidt numbers of 1.0 to 1000.0, for ranges of the parameter Gr/Re^2 were calculated.

The range of values of this parameter for which solutions could be calculated was dependent upon the value of the Schmidt number. For a Schmidt number of 2.4 the ^{excluded} range _^ was $0.2 \leq Gr/Re^2 \leq 2.0$, whereas for a Schmidt number of 100.0 the ^{excluded} range _^ was $0.2 \leq Gr/Re^2 \leq 80.0$. ^{within} Outside these limits, the method broke down and could not be used to predict transfer rates. The break - down of the integral method occurred at approximately the same value of the parameter Gr/Re^2 for the bluff bodies and the vertical flat plate.

5) A number of attempts were made to extend the range of values of the parameter Gr/Re^2 over which the approximate method could be used. The results of this investigation showed that:-

i) The order of the polynomials which represented the velocity and concentration profiles had no effect on the break - down.

ii) No improvement was found when two velocity profiles were used, one profile over the region of the transfer boundary layer, and the other over the remainder of the hydrodynamic boundary layer.

iii) The range of values of the parameter Gr/Re^2 over which the approximate method could be used was improved by the introduction of scaling in terms of the Schmidt number. This scaling removed the Schmidt number from the transfer equation and introduced it into the hydrodynamic equation. The hydrodynamic boundary layer equation over the remainder of the hydrodynamic boundary layer was left unscaled. A double velocity profile was again used.

This modified method improved the range over which solutions could be obtained, and whereas for a Schmidt number of 100.0 the original method failed at a value of the parameter Gr/Re^2 of approximately 0.2, the modified method was used up to a value of approximately 50.0.

6) The series method was used to predict mass transfer rates from a bluff body and the vertical flat plate in a uniform free stream. The first equation of each of the two infinite sets of ordinary differential equations for the bluff body were independent of shape, and the solution of these equations represented that for the region of the front stagnation point.

7) A quasilinearization technique was used to solve the two point boundary value problem associated with the

solution of these two ordinary differential equations. The equations were solved for a range of Schmidt numbers of 0.72 to 1000.0 and for values of the parameter Gr/Re^2 of 0.0 to 100.0 for both the stagnation point solution and the vertical flat plate. This technique was shown to predict accurate solutions of the original non - linear equations even when very approximate initial conditions were employed.

8) The value of the parameter Gr/Re^2 at which either free or forced convection was negligible was found to be dependent upon the Schmidt number and, to some extent, the body shape. In general, for both the stagnation point and the vertical flat plate, when the value of the parameter Gr/Re^2 had a value less than approximately 0.1, the effects of free convection were negligible.

A further increase of the parameter then resulted in an increase of the mass transfer number $Sh/Re^{1/2}$.

In the interaction region, the mass transfer number was never greater than 25% of that for the predominant pure convective flow. For large Schmidt numbers, the difference was reduced to 10 - 15%. Above a value of the parameter Gr/Re^2 of approximately 30.0 for small Schmidt numbers and 50.0 for large Schmidt numbers, the effects of forced convection were becoming negligible.

9) When free and forced convective forces were acting in opposition the buoyancy force was seen to act as an adverse pressure gradient. Once a certain value of the parameter Gr/Re^2 was reached a backflow within the boundary layer

was observed.

The effect of opposing flows in the investigation of local transfer rates was observed to cause a movement of the separation point towards the front of the body compared with the separation point for aiding flows.

10) The mass transfer numbers for aiding flows were higher than those for opposing flows for the same value of the parameter Gr/Re^2 . Those for opposing flows were depressed below the mass transfer number for pure forced convective flow.

11) For both the stagnation point solution and the vertical flat plate, the approximate solutions were shown to be in excellent agreement with the exact solutions whenever the integral method could be used to predict transfer rates.

12) The modified approximate method has been shown to be very complicated, even when applied only to the front stagnation point of a bluff body. Since this method also failed over a range of values of the parameter Gr/Re^2 , and because further modifications to the integral method would probably lead to even more complicated equations, it is thought that an alternative method should be used for the prediction of local transfer rates, where the simple integral method cannot be used. It is thought that the use of a numerical step - by - step method should be investigated. The exact stagnation point solution could then be used as an initial condition to start the step - by - step method.

APPENDIX A.

Orthogonal Curvilinear Coordinates.

The curvilinear coordinate system is described in detail by Spiegel⁶⁸, but the necessary relationships for the present work are reproduced for ease of reference.

The curvilinear coordinates are defined as x_1, x_2, x_3 with the unit vectors $\underline{e}_1, \underline{e}_2, \underline{e}_3$ in the directions of increasing x_1, x_2, x_3 respectively. The magnitudes of the tangent vectors are called the scale factors h_1, h_2, h_3 where $h_i = \partial \underline{r} / \partial x_i$, $i = 1, 2, 3$, and where $\underline{r}$ is the position vector at a point in the system.

The vector operator del, ∇ , is shown by Spiegel to have the following form in the orthogonal curvilinear coordinate system x_1, x_2, x_3 :

$$\nabla = \frac{\underline{e}_1}{h_1} \frac{\partial}{\partial x_1} + \frac{\underline{e}_2}{h_2} \frac{\partial}{\partial x_2} + \frac{\underline{e}_3}{h_3} \frac{\partial}{\partial x_3} \quad (\text{A-1})$$

The system is orthogonal if the three surfaces of $x_1 = \text{constant}$, $x_2 = \text{constant}$, $x_3 = \text{constant}$ intersect at right angles.

If ϕ is a scalar and $\underline{a} = a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3$, a vector function of the orthogonal curvilinear coordinates x_1, x_2, x_3 , then the following results are valid.

$$\nabla \phi = \text{grad } \phi = \frac{\underline{e}_1}{h_1} \frac{\partial \phi}{\partial x_1} + \frac{\underline{e}_2}{h_2} \frac{\partial \phi}{\partial x_2} + \frac{\underline{e}_3}{h_3} \frac{\partial \phi}{\partial x_3} \quad (\text{A-3})$$

$$\nabla \cdot \underline{a} = \text{div } \underline{a} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x_1} (h_2 h_3 a_1) + \frac{\partial}{\partial x_2} (h_3 h_1 a_2) + \frac{\partial}{\partial x_3} (h_1 h_2 a_3) \right] \quad (\text{A-3})$$

$$\nabla \times \underline{a} = \text{curl } \underline{a} = \frac{1}{h_1 h_2 h_3} \begin{bmatrix} h_1 e_1 & h_2 e_2 & h_3 e_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1 a_1 & h_2 a_2 & h_3 a_3 \end{bmatrix} \quad (\text{A-4})$$

Vector Relationships.

In order to arrange the Navier Stokes equations into a more manageable form, the following standard vector relationships are required:

$$\nabla (\underline{a} \cdot \underline{b}) = (\underline{b} \cdot \nabla) \underline{a} + (\underline{a} \cdot \nabla) \underline{b} + \underline{b} \times (\nabla \times \underline{a}) + \underline{a} \times (\nabla \times \underline{b}) \quad (\text{A-5})$$

$$\nabla \times (\nabla \times \underline{a}) = \nabla (\nabla \cdot \underline{a}) - \nabla^2 \underline{a} \quad (\text{A-6})$$

$$(\underline{a} \cdot \nabla) \phi = \underline{a} \cdot \nabla \phi \quad (\text{A-7})$$

For the present work, $\underline{a}$ and $\underline{b}$ above are replaced by the velocity vector $\underline{v}$ and the scalar ϕ by the concentration C .

Thus $\nabla^2 \underline{v}$, $(\underline{v} \cdot \nabla) \underline{v}$ and $(\underline{v} \cdot \nabla) C$ may be written as :

$$\nabla^2 \underline{v} = \nabla (\nabla \cdot \underline{v}) - \nabla \times (\nabla \times \underline{v}) \quad (\text{A-8})$$

but from the continuity equation, $\nabla \cdot \underline{v} = 0$, therefore equation (A-8) becomes:

$$\nabla^2 \underline{v} = - \nabla \times (\nabla \times \underline{v}) \quad (\text{A-9})$$

$$(\underline{v} \cdot \nabla) \underline{v} = \frac{1}{2} \nabla (\underline{v} \cdot \underline{v}) - \underline{v} \times (\nabla \times \underline{v}) \quad (\text{A-10})$$

$$(\underline{v} \cdot \nabla) C = \underline{v} \cdot \nabla C \quad (\text{A-11})$$

Equations (A-9) to (A-11) may now be used in Chapter 3 to transform equations (3-5) to (3-7) into equations (3-9) to (3-11).

APPENDIX B.

The Particular and Homogeneous Equations.

The sets of equations required in the quasilinearization technique to solve the two point boundary value problem for the stagnation point solution were derived in Chapter 5.

The corresponding sets of linearized non-homogeneous and homogeneous equations for each of the other problems studied are presented here without further discussion. They are all derived in exactly the same manner.

The initial conditions for all the problems are the same and have been presented in Chapter 5 as equations (5-79) to (5-81).

For completeness, the equations already derived in Chapter 5 for the stagnation point solution will be repeated.

B.1. Stagnation Point Solution ; $Sc < 100$

The non - linear equations are:

$$f_1''' = - f_1 f_1'' - 0.5 \left(1 + \frac{\overline{Gr}}{Re^2} F_0 - f_1' f_1' \right)$$

$$F_0 = - Sc f_1 F_0' \quad (B.1.1)$$

The linearized non - homogeneous equations are:

$$\frac{dY(1)^{k+1}}{d\eta} = Y(2)^{k+1}$$

$$\frac{dY(2)^{k+1}}{d\eta} = Y(3)^{k+1} \quad (B.1.2)$$

$$\begin{aligned} \frac{dY(3)^{k+1}}{d\gamma} &= - (Y(3)^k Y(1)^{k+1} + Y(1)^k Y(3)^{k+1} - Y(1)^k Y(3)^k \\ &\quad + 0.5 \left(1 + \frac{\overline{Gr}}{Re^2} \right) Y(4)^{k+1} + Y(2)^k Y(2)^k \\ &\quad - 2 Y(2)^k Y(2)^{k+1}) \end{aligned}$$

$$\frac{dY(4)^{k+1}}{d\gamma} = Y(5)^{k+1} \quad (B.1.2)$$

$$\frac{dY(5)^{k+1}}{d\gamma} = - Sc (Y(1)^{k+1} Y(5)^k + Y(1)^k Y(5)^{k+1} - Y(1)^k Y(5)^k)$$

The homogeneous equations are:

$$\frac{dY(1)^{k+1}}{d\gamma} = Y(2)^k$$

$$\frac{dY(2)^{k+1}}{d\gamma} = Y(3)^{k+1}$$

$$\begin{aligned} \frac{dY(3)^{k+1}}{d\gamma} &= - (Y(3)^k Y(1)^{k+1} + Y(1)^k Y(3)^{k+1} \\ &\quad + 0.5 \left(+ \frac{\overline{Gr}}{Re^2} \right) Y(4)^{k+1} - 2 Y(2)^k Y(2)^{k+1}) \end{aligned}$$

$$\frac{dY(4)^{k+1}}{d\gamma} = Y(5)^{k+1} \quad (B.1.3)$$

$$\frac{dY(5)^{k+1}}{d\gamma} = - Sc (Y(1)^k Y(5)^{k+1} + Y(1)^{k+1} Y(5)^k)$$

B.2. Stagnation Point Solution ; $Sc \geq 100$

The non - linear equations are:

$$\begin{aligned} f_1''' &= \frac{f_1 f_1''}{Sc} - 0.5 \left[\frac{1}{Sc^{\frac{1}{3}}} + \frac{\overline{Gr}}{Re^2 Sc^{\frac{1}{3}}} F_0 - \frac{f_1'^2}{Sc} \right] \\ F_0'' &= - f_1 F_0' \end{aligned} \quad (B.2.1)$$

The linearized non-homogeneous equations are:

$$\begin{aligned}
 \frac{dY(1)^{k+1}}{d\bar{\eta}} &= Y(2)^{k+1} \\
 \frac{dY(2)^{k+1}}{d\bar{\eta}} &= Y(3)^{k+1} \\
 \frac{dY(3)^{k+1}}{d\bar{\eta}} &= - \left((Y(3)^k Y(1)^{k+1} + Y(1)^k Y(3)^{k+1} - Y(1)^k Y(3)^k) / Sc \right. \\
 &\quad + 0.5 \left(\frac{1}{Sc^3} + \frac{\overline{Gr}}{Re^2 Sc^3} Y(4)^{k+1} + (Y(2)^k Y(2)^k \right. \\
 &\quad \left. \left. - 2 Y(2)^k Y(2)^{k+1} \right) / Sc \right) \\
 \frac{dY(4)^{k+1}}{d\bar{\eta}} &= Y(5)^{k+1} \quad (B.2.2) \\
 \frac{dY(5)^{k+1}}{d\bar{\eta}} &= - \left(Y(1)^{k+1} Y(5)^k + Y(1)^k Y(5)^{k+1} - Y(1)^k Y(5)^k \right)
 \end{aligned}$$

The homogeneous equations are:

$$\begin{aligned}
 \frac{dY(1)^{k+1}}{d\bar{\eta}} &= Y(2)^{k+1} \\
 \frac{dY(2)^{k+1}}{d\bar{\eta}} &= Y(3)^{k+1} \\
 \frac{dY(3)^{k+1}}{d\bar{\eta}} &= - \left((Y(3)^k Y(1)^{k+1} + Y(1)^k Y(3)^{k+1}) / Sc \right. \\
 &\quad \left. - Y(2)^k Y(2)^{k+1} / Sc + 0.5 \left(\frac{\overline{Gr}}{Re^2 Sc^3} Y(4)^{k+1} \right) \right) \\
 \frac{dY(4)^{k+1}}{d\bar{\eta}} &= Y(5)^{k+1} \quad (B.2.3) \\
 \frac{dY(5)^{k+1}}{d\bar{\eta}} &= - \left(Y(1)^{k+1} Y(5)^k + Y(1)^k Y(5)^{k+1} \right)
 \end{aligned}$$

B.3. Vertical Flat Plate ; $Sc < 100$

The dimensionless distance, η , is that which is normally associated with forced convection:

$$\eta = y \left(\frac{U_{\infty}}{\nu_x} \right)^{\frac{1}{2}}$$

The non - linear equations are:

$$f_1''' = - 0.5 f_1 f_1'' - \frac{Gr}{Re^2} F_0$$

$$F_0'' = - \frac{Sc}{2} f_1 F_0'$$

(B.3.1)

The linearized non - homogeneous equations are:

$$\frac{dY(1)^{k+1}}{d\eta} = Y(2)^{k+1}$$

$$\frac{dY(2)^{k+1}}{d\eta} = Y(3)^{k+1}$$

$$\begin{aligned} \frac{dY(3)^{k+1}}{d\eta} = & - 0.5 (Y(1)^k Y(3)^{k+1} + Y(1)^{k+1} Y(3)^k - Y(1)^k Y(3)^k) \\ & - \frac{Gr}{Re^2} Y(4)^{k+1} \end{aligned}$$

$$\frac{dY(4)^{k+1}}{d\eta} = Y(5)^{k+1}$$

(B.3.2)

$$\begin{aligned} \frac{dY(5)^{k+1}}{d\eta} = & - 0.5 Sc (Y(1)^k Y(5)^{k+1} + Y(1)^{k+1} Y(5)^k \\ & - Y(1)^k Y(5)^k) \end{aligned}$$

The homogeneous equations are:

$$\frac{dY(1)^{k+1}}{d\eta} = Y(2)^{k+1}$$

$$\frac{dY(2)^{k+1}}{d\eta} = Y(3)^{k+1}$$

$$\frac{dY(3)^{k+1}}{d\eta} = -0.5 (Y(1)^k Y(3)^{k+1} + Y(1)^{k+1} Y(3)^k) + \frac{Gr}{Re^2} Y(4)^{k+1}$$

$$\frac{dY(4)^{k+1}}{d\eta} = Y(5)^{k+1} \quad (B.3.3)$$

$$\frac{dY(5)^{k+1}}{d\eta} = -0.5 Sc (Y(1)^k Y(5)^{k+1} + Y(1)^{k+1} Y(5)^k)$$

B.4. Vertical Flat Plate ; $Sc \geq 100$

The dimensionless distance, η , is that which is normally associated with forced convection:

$$\eta = y \left(\frac{U_\infty}{\nu_x} \right)^{\frac{1}{2}}$$

The non-linear equations are:

$$f_1''' = \frac{0.5}{Sc} f_1 f_1'' + \frac{Gr}{Re^2 Sc^3} F_0$$

(B.4.1)

$$F_0'' = -0.5 f_1 F_0'$$

The linearized non-homogeneous equations are:

$$\frac{dY(1)^{k+1}}{d\bar{\eta}} = Y(2)^{k+1}$$

$$\frac{dY(2)^{k+1}}{d\bar{\eta}} = Y(3)^{k+1}$$

$$\begin{aligned} \frac{dY(3)^{k+1}}{d\bar{\eta}} = & - 0.5 (Y(1)^k Y(3)^{k+1} + Y(1)^{k+1} Y(3)^k \\ & - Y(1)^k Y(3)^k) / Sc + \frac{Gr}{Re^2 Sc^3} Y(4)^{k+1} \end{aligned}$$

$$\frac{dY(4)^{k+1}}{d\bar{\eta}} = Y(5)^{k+1} \quad (B.4.2)$$

$$\frac{dY(5)^{k+1}}{d\bar{\eta}} = - 0.5 (Y(1)^k Y(5)^{k+1} + Y(1)^{k+1} Y(5)^k - Y(1)^k Y(5)^k)$$

The homogeneous equations are:

$$\frac{dY(1)^{k+1}}{d\bar{\eta}} = Y(2)^{k+1}$$

$$\frac{dY(2)^{k+1}}{d\bar{\eta}} = Y(3)^{k+1}$$

$$\begin{aligned} \frac{dY(3)^{k+1}}{d\bar{\eta}} = & - 0.5 (Y(1)^k Y(3)^{k+1} + Y(1)^{k+1} Y(3)^k) / Sc \\ & + \frac{Gr}{Re^2 Sc^3} Y(4)^{k+1} \end{aligned}$$

$$\frac{dY(4)^{k+1}}{d\bar{\eta}} = Y(5)^{k+1} \quad (B.4.3)$$

$$\frac{dY(5)^{k+1}}{d\bar{\eta}} = - 0.5 (Y(1)^k Y(5)^{k+1} + Y(1)^{k+1} Y(5)^k)$$

B.5. Vertical Flat Plate.

The dimensionless distance, η , is that which is normally associated with free convection.

$$\eta = \left(\frac{g \beta (C_w - C_\infty)}{4 \nu^2} \right)^{\frac{1}{4}} \frac{y}{x^{\frac{1}{4}}}$$

The non - linear equations are:

$$\begin{aligned} \mathcal{J}_1''' &= 2 \mathcal{J}_1'^2 - \Omega - 3 \mathcal{J}_1 \mathcal{J}'' \\ \Omega'' &= - 3 \text{Sc} \mathcal{J}_1 \Omega' \end{aligned} \quad (\text{B.5.1})$$

The linearized non - homogeneous equations are:

$$\begin{aligned} \frac{dY(1)^{k+1}}{d\eta} &= Y(2)^{k+1} \\ \frac{dY(2)^{k+1}}{d\eta} &= Y(3)^{k+1} \\ \frac{dY(3)^{k+1}}{d\eta} &= 4 Y(2)^{k+1} Y(2)^k - 2 Y(2)^k Y(2)^k - 3 (- Y(1)^k Y(3)^k \\ &\quad + Y(1)^{k+1} Y(3)^k + Y(3)^{k+1} Y(1)^k) - Y(4)^{k+1} \\ \frac{dY(4)^{k+1}}{d\eta} &= Y(5)^{k+1} \\ \frac{dY(5)^{k+1}}{d\eta} &= - 3 \text{Sc} (- Y(1)^k Y(5)^k + Y(1)^{k+1} Y(5)^k \\ &\quad + Y(1)^k Y(5)^{k+1}) \end{aligned} \quad (\text{B.5.2})$$

The homogeneous equations are:

$$\begin{aligned}
 \frac{dY(1)^{k+1}}{d\gamma} &= Y(2)^{k+1} \\
 \frac{dY(2)^{k+1}}{d\gamma} &= Y(3)^{k+1} \\
 \frac{dY(3)^{k+1}}{d\gamma} &= 4 Y(2)^{k+1} Y(2)^k - 3 (Y(1)^{k+1} Y(3)^k \\
 &\quad + Y(1)^k Y(3)^{k+1}) - Y(4)^{k+1} \\
 \frac{dY(4)^{k+1}}{d\gamma} &= Y(5)^{k+1} \quad (B.5.3) \\
 \frac{dY(5)^{k+1}}{d\gamma} &= - 3 Sc (Y(1)^{k+1} Y(5)^k + Y(5)^{k+1} Y(1)^k)
 \end{aligned}$$

The superposition constants are calculated using a slightly different formula than the others since the boundary condition $Y(2) (\infty) \Rightarrow \frac{1}{2} \left(\frac{Gr}{Re^2} \right)^{-\frac{1}{2}}$

$$\text{Thus if } \sigma = \frac{1}{2} \left(\frac{Gr}{Re^2} \right)^{-\frac{1}{2}}$$

$$\tau_1^{k+1} = \frac{\sigma Y(4)_{h,2}^{k+1} + Y(4)_p^{k+1} Y(2)_{h,2}^{k+1} - Y(2)_p^{k+1} Y(4)_{h,2}^{k+1}}{Y(2)_{h,1}^{k+1} Y(4)_{h,2}^{k+1} - Y(4)_{h,1}^{k+1} Y(2)_{h,2}^{k+1}} \quad (B.5.4)$$

$$\tau_2^{k+1} = \frac{\sigma Y(4)_{h,1}^{k+1} + Y(4)_p^{k+1} Y(2)_{h,1}^{k+1} - Y(2)_p^{k+1} Y(4)_{h,1}^{k+1}}{Y(2)_{h,2}^{k+1} Y(4)_{h,1}^{k+1} - Y(4)_{h,2}^{k+1} Y(2)_{h,1}^{k+1}} \quad (B.5.5)$$

APPENDIX C.TABLES OF RESULTS.

Table C.1. The Effect of Integration Step Length on the
Predicted Values of $f_1''(0)$ and $F_0'(0)$ for the
Stagnation Point Solution.

Table C.1.(a).

$\frac{\overline{Gr}}{Re^2} = 1.0$	$Sc = 1.0$	Convergence Criteria 1×10^{-6}
H	$f_1''(0)$	$F_0'(0)$
0.5	1.27616	-0.59370
0.25	1.24531	-0.57800
0.1	1.23652	-0.57370
0.05	1.23523	-0.57309
0.025	1.23491	-0.57293
0.01	1.23482	-0.57289
0.005	1.23482	-0.57289

Table C.1.(b).

$\frac{\overline{Gr}}{Re^2} = 5.0$	$Sc = 1.0$	Convergence Criteria 1×10^{-6}
H	$f_1''(0)$	$F_0'(0)$
0.1	2.28274	-0.66581
0.05	2.28005	-0.66496
0.025	2.27937	-0.66475
0.01	2.27918	-0.66469
0.005	2.27917	-0.66469

Table C.1.(c).

$$\frac{\overline{Gr}}{\overline{Re}^2} = 0.00001 \quad Sc = 10.0 \quad \begin{array}{l} \text{Convergence} \\ \text{Criteria } 1 \times 10^{-6} \end{array}$$

<u>H</u>	<u>f₁^{''}(0)</u>	<u>F₀['](0)</u>
0.1	0.929036	-1.24636
0.05	0.92814	-1.24145
0.025	0.92776	-1.23937
0.01	0.92769	-1.23897
0.005	0.92769	-1.23897

Table C.1.(d).

$$\frac{\overline{Gr}}{\overline{Re}^2} = 0.00001 \quad Sc = 100 \quad \begin{array}{l} \text{Convergence} \\ \text{Criteria } 1 \times 10^{-6} \end{array}$$

<u>H</u>	<u>f₁^{''}(0)</u>	<u>F₀['](0)</u>
0.2	0.92793	-2.75119
0.1	0.92774	-2.74040
0.05	0.92770	-2.73733
0.1	0.92769	-2.73695
0.005	0.92769	-2.73695

Table C.2. The Effect of the Integration Step Length on the
Integration of the Non - Linear Equations for the
Stagnation Point Solution.

Table C.2.(a).

$$\frac{\overline{Gr}}{Re^2} = 0.5 \quad Sc = 2.4 \quad \text{Convergence Criteria } 1 \times 10^{-6}$$

<u>H</u>	<u>$f_1''(0)$</u>	<u>$F_0'(0)$</u>
0.01	1.057712	-0.766351
0.05	1.057712	-0.766351
0.1	1.057712	-0.766352
0.5	1.057708	-0.766367

<u>X</u>	<u>$f_1'(\eta)_{0.01}$</u>	<u>$f_1'(\eta)_{0.05}$</u>	<u>$f_1'(\eta)_{0.1}$</u>	<u>$f_1'(\eta)_{0.2}$</u>
0.0	0.000000	0.000000	0.000000	0.000000
1.0	0.719065	0.719065	0.719064	0.719057
2.0	0.953508	0.953508	0.953507	0.953501
3.0	0.996168	0.996168	0.996168	0.996172
4.0	0.999861	0.999861	0.999861	0.999876
5.0	0.999998	0.999998	0.999997	1.000020
6.0	0.999999	0.999999	0.999999	1.000029
7.0	1.000000	1.000000	1.000000	1.000035
8.0	1.000000	1.000000	1.000000	1.000042

<u>X</u>	<u>$F_0(\eta)_{0.01}$</u>	<u>$F_0(\eta)_{0.05}$</u>	<u>$F_0(\eta)_{0.1}$</u>	<u>$F_0(\eta)_{0.2}$</u>
0.0	1.000000	1.000000	1.000000	1.000000
1.0	0.297156	0.297156	0.297157	0.297156
2.0	0.020491	0.020491	0.020492	0.020496
3.0	0.000185	0.000185	0.000186	0.000173
4.0	0.000000	0.000000	0.000000	-0.000021
5.0	0.000000	0.000000	0.000000	-0.000021
6.0	0.000000	0.000000	-0.000000	-0.000021
7.0	0.000000	0.000000	-0.000000	-0.000021
8.0	0.000000	0.000000	-0.000000	-0.000021

Table C.2.(b).

$$\frac{\overline{Gr}}{Re^2} = 50.0 \quad Sc = 2.4 \quad \text{Convergence Criteria } 1 \times 10^{-6}$$

<u>H</u>	<u>$f_1''(0)$</u>	<u>$F_0'(0)$</u>
0.01	8.690901	-1.363127
0.05	8.690898	-1.363129
0.1	8.690854	-1.363147
0.2	8.688960	-1.363429

<u>X</u>	<u>$f_1'(\eta)_{0.01}$</u>	<u>$f_1'(\eta)_{0.05}$</u>	<u>$f_1'(\eta)_{0.1}$</u>	<u>$f_1'(\eta)_{0.2}$</u>
0.0	0.000000	0.000000	0.000000	0.000000
1.0	1.531205	1.531206	1.531216	1.529872
2.0	1.035661	1.035678	1.035917	1.037152
3.0	1.000846	1.000877	1.001323	1.005501
4.0	1.000008	1.000052	1.000696	1.007801
5.0	1.000000	1.000058	1.000899	1.010939
6.0	1.000000	1.000072	1.001110	1.014086
7.0	1.000000	1.000086	1.001322	1.017232
8.0	1.000000	1.000099	1.001533	1.020359

<u>X</u>	<u>$F_0(\eta)_{0.01}$</u>	<u>$F_0(\eta)_{0.05}$</u>	<u>$F_0(\eta)_{0.1}$</u>	<u>$F_0(\eta)_{0.2}$</u>
0.0	1.000000	1.000000	1.000000	1.000000
1.0	0.062212	0.062212	0.062207	0.062127
2.0	0.000215	0.000215	0.000204	-0.000030
3.0	0.000000	-0.000000	-0.000007	-0.000284
4.0	-0.000000	-0.000000	-0.000012	-0.000284
5.0	-0.000000	-0.000000	-0.000012	-0.000284
6.0	-0.000000	-0.000000	-0.000012	-0.000284
7.0	-0.000000	-0.000000	-0.000012	-0.000284
8.0	-0.000000	-0.000000	-0.000012	-0.000285

Table C.3. The Effect of the Value of γ_{MAX} upon the Values
 $f_1''(0)$ and $F_0'(0)$.

Table C.3.(a).

$\frac{\overline{Gr}}{\overline{Re}^2} = 1.0$	$Sc = 1000$	Convergence Criteria 1×10^{-6}
γ_{MAX}	$f_1''(0)$	$F_0'(0)$
10.0	1.285283	-0.663509
15.0	1.067600	-0.621925
20.0	0.999999	-0.607777
25.0	0.979808	-0.603417
30.0	0.974514	-0.602263
35.0	0.973339	-0.602006
40.0	0.973122	-0.601959
45.0	0.973089	-0.601952
50.0	0.973085	-0.601951
55.0	0.973085	-0.601951
60.0	0.973085	-0.601951
65.0	0.973085	-0.601951
70.0	0.973085	-0.601951

Table C.3.(b).

$\frac{\overline{Gr}}{\overline{Re}^2} = 1.0$	$Sc = 2.4$	Convergence Criteria 1×10^{-6}
γ_{MAX}	$f_1''(0)$	$F_0'(0)$
2.0	1.195802	-0.803863
2.5	1.185846	-0.788297
3.0	1.183213	-0.785834
3.5	1.182603	-0.785510
4.0	1.182490	-0.785465
4.5	1.182473	-0.785460
5.0	1.182471	-0.785459
6.0	1.182471	-0.785459
7.0	1.182471	-0.785459
8.0	1.182471	-0.785459

Table C.4. The Effect of Variation in the Position at which
 $F''_0(\tilde{\eta})$ is Set to Zero.

$$\frac{\overline{Gr}}{Re^2} = 0.1 \quad Sc = 10000 \quad H = 0.01, 0.05, 0.1, \text{Convergence} \\ 0.5, 0.5. \quad \text{Criteria } 1 \times 10^{-6}$$

$$ZMAX = 2, 6, 9, \\ 10, 90.$$

<u>XTOP = 5</u>	<u>$f''_1(0)$</u>	<u>$F'_0(0)$</u>	<u>$\overline{\eta}_{MAX}$</u>
	1.249338	-0.659684	10.0
	0.933820	-0.596882	30.0
	0.932358	-0.596558	50.0
	0.932358	-0.596558	70.0
	0.932357	-0.596558	90.0

XTOP = 6

1.249338	-0.659684	10.0
0.933820	-0.596882	30.0
0.932358	-0.596559	50.0
0.932358	-0.596559	70.0
0.932357	-0.596558	90.0

XTOP = 7

1.249338	-0.659684	10.0
0.933820	-0.596882	30.0
0.932358	-0.596559	50.0
0.932358	-0.596559	70.0
0.932357	-0.596558	90.0

Table C.5. The Effect of Variation of the Convergence Criteria.Table C.5.(a).

$$\frac{\overline{Gr}}{Re^2} = 5.0 \quad Sc = 1.0 \quad H = 0.05$$

$\eta_{MAX} = 4$	$f_1''(0)$	$F_0'(0)$	<u>Convergence Criteria</u>
	2.279920177	-0.6650283353	1×10^{-2}
	2.279901942	-0.6650163091	1×10^{-4}
	2.279901730	-0.6650165171	1×10^{-6}
	2.279901731	-0.6650165188	1×10^{-8}
	2.279901731	-0.6550165188	1×10^{-10}

$$\eta_{MAX} = 7$$

	2.279410915	-0.6651573933	1×10^{-2}
	2.280054935	-0.6649660856	1×10^{-4}
	2.280054902	-0.6649660390	1×10^{-6}
	2.280054902	-0.6649660393	1×10^{-8}
	2.280054902	-0.6649660393	1×10^{-10}

$$\eta_{MAX} = 13$$

	2.279972876	-0.6649873471	1×10^{-2}
	2.280054901	-0.6649660397	1×10^{-4}
	2.280054902	-0.6649660393	1×10^{-6}
	2.280054902	-0.6649660393	1×10^{-8}
	2.280054902	-0.6649660393	1×10^{-10}

Table C.5.(b).

$$\frac{\overline{Gr}}{Re^2} = 0.1 \quad Sc = 1000 \quad H = 0.01, 0.05, \quad X_{TOP} = 6 \\ 0.1, 0.5, 0.5.$$

$$Z_{MAX} = 2, 6, 9, \\ 10, 90.$$

$\overline{\eta}_{MAX} = 10$	$f_1''(0)$	$F_0'(0)$	Convergence Criteria
	1.249338273	-0.6596790016	1×10^{-2}
	1.249338056	-0.6596847010	1×10^{-4}
	1.249338058	-0.6596846941	1×10^{-6}
	1.249338058	-0.6596846941	1×10^{-8}
	1.249338058	-0.6596846938	1×10^{-10}
$\overline{\eta}_{MAX} = 30$			
	0.9338203740	-0.5968828464	1×10^{-2}
	0.9338203740	-0.5968828463	1×10^{-4}
	0.9338204770	-0.5968825279	1×10^{-6}
	0.9338204776	-0.5968825319	1×10^{-8}
	0.9338204776	-0.5968825319	1×10^{-10}
$\overline{\eta}_{MAX} = 90$			
	0.9324482712	-0.5965811248	1×10^{-2}
	0.9323579614	-0.5965589471	1×10^{-4}
	0.9323579635	-0.5965589710	1×10^{-6}
	0.9323580594	-0.5965589710	1×10^{-8}
	0.9323580594	-0.5965589710	1×10^{-10}

Table C.6.(a). Comparison of $F'_0(0)$ in the present work with that of Green⁷³ at a value of the parameter $Gr/Re^2 = 0.0$ for the stagnation point solution.

<u>Sc</u>	<u>$F'_0(0)$</u> <u>Present</u>	<u>$F'_0(0)$</u> <u>Green</u>
0.7	-0.4704930	-0.4704934
1.0	-0.5389789	-0.5389792
2.4	-0.7453304	-0.7453308
5.0	-0.9702980	-0.9702980
10.0	-1.2388910	-1.2388920
20.0	-1.5766100	-1.5766100
50.0	-2.1605200	-2.1605210

Table C.6.(b). Comparison of $F'_0(0)$ in the present work with that of Lloyd and Sparrow⁵⁴ for the Vertical Flat Plate.

<u>$\frac{Gr}{Re^2}$</u>	<u>Prandtl or</u> <u>Schmidt No.</u>		<u>Prandtl or</u> <u>Schmidt No.</u>	
	0.72		10.0	
	<u>Lloyd/</u> <u>Sparrow</u>	<u>Present</u>	<u>Lloyd/</u> <u>Sparrow</u>	<u>Present</u>
0.0	-0.2956	-0.2956	-0.7281	-0.7281
0.1	-0.3158	-0.3157	-0.7574	-0.7574
1.0	-0.4058	-0.4057	-0.9212	-0.9211
2.0	-0.4584	-0.4584	-1.0290	-1.0293

Table C.7. Comparison Between Quasilinearization Solution (Q/L) and Solution to Non - Linear (N/L) Equations using Constants Predicted by Quasilinearization.

Table C.7.(a).

$Sc = 2.4$		$Gr/Re^2 = 0.0$		$H = 0.01$
X	$f_1'(\gamma)_{Q/L}$	$f_1'(\gamma)_{N/L}$	$F_o(\gamma)_{Q/L}$	$F_o(\gamma)_{N/L}$
0.0	0.0000000	0.0000000	1.0000000	1.0000000
0.5	0.4014550	0.4014550	0.6313731	0.6313731
1.0	0.6811150	0.6811150	0.3110725	0.3110725
1.5	0.8526213	0.8526213	0.1073753	0.1073753
2.0	0.9422495	0.9422495	0.0237534	0.0237534
2.5	0.9811658	0.9811658	0.0031685	0.0031685
3.0	0.9949594	0.9949594	0.0002458	0.0002458
3.5	0.9989045	0.9989045	0.0000108	0.0000108
4.0	0.9998081	0.9998081	0.0000002	0.0000002
4.5	0.9999731	0.9999731	0.0000000	0.0000000
5.0	0.9999969	0.9999969	0.0000000	-0.0000000
6.0	0.9999999	0.9999999	0.0000000	-0.0000000
7.0	1.0000000	1.0000000	0.0000000	-0.0000000
8.0	1.0000000	1.0000000	0.0000000	-0.0000000

Table C.7.(b).

$Sc = 2.4$		$Gr/Re^2 = 50.0$		$H = 0.01$
X	$f_1'(\gamma)_{Q/L}$	$f_1'(\gamma)_{N/L}$	$F_o(\gamma)_{Q/L}$	$F_o(\gamma)_{N/L}$
0.0	0.0000000	0.0000000	1.0000000	1.0000000
0.5	1.8859787	1.8859787	0.3687771	0.3687771
1.0	1.5312053	1.5312053	0.0622124	0.0622124
1.5	1.1621541	1.1621541	0.0050253	0.0050253
2.0	1.0356617	1.0356617	0.0002158	0.0002158
2.5	1.0061537	1.0061538	0.0000051	0.0000051
3.0	1.0008464	1.0008465	0.0000000	0.0000000
3.5	1.0000927	1.0000927	0.0000000	-0.0000000
4.0	1.0000080	1.0000081	0.0000000	-0.0000000
4.5	1.0000005	1.0000006	-0.0000000	-0.0000000
5.0	1.0000000	1.0000001	-0.0000000	-0.0000000
6.0	1.0000000	1.0000001	-0.0000000	-0.0000000
7.0	1.0000000	1.0000001	-0.0000000	-0.0000000
8.0	1.0000000	1.0000001	-0.0000000	-0.0000000

Table C.7.(c).

$$Sc = 1000 \quad \overline{Gr}/Re^2 = 0.0$$

<u>X</u>	<u>$f_1'(\gamma)_{Q/L}$</u>	<u>$f_1'(\gamma)_{N/L}$</u>	<u>$F_o(\gamma)_{Q/L}$</u>	<u>$F_o(\gamma)_{N/L}$</u>
0.00	0.0000000	0.0000000	1.0000000	1.0000000
0.04	0.0367072	0.0367072	0.7622417	0.7622417
0.08	0.0726144	0.0726144	0.5324595	0.5324595
0.12	0.1077217	0.1077217	0.3287737	0.3287737
0.16	0.1420292	0.1420292	0.1720647	0.1720647
0.20	0.1755373	0.1755373	0.0728450	0.0728450
0.30	0.2558133	0.2558133	0.0024149	0.0024149
0.40	0.3311103	0.3311103	0.0000069	0.0000069
0.50	0.4014553	0.4014553	0.0000000	0.0000000
1.00	0.6811156	0.6811156	0.0000000	0.0000000
2.00	0.9422507	0.9422507	0.0000000	0.0000000
3.00	0.9949614	0.9949614	0.0000000	0.0000000
4.00	0.0008110	0.9998110	0.0000000	0.0000000
5.00	1.0000007	1.0000007	0.0000000	0.0000000
6.00	1.0000036	1.0000045	0.0000000	0.0000000
7.00	1.0000000	1.0000054	0.0000000	0.0000000

Table C.7.(d).

$$Sc = 1000 \quad \overline{Gr}/Re^2 = 5.0$$

<u>X</u>	<u>$f_1'(\gamma)_{Q/L}$</u>	<u>$f_1'(\gamma)_{N/L}$</u>	<u>$F_o(\gamma)_{Q/L}$</u>	<u>$F_o(\gamma)_{N/L}$</u>
0.00	0.0000000	0.0000000	1.0000000	1.0000000
0.04	0.0436481	0.0436481	0.7511851	0.7511851
0.08	0.0834885	0.0834885	0.5123108	0.5123108
0.12	0.1204696	0.1204696	0.3048754	0.3048754
0.16	0.1554137	0.1554137	0.1513538	0.1513538
0.20	0.1889330	0.1889330	0.0596965	0.0596965
0.30	0.2684753	0.2684753	0.0015216	0.0015216
0.40	0.3429238	0.3429238	0.0000029	0.0000029
0.50	0.4124317	0.4124317	0.0000000	0.0000000
1.00	0.6881933	0.6881933	0.0000000	0.0000000
2.00	0.9440952	0.9440952	0.0000000	0.0000000
3.00	0.9951789	0.9951788	0.0000000	0.0000000
4.00	0.9998189	0.9998188	0.0000000	0.0000000
5.00	0.9999971	0.9999970	0.0000000	0.0000000
7.00	0.9999999	0.9999999	0.0000000	0.0000000

Table C.8. The Effect of the Parameter $\overline{Gr}/\overline{Re}^2$ on the Values of $f_1''(0)$ and $F_0'(0)$ for the Stagnation Point Solution for Aiding Flows.

$\overline{Gr}/\overline{Re}^2$	<u>Table C.8(a), $Sc=0.72$</u>		<u>Table C.8(b), $Sc=1.0$</u>	
	$f_1''(0)$	$F_0'(0)$	$f_1''(0)$	$F_0'(0)$
0.00000	0.927680	-0.475613	0.927680	-0.538978
0.00001	0.927683	-0.475614	0.927683	-0.538979
0.001	0.928366	-0.475783	0.928003	-0.539017
0.01	0.931473	-0.476103	0.931251	-0.539545
0.05	0.944893	-0.477379	0.943788	-0.540872
0.1	0.962355	-0.479256	0.960154	-0.542928
0.2	0.995749	-0.484289	0.991573	-0.546407
0.5	1.095480	-0.492257	1.084932	-0.556897
1.0	1.255518	-0.507179	1.235239	-0.573091
2.0	1.556217	-0.532748	1.518284	-0.600967
5.0	2.362147	-0.590697	2.280055	-0.664966
10.0	3.533077	-0.658566	3.387236	-0.739861
20.0	5.555841	-0.749392	5.301841	-0.841314
50.0	10.563730	-0.909837	10.045690	-1.020922
100.0	17.483552	-1.065430	16.600843	-1.195440

$\overline{Gr}/\overline{Re}^2$	<u>Table C.8(c), $Sc=2.4$</u>		<u>Table C.8(d), $Sc=10.0$</u>	
	$f_1''(0)$	$F_0'(0)$	$f_1''(0)$	$F_0'(0)$
0.00000	0.927680	-0.745330	0.927680	-1.238891
0.00001	0.927682	-0.745330	0.927680	-1.238891
0.001	0.927685	-0.745389	0.927865	-1.238943
0.01	0.930680	-0.746222	0.929908	-1.241338
0.05	0.940952	-0.747532	0.936922	-1.241476
0.1	0.954172	-0.749723	0.946492	-1.245975
0.2	0.980429	-0.754011	0.964460	-1.249113
0.5	1.057720	-0.766375	1.018742	-1.263973
1.0	1.182816	-0.785473	1.107189	-1.289227
2.0	1.419493	-0.819355	1.276566	-1.331871
5.0	2.063256	-0.899800	1.744671	-1.434992
10.0	3.004043	-0.994964	2.438427	-1.570165
20.0	4.638465	-1.127895	3.654325	-1.760039
50.0	8.694116	-1.365949	6.689411	-2.111143
100.0	14.300450	-1.596982	10.892710	-2.461340

Table C.8(e), $Sc=100.0$ Table C.8(f), $Sc=1000.0$

$\overline{Gr}/Re^2$	$\underline{f_1''(0)}$	$\underline{F_0'(0)}$	$\underline{f_1''(0)}$	$\underline{F_0'(0)}$
0.00000	0.927680	-2.736690	0.92768	-5.95859
0.00001	0.927680	-2.736690	0.92768	-5.95859
0.001	0.927774	-2.736745	0.92772	-5.95865
0.01	0.928626	-2.737270	0.92813	-5.95920
0.05	0.932406	-2.739640		
0.1	0.937125	-2.742515	0.93236	-5.96559
0.2	0.946535	-2.748298	0.93680	-5.97078
0.5	0.974640	-2.765755	0.95054	-5.98978
1.0	1.020651	-2.793300	0.97308	-6.01951
2.0	1.110692	-2.845864	1.01761	-6.07738
5.0	1.366331	-2.985193	1.14712	-6.23977
10.0	1.757857	-3.176423	1.35179	-6.48047
20.0	2.460333	-3.471445	1.73058	-6.88414
50.0	4.247100	-4.056475	2.72396	-7.76522
100.0	6.745833	-4.666654	4.13837	-8.75302

Table C.9. The Effect of the Parameter $\overline{Gr}/Re^2$ on the Values of $\underline{f_1''(0)}$ and $\underline{F_0'(0)}$ for the Stagnation Point Solution for Opposing Flows.

Table C.9(a), $Sc=0.72$ Table C.9(b), $Sc=1.0$

$\overline{Gr}/Re^2$	$\underline{f_1''(0)}$	$\underline{F_0'(0)}$	$\underline{f_1''(0)}$	$\underline{F_0'(0)}$
0.0000	0.92768	-0.47561	0.92768	-0.53897
0.0001	0.92764	-0.47561	0.92764	-0.53897
0.01	0.92456	-0.47539	0.92478	-0.53878
0.1	0.89293	-0.47200	0.89518	-0.53511
0.2	0.85772	-0.46828	0.86228	-0.53114
0.5	0.74909	-0.45638	0.76094	-0.51848
1.0	0.55526	-0.43326	0.58141	-0.49438
2.0	0.06487	-0.35718	0.14324	-0.42065
2.2	-0.09432	-0.32166	0.02348	-0.39436
2.3	-	-	-0.05017	-0.37593

Table C.9(c), Sc=2.4Table C.9(d), Sc=10.0

$\frac{\overline{Gr}}{Re^2}$	$\underline{f_1''(0)}$	$\underline{F_0'(0)}$	$\underline{f_1''(0)}$	$\underline{F_0'(0)}$
0.0000	0.92768	-0.74533	0.92768	-1.23889
0.0001	0.92765	-0.74532	0.92766	-1.23888
0.01	0.92535	-0.74520	0.92582	-1.23837
0.1	0.90093	-0.74085	0.90909	-1.23366
0.2	0.87392	-0.73627	0.89038	-1.22835
0.5	0.79111	-0.72184	0.83338	-1.21188
1.0	0.64644	-0.69544	0.73578	-1.18424
2.0	0.31530	-0.62436	0.52429	-1.11500
2.3	0.19639	-0.59413	0.45540	-1.09047
2.703	0.00021	-0.53544	-	-
3.0	-	-	0.27932	-1.02217
3.86	-	-	0.00038	-0.88948

Table C.9(e), Sc=100.0Table C.9(f), Sc=1000.0

$\frac{\overline{Gr}}{Re^2}$	$\underline{f_1''(0)}$	$\underline{F_0'(0)}$	$\underline{f_1''(0)}$	$\underline{F_0'(0)}$
0.0	0.92768	-2.73669	0.92768	-5.95859
0.1			0.92321	-5.95335
0.5			0.90483	-5.92859
1.0	0.83123	-2.67596	0.88166	-5.89709
2.0	0.73057	-2.60922	0.83463	-5.83209
4.0	0.51215	-2.45136		
7.0	0.09518	-2.07581		
7.48	0.00038	-1.96854		
10.0			0.41043	-5.16652
15.0			0.04115	-4.40257

Table C.10. The Effect of the Parameter Gr/Re^2 on the Values of $f_1''(0)$ and $F_0'(0)$ for the Vertical Flat Plate for Aiding Flows.

<u>Table C.10(a), $Sc=0.72$</u>			<u>Table C.10(b), $Sc=1.0$</u>	
<u>$\frac{Gr}{Re^2}$</u>	<u>$f_1''(0)$</u>	<u>$F_0'(0)$</u>	<u>$f_1''(0)$</u>	<u>$F_0'(0)$</u>
0.00000	0.332059	-0.295659	0.332059	-0.332059
0.00001	0.332074	-0.295661	0.332073	-0.332061
0.001	0.333564	-0.195893	0.333444	-0.332307
0.01	0.346990	-0.297963	0.345811	-0.334506
0.05	0.404390	-0.306451	0.398831	-0.343574
0.1	0.471936	-0.315777	0.461454	-0.353619
0.2	0.596999	-0.331525	0.577825	-0.370725
0.5	0.925026	-0.366285	0.884299	-0.408890
1.0	1.391825	-0.405753	1.321556	-0.452592
2.0	2.188194	-0.458387	2.068325	-0.511156
5.0	4.141070	-0.550895	3.900165	-0.714746
10.0	6.823012	-0.640703	6.415651	-0.714746
20.0	11.331525	-0.750325	10.643554	-0.837278
50.0	22.310807	-0.931086	20.937267	-1.039329
100.0	37.358290	-1.100090	35.042833	-1.228221

<u>Table C.10(c), $Sc=2.4$</u>			<u>Table C.10(d), $Sc=10.0$</u>	
<u>$\frac{Gr}{Re^2}$</u>	<u>$f_1''(0)$</u>	<u>$F_0'(0)$</u>	<u>$f_1''(0)$</u>	<u>$F_0'(0)$</u>
0.00000	0.332059	-0.449517	0.332059	-0.728142
0.00001	0.332059	-0.449517	0.332066	-0.728145
0.001	0.333157	-0.449797	0.332787	-0.728459
0.01	0.342978	-0.452291	0.339315	-0.731285
0.05	0.385378	-0.462723	0.367765	-0.743344
0.1	0.435944	-0.474527	0.402175	-0.757405
0.2	0.530861	-0.495100	0.467821	-0.782815
0.5	0.783925	-0.542462	0.646909	-0.844615
1.0	1.148105	-0.598144	0.909571	-0.921172
2.0	1.772663	-0.673983	1.365093	-1.029322
5.0	3.307645	-0.809162	2.492303	-1.227552
10.0	5.416237	-0.941276	4.044997	-1.424335
20.0	8.959658	-1.102930	6.656549	-1.666854
50.0	17.584010	-1.369743	13.014819	-2.068843
100.0	29.398607	-1.619237	21.725450	-2.445492

Table C.10(e), $Sc=100.0$ Table C10(f), $Sc=1000.0$

$\frac{Gr}{Re^2}$	$f_1''(0)$	$F_0'(0)$	$f_1''(0)$	$F_0'(0)$
0.00000	0.332059	-1.571832	0.332059	-3.387084
0.00001	0.332061	-1.571832	0.332059	-3.387084
0.001	0.332411	-1.572171	0.332262	-3.387567
0.01	0.335591	-1.575243	0.333759	-3.390711
0.05	0.349586	-1.588625	0.340379	-3.404551
0.1	0.366785	-1.604759	0.348586	-3.421555
0.2	0.400300	-1.635296	0.364786	-3.454628
0.5	0.495104	-1.715962	0.411816	-3.547175
1.0	0.639483	-1.825815	0.486059	-3.684189
2.0	0.897008	-1.993984	0.623141	-3.913295
5.0	1.547912	-2.557385	0.981499	-4.411534
10.0	2.453866	-2.669558	1.490321	-4.968528
20.0	3.984089	-3.103867	2.357592	-5.701576
50.0	7.717991	-3.834820	4.484305	-6.972534
100.0	12.838089	-4.525200	7.406561	-8.192922

Table C.11. The Effect of the Parameter Gr/Re^2 on the Values of $f_1''(0)$ and $F_0'(0)$ for the Vertical Flat Plate for Opposing Flows.

Table C.11(a), $Sc=0.72$ Table C.11(b), $Sc=1.0$

$\frac{Gr}{Re^2}$	$f_1''(0)$	$F_0'(0)$	$f_1''(0)$	$F_0'(0)$
0.00000	0.332059	-0.295659	0.332059	-0.332059
0.00001	0.332044	-0.295656	0.332045	-0.332056
0.0001	0.331908	-0.295635	0.331920	-0.332034
0.001	0.330551	-0.295424	0.330671	-0.331809
0.01	0.316858	-0.293267	0.318078	-0.329527
0.05	0.252814	-0.282637	0.259462	-0.318379
0.1	0.162000	-0.265642	0.177647	-0.301072
0.176	-	-	0.000973	-0.251823

Table C.11(c), $Sc=2.4$ Table C.11(d), $Sc=10.0$

$\frac{Gr}{Re^2}$	$\underline{f_1''(0)}$	$\underline{F_0'(0)}$	$\underline{f_1''(0)}$	$\underline{F_0'(0)}$
0.00000	0.332059	-0.449517	0.332059	-0.728142
0.00001	0.332048	-0.449514	0.332051	-0.728139
0.0001	0.331949	-0.449489	0.331986	-0.728110
0.001	0.330959	-0.449235	0.331330	-0.727825
0.01	0.320997	-0.446667	0.324741	-0.724943
0.05	0.275135	-0.434403	0.294799	-0.711537
0.1	0.213155	-0.416493	0.255626	-0.693156
0.2	0.056950	-0.361290	0.169079	-0.648437
0.22	0.010750	-0.340573		
0.32	-	-	0.033563	-0.561630
0.34	-	-	0.000631	-0.535424

Table C.11(e), $Sc=100.0$ Table C.11(f), $Sc=1000.0$

$\frac{Gr}{Re^2}$	$\underline{f_1''(0)}$	$\underline{F_0'(0)}$	$\underline{f_1''(0)}$	$\underline{F_0'(0)}$
0.00000	0.332059	-1.571832	0.332059	-3.387084
0.00001	0.332054	-1.571827	0.332062	-3.387084
0.0001	0.332022	-1.571795	0.332079	-3.387182
0.001	0.331703	-1.571488	0.331929	-3.386867
0.01	0.328509	-1.568388	0.330430	-3.383709
0.05	0.314169	-1.554310	0.323734	-3.369535
0.1	0.295885	-1.535976	0.315290	-3.351489
0.2	0.257951	-1.542878	0.298138	-3.314222

Table C.12. Comparison Between Predicted Mass Transfer Rates
for the Vertical Flat Plate, up to the Value
of the Parameter Gr/Re^2 at which the Method Failed.

$Sc = 10.0$

$\frac{Gr}{Re^2}$	A $\sqrt{2}$	B $\sqrt{2}$	C $\sqrt{2}$	D $\sqrt{2}$	E $\sqrt{2}$
0.002	16.581	17.526	16.881	17.961	16.510
0.005	10.345	10.939	10.532	11.190	10.284
0.01	7.301	7.726	7.432	7.876	7.365
0.02	5.182	5.491	5.275	5.562	5.071
0.05	3.334	3.546	3.339	3.525	3.163
0.1	2.427	2.598	2.468	2.590	2.232
0.15	2.039	2.197	2.073	2.173	
0.2	1.844	1.999	1.861	1.805	

$Sc = 100.0$

$\frac{Gr}{Re^2}$	A $\sqrt{2}$	B $\sqrt{2}$	C $\sqrt{2}$	D $\sqrt{2}$
0.002	35.956	38.048	36.632	37.186
0.005	22.437	23.753	22.859	22.453
0.01	15.841	16.781	16.138	15.661
0.02	11.252	11.936	11.462	11.179
0.05	7.254	7.726	7.388	6.967
0.1	5.304	5.686	5.400	5.201
0.15	4.482	4.838	4.562	4.316

- A - Cubic concentration profile, single quartic velocity profile.
- B - Quartic concentration profile, single quartic velocity profile.
- C - Sextic concentration profile, single quartic velocity profile.
- D - Quartic concentration, double velocity profile; inner sextic, outer quartic.
- E - Cubic concentration, double velocity profile, inner quintic, outer quartic.

APPENDIX D.COMPUTER PROGRAMMES.D.1.1 Introduction.

The numerical techniques which have been employed to solve the boundary layer equations have been presented in Chapters 4 and 5. The numerical solutions of a number of problems have been calculated with the aid of a digital computer.

Two computer programmes have been developed. The first solves the boundary layer equations by the approximate method discussed in Chapter 4. The programme is written in a general form so that it may be readily adapted for different body shapes from those which are specifically considered in the present work. The second programme solves the exact equations for the stagnation point and the vertical flat plate by the use of the quasilinearization technique described in Chapter 5. This programme is also written in a general form and may readily be adapted to solve other two point boundary value problems, not necessarily associated with the interaction of free and forced convection.

The alterations which are required to modify either programme are discussed in the relevant sections describing the individual subroutines.

The programmes are written in FORTRAN IV language, and in their present form may be run on a CDC 6600 computer. In order to run them on other machines some modifications may be necessary. Both programmes have been written so that the only changes required from run to run are in the data input. All of the data are read in from data cards. This enables the user to obtain binary decks for the entire programme. Thus considerable compilation time will be saved if the programmes are run many

times.

In order that solutions for values of any of the parameters which are different from those presented in Chapter 6, the programmes are presented in detail.

Sections D.2. and D.3. describe the computer programmes for the approximate and exact methods respectively. Both of these sections are sub-divided in the same manner.

Sub - section 1 presents a list of the FØRTRAN symbols which are used in the programme together with their meanings. Some variables, although different, have the same name in the two programmes. This occurs infrequently and was allowed for continuity within each programme. Since the variables are clearly defined, there should be no confusion when this occurs.

In sub - section 2 the flow charts are presented for the overall programme and the important sub - routines. A description of each sub - routine is presented which explains its function. The input of data is described in the appropriate sub - routine, and the control variables are described so that alterations may readily be made to the programme.

The complete programme listings are presented in sub - section 3.

D.2.1. List of Symbols used in the Computer Programme.

<u>FØRTRAN</u>	<u>Meaning</u>
<u>Symbol</u>	
A	$(\bar{d}^2 - \bar{r}^2)^{\frac{1}{2}}$
A1	$Sh / Re^{\frac{1}{2}} (Gr/Re^2)^{\frac{1}{2}}$
A2	$Sh / Re^{\frac{1}{2}}$
A3	$Sh / Re^{\frac{1}{2}} Sc^{\frac{1}{3}} (Gr/Re^2)^{\frac{1}{2}}$
A4	$Sh / Re^{\frac{1}{2}} Sc^{\frac{1}{3}}$
A5	$Sh / Gr^{\frac{1}{4}}$
AAA	$\partial u / \partial y \big _{y=0}$
AKAP	$\times$
AKNEW	x^{k+1}
ALA	$\epsilon \frac{Re^2}{Gr} \frac{dU}{d\Theta} Z$
AMAX4	Maximum value of A4.
ANUMLS	The right hand side of equation (4-87).
AR	Current angular position around the body from the front stagnation point, radians.
ATHETA(50)	Array containing the angles at which the velocity and concentration profiles are to be printed out.
B	$\sinh^2 \xi_0 + \cos^2 \Theta$
C	$\cos \Theta$
C1	$x Sc^{\frac{1}{3}}$
C2	$\cos^2 \Theta$
C3	$x Sc^{\frac{1}{4}}$
CC	$\cos \varphi$
CHD	C_D
CØH	$\cosh \xi_0$

FØRTRANMeaningSymbol

CØH2	$\cosh^2 \xi_0$
CØN	$(\cosh^2 \xi_0 \tan^{-1} (1 / \sinh \xi_0)) - \sinh \xi_0$
CØNT	$((\cosh^2 \xi_0 \tan^{-1} (1 / \sinh \xi_0)) - \sinh \xi_0) \times (\sinh^2 \xi_0 + 1)$
D	$\bar{d}$
DD	d'
DEL(10)	Array containing the variable integration step lengths measured in degrees.
DEN	The left hand side of equation (4-88).
DENMLS	The left hand side of equation (4-87).
DIFF	Accuracy with which the initial conditions are calculated in Newton - Raphson procedure.
DJ	Denominator of the second term of equations (4-93) and (4-94).
DX	Δ
DZ	δ
ECC	$\bar{f} / \bar{d}$
EPS	$\bar{\epsilon}$
ERG	$\bar{\epsilon} \text{Re}^2 / \text{Gr}$
ETA	$\frac{\bar{f}}{\bar{d}} \frac{\cos^2 \psi}{\cos^2 \Theta} \bar{\epsilon} = \frac{\text{Re}^2}{\text{Gr}}$
F	$\bar{f}$
F1, ... F6.	Variable names given to equations (4-66) to (4-71).
F1D, ... F6D	Variable names given to the differential of equations (4-66) to (4-71) w.r.t. Z.
FA	L_Z

<u>FØRTTRAN</u>	<u>Meaning</u>
<u>Symbol</u>	
FK	$dL_Z / d\alpha$
FZ	dL_Z / dZ
G	L_α
GK	$dL_\alpha / d\alpha$
GZ	dL_α / dZ
GR	Gr / Re^2
GRS	$Gr / Re^2 Sc^{1/3}$
H	Integration step length in Runge - Kutta integration.
H1,H2	Variable names given to equations (4-76) and (4-77).
H3,H4	Variable names given to the differential of equations (4-76) and (4-77) w.r.t. α .
I1	Number of changes of integration step length.
I2	Number of complete solutions to be calculated during the present run.
I4	Counter for the number of angles at which the velocity and concentration profiles are printed out.
I5	Counter for the number of profile plots.
INDEX	Counter for the angular position measured in an integer number of degrees.
IPLØT	Set to 0 if no profile plots are required.
IPRØFL	Set to 0 if no profile print outs are required.
ISTØP	Error control variable.
IVEL(15)	The angles at which profile plots are required.

<u>FORTTRAN</u>	<u>Meaning</u>
<u>Symbol</u>	
M1(10)	Array containing the I1 values of N1.
M2(10)	Array containing the I1 values of N2.
M3(10)	Array containing the I1 values of N3.
M4(10)	Array containing the I1 values of N4.
M5(10)	Array containing the I1 values of N5.
M6(10)	Array containing the I1 values of N6.
N	The number of first order differential equations to be solved.
NVL	The number of profile plots which are required.
P1	ψ
PHI	$90^\circ - \Theta$
	$\pi / 180$
R	$Sh_\Theta / Sh_{\Theta=0}$
R1	$Z^{k+1} - Z^k$
R2	$\chi^{k+1} - \chi^k$
RG	Re^2 / Gr
S	$\sin \Theta$
S1	$Sc^{\frac{1}{3}}$
S14	$Sc^{\frac{1}{4}}$
SCH	Sc
SIH	$\sinh \xi_0$
SIH2	$\sinh^2 \xi_0$
SQGR	$(Gr/Re^2)^{\frac{1}{2}}$
SQ2GR	$(Gr/Re^2)^{\frac{1}{4}}$
SS	$\sin \psi$
T	$\sin (90 - \Theta) / \cos (90 - \Theta)$

<u>FØRTRAN</u>	<u>Meaning</u>
<u>Symbol</u>	
THETA	Θ
U, TRU	U
UD, TRUD	$dU / d\Theta$
UDD, TRUDD	$d^2U / d\Theta^2$
V	$(\sinh^2 \zeta_o + 1)$
W	$\cosh^2 \zeta_o / ((\cosh^2 \zeta_o \tan^{-1} (1/\sinh \zeta_o)) - \sinh \zeta_o) (\sinh^2 \zeta_o + 1)^{1/2}$
X	Variable name given to AR in subroutine RKINT.
XTSEC	Central Processor time in seconds.
Y(1)	Z in subroutine RKINT.
Y(2)	χ
YG(1)	$dZ / d\Theta$
YG(2)	$d\chi / d\Theta$

(i) MAIN (Approximate Method)

The MAIN subroutine acts almost solely in an executive capacity. Its principal function is to set the control parameters and to transfer control to the different subroutines, each of which performs a separate function. The flow diagram is shown in Fig. D.2.1. The control variables which are read in from data cards define: the number of problems to be solved, the number of step length changes, and whether plots are required for the local mass transfer results, and the velocity and concentration profiles. If the latter plots are required, details of the angles for which the plots are required must also be supplied.

The clock is called so that the central processor time for each solution may be determined.

Subroutine CASE is called to define the problem to be solved. The information which is needed defines the physical parameters and the body shape.

If approximate initial conditions for the boundary layer thicknesses have been supplied, the executive calls INIT to refine them and, if no convergence is achieved, control is transferred to an error printout and computation is stopped. Otherwise, on returning to the executive, a loop is entered which controls the variable step lengths.

The step length, H , is defined together with the number of steps to be taken at this step length, $N5$; the number of steps between each calculation and printout of results, $N1$; and between the calculation of the velocity and concentration profiles, $N6$. A set of counters, with which to compare the control variables, are set to zero.

When the control variables have been defined, another

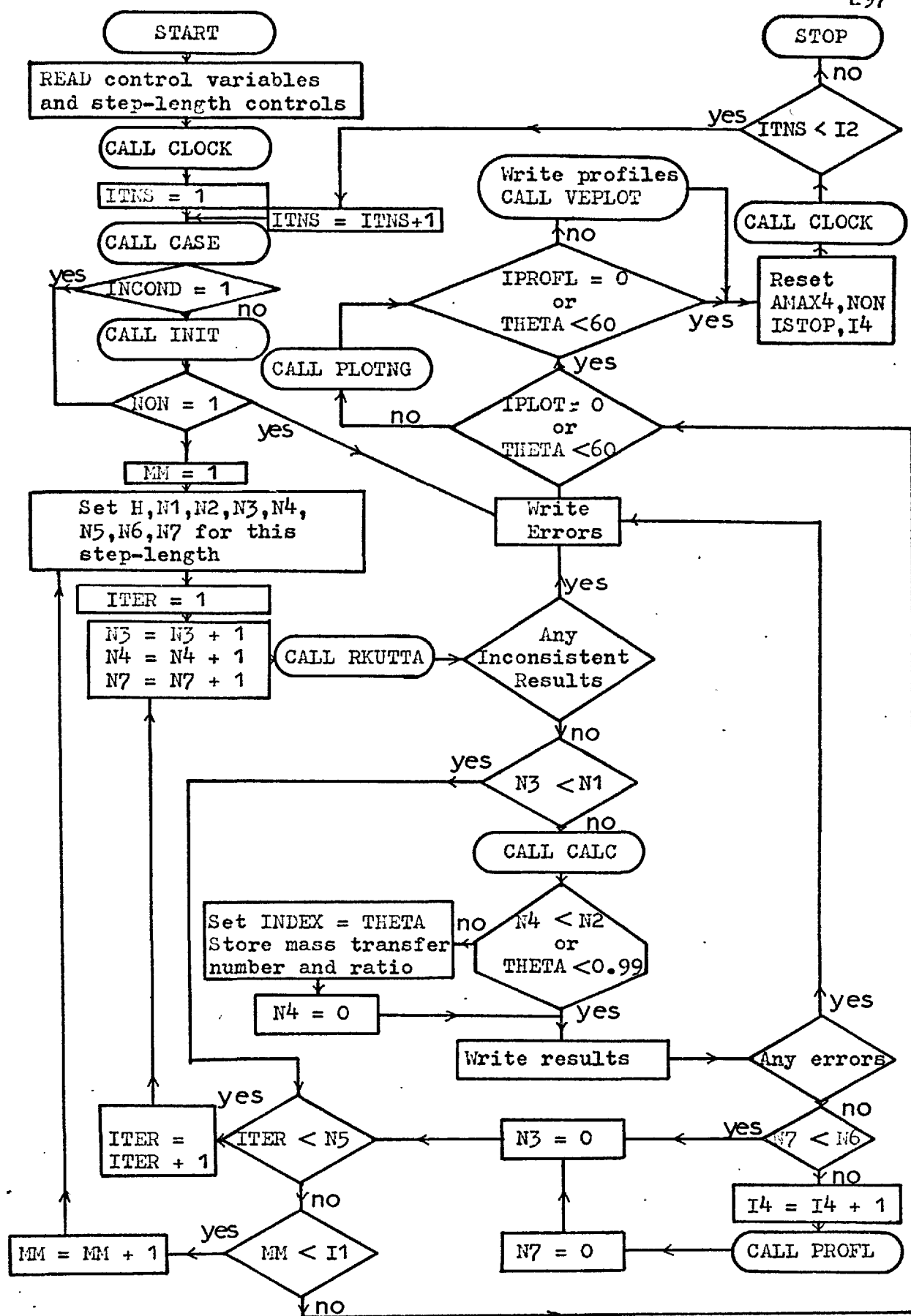


Fig. D.2.1. Flow Diagram of Subroutine MAIN (Approximate Method).

loop is entered to integrate the required number of steps at the specified step length. The counters N3, N4 and N7 are all updated by unity and a single step is made by calling the Runge - Kutta integration routine, RKUTTA. After this step has been taken, the values of Y(1) and Y(2) are checked for any obvious errors, which, if present are printed out.

If no errors are found, a check is made to see whether the mass transfer numbers have to be calculated at this step. If not, control is transferred to the end of the loop.

A further test is made to see whether the velocity and concentration profiles are to be calculated at the current angular position. If the profiles are not to be calculated, control is transferred to the end of the loop. At this stage, either another step is made with the current step length, or the step length is changed. If no more changes of step are required, control is transferred from the integration loops.

The local transfer rates are now plotted if required, and the velocity and concentration profiles are tabulated. Plots of the profiles, at the prespecified angles, are also made.

In case a further solution is required, the control parameters are re - set to their initial conditions. The clock is recalled for a final central processor time to be printed and then the end of the loop, for one complete solution, is reached. If another problem is to be solved, control is transferred back to the beginning of the programme. Otherwise the computation is stopped.

(ii) CASE and ECCEN

Subroutine CASE is called once for each complete problem solved. It reads in all the necessary data to describe the physical system to be solved, Y(1), Y(2), RG, SCH, THETA, SPHERE; and the control variables necessary for the run, QZ, ICH, WHICH1, INCOND.

Subroutine ECCEN is called from CASE. The shape variables are calculated for either the sphere or the oblate spheroid. If the body is an oblate spheroid and the shape is different from the previous problem, or if this is the first problem, information is read to describe the spheroidal shape; F, D, CHD, DD.

The headings for the problem are printed out and control returns through CASE to MAIN.

To treat any other body shape, the appropriate subroutine ECCEN must be written to describe the required shape variables.

(iii) INIT

Subroutine INIT is called from MAIN if approximations to the initial boundary layer thicknesses have been read in and improved values are required. Fig. D.2.2 shows the flow diagram for subroutine INIT.

The Newton - Raphson method is used, and convergence is usually very fast from a reasonable initial guess. The subroutine calculates the values of the two functions, defined by equations (4-88) and (4-89), with the approximate values for the boundary layer thicknesses supplied. The two derivatives for each function are also calculated.

A new estimate of the values is then calculated and compared with the old values. If these are within the required

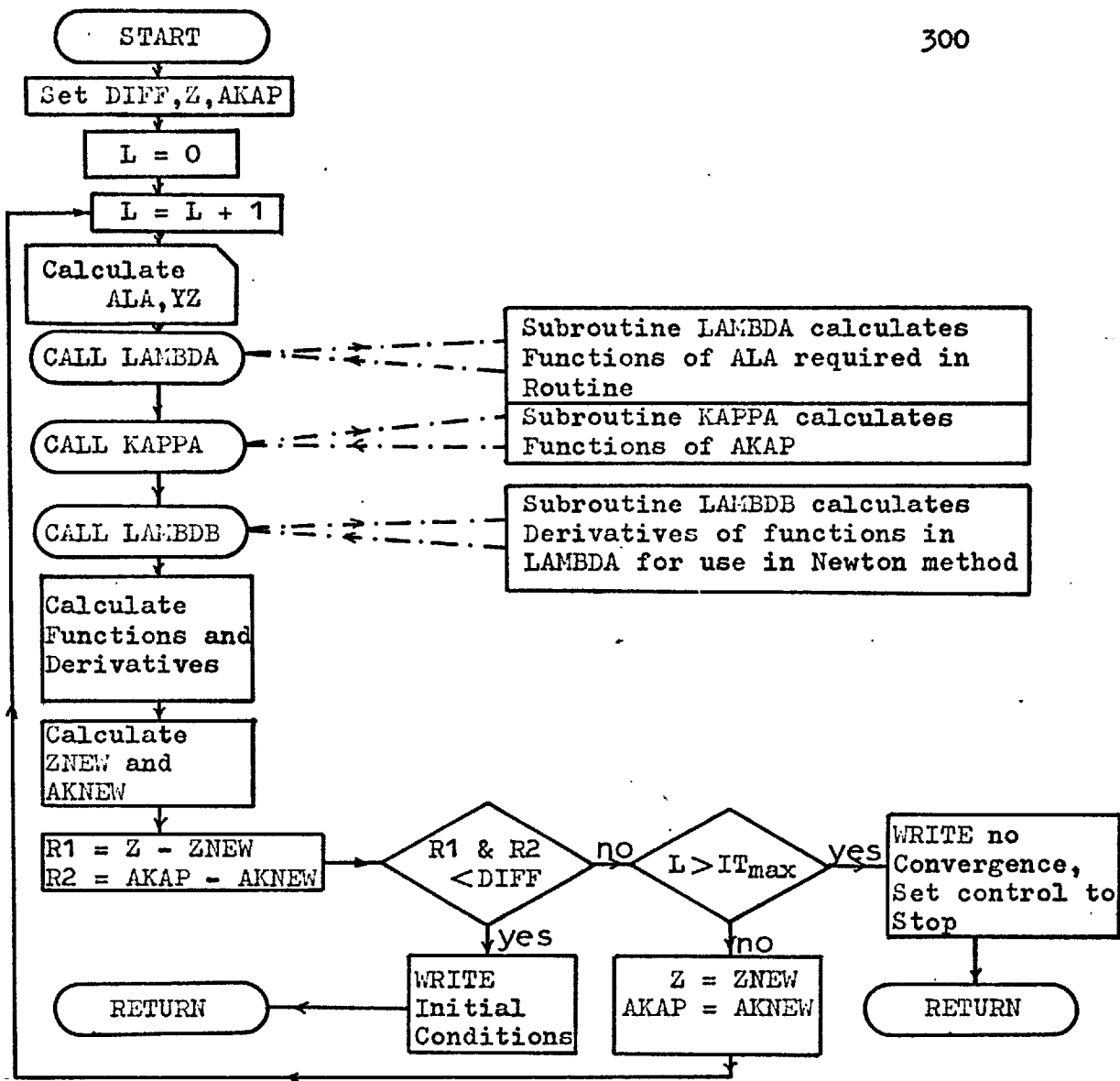


Fig. D.2.2. Flow Diagram of Subroutine INIT

accuracy, the results are printed out and control returns to MAIN.

If convergence has not been achieved, a test is made to see whether the pre-set iteration limit has been reached. If this is the case the computation is stopped. Otherwise the old values are replaced by those just calculated, and a further iteration is made.

If other velocity and concentration profiles are assumed, the subroutine remains unchanged, but the cards representing the calculation of the two functions and their derivatives must be replaced by those for the new case.

(iv) YGRD

Subroutine YGRD is called by RKUTTA four times in every integration step. The flow diagram is shown in Fig. D.2.4.

The routine is programmed to calculate the derivatives of $Y(1)$ and $Y(2)$ with respect to THETA, for both the sphere and the oblate spheroid.

On entering the subroutine, a test is made to see whether the body is a sphere or an oblate spheroid. The appropriate shape values are calculated, and the routine which calculates the potential flow for the specific shape is called. Subroutines LAMBDA and KAPPA are called to calculate functions of ALA and $AKAP$ respectively. The latter two subroutines are independent of body shape.

The derivatives $YG(1)$ and $YG(2)$ are now calculated. Each calculation is split up into sections to enable checks to be made easily if they are required. Once the derivatives have been evaluated, control returns to RKUTTA.

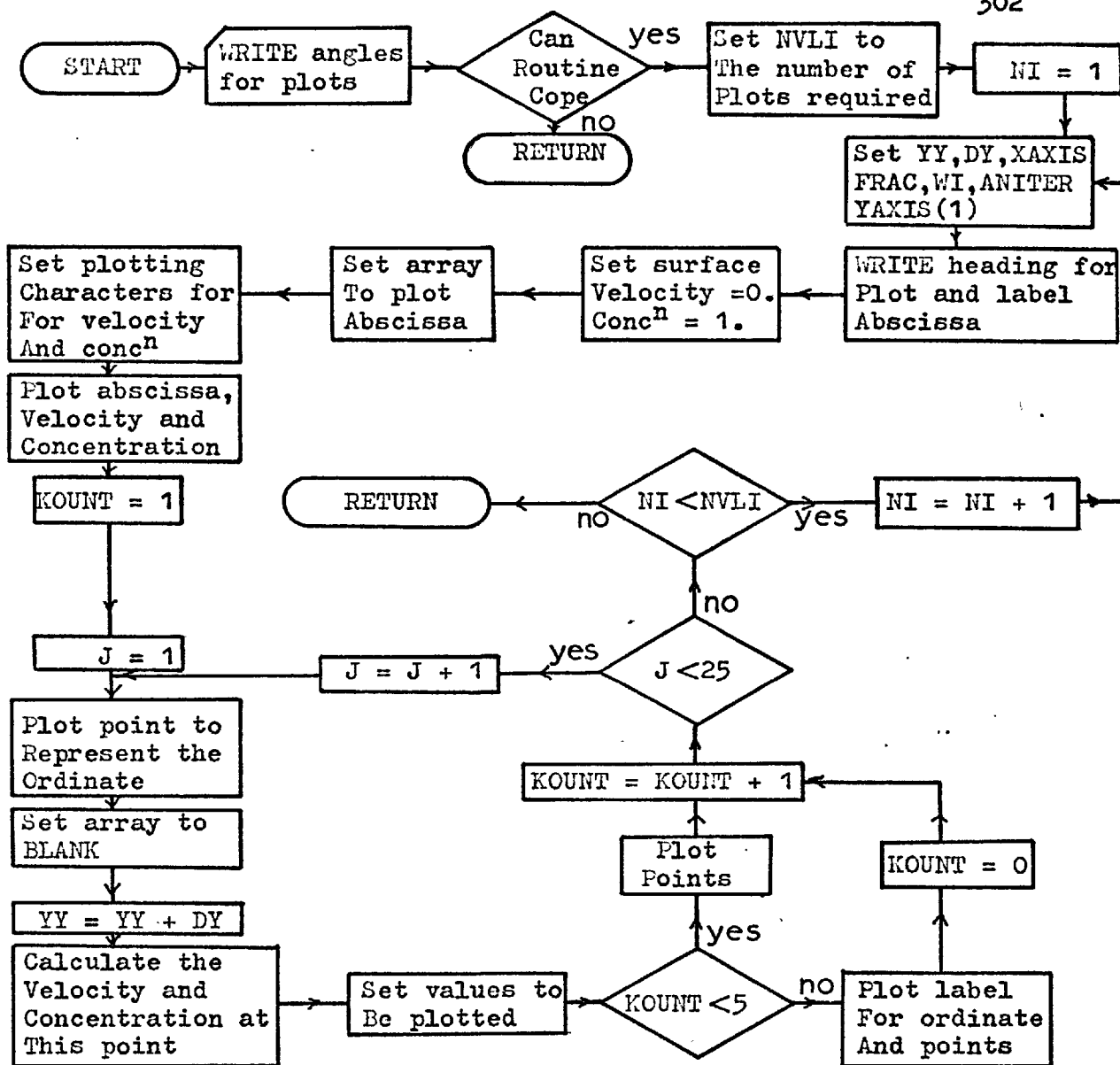


Fig. D.2. 3. Flow Diagram of Subroutine VEPLLOT

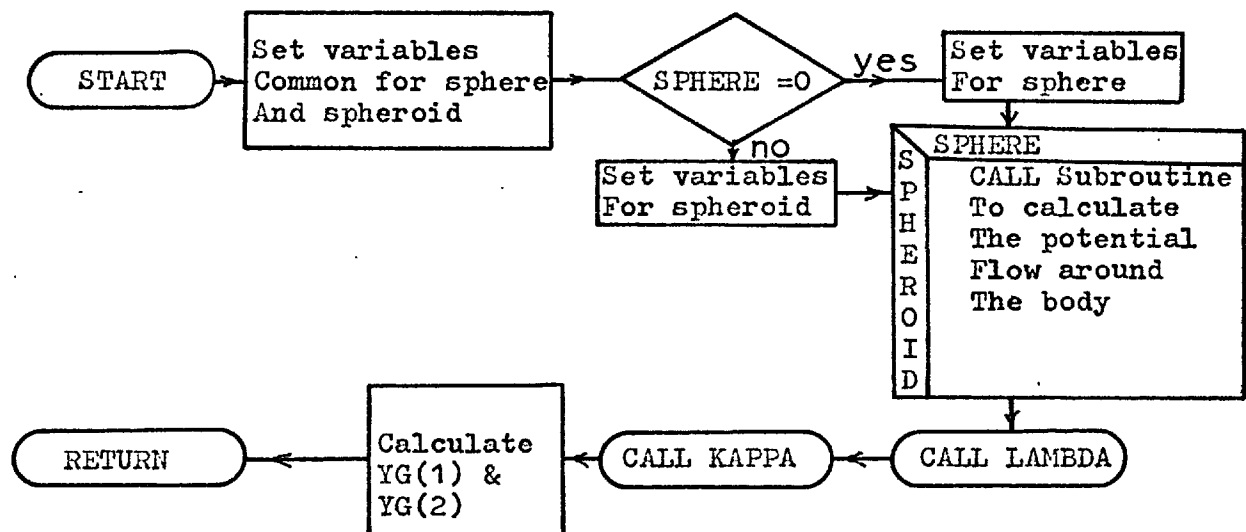


Fig. D.2.4. Flow Diagram of Subroutine YGRD

To treat different assumed profiles, YGRD must be completely replaced by a new subroutine YGRD. This calculates the YG's defined by the new equations.

(v) CALC

Subroutine CALC is called from MAIN whenever calculation and print-out of the results, at the present position on the body surface, is required.

The five mass transfer numbers, A_1 , A_2 , A_3 , A_4 , and A_5 are computed together with the dimensionless velocity gradient at the surface, AAA. The latter calculation is made so that an estimate of the separation point may be determined. Control is then transferred back to MAIN.

Alterations to this subroutine may be necessary if different transformations are employed in the derivation of the dimensionless equations.

(vi) KAPPA, LAMBDA, LAMBDB

These subroutines calculate functions of AKAP and ALA, defined by equations (4-76), (4-77) and (4-66) to (4-71). These are required for the computation of the gradients in subroutines INIT and YGRD.

All three routines would definitely need to be replaced if different velocity and concentration profiles were assumed.

(vii) THEORY

Subroutine THEORY calculates the potential flow velocity functions at an angle THETA from the front stagnation point

of an oblate spheroid. The formulae used are those presented in Table 4-1.

If an experimentally measured profile is available to represent the potential flow, this could very easily be incorporated into the programme.

(viii) THEORI

Subroutine THEORI calculates the potential flow velocity functions for a sphere, at an angle THETA from the front stagnation point. The equations used were presented earlier in Table 4-1. The subroutine can be readily replaced by one which represents an experimentally determined profile.

(ix) PROFL

Subroutine PROFL is called from MAIN and calculates the velocity and concentration profiles at the current angular position. The flow diagram for subroutine PROFL is shown in Fig. D.2.5.

The hydrodynamic boundary layer thickness is used to calculate the coefficients of the velocity profile, AZ, BZ, CZ, and EZ. The transfer boundary layer thickness, DX, is calculated and then the dimensionless velocity and concentration, at a number of points within the boundary layer, are calculated. These are stored in arrays PR and CPR respectively. The present programme calculates these values at five points within the hydrodynamic boundary layer and five points outside it. The values which are calculated are stored for printing later in the programme.

If a plot of the profiles is required, the coefficients

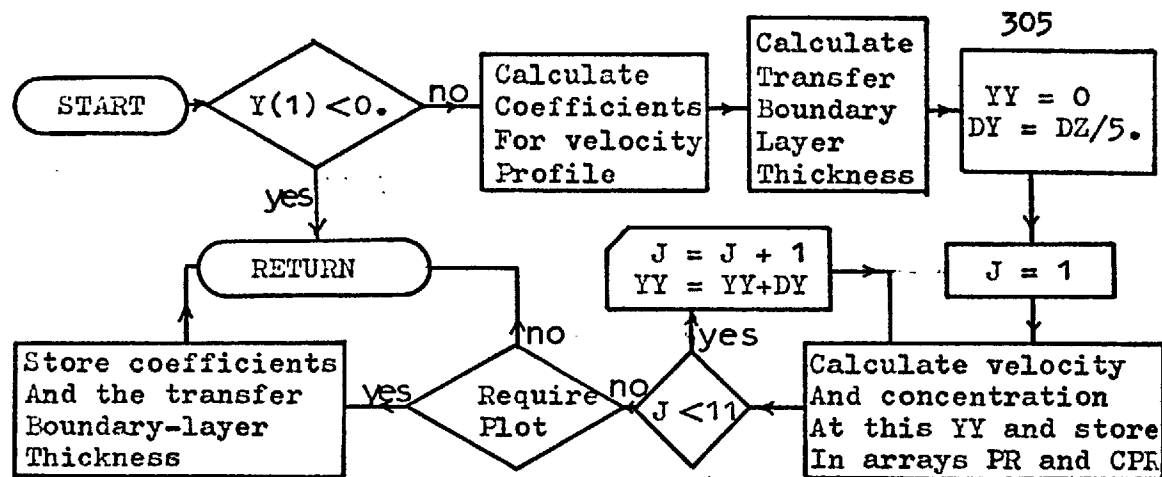


Fig. D.2.5 Flow Diagram for Subroutine PROFL

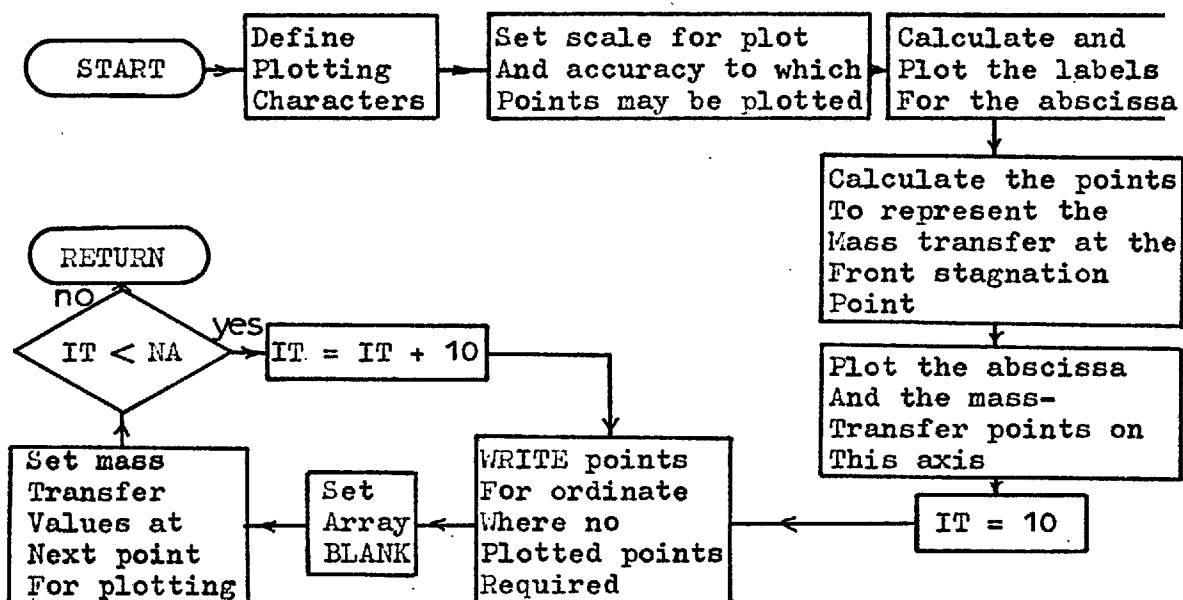


Fig. D.2.6. Flow Diagram for Subroutine PLOTNG

which have been calculated at this angle are stored.

To modify the programme when other assumed profiles are used, new expressions must be substituted in order to calculate the coefficients of the velocity profile and the dimensionless velocity and concentration.

(x) PLOTNG

Subroutine PLOTNG plots the local mass transfer results. The flow diagram for the subroutine is Fig. D.2.6.

Four print characters are defined: BLANK , PLOT * , SPLOT . , and RPLOT + . The scale of the plot is set depending upon the largest value of the mass transfer number to be plotted. The abscissa is represented by 101 points. The value WI is the accuracy with which the local transfer rates may be plotted. The mass transfer results at the front stagnation point are plotted together with the abscissa. The first member of the plotting array is kept as SPLOT, to plot the ordinate, whilst the remainder are made BLANK.

The values of the mass transfer numbers at the next angle, for which points are required, are converted for plotting. The appropriate members of the array are then set to the plotting characters and the array is plotted. This procedure is then repeated for all the values which are to be plotted.

(xi) VEPLOT

The subroutine to plot the velocity and concentration profiles, VEPLOT, is very similar in principle to that used for plotting the local mass transfer results.

The previous subroutine simply had to plot stored information. For this plot, however, only the coefficients of the velocity profile, and the transfer boundary layer thickness, at the angle for which the plot is required, have been stored. The points to be plotted must therefore be generated within the subroutine.

The maximum value of the abscissa has been set to 1.0 . This means that whenever the velocity within the boundary layer becomes greater than that in the free stream, the plotting routine as it stands cannot cope. In general, this restriction only allows plots to be made when the value of the parameter Gr/Re^2 is less than unity. The plotting procedure is the same as that for the previous subroutine, and may be followed with the aid of the flowchart Fig. D.2.3.

JOB(UMEC023,T15 ,CM25000) TEST

P.A.DIPLOCK,IMPERIAL COLLEGE

FUN(S)

REDUCE.

LGO.

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PROGRAM MAIN (INPUT,OUTPUT,TAPES=INPUT,TAPE6=OUTPUT)
DIMENSION Y(5),YG(5),YH(5),YD(5),DEL(10),M1(10),M2(10),M3(10),M4(
110),M5(10),M6(10),ANS(200),RA(200),ATHETA(50),PR(50,12),CPR(50,
212),IVEL(15)
INTEGER SPHERE,WHICH1
COMMON /BK1/ A1,A2,A3,A4,A5,C1,C3,AAA
COMMON /BK2/ H1,H2,H3,H4
COMMON /BK3/ F1,F2,F3,F4,F5,F6
COMMON /BK4/ C,S,C2
COMMON /BK5/ SIH,SIH2,COH,COH2,CON
COMMON /BK6/ Y,DZ
COMMON /BK7/ SCH,RG,RG2,S1,S14,SQGR,SQ2GR,QZ,Q,SQRGS1,SQR2S1,SQRG
COMMON /BK8/ F,D,CHD,DD,SPHERE,WHICH1,ÇDD
COMMON /BK9/ ECC,CONT,V,W,B,EPS
COMMON /BK11/ANS,RA,AMAX4
COMMON /BK12/U,UD,SS
C *****
YG(2) = 1.E-10
READ(5,400)I1,I2,I3,I4,IPOFL,IPL0T,NON,INDEX,N,AMAX4,NVL,15,IVELP
READ (5,600) (IVEL(I),I=1,NVL)
DO 10 M = 1,I1
READ (5,500) DEL(M),M1(M),M2(M),M3(M),M4(M),M5(M),M6(M)
C**** READ ALL DATA CONTROLLING THE GENERAL RUNNING OF THE PROGRAM ..
C**** INCLUDING INTEGRATION STEP CHANGES, PRINT OUT CONTROLS,PLOTTING
C**** CONTROLS AND INITIALIZATION OF COUNTERS.
10 WRITE (6,1000) DEL(M),M1(M),M2(M),M3(M),M4(M),M5(M),M6(M)
CALL SECOND (XTSEC)
WRITE (6,9000) XTSEC
DO 120 ITNS = 1,I2
C**** DO LOOP CONTROLS THE NUMBER OF COMPLETE SOLUTIONS REQUIRED.
C
14 = 0
C *****
CALL CASE (AR,THETA,INCOND)
C**** SUBROUTINE CASE CALLED TO DESCRIBE THE PHYSICAL PROBLEM.
IF (INCOND.EQ.1) GO TO 20
C *****
CALL INIT (NON,WHICH1)
C**** SUBROUTINE INIT IS CALLED TO CALCULATE THE INITIAL CONDITIONS FOR
C**** THE INTEGRATION EXCEPT WHEN ACCURATE VALUES HAVE BEEN READ IN.
C *****
20 IF (NON.EQ.1) GO TO 110
C**** IF NO INITIAL CONDITIONS HAVE BEEN FOUND, COMPUTATION IS
C**** TERMINATED FOR THIS PROBLEM.
WRITE (6,8000)
C**** WRITE HEADING FOR TABLE OF RESULTS.
DO 60 MM = 1,I1
C**** BEGINNING OF LOOP TO CONTROL THE VARIABLE STEP.
H = DEL(MM) * Q
C**** INTEGRATION STEP LENGTH.
N1 = M1(MM)
C**** CALCULATION AND PRINT-OUT OF RESULTS EVERY N1 STEPS.
N2 = M2(MM)
C**** NUMBER OF STEPS PER ONE DEGREE.

```

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      N3 = M3(MM)
C**** COUNTER FOR COMPARISON WITH N1.
      N4 = M4(MM)
C**** COUNTER FOR COMPARISON WITH N2.
      N5 = M5(MM)
C**** NUMBER OF STEPS TO BE TAKEN FOR THE PARTICULAR STEP-LENGTH.
      N6 = M6(MM)
C**** CALCULATION OF THE VELOCITY AND CONCENTRATION PROFILES EVERY N6 STEP
      N7 = 0
C**** COUNTER FOR COMPARISON WITH N6.
      DO 50 ITER = 1,N5
C**** LOOP TO STEP N5 STEPS AT STEP LENGTH H.
      N7 = N7 + 1
      N3 = N3 + 1
      N4 = N4 + 1
C**** UPDATE COUNTERS.
C      *****
      CALL RKUTTA (AR,N,H,Y,YG,YH,YD)
C**** STEP FROM AR TO AR + H
C      *****
      IF (Y(1).LT.0..OR.Y(2).LT.0..OR.Y(1).GT.1.E+6) ISTOP=1
      IF (ISTOP.EQ.1) GO TO 70
C**** CHECK WHETHER VALUES OF Y(1) OR Y(2) ARE IN ERROR. IF SO, STOP
C**** COMPUTATION.
      THETA = THETA + DEL(MM)
C**** UPDATE ANGLE
      IF (N3.LT.N1) GO TO 50
C**** COMPARE N3 WITH N1 TO SEE WHETHER CALCULATION AND PRINT-OUT OF
C**** RESULTS REQUIRED.
C      *****
      CALL CALC (WHICH1)
C      *****
      IF (N4.LT.N2..OR.THETA.LT.0.9999) GO TO 30
C**** CHECK WHETHER STEP HAS REACHED AN INTEGER NUMBER OF DEGREES.
      INDEX = THETA + 0.5
C**** SET INDEX TO THIS NUMBER OF DEGREES.
      IF (INDEX.NE.IVEL(I5)) GO TO 21
C**** CHECK TO SEE WHETHER PLOT OF PROFILES IS REQUIRED. IF SO, SET THE
C**** RELEVANT CONTROL PARAMETERS.
      IVELP = 1
      I5 = I5 + 1
      21 ANS(INDEX) = A4
C**** STORE VALUE OF THE MASS TRANSFER NUMBER TO BE PLOTTED AND ARRANGE
C**** FOR THE MAXIMUM VALUE OF THE ARRAY TO BE STORED IN LOCATION AMAX4.
      IF (ANS(INDEX).GT.AMAX4) AMAX4 = ANS(INDEX)
      RA(INDEX) = A4/ANS(1)
C**** STORE THE RATIO OF THE MASS TRANSFER NUMBER AT THIS ANGLE TO THAT
C**** AT THE FRONT STAGNATION POINT.
      R = RA(INDEX)
      N4 = 0
      30 WRITE (6,2000) Y(1),Y(2),AR,THETA,C1,C3,A1,A2,A3,A4,ALA,AAA,A5,R
C**** PRINT OUT RESULTS.
      IF (Y(1).GT.100..AND.RG.GT.1.) GO TO 69
      IF (Y(2).GT.1.6) GO TO 120
      IF (AAA.LT.0.) GO TO 71
C**** CHECK FOR ANY INCONSISTANCIES OR FOR SEPARATION.
C**** GO TO 40
      IF (N7.LT.N6) GO TO 40
C**** CHECK WHETHER THE VELOCITY AND CONCENTRATION PROFILES ARE REQUIRED

```

```

C**** AT THIS ANGLE.
      I4 = I4 + 1
C *****
      CALL PROFL (U,UD,I4,PR,CPR,IVELP,I5,IVEL,WHICH1)
C**** CALL SUBROUTINE TO CALCULATE THE PROFILES.
C *****
      ATHETA(I4) = THETA
      N7 = 0
      40 N3 = 0
C**** RESET COUNTERS TO ZERO
      IF (THETA.GT.180..OR.Y(2).GT.3.) GO TO 70
      50 CONTINUE
      60 CONTINUE
      69 WRITE (6,9001)
      GO TO 72
      70 IF (ISTOP.EQ.1) WRITE (6,3000) THETA
      GO TO 72
      71 WRITE(6,7000)
C**** ERROR MESSAGE PRINT OUTS.
      72 IF (IPLT.EQ.0..OR.THETA.LT.60.) GO TO 80
C**** CHECK WHETHER PLOT OF LOCAL TRANSFER RATES REQUIRED.
      NA = 90
      IF (THETA.LT.90.) NA = 80
C *****
C**** GO TO 120
      CALL PLOTNG (NA)
C**** CALL SUBROUTINE TO PLOT GRAPH.
C *****
      80 IF (IPROFL.EQ.0..OR.THETA.LT.60.) GO TO 110
C**** CHECK WHETHER TABLES OF VELOCITY AND CONCENTRATION PROFILES ARE
C**** REQUIRED. IF SO, PRINT OUT AND THEN CALL PLOTTING ROUTINE.
      WRITE (6,4000)
      DO 90 I = 1,I4
      90 WRITE (6,5000) ATHETA(I),(PR(I,J),J=1,11)
      WRITE (6,6000)
      DO 100 I = 1,I4
      100 WRITE (6,5000) ATHETA(I),(CPR(I,J),J=1,11)
C *****
C**** CALL VEPLT (NVL,RG,IVEL)
C *****
      110 AMAX4 = 0.
      NON = 0
      ISTOP = 0
      I4 = 0
C**** RESET CONTROL VARIABLES.
      CALL SECOND (XTSEC)
      WRITE (6,9000) XTSEC
      120 CONTINUE
C**** END OF LOOP FOR COMPLETE CALCULATION.

```

```
STOP
400 FORMAT (9I2,F5.2,3I2) .
500 FORMAT (F10.5,6I5)
600 FORMAT (15I3)
1000 FORMAT (1X,F10.5,6(2X,I5))
2000 FORMAT (1X,F10.5,2X,2(F8.5,2X),F6.2,2(2X,F7.5),4(2X,F9.4),2X,F5.1,
12X,F6.1,2X,F8.4,1X,F5.3)
3000 FORMAT (1X,41H CALCULATION STOPPED B/L NEGATIVE AT THETA,F6.2)
4000 FORMAT (1H1,1X,34H VELOCITY PROFILES AT VARIOUS THETA,/1X,6H THETA,
154X,3HB/L)
5000 FORMAT (1X,F6.2,11(2X,F8.4))
6000 FORMAT (1H1,1X,39H CONCENTRATION PROFILES AT VARIOUS THETA,/1X,6H T
1HETA,54X,3HB/L)
7000 FORMAT (///1X,23H SEPARATION,AAA NEGATIVE)
8000 FORMAT (5X,1HZ,9X,4HAKAP,6X,2HAR,8X,5H THETA,2X,6HCONST1,3X,6HCONST
12,5X,4HMTN1,7X,4HMTN2,7X,4HMTN3,7X,4HMTN4,6X,3HALA,5X,3HAAA,4X,6HA
2LTFRE,2X,1HR,/)
9000 FORMAT (1X,6H TIME =,F9.3)
9001 FORMAT (1X,32HY(1) GT 100. CALCULATION STOPPED)
END
```

```

SUBROUTINE CASE (AR,THETA,INCOND)
DIMENSION Y(5)
INTEGER SPHERE,WHICH1
COMMON /BK4/ C,S,C2
COMMON /BK5/ SIH,SIH2,COH,COH2,CON
COMMON /BK6/ Y,DZ
COMMON /BK7/ SCH,RG,RG2,S1,S14,SQGR,SQ2GR,QZ,Q,SQRGS1,SQR2S1,SQGR
COMMON /BK8/ F,D,CHD,DD,SPHERE,WHICH1,CDD
COMMON /BK9/ ECC,CONT,V,W,B,EPS
READ (5,600) Y(1),Y(2),RG,SCH,THETA,SPHERE,QZ,ICH,WHICH1,INCOND
*****
C   Y(1)   = INITIAL VALUE OF Z.
C   Y(2)   = INITIAL VALUE OF AKAP.
C   RG      = VALUE OF REYNOLDS NO. SQUARED/GRASHOF NO.FOR METHOD 1
C            = VALUE OF GRASHOF NO./((REYNOLDS NO. SQUARED * SCHMIDT NO.
C            ** 1/3) FOR METHODS 2 AND3
C   SCH      = VALUE OF SCHMIDT NO.
C
C   THETA    = ANGLE (IN DEGREES) FROM FRONT STAGNATION POINT AT WHICH
C              THE CALCULATION IS INITIATED.
C   SPHERE   = SET TO 0 FOR SPHERE
C              SET TO 1 FOR SPHEROID
C   QZ       = SET TO +1.0 FOR AIDING FLOWS
C              SET TO -1.0 FOR OPPOSING FLOWS \
C
C   ICH      = SET TO 1 IF BODY SHAPE IS CHANGED FROM PREVIOUS CASE OR
C              IF WE ARE CONSIDERING THE FIRST CASE OF A PARTICULAR RUN.
C              SET TO 0 IF BODY SHAPE IS UNCHANGED FROM PREVIOUS CASE
C              CONSIDERED.
C              N.B. IF ICH = 1 THIS CARD MUST BE FOLLOWED BY A SHAPE
C                   DEFINING CARD
C *****
C   WHICH1   = NUMBER OF METHOD TO BE USED FOR SOLUTION
C   INCOND   = SET TO 0 IF WE HAVE ONLY SUPPLIED APPROXIMATE VALUES FOR
C              Y(1) AND Y(2) AND WE WISH TO CALL SUBROUTINE INIT.
C              SET TO 1 IF CORRECT INITIAL Y(2) AND Y(1) HAVE BEEN
C              SUPPLIED.
C *****
C *****
C   CALL ECCEN (ICH)
C**** CALL SUBROUTINE ECCEN TO SET THE VARIABLES WHICH DESCRIBE THE
C**** BODY SHAPE.
C *****
C**** SET VARIABLES WHICH ARE REQUIRED IN LATER COMPUTATION.
      S1 = SCH ** 0.333333
      S14 = SCH ** 0.25
      1 GR = 1./RG
      RG2 = RG * RG
      SQGR = SQRT(GR)
      SQ2GR = SQRT(SQGR)
      3 Q = 3.14159/180.
      AR = THETA * Q
C**** CHANGE ANGLE FROM DEGREES TO RADIANS.
      C = COS(AR)
      S = SIN(AR)
      600 FORMAT (5F10.5,I1,F5.2,3I1)
      RETURN
      END

```



```

SUBROUTINE ECCEN (ICH)
  INTEGER SPHERE,WHICH1.
  COMMON /BK5/ SIH,SIH2,COH,COH2,CON
  COMMON /BK7/ SCH,RG,RG2,S1,S14,SQGR,SQ2GR,QZ,Q,SQGRS1,SQR2S1,SQGR
  COMMON /BK8/ F,D,CHD,DD,SPHERE,WHICH1,CDD
  COMMON /BK9/ ECC,CONT,V,W,B,EPS
  IF (SPHERE.EQ.0) GO TO 1
C**** CHECK WHETHER BODY IS A SPHERE OR AN OBLATE SPHEROID.
  IF (ICH.EQ.0) GO TO 3
  READ (5,100) F,DD,D,CHD
C**** IF SHAPE HAS CHANGED,READ DATA TO DESCRIBE THE NEW SHAPE.
  100 FORMAT (4F10.6)
C**** CALCULATE PARAMETERS FOR THE OBLATE SPHEROID.
  ECC = F / D
  CDD = CHD / DD
  A = ((D*D)-(F*F))**.5
  COH = D / A
  SIH = F / A
  COH2 = COH * COH
  SIH2 = SIH * SIH
  CON = (COH2*ATAN(1./SIH))-SIH
  V = SIH2 + 1.
  CONT = CON * SQRT(V)
  W = COH2/CONT
  WRITE (6,1002) WHICH1
  1002 FORMAT (1X,8HWHICH1 =,I5)
  WHICH1 = 1
  3 WRITE (6,1000) SCH,QZ,F,RG,WHICH1
C**** WRITE HEADING FOR RUN.
  5 EPS = CHD/DD
  B = SIH2+C2
  RETURN
C**** DEFINE PARAMETERS FOR SPHERE.
  1 EPS = 2.
  CONT = 2./3.
  W = 3.
  V = 2.
  ECC = 1.
  6 WRITE (6,2000) SCH,QZ,RG,WHICH1
C**** WRITE HEADING FOR RUN..
  RETURN
  1000 FORMAT (1H1,59X,8HSPHEROID,/60X,8(1H*),//1X,20HTHE SCHMIDT NUMBER
  1=F12.3,///1X,19HTHEORETICAL PROFILE,5X,F5.2,///1X,10HSPHEROID ,
  2F4.1,4H/16,///1X,47HTHE RATIO REYNOLDS NO. SQUARED / GRASHOF NO. 1
  3S,F11.5,/1X,58(1H-),//1X,24HMETHOD USED IS NUMBER ...,I4,/)
  2000 FORMAT (1H1,59X,6HSPHERE,/60X,6(1H*),//1X,17H SCHMIDT NUMBER =,F12
  1.3,///1X,19HTHEORETICAL PROFILE,5X,F5.2,///1X,47HTHE RATIO REYNOLD
  2S NO. SQUARED / GRASHOF NO. IS,F11.5,/1X,58(1H-),//1X,24HMETHOD US
  3ED IS NUMBER ...,I4,/)
  END

```

```

SUBROUTINE INIT (NON,WHICH1)
DIMENSION Y(5)
INTEGER SPHERE,WHICH1
COMMON /BK2/ H1,H2,H3,H4
COMMON /BK3/ F1,F2,F3,F4,F5,F6
COMMON /BK6/ Y,DZ
COMMON /BK7/ SCH,RG,RG2,S1,S14,SQGR,SQ2GR,QZ,Q,SQRGS1,SQR2S1,SQRG
COMMON /BK9/ ECC,CONT,V,W,B,EPS
COMMON /BK10/ F1D,F2D,F3D,F4D,F5D,F6D
DIFF = 1.E-4
Z = Y(1)
AKAP = Y(2)
C**** Y(1) AND Y(2) ARE THE APPROXIMATIONS TO THE VALUES AT THE FRONT
C**** STAGNATION POINT.
S1 = SCH**0.333333$S2=S1*S1$S3=SCH
GRS=RG*QZ $W2=W*W$V2=V*V
Y12=Y(1)*Y(2)$Y11=Y(1)*Y(1)$Y22=Y(2)*Y(2)
10 L = 0
C**** SET COUNTER TO ZERO
1 L = L + 1
C**** UPDATE COUNTER.
ERG = EPS * RG
ALA = Z*ERG*W/V
C *****
C**** CALL ROUTINES TO CALCULATE RELEVANT FUNCTIONS.
CALL LAMBDA (ALA)
C *****
C CALL KAPPA (AKAP)
C *****
YZ = ALA/Z
C *****
CALL LAMDBD (ALA,YZ)
C *****
Z2 = Z * Z
Z3 = Z2 * Z
AK2 = AKAP * AKAP
FA=F1+F3+Z*CONT*QZ*(F4-3.*AKAP/5.)+ERG*Z2*(F2*QZ*(1.+ECC)+Z*CONT*
1(1.+2.*ECC)/4536.)
FZ = F1D+F3D+CONT*QZ*(F4-3.*AKAP/5.+F4D*Z)+ERG*Z*(QZ*(2.*F2+Z*F2D)
1*(1.+ECC)+Z*CONT*(1.+2.*ECC)/1512.)
FK = -3.*Z*CONT *QZ/5.
W1 = 1./CONT + W/V
W2 = QZ*(1.+ECC)+EPS*RG*W*W1/V
G = (2./(SCH*AKAP))-(ERG*Z*(H1*Z*W2+H3*W1))
GZ = -ERG*(2.*H1*Z*W2+H3*W1)
GK = (-2./(SCH*AK2))-(ERG*Z*(H2*Z*W2+H4*W1))
13 DJ = FZ*GK-FK*GZ
ZNEW = Z -(FA*GK-G*FK)/DJ
AKNEW = AKAP + (FA*GZ-G*FZ)/DJ
C**** CALCULATE NEW APPROXIMATION TO VALUES OF Y(1) AND Y(2) BY NEWTON
C**** RAPHSON METHOD.
R1 = ABS(ZNEW-Z)
R2 = ABS(AKNEW-AKAP)
IF (R1.LE.DIFF.AND.R2.LE.DIFF) GO TO 2
C**** CHECK WHETHER TWO CONSECUTIVE VALUES OF BOTH Y(1) AND Y(2) HAVE
C**** CONVERGED TO THE REQUIRED ACCURACY. IF THEY HAVE WRITE OUT THE
C**** VALUES AND RETURN. IF NOT, CHECK WHETHER THE ITERATION LIMIT HAS BE
C**** REACHED. IF IT HAS, PRINT MESSAGE, OTHERWISE PERFORM FURTHER
C**** ITERATION.

```

```
Z = ZNEW
AKAP = AKNEW
Y(1) = ZNEW
Y(2) = AKNEW
Y12 = Y(1) * Y(2) $Y11 = Y(1) * Y(1) $ Y22 = Y(2) * Y(2)
IF (L.GE.50) GO TO 3
GO TO 1
2 WRITE (6,1000) ZNEW,AKNEW,L
1000 FORMAT (1X,32HINITIAL CONDITIONS FOUND ... Z:=,F10.5,3X,6HAKAP =,
1F10.5,5X,I3)
GO TO 4
3 WRITE (6,2000)
NON = 1
2000 FORMAT (1X,38H 50 ITERATIONS REACHED NO CONVERGENCE )
4 RETURN
END
```

```

SUBROUTINE YGRD (X,N,Y,YG)
DIMENSION Y(N),YG(N)
INTEGER SPHERE,WHICH1
COMMON /BK2/ H1,H2,H3,H4
COMMON /BK3/ F1,F2,F3,F4,F5,F6
COMMON /BK4/ C,S,C2
COMMON /BK5/ SIH,SIH2,COH,COH2,CON
COMMON /BK7/ SCH,RG,RG2,S1,S14,SQGR,SQ2GR,QZ,Q,SQGRS1,SQR2S1,SQGR
COMMON /BK8/ F,D,CHD,DD,SPHERE,WHICH1,CDD
COMMON /BK9/ ECC,CONT,V,W,B,EPS
COMMON /BK12/ U,UD,SS
COMMON /BK14/ AST,BETA,CST,DST
C**** SUBROUTINE YGRD SIMPLY CALCULATES THE DERIVATIVES OF Y(1) AND Y(2)
C**** AT THE PARTICULAR ANGLE. THUS WHEN FIRST CALLED FROM RKUTTA THE
C**** DERIVATIVES ARE CALCULATED AT ANGLE,AR. THE NEXT TWO CALLS AT
C**** AR + H/2 AND FINALLY AT AR + H
C = COS(X)
S = SIN(X)
Z = Y(1)
AKAP = Y(2)
Z2 = Z * Z
Z3 = Z2 * Z
IF (SPHERE.EQ.0) GO TO 1
EPS = CDD*(1./SQRT(1.-S*S/COH2))
C**** IF THE BODY IS A SPHEROID, CALCULATE THE ANGLE PSI AT THE
C**** PARTICULAR THETA.
PHI = ABS((3.14159/2.)-X)
T = SIN(PHI)/COS(PHI)
P1 = ATAN(ECC/T)
CC = COS(P1)
SS = SIN(P1)
C2 = C * C
ETA = ECC*CC*CC/C2
C *****
C**** CALL SUBROUTINE FOR THE SPECIFIC SHAPE TO CALCULATE THE POTENTIAL
CALL THEORY (U,UD,UD)
C *****
GO TO 2
1 CC = C
SS = S
C *****
CALL THEORI (U,UD,UD)
C *****
ETA = 1.
2 EPS2 = EPS * EPS
ERG = EPS*RG
ERG2 = ERG * ERG
ETA = ETA * ERG
ALA = ERG*UD*Z.
C *****
C**** CALL ROUTINES TO CALCULATE REQUIRED FUNCTIONS.
CALL LAMBDA (ALA)
C *****
CALL KAPPA (AKAP)
C *****
ANUMLS = F2*Z2*(QZ*CC*ETA+U*UD*ERG2+(ERG*QZ*C*SS)/S)+F1+Z3*SS*
1((CC*ETA+(ERG*C*SS)/(2.*S))/U+ERG2*QZ*UD)/2268.+Z*SS*(QZ*F4-(QZ*3
2.*AKAP)/5.)/U+(C*U*F3)/(S*UD)
DENMLS = (U*F5+SS*(Z*QZ*F6-(5.*Z2*SS)/(9072.*U)))*ERG

```

```

C**** CALCULATE THE REQUIRED DERIVATIVES AND RETURN.
YG(1) = ANUMLS/DENMLS
BA = (C*ERG/S)*(Z*(H1*(Z*QZ*SS+U*ALA)+U*H3))
BB = ERG*YG(1)*0.5*(H1*3.*(Z*SS*QZ+U*ALA)+U*H3)
BC = Z2*H1*(CC*QZ*ETA+U*ERG2*UDD)+ALA*(H3+ALA*H1)
BD = 2./(SCH*AKAP)
DEN = ERG*Z*(H2*(Z*SS*QZ+U*ALA)+U*H4)
YG(2) = (BD-(BA+BB+BC))/DEN
AST = (SS*Y(1)+U*ALA+12.*U)/6.
RETURN
END

```

```

SUBROUTINE RKUTTA (X,N,H,Y,YG,YH,YD)
DIMENSION Y(N),YG(N),YH(N),YD(N)
C**** RUNGE-KUTTA SINGLE STEP INTEGRATION ROUTINE. THIS ROUTINE IS
C**** IDENTICAL TO THAT IN THE EXACT PROGRAM.
DO 1 J = 1,N
1 YD(J) = Y(J)
C *****
C CALL YGRD (X,N,Y,YG)
C *****
DO 2 J = 1,N
YG(J) = YG(J)*H
2 YH(J) = YG(J)
X = X + 0.5*H
DO 4 I = 1,2
DO 3 J = 1,N
3 Y(J) = YD(J) + 0.5*YG(J)
C *****
C CALL YGRD (X,N,Y,YG)
C *****
DO 4 J = 1,N
YG(J) = YG(J) * H
4 YH(J) = YH(J) + 2.*YG(J)
X = X + 0.5*H
DO 5 J = 1,N
5 Y(J) = YD(J) + YG(J)
C *****
C CALL YGRD (X,N,Y,YG)
C *****
DO 6 J = 1,N
YH(J) = (YH(J)+ H*YG(J))/6.
YD(J) = YD(J)+YH(J)
6 Y(J) = YD(J)
RETURN
END

```

```

SUBROUTINE THEORY (TRU,TRUD,TRUDD)
COMMON /BK4/ C,S,C2
COMMON /BK5/ SIH,SIH2,COH,COH2,CON
C**** SUBROUTINE THEORY CALCULATES THE POTENTIAL FLOW AND ITS FIRST AND
C**** SECOND DERIVATIVES, WITH RESPECT TO THETA, AT AN ANGLE THETA FROM
C**** THE FRONT STAGNATION POINT OF AN OBLATE SPHEROID.
C**** TRU IS THE VELOCITY
C**** TRUD ITS FIRST DERIVATIVE
C**** TRUDD ITS SECOND DERIVATIVE
      B = SIH2 + C2
      TRU = S/(CON*B**0.5)
      TRUD = COH2*C/(CON*B**1.5)
      TRUDD = (COH2*S*((2.*C2) - SIH2 ))/(CON*B**2.5)
      RETURN
      END

```

```

SUBROUTINE THEORI (TRU,TRUD,TRUDD)
COMMON /BK4/ C,S,C2
C**** SUBROUTINE THEORI CALCULATES THE POTENTIAL FLOW VELOCITY, AND ITS
C**** FIRST AND SECOND DERIVATIVES, WITH RESPECT TO THETA, AT AN ANGLE
C**** THETA FROM THE FRONT STAGNATION POINT OF A SPHERE.
C**** TRU IS THE VELOCITY
C**** TRUD ITS FIRST DERIVATIVE
C**** TRUDD ITS SECOND DERIVATIVE
      TRU = 1.5*S
      TRUD = 1.5*C
      TRUDD = -TRU
      RETURN
      END

```

```

SUBROUTINE CALC (WHICH1)
DIMENSION Y(5)
INTEGER SPHERE,WHICH1
COMMON /BK1/ A1,A2,A3,A4,A5,C1,C3,AAA
COMMON /BK6/ Y,DZ
COMMON /BK7/ SCH,RG,RG2,S1,S14,SQGR,SQ2GR,QZ,Q,SQRGS1,SQR2S1,SQGR
COMMON /BK12/ U,UD,SS
COMMON /BK14/ AST,BETA,CST,DST
C**** SUBROUTINE CALC CALCULATES THE MASS TRANSFER NUMBERS WHICH ARE TO
C**** BE PRINTED OUT AND ALSO THE VELOCITY GRADIENT AT THE SURFACE SO TH
C**** THE POINT OF SEPARATION MAY BE DETERMINED.
      DZ= SQRT(Y(1))
      C1 = Y(2)*S1
      C3 = Y(2) * S14
1  A1 = 2./((Y(2)*DZ)
      A2 = A1*SQGR
      A3 = 2./((C1*DZ)
      A4 = A3*SQGR
      A5 = A1 * SQ2GR
      ALA = EPS * RG * UD * Y(1)
      AAA = AST/DZ
      RETURN
      END

```

```

SUBROUTINE KAPPA (AKAP)
COMMON /BK2/ H1,H2,H3,H4
C**** SUBROUTINE KAPPA CALCULATES A NUMBER OF FUNCTIONS OF Y(2) WHICH
C**** ARE REQUIRED IN SUBROUTINE INIT AND SUBROUTINE YGRD.
AK2 = AKAP * AKAP
AK3 = AKAP * AK2
AK4 = AKAP * AK3
AK5 = AKAP * AK4
H1 = ((AK2/90.) - (AK3/84.) + ((3.*AK4)/560.) - (AK5/1080.))
H2 = ((AKAP/45.) - (AK2/28.) + ((3.*AK3)/140.) - (AK4/216.))
H3 = (((2.*AK2)/15.) - ((3.*AK4)/140.) + (AK5/180.))
H4 = (((4.*AKAP)/15.) - ((3.*AK3)/35.) + (AK4/36.))
RETURN
END

```

```

SUBROUTINE LAMBDA (ALA)
COMMON /BK3/ F1,F2,F3,F4,F5,F6
C**** SUBROUTINE LAMBDA CALCULATES A NUMBER OF FUNCTIONS OF ALA WHICH
C**** ARE REQUIRED IN SUBROUTINE INIT AND SUBROUTINE YGRD.
F1 = 4.+ALA*(-232./315.+ALA*(79./3780.+ALA/2268.))
F2 = 2./945.+ALA/2268.
F3 = ALA*(-74./315.+ALA*(2./945.+ALA/4536.))
F4 = 1./3.+ALA*(71./3780.+ALA/2268.)
F5 = 37./315.+ALA*(-1./315.-5.*ALA/9072.)
F6 = 1./315.-5.*ALA/4536.
RETURN
END

```

```

SUBROUTINE LAMBD8 (ALA,Y)
COMMON /BK10/ F1D,F2D,F3D,F4D,F5D,F6D
C**** SUBROUTINE LAMBD8 CALCULATES THE DERIVATIVES OF THE FUNCTIONS IN
C**** SUBROUTINE LAMBDA WITH RESPECT TO Y(1) FOR USE IN SUBROUTINE INIT
F1D = Y*(-232./315.+ALA*(79./1890.+ALA/756.))
F2D = Y/2268.
F3D = Y*(-74./315.+ALA*(4./945.+ALA/1512.))
F4D = Y*(71./3780.+4.*ALA/4536.)
F5D = Y*(-1./315.-10.*ALA/9072.)
F6D = -5.*Y/4536.
RETURN
END

```

```

SUBROUTINE PROFL (U,UD,INTG,PR,CPR,IVELP,15,IVEL,WHICH1)
  DIMENSION PR(50,12),CPR(50,12)
  DIMENSION Y(5)
  DIMENSION AZZ(15),BZZ(15),CZZ(15),EZZ(15),UZ(15),DXZ(15),DX3Z(15),
1DX4Z(15),IVEL(15),DZZ(15),CHAR(101)
  INTEGER SPHERE,WHICH1
  COMMON /BK4/ C,S,C2
  COMMON /BK6/ Y,DZ
  COMMON /BK7/ SCH,RG,RG2,S1,S14,SQGR,SQ2GR,QZ,Q,SQRGS1,SQR2S1,SQRG
  COMMON /BK9/ ECC,CONT,V,W,B,EPS
  COMMON/BK13/AZZ,BZZ,CZZ,EZZ,UZ,DXZ,DX3Z,DX4Z,DZZ
  COMMON /BK14/ AST,BETA,CST,DST
C**** SUBROUTINE PROFL CALCULATES AND STORES THE VELOCITY AND CONCENTRATI
C**** PROFILES FROM THE SURFACE TO A DISTANCE OF TWICE THE CALCULATED
C**** BOUNDARY LAYER THICKNESS
  IF (Y(1).LT.0.) RETURN
1  X1 = S*QZ + U*UD*EPS*RG
  X2 = X1*Y(1)
  AZ = (X2+12.*U)/(6.*DZ)
  BZ = -X1/2.
  CZ = (X2-4.*U)/(2.*Y(1)*DZ)
  EZ = (6.*U-X2)/(6.*Y(1)*Y(1))
C**** CALCULATE THE COEFFICIENTS IN ASSUMED VELOCITY PROFILE.
C**** CALCULATE THE TRANSFER BOUNDARY LAYER THICKNESS.
3  DX = Y(2)*DZ
  DX2 = DX * DX
  DX3 = DX * DX2
  DX4 = DX * DX3
  YY = 0.
  DY = DZ/5.
  DO 5 J = 1,11
C**** LOOP TO CALCULATE THE DIMENSIONLESS VELOCITY AND CONCENTRATION
C**** AT FIVE POINTS WITHIN THE HYDRODYNAMIC BOUNDARY LAYER, AND FIVE
C**** POINTS OUTSIDE.
4  YY2 = YY * YY
  YY3 = YY * YY2
  YY4 = YY * YY3
  PR(INTG,J) = (AZ*YY + BZ*YY2 + CZ*YY3 + EZ*YY4)/U
  CPR(INTG,J) = 1.-2.*YY/DX+2.*YY3/DX3-YY4/DX4
5  YY = YY + DY
  IF (IVELP.NE.1) RETURN
C**** CHECK WHETHER PLOTS OF THE PROFILES ARE REQUIRED AT THIS ANGLE.
C**** IF SO, STORE THE COEFFICIENTS. IF NOT .. RETURN.
  AZZ(15-1) = AZ
  BZZ(15-1) = BZ
  CZZ(15-1) = CZ
  DZZ(15-1) = DZ
  EZZ(15-1) = EZ
  UZ(15-1) = U
  DXZ(15-1) = DX
  DX3Z (15-1) = DX3
  DX4Z(15-1) = DX4
  IVELP = 0
  RETURN
END

```



```

SUBROUTINE PLOTNG (NA)
  DIMENSION CHAR(101), ANG(100), ANS(200), ANW(50), RA    (200)
  INTEGER SPHERE
  COMMON /BK7/ SCH, RG, RG2, S1, S14, SQGR, SQ2GR, QZ, Q, SQGRS1, SQR2S1, SQRG
  COMMON /BK8/ F, D, CHD, DD, SPHERE, WHICH1, CDD
  COMMON /BK11/ ANS, RA, AMAX4
  C**** SUBROUTINE PLOTNG PLOTS THE LOCAL MASS TRANSFER NUMBER AND THE
  C**** RATIO OF THE TRANSFER AT ANY ANGLE THETA TO THAT AT THE FRONT
  C**** STAGNATION POINT. THE PLOT BEING FROM THE FRONT STAGNATION POINT
  C**** TO THE EQUATOR.
  DATA BLANK, PLOT, SPLOT, RPLLOT/1H ,1H*,1H.,1H+/
  C**** DEFINE THE PLOTTING CHARACTERS.
  ANG(1) = 0.0
  FRAC = 10.
  IF (AMAX4.LT.5.) FRAC = 5.
  IF (AMAX4.LT.2.) FRAC = 2.
  ANITER = FRAC/20.
  WI = FRAC / 100.
  C**** SET THE SCALE AND LABELLING OF THE Y-AXIS DEPENDING ON THE
  C**** VALUE OF AMAX4.
  ANW(1) = 0.0
  DO 10 JI = 2,21
    10 ANW(JI) = ANW(JI-1) + ANITER
    IF (SPHERE.EQ.0) GO TO 20
    WRITE (6,1000) SCH, RG, F, D
    GO TO 30
  20 WRITE (6,2000) SCH, RG
  C**** PRINT HEADINGS AND LABELS FOR Y-AXIS.
  30 WRITE (6,3000) (ANW(JJ) ,JJ=1,21)
  DO 40 K = 1,101
    40 CHAR(K) = SPLOT
  C**** SET ARRAY TO PLOT Y-AXIS.
  I = ANS(1)/WI + 1.1
  J = RA(1)/WI + 1.1
  C**** CALCULATE THE POSITION OF POINT FOR ANS AND RA AT THIS POINT
  C**** (I.E. THE FRONT STAGNATION POINT).
  IF (I.GT.101) I = 0
  CHAR(I) = PLOT
  CHAR(J) = RPLLOT
  C**** SET THESE VALUES TO BE PLOTTED.
  WRITE (6,4000) ANG(1) ,CHAR
  C**** PLOT Y-AXIS.
  DO 60 IT = 10,NA,10
  C**** LOOP TO PLOT AT SPECIFIED INTERVALS ALONG X-AXIS.
  WRITE (6,5000)
  C**** WRITE STATEMENT TO PLOT X-AXIS IF NO POINT IS TO BE PLOTTED.
  I = ANS(IT)/WI + 1.1
  J = RA(IT)/WI + 1.1
  C**** CALCULATE POSITION OF POINT.
  IF (I.GT.101) I = 0
  DO 50 K = 2,101
    50 CHAR(K) = BLANK
  C**** SET THE WHOLE ARRAY TO BLANKS.
  ANG(IT) = IT
  CHAR(J) = RPLLOT
  CHAR(I) = PLOT
  C**** SET PLOTTING POINTS.
  60 WRITE (6,4000) ANG(IT),CHAR
  C**** PLOT LABEL FOR X-AXIS AND PLOT THE POINTS.

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```

      WRITE (6,6000)
1000 FORMAT (1H1,9X,14HPLOT -- SCH =,F9.2,5X,4HRG =,F12.6,5X,F4.1,1H/,
      1F4.1,3X,8HSPHEROID,//33X,22HMASS TRANSFER NO. MTN4)
2000 FORMAT (1H1,9X,14HPLOT -- SCH =,F9.2,5X,4HRG =,F12.6,3X,6HSPHERE,
      1//33X,22HMASS TRANSFER NO. MTN4)
3000 FORMAT (1X,10X,21(F3.1,2X))
4000 FORMAT (1X,5X,F4.1,2X,101A1)
5000 FORMAT (1X,11X,1H.,3(/,1X,11X,1H.))
6000 FORMAT (///1X,10X,102HTHE SYMBOL + REPRESENTS A PLOT OF THE RAT
      110 OF MTN4 AT A PARTICULAR ANGLE TO THE VALUE AT THE F.S.P.)
      RETURN
      END

```

```

SUBROUTINE VEPLIT (NVL, RG, IVEL)
  DIMENSION AZZ(15), BZZ(15), CZZ(15), EZZ(15), UZ(15), DXZ(15), DX3Z(15),
  1DX4Z(15), IVEL(15), DZZ(15), CHAR(101), YAXIS(50)
  COMMON/BK13/AZZ, BZZ, CZZ, EZZ, UZ, DXZ, DX3Z, DX4Z, DZZ
C****SUBROUTINE VEPLIT PLOTS THE VELOCITY AND CONCENTRATION PROFILES
C**** AT ANY ANGLE SPECIFIED BY THE USER.
  DATA BLANK, PLOT, SPLOT, RPLLOT/1H ,1H*,1H.,1H+/
C**** WRITE THE ANGLES AT WHICH PLOTS ARE TO BE PRESENTED.
  WRITE (6,1001) (IVEL(I), I=1, NVL)
  1001 FORMAT (1X,15I3)
  IF (WHICH1.EQ.1.AND.RG.LT.1..OR.WHICH1.NE.1.AND.RG.GT.0.1) RETURN
C**** CHECK THAT THE SUBROUTINE CAN COPE WITH THE PLOTTING REQUIREMENTS.
  NVLI = NVL - 1
C**** SET NVLI TO THE NUMBER OF PLOTS REQUIRED.
  DO 60 NI = 1, NVLI
  YY = 0.
C**** SET YY TO ZERO -- SURFACE.
  DY = DZZ(NI)/25.
C**** SET DY TO 1/25TH OF THE BOUNDARY LAYER THICKNESS.
  XAXIS = 0.
  FRAC = 1.
  WI = FRAC/100.
  ANITER = FRAC/20.
C**** SET THE SCALE , PLOTTING INCREMENTS AND LABELS.
  YAXIS(1) = 0.
  DO 10 JI = 2, 21
  10 YAXIS(JI) = YAXIS(JI-1) + ANITER
  WRITE (6,1000) IVEL(NI), (YAXIS(JJ), JJ=1, 21)
C**** WRITE THE HEADING AND THE LABELS FOR Y-AXIS.
  VDOT = 0.
  CDOT = 1.
C**** SET THE VALUES OF THE VELOCITY AND CONCENTRATION AT THE SURFACE.
  DO 20 K = 1, 101
  20 CHAR(K) = SPLOT
C**** SET ARRAY TO PLOT THE Y-AXIS.
  IV = VDOT/WI + 1.1
  IC = CDOT/WI + 1.1
  CHAR(IV) = PLOT
  CHAR(IC) = RPLLOT
C**** SET THE SURFACE VALUES TO BE PLOTTED.
  WRITE (6,2000) XAXIS, CHAR
C**** PLOT Y-AXIS TOGETHER WITH X-AXIS LABEL.
  XAXIS = XAXIS + 0.2
C**** UPDATE THE LABEL FOR X-AXIS.
  KOUNT = 1
C**** SET COUNTER TO UNITY.
  DO 30 JJ = 1, 25
C**** LOOP TO PLOT THE VALUES AT 25 POINTS.
  WRITE (6,3000)
  DO 40 K = 2, 101
  40 CHAR(K) = BLANK
C**** SET THE ARRAY TO BLANKS.
  YY = YY + DY
  YY2 = YY * YY
  YY3 = YY2 * YY
  YY4 = YY3 * YY
  VDOT = (AZZ(NI)*YY+BZZ(NI)*YY2+CZZ(NI)*YY3 +EZZ(NI)*YY4)/UZ(NI)
  CDOT = 1.-2.*YY/DXZ(NI)+2.*YY3/DX3Z(NI)-YY4/DX4Z(NI)
C**** CALCULATE THE VELOCITY AND CCNCENTRATION VALUES AT THE POINT.

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      IV = VDOT/WI + 1.1
      IC = CDOT/WI + 1.1
      IF (IV.GT.101) IV=101
      IF (IC.LE.0) IC = 1
      CHAR(IV) = PLOT
      CHAR(IC) = RPLLOT
C**** SET PLOTTING POINTS.
      IF (KOUNT.LT.5) GO TO 50
C**** CHECK WHETHER LABEL FOR X-AXIS IS REQUIRED.
C**** IF IT IS, PLOT LABEL AND POINTS, THEN UPDATE LABEL. OTHERWISE
C**** JUST PLOT THE POINTS.
      WRITE (6,2000) XAXIS,CHAR
      XAXIS = XAXIS + 0.2
      KOUNT = 0
      GO TO 30
    50 WRITE (6,4000) CHAR
    30 KOUNT = KOUNT + 1
    60 CONTINUE
C**** NEXT ANGLE OR RETURN.
      RETURN
1000 FORMAT (1H1,1X,44HVELOCITY AND CONCENTRATION PROFILES AT ANGLE,
116  ,3X,27HFROM FRONT STAGNATION POINT//1X,56H* DIMENSIONLESS VELO
2CITY, + DIMENSIONLESS CONCENTRATION.//1X,84HPLOT FROM BODY SURFACE
3 TO EDGE OF HYDRODYNAMIC B/L,DIMENSIONLESS DISTANCE ON X-AXIS.///
41X, 9X,21(F5.2))
2000 FORMAT (1X,5X,F4.1,2X,101A1)
3000 FORMAT (1X,11X,1H.)
4000 FORMAT (1X,11X,101A1)
      END

```

D.3.1. List of Symbols used in the Computer Programme:-

<u>FØRTRAN</u>	<u>Meaning</u>
<u>Symbol</u>	
A	$\gamma_1^k - \gamma_1^{k+1}$
ABC	$ \gamma_1^{k+1} $
B	$\gamma_2^k - \gamma_2^{k+1}$
C1	γ_1^k
C2	γ_2^k
CØNØ	The last positive value of γ_1^k
CØNST(2)	$\gamma_1^{k+1}, \gamma_2^{k+1}$
DELMAX	Stepwise increase of the effective infinity each time required.
ERMINA	Prescribed accuracy for A.
ERMINB	Prescribed accuracy for B.
GR	$\bar{G}r/\bar{R}e^2$
GRH	Current change made in the value of GR when searching for separation.
GRØ	Value of $\bar{G}r/\bar{R}e^2$ corresponding to CØNØ.
GUESS(5)	Original guesses to solution if no approximation is available.
H	Current integration step length.
HS(10)	Array which contains the variable integration steps.
HY2IN(2)	Array which contains the final $Y(2)$ for each homogeneous set $Y(2)_h^{k+1} (\infty)$.
HY4IN(2)	Array which contains the final $Y(4)$ for each homogeneous set $Y(4)_h^{k+1} (\infty)$.

FØRTRANMeaningSymbol

IABC	$\tau_1^k \times 10 / (\tau_1^k + \tau_1^{k+1})$
ICØNTRL	Control variable which has the value 1 if a solution has been obtained, 2 if a further iteration is required and 3 if an error has been diagnosed.
IERRØR	This has a value of 0 usually, but is given a value of from 1 to 6 if an error is diagnosed during the computation.
IEXTEND	Set to 1 if the effective infinity is to be increased at any time.
IPARHØM	Set to 1 if the print out of the particular solution and two homogeneous solutions are required.
IPRINT	If a print out of the solution is required every iteration, set to 0.
ISEP	Set to 1 if the programme is to solve the case for which the prediction of separation is required.
ISETUP	Set to 1 to extend the existing solution, 2 to set up an initial guess, 3 to solve the non-linear equations.
ISKIP	Set to 1 if the array YK already contains an approximation to the solution.
IX	The number of step changes specified for the numerical integration.
JWRITE(10)	Print out control; The results are printed every JWRITE(I) steps at a step length HS(I).

FØRTRANMeaningSymbol

KU	Control variable which adjusts the change in Gr/Re^2 for the prediction of separation
METHØDQ	Control variable which describes the problem to be solved: 1. Stagnation point solution $Sc < 100$ 2. Stagnation point solution $Sc \geq 100$ 3. Vertical flat plate (Forced) $Sc < 100$ 4. Vertical flat plate (Forced) $Sc \geq 100$ 5. Vertical flat plate (Free)
MM	The number of separate problems to be solved for the current run.
MMAX(5)	Current maximum number of integration steps.
N	Number of first order equations to be solved.
NC	Iteration counter.
NCMAX	Maximum number of iterations allowed.
NHMG	Number of Homogeneous sets to be solved.
NMAX(10)	The total number of integration steps after each change of step length.
PY2INF	$Y(2)_p(\infty)$
PY4INF	$Y(4)_p(\infty)$
S1	$Sc^{\frac{1}{3}}$
SCH	Sc
SIG	$Y(2)(\infty)$
X	η
XMAX	η_{∞}
XXMAX	$\eta_{\infty MAX}$
XTOP	The value of η at which F''_0 is set to zero
Y(5)	$Y(I)$, $I = 1, 5$.
YD(5)	Dummy array in subroutine RKINT.

FØRTRANMeaningSymbol

YG(5)	$Y'(I), \quad I = 1,5 \quad .$
YH(5)	Dummy array in subroutine RKINT.
YHØ(2,5,1000)	$Y(I)_h^{k+1}(\eta), \quad I = 1,5 ; h = 1, \text{NHMG}.$ $\eta = 1, \text{XMAX}.$
YI(5,5)	Initial conditions for the homogeneous sets of equations.
YK(5,1000)	$Y(I)^k(\eta) \quad .$
YL(5)	$Y(I)^k$ at the point where the function is evaluated.
YP(2,1000)	$Y(I)_p^{k+1}(\eta), \quad I = 1,5 ; \eta = 1, \text{XMAX}.$
ZMAX(10)	The value of X at which the change of integration step length is to be made.

(i) MAIN (Exact Method)

This routine is the executive part of the programme, and transfers control either within the executive itself, or to the subroutines, each of which performs a specific function. The flow diagram is shown by Fig. D.3.1.

The computer programme may be used to solve any number of complete problems during the course of each run. The information on the first data card provides the value MM, which is the number of complete problems which are to be solved. Subroutine SETUP is called immediately, and all the necessary physical parameters of the problem are defined together with a number of control parameters. A check is made to see whether there has been any error in the data input. If an error is found, a diagnostic message is printed out and the computation, for the particular problem, is terminated.

Subroutine SOLVE is called and returns a "new" solution. This is checked with the previous solution to ascertain whether convergence has been achieved. If convergence has not been obtained, a check is made to see whether the maximum number of permitted iterations has been reached. If so, the computation is stopped. Otherwise SOLVE is called again to improve the solution.

Once a converged solution has been returned from SOLVE, WRITOUT is called to print this solution in tabular form.

Subroutine UPDATE is called to see whether the solution requires any modifications or extensions. Control then returns to the executive and is then either transferred back to obtain a "new" solution, if the effective infinity has been increased, or is transferred to the end of the solution loop, if a final solution has been returned.

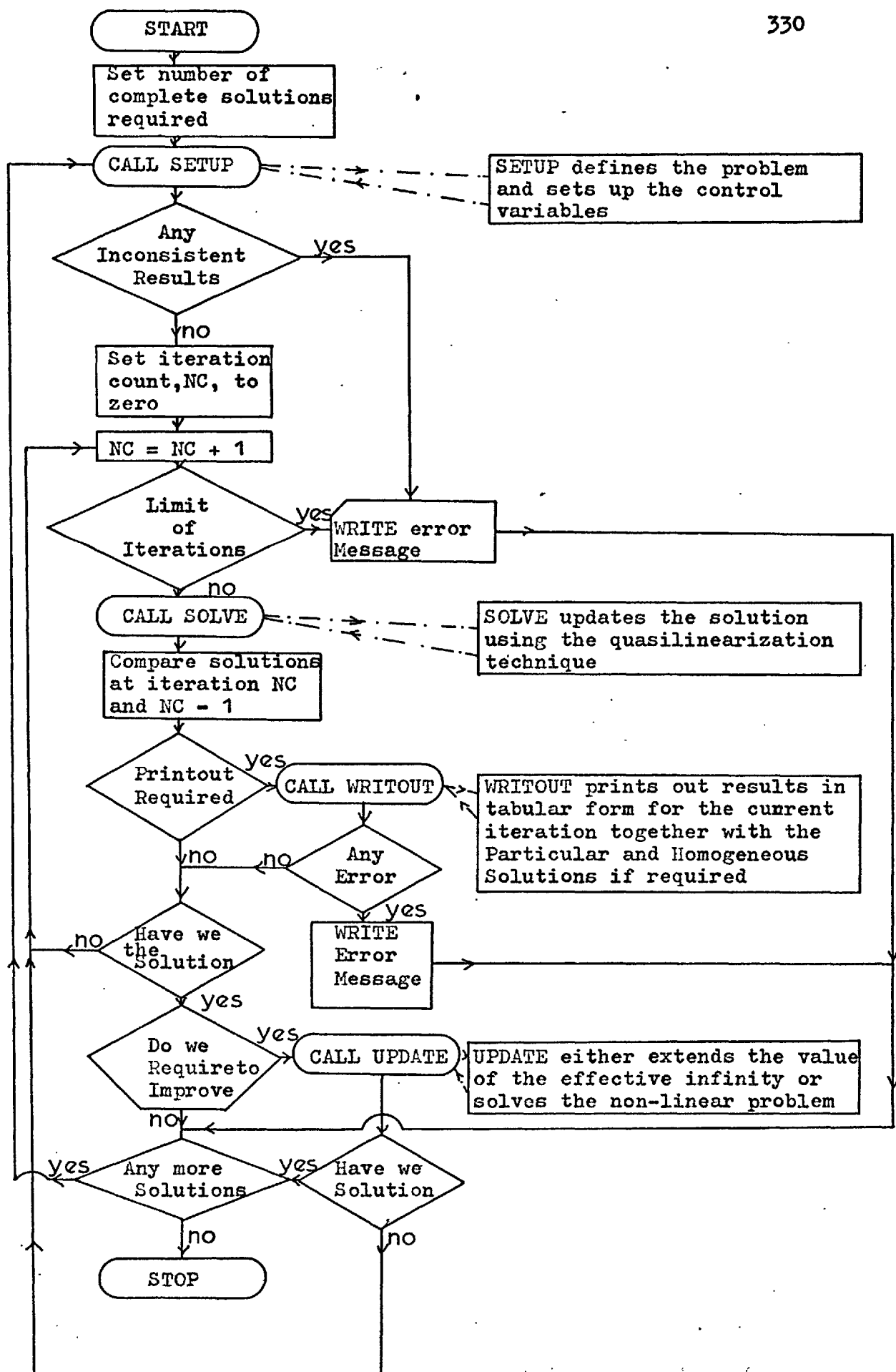


Fig. D.3.1. Flow Diagram for Subroutine EXACT

Further intermediate checks are made for errors, which, if found, cause a diagnostic message to be printed and execution to be terminated for the particular problem.

(ii) SETUP

Subroutine SETUP is called once for every complete solution. The flow diagram for this subroutine is shown in Fig. D.3.2.

The data which define the problem are read in. The first value read is the value of the parameter Grashof number divided by the Reynolds number squared, GR. The Schmidt number, SCH, (or Prandtl number for heat transfer) is specified. The third real number, SIG, must be set to unity, and this is the value of the dimensionless velocity in the free stream. Changes may be made to this value, within the programme, depending upon the physical problem which is to be solved. XXMAX is the maximum value of the effective infinity. If the solution is to be extended from a smaller value for infinity, specified within the programme, the extension is made by steps of length DELMAX.

The remaining variables control the general running of the programme. ISKIP is set to 1 if an approximate solution to the problem is stored in the array YK. If such a solution is not available, ISKIP must be set to 0.

If IPRINT is set to 0 the solution will be printed out for every iteration. If only the final solution is required, IPRINT is set to 1.

If IPARHOM is set to 1 the particular and homogeneous solutions are printed out for the final solution. If these are not required, the variable is set to 0.

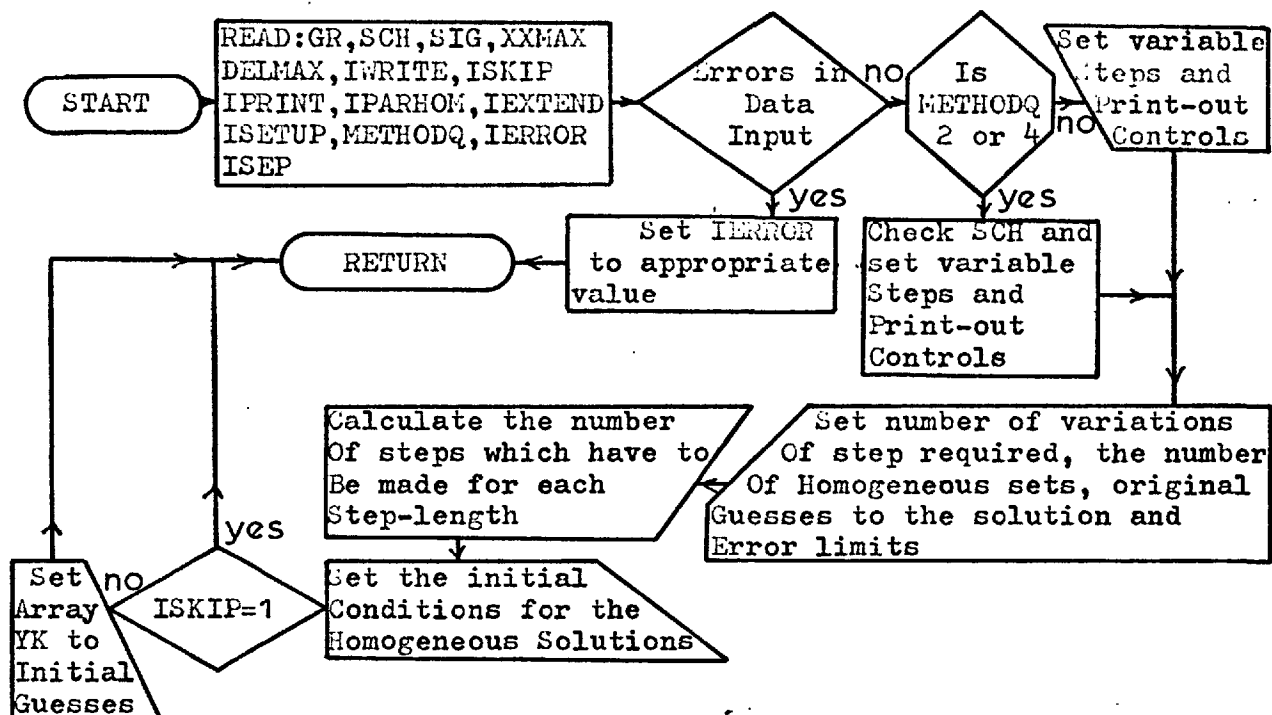
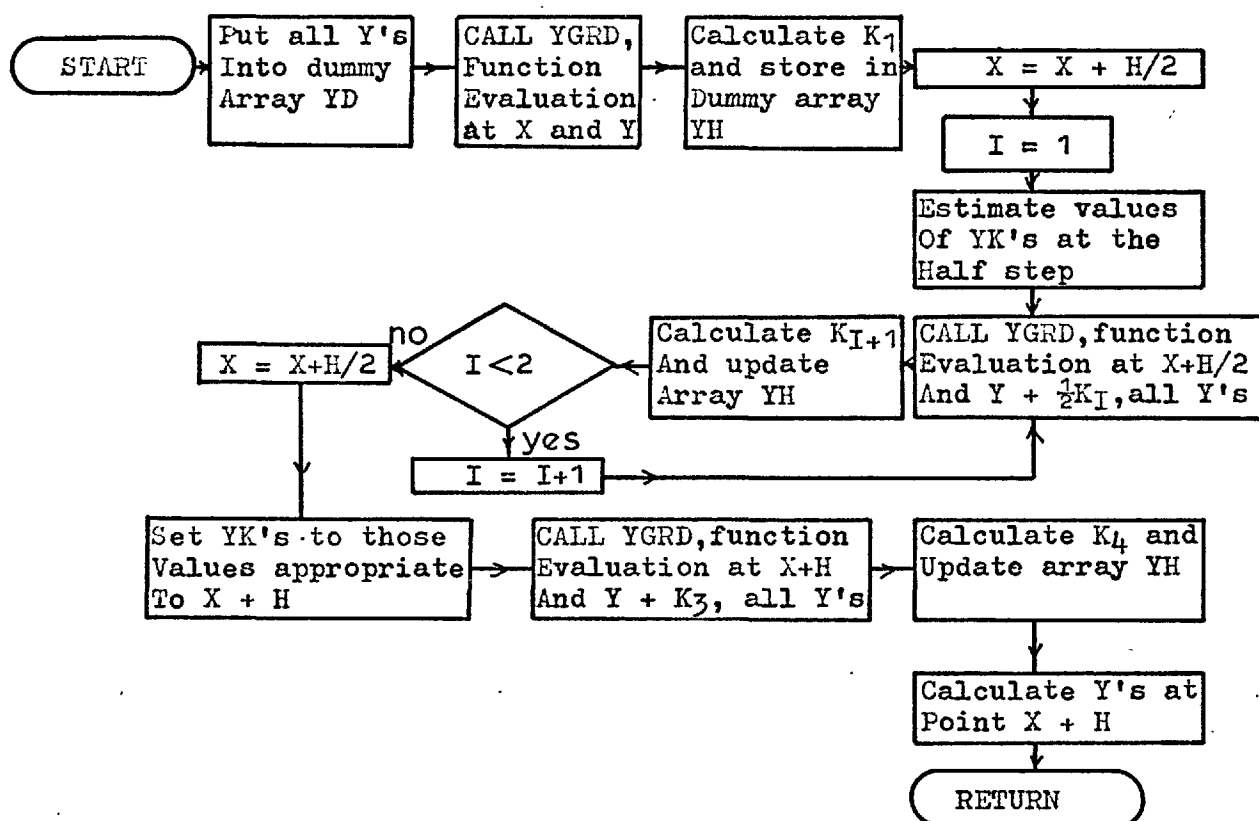


Fig. D.3.2. Flow Diagram of Subroutine SETUP

Fig. D.3.3. Flow Diagram of Subroutine RKINT
(Exact and Approximate Solutions)

IEXTEND is set to 1 if the final solution can only be obtained by stepping out from a small effective infinity to one which is much larger. If the solution is stable without this sophistication, IEXTEND is set to 0.

An approximate solution will be generated, and stored in array YK, when ISETUP is set to 2. The array YK is generated by the integration of the non-linear equations with the known initial conditions of a similar problem.

If ISETUP is set to 1, an existing solution may be extended from a small effective infinity to a larger one. The non-linear equations are again used with the initial conditions predicted for the smaller infinity.

The solution of the non-linear problem will be calculated and printed out by the use of the initial conditions predicted by the quasilinearization technique when ISETUP is set to 3. When the non-linear equations are not to be used, ISETUP is set to 0.

The value of the variable METHODQ defines the physical system as follows:-

- 1 . . . Stagnation Flow $Sc < 100$
- 2 . . . Stagnation Flow $Sc \geq 100$
- 3 . . . Vertical Flat Plate
(Forced Limit) $Sc < 100$
- 4 . . . Vertical Flat Plate
(Forced Limit) $Sc \geq 100$
- 5 . . . Vertical Flat Plate
(Free Limit)

The reasons for the use of different equations for cases when the Schmidt number is less than 100, or greater than or equal to 100, have been discussed in Chapter 5.

IERROR is the variable which controls the error diagnostic messages and must be set initially to 0.

By setting ISEP to 1 the opposing flow situation is investigated, and the parameter Grashof number divided by Reynolds number squared is increased until backflow is found. At this point, computation for the problem is stopped. If the aiding flow case is under investigation, ISEP is set to 0.

Depending upon which physical problem is to be solved, the appropriate variable step lengths, HS(I), and the write out instructions, JWRITE(I), are specified.

The assumed values for each variable are set. For the present study, these were taken as constant values over the whole range of integration.

The convergence limits ERMINA and ERMINB are set. In the present programme these variables have been given the value 1.E-6. The effect of varying these convergence criteria has been discussed earlier.

The number of steps, NMAX(I), for each step length, HS(I), is now calculated. The total of these NMAX(I)'s must not exceed 1,000, since this is the limit allowed by the present dimension statements YK(5,1000), YP(5,1000) and YHO(2,5,1000). If more steps are required, all these dimension statements must be altered.

The initial conditions for the two homogeneous sets are defined according to equations (5-80) and (5-81). If an approximate solution is already stored control is returned to the executive. Otherwise the array is set up using the assumed values defined above.

(iii) SOLVE

A "new" approximation to the solution is calculated, using the quasilinearization technique, each time subroutine SOLVE is called. The form of the subroutine is very straightforward to follow, with the aid of the flow chart Fig. D.3.4. and the description of the quasilinearization technique in Chapter 5.

The linearized equations are integrated from the surface to the specified infinity by the following method. The initial conditions are set, together with the initial step length and the number of steps which are to be made at this step length. A step is made using the Runge-Kutta method and the values of the variables, $Y(I)$, are stored in array YP. Further steps are made, changing the step length where specified, until the current effective infinity is reached. The final values of $Y(2)$ and $Y(4)$ are stored in PY2INF and PY4INF respectively, for use in the calculation of the superposition constants.

The two sets of homogeneous equations are integrated in the same way, and the values stored in array YHO. The final values of $Y(2)$ and $Y(4)$ are stored in arrays HY2IN and HY4IN respectively.

The superposition constants are now calculated, and the new estimate of the solution is calculated and stored in array YK. Control is then returned to the executive.

(iv) WRITOUT

Subroutine WRITOUT is the routine which controls the main part of the print out of the results. These may be either intermediate results, or the final solution.

The subroutine prints the final solution, and the

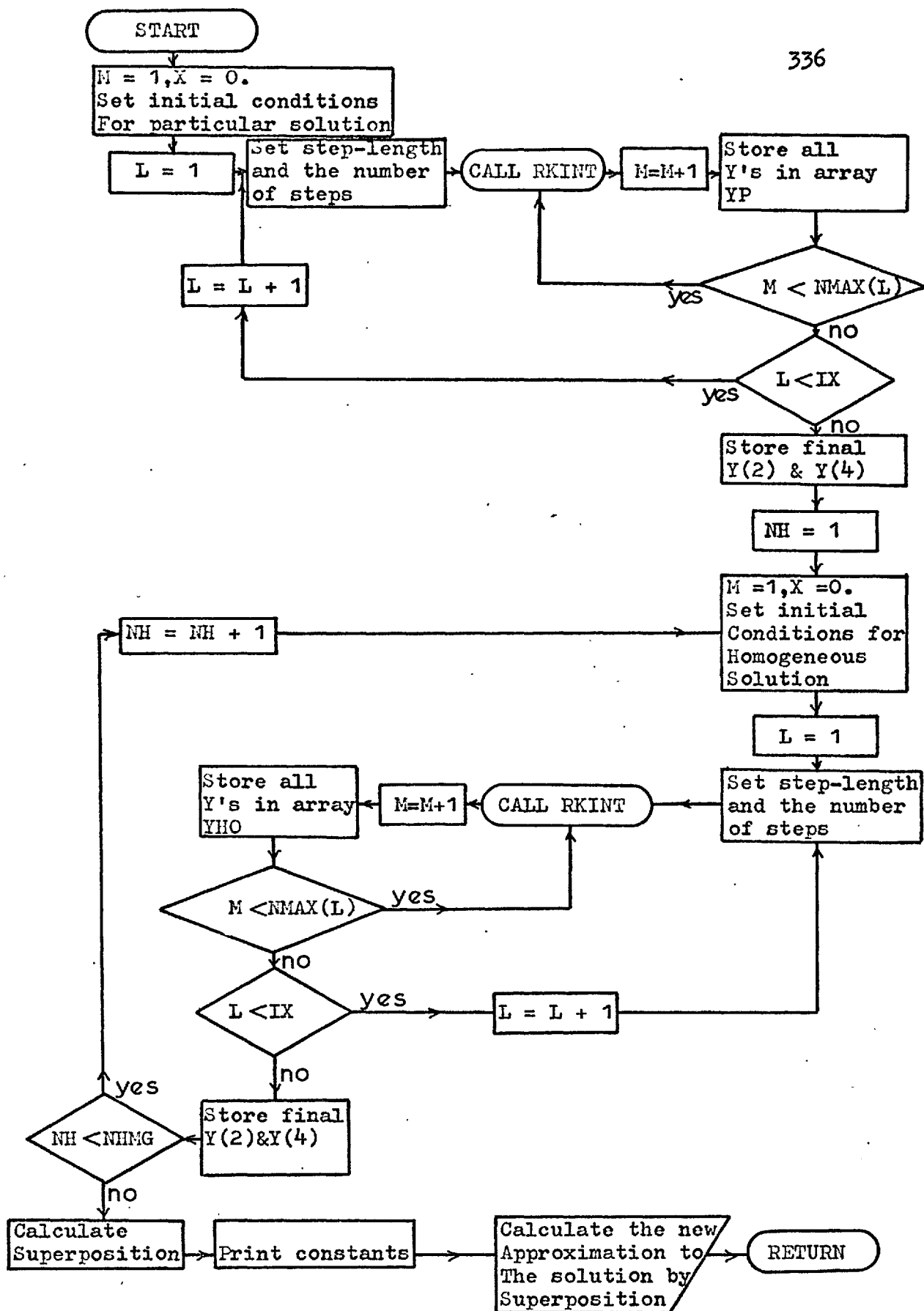


Fig. D.3.4. Flow Diagram for Subroutine SOLVE

particular and homogeneous solutions, if required. Each section is programmed in the same way and therefore only one will be discussed in detail.

The solution at the surface is printed out, and then at every JWRITE(L) steps for each step length HS(L). This continues for each of the variable steps up to the final step length HS(IX). The final value is NMAX(IX) steps from the surface, i.e. at the effective infinity.

The print out control can be readily changed by the user if required. The necessary changes are made in subroutine SETUP.

(v) UPDATE

The flow diagram for subroutine UPDATE is shown in Fig.D.3.5. The subroutine performs four distinct functions, which are as follows.

The non-linear problem may be solved, if required, by setting ISETUP equal to 3. The initial conditions, which have been predicted by the quasilinearization method, are then used as the starting values for the integration of the non-linear equations.

The non-linear equations may also be used to set up an initial approximation to the solution. If the solution of a similar problem to the one which is to be solved is known, the initial conditions of the former problem may be used to integrate the non-linear equations to produce the initial matrix YK. In this situation, ISETUP must be put equal to 2.

An extension of an existing solution may be obtained by increasing the effective infinity in a similar manner. In this case, the initial conditions which were predicted for the "old" infinity are used to integrate the non-linear equations to the

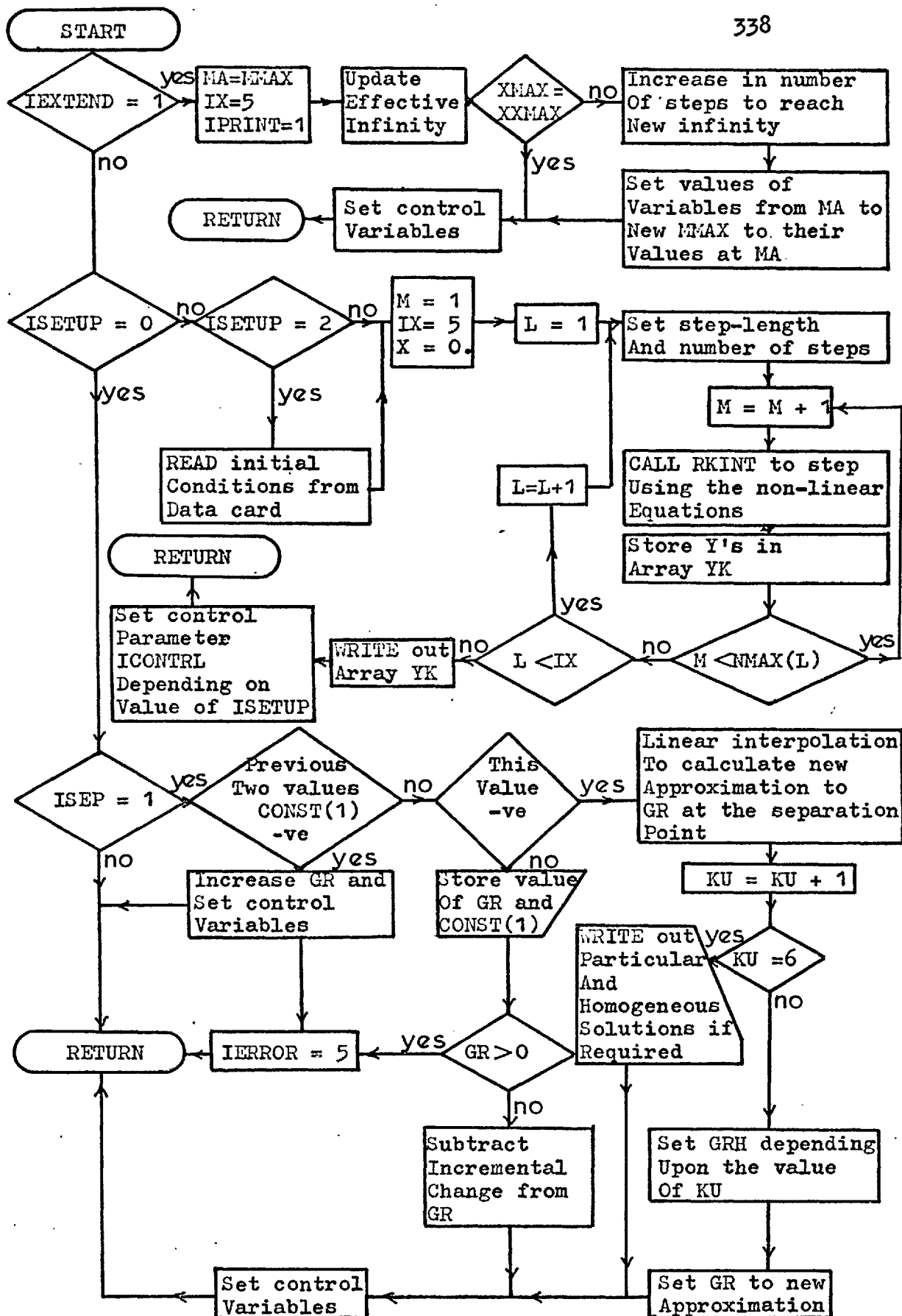


Fig. D.3.5. Flow Diagram of Subroutine UPDATE

"new"infinity. For this extension to be made, ISETUP must be set to 1.

If IEXTEND is set to 1 an existing solution is extended in a different way. The procedure used here is to set the values of all the variables, from the "old" infinity to the "new" infinity, to their values at the "old" infinity.

Thus the YK matrix is enlarged, to take account of the larger infinity, for use in the next iteration involving the quasilinearization procedure.

The final function of this subroutine is to predict the point where there is imminent backflow within the boundary layer for the opposing flow regime. For this prediction to be made, ISEP must be set to 1.

The method used is to find two values of the parameter Gr/Re^2 , one where the velocity gradient at the surface is positive, and the other where it is negative. A linear interpolation is then made between the two values to obtain an improved prediction of separation. The procedure is repeated until the required accuracy is obtained.

If a final solution to the problem has been obtained and no further computation is required, the control variable, ICTRL, returns to the executive with a value of 1. A further iteration, using the quasilinearization technique, is carried out if ICTRL is set to 2. A value of 3 for ICTRL signifies that an error has been found and an error message will be printed.

(vi) RKINT and RKUTTA

These two subroutines, the former used in the programme for the exact method and the latter in the programme for the

approximate method, are virtually identical and will be treated only once. The flow diagram is shown in Fig. D.3.3.

The subroutine represents the standard fourth order Runge-Kutta numerical integration routine. The values of all of the Y's are supplied at a point X, and the routine then predicts the values at the next step, $X + H$.

Four function evaluations, which are obtained by calling subroutine YGRD, are required at each step. One function evaluation is calculated at the point X, two at the half step and one at $X + H$.

The values of the Y's for the previous iteration, i.e. the YK's are required in the programme of the exact method. This is the only difference between the two subroutines. The values of the variable YK are taken to be those values at the point where the function evaluation is made. Those at the half-step are a simple average of the values at X and $X + H$.

In the programme of the exact method the subroutine YGRD which is called may be any one of three. The appropriate subroutine is specified in the arguments of the CALL statement for RKINT. Since the argument is a subroutine, an EXTERNAL statement must appear in the calling routine.

(vii) YGRD1, YGRD2, YGRD3

The form of all these three subroutines is identical. Subroutine YGRD1 provides the five first order, linearized, non-homogeneous equations which have to be solved simultaneously for each of the five different problems which the programme is capable of solving. Subroutine YGRD2 provides the homogeneous sets of equations and Subroutine YGRD3 provides the non-linear equations.

```

PROGRAM MAIN (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
DIMENSION CONST(10),HS(10),NMAX(10),YI(5,5)
DIMENSION YK(5,1000),YH(5),Y(5),YG(5),YD(5),YL(5)
COMMON /SET1/ A,ERMINA,ERMINB,NC,NCMAX
COMMON /SET2/ DELMAX,ICONTRL,IEXTEND,IPARHOM,IPRINT,ISEP,ISSETUP,
1      XXMAX,XMAX,JWRITE(10),ZMAX(10)
COMMON /SET3/ CONST,HS,NMAX,YI,IX
COMMON /SET4/ YK,YH,Y,YG,YD,YL
COMMON /SET6/ SIG,SCH,S1,X,NHMG,GR,IERROR,M,METHODQ,N
EXTERNAL YGRD1,YGRD2
N = 5 $ NHMG = 2
READ (5,100) MM
C**** MM IS THE NUMBER OF COMPLETE SOLUTIONS TO BE SOLVED DURING THE
C**** PRESENT COMPUTER RUN.
DO 4 NUMB = 1,MM
CALL SETUP
C**** SETUP DEFINES A NUMBER OF CONTROL VARIABLES AND PHYSICAL PARAMETERS
C**** WHICH DESCRIBE THE SYSTEM.
IF (IERROR.NE.0) GO TO 3
C**** ERROR CHECK FOR INCONSISTENT DATA INPUT.
ICONTRL = 2
IERROR = 0
1 NC = NC + 1
C**** UPDATING OF THE ITERATION COUNT FOR THE CURRENT SOLUTION.
IF (NC.GT.NCMAX) IERROR = 4
C**** CHECK CURRENT ITERATION NUMBER AGAINST THE MAXIMUM ACCEPTABLE.
IF (IERROR.NE.0) GO TO 3
CALL SOLVE
IF (NC.EQ.1) GO TO 2
C**** SOLVE CALCULATES A NEW APPROXIMATION TO THE SOLUTION USING THE
C**** QUASILINEARIZATION TECHNIQUE.
A = CONST(1) - C1
IF (A.LT.0.) A = -A
B = CONST(2) - C2
IF (B.LT.0.) B = -B
WRITE (6,1006) A,B
IF (A.LE.ERMINA.AND.B.LE.ERMINB) IPRINT = 0
C**** CONVERGENCE CHECK AND PRINTOUT CONTROL.
2 C1 = CONST(1)
C2 = CONST(2)
X = 0.
YK(1,1) = 0.
YK(2,1) = 0.
YK(3,1) = CONST(1)
YK(4,1) = 1.
YK(5,1) = CONST(2)
IF (IPRINT.EQ.0) CALL WRITOUT
C**** WRITOUT CALLED IF REQUIRED TO PRINT OUT THE TABLE OF RESULTS.
IF (IERROR.NE.0) GO TO 3
C**** ERROR CHECK.
IF (A.LE.ERMINA.AND.B.LE.ERMINB) CALL UPDATE
C**** UPDATE IS CALLED IF A SOLUTION HAS BEEN OBTAINED TO CHECK WHETHER
C**** THE SOLUTION REQUIRES UPDATING OR WHETHER THE NON-LINEAR SOLUTION
C**** IS REQUIRED.
GO TO (4,1,3) ICONTRL
C**** ICONTRL IS A CONTROL PARAMETER WHICH TRANSFERS CONTROL WITHIN THE
C**** MAIN PROGRAM.
3 GO TO (31,32,33,34,35,36),IERROR
C**** ERROR MESSAGE DIAGNOSTIC SET.

```

```
31 WRITE (6,1000) METHODQ  
   GO TO 4  
32 WRITE (6,1001)  
   GO TO 4  
33 WRITE (6,1002)  
   GO TO 4  
34 WRITE (6,1003)  
   GO TO 4  
35 WRITE (6,1004)  
   GO TO 4  
36 WRITE (6,1005)  
   4 CONTINUE  
   STOP  
100 FORMAT (15)  
1000 FORMAT (1X,* ERROR MESSAGE NO.1... VALUE OF METHODQ READ IS INCORR  
   1ECT, EITHER ZERO OR GE 6, ... =*,15)  
1001 FORMAT (1X,* ERROR MESSAGE NO.2...NO INITIAL GUESS SET FOR FIRST D  
   1ATA CARD *)  
1002 FORMAT (1X,* ERROR MESSAGE NO.3...NO PRINTOUT INSTRUCTIONS ..KM=0  
   1 *)  
1003 FORMAT (1X,* ERROR MESSAGE NO.4...NO CONSTANTS FOUND AFTER 15 ITER  
   1ATIONS *)  
1004 FORMAT (1X,* ERROR MESSAGE NO.5... GR HAS GONE POSITIVE DURING CAL  
   1CULATION OF SEPARATION*)  
1005 FORMAT (1X,* ERROR MESSAGE NO.6... SEPARATION CALCULATION, GRH SET  
   1TO ZERO ON DATA CARD*)  
1006 FORMAT (6X,*A          =*,E17.10,5X,*B          =*,E17.10/)  
   END
```

```

SUBROUTINE SETUP
  DIMENSION GUESS(10)
  DIMENSION CONST(10),HS(10),NMAX(10),YI(5,5)
  DIMENSION YK(5,1000),YH(5),Y(5),YG(5),YD(5),YL(5)
  COMMON /SET1/ A,ERMINA,ERMINB,NC,NCMAX
  COMMON /SET2/ DELMAX,ICONTROL,IEXTEND,IPARHOM,IPRINT,ISEP,ISSETUP,
1    XXMAX,XMAX,JWRITE(10),ZMAX(10)
  COMMON /SET3/ CONST,HS,NMAX,YI,IX
  COMMON /SET4/ YK,YH,Y,YG,YD,YL
  COMMON /SET6/ SIG,SCH,S1,X,NHMG,GR,IERROR,M,METHODQ,N
  COMMON /SET7/ GRH,KU
  READ (5,100) GR,SCH,SIG,XXMAX,DELMAX,GRH,KU,ISKIP,IPRINT,IPARHOM,
1 IEXTEND,ISSETUP,METHODQ,IERROR,ISEP,NCMAX
C**** GR IS THE VALUE OF THE GRASHOF NUMBER DIVIDED BY THE REYNOLDS NUMBER
C**** SQUARED.
C**** SCH IS THE SCHMIDT NUMBER.
C**** SIG IS NORMALLY SET TO 1.0 BEING THE DIMENSIONLESS
C**** VELOCITY AT INFINITY.
C**** XXMAX IS THE EFFECTIVE INFINITY FOR THE INTEGAITION.
C**** DELMAX IS THE STEPWISE INCREASE IN THE EFFECTIVE INFINITY IF WE
C**** GRADUALLY INCREASE THE VALUE OF INFINITY.
C**** ISKIP IS SET TO 1 IF WE ALREADY HAVE AN APPROXIMATION TO THE
C**** SOLUTION IN STORE.
C**** IPRINT IS SET TO 1 IF WE ONLY REQUIRE PRINTOUT OF THE FINAL SOLUTION.
C**** IPARHOM IS SET TO 1 IF WE REQUIRE PRINTOUT OF THE PARTICULAR AND
C**** THE TWO HOMOGENEOUS SOLUTIONS.
C**** IEXTEND IS SET TO 1 IF THE SOLUTION MUST BE UPDATED BY EXTENDING
C**** THE EFFECTIVE INFINITY.
C**** ISSETUP SET TO 1 TO EXTEND EXISTING SOLUTION.
C**** SET TO 2 TO SET UP AN INITIAL GUESS.
C**** SET TO 3 TO SOLVE THE N/N-LINEAR PROBLEM USING THE CONSTANTS
C**** SUPPLIED FROM THE QUASILINEARIZATION ROUTINE.
C****METHODQ SET TO 1 STAGNATION FLOW AT LOW SCH.
C**** SET TO 2 STAGNATION FLOW AT HIGH SCH (GT.100.)
C**** SET TO 3 FLAT PLATE, FORCED LIMIT LOW SCH.
C**** SET TO 4 FLAT PLATE, FORCED LIMIT HIGH SCH (GT.100.)
C**** SET TO 5 FLAT PLATE, FREE LIMIT.
C**** IERROR IS THE ERROR CONTROL VARIABLE SET INITIALLY TO 0.
C**** ISEP IS SET TO 1 IF THE PREDICTION OF SEPARATION FOR OPPOSING FLOWS
C**** IS REQUIRED.
C**** IF (METHODQ.EQ.0.OR.METHODQ.GE.6) IERROR = 1
C**** IF (ISEP.EQ.1.AND.GRH.EQ.0.) IERROR = 6
C**** IF (NUMB.EQ.1.AND.ISKIP.EQ.1) IERROR = 2
C**** IF (IERROR.NE.0) RETURN
C**** CHECKS FOR ERRORS IN DATA INPUT.
C**** CUBE = 1./3.
C**** S1 = SCH ** CUBE
C**** SET THE CUBE ROOT OF THE SCHMIDT NUMBER.
C**** GO TO (1,2,1,2,1),METHODQ
C**** TRANSFER OF CONTROL DEPENDS UPON THE METHOD USED.
C**** SET UP THE VARIABLE INTEGRATION STEPS FOR THE SYSTEM TOGETHER WITH
C**** PRINTOUT CONTROL.
1 HS(1)=HS(2)=HS(3)=HS(4)=HS(5)=0.01
  ZMAX(1)=1,$ZMAX(2)=2,$ZMAX(3)=3,$ZMAX(4)=4,$ZMAX(5)=XXMAX
  JWRITE(1)=5$JWRITE(2)=JWRITE(3)=JWRITE(4)=JWRITE(5)=50
  GO TO 5
2 SIG = S1
  IF (SCH.GE.900.) GO TO 3
  IF (SCH.GE.90.) GO TO 4

```

```

HS(1)=HS(2)=HS(3)=HS(4)=HS(5)=0.05
ZMAX(1)=1.$ZMAX(2)=2.$ZMAX(3)=3.$ZMAX(4)=4.$ZMAX(5)=XXMAX
JWRITE(1)=JWRITE(2)=JWRITE(3)=JWRITE(4)=JWRITE(5)=2
GO TO 5
3 HS(1)=0.01$HS(2)=0.05$HS(3)=0.1$HS(4)=0.5$HS(5)=0.5
ZMAX(1)=2.$ZMAX(2)=6.$ZMAX(3)=9.$ZMAX(4)=10.$ZMAX(5)=XXMAX
JWRITE(1)=JWRITE(2)=20$JWRITE(3)=10$JWRITE(4)=2$JWRITE(5)=10
GO TO 5
4 HS(1)=0.01$HS(2)=0.05$HS(3)=0.05$HS(4)=0.1$HS(5)=0.1
ZMAX(1)=2.$ZMAX(2)=6.$ZMAX(3)=9.$ZMAX(4)=10.$ZMAX(5)=XXMAX
JWRITE(1)=50$JWRITE(2)=JWRITE(3)=20$JWRITE(4)=10$JWRITE(5)=50
5 IX = 5
XMAX = ZMAX(4)
A = 1.
IF (METHODQ.EQ.5) SIG = 1./(2.*SQRT(GR))
NHMG = 2
GUESS(1)=3.$GUESS(2)=0.5$GUESS(3)=0.5$GUESS(4)=0.$GUESS(5)=-0.3
C**** GUESSED VALUES TO BE SUPPLIED IF REQUIRED.
ERMINA = 1.E-6 $ ERMINB = 1.E-6
C**** LIMITS FOR CONVERGENCE FOR ACCEPTABLE SOLUTION.
C**** CALCULATIONS OF THE NUMBER OF STEPS TO BE MADE AT EACH STEPLENGTH.
NMAX(1) = (((ZMAX(1)+1.E-5)/HS(1))+1.00001
C**** PRINTOUT OF STEP DATA FOR CHECK.
WRITE (6,1000) METHODQ,GR,SCH,HS(1),ZMAX(1),NMAX(1)
DO 6 K = 2,IX
NMAX(K) = (((ZMAX(K)-ZMAX(K-1))+1.E-5)/HS(K))+NMAX(K-1)
6 WRITE(6,2000) HS(K),ZMAX(K),NMAX(K)
IF (IEXTEND.EQ.1.OR.ISETUP.EQ.1) IX = 4
C**** INITIAL CONDITIONS FOR THE TWO HOMOGENEOUS SETS.
YI(1,1)=0.$YI(1,2)= 0.$YI(1,3)=1.$YI(1,4)=0.$YI(1,5)=0.
YI(2,1)=0.$YI(2,2)=0.$YI(2,3)=0.$YI(2,4)=0.$YI(2,5)=1.
NC = 0
C**** SET THE ITERATION COUNT.
IF (ISKIP.EQ.1) RETURN
C**** IF WE ALREADY HAVE AN APPROXIMATE SOLUTION RETURN, OTHERWISE SET
C**** VALUES TO THOSE GUESSED.
JK = NMAX(IX)
WRITE (6,1002) JK
DO 7 M = 1,JK
DO 7 I = 1,5
7 YK(I,M)=GUESS(I)
RETURN
100 FORMAT (6F10.5,9I1,I2)
1000 FORMAT(1H1,1X,* METHOD NUMBER ... * ,I5,* GR = *,F10.5, * SCH
1= *,F10.5/1X,2(F10.5,5X),I5/)
2000 FORMAT (1X,2(F10.5,5X),I5/)
1002 FORMAT (1X,* JK = *,I5)
END

```



```

SUBROUTINE SOLVE
  DIMENSION CONST(10),HS(10),NMAX(10),YI(5,5)
  DIMENSION YK(5,1000),YH(5),Y(5),YG(5),YD(5),YL(5),HY2IN(5),HY4IN(5)
  COMMON /SET3/ CONST,HS,NMAX,YI,IX
  COMMON /SET4/ YK,YH,Y,YG,YD,YL
  COMMON /SET6/ SIG,SCH,S1,X,NHMG,GR,IERROR,M,METHODQ,N
  COMMON /SET5/ YP(5,1000),YHO(2,5,1000)
  EXTERNAL YGRD1,YGRD2
  M = 1
C**** M IS THE STEP COUNTER. M=1 AT THE BODY SURFACE.
  DO 1 I = 1,N
    1 Y(I) = 0.
    Y(4) = 1.
C**** SET THE INITIAL CONDITIONS FOR THE PARTICULAR SOLUTION.
    X = 0.
    DO 4 L = 1,IX
      H = HS(L)
      MMAX = NMAX(L)
C**** SET THE STEP LENGTH AND THE NUMBER OF STEPS AT THIS STEP LENGTH.
      2 CALL RKINT (H,YGRD1)
C**** RKINT STEPS FROM X TO X + H .
      M = M + 1
      DO 3 I = 1,N
        3 YP(I,M) = Y(I)
C**** STORE VALUES OF THE PARTICULAR SOLUTION FOR ALL Y .
      IF (M.LT.MMAX) GO TO 2
C**** IF WE HAVE TO MAKE A FURTHER STEP WITH THIS STEP LENGTH, BACK TO
C**** RKINT.
      4 CONTINUE
      PY2INF = YP(2,MMAX)
      PY4INF = YP(4,MMAX)
C**** STORE FINAL VALUES OF Y(2) AND Y(4) FOR LATER CALCULATION OF
C**** THE INTEGRATION CONSTANTS.
      DO 9 NH = 1,NHMG
C**** CALCULATIONS FOR THE TWO HOMOGENEOUS SETS,NH INITIALLY 1 THEN 2.
      M = 1
      DO 5 I = 1,N
        5 Y(I) = YI(NH,I)
        X = 0.
        DO 8 L = 1,IX
          H = HS(L)
          MMAX = NMAX(L)
C**** SET THE STEP LENGTH AND THE NUMBER OF STEPS AT THIS STEP LENGTH.
          6 CALL RKINT (H,YGRD2)
C**** RKINT STEPS FROM X TO X + H .
          M = M + 1
          DO 7 I = 1,N
            7 YHO(NH,I,M) = Y(I)
C**** STORE THE VALUES OF THE HOMOGENEOUS SOLUTIONS.
          IF (M.LT.MMAX) GO TO 6
C**** IF WE HAVE TO MAKE A FURTHER STEP WITH THIS STEP LENGTH, BACK TO
C**** RKINT.
          8 CONTINUE
          HY2IN(NH) = YHO(NH,2,MMAX)
          HY4IN(NH) = YHO(NH,4,MMAX)
C**** STORE FINAL VALUES OF Y(2) AND Y(4) FOR LATER CALCULATION OF
C**** THE INTEGRATION CONSTANTS.
          9 CONTINUE
          CONST(1)=(SIG*HY4IN(2)+PY4INF*HY2IN(2)-PY2INF*HY4IN(2))/(HY2IN(1)*

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1HY4IN(2)-HY4IN(1)*HY2IN(2))
  CONST(2)=(SIG*HY4IN(1)+PY4INF*HY2IN(1)-PY2INF*HY4IN(1))/(HY2IN(2)*
1HY4IN(1)-HY4IN(2)*HY2IN(1))
  WRITE (6,1003) CONST(1),CONST(2)
C**** CALCULATION OF THE CONSTANTS.
C**** PRINT OUT THE VALUES OF THE CONSTANTS.
  DO 10 M = 2,MMAX
  DO 10 I = 1,N
C**** CALCULATE THE NEW SOLUTION BY SUPERPOSITION.
  YK(I,M) = YP(I,M)+CONST(1)*YHO(1,I,M)+CONST(2)*YHO(2,I,M)
  10 IF (YK(4,M).GT.0.01) NMAX(10) = M
  RETURN
1003 FORMAT (6X,*CONST(1) =*,E17.10,5X,*CONST(2) =*,E17.10)
END

```

```

SUBROUTINE WRITOUT
  DIMENSION CONST(10),HS(10),NMAX(10),YI(5,5)
  DIMENSION YK(5,1000),YH(5),Y(5),YG(5),YD(5),YL(5)
  COMMON /SET1/ A,ERMINA,ERMINB,NC,NCMAX
  COMMON /SET2/ DELMAX,ICONTRL,IEXTEND,IPARHOM,IPRINT,ISEP,ISSETUP,
1      XXMAX,XMAX,JWRITE(10),ZMAX(10)
  COMMON /SET3/ CONST,HS,NMAX,YI,IX
  COMMON /SET4/ YK,YH,Y,YG,YD,YL
  COMMON /SET6/ SIG,SCH,S1,X,NHMG,GR,IERROR,M,METHODQ,N
  COMMON /SET5/ YP(5,1000),YHO(2,5,1000)
C**** WRITOUT PRINTS OUT THE VALUES OF THE SOLUTION ARRAY ,YK, AND THE
C**** PARTICULAR AND HOMOGENEOUS SOLUTIONS IF REQUIRED.
C**** KN IS THE NUMBER OF THE GRID LINE AT WHICH THE PRINT OUT BEGINS.
C**** KM IS THE FINAL GRID NUMBER.
C**** KL IS THE STEPWISE INCREASE AT WHICH INTERMEDIATE VALUES ARE
C**** PRINTED
C**** IF IPARHOM IS SET TO 0 NO PRINT OUT OF THE PARTICULAR AND
C**** HOMOGENEOUS SOLUTIONS WILL BE MADE.
  IF (ISEP.EQ.1.AND.IPARHOM.EQ.1) GO TO 3
  WRITE (6,1000) NC
  WRITE (6,1001)
  KN = 1
  X = -HS(1) * JWRITE(1)
  DO 2 L = 1,IX
    KL = NMAX(L)
    KM = JWRITE(L)
    IF (KM.EQ.0) IERROR=3
    IF (IERROR.NE.0) RETURN
    DO 1 M = KN,KL,KM
      X = X + HS(L) * KM
1  WRITE (6,1002) X,(YK(I,M),I=1,5)
    KN = NMAX(L)+JWRITE(L+1)
2  CONTINUE
  WRITE (6,1102) NMAX(10)
  1102 FORMAT (///1X,*Y(4).LT.1 PER CENT AT STEP *,I5)
3  IF (IPARHOM.EQ.0) RETURN
  KN = 1
  YP(1,1) = 0.
  YP(2,1) = 0.
  YP(3,1) = 0.
  YP(4,1) = 1.
  YP(5,1) = 0.
  WRITE (6,1100)
  WRITE (6,1001)
  X = -HS(1) * JWRITE (1)
  DO 4 L = 1,IX
    KL = NMAX(L)
    KM = JWRITE(L)
    IF(KM.EQ.0) IERROR = 3
    IF (IERROR.NE.0) RETURN
    DO 5 M = KN,KL,KM
      X = X + HS(L)*KM
5  WRITE (6,1002) X,(YP(I,M),I=1,5)
4  KN = NMAX(L) + JWRITE(L+1)
  DO 6 NH = 1,NHMG
    KN = 1
    X = - HS(1) * JWRITE (1)
    DO 7 I = 1,N
7  YHO(NH,I,1) = YI(NH,I)

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```
WRITE (6,1101) NH
WRITE (6,1001)
DO 8 L = 1,IX
  KL = NMAX(L)
  KM = JWRITE(L)
  DO 9 M = KN,KL,KM
    X = X + HS(L) * KM
  9 WRITE (6,1002) X,(YHO(NH,I,M),I=1,5)
  8 KN = NMAX(L) + JWRITE(L+1)
  6 CONTINUE
  RETURN
1000 FORMAT(1X,25HTHIS IS ITERATION NUMBER, 13)
1001 FORMAT (1X,10X,1HX,13X,4HY(1),18X,4HY(2),18X,4HY(3),18X,4HY(4),18X
1,4HY(5))
1002 FORMAT (10X,F8.5,5(5X,E17.10))
1100 FORMAT (1H1,1X,* PARTICULAR SOLUTION *)
1101 FORMAT (1H1,1X,* HOMOGENEOUS SOLUTION NO. *,15)
END
```

```

SUBROUTINE UPDATE
  DIMENSION CONST(10),HS(10),NMAX(10),Y1(5,5)
  DIMENSION YK(5,1000),YH(5),Y(5),YG(5),YD(5),YL(5)
  COMMON /SET2/ DELMAX,ICONTRL,IEXTEND,IPARHOM,IPRINT,ISEP,ISSETUP,
1      XXMAX,XMAX,JWRITE(10),ZMAX(10)
  COMMON /SET3/ CONST,HS,NMAX,Y1,IX
  COMMON /SET4/ YK,YH,Y,YG,YD,YL
  COMMON /SET6/ SIG,SCH,S1,X,NHMG,GR,IERROR,M,METHODQ,N
  COMMON /SET7/ GRH,KU
  EXTERNAL YGRD3
  IF (IEXTEND.EQ.1) GO TO 7
C**** EXTENSION OF EXISTING SOLUTION BY SIMPLE EXTRAPOLATION.
  ICONTRL = 1
C**** PREDICTION OF THE SEPARATION POINT, I.E. THE REGION OF BACKFLOW.
  IF (ISEP.EQ.1) GO TO 10
  IF (ISSETUP.NE.0) GO TO 1
  RETURN
C**** THE NON-LINEAR EQUATIONS ARE USED TO ....
C****
C****          (A) EXTEND THE EXISTING SOLUTION .. (ISSETUP=1)
C****
C****          (B) SET UP THE INITIAL GUESS .. (ISSETUP=2)
C****
C****          (C) SOLVE THE NON-LINEAR PROBLEM WITH THE
C****              BOUNDARY CONDITIONS CALCULATED USING THE
C****              QUASILINEARIZATION TECHNIQUE ....(ISSETUP=3)
  1 IF (ISSETUP.NE.2) GO TO 2
  READ (5,100) YK(3,1),YK(5,1)
C**** IF WE WISH TO SET UP AN INITIAL APPROXIMATION TO THE SOLUTION WE
C**** MUST SUPPLY THE VALUES OF Y(3) AND Y(5) AT THE SURFACE.
  YK(1,1)=0.$YK(2,1)=0.$YK(4,1)=1.
  2 DO 3 I = 1,N
  3 Y(I)=YK(I,1)
C**** SET ALL VALUES OF THE Y'S AT THE SURFACE
  M = 1
  IX = 5
C**** IX IS THE NUMBER OF CHANGES OF STEP.
  X = 0.
  DO 6 L = 1,IX
  H = HS(L)
  MMAX = NMAX(L)
C**** SET STEP LENGTH AND THE NUMBER OF STEPS.
  4 M = M + 1
  CALL RKINT (H,YGRD3)
C**** STEP TO NEXT POINT USING THE NON-LINEAR EQUATIONS.
  DO 5 I = 1,N
  5 YK(I,M) = Y(I)
C**** SET UP THE STORED FIRST APPROXIMATION TO THE SOLUTION.
  IF (M.LT.MMAX) GO TO 4
C**** IF WE HAVE NOT REACHED THE LIMIT OF STEPS, RETURN TO RKINT.
  6 CONTINUE
  WRITE (6,1009) ISSETUP
  WRITE (6,1005) GR,SCH
  WRITE (6,1001)
C**** PRINT OUT THE SOLUTION.
  KN.= 1
  X = -HS(1)*JWRITE(1)
  DO 61 L = 1,IX
  KL = NMAX(L)

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      KM = JWRITE(L)
      DO 62 M = KN,KL,KM
      X = X + HS(L)*KM
62  WRITE(6,1002) X,(YK(I,M),I=1,5)
      KN = NMAX(L) + JWRITE(L+1)
61  CONTINUE
      ICONTRL = 1
      IF (ISETUP.EQ.1) ICONTRL=2
      ISETUP = 0
C**** SET THE CONTROL PARAMETER FOR MAIN PROGRAM CONTROL.
      RETURN
C**** EXISTING SOLUTION EXTENDED BY SIMPLE EXTRAPOLATION.
      7 MA = NMAX(IX)
      IX = 5
      IPRINT = 1
      XMAX = XMAX + DELMAX
C**** UPDATE THE EFFECTIVE INFINITY.
      IF (XMAX.LE.XXMAX) GO TO 8
      ICONTRL = 1
      IF (ISETUP.NE.0) GO TO 1
C**** IF WE HAVE REACHED THE INFINITY LIMIT RETURN TO MAIN PROGRAM HAVING
C**** SET THE CONTROL PARAMETER.
      RETURN
      8 NMAX(IX) = ((XMAX-ZMAX(4))+1.E-5)/HS(5) + NMAX(4)
      MMAX = NMAX(IX)
C**** UPDATE NUMBER OF STEPS AT STEP LENGTH 4
      DO 9 I = 1,N
      DO 9 M = MA,MMAX
      9 YK(I,M) = YK(I,MA)
C**** SET ALL VALUES OF THE VARIABLES Y FROM THE OLD INFINITY TO THE
C**** NEW INFINITY TO THEIR VALUES AT THE OLD INFINITY.
      ICONTRL = 2
      RETURN
      10 IF (CONST(1).LT.0..AND.NZ.EQ.1) GO TO 20
C**** IF THE LAST TWO VALUES OF CONST(1) WERE NEGATIVE WE NEED TO MAKE
C**** THE VALUE OF GR LARGER.
      IF (CONST(1).LT.0.) GO TO 11
C**** IF THE VALUE OF CONST(1) IS NEGATIVE BUT THE LAST WAS POSITIVE
C**** EXTRAPOLATE BETWEEN THEM TO PREDICT VALUE AT WHICH CONST(1)=0.
C**** IF CONST(1) IS POSITIVE, STORE ITS VALUE AND THE CORRESPONDING
C**** VALUE OF GR.
      GRO = GR
      CONO = CONST(1)
      GO TO 21
      11 ABC = ABS(CONST(1))
      IABC = CONO*10./(ABC+CONO)
C**** LINEAR EXTRAPOLATION TO PREDICT VALUE OF GR AT WHICH CONST(1) = 0.
      NQ = 1
      KU = KU + 1
C**** VALUE OF KU DETERMINES THE MAGNITUDE OF THE CHANGE TO GR.
      WRITE (6,1010) ABC,CONO,IABC
      GO TO (12,13,14,15,16,22),KU
C**** IF KU HAS THE VALUE 6 THE SEPARATION HAS BEEN PREDICTED TO 1.E-6.
      12 GRH = 0.1
      IABC = IABC/2
      GO TO 17
      13 GRH = 0.01
      GO TO 17
      14 GRH = 0.001

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        GO TO 17
15  GRH = 0.0001
    GO TO 17
16  GRH = 0.00001
C**** GRH IS THE INCREMENTAL CHANGE OF GR.
17  GR = GRO - (GRH*IABC)
    NZ = 1
19  NC = 0
    NQ = 0
    IPRINT = 1
    ICONTRL = 2
C**** THE NEW VALUE OF GR IS RETURNED TO MAIN AND THE NORMAL SOLUTION
C**** PROCEDURE IS FOLLOWED.
    RETURN
20  GR = GR + GRH
    NC = 1
    IPRINT = 1
    ICONTRL = 2
    RETURN
21  IF (GR.GT.0.) GO TO23.
C**** IF GR HAS GONE POSITIVE ARRANGE FOR ERROR PRINTOUT.
    NZ = 0
    GR = GRO - GRH
    NC = 0
    IPRINT = 1
    ICONTRL = 2
    RETURN
22  IF (IPARHOM.EQ.1) CALL WRITOUT
C**** IF WE HAVE A SOLUTION AND REQUIRE A PRINTOUT OF THE PARTICULAR
C**** AND HOMOGENEOUS SOLUTIONS, CALL WRITOUT.
    ICONTRL = 1
    RETURN
23  IERROR = 5
    ICONTRL = 3
    RETURN
100  FORMAT (2F15.10)
1001  FORMAT (1X,10X,1HX,13X,4HY(1),18X,4HY(2),18X,4HY(3),18X,4HY(4),18X,
1,4HY(5))
1002  FORMAT (10X,F8.5,5(5X,E17.10))
1005  FORMAT (1X,*GR =*,F10.5,5X,* SCH =*,F10.5)
1009  FORMAT (1H1,1X,* NON-LINEAR INITIAL VALUE PROBLEM *,5X,* ISETUP =*
1,15)
1010  FORMAT (1X,*ABC =*,F10.5,5X,*CONO =*,F10.5,5X,*IABC =*,15)
    END

```

```

SUBROUTINE RKINT (H,YGRD)
DIMENSION YK(5,1000),YH(5),Y(5),YG(5),YD(5),YL(5)
COMMON /SET4/ YK,YH,Y,YG,YD,YL
COMMON /SET6/ SIG,SCH,S1,X,NHMG,GR,IERROR,M,METHODQ,N
C**** RUNGE-KUTTA SINGLE STEP INTEGRATION ROUTINE.
DO 1 J=1,N
  YL(J) = YK(J,M)
  1 YD(J)=Y(J)
C**** FUNCTION EVALUATION AT POINT X.
  CALL YGRD
  DO 2 J=1,N
    YG(J)=YG(J)*H
  2 YH(J)=YG(J)
    X=X+0.5*H
    DO 4 I=1,2
      DO 3 J=1,N
        YL(J) = (YK(J,M)+YK(J,M+1))/2.
      3 Y(J)=YD(J)+0.5*YG(J)
C**** FUNCTION EVALUATION AT HALF STEP.
      CALL YGRD
      DO 4 J=1,N
        YG(J)=YG(J)*H
      4 YH(J)=YH(J)+2.*YG(J)
        X=X+0.5*H
        DO 5 J=1,N
          YL(J) = YK(J,M+1)
        5 Y(J)=YD(J)+YG(J)
C**** FUNCTION EVALUATION AT X + H .
      CALL YGRD
      DO 6 J=1,N
C**** CALCULATION OF THE CHANGE IN EACH Y.
        YH(J)=(YH(J)+H*YG(J))/6.
        YD(J)= YD(J)+YH(J)
C**** VALUE OF Y'S AT X + H .
      6 Y(J)=YD(J)
      RETURN
    END

```



```

SUBROUTINE YGRD1
DIMENSION YK(5,1000),YH(5),Y(5),YG(5),YD(5),YL(5)
COMMON /SET4/ YK,YH,Y,YG,YD,YL
COMMON /SET6/ SIG,SCH,S1,X,NHMG,GR,IERROR,M,METHODQ,N
C**** LINEARIZED NON-HOMOGENEOUS EQUATIONS.
GO TO (1,2,3,4,5),METHODQ
C**** STAGNATION FLOW SCH.LT.100.
1 YG(1) = Y(2)
  YG(2) = Y(3)
  YG(3) = -(YL(3)*Y(1)+YL(1)*Y(3)-YL(1)*YL(3)+0.5*(1.+GR*Y(4)+YL(2)*
1 YL(2)-2.*YL(2)*Y(2)))
  YG(4) = Y(5)
  YG(5) = -SCH*(Y(1)*YL(5)+YL(1)*Y(5)-YL(1)*YL(5))
  RETURN
C**** STAGNATION FLOW SCH.GT.100.
2 YG(1) = Y(2)
  YG(2) = Y(3)
  YG(3) = -((YL(3)*Y(1)+YL(1)*Y(3)-YL(1)*YL(3))/SCH+0.5*(1./S1+GR*Y(4
1)/S1+(YL(2)*YL(2)-2.*YL(2)*Y(2))/SCH))
  YG(4) = Y(5)
  YG(5) = -(Y(1)*YL(5)+YL(1)*Y(5)-YL(1)*YL(5))
  IF (X.GT.6.) YG(5) = 0.
  RETURN
C**** VERTICAL FLAT PLATE (FORCED LIMIT) SCH.LT.100.
3 YG(1) = Y(2)
  YG(2) = Y(3)
  YG(3) = -0.5*(YL(1)*Y(3)+Y(1)*YL(3)-YL(1)*YL(3))-GR*Y(4)
  YG(4) = Y(5)
  YG(5) = -0.5*SCH*(YL(1)*Y(5)+Y(1)*YL(5)-YL(1)*YL(5))
  RETURN
C**** VERTICAL FLAT PLATE (FORCED LIMIT) SCH.GT.100.
4 YG(1) = Y(2)
  YG(2) = Y(3)
  YG(3) = -0.5*(YL(1)*Y(3)+Y(1)*YL(3)-YL(1)*YL(3))/SCH-GR*Y(4)/S1
  YG(4) = Y(5)
  YG(5) = -0.5*(YL(1)*Y(5)+Y(1)*YL(5)-YL(1)*YL(5))
  IF (X.GT.10.) YG(5) = 0.
  RETURN
C**** VERTICAL FLAT PLATE (FREE LIMIT)
5 YG(1) = Y(2)
  YG(2) = Y(3)
  YG(3) = 4.*Y(2)*YL(2)-2.*YL(2)*YL(2)-3.*(-YL(1)*YL(3)+Y(1)*YL(3)+
1 Y(3)*YL(1))-Y(4)
  YG(4) = Y(5)
  YG(5) = -3.*SCH*(-YL(1)*YL(5)+Y(1)*YL(5)+YL(1)*Y(5))
  RETURN
END

```

```

SUBROUTINE YGRD2
DIMENSION YK(5,1000),YH(5),Y(5),YG(5),YD(5),YL(5)
COMMON /SET4/ YK,YH,Y,YG,YD,YL
COMMON /SET6/ SIG,SCH,S1,X,NHMG,GR,IERRORE,M,METHODQ,N
C**** HOMOGENEOUS EQUATIONS.
GO TO (1,2,3,4,5),METHODQ
C**** STAGNATION FLOW SCH.LT.100.
1 YG(1) = Y(2)
  YG(2) = Y(3)
  YG(3) = -(YL(3)*Y(1)+YL(1)*Y(3)+0.5*(GR*Y(4)-2.*YL(2)*Y(2)))
  YG(4) = Y(5)
  YG(5) = -SCH*(Y(1)*YL(5)+YL(1)*Y(5))
  RETURN
C**** STAGNATION FLOW SCH.GT.100.
2 YG(1) = Y(2)
  YG(2) = Y(3)
  YG(3) = -((YL(3)*Y(1)+YL(1)*Y(3))/SCH+0.5*(GR*Y(4)/S1-2.*YL(2)*Y(2)
1/SCH))
  YG(4) = Y(5)
  YG(5) = -(Y(1)*YL(5)+Y(5)*YL(1))
  IF (X.GT.6.) YG(5) = 0.
  RETURN
C**** VERTICAL FLAT PLATE (FORCED LIMIT) SCH.LT.100.
3 YG(1) = Y(2)
  YG(2) = Y(3)
  YG(3) = -0.5*(YL(1)*Y(3)+Y(1)*YL(3))-GR*Y(4)
  YG(4) = Y(5)
  YG(5) = -0.5*SCH*(YL(1)*Y(5)+YL(5)*Y(1))
  RETURN
C**** VERTICAL FLAT PLATE (FORCED LIMIT) SCH.GT.100.
4 YG(1) = Y(2)
  YG(2) = Y(3)
  YG(3) = -0.5*(YL(1)*Y(3)+Y(1)*YL(3))/SCH-GR*Y(4)/S1
  YG(4) = Y(5)
  YG(5) = -0.5*(YL(1)*Y(5)+Y(1)*YL(5))
  IF (X.GT.10.) YG(5) = 0.
  RETURN
C**** VERTICAL FLAT PLATE (FREE LIMIT)
5 YG(1) = Y(2)
  YG(2) = Y(3)
  YG(3) = 4.*Y(2)*YL(2)-3.*(Y(1)*YL(3)+Y(3)*YL(1))-Y(4)
  YG(4) = Y(5)
  YG(5) = -3.*SCH*(Y(1)*YL(5)+Y(5)*YL(1))
  RETURN
END

```

```

SUBROUTINE YGRD3
DIMENSION YK(5,1000),YH(5),Y(5),YG(5),YD(5),YL(5)
COMMON /SET4/ YK,YH,Y,YG,YD,YL
COMMON /SET6/ SIG,SCH,S1,X,NHMG,GR,IERROR,M,METHODQ,N
C**** NON-LINEAR EQUATIONS.
GO TO (1,2,3,4,5),METHODQ
C**** STAGNATION FLOW SCH.LT.100.
1 YG(1) = Y(2)
  YG(2) = Y(3)
  YG(3) = -Y(1)*Y(3)-0.5*(1.+GR*Y(4)-Y(2)*Y(2))
  YG(4) = Y(5)
  YG(5) = -SCH*Y(1)*Y(5)
  RETURN
C**** STAGNATION FLOW SCH.GT.100.
2 YG(1) = Y(2)
  YG(2) = Y(3)
  YG(3) = -Y(1)*Y(3)/SCH-0.5*((1.+GR*Y(4))/S1-Y(2)*Y(2)/SCH)
  YG(4) = Y(5)
  YG(5) = -Y(1)*Y(5)
  IF (X.GT.6.) YG(5) = 0.
  RETURN
C**** VERTICAL FLAT PLATE (FORCED LIMIT) SCH.LT.100.
3 YG(1) = Y(2)
  YG(2) = Y(3)
  YG(3) = -0.5*Y(1)*Y(3) - GR*Y(4)
  YG(4) = Y(5)
  YG(5) = -SCH*Y(1)*Y(5)*0.5
  RETURN
C**** VERTICAL FLAT PLATE (FORCED LIMIT) SCH.GT.100.
4 YG(1) = Y(2)
  YG(2) = Y(3)
  YG(3) = -0.5*Y(1)*Y(3)/SCH - GR*Y(4)/S1
  YG(4) = Y(5)
  YG(5) = -0.5*Y(1)*Y(5)
  IF (X.GT.10.) YG(5) = 0.
  RETURN
C**** VERTICAL FLAT PLATE (FREE LIMIT)
5 YG(1) = Y(2)
  YG(2) = Y(3)
  YG(3) = 2.*Y(2)*Y(2)-Y(4)-3.*Y(1)*Y(3)
  YG(4) = Y(5)
  YG(5) = -3.*SCH*Y(1)*Y(5)
  RETURN

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APPENDIX E.Integral Equations.

In Chapter 4, a number of modifications which were made to the integral method were discussed. In Section 4.5, the equations were presented for the integral method which employed a double velocity profile, and in which the equations over the region of the transfer boundary layer were scaled.

The velocity and concentration profiles which were employed are presented here together with the final equations which were used to calculate the stagnation point solution.

The boundary conditions which were used to calculate the profiles were presented in Chapter 4 as equations (4-95) to (4-97). The velocity and concentration profiles are as follows:

The inner velocity is:

$$u_i = \frac{1}{(1 - \chi^2)} \left\{ \bar{\beta} \frac{\delta Sc^{\frac{1}{3}}}{6} (1 - \chi^3) + \frac{U}{\delta} (2 - 3\chi^2 + \chi^3) \right\} y_i$$

$$- \frac{\bar{\beta}}{2} y_i^2 + \frac{1}{(1 - \chi^2)} \left\{ \bar{\beta} \frac{(1 - \chi)}{2\delta Sc^{\frac{1}{3}}} + \frac{U (-2 + 3\chi - \chi^2)}{\delta^3 Sc^{\frac{2}{3}}} \right\} y_i^3$$

(E-1)

The outer velocity is:

$$u_o = \frac{1}{(1 - \chi^2)} \left\{ \left(\frac{U (2 - 3\chi^2)}{\delta} + \bar{\beta} \frac{\delta Sc^{\frac{1}{3}}}{6} \right) y_o \right.$$

$$+ \left(\frac{3U\chi^2}{\delta^2} - \bar{\beta} \frac{Sc^{\frac{1}{3}}}{2} \right) y_o^2 + \left(\bar{\beta} \frac{Sc^{\frac{1}{3}}}{2\delta} - \frac{U (2 + \chi^2)}{\delta^3} \right) y_o^3$$

$$\left. + \left(\frac{U}{\delta^4} - \bar{\beta} \frac{Sc^{\frac{1}{3}}}{6\delta^2} \right) y_o^4 \right\}$$

(E-2)

The concentration profile is:

$$C = 1 - \frac{2y_i}{\chi \delta Sc^{1/3}} + \frac{2y_i}{\chi \delta^3 Sc} - \frac{y_i^4}{\chi \delta^4 Sc^{4/3}} \quad (E-3)$$

$$\text{where } \bar{\beta} = \frac{Gr}{Re^2 Sc^{1/3}} \sin \psi + \frac{U \bar{\epsilon}}{Sc^{1/3}} \frac{dU}{d\Theta} \quad (E-4)$$

These velocity and concentration profiles are substituted into the integral equations (4-100) and (4-101). The limiting equations for the front stagnation point solution are obtained after a considerable amount of algebra. The limiting equations which correspond to equations (4-89) and (4-90) for the original method are:

$$\begin{aligned} 0 = & \bar{\beta}_L \left\{ U_L Z^2 Sc^{1/3} (B(\chi) + G(\chi)) + \frac{2\bar{\beta}_L Z^3 Sc^{2/3}}{(1-\chi^2)^2} (C(\chi) + H(\chi)) \right. \\ & - \frac{U_L Z^2 Sc^{1/3}}{(1+\chi)} (E(\chi) + J(\chi) / Sc^{1/3}) \left. \right\} \\ & + U_L' \bar{\epsilon} \left\{ 2U_L Z (A(\chi) + F(\chi)) + \frac{\bar{\beta}_L Z^2 Sc^{1/3}}{(1-\chi^2)^2} (B(\chi) + G(\chi)) \right. \\ & - \frac{UZ}{(1+\chi)} (D(\chi) + L(\chi) / Sc^{1/3}) \left. \right\} \\ & + \bar{\epsilon} \left\{ \frac{(G_1 + G_3/Sc)}{(1-\chi^2)^2} - \frac{U_L (G_2 + G_4/Sc)}{(1+\chi)} \right\} \\ & - \left\{ \frac{Gr}{Re^2 Sc^{1/3}} \frac{3\chi Z Sc^{1/3}}{10} + U_L U_L' Z - \frac{\bar{\beta}_L Z Sc^{1/3}}{(1-\chi^2)} \left\{ \frac{\bar{\beta}_L Z Sc^{1/3}}{6} (1-\chi^3) \right. \right. \\ & \left. \left. + U_L (2 - 3\chi^2 + \chi^3) \right\} \right\} \quad (E-5) \end{aligned}$$

$$\begin{aligned} 0 = & \frac{2}{\chi Sc^{1/3}} - \left\{ \bar{\beta}_L' Z Sc M(\chi) + U_L' Z Sc^{2/3} N(\chi) \right. \\ & \left. + \frac{\bar{\epsilon}}{(1-\chi^2)^2} \left(\bar{\beta}_L Z^2 Sc M(\chi) + U_L Z Sc^{2/3} N(\chi) \right) \right\} \quad (E-6) \end{aligned}$$

$$\text{where } \bar{\beta}_L = \frac{Gr}{Re^2 Sc^3} + U_L U_L' \bar{\epsilon} \quad (E-7)$$

$$\text{and } \bar{\beta}_L' = (d\bar{\beta}/d\Theta)_L \quad (E-8)$$

$$G_1 = U_L^2 Z A(\chi) + \bar{\beta}_L U_L Z^2 Sc^{\frac{1}{3}} B(\chi) + \bar{\beta}_L^2 Z^3 Sc^{\frac{2}{3}} C(\chi) \quad (E-9)$$

$$G_2 = U_L Z D(\chi) + \bar{\beta}_L Z^2 Sc^{\frac{1}{3}} E(\chi) \quad (E-10)$$

$$G_3 = U_L^2 Z Sc F(\chi) + \bar{\beta}_L U_L Z^2 Sc^{4/3} G(\chi) + \bar{\beta}_L^2 Z^3 Sc^{5/3} H(\chi) \quad (E-11)$$

$$G_4 = \bar{\beta}_L Z^2 Sc J(\chi) + U_L Z Sc^{\frac{2}{3}} L(\chi) \quad (E-12)$$

and where $A(\chi)$ to $N(\chi)$ are polynomials in χ and are:

$$\begin{aligned} A(\chi) = & 0.582 - 1.221\chi^2 + 1.333\chi^3 + 2.642\chi^4 + 5.6\chi^5 \\ & - 3.666\chi^6 - 5.171\chi^7 + 7.75\chi^8 - 4.539\chi^9 - 0.75\chi^{10} - 0.142\chi^{11} \end{aligned} \quad (E-13)$$

$$\begin{aligned} B(\chi) = & 0.009 - 0.0107\chi^2 - 0.222\chi^3 + 0.5\chi^4 + 0.066\chi^5 - 1.277\chi^6 \\ & + 1.695\chi^7 - 1.04\chi^8 + 0.322\chi^9 - 0.041\chi^{10} \end{aligned} \quad (E-14)$$

$$\begin{aligned} C(\chi) = & 0.0001 - 0.009\chi^3 + 0.041\chi^4 - 0.083\chi^5 - 0.074\chi^6 \\ & - 0.059\chi^7 - 0.020\chi^8 - 0.003\chi^9 \end{aligned} \quad (E-15)$$

$$D(\chi) = \frac{1}{20} (14 + 14\chi - 21\chi^2 - 21\chi^3 + 19\chi^4 - 5\chi^5) \quad (E-16)$$

$$E(\chi) = \frac{1}{120} (1 + \chi - 9\chi^2 + 11\chi^3 - 4\chi^4) \quad (E-17)$$

$$\begin{aligned} F(\chi) = & 1.333\chi^3 - 5.6\chi^5 + 3.733\chi^6 + 5.171\chi^7 - 8.114\chi^8 \\ & + 4.590\chi^9 - 0.457\chi^{10} + 0.142\chi^{11} \end{aligned} \quad (E-18)$$

$$\begin{aligned} G(\chi) = & 0.222\chi^3 - 0.5\chi^4 - 0.066\chi^5 + 1.272\chi^6 - 1.452\chi^7 \\ & + 0.814\chi^8 - 0.832\chi^9 + 0.042\chi^{10} \end{aligned} \quad (E-19)$$

$$\begin{aligned} H(\chi) = & 0.009\chi^3 - 0.25\chi^4 + 0.08\chi^5 + 0.114\chi^6 + 0.2\chi^7 \\ & + 0.062\chi^8 - 0.205\chi^9 \end{aligned} \quad (E-20)$$

$$J(\chi) = \frac{\chi^2}{6} \left(1 - \chi + \frac{\chi^2}{2} \right) \quad (\text{E-21})$$

$$L(\chi) = \chi^2 \left(1 + \chi - \chi^2 + \frac{\chi^3}{4} \right) \quad (\text{E-22})$$

$$M(\chi) = \chi^2 \left(1/90 + \chi \left(-0.023 + \chi (3/560 + 0.006\chi) \right) \right) \quad (\text{E-23})$$

$$N(\chi) = \chi^2 \left(2/9 + \chi^2 \left(-0.221 + \chi (0.098 - 3\chi/280) \right) \right) \quad (\text{E-24})$$

These equations were used to calculate the stagnation point solutions which are plotted in figure 6.60. It is clear from the limiting equations that a considerable amount of extra work is involved when the modified integral method is used. The two ordinary differential equations themselves would be considerably more cumbersome.

A large number of forms of the velocity and concentration profiles were used in the integral equations to attempt to improve the range of values of the parameter Gr/Re^2 over which the approximate method could be used. Some of these are presented below for the axisymmetric flow body, and the vertical flat plate. The boundary conditions are those discussed in Chapter 4.

Vertical Flat Plate:

A

Quintic Concentration Profile:

$$C = 1 - \frac{5y}{12\Delta} + \frac{5y^3}{6\Delta^3} - \frac{5y^4}{6\Delta^4} + \frac{y^5}{4\Delta^5} \quad (\text{E-25})$$

Quintic Velocity Profile:

$$u = \left(\frac{5}{5} + \frac{3}{20\delta} \right) y - \frac{y^2}{2} + \left(\frac{23}{40\delta} - \frac{7}{10\delta^3} \right) y^3 + \left(\frac{4}{5\delta^4} - \frac{3}{10\delta^2} \right) y^4 + \left(\frac{1}{16\delta^3} - \frac{1}{4\delta^5} \right) y^5 \quad (\text{E-26})$$

where the additional boundary condition was:

$$\frac{\partial^3 u}{\partial y^3} = 0 \text{ at } y = 0 \quad (\text{E-27})$$

B

Quartic Concentration Profile:

$$C = 1 - \frac{2y}{\Delta} + \frac{2y^3}{\Delta^3} - \frac{y^4}{\Delta^4} \quad (\text{E-28})$$

Sextic Velocity Profile:

$$u = \left(\frac{\delta}{5} + \frac{2}{\delta} \right) y - \frac{y^2}{2} + \left(\frac{1}{\delta^2} - \frac{5}{\delta^4} \right) y^4 + \left(-\frac{1}{\delta^3} + \frac{6}{\delta^5} \right) y^5 + \left(\frac{3}{10\delta^4} - \frac{2}{\delta^6} \right) y^6 \quad (\text{E-29})$$

Where the additional boundary conditions were:

$$\frac{\partial^3 u}{\partial y^3} = 0 \text{ at } y = 0 \text{ and at } y = \delta \quad (\text{E-30})$$

C

Quartic Velocity Profile:

$$u = \frac{2y}{\delta} - \frac{2y^3}{\delta^3} + \frac{y^4}{\delta^4} + \frac{\delta^2 y}{6} \left(1 - \frac{3y}{\delta} + \frac{3y^2}{\delta^2} - \frac{y^3}{\delta^3} \right) \quad (\text{E-31})$$

Sextic Concentration Profile:

$$C = 1 - \frac{2y}{\Delta} + \frac{5y^4}{\Delta^4} - \frac{6y^5}{\Delta^5} + \frac{2y^6}{\Delta^6} \quad (\text{E-32})$$

D

Double Velocity Profile:

Simple inner velocity profile, over the transfer boundary layer:

$$u_i = \frac{1}{3\delta} \left[(6 - 4\chi + \chi^2) + \frac{\delta}{6} (1 + 2\chi) \right] y - \frac{y^2}{2} \quad (\text{E-33})$$

Quartic Outer Velocity Profile:

$$u_{\text{outer}} = \left(2\delta - \frac{16\Delta}{3} + \frac{3\Delta^2}{\delta} + \frac{\delta^3}{6} \right) \frac{y}{(\delta - \Delta)^2} + \left(\frac{4}{\delta} - \frac{3\Delta^2}{\delta^2} - \frac{\delta^2}{2} \right) \frac{y^2}{(\delta - \Delta)^2} + \left(-\frac{2}{\delta} + \frac{\Delta^2}{\delta^3} + \frac{\delta}{2} \right) \frac{y^3}{(\delta - \Delta)^2} + \left(\frac{1}{\delta^2} - \frac{2\Delta}{3\delta^3} - \frac{1}{6} \right) \frac{y^4}{(\delta - \Delta)^2} \quad (\text{E-34})$$

Cubic Concentration Profile:

$$C = 1 - \frac{3}{2} \frac{y}{\Delta} + \frac{y^3}{2\Delta^3} \quad (\text{E-35})$$

E

Double Profile, Thin Transfer Boundary Layer;

Quartic Concentration:

$$C = 1 - \frac{2y}{\Delta} + \frac{2y^3}{\Delta^3} - \frac{y^4}{\Delta^4} \quad (\text{E-36})$$

Calculate the inner velocity profile from the boundary condition, $\delta^2 u / dy^2 = -C$ (E-37), very close to the surface.

$$u_i = \frac{1}{3\delta} \left[(\kappa^2 - 4\kappa + 6) + \frac{\delta^3}{90} (19\kappa + 2) \right] y - \frac{y^2}{2} + \frac{y^3}{3\Delta} - \frac{y^5}{10\Delta^3} + \frac{y^6}{30\Delta^4} \quad (\text{E-38})$$

Quartic Outer Velocity Profile:

$$u_{\text{outer}} = \left(2\delta + \frac{\delta^3}{45} + \frac{3\Delta^2}{\delta} - \frac{16\Delta}{3} \right) \frac{y}{(\delta - \Delta)^2} + \left(\frac{4\Delta}{\delta} - \frac{3\Delta}{\delta^2} - \frac{\delta^2}{15} \right) \frac{y^2}{(\delta - \Delta)^2} + \left(\frac{\Delta^2}{\delta^3} - \frac{2}{\delta} + \frac{\delta}{15} \right) \frac{y^3}{(\delta - \Delta)^2} + \left(\frac{1}{\delta^2} - \frac{1}{45} - \frac{2\Delta}{3\delta^3} \right) \frac{y^4}{(\delta - \Delta)^2} \quad (\text{E-39})$$

Body of Revolution.

F

Linear Velocity Profile:

$$u = \frac{\delta^2 \sin \psi + U\lambda + 12U}{6\delta} \quad (\text{E-40})$$

Quartic Concentration Profile:

$$C = 1 - \frac{2y}{\Delta} + \frac{2y^3}{\Delta^3} - \frac{y^4}{\Delta^4} \quad (\text{E-41})$$

G

Two velocity profiles, scaled equations over the region of the transfer boundary layer:

$$u_{\text{inner}} = \left(\frac{\sqrt{g}\Delta}{4} + \frac{3U}{\delta} - \frac{3U\Delta}{2\delta^2 Sc^{1/3}} \right) y_i - \frac{\sqrt{g}y_i^2}{2} + \left(\frac{\sqrt{g}}{4} - \frac{3U}{2\Delta\delta^2 Sc^{1/3}} + \frac{U}{\delta^3 Sc^{1/3}} \right) y_i^3 \quad (\text{E-42})$$

$$u_{\text{outer}} = \frac{3Uy}{\delta} - \frac{3Uy^2}{\delta^2} + \frac{Uy^3}{\delta^3} \quad (\text{E-43})$$

Quartic Concentration Profile:

$$C = 1 - \frac{2y_i}{\Delta} + \frac{2y_i^3}{\Delta^3} - \frac{y_i^4}{\Delta^4} \quad (\text{E-44})$$

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