

Computational Study of the Stability of Annular Couette-Poiseuille Flow to Axisymmetric Travelling Wave Perturbations

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To my family.

Abstract

This thesis considers the axisymmetric stability of a flow, subject to an axial pressure gradient, between concentric sliding cylinders, or annular Couette-Poiseuille flow (ACPF). The flow geometry and governing equations are set out and we analyse the linear and nonlinear stability of ACPF for a range of inner cylinder velocity, radius ratio and disturbance amplitude. We establish that the flow is inviscidly stable and introduce an axisymmetric travelling wave disturbance to the flow. We find that ACPF is linearly unstable and comparisons are made with published work such as those of Walton (2004) and Webber (2008). Using a continuation method, the linearly unstable solutions provide a starting point to compute finite-amplitude, nonlinear axisymmetric travelling wave solutions. These nonlinear solutions also exist in the wavenumber-Reynolds number, or neutral curve, space where linear stability predicts no solutions. The nonlinear travelling wave solutions result in lower critical Reynolds number and higher critical inner cylinder velocity compared to that of linear stability predictions. The nonlinear wave disturbance influence is strongest near the inner and outer cylinder walls and the mean flow distortion has the general effect of slowing the basic flow velocity. We also present the existence of multiple neutral curves in nonlinear space and these multiple neutral curves appear to unite as the disturbance amplitude increases. In an effort to link our results to Hagen-Poiseuille flow (HPF), we analyse the special case of ACPF where the maximum velocity of the flow is always achieved on the inner cylinder and find that this flow is linearly stable.

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Andrew W. H. Wong

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Chapter 1

Introduction

Understanding the behaviour of fluid through a circular pipe, or pipe flow, is a long standing fluid dynamics problem that is unresolved and an active area of research. Fluid flow through a pipe subject to an axial pressure gradient is also known as Hagen-Poiseuille flow (HPF), so named in honour of the two researchers, Gotthilf Hagen and Jean Poiseuille, who independently conducted experimental work on the stability of fluid flow through circular pipes in 1839 and 1840 respectively. Hagen (1839) discovered laminar (ordered flow in space and time) and turbulent pipe flows (disordered flow in space and time) through his experiments whilst Poiseuille (1840) found laminar pipe flow (ordered flow in space and time). Although both Hagen and Poiseuille conducted experiments into the stability of flow between circular pipes, their findings differed due to the diameters of pipes used in their respective experiments - Hagen had used larger-diameter pipes and managed to observe both laminar and turbulent flows whilst Poiseuille used smaller-diameter pipes and observed only laminar flow. The link between laminar and turbulent flows through circular pipes was made when Osborne Reynolds conducted his famous experiment in 1883. Reynolds' experiment involved introducing a fluid containing dye, through trumpet mouthpieces, into pipes of three different diameters. Trumpet mouthpieces were used to reduce or remove disturbances in the fluid prior to entering the pipes. The fluid would be introduced into the pipes for various velocities and the dye would allow the fluid behaviour in the pipe to be observed. Reynolds observed that for low velocities, the dye would form a smooth and straight line through the pipe (laminar flow). The onset of instability in the fluid was noted

when the fluid was at a sufficiently high velocity; as fluid velocity was increased, the flow would undergo transition from laminar to turbulent flow and the transition between laminar and turbulent flow was observed by way of turbulent “spots”. These turbulent spots only materialised downstream of the inlet and were found to exist intermittently and were separated by regions of laminar flow. In addition, Reynolds also found that if the fluid introduced in the pipe was “unsettled” sufficiently, then a lower fluid velocity would generate instability in the flow. Reynolds experiments also suggested that the fluid itself may inherently possess perturbations prior to being introduced to the pipe which may cause the onset of instability to be observed at lower flow velocities than anticipated. Reynolds’ had proposed that a non-dimensional (Reynolds, $R = UD/\nu$) number as a function of pipe diameter (D), mean fluid flow (U) and kinematic viscosity (ν) of the fluid would be able to categorise whether a fluid in a circular pipe is laminar, transitional, or turbulent flow.

1.1 Pipe Flow - Experimental

Experimental studies such as Wygnanski & Champagne (1973) have shown transition between laminar to turbulent flow is at $R \approx 2,000$ but Darbyshire & Mullin (1995) noted that the flow transitions at $R \approx 17,000$ and this discrepancy has been the focus of many further studies. In particular, Wygnanski et al. (1973, 1975) conducted pressure-driven circular pipe flow experiments and found ‘slugs’ and ‘puffs’, broadening the understanding of turbulence and flow transition. The experiment of Wygnanski et al. (1973) involved introducing disturbances at the inlet of the pipe which triggered transition in the flow. A slug was defined as a turbulent region that would occupy the cross section of the pipe and would have leading and trailing fronts with turbulent flow in between these fronts. By contrast, a puff has a leading front but no defined trailing edge and the turbulence dissipates, but does not fully disappear, downstream. The discovery of slugs and puffs demonstrated the intricacies of transition and turbulence in the pipe flow. Darbyshire & Mullin (1995) took a different experimental approach to investigating the stability of flow in a circular pipe by using a constant mass flux on the flow instead of a constant pressure gradient driven flow that Reynolds, Hagen, Poiseuille and Wygnanski et al. investigated. Darbyshire & Mullin disturbed the flow through an injection of mass which gave a measure of distur-

bance ‘amplitude’. They found that an amplitude threshold could be achieved such that disturbance amplitudes above this threshold would cause transition from laminar to turbulent flow in the pipe, and disturbance amplitudes below this threshold would not result in turbulent pipe flow and the disturbances would decay downstream. The studies by Wygnanski et al. (1973, 1975) and Darbyshire & Mullin (1995) clearly illustrate the importance of disturbance amplitude on the stability of HPF.

Draad et al. (1998) showed that pipe flow is not only sensitive to disturbance amplitude but also to the frequency of the disturbance applied to the flow. First, they determined the transition Reynolds number of pipe flow with no external disturbances introduced. They then applied disturbances to the flow using a mechanism comprising of two oscillating syringes positioned directly across each other; one syringe would inject whilst the other would extract fluid from the pipe, suggesting that the disturbance is of azimuthal wavenumber of one ($N = 1$). The apparatus set-up also allowed for control over the frequency and magnitude of the disturbance through the injection and extraction of fluid from the pipe, as a function of maximum piston velocity. Draad et al. discovered multiple transition points for a constant disturbance frequency (or non-dimensional wavenumber) beyond $R \approx 30,000$ where the flow would undergo transition from laminar to turbulent, back to laminar, and then back to turbulent flow if the disturbance introduced was increased sufficiently. Also in 1998, Eliahou et al. (1998) conducted an interesting experiment on pipe flow where the disturbances were introduced to the flow for azimuthal wavenumbers 2, -2 and 4 for three different disturbance amplitudes settings. Their experiment illustrated the correlation between azimuthal and amplitude disturbances on the stability of HPF, that is, a lower disturbance amplitude is required to cause transition for HPF with a higher azimuthal number. The findings of Draad et al. (1998) and Eliahou et al. (1998) are examples of the importance of disturbance amplitude and that azimuthal disturbances play a critical role in the stability of HPF.

1.2 Pipe Flow - Theoretical

Having discussed some of the experimental studies on HPF, let us turn our attention to the theoretical studies on the stability of HPF. Lord Rayleigh (1892) established that HPF

was inviscidly stable to infinitesimal disturbances (linearly stable). In 1927, Searl took viscosity into account and found that HPF continued to be linearly stable to axisymmetric disturbances at high Reynolds number. With the improved performance of digital computers, Davey & Drazin (1969)'s numerical calculations showed that HPF is indeed linearly stable to all axisymmetric perturbations, at all Reynolds numbers. Salwen & Grosch (1972)'s numerical analysis of the linear stability of HPF supported the findings of Davey & Drazin (1969) that HPF is linearly stable to all axisymmetric perturbations. However, Salwan & Grosch (1972) also found that HPF was linearly stable to all azimuthal variations ($N = 1 - 5$) simulated. These are merely a handful of studies which have shown that HPF is linearly stable to axisymmetric and non-axisymmetric disturbances. Given that we know, from experimental data, that HPF can and indeed does undergo transition from laminar to turbulent flow, theoretical and numerical investigations looked towards other factors which could generate instability in HPF such as nonlinearity and disturbance amplitude. The approach of weakly nonlinear stability as applied to HPF for axisymmetric disturbances was undertaken by Davey & Nguyen (1971) who suggested instability could be found for HPF. However, a study by Itoh (1977) contradicted the findings of Davey & Nguyen (1971) and this direct contradiction did not inspire confidence in the weakly nonlinear theory approach.

Smith & Bodonyi (1982) took a critical layer theory approach to investigate the nonlinear stability of HPF. They looked at HPF at large Reynolds number subjected to a disturbance in three dimensions, for the specific case of azimuthal wavenumber of $N = 1$. They discovered neutral stability to finite-amplitude disturbances in HPF for $N = 1$, suggesting that this solution could only be found for $N = 1$, and Walton (2001) subsequently verified that the solution only exists for $N = 1$. Walton (2002) demonstrated that the solution found by Smith & Bodonyi (1982) could be achieved from the linear instability of impulsively started HPF. These theoretical investigations suggested that azimuthal variation and disturbance amplitude could lead to instability in HPF.

More recently, Faisst & Eckhardt (2003) and Wedin & Kerswell (2004) conducted numerical studies into the stability of HPF and discovered nonlinear travelling waves which has, at the heart of it, the self-sustaining process first proposed by Waleffe (1995), albeit for Couette flow. Guided by the direct numerical simulations (DNS) of Hamilton, Kim & Waleffe (1995), the self-sustaining process (SSP) proposed by Waleffe (1995) offered a

different approach to understanding the stability of shear flow and these ideas have since been used in flows in other geometries such as HPF. Waleffe (1995) suggested that there are three main components to the SSP: streaks, rolls and waves. The SSP can be described as follows: the radial and azimuthal components (rolls) transfer energy to the axial component (streaks), and the streaks in turn transmit energy to the nonlinear disturbance wave. Finally, the wave drives the rolls via the nonlinear wave interaction. Waleffe (1995) had backed up the proposal of rolls, streaks and waves with numerical calculations which added weight to the theory being proposed. These nonlinear steady state solutions consisting of streaks, rolls and waves were termed exact coherent structures by Waleffe (2001). In 1997, Waleffe applied SSP to plane Couette flow and demonstrated that SSP is insensitive to either no-slip or free-slip boundary conditions being applied. Waleffe (1997) also suggested that the SSP was the cause for low Reynolds number transition and for high Reynolds number, SSP plays a part in controlling turbulent flow in the near-wall region. An alternative to the SSP energy transfer mechanism is a nonlinear vortex-wave interaction argument with one of the more recent findings presented by Hall & Sherwin (2010) for plane Couette flow whereby energy transfer was found between vortex and nonlinear travelling wave solutions. These cylindrical geometry solutions all take into consideration asymmetric disturbances which form the basis of energy transfer between the various flow components and the nonlinear travelling wave solutions.

1.3 Other Types of Shear Flows

So far, we have focussed on studies into HPF but as we have seen with Waleffe's self-sustaining process, there is much to be acquired from studies of shear flows. We now turn our attention to the theoretical studies of Plane Couette and Plane Poiseuille flows as these are the most relevant shear flows to this thesis. For a basic flow $U(y)$ between two planes in a domain $-1 \leq y \leq 1$, a plane Couette flow is defined as $U(y) = y$ and a plane Poiseuille flow as $U(y) = 1 - y^2$ (Drazin (2002)). Let us first look at the theoretical studies for Plane Couette flow. It can be shown that Plane Couette flow is inviscidly stable (see Drazin & Reid (1981)) by employing Rayleigh's stability criterion of requiring the second derivative of the basic flow to change sign somewhere between the planes. All linear stability studies

indicate that plane Couette flow is linearly stable (Drazin & Reid (1981)) for all Reynolds number. Nagata (1990) suggested that instability may exist in plane Couette flow for a finite disturbance amplitude and used the idea of “homotopy” to uncover these neutral states. Nagata’s study of finding nonlinear plane Couette flow solutions from rotating plane Couette flow illustrates the concept of homotopy very well. The starting point for Nagata was to obtain an axisymmetric finite-amplitude solution of a flow between rotating cylinders with a narrow gap for a viscous incompressible fluid. Nagata took the solutions, which were called “Taylor vortex flow” (TVF) solutions, and superimposed three dimensional perturbations on the TVF solutions, noting that Squire’s theorem did not apply to the rotating system being resolved. The resulting three dimensional problem was solved numerically by Nagata by taking the specific case of an average rotation rate of zero between the rotating cylinders, Nagata discovered solutions in plane Couette flow for finite-amplitude disturbances. Faisst & Eckhardt (2000) extended Nagata’s (1990) analysis slightly to the full Taylor-Couette flow problem and came to the same conclusion as Nagata did, confirming that nonlinear finite-amplitude neutral states could be found using homotopy in plane Couette flow. Further evidence of successful homotopy studies include Clever & Busse (1992) from Rayleigh-Bernard convection to plane Couette flow and Nagata (1998), again for the rotating plane Couette flow to plane Couette flow but also discovering so called tertiary neutral solutions of three dimensions and with non-zero streamwise wavenumber. In view of the rather successful application of homotopy to mapping out three dimensional finite-amplitude solutions relating to plane Couette flow, Barnes & Kerswell (2000) conducted a numerical study into the stability of rotating Hagen-Poiseuille flow with a view to finding nonlinear solutions in HPF but were unsuccessful, confirming the findings of Toplosky & Akylas (1988).

Plane Poiseuille flow by contrast is unstable to infinitesimal disturbances, and the critical Reynolds number is $R_c \approx 5,772$ (Orzag (1971), Drazin (2002)). However, experimental evidence found by Patel & Head (1969) found a minimum $R_c \approx 1,000$ suggesting that nonlinearity and finite-amplitude disturbances have a role to play in bridging the gap between experimental and theoretical results. Incidentally, Patel & Head’s experimental investigation looked at both pipe and channel flows driven by a pressure gradient and highlighted significant differences between these flows such as the difference in critical

Reynolds number. They also drew comparisons with the (empirical) power law widely used in engineering but fundamentally flawed since no nonlinearity, finite-amplitude disturbance or other physical science is taken into consideration. Studies by Herbert (1988) and Bayly et al. (1988) showed promising signs that finite-amplitude disturbances reduced critical Reynolds number in plane Poiseuille flow but differences remained with experimental results. Reddy et al. (1998) investigated the effects of streak instability for plane Poiseuille and plane Couette flows. They also highlighted, through an energy argument, that energy required for transition in plane Poiseuille flow from streaks was two orders of magnitude lower than that predicted by nonlinear travelling waves. Waleffe (2003) computed three dimensional exact coherent structures from two dimensional steady states using Newton's iterative procedure. Waleffe (2003) then went on to use homotopy for both plane Couette and plane Poiseuille flows with different boundary conditions (free-free, rigid-free, rigid-rigid) and found steady state solutions with relative ease, echoing the findings of Waleffe (1997, 2001). These studies into plane Poiseuille flow all point towards the fact that nonlinearity has a very important role to play in determining the stability of the flow.

Another flow that has recently been studied is annular sliding Couette flow, or sometimes referred to simply as sliding Couette flow. This is the flow through concentric cylinders, assumed to be of infinite length with a sliding inner cylinder. One of the early studies into sliding Couette flow was conducted by Gittler (1993) as a unique case of annular Poiseuille-Couette flow, where the pressure gradient is removed, resulting in pure annular sliding Couette flow. Gittler (1993) found that sliding Couette flow is linearly stable to axisymmetric disturbances and identified a radius ratio (inner to outer radius ratio) of 0.1415 above which no instability is predicted. This result was subsequently confirmed by Deguchi & Nagata (2011). They went further and extended Gittler's (1993) analysis to include non-axisymmetric and nonlinear perturbations and found that finite-amplitude axisymmetric steady state solutions exist below the critical Reynolds number found by Gittler for radius ratio of 0.1415. Deguchi & Nagata (2011) also concluded that nonlinear and non-axisymmetric travelling wave solutions can be found for radius ratios greater than 0.1415 where linear analysis had previously predicted no instability. In order to uncover the existence of instability in nonlinear space (nonlinear solutions in the parameter regime) where linear stability predicts none, Deguchi & Nagata employed the homotopy approach

for three dimensional plane Couette flow as in Nagata (1990) and pushed radius ratio to a narrow-gap of unity before continuing the solution to the case of radius ratio below unity.

1.4 Annular Couette-Poiseuille Flow

The case of annular Couette Poiseuille flow, a pressure driven gradient flow between concentric cylinders, is also referred to as thread-annular flow in relation to a potentially promising medical implantation technique. Numerical results for ACPF have largely been presented in the form of axial wavenumber - Reynolds number results. Earlier research into this area include studies conducted by Mott & Joseph (1968) and Sadeghi & Higgins (1991) concluding that ACPF was linearly unstable given the right conditions for the axisymmetric disturbances in the former and both axisymmetric and non-axisymmetric disturbances in the latter, both noting that a unique neutral curve existed for a given set of ACPF conditions such as inner cylinder velocity and radius ratio. Gittler (1993) confirmed that under the right conditions, it was possible to find multiple neutral curves, which contradicted the findings of Sadeghi & Higgins who found the existence of unique neutral curves only for a given set of ACPF conditions. However, subsequent computational investigations by Walton (2004), Webber (2008) and most recently Wong & Walton (2012), have confirmed beyond doubt that multiple neutral curves can and do exist in the same axial wavenumber - Reynolds number, (α, R) , space.

In 2000, Frei et al. conducted experiments simulating ACPF, with medical implantation in mind. The experiments were conducted using various outer cylinder diameters with a rubber monofilament (simulating thread injection) as the inner cylinder with a pressure gradient driving the thread and a motor controlling the velocity of the thread. Observations of the thread behaviour were made using a video camera filming the cross-section, downstream, of the pipe. Strain gauges were placed on a beam which recorded the axial force applied on the thread with mass flow measure at the exit of the pipe. In a similar manner to Patel & Head (1969), Frei et al. (2000) plotted graphs of friction factor against Reynolds number, drawing comparisons with empirical power laws noting that for narrow-gap (large radius ratios), there was more discrepancy than for a larger-gap for the case of thread velocity of zero. For the case of non-zero thread velocity, there was more discrepancy in friction

factor (compared against engineering power laws) for the small-gap tests than that of the larger-gap scenario.

From our discussion earlier on the experiments conducted into HPF, we note that Frei et al.'s apparatus did not include a settling-chamber to reduce (or even remove) disturbances in the fluid prior to introduction into the pipe nor is there any kind of “tripping” mechanism to force transition earlier from laminar to turbulent flow. There is also an underlying assumption that fully developed flow is achieved between the thread and pipe downstream of the pipe inlet but no figures or illustrations have been offered as evidence of fully developed flow being reached by Frei et al. - perhaps (in future) a dye could be introduced into the fluid when entering the pipe, with the video camera recording the flow behaviour. As noted by Frei et al., the eccentricity of the thread relative to the pipe walls is one other factor that has an effect on the experimental results, and indeed on theoretical predictions, of ACPF and from figure 8 of Frei et al. (2000), it is very clear that the rubber thread was not concentric to the pipe. With the potential of applying ACPF to medical implants and oil pipelines, Frei et al. (2000)'s discovery of discrepancies between theoretical predictions and experimental data for ACPF sparked renewed interest into the analysis of ACPF.

Walton (2003) took an asymptotic approach to ACPF at high Reynolds number which resulted in better agreement with Frei et al.'s (2000) experimental findings, although discrepancies still existed between linear stability predictions of ACPF and Frei et al.'s experimental results. Walton (2004) conducted a more comprehensive review of the linear stability of ACPF subject to axisymmetric disturbance using both asymptotic and numerical techniques to uncover many hidden and complex areas on the linear stability of ACPF. One of the differences highlighted between the linear analysis by Walton (2004) and experimental results of Frei et al. was that critical Reynolds number differed by a factor of approximately 1×10^1 , suggesting that nonlinear effects such as disturbance amplitude could help reconcile this difference. Webber (2008) suggested that a “slip-length” on the boundary conditions would have a stabilizing effect on the critical Reynolds number for ACPF and rotation on the inner cylinder has a destabilizing impact on ACPF. It is worth pointing out that the vast majority of research conducted typically determine the stability conditions for a given disturbance amplitude, or determine the disturbance amplitude as part of the solution for given conditions such that the flow is marginally stable.

1.5 Outline of Thesis

In this thesis, we aim to bridge the gap between the experimental findings of Frei et al. (2000) and theoretical predictions of ACPF by introducing nonlinear influences to ACPF, such as a mean flow distortion, and assess the impact of nonlinear effects on the stability of ACPF. We also introduce disturbance amplitude as a variable such that for each disturbance amplitude, the marginal stability of ACPF is determined. We would like to see if the effects of nonlinearity bring us closer to the experimental findings of Frei et al. In addition to the traditional axial wavenumber - Reynolds number neutral curves at finite amplitudes, we also include contours and velocity profiles across the annulus which aim to expose some of the detail in relation to how the flow is affected by nonlinearity and disturbance amplitude in order to expose some of the detail in relation to the effects of nonlinearity and disturbance amplitude. We illustrate the relationship and effects of nonlinearity to ACPF through critical disturbance amplitude, Reynolds number and inner cylinder velocity associations and use the linearly unstable ACPF as our starting point for nonlinear analysis of ACPF. As we will see later in this thesis, one of the key results is the existence of instability in nonlinear space where linear analysis predicts no instability.

The outline of the remainder of this thesis is as follows.

- We begin with the governing Navier-Stokes equations to a general annular flow problem and introduce an axisymmetric perturbation to obtain a set of nonlinear equations governing the evolution of the disturbances. We then present a graphical representation of ACPF and apply the ACPF model to our equations to derive the basic flow profile, noting that the pressure gradient is Reynolds number dependent, and introduce the no-slip boundary conditions.
- Linear stability of ACPF allows some simplification of the generic governing equations for ACPF. We simplify the equations, assuming small disturbance amplitude, and obtain the well known Orr-Sommerfeld equation and present an inviscid stability argument to demonstrate that instability in ACPF arises from viscous effects. We proceed to give an account of the computational method that we utilise to solve the ACPF/Orr-Sommerfeld problem and present our results and summarise the findings

on the linear stability of ACPF.

- We then assume that the disturbance amplitude is no longer “small” and derive another set of equations, which are fully nonlinear in nature and includes a mean flow distortion to the basic flow. In order to solve the nonlinear ACPF problem, we deploy a Newton-Raphson technique and provide an account of this, including some detail into the Jacobian matrix of derivatives. Having established the nonlinear equations and computational techniques, we present the results for a limited number of ACPF conditions as the ACPF-space is far too wide to encompass completely. The conditions of ACPF which we review are confined to the following cases where (i) linear stability predicts a minimum critical Reynolds number where radius ratio is approximately 0.5, (ii) multiple neutral curves exist for the same ACPF conditions, and (iii) a closed neutral curve exists and eventually shrinks to a point. These conditions of ACPF are of great interest and builds on the linear stability findings. We also discover instability which exists solely in nonlinear space, requiring nonlinear solutions to jump-start the computational process.
- We then conduct a full discussion on the linear and nonlinear stability of ACPF, drawing comparisons and conclusions on the influences of the disturbance and mean flow distortion to ACPF. In an attempt to connect the current ACPF research with a pressure driven flow in a circular pipe, or annular Hagen-Poiseuille flow, we analyse the linear stability of a special case of ACPF where the basic flow reaches a maximum on the inner cylinder for any given radius ratio. We call this special case of ACPF sliding annular flow and report that no instability was found for the set of conditions specified.
- Finally, we draw conclusions and offer suggestions on potential avenues of further research.

Some of the ACPF research and findings contained in this thesis have recently been published in the Quarterly Journal of Mechanics and Applied Mathematics, see Wong & Walton (2012).

Chapter 2

Governing Equations for Pressure-Driven Annular Couette Poiseuille Flow (ACPF)

For our annular Couette-Poiseuille flow, it is convenient to work in the cylindrical polar coordinate system, (ax, ar, θ) , where a is the radius of the outer cylinder, x is the axial component, r is the radial component and θ is the azimuthal angle about the x -component of the system. Before stating the Navier-Stokes (NS) equations, we confine our analysis to a flow with no swirl. This reduces our cylindrical system by one degree of freedom to (ax, ar) . We now introduce non-dimensionalization and definition of several variables to the NS equations.

We define the non-dimensional cylindrical coordinates (x, r) such that the velocity components can be written as $(ga^2/\rho\nu)(\hat{u}, \hat{v})$. We then define the unperturbed constant axial pressure gradient, $\partial\hat{p}/\partial x$ as $-4g$. The pressure is given by $(g^2a^4/\rho\nu^2)\hat{p}$ where ρ and ν are the density and kinematic viscosity of the incompressible fluid respectively. We scale time by $(\rho\nu/ga)t$, since we have non-dimensionalized our lengths with respect to a . The inner cylinder velocity is similarly scaled by $(ga^2/\rho\nu)V$. By applying these factors, we obtain the non-dimensional, unsteady, axisymmetric NS equations for an incompressible fluid. Note that the Reynolds number, R , is represented by $ga^3/\rho\nu^2$. In non-dimensional terms, the Reynolds number-dependent axial pressure gradient is $\partial\hat{p}/\partial x = -4/R$. The

non-dimensionalized NS equations can now be written as

- Conservation of Mass

$$\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial r} + \frac{\hat{v}}{r} = 0, \quad (2.1)$$

- Axial Momentum

$$\frac{\partial \hat{u}}{\partial t} + \hat{u} \frac{\partial \hat{u}}{\partial x} + \hat{v} \frac{\partial \hat{u}}{\partial r} = -\frac{\partial \hat{p}}{\partial x} + \frac{1}{R} \left(\frac{\partial^2 \hat{u}}{\partial r^2} + \frac{1}{r} \frac{\partial \hat{u}}{\partial r} + \frac{\partial^2 \hat{u}}{\partial x^2} \right), \quad (2.2)$$

- Radial Momentum

$$\frac{\partial \hat{v}}{\partial t} + \hat{u} \frac{\partial \hat{v}}{\partial x} + \hat{v} \frac{\partial \hat{v}}{\partial r} = -\frac{\partial \hat{p}}{\partial r} + \frac{1}{R} \left(\frac{\partial^2 \hat{v}}{\partial x^2} + \frac{\partial^2 \hat{v}}{\partial r^2} + \frac{1}{r} \frac{\partial \hat{v}}{\partial r} - \frac{\hat{v}}{r^2} \right), \quad (2.3)$$

where the Reynolds number is linked to the pressure gradient in the flow, as defined above. We could have used different non-dimensionalizations such as defining the Reynolds number based on the radius of the inner cylinder, as Gittler (1993) or as a function of maximum flow velocity by Sadeghi & Higgins (1991) for ACPF. However, given that our outer cylinder location is fixed ($r = 1$, in non-dimensional terms) and the flow between the concentric cylinders is subject to a constant pressure gradient, basing our Reynolds number on pressure gradient and outer cylinder radius is the most practical and appropriate non-dimensionalization for our analysis.

We introduce our travelling-wave perturbation to the Navier-Stokes equations. First we write

$$\hat{u} = U_0(r) + u(r, x, t), \quad (2.4)$$

$$\hat{v} = v(r, x, t), \quad (2.5)$$

$$\hat{p} = -\frac{4}{R}x + p(r, x, t), \quad (2.6)$$

where U_0 is the basic flow (the exact solution of ACPF which will be derived in the next section) and (u, v, p) represent a perturbation introduced to the system. We define (u, v, p) in the next chapter.

Substituting (2.4), (2.5) & (2.6) into equations (2.1), (2.2) & (2.3), we obtain the following exact, fully nonlinear, disturbance equations:

- Conservation of Mass

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0, \quad (2.7)$$

- Axial Momentum

$$\frac{\partial u}{\partial t} + U_0 \frac{\partial u}{\partial x} + v U'_0 + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = -\frac{\partial p}{\partial x} + \frac{1}{R} \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2} \right), \quad (2.8)$$

- Radial Momentum

$$\frac{\partial v}{\partial t} + U_0 \frac{\partial v}{\partial x} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial r} = -\frac{\partial p}{\partial r} + \frac{1}{R} \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial x^2} \right). \quad (2.9)$$

The relevant no-slip boundary conditions are applied at the inner and outer cylinder walls, $r = \delta$ and $r = 1$. Further details on the exact boundary conditions applied are detailed in the remaining sections of this chapter. Before we progress and computationally solve the equations above, let us first establish the mathematical models for the annular flows which we wish to solve for in the remainder of this chapter.

2.1 ACPF Model & Flow Equation

Figures 2.1 and 2.2 show the Annular Couette-Poiseuille Flow model and one possible application of thread injection. The outer cylinder is fixed and remains stationary whilst the inner cylinder is allowed to move in the axial direction, with no rotation. In the thread injection application, the thread is represented by the inner cylinder and there is a control-mechanism, in this case the coil motor, which dictates the axial velocity of the thread entering the body, through a syringe. The inner cylinder is assumed to be infinitely long, rigid and remains concentric at all times. In addition, we assume that the basic flow is fully developed.

The exact solution of ACPF can be derived from the Navier-Stokes equations as follows. We seek a steady, parallel solution to the flow subjected to a constant axial pressure

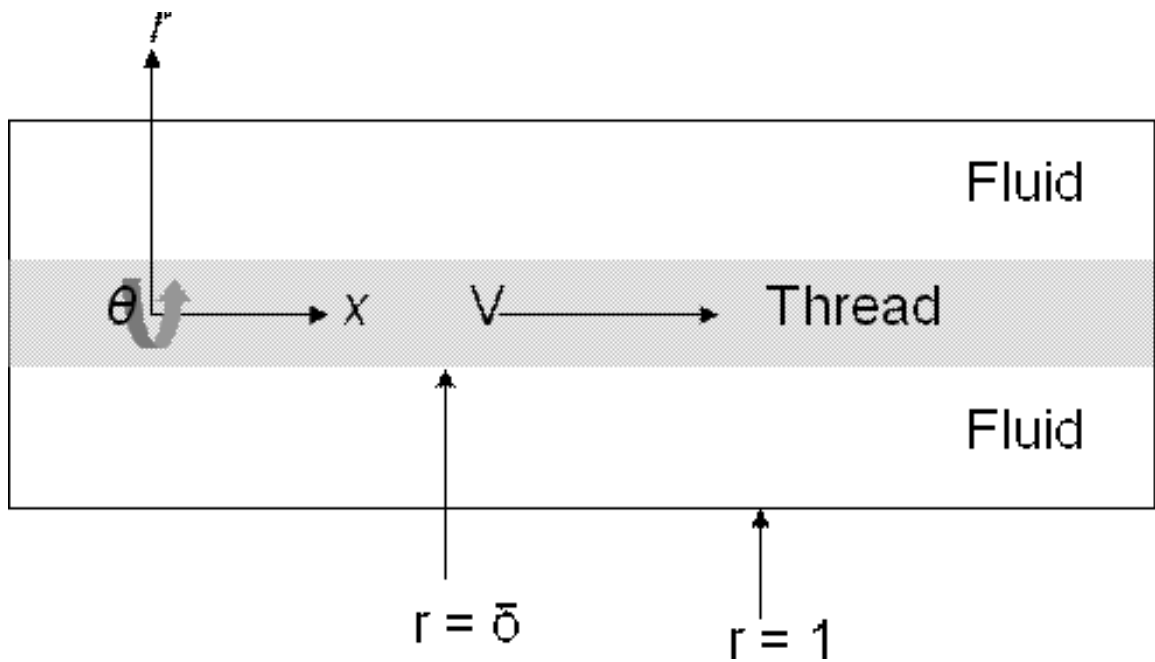


Figure 2.1: Annular Couette-Poiseuille flow mathematical model

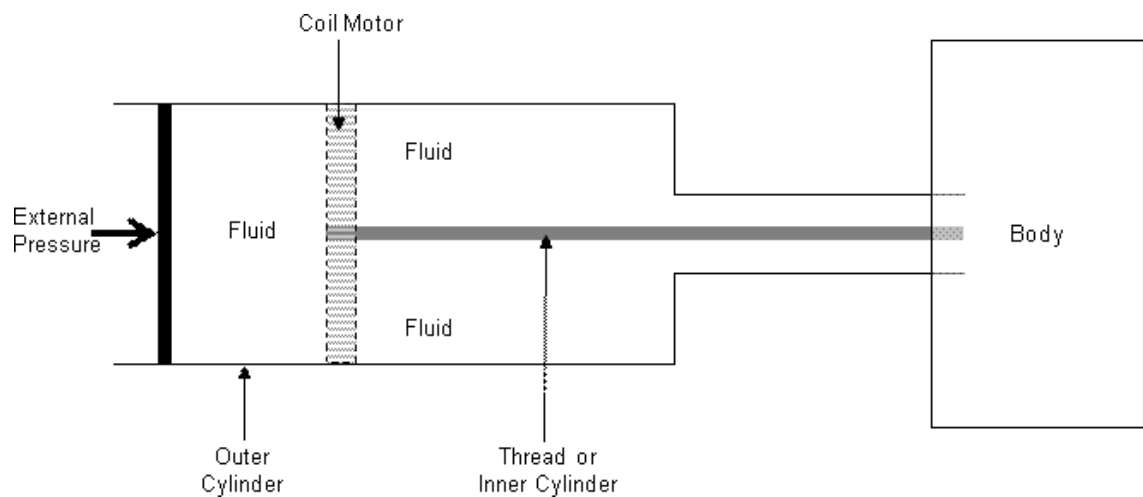


Figure 2.2: Thread injection model, one application of ACPF

gradient:

$$u = U_0(r), \quad (2.10)$$

$$v = 0, \quad (2.11)$$

$$\frac{\partial p}{\partial x} = -\frac{4}{R}, \quad (2.12)$$

and substitute this into equations (2.1), (2.2) and (2.3) to obtain a second-order ordinary differential equation

$$U_0'' + \frac{1}{r}U_0' = -4. \quad (2.13)$$

On application of the no-slip conditions on the inner and outer cylinder walls, the exact solution of ACPF can be found. The physical no-slip criteria are: $U_0 = V$ on the inner cylinder wall $r = \delta$ and $U_0 = 0$ on the outer cylinder wall $r = 1$. We find that the exact solution of ACPF is

$$U_0 = 1 - r^2 + \frac{(V - 1 + \delta^2)}{\ln(\delta)} \ln(r), \quad (2.14)$$

where $\delta \leq r \leq 1$ and V is the axial non-dimensional velocity of the inner cylinder. Figure 2.3 shows the velocity profile of ACPF when subjected to different inner cylinder velocities for various radius ratios δ as an illustration of the basic flow behaviour with no disturbances.

Our analysis will focus wholly on the case where the inner cylinder velocity is positive. This is because we are modelling the stability of ACPF / thread-annular flow as the thread is being injected into the body, rather than the thread being extracted; we note in passing that the negative- V configuration is more unstable in general as Walton (2004) demonstrated (on the basis of linear stability theory). For positive- V , the most unstable configuration always occurs when the inner cylinder is at rest as we will demonstrate in the next chapter.

Building on the governing equations pertaining to ACPF, in the next few chapters we analyse the stability of ACPF to linear and nonlinear travelling wave perturbations. We compare the results with published work where possible and draw conclusions from the analysis that we carry out.

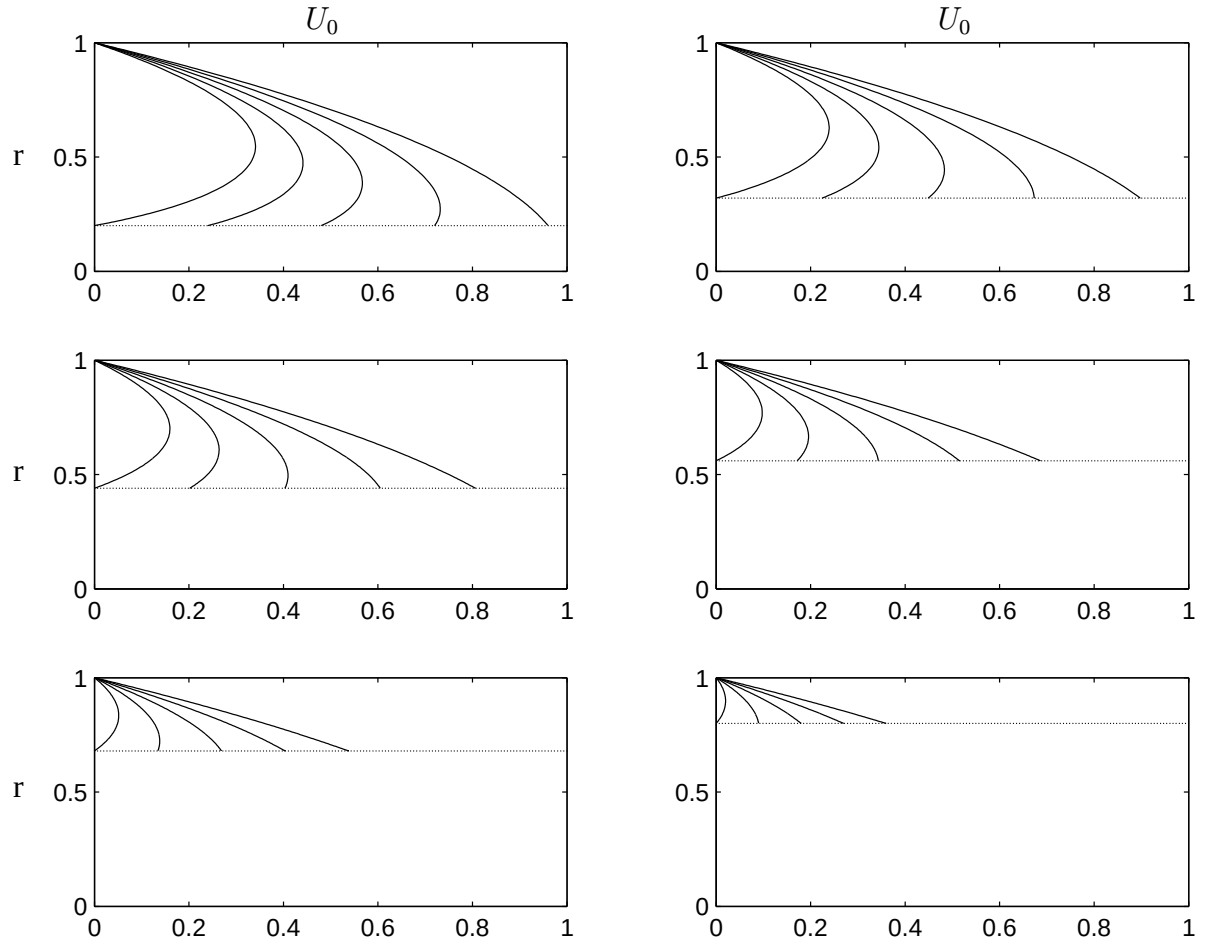


Figure 2.3: Basic ACPF for 5 equally spaced inner cylinder velocities between 0 and $1 - \delta^2$, from left to right, for $\delta = 0.2, 0.32, 0.44, 0.56, 0.68, 0.8$.

Chapter 3

Linear Stability of ACPF

In this chapter, we analyse the linear stability of ACPF, prove that ACPF is inviscidly stable, and therefore any instability must stem from viscous effects of the flow. We derive the cylindrical geometry equivalent of the Orr-Sommerfeld equation, which governs the linear stability of ACPF, and present the numerical method and results from solving this equation. We also summarise the linear results which are most relevant to this thesis, limiting our analysis to specific conditions of ACPF such as radius ratio and inner cylinder velocities. We begin by deriving the Orr-Sommerfeld equation by introducing a travelling wave disturbance to our flow.

3.1 Governing Equations for Linear Stability of ACPF

We introduce a perturbation, (u, v, p) , to ACPF in the form of

$$u(r, x, t) = \hat{u}_0(r) + \sum_{n=1}^{\infty} u_n(r) E^n + c.c., \quad (3.1)$$

$$v(r, x, t) = \sum_{n=1}^{\infty} v_n(r) E^n + c.c., \quad (3.2)$$

$$p(r, x, t) = \hat{p}_0(r) + \sum_{n=1}^{\infty} p_n(r) E^n + c.c., \quad (3.3)$$

where $E \equiv e^{i\alpha(x-ct)}$, α is the wavenumber, c is the wavespeed, n is the number of wave harmonics, $c.c.$ represents the complex conjugate of the wave and $\hat{u}_0(r)$ is the mean flow correction. For computational purposes, we will truncate the number of harmonics and we discuss this in more detail in chapter 4. In our analysis throughout this thesis, we take a temporal approach and allow c to be complex and requiring that the wavenumber α be real.

For our linear analysis, we assume that the travelling wave perturbation is small and therefore keep only the first terms of (3.1), (3.2) & (3.3), ignoring higher wave harmonics and their nonlinear influences for now. We obtain the following linearized equations

$$i\alpha u + v' + \frac{v}{r} = 0, \quad (3.4)$$

$$i\alpha(U_0 - c)u + vU_0' = -i\alpha p + R^{-1}(u'' + u'/r - \alpha^2 u), \quad (3.5)$$

and

$$i\alpha(U_0 - c)v = -p' + R^{-1}(v'' + v'/r - v/r^2 - \alpha^2 v), \quad (3.6)$$

noting that U_0 is defined in equation (2.14) in the previous chapter. The axial equation can be differentiated with respect to r :

$$i\alpha U_0' u + i\alpha(U_0 - c)u' + v'U_0' + vU_0'' = -i\alpha p' + R^{-1}(u''' + u''/r - u'/r^2 - \alpha^2 u'), \quad (3.7)$$

allowing us to eliminate the pressure-term, p' , to reach the following equation:

$$\begin{aligned} (i\alpha R)^{-1}(u''' + u''/r - u'/r^2 - \alpha^2 u') - R^{-1}(v'' + v'/r - v/r^2 - \alpha^2 v) \\ = (U_0 - c)(u' - i\alpha v) + U_0' u + (i\alpha)^{-1}(v'U_0' + vU_0''). \end{aligned} \quad (3.8)$$

We note from the continuity equation that the axial velocity (u) can be written in terms of the radial velocity (v) as

$$u = -\frac{1}{i\alpha} \left(v' + \frac{v}{r} \right). \quad (3.9)$$

Thus, we obtain an equation in terms of (r, v) . Finally, we introduce the substitution $\phi = rv$

and arrive at the following equation:

$$\begin{aligned} (i\alpha R)^{-1}(\phi^{iv} - 2\phi'''/r + (3/r^2 - 2\alpha^2)\phi'' + (2\alpha^2/r - 3/r^3)\phi' + \alpha^4\phi) \\ = (U_0 - c)(\phi'' - \phi'/r - \alpha^2\phi) - (U_0'' - U_0'/r)\phi. \end{aligned} \quad (3.10)$$

This is the cylindrical geometry equivalent of the Orr-Sommerfeld equation (see Drazin (2002), Walton (2004, 2005)). The no-slip boundary conditions in terms of ϕ are $\phi = \phi' = 0$ on $r = \delta$ and $r = 1$.

We are now in a position to introduce numerical procedures to solve the Orr-Sommerfeld equation. However, before we do so, we investigate the stability of ACPF analytically by simplifying our Orr-Sommerfeld equation to model an inviscid fluid.

3.2 Inviscid Stability of ACPF

An inviscid fluid is one which possesses no viscosity. Although all real fluids have some viscosity, our analysis of the Orr-Sommerfeld equation can be simplified by considering an inviscid fluid to begin with, conscious that this is only a part of the bigger picture. We look at the inviscid stability of ACPF to determine if (i) ACPF is inviscidly stable which will indicate that any instability will be due to viscous effects, or (ii) ACPF is inviscidly unstable, in which case viscosity influences the stability of ACPF either by amplifying or reducing instability of ACPF.

In order to determine if ACPF is inviscidly stable, we utilise Rayleigh's stability criterion. Rayleigh's condition for a plane parallel flow requires the second derivative of the basic flow to change its sign between the inner and outer planes as a necessary condition for instability to exist. We therefore need to obtain the ACPF-equivalent of Rayleigh's stability criterion for a cylindrical geometry.

The ACPF-equivalent of Rayleigh's stability equation can be derived from equation (3.10) by applying $R = \infty$:

$$(U_0 - c)(\phi'' - \frac{\phi'}{r} - \alpha^2\phi) - (U_0'' - U_0'/r)\phi = 0. \quad (3.11)$$

Bearing in mind that we are taking a temporal approach to solving the ACPF flow, the neutral stability of the flow is attained when the imaginary-component of the wavespeed is zero. We multiply the equation by the complex conjugate factor $\phi^*/r(U_0 - c)$ and integrate between the cylinder walls $r = \delta$ and $r = 1$.

$$\int_{\delta}^1 \left(\frac{\phi^* \phi''}{r} - \frac{\phi^* \phi'}{r^2} - \alpha^2 \frac{|\phi|^2}{r} \right) - \left(\frac{U_0''}{r} - \frac{U_0'}{r^2} \right) \frac{|\phi|^2}{(U_0 - c)} dr = 0. \quad (3.12)$$

We then expand the wavespeed in terms of its real and imaginary parts and integrate the first two terms by parts and find

$$\begin{aligned} \int_{\delta}^1 \left(-\frac{|\phi'|^2}{r} - \alpha^2 \frac{|\phi|^2}{r} \right) - \left(\frac{U_0''}{r} - \frac{U_0'}{r^2} \right) \left(\frac{(U_0 - c_r)|\phi|^2}{|U_0 - c|^2} \right) \\ + i c_i \left(\frac{U_0''}{r} - \frac{U_0'}{r^2} \right) \left(\frac{|\phi|^2}{|U_0 - c|^2} \right) dr = 0. \end{aligned} \quad (3.13)$$

Focussing on the imaginary part of the equation, we find

$$c_i \int_{\delta}^1 \left(\frac{U_0''}{r} - \frac{U_0'}{r^2} \right) \frac{|\phi|^2}{|U_0 - c|^2} dr = 0. \quad (3.14)$$

Since the absolute-terms are always positive, it falls to the basic-flow terms that need to change signs in order that ACPF becomes inviscidly unstable, and this must occur between the wall boundaries $r = \delta$ and $r = 1$. Expanding the basic-flow terms,

$$c_i \int_{\delta}^1 \left(-\frac{2}{r^3} \frac{V - 1 + \delta^2}{\ln(\delta)} \right) \frac{|\phi|^2}{|U_0 - c|^2} dr = 0. \quad (3.15)$$

We can see that all the terms are single-signed and therefore there is no change in sign between $\delta \leq r \leq 1$. Figure 3.1 illustrates that the basic-flow terms do not change its sign between the cylinder walls.

Therefore, ACPF is inviscidly stable and any instability stems from viscous effects. We now proceed to analyse the stability of ACPF to a small-amplitude wave disturbance,

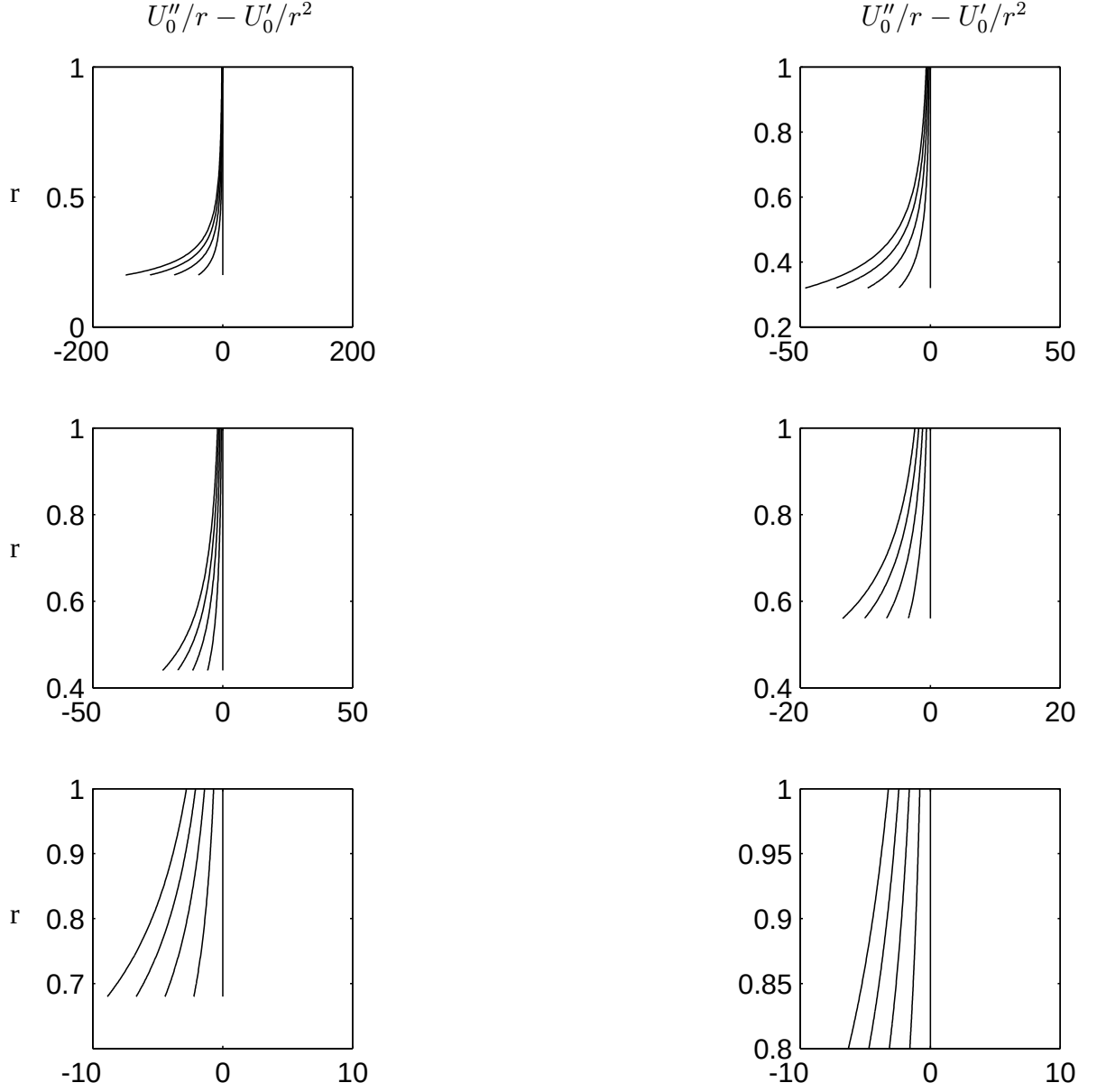


Figure 3.1: Illustration of $U''_0/r - U'_0/r^2$ for 5 equally spaced inner cylinder velocities between 0 and $1 - \delta^2$, from left to right, for $\delta = 0.2, 0.32, 0.44, 0.56, 0.68, 0.8$.

ignoring higher wave harmonics for the moment.

3.3 Numerical Method for Linear Stability

The Orr-Sommerfeld equation (3.10) governs the linear stability of ACPF and we focus on solving this equation numerically in this section. In order to solve the Orr-Sommerfeld equation numerically, we employ a spectral method which involves introducing Chebyshev polynomials to the equation and solving the resulting eigenvalue problem using a software package such as Matlab. There are other numerical methods which could be employed such as finite differences, however, the combination of Chebyshev polynomials and an eigenvalue-solving method seems to be the widely accepted numerical procedure for solving the Orr-Sommerfeld problem in relation to the linear stability problem on a finite domain. We will demonstrate the accuracy and speed of this spectral method applied to ACPF later in this section. The reader is referred to many authors who have documented the numerical procedure for this method of solving the Orr-Sommerfeld equation computationally such as Schmid & Henningson (2001) and Walton (2004). For completeness, we provide an explanation of the numerical method here. We begin by discretizing our ACPF model by mapping r going from $\delta \leq r \leq 1$ to $-1 \leq y \leq 1$ as

$$r = \frac{(1 + \delta) - (1 - \delta)y}{2}, \quad (3.16)$$

and discretize y with

$$y_j = \cos\left(\frac{\pi(j-1)}{M}\right) \quad (3.17)$$

where y is evaluated at Gauss points, $j = 1, 2, \dots, M+1$ and $M+1$ is the number of collocation points used to discretize the ACPF model.

We now introduce Chebyshev polynomials to the Orr-Sommerfeld equation by expanding ϕ in the following fashion

$$\phi = \sum_{n=1}^{M+1} a_n T_{n-1}(y), \quad (3.18)$$

where $T_{n-1}(y)$ is the $(n-1)^{th}$ Chebyshev polynomials evaluated at Gauss points y . Let us digress slightly to discuss what and how we use Chebyshev polynomials. Chebyshev

polynomials of the first kind, for a variable x , are defined as

$$T_n(x) = \cos(n \arccos(x)). \quad (3.19)$$

Using the substitution of $\varphi = \arccos(x)$ and applying this to our ACPF model, we can obtain a square matrix composed of $n = 1, 2, \dots, M, M + 1$ Chebyshev polynomials evaluated at Gauss points y_j where $j = 1, 2, \dots, M, M + 1$, thus allowing us to pose an eigenvalue problem for our Orr-Sommerfeld equation. In order to more readily use the expansion for ϕ , we re-arrange the Orr-Sommerfeld equation in the following manner

$$\begin{aligned} & R^{-1}(\phi^{iv} - 2\phi'''/r + (3/r^2 - 2\alpha^2)\phi'' + (2\alpha^2/r - 3/r^3)\phi' + \alpha^4\phi) \\ & - i\alpha U_0(\phi'' - \phi'/r - \alpha^2\phi) + i\alpha(U_0'' - U_0'/r)\phi = i\alpha c(\phi'' - \phi'/r - \alpha^2\phi). \end{aligned} \quad (3.20)$$

We then define Chebyshev differentiation matrices (D0, D1, D2, D3, D4), detailed in Appendix C, allowing us to write our discrete Orr-Sommerfeld equation as

$$\begin{aligned} & (R^{-1}(D4(y) - 2D3(y)/r + (3/r^2 - 2\alpha^2)D2(y) + (2\alpha^2/r - 3/r^3)D1(y) + \alpha^4D0(y)) \\ & - i\alpha U_0(D2(y) - D1(y)/r - \alpha^2D0(y)) + i\alpha(U_0'' - U_0'/r)D0(y))\underline{a} \\ & = i\alpha c(D2 - D1/r - \alpha^2D0)\underline{a}, \end{aligned} \quad (3.21)$$

and the Chebyshev derivative matrices (D0, D1, D2, D3, D4) are all evaluated at Gauss points y_j . The discretization of the above allows us to state our eigenvalue problem as

$$B\underline{a} = (i\alpha c)C\underline{a}, \quad (3.22)$$

where the matrices B and C are functions of Chebyshev derivative matrices and are square matrices of size $M + 1$. The vector $\underline{a}$ is length $M + 1$. This is a generalized eigenvalue problem in the form of $B\underline{a} = \lambda C\underline{a}$, where $\lambda = i\alpha c$ and we can solve for λ , specifically to obtain the marginally stable eigenvalues where the imaginary part of the wavespeed is zero ($c_i = 0$). Using a computational package such as Matlab or LAPACK, the eigenvalues, λ , and the corresponding eigenvectors, $\underline{a}$, can be found easily. For illustrative purposes,

matrix C (including the boundary conditions) is written as

$$\begin{bmatrix} D0_1(y_1) & \cdots & D0_{M+1}(y_1) \\ D1_1(y_2) & \cdots & D1_{M+1}(y_2) \\ f_n(y_3) & \cdots & f_n(y_3) \\ \vdots & \ddots & \vdots \\ D1_1(y_M) & \cdots & D1_{M+1}(y_M) \\ D0_1(y_{M+1}) & \cdots & D0_{M+1}(y_{M+1}) \end{bmatrix}, \quad (3.23)$$

where $n = 1, 2, \dots, M, M + 1$ and the function f_n is evaluated at Gauss points y which can be written as

$$f_n(y) = D2(y) - D1(y)/r(y) - \alpha^2 D0(y), \quad (3.24)$$

and y is evaluated over the number of discrete collocation points, $1 \leq j \leq M + 1$ in our ACPF model. For example, the third row of matrix C is

$$f_1(y_3), \dots, f_{M+1}(y_3). \quad (3.25)$$

The no-slip boundary conditions need to be applied to our eigenvalue problem. Noting that the boundary conditions are $\phi = \phi' = 0$ on $r = \delta$ and $r = 1$, we implement these boundary conditions on the first and last two rows of our matrices B and C . Our eigenvalue problem can be written as:

$$\begin{bmatrix} D0_1(y_1) & \cdots & D0_{M+1}(y_1) \\ D1_1(y_2) & \cdots & D1_{M+1}(y_2) \\ g_n(y_3) & \cdots & g_n(y_3) \\ \vdots & \ddots & \vdots \\ D1_1(y_M) & \cdots & D1_{M+1}(y_M) \\ D0_1(y_{M+1}) & \cdots & D0_{M+1}(y_{M+1}) \end{bmatrix} \underline{a} = \lambda \begin{bmatrix} D0_1(y_1) & \cdots & D0_{M+1}(y_1) \\ D1_1(y_2) & \cdots & D1_{M+1}(y_2) \\ f_n(y_3) & \cdots & f_n(y_3) \\ \vdots & \ddots & \vdots \\ D1_1(y_M) & \cdots & D1_{M+1}(y_M) \\ D0_1(y_{M+1}) & \cdots & D0_{M+1}(y_{M+1}) \end{bmatrix} \underline{a}, \quad (3.26)$$

where functions g_n and f_n represent the Chebyshev polynomials, evaluated at y , in matrices

B and C respectively. The function g_n evaluated at Gauss points y can be written as

$$g_n(y) = \frac{1}{R} \left(D4(y) - \frac{2}{r} D3(y) + \left(\frac{3}{r^2} - 2\alpha^2 \right) D2(y) + \left(\frac{2}{r} \alpha^2 - \frac{3}{r^3} \right) D1(y) + \alpha^4 D0(y) \right) - i\alpha U_0 \left(D2(y) - \frac{1}{r} D1(y) - \alpha^2 D0(y) \right) + i\alpha \left(U_0'' - \frac{1}{r} U_0' \right) D0(y). \quad (3.27)$$

where the third row of g_n is

$$g_1(y_3), \dots, g_{M+1}(y_3). \quad (3.28)$$

Once the eigenvalues are found, corresponding to values of wavespeed c , we need to filter out spurious and stable modes. This can be done visually, but is rather time consuming. The filter to be applied requires, at the very least, that the real part of the wavespeed, c_r , is between the minimum and maximum of the basic flow, U_0 , i.e. $\min(U_0) \leq c_r \leq \max(U_0)$. It is then very easy to identify unstable eigenvalues where $c_i > 0$, and to find the corresponding eigenvectors $\underline{a}$. We now have a method of determining whether for a given pair of Reynolds number and wavenumber the flow is stable or unstable and can program the above using a software package. We use the software package Matlab for our programming and computations in this thesis.

We now turn our attention to finding the neutrally-stable combination of Reynolds number and wavenumber (neutral curve points) for a given ACPF configuration of radius ratio and inner cylinder velocity. In order to do this, we exploit a built-in function in Matlab, *fzero*, which uses a combination of bisection and interpolation methods to locate a change of sign for a specified variable. In our case, we wish to find a change of sign in the imaginary part of the wavespeed, c_i , and use *fzero* to locate the neutrally stable point on the neutral curve. For a given inner cylinder velocity and radius ratio of ACPF, we specify the Reynolds number and provide an initial wavenumber guess and the resulting output from the program is the neutrally stable wavenumber on the neutral curve.

In this thesis, we use 150 collocation points ($M = 150$) which proves sufficient for computational purposes to reach good accuracy, striking a balance between the computational time required and the accuracy of the numerical results generated by the program. For the same initial guess we use in *fzero*, Figure 3.2 demonstrates, for linear stability of ACPF, that there is practically no change in the computation of the wavenumber, α , beyond

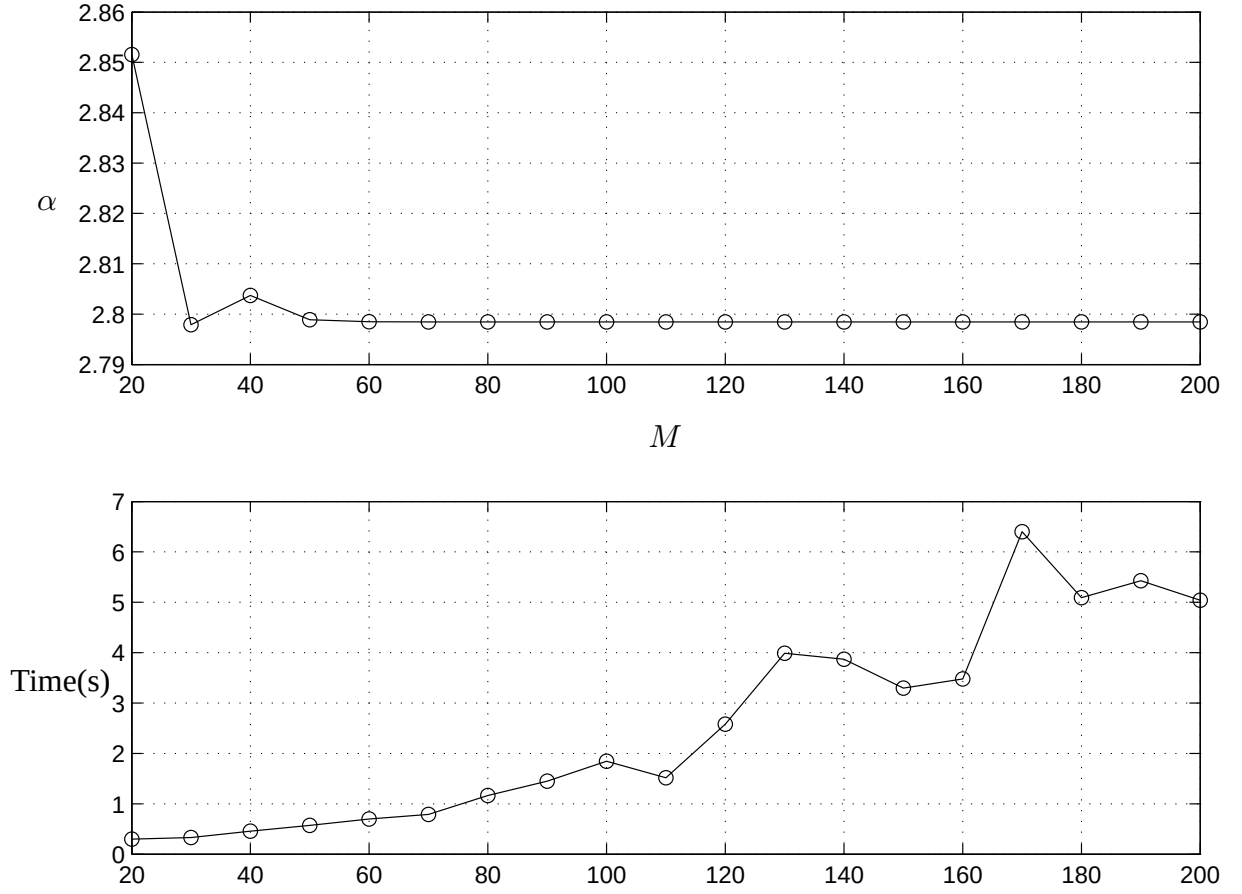


Figure 3.2: Resolution of neutral curve solution for $\delta = 0.5$ at $R = 1 \times 10^6$ for $V = 0$. Top: wavenumber against number of Chebyshev points, $20 \leq M \leq 200$. Bottom: Time (seconds) against number of Chebyshev points, $20 \leq M \leq 200$, for *fzero* to converge the neutral curve solution.

$M = 50$ with the duration to find the solution taking under one second. Increasing the number of collocation points will require more computational time and memory. However, as we will demonstrate in the next chapter, more collocation points are required to attain sufficient numerical accuracy for the nonlinear stability computations. For $M = 150$, we have sufficient number of collocation points that yields an accurate numerical result within a reasonable time.

Having established our approach to solving the Orr-Sommerfeld equation which governs the linear stability of ACPF, we apply our program to ACPF with specific parameter values and present our findings in the remainder of this chapter.

3.4 Linear Stability Numerical Results

We present results on the linear stability of ACPF and seek to validate the linear ACPF program results against published work such as Walton (2004) & Webber (2008). We also focus on areas where Walton 2004 highlighted interesting features of ACPF. The findings are presented in the (α, R) -plane in the form of neutral curves. Generally, the area within the upper and lower branches of a neutral curve indicates the regions where ACPF is linearly unstable. As we will see later in Chapter 4, the critical Reynolds number, R_{crit} , and the bandwidth of instability (range of wavenumbers) changes significantly when ACPF is subjected to nonlinear perturbations.

We begin by presenting the neutral curve in figure 3.3 for the case $V = 0$ and $\delta = 0.5$ where the minimum critical Reynolds number is reached for ACPF. The minimum critical Reynolds number is approximately 4.7×10^5 and is located at the nose of the neutral curve. The band of unstable wavenumbers is between $\alpha = 2.4$ and $\alpha = 4.1$ for Reynolds numbers up to 2×10^6 with the maximum bandwidth (difference between maximum and minimum wavenumbers) approximately 1.5. This case also exhibits the typical neutral curve shape, i.e. has a minimum R_{crit} with an upper and lower branch curve and the gap between the upper and lower branches reduces towards 0 as $R \rightarrow \infty$.

ACPF is least stable with the inner cylinder at rest and figures 3.4 and 3.5 demonstrate, for the case of a stationary inner cylinder, the critical Reynolds number variation and neutral curves as a function of radius ratio respectively. The critical Reynolds number increases

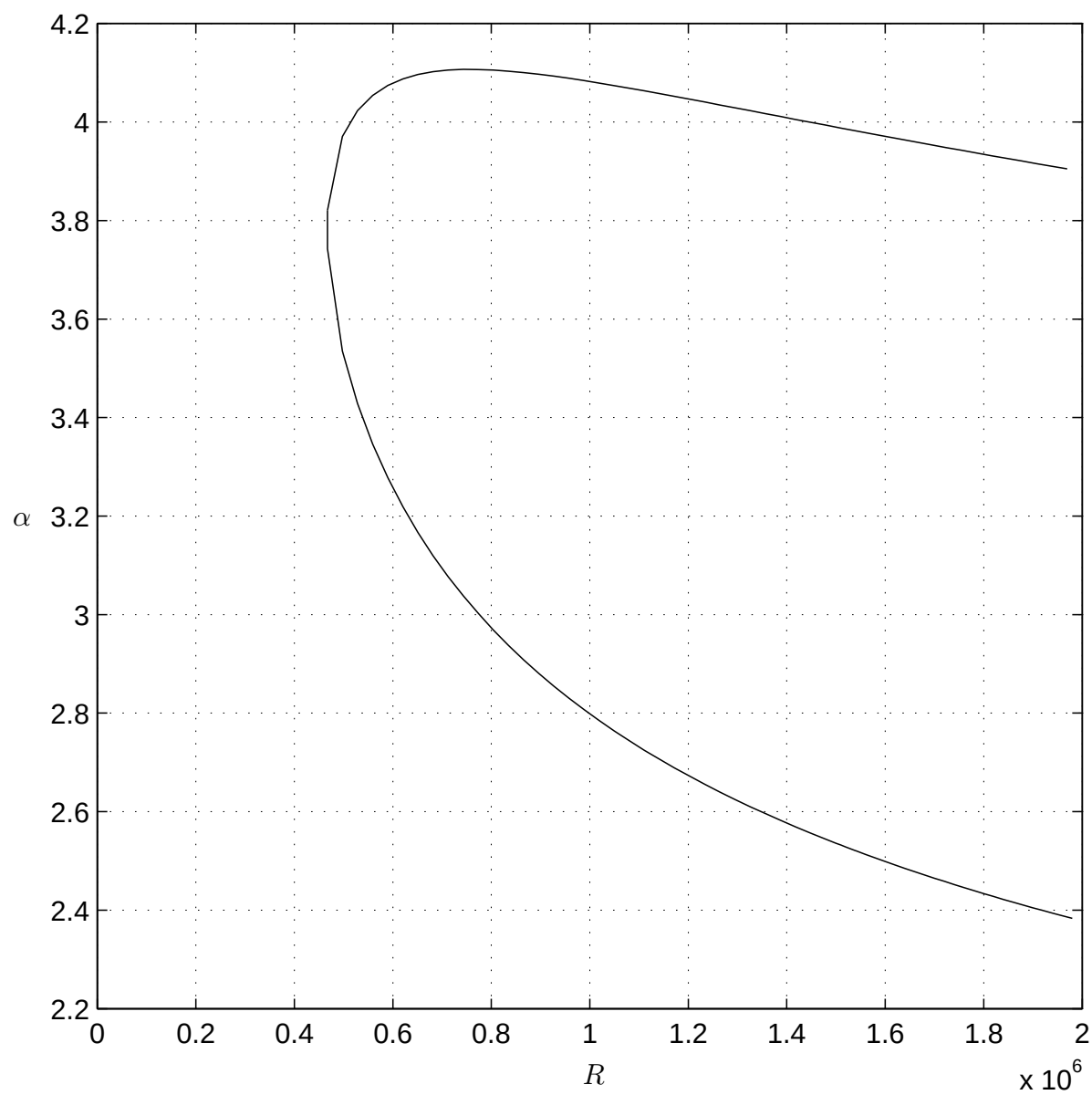
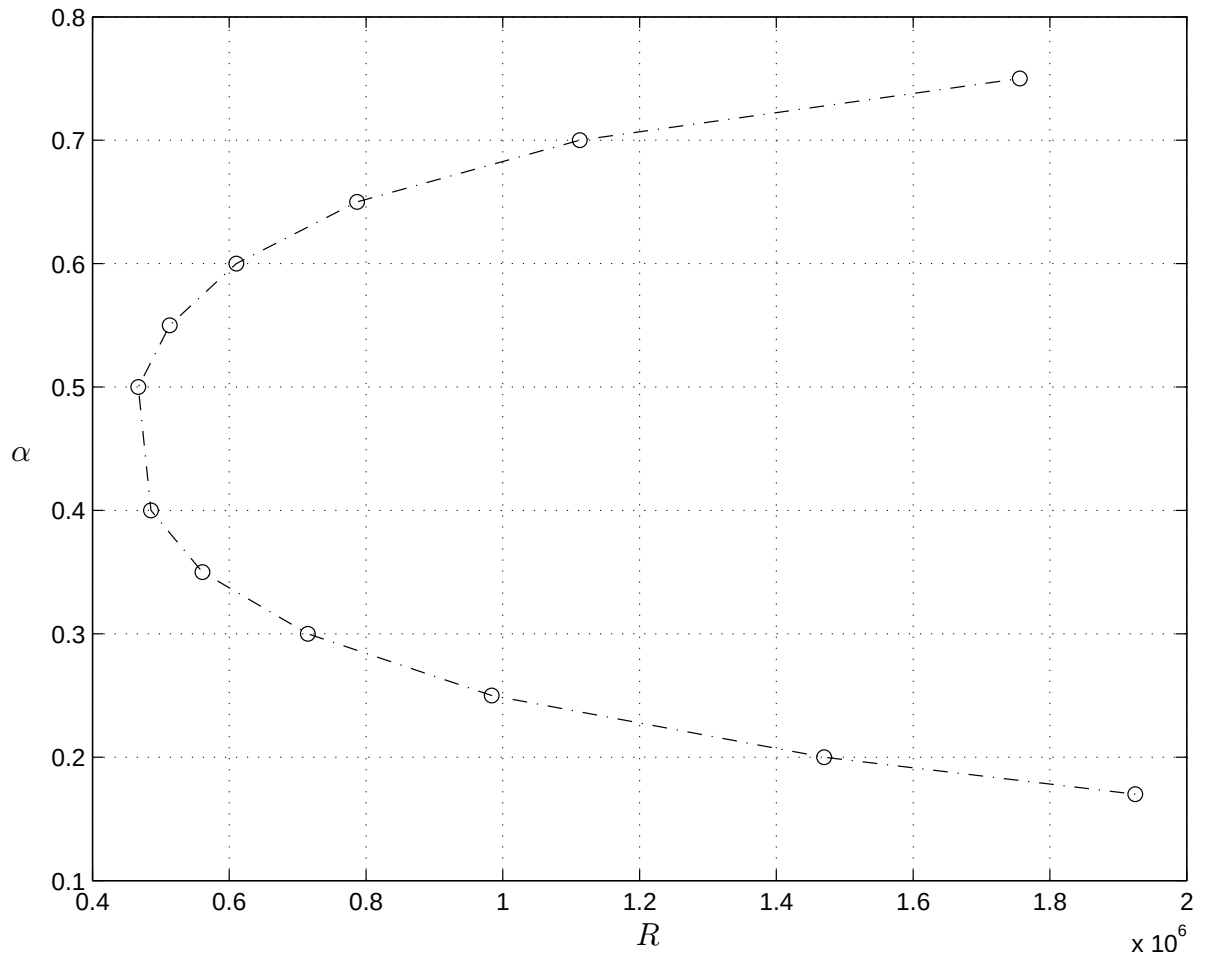


Figure 3.3: Neutral curve (α against R) for $\delta = 0.5$ and $V = 0.0$

Figure 3.4: δ against R_{crit} for various δ at $V = 0$

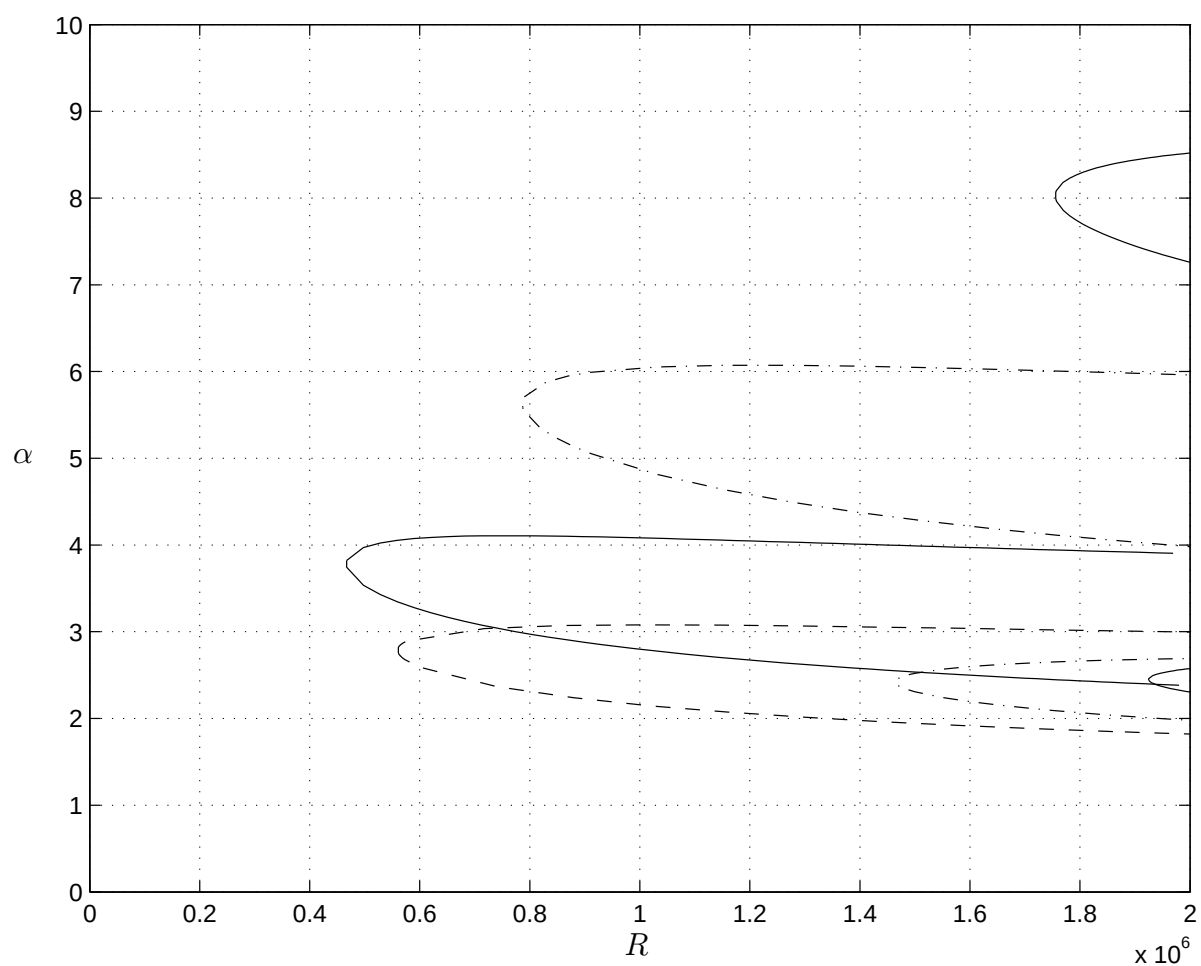


Figure 3.5: Neutral curves (α against R) for various δ with $V = 0.0$. From top to bottom: $\delta = 0.75$ (solid), 0.65 (dot-dashed), 0.5 (solid), 0.35 (dashed), 0.2 (dot-dashed), 0.17 (solid)

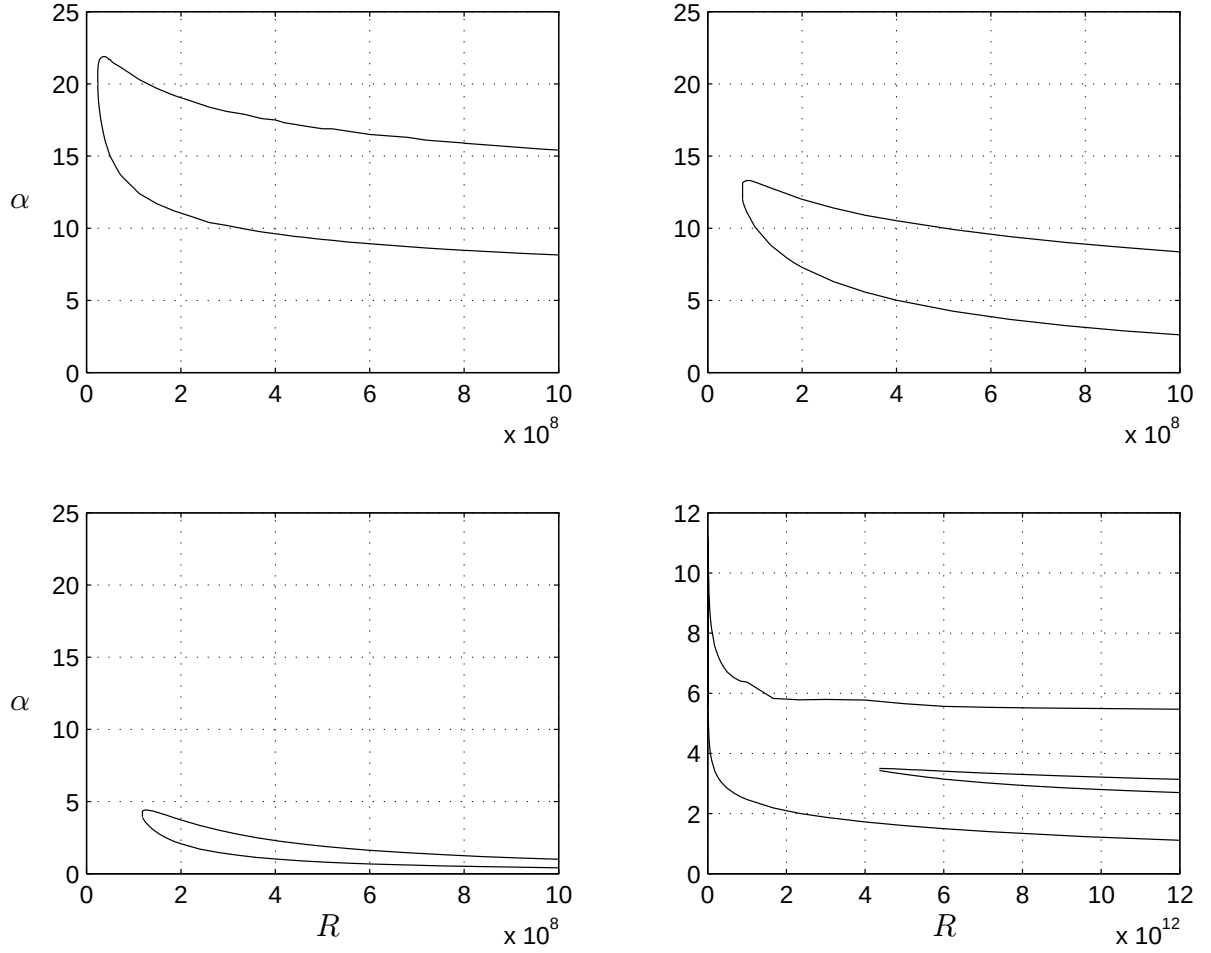


Figure 3.6: Neutral curves (α against R) for $\delta = 0.9$ for $V = 0.0$ (top left), $V = 0.001$ (top right), $V = 0.003$ (bottom left) and $V = 4.4 \times 10^{-5}$ (bottom right)

monotonically on either side of $\delta = 0.5$ supporting the suggestion further that $\delta \approx 0.5$ is the most unstable configuration when the inner cylinder is at rest. The neutral curves show that the band of unstable wavelengths increases as δ is increased for values of $\delta > 0.5$ but decreases as δ is decreased for $\delta < 0.5$ again indicating that for a stationary inner cylinder, the case of $\delta = 0.5$ is a unique one.

A summary of results for the case of $\delta = 0.9$ for various V is presented in figure 3.6. As V is increased, the neutral curve shrinks and retreats to $R = \infty$ and curve splitting behaviour is uncovered as V increases from $V = 0.0$. The upper curve (or hybrid curve as

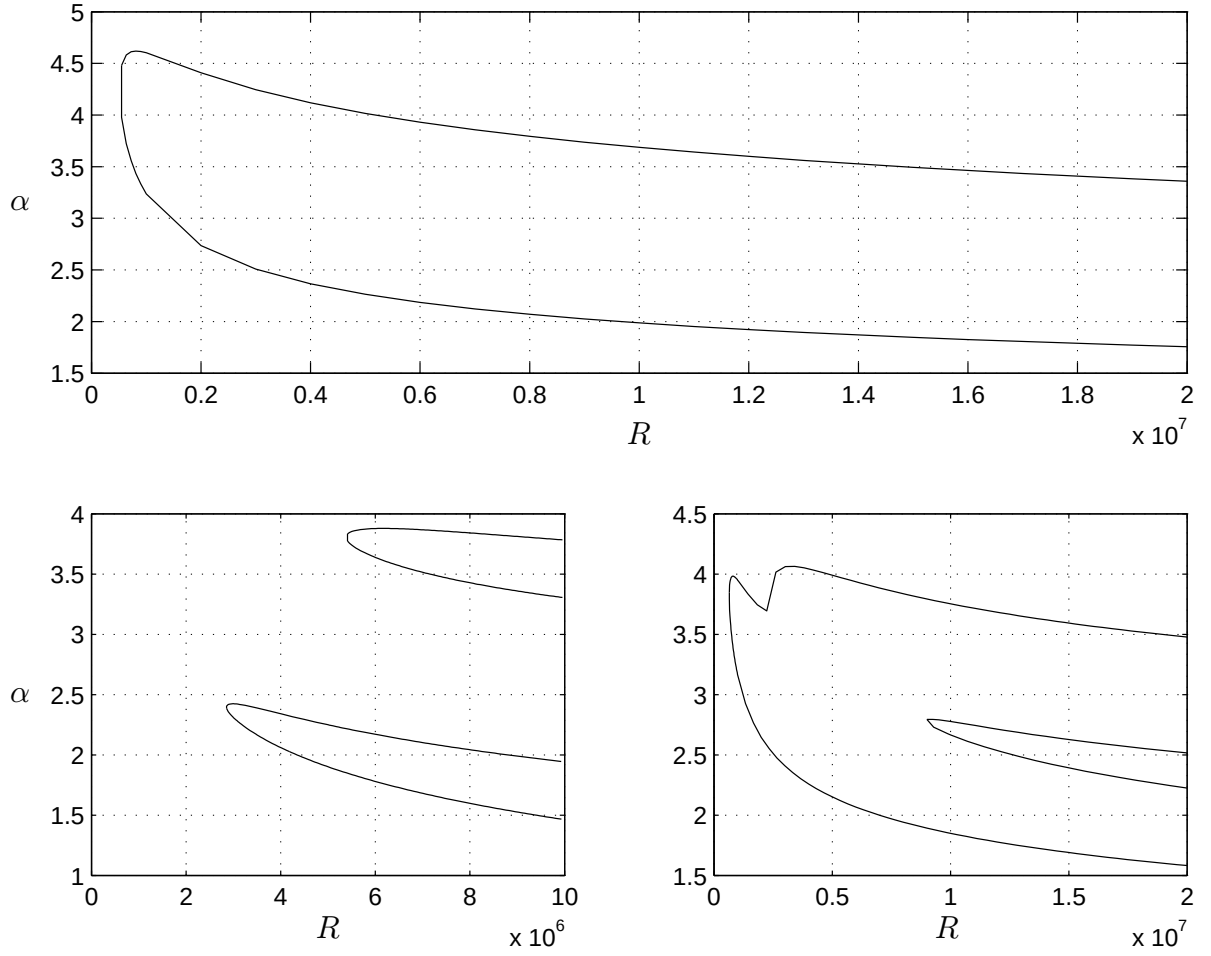


Figure 3.7: Multiple neutral curves (α against R) for $\delta = 0.55$ for $V = 0.0$ (top), $V = 0.005$ (bottom right) and $V = 0.01$ (bottom left).

referred to in Walton 2004) closes up at finite R whilst the lower curve persists at higher R and eventually retreats to $R = \infty$. The author can also confirm Walton (2004)'s findings for the case of $\delta = 0.9$ using the linear program for ACPF, verifying the functionality of the program.

In figure 3.7, we present the case where curve splitting behaviour is present leading to two distinct neutral curves, of approximately equal size, for $\delta = 0.55$ and $V = 0.01$. These neutral curves originate from one neutral curve when $V = 0.0$ but as V increases, a region of instability penetrates the neutral curve eventually slicing the curve into two parts. In

this instance, R_{crit} is governed by the lower curve. The lower curve continues to persist at much higher Reynolds number whilst the upper curve shrinks to a point, eventually disappearing completely, as V increases. Note that the bands of unstable wavelengths are also of equal size in the two curves.

The next case we present is for a radius ratio of $\delta = 0.4$ for various inner cylinder velocities V . Figure 3.8 shows the neutral curve behaviour as V increases: the neutral curve shrinks to a point, with minimal change to R_{crit} . This neutral curve does not show different regions of instability or give any indication that curve-splitting behaviour is present and this neutral curve does not retreat to $R = \infty$ and eventually disappear. In addition, the bandwidth of the wavelength decreases significantly, eventually disappearing at $V \approx 0.0355$.

Finally, we present the numerical results for a small radius ratio of $\delta = 0.1$ and figures 3.9 to 3.11 show the progression of neutral curves as inner cylinder velocity is increased from $V = 0.0$ to $V = 0.6$. Instability in ACPF is found to exist at much larger inner cylinder velocities compared to the previous cases ($\delta = 0.4, 0.5, 0.55, 0.9$). In addition, we discover the existence of multiple regions of instability for a given V which is consistent with findings for the previous $\delta = 0.55, 0.9$ cases. Indications of multiple regions of instability can be seen for $V = 0.1, 0.4$ and $V = 0.45$ where the neutral curve's upper or lower branch is not smooth either for a range of Reynolds number or beyond a certain Reynolds number. We observe (typical) curve-splitting behaviour for the sequence of inner cylinder velocities $V = 0.475 - 0.6$ where an intrusion splits the neutral curve from right to left, the upper branch curve develops into a closed neutral curve eventually shrinking to a point which leaves the lower branch curve to govern the stability of ACPF. This behaviour is consistent with what we have found for the case of $\delta = 0.55$ as V is increased from 0.0.

Focusing on an inner cylinder velocity of $V = 0.1$ and $\delta = 0.1$ in figure 3.9, a bubble appears to form at the nose of the single neutral curve. This bubble quickly disappears, and a single neutral curve again exists for $V = 0.325$ as shown in figure 3.10. However, the appearance of this bubble suggests that it is possible for another region of instability to exist - a region of instability that is not due solely by the formation of an intrusion resulting in the splitting of a neutral curve. This observation is confirmed for an inner cylinder velocity $V = 0.475$, which shows two separate neutral curves, one closed neutral curve which dictates the critical Reynolds number, and a typical neutral curve which eventually

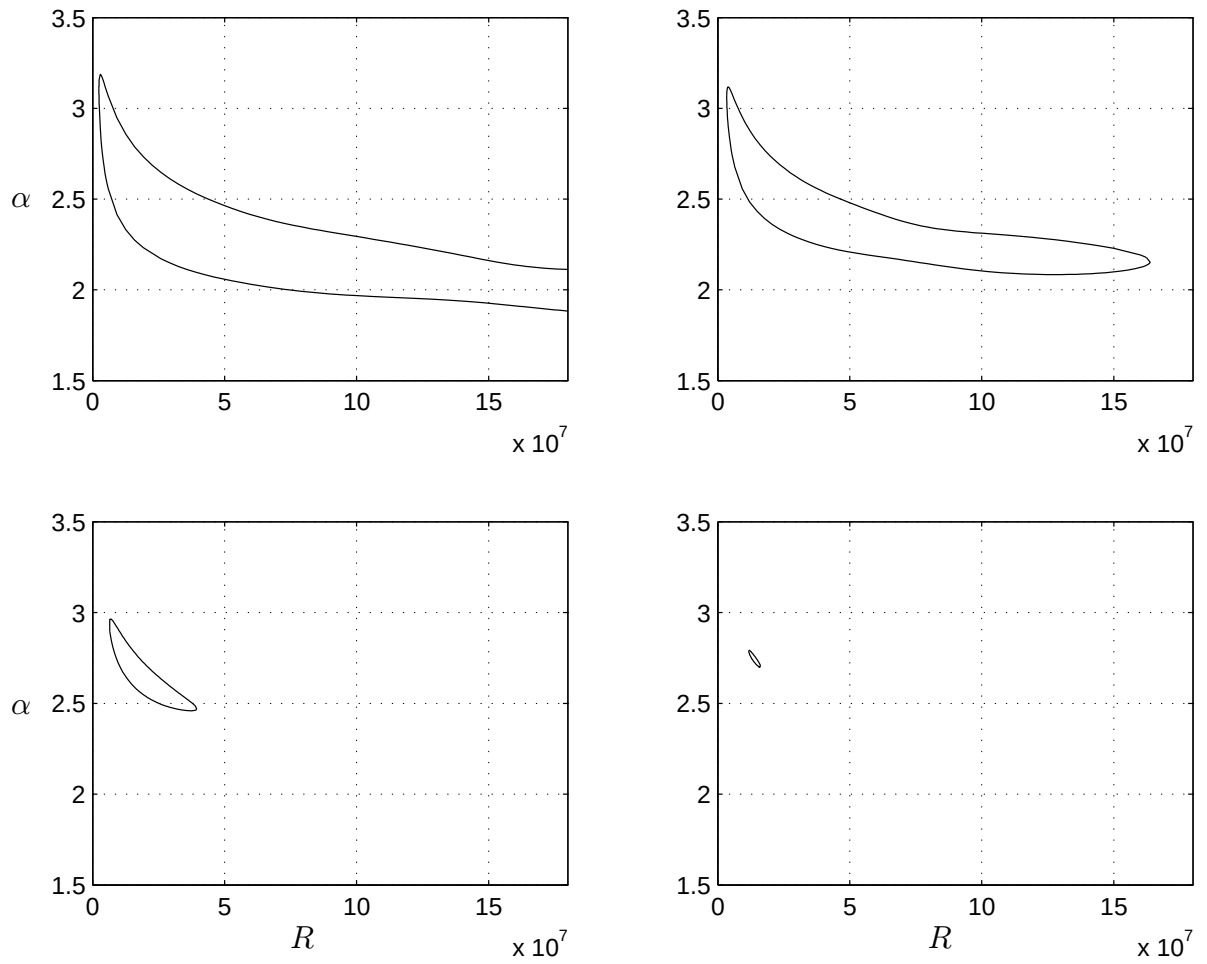


Figure 3.8: Neutral curves (α against R) for $\delta = 0.4$ for $V = 0.026$ (top left), $V = 0.030$ (top right), $V = 0.034$ (bottom left) and $V = 0.03534$ (bottom right).

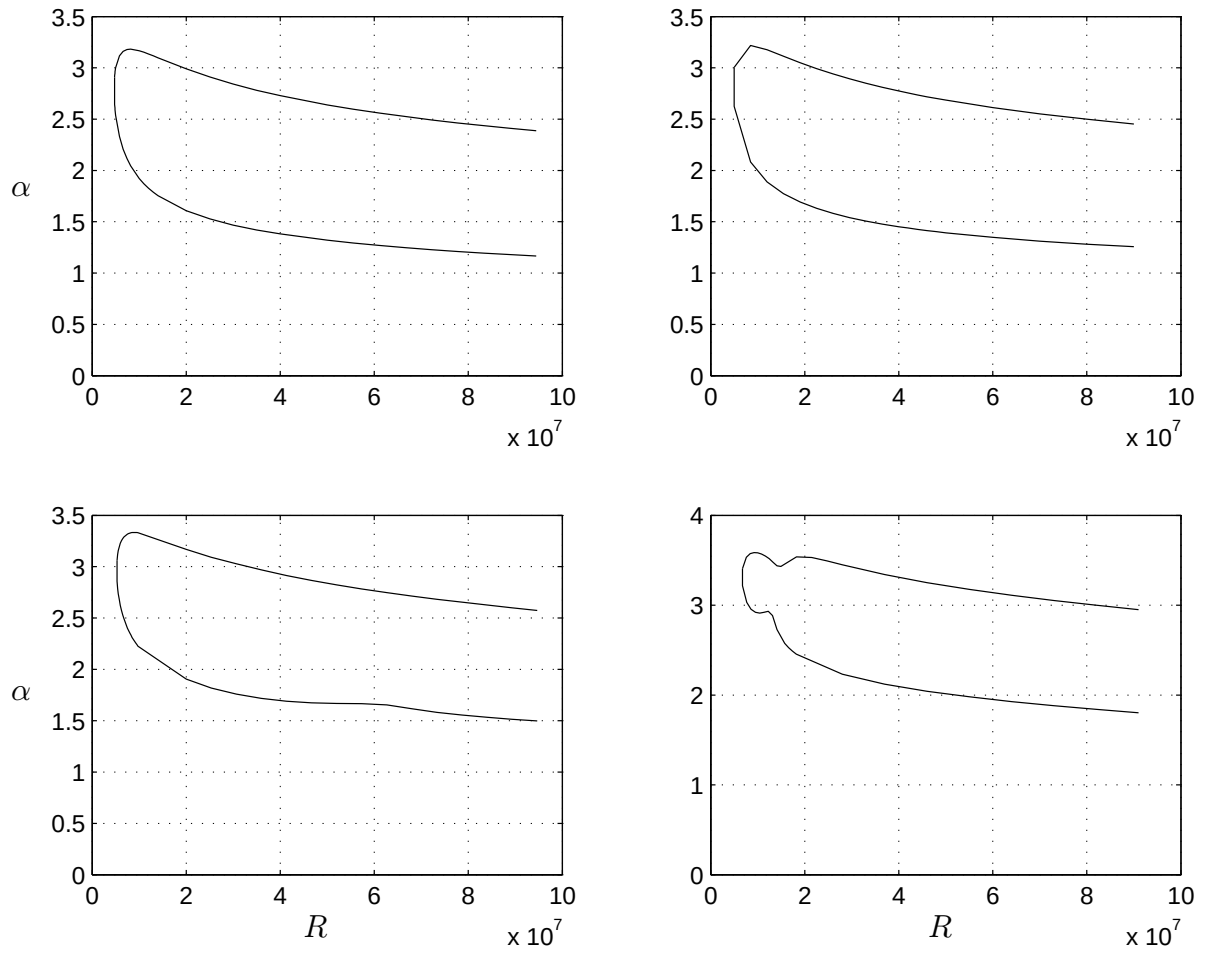


Figure 3.9: Neutral curves (α against R) for $\delta = 0.1$ for $V = 0.0$ (top left), $V = 0.025$ (top right), $V = 0.1$ (bottom left), $V = 0.25$ (bottom right)

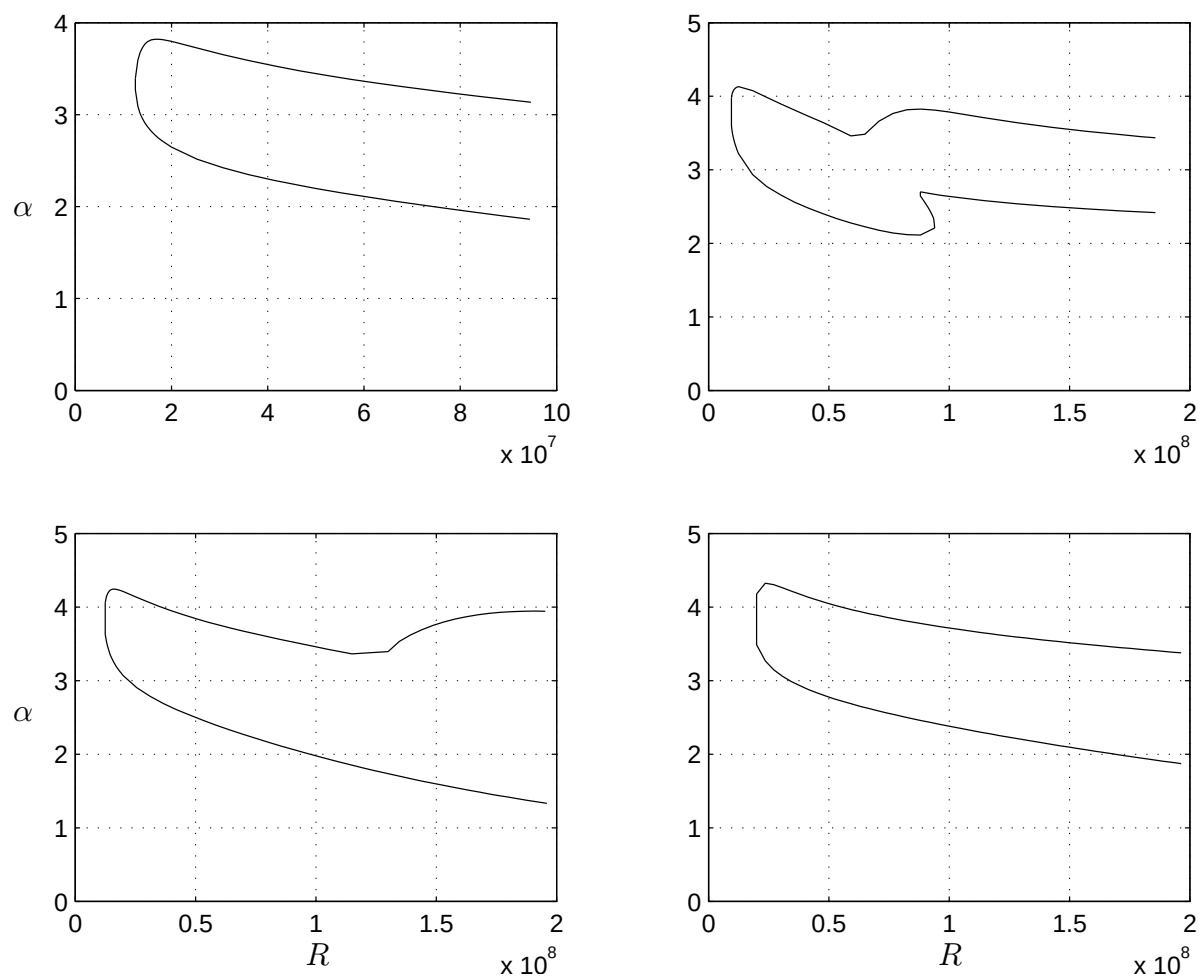


Figure 3.10: Neutral curves (α against R) for $\delta = 0.1$ for $V = 0.325$ (top left), $V = 0.4$ (top right), $V = 0.425$ (bottom left), $V = 0.45$ (bottom right)

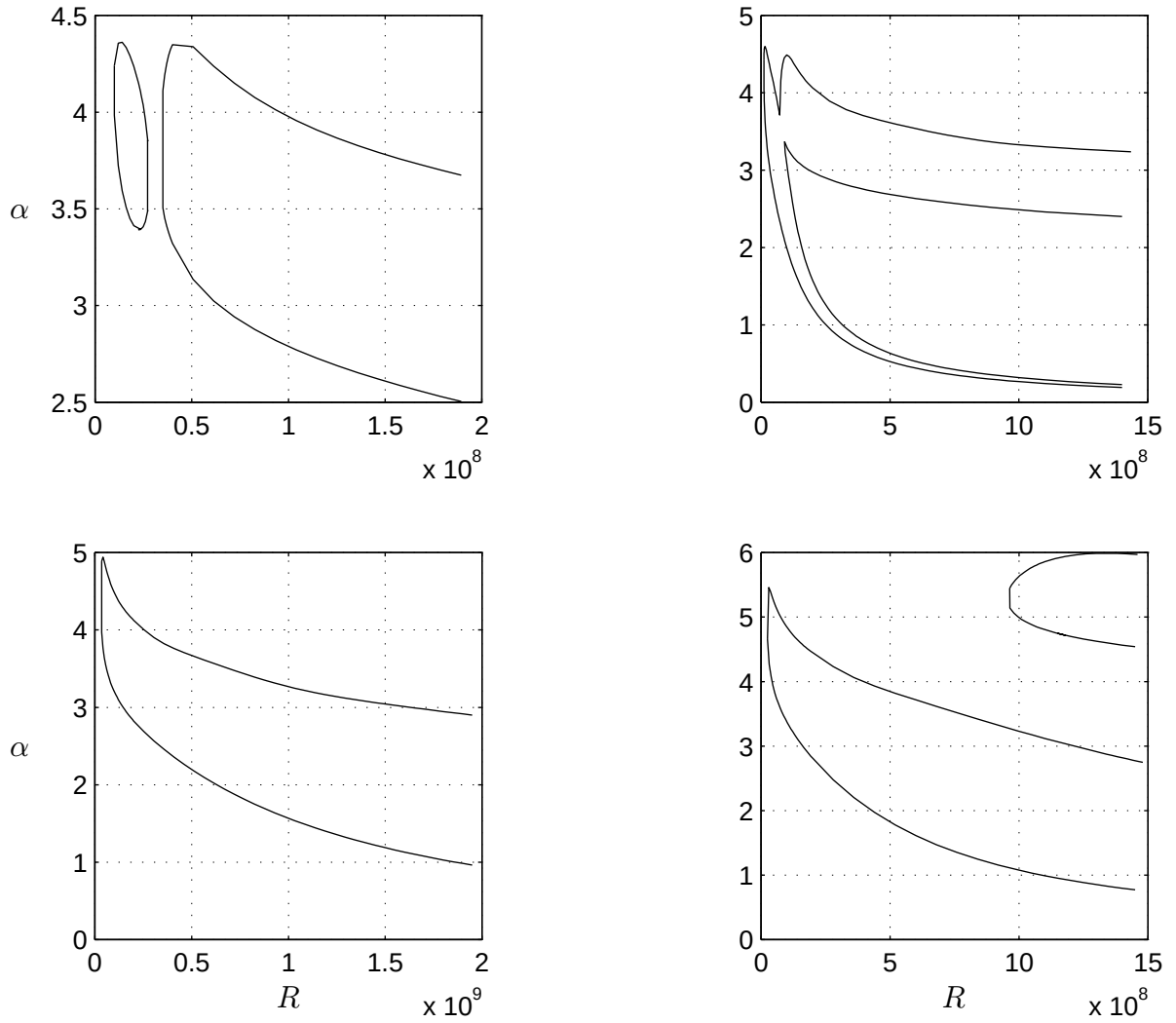


Figure 3.11: Neutral curves (α against R) for $\delta = 0.1$ for $V = 0.475$ (top left), $V = 0.5$ (top right), $V = 0.55$ (bottom left), $V = 0.6$ (bottom right)

develops curve-splitting behaviour. The observations from this is that we can have multiple regions of instability forming in different ways and that the closed neutral curve governs the critical Reynolds number, if only for a very narrow range of inner cylinder velocities.

3.5 Summary of Linear Stability of ACPF

We see that the linear stability program produces the same results (figures 3.3 to 3.8) to that of published work such as Walton (2004, 2005) and Webber (2008), indicating that the linear program functions correctly and is producing accurate results. This also acts as part of the process of the linear program validation.

We confirm the findings of Walton (2004) in that for some conditions of ACPF, multiple regions of instability can exist. In addition, we have found that neutral curves will either eventually retreat to infinity as $R \rightarrow \infty$ or close up to a point. The ultimate fate of the neutral curves (disappearing at a point or retreating to infinity) is to leave ACPF linearly stable for given ACPF conditions of inner radius ratio and inner cylinder velocity.

We have demonstrated that for a sufficiently small radius ratio ($\delta = 0.1$), multiple regions of instability can exist over a range of inner cylinder velocities through the formation of a “bubble” before the typical curve-splitting behaviour is observed. In addition, instability is found to exist for much higher inner cylinder velocities compared to that of other radius ratios.

In the following section, we analyse the effects of nonlinear axisymmetric perturbations to ACPF. In particular, we focus on areas where linear analysis predicts minimum R_{crit} , two almost-equally sized neutral curves exist and a single neutral curve closes up at finite V .

Chapter 4

Nonlinear Stability of ACPF - Wong and Walton 2012

In this chapter, we analyse the nonlinear stability of ACPF. We tailor the fully nonlinear disturbance equations (2.7), (2.8), and (2.9) in order to solve these equations computationally. We present our numerical method and results for ACPF with specific parameter conditions building on the findings of the previous section.

4.1 Governing Equations for Nonlinear Stability of ACPF

We are now interested in analyzing the interaction of various wave harmonics from equations (3.1), (3.2) & (3.3). For computational purposes, we must truncate our series to N -modes, where N is chosen to give sufficient accuracy of a solution on a neutral curve for specific ACPF conditions such as R, δ, V and disturbance amplitude, A . In order to appreciate the effect of truncating our series with N modes, results for a given neutral curve point will be computed for various values of N and M Chebyshev polynomials later in this chapter. We will demonstrate that using $M = 150$ and $5 \leq N \leq 25$ proves to be a good balance between computational time and resource demanded and the accuracy of the numerical results that we achieve.

We introduce our truncation to the equations (2.7), (2.8), and (2.9) and keep the nonlinear (inertial) terms. The resulting nonlinear equations can be re-written as

Mean Flow Equation

$$\frac{1}{R} \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \hat{u}_0 = \sum_{k=1}^N \left(v_k \frac{du_k^*}{dr} + v_k^* \frac{du_k}{dr} \right), \quad (4.1)$$

Conservation of Mass Equation

$$in\alpha u_n + v_n' + \frac{1}{r} v_n = 0, \quad (4.2)$$

Axial Momentum Equation

$$\begin{aligned} & \frac{1}{R} \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - n^2 \alpha^2 - in\alpha R(U_0 - c) \right) u_n \\ &= in\alpha p_n + \frac{dU_0}{dr} v_n + \frac{du_0}{dr} v_n + in\alpha u_0 u_n \\ &+ \sum_{k=1}^{n-1} \left(v_k \frac{du_{n-k}}{dr} + i(n-k)\alpha u_{n-k} u_k \right) \\ &+ \sum_{k=n+1}^N \left(v_k \frac{du_{k-n}^*}{dr} - i(k-n)\alpha u_{k-n}^* u_k \right) \\ &+ \sum_{k=1}^{N-n} \left(v_k^* \frac{du_{n+k}}{dr} + i(n+k)\alpha u_{n+k} u_k^* \right), \end{aligned} \quad (4.3)$$

Radial Momentum Equation

$$\begin{aligned} & \frac{1}{R} \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - n^2 \alpha^2 - \frac{1}{r^2} - in\alpha R(U_0 + \hat{u}_0 - c) \right) v_n \\ &= \frac{dp_n}{dr} + \sum_{k=1}^{n-1} \left(v_k \frac{dv_{n-k}}{dr} + i(n-k)\alpha u_k v_{n-k} \right) \\ &+ \sum_{k=n+1}^N \left(v_k \frac{dv_{k-n}^*}{dr} - i(k-n)\alpha u_k v_{k-n}^* \right) \\ &+ \sum_{k=1}^{N-n} \left(v_k^* \frac{dv_{n+k}}{dr} + i(k+n)\alpha u_k^* v_{n+k} \right), \end{aligned} \quad (4.4)$$

for $n = 1, 2, \dots, N$ where N is the number of nonlinear modes and (*) superscript denotes

complex conjugates. We now have our full set of nonlinear disturbance equations to solve numerically for N modes. The usual no-slip boundary conditions we apply is

$$\hat{u}_0 = u_n = v_n = 0 \quad (4.5)$$

on $r = \delta$ and $r = 1$ for $n = 1, 2, \dots, N$. Next, let us introduce the numerical method which we apply to solve the above equations.

4.2 Numerical Method for Nonlinear Stability

In order to solve equations (4.1), (4.3) and (4.4), we use the same discretization as for the linear ACPF problem in (3.17) and (3.16). However, instead of $\phi = rv$ that we utilised for our linear stability analysis, we use the following representation for our nonlinear analysis:

$$\frac{1}{R}\phi_n = rv_n. \quad (4.6)$$

We reduce the number of equations from $3N + 1$ (from the conservation of mass, axial & radial momentum and mean-flow equations) to $N + 1$ with the introduction of (4.6). After some algebra and simplification, we find (in terms of ϕ_n), the following fully nonlinear set of N equations

$$\begin{aligned} & (U_0 - c)(\phi_n'' - \phi_n'/r - n^2\alpha^2\phi_n) - (U_0'' - U_0'/r)\phi_n - (\hat{u}_0'' - \hat{u}_0'/r)\phi_n \\ & - \frac{1}{in\alpha R} (\phi_n^{iv} - 2\phi_n'''/r + (3/r^2 - 2n^2\alpha^2)\phi_n'' + (2n^2\alpha^2/r - 3/r^3)\phi_n' + n^4\alpha^4\phi_n) \\ & + \sum_{k=1}^{n-1} \mathfrak{a} + \sum_{k=1}^{N-n} \mathfrak{b} + \sum_{k=n+1}^N \mathfrak{c} = 0, \end{aligned} \quad (4.7)$$

where $n = 1, 2, \dots, N$. The sum-terms contain the inertial terms, and together with the mean flow equation, we solve the above for a set of given conditions of ACPF, such as R , V and δ . The sum-terms are rather lengthy to detail here and the reader is directed to

Appendix A for further details. However, for illustrative purposes, the first sum term, $\mathbf{a}$ is

$$\begin{aligned} \mathbf{a} = & \left(\frac{n}{n-k} \right) \left(3 \frac{\phi_k \phi_{n-k}''}{r^2} - 3 \frac{\phi_k \phi_{n-k}'}{r^3} - \frac{\phi_k \phi_{n-k}'''}{r} + \frac{\phi_k' \phi_{n-k}'}{r^2} - \frac{\phi_k' \phi_{n-k}''}{r} \right) \\ & + \left(\frac{n}{k} \right) \left(\frac{\phi_k' \phi_{n-k}''}{r} + \frac{\phi_k'' \phi_{n-k}'}{r} - 2 \frac{\phi_k' \phi_{n-k}'}{r^2} \right) \\ & - n^2 \alpha^2 \left(\frac{\phi_k \phi_{n-k}}{r^2} - \frac{\phi_n \phi_{n-k}'}{r} + \frac{n-k}{k} \frac{\phi_k' \phi_{n-k}}{r} \right). \end{aligned} \quad (4.8)$$

The final equation we require is the mean flow distortion equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \hat{u}_0 = \sum_{k=1}^N \frac{1}{ik\alpha R} \left(\frac{\phi_k^* \phi_k''}{r^2} - \frac{\phi_k^* \phi_k'}{r^3} \right) + c.c., \quad (4.9)$$

where $c.c.$ is the complex conjugate. We note that ϕ_n is complex in nature and we split ϕ_n in the following manner:

$$\phi_n = \Phi_n + i\Psi_n, \quad (4.10)$$

with the aim of obtaining a set of equations from considering the real and imaginary coefficients of ϕ_n . The resulting equations in terms of Φ_n and Ψ_n allow us to apply a discretization procedure similar to that applied for the linear stability of ACPF in the previous chapter. To be consistent with our linear analysis as far as possible, we expand Φ , Ψ and the mean flow distortion, $\hat{u}_0$, in terms of Chebyshev polynomials as we did for ϕ in (3.18) for the linear stability case:

$$\Phi_n = \sum_{m=1}^{M+1} a_m^{(n)} T_{m-1}(y), \quad (4.11)$$

$$\Psi_n = \sum_{m=1}^{M+1} b_m^{(n)} T_{m-1}(y), \quad (4.12)$$

$$\hat{u}_0 = \sum_{m=1}^{M+1} d_m T_{m-1}(y), \quad (4.13)$$

where y is defined in (3.17), $n = 1, 2, \dots, N$ for N nonlinear modes and $m = 1, 2, \dots, M+1$ for $M+1$ Chebyshev polynomials.

As we are dealing with a nonlinear set of equations, it is not possible to simplify the

equations into an eigenvalue-type solution as we did with the linear case. Instead, we need a procedure to solve for the coefficients $a_m^{(n)}, b_m^{(n)}$ and d_m by a nonlinear computational method. We note that there will be $(2N + 1)(M + 1)$ equations to solve for the coefficients from Φ, Ψ and $\hat{u}_0$. For a given set of ACPF conditions of inner cylinder velocity, V , radius ratio, δ , and Reynolds number (which we specify), we wish to compute the neutrally stable point on a neutral curve. We are solving for the coefficients $a_m^{(n)}, b_m^{(n)}, d_m$, wavenumber α and wavespeed c and require $(2N + 1)(M + 1) + 2$ equations.

We therefore need to introduce two additional equations in order to compute solutions for α and c in addition to the coefficients $a_m^{(n)}, b_m^{(n)}$ and d_m . We are free to impose two additional equations and the first condition we implement is on the disturbance amplitude (A), since we are interested in analysing the effects of disturbance amplitude on the stability of ACPF. The second equation that we impose is based on the phase condition (Θ) which arises from the observation that equations (4.1), (4.2), (4.3), (4.4) and (4.7) do not change under the transformation:

$$\phi_n \rightarrow \phi_n e^{in\Theta},$$

for any real Θ .

We are free to set the amplitude condition and can define disturbance amplitude as a fixed value or based upon specific ACPF parameters such as inner cylinder velocity. However, it seems reasonable to link the amplitude of the wave perturbation to the inertial terms and number of modes in our computational analysis. In addition, we require that for a sufficiently small disturbance amplitude, we reach the Orr-Sommerfeld equation which dictates the linear stability of ACPF. Our amplitude and phase equations are defined as follows:

$$\begin{aligned} A^2 &= \sum_{n=1}^N (\Phi_n(y_{J1})^2 + \Psi_n(y_{J1})^2), \\ \Phi_1(y_{J2}) &= 0, \end{aligned} \tag{4.14}$$

where $J1$ and $J2$ are arbitrary locations and we are free to set $J1$ and $J2$ at any point in our domain. The amplitude condition is set at a point $J1$ where the linear solution, ϕ_1 , is maximum whilst the point $J2$ is set where ϕ_1 of the linear solution crosses zero. The

number of equations now matches the number of variables to be computed.

For convenience and in anticipation of the next section, let us label our nonlinear equations. From the substitution (4.10), we obtain two equations in terms of Φ and Ψ corresponding to the real and imaginary coefficients of ϕ . We write the $\text{real}(\phi)$ and $\text{imag}(\phi)$ in terms of Φ and Ψ as follows

$$\begin{aligned} f_n \equiv & \frac{1}{n\alpha R} \left(\Psi_n'''' - \frac{2}{r} \Psi_n''' + \left(\frac{3}{r^2} - 2n^2\alpha^2 \right) \Psi_n'' + \left(\frac{2}{r} n^2\alpha^2 - \frac{3}{r^3} \right) \Psi_n' + n^4\alpha^4 \Psi_n \right) \\ & + \left(\hat{u}_0'' - \frac{1}{r} \hat{u}_0' + n^2\alpha^2 \hat{u}_0 \right) \Phi_n - \left(\Phi_n'' - \frac{1}{r} \Phi_n' \right) \hat{u}_0 - (U_0 - c) \left(\Phi_n'' - \frac{1}{r} \Phi_n' - n^2\alpha^2 \Phi_n \right) \\ & + \sum_{k=1}^{n-1} \mathfrak{a} + \sum_{k=n+1}^N \mathfrak{b} + \sum_{k=1}^{N-n} \mathfrak{c} = 0, \end{aligned}$$

and

$$\begin{aligned} g_n \equiv & \frac{1}{n\alpha R} \left(\Phi_n'''' - \frac{2}{r} \Phi_n''' + \left(\frac{3}{r^2} - 2n^2\alpha^2 \right) \Phi_n'' + \left(\frac{2}{r} n^2\alpha^2 - \frac{3}{r^3} \right) \Phi_n' + n^4\alpha^4 \Phi_n \right) \\ & + \left(\hat{u}_0'' - \frac{1}{r} \hat{u}_0' + n^2\alpha^2 \hat{u}_0 \right) \Psi_n - \left(\Psi_n'' - \frac{1}{r} \Psi_n' \right) \hat{u}_0 - (U_0 - c) \left(\Psi_n'' - \frac{1}{r} \Psi_n' - n^2\alpha^2 \Psi_n \right) \\ & + \sum_{k=1}^{n-1} \mathfrak{d} + \sum_{k=n+1}^N \mathfrak{e} + \sum_{k=1}^{N-n} \mathfrak{f} = 0, \end{aligned}$$

where $n = 1, 2, \dots, N$ and the sum-terms for $\mathfrak{a} - \mathfrak{f}$ above contain inertial terms which are written as a function of (Φ, Ψ) . The reader is referred to Appendix B for the detail of these sum-terms as these terms are too bulky to detail in this section.

We also write (4.9), as a function of Φ and Ψ , as

$$h \equiv \hat{u}_0'' + \frac{1}{r} \hat{u}_0' - \frac{2}{\alpha R} \sum_{k=1}^N \left(\frac{\Phi_k \Phi_k'' + \Psi_k \Psi_k''}{kr^2} + \frac{\Phi_k \Phi_k' + \Psi_k \Psi_k'}{kr^3} \right) = 0, \quad (4.15)$$

where $k = 1, 2, \dots, N$. Finally, we take our amplitude and phase conditions and label these as

$$\tau \equiv A - \sqrt{\sum_{n=1}^N (\Phi_n(y_{J1})^2 + \Psi_n(y_{J1})^2)} = 0, \quad (4.16)$$

for our amplitude condition where $n = 1, 2, \dots, N$ and

$$\sigma \equiv \Phi_1(y_{J2}) = 0, \quad (4.17)$$

for the phase condition. We use the expansions in (4.11), (4.12) and (4.13) for our nonlinear equations, and write our nonlinear equations in vector form as

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}_m^n \\ \mathbf{g}_m^n \\ \mathbf{h}_m^n \\ \tau \\ \sigma \end{bmatrix}, \quad (4.18)$$

which is of length $(2N + 1)(M + 1) + 2$ where N is the number of nonlinear modes and M is the number of Chebyshev polynomials.

Turning our attention to the boundary conditions to be implemented in our computations, the no-slip boundary conditions in equation (4.5), in terms of ϕ_n , are $\phi_n = \phi'_n = 0$ on $r = \delta$ and $r = 1$, at the inner and outer cylinder walls respectively. We also have a mean flow correction equation (4.1), and the boundary conditions on this equation are $\hat{u} = 0$ on $r = \delta$ and $r = 1$ which corresponds to the no-slip condition on the inner and outer cylinder walls of ACPF.

Before proceeding to describe the computational method that we apply to solve these nonlinear equations, we can perform a check to see if we do indeed recover the Orr-Sommerfeld equation (3.10) from (4.7). If the disturbance amplitude is sufficiently small (in the zero-limit), the result is that any nonlinear/inertial terms will have a negligible effect on the stability of ACPF, and we should recover (3.10) which governs the linear stability of ACPF. In the zero-amplitude limit

$$A \rightarrow 0,$$

we find that the following inertial terms approach zero as well

$$\begin{aligned}\sum_{k=1}^{n-1} a &\rightarrow 0, \\ \sum_{k=1}^{N-n} b &\rightarrow 0, \\ \sum_{k=n+1}^N c &\rightarrow 0, \\ \hat{u}_0 &\rightarrow 0,\end{aligned}$$

and we reach our Orr-Sommerfeld equation (3.10) that governs the linear stability of ACPF. This acts as a partial check that our nonlinear equations are correct. We are now in a position to introduce a computational method to solve our nonlinear set of equations.

4.3 Multi-Dimensional Newton-Raphson Computational Method

Our choice of using the Newton-Raphson method, or Newton method, is due to the fact that we can compute, exactly, the derivatives of the nonlinear equations and we have a suitable initial guess from the linear stability results. Although the Newton method requires additional resources to be invested (to obtain the Jacobian terms and programming), the method is very robust and has a fast-convergence rate compared to other methods of computations for nonlinear equations. Also, because we are able to calculate the exact Jacobian and function at each step, we are able to use larger step sizes than we normally would if we chose a different computational method.

Let us begin by setting out the standard Newton method in one-dimension for a variable x , and function f , before generalizing to a multi-dimensional problem. Newton's method aims to find a solution for

$$f(x) = 0,$$

for a specified tolerance on the numerical result. First, we take an initial guess or approximate value of x_0 , and compute the function $f(x_0)$ and derivative $f'(x_0)$. We then compute

the step change, Δx , from this initial guess:

$$\Delta x = -\frac{f(x_0)}{f'(x_0)},$$

and Δx is the correction to x_0 for the next step of the iterative procedure. The next value of x , say x_1 is

$$x_1 = x_0 + \Delta x,$$

which we use to compute $f(x_1)$. This procedure can be generalised for i -iterations and where the generalized form of x is

$$x_{i+1} = x_i + \Delta x$$

and Δx can be written in general form as

$$\Delta x = -\frac{f(x_i)}{f'(x_i)}.$$

The process above will be repeated for i -iterations until we reach a stage where an appropriate convergence criterion is satisfied. Implementation of the convergence criterion is typically to set Δx to be less than or equal to some numerical value, for example 1×10^{-5} , determined by experience and dependent on the accuracy required for a mathematical problem. We therefore typically write

$$\Delta x = x_{i+1} - x_i \leq 1 \times 10^{-5}$$

for our convergence criteria. Newton's method can be summarised as follows

1. Define function f and derivative f' ,
2. Use initial guess x_0 to compute f and f' ,
3. Compute Δx using x_0 ,
4. Compute next point x_1 by adjusting x_0 with Δx ,
5. Compute convergence criterion $\Delta x \leq 1 \times 10^{-5}$ say,

6. Repeat steps 2 – 5 until the convergence criterion in step 5 is met,
7. If number of iterations is beyond say 10 or 15, stop computations as either the initial guess is not sufficiently close to the solution, or no solution exists.

Step 7 is essential to ensure that computational resources are being utilised sensibly since it is entirely possible that Newton's method does not find a solution for $f(x) = 0$, resulting in a never-ending iterative cycle. This may be due to an inappropriate initial guess x_0 or non-existence of any roots. Whilst Newton's method of root finding is very efficient and accurate, this is reliant upon having a sufficiently good initial guess to kick start the iterative procedure in order to locate the roots of the function. Also, if the tolerance on Δx is set too small, say 1×10^{-25} , it is possible that the convergence criterion is not met.

Now that we have established Newton's method for one variable, we can extend our analysis to a multi-variable one to solve our nonlinear set of equations (4.7) along with the amplitude and phase conditions (4.14). The numerical procedure is exactly the same as for a single variable, but the variable x , function f and the derivative of f' are more complicated. The variable x becomes a vector containing multiple variables, $\mathbf{x}$, the function $\mathbf{f}$ now includes more than one equation and the derivative, $\mathbf{f}'$ in the multi-dimensional Newton method is called the *Jacobian*, J . The Jacobian (J) is a (very) large square matrix. We can represent our multi-dimensional problem in vector form as

$$\mathbf{f}(\mathbf{x}_q) = 0,$$

where $\mathbf{f}$ contains the functions of multiple equations evaluated for vector $\mathbf{x}$ at the q -th iteration where $1 \leq q \leq 10$ typically. The step change is now

$$\Delta \mathbf{x} = -\frac{\mathbf{f}(\mathbf{x}_q)}{J(\mathbf{x}_q)}, \quad (4.19)$$

where J is the Jacobian, containing derivatives of the functions in equations (4.7) and (4.14) and we will provide more detail on the Jacobian in the next section. The convergence criteria we apply is $\Delta \mathbf{x} = (\mathbf{x}_{q+1} - \mathbf{x}_q)$. Typically, we set $|\Delta \mathbf{x}| \approx 1 \times 10^{-5}$. The vector $\mathbf{x}$ contains the nonlinear set of equations evaluated at Gauss points 1, ..., $M + 1$ for N-modes,

is of size $((2N + 1)(M + 1) + 2, 1)$ and can be illustrated as

$$\begin{bmatrix} a(m = 1, n = 1) \\ a(m = 2, n = 1) \\ \vdots \\ a(m = M, n = 1) \\ a(m = M + 1, n = 1) \\ a(m = 1, n = 2) \\ \vdots \\ a(m = M + 1, n = 2) \\ \vdots \\ a([m = 1 : M + 1, n = N])^T \\ b([m = 1 : M + 1, n = 1 : N])^T \\ d([m = 1 : M + 1]^T) \\ \alpha \\ c \end{bmatrix}, \quad (4.20)$$

where we solve for the coefficients a_m^n, b_m^n, d_m , wavenumber α , wavespeed c where $m = 1, 2, \dots, M + 1$ and $n = 1, 2, \dots, N$.

The step change $\Delta \mathbf{x}$ computed will be the corrections to the variables $a_m^n, b_m^n, d_m, \alpha$ and c and Δx is a vector of the same vector-size as $\mathbf{x}$ above. The corresponding equations to solve are $\mathbf{f} = (f_n, g_n, h, \sigma, \tau)$ as given in equation (4.18).

The initial conditions to start our Newton iteration are taken from the linear stability results, such as α, c, ϕ for a certain set of ACPF parameters. Once we obtain a set of finite, small-amplitude results, we can use these results for subsequent computations at larger disturbance amplitude, A . Our computations are presented for up to $A \approx 500$. Beyond this, limitations on computational resources such as memory make it tricky to obtain accurate numerical results.

To determine the number of Chebyshev polynomials $(M + 1)$ and number of nonlinear modes (N) to use, we present figure 4.1 to illustrate the effect of both M and N on the numerical accuracy and duration of the computation for one point of a neutral curve. We can

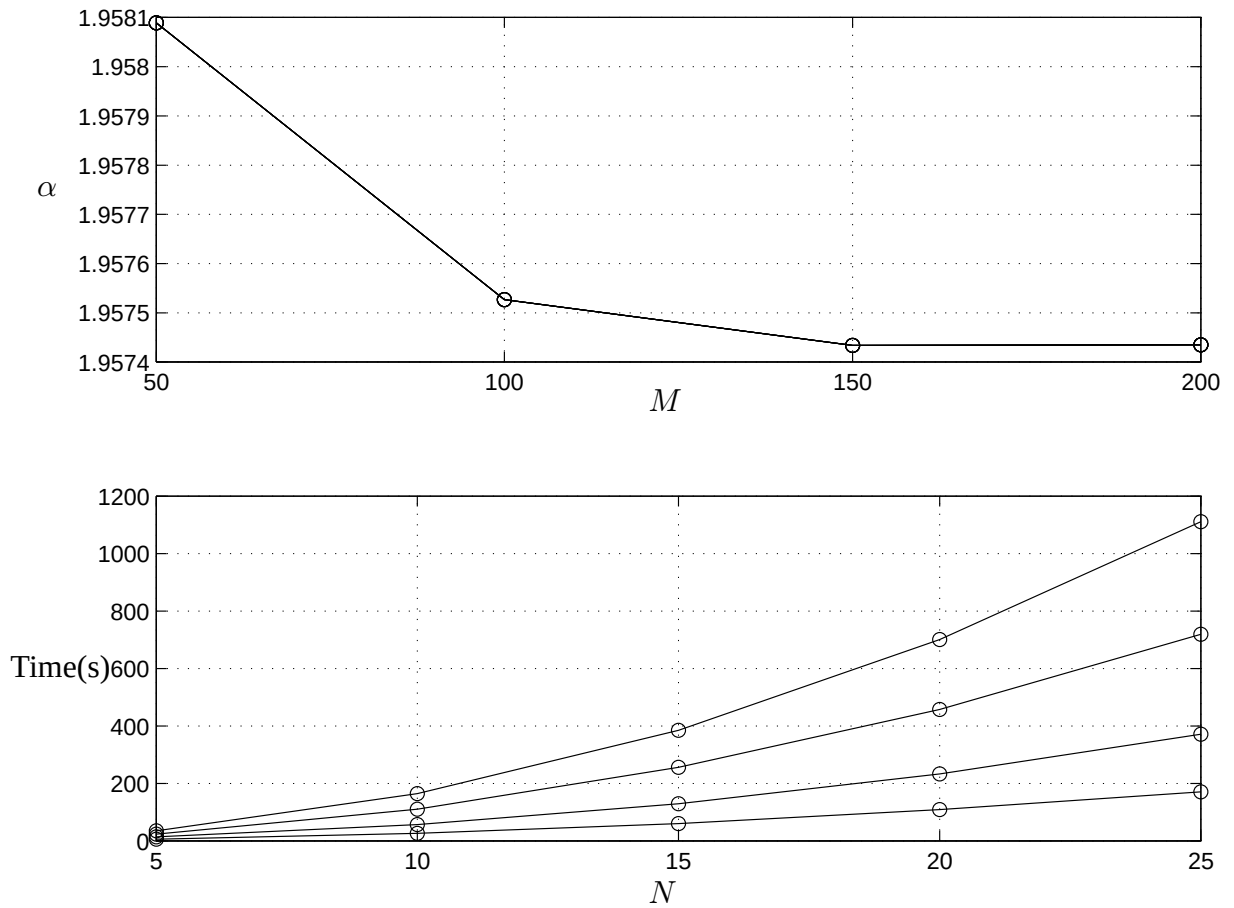


Figure 4.1: Resolution of neutral curve solution for $\delta = 0.5$ at $R = 1 \times 10^6$ for $V = 0$ at $A = 15$. Top: wavenumber against number of Chebyshev points for number of modes of $N = 5, 10, 15, 20, 25$ and $M = 50, 100, 150, 200$. Bottom: Time (seconds) against $N = 5, 10, 15, 20, 25$ for $M = 50, 100, 150, 200$

confirm that there is no change, to 5 decimal places, to the numerical solution of wavenumber by any increase in N and it is clear that increasing N has a significant impact on the time required to compute each point on a neutral curve. At $M = 150$, there is minimal change in numerical accuracy of wavenumber and we use $M = 150$ for our computations. The number of nonlinear modes required depends upon the disturbance amplitude. However, $5 \leq N \leq 25$ proves to be sufficient and the higher the disturbance amplitude, the more nonlinear modes required.

Now that we have highlighted the numerical method to solve the nonlinear ACPF problem, we look at the Jacobian in more detail as the Jacobian plays a key role in Newton's method. In addition, it is not immediately obvious how each term of Φ 's and Ψ 's can be differentiated with respect to the coefficients of say a_i^j , which we explain in the next section.

4.4 The Jacobian Revealed

The Jacobian, J , can be computed explicitly at each point of the Newton iteration and this improves convergence of the computation. Let us illustrate how the derivative with respect to a_i^j of say Φ'_n can be computed.

We have

$$\Phi'_n = \sum_{m=1}^{M+1} a_m^{(n)} T'_{m-1}(y),$$

and therefore

$$\frac{\partial(\Phi'_n)}{\partial a_i^j} = \begin{cases} T'_{i-1}(y), & \text{if } n = j, \\ 0 & \text{otherwise,} \end{cases} \quad (4.21)$$

where $n = 1, 2, \dots, N$, $i = 1, 2, \dots, M + 1$, y is defined in (3.17) and the derivative, say $T'_{m-1}(y)$, is differentiated with respect to y . The inertial terms with the sums are more complicated, containing more than one sum for each term. Let us take the example of

$$\sum_{k=1}^{n-1} \Phi'_k \Psi''_{n-k}, \quad (4.22)$$

to demonstrate how the derivative of this term can be found with respect to the coefficient

a_i^j . For convenience, we will use slightly different subscripts and superscripts in order to identify clearly which modes and Gauss points we are referring to. We expand (Φ'_k) and (Ψ''_{n-k}) in terms of Chebyshev polynomials

$$\Phi'_k = \sum_{m=1}^{M+1} a_m^k T'_{m-1}(y) \quad \text{and} \quad \Psi''_{n-k} = \sum_{s=1}^{M+1} b_s^{n-k} T''_{s-1}(y). \quad (4.23)$$

We wish to differentiate this with respect to the coefficient a_i^j . This is written as

$$\frac{\partial}{\partial a_i^j} \left(\sum_{k=1}^{n-1} \Phi'_k \Psi''_{n-k} \right),$$

or

$$\frac{\partial}{\partial a_i^j} \left(\sum_{k=1}^{n-1} \left[\sum_{m=1}^{M+1} a_m^k T'_{m-1}(y) \sum_{s=1}^{M+1} b_s^{n-k} T''_{s-1}(y) \right] \right).$$

We can generalise the equation (4.21) by defining $\kappa_{j,k}$ such that we get

$$\kappa_{j,k} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases} \quad (4.24)$$

Then, it follows that

$$\frac{\partial a_m^k}{\partial a_i^j} = \kappa_{i,m} \kappa_{j,k}. \quad (4.25)$$

Implementing the above to our example, we find that

$$\frac{\partial}{\partial a_i^j} \left(\sum_{k=1}^{n-1} \Phi'_k \Psi''_{n-k} \right) = \frac{\partial}{\partial a_i^j} \sum_{k=1}^{n-1} \sum_{m=1}^{M+1} \sum_{s=1}^{M+1} (a_m^k T'_{m-1}(y) b_s^{(n-k)} T''_{s-1}(y)) \quad (4.26)$$

becomes

$$\sum_{k=1}^{n-1} \sum_{m=1}^{M+1} \sum_{s=1}^{M+1} \left(\frac{\partial a_m^k}{\partial a_i^j} T'_{m-1}(y) b_s^{(n-k)} T''_{s-1}(y) \right), \quad (4.27)$$

and we apply (4.24)

$$\sum_{k=1}^{n-1} \sum_{m=1}^{M+1} \sum_{s=1}^{M+1} (\kappa_{i,m} \kappa_{j,k} T'_{m-1}(y) b_s^{(n-k)} T''_{s-1}(y)) \quad (4.28)$$

we thus obtain

$$\sum_{s=1}^{M+1} T'_{i-1}(y) b_s^{(n-j)} T''_{s-1}(y), \quad (4.29)$$

where previously $k = 1 : n - 1$, we now have $j = 1 : n - 1$ and where $m = 1 : M + 1$, we have $i = 1 : M + 1$ instead. Noting that

$$\Psi''_{n-j} = \sum_{s=1}^{M+1} b_s^{(n-j)} T''_{s-1}(y), \quad (4.30)$$

therefore we reach our derivative

$$\frac{\partial}{\partial a_i^j} \left(\sum_{k=1}^{n-1} \Phi'_k \Psi''_{n-k} \right) = T'_{i-1}(y) \Psi''_{n-j}. \quad (4.31)$$

We apply the above to all the nonlinear terms in our systems of equations and compute the resulting Jacobian exactly. The Jacobian is a square matrix of size $((2N + 1)(M + 1) + 2)^2$ and can get very large with an increase in N , as we have demonstrated in the previous section. The number of elements contained in the Jacobian increases rapidly if N is increased, as figure 4.2 demonstrates.

Finally, we can form the Jacobian matrix as a combination of matrices:

$$\begin{bmatrix} \frac{\partial \mathbf{f}_m^{(n)}}{\partial a_i^{(j)}} & \frac{\partial \mathbf{f}_m^{(n)}}{\partial b_i^{(j)}} & \frac{\partial \mathbf{f}_m^{(n)}}{\partial d_i} & \frac{\partial \mathbf{f}_m^{(n)}}{\partial A} & \frac{\partial \mathbf{f}_m^{(n)}}{\partial c} \\ \frac{\partial \mathbf{g}_m^{(n)}}{\partial a_i^{(j)}} & \frac{\partial \mathbf{g}_m^{(n)}}{\partial b_i^{(j)}} & \frac{\partial \mathbf{g}_m^{(n)}}{\partial d_i} & \frac{\partial \mathbf{g}_m^{(n)}}{\partial A} & \frac{\partial \mathbf{g}_m^{(n)}}{\partial c} \\ \frac{\partial \mathbf{h}_m}{\partial a_i^{(j)}} & \frac{\partial \mathbf{h}_m}{\partial b_i^{(j)}} & \frac{\partial \mathbf{h}_m}{\partial d_i} & \frac{\partial \mathbf{h}_m}{\partial A} & \frac{\partial \mathbf{h}_m}{\partial c} \\ \frac{\partial \tau}{\partial a_i^{(j)}} & \frac{\partial \tau}{\partial b_i^{(j)}} & \frac{\partial \tau}{\partial d_i} & \frac{\partial \tau}{\partial A} & \frac{\partial \tau}{\partial c} \\ \frac{\partial \sigma}{\partial a_i^{(j)}} & \frac{\partial \sigma}{\partial b_i^{(j)}} & \frac{\partial \sigma}{\partial d_i} & \frac{\partial \sigma}{\partial A} & \frac{\partial \sigma}{\partial c} \end{bmatrix}$$

where $\mathbf{f}_m^{(n)}$, $\mathbf{g}_m^{(n)}$, $\mathbf{h}_m$, σ , τ represent the nonlinear equations (4.18) detailed in the previous section.

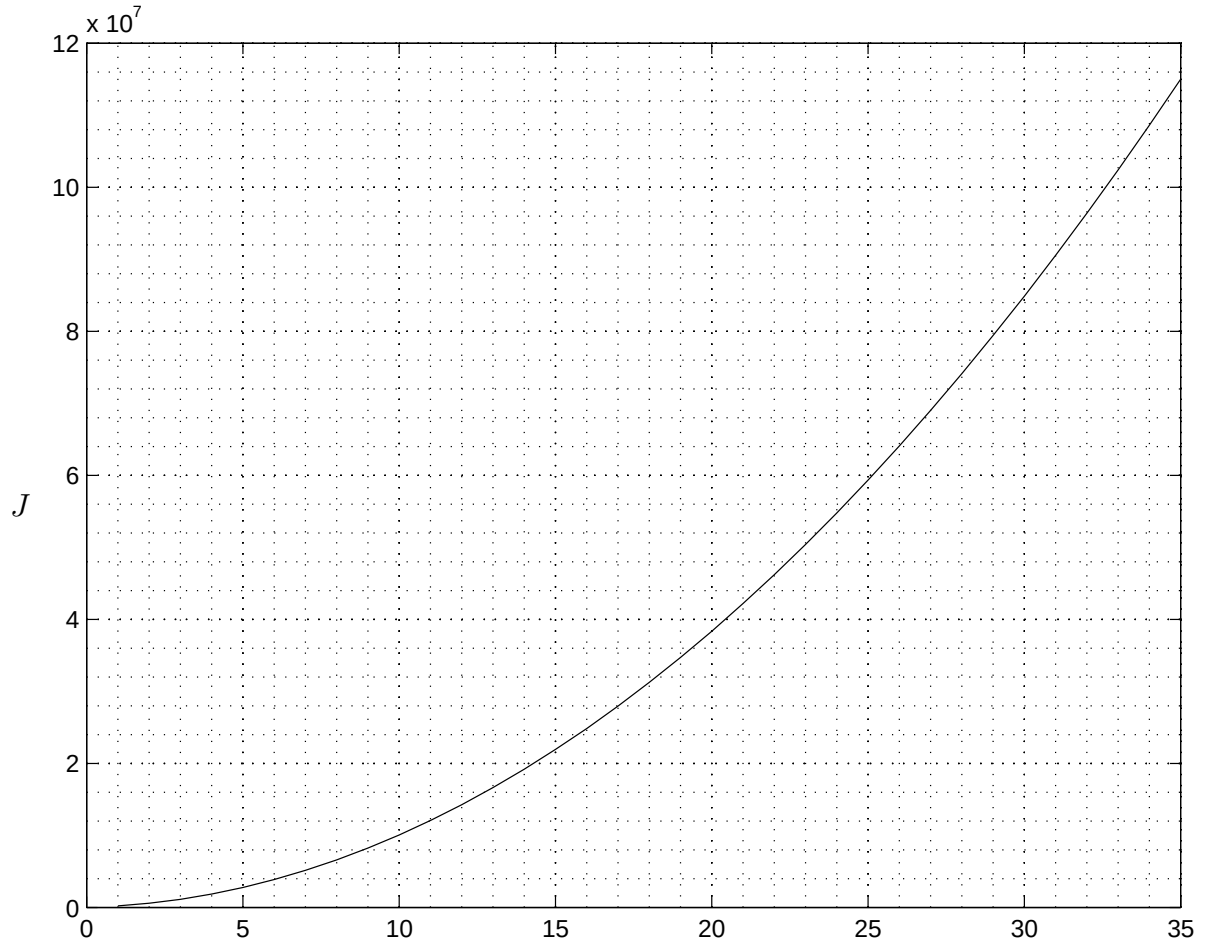


Figure 4.2: Number of elements of the Jacobian, J , against the number of modes, N , illustrating the quadratic relationship between J and N through $((2N + 1)(M + 1) + 2)^2$, for $M = 150$

4.5 Nonlinear Stability of ACPF: Style and Presentation of Results

We present our results mostly in the form of neutral curves, but also look at the behaviour of the disturbance wave and mean flow distortion to uncover the influences of the disturbance on the overall flow. The α, R graphs are a clear indication of whether a flow is more, or less stable, as a result of changes to the disturbance and other flow conditions. The mean flow distortion, which is part of the solution found from our computational analysis, will give an indication as to where the mean flow disturbance is concentrated, and the overall effect on the basic flow.

To appreciate the behaviour of the disturbance, it is useful to present contour plots with velocity arrows. In order to do so, we note that ϕ is:

$$\phi = \sum_n \phi_n(r) e^{in\alpha(x-ct)} + c.c., \quad (4.32)$$

where $c.c.$ is the complex conjugate. In this instance, we are interested in understanding the disturbance wave behaviour and we take the *travelling wave coordinate* of $\xi = \alpha(x - ct)$. We expand $\phi_n = \Phi_n + \Psi_n$ and $e^{in\xi} = \cos(n\xi) + i \sin(n\xi)$. The radial velocity is therefore represented by:

$$v(r, \xi) = \frac{2}{r} \sum_n (\Phi_n(r) \cos n\xi - \Psi_n(r) \sin n\xi),$$

and the axial velocity, through the continuity equation, is

$$u(r, \xi) = -\frac{2}{r} \frac{2}{1 - \delta} \sum_n \left(-\Phi'_n(r) \frac{\sin n\xi}{n} - \Psi'_n(r) \frac{\cos n\xi}{n} \right).$$

Finally, the stream function is:

$$\varphi(r, \xi) = -2 \sum_n \left(\Phi_n(r) \frac{\sin n\xi}{n} + \Psi_n(r) \frac{\cos n\xi}{n} \right).$$

The disturbance stream function will reveal the location, between the inner and outer cylinder walls, where the basic flow is most affected. The closer the contour lines the stronger

the disturbance influence. As we will see, the disturbance influence is usually concentrated near the inner and outer cylinder walls. The size and direction of the arrows in our figures also show the location and potency of the disturbance.

4.6 Nonlinear Stability Numerical Results

In the next few sections, we present numerical results for several conditions of ACPF of radius ratio, inner cylinder velocity and disturbance amplitude. These are largely in the form of neutral curves, with contour and velocity figures to illustrate the behaviour of the flow at the nose of the neutral curve, i.e. at or near the critical Reynolds number of the flow.

We look specifically at the cases for $\delta = (0.4, 0.5, 0.55)$ for both zero and non-zero inner cylinder velocities at various disturbance amplitudes. We begin with the case of minimum Reynolds number found in the linear regime where $\delta = 0.5$ with inner cylinder velocity $V = 0.0$.

4.7 Minimum Critical Reynolds number case $\delta = 0.5$ & $V = 0.0$

In figure 4.3, the effect of increasing disturbance amplitude, A , is presented for the case of minimum R_{crit} where $\delta = 0.5$ and $V = 0.0$ and the linear neutral curve (inner most curve) is also plotted for comparison. There is minimal change to the lower branch of the neutral curve, whilst the upper curve varies significantly and is much more sensitive to changes in A . In addition, the bandwidth of unstable wavelength increases. A minimum critical Reynolds number of approximately 1.15×10^5 is reached when $A = 500$. The author finds that increasing the amplitude above $A \approx 400$ does not yield any further significant reduction in R_{crit} suggesting that there is a *critical disturbance amplitude*, $A_{crit} \approx 400$ beyond which R_{crit} does not alter. Also, the number of nonlinear modes required to compute neutral curve solutions at higher A becomes excessively expensive rendering the computational analysis inefficient. The number of modes used to compute the neutral curve solution at $A = 500$ is $N = 10$. Given these issues, no further computational analysis is carried out for higher A nor N .

Figure 4.4 demonstrates the overall behaviour of R_{crit} for increasing disturbance am-

plitude, A . We note that there is initially a stabilising effect, where R_{crit} increases between $A = 0$ to $A \approx 10$. However, this behaviour is short lived and R_{crit} decreases rapidly until $A \approx 250$. At $A = 400$, $R_{crit} = 1.15 \times 10^5$ and there appears to be minimal change to this value of R_{crit} for $A \geq 400$ suggesting that an asymptote has been reached, although we cannot be certain as our computations have been limited to $A \approx 500$. Comparing this to linear ACPF of 4.7×10^5 , it is clear that nonlinear effects contribute significantly to the stability of ACPF.

In order to understand the effects of the nonlinear disturbances on the stability of ACPF, we present figures 4.5 and 4.6. Figure 4.5(a), shows the basic flow and wavespeed, c (relatively small), where c cuts the basic flow curve at two points: close to the inner and outer cylinder which corresponds to two critical layers. Figure 4.5(b) shows the mean flow distortion, $\hat{u}_0$, to the basic flow. The mean flow distortion is largely negative ($O(10^{-2})$), slowing the total flow with the exception of a small positive distortion close to the outer cylinder wall. Figure 4.5(c) demonstrates very clearly that the $n = 1$ mode is the most influential mode and dictates the nonlinear effects for this case and this is a trend that remains constant throughout our nonlinear analysis of the stability of ACPF.

Figure 4.6 shows the contours of radial velocity and stream function of the disturbance wave as a function of radius, r , and the travelling-wave coordinate, $\alpha(x - ct)$ going from $0 \leq \alpha(x - ct) \leq 2\pi$. The contours show that the disturbance wave is most potent near the inner and outer cylinder walls and is relatively calm in between the cylinders. The disturbance is stronger near the inner cylinder wall than the outer cylinder wall.

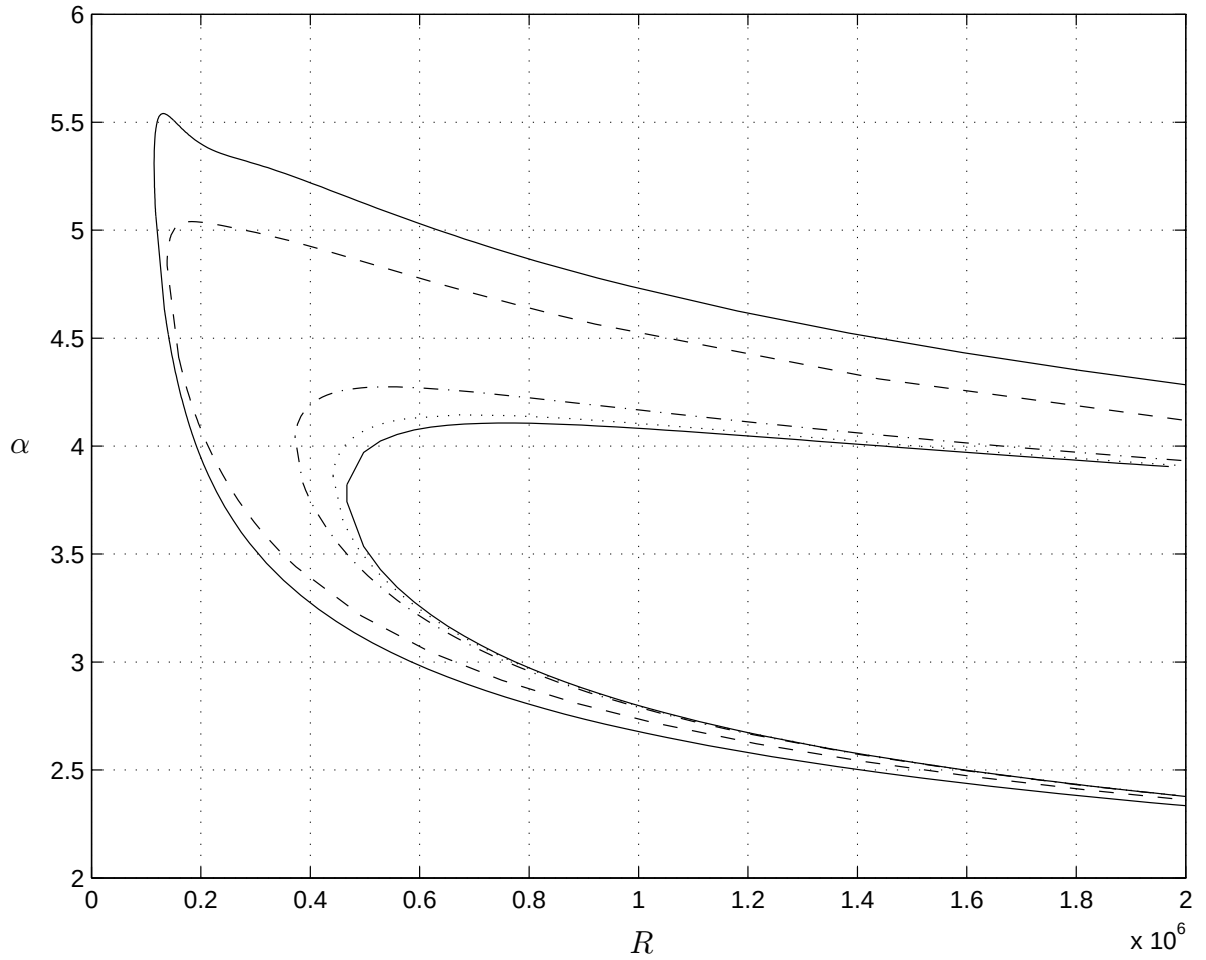


Figure 4.3: Neutral curves (α against R) for $\delta = 0.5, V = 0.0$ for various disturbance amplitudes. From outer to inner: $A = 500$ (solid), 300 (dashed), 100 (dot-dashed), 0 (solid, linear solution).

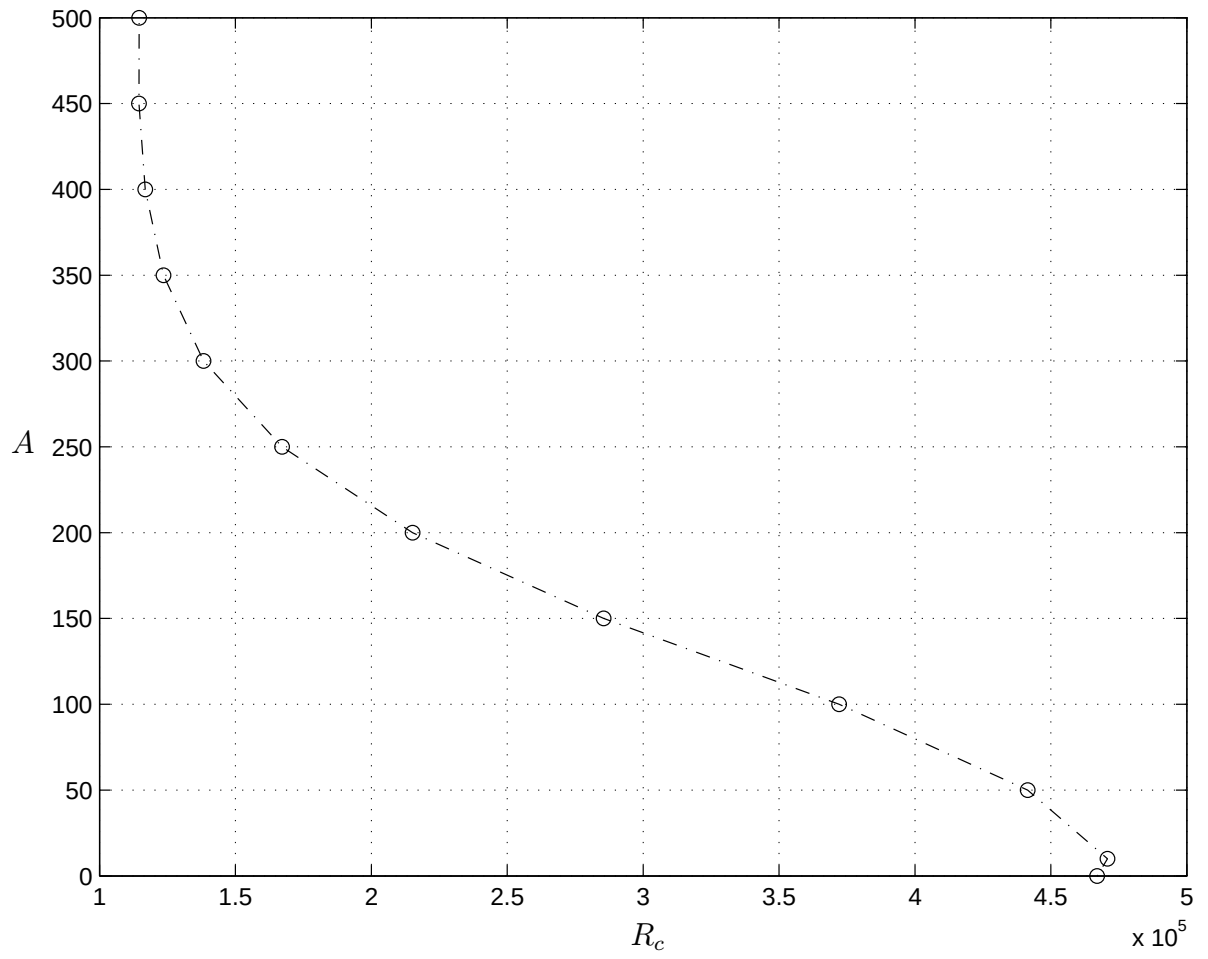


Figure 4.4: Disturbance amplitude A against R_{crit} for $\delta = 0.5, V = 0.0$.

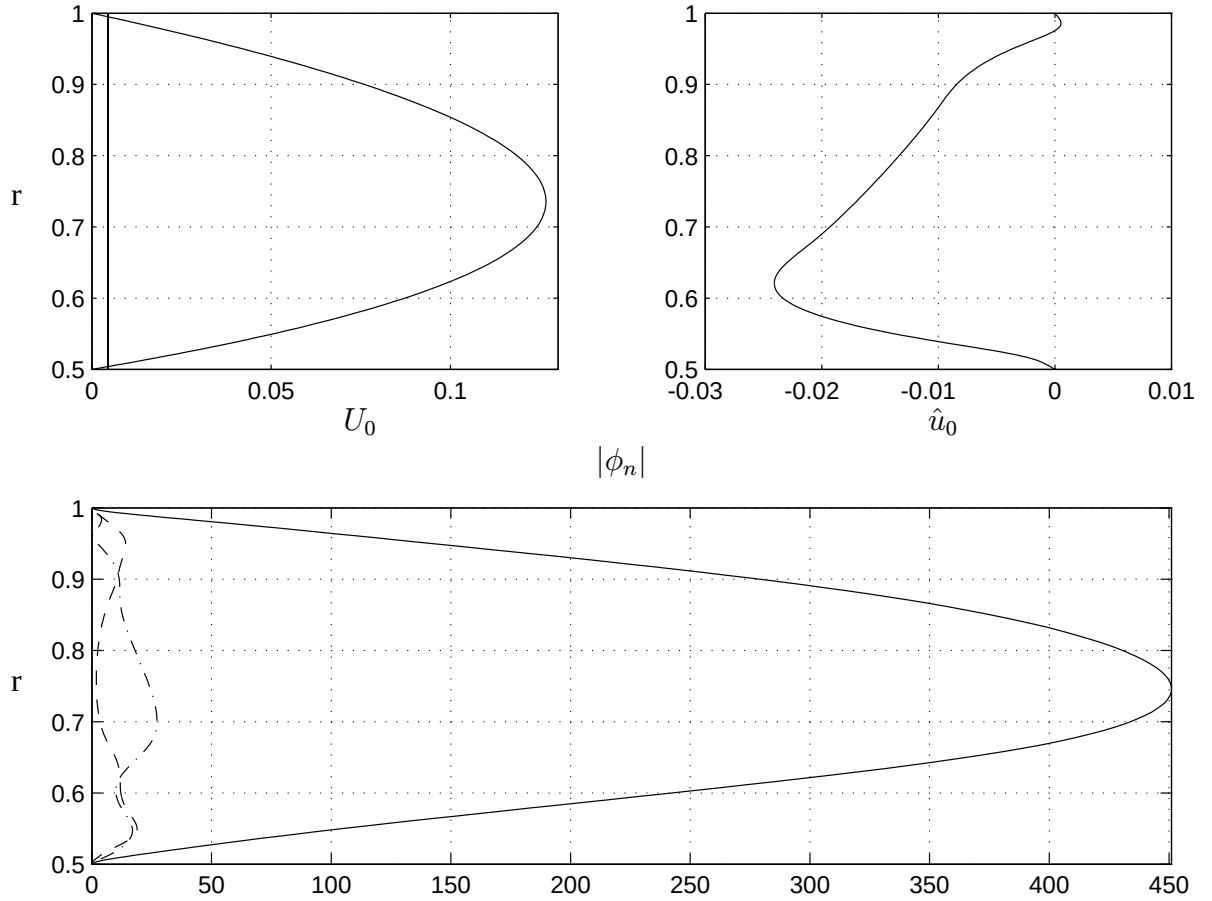


Figure 4.5: For $\delta = 0.5, V = 0.0, A = 500$ at R_{crit} : (a) Profile of the basic flow and wavespeed (vertical line) (top left). (b) Profile of the mean flow distortion (top right). (c) Profile of the intensity of nonlinear modes for $N = 1$ (solid), $N = 2$ (dot-dashed) and $N = 3$ (dashed) (bottom).

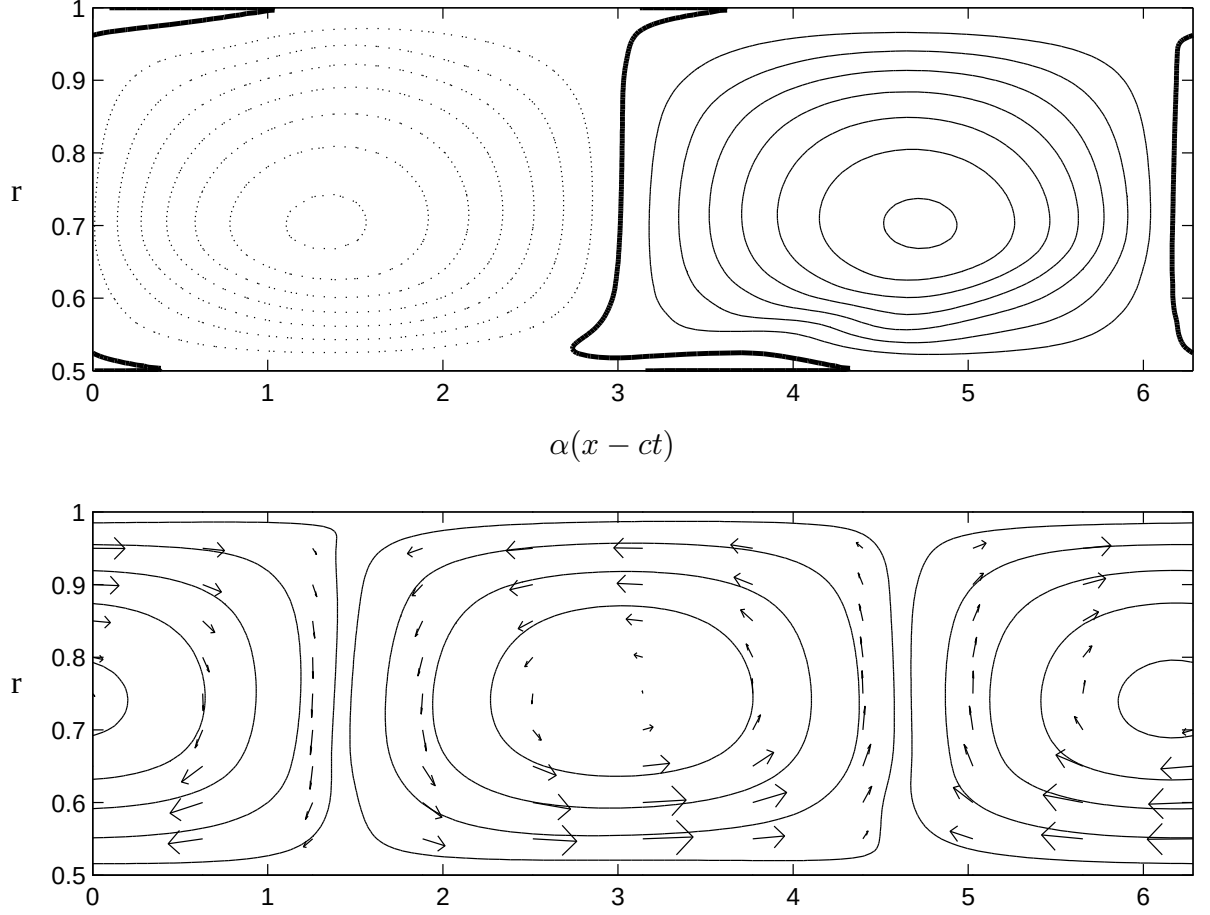


Figure 4.6: Contours of the radial velocity and stream function of disturbance as a function of radius, r , and travelling-wave coordinate $\alpha(x - ct)$ for $\delta = 0.5$, $V = 0.0$ at R_{crit} for $A = 500$. (a) Contours of radial velocity at $(\pm 200 \pm 400 \pm 600 \pm 800 \pm 1000 \pm 1200 \pm 1400)/R_{crit}$ (top). (b) Disturbance stream function with disturbance velocity vector field superimposed (bottom).

4.8 Multiple neutral curves existence, $\delta = 0.55$

Next, we turn our attention to the case of ACPF for $\delta = 0.55$ and $V = 0.01$ where the linear analysis of ACPF shows the existence of two neutral curves of approximately equal sizes in figure 4.7. We vary the disturbance amplitude with the resulting neutral curves shown in figure 4.7. Looking at the upper curve, there appears to be a steady decrease in the minimum Reynolds number. The lower curve on the other hand shows very little movement until $A \simeq 400$. The lower curve behaviour is similar to that of the lower branch for $\delta = 0.5, V = 0.0$. Indeed, this is what Walton (2004) predicted; the lower curve for $\delta = 0.55, V = 0.01$ takes on the lower branch properties similar to the case of $\delta = 0.5, V = 0.0$. However, taking the critical Reynolds number across both neutral curves, we find that the increasing A has an ultimately destabilising effect on the flow in that R_{crit} decreases and the band of wavenumbers increases for both the lower and upper neutral curves, increasing the unstable area of these neutral curves.

From figure 4.8, we observe that R_{crit} taken across both curves initially *increases* up to $A \simeq 300$ and decreases rapidly up to $A \simeq 500$. Beyond $A \simeq 500$, there is little change to R_{crit} . Also, it becomes computationally challenging to obtain accurate results near the nose of the neutral curves for $A \geq 600$ due to an increase in the number of nonlinear modes required and the nose of the neutral curves appears to reach a junction, where the suggestion is these curves re-join to form one neutral curve. The reader may note that the initial increase, before a rapid decrease, in R_{crit} was observed in the case for $\delta = 0.5, V = 0.0$, albeit for much lower A in the latter. It does not seem that, even if the curves re-attached at the nose, that R_{crit} would be further reduced for any increase in A . Indeed, this is what the author found in conducting computations at disturbance amplitudes of $A \geq 600$. This suggests that $A_{crit} \approx 500$ in this instance.

We analyse the disturbance wave behaviour from figures 4.9 to 4.12 by reviewing the critical layer location, mean flow distortion, nonlinear mode influence on the computations, the radial velocity and stream function of the disturbance. Firstly, the wavespeed, c , is smaller in the upper curve and indicates that there is one critical layer for this case. The location of the critical layer is close to the outer cylinder wall. Next, the mean flow distortion, $\hat{u}_0$ is very different for the upper and lower branches of the neutral curves: (i)

$\hat{u}_0$ for the upper curve shows drastic variation close to the inner and outer cylinder walls whereas there is a much smoother transition in the lower curve, (ii) the magnitude of $\hat{u}_0$ for the upper branch solution is $O(10^{-3})$ whereas this is $O(10^{-2})$ in the lower branch solution, and (iii) the region of $\hat{u}_0 \geq 0$ is located near the inner cylinder wall for the upper curve whilst the opposite is true for the lower curve. As with the $\delta = 0.5, V = 0.0$ case, the first mode is the most dominant in both upper and lower branch solutions.

Figures 4.11 and 4.12 show the contours of radial velocity and stream function of the disturbance wave for the upper and lower curves respectively, at the minimum Reynolds number, i.e. at the nose of the curves. From observing the size of the velocity arrows and contours, the disturbance for both curves are of similar magnitude with the intensity reaching a maximum at the inner wall. The disturbance close to the outer wall is larger in the upper curve than it is for the lower curve. Therefore, the upper curve shows a shift of the region of relatively-calm flow (approximately) midway between the inner and outer walls.

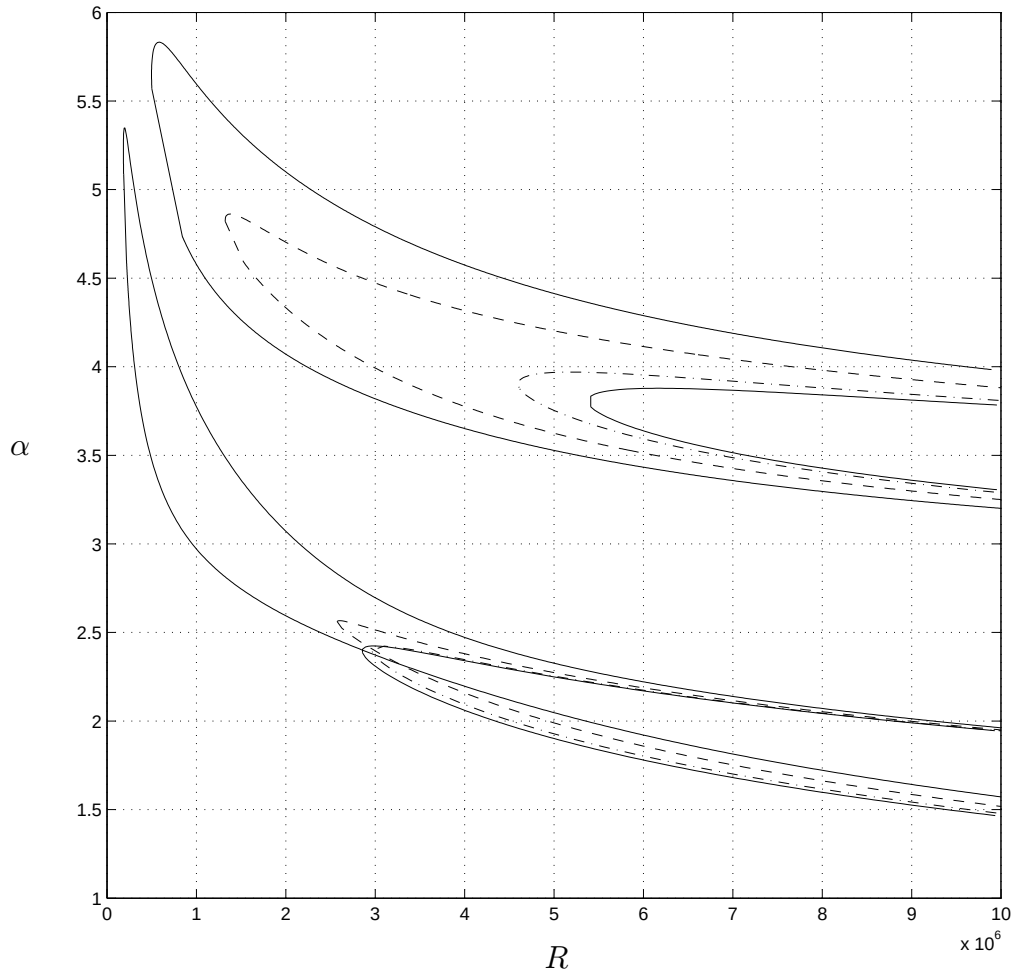


Figure 4.7: Multiple neutral curves (α against R) at various disturbance amplitudes for $\delta = 0.55, V = 0.01$. For each curve, outer to inner: $A = 600$ (solid), 400 (dashed), 200 (dot-dashed), 0 (solid, linear solution).

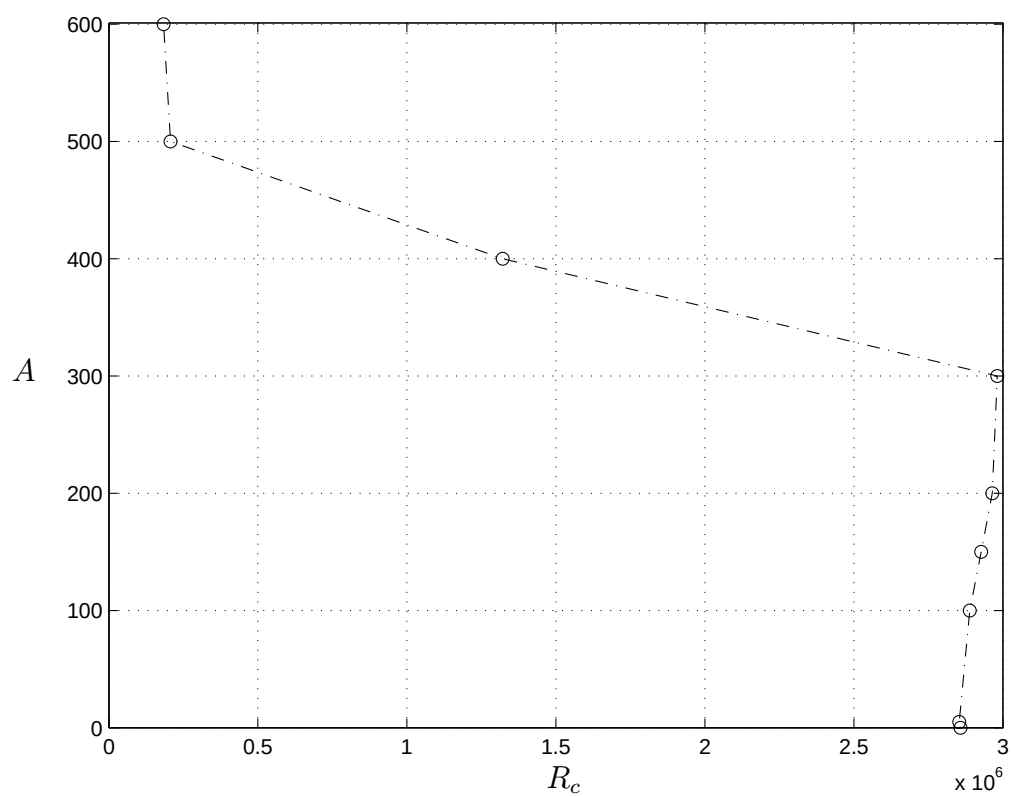


Figure 4.8: Disturbance amplitude A against R_{crit} for $\delta = 0.55, V = 0.01$ across multiple neutral curves.

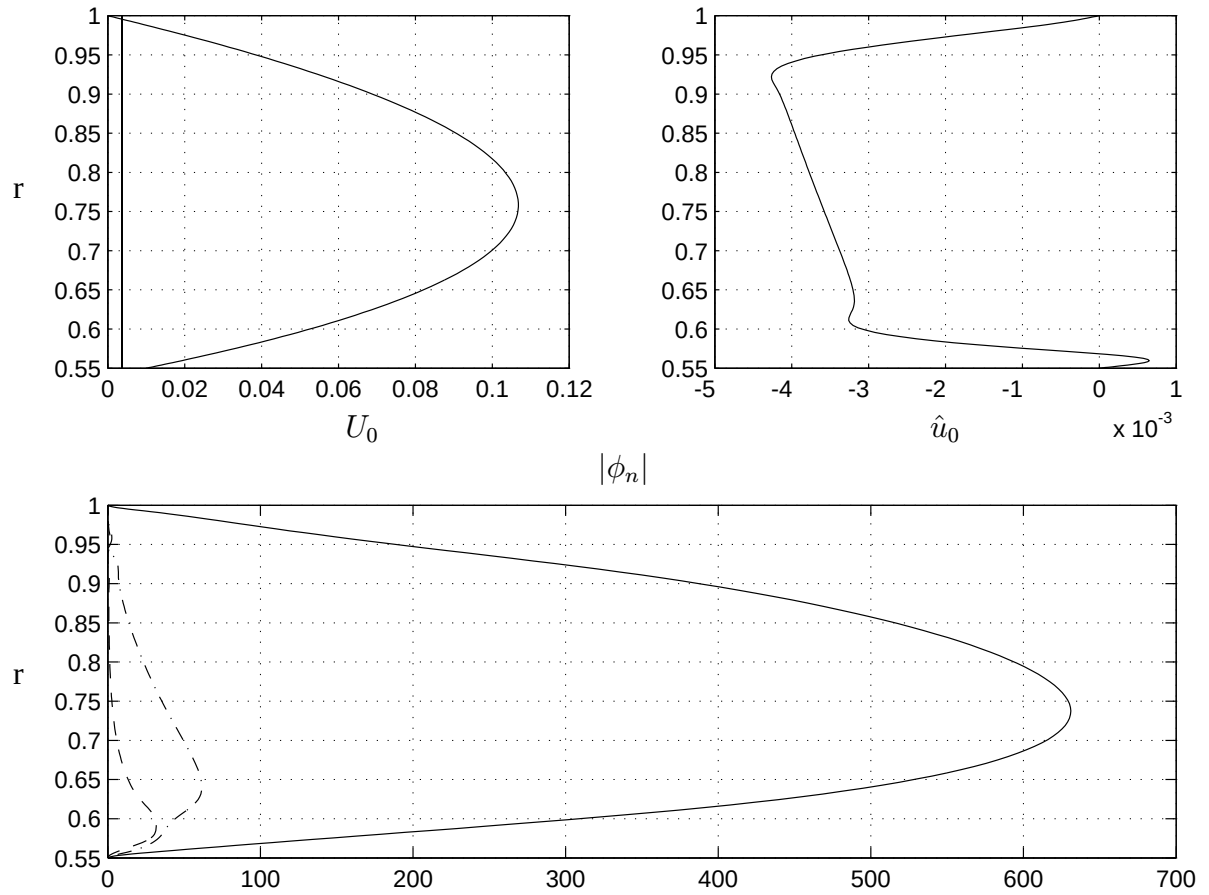


Figure 4.9: For upper curve of $\delta = 0.55$, $V = 0.01$, $A = 600$ at R_{crit} : (a) Profile of the basic flow and wavespeed (vertical line) (top left). (b) Profile of the mean flow distortion (top right). (c) Profile of the intensity of nonlinear modes for $N = 1$ (solid), $N = 2$ (dot-dashed) and $N = 3$ (dashed) (bottom).

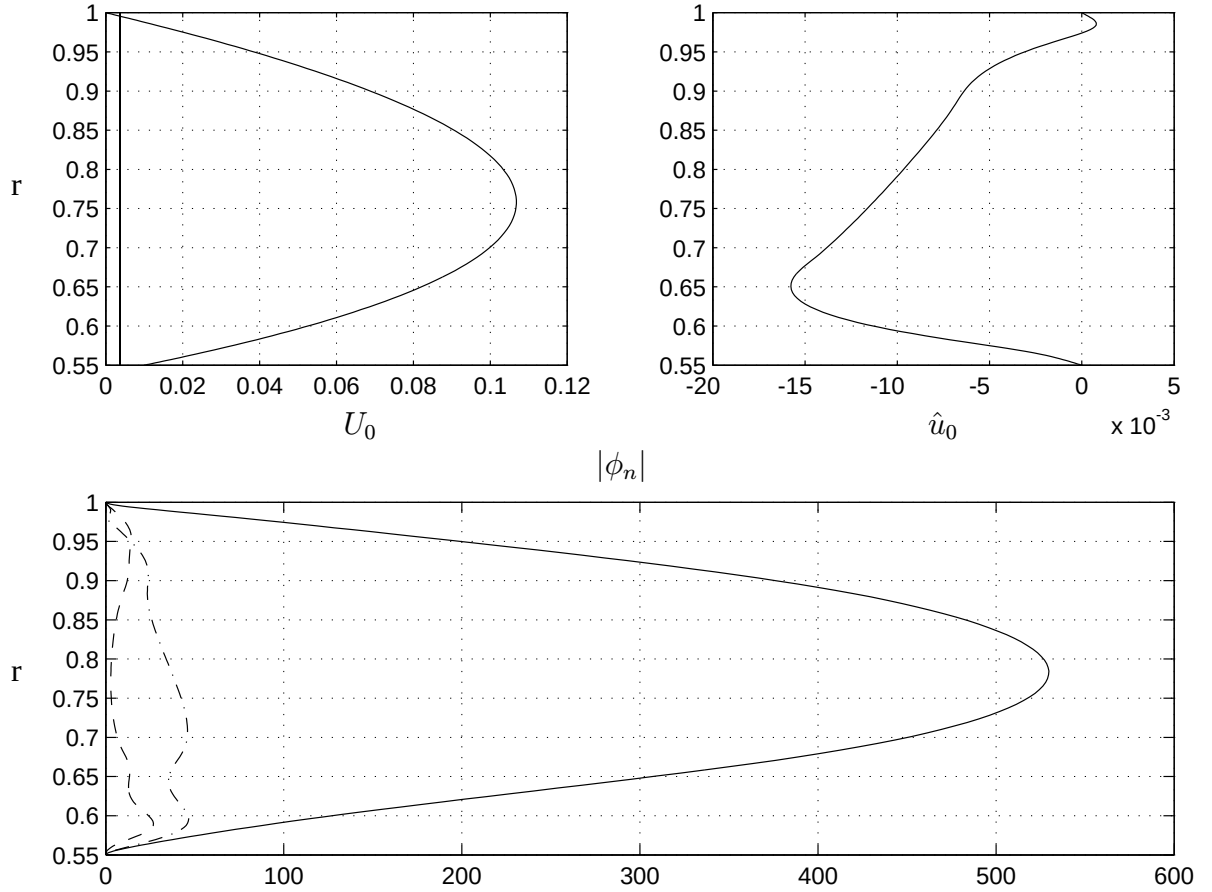


Figure 4.10: For lower curve of $\delta = 0.55$, $V = 0.01$, $A = 600$ at R_{crit} : (a) Profile of the basic flow and wavespeed (vertical line) (top left). (b) Profile of the mean flow distortion (top right). (c) Profile of the intensity of nonlinear modes for $N = 1$ (solid), $N = 2$ (dot-dashed) and $N = 3$ (dashed) (bottom).

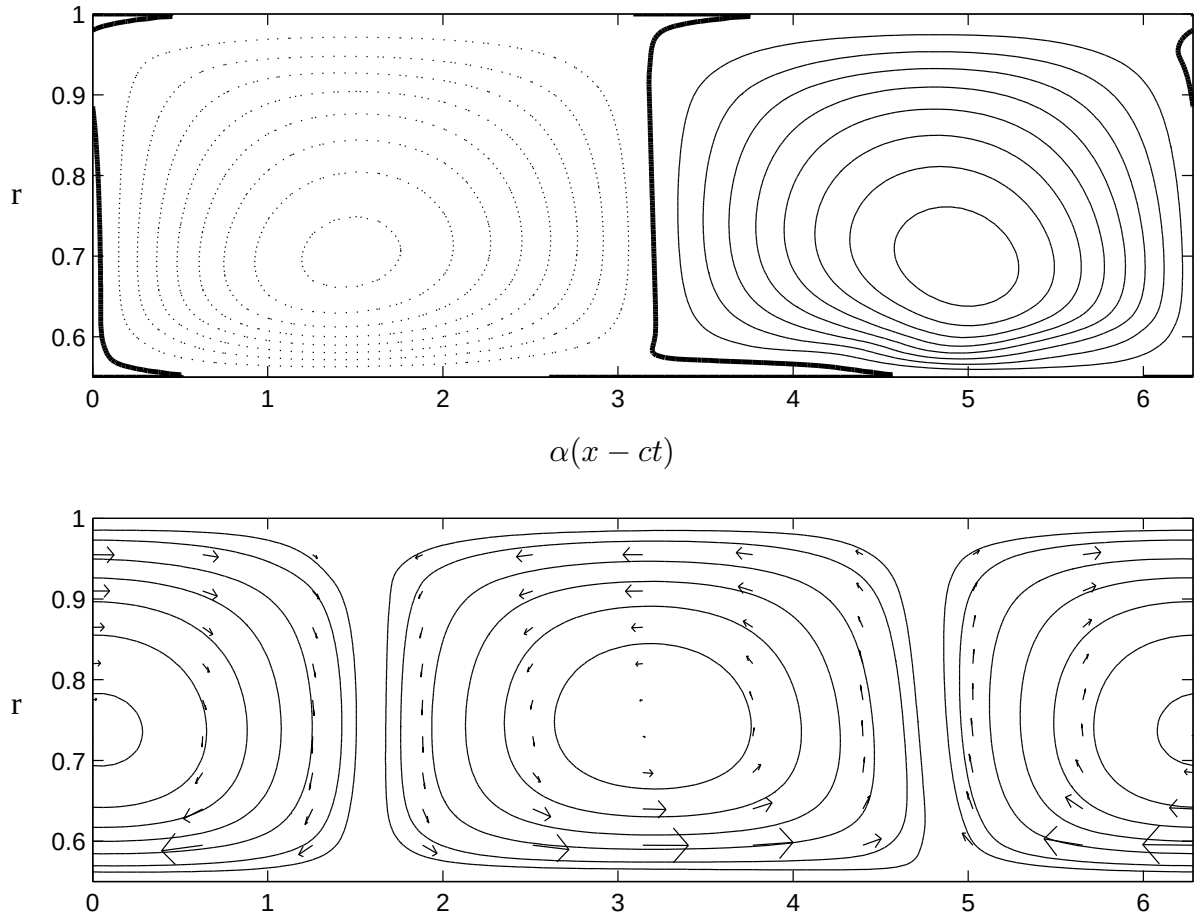


Figure 4.11: Contours of the radial velocity and stream function for upper curve for $\delta = 0.55, V = 0.01$ at R_{crit} for $A = 600$. (a) Contours of radial velocity at $(\pm 200 \pm 400 \pm 600 \pm 800 \pm 1000 \pm 1200 \pm 1400 \pm 1600)/R_{crit}$ (top). (b) Disturbance stream function with disturbance velocity vector field superimposed (bottom) .

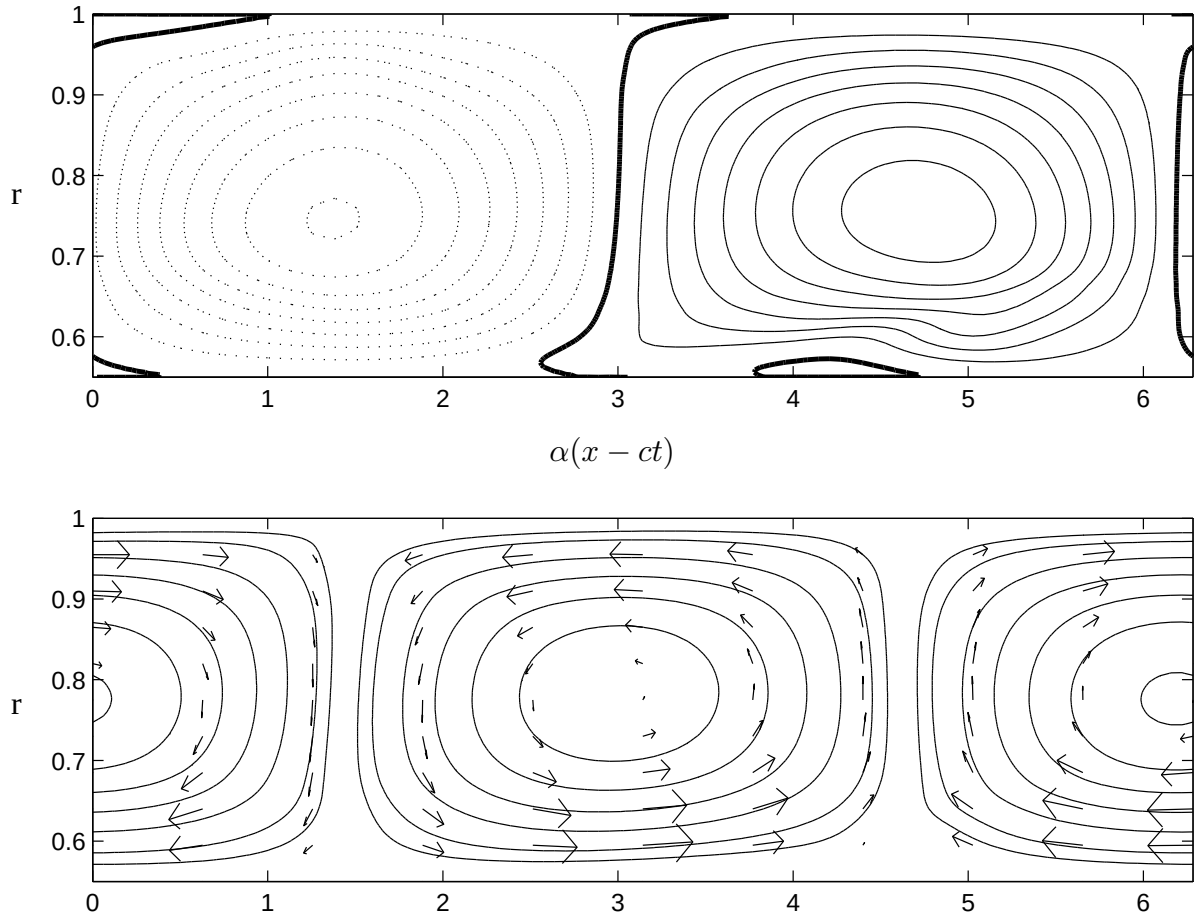


Figure 4.12: Contours of the radial velocity and stream function for lower curve for $\delta = 0.55$, $V = 0.01$ at R_{crit} for $A = 600$. (a) Contours of radial velocity at $(\pm 200 \pm 400 \pm 600 \pm 800 \pm 1000 \pm 1200 \pm 1400 \pm 1600)/R_{crit}$ (top). (b) Disturbance stream function with disturbance velocity vector field superimposed (bottom).

4.9 Closed neutral curve $\delta = 0.4$

Figure 4.13 presents neutral curves for the case of $\delta = 0.4$ with an inner cylinder velocity $V = 0.0356$, for various disturbance amplitudes. At this value of V , ACPF is linearly stable and the linear program predicts no instability - the minimum value of V where there is evidence of linear instability is slightly under $V \approx 0.0355$ for $\delta = 0.4$ (see figure 3.8) and nonlinear effects have clearly generated instability. In order to uncover the detail of this behaviour, we use the procedure below:

- start from an inner cylinder velocity where the linear solution almost disappears, V_{min} ,
- increase the disturbance amplitude, A , for V_{min} up to larger values of A . In this case, we increase A to $A = 200$,
- at $A = 200$, increase V from V_{min} to larger values of V where linear space predicts no instability. In this case, we increase V to 0.0356,
- at $V = 0.0356$, we then decrease A back down from $A = 200$ to lower values of A until we discover a critical or cut-off amplitude, A_{crit} and find that $A_{crit} \approx 94$ in this case,
- at each stage of V or A , we map out a neutral curve in order to understand (and gauge the trend of) ACPF from our nonlinear results.

We find that as A increases, the region for instability increases with the range of unstable wavelengths increasing very quickly and R_{crit} reduces at the same time. The neutral curves show that the curves remains closed and the “tail” does not open up nor attach to infinity, at least up to $A = 350$ in this instance.

From figure 4.14, although we do not have a linear comparison to make for R_{crit} in this case (as ACPF is linearly stable for these conditions), we note that R_{crit} varies greatly between $94 \leq A \leq 350$, reaching a minimum of $R_{crit} \approx 3.81 \times 10^5$, a difference of a couple of orders of magnitude. Also, it appears that we reach an asymptote when $A_{crit} \approx 350$ as there appears to be minimal reduction in R_{crit} beyond $A \approx 350$. The effect of nonlinearity is to generate instability where linear theory predicts none.

It would be interesting to perform further analysis to see if a plateau is reached in terms of increasing V and A to an extent where any further increase in either will not yield any decrease in R_{crit} or V_{crit} .

Looking at the disturbance behaviour at the nose of the neutral curve for $A = 350$ in figure 4.15, we find that there is one critical layer located close to the outer cylinder wall and also that the wavespeed c is relatively larger (compared with the overall basic flow) than for the previous cases of $\delta = 0.5, V = 0.0$ and $\delta = 0.55, V = 0.01$. The mean flow now varies very dramatically between the inner and outer cylinders. Although $\hat{u}_0$ remains largely negative, there is now a region of significant magnitude where $\hat{u}_0$ is positive, close to the inner wall. Also, the sudden variation in $\hat{u}_0$ near the inner and outer walls is similar to the behaviour encountered in the $\delta = 0.55, V = 0.01$ upper curve in order to incorporate the viscous and nonlinear effects of the disturbance. The order of magnitude of the mean flow is much lower compared to the previous two cases and is $O(10^{-5})$. We note that the first mode is the most significant in the solution.

Figure 4.16 shows contours of the disturbance at the nose of the neutral curve for $\delta = 0.4$ and $V = 0.0356$. The disturbance is most pronounced in the region close to the inner wall, with the outer wall disturbance being smaller in magnitude than that of the inner wall. This is in keeping with the upper curve for $\delta = 0.55, V = 0.01$ where the difference in magnitude of disturbance between the inner and outer wall is less than that of the lower curve. The region of relatively undisturbed flow is located off-centre (along the radial axis), closer to the inner cylinder.

In our next case study, we look at the behaviour of the neutral curve for $\delta = 0.4$ for a disturbance amplitude $A = 200$, increasing the inner cylinder velocity, V , this time. This case is different to the previous case in that we can readily trace out neutral curves at $V = 0.034$ from the linear solution. In figure 4.17, we observe that as V increases, the neutral curve shrinks to a point and eventually disappears completely leaving the flow completely stable. This behaviour is consistent, at this disturbance amplitude, with linear theory in that as V is increases, the neutral curve shrinks to a point before disappearing (Walton (2004)). The range of wavenumbers also decreases considerably as V increases.

We illustrate in figure 4.18 that at $A = 200$, increasing V leads to a cut-off velocity where no instability is predicted beyond this V , as in the linear solution for this case. The

cut off velocity is $V \approx 0.0366$ at $A = 200$ and R_{crit} increases as V increases.

Figure 4.19 shows the basic and mean flow distortion for the nose of the neutral curve for this case of $\delta = 0.4$, $V = 0.034$, $A = 200$ is very similar to that of the case $\delta = 0.4$, $V = 0.0356$, $A = 350$ discussed earlier. The maximum magnitude of $\hat{u}_0$ however is smaller in this case. There is one critical layer, located close to the outer wall with $\hat{u}_0$ varying dramatically in the regions close to the inner and outer cylinder walls to accommodate the nonlinear effects of the disturbance to the flow.

In figure 4.20, the disturbance is largest close to the inner wall and weaker in the region close to the outer wall. There is a region of quiet flow slightly off-centre (radial axis), closer to the inner wall.

For completeness, we present our final case for $\delta = 0.4$ for an inner cylinder of $V = 0.034$ varying the disturbance amplitude, A , in figure 4.21. In the previous cases, we predict instability in the $(\alpha - R)$ -space for ACPF where our linear analysis suggests no instability and also look at the behaviour of the neutral curve fixing A whilst varying V . In this instance, we fix V and vary A . As A increases, the neutral curve “expands” with the tail elongating and the range of unstable wavelengths increasing with a decrease in R_{crit} as shown in figure 4.22.

Figure 4.22 demonstrates, in 3-dimensional space of (α, A, R) -space how the neutral curves vary with changes to these conditions. The area within each neutral curve represents the conditions under which ACPF would be unstable.

Figure 4.23 shows R_{crit} decreasing as A increases, exhibiting the same trend as observed earlier, for a fixed V . The rate of decrease of R_{crit} slows between $A = 300 - 400$ as a critical disturbance amplitude is approached. Given the experience of previous cases, it is reasonable to assume that there will be minimal decrease in R_{crit} beyond $A = 400$. The tail does not open up or re-attach to $R = \infty$. Indeed, the author finds that there is little decrease in R_{crit} above $A = 400$.

The basic, mean flow distortion and intensity of nonlinear influence are very similar in terms of the nature and shape of the curves in figure 4.24 compared with figures 4.15 and 4.19. The critical layer is again located close to the outer cylinder wall. The mean flow disturbance magnitude is similar to that of $\delta = 0.4$, $V = 0.0356$, i.e. $\hat{u}_0 \sim O(10^{-5})$ and varies rapidly between the inner and outer cylinders to accommodate nonlinear effects.

The resulting mean flow disturbance has a largely negative value but has a region of positive value very close to the inner cylinder.

The contours and magnitude of velocity from figure 4.25 show that the disturbance affects the flow close to the inner cylinder the most. The magnitude of the disturbance in the outer wall region is much smaller compared to that near the inner wall. Once again, we note that there is an area of almost unperturbed flow located slightly off-centre, closer to the inner cylinder.

Figure 4.26 demonstrates, at $A = 200$, that the closed neutral curve does re-attach to $R = \infty$ as V increases. The contour, velocity and mean-flow distortion figures at the nose of $V = 0.026$ are similar to that of the other $\delta = 0.4$ cases.

In the next chapter, we discuss and compare the nonlinear and linear stability of ACPF and draw some conclusions on the stability of ACPF. We also take a look at the special case of ACPF where a specific choice of inner cylinder velocity results in the maximum basic flow velocity being achieved on the inner cylinder and comment on the stability of this flow that we call sliding annular flow (SAF).

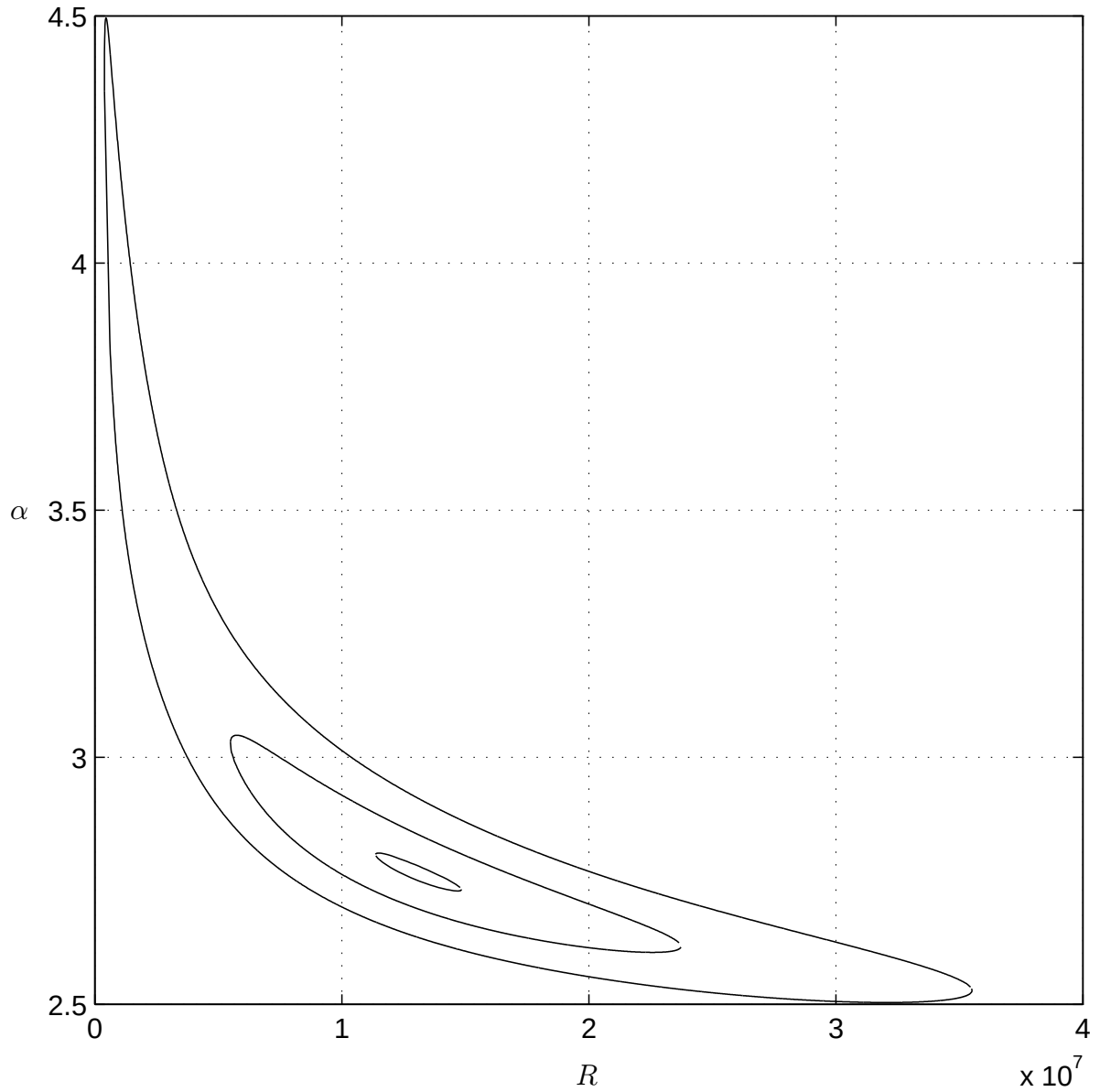


Figure 4.13: Neutral curves (α against R) for $\delta = 0.4$, $V = 0.0356$ for various disturbance amplitudes. From outer to inner: $A = 350$, 200 and 100 .

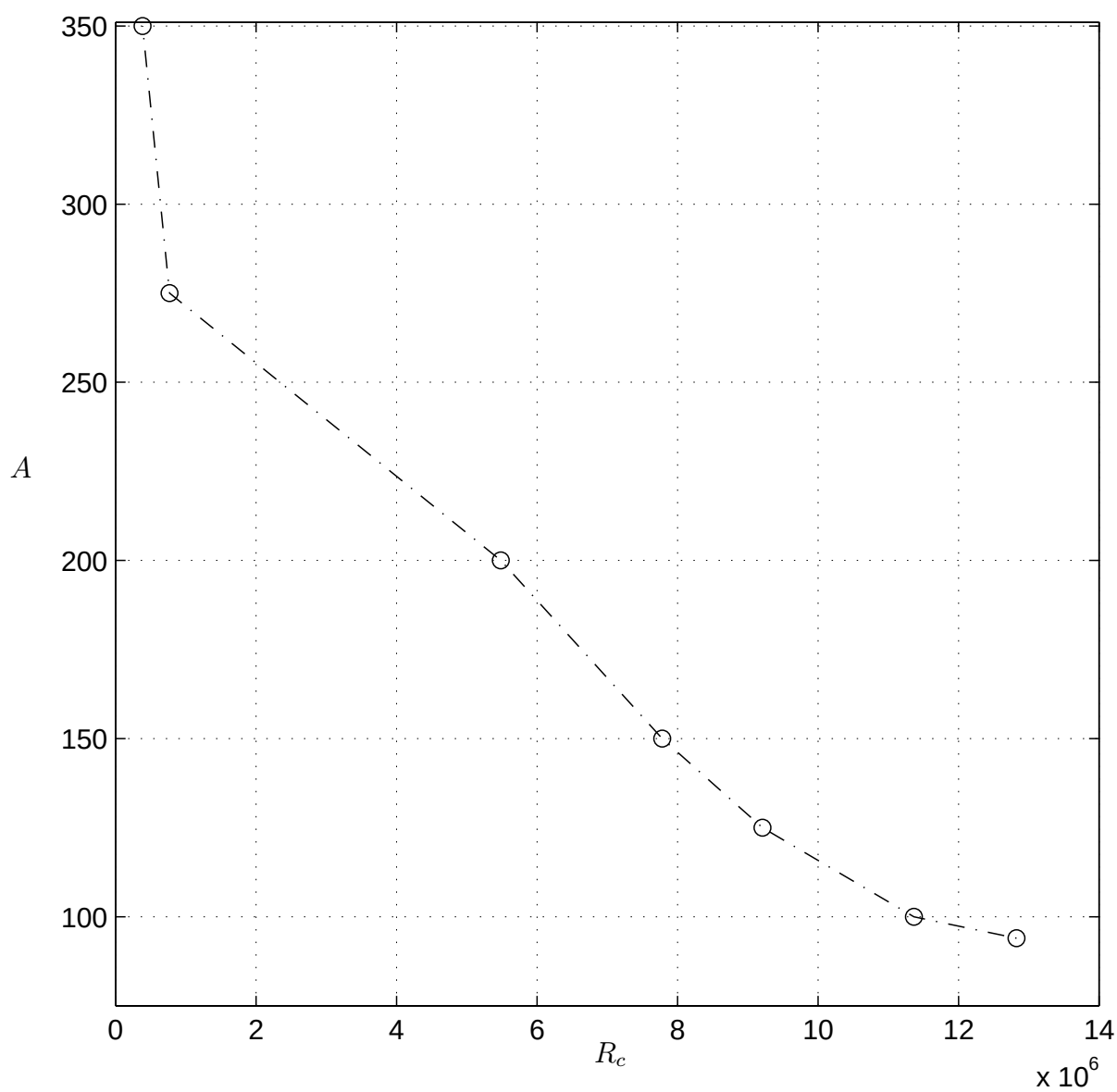


Figure 4.14: Disturbance amplitude A against R_{crit} for $\delta = 0.4$, $V = 0.0356$.

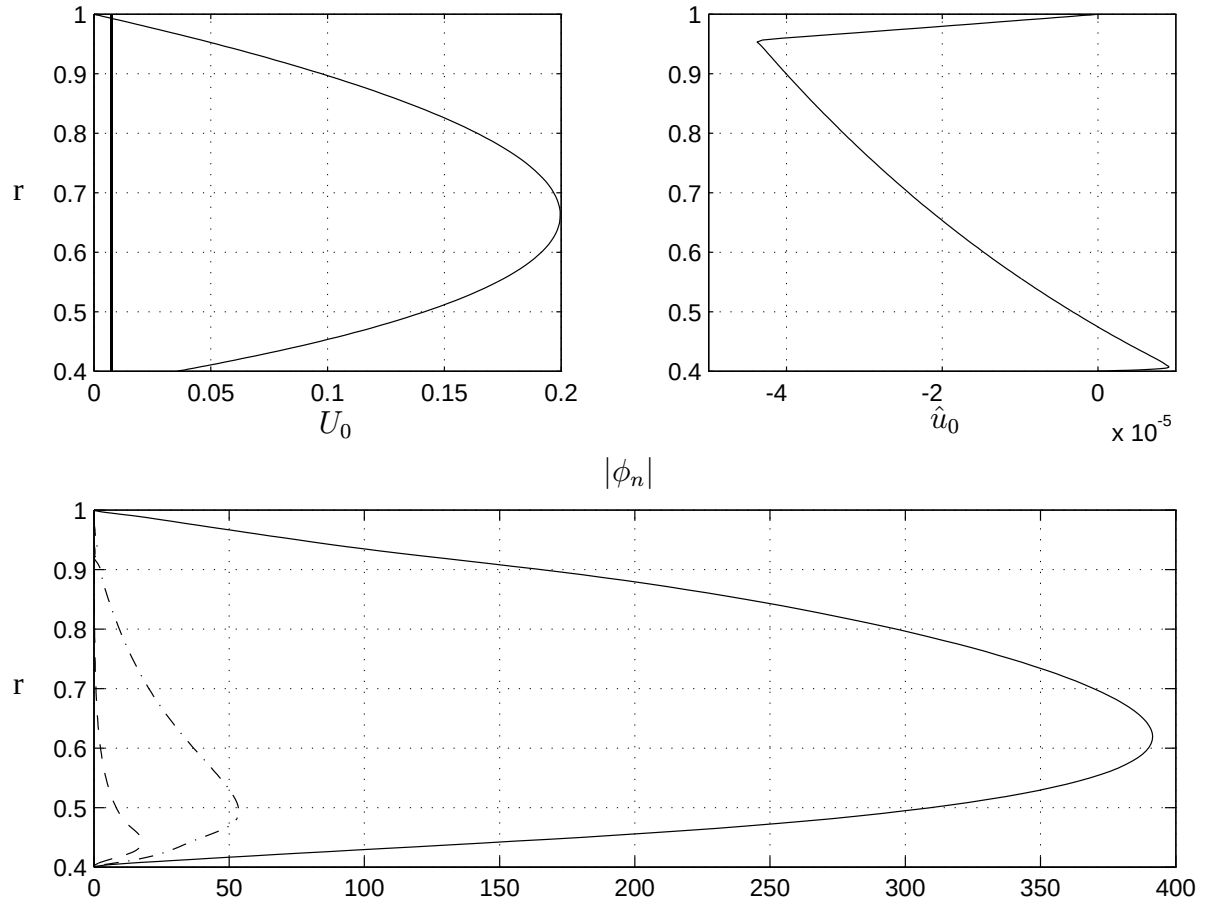


Figure 4.15: For $\delta = 0.4$, $V = 0.0356$, $A = 350$ at R_{crit} : (a) Radius against basic flow and wavespeed (vertical line) (top left). (b) Radius against mean flow distortion (top right). (c) Radius against intensity of nonlinear modes for $N = 1$ (solid), $N = 2$ (dot-dashed) and $N = 3$ (dashed) (bottom).

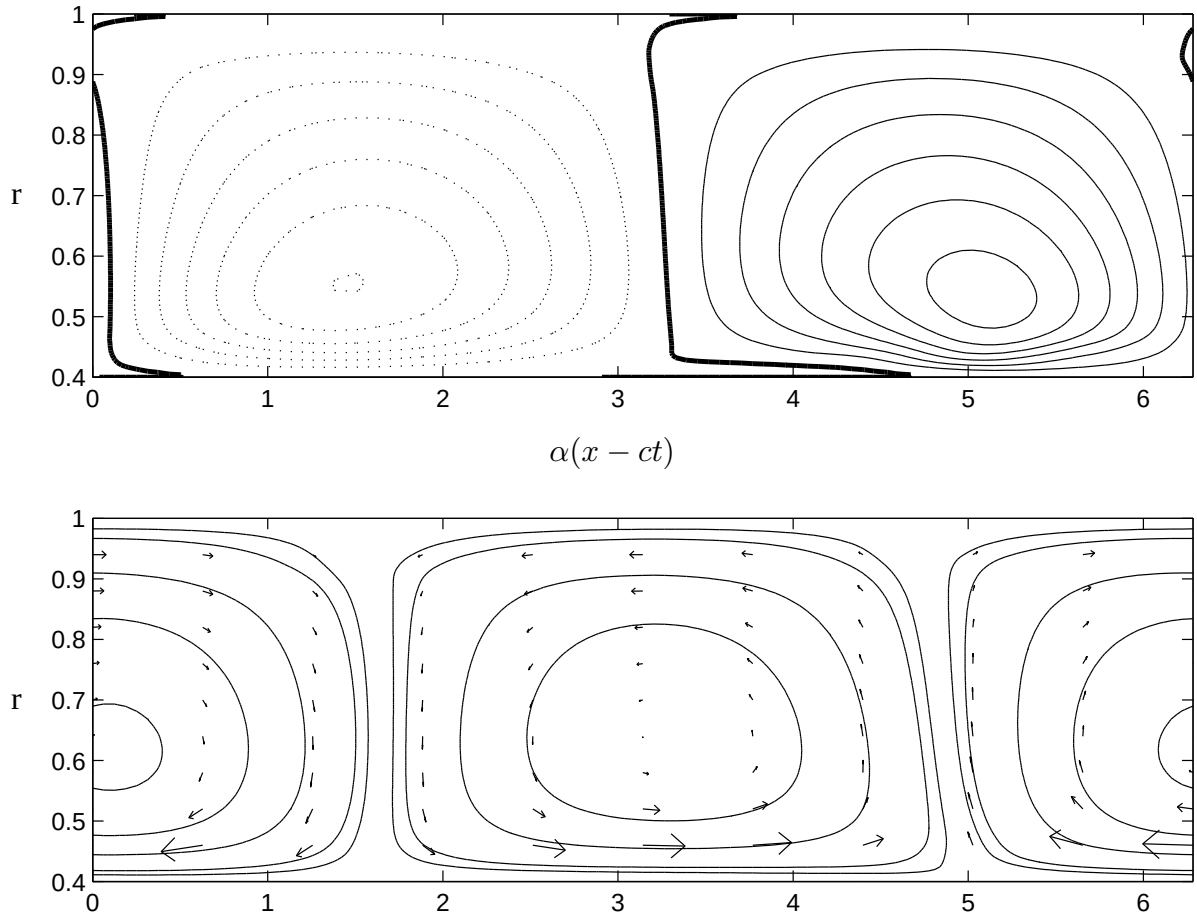


Figure 4.16: Contours of the radial velocity and stream function for $\delta = 0.4$, $V = 0.0356$ at R_{crit} for $A = 350$. (a) Contours of radial velocity at $(\pm 200 \pm 400 \pm 600 \pm 800 \pm 1000 \pm 1200 \pm 1300)/R_{crit}$ (top). (b) Disturbance stream function with disturbance velocity vector field superimposed (bottom).

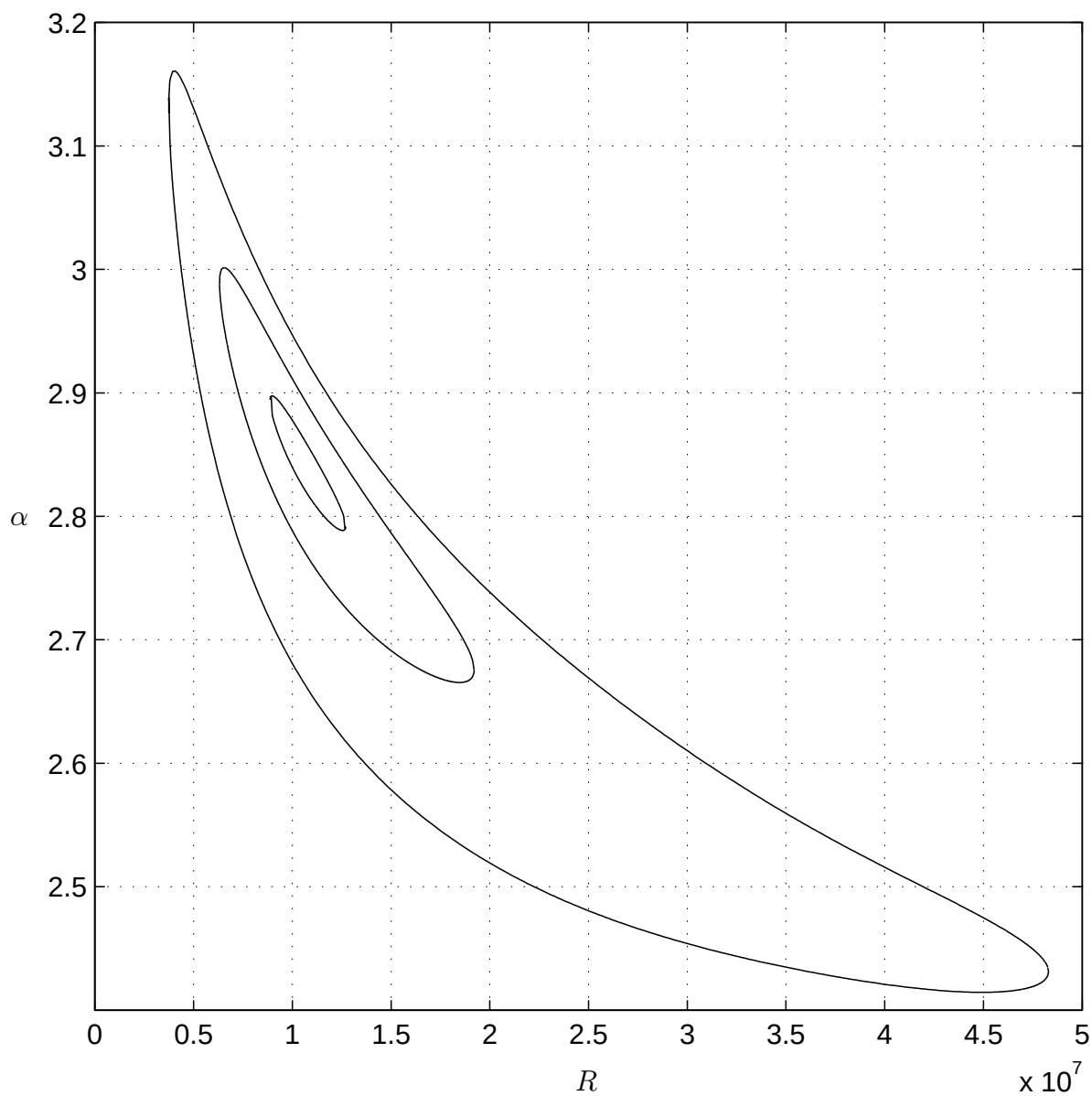


Figure 4.17: Neutral curves (α against R) for $\delta = 0.4$, $A = 200$ for various inner cylinder velocities, V . From outer to inner: $V = 0.034, 0.036$ and 0.0365 .

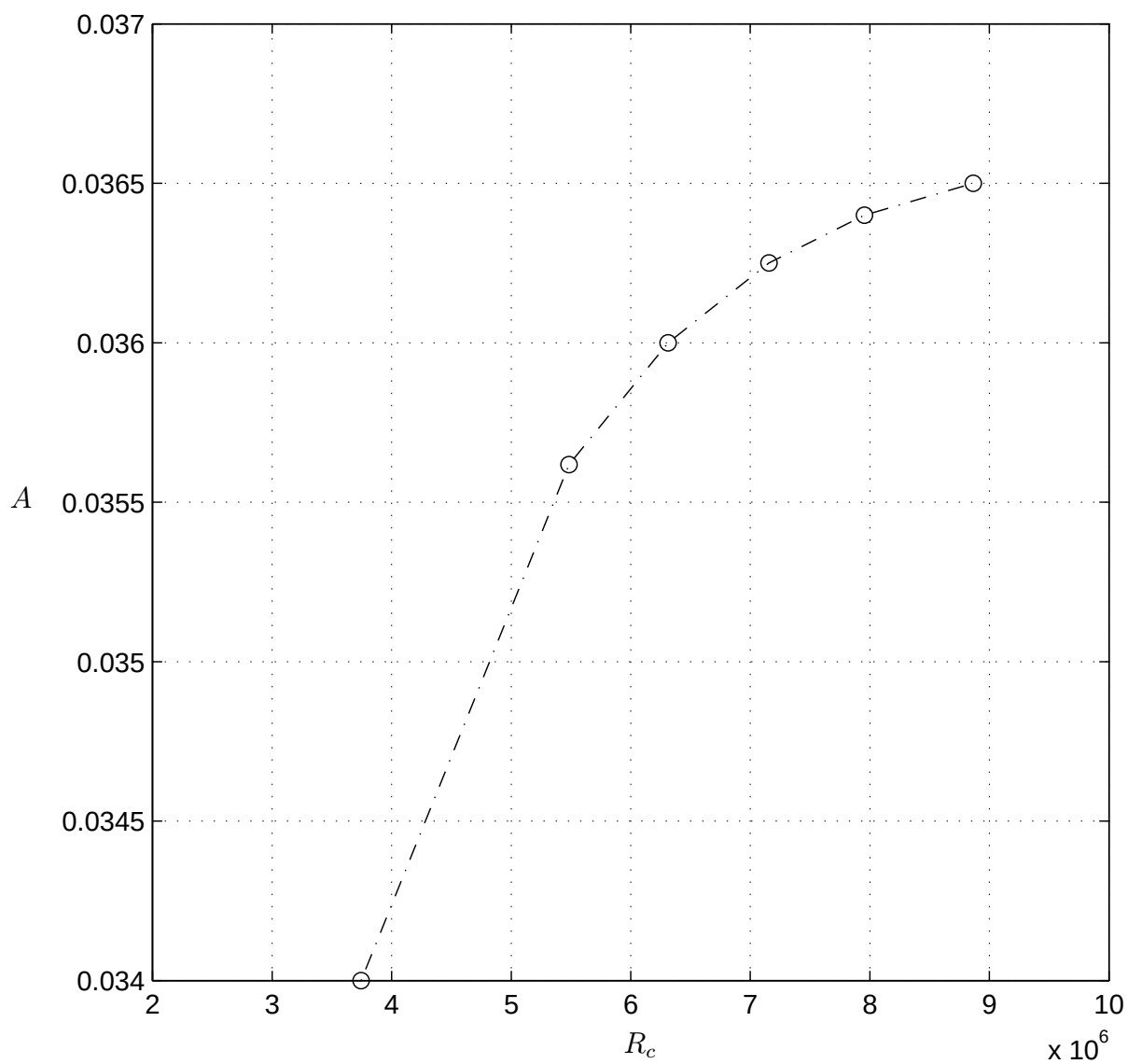


Figure 4.18: Inner cylinder velocity V against R_{crit} for $\delta = 0.4$ at amplitude $A = 200$.

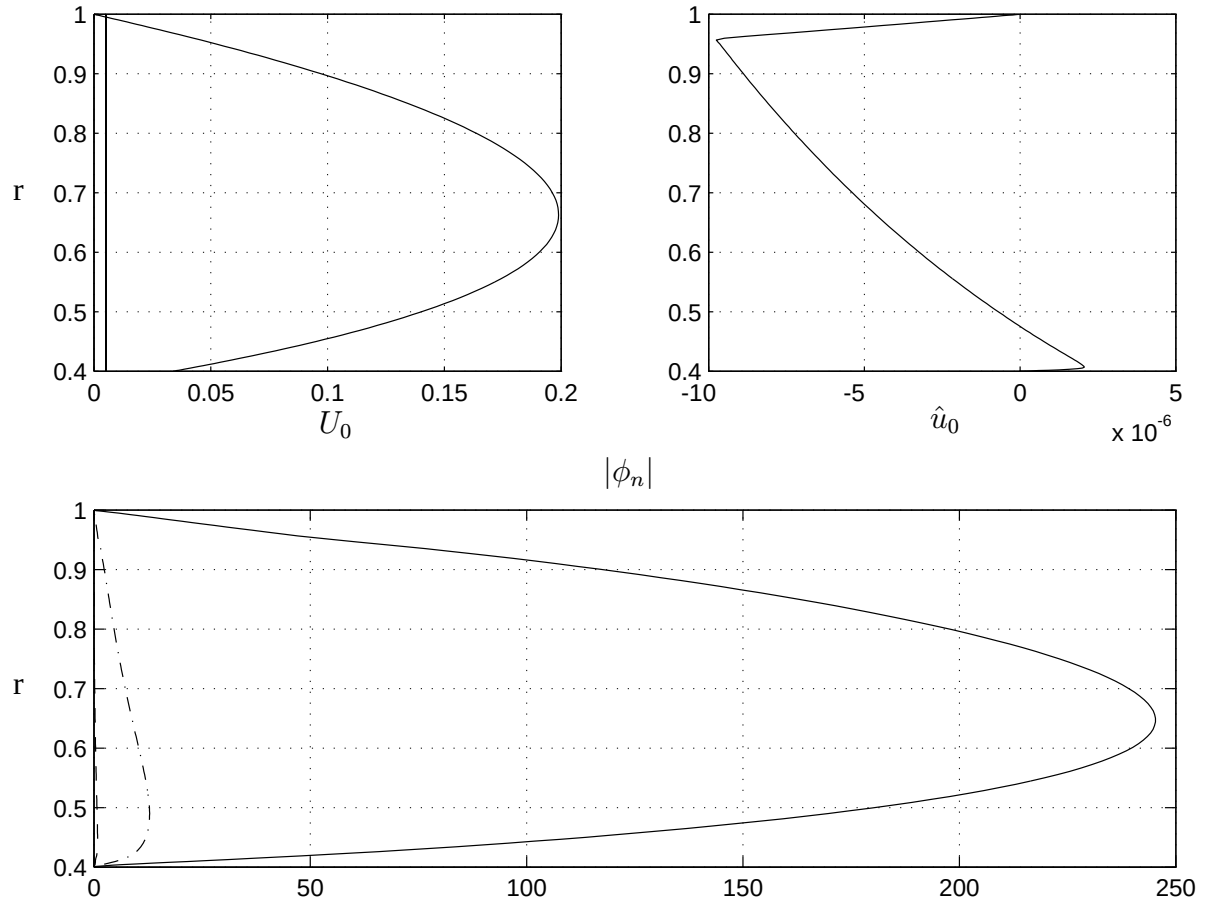


Figure 4.19: For $\delta = 0.4$, $V = 0.034$, $A = 200$ at R_{crit} : (a) Radius against basic flow and wavespeed (vertical line) (top left). (b) Radius against mean flow distortion (top right). (c) Radius against intensity of nonlinear modes for $N = 1$ (solid), $N = 2$ (dot-dashed) and $N = 3$ (dashed) (bottom).

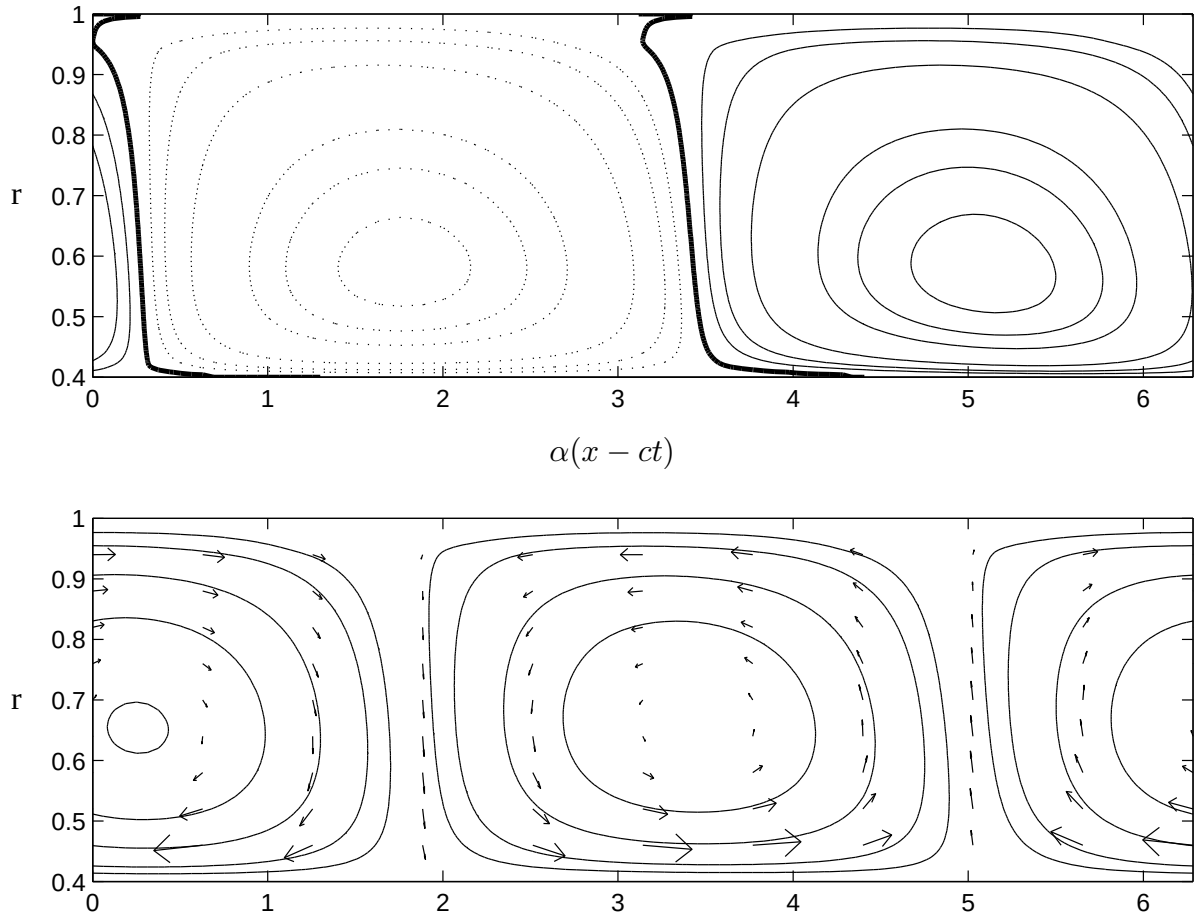


Figure 4.20: Contours of the radial velocity and stream function for $\delta = 0.4, V = 0.034$ at R_{crit} for $A = 200$. (a) Contours of radial velocity at $(\pm 50 \pm 100 \pm 200 \pm 300 \pm 400 \pm 500 \pm 600 \pm 675)/R_{crit}$ (top). (b) Disturbance stream function with disturbance velocity vector field superimposed (bottom).

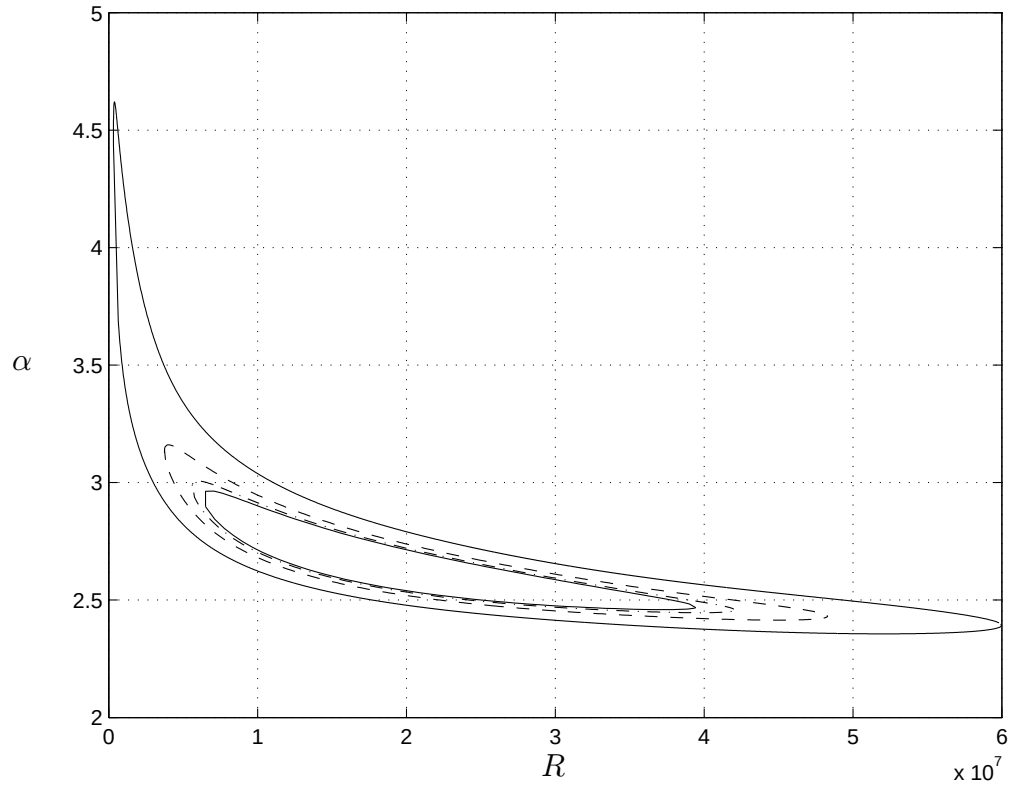


Figure 4.21: Neutral curves (α against R) for $\delta = 0.4$, $V = 0.034$ for various disturbance amplitudes. From outer to inner: $A = 400$ (solid), 200 (dashed), 100 (dot-dashed), 0 (solid, linear solution).

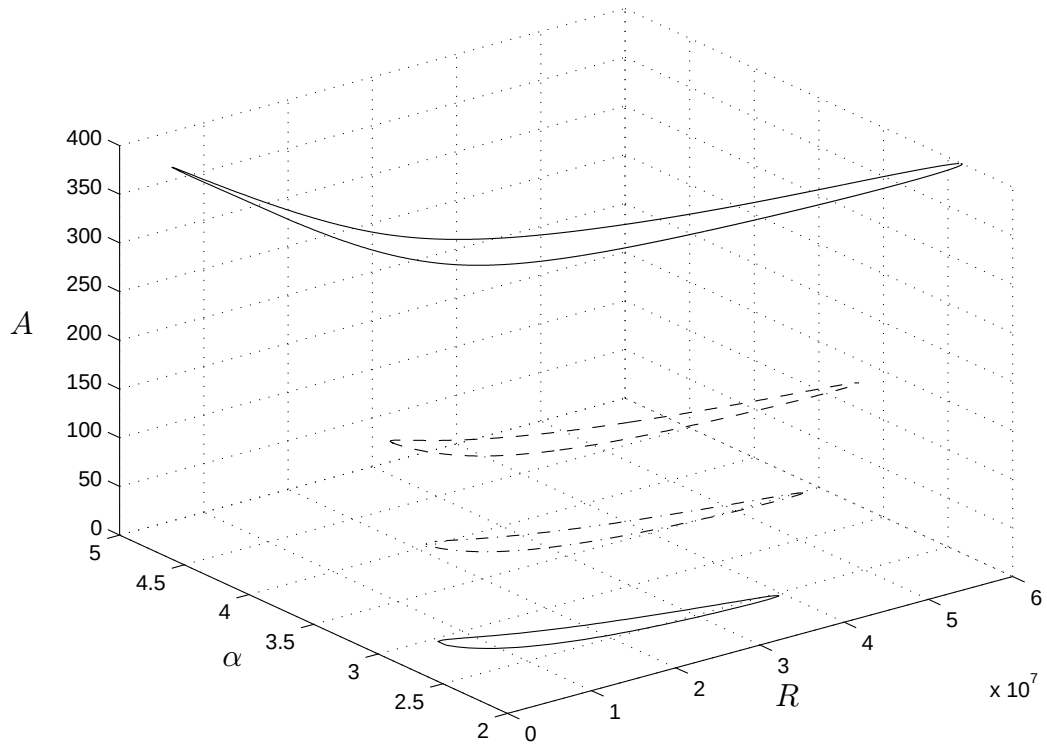


Figure 4.22: 3D Neutral curves (α against R) for $\delta = 0.4$, $V = 0.034$ for various disturbance amplitudes. From top to bottom: $A = 400$ (solid), 200 (dashed), 100 (dot-dashed), 0 (solid, linear solution).

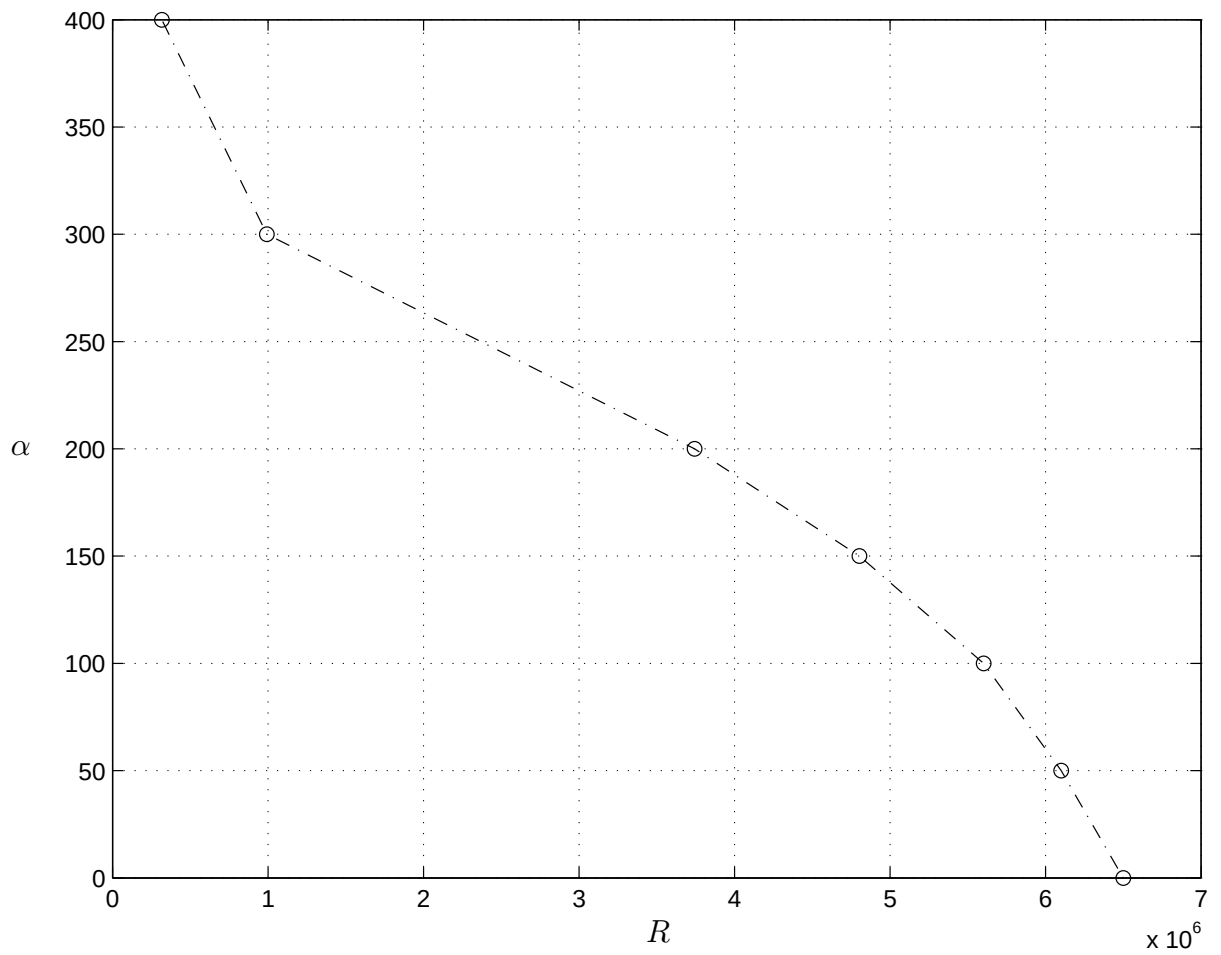


Figure 4.23: Disturbance amplitude A against R_{crit} for $\delta = 0.4$, $V = 0.034$.

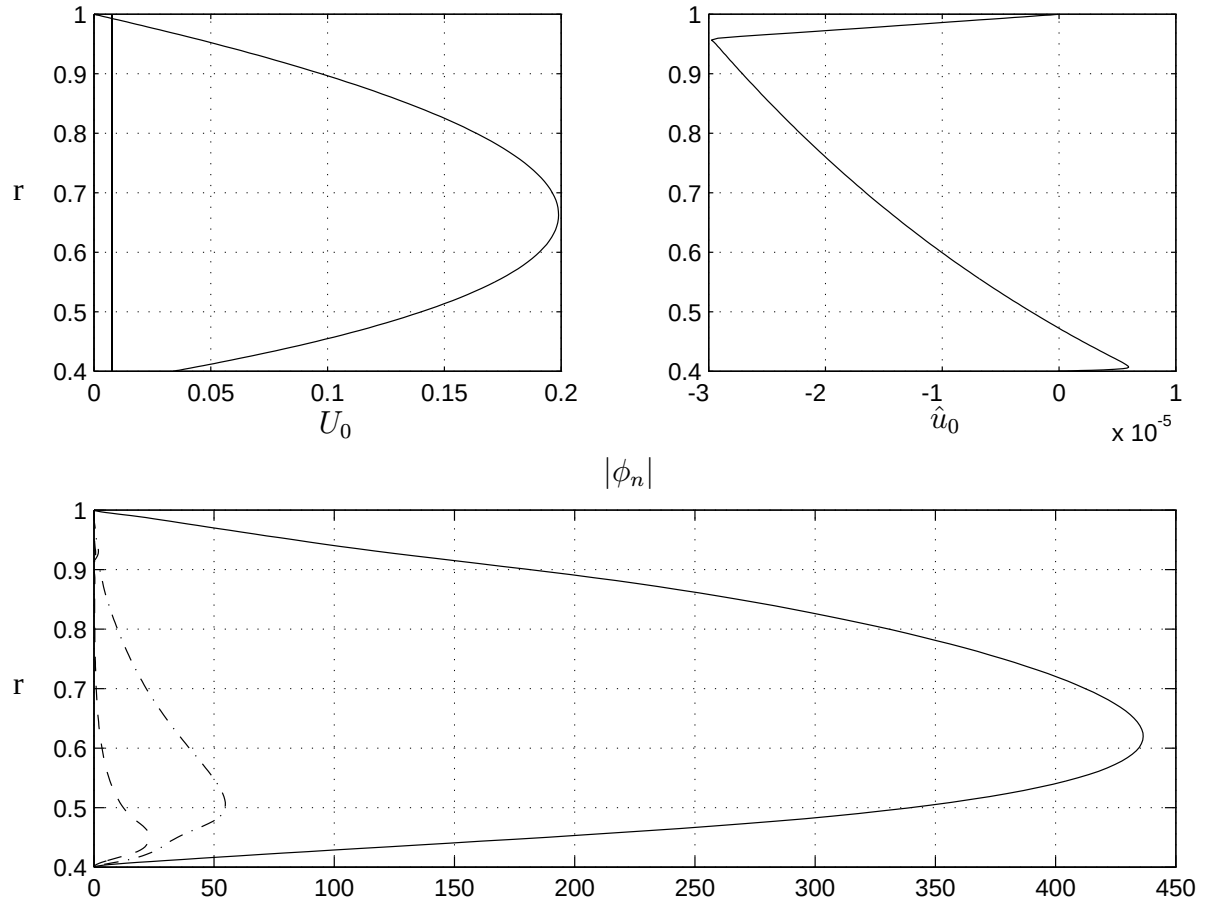


Figure 4.24: For $\delta = 0.4$, $V = 0.034$, $A = 400$ at R_{crit} : (a) Radius against basic flow and wavespeed (vertical line) (top left). (b) Radius against mean flow distortion (top right). (c) Radius against intensity of nonlinear modes for $N = 1$ (solid), $N = 2$ (dot-dashed) and $N = 3$ (dashed) (bottom).

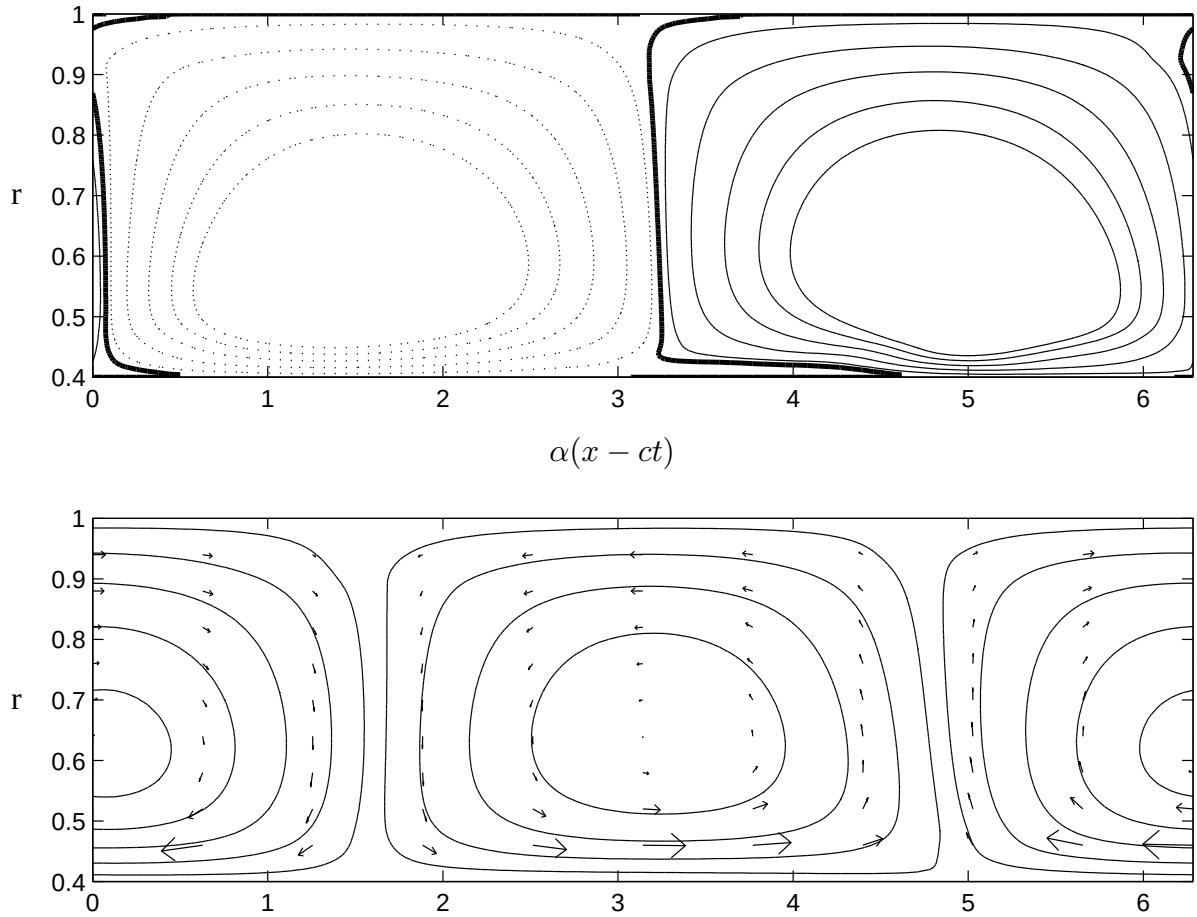


Figure 4.25: Contours of the radial velocity and stream function for $\delta = 0.4$, $V = 0.034$ at R_{crit} for $A = 400$. (a) Contours of radial velocity at $(\pm 50 \pm 200 \pm 400 \pm 600 \pm 775)/R_{crit}$ (top). (b) Disturbance stream function with disturbance velocity vector field superimposed (bottom).

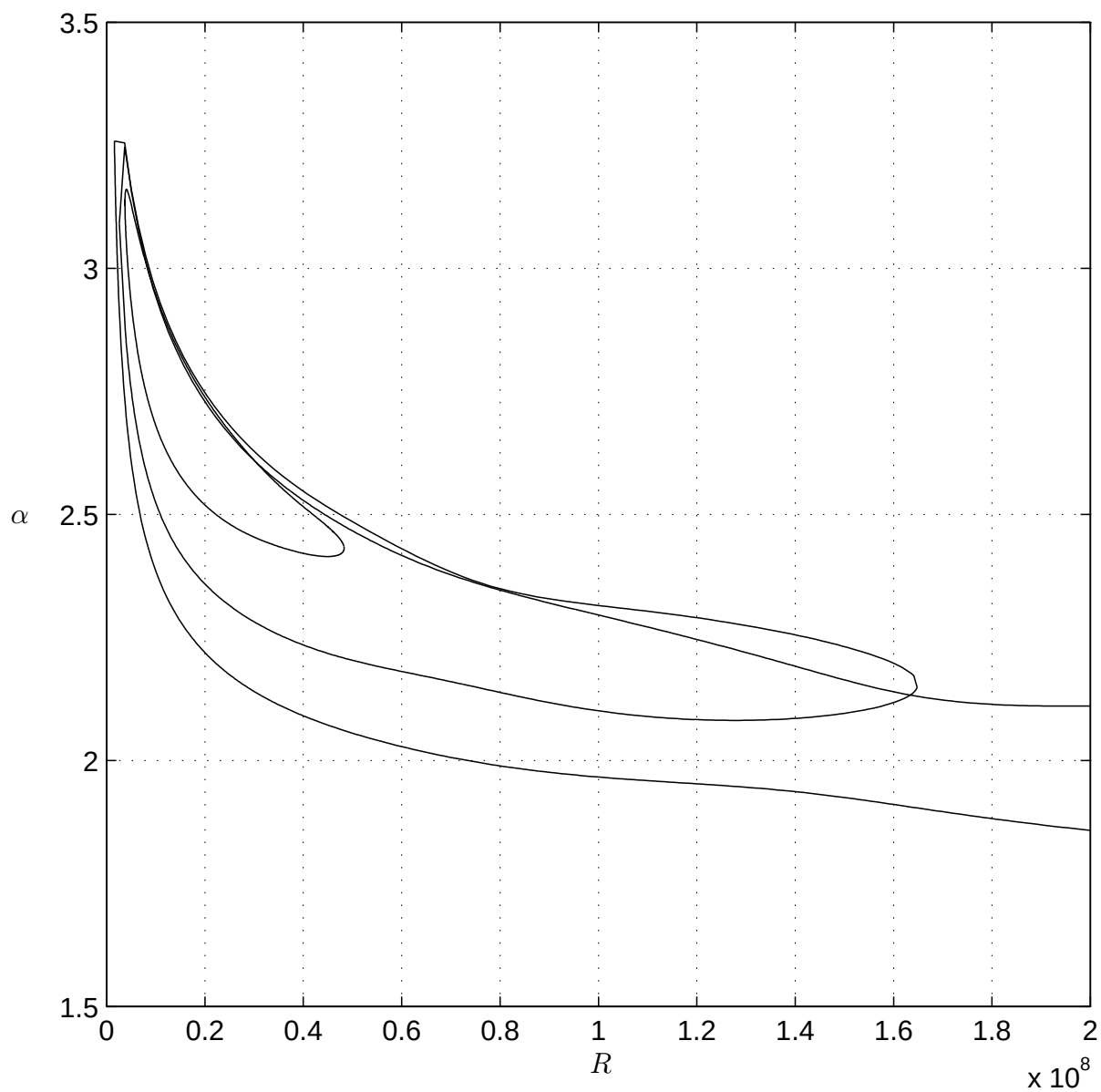


Figure 4.26: Neutral curves (α against R) for $\delta = 0.4$, $A = 200$ for various inner cylinder velocities, V . From outer to inner: $V = 0.026, 0.030$ and 0.034 .

Chapter 5

Discussion on ACPF - Linear & Nonlinear Findings

We have presented findings from linear and nonlinear computations for ACPF. For our linear results, we confirm the findings of published work such as Walton 2004's computational results, and utilise these linear results as the initial data set for our nonlinear computations. The linear stability of ACPF results shows that for $V = 0$, a unique neutral curve exists, with a distinct upper and lower branch and a critical R , for a given set of ACPF conditions such as V and δ . As V is increased, multiple neutral curve can and do exist for certain radius ratios such as $\delta = 0.55$. The behaviour and fate of each neutral curve is to disappear, either by shrinking to a point or retreating as $R \rightarrow \infty$. Ultimately, at sufficiently large V , ACPF is linearly stable and no neutral curve exists. The introduction of nonlinearity and hence a finite disturbance amplitude modifies the behaviour of the flow, drastically in some cases. We focus on several cases for specific ACPF parameter values to enable us to draw more specific comparisons between the linear and nonlinear computations.

Our linear and nonlinear results have been presented as neutral curves in the (α, R) -space. We have also presented figures of contours, streamlines, and the mean flow distortion to understand the influences of the nonlinear travelling-wave solutions at or near the critical Reynolds number, R_{crit} . We have found that the lower branch of the neutral curve is more stable to changes in disturbance amplitudes A up to a point. There appears to be a turning point in A , where R_{crit} reduces above a certain value of A and R_{crit} increases

below this value of A as seen in cases where $\delta = 0.5, 0.55$.

The general observation, at a fixed disturbance amplitude A , is that the linear behaviour of ACPF is mirrored in the nonlinear results. For example, at fixed A , we observe in all cases that the neutral curves either shrink to a point or shift to high Reynolds numbers, longer wavelengths and eventually extends to $R = \infty$ as inner cylinder velocity V is increased. This behaviour has been consistent with the findings of our linear results.

We have also found that for a fixed V , increasing A sufficiently will generally reduce R_{crit} , very significantly in some cases. This result, coupled with the knowledge that increasing V leads to an increase in R_{crit} indicates that there are potentially a set of values for (A, V) where there is no change in the stability for a given δ . This comes from the following observations: (i) in our nonlinear analysis, we have noted that there is a critical amplitude, A_{crit} , above which there is no change in R_{crit} and (ii) we have also found that there is a critical inner cylinder velocity, V_{crit} , beyond which R_{crit} does not vary. It would be worthwhile conducting further analysis to see if a correlation between (A, V) and R_{crit} can be found and if a “plateau” from (A, V) can be attained where any change in A or V will yield no further change in R_{crit} .

It is worth noting that increasing A , with R, V, δ fixed, also increases the band of unstable wavelengths of neutral curves. For the parameter space that we have explored, there has not been any case where an increase in amplitude reduced, in any manner, the width of a neutral curve. Therefore, ACPF is more unstable when $A \neq 0$ than for the corresponding $A = 0$ case for a given values of V and δ .

The contour and streamline figures demonstrate the location and magnitude of where the basic flow is most affected. We note that the disturbance influence is concentrated mostly near the inner and outer walls, with the magnitude being higher near the inner cylinder for the majority of cases. The mean flow distortion is also concentrated near the inner and outer walls, with the mean flow having a predominantly negative value on ACPF. In order to computationally model ACPF, we have truncated the number of nonlinear travelling-wave solutions to N modes and demonstrate (through plots of $|\phi_n|$) that the first mode is always most dominant. The influence of the secondary, tertiary and other modes are certainly important, but it is clear that most of the influence is down to the first mode.

In the first case, we find that R_{crit} in the linear regime arises for $\delta = 0.5$ with a stationary

inner cylinder. We therefore anticipate that R_{crit} in the nonlinear regime would also be for these ACPF conditions given that we have observed linear behaviour being mirrored in the nonlinear case, for a fixed A . We find that nonlinearity decreases R_{crit} to approximately 1.15×10^5 at higher disturbance amplitudes. Frei et al. (2000) found R_{crit} to be $R \approx 3 \times 10^4$. Nonlinear effects clearly have a major influence in bridging the gap between the linear stability prediction (4.7×10^5 , see section 3.4, Linear Stability Numerical Results) and experimental results, although not quite accounting for the entire difference. In addition, we discover that there is no further change in R_{crit} beyond a critical disturbance amplitude, A_{crit} . For higher values of A or N , there are computational limitations and difficulties in solving the set of nonlinear equations, rendering the process inefficient and expensive; the number of elements in the Jacobian is quadratic in nature and more computational memory will be required extending the required computational time and this assumes that convergence is actually achieved which may not always be the case.

For our second case, $\delta = 0.55$ and $V = 0.01$, we find two neutral curves of approximately equal size for the linear stability of ACPF (see section 3.4, Linear Stability Numerical Results section). We also demonstrate that these neutral curves continue to exist in nonlinear space as A is increased. We find that, up to a disturbance amplitude of approximately 300, R_{crit} is governed by the lower curve and R_{crit} increases between $0 \leq A \leq 300$. For $300 \leq A \leq 500$, we notice that there is a transition stage where R_{crit} is dictated by the upper curve. Above $A \approx 500$, the critical disturbance amplitude, there is little or no change in R_{crit} . Also, the two nonlinear neutral curves appear to merge into one. Unfortunately, increasing $A > 600$ causes numerical difficulties as mentioned previously, making results either difficult to obtain or unreliable. It appears, however, that we approach a pre-split neutral curve stage.

In 2004, Walton suggested that although multiple neutral curves exist for the same ACPF conditions, each curve extracts energy from the ACPF system in very different ways. We therefore expect the behaviour of the multiple neutral curves, through contour, velocity and mean flow figures, to reflect that there is indeed a difference in the way each of these curves behaves. By looking at the structure of the modes at the nose of the neutral curves for $A = 600$, we note that the disturbance is strongest close to the inner cylinder and weaker near the outer cylinder. However, for the lower branch solution, the difference in

magnitude between the inner and outer cylinders is much less pronounced. The mean flow distortion is also very different for these two curves. The mean-flow distortion for the lower curve is much closer in shape to that of the $\delta = 0.5$ case whereas the upper curve has large variations in localised regions near the inner and outer cylinder walls. The first mode, for both curves, is the main contributor of nonlinearity.

The last case we have analysed is $\delta = 0.4$ for various inner cylinder velocities and disturbance amplitudes. The linear neutral curves found agree with published work. For our nonlinear analysis, we confirm that instability continues to be predicted by nonlinear calculation as disturbance amplitude is increased and confirm that as V is increased, the neutral curve shrinks to a point, reflecting linear stability behaviour in this regard. Also, as A is increased, the closed neutral curves tend to elongate or stretch at either end of the neutral curve, with an increase in the range of unstable wavelengths.

The velocity and mean flow distortion plots for $\delta = 0.4$ show similar behaviour for various disturbance amplitudes and inner cylinder velocities. The magnitude of disturbance is largest in the region close to the inner cylinder, and the mean flow distortion varies in much the same way for the examples seen in the previous $\delta = 0.5, 0.55$ cases. The mean flow distortion is largely negative and varies rapidly near the inner and outer cylinder walls suggesting that the viscous effects are concentrated in these regions.

In nonlinear calculations, we demonstrated that there exists a critical velocity, beyond which there is no change to R_{crit} . We also find that, for $\delta = 0.4$, nonlinear calculations predicts instability where ACPF is linearly stable. We have demonstrated that predicting instability in nonlinear calculations, where ACPF is linearly stable, requires care and can be time consuming as we must first find the critical velocity in the linear regime in order to obtain our starting point for the nonlinear continuation approach. In addition to proving instability existing for ACPF in nonlinear calculations, we have also increased V_{crit} and could easily continue this process to higher values of inner cylinder velocities in the nonlinear regime. Once in the nonlinear regime, we can increase both disturbance amplitude and inner cylinder velocity, which would almost certainly yield an increase in V_{crit} and most likely increase the band of unstable wavelengths whilst the behaviour of R_{crit} would depend specifically on the amplitude and velocity.

Finally, we confirm with examples that we do observe linear behaviour, for given

R, V, δ , if we fix the disturbance amplitude. This is demonstrated for $\delta = 0.55$ where we fix the disturbance amplitude and reduce V from 0.01 to 0.0, and for $\delta = 0.4$ where disturbance amplitude A is fixed at 200 and V is varied.

5.1 A Special case of ACPF - Sliding Annular Flow

Sliding Annular Flow (SAF) is a special case of ACPF where the maximum basic velocity is attained on the inner cylinder for an inner cylinder velocity of $V(\delta)$. Before we discuss the stability of SAF, we derive the basic flow of SAF, U_{SAF} , and the inner cylinder velocity, $V(\delta)$, that we require. The basic ACPF flow is

$$U_0 = 1 - r^2 + \frac{V - 1 + \delta^2}{\ln(\delta)} \ln(r), \quad (5.1)$$

it follows that

$$U'_0 = -2r + \frac{V - 1 + \delta^2}{r \ln(\delta)}. \quad (5.2)$$

We achieve the maximum basic flow velocity when $U'_0 = 0$, which we apply on the inner cylinder where $r = \delta$ for SAF. We find a δ -dependent $V(\delta)$:

$$V(\delta) = 1 - \delta^2 + 2\delta^2 \ln(\delta). \quad (5.3)$$

Therefore, the resulting basic flow for SAF is dependent only on δ :

$$U_{SAF} = 1 - r^2 + 2\delta^2 \ln(r), \quad (5.4)$$

where $\delta \leq r \leq 1$. This velocity profile ensures that the maximum velocity is always achieved on the inner cylinder. For illustrative purposes, we show the velocity profiles for various inner cylinder velocity as δ changes for SAF.

SAF is of interest because as $\delta \rightarrow 0$, SAF approaches a constant-pressure-gradient-driven flow in a circular pipe or Hagen-Poiseuille flow, which is linearly stable (see Chapter 1, Introduction). In our linear stability computations of SAF, we do not find any instability for all ranges of δ and V . From our ACPF analysis, the maximum non-dimensional inner

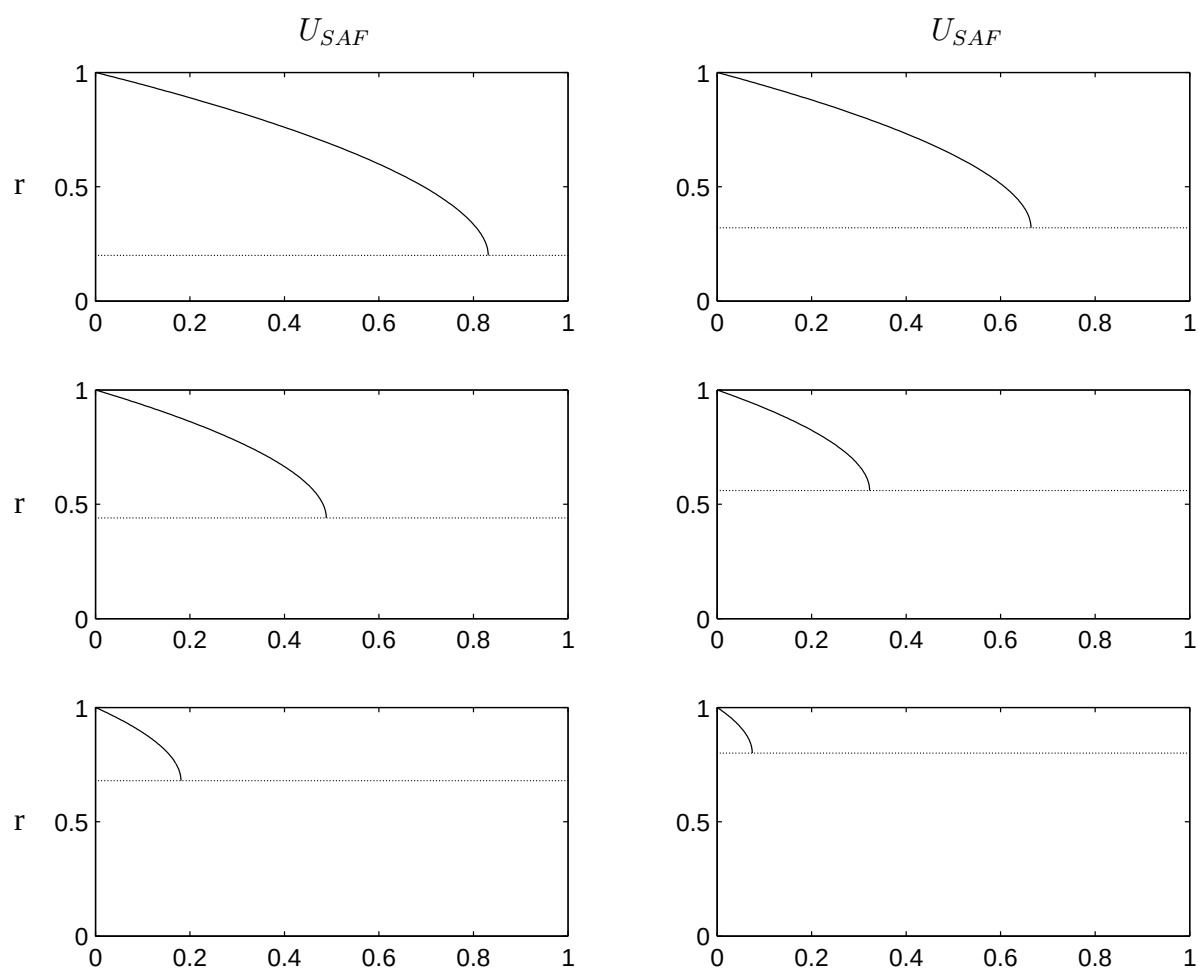


Figure 5.1: Sliding Annular Flow: U_{SAF} velocity profile plots, from left to right, for $\delta = 0.2, 0.32, 0.44, 0.56, 0.68, 0.8$.

cylinder velocity is $V \approx 0.6$ where we find instability to exist for $\delta = 0.1$ and disappears for slightly larger increments of V . However, the corresponding $V(\delta)$ for U_{SAF} is much larger than $V \approx 0.6$ and it therefore seems unlikely that any instability would exist for SAF.

5.2 Conclusion - Annular Couette-Poiseuille Flow

We have presented numerical findings for annular Couette-Poiseuille flow subjected to both linear and nonlinear axisymmetric travelling wave perturbations. The linear findings are consistent with published works and we have found that disturbance amplitude plays a very significant role in the stability of ACPF, lowering the critical Reynolds number by orders of magnitudes in some cases.

We have analysed the stability of the flows to axisymmetric perturbations only and the natural progression of this would be to include non-axisymmetric disturbances and eccentricity to the model, amongst other conditions. Noting that although disturbance amplitude and mean flow distortion has a very significant impact on the stability of ACPF, this has not completely bridged the gap with the critical Reynolds number predicted by the thread injection experiments conducted by Frei et al., with a difference of approximately 5×10^1 remaining.

We have also looked at the linear stability of sliding annular flow with a view to find a connection to Hagen Poiseuille flow. However, no linear instability was found for SAF. It is possible that instability could be found for a nonlinear disturbance of finite amplitude. Drawing on the experience from our ACPF analysis, it is possible that instability could be found, given the right conditions, for SAF due to nonlinear effects. However, a different numerical method to that used for ACPF will be required given that there is no starting-point or initial-condition to kick start the Newton method for SAF.

Our analysis should be viewed as stepping stone, adding to the work that has been conducted thus far in understanding the stability of annular Couette-Poiseuille flow and sliding annular flow subjected to various configurations, with the recent experiments such as that of Frei et al. (2000) providing experimental data for comparison. Walton's (2004) asymptotic analysis was hugely successful in mapping and predicting linear instability for ACPF and it would stand to reason that asymptotic analysis may bring some success with

sliding annular flow to give an indication as to what conditions may generate instability in SAF.

Chapter 6

Appendix A - Fully Nonlinear Equations

The fully nonlinear equation, in terms of ϕ , is detailed in this section. Re-capping briefly, we define a Reynolds number defined based on a constant axial pressure gradient for annular Couette Poiseuille flow:

$$\frac{\partial p}{\partial x} = -\frac{4}{R},$$

and we set

$$\frac{1}{R}\phi_n = rv_n,$$

where $n = 1, 2, \dots, N$ for N modes and $v_n(r, x, t)$ is the travelling wave disturbance in the radial direction of our cylindrical coordinate system of the Navier-Stokes equations. Substituting the above into the nonlinear Navier-Stokes equations in cylindrical coordinates,

we reach the following equation in terms of ϕ_n

$$\begin{aligned}
& \phi_n'''' - \frac{2}{r} \phi_n''' + \left(\frac{3}{r^2} - 2n^2 \alpha^2 \right) \phi_n'' + \left(\frac{2}{r} n^2 \alpha^2 - \frac{3}{r^2} \right) \phi_n' + n^4 \alpha^4 \phi_n \\
& - in \alpha R (U_0 + \hat{u}_0 - c) \left(\phi_n'' - \frac{1}{r} \phi_n' - n^2 \alpha^2 \phi_n \right) \\
& + in \alpha R \left((U_0 + \hat{u}_0)'' - \frac{1}{r} (U_0 + \hat{u}_0)' \right) \phi_n \\
& + n^2 \alpha^2 \sum_{k=1}^{n-1} \left(-\frac{n-k}{k} \frac{1}{r} \phi_k' \phi_{n-k} + \frac{1}{r} \phi_k \phi_{n-k}' - \frac{1}{r^2} \phi_k \phi_{n-k} \right) \\
& + n^2 \alpha^2 \sum_{k=1}^{N-n} \left(\frac{n+k}{k} \frac{1}{r} (\phi_k^*)' \phi_{n+k} + \frac{1}{r} (\phi_k^*) \phi_{n+k}' - \frac{1}{r^2} (\phi_k^*) \phi_{n+k} \right) \\
& + n^2 \alpha^2 \sum_{k=n+1}^N \left(\frac{k-n}{k} \frac{1}{r} \phi_k' (\phi_{k-n}^*) + \frac{1}{r} \phi_k (\phi_{k-n}^*)' - \frac{1}{r^2} \phi_k (\phi_{k-n}^*) \right) \\
& + \frac{n}{n-k} \sum_{k=1}^{n-1} \left(\frac{3}{r^2} \phi_k \phi_{n-k}'' - \frac{3}{r^3} \phi_k \phi_{n-k}' - \frac{1}{r} \phi_k \phi_{n-k}''' + \frac{1}{r^2} \phi_k' \phi_{n-k}' - \frac{1}{r} \phi_k' \phi_{n-k}'' \right) \\
& + \frac{n}{k+n} \sum_{k=1}^{N-n} \left(\frac{3}{r^2} (\phi_k^*) \phi_{k+n}'' - \frac{3}{r^3} (\phi_k^*) \phi_{k+n}' - \frac{1}{r} (\phi_k^*) \phi_{k+n}''' + \frac{1}{r^2} (\phi_k^*)' \phi_{k+n}' - \frac{1}{r} (\phi_k^*)' \phi_{k+n}'' \right) \\
& - \frac{n}{k-n} \sum_{k=n+1}^N \left(\frac{3}{r^2} \phi_k (\phi_{k-n}^*)'' - \frac{3}{r^3} \phi_k (\phi_{k-n}^*)' - \frac{1}{r} \phi_k (\phi_{k-n}^*)''' + \frac{1}{r^2} \phi_k' (\phi_{k-n}^*)' - \frac{1}{r} \phi_k' (\phi_{k-n}^*)'' \right) \\
& + \frac{n}{k} \sum_{k=1}^{n-1} \left(-\frac{2}{r^2} \phi_k' \phi_{n-k}' + \frac{1}{r} \phi_k' \phi_{n-k}'' + \frac{1}{r} \phi_k'' \phi_{n-k}' \right) \\
& + \frac{n}{k} \sum_{k=1}^{N-n} \left(\frac{2}{r^2} (\phi_k^*)' \phi_{k+n}' - \frac{1}{r} (\phi_k^*)' \phi_{k+n}'' - \frac{1}{r} (\phi_k^*)'' \phi_{k+n}' \right) \\
& - \frac{n}{k} \sum_{k=n+1}^N \left(\frac{2}{r^2} \phi_k' (\phi_{k-n}^*)' - \frac{1}{r} \phi_k' (\phi_{k-n}^*)'' - \frac{1}{r} \phi_k'' (\phi_{k-n}^*)' \right) = 0,
\end{aligned}$$

where $(^*)$ denotes complex conjugate and $(')$ is differentiated with respect to radius, r . The mean flow correction equation is given in equation 4.9, in terms of ϕ , which completes our set of nonlinear equations.

Chapter 7

Appendix B - Fully Nonlinear Equations Expanded

In this section, we write ϕ_n in terms of its real and imaginary parts

$$\phi_n = \Phi_n + i\Psi_n,$$

where $n = 1, 2, \dots, N$ for N modes. After some algebra and simplification, we find the equations in (4.18). We introduce some substitution of terms to simplify our analysis as many terms are repeated in the algebra.

$$r = \frac{1}{2} ((1 + \delta) - (1 - \delta)y) \equiv \frac{1}{2}t(y),$$

where δ is inner cylinder radius and y is discretized to a finite domain. We use y and t in this instance to avoid notational confusion and all derivatives in this section are with respect to t . We also define the following parameters which are convenient to use:

$$\begin{aligned}\bar{\beta} &= \frac{\alpha R}{4}, \\ a^2 &= \frac{\alpha^2}{4}, \\ \bar{\Omega} &= 10^{-5}\bar{\beta}c,\end{aligned}$$

and

$$\Delta = \frac{1}{1 - \delta}.$$

The set of nonlinear equations is given in 4.18, reproduced below.

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}_m^n \\ \mathbf{g}_m^n \\ \mathbf{h}_m^n \\ \tau \\ \sigma \end{bmatrix}.$$

We expand equations $\mathbf{f}_m^n$ and $\mathbf{g}_m^n$ below as $\mathbf{h}_m, \tau$ and σ are produced in chapter 4.

$$\begin{aligned} f_n \equiv & \Delta^4 \Phi_n'''' + \frac{2}{t} \Delta^3 \Phi_n''' + \left(\frac{3}{t^2} - 2n^2 \alpha^2 \right) \Delta \Phi_n'' + \left(\frac{3}{t^3} - \frac{2}{t} n^2 \alpha^2 \right) \Delta \Phi_n' + n^4 \alpha^4 \Phi_n \\ & - n \bar{\beta} \left(-(U_0 + \hat{u}_0 - c)(\Delta^2 \Psi_n'' + \frac{\Delta}{t} \Psi_n' - n^2 \alpha^2 \Psi_n) + \Delta^2 (U_0 + \hat{u}_0)'' \Psi_n + \frac{\Delta}{t} (U_0 + \hat{u}_0)' \Psi_n \right) \\ & + \sum_{k=1}^{n-1} A + \sum_{k=1}^{N-n} B + \sum_{k=n+1}^N C = 0, \end{aligned}$$

where the terms A, B and C are detailed below. The term $\sum A$ is:

$$\begin{aligned} \sum_{k=1}^{n-1} A = & n^2 \alpha^2 \sum_{k=1}^{n-1} \left(\frac{n-k}{k} \frac{\Delta}{t} (\Phi_k' \Phi_{n-k} - \Psi_k' \Psi_{n-k}) - \frac{\Delta}{t} (\Phi_k \Phi_{n-k}' - \Psi_k \Psi_{n-k}') \right) \\ & + n^2 \alpha^2 \sum_{k=1}^{n-1} \left(-\frac{1}{t^2} (\Phi_k \Phi_{n-k} - \Psi_k \Psi_{n-k}) \right) \\ & + \sum_{k=1}^{n-1} \frac{n}{n-k} \left(\frac{3\Delta^2}{t^2} (\Phi_k \Phi_{n-k}'' - \Psi_k \Psi_{n-k}'') + \frac{3\Delta}{t^3} (\Phi_k \Phi_{n-k}' - \Psi_k \Psi_{n-k}') \right) \\ & + \sum_{k=1}^{n-1} \frac{n}{n-k} \left(\frac{\Delta^3}{t} (\Phi_k \Phi_{n-k}''' - \Psi_k \Psi_{n-k}''') + \frac{\Delta^2}{t^2} (\Phi_k' \Phi_{n-k}'' - \Psi_k' \Psi_{n-k}'') + \frac{\Delta^2}{t^2} (\Phi_k' \Phi_{n-k}' - \Psi_k' \Psi_{n-k}') \right) \\ & - \sum_{k=1}^{n-1} \frac{n}{k} \left(\frac{2\Delta^2}{t^2} (\Phi_k' \Phi_{n-k}' - \Psi_k' \Psi_{n-k}') + \frac{\Delta^3}{t} (\Phi_k' \Phi_{n-k}'' - \Psi_k' \Psi_{n-k}'') + \frac{\Delta^3}{t} (\Phi_k' \Phi_{n-k}'' - \Psi_k' \Psi_{n-k}') \right). \end{aligned}$$

The term $\sum B$ is:

$$\begin{aligned}
\sum_{k=1}^{N-n} B &= n^2 \alpha^2 \sum_{k=1}^{N-n} \left(-\frac{n+k}{k} \frac{\Delta}{t} (\Phi'_k \Phi_{n+k} + \Psi'_k \Psi_{n+k}) - \frac{\Delta}{t} (\Phi_k \Phi'_{n+k} + \Psi_k \Psi'_{n+k}) \right) \\
&+ n^2 \alpha^2 \sum_{k=1}^{N-n} \left(-\frac{1}{t^2} (\Phi_k \Phi_{n+k} - \Psi_k \Psi_{n+k}) \right) \\
&+ \sum_{k=1}^{N-n} \frac{n}{k+n} \left(\frac{3\Delta^2}{t^2} (\Phi_k \Phi''_{k+n} + \Psi_k \Psi''_{k+n}) + \frac{3\Delta}{t^3} (\Phi_k \Phi'_{k+n} + \Psi_k \Psi'_{k+n}) \right) \\
&+ \sum_{k=1}^{N-n} \frac{n}{k+n} \left(\frac{\Delta^3}{t} (\Phi_k \Phi'''_{k+n} + \Psi_k \Psi'''_{k+n} + \Phi'_k \Phi''_{k+n} + \Psi'_k \Psi''_{k+n}) + \frac{\Delta^2}{t^2} (\Phi'_k \Phi'_{k+n} + \Psi'_k \Psi'_{k+n}) \right) \\
&+ \sum_{k=1}^{N-n} \frac{n}{k} \left(\frac{2\Delta^2}{t^2} (\Phi'_k \Phi'_{k+n} + \Psi'_k \Psi'_{k+n}) + \frac{\Delta^3}{t} (\Phi'_k \Phi''_{k+n} + \Psi'_k \Psi''_{k+n} + \Phi''_k \Phi'_{k+n} + \Psi''_k \Psi'_{k+n}) \right).
\end{aligned}$$

The term $\sum C$ is:

$$\begin{aligned}
\sum_{k=n+1}^N C &= n^2 \alpha^2 \sum_{k=n+1}^N \left(-\frac{k-n}{k} \frac{\Delta}{t} (\Phi'_k \Phi_{k-n} + \Psi'_k \Psi_{k-n}) - \frac{\Delta}{t} (\Phi_k \Phi'_{k-n} + \Psi_k \Psi'_{k-n}) \right) \\
&+ n^2 \alpha^2 \sum_{k=n+1}^N \left(-\frac{1}{t^2} (\Phi_k \Phi_{k-n} - \Psi_k \Psi_{k-n}) \right) \\
&+ \sum_{k=n+1}^N \frac{-n}{k-n} \left(\frac{3\Delta^2}{t^2} (\Phi_k \Phi''_{k-n} + \Psi_k \Psi''_{k-n}) + \frac{3\Delta}{t^3} (\Phi_k \Phi'_{k-n} + \Psi_k \Psi'_{k-n}) \right) \\
&+ \sum_{k=n+1}^N \frac{-n}{k-n} \left(\frac{\Delta^3}{t} (\Phi_k \Phi'''_{k-n} + \Psi_k \Psi'''_{k-n} + \Phi'_k \Phi''_{k-n} + \Psi'_k \Psi''_{k-n}) + \frac{\Delta^2}{t^2} (\Phi'_k \Phi'_{k-n} + \Psi'_k \Psi'_{k-n}) \right) \\
&- \sum_{k=n+1}^N \frac{n}{k} \left(\frac{2\Delta^2}{t^2} (\Phi'_k \Phi'_{k-n} + \Psi'_k \Psi'_{k-n}) + \frac{\Delta^3}{t} (\Phi'_k \Phi''_{k-n} + \Psi'_k \Psi''_{k-n} + \Phi''_k \Phi'_{k-n} + \Psi''_k \Psi'_{k-n}) \right).
\end{aligned}$$

The equation g_n is

$$\begin{aligned}
g_n \equiv & \Delta^4 \Psi_n'''' + \frac{2}{t} \Delta^3 \Psi_n''' + \left(\frac{3}{t^2} - 2n^2 \alpha^2 \right) \Delta \Psi_n'' + \left(\frac{3}{t^3} - \frac{2}{t} n^2 \alpha^2 \right) \Delta \Psi_n' + n^4 \alpha^4 \Psi_n \\
& - n \bar{\beta} \left(-(U_0 + \hat{u}_0 - c)(\Delta^2 \Phi_n'' + \frac{\Delta}{t} \Phi_n' - n^2 \alpha^2 \Phi_n) + \Delta^2 (U_0 + \hat{u}_0)'' \Phi_n + \frac{\Delta}{t} (U_0 + \hat{u}_0)' \Phi_n \right) \\
& + \sum_{k=1}^{n-1} D + \sum_{k=1}^{N-n} E + \sum_{k=n+1}^N F = 0,
\end{aligned}$$

where the terms D , E and F are detailed below. The term $\sum D$ is:

$$\begin{aligned}
\sum_{k=1}^{n-1} D = & n^2 \alpha^2 \sum_{k=1}^{n-1} \left(\frac{n-k}{k} \frac{\Delta}{t} (\Phi_k' \Psi_{n-k} + \Psi_k' \Phi_{n-k}) - \frac{\Delta}{t} (\Phi_k \Psi_{n-k}' + \Psi_k \Phi_{n-k}') \right) \\
& + n^2 \alpha^2 \sum_{k=1}^{n-1} \left(-\frac{1}{t^2} (\Phi_k \Psi_{n-k} + \Psi_k \Phi_{n-k}) \right) \\
& + \sum_{k=1}^{n-1} \frac{n}{n-k} \left(\frac{3\Delta^2}{t^2} (\Phi_k \Psi_{n-k}'' + \Psi_k \Phi_{n-k}'') + \frac{3\Delta}{t^3} (\Phi_k \Psi_{n-k}' + \Psi_k \Phi_{n-k}') \right) \\
& + \sum_{k=1}^{n-1} \frac{n}{n-k} \left(\frac{\Delta^3}{t} (\Phi_k \Psi_{n-k}''' + \Psi_k \Phi_{n-k}''' + \Phi_k' \Psi_{n-k}'' + \Psi_k' \Phi_{n-k}'') + \frac{\Delta^2}{t^2} (\Phi_k' \Psi_{n-k}' + \Psi_k' \Phi_{n-k}') \right) \\
& - \sum_{k=1}^{n-1} \frac{n}{k} \left(\frac{2\Delta^2}{t^2} (\Phi_k' \Psi_{n-k}' + \Psi_k' \Psi_{n-k}') + \frac{\Delta^3}{t} (\Phi_k' \Psi_{n-k}'' + \Psi_k' \Phi_{n-k}'' + \Phi_k' \Psi_{n-k}'' + \Phi_k'' \Psi_{n-k}') \right).
\end{aligned}$$

The term $\sum E$ is:

$$\begin{aligned}
\sum_{k=1}^{N-n} E &= n^2 \alpha^2 \sum_{k=1}^{N-n} \left(-\frac{n+k}{k} \frac{\Delta}{t} (\Phi'_k \Psi_{n+k} - \Psi'_k \Phi_{n+k}) - \frac{\Delta}{t} (\Phi_k \Psi'_{n+k} - \Psi_k \Phi'_{n+k}) \right) \\
&+ n^2 \alpha^2 \sum_{k=1}^{N-n} \left(-\frac{1}{t^2} (\Phi_k \Psi_{n+k} - \Psi_k \Phi_{n+k}) \right) \\
&+ \sum_{k=1}^{N-n} \frac{n}{k+n} \left(\frac{3\Delta^2}{t^2} (\Phi_k \Psi''_{k+n} - \Psi_k \Phi''_{k+n}) + \frac{3\Delta}{t^3} (\Phi_k \Phi'_{k+n} - \Psi_k \Psi'_{k+n}) \right) \\
&+ \sum_{k=1}^{N-n} \frac{n}{k+n} \left(\frac{\Delta^3}{t} (\Phi_k \Psi'''_{k+n} - \Psi_k \Phi'''_{k+n} + \Phi'_k \Psi''_{k+n} - \Psi'_k \Phi''_{k+n}) + \frac{\Delta^2}{t^2} (\Phi'_k \Psi'_{k+n} - \Psi'_k \Phi'_{k+n}) \right) \\
&+ \sum_{k=1}^{N-n} \frac{n}{k} \left(\frac{2\Delta^2}{t^2} (\Phi'_k \Psi'_{k+n} - \Psi'_k \Phi'_{k+n}) + \frac{\Delta^3}{t} (\Phi'_k \Psi''_{k+n} - \Psi'_k \Phi''_{k+n} + \Phi''_k \Psi'_{k+n} + \Psi''_k \Phi'_{k+n}) \right).
\end{aligned}$$

And finally, the term $\sum F$ is:

$$\begin{aligned}
\sum_{k=n+1}^N F &= n^2 \alpha^2 \sum_{k=n+1}^N \left(-\frac{k-n}{k} \frac{\Delta}{t} (\Psi'_k \Phi_{k-n} - \Phi'_k \Psi_{k-n}) - \frac{\Delta}{t} (\Psi_k \Phi'_{k-n} - \Phi_k \Psi'_{k-n}) \right) \\
&+ n^2 \alpha^2 \sum_{k=n+1}^N \left(-\frac{1}{t^2} (\Psi_k \Phi_{k-n} - \Phi_k \Psi_{k-n}) \right) \\
&+ \sum_{k=n+1}^N \frac{-n}{k-n} \left(\frac{3\Delta^2}{t^2} (\Psi_k \Psi''_{k-n} - \Phi_k \Psi''_{k-n}) + \frac{3\Delta}{t^3} (\Psi_k \Phi'_{k-n} - \Phi_k \Psi'_{k-n}) \right) \\
&+ \sum_{k=n+1}^N \frac{-n}{k-n} \left(\frac{\Delta^3}{t} (\Psi_k \Phi'''_{k-n} - \Phi_k \Psi'''_{k-n} + \Psi'_k \Phi''_{k-n} - \Phi'_k \Psi''_{k-n}) + \frac{\Delta^2}{t^2} (\Psi'_k \Phi'_{k-n} - \Phi'_k \Psi'_{k-n}) \right) \\
&- \sum_{k=n+1}^N \frac{n}{k} \left(\frac{2\Delta^2}{t^2} (\Psi'_k \Phi'_{k-n} - \Phi'_k \Psi'_{k-n}) + \frac{\Delta^3}{t} (\Psi'_k \Phi''_{k-n} - \Phi'_k \Psi''_{k-n} + \Psi''_k \Phi'_{k-n} - \Phi''_k \Psi'_{k-n}) \right).
\end{aligned}$$

Chapter 8

Appendix C - Chebyshev Differentiation Matrices

In this section, we provide a sample Matlab code (freely available) that calculates the first few Chebyshev derivative matrices. The reader is referred to Schmid and Henningson (2001) for one such source of this code.

```

N = Number of modes
D0 = Zeroth derivative matrix
D1 = First derivative matrix
D2 = Second derivative matrix
D3 = Third derivative matrix
D4 = Fourth derivative matrix
num=round(abs(N));
D0=[ ];
vec= (0 : 1 : num)';
for j=0:1:num
    D0=[D0 cos(j*pi*vec/num)];
end ;
lv=length(vec);
D1=[zeros(lv,1) D0(:,1) 4*D0(:,2)];

```

```

D2=[zeros(lv,1) zeros(lv,1) 4*D0(:,1)];
D3=[zeros(lv,1) zeros(lv,1) zeros(lv,1)];
D4=[zeros(lv,1) zeros(lv,1) zeros(lv,1)];
    for j=3:num
        D1=[D1 2*j*D0(:,j)+j*D1(:,j-1)/(j-2)];
        D2=[D2 2*j*D1(:,j)+j*D2(:,j-1)/(j-2)];
        D3=[D3 2*j*D2(:,j)+j*D3(:,j-1)/(j-2)];
        D4=[D4 2*j*D3(:,j)+j*D4(:,j-1)/(j-2)];
    end ;

```

Chapter 9

List of Publications

Research and findings that the author supplies in this thesis can also be obtained in the following publication:

- [46] A.G. Walton & A.W.H. Wong. Axisymmetric Travelling Waves In Annular Couette-Poiseuille Flow *The Quarterly Journal of Mechanics and Applied Mathematics*, 2012,

which has the abstract:

“A computational study of the stability of the axial pressure-gradient-driven viscous flow between concentric sliding cylinders is presented. Nonlinear axisymmetric travelling wave solutions, which bifurcate from the linear neutral curve, are found for a range of radius ratio, sliding velocity and disturbance amplitude. Solutions which come into existence at finite amplitude and are not connected to the linear state in wavenumber-Reynolds number space are also found. The critical Reynolds numbers associated with these solutions are found to be substantially lower than those predicted by linear theory. For some combinations of parameters, two nonlinear solutions with different wavelengths are found to coexist and appear to merge into one as the amplitude is increased.”

The full article can be downloaded from:

<http://qjmam.oxfordjournals.org/cgi/reprint/hbs002?ijkey=06PjHrjqNQ7zdf8&keytype=ref>

Digital Object Identifier (DOI): 10.1093/qjmam/hbs002

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