

Imperial College London
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Macro-prudential Policy and Asset Pricing

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Abstract

I study questions related to the general equilibrium effects of macro-prudential policy designed to restrict speculation driven by belief dispersion among investors. First, I explore the asset pricing implications of differences in beliefs, and I find that belief dispersion caused by overconfidence can explain the downward sloping equity premium term structure and upward sloping bond yield curve. Second, I investigate the impact of speculation driven by belief dispersion on asset prices, social welfare and the aggregate economy. I find speculation reduces social welfare and creates a negative output gap, creating a role for macro-prudential policy. I conclude by studying a general equilibrium asset pricing model with a macro-prudential policy constraint that dampens the leverage cycle. My results show that macro-prudential policy can help to mitigate the externalities caused by speculation, enhance economic growth, and increase social welfare even when policy makers do not know the correct physical distribution of exogenous technology shocks.

Contents

1	Difference in Beliefs, Overconfidence and Equity Premium Term Structure	9
I	Introduction	9
II	Theoretical Framework	12
A	Heterogeneous Expectation and Learning	12
B	‘Catching Up with Joneses’ Preferences	14
C	The Equilibrium	15
D	The State Price Density	16
III	Asset Prices	17
A	Dividend Strip and Bond Strip Prices	18
B	Dividend Strips Volatility and Risk Premium Term Structure	19
C	Bond Yield	22
IV	Risk Price Term Structure	22
V	Conclusion	25
VI	Appendix: Tables	26
VII	Appendix: Proofs	27
A	Proof of Corollary 1 and Corollary 2	27
B	Proof of Corollary 3	27
C	Proof of Proposition 1	27
D	Proof of Proposition 2	35
E	Proof of Lemma 1	36
F	Proof of Proposition 3	37
G	Proof of Corollary 7	38
VIII	Appendix: Figures	39
2	Speculation, Differences in Beliefs and Growth	46
I	Introduction	46
II	Related Literature	48
III	Model	48
A	Heterogeneous Expectation and Learning	49
B	Preferences and Optimization	50
IV	The Equilibrium	51
A	Asset Prices	53
B	Aggregate Output and Difference in Beliefs	54
C	Belief Neutral Welfare Analysis	55
V	Numerical Results: Effects of Speculation	56
A	Speculation and Asset Prices	57

	B	Speculation and Social Welfare	58
	C	Aggregate Output	59
VI		Conclusion	60
VII		Appendix: Tables	61
VIII		Appendix: Proofs	62
	A	Proof of Proposition 4	62
	B	Proof of Proposition 5	62
	C	Proof of Proposition 6	63
	D	Speculation and Output Dynamics	63
	E	Proof of Proposition 7	64
IX		Appendix: Figures	66
3		Macro-prudential Policy, Differences in Beliefs and Growth	69
I		Introduction	69
II		Related Literature	72
III		Model	73
	A	Learning and Difference in Beliefs	74
	B	Preferences and Optimization with Policy Constraint	75
IV		The Equilibrium	78
	A	Macprudential Policy and Asset Prices	80
	B	Welfare Analysis	83
	C	Macro-prudential Policy and Aggregate Economy	83
V		Numerical Results: Effects of Macro-prudential Policy	84
	A	Results of the Asset Prices and Social Welfare	84
	B	Macro-prudential Policy and Aggregate Output	85
VI		Discussion	86
	A	Speculation and Risk Sharing	86
	B	Preference Assumption, Policy and Output Growth	87
VII		Conclusion	87
VIII		Appendix: Tables	88
IX		Appendix: Proofs	89
	A	Proof of Proposition 9	89
	B	Proof of Corollary 8	89
	C	Proof of Proposition 9	90
	D	Proof of Proposition 11	90
	E	Proof of Proposition 10	90
	F	Dynamics of the Output	91
	G	Proof of Corollary 11	92
	H	Extension: Speculation and Risk Sharing	93
X		Appendix: Figures	96

List of Figures

1.1	Risk free rate	39
1.2	Dividend Strip Volatility Term Structure	40
1.3	Equity Premium Term Structure and Difference in Beliefs	41
1.4	Equity Premium Term Structure and Overconfidence	42
1.5	Bond Yield Curve and Overconfidence	43
1.6	Price of Risk	44
1.7	Survival of the Overconfident Agent	45
2.1	Effects of Speculation on Asset Prices	66
2.2	Effects of Speculation on Individual Welfare and Social Welfare	67
2.3	Effects of Speculation on Aggregate Labor Supply	68
3.1	A Special Parametrization: Discount Rate Channel	96
3.2	Impacts of Macro-prudential Policy: Baseline Calibration	97
3.3	Impacts of Macro-prudential Policy: An Alternative Calibration	98
3.4	Model Extension: Risk Sharing and Speculation	99
3.5	Risk Sharing and Speculation: An Alternative Utility Function	100

List of Tables

1.1	Parameter Choice of Chapter 1	26
2.1	Chapter 2 Production Economy Calibration	61
3.1	Choice of Parameter Values of Chapter 3	88

Declaration of Originality

‘I herewith certify that this thesis constitutes my own work and that all material, which is not my own work, has been properly acknowledged’

Daren Wei

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Chapter 1

Difference in Beliefs, Overconfidence and Equity Premium Term Structure

I. Introduction

Empirical studies suggest that the equity premium term structure is downward sloping (see [Van Binsbergen, Brandt, and Koijen \(2012\)](#) and [Van Binsbergen, Hueskes, Koijen, and Vrugt \(2013\)](#)). This finding is consistent with the value premium ([Fama and French \(1992\)](#)), i.e. value stocks earn higher returns than growth stocks, because the dividend flow duration of value stocks is shorter than that of growth stocks.

Popular asset pricing models such as the habit formation model in [Campbell and Cochrane \(1999\)](#), the rare disaster model in [Barro \(2009\)](#) and long-run risk model in [Bansal and Yaron \(2004\)](#) successfully capture aggregate asset pricing facts such as high excess returns and volatility for stocks, low level and volatility for the risk-free interest rate; however, they predict a flat or strongly upward sloping equity premium term structure. Newly developed asset pricing models characterize the higher short-term risk premium by increasing short-term dividend volatility (cash flow channel, [Nakamura, Steinsson, Barro, and Ursúa \(2013\)](#) and [Belo, Collin-Dufresne, and Goldstein \(2015\)](#)) and modifying agents' utility functions (discount rate channel, [Andries \(2014\)](#) and [Curatola \(2015\)](#)). However, dividend growth volatility is too small to induce a reasonable level equity premium (see [Shiller \(1981\)](#)) and modifying preferences cannot explain the pro-cyclical slope of the equity premium term structure. In this paper, I present a general equilibrium model featuring differences in beliefs and overconfidence, which determine the equity risk premium term structure via the discount rate channel without any ad-hoc modification of preferences.

In our endowment economy, stochastic expected dividend growth is latent to agents. Thus agents have to estimate this dividend growth in order to optimize their portfolios. Apart from observing the realized dividend flow to estimate the dividend growth, some agents may misjudge the informativeness of a public noise signal and obtain an over-precise estimation, which is referred as overconfidence. Overconfidence prevents them from understanding the informativeness of the signal. Hence, the overconfident agent gives too much weight to the public signal when she update her prior estimates, causing persistent disagreement and fluctuations in consumption allocations across agents.¹ Two agents agree to disagree and speculate against each other by betting on their estimation precision.

[Bhamra and Uppal \(2014\)](#) point out that asset prices of a heterogeneous agent economy can be

¹This overconfidence modeling approach is proposed by [Scheinkman and Xiong \(2003\)](#) and also followed by [Dumas, Kurshev, and Uppal \(2009\)](#)

represented by the linear combination of weighted asset prices from single agent economies, in which the weights are functions of consumption good allocations between the rational agent and the overconfident agent in my setting. The consumption share, one of the four state variables, is a function of sentiment², whose diffusion term depends on the difference in beliefs across agents. Hence, any perceived fundamental shock not only impacts the current consumption shares and the short-term discount rate, but is also expected to reduce the consumption share of the overconfident agent and thus gradually reduces the market weight of the overconfident agent, resulting a lower long-term risk premium.

I solve the risk premium and risk prices in closed form and find that when the overconfident agent is pessimistic about the market, typically in a bear market, 1) the short-maturity dividend strip risk premium is higher than the long-maturity dividend strip risk premium and the aggregate equity premium; 2) the dividend strip price volatility term structure is downward sloping with maturity. These results imply that the equity premium term structure is downward sloping and its slope is pro-cyclical. I also solve bond prices in closed form, and we obtain an upward sloping bond yield term structure.

The risk premium equals the quantity of risks multiplied by the price of risks. We can explore the reasons of downward sloping equity premium term structure from looking at the term structure of quantity and price of risks. Firstly, I show that difference in beliefs and over confidence can explain the downward sloping quantity of risk-term structure. Intuitively, a financial market is affected by fluctuations of the market sentiment so that sentiment volatility causes prices to be more volatile than those which would be justified by dividend volatility alone. Overconfidence bias increases the sentiment volatility and thus the short term dividend strips volatilities. Meanwhile, the sentiment volatility depends on the difference in beliefs which has a zero long-term expectation. Therefore, long maturity dividend strip prices are less volatile than short maturity dividend strips.

Differences in beliefs and overconfidence can also explain the downward sloping risk prices. I decompose the risk price into an instantaneous and a long-term component, and these two components illustrate that overconfidence and differences in beliefs generate a downward sloping risk price term structure. The instantaneous component describes the instantaneous response of the SDF to the shocks today, while the long-term component measures the long-run response of the SDF to those shocks. Two effects drive the term structure of the market price of risk. First, the level of the market price of risk is determined by the short-term market price of risk, which is independent of overconfidence and solely determined by current views of all market participants³ and their market weights. This is because the overconfidence only impacts the way market participants react to new information and forecasts about economic and market fundamentals, therefore it only affects the path of state variables in the future but not their current values when investors have same risk aversion. Second, the long-term market price of risk depends on how the current shocks impact the dividend growth forecast and the market shares of agents. From her perspective, the rational agent is able to arbitrage away pricing errors due to overconfidence behavior bias, and the share of the irrational agent is expected to decline. In a bear market, the overconfident agent is more pessimistic as she over-reacts to negative news. The bearish sentiment in an aggregate market

²Dumas, Kurshev, and Uppal (2009) give the name of "sentiment" to the Radon-Nikodym derivative that defines the change of measure from the subjective measure of one agent to that of the other agent.

³The market price of risk is also determined by the risk aversion of market participants.

induces high risk prices, which is further distorted by overconfidence. However, the rational agent expects that she will obtain speculative profit and reduce the overconfident pessimists market share; as a result, she will increase the market sentiment and lower the market price of risk. In addition, the rational agent can obtain speculative gains from the overconfident agent in the long-term, so she requires less long-term risk premium. If the pessimistic agent is more irrational, she will lose her wealth and market share faster in the eyes of the rational agent, and the risk-price term structure becomes steeper. The response of the long-term SDF to current shocks is little if there is no overconfidence and differences in beliefs, and hence the long-term effect disappears. This mechanism implies that the slope of equity premium term structure is pro-cyclical, because the long-term effect is more pronounced in recessions than booms when the market weight of overconfident pessimist is high.

This research contributes to the cross-sectional asset pricing literature. [Lettau and Wachter \(2007\)](#) argue that all leading asset pricing models such as the habit formation models in [Campbell and Cochrane \(1999\)](#), the long-run risk model in [Bansal and Yaron \(2004\)](#), disaster risk model in [Barro \(2009\)](#), fail to produce a downward sloping equity premium. These models generate upward sloping equity premium term structure, because shocks to the discount factor are priced. Firms with long-duration cash flows have a high exposure to such shocks and should therefore have a higher risk premium than firms with short- duration cash flows. If one adheres to the view that value firms have short-duration cash flows and growth firms have long-duration cash flows, this implies that there is a growth premium, not a value premium. [Lettau and Wachter \(2007\)](#) and [Lettau and Wachter \(2011\)](#) propose a reduced-form model with an exogenous stochastic discount factor that generates higher expected returns for short-term dividend strips. They illustrate the correlation structure between (un)expected cash flow shocks and shocks to the price of risk, which is sufficient to generate a value premium in their model. This paper contributes to this literature by explaining the downward sloping market price of risk and risk premia without ad-hoc modification of preferences. Several studies within general equilibrium frameworks either modify preferences to increase short-term effective risk aversion, or adjust the technology process to increase short-term volatility. [Eisenbach and Schmalz \(2015\)](#) and [Andries, Eisenbach, and Schmalz \(2014\)](#) modify standard preferences so that the representative agent is more risk averse for imminent than for distant risk. [Andries \(2014\)](#) and [Curatola \(2015\)](#) consider loss-aversion preferences. [Berrada, Detemple, and Rindisbacher \(2014\)](#) induce a regime-switching model where the preference changes with the regimes. [Nakamura, Steinsson, Barro, and Ursúa \(2013\)](#) and [Hasler and Marfe \(2015\)](#) extend the disaster model and assume that the short dividend flow is subject to disaster risk, but the long-term dividend flow isn't because the economy will recover in the long-term. [Belo, Collin-Dufresne, and Goldstein \(2015\)](#) and [Marfè \(2015\)](#) generate more volatile short-term dividend flow by forcing the shareholders to divest (invest) when leverage is low (high), which shifts risk from long- to short-horizon dividend strips. [Marfè \(2014\)](#) develops a heterogeneous agents (workers and shareholders) model to increase the riskiness of short-term dividend. The other type of models consider alternative beliefs, such as bounded rationality in [Croce, Lettau, and Ludvigson \(2014\)](#).

This paper is also related to the literature of heterogeneous beliefs. There are two ways to generate heterogeneity in beliefs. In the first approach, agents receive difference information ([Grossman and](#)

Stiglitz (1980), Hellwig (1980), Wang (1993), and Shefrin and Statman (1994)). In this class of models, informed traders receive private signals strategically bid assets against noise traders to prevent the private information being fully revealed from quotes. The second class approach to generate heterogeneity, which is the one I adopt, is to have agents who “agree to disagree” about some aspect of the underlying economy. Recent papers using this approach include Zapatero (1998), Veronesi (1999), Basak (2000), Cecchetti, Lam, and Mark (2000), Duffie, Garleanu, and Pedersen (2002), David and Veronesi (2002), Scheinkman and Xiong (2003), Buraschi and Jiltsov (2006), David (2008), Dumas, Kurshev, and Uppal (2009), Gallmeyer and Hollifield (2008), Yan (2008), Xiong and Yan (2009), Chabakauri (2015) and Borovička (2015). In terms of methodology, this chapter is closely related to Dumas, Kurshev, and Uppal (2009) with two main differences. First, the two representative agents are featured as a “Keep up with the Joneses” preference, which allows to match asset prices better. Second, my objective is different, since I want to explain the downward sloping equity premium term structure.

The rest of this paper is arranged as follows. In Section 2 I firstly describe the model of an exchange economy and how agents’ beliefs are formulated and updated, then show the solution of stochastic discount factor. Section 3 illustrates the main result: the closed form solutions for securities prices and the equity premium term structure and bond yield structure. In Section 4 I discuss the term structure of risk prices. Section 5 concludes. I highlight all results in propositions, and detailed proofs and plots for all the results are provided in the appendix.

II. Theoretical Framework

I assume a Lucas endowment economy, in which the dividend flow follows a diffusion process:

$$\frac{dD_t}{D_t} = \mu_{D,t}dt + \sigma_D dZ_{D,t}. \quad (1.1)$$

where σ_D is a constant and $Z_{D,t}$ is a standard Brownian motion. The expected dividend growth follows an Ornstein-Uhlenbeck process:

$$d\mu_{D,t} = \zeta (\bar{\mu}_D - \mu_{D,t}) dt + \sigma_\mu dZ_{\mu,t}, \quad (1.2)$$

where $\bar{\mu}_D$ is the long-term mean of expected dividend growth rate, ζ is the mean reverting speed coefficient, and $Z_{\mu,t}$ is a standard Brownian motion and independent of the dividend Brownian motion shocks. Time-varying property expected dividend growth rate keeps the difference in beliefs persistent if agents have different interpretation of the signal informativeness. In addition, the affine structure assumption gives the tractability of asset prices. There are three assets in the market: a stock, a perpetual bond and instantaneous risk-free lending.

A. Heterogeneous Expectation and Learning

In this endowment economy, investors may hold different views about the economy growth and keep on updating their views by observing the dividend flow and public signals. They may also overestimate the

informativeness of the public noise signal and put some weight on this noise signal when they forecast the dividend growth. More specifically, I assume that agents observe the following signal flow:

$$dS_t = dZ_{S,t} \quad (1.3)$$

where $Z_{S,t}$ is a standard Brownian motion independent of all Brownian motions introduced earlier. S_t is a pure noise.

There are two types of agents. Type i agent thinks the innovations $dZ_{S,t}$ in the signal S_t are correlated with the innovations $dZ_{\mu,t}$ in the expected dividend growth process, with ϕ_i as the correlation parameter which characterizes the informativeness of this public signal. Because $\mu_{D,t}$ is a latent process, agents misperceive the volatility of $\mu_{D,t}$ by a factor ω_i . $\omega_i = 1$ corresponds to the case in which Type- i agent correctly perceives the volatility. From agent i 's point of view, the expected dividend growth follows

$$d\mu_{D,t} = \zeta (\bar{\mu}_D - \mu_{D,t}) dt + \omega_i \sigma_\mu dZ_{\mu,t}; \quad (1.4)$$

the noise signal S_t , which she believes is correlated the shocks to $\mu_{D,t}$, has the following equivalent dynamics

$$dS_t^i = \phi_i dZ_{\mu,t} + \sqrt{1 - \phi_i^2} dZ_{S,t} \quad (1.5)$$

in which $dZ_{S,t}dZ_{D,t} = dZ_{S,t}dZ_{\mu,t} = 0$. By [Lipster and Shiryaev \(2001\)](#), the posterior for the estimation of $\mu_{D,t}$, $\hat{\mu}_{D,t}^i$, after the belief variance reaches its stationary level, follows

$$d\hat{\mu}_{D,t}^i = \zeta (\bar{\mu}_D - \hat{\mu}_{D,t}^i) dt + \frac{1}{\sigma_D^2} \bar{v}_i \left(\frac{dD_t}{D_t} - \hat{\mu}_{D,t} dt \right) + \phi_i \omega_i \sigma_\mu dS_t^i \quad (1.6)$$

where $\bar{v}_i$ represents the variance of the posterior at steady state and is the positive root of following quadratic equation:

$$\frac{1}{\sigma_D^2} \bar{v}_i + 2\lambda_\mu \bar{v}_i - (1 - \phi_i^2) \omega_i^2 \sigma_\mu^2 = 0. \quad (1.7)$$

Without loss of generality, I assume Type 1 agent is rational (Bayesian) and Type 2 agent is overconfident, namely $\phi_1 = 0$ and $\phi_2 \neq 0$. The quadratic equation above implies that, compared to the rational agent, the overconfident agent reaches a lower level of steady posterior variance, i.e. $\bar{v}_1 > \bar{v}_2$, therefore she has an overprecise estimation (overconfident). Due to behaviour bias, the overconfident agent insists on her incorrect judgement on the informativeness of the noise signal (ϕ_2).

I define their subjective probability measure $\mathbb{P}^i$ against the objective probability $\mathbb{P}$ measure by the following exponential martingale

$$\frac{d\mathbb{P}^i}{d\mathbb{P}} = \xi_{t,i}, \quad (1.8)$$

where

$$\xi_{t,i} = \xi_{t,0} \exp \left[\int_0^t -\frac{1}{2} \sigma_{\xi,i}^2 ds + \int_0^t \sigma_{\xi,i} dZ_{D,t} \right] \quad (1.9)$$

$$\sigma_{\xi,i} = \frac{\hat{\mu}_{D,t}^i - \mu_{D,t}}{\sigma_D}. \quad (1.10)$$

Due to overconfidence, difference in beliefs is being regenerated even after long periods of updating, given heterogeneous priors and different signal interpretation. The difference in beliefs $g_t = \hat{\mu}_{D,t}^1 - \hat{\mu}_{D,t}^2$ follows

$$dg_t = - \left(\zeta + \frac{\bar{v}_2}{\sigma_D^2} \right) g_t dt + \frac{1}{\sigma_D} (\bar{v}_1 - \bar{v}_2) dZ_t^1 + (\phi_1 \omega_1 - \phi_2 \omega_2) \sigma_\mu dS_t. \quad (1.11)$$

I quantify the level of disagreement by the process ξ_t , which is called “sentiment” by [Dumas, Kurshev, and Uppal \(2009\)](#) and defined by:

$$\xi_t = \frac{\xi_{t,2}}{\xi_{t,1}}, \quad (1.12)$$

and hence

$$\frac{d\xi_t}{\xi_t} = \mu_\xi dt + \sigma_\xi dZ_{D,t}, \quad (1.13)$$

where

$$\mu_\xi = -\sigma_{\xi,1} (\sigma_{\xi,2} - \sigma_{\xi,1}) \quad (1.14)$$

$$\sigma_\xi = \sigma_{\xi,2} - \sigma_{\xi,1} = -\frac{g_t}{\sigma_D}. \quad (1.15)$$

B. ‘Catching Up with Joneses’ Preferences

Both agents have ‘Catching up with Joneses’ preferences. The consumption of Agent i at instant u is denoted by $C_{i,u}$, and the instantaneous utility from consumption is given by the following power function that depends on the external habit, X_u :

$$U_i(C_{i,u}, X_u) \equiv e^{-\rho u} \frac{1}{1-\gamma} \left(\frac{C_{i,u}}{X_u} \right)^{1-\gamma}, \quad (1.16)$$

where the constant subjective discount rate ρ and the risk aversion coefficient γ are assumed to be identical for two types of agents. The quantity X_u in Equation 3.14 can be interpreted as the historical standard of living in the economy, which is the same as [Chan and Kogan \(2002\)](#). Under this specification, let $x_u \equiv \log X_u$, which can be defined as the weighted geometric average of past realizations of the logarithm of the aggregate endowment process:

$$x_t = x_0 e^{-\lambda_x t} + \lambda_x \int_0^t e^{-\lambda(t-u)} d_u du, \quad (1.17)$$

where d_u denotes the logarithmic aggregate endowment at time u . Define the distance between the aggregate endowment and its weight average as

$$\Omega_t = e^{\omega t} \equiv \frac{D_t}{X_t}, \quad (1.18)$$

so X_t and Ω_t have the dynamics

$$\frac{dX_t}{X_t} = \lambda_x \omega_t dt, \quad (1.19)$$

$$\frac{d\Omega_t}{\Omega_t} = (\mu_{D,t} - \lambda_x \omega_t) dt + \sigma_D dZ_{D,t}. \quad (1.20)$$

Given agent i 's belief which is represented by the probability measure $\mathbb{P}^k$, her expected lifetime utility at time t from consuming $C_{i,u}$ is given by

$$V_{i,t} = E_t^i \left[\int_t^\infty e^{-\rho u} \frac{1}{1-\gamma} \left(\frac{C_{i,u}}{X_u} \right)^{1-\gamma} du \right]. \quad (1.21)$$

Given the complete market assumption and separable utility assumption, [Cox and Huang \(1989\)](#) show that the dynamic consumption-portfolio choice problem simplifies to a static problem. The agents optimize their life time utility subject to a static budget constraint, which restricts the present value of all future consumption to be no more than the initial wealth of each agent, denoted by $W_{k,0}$:

$$E_0^i \left[\int_t^\infty \frac{\Lambda_{i,u}}{\Lambda_{i,0}} C_{i,u} du \right] \leq W_{i,0} = a_i S_0 \quad (1.22)$$

where S_0 is the initial wealth of the economy, a_i is her share of aggregate wealth, and her Stochastic Discount Factor (SDF) is

$$\Lambda_{i,u} \equiv \frac{\partial U(C_{i,u})}{\partial C_{i,u}} = e^{-\rho_i u} \left(\frac{1}{X_t} \right)^{1-\gamma} C_{i,u}^{-\gamma}. \quad (1.23)$$

C. The Equilibrium

The usual definition of equilibrium applies in this model: both agents optimize their expected utility and all markets are clear. If the agents have identical beliefs, we can construct a simple central planner's problem whose utility is the weighted utility of two classes of agents, and choose the optimal consumption weight allocation such that the utility of the social planner is maximized. In the presence of heterogeneous beliefs, [Basak \(2000\)](#) shows that the central planner's utility can be represented by stochastically weighted utility of agents:

$$\sup_{C_1+C_2 \leq Y} \sum_{k=1}^2 \lambda_{i,t} U_i(C_{i,t}), \quad \text{where } \lambda_{i,t} = \lambda_{i,0} \xi_{i,t} \quad (1.24)$$

The first order condition of the central planner's problem (1.24) gives the consumption sharing between two agents:

$$\lambda_{1,0} \xi_{1,t} e^{-\rho t} \left(\frac{1}{X_t} \right)^{1-\gamma} C_{1,t}^{-\gamma} = \lambda_{2,0} \xi_{2,t} e^{-\rho t} \left(\frac{1}{X_t} \right)^{1-\gamma} C_{2,t}^{-\gamma}. \quad (1.25)$$

We can write $\nu_{i,t} = \frac{C_{i,t}}{C_t}$ and simplify the equation above

$$\hat{\Lambda}_{1,t} \nu_{1,t}^{-\gamma} = \hat{\Lambda}_{2,t} \nu_{2,t}^{-\gamma} \quad (1.26)$$

where

$$\hat{\Lambda}_{i,t} = \lambda_{i,0} \xi_{i,t} e^{-\rho_i t} \left(\frac{1}{X_t} \right)^{1-\gamma} D_t^{-\gamma}, \quad (1.27)$$

which is the SDF of the dominating agent in the sole agent economy. Given identical preference assumption, the equilibrium consumption share is a function of the sentiment:

$$\nu_{1,t} = \left(1 + \xi_t^{\frac{1}{\gamma}} \right)^{-1}. \quad (1.28)$$

In the following proposition I show the dynamics of the consumption sharing rule.

Corollary 1. *If the agents have identical preference but heterogeneous beliefs, their consumption share evolution is given by*

$$\frac{d\nu_{1,t}}{\nu_{1,t}} = \mu_{\nu_1} dt + \sigma_{\nu_1, D} dZ_t \quad (1.29)$$

where

$$\sigma_{\nu_1} = -\frac{\nu_{2,t}}{\gamma} \sigma_{\xi,t} \quad (1.30)$$

$$\mu_{\nu_1} = \frac{\nu_{2,t}}{\gamma} \left[\frac{\frac{1}{2} (\hat{\mu}_{D,t}^1 + \hat{\mu}_{D,t}^2) - \mu_{D,t}}{\sigma_D} \right] \sigma_{\xi,t} + \frac{\nu_{2,t}}{2\gamma^2} (\nu_{2,t}^2 - \nu_{1,t}^2) \sigma_{\xi,t}^2 \quad (1.31)$$

From Corollary 1, we see that the volatility of consumption share is driven by the difference in beliefs: greater difference in beliefs leads to a higher consumption share volatility. Equation 1.31 shows how the drift of consumption sharing rule is a quadratic in the difference in beliefs, which is nonzero almost surely as both overconfident agent is overreacting to the signal.

D. The State Price Density

Bhamra and Uppal (2014) show that the aggregate SDF of two agents economy with constant but heterogeneous beliefs can be written as the linear combination of weighted product of two SDFs of single agent underlying economy. This property still holds when the difference in beliefs is stochastic.

Corollary 2. *Suppose agents have identical risk aversion, namely $\gamma_1 = \gamma_2 = \gamma$, but different beliefs. Then the state-price density can be written as*

$$\Lambda_t = \left(\hat{\Lambda}_{1,t}^{\frac{1}{\gamma}} + \hat{\Lambda}_{2,t}^{\frac{1}{\gamma}} \right)^{\gamma} \quad (1.32)$$

Using binomial expansion, if γ is a natural number, we can also write

$$\Lambda_t = \sum_{n=0}^{\gamma} \binom{\gamma}{n} \hat{\Lambda}_{1,t}^{\frac{n}{\gamma}} \hat{\Lambda}_{2,t}^{1-\frac{n}{\gamma}}. \quad (1.33)$$

From the aggregate state price density we can obtain the instantaneous market price of risk and the

risk-free rate.

Corollary 3. *The instantaneous market price of risk perceived by the central planner θ_t is*

$$\theta_t = \gamma\sigma_D + \frac{\mu_{D,t} - \hat{\mu}_{D,t}}{\sigma_D} \quad (1.34)$$

and the instantaneous market price of risk for the dividend shock perceived by the two agents are

$$\theta_{1,t} = \gamma \left(\sigma_D + \frac{\nu_{2,t}}{\gamma} \frac{\hat{\mu}_{D,t}^1 - \hat{\mu}_{D,t}^2}{\sigma_D} \right) = \gamma\sigma_D + \frac{\hat{\mu}_{D,t}^1 - \hat{\mu}_{D,t}}{\sigma_D}, \quad (1.35)$$

$$\theta_{2,t} = \gamma \left(\sigma_D + \frac{\nu_{1,t}}{\gamma} \frac{\hat{\mu}_{D,t}^2 - \hat{\mu}_{D,t}^1}{\sigma_D} \right) = \gamma\sigma_D + \frac{\hat{\mu}_{D,t}^2 - \hat{\mu}_{D,t}}{\sigma_D}. \quad (1.36)$$

where

$$\hat{\mu}_{D,t} = \nu_{1,t}\hat{\mu}_{D,t}^1 + \nu_{2,t}\hat{\mu}_{D,t}^2 \quad (1.37)$$

$$\nu_{1,t} = \frac{C_{1,t}}{D_t} \quad (1.38)$$

$$\nu_{2,t} = 1 - \nu_{1,t}. \quad (1.39)$$

The risk-free rate is given by

$$r_t = \rho + \gamma\hat{\mu}_{D,t} - \frac{1}{2}\gamma(1+\gamma)\sigma_D^2 - (\gamma-1)\lambda_x\omega_t + \frac{1}{2}\nu_{1,t}\nu_{2,t}\left(1 - \frac{1}{\gamma}\right)\sigma_{\xi,t}^2 \quad (1.40)$$

Corollary 3 shows that the market price of risk and risk-free rate are determined by the four state variables in this economy, consumption surplus ω_t , difference in beliefs, g_t , the view of one type of agent, e.g. $\hat{\mu}_{D,t}$ and the consumption share between two types of agents. This result is consistent with [Bhamra and Uppal \(2014\)](#). Note that the current risk-free rate and market price of risk does not depend on overconfidence (sentiment system). This is because signal shocks enters the pricing system by impacting the volatilities of relevant state variables ($\nu_{1,t}$ and ξ_t) but not their level. However, the overconfidence generates a path dependent effect which has long-term indirect impact on asset prices, which I will show later Figure 1.1 plots the risk-free rate against the consumption share $\nu_{1,t}$ and the difference in beliefs g_t . The figure shows a reasonable level of instantaneous risk-free rate.

III. Asset Prices

As mentioned in the beginning of Section II, there are three assets: instantaneous lending, a perpetual bond and an equity. The risky consumption claims (the perpetual bond and the equity) can be decomposed into bond strips and dividend strips over various maturities. I derive closed-form pricing formulas of bond and dividend strips, and hence obtain the closed form solutions of stock prices, bond prices, the equity risk premium, and the volatility of stock market returns, volatilities of dividend strip return, and the term

structure of equity premium. Then I show how the heterogeneous beliefs and the overconfidence impacts asset prices.

A. Dividend Strip and Bond Strip Prices

The following proposition shows the closed form pricing formula of bond strips and dividend strips prices, which allows to compute the interest rate term structure and equity premium term structure in closed form.

Proposition 1 (Prices of Dividend Strips and Bond Strips). *The dividend strip, which is defined as a single dividend cash flow in terms of D_u unit of consumption good that realizes at time $u > t$, has a price as a function of $\nu_{1,t}$, ω_t , $\hat{\mu}_{D,t}^1$ and g_t :*

$$p_{t,u} = \frac{P_{u,t}}{D_t} = E_t \left[\frac{\Lambda_u}{\Lambda_t} \frac{D_u}{D_t} \right] = \sum_{n=0}^{\gamma} w_{n,t} \zeta_{n,u,t}^D \quad (1.41)$$

where

$$w_{n,t} = \binom{\gamma}{n} \left(\nu_{1,t}^{1-\frac{n}{\gamma}} \nu_{2,t}^{\frac{n}{\gamma}} \right)^{\gamma} \quad (1.42)$$

$$\zeta_{n,u,t}^D = H_1 \left((1-\gamma), -(1-\gamma), \omega_t, \hat{\mu}_{D,t}^1, u-t \right) H_2 \left((1-\gamma), -(1-\gamma), \frac{n}{\gamma}, g_t, u-t \right). \quad (1.43)$$

The bond strip, which delivers a unit of consumption good at time $u > t$, has the price

$$B_t = E_t \left[\frac{\Lambda_u}{\Lambda_t} \right] = \sum_{n=0}^{\gamma} w_{n,t} \zeta_{n,u,t}^B \quad (1.44)$$

where

$$\zeta_{n,u,t}^B = H_1 \left(-\gamma, -(\gamma-1), \omega_t, \hat{\mu}_{D,t}^1, u-t \right) H_2 \left(-\gamma, -(\gamma-1), \frac{n}{\gamma}, g_t, u-t \right) \quad (1.45)$$

and

$$H_1(\epsilon, \eta, \hat{\mu}_{D,t}^1, \omega_t, \tau) = \exp \{ h_0(\tau) + h_1(\tau)(\hat{\mu}_{D,t}^1 - \bar{\mu}_D) + h_2(\tau)\omega_t \} \quad (1.46)$$

$$H_2(\epsilon, \eta, \chi, g_t, \tau) = \exp \{ A_1(\tau) + (1-\gamma)^2 A_2(\tau) + (1-\gamma)g_t B(\tau) + g_t^2 C(\tau) \}. \quad (1.47)$$

and the expression h_0 , h_1 , h_2 , A_1 , A_2 , B and C are given in the proof.

From Proposition 1 we see that the price dividend ratio is a weighted sum of the price-dividend ratios in economies where the SDF is a scaled product of the SDF of a sole agent economy, e.g. $\Lambda_{n,t} = \hat{\Lambda}_{1,t}^{1-\frac{n}{\gamma}} \hat{\Lambda}_{2,t}^{\frac{n}{\gamma}}$. The expression shows that the price-dividend ratio $p_{t,u}$ depends on the distance between the logarithm of the aggregate endowment and its weighted geometric average ω_t , the two agent's beliefs, the consumption share of two agents, and also the duration of the dividend payment. Under this specification, change of overconfidence level does not shift the current consumption share or the weighting function $w_{n,t}$, which

solely depends on the current difference in beliefs. But it does change the path of consumption (and SDF process) in the future, and the effect is accumulating over time, which is implied in the function $\zeta_{n,u,t}^B$ and $\zeta_{n,u,t}^D$. Due to the affine structure of learning system, I can decompose the effect of learning and the difference in beliefs by the function H_1 and H_2 . H_1 represents impact of the expected dividend growth estimation and the expected growth of surplus ratio ω_t (learning effect). H_2 represents the effect of difference in beliefs via the discount rate channel. There are three reasons why Both H_1 and H_2 are duration relevant. First, the expected dividend growth is a mean reverting process, and its expectation is duration relevant. The difference in beliefs is also mean reverting process which fluctuates around zero. Investors would expect the expected dividend growth and the difference in beliefs to go back to their long-run mean in sufficient time. Second, prices of dividend strips with longer duration are more sensitive to the discount rate variation. Third, due to different priors and updating rule, the difference in beliefs is persistent in the economy. Any perceived dividend and signal shocks generate innovations in difference in beliefs, which determines the volatility of the sentiment and the consumption share. Hence, the price of a dividend strip is a path dependent process, which is determined by the path of difference in beliefs. More details about the third reason are available in Proposition 3. The following corollary shows the equity price and perpetual bond price by summing up dividend strips price over various maturities.

Corollary 4 (Equity Price). *If both agents have identical risk aversion and their belief updating processes follow an affine process, the price of claim that delivers D_t units of consumption goods continuously is:*

$$p_t = \frac{P_t}{D_t} = E_t \left[\int_t^\infty \frac{\Lambda_u}{\Lambda_t} \frac{D_u}{D_t} du \right] = \sum_{n=0}^{\gamma} w_{n,t} \zeta_{n,t}^D \quad (1.48)$$

where

$$\zeta_{n,t}^D = \int_t^\infty H_1((1-\gamma), -(1-\gamma), \omega_t, \hat{\mu}_t^1, u-t) H_2\left((1-\gamma), -(1-\gamma), \frac{n}{\gamma}, g_t, u-t\right) du \quad (1.49)$$

$$g_t = \hat{\mu}_{D,t}^1 - \hat{\mu}_{D,t}^2. \quad (1.50)$$

B. Dividend Strips Volatility and Risk Premium Term Structure

The following corollary shows the closed form solution of the term structure of equity volatility and equity premium.

Proposition 2 (Term Structure of Volatility and Equity Risk Premium). *Denote the price of the dividend strip which delivers D_u units of consumption goods by the following*

$$\frac{dp_{t,u}}{p_{t,u}} = \mu_{p_{u,t}} dt + \sigma_{p_{u,D}} dZ_{D,t} + \sigma_{p_{u,S}} dZ_{S,t}, \quad (1.51)$$

then the diffusion part follows

$$\sigma_{p_{u,D}} = \sigma_D + \frac{1}{p_{t,u}} \left(\frac{\partial p_{t,u}}{\partial \omega_t} \sigma_D + \frac{\partial p_{t,u}}{\partial \xi_t} \xi_t \sigma_\xi + \frac{\partial p_{t,u}}{\partial \hat{\mu}_{D,t}^1} \frac{\bar{v}_1}{\sigma_D} + \frac{\partial p_{t,u}}{\partial g_t} \frac{\bar{v}_1 - \bar{v}_2}{\sigma_D} \right) \quad (1.52)$$

$$\sigma_{p_{u,S}} = \frac{1}{p_{t,u}} \left(\frac{\partial p_{t,u}}{\partial \hat{\mu}_{D,t}^1} \phi_1 \omega_1 \sigma_\mu + \frac{\partial p_{t,u}}{\partial g_t} (\phi_1 \omega_1 - \phi_2 \omega_2) \sigma_\mu \right). \quad (1.53)$$

The volatility of dividend strips is

$$\sigma_{p_{u,t}} = \sqrt{\sigma_{p_{u,D}}^2 + \sigma_{p_{u,S}}^2}, \quad (1.54)$$

and the equity premium of dividend strips is

$$RP_{u,t} = \sigma_{p_{u,D}} \left(R_t \sigma_D + \frac{\mu_{D,t} - \hat{\mu}_{D,t}}{\sigma_D} \right), \quad (1.55)$$

where $\mu_{p_{u,t}}$ and the derivatives are given in the Appendix.

In the classical Lucas representative agent model, the stock price and the dividend strip price volatility is equal to the fundamental volatility σ_D . From Proposition 2 we can see that the excess volatility consists of two parts: the first part stems from the belief heterogeneity while the second part is due to the misjudgement of the informativeness of the noise signal. The first part $\frac{1}{p_t} \left(\frac{\partial p_t}{\partial \xi_t} \xi_t \sigma_\xi + \frac{\partial p_t}{\partial \hat{\mu}_{1,t}} \frac{\bar{v}_1}{\sigma_D} + \frac{\partial p_t}{\partial g_t} \frac{\bar{v}_1 - \bar{v}_2}{\sigma_D} \right)$ can be interpreted as the elasticity of the price-dividend ratio with respect to the sentiment (the consumption share equivalent), the expected dividend growth estimation, and the difference in beliefs. The second part of excess volatility $\frac{1}{p_t} \left(\frac{\partial p_t}{\partial \hat{\mu}_{1,t}} \phi_1 \omega_1 \sigma_\mu + \frac{\partial p_t}{\partial g_t} (\phi_1 \omega_1 - \phi_2 \omega_2) \sigma_\mu \right)$ is due to mis-perceived signal. As in the overconfident agent's subjective view, the noise signal is correlated with the dividend growth rate shock, hence impacts the stock return. This impact also depends on the reaction of agents, e.g. if Agent 1 over-reacts to the noise signal more actively, in other words, she is more irrational, the contribution of irrationality to the excess volatility is greater. The different interpretation of noise signal also drives the belief dispersion, which is also shown in the second part.

However, only the diffusion associated with the fundamental shocks is priced instantaneously, as the signal innovations have a zero instantaneous correlation with consumption share or the SDF innovations. Therefore the aggregate equity risk premium depends on the instantaneous covariance of stock return and SDF, which is solely determined by the fundamental shocks. But the risk of signal shocks is priced in the longer term. Consider a discrete time case where the difference beliefs is zero at time t : a noise signal at time $t + 1$ changes g_{t+1} ; $t + 1$ noise shock does not change the consumption allocation or SDF at time $t + 1$, which remains constant as it depends on g_t only. But time t noise shock has non-zero correlation with the SDF at time $t + 2$, as SDF at time $t + 2$ is determined by g_{t+1} and the realized dividend shocks. The signal shock is priced, and its pricing effect via the discount rate channel is contained in the price volatility component driven by the dividend shocks.

I calibrate the model and show that the difference in beliefs and the overconfidence contribute to the downward sloping the volatility and risk premium term structure of the dividend strips. Figure 1.2 shows the volatility term structure of dividend strip price with different maturity. The upper panel shows the downward sloping surface of the volatility with different level of misjudgement on the noise signal informativeness, while in the bottom panel shows three representative cases in the surface. Those two figures show that if the Agent 2 has higher level of misjudgement (overconfidence), the slope of dividend strip price volatility is steeper.

Two main effects determine the volatility term structure: the difference in beliefs and the pricing weight of two agents. Compared to the Lucas economy, beliefs heterogeneity is such an ingredient that distorts the aggregate SDF and increases the volatilities. The reason is that investors hold different portfolio in the presence of beliefs heterogeneity, and the aggregate shocks impact their wealth asymmetrically. This distortion is most pronounced when two types of investors have equal pricing weight. But the noise signal shocks also contribute to the equity volatility because the overconfident agent believes the signal can forecast the dividend growth, thus the volatility component caused by the noise signal is increasing the pricing weight of the overconfident agent, but drops sharply when the weight is getting close to 1 as the overconfident agent is dominating the market and no one can trade with her. The pricing weight plays a role to determine the term structure. In the view of the rational agent or under the true probability measure, the pricing weight of the overconfident agent is decreasing over time, faster with the a higher overconfidence level. Hence, the dividend strips price volatility is decreasing in the security maturity.

The return of the equity can be described by its capital gain and the dividends

$$dR_t = \frac{dP_t + D_t dt}{P_t} = \mu_{R,t} dt + \sigma_{R,D,t} dZ_{D,t} + \sigma_{R,S,t} dS_t, \quad (1.56)$$

and its volatility can be obtained in similar way.

Corollary 5 (Volatility of stock return and the equity premium). *The volatility of the stock market return is given by*

$$\sigma_{R,t} = \sqrt{\sigma_{R,D,t}^2 + \sigma_{R,S,t}^2}, \quad (1.57)$$

where

$$\sigma_{R,D,t} = \sigma_D + \frac{1}{p_t} \left(\frac{\partial p_t}{\partial \xi_t} \xi_t \sigma_\xi + \frac{\partial p_t}{\partial \hat{\mu}_{1,t}} \frac{\bar{v}_1}{\sigma_D} + \frac{\partial p_t}{\partial g_t} \frac{\bar{v}_1 - \bar{v}_2}{\sigma_D} + \frac{\partial p_t}{\partial \omega_t} \sigma_D \right), \quad (1.58)$$

$$\sigma_{R,S,t} = \frac{1}{p_t} \left(\frac{\partial p_t}{\partial \hat{\mu}_{1,t}} \phi_1 \omega_1 \sigma_\mu + \frac{\partial p_t}{\partial g_t} (\phi_1 \omega_1 - \phi_2 \omega_2) \sigma_\mu \right). \quad (1.59)$$

And the risk premium on equity is

$$RP_t = \sigma_{R,D,t} \left(\mathbf{R}_t \sigma_D + \frac{\mu_{D,t} - \hat{\mu}_{D,t}}{\sigma_D} \right). \quad (1.60)$$

Figure 1.3 shows the equity premium term structure with different levels of beliefs heterogeneity. The upper panel shows the downward sloping surface of the equity premium, while the bottom panel shows three representative cases in the surface. The aggregate equity premium can be obtained by integrating the surface over the maturity axis. The level equity premium is decreasing in belief heterogeneity, where the equity premium is around 6% when the different in beliefs g_t is 2 %, but it is 3% as the difference in beliefs g_t is 1 %. This result is consistent with [Dumas, Kurshev, and Uppal \(2009\)](#) and [Bhamra and Uppal \(2014\)](#). More interesting is that the difference in beliefs also affects the slope of the equity premium term structure: belief heterogeneity is greater, the slope is steeper. The reason is that the difference in beliefs g_t is a mean reverting process hence greater g_t brings greater short-term distortions in short-term

SDF; however, the long-term distortion is small as g_t is expected to go towards zero.

Figure 1.4 shows the equity premium term structure with different levels of overconfidence level. The upper panel shows the downward sloping surface of the volatility with different level of misjudgement on the noise signal informativeness, while the bottom panel shows three representative cases surfaces. The level of equity premium term structure is determined by the difference in beliefs. As we can see, the those three curve at the near zero maturity are close. But the slope of the equity premium term structure is steeper when the level of misjudgement or the confidence level is higher. As I mentioned above, the mechanism is related to the pricing weight. From the viewpoint of the rational agent, the overconfident agent has incorrect beliefs almost certainly, so any realized dividend shocks will reduce the pricing weight of the overconfident agent overtime. Therefore, in a bear market when the overconfident agent is more pessimistic, the short-term equity premium is high, but as with the expectation of decreasing pricing weight of the pessimist, the equity premium decreases over the maturity. This mechanism also implies that the slope of the equity premium term structure is pro-cyclical, which is in line with the findings in Van Binsbergen and Kojen (2017).

C. Bond Yield

In this subsection, I present the pricing formula of the perpetual bond and show that the interest rate term structure implied by this model is reasonable.

Corollary 6 (Bond Price). *The perpetual bond, which delivers a unit of consumption good perpetually, has the price*

$$B_t = E_t \left[\int_t^\infty \frac{\Lambda_u}{\Lambda_t} du \right] = \sum_{n=0}^{\gamma} w_{n,t} \zeta_{n,t}^B \quad (1.61)$$

where

$$\zeta_{n,t}^B = \int_t^\infty H_1(-\gamma, -(\gamma-1), \omega_t, \hat{\mu}_{D,t}^1, u-t) H_2\left(-\gamma, -(\gamma-1), \frac{n}{\gamma}, g_t, u-t\right) du. \quad (1.62)$$

Figure 1.5 shows the bond yield curve with different levels of overconfidence level. The upper panel shows the downward sloping surface of the volatility with different level of misjudgement on the noise signal informativeness, while in the bottom panel shows three representative cases in the surface. Overall the yield curve has a reasonable level around 0% in the short-term and around 2% at 10-year maturity. The yield curve is upward sloping, matching with the empirical facts. The overconfidence level impacts the level of the yield curve. The intuition is that in my calibration the overconfident agent is relative pessimistic, and the level of misjudgement on the noise signal informativeness determines the speculative profit for the rational optimist. Hence the optimist tend to borrow more (less) if ϕ_2 is higher (lower), because she expects to speculate more against the overconfident agent.

IV. Risk Price Term Structure

The risk premium is determined by the quantity of risk and the price of risk. My model shows that the market price of risk term structure is also downward sloping in my calibration, which is consistent with the empirical facts in Van Binsbergen, Brandt, and Kojen (2012).

Two distinct effects drive the term structure of the Sharpe ratio. First, the level of the Sharpe ratio is determined by the short-term market price of risk. The short-term market price of risk reflects the current view of both types of agents and their market pricing power (consumption share). Second, the longer term SDF is less sensitive to instantaneous shocks. As I have shown in the previous section, the aggregate SDF is a consumption share weighted sum of sole agent economy SDFs. But any temporary shocks, both perceived fundamental shocks or signal shocks, generates a path-dependent long-run impact on the behaviour of long-term aggregate SDF. I characterize the effects by using the following proposition. First, I show the Malliavin derivatives of relevant state variables to show the long-term impacts of instantaneous shocks.

Lemma 1. *The Malliavin derivatives for the five state variables are*

$$\mathcal{D}_t^1 \hat{\mu}_{D,u}^1 = e^{-\zeta(u-t)} \begin{bmatrix} \frac{\bar{v}_1}{\sigma_D} & 0 \end{bmatrix} \quad (1.63)$$

$$\mathcal{D}_t^1 g_u = e^{-\left(\zeta + \frac{\bar{v}_2}{\sigma_D^2}\right)(u-t)} \begin{bmatrix} \frac{\bar{v}_1 - \bar{v}_2}{\sigma_D} & (\phi_1 \omega_1 - \phi_2 \omega_2) \sigma_\mu \end{bmatrix} \quad (1.64)$$

$$\frac{\mathcal{D}_t^1 D_u}{D_u} = \begin{bmatrix} \sigma_D & 0 \end{bmatrix} + \frac{1}{\zeta} \left(1 - e^{-\zeta(u-t)}\right) \begin{bmatrix} \frac{\bar{v}_1}{\sigma_D} & 0 \end{bmatrix} \quad (1.65)$$

$$\frac{\mathcal{D}_t^1 \xi_u}{\xi_u} = \begin{bmatrix} -\frac{g_t}{\sigma_D} & 0 \end{bmatrix} - \int_t^u e^{-\left(\zeta + \frac{\bar{v}_2}{\sigma_D^2}\right)(s-t)} \left(\frac{g_s}{\sigma_D^2} ds + \frac{1}{\sigma_D} dZ_{D,t}^1 \right) \begin{bmatrix} \frac{\bar{v}_1 - \bar{v}_2}{\sigma_D} & (\phi_1 \omega_1 - \phi_2 \omega_2) \sigma_\mu \end{bmatrix} \quad (1.66)$$

$$\mathcal{D}_t^1 \omega_u = e^{-\lambda_x(u-t)} \begin{bmatrix} \sigma_D & 0 \end{bmatrix} - \frac{1}{\lambda_x + \zeta} \left(1 - e^{-(\lambda_x + \zeta)(u-t)}\right) \begin{bmatrix} \frac{\bar{v}_1}{\sigma_D} & \phi_1 \omega_1 \sigma_\mu \end{bmatrix} \quad (1.67)$$

Proposition 3 (Long Term Risk Prices). *Using Malliavin derivatives and the marginal utility of rational agent, the long-term market risk price $\theta_{1,t,T}$ can be decomposed into the short-term risk price and the long-term risk price which is a function of the future consumption share change.*

$$\begin{aligned} \theta_{1,t,T} = -E_t^1 \left[\frac{\mathcal{D}_t^1 \Lambda_{1,T}}{\Lambda_{1,T}} \right] &= \begin{bmatrix} \theta_t & 0 \end{bmatrix} - (1 - \gamma) \left(\left(e^{-\lambda_x(T-t)} - 1 \right) \mathcal{D}_t^1 \omega_t + \int_t^T \left(e^{-\lambda_x(T-u)} - 1 \right) \mathcal{D}_t^1 \hat{\mu}_{D,u}^1 du \right) \\ &- \left(E_t^1 [\nu_{2,T}] - \nu_{2,t} \right) \frac{\mathcal{D}_t^1 \xi_t}{\xi_t} + E_t^1 \left[\nu_{2,T} \left(\int_t^T (\mathcal{D}_t^1 g_u) \left[\frac{g_u}{\sigma_D^2} du + \frac{1}{\sigma_D} dZ_{D,t}^1 \right] \right) \right]. \end{aligned} \quad (1.68)$$

Proposition 3 decomposes the expected response of the long-term SDF to perceived dividend shocks and signal shocks, which implies the term structure of the market price of risk. We can see the long-term risk prices equal to the short-term risk prices, plus a weighted impact on the current surplus ratio, and weighted impact on $\mathcal{D}_t \hat{\mu}_{D,u}^1$ future change in beliefs of rational agents, plus the short-term impact of the sentiment weighted by the future change in the consumptions share of overconfident agents, and the impact $\mathcal{D}_t \hat{\mu}_{D,u}^1$ of future changes in the difference in beliefs. A numerical result is provided in Figure 1.6, showing that the price of risk is downward sloping.

I should stress two important points. First, the long-term market price of risk is strongly impacted by the term $(\nu_{2,T} - \nu_{2,t}) \frac{\mathcal{D}_t^1 \xi_t}{\xi_t}$ due to a relative greater value of $\frac{\mathcal{D}_t^1 \xi_t}{\xi_t} = -\frac{g_t}{\sigma_D}$. Hence the slope of risk prices is mainly determined by the rational agent's expectation of overconfident agent's consumption share change.

As the overconfident agent overreacts to the signal and she would lose the speculative game against the rational agent, the term $E_t^1[(\nu_{2,T} - \nu_{2,t})] \frac{\mathcal{D}_t^1 \xi_t}{\xi_t}$ is positive and increasing in maturity if $g_t > 0$. The greater g_t and ϕ_2 are, the term $E_t^1[\nu_{2,T}] - \nu_{2,t}$ is greater in a short run, and the slope of the risk price is steeper. Second, the long-term risk prices of the noise shocks are non-zero as long as $\phi_2 \neq 0$, because the term $\nu_{2,T} \left(\int_t^T (\mathcal{D}_t^1 g_u) \left[\frac{g_u}{\sigma_D^2} du + \frac{1}{\sigma_D} dZ_{D,t}^1 \right] \right)$ is greater than zero if $T > t$. The intuition is that any signal shocks don't drive the consumptions share instantaneously, but as long as the overconfident agent believes the signal is correlated to the economic growth, she is losing in speculation and her consumption saving plan bias gradually shows up and accumulates over time. Due to our general equilibrium framework, the pricing effect would show up over time, and therefore signal shocks impact her consumption saving plan and discount rate in the long-term in the eye of rational agent.

I should also highlight the fact that, if both two types of agents are rational, the term structure of risk prices is upward sloping. This is because when $\phi_2 = 0$ and $g_t = 0$, $\nu_{2,T} \left(\int_t^T (\mathcal{D}_t^1 g_u) \left[\frac{g_u}{\sigma_D^2} du + \frac{1}{\sigma_D} dZ_{D,t}^1 \right] \right)$ and the term $-(E_t^1[\nu_{2,T}] - \nu_{2,t}) \frac{\mathcal{D}_t^1 \xi_t}{\xi_t}$ are zero. The long-term SDF is more sensitive than the short-term SDF to the fundamental shocks, generating a upward sloping market price of risk term structure⁴. If $\phi_2 = 0$ but $g_t \neq 0$, the relative consumption share ν_2 is expected to change until the difference in beliefs converge to zero, generating a sharp change in discount rate in a very short run period.

The decomposition of the long-term market price of risk also implies that the slope of the market price of risk term structure is pro-cyclical, and so is the equity premium term structure. The reason is that the consumption share shortfall of the overconfident agent is pro-cyclical. During economic or market booms, the proportion of overconfident pessimists is low, so $\nu_{2,t}$ is small. From the viewpoint of the rational agent, the consumption share of overconfident pessimists won't change by a great enough amount to generate enough effect to offset the greater sensitivity of long-term SDF to the fundamental shocks, which brings up the term structure. On the other hand, during economic or market recession, the percentage of overconfident pessimists is high, so is $\nu_{2,t}$. As the expected shortfall of $\nu_{2,t}$ is large, the "natural selection" effect dominates the long-term SDF's greater sensitivity, reaching a downward sloping term structure. Also, the more overconfident the irrational agent is, the steeper is the market price of risk term structure, due to a faster declining rate of ν_2 .

The argument above is strongly based on the declining consumption share of overconfident over time. The following Corollary shows her long-term survival described by her consumption share.

Corollary 7 (Survival of Overconfident agent). *The survival of overconfident agent at a future time T , represented by her consumption share $\nu_{2,T}$. In the eye of rational agent, has an expectation:*

$$E_t^1[\nu_{2,T}] = \int_0^\infty \left(1 + \tilde{\xi}^{-\frac{1}{\gamma}}\right)^{-1} \left[\frac{1}{2\pi} \int_{-\infty}^\infty \left(\frac{\tilde{\xi}}{(\nu_{2,t}^{-1} - 1)^{-\gamma}} \right)^{-i\chi} H_2(0, 0, i\chi, g_t, T - t) d\chi \right] \frac{d\tilde{\xi}}{\tilde{\xi}}. \quad (1.69)$$

Apart from its exact solution in Corollary 7, We can also see this decrease in long-term from the

⁴This mechanism is discussed in Hansen, Heaton, and Li (2008).

function form of consumption sharing rule $\nu_{2,t} = \left(1 + \xi_t^{-\frac{1}{\gamma}}\right)^{-1}$, where

$$\xi_t = \exp \left[-\frac{1}{2} \int_0^t \frac{g_u^2}{\sigma_D} du - \int_0^t \frac{g_u}{\sigma_D} dZ_{D,t} \right] \quad (1.70)$$

is a positive martingale which decreases at the rate of g^2 . Hence, if the initial value of g_t or the overconfidence level is (or both) higher, the consumption share $\nu_{2,t}$ is expected to decrease at a faster rate.

To sum up, my model shows that in the equity premium term structure and volatility term structure are downward sloping, and the risk premium is higher for short-maturity claims than for the aggregate stock market. The decomposition of the long-term market price risk also implies that the slope of equity premium term structure is pro-cyclical. These findings are consistent with the recent empirical study.

V. Conclusion

In this chapter, I study a heterogeneous agent asset pricing model in which agents have a "Keep up with Joneses" preference. Two agents are heterogeneous in their beliefs and confidence level. I solve in closed form for the price and volatility of dividend strips and identify the equity premium term structures. By decomposing the overall effect of perceived shocks, I also show that the risk prices can be decomposed into short-term and long-term components. Belief heterogeneity generates distortions in the aggregate short-term SDF due to antisymmetric impact over agents as they are holding different portfolios, but those distortions are smoothed away on the long-term SDF.

I find that when the share of overconfident pessimists is large, i.e. during economic downturn, 1) risk premia is higher for short-maturity claims than for the aggregate stock market; 2) the volatility and equity premium term structures are downward sloping in maturities; 3) the slope of equity premium and risk prices term structures are pro-cyclical.

VI. Appendix: Tables

Table 1.1. Parameter Choice of Chapter 1

Name	Symbol	Value
Long term mean of aggregate endowment growth rate	$\bar{\mu}_D$	0.02
Volatility of aggregate endowment growth	σ_D	0.03
Volatility of expected growth rate of endowment	σ_μ	0.015
Mean reversion Speed	ζ	0.1
Habit sensitivity	λ_x	0.28
Agent 1's correlation between signal and mean growth rate	ϕ_1	0
Agent 2's correlation between signal and mean growth rate	ϕ_2	Varies
Time preference of both agents	ρ	0.02
Risk Aversion for both agents	γ	2
Agent A's expected growth rate of endowment	$\hat{\mu}_{D,t}^1$	0.005
Real expected growth rate of endowment	$\mu_{D,t}$	0.01
Difference of Agent B's belief and Agent A's	$g_t = \hat{\mu}_{D,t}^1 - \hat{\mu}_{D,t}^2$	0.02
Consumption share	$\nu_{1,t}$	0.25

VII. Appendix: Proofs

A. Proof of Corollary 1 and Corollary 2

Follow the proof in [Bhamra and Uppal \(2014\)](#) with identical risk aversion.

B. Proof of Corollary 3

Given $\Lambda_t = e^{-\rho_1} C_{1,t}^{-\gamma_1} = \hat{\Lambda}_{1,t} \nu_{1,t}^{-\gamma_1}$, from Ito's Lemma,

$$d\Lambda_t/\Lambda_t = - \left[\rho_1 + \gamma_1(\hat{\mu}_{D,t}^1 + \mu_{\nu_1,t}^{\mathbb{P}^1} \sigma_D \sigma_{\nu_1,D}) - \frac{1}{2} \gamma_1(1 + \gamma_1) (\sigma_D + \sigma_{\nu_1,D})^2 \right] dt - \gamma_1 (\sigma_D + \sigma_{\nu_1,D}) dZ_t, \quad (1.71)$$

hence the risk free rate and the market price of risk

$$r_t = \rho_1 + \gamma_1(\hat{\mu}_{D,t}^1 + \mu_{\nu_1,t}^{\mathbb{P}^1} \sigma_D \sigma_{\nu_1,D}) - \frac{1}{2} \gamma_1(1 + \gamma_1) (\sigma_D + \sigma_{\nu_1,D})^2 - \frac{1}{2} \gamma_1(1 + \gamma_1) \sigma_{\nu_1,s}^2 \quad (1.72)$$

$$\theta_t = \gamma_1 (\sigma_D + \sigma_{\nu_1,D}) \quad (1.73)$$

Substitute other variables into gives the result.

C. Proof of Proposition 1

We want to derive the asset prices in terms of the two state variables $\hat{\mu}_{D,t}^1$ and $\hat{\mu}_{D,t}^2$.

The PD ratio of the equity,

$$\frac{P_t}{D_t} = E_t \left[\int_t^\infty \frac{\Lambda_u}{\Lambda_t} \frac{D_u}{D_t} du | \mathcal{F}_t \right] \quad (1.74)$$

$$= E_t \left[\int_t^\infty \frac{\sum_{n=0}^\gamma \binom{\gamma}{n} \hat{\Lambda}_{1,u}^{1-\frac{n}{\gamma}} \hat{\Lambda}_{2,u}^{\frac{n}{\gamma}}}{\Lambda_t} \frac{D_u}{D_t} du | \mathcal{F}_t \right] \quad (1.75)$$

$$= \sum_{n=0}^\gamma \binom{\gamma}{n} \frac{\hat{\Lambda}_{1,t}^{1-\frac{n}{\gamma}} \hat{\Lambda}_{2,t}^{\frac{n}{\gamma}}}{\Lambda_t} E_t \left[\int_t^\infty \frac{\hat{\Lambda}_{1,u}^{1-\frac{n}{\gamma}} \hat{\Lambda}_{2,u}^{\frac{n}{\gamma}}}{\hat{\Lambda}_{1,t}^{1-\frac{n}{\gamma}} \hat{\Lambda}_{2,t}^{\frac{n}{\gamma}}} \frac{D_u}{D_t} du | \mathcal{F}_t \right] = \sum_{n=0}^\gamma w_{n,t} \zeta_{n,t}^D \quad (1.76)$$

where

$$w_{n,t} = \binom{\gamma}{n} \frac{\hat{\Lambda}_{1,t}^{1-\frac{n}{\gamma}} \hat{\Lambda}_{2,t}^{\frac{n}{\gamma}}}{\Lambda_t} = \binom{\gamma}{n} \left(\nu_{1,t}^{1-\frac{n}{\gamma}} \nu_{2,t}^{\frac{n}{\gamma}} \right)^\gamma \quad (1.77)$$

$$\zeta_{n,t}^D = E_t \left[\int_t^\infty \frac{\hat{\Lambda}_{1,u}^{1-\frac{n}{\gamma}} \hat{\Lambda}_{2,u}^{\frac{n}{\gamma}}}{\hat{\Lambda}_{1,t}^{1-\frac{n}{\gamma}} \hat{\Lambda}_{2,t}^{\frac{n}{\gamma}}} \frac{D_u}{D_t} du | \mathcal{F}_t \right] \quad (1.78)$$

We want to derive

$$\zeta_{n,t}^D = E_t \left[\int_t^\infty \frac{\hat{\Lambda}_{1,u}^{1-\frac{n}{\gamma}} \hat{\Lambda}_{2,u}^{\frac{n}{\gamma}}}{\hat{\Lambda}_{1,t}^{1-\frac{n}{\gamma}} \hat{\Lambda}_{2,t}^{\frac{n}{\gamma}}} \frac{D_u}{D_t} du | \mathcal{F}_t \right], \quad (1.79)$$

and more specifically we want to compute

$$E_t \left[\frac{\hat{\Lambda}_{1,u}^{1-\frac{n}{\gamma}} \hat{\Lambda}_{2,u}^{\frac{n}{\gamma}}}{\hat{\Lambda}_{1,t}^{1-\frac{n}{\gamma}} \hat{\Lambda}_{2,t}^{\frac{n}{\gamma}}} \frac{D_u}{D_t} | \mathcal{F}_t \right]. \quad (1.80)$$

As

$$\frac{\hat{\Lambda}_{k,u}}{\hat{\Lambda}_{k,t}} \frac{D_u}{D_t} = \frac{\xi_{k,u}}{\xi_{k,t}} e^{-\rho(u-t) + (1-\gamma) \left(\ln \left(\frac{D_u}{X_u} \right) - \ln \left(\frac{D_t}{X_t} \right) \right)} \quad (1.81)$$

$$\ln \left[\frac{\hat{\Lambda}_{k,u}}{\hat{\Lambda}_{k,t}} \frac{D_u}{D_t} \right] = -\frac{1}{2} \int_t^u \sigma_{\xi,k}^2 ds + \int_t^u \sigma_{\xi,k} dZ_s - \rho(u-t) + (1-\gamma) \left(\ln \left(\frac{D_u}{X_u} \right) - \ln \left(\frac{D_t}{X_t} \right) \right) \quad (1.82)$$

Therefore

$$E \left[\frac{\hat{\Lambda}_{k,u}}{\hat{\Lambda}_{k,t}} \frac{D_u}{D_t} \right] = E \left[\frac{\Lambda_{k,u}}{\Lambda_{k,t}} \frac{D_u}{D_t} \frac{\xi_{k,u}}{\xi_{k,t}} \right] = E^k \left[\frac{\Lambda_{k,u}}{\Lambda_{k,t}} \frac{D_u}{D_t} \right] \quad (1.83)$$

where under $\mathbb{P}^k$

$$\frac{dD_t}{D_t} = \hat{\mu}_{D,t}^k dt + \sigma_D dZ_t^k \quad (1.84)$$

and $\Lambda_{k,t}$ is the solo agent's SDF when she holds the correct belief, and $\Lambda_{1,t} = \Lambda_{2,t} = e^{-\rho t} D_t^{-\gamma}$ are identical given the same risk aversion coefficients. Hence

$$E_t \left[\hat{\Lambda}_{1,u}^{1-\frac{n}{\gamma}} \hat{\Lambda}_{2,u}^{\frac{n}{\gamma}} D_u \right] = E_t \left[\Lambda_{1,u}^{1-\frac{n}{\gamma}} \xi_{1,u}^{1-\frac{n}{\gamma}} \Lambda_{2,u}^{\frac{n}{\gamma}} \xi_{2,u}^{\frac{n}{\gamma}} D_u \right] \quad (1.85)$$

$$= E_t \left[\Lambda_{1,u}^{1-\frac{n}{\gamma}} \xi_{1,u}^{1-\frac{n}{\gamma}} \Lambda_{2,u}^{\frac{n}{\gamma}} \xi_{2,u}^{\frac{n}{\gamma}} D_u \right] \quad (1.86)$$

$$= E_t \left[\Lambda_{1,u} \xi_{1,u}^{1-\frac{n}{\gamma}} \xi_{2,u}^{\frac{n}{\gamma}} D_u \right] \quad (1.87)$$

$$= E_t^1 \left[\Lambda_{1,u} \xi_{1,u}^{1-\frac{n}{\gamma}} \xi_{2,u}^{\frac{n}{\gamma}} D_u \right] \xi_{1,t} = E_t^1 \left[e^{-\rho u} \xi_u^{\frac{n}{\gamma}} \left(\frac{D_u}{X_u} \right)^{1-\gamma} \right] \xi_{1,t} \quad (1.88)$$

$$= E_t^2 \left[\Lambda_{1,u} \xi_{1,u}^{1-\frac{n}{\gamma}} \xi_{2,u}^{\frac{n}{\gamma}-1} D_u \right] \xi_{2,t} = E_t^2 \left[e^{-\rho u} \xi_u^{\frac{n}{\gamma}-1} \left(\frac{D_u}{X_u} \right)^{1-\gamma} \right] \xi_{2,t} \quad (1.89)$$

To evaluate $E_t^1 \left[e^{-\rho u} \xi_u^{\frac{n}{\gamma}} \left(\frac{D_u}{X_u} \right)^{1-\gamma} \right]$, define

$$h_t(\xi_t, D_t, X_t, \chi, \epsilon, \eta) = E_t^1 \left[e^{-\rho u} \xi_u^\chi D_u^\epsilon X_u^\eta \right] \quad (1.90)$$

which is a function of time, t and two other state variables $\hat{\mu}_t^1$ and $\hat{\mu}_t^2$.

Under measure $\mathbb{P}$, the dynamics of $\hat{\mu}_{D,t}^i$ follow

$$d\hat{\mu}_{D,t}^i = \zeta (\bar{\mu}_D - \hat{\mu}_{D,t}^i) dt + \frac{\bar{v}_i}{\sigma_D^2} (\mu_{D,t} - \hat{\mu}_{D,t}^i) dt + \frac{\bar{v}_i}{\sigma_D} dZ_t + \phi_i \omega_i \sigma_\mu dS_t \quad (1.91)$$

While under measure $\mathbb{P}^1$, the dynamics of $\hat{\mu}_{D,t}^i$ follow

$$d\hat{\mu}_{D,t}^i = \zeta (\bar{\mu}_D - \hat{\mu}_{D,t}^i) dt + \frac{\bar{v}_i}{\sigma_D^2} (\mu_{D,t} - \hat{\mu}_{D,t}^i) dt + \frac{\bar{v}_i}{\sigma_D} \left(\left(\frac{\hat{\mu}_{D,t}^1 - \mu_{D,t}}{\sigma_D} + \right) dt + dZ_t^1 \right) + \phi_i \omega_i \sigma_\mu dS_t \quad (1.92)$$

$$= \zeta (\bar{\mu}_D - \hat{\mu}_{D,t}^i) dt + \frac{\bar{v}_i}{\sigma_D^2} (\hat{\mu}_{D,t}^1 - \hat{\mu}_{D,t}^i) dt + \frac{\bar{v}_i}{\sigma_D} dZ_t^1 + \phi_i \omega_i \sigma_\mu dS_t \quad (1.93)$$

Let $g_t = \hat{\mu}_{D,t}^1 - \hat{\mu}_{D,t}^2$, the dynamics of g_t follows:

$$dg_t = - \left(\zeta + \frac{\bar{v}_2}{\sigma_D^2} \right) g_t dt + \frac{1}{\sigma_D} (\bar{v}_1 - \bar{v}_2) dZ_t^1 + (\phi_1 \omega_1 - \phi_2 \omega_2) \sigma_\mu dS_t. \quad (1.94)$$

The change of measure $\xi_t = \frac{\xi_{2,t}}{\xi_{1,t}}$, and its dynamics follow

$$\frac{d\xi_t}{\xi_t} = - \left(\frac{\hat{\mu}_{D,t}^1 - \mu_{D,t}}{\sigma_D} \right) \left(\frac{\hat{\mu}_{D,t}^2 - \hat{\mu}_{D,t}^1}{\sigma_D} \right) dt + \frac{\hat{\mu}_{D,t}^2 - \hat{\mu}_{D,t}^1}{\sigma_D} dZ_t \quad (1.95)$$

$$= - \left(\frac{\hat{\mu}_{D,t}^1 - \mu_{D,t}}{\sigma_D} \right) \left(\frac{\hat{\mu}_{D,t}^2 - \hat{\mu}_{D,t}^1}{\sigma_D} \right) dt + \frac{\hat{\mu}_{D,t}^2 - \hat{\mu}_{D,t}^1}{\sigma_D} \left(\left(\frac{\hat{\mu}_{D,t}^1 - \mu_{D,t}}{\sigma_D} \right) dt + dZ_t^1 \right) \quad (1.96)$$

$$= \frac{\hat{\mu}_{D,t}^2 - \hat{\mu}_{D,t}^1}{\sigma_D} d\hat{Z}_t^1 = -g_t \frac{1}{\sigma_D} d\hat{Z}_t^1 \quad (1.97)$$

The Feynman-Kac Theorem implies that the following PDE exists

$$\begin{aligned} 0 = & \frac{\partial H}{\partial t} + \frac{\partial H}{\partial D_t} D_t \hat{\mu}_{D,t}^1 + \frac{\partial H}{\partial X_t} X_t \lambda_x \omega_t - \frac{\partial h}{\partial \hat{\mu}_t^1} \zeta (\hat{\mu}_{D,t}^1 - \bar{\mu}_D) + \frac{\partial H}{\partial \omega_t} \left(\hat{\mu}_t^1 - \frac{1}{2} \sigma_D^2 - \lambda_x \omega_t \right) + \frac{1}{2} \frac{\partial^2 H}{\partial D_t^2} D_t^2 \sigma_D^2 \\ & - \frac{1}{2} \frac{\partial^2 H}{\partial \partial (\hat{\mu}_t^1)^2} \left(\frac{\bar{v}_1^2}{\sigma_D^2} + \phi_1^2 \omega_1^2 \sigma_\mu^2 \right) + \frac{1}{2} \frac{\partial^2 H}{\partial \omega_t^2} \sigma_D^2 + \frac{\partial^2 H}{\partial \omega_t \partial \hat{\mu}_t^1} \bar{v}_1 + \frac{\partial^2 H}{\partial D_t \partial \hat{\mu}_t^1} D_t \sigma_D \frac{\bar{v}_1}{\sigma_D} + \frac{\partial^2 H}{\partial D_t \partial \omega_t} D_t^2 \sigma_D^2 \\ & - \frac{\partial h}{\partial g_t} \left(\zeta + \frac{\bar{v}_2}{\sigma_D^2} \right) g_t + \frac{1}{2} \frac{\partial^2 h}{\partial g_t^2} \left(\frac{(\bar{v}_1 - \bar{v}_2)^2}{\sigma_D^2} + (\phi_1 \omega_1 - \phi_2 \omega_2)^2 \sigma_\mu^2 \right) + \frac{1}{2} \frac{\partial^2 h}{\partial \xi_t^2} \xi_t^2 g_t^2 \frac{1}{\sigma_D^2} + \frac{\partial^2 h}{\partial \hat{\mu}_t^1 \partial \xi_t} \left(-g_t \xi_t \frac{\bar{v}_1}{\sigma_D^2} \right) \\ & + \frac{\partial^2 h}{\partial \hat{\mu}_t^1 \partial g_t} \left(\frac{\bar{v}_1 (\bar{v}_1 - \bar{v}_2)}{\sigma_D^2} + \phi_1 \omega_1 (\phi_1 \omega_1 - \phi_2 \omega_2) \sigma_\mu^2 \right) + \frac{\partial^2 h}{\partial g_t \partial \xi_t} \left(-\xi_t g_t \frac{(\bar{v}_1 - \bar{v}_2)}{\sigma_D^2} \right) + \frac{\partial^2 h}{\partial g_t \partial D_t} D_t \sigma_D \frac{(\bar{v}_1 - \bar{v}_2)}{\sigma_D} \\ & - \frac{\partial^2 h}{\partial \xi_t \partial D_t} D_t \sigma_D \xi_t g_t \frac{1}{\sigma_D} - \frac{\partial^2 h}{\partial \xi_t \partial \omega_t} \xi_t g_t \frac{1}{\sigma_D} \sigma_D + \frac{\partial^2 h}{\partial g_t \partial \omega_t} \frac{(\bar{v}_1 - \bar{v}_2)}{\sigma_D} \sigma_D \end{aligned} \quad (1.98)$$

and that the solution takes the form

$$H(\tau, \xi_t, \hat{\mu}_t^1, \omega_t, g_t, \chi, \epsilon, \eta) = \xi_t^x D_t^x X_t^\eta H_1(\epsilon, \eta, \hat{\mu}_t^1, \omega_t, \tau) H_2(\epsilon, \eta, \chi, g_t, \tau) \quad (1.99)$$

where

$$H_1(\epsilon, \eta, \hat{\mu}_t^1, \omega_t, \tau) = \exp \{ h_0(\tau) + h_1(\tau)(\hat{\mu}_t^1 - \bar{\mu}_D) + h_2(\tau)\omega_t \} \quad (1.100)$$

$$H_2(\epsilon, \eta, \chi, g_t, \tau) = \exp \{ A_1(\tau) + (1 - \gamma)^2 A_2(\tau) + (1 - \gamma)g_t B(\tau) + g_t^2 C(\tau) \}. \quad (1.101)$$

We can solve H_1 firstly by assuming two types of agents hold homogeneous beliefs hence we can ignore the ξ_t and $H_2(\epsilon, \chi, g_t, \tau)$ term, and the following PDE holds

$$\begin{aligned} 0 = & \frac{\partial H}{\partial t} + \frac{\partial H}{\partial D_t} D_t \hat{\mu}_{D,t}^1 + \frac{\partial H}{\partial X_t} X_t \lambda_x \omega_t - \frac{\partial h}{\partial \hat{\mu}_t^1} \zeta (\hat{\mu}_{D,t}^1 - \bar{\mu}_D) + \frac{\partial H}{\partial \omega_t} \left(\hat{\mu}_t^1 - \frac{1}{2} \sigma_D^2 - \lambda_x \omega_t \right) + \frac{1}{2} \frac{\partial^2 H}{\partial D_t^2} D_t^2 \sigma_D^2 \\ & + \frac{1}{2} \frac{\partial^2 H}{\partial (\hat{\mu}_t^1)^2} \left(\frac{\bar{v}_1^2}{\sigma_D^2} + \phi_1^2 \omega_1^2 \sigma_\mu^2 \right) + \frac{1}{2} \frac{\partial^2 H}{\partial \omega_t^2} \sigma_D^2 + \frac{\partial^2 H}{\partial \omega_t \partial \hat{\mu}_t^1} \bar{v}_1 + \frac{\partial^2 H}{\partial D_t \partial \hat{\mu}_t^1} D_t \sigma_D \frac{\bar{v}_1}{\sigma_D} + \frac{\partial^2 H}{\partial D_t \partial \omega_t} D_t \sigma_D^2. \end{aligned} \quad (1.102)$$

By Ito's lemma an matching the terms,

$$h_2'(\tau) = \eta \lambda_x - \lambda_x h_2(\tau) \quad (1.103)$$

$$h_1'(\tau) = h_2(\tau) - \zeta h_1(\tau) + \epsilon \quad (1.104)$$

$$\begin{aligned} h_0'(\tau) = & \epsilon \bar{\mu}_D + h_2(\tau)(\bar{\mu}_D - \frac{1}{2} \sigma_D^2) + \frac{1}{2} \left(h_1(\tau) \frac{\bar{v}_1}{\sigma_D} + h_2(\tau) \sigma_D \right)^2 + \frac{1}{2} h_1(\tau)^2 \phi_1^2 \omega_1^2 \sigma_\mu^2 \\ & + \frac{1}{2} \epsilon (\epsilon - 1) \sigma_D^2 + \epsilon h_1(\tau) \bar{v}_1 + \epsilon h_2(\tau) \sigma_D^2. \end{aligned} \quad (1.105)$$

Solving the above ODEs gives

$$h_2(\tau) = \eta(1 - e^{-\lambda_x \tau}) \quad (1.106)$$

$$h_1(\tau) = \frac{\zeta (\eta(1 - e^{-\lambda_x \tau}) + \epsilon(1 - e^{-\zeta \tau})) - (1 - e^{-\zeta \tau}) \lambda_x (\eta + \epsilon)}{\zeta (\zeta - \lambda_x)} \quad (1.107)$$

$$= \frac{\eta}{\zeta - \lambda_x} (1 - e^{-\lambda_x \tau}) + \frac{\zeta \epsilon - \lambda_x (\eta + \epsilon)}{\zeta (\zeta - \lambda_x)} (1 - e^{-\zeta \tau}) \quad (1.108)$$

Let $\kappa_1 = \frac{\zeta \epsilon - \lambda_x (\eta + \epsilon)}{\zeta (\zeta - \lambda_x)}$ and $\kappa_2 = \frac{\eta}{\zeta - \lambda_x}$, then

$$h_1(\tau) = \kappa_1 (1 - e^{-\zeta \tau}) + \kappa_2 (1 - e^{-\lambda_x \tau}). \quad (1.109)$$

We can solve $H_2(\epsilon, \chi, g_t, \tau)$ by matching terms.

$$C'(u - t) = aC^2(u - t) - 2bC(u - t) + c \quad (1.110)$$

$$B'(u - t) = B(u - t) [aC(u - t) - b] + 2m(u - t)C(u - t) + n(u - t) \quad (1.111)$$

$$\begin{aligned} B'(u - t) = & B(u - t) [aC(u - t) - b] + k + l_1 e^{-\zeta(u-t)} + l_2 e^{-\lambda_x(u-t)} \\ & + 2 \left[m + n_1 e^{-\zeta(u-t)} + n_2 e^{-\lambda_x(u-t)} \right] C(u - t) \end{aligned} \quad (1.112)$$

$$A'_1(u-t) = \frac{a}{2}C(u-t) \quad (1.113)$$

$$A'_2(u-t) = B(u-t) \left[m + n_1 e^{-\zeta(u-t)} + n_2 e^{-\lambda_x(u-t)} + \frac{a}{4}B(u-t) \right] \quad (1.114)$$

where

$$a = 2 \left(\frac{(\bar{v}_1 - \bar{v}_2)^2}{\sigma_D^2} + (\phi_1 \omega_1 - \phi_2 \omega_2)^2 \sigma_\mu^2 \right) \quad (1.115)$$

$$b = \zeta + \frac{\bar{v}_2}{\sigma_D^2} + \chi \frac{(\bar{v}_1 - \bar{v}_2)}{\sigma_D^2} \quad (1.116)$$

$$c = \frac{1}{2\sigma_D^2} \chi (\chi - 1) \quad (1.117)$$

$$k = -\chi \left(1 + \frac{\bar{v}_1}{\sigma_D^2} \frac{\kappa_1 + \kappa_2}{\epsilon} + \frac{\eta}{\epsilon} \right) \quad (1.118)$$

$$l_1 = \chi \frac{\kappa_1}{\epsilon} \frac{\bar{v}_1}{\sigma_D^2} \quad (1.119)$$

$$l_2 = \chi \left(\frac{\kappa_2}{\epsilon} \frac{\bar{v}_1}{\sigma_D^2} + \frac{\eta}{\epsilon} \right) \quad (1.120)$$

$$m = (\bar{v}_1 - \bar{v}_2) + \frac{\eta}{\epsilon} (\bar{v}_1 - \bar{v}_2) + \frac{\kappa_1 + \kappa_2}{\epsilon} \left[\frac{\bar{v}_1(\bar{v}_1 - \bar{v}_2)}{\sigma_D^2} + \phi_1 \omega_1 (\phi_1 \omega_1 - \phi_2 \omega_2) \sigma_\mu^2 \right] \quad (1.121)$$

$$n_1 = -\frac{\kappa_1}{\epsilon} \left[\frac{\bar{v}_1(\bar{v}_1 - \bar{v}_2)}{\sigma_D^2} + \phi_1 \omega_1 (\phi_1 \omega_1 - \phi_2 \omega_2) \sigma_\mu^2 \right] \quad (1.122)$$

$$n_2 = -\frac{\kappa_2}{\epsilon} \left[\frac{\bar{v}_1(\bar{v}_1 - \bar{v}_2)}{\sigma_D^2} + \phi_1 \omega_1 (\phi_1 \omega_1 - \phi_2 \omega_2) \sigma_\mu^2 \right] - \frac{\eta}{\epsilon} (\bar{v}_1 - \bar{v}_2) \quad (1.123)$$

$$(1.124)$$

Denote

$$q = \sqrt{b^2 - ac} \quad (1.125)$$

$$\mathcal{V}_1 = \frac{2cm + k(b+q)}{q} \quad \mathbf{v}_1 = 0 \quad (1.126)$$

$$\mathcal{V}_2 = \frac{2cm + k(b-q)}{q} \quad \mathbf{v}_2 = 2q \quad (1.127)$$

$$\mathcal{V}_3 = \frac{2cn_1 + l_1(b+q)}{q - \zeta} \quad \mathbf{v}_3 = \zeta \quad (1.128)$$

$$\mathcal{V}_4 = \frac{2cn_1 + l_1(b-q)}{q + \zeta} \quad \mathbf{v}_4 = 2q + \zeta \quad (1.129)$$

$$\mathcal{V}_5 = \frac{2cn_2 + l_2(b+q)}{q - \lambda_x} \quad \mathbf{v}_5 = \lambda_x \quad (1.130)$$

$$\mathcal{V}_6 = \frac{2cn_2 + l_2(b-q)}{q + \lambda_x} \quad \mathbf{v}_6 = 2q + \lambda_x \quad (1.131)$$

$$\mathcal{V}_7 = -(\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3 + \mathcal{V}_4 + \mathcal{V}_5 + \mathcal{V}_6) \quad \mathbf{v}_7 = q \quad (1.132)$$

$$C(u-t) = \frac{c(1 - e^{-2q(u-t)})}{q+b+(q-b)e^{-2q(u-t)}} \quad (1.133)$$

$$B(u-t) = \frac{\sum_{i=1}^7 \mathcal{V}_i e^{-\mathbf{v}_i(u-t)}}{q+b+(q-b)e^{-2q(u-t)}} \quad (1.134)$$

$$A_1(u-t) = \frac{a}{2} \int_t^u C(\tau-t) d\tau \quad (1.135)$$

$$= \frac{1}{2} \left[(b-q)(u-t) + \ln(2q) - \ln \left(q+b+(q-b)e^{-2q(u-t)} \right) \right] \quad (1.136)$$

$$A_2(u-t) = \int_t^u B(\tau-t) \left[m + n_1 e^{-\zeta(\tau-t)} + n_2 e^{-\lambda_x(\tau-t)} + \frac{a}{4} B(\tau-t) \right] d\tau \quad (1.137)$$

$$= \sum_{j=1}^7 \mathcal{V}_j [m D_1(\mathbf{v}_j, u-t) + n_1 D_1(\mathbf{v}_j + \zeta, u-t) + n_2 D_1(\mathbf{v}_j + \lambda_x, u-t)] + \frac{a}{4} \sum_{i,j=1}^7 \mathcal{V}_i \mathcal{V}_j D_2(\mathbf{v}_i + \mathbf{v}_j, u-t) \quad (1.138)$$

where

$$D_1(p; u-t) = \int_t^u \frac{e^{-p(s-t)}}{q+b+(q-b)e^{-2q(s-t)}} ds \quad (1.139)$$

$$= \begin{cases} \frac{2q(u-t) - \ln(2q) + \ln(q+b+(q-b)e^{-2q(u-t)})}{2q(q+b)} & , p=0 \\ \frac{1}{p(q+b)} \left[\mathcal{H} \left(1, \frac{p}{2q}, 1 + \frac{p}{2q}, -\frac{q-b}{q+b} \right) - e^{-p(u-t)} \times \mathcal{H} \left(1, \frac{p}{2q}, 1 + \frac{p}{2q}, -\frac{q-b}{q+b} \right) e^{-2q(u-t)} \right] & , p>0 \end{cases} \quad (1.140)$$

and

$$D_2(p; u-t) = \int_t^u \frac{e^{-p(s-t)}}{(q+b+(q-b)e^{-2q(s-t)})^2} ds \quad (1.141)$$

$$= \frac{1}{2q(q+b)} \left[\frac{1}{2q} - \frac{e^{-p(u-t)}}{q+b+(q-b)e^{-2q(u-t)}} + (2q-p) D_1(p; u-t) \right] \quad (1.142)$$

and

$$E_t \left[\frac{\hat{\Lambda}_{1,u}^{1-\frac{n}{\gamma}} \hat{\Lambda}_{2,u}^{\frac{n}{\gamma}}}{\hat{\Lambda}_{1,t}^{1-\frac{n}{\gamma}} \hat{\Lambda}_{2,t}^{\frac{n}{\gamma}}} \frac{D_u}{D_t} \middle| \mathcal{F}_t \right] = H_1 \left((1-\gamma), -(1-\gamma), \hat{\mu}_t^1, \omega_t, u, t \right) H_2 \left((1-\gamma), -(1-\gamma), \frac{n}{\gamma}, g_t, u, t \right) \quad (1.143)$$

$$\zeta_{n,t}^D = E_t \left[\int_t^\infty \frac{\hat{\Lambda}_{1,u}^{1-\frac{n}{\gamma}} \hat{\Lambda}_{2,u}^{\frac{n}{\gamma}}}{\hat{\Lambda}_{1,t}^{1-\frac{n}{\gamma}} \hat{\Lambda}_{2,t}^{\frac{n}{\gamma}}} \frac{D_u}{D_t} du | \mathcal{F}_t \right] = \int_t^\infty H_1((1-\gamma), \hat{\mu}_t^1, u, t) H_2\left((1-\gamma), \frac{n}{\gamma}, g_t, u, t\right) du. \quad (1.144)$$

If there is no external habit, $\eta = 0$, h_1 is reduced to $\frac{\epsilon}{\zeta}(1 - e^{-\zeta\tau})$, and $\kappa_1 = \frac{\epsilon}{\zeta}$, $\kappa_2 = 0$, and all expressions above are reduced to the case of Dumas, Kurshev and Uppal (2009).

$$C'(u-t) = aC^2(u-t) - 2bC(u-t) + c \quad (1.145)$$

$$B'(u-t) = B(u-t) [aC(u-t) - b] + k + le^{-\zeta(u-t)} + 2 \left[m + ne^{-\zeta(u-t)} \right] C(u-t) \quad (1.146)$$

$$A'_1(u-t) = \frac{a}{2} C(u-t) \quad (1.147)$$

$$A'_2(u-t) = B(u-t) \left[m + ne^{-\zeta(u-t)} + \frac{a}{4} B[u-t] \right] \quad (1.148)$$

where

$$a = 2 \left(\frac{(\bar{v}_1 - \bar{v}_2)^2}{\sigma_D^2} + (\phi_1\omega_1 - \phi_2\omega_2)^2 \sigma_\mu^2 \right) \quad (1.149)$$

$$b = \zeta + \frac{\bar{v}_2}{\sigma_D^2} + \chi \frac{(\bar{v}_1 - \bar{v}_2)}{\sigma_D^2} \quad (1.150)$$

$$c = \frac{1}{2\sigma_D^2} \chi(\chi - 1) \quad (1.151)$$

$$k = -\chi \left(\frac{1}{\zeta} \frac{\bar{v}_1}{\sigma_D^2} + 1 \right) \quad (1.152)$$

$$l = \chi \frac{1}{\zeta} \frac{\bar{v}_1}{\sigma_D^2} \quad (1.153)$$

$$m = (\bar{v}_1 - \bar{v}_2) + \frac{1}{\zeta} \left[\frac{\bar{v}_1(\bar{v}_1 - \bar{v}_2)}{\sigma_D^2} + \phi_1\omega_1 (\phi_1\omega_1 - \phi_2\omega_2) \sigma_\mu^2 \right] \quad (1.154)$$

$$n = -\frac{1}{\zeta} \left[\frac{\bar{v}_1(\bar{v}_1 - \bar{v}_2)}{\sigma_D^2} + \phi_1\omega_1 (\phi_1\omega_1 - \phi_2\omega_2) \sigma_\mu^2 \right] \quad (1.155)$$

Denote

$$q = \sqrt{b^2 - ac} \quad (1.156)$$

$$\mathcal{V}_1 = \frac{2cm + k(b+q)}{q} \quad \mathbf{v}_1 = 0 \quad (1.157)$$

$$\mathcal{V}_2 = \frac{2cm + k(b-q)}{q} \quad \mathbf{v}_2 = 2q \quad (1.158)$$

$$\mathcal{V}_3 = \frac{2cn + l(b+q)}{q - \zeta} \quad \mathbf{v}_3 = \zeta \quad (1.159)$$

$$\mathcal{V}_4 = \frac{2cn + l(b-q)}{q + \zeta} \quad \mathbf{v}_4 = 2q + \zeta \quad (1.160)$$

$$\mathcal{V}_5 = -(\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3 + \mathcal{V}_4) \quad \mathbf{v}_5 = q \quad (1.161)$$

$$C(u-t) = \frac{c(1 - e^{-2q(u-t)})}{q+b+(q-b)e^{-2q(u-t)}} \quad (1.162)$$

$$B(u-t) = \frac{\sum_{i=1}^5 \mathcal{V}_i e^{-\mathbf{v}_i(u-t)}}{q+b+(q-b)e^{-2q(u-t)}} \quad (1.163)$$

$$A_1(u-t) = \frac{a}{2} \int_t^u C(\tau-t) d\tau \quad (1.164)$$

$$= \frac{1}{2} \left[(b-q)(u-t) + \ln(2q) - \ln \left(q+b+(q-b)e^{-2q(u-t)} \right) \right] \quad (1.165)$$

$$A_2(u-t) = \int_t^u B(\tau-t) \left[m + ne^{-\zeta(\tau-t)} + \frac{a}{4} B(\tau-t) \right] d\tau \quad (1.166)$$

$$= \sum_{j=1}^5 \mathcal{V}_j [mD_1(\mathbf{v}_j, u-t) + nD_1(\mathbf{v}_j + \zeta, u-t)] + \frac{a}{4} \sum_{i,j=1}^5 \mathcal{V}_i \mathcal{V}_j D_2(\mathbf{v}_i + \mathbf{v}_j, u-t) \quad (1.167)$$

where

$$D_1(p; u-t) = \int_t^u \frac{e^{-p(s-t)}}{q+b+(q-b)e^{-2q(s-t)}} ds \quad (1.168)$$

$$= \begin{cases} \frac{2q(u-t) - \ln(2q) + \ln(q+b+(q-b)e^{-2q(u-t)})}{2q(q+b)} & , p = 0 \\ \frac{1}{p(q+b)} \left[\mathcal{H} \left(1, \frac{p}{2q}, 1 + \frac{p}{2q}, -\frac{q-b}{q+b} \right) - e^{-p(u-t)} \times \mathcal{H} \left(1, \frac{p}{2q}, 1 + \frac{p}{2q}, -\frac{q-b}{q+b} \right) e^{-2q(u-t)} \right] & , p > 0 \end{cases} \quad (1.169)$$

and

$$D_2(p; u-t) = \int_t^u \frac{e^{-p(s-t)}}{(q+b+(q-b)e^{-2q(s-t)})^2} ds \quad (1.170)$$

$$= \frac{1}{2q(q+b)} \left[\frac{1}{2q} - \frac{e^{-p(u-t)}}{q+b+(q-b)e^{-2q(u-t)}} + (2q-p)D_1(p; u-t) \right] \quad (1.171)$$

and

$$E_t \left[\frac{\hat{\Lambda}_{1,u}^{1-\frac{n}{\gamma}} \hat{\Lambda}_{2,u}^{\frac{n}{\gamma}}}{\hat{\Lambda}_{1,t}^{1-\frac{n}{\gamma}} \hat{\Lambda}_{2,t}^{\frac{n}{\gamma}}} \frac{D_u}{D_t} | \mathcal{F}_t \right] = H_1((1-\gamma), (1-\gamma), \hat{\mu}_t^1, \omega_t, u-t) H_2 \left((1-\gamma), (1-\gamma), \frac{n}{\gamma}, g_t, u-t \right) \quad (1.172)$$

$$\begin{aligned} \zeta_{n,t}^D &= E_t \left[\int_t^\infty \frac{\hat{\Lambda}_{1,u}^{1-\frac{n}{\gamma}} \hat{\Lambda}_{2,u}^{\frac{n}{\gamma}}}{\hat{\Lambda}_{1,t}^{1-\frac{n}{\gamma}} \hat{\Lambda}_{2,t}^{\frac{n}{\gamma}}} \frac{D_u}{D_t} du | \mathcal{F}_t \right] \\ &= \int_t^\infty H_1((1-\gamma), (1-\gamma), \hat{\mu}_t^1, \omega_t, u-t) H_2 \left((1-\gamma), (1-\gamma), \frac{n}{\gamma}, g_t, u-t \right) du. \end{aligned} \quad (1.173)$$

and the PD ratio of the equity is

$$p_t = \frac{P_t}{D_t} = \int_t^\infty \frac{\Lambda_u}{\Lambda_t} \frac{D_u}{D_t} du = \sum_{n=0}^{\gamma} w_{n,t} \zeta_{n,t}^D. \quad (1.174)$$

D. Proof of Proposition 2

The dynamics of dividend strips prices can be also obtained by apply Ito's lemma to the dividend strips.

$$\frac{dp_{t,u}}{p_{t,u}} = \mu_{p_{u,t}} dt + \sigma_{p_{u,D}} dZ_{D,t} + \sigma_{p_{u,S}} dZ_{S,t}, \quad (1.175)$$

the drift part follows

$$\begin{aligned} \mu_{p_{u,t}} = & \mu_{D,t} + \frac{1}{p_{t,u}} \left[\frac{\partial p_{t,u}}{\partial \omega_t} \left(\mu_{D,t} - \lambda_x - \frac{1}{2} \sigma_D^2 \right) + \frac{\partial p_{t,u}}{\partial \nu_{1,t}} \nu_{1,t} \mu_{\nu_1} + \frac{\partial p_{t,u}}{\partial \hat{\mu}_{D,t}^1} \left(\zeta (\bar{\mu}_D - \hat{\mu}_{D,t}^1) + \frac{\bar{v}_1}{\sigma_D^2} (\mu_{D,t} - \hat{\mu}_{D,t}^1) \right) \right. \\ & + \frac{\partial p_{t,u}}{\partial g_t} \left(-\zeta g_t + \left(\frac{\bar{v}_1}{\sigma_D^2} + \frac{\bar{v}_2}{\sigma_D^2} \right) \mu_{D,t} - \left(\frac{\bar{v}_1}{\sigma_D^2} \hat{\mu}_{D,t}^1 + \frac{\bar{v}_2}{\sigma_D^2} \hat{\mu}_{D,t}^2 \right) \right) + \frac{1}{2} \frac{\partial^2 p_{t,u}}{\partial \omega_t^2} \sigma_D^2 + \frac{1}{2} \frac{\partial^2 p_{t,u}}{\partial \nu_{1,t}^2} \nu_{1,t} \sigma_{\nu_1} \\ & + \frac{1}{2} \frac{\partial^2 p_{t,u}}{\partial \left(\hat{\mu}_{D,t}^1 \right)^2} \left(\frac{\bar{v}_1^2}{\sigma_D^2} + \phi_1^2 \omega_1^2 \sigma_\mu^2 \right) + \frac{1}{2} \frac{\partial^2 p_{t,u}}{\partial g_t^2} \left(\left(\frac{\bar{v}_1}{\sigma_D} - \frac{\bar{v}_2}{\sigma_D} \right)^2 + (\phi_1 \omega_1 - \phi_2 \omega_2)^2 \sigma_\mu^2 \right) + \frac{\partial^2 p_{t,u}}{\partial \omega_t \partial \nu_{1,t}} \sigma_D \nu_{1,t} \sigma_{\nu_1} \\ & + \frac{\partial^2 p_{t,u}}{\partial \omega_t \partial \hat{\mu}_{D,t}^1} \sigma_D \frac{\bar{v}_1}{\sigma_D} + \frac{\partial^2 p_{t,u}}{\partial \omega_t \partial g_t} \sigma_D \left(\frac{\bar{v}_1}{\sigma_D} - \frac{\bar{v}_2}{\sigma_D} \right) + \frac{\partial^2 p_{t,u}}{\partial \nu_{1,t} \partial \hat{\mu}_{D,t}^1} \nu_{1,t} \sigma_{\nu_1} \frac{\bar{v}_1}{\sigma_D} + \frac{\partial^2 p_{t,u}}{\partial \nu_{1,t} \partial g_t} \nu_{1,t} \sigma_{\nu_1} \left(\frac{\bar{v}_1}{\sigma_D} - \frac{\bar{v}_2}{\sigma_D} \right) \\ & \left. + \frac{\partial^2 p_{t,u}}{\partial \hat{\mu}_{D,t}^1 \partial g_t} \left(\frac{\bar{v}_1}{\sigma_D} \left(\frac{\bar{v}_1}{\sigma_D} - \frac{\bar{v}_2}{\sigma_D} \right) + \phi_1 \omega_1 (\phi_1 \omega_1 - \phi_2 \omega_2) \sigma_\mu^2 \right) \right] \end{aligned} \quad (1.176)$$

where

$$\frac{\partial p_{t,u}}{\partial \omega_t} = h_2(\tau) p_{t,u} \quad (1.177)$$

$$\frac{\partial^2 p_{t,u}}{\partial \omega_t^2} = (h_2(\tau))^2 p_{t,u} \quad (1.178)$$

$$\frac{\partial p_{t,u}}{\partial \hat{\mu}_{D,t}^1} = h_1(\tau) p_{t,u} \quad (1.179)$$

$$\frac{\partial^2 p_{t,u}}{\partial \left(\hat{\mu}_{D,t}^1 \right)^2} = (h_1(\tau))^2 p_{t,u} \quad (1.180)$$

$$\frac{\partial p_{t,u}}{\partial g_t} = \sum_{n=0}^{\gamma} w_{n,t} ((1-\gamma)B(\tau) + 2g_t C(\tau)) \zeta_{n,u,t}^D \quad (1.181)$$

$$\frac{\partial^2 p_{t,u}}{\partial g_t^2} = \sum_{n=0}^{\gamma} w_{n,t} \left[2C(\tau) + ((1-\gamma)B(\tau) + 2g_t C(\tau))^2 \right] \zeta_{n,u,t}^D \quad (1.182)$$

$$\frac{\partial p_{t,u}}{\partial \nu_{1,t}} = \sum_{n=0}^{\gamma} w_{n,t} \left[\frac{\gamma - n}{\nu_{1,t}} + \frac{n}{1 - \nu_{1,t}} \right] \zeta_{n,u,t}^D \quad (1.183)$$

$$\frac{\partial^2 p_{t,u}}{\partial \nu_{1,t}^2} = \sum_{n=0}^{\gamma} w_{n,t} \left[\frac{(\gamma - n)(\gamma - n - 1)}{\nu_{1,t}^2} + \frac{2n(\gamma - n)}{\nu_{1,t}(1 - \nu_{1,t})} + \frac{n(n - 1)}{(1 - \nu_{1,t})^2} \right] \zeta_{n,u,t}^D \quad (1.184)$$

E. Proof of Lemma 1

As the dynamics of $\hat{\mu}_{D,t}^1$ follow

$$d\hat{\mu}_{D,t}^1 = \zeta (\bar{\mu}_D - \hat{\mu}_{D,t}^1) dt + \frac{\bar{v}_1}{\sigma_D} dZ_t^1 + \phi_1 \omega_1 \sigma_\mu dS_t \quad (1.185)$$

$$\hat{\mu}_{D,u}^1 = \hat{\mu}_{D,t}^1 + (\bar{\mu}_D - \hat{\mu}_{D,t}^1) \left(1 - e^{-\zeta(u-t)} \right) + \int_t^u e^{-\zeta(u-s)} \frac{\bar{v}_1}{\sigma_D} dZ_s^1 + \int_t^u e^{-\zeta(u-s)} \phi_1 \omega_1 \sigma_\mu dS_s \quad (1.186)$$

Its Malliavin derivative is

$$\mathcal{D}_t^1 \hat{\mu}_{D,u}^1 = e^{-\zeta(u-t)} \left[\frac{\bar{v}_1}{\sigma_D} \quad \phi_1 \omega_1 \sigma_\mu \right]. \quad (1.187)$$

Using the same technique,

$$d\omega_t = \left(\hat{\mu}_{D,t}^1 - \frac{1}{2} \sigma_D^2 - \lambda_x \omega_t \right) dt + \sigma_D dZ_t^1 \quad (1.188)$$

$$\omega_u = e^{-\lambda_x(u-t)} \omega_t + \int_t^u e^{-\lambda_x(u-s)} \left(\hat{\mu}_{D,s}^1 - \frac{1}{2} \sigma_D^2 \right) ds + \int_t^u e^{-\lambda_x(u-s)} \sigma_D dZ_s^1 \quad (1.189)$$

so

$$\mathcal{D}_t^1 \omega_u = e^{-\lambda_x(u-t)} \left[\sigma_D \quad 0 \right] + \int_t^u e^{-\lambda_x(u-s)} (\mathcal{D}_t^1 \hat{\mu}_{D,s}^1) ds \quad (1.190)$$

$$= e^{-\lambda_x(u-t)} \left[\sigma_D \quad 0 \right] + \int_t^u e^{-\lambda_x(u-s)} e^{-\zeta(u-s)} ds \left[\frac{\bar{v}_1}{\sigma_D} \quad \phi_1 \omega_1 \sigma_\mu \right] \quad (1.191)$$

$$= e^{-\lambda_x(u-t)} \left[\sigma_D \quad 0 \right] + \frac{1}{\lambda_x + \zeta} \left(1 - e^{-(\lambda_x + \zeta)(u-t)} \right) \left[\frac{\bar{v}_1}{\sigma_D} \quad \phi_1 \omega_1 \sigma_\mu \right]. \quad (1.192)$$

The dynamics of g_t follow

$$dg_t = - \left(\zeta + \frac{\bar{v}_2}{\sigma_D^2} \right) g_t dt + \frac{1}{\sigma_D} (\bar{v}_1 - \bar{v}_2) dZ_t^1 + (\phi_1 \omega_1 - \phi_2 \omega_2) \sigma_\mu dS_t, \quad (1.193)$$

$$g_u = g_t e^{-\left(\zeta + \frac{\bar{v}_2}{\sigma_D^2} \right)(u-t)} + \int_t^u e^{-\left(\zeta + \frac{\bar{v}_2}{\sigma_D^2} \right)(u-s)} \frac{1}{\sigma_D} (\bar{v}_1 - \bar{v}_2) ds + \int_t^u e^{-\left(\zeta + \frac{\bar{v}_2}{\sigma_D^2} \right)(u-s)} (\phi_1 \omega_1 - \phi_2 \omega_2) \sigma_\mu dZ_s^1 \quad (1.194)$$

so i

$$\mathcal{D}_t^1 g_u = e^{-\left(\zeta + \frac{\bar{v}_2}{\sigma_D^2} \right)(u-t)} \left[\frac{1}{\sigma_D} (\bar{v}_1 - \bar{v}_2) \quad (\phi_1 \omega_1 - \phi_2 \omega_2) \sigma_\mu \right]. \quad (1.195)$$

For the dividend D_t

$$\frac{dD_t}{D_t} = \hat{\mu}_{D,t}^1 dt + \sigma_D dZ_t^1 \quad (1.196)$$

$$D_u = D_t \exp \left[\int_t^u \left(\hat{\mu}_{D,s}^1 - \frac{1}{2} \sigma_D^2 \right) ds + \int_t^u \sigma_D dZ_s^1 \right] \quad (1.197)$$

so

$$\frac{\mathcal{D}_t^1 D_u}{D_u} = \left[\sigma_D \quad 0 \right] + \int_t^u \mathcal{D}_t^1 \hat{\mu}_{D,s}^1 ds = \left[\sigma_D + \frac{1}{\xi} (1 - e^{-\zeta(u-t)}) \frac{\bar{v}_1}{\sigma_D} \quad 0 \right]. \quad (1.198)$$

The change of measure ξ_t :

$$\frac{d\xi_t}{\xi_t} = -\frac{g_t}{\sigma_D} dZ_t^1, \quad (1.199)$$

$$\xi_u = \xi_t \exp \left[-\int_t^u \frac{1}{2} \frac{g_s^2}{\sigma_D^2} ds - \int_t^u \frac{g_s}{\sigma_D} dZ_s^1 \right]. \quad (1.200)$$

Its Malliavin derivative is

$$\frac{\mathcal{D}_t^1 \xi_u}{\xi_u} = \left[-\frac{g_t}{\sigma_D} \quad 0 \right] - \int_t^u \mathcal{D}_t^1 g_s \left[\frac{g_s}{\sigma_D^2} ds + \frac{1}{\sigma_D} dZ_s^1 \right] \quad (1.201)$$

$$= \left[-\frac{g_t}{\sigma_D} \quad 0 \right] - \int_t^u e^{-\zeta(s-t)} \left[\frac{g_s}{\sigma_D^2} ds + \frac{1}{\sigma_D} dZ_s^1 \right] \left[\frac{1}{\sigma_D} (\bar{v}_1 - \bar{v}_2) \quad (\phi_1 \omega_1 - \phi_2 \omega_2) \sigma_\mu \right]. \quad (1.202)$$

F. Proof of Proposition 3

The short term market of risk is

$$\frac{\mathcal{D}_t \Lambda_{1,t}}{\Lambda_{1,t}} = (1 - \gamma) \mathcal{D}_t \omega_t - \frac{\mathcal{D}_t^1 D_t}{D_t} + \nu_{2,t} \frac{\mathcal{D}_t^1 \xi_t}{\xi_t}. \quad (1.203)$$

The long term market of risk is

$$\frac{\mathcal{D}_t^1 \Lambda_{1,T}}{\Lambda_{1,T}} = (1 - \gamma) \mathcal{D}_t^1 \omega_T - \frac{\mathcal{D}_t^1 D_T}{D_T} + \nu_{2,T} \frac{\mathcal{D}_t^1 \xi_T}{\xi_T} \quad (1.204)$$

$$\begin{aligned} & (1 - \gamma) \left(e^{-\lambda_x(T-t)} \mathcal{D}_t^1 \omega_t + \int_t^T e^{-\lambda_x(T-u)} (\mathcal{D}_t^1 \hat{\mu}_{D,u}^1) du \right) - \left(\frac{\mathcal{D}_t^1 D_t}{D_t} - \int_t^T \mathcal{D}_t^1 \hat{\mu}_{D,u} du \right) \\ &= + \nu_{2,T} \left(\frac{\mathcal{D}_t^1 \xi_t}{\xi_t} - \int_t^T (\mathcal{D}_t^1 g_u) \left[\frac{g_u}{\sigma_D^2} du + \frac{1}{\sigma_D} dZ_{D,t}^1 \right] \right) \end{aligned} \quad (1.205)$$

$$\begin{aligned} & \frac{\mathcal{D}_t^1 \Lambda_{1,t}}{\Lambda_{1,t}} + (1 - \gamma) \left(\left(e^{-\lambda_x(T-t)} - 1 \right) \mathcal{D}_t^1 \omega_t + \int_t^T e^{-\lambda_x(T-u)} (\mathcal{D}_t^1 \hat{\mu}_{D,u}^1) du \right) + \int_t^T \mathcal{D}_t^1 \hat{\mu}_{D,u} du \\ &= + (\nu_{2,T} - \nu_{2,t}) \frac{\mathcal{D}_t^1 \xi_t}{\xi_t} - \nu_{2,T} \left(\int_t^T (\mathcal{D}_t^1 g_u) \left[\frac{g_u}{\sigma_D^2} du + \frac{1}{\sigma_D} dZ_{D,t}^1 \right] \right) \end{aligned} \quad (1.206)$$

$$\begin{aligned}
& \frac{\mathcal{D}_t^1 \Lambda_{1,t}}{\Lambda_{1,t}} + (1 - \gamma) \left(e^{-\lambda_x(T-t)} - 1 \right) \begin{bmatrix} \sigma_D & 0 \end{bmatrix} \\
& = + \left((1 - \gamma) \frac{1}{\lambda_x + \zeta} \left[1 - e^{-(\lambda_x + \zeta)(T-t)} \right] + \frac{1}{\zeta} \left[1 - e^{-\zeta(T-t)} \right] \right) \begin{bmatrix} \frac{\bar{v}_1}{\sigma_D} & \phi_1 \omega_1 \sigma_\mu \end{bmatrix} \\
& + (\nu_{2,T} - \nu_{2,t}) \begin{bmatrix} -\frac{g_t}{\sigma_D} & 0 \end{bmatrix} - \nu_{2,T} \left(\int_t^T (\mathcal{D}_t^1 g_u) \left[\frac{g_u}{\sigma_D^2} du + \frac{1}{\sigma_D} dZ_{D,t}^1 \right] \right)
\end{aligned} \tag{1.207}$$

Taking expectations at both sides leads to the result.

G. Proof of Corollary 7

From the proof of 4, we can set the first and second argument of function H_1 and H_2 to be zero to get the moment generating function of $\frac{\xi_u}{\xi_t}, \forall u \geq t$ simply

$$E_t^1 \left[\frac{\xi_u^\chi}{\xi_t^\chi} \right] = H_2(0, 0, g_t, u, t). \tag{1.208}$$

By inverse Fourier transform and the consumption sharing rule

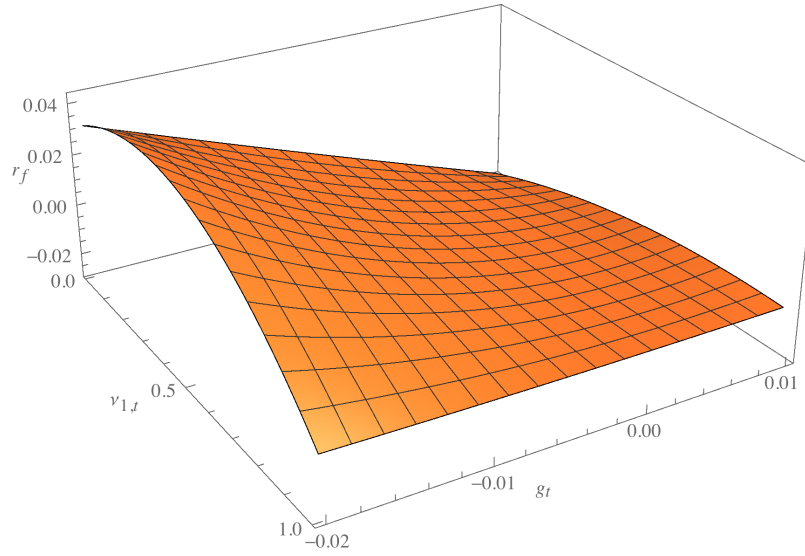
$$\nu_2(\xi) = \left(1 + \xi^{-\frac{1}{\gamma}} \right)^{-1} \tag{1.209}$$

the conditional expectation of the consumption share is

$$E_t^1[\nu_{2,T}] = \int_0^\infty \nu_2(\tilde{\xi}) \left[\frac{1}{2\pi} \int_{-\infty}^\infty \left(\frac{\tilde{\xi}}{\xi} \right)^{-i\chi} H_2(0, 0, i\chi, g_t, T-t) d\chi \right] \frac{d\tilde{\xi}}{\tilde{\xi}} \tag{1.210}$$

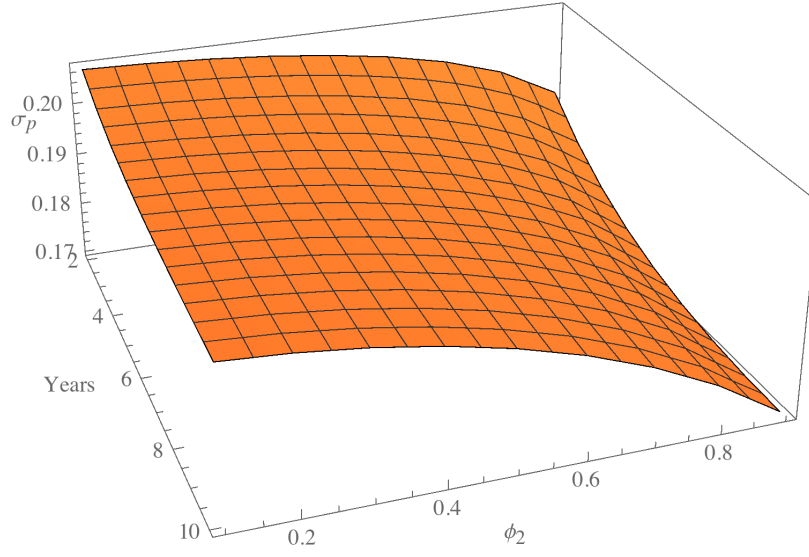
$$= \int_0^\infty \left(1 + \tilde{\xi}^{-\frac{1}{\gamma}} \right)^{-1} \left[\frac{1}{2\pi} \int_{-\infty}^\infty \left(\frac{\tilde{\xi}}{(\nu_{2,t}^{-1} - 1)^{-\gamma}} \right)^{-i\chi} H_2(0, 0, i\chi, g_t, T-t) d\chi \right] \frac{d\tilde{\xi}}{\tilde{\xi}}. \tag{1.211}$$

VIII. Appendix: Figures

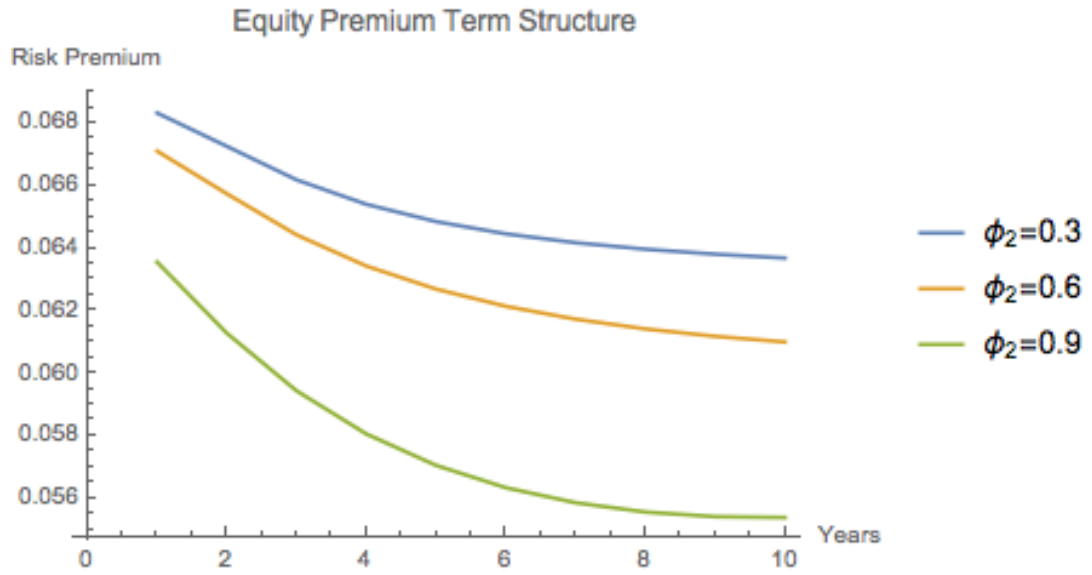


These two figures illustrate the risk free rate. The x-axis is current consumption share $\nu_{1,t} = \frac{C_{1,t}}{D_t}$ and the y-axis is the difference in beliefs g_t .

Figure 1.1. Risk free rate



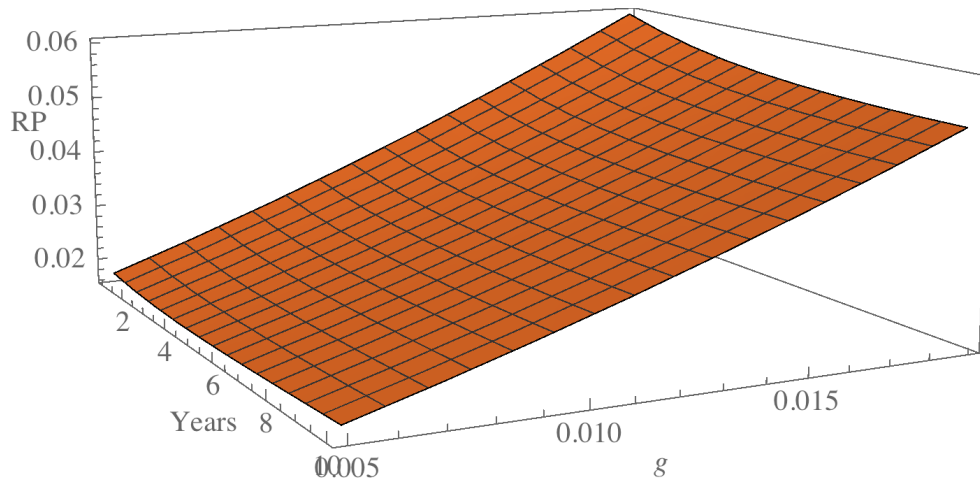
(a) Dividend Strips volatility term structure Surface



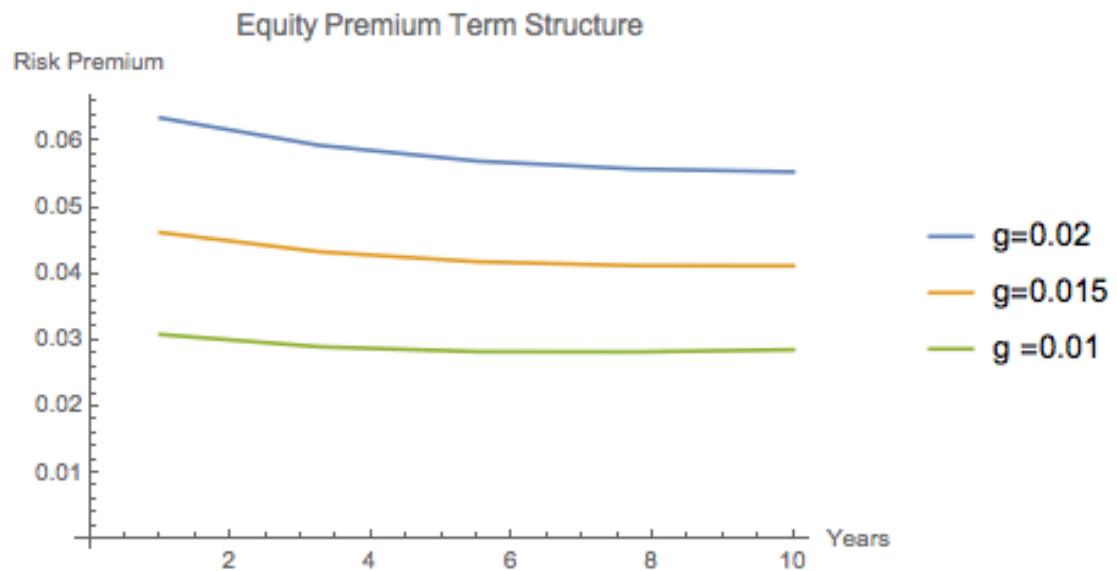
(b) Dividend Strips volatility term structure

These two figures illustrate the dividend strip volatility term structure against the over-confidence level. The top panel shows the surface of the dividend strip volatility term structure while the bottom panel shows three curves on the surface where the Agent 2's subjective informativeness of the noise signal changes.

Figure 1.2. Dividend Strip Volatility Term Structure



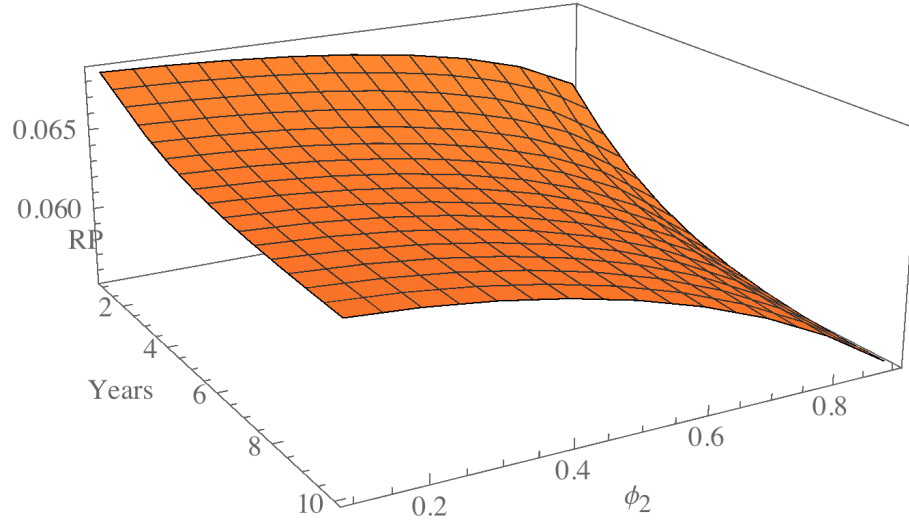
(a) Equity Premium Term Structure and Difference in Beliefs Surface



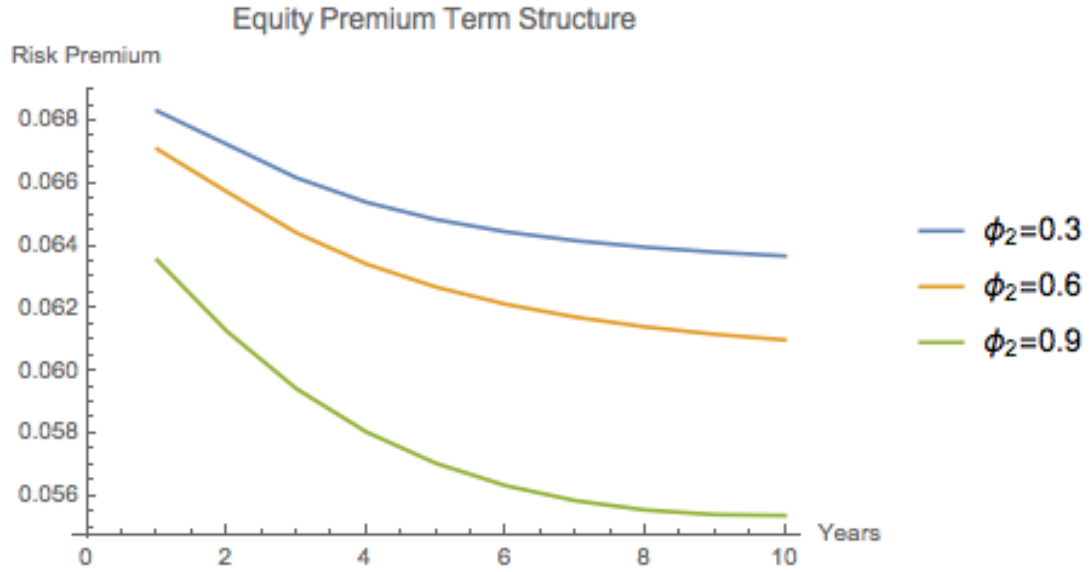
(b) Equity Premium Term Structure and Difference in Beliefs

These two figures illustrate the equity premium term structure against the difference in beliefs. The top panel shows the surface of equity premium term structure while the bottom panel shows three curves on the surface with three levels of difference in beliefs.

Figure 1.3. Equity Premium Term Structure and Difference in Beliefs



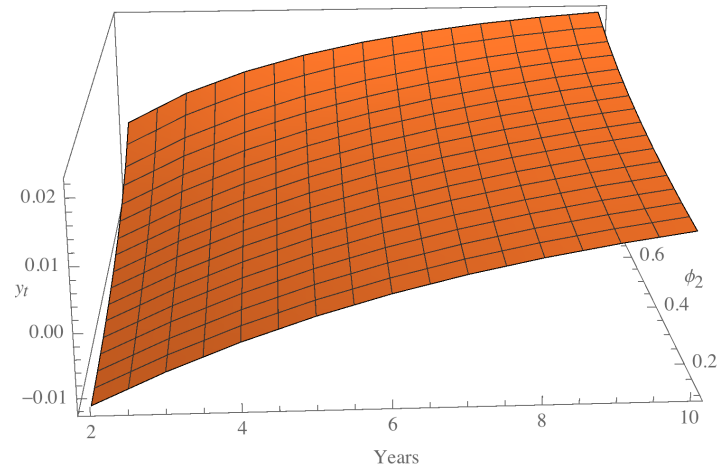
(a) Equity Premium Term Structure and Overconfidence Surface



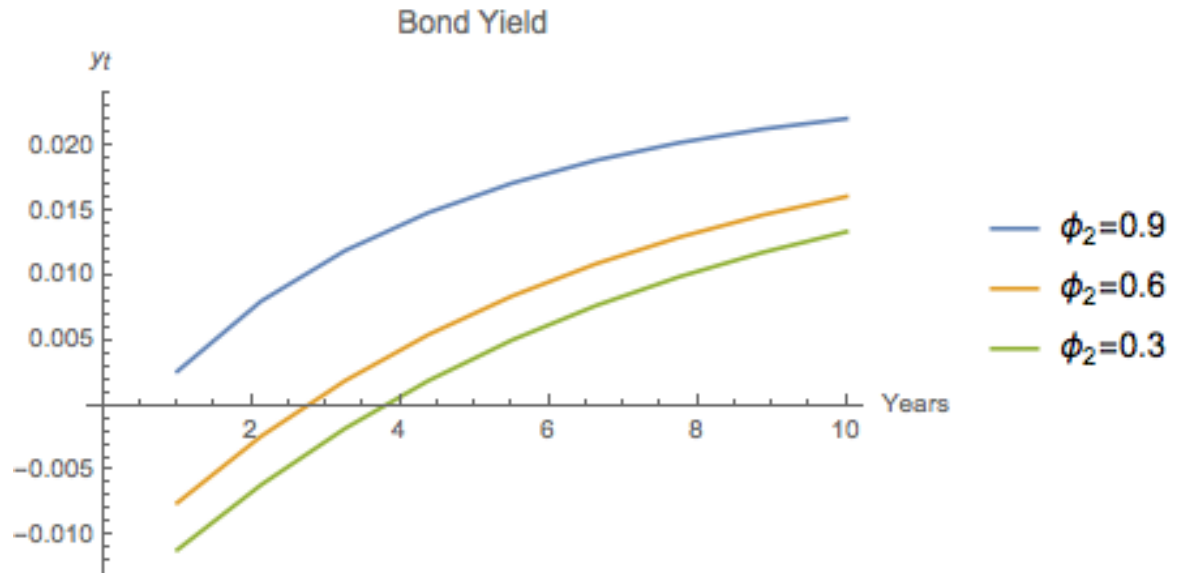
(b) Equity Premium Term Structure and Overconfidence

These two figures illustrate the equity premium term structure against the overconfidence level. The top panel shows the surface of equity premium term structure while the bottom panel shows three curves on the surface where the Agent 2's subjective informativeness of the noise signal changes. The parameters used in the plot follows Table 1.1.

Figure 1.4. Equity Premium Term Structure and Overconfidence



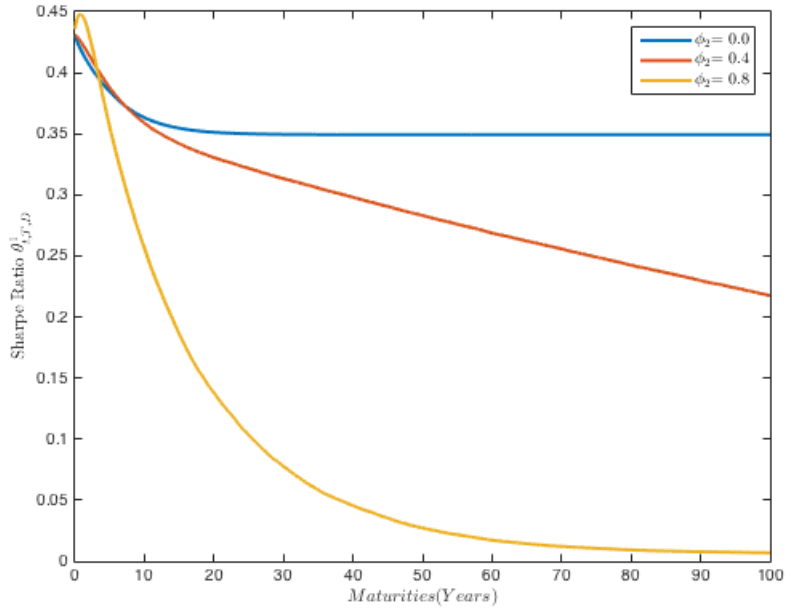
(a) Bond Yield Curve and Overconfidence (Surface)



(b) Bond Yield Curve and Overconfidence

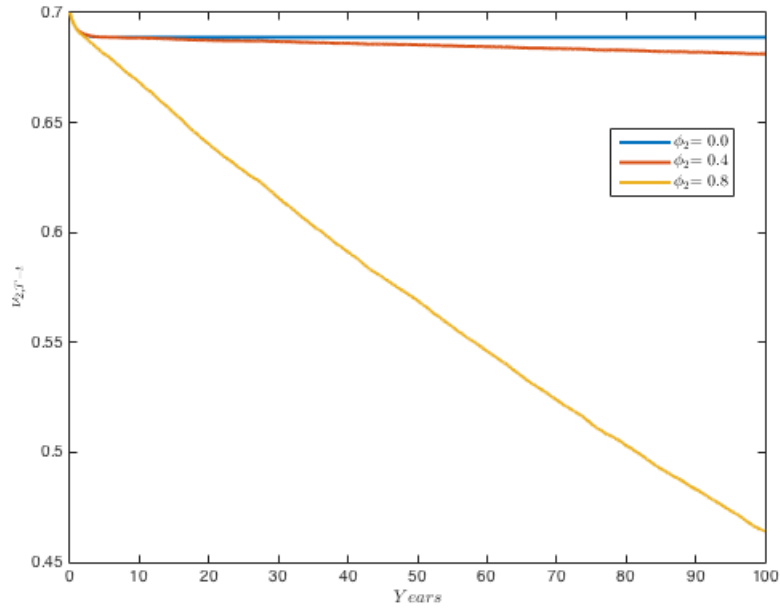
These two figures illustrate the bond yield term structure against the difference in beliefs. The top panel shows the surface of bond yield term structure while the bottom panel shows three yield curves on the curve with three levels of difference in beliefs.

Figure 1.5. Bond Yield Curve and Overconfidence



This figure illustrates the term structure of market price of risk in the eye of the rational agent under different level of overconfidence $\phi_2 = 0$, $\phi_2 = 0.4$ and $\phi_2 = 0.8$. The x-axis is the maturities that cash flows realize. The figure shows that if the agent is more overconfident, the market price of risk is steeper.

Figure 1.6. Price of Risk



This figure illustrates the expected consumption share of overconfident agent under objective probability measure with different level of overconfidence $\phi_2 = 0$, $\phi_2 = 0.4$ and $\phi_2 = 0.8$. The x-axis is the maturity that cash flows realize. The other parameters used in the plot follows Table 1.1. The figure shows that if the overconfident agent will not die in reasonable long run, especially if the overconfidence level is modest.

Figure 1.7. Survival of the Overconfident Agent

Chapter 2

Speculation, Differences in Beliefs and Growth

I. Introduction

If two speculators want to bet against each other on future stock prices, should they be protected from themselves? The answer from most economists and regulators was "NO" for a long time, as speculators can reveal valuable new information, bear market risk and increase market liquidity. However, after financial crises, economists have started to question the premise that speculation creates more good than harm. Politicians argue that speculation caused the recent financial crisis and as a result too big to fail financial intermediaries have had to rely on tax payers' largesse. US lawmakers passes the Volcker Rule to restrict speculation. Given this background, we should remain more prudent and understand the general equilibrium effects of speculation.

Belief dispersion has long been recognized as an important ingredient of speculation, and it appears the reason is that investors are overconfident, or because they have different prior beliefs, and it appears this is due to distorted information transmission. In this chapter I try to argue that speculation caused by differences in beliefs is a negative sum game because it exposes speculators to risk without fully compensating them and can even have negative consequences for a third party, namely ordinary people outside speculative games. This chapter studies those negative consequences of speculation caused by beliefs dispersion on financial markets, the real economy, and social welfare.

I develop a continuous-time asset pricing model in which households purchase the equity of firms and provide labor for production. Firms pay dividends to shareholders after deducting labor costs from revenues generated by production good sales. My model makes one key assumption. I assume that there are two types of households, who have different priors about latent technological growth (specifically, agents have different variances of their estimates for technological growth), and thus less confident households (large variance of the estimate) react more to the realizations of technological progress. Belief dispersion among households results in speculation in financial markets. This assumption implies that less confident households tend to hold excess leverage in booms, but insufficient leverage in downturns, and that aggregate leverage is pro-cyclical. Those households are fully rational and maximize their utility by choosing their optimal consumption, working hours and asset allocations. Since households gain their utility from consumption and leisure, their portfolio choices affect their intertemporal consumption and working choices. Speculation is a part of the asset allocation and impacts consumption and labor choices, and therefore plays a role in determining aggregate output and social welfare. By introducing endogenous labor supply in a general equilibrium heterogeneous-agent asset-pricing model, I can explore not only the

effects of speculation on financial market, but also on the real economy.

The main results of the paper are as follows. First, speculation increases excess stock return volatility and the equity risk premium, and therefore reduces risky asset price levels, similar to the results of David (2008) and Dumas, Kurshev, and Uppal (2009). The intuition is as follows: in the presence of heterogeneous priors and learning, differences in beliefs fluctuate over the long-term, exposing the financial market to sentiment risk (Dumas, Kurshev, and Uppal (2009)). In a complete market, where there are no constraints and households can freely speculate against each other, they are fully exposed to sentiment risk (speculative risk), which increases the equity premium and creates greater return volatility relative to the case where there is no belief heterogeneity.

Second, speculative trades due to belief dispersion do more harm than good under every possible reasonable belief specified by the belief-neutral welfare criterion of Brunnermeier, Simsek, and Xiong (2014). Given heterogeneous beliefs, the probability measures of different types of households are different. My model shows that speculation improves a speculator's welfare under his own beliefs. In contrast, Posner and Weyl (2013a) and Posner and Weyl (2013b) argue that the speculation is welfare reducing, because it exposes speculators to increased overall risk. In terms of social welfare (the welfare of the group of all agents) or the effect of the third party, it is tempting to evaluate welfare under correct beliefs. However, even a central planner may not have correct beliefs. Therefore, instead of taking a stand on which (or whose) beliefs are correct, a belief-neutral welfare criterion asserts that an allocation is belief-neutral efficient (inefficient) if it is efficient (inefficient) under every possible reasonable probability measure. My result shows that the resource allocation without speculation is belief neutral superior to the allocation with speculation. In other words, speculation reduces social welfare under any reasonable probability measure. The reason is the negative externalities caused by speculation, i.e. the aggregate labor supply and output distortions, and greater consumption growth and leisure volatility.

Third, I show that speculation creates an output gap¹ and destabilizes economic growth. The intuition is as follows. First, speculation creates distortions in portfolio choices, and hence it induces suboptimal consumption and leisure allocations, and further distorts the aggregate labor supply and economic growth. In other words, the belief heterogeneity results in an output gap in the economy. Second, in Proposition 2.37 I show that aggregate labor supply is a function of the consumption allocation and that any deviation from the symmetric consumption allocation reduces the aggregate labor supply. After the speculation profit is realized, either of the two parties would lose their wealth, at least in the short run. An asymmetric wealth distribution caused by speculation further impacts consumption choice (allocation) and hence the aggregate output. In the long run, speculators with more accurate beliefs wipe out their unlucky opponents. However, winners sit on their speculation profits and work less, reducing the aggregate labor supply even when losers increase working hours. Third, speculation increases the volatility of aggregate output growth and destabilizes the economy. Output volatility depends on exogenous technological growth volatility as well as the sensitivity of aggregate labor supply to belief dispersion, which becomes greater when the consumption share deviates from equal allocation. Hence, the two effects mentioned above cause the consumption allocation deviation and hence the greater sensitivity of aggregate labor (output)

¹Analogously to the New Keynesian literature, we can refer to the *output gap* as the difference between distorted output in the presence of speculation and the output (*natural output*) in the economy where there is no belief heterogeneity.

to sentiment shocks. The impact is even more significant after a negative aggregate shock, because optimists are hit heavily and the market sentiment tends to change dramatically.

The rest of Chapter 2 is organized as follow: Section II reviews the literature, Section III lays out the model, Section IV characterizes the equilibrium, Section V presents a numerical example of the impact of speculation on asset prices, social welfare and aggregate economy. I conclude in Section VI.

II. Related Literature

This paper is closest to the literature that studies risk sharing, speculation and savings decisions of investors in the presence of heterogeneous beliefs, which was started by [Harrison and Kreps \(1978\)](#).

[Xiong \(2013\)](#) reviews literature that builds on heterogeneous beliefs, a widely observed attribute of individuals, to explain bubbles, crises, and endogenous risk in financial markets. [Blume, Cogley, Easley, Sargent, Tsyrennikov, et al. \(2014\)](#) study an exchange economy and show that welfare might be higher in an incomplete market than a complete market if measured under a set of objective beliefs instead of the investor's beliefs. [Heyerdahl-Larsen and Walden \(2014\)](#) study hedging and speculation in a static model and show that the competitive equilibrium is always inefficient under their own efficiency criterion. [Posner and Weyl \(2013a\)](#) and [Posner and Weyl \(2013b\)](#) argue that the speculative trade should be prevented. Their central concern here is that belief-motivated trade unnecessarily increases the consumption volatility of both investors, under any probability measure, without clear social benefit. [Simsek \(2013a\)](#) and [Simsek \(2013b\)](#) show that differences in beliefs provide an incentive for financial innovation whose purpose is facilitate trade based on differences in beliefs, and which has a tendency to raise the volatility of consumption. [Duffie \(2014\)](#) challenges the necessity of prohibiting speculation and argues that the preference of gambling by speculators is completely rational, leaving aside the potential and possible third-party harm. Furthermore, he stresses the benefits of speculation via increasing the informativeness of financial markets. [Detemple and Rindisbacher \(2016\)](#) discuss the necessary and sufficient conditions for speculation to be socially beneficial.

On the methodological side, this research draws on the heterogeneous agent asset pricing literature. I model an economy where two types of agents with different priors, which gives rise to a credit market as in [Scheinkman and Xiong \(2003\)](#), [Dumas, Kurshev, and Uppal \(2009\)](#) and [Xiong and Yan \(2009\)](#). [David \(2008\)](#) resolves the equity premium puzzle by exploring the speculation on a more precise estimation.

III. Model

I model a finite-horizon continuous time production economy in which a competitive final goods producer aggregates a continuum of intermediate inputs

$$Y_t = \left(\int_0^1 Y_{j,t}^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}, \quad (2.1)$$

where the intermediate goods $Y_{j,t}$ output by a continuum of monopolistically competitive intermediate goods producers. Firm j employs $n_{j,t}$ units of labor and technology A_t to produce

$$Y_{j,t} = A_t n_{j,t}. \quad (2.2)$$

The technology level is exogenous, and its dynamics follow

$$\frac{dA_t}{A_t} = \mu_A dt + \sigma_A dZ_{A,t}. \quad (2.3)$$

where σ_A and μ_A are constant, and $Z_{A,t}$ is a standard Brownian motion. μ_A is assumed to be latent to households. The following subsection shows how households infer the true μ_A and how the difference in beliefs is generated.

A. *Heterogeneous Expectation and Learning*

The economics and finance literature has widely adopted the Bayesian inference framework to model investors' learning process about unobservable economic variables, such as the productivity or profitability of a firm. In one line of literature (see Basak (2000)), the difference in beliefs deterministically converges to zero in the long run if agents' priors are different².

Following this approach, I assume that this production economy is populated by a continuum of measure one of households who don't observe the expected technological growth rate μ_A and therefore have to use public signals to estimate its true value.

More specifically, I assume that there are two types of households (two agents) whose Gaussian priors are different, i.e. for agent i , $\mu_{A,t}^i \sim Normal(\hat{\mu}_{A,t}^i, v_t^i)$. Their forecast $\hat{\mu}_{A,t}^i = E^i[\mu_A^i | \mathcal{F}_t^A]$ follows

$$d\hat{\mu}_{A,t}^i = \frac{\bar{v}_i}{\sigma_A} dZ_{A,t}^i \quad (2.4)$$

where

$$dZ_{A,t}^i = \frac{1}{\sigma_A} \left(\frac{dA_t}{A_t} - \hat{\mu}_{A,t}^i dt \right) \quad (2.5)$$

is the shocks agent i perceives and $\bar{v}_{i,t}$ is the variance of the estimation :

$$\bar{v}_t^i = \frac{\bar{v}_0^i \sigma_A^2}{\bar{v}_0^i t + \sigma_A^2}. \quad (2.6)$$

Without loss of generality, I assume $v_0^1 \leq v_0^2$. Therefore although same signal (technology growth realization) is observed, the heterogeneous prior knowledge leads investors to react differently to the signal flow and form different views towards the economy: the Type 2 agent reacts more to perceived shocks, and the difference in beliefs is stochastic.

²Another strand of literature (Scheinkman and Xiong (2003), Dumas, Kurshev, and Uppal (2009), David (2008), Buraschi and Jiltsov (2006)) assumes difference in beliefs arises due to the different prior knowledge about the informativeness of signals and the dynamics of unobservable economic variable, making difference in beliefs persistent in the long run.

Though the agents' estimation of the expected technological growth is time varying, we can still define their subjective probability measure $\mathbb{P}^i$ against the objective probability measure $\mathbb{P}$ by the following exponential martingale

$$\frac{d\mathbb{P}^i}{d\mathbb{P}} = \xi_{t,i}, \quad (2.7)$$

where

$$\xi_{t,i} = \xi_{t,0} \exp \left[\int_0^t -\sigma_{\xi,i}^2 ds + \int_0^t \sigma_{\xi,i} dZ_{A,t} \right] \quad (2.8)$$

$$\sigma_{\xi,i} = \frac{\hat{\mu}_{A,t}^i - \mu_{A,t}}{\sigma_A}. \quad (2.9)$$

We quantify the level of disagreement by the process ξ_t , which is defined as:

$$\xi_t = \frac{\xi_{t,2}}{\xi_{t,1}}, \quad (2.10)$$

and hence

$$\frac{d\xi_t}{\xi_t} = \mu_\xi dt + \sigma_\xi dZ_{A,t}, \quad (2.11)$$

where

$$\mu_\xi = -\sigma_{\xi,1} (\sigma_{\xi,2} - \sigma_{\xi,1}) \quad (2.12)$$

$$\sigma_\xi = \sigma_{\xi,2} - \sigma_{\xi,1}. \quad (2.13)$$

As the difference in beliefs is stochastic in this economy, the level of disagreement and its volatility are stochastic, constituting the source of the sentiment risk (see [Dumas, Kurshev, and Uppal \(2009\)](#)).

B. Preferences and Optimization

Both agents work and consume their capital and labor income. The consumption and leisure of Agent i at instant time t is denoted by $C_{i,t}$, and the instantaneous utility from consumption and leisure is given by:

$$U_i(C_{i,t}, L_{i,t}) \equiv e^{-\rho t} \left(\frac{C_{i,t}^{1-\gamma}}{1-\gamma} + \frac{L_{i,t}^{1-\eta}}{1-\eta} \right). \quad (2.14)$$

Given agent i 's belief which is represented by the probability measure $\mathbb{P}^k$, her expected utility at time 0 from consuming $C_{i,t}$ and leisure $L_{i,t}$ is given by

$$V_{i,0} = E_0^i \left[\int_0^T U_i(C_{i,t}, L_{i,t}) dt \right]. \quad (2.15)$$

The agents optimize their life time utility subject to a dynamic budget constraint

$$dG_{i,t} = r_t G_{i,t} + \theta_{i,t} (dJ_t - r_t J_t + D_t dt) + \frac{W_t}{P_t} (\bar{N}_i - L_{i,t}) dt - C_{i,t} dt, \quad (2.16)$$

and the law of motion of the consumption allocation

$$\frac{d\nu_{1,t}}{\nu_{1,t}} = \mu_{\nu_1} dt + \sigma_{\nu_1} dZ_{A,t}^i, \quad (2.17)$$

where $\nu_{1,t} \equiv \frac{C_{1,t}}{Y_t}$ is the consumption allocation of agent 1, W_t is the wage, P_t is the consumption good price level, J_0 is the stock market value. According to [Cox and Huang \(1989\)](#), the constraints above are equivalent to the static budget constraint, which restricts the present value of all future consumption less the labor income to be no more than the initial wealth of each agent:

$$E_0^i \left[\int_0^T \Lambda_{i,t} \left(C_{i,t} - \frac{W_t}{P_t} N_{i,t} \right) dt \right] \leq x_i J_0, \quad (2.18)$$

where $\Lambda_{i,t}$ is agent i 's state-price density in his own probability measure and follows

$$\frac{d\Lambda_{i,t}}{\Lambda_{i,t}} = -r_t dt + \kappa_{i,t} dZ_{A,t}^i, \quad \Lambda_{i,0} = 1. \quad (2.19)$$

The FOCs give the optimal consumption and leisure choice

$$C_{i,t} = y_i^{-\frac{1}{\gamma}} e^{-\frac{\rho}{\gamma} t} \Lambda_{i,t}^{-\frac{1}{\gamma}} \quad (2.20)$$

$$L_{i,t} = y_i^{-\frac{1}{\eta}} e^{-\frac{\rho}{\eta} t} \left(\Lambda_{i,t} \frac{W_t}{P_t} \right)^{-\frac{1}{\eta}} \quad (2.21)$$

where y_i is the Lagrange multiplier from the static budget constraint and solves the following equation

$$E_0^i \left[\int_0^T e^{-\frac{\rho}{\gamma} t} \Lambda_{i,t}^{\frac{\gamma-1}{\gamma}} dt \right] y_i^{-\frac{1}{\gamma}} + E_0^i \left[\int_0^T e^{-\frac{\rho}{\eta} t} \Lambda_{i,t}^{\frac{\eta-1}{\eta}} \left(\frac{W_t}{P_t} \right)^{\frac{\eta-1}{\eta}} dt \right] y_i^{-\frac{1}{\eta}} = E_0^i \left[\int_0^T \Lambda_{i,t} \frac{W_t}{P_t} \bar{N}_i dt \right] + x_i J_0 \quad (2.22)$$

IV. The Equilibrium

The difference in beliefs induces speculative trading among investors who need to use only one non-redundant risky security to hedge the perceived fundamental risk $dZ_{A,t}$. I assume there are two linear independent securities: a riskless, instantaneous bank deposit with a rate of interest rate r_t and the equity which pays the aggregate dividend perpetually. The usual definition of the equilibrium applies: both types of investors make their optimal consumption, leisure, and portfolio choice based on their beliefs, and both consumption goods and asset markets clear.

To determine the equilibrium price system, it suffices to find the two SDF processes that clear the consumption good market. Define

$$\lambda_t = \frac{y_1 \Lambda_{1,t}}{y_2 \Lambda_{2,t}} \quad (2.23)$$

where $\lambda_0 = \frac{y_1}{y_2}$. Its dynamics follow

$$\frac{d\lambda_t}{\lambda_t} = -\sigma_\lambda dZ_{D,t}^1, \quad (2.24)$$

where

$$\sigma_{\lambda,t} = \kappa_{1,t} - \kappa_{2,t} = \frac{g_t}{\sigma_A} \quad (2.25)$$

λ_t is the only state variable in the economy besides agents' beliefs and essentially equivalent to the share of consumption as the individual SDF is just the marginal utility with respect to consumption. The dynamics of λ_t depend only on the difference in beliefs $\kappa_{1,t} - \kappa_{2,t} = \frac{\hat{\mu}_{A,t}^1 - \hat{\mu}_{A,t}^2}{\sigma_A}$. The volatility of λ_t is stochastic and thus induces the sentiment risk, impacting optimal consumption and leisure choices, asset prices and social welfare. I characterize the equilibrium by the following proposition.

Proposition 4. *Assume that an equilibrium exists. The SDFs in the fictitious for two types of households are*

$$\Lambda_{1,t} = e^{-\rho t} \frac{U_c(Y_t, \lambda_t)}{U_c(Y_0, \lambda_0)}, \quad \Lambda_{2,t} = \frac{\lambda_t}{\lambda_0} \Lambda_{1,t}, \quad (2.26)$$

where $U_c(Y_t, \lambda_t)$ is the marginal utility of a social planner which is defined as

$$U(Y_t, \lambda_t) = \max_{C_{1,t} + C_{2,t} \leq A_t (\bar{N} - L_{1,t} - L_{2,t}) = Y_t} U(C_{1,t}, L_{1,t}) + \lambda_t U(C_{2,t}, L_{2,t}). \quad (2.27)$$

The stock price J_0 is computed from the more confident agent's probability measure

$$J_0 = \frac{1}{\epsilon} E_0^1 \left[\int_0^T e^{-\rho s} \frac{U_c(Y_s, \lambda_s)}{U_c(Y_0, \lambda_0)} Y_s ds \right]. \quad (2.28)$$

The equilibrium consumption allocations are

$$C_{1,t} = \frac{1}{\epsilon U_c(Y_t, \lambda_t)}, \quad C_{2,t} = \frac{\lambda_t}{\epsilon U_c(Y_t, \lambda_t)}. \quad (2.29)$$

The stochastic weighting process λ_t has the dynamics given by (3.36) where λ_0 clears the markets at time $t = 0$, namely

$$C_1(\lambda_0, A_0) + C_2(\lambda_0, A_0) = A_0 (\bar{N} - L_1(\lambda_0, A_0) - L_2(\lambda_0, A_0)). \quad (2.30)$$

Conversely, if there exists processes $\Lambda_{1,t}$, $\Lambda_{2,t}$, λ_t and J_t satisfying equations (3.39) - (3.43), with $\sigma_Y > 0$, the associated optimal consumption portfolio policies clear all markets.

The proposition above shows the conditions of the existence of the equilibrium and the pricing formula of the dividend claims. In the following three subsections, I show the impact of belief dispersion in asset prices, aggregate economy and social welfare.

A. Asset Prices

Equity premium literature (e.g. [Dumas, Kurshev, and Uppal \(2009\)](#)) shows the difference in beliefs and the sentiment risk explain the equity premium puzzle and the excess volatility puzzle. In this part, I show that the model in this chapter with an endogenous labor supply generates consistent but slightly different implications with the literature due to endogenous labor.

Proposition 5. *Suppose $\bar{J}_0$ is the representative agent's marginal valuation of the stock without different in beliefs, then*

$$J_0 < \bar{J}_0. \quad (2.31)$$

Proof: see Appendix [B](#).

Proposition 5 shows that the stock price without the sentiment risk is higher. The intuition is that the speculation due to the difference in beliefs and the sentiment risk increase the variation of the aggregate SDF and the stock price volatility, and thus increase the equity premium and lowers the stock price.³ This result is consistent with [Dumas, Kurshev, and Uppal \(2009\)](#).

Two types of agents know that they don't have the same estimation about the growth of technology. So two agents 'agree to disagree' on fundamentals. However, in the financial market they have to agree to the stock price and bond price. As they agree to the bond price, the subject risk-free rates for them are equal. However, they don't agree on the market price of risk as they have difference in beliefs of the expected return. The risk-free rate and the subjective price of risk for two types of agents are given in the following proposition.

Proposition 6. *The market prices of risk for two agents are*

$$\kappa_{1,t} = \sigma_{Y,t} + \nu_{2,t}\bar{\mu} \quad (2.32)$$

$$\kappa_{2,t} = \sigma_{Y,t} - \nu_{1,t}\bar{\mu} \quad (2.33)$$

The instantaneous risk-free rate is

$$r_t = \rho + \boldsymbol{\mu}_{Y,t} - \sigma_{Y,t}^2, \quad (2.34)$$

where $\boldsymbol{\mu}_{Y,t}$ is the aggregate belief of output growth rate defined by

$$\boldsymbol{\mu}_{Y,t} = \nu_{1,t}\mu_{Y,t}^1 + \nu_{2,t}\mu_{Y,t}^2; \quad (2.35)$$

$\mu_{Y,t}^i$ and $\sigma_{Y,t}$ are endogenous and given in the proof.

Proof: see Appendix [C](#).

Proposition 6 shows that the risk prices and the risk-free rate depend on the difference in beliefs and the consumption allocation $\nu_{i,t}$, consistent with [Basak \(2000\)](#) and [Bhamra and Uppal \(2014\)](#). The

³ As [Cochrane \(2017a\)](#) points out, the existence of the sentiment risk induces the time varying risk bearing capacity and thus increases the variation of the aggregate SDF.

equilibrium risk-free rate in our economy is endogenous, and it depends on the difference in beliefs as well as the consumption allocation, which is consistent with Basak (2000) in terms of expression. But this result is slightly different from them as the previous two papers are built on an endowment economy where the output (the dividend) process and hence its volatility component are exogenous. While in my model the aggregate output depends on the exogenous technology as well as the endogenous aggregate labor supply. In the appendix, I show that the output volatility is determined by the difference in beliefs as well as the consumption allocation.

With the price of risk, I can obtain the subjective equity premium and the true equity premium (only if the true probability measure is specified) by computing the stock volatility which is reported in the following proposition.

Proposition 7. *The volatility of the equity is given by*

$$\sigma_{J,t} = \sigma_Y + \left(\frac{\lambda_t}{1 + \lambda_t} - \frac{\lambda_t}{N_t} \frac{\partial N_t}{\partial \lambda_t} \right) \sigma_\lambda + \frac{\rho E_t^1 \left[\int_t^T e^{-\rho s} \mathcal{D}_t^1 \lambda_s ds \right]}{(e^{-\rho t} - e^{-\rho T}) (1 + \lambda_t)} \quad (2.36)$$

where $\mathcal{D}_t \lambda_s$ is the Malliavin derivative for λ_s and defined in the proof of this proposition.

Proposition 7 shows how the stock price reacts to technological shocks through changes in λ_t with beliefs dispersion. The first term of (3.58) measures the effect of a technological shock on the stock price through Y_t output. The second term shows the instantaneous impact of a technological shock on the SDF through λ_t , and illustrates the effect of difference in beliefs on the volatility through its impact on the SDF as well as the cash flow (the aggregate labor supply). The third term measures a path dependent the long-term effect of the technological shocks on the SDF in the future. The derivative $\mathcal{D}_t \lambda_s$ measures how perturbing $Z_{A,t}^1$ affects instantaneous change of λ_s for $s \geq t$. The second and third terms shows how difference in beliefs impacts the volatility. When beliefs are identical, the λ_t is constant, the second and the third terms vanish, and the stock price volatility reduces to the output volatility which is essentially equal to the technological volatility σ_A as the consumption allocation is constant due to identical preference setting.

B. Aggregate Output and Difference in Beliefs

Based on their Bayesian rule of updating, two types of households potentially have a different view towards the economic growth due to different variances of their estimation. Speculation caused by difference in beliefs distorts the portfolio-consumption-leisure choice. In this part I show how speculation impacts the aggregate economy by affecting the aggregate labor supply.

Proposition 8. *In equilibrium, the aggregate labor supply is a function of consumption allocation $\nu_{i,t}$, and it satisfies the following polynomial equation*

$$A_t^{\frac{\gamma-1}{\eta}} N_t^{\frac{\gamma}{\eta}} + \left(\sum_{i=1,2} \nu_{i,t}^{\frac{\gamma}{\eta}} \right)^{-1} \left(\frac{W_t}{A_t P_t} \right)^{\frac{1}{\eta}} N_t - \left(\sum_{i=1,2} \nu_{i,t}^{\frac{\gamma}{\eta}} \right)^{-1} \left(\frac{W_t}{A_t P_t} \right)^{\frac{1}{\eta}} \bar{N} = 0. \quad (2.37)$$

Proof: see Appendix D

Proposition 8 reveals that the aggregate labor supply is determined by the technology level and the consumption allocation. Equation (2.37) can be solved in closed form by the polynomial theory if $\frac{\gamma}{\eta} \in \{1, 2, 3, 4, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\}$, or solved in closed-form by using the hypergeometric function if $\frac{\gamma}{\eta} \in \mathbb{N}$. As $\frac{W_t}{A_t P_t} = 1 - \frac{1}{\epsilon}$ in equilibrium, the closed form solution is a function of technology level A_t and consumption allocation $\nu_{i,t}$.

First, the aggregate labor supply is a non-increasing (increasing) function of technology if $\gamma \geq 1$ ($\gamma < 1$). The changes in aggregate labor supply are characterized through classical income and substitution effect. The income effect is that technological progress improves household wealth, and so leads them to consume more and enjoy more leisure time. The substitution effect is that technology progress improves the price of labor relative to consumption goods ($\frac{W_t}{A_t P_t} = 1 - \frac{1}{\epsilon}$) and so encourages households to work more. The net effect of technology progress depends on the strength of income and substitution effect. If the income effect is stronger than the substitution effect $\gamma \geq 1$, then technology progress reduces the aggregate labor supply; and vice versa. But I will assume $\gamma \geq 1$, namely the dominance of the income effect over the substitution effect, as its result is more intuitive in the real world.

Second, the equilibrium labor supply is concave and symmetric in consumption share when $\frac{\gamma}{\eta} > 1$, and it reaches maximum when $\nu_{1,t} = 0.5$. Note that in a representative agent economy, say two types of households are identical, the consumption goods should be equally allocated, and aggregate labor supply is maximized. I will show later with a numerical example, that speculation introduces inefficiency in this economy, i.e. an output gap. The intuition is that speculation and the difference in beliefs distort the individual optimal choices and aggregate risk, making the consumption allocation deviate from the efficient level (equal allocation) given identical preferences among households. In addition, speculation between two types of households is reshaping the wealth distribution over time, affecting the consumption sharing and hence the aggregate output.

The parameter choice affects the function that links the consumption sharing and the aggregate labor supply. It's possible that the output is maximized when the consumption share is equal to 0 or 1 (the aggregate labor supply is a convex function of $\nu_{1,t}$) under a specific parameterization, which implies that the wealth inequality increases the aggregate output. While income and wealth inequality is well documented and argued to be inefficient for the economy or even a threat to the stability of a society (see (Piketty (2014))), the discussion of economic implication of income and inequality is beyond my research. However, we can interpret the increase in aggregate output induced by difference in beliefs and consequential consumption share inequality as positive output gap, or overheat in the economy.

C. Belief Neutral Welfare Analysis

I have so far analyzed the implications of difference in beliefs on asset prices and aggregate economy. In this part I evaluate its impact on the social welfare under the belief neutral welfare criterion. In the economy, the optimal choices of consumption and leisure under individual probability measure are affected as a result of the speculation motive and the sentiment risk. Whether the speculation is beneficial to social welfare becomes an important question. While it is attempting to evaluate social welfare under the

true probability measure, which however is unlikely to be known even for the social planner. Therefore we evaluate the speculation impact by a belief neutral criterion.

Brunnermeier, Simsek, and Xiong (2014) propose a welfare criterion for economies in which agents have heterogeneously distorted beliefs. This welfare criterion is defined by the following.

Definition 1. *Consider two social allocations, x and y . If the expected social welfare of allocation x dominates that of the allocation y for every reasonable probability measure $\mathbb{P}^h$, where $\mathbb{P}^h$ denotes the convex combination of the two agents' beliefs with weight $h > 0$ and $1 - h > 0$,*

$$W\left(E_0^h[V_1(x)], E_0^h[V_2(x)]\right) \geq W\left(E_0^h[V_1(y)], E_0^h[V_2(y)]\right) \quad (2.38)$$

with the inequality holding strictly for at least one reasonable measure, then allocation x is belief-neutral superior to allocation y .

As both types of agents are endowed with identical wealth, labor (time) and preference, we construct a simple social welfare function as a utilitarian welfare function, and it is as following:

$$W\left(E_t^h[\mathcal{V}_{1,t}], E_t^h[\mathcal{V}_{2,t}]\right) = E_t^h[\mathcal{V}_{1,t}] + E_t^h[\mathcal{V}_{2,t}], \quad (2.39)$$

where

$$\mathcal{V}_{i,t} = \int_t^T e^{-\rho(u-t)} U(C_{i,u}, L_{i,u}) du. \quad (2.40)$$

Without taking a stand on which belief is correct, the central planner can categorically determine that the speculation leads to an inefficient social outcome. The key is the existence of uniformly negative externality. In fact, in our model, the conflicting beliefs of the more confident agent and the less confident agent induce a form of externality. First, the difference in priors (or simply confidence level in presence of learning) induces a bias in consumption leisure choice in our model, resulting a deviation in aggregate labor supply and total output. Second, consumption and leisure become more volatile due to heterogeneous risk exposure in their investment. These externalities make the trading between two agents suboptimal for the social welfare even under any probability measure. Given the true probability measure is hardly to be known even for the central planner, a policy is preferred such that it mitigates these externalities and increases social welfare at every possible reasonable probability measure.

V. Numerical Results: Effects of Speculation

In this section I further analyze the effects of speculation and beliefs dispersion by setting parameter values and solving for the resulting equilibrium. This requires solving optimization problem of two types of agents simultaneously. I apply numerical methods, specifically Monte Carlo simulation. Conditional on an initial guess for λ_0 , we simulate the weighting process λ_t and output process to solve for the stock price. I iterate to find λ_0 satisfying the equilibrium conditions. The parameter choice can be found in the Appendix. I set the volatility of technological development to 2% to match the volatility of aggregate GDP growth. We can set the $g_0 = \hat{\mu}_{A,0}^1 - \hat{\mu}_{A,0}^2$ to be a slightly positive number, i.e. 0.01 to match the

equity volatility and risk-free rate. The disagreement process g_t is time varying due to learning under different confidence level, both of which will converge to zero after learning for a sufficiently long time. I fix the variance of Agent 2's prior $\bar{v}_{2,0}$ to be 0.01 and allow the variance of Agent 1's prior $\bar{v}_{1,0}$ varies. The difference between $\bar{v}_{1,0}$ and $\bar{v}_{2,0}$ characterizes the beliefs dispersion as both agents react differently to the perceived shock according to their variance of estimation. Appendix IX plots the impacts of speculation and difference beliefs on asset prices, aggregate output and social welfare. The following subsections illustrate those results in details.

A. Speculation and Asset Prices

The behaviour of equilibrium stock prices is consistent with Proposition 5, as seen in Figure 2.1. In presence of difference in confidence level, the difference in beliefs is persistent. Speculation on beliefs is also persistent over long time until both agents reach a consensus on their forecast. I plot the effect of speculation on asset prices in terms of P/D ratio and volatility. I fix $v_{0,2}$ at 0.01, and allow $v_{0,1}$ to vary, and the different between $v_{0,1}$ and $v_{0,2}$ characterizes the reaction to perceived shocks: if the difference is larger, the beliefs dispersion is greater and speculation is more severe. This figure clearly shows the beliefs dispersion and speculation reduce P/D ratio. When $v_{0,1} = v_{0,2}$ and households react to shocks in a similar manner and eliminate the existence of difference in beliefs quickly. As a result, the economy is very close to the representative agent economy. The P/D ratio is above 16.5, which is a similar result in a representative agent economy. As $v_{0,1}$ decreases and the potential dispersion increases, the P/D ratio is decreasing from above 16.5 to 13, which is economically significant.

The lower panel illustrates the impact of speculation on equity volatility. When $v_{0,1} = v_{0,2}$ (in a representative agent economy), σ_λ , the sum of the last two components is around to 0 in Equation 3.58 which shows three components on which the stock price volatility depends, and the equity volatility is simply around 2% which is equal to the component fundamental volatility σ_Y . The term $\frac{\lambda_0}{N_0} \frac{\partial N_0}{\partial \lambda_0}$ is around 0 as the consumption allocation is around 0.5 and therefore the sensitivity of labor supply to consumption allocation $\frac{\partial N_0}{\partial \lambda_0}$ is close to zero. In the calibration, I set $\sigma_\lambda = \frac{g_0}{\sigma_A} = 0.5$, but any new shocks reduces the beliefs dispersion in expectation if two types of households interpret the new information and update their beliefs similarly, which is the last term represents. As a result, the sum of the second term and the last term is around zero, and the equity volatility is close to the output volatility σ_Y . The output volatility is not necessarily the technological growth volatility σ_A ; however, if $\gamma = 1$ (in my parameterization), the aggregate supply is the function of consumption allocation only but not the technological level, the sensitivity of aggregate labor supply to consumption allocation around $\nu_{1,t} \approx 0.5$ is around 0, hence $\sigma_Y \approx \sigma_A$. The equity risk premium can be obtained by the product of the equity volatility and the true market price of risk if the true belief can be specified. However, as to which level of the true market price of risk (any positive level), the effect on the risk premium is similar to that on the volatility: speculation increases the equity premium.

Those numerical results confirm that the speculation due to beliefs dispersion increases the equity risk premium in a production economy where the output is endogenous. The equity premium puzzle literature, e.g. Dumas, Kurshev, and Uppal (2009), Buraschi and Jiltsov (2006), show that the difference in beliefs

can explain the high equity premium and excess volatility as well as the low risk-free rate without assuming agents have high risk aversion. Analogous to the literature, speculation due to beliefs dispersion also causes the sentiment risk. While in contrast, the output process in my model depends on the endogenous labor supply thus the existence of beliefs dispersion impacts not only the consumption sharing rule among households but also the aggregate labor supply, hence the marginal aggregate consumption and dividend growth. More details are analyzed in the next subsection.

B. Speculation and Social Welfare

Figure 2.2 shows the welfare implication of speculation due to belief dispersions. The lower two panels plot the individual welfare which is computed as individuals' expected utility under their own beliefs. In contrast to Posner and Weyl (2013b) who argue that those households wouldn't engage in speculation in the first place⁴, I show that rational people will engage in speculation without assuming risk seeking behaviour, confused parties or regulatory arbitrage. First, two agents in my model are fully Bayesian learners and fully rational. They hold different views of economic growth not because one of them (or both) is confused, but because they have different prior beliefs. Difference in priors rises for many reason, some times they simply observe a different time of history, e.g. young and old generations. One cannot easily judge a investor just because he holds a different view. Second, both agents are risk averse, the target of their speculative trading is to smooth their consumption growth and leisure. Third, no regulatory measures in my model. However, even without the three points mentioned above, a fully rational risk-averse investor would want to speculate as: (1) investors agree to disagree; they don't believe their trading counter party is right but they agree upon the difference; (2) the difference among investors is stochastic hence induces the sentiment risk; so speculation provides protection for themselves from the sentiment risk and helps smooth their consumption growth and leisure under their own subjective beliefs; (3) as the difference beliefs becomes greater, each agent thinks the rest of the market participants are making mistakes hence he can take better on them. Therefore, as we can see from Figure 2.2 in the bottom two pictures, the welfare (expected utility) of both Type 1 agent and Type 2 agent are increasing in the difference of $v_{0,1}$ and $v_{2,0}$ in their own probability measure, as (1) his speculative profit is increasing (2) he can hedge his consumption and leisure uncertainty.

However, on the other hand, for a social planner the speculation is not necessarily beneficial, as the top panel shows that the social welfare is decreasing in the difference of $v_{0,1}$ and $v_{2,0}$, opposite to the individual welfare effect. I measure the social welfare by a utilitarian function, namely, I assign the same weights for two agents and calculate the sum of expected individual utility. Identical weighting is reasonable as

⁴Posner and Weyl (2013b) write: "By contrast, when a person speculates, that person exposes herself to increased net risk without offsetting a risk faced by a counterparty: she merely gambles in hopes of gaining at the expense of her counterparty or her counterparty's regulator. Speculation is a zero-sum activity, which, in the aggregate, harms the people who engage in it, and which can also produce negative third-party effects by increasing systemic risk in the economy." Further, they write: "When two people bet over whether a coin will turn up heads, they each incur the risk that they will be poorer in the future, when, assuming that they are risk-averse, the gain will not be sufficient to outweigh the loss in terms of utility. Thus, rational people will not engage in speculation in the first place unless (1) they like to gamble (in which case there are cheaper ways, like casinos, to satisfy this preference), (2) at least one party is confused (which we believe is extremely common), or (3) they are engaging in regulatory arbitrage (which is also extremely common). Thus, there is no social gain from permitting speculation."

two agents have the same initial endowment and maximum labor supply. The top panel plots two lines: the blue line represents the social welfare evaluated under the Type 1 Agent's probability measure, and the red line represents the social welfare under the Type 2 Agent's probability measure. In the previous section I introduced the belief neutral social welfare criteria. These two lines draw the two boundaries of the social welfare under any possible reasonable probability measure, as the convex combination of the social welfare under two extreme beliefs simply lies between those two lines. In other words, the social welfare specified by the belief neutral criterion is decreasing in speculation.

There are three important points about this result. First, the welfare loss or externality induced by speculation is uniformly negative across the set of reasonable beliefs. This is because trades initiated by the speculation increase consumption volatility in the convex combination of the two agents' beliefs. Second, the criterion to evaluate the social welfare ignores agents' preferences under their own beliefs, hence it is naturally paternalistic. [Brunnermeier, Simsek, and Xiong \(2014\)](#) argue this criterion features a specific and discipline form of paternalism, as it is designed to detect inefficiencies based on disagreements between agents in a group setting rather than to analyse the inefficiencies induced by irrational behaviour of a individual agent. That said, I would not argue whether the speculation is efficient for individual agent (even speculation trade is optimal under each agent's belief). Third, I consider the set of reasonable beliefs as convex combinations of agents' beliefs. This is reasonable because (1) the convex set is large enough to include all the extreme beliefs of agents in the economy; (2) the set of convex combination is appropriate given the focus on detecting the inefficiencies due to belief dispersion as opposed to belief mistakes shared by all agents.

C. Aggregate Output

As the total output depends on the exogenous technology and the endogenous labor supply, the speculation impacts the aggregate economy through its effect on the individual consumption and leisure choice. The speculation driven by belief dispersion increases the aggregate risk (e.g. the sentiment risk), distorts households' portfolio choice and thus working choice. I plot the results in [Figure 2.3](#). In the upper panel I plot the closed form solution of the aggregate labor supply as a function of consumption allocation ν_1 , which is also shown in [Proposition 8](#). Given the identical individual preference setting, if there are no differences in beliefs, the endowment (wealth) allocation and the consumption allocation should be equal, and the aggregate labor should reach maximum. I refer the output when there is no belief dispersion as the *natural* level of output, and the deviation from the natural output level as *output gap*.

In the lower panel of [Figure 2.3](#) I show the speculation induced by dispersion in beliefs reduces the output, creating a negative output gap. As the difference between the variances of agents' estimation increases, the output gap decreases to around -0.7% , which is economic significant in comparison to the subjective technological growth (2% for Agent 1 and 1% for Agent 2) and to the 2% real GDP growth in U.S. and Europe.

VI. Conclusion

In this chapter I discuss the general equilibrium effects of the speculation driven by the dispersion in beliefs. I build a continuous time production based heterogeneous-agent asset pricing model and show that speculation induces sentiment risk and hence distorts the aggregate financial risks. By solving the model numerically, I show that speculation increases equity risk premium and excess volatility. In the presence of differences in beliefs, I show that speculation driven by difference in beliefs is Pareto optimal; however, it reduces the social welfare without taking a stand on which belief is correct. Speculation also has negative consequence on aggregate economy by distorting households' consumption working choice. It can reduce the aggregate output by around 0.7%, which is economic significant. Those effects on asset prices, social welfare and aggregate output rationalize regulation on the speculation. In Chapter 3, I will discuss the macro-prudential policy and the regulation of speculation.

VII. Appendix: Tables

Table 2.1. Chapter 2 Production Economy Calibration

Name	Symbol	Value
Volatility of technological development	σ_A	0.02
Time preference of both agents	ρ	0.02
Risk Aversion	γ	1
Curvature Parameter	η	0.25
Agent 1's expected technological growth rate	$\hat{\mu}_{A,0}^1$	0.02
Variance of Agent 1's prior	$\bar{v}_{1,0}$	Varies
Variance of Agent 2's prior	$\bar{v}_{2,0}$	0.01
Difference of Agent 2's belief and Agent 1's	$g_t = \hat{\mu}_{A,0}^1 - \hat{\mu}_{A,0}^2$	0.02

VIII. Appendix: Proofs

A. Proof of Proposition 4

The proof follows Karatzas, Lehoczky, and Shreve (1990) with the appropriate modifications taken to accommodate for investors facing different state prices. From clearing in the consumption good market, state prices follow from the representative agent construction. And the equivalent SDF using the marginal utility function of the planner is:

$$U_c(Y_t, \lambda_t) = \frac{\partial U_1}{\partial C_{1,t}} \frac{\partial C_{1,t}}{\partial Y_t} + \lambda_t \frac{\partial U_2}{\partial C_{2,t}} \frac{\partial C_{2,t}}{\partial Y_t} + \frac{\partial U_1}{\partial L_{1,t}} \frac{\partial L_{1,t}}{\partial Y_t} + \lambda_t \frac{\partial U_2}{\partial L_{2,t}} \frac{\partial L_{2,t}}{\partial Y_t} \quad (2.41)$$

$$= \frac{\partial U_1}{\partial C_{1,t}} + \frac{\partial U_1}{\partial L_{1,t}} \frac{\partial (L_{1,t} + L_{2,t})}{\partial Y_t} = \frac{\partial U_1}{\partial C_{1,t}} - \frac{1}{A_t} \frac{\partial U_1}{\partial L_{1,t}} = \frac{\partial U_1}{\partial C_{1,t}} - \frac{W_t}{A_t P_t} \frac{\partial U_1}{\partial C_{1,t}} \quad (2.42)$$

$$= \frac{1}{\epsilon} \frac{\partial U_1}{\partial C_{1,t}}. \quad (2.43)$$

Inserting the marginal utility yields the individual SDF. The Dividend flow in a monopolistic competition equilibrium is

$$D_t = Y_t - \frac{W_t}{P_t} N_t = Y_t \left(1 - \frac{W_t}{A_t P_t} \right) = \frac{1}{\epsilon} Y_t \quad (2.44)$$

Inserting into the fundamental pricing formula the stock price.

To show the converse, assume that there exists $\Lambda_{i,t}$, λ_t and J_t satisfying the pricing equation. Clearing in the consumption goods market follows from the solutions to the investors' consumption-portfolio problems. Clearing in bond market follows from the fact that the discount total wealth, $\Lambda_{2,t} (X_{1,t} + X_{2,t})$ where $X_{i,t}$ is the financial wealth of agent i , is a martingale since agents' optimal wealth process is a tradeable strategy, and they must agree on the prices of all tradeable strategies.

$$X_{1,t} + X_{2,t} = \frac{1}{\Lambda_{2,t}} E^1 \left[\int_t^T \Lambda_{2,s} \left(C_{1,s} - \frac{W_s}{P_s} N_{1,s} + C_{2,s} - \frac{W_s}{P_s} N_{2,s} \right) ds \right]. \quad (2.45)$$

Substitute clearing in the consumption goods market, clearing in the bond market results. Finally, clearing in the stock market follows by comparing the diffusion dynamics of $\Lambda_{2,t} J_t$ with $\Lambda_{2,t} (X_{1,t} + X_{2,t})$ using clearing in both consumption goods market and the bond market.

B. Proof of Proposition 5

Without difference in beliefs, the representative agent's marginal valuation for the stock is given by

$$\frac{J_0}{D_0} = E_t^1 \left[\int_0^T \frac{U_C(Y_s, \lambda_s)}{U_C(Y_0, \lambda_0)} \frac{D_s}{D_0} ds \right] = E_t^1 \left[\int_0^T e^{-\rho s} \frac{1 + \lambda_{1,s}}{1 + \lambda_{1,0}} ds \right] \leq E_t^1 \left[\int_0^T e^{-\rho s} ds \right] = \frac{\bar{J}_0}{D_0}. \quad (2.46)$$

The inequality is due to the fact that λ_t is a positive martingale and therefore a decreasing process for which the long run decreasing rate is $(\hat{\mu}_{A,t}^1 - \hat{\mu}_{A,t}^2)^2$.

C. Proof of Proposition 6

Apply Ito's lemma to each agent's first order conditions

$$e^{-\rho t} \frac{\partial U_i}{\partial C_{i,t}} = y_i \Lambda_{i,t} \quad (2.47)$$

and match the diffusion term,

$$-\frac{\frac{\partial^2 U_i}{\partial C_{i,t}^2}}{\frac{\partial U_i}{\partial C_{i,t}}} \sigma_{C_i} = \kappa_{i,t} \quad (2.48)$$

where $C_{i,t}$ satisfies

$$dC_{i,t} = \mu_{C_i} dt + \sigma_{C_i} dZ_{A,t}^i. \quad (2.49)$$

Market clearing condition in consumption goods implies that

$$\sigma_{C_1} + \sigma_{C_2} = \sigma_{Y,t} Y_t, \quad (2.50)$$

which implies (3.54) and (3.55).

The risk free rate follows from applying Ito's lemma to the state-price density for each agent, changing to the true probability measure, matching the deterministic terms and rearranging. Risk premium can be calculated using the consumption share weighted risk prices times the stock volatility. And the $\mu_{Y,t}^1$ and $\mu_{Y,t}^2$ comes from applying Ito's lemma to $Y_t = A_t N_t$ with respect to agent 1 and agent 2's probability measure.

D. Speculation and Output Dynamics

The output depends on the technological development as well as the aggregate labor supply. The individual optimization yields the optimal consumption and leisure choice:

$$C_{i,t}^\gamma = \frac{e^{-\rho t}}{y_i \Lambda_{i,t}} \quad (2.51)$$

$$L_{i,t}^\eta = \frac{e^{-\rho t}}{y_i \Lambda_{i,t} \frac{W_t}{P_t}} \quad (2.52)$$

Consumption goods market clearing condition gives:

$$\bar{N} - \sum_{i=1,2} N_{i,t} = \left(\frac{W_t}{P_t} \right)^{-\frac{1}{\eta}} \sum_{i=1,2} C_{i,t}^{\frac{\gamma}{\eta}} \quad (2.53)$$

$$\bar{N} - N_t = \left(\frac{W_t}{P_t} \right)^{-\frac{1}{\eta}} Y_t^{\frac{\gamma}{\eta}} \sum_{i=1,2} \nu_{i,t}^{\frac{\gamma}{\eta}} = \left(\frac{W_t}{A_t P_t} \right)^{-\frac{1}{\eta}} A_t^{\frac{\gamma-1}{\eta}} N_t^{\frac{\gamma}{\eta}} \sum_{i=1,2} \nu_{i,t}^{\frac{\gamma}{\eta}}. \quad (2.54)$$

By Ito's lemma, the diffusion of Y_t follows

$$\sigma_Y = \sigma_A - \frac{\lambda_t}{N_t} \sigma_\lambda \frac{\partial N_t}{\partial \lambda_t} = \sigma_A - \frac{\lambda_t}{N_t} \bar{\mu}_t \frac{\partial N_t}{\partial \lambda_t}. \quad (2.55)$$

Assume that an equilibrium exists. The SDFs for the two agents are

$$\Lambda_{1,t} = e^{-\rho t} \frac{U_c(Y_t, \lambda_t)}{U_c(Y_0, \lambda_0)}, \quad \Lambda_{2,t} = \frac{\lambda_t}{\lambda_0} \Lambda_{1,t}, \quad (2.56)$$

where $U_c(Y_t, \lambda_t)$ is the marginal utility of a social planner which is defined as

$$U(Y_t, \lambda_t) = \max_{C_{1,t} + C_{2,t} \leq A_t, (\bar{N} - L_{1,t} - L_{2,t}) = Y_t} \log C_{1,t} + \frac{L_{1,t}^{1-\eta}}{1-\eta} + \lambda_t \left(\log C_{2,t} + \frac{L_{2,t}^{1-\eta}}{1-\eta} \right) \quad (2.57)$$

The stock price J_t is computed from the unconstrained agent's probability measure

$$J_t = E_t^1 \left[\int_t^T e^{-\rho s} \frac{U_c(Y_s, X_s, \lambda_s)}{U_c(Y_0, X_0, \lambda_0)} D_s ds \right]. \quad (2.58)$$

E. Proof of Proposition 7

We use the following lemma to prove Proposition Volatility.

Lemma 2 (Clark-Ocone Theorem - [Nualart \(2006\)](#)). *Let $F \in D^{1,2}$, where the space $D^{1,2}$ is the closure of the class of smooth random variables S with respect to the norm*

$$\|F\|_{1,2} = \left[E[|F|^2] + E\left(\|\mathcal{D}F\|_{L^2(T)}^2\right) \right]^{1/2}. \quad (2.59)$$

Suppose that W_t is a multi-dimensional Brownian Motion. Then

$$F = E[F] + \int_0^T E(\mathcal{D}_s F | \mathcal{F}_s) dW_s. \quad (2.60)$$

Taking conditional expectations,

$$E[F | \mathcal{F}_t] = E[F] + \int_0^t E(\mathcal{D}_s F | \mathcal{F}_s) dW_s. \quad (2.61)$$

Proof of Lemma 2. See [Nualart \(2006\)](#).

Next we consider the equity pricing equation respect to probability measure of the Type 1 Agent:

$$J_t = \frac{1}{U_C(C_t, \lambda_t)} E^1 \left[\int_s^T e^{-\rho s} U_C(C_t, \lambda_t) \frac{Y_s}{\epsilon} ds | \mathcal{F}_t \right]. \quad (2.62)$$

Define the $\mathcal{L}^2(\mathcal{P}^1)$ martingale M_t

$$M_t = E_t^1 \left[\int_t^T e^{-\rho s} U_C(C_s, \lambda_s) Y_s ds \right]. \quad (2.63)$$

By the martingale representation theorem, there exist a process $\phi(t)$ such that $E_t^1 \left[\int_t^T |\phi(t)|^2 ds \right] < \infty$ and $M_t = M_0 + \int_0^T \phi(t) dZ_{A,t}^1$.

The value of the stock can be rewritten as

$$J_t = \frac{1}{\epsilon U_C(C_t, \lambda_t)} \left[M_t - \int_0^t e^{-\rho s} U_C(C_s, \lambda_s) Y_s ds \right] \quad (2.64)$$

Applying Ito's lemma to equation above and comparing the diffusion term earns the volatility

$$\sigma_{J,t} = -\frac{U_{CC}(C_t, \lambda_t)}{U_C(C_t, \lambda_t)} Y_t \sigma_Y + \frac{U_{C\lambda}(C_t, \lambda_t)}{U_C(C_t, \lambda_t)} \lambda_t \sigma_\lambda + \frac{1}{E_t^1 \left[\int_t^T e^{-\rho s} U_C(C_s, \lambda_s) Y_s ds \right]} \phi(t) \quad (2.65)$$

$$= \sigma_Y + \left(\frac{\lambda_t}{1 + \lambda_t} - \frac{\lambda_t}{N_t} \frac{\partial N_t}{\partial \lambda_t} \right) \sigma_\lambda + \frac{1}{E_t^1 \left[\int_t^T e^{-\rho s} (1 + \lambda_s) ds \right]} \phi(t), \quad (2.66)$$

in which $\{\lambda_t\}$ is a martingale under Type 1 agent's probability measure.

To complete the proof we need to use Malliavin derivatives to calculate the the response of M_t to fundamental shocks. By Lemma 2, the diffusion term of dynamics of M_t is

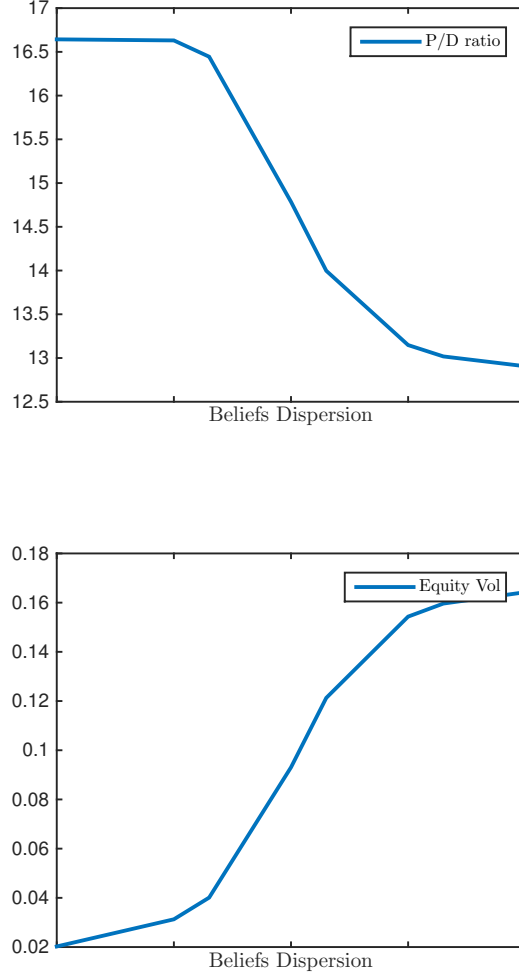
$$\phi(t) = E_t^1 \left[\int_t^T e^{-\rho s} \mathcal{D}_t^1 \lambda_s ds \right] \quad (2.67)$$

where the Malliavin derivatives are

$$\frac{\mathcal{D}_t^1 \lambda_s}{\lambda_s} = -\sigma_\lambda(t) - \int_t^s \mathcal{D}_t^1 \bar{\mu}(z) [\sigma_\lambda(z) dz + dZ_{A,z}^1] \quad (2.68)$$

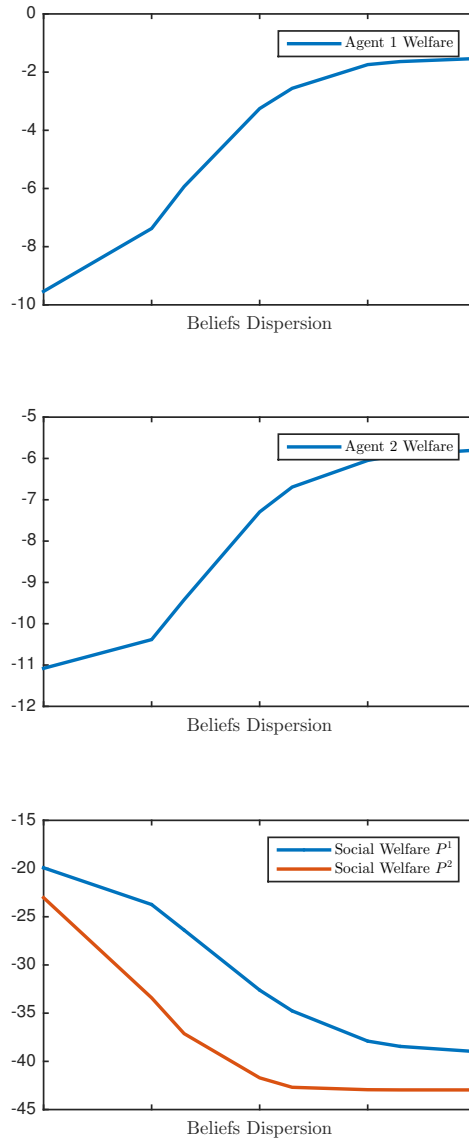
$$\mathcal{D}_t^1 \bar{\mu}(z) = e^{-\int_t^z \frac{\bar{v}_u^2}{\sigma_A^2} du} \frac{1}{\sigma_A^2} (\bar{v}_t^1 - \bar{v}_t^2) \quad (2.69)$$

IX. Appendix: Figures



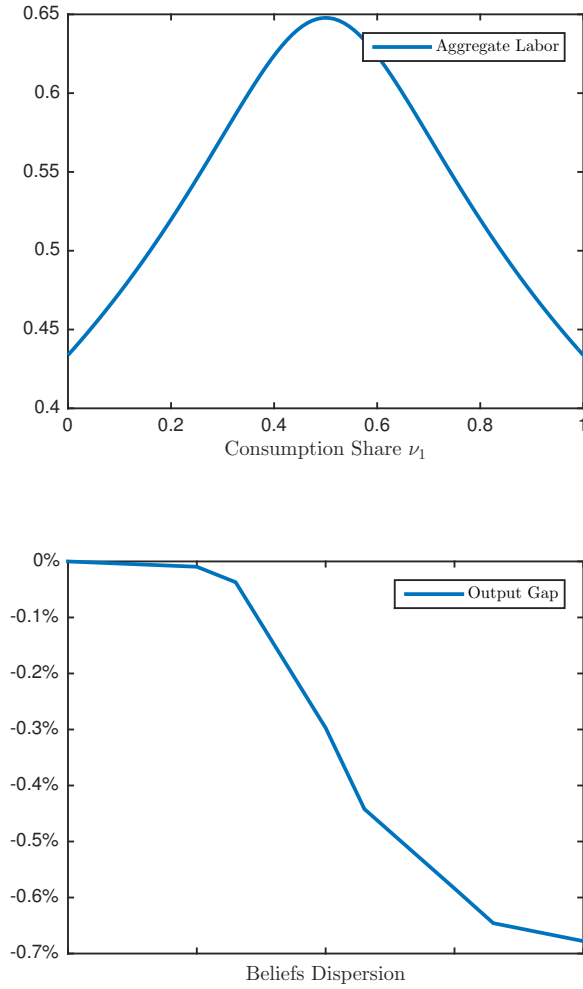
This figure plots Price-dividend ratio and volatility of equity, and output level against different confidence level of Type 1 households. ($v_{0,1}$ is smaller, agent is more confidence, and the difference in beliefs is potentially larger). We set $\eta = 0.5$ and $g_t = \hat{\mu}_{A,t}^1 - \hat{\mu}_{A,t}^2 = 0.01$. The upper and lower panels show that speculation reduces the asset prices and increases the equity volatility.

Figure 2.1. Effects of Speculation on Asset Prices



This figure plots speculators' welfare under their subjective beliefs and the social welfare under two beliefs against different confidence level of Type 1 households. The upper panel plots the social welfare is decreasing speculation as the convex combination of the blue line and the red line, which represent the social welfare under two extreme boundary beliefs, is decreasing.

Figure 2.2. Effects of Speculation on Individual Welfare and Social Welfare



This figure summarizes the effect of speculation over the aggregate economy. The upper panel plots the closed form solution of aggregate labor supply against the consumption allocation. The aggregate output reaches maximum when the consumption allocation between two types of households is identical in absence of speculation. The lower panel shows that speculation makes the consumption allocation deviate from the identical allocation, hence reducing the aggregate labor supply and output.

Figure 2.3. Effects of Speculation on Aggregate Labor Supply

Chapter 3

Macro-prudential Policy, Differences in Beliefs and Growth

I. Introduction

As the ultimate asset holders, households process new information and decide on their trades using competing models of economic fundamentals through financial intermediaries, e.g. commercial banks, hedge funds, ETFs, etc. Some of those households do not always attempt to share risks with others; rather, in periods of large disagreement of views they take bets on the relative accuracy of their models' predictions, namely speculate. However, speculative trading can be a negative sum game because it may cause financial imbalances ([Scheinkman and Xiong \(2003\)](#)) or trigger financial crisis (e.g. 1997 Asian financial crises).

In the previous chapter I show that speculation due to financial market participants “agree to disagree” generates negative externalities for the aggregate output and the labor market? In this chapter I explore the general equilibrium effects of macro-prudential policy to restrict speculation. I address the following questions. While macro-prudential policy can limit speculation and dampen procyclical leverage cycle by offsetting speculative motive due to differences in beliefs, would tighter regulation result a more stable financial market? What will be the impact of this regulation on the spillover effects on the labor market? Does tighter regulation improve the social welfare when policy makers are not sure which is the true belief?

To illustrate the necessity of macro-prudential policy, I show an example why speculation due to the difference in beliefs can cause negative externalities. For this reason I use following example from [Brunnermeier, Simsek, and Xiong \(2014\)](#)¹. Consider a bet between Joe and Bob who argue over the filling material of a pillow. Joe thinks the filling is natural down with probability 0.9 while Bob thinks it is artificial down with the same probability. They decide to bet by cutting this pillow: if the filling is natural down, then Bob gives Joe \$100; if it is artificial down, Joe gives Bob \$100; the winner buys a new pillow with a cost of \$50². Clearly, both Joe and Bob will choose to enter the bet as their expectation of profit is $0.9 \times (\$100 - \$50) + 0.1 \times (-\$100) = \35 after the replacement cost for the destroyed pillow. This bet is desirable from either Bob or Joe's perspective, hence the betting is Pareto optimal. However, the bet generates only a wealth transfer and a destroyed pillow. Of course this bet may be taken just for entertainment purposes, but a more realistic trigger is that each bettor thinks he would win and the other side would lose. If this bet is triggered by the beliefs distortion, obviously it is a negative sum game, no

¹The original example is from p. 193 in [Kreps \(2012\)](#).

²The replacement cost may be state dependent, e.g. \$50 for a natural down pillow and \$20 for a artificial down pillow.

matter which probability measure the central planner takes to evaluate social welfare. The welfare loss, or the externality caused by the bet, is the destroyed pillow.

Like the pillow betting example, speculation can be a negative sum game in financial markets. In addition, what those speculators disagree on should be revealed to the public at some point in the future through some policy announcement or other news, and the social planner is aware of this information. Therefore policy makers may help investors to approach this even with more wealth (see [Chabakauri \(2015\)](#)) by setting regulatory measures. Moreover, the recent financial crisis and its consequential damaging real effects have generated a broad agreement among academics and policy makers that financial regulation needs to acquire a macro-prudential dimension, e.g. a leverage constraint, which ultimately aims to prevent financial risks from spreading across the financial system and the real sector ([Hanson, Kashyap, and Stein \(2011\)](#), [Borio and Shim \(2007\)](#) and [Farhi and Werning \(2013\)](#)).

By developing a dynamic general equilibrium asset pricing model in a production economy with endogenous aggregate labor supply, I explore the negative consequences of speculation due to belief dispersion and the general equilibrium effects of macro-prudential policy that restricts speculative trading. More specifically, I assume that the expected technological growth is latent to households (agents), who have to rely on heuristics to form their priors and observe a public signal to update their beliefs. Even if almost inevitable biases (e.g. heuristic simplification, see [Hirshleifer \(2001\)](#)) and consequential belief heterogeneity can cancel out across agents, and even if the difference in beliefs can converge to zero in the long run as agents update their estimation in a Bayesian manner, speculation triggered by belief dispersion would cause negative consequences when agents agree to disagree. Analogously to [Drechsler, Savov, and Schnabl \(2014\)](#), one can think of the optimistic agents as pooling their wealth into the net worth (equity capital) of financial institutions, or banks for short.

In equilibrium, banks take levered positions in risky assets by borrowing from the pessimistic agents using short-term risk-free claims, which we think of as taking deposits. My view of banks as levered risk-takers is purposely simplified, abstracting from other functions such as screening and monitoring in order to focus on risk taking and speculation. It has the advantage of accommodating a diverse set of financial institutions, including commercial banks, broker-dealers, and hedge funds, whose unifying characteristic is that they take leverage using short-term debt. Procyclical leverage and speculation due to belief dispersion induce greater endogenous wealth fluctuations, increasing the excess stock price volatility and the equity risk premium and making individual consumption and leisure more volatile. Moreover, expected speculative profit in subjective beliefs makes agents work less and hence distorts the aggregate labor supply and the aggregate output. These negative externalities caused by speculation are not internalized by households.

To dampen the procyclical leverage and mitigate the negative externalities which are caused by speculation due to belief heterogeneity, central banks can regulate risk taking by restricting short term borrowing. In this paper, I consider the leverage constraint as the representative constraint of various macro-prudential policy tools.³ I explore the effects of macro-prudential policy on asset prices, output

³Those intermediaries face various regulations or constraints, among which some common market based instruments are

$$\text{Leverage Ratio (LR)} = \frac{\text{Total exposure}}{\text{Equity}},$$

level and social welfare by comparing the outcomes with different tightness of leverage constraint.

First, macro-prudential policy reduces the stock price volatility and the equity risk premium, therefore it raises stock prices in terms of the price dividend ratio. In the presence of belief heterogeneity and learning, agents can freely speculate against each other without any constraints, and they are fully exposed to the speculative risk and the leverage amplifies the fundamental shocks. Thus the equity premium and the stock price volatility is higher compared to the case where there is no belief heterogeneity (similar to [Dumas, Kurshev, and Uppal \(2009\)](#) and [David \(2008\)](#)). By restricting procyclical leverage, macro-prudential policy reduces the risk in endogenous wealth fluctuation, and therefore reduces the equity premium and stock price volatility, boosting up asset prices in terms of the price dividend ratio.

Second, macro-prudential policy helps to increase the aggregate output level. In this heterogeneous-agent production economy, speculation distorts households' consumption and leisure choices and hence the aggregate labor supply and the output level. macro-prudential policy, in contrast, enhances the output level in the following two ways. First, by restricting potential extreme portfolio choices, macro-prudential policy offsets the distortions in individual consumption and leisure choice. Second, by setting a portfolio constraint, macro-prudential policy prevents the wealth inequality among households by mitigating their risk exposures in financial markets, achieving balanced consumption and leisure allocations and a more stable economic growth path.

Is tighter policy better for the aggregate output? The model in [Buss, Dumas, Uppal, and Vilkov \(2016\)](#) predicts a monotonic relationship among economic growth, asset prices and policy tightness, as in their AK economy higher marginal return on capital expenditures implies higher output growth. However, this relationship is not monotonic in my model: over-tight policy impairs the aggregate output level. Intuitively, over-tight policy may impair households' intertemporal consumption-leisure-portfolio choice, creating aggregate labor supply distortions.

How about social welfare? Individual and social welfare are difficult to evaluate given that the objective belief may differ from agents subjective beliefs. It is tempting to evaluate macro-prudential policy under the true probability measure, but the true measure is often extremely hard to be known even by the central planner. If a policy maker is not sure about the true belief, it becomes difficult to evaluate a policy's welfare implications, as this policy is likely to be good for social welfare under the probability measure of one household type but bad under the others. Evaluating social welfare implications with heterogeneous beliefs is therefore controversial in the literature⁴ and devising a new methodology to address this interesting topic is beyond the scope of my research. To make progress, I therefore use the belief neutral welfare

$$\text{Haircut} = \frac{\text{Assets} \times \text{Price} - \text{Debt}}{\text{Assets} \times \text{Price}} = \frac{\text{Equity}}{\text{Assets} \times \text{Price}},$$

$$\text{Loan to Value (LTV)} = \frac{\text{Debt}}{\text{Assets} \times \text{Price}}.$$

Even though commercial banks, hedge funds, shadow banks and mortgagors have different types of constraints, comparing the different yardsticks for leverage, we find $\frac{1}{\text{LR}} = \text{Haircut} = 1 - \text{LTV}$ and we observe a wide range of LTV ratios across the financial system from 99 percent for special purpose vehicles to 80 percent for mortgages ([Schoenmaker and Wiertz \(2015\)](#)). No matter what the cross-sectional difference in ratios requirement, macro-prudential policy tools share the same target to regulate speculation and stabilize financial markets.

⁴Those studies include [Blume, Cogley, Easley, Sargent, Tsyrennikov, et al. \(2014\)](#), [Brunnermeier, Simsek, and Xiong \(2014\)](#) and [Gilboa, Samuelson, and Schmeidler \(2014\)](#).

criterion specified in [Brunnermeier, Simsek, and Xiong \(2014\)](#), as in the previous chapter. This criterion, instead of taking a stand on which (whose) belief is correct, asserts that an allocation is belief-neutral efficient (inefficient) if it is efficient (inefficient) under any every possible reasonable probability measure.

According to the belief neutral welfare criterion, I show that the resource allocation under the macro-prudential policy that constrains speculation is belief neutral superior to the allocation without the policy. In other words, macro-prudential policy increases social welfare under any possible reasonable probability measure. Intuitively the negative externalities caused by speculation, e.g. the aggregate labor supply and output distortions, and greater consumption growth and leisure volatility, are mitigated by macro-prudential policy.

The rest of the chapter is organized as follows: Section II reviews the literature, Section III lays out the baseline model, Section IV characterizes the equilibrium, Section V presents a numerical example of the impact of macro-prudential policy on asset prices, social welfare and the aggregate economy. Various cases are discussed in Section VI and I conclude in Section VII.

II. Related Literature

This research is closest to the literature that studies risk sharing, speculation and savings decisions of investors in presence of heterogeneous beliefs. [Blume, Cogley, Easley, Sargent, Tsyrennikov, et al. \(2014\)](#) study an exchange economy and show that, the welfare might be higher in an incomplete market than a complete market if measured under a set of objective beliefs instead of the investor's beliefs. [Heyerdahl-Larsen and Walden \(2014\)](#) study hedging and speculation in a static model and show that the competitive equilibrium is always inefficient under their own efficiency criterion. [Buss, Dumas, Uppal, and Vilkov \(2016\)](#) also study the AK production economy in discrete time and show similar results. this research contributes to this literature by endogenizing the aggregate labor supply and studying the real effects of macro-prudential policy and speculation due to belief heterogeneity.

'Pecuniary externality' is one of the main existing theoretical justification for macro-prudential policy. In the pecuniary externality literature, it simply rises due to the market incompleteness introduced to the Arrow-Debreu construct. The relative price change which is induced by the redistribution of asset holding given incomplete asset market affects the spanning properties of the limited existing set of assets. This pecuniary externality is not internalized by competitive agents, resulting a constrained inefficient equilibrium. The planner can improve the equilibrium outcome by intervening in the financial markets (see e.g. [Stiglitz \(1982\)](#), [Geanakoplos and Polemarchakis \(1985\)](#) [Geanakoplos, Magill, Quinzii, and Dreze \(1990\)](#)). [Farhi and Werning \(2013\)](#) consider the nominal rigidities friction and zero lower bound of monetary policy. Instead of pecuniary externalities, they emphasize aggregate demand externalities. [Elenev, Landvoigt, and Van Nieuwerburgh \(2016\)](#) consider the leverage constraints of financial intermediary as the main friction, and evaluate the asset prices and output with the implicit support from governments. I consider the behavioural heterogeneity and stresses on suboptimal individual choice and aggregate supply externalities.

This research is also related to the financial sector modeling of the macro economy. Investors who have conflict beliefs should borrow (lend) from (to) the other side through financial intermediaries, which is

not modelled explicitly in my economy in order to focus on behaviour bias rather than the financial sector friction, but in fact those financial intermediaries are the entities regulated by macroprudential policy in the real world. I show in the result that macroprudential policy improves the social welfare without the frictions in financial intermediaries, while incorporating the financial sector should strengthen my results and is of great interest. Earlier works before financial crisis emphasis ‘financial accelerator’ and the balance sheet channel of monetary policy. Those works include [Bernanke and Gertler \(1989\)](#), [Bernanke and Gertler \(1995\)](#), [Gertler and Karadi \(2011\)](#), [Bernanke, Gertler, and Gilchrist \(1996\)](#), [Bernanke, Gertler, and Gilchrist \(1999\)](#), and many others. More recent research allows a richer setting of financial sector. [He and Krishnamurthy \(2013\)](#) study the asset pricing effect when the marginal investor is a financial intermediary who is facing an equity capital constraint, which is similar to us. [Brunnermeier and Sannikov \(2014\)](#) study the persistent endogenous risk driven by asset liquidity and its non-linear implication effects and show that the economy is prone to instability and occasionally enters volatile crisis episodes. [He and Krishnamurthy \(2014\)](#) consider systemic risk arises when shocks lead to states where a disruption in financial intermediation adversely affects the economy and feeds back into further disrupting financial intermediation. [Drechsler, Savov, and Schnabl \(2014\)](#) consider a dynamic heterogeneous agents asset pricing model in which monetary policy affects the risk premium component of the cost of capital. The effect of policy shocks are exemplified via bank balance sheet effect.

On the methodological side, this research draws from the heterogeneous agent asset pricing literature. I model an economy where two types of agents with different priors, which gives rise to a credit market as in [Scheinkman and Xiong \(2003\)](#), [Dumas, Kurshev, and Uppal \(2009\)](#) and [Xiong and Yan \(2010\)](#). [David \(2008\)](#) resolves the equity premium puzzle by exploring the speculation on preciser estimation. Our model is also related the literature on collateral or margin constraints and their effect on asset prices (see [Gromb and Vayanos \(2002\)](#), [Brunnermeier and Pedersen \(2009\)](#), [Gorton and Ordoñez \(2014\)](#), [Moreira and Savov \(2014\)](#)). [Basak and Cuoco \(1998\)](#) solve the asset pricing implication with limited stock market participation, and [Gallmeyer and Hollifield \(2008\)](#) also examine asset prices in a heterogeneous-agent model with a short constraint.

III. Model

I model a finite-horizon continuous time production economy in which a competitive final goods producer aggregates a continuum of intermediate inputs

$$Y_t = \left(\int_0^1 Y_{j,t}^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}, \quad (3.1)$$

where the intermediate goods $Y_{j,t}$ are produced by a continuum of producers who face monopolistic competition. Firm j employs $n_{j,t}$ units of labor and technology A_t to produce

$$Y_{j,t} = A_t n_{j,t}. \quad (3.2)$$

I assume the technology level is exogenous, and its dynamics follow

$$\frac{dA_t}{A_t} = \mu_A dt + \sigma_A dZ_{A,t}, \quad (3.3)$$

where μ_A and σ_A are constant, and $Z_{A,t}$ is a standard Brownian motion.

A. Learning and Difference in Beliefs

I borrow the production economy I develop in the previous chapter, but I shall restate the key assumptions in the following. This finite horizon production economy is populated by a continuum of households who don't observe the expected technological growth rate μ_A and therefore have to use public signals to estimate its true value. I assume at time 0, households (agents) have to form an initial guess by heuristics. Such heuristics are effective when applied to appropriate problems. But their inevitable biases can become flagrant when used outside their ideal domain of applicability. Due to heuristic behavior bias,⁵ agents obtain different priors and they know each other's priors, that is, they agree to disagree.⁶ Without loss of generality, I assume there are two agents who have different Gaussian priors,⁷ i.e. for agent i , $\mu_{A,t}^i \sim N(\hat{\mu}_{A,t}^i, v_t^i)$. Their forecast $\hat{\mu}_{A,t}^i = E^i[\mu_A^i | \mathcal{F}_t^A]$ follows

$$d\hat{\mu}_{A,t}^i = \frac{\bar{v}_i}{\sigma_A} dZ_{A,t}^i, \quad (3.4)$$

where

$$dZ_{A,t}^i = \frac{1}{\sigma_A} \left(\frac{dA_t}{A_t} - \hat{\mu}_{A,t}^i dt \right) \quad (3.5)$$

is the shocks agent i perceives and $\bar{v}_{i,t}$ is the variance of the estimation

$$\bar{v}_t^i = \frac{\bar{v}_0^i \sigma_A^2}{\bar{v}_0^i t + \sigma_A^2}. \quad (3.6)$$

Without loss of generality, I assume $v_0^1 \leq v_0^2$, therefore the Type 2 agent reacts more to perceived shocks, and the difference in beliefs is stochastic. This assumption also makes the aggregate leverage pro-cyclical as the Type 2 agent is too optimistic in good times but too pessimistic in bad times.

I define their subjective probability measure $\mathbb{P}^i$ against the objective probability measure $\mathbb{P}$ by the

⁵Hirshleifer (2001) provides a detailed survey of investor psychology and the impact of behavior bias on asset prices.

⁶This assumption provides a simple model of trade based on belief dispersion. To the extent that prices do not fully reveal information, my model can be viewed as a reduced form for a setting in which belief disagreements are driven by differences in information rather than priors.

⁷A related literature concerns the plausibility of heterogeneous prior beliefs in financial markets. Morris (1996) argues that heterogeneous priors should be selectively incorporated into economic analysis. Another concern is whether belief disagreement would disappear in view of trader's Bayesian learning. Acemoglu, Chernozhukov, and Wold (2016) argue that, the differences in beliefs with Bayesian updating might fail to converge to zero even in the long run. But by the assumption in this paper, the difference in beliefs will converge to zero in the long run, but the belief dispersion in the finite horizon is sufficient to generate significant results.

following exponential martingale

$$\frac{d\mathbb{P}^i}{d\mathbb{P}} = \xi_{t,i}, \quad (3.7)$$

where

$$\xi_{t,i} = \xi_{t,0} \exp \left[\int_0^t -\sigma_{\xi,i}^2 ds + \int_0^t \sigma_{\xi,i} dZ_{A,t} \right] \quad (3.8)$$

$$\sigma_{\xi,i} = \frac{\hat{\mu}_{A,t}^i - \mu_{A,t}}{\sigma_A}. \quad (3.9)$$

I can quantify the level of disagreement by the process ξ_t , which is defined as:

$$\xi_t = \frac{\xi_{t,2}}{\xi_{t,1}}, \quad (3.10)$$

and hence

$$\frac{d\xi_t}{\xi_t} = \mu_\xi dt + \sigma_\xi dZ_{A,t}, \quad (3.11)$$

where

$$\mu_\xi = -\sigma_{\xi,1} (\sigma_{\xi,2} - \sigma_{\xi,1}) \quad (3.12)$$

$$\sigma_\xi = \sigma_{\xi,2} - \sigma_{\xi,1}. \quad (3.13)$$

Analogously to [Drechsler, Savov, and Schnabl \(2014\)](#), we can think of the optimistic agent as pooling her wealth into the net worth (equity capital) of financial institutions, or banks for short. In equilibrium, banks take levered positions in risky assets by borrowing from the pessimistic agent using short-term risk-free claims, which we think of as taking deposits. My view of banks as levered risk-takers is purposely simplified, abstracting from other functions such as screening and monitoring in order to focus on risk taking and speculation. It has the advantage of accommodating a diverse set of financial institutions, including commercial banks, broker-dealers, and hedge funds, whose unifying characteristic is that they take leverage using short-term debt. As the difference in beliefs is persistent and stochastic in this economy, the level of disagreement and its volatility are stochastic, constituting the source of the speculative risk⁸.

B. Preferences and Optimization with Policy Constraint

Instantaneous utility of type i representative agent is given by the following function that depends on the consumption $C_{i,t}$ and leisure $L_{i,t}$:

$$U(C_{i,t}, L_{i,t}) \equiv \log C_{i,t} + \frac{L_{i,t}^{1-\eta}}{1-\eta} \quad (3.14)$$

⁸[Dumas, Kurshev, and Uppal \(2009\)](#) term the fluctuation in market sentiment as the “sentiment risk”, which is similar to the speculative risk in this chapter, as the speculation is driven by agents’ market sentiment.

where η catches the smoothing motive for leisure component and takes value of 1 in our baseline model to match the standard log preference in the real business cycle literature (see [Ljungqvist and Sargent \(2004\)](#)). The leisure is defined as the total hours minus the working hours: $L_{i,t} \equiv \bar{N}_i - N_{i,t}$, and the aggregate total hours is $\bar{N} = \bar{N}_1 + \bar{N}_2$, in which we assume $\bar{N}_i = \beta_i \bar{N}$ and $\beta_1 = \beta_2 = \frac{1}{2}$.

In this economy, there are no frictions other than beliefs heterogeneity and policy intervention, and we assume that two types of agents shall invest their financial wealth into two securities: a real bond and a claim to dividend flow (equity). Both agents optimize the consumption, leisure and portfolio to maximize their expected utility, which is given by

$$V_{i,t} = \max_{C_{i,u}, N_{i,u}, \theta_{i,u} \in \Theta} E_t^i \left[\int_t^\infty e^{-\rho(u-t)} U(C_{i,u}, L_{i,u}) du \right] \quad (3.15)$$

subject to a dynamic budge constraint

$$dG_{i,t} = r_t G_{i,t} dt + \theta_{i,t} (dJ_t - r_t J_t + D_t dt) + \frac{W_t}{P_t} (\bar{N}_i - L_{i,t}) dt - C_{i,t} dt, \quad (3.16)$$

and the dynamics of consumption share

$$\frac{d\nu_{1,t}}{\nu_{1,t}} = \mu_{\nu_1} dt + \sigma_{\nu_1} dZ_{A,t}^i, \quad (3.17)$$

where $\nu_{1,t} = \frac{C_{1,t}}{Y_t}$, ρ is the constant subjective discount rate, P_t is the price level, $D_t = Y_t - \frac{W_t}{P_t} N_t$ is the dividend and $N_t = N_{1,t} + N_{2,t}$ is the aggregate labor supply, J_t is the price of the dividend claim, and $G_{i,t}$ represents their total wealth includes the financial assets and the human capital. $\theta_{i,u}$ is the portfolio weight of stock, and Θ represents the set of admissible portfolio strategies.

Both types of agents are regulated by the LTV constraint. But the more confident agent reacts less to the shocks, and she is less likely to be bounded by the constraint. Therefore for tractability reason, I assume she is not subject to LTV constraint. This assumption allows us to apply the standard martingale approach to characterize the optimal consumption and portfolio for general price systems as in [Cox and Huang \(1989\)](#). The SDF of the more confident agent, summarizing the stock and bond price system, is given by

$$\frac{d\Lambda_{1,t}}{\Lambda_{1,t}} = -r_t dt - \kappa_{1,t} dZ_{A,t}^1, \quad (3.18)$$

$$\Lambda_{1,0} = 1. \quad (3.19)$$

Her consumption-leisure portfolio problem is

$$\max_{C_{1,t}, L_{1,t}} E_0^1 \left[\int_0^T e^{-\rho t} \left(\log C_{1,t} + \frac{L_{1,t}^{1-\eta}}{1-\eta} \right) dt \right] \quad (3.20)$$

$$\text{subject to } E^1 \left[\int_0^T \Lambda_{1,t} \left(C_{1,t} - \frac{W_t}{P_t} N_{1,t} \right) dt \right] \leq x_1 J_0. \quad (3.21)$$

The FOCs give the optimal consumption and leisure choice:

$$\hat{C}_{1,t} = \frac{e^{-\rho t}}{y_1 \Lambda_{1,t}}, \quad (3.22)$$

$$\hat{L}_{1,t} = \left(\frac{e^{-\rho t}}{y_1 \Lambda_{1,t} \frac{W_t}{P_t}} \right)^{\frac{1}{\eta}}, \quad (3.23)$$

where $\kappa_{1,t}$ is the instantaneous risk price and y_1 is the Lagrange multiplier from the static budget constraint and solves following equation

$$E_0^1 \left[\int_0^T e^{-\frac{\rho}{\eta} t} \Lambda_{1,t}^{\frac{\eta-1}{\eta}} \left(\frac{W_t}{P_t} \right)^{\frac{\eta-1}{\eta}} dt \right] y_1^{-\frac{1}{\eta}} + \frac{1}{\rho} (1 - e^{-\rho T}) y_1^{-1} = E_0^1 \left[\int_0^T \Lambda_{1,t} \frac{W_t}{P_t} \bar{N}_1 dt \right] + x_1 J_0. \quad (3.24)$$

The less confident agent faces an incomplete financial market, as her portfolio optimization is complicated by the LTV constraint. Similar to [Cvitanic and Karatzas \(1992\)](#) and [Chabakauri \(2015\)](#), we solve the constrained agent's optimization problem in a fictitious unconstrained economy, in which the price of the instantaneous bond and equity follow

$$\frac{dB_t}{B_t} = (r_t + \delta(v_t^*)) dt \quad (3.25)$$

$$\frac{dJ_t}{J_t} = (\mu_{J,t}^2 + v_t^* + \delta(v_t^*)) dt + \sigma_{J,t} dZ_{A,t}^2 \quad (3.26)$$

where $\delta(\cdot)$ is the *support function* for the set of admissible portfolio weights Θ , defined as

$$\delta(v) = \sup_{\theta \in \Theta} (-v\theta) \quad (3.27)$$

which is obtained from complementary slackness condition $v_t^* \theta_t^* + \delta(v^*) = 0$ (see [Cvitanic and Karatzas \(1992\)](#) and [Karatzas and Shreve \(1998\)](#)). The *adjustment* v_t^* can be interpreted as the shadow cost of the constraint. Therefore, the SDF of the Type 2 agent in the fictitious economy is:

$$\frac{d\Lambda_{2,t}}{\Lambda_{2,t}} = -(r_t + \delta(v_t^*)) dt - \left(\kappa_{2,t} + \frac{v_t^*}{\sigma_{J,t}} \right) dZ_{A,t}^2, \quad (3.28)$$

$$\Lambda_{2,0} = 1, \quad (3.29)$$

and the less confident agent's consumption portfolio problem is

$$\max_{C_{2,t}, L_{2,t}} E^2 \left[\int_0^T e^{-\rho t} \left(\log C_{2,t} + \frac{L_{2,t}^{1-\eta}}{1-\eta} \right) dt \right] \quad (3.30)$$

subject to the static budget constraint

$$E^2 \left[\int_0^T \Lambda_{2,t} \left(C_{2,t} - \frac{W_t}{P_t} N_{2,t} \right) dt \right] \leq x_2 J_0. \quad (3.31)$$

The FOCs give the optimal consumption and leisure choice:

$$\hat{C}_{2,t} = \frac{e^{-\rho t}}{y_2 \Lambda_{2,t}}, \quad (3.32)$$

$$\hat{L}_{2,t} = \left(\frac{e^{-\rho t}}{y_2 \Lambda_{2,t} \frac{W_t}{P_t}} \right)^{1/\eta}, \quad (3.33)$$

where $\kappa_{2,t}$ is the instantaneous risk price and y_2 is the Lagrange multiplier from the static budget constraint and solves following equation

$$E_0^2 \left[\int_0^T e^{-\frac{\rho}{\eta} t} \Lambda_{2,t}^{\frac{\eta-1}{\eta}} \left(\frac{W_t}{P_t} \right)^{\frac{\eta-1}{\eta}} dt \right] y_2^{-\frac{1}{\eta}} + \frac{1}{\rho} (1 - e^{-\rho T}) y_2^{-1} = E_0^2 \left[\int_0^T \Lambda_{2,t} \frac{W_t}{P_t} \bar{N}_2 dt \right] + x_2 J_0. \quad (3.34)$$

IV. The Equilibrium

To determine the equilibrium price system, it is sufficient to find the two SDF processes that clear the consumption market. Define

$$\lambda_t = \frac{y_1 \Lambda_{1,t}}{y_2 \Lambda_{2,t}} \quad (3.35)$$

where $\lambda_0 = \frac{y_1}{y_2}$. Its dynamics follow

$$\frac{d\lambda_t}{\lambda_t} = -\mu_\lambda dt - \sigma_\lambda dZ_{D,t}^1, \quad (3.36)$$

where

$$\mu_{\lambda,t} = -\kappa_{2,t} \frac{v_t^*}{\sigma_{J,t}} - \left(\frac{v_t^*}{\sigma_{J,t}} \right)^2 - \delta(v_t^*), \quad (3.37)$$

$$\sigma_{\lambda,t} = \kappa_{1,t} - \kappa_{2,t} - \frac{v_t^*}{\sigma_{J,t}} \quad (3.38)$$

λ_t is the only state variable in the economy and essentially equivalent to the consumption allocation as the individual SDF is just the marginal utility with respect to consumption. It summarises the impact of difference in beliefs and the policy. In the absence of policy, the shadow price of constraint is 0, and the dynamics of λ_t depend only on the difference in beliefs $\kappa_{1,t} - \kappa_{2,t} = \frac{\hat{\mu}_{A,t}^1 - \hat{\mu}_{A,t}^2}{\sigma_A}$. While the policy affects the drift and the volatility of λ_t and thus impacts optimal choices, asset prices and social welfare. I characterize the equilibrium by following proposition.

Proposition 9. *Assume that an equilibrium exists. The SDFs in the fictitious for two types of households are*

$$\Lambda_{1,t} = e^{-\rho t} \frac{U_c(Y_t, \lambda_t)}{U_c(Y_0, \lambda_0)}, \quad \Lambda_{2,t} = \frac{\lambda_t}{\lambda_0} \Lambda_{1,t}, \quad (3.39)$$

where $U_c(Y_t, \lambda_t)$ is the marginal utility of a social planner which is defined as

$$U(Y_t, \lambda_t) = \max_{C_{1,t} + C_{2,t} \leq A_t, (\bar{N} - L_{1,t} - L_{2,t}) = Y_t} U(C_{1,t}, L_{1,t}) + \lambda_t U(C_{2,t}, L_{2,t}). \quad (3.40)$$

The stock price J_0 is computed from the more confident agent's probability measure

$$J_0 = \frac{1}{\epsilon} E_0^1 \left[\int_0^T e^{-\rho s} \frac{U_c(Y_s, \lambda_s)}{U_c(Y_0, \lambda_0)} Y_s ds \right]. \quad (3.41)$$

The equilibrium consumption allocations are

$$C_{1,t} = \frac{1}{\epsilon U_c(Y_t, \lambda_t)}, \quad C_{2,t} = \frac{\lambda_t}{\epsilon U_c(Y_t, \lambda_t)}. \quad (3.42)$$

The stochastic weighting process λ_t has the dynamics given by (3.36) where λ_0 clears the markets at time $t = 0$, namely

$$C_1(\lambda_0, A_0) + C_2(\lambda_0, A_0) = A_0 (\bar{N} - L_1(\lambda_0, A_0) - L_2(\lambda_0, A_0)). \quad (3.43)$$

Conversely, if there exists processes $\Lambda_{1,t}$, $\Lambda_{2,t}$, λ_t and J_t satisfying equations (3.39) - (3.43), with $\sigma_Y > 0$, the associated optimal consumption portfolio policies clear all markets.

Proof: see Appendix A.

The equilibrium consists of two regions which depend on whether the difference in beliefs is large enough to trigger the policy constraint. In the first region, the difference in beliefs is small enough so that Agent 2 is willing to hold only a small portion of stock without any leverage. Here the constraint does not bind. The dynamics of the weighting process depend on the difference in beliefs only. In the second region, the difference in beliefs is large enough so that Agent 2 is eager to lever up to hold more equity. In this case the leverage constraint binds; the dynamics of the weighting process depends on the tightness of the leverage constraint as well as the difference in beliefs. The next proposition reports the shadow price of the constraint in these two regions.

Proposition 10. *In the above equilibrium, for LTV constraint $\theta \leq \tilde{\theta}$ with $\tilde{\theta} > 1$, adjustment v_t^* satisfies Kuhn-Tucker optimality conditions*

$$v_t^* (\theta^* - \tilde{\theta}) = 0, \quad (3.44)$$

$$v_t^* \tilde{\theta} \leq 0, \quad (3.45)$$

$$\theta^* \leq \tilde{\theta} \quad (3.46)$$

and is given by the following expression

$$v_t^* = \begin{cases} 0, & \text{if } \theta^* < \tilde{\theta} \\ \tilde{\theta} \sigma_{J,t}^2 - \kappa_{2,t} \sigma_{J,t} & \text{if } \theta^* = \tilde{\theta}, \end{cases} \quad (3.47)$$

and

$$\delta(v_t^*) = -\tilde{\theta} v_t^*. \quad (3.48)$$

Proposition 10 shows that the shadow price of the policy constraint is determined by the tightness of policy and the endogenous stock price volatility. For tractability purpose I further assume that the

constraint is the inverse of the volatility to disentangle the shadow price and stock price volatility, and the result is given by the following corollary.

Corollary 8. *If the leverage constraint takes the form of inverse of the volatility $\tilde{\theta} = \frac{\bar{\kappa}}{\sigma_{J,t}}$. The adjustment*

$$v_t^* = \begin{cases} 0, & \text{if } \theta^* < \tilde{\theta} \\ \bar{\kappa}\sigma_{J,t} - \kappa_{2,t}\sigma_{J,t} & \text{if } \theta^* = \tilde{\theta}, \end{cases} \quad (3.49)$$

and its support function is

$$\delta(v_t^*) = \begin{cases} 0, & \text{if } \theta^* < \tilde{\theta} \\ \kappa_{2,t}\bar{\kappa} - \bar{\kappa}^2 & \text{if } \theta^* = \tilde{\theta}. \end{cases} \quad (3.50)$$

Proof: see Appendix B.

The closed form solutions for the support function depends on the assumption in Corollary 8, which we also keep in the following sections.

A. Macprudential Policy and Asset Prices

In this part, I discuss the asset pricing implications of macro-prudential policy. First I compare the stock price with and without the speculative risk (sentiment risk).

Proposition 11. *Suppose $\bar{J}_0$ is the representative agent's marginal valuation of the stock without difference in beliefs, then*

$$J_0 < \bar{J}_0. \quad (3.51)$$

Proof: see Appendix D.

Proposition 11 shows that the stock price without the speculative risk is higher. The intuition is that the speculation due to the difference in beliefs increases the fluctuations in individual wealth and the variation of the aggregate SDF, hence it increases the stock price volatility and the equity premium and lowers the stock price.⁹ The macro-prudential policy offsets with the speculative risk, and thus raises the stock price.

Corollary 9. *Suppose $\tilde{J}_0$ is the constrained agent's marginal valuation of the stock defined as*

$$\tilde{J}_0 = E_0^2 \left[\int_0^T \frac{\Lambda_{2,s}}{\Lambda_{2,0}} D_s ds \right]. \quad (3.52)$$

Then

$$\begin{cases} J_0 > \tilde{J}_0, & \text{if the constraint binds with positive probability for some } t' \geq t, \\ J_0 = \tilde{J}_0, & \text{else.} \end{cases} \quad (3.53)$$

⁹ Like [Cochrane \(2017b\)](#) points out, the existence of the speculative risk induces the time varying risk bearing capacity and thus increases the variation of the aggregate SDF. [Dumas, Kurshev, and Uppal \(2009\)](#) argue the sentiment risk explain the excess equity return volatility and derive the optimal portfolio strategy to hedge against the sentiment risk.

Proof: see Appendix C.

If the constraint does not bind the less confident agent (Agent 2), her stock holding varies accordingly with the speculation motives as the difference in beliefs fluctuates. If there is positive probability that the leverage constraint will bind at some time in the future, the stock price is higher than the less confident agent's valuation for the stock. The difference between the stock prices and the less confident agent's valuation can be interpreted as the discount value for speculative profit loss due to the policy constraint in her subjective probability measure.¹⁰

The corollary above characterizes the consumption allocations and the stock price in equilibrium, then the following corollary shows the subjective risk prices for two agents and risk-free rate.

Corollary 10. *The market prices of risk for two agents are*

$$\kappa_{1,t} = \begin{cases} \frac{1}{\nu_{1,t}}\sigma_{Y,t} - \frac{\nu_{2,t}}{\nu_{1,t}}\bar{\kappa} & \text{if the constraint binds,} \\ \sigma_{Y,t} + \nu_{2,t}\bar{\mu}, & \text{otherwise,} \end{cases} \quad (3.54)$$

$$\kappa_{2,t} = \begin{cases} \frac{1}{\nu_{1,t}}\sigma_{Y,t} - \frac{\nu_{2,t}}{\nu_{1,t}}\bar{\kappa} - \bar{\mu}_t & \text{if the constraint binds,} \\ \sigma_{Y,t} - \nu_{1,t}\bar{\mu}, & \text{otherwise.} \end{cases} \quad (3.55)$$

The instantaneous risk-free rate is

$$r_t = \begin{cases} \rho + \mu_{Y,t}^1 - (\nu_{1,t}\kappa_{1,t}^2 + \nu_{2,t}\bar{\kappa}^2) - \nu_{1,t}\delta(v_t^*), & \text{if the constraint binds,} \\ \rho + \mu_{Y,t} - \sigma_{Y,t}^2, & \text{otherwise,} \end{cases} \quad (3.56)$$

where $\mu_{Y,t}$ is the aggregate belief of output growth rate defined by

$$\mu_{Y,t} = \nu_{1,t}\mu_{Y,t}^1 + \nu_{2,t}\mu_{Y,t}^2; \quad (3.57)$$

$\mu_{Y,t}^i$ and $\sigma_{Y,t}$ are endogenous and given in the proof.

Proof: see Appendix E.

Corollary 10 shows that the risk prices and the risk-free rate in the unconstrained region depend on the difference in beliefs and the consumption allocation $\nu_{i,t}$, consistent with Basak (2000) and Bhamra and Uppal (2014). When the constraint is binding, the subjective risk price of the unconstrained agent depends on her own consumption share and the tightness of policy. Conditional on the consumption allocation, the subjective risk price of the unconstrained agent is higher in the constrained region, because the policy forces the unconstrained agent to bear additional risk despite her belief. As the output is endogenous, the macro-prudential policy impacts the aggregate output volatility and the prices of risk as a consequence of trading constraint. In the appendix, we show that the output volatility is determined by the difference in beliefs as well as the consumption allocation. The portfolio of less confident agent has a highly volatile

¹⁰Morris (1996), Scheinkman and Xiong (2003) and Gallmeyer and Hollifield (2008) explain the difference in subjective pricing with a short sale constraint as the speculative premium for resale opportunities. This is similar to ours, as in this model the constrained agent expects that she can buy the stock from the unconstrained agent at a better price in the future.

exposure to the fundamental risk, as she put more weight on the shocks when she updates her estimation. While if she is constrained by the LTV cap, her risk exposure is also capped, making consumption allocation and labor supply less volatile and achieving stability in economic growth.

The equilibrium risk-free rate in our economy is endogenous. The rate in the unconstrained region depends on the difference in beliefs as well as the consumption allocation, consistent with Basak (2000) in terms of expression. The risk-free rate is lower when the LTV constraint caps the leverage of the less confident but more optimistic agent. Unlike the short sale constraint case in Gallmeyer and Hollifield (2008) in which the risk-free rate depends solely on the beliefs of one agent, it still depends on the difference in beliefs besides the consumption allocation. The reason is that the pessimist is completely banned from the market if she is short sale constrained, however when facing a macro-prudential constraint, she can still partially share the risk.

With the price of risk, we can obtain the equity premium by computing the stock volatility which is reported in the following proposition.

Corollary 11. *The volatility of the equity is given by*

$$\sigma_{J,t} = \sigma_Y + \left(\frac{\lambda_t}{1 + \lambda_t} - \frac{\lambda_t}{N_t} \frac{\partial N_t}{\partial \lambda_t} \right) \sigma_\lambda + \frac{E_t^1 \left[\int_t^T e^{-\rho s} \mathcal{D}_t^1 \lambda_s ds \right]}{E_t^1 \left[\int_t^T e^{-\rho s} (1 + \lambda_s) ds \right]} \quad (3.58)$$

where $\mathcal{D}_t \lambda_s$ is the Malliavin derivative for λ_s and defined in the proof of this proposition.

Proof: see Appendix G.

Corollary 11 shows how the stock price reacts to technological shocks through changes in λ_t under macro-prudential policy. The first term of (3.58) measures the effect of a technological shock on the stock price through Y_t output. The second term shows the instantaneous impact of a technological shock on the SDF through λ_t , and illustrates the effect of macro-prudential policy on the volatility. The third term measures a path dependent the long term effect of the technological shocks on the SDF in the future. The derivative $\mathcal{D}_t \lambda_s$ measures how perturbing $Z_{A,t}^1$ affects instantaneous change of λ_s for $s \geq t$. The second and third terms shows how difference in beliefs and macro-prudential policy impacts the volatility. When beliefs are identical and the constraint does not bind, the λ_t is constant, the second and the third terms vanish, and the stock price volatility reduces to the output volatility. In the presence of modest heterogeneity in beliefs, the equity volatility depends on the difference in beliefs only; while the equity volatility is affected by macro-prudential constraint if the difference in beliefs is large enough.

Given that the expectation structure in (3.58), the effect of macro-prudential policy on the asset prices cannot be characterized in closed form. However intuitively, the macro-prudential policy prevents the speculation and reduces the volatility of λ_t (and the volatility of consumption allocation and wealth distribution) and therefore the equity and output volatility, achieving a more resilient financial market and economy.

B. Welfare Analysis

I have so far analyzed the asset pricing implications of macro-prudential policy. In this subsection I evaluate its impact on the social welfare.

In our economy, the optimal choices of consumption and leisure under individual probability measure are affected by macro-prudential policy as a result of offsetting the speculation motive and the sentiment risk. Whether the policy is beneficial to the social welfare becomes an important question. While it is attempting to evaluate the social welfare under the true probability measure, which however is unlikely to be known even for the social planner. Therefore we evaluate the policy impact by a belief neutral welfare criterion.

In the previous chapter have shown the belief neutral welfare criterion proposed by [Brunnermeier, Simsek, and Xiong \(2014\)](#) for economies in which agents have heterogeneously distorted beliefs. I also have shown that the conflicting beliefs of the more confident agent and the less confident agent induce a form of externality. First, the difference in priors (or simply confidence level in presence of learning) induces a bias in consumption leisure choice, resulting in a deviation in aggregate labor supply and total output. Second, consumption and leisure become more volatile due to heterogeneous risk exposure in their investment. These externalities makes the trading between two agents suboptimal for the social welfare even under any probability measure. Given the true probability measure is hardly to be known even for the central planner, a policy is preferred such that it mitigates these externalities and increases the social welfare at every possible reasonable probability measure. This welfare criteria, without taking a view on whose belief is correct, can identify the welfare loss to the group of two types of agents. As those agents are endowed with identical wealth, labor (time) and preference, I use a simple utilitarian welfare function, and it is as following:

$$W \left(E_0^h [\mathcal{V}_1], E_0^h [\mathcal{V}_2] \right) = E_0^h [\mathcal{V}_1] + E_0^h [\mathcal{V}_2], \quad (3.59)$$

where

$$\mathcal{V}_i = \int_0^T e^{-\rho u} U(C_{i,u}, L_{i,u}) du. \quad (3.60)$$

I compute the social welfare to show the macro-prudential policy is belief neutral superior to no policy intervention, and the numerical details are in the next section.

C. Macro-prudential Policy and Aggregate Economy

To endogenize the economic growth and capture the real effect of policy, the next proposition characterizes the linkage between households consumption-leisure-portfolio choice and the aggregate labor supply.

Corollary 12. *In equilibrium, the aggregate labor supply is a function of consumption allocation $\nu_{i,t}$, and it satisfies the following polynomial equation*

$$\left(\nu_{1,t}^{\frac{1}{\eta}} + \nu_{2,t}^{\frac{1}{\eta}} \right) N_t^{\frac{1}{\eta}} + \left(\frac{W_t}{A_t P_t} \right)^{\frac{1}{\eta}} N_t - \left(\frac{W_t}{A_t P_t} \right)^{\frac{1}{\eta}} \bar{N} = 0. \quad (3.61)$$

Proof: similar to Chapter 2.

Corollary 12 shows that the aggregate labor supply depends on the consumption share, which is determined by the state variable λ_t . The equilibrium labor is concave and symmetric in consumption share when $0 < \eta < 1$, and it is at maximum when $\nu_{1,t} = 0.5$. Hence the equilibrium labor and the aggregate economic output are maximized if the consumption goods are equally allocated. Individual agent's labor supply is bounded by the maximum working hours, and it is decreasing in her consumption share. This is intuitive as if she is poor and her consumption level is low, she needs to work harder to earn salary to cover the expense for consumption. While she is rich and her consumption level is high, she can mainly rely on her capital income and enjoy the leisure hours.

More importantly, this proposition reveals the risk premia channel via which the macro-prudential policy impacts the aggregate labor supply and the real economy. Macro-prudential policy constrains the portfolio weight and the effective risk premium for the type 2 agent. As agents are forward looking, the restriction on speculation also corrects the initial consumption-leisure choice. In addition, the long run impact of macro-prudential is summarized by the dynamics of λ_t , in which the volatility component and thus the evolution path are affected by the LTV constraint. As λ_t is a positive martingale and is expected to decrease in the probability measure of the more confident agent. By reducing its volatility component, macro-prudential policy impacts expected long term trend of λ_t which determines the long run consumption allocation and thus affects the expected long term trend of aggregate labor and output.

V. Numerical Results: Effects of Macro-prudential Policy

In this section, I show the effects of macro-prudential policy by setting parameter values and solving for the resulting equilibrium. This requires solving optimization of the two types of agents simultaneously. Since these results rely on solving a path dependent problem and do not permit closed form solutions, I apply numerical methods, specifically Monte Carlo simulation. Conditional on an initial guess for λ_0 , I simulate the weighting process λ_t and technology process A_t to solve for the stock price through (3.41). I iterate to find a λ_0 satisfying the equilibrium conditions.

A. Results of the Asset Prices and Social Welfare

In baseline exercise I set the curvature parameter for the leisure component to be 1 so that agents have logarithmic utility flow over both consumption and leisure. This preference is also consistent with the textbook setting in neo-classical growth literature. The reason to do so is to focus on the difference in beliefs and the policy implication in discount rate channel by shutting down the impact on the aggregate output and the dividend flow. Table 3.1 displays benchmark parameters. I set volatility of the technological development to 2% to match the volatility of aggregate GDP growth. I can set the $g_t = \hat{\mu}_{A,0}^1 - \hat{\mu}_{A,0}^2$ to be a slightly positive number, i.e. 0.02 to match the equity volatility and risk-free rate. The disagreement process g_t is time varying due to learning under different confidence level, which will converge to zero after learning for a period of time but is sufficient to generate significant results.

Figure 3.1 shows that the macro-prudential policy increases the asset prices and the social welfare

in baseline parameterization. The upper two panels illustrate the impact over asset prices, specifically the PD ratio and the equity volatility. I characterize the policy tightness by parameter $\bar{\kappa}$: the policy is tighter $\bar{\kappa}$ is smaller. The upper left panel shows that the tighter policy increases the asset prices in terms of PD ratio. The intuition behind the increase in asset prices is following. In the presence of difference in confidence level, the difference in beliefs and sentiment risk are persistent. Speculation on beliefs is also persistent over long time until both agents reach a consensus on their forecast. It's well known that the sentiment risk increases the risk premium (e.g. [Dumas, Kurshev, and Uppal \(2009\)](#), [Buraschi and Jiltsov \(2006\)](#)), as the trading triggered by the difference in beliefs increases the variation in aggregate SDF; while by overturning the portfolio, macro-prudential policy prevents agents to harvest their expected risk premium in their subjective probability measure and avoid large wealth gains and losses due to speculation, and thus it reduces large variation of the consumption allocation and the aggregate discount factor. In other words, macro-prudential policy mitigates the impact of sentiment risk and reduces the equity premium, and thus increases the asset prices. The upper-right panel shows a very similar pattern to the equity volatility. With tighter macro-prudential policy to offset against the speculation motive, the equity volatility decreases.

The bottom-left panel shows how macro-prudential policy impacts the social welfare by plotting the two extreme boundaries of social welfare at all possible reasonable probability measures against macro-prudential policy with different tightness levels. The blue dash line represents the social welfare evaluated under the Agent 1's probability measure, while the red solid line is the social welfare evaluated under the Agent 2's probability measure. As the social welfare under any possible reasonable probability measure should be a convex combination of probability measures of two agents, these two lines represent the two extreme values of the social welfare against different tightness of macro-prudential policy. The figure shows that the social welfare increase in policy tightness in every possible probability measure. In the baseline model, a decrease in the consumption and leisure volatilities is the contributing factor to welfare improve, due to the aggregate labor supply is fixed given log preference over leisure, as shown in the bottom-right panel in [Figure 3.1](#). I shall show my full calibration and an alternative parameterization in the following section, in which case the policy impacts the aggregate output while the results about social welfare and asset prices are the same as the baseline exercise where I shut down the cash flow channel.

B. Macro-prudential Policy and Aggregate Output

In the baseline parameterization I have shown the case $\eta = 1$ to focus on the discount rate channel. While $\eta \neq 1$, [Corollary 12](#) shows the aggregate labor supply as the function of consumption allocation. I set η to be $\frac{1}{4}$ in my main calibration and $\frac{1}{2}$ as an alternative parameterization and plot the results in [Figure 3.2](#) and [Figure 3.3](#) from which we can see the aggregate labor supply is at maximum in both parameterizations when the consumption allocations for two agents are equal. I define the output level in absence of beliefs dispersion as the natural level of output analogous to the New Keynesian literature.

In comparison to the previous case, macro-prudential policy affects the aggregate output when the aggregate labor supply is a function of consumption allocation. Given the identical individual preference setting, if there is no difference in beliefs through the path, the endowment (wealth) allocation and

the consumption allocation should be equal, and the aggregate labor should be at maximum. The heterogeneity of beliefs, however, affects the consumption-leisure-portfolio choice over time and makes the consumption allocation deviated from equal and the aggregate labor supply and output away from the maximum. By restricting the procyclical leverage of the less confident agent who has stronger reaction to perceived economic shocks, the macro-prudential policy overturns the consumption allocation and impacts the aggregate economy. The result is shown in the bottom-right panel in Figure 3.2. Without policy, the speculation due to beliefs dispersion results in an output gap as large as 0.7%, which is considered significant output loss given that the annual economic growth is 2%. The policy can increase the aggregate labor supply and reduce the output gap. However, we need to note that as the policy becomes tighter, the aggregate labor supply and aggregate output increases then decreases. The intuition is as follow: moderate policy helps to reduce the distortion in portfolio selection thus the impact of the heterogeneity of beliefs. However, the over-tight policy over-corrects the suboptimal portfolio choice, and it may set too much restriction over the less confident agent who would have to reduce his consumption. An over-tight policy creates an imbalanced consumption allocation and is bad for output minimization.

Figure 3.3 also shows the impact of macro-prudential policy over the asset prices and the social welfare with an alternative parametrization. The tighter policy increases the P/D ratio and reduces the equity volatility, and it enhances the social welfare in any possible reasonable probability measure same as the baseline case. Hence, the results show a trade-off of policy conduction: tighter policy enhances the social welfare, but does not always increase the aggregate output and thus the economic growth.

VI. Discussion

In this section I now discuss how the results in previous section are affected by changes in various model setting.

A. Speculation and Risk Sharing

With the identical preference setting, trades between two types of households are due to heterogeneity in their prospects. The difference in beliefs does indeed contribute a significant proportion of daily trades, meanwhile the risk sharing should also be an important motive for different portfolio holding. Risk sharing in the presence of heterogeneity in risk aversion also increase the social welfare, while restricting leverage may harm the risk sharing and therefore reduce the social welfare. For this reason, I extend the baseline model by assuming two agents also differ in their risk aversion besides the priors (see Appendix H). For robustness I also use two sets of utility function choices. Agent 1 has the utility flow of leisure is simply $\log L_{1,t}$ in the first case but is $\frac{L_{1,t}^{1-\gamma}}{1-\gamma}$. Given two different utility function choices and the utilitarian function of aggregate utility to evaluate the social welfare, more strictly speaking I assign more weight to Type 1 agent in the first case¹¹.

¹¹In fact the utility flow assign more weight to any distortion in $L_{1,t}$ in $U = \frac{C_{1,t}^{1-\gamma}}{1-\gamma} + \frac{L_{1,t}^{1-\gamma}}{1-\gamma} + \log C_{1,t} + \log L_{1,t}$ than $U = \frac{C_{1,t}^{1-\gamma}}{1-\gamma} + \log L_{1,t} + \log C_{2,t} + \log L_{2,t}$, as innovation in $L_{1,t}$ results more welfare loss evaluated by $\frac{L_{1,t}^{1-\gamma}}{1-\gamma}$ than $\log L_{1,t}$ if $\gamma > 1$. For the same reason, any choice distortions for the more risk averse agent are more pronounced

The results are shown in Figure 3.4 and Figure 3.5, and the main implications still hold with a conservative social welfare function construction: moderate macro-prudential policy improves the social welfare unambiguously. Figure 3.4 shows that policy increases the social welfare unambiguously, while Figure 3.5 shows that only moderate policy enhance social welfare. The intuition behind this result is as follows. The macro-prudential policy has the following three effects: first it reduces the speculative profit for both agents in their own subjective probability measure thus reduce the welfare; second, it smooths the consumption and leisure volatility hence increases the welfare; third, tight policy may impair the risk sharing and could harm the social welfare. Figure 3.5 highlight a conservative case in which utilitarian social welfare construction amplifies the welfare loss of the more risk averse agent. In this case, moderate macro-prudential policy brings more welfare gains from consumption and leisure smoothing than the negative welfare losses from impairing the risk sharing and banning their subjective speculative profit. However, very tight policy reduces the speculative profit and risk sharing, with more welfare loss pronounced in Agent 1 who is more risk averse, thus it does not increase the social welfare under the Agent 1's subjective probability measure. This result is conservative, and the tighter policy could increase the social welfare if fairer weight is assigned to the less risk averse agent.

B. Preference Assumption, Policy and Output Growth

The previous results have shown the impact of macro-prudential policy on aggregate labor supply and output depends on the agents' preferences. Corollary 12 shows the aggregate labor as the function of the consumption allocation between two types of households if we assume $0 < \eta < 1$. This assumption is reasonable in that η determines the desire for leisure smoothing, which is reasonably less important than consumption smoothing for households. While if $\eta > 1$, the aggregate labor supply and thus the aggregate output are concave and non-monotonic functions of the consumption allocation, implying the income and wealth inequality boosts up aggregate labor supply and output growth. This discussion regarding inequality and growth is beyond this paper. But even if we assume $\eta > 1$, the distortion caused by speculation still exists. The consumption and leisure allocation distortion would allow the economy to overheat; while macro-prudential policy offsets the sentiment risk and the speculation motive, achieving balanced economic growth.

VII. Conclusion

In this chapter, I propose a model of macroprudential policy interventions in financial markets. I solve a heterogeneous-agent asset pricing model and show that speculation caused by differences in beliefs about expected technology growth increases asset returns volatility and the equity risk premium, reduces aggregate output and makes individual consumption growth and leisure more volatile. Macroprudential policy helps to offset the distortions caused by speculation, reducing the asset return volatility and the equity risk premium. It also mitigates output distortion and smooths consumption and leisure. Therefore, under every possible reasonable probability measure specified by the belief-neutral welfare criteria proposed by Brunnermeier, Simsek, and Xiong (2014), macroprudential policy raises social welfare.

VIII. Appendix: Tables

Table 3.1. Choice of Parameter Values of Chapter 3

Name	Symbol	Value
Volatility of technological development	σ_A	0.02
Time preference of both agents	ρ	0.02
Curvature Parameter	η	1, 1/2/, 1/4
Agent 1's expected technological growth rate	$\hat{\mu}_{A,0}^1$	0.02
Variance of Agent 1's prior	$\bar{v}_{1,0}$	0.00001
Variance of Agent 2's prior	$\bar{v}_{2,0}$	0.01
Difference of Agent 2's belief and Agent 1's	$g_t = \hat{\mu}_{A,0}^1 - \hat{\mu}_{A,0}^2$	0.02

IX. Appendix: Proofs

A. Proof of Proposition 9

The proof follows Karatzas, Lehoczky, and Shreve (1990) with the appropriate modifications taken to accommodate for investors facing different state prices. From clearing in the consumption good market, state prices follow from the representative agent construction. And the equivalent SDF using the marginal utility function of the planner is:

$$U_c(Y_t, \Omega_t, \lambda_t) = \frac{\partial U_1}{\partial C_{1,t}} \frac{\partial C_{1,t}}{\partial Y_t} + \lambda_t \frac{\partial U_2}{\partial C_{2,t}} \frac{\partial C_{2,t}}{\partial Y_t} + \frac{\partial U_1}{\partial L_{1,t}} \frac{\partial L_{1,t}}{\partial Y_t} + \lambda_t \frac{\partial U_2}{\partial L_{2,t}} \frac{\partial L_{2,t}}{\partial Y_t} \quad (3.62)$$

$$= \frac{\partial U_1}{\partial C_{1,t}} + \frac{\partial U_1}{\partial L_{1,t}} \frac{\partial (L_{1,t} + L_{2,t})}{\partial Y_t} = \frac{\partial U_1}{\partial C_{1,t}} - \frac{1}{A_t} \frac{\partial U_1}{\partial L_{1,t}} = \frac{\partial U_1}{\partial C_{1,t}} - \frac{W_t}{A_t P_t} \frac{\partial U_1}{\partial C_{1,t}} \quad (3.63)$$

$$= \frac{1}{\epsilon} \frac{\partial U_1}{\partial C_{1,t}}. \quad (3.64)$$

Inserting the marginal utility yields the individual SDF. The Dividend flow in a monopolistic competition equilibrium is

$$D_t = Y_t - \frac{W_t}{P_t} N_t = Y_t \left(1 - \frac{W_t}{A_t P_t} \right) = \frac{1}{\epsilon} Y_t \quad (3.65)$$

By inserting into the fundamental pricing formula we obtain the stock price.

To show the converse, assume that there exists $\Lambda_{i,t}$, λ_t and J_t satisfying the pricing equation. Clearing in the consumption goods market follows from the solutions to the investors' consumption-portfolio problems. Clearing in bond market follows from the fact that the discount total wealth, $\Lambda_{2,t} (X_{1,t} + X_{2,t})$ where $X_{i,t}$ is the financial wealth of agent i , is a martingale since agents' optimal wealth process is a tradeable strategy, and they must agree on the prices of all tradeable strategies.

$$X_{1,t} + X_{2,t} = \frac{1}{\Lambda_{2,t}} E^1 \left[\int_t^T \Lambda_{2,s} \left(C_{1,s} - \frac{W_s}{P_s} N_{1,s} + C_{2,s} - \frac{W_s}{P_s} N_{2,s} \right) ds \right]. \quad (3.66)$$

Finally, clearing in the stock market follows by comparing the diffusion dynamics of $\Lambda_{2,t} J_t$ with $\Lambda_{2,t} (X_{1,t} + X_{2,t})$ using clearing in both consumption goods market and the bond market.

B. Proof of Corollary 8

The risky asset portfolio weight of the constrained agent with log preference in the fictitious economy is

$$\theta_t = \frac{1}{\sigma_{J,t}} \left(\kappa_{2,t} + \frac{v_t^*}{\sigma_{J,t}} \right) \leq \tilde{\theta} \quad (3.67)$$

$$v_t^* = \begin{cases} 0, & \text{if } \theta^* < \tilde{\theta} \\ \sigma_{J,t} (\bar{\kappa} - \kappa_{2,t}) & \text{if } \theta^* = \tilde{\theta} \end{cases} \quad (3.68)$$

and

$$\delta(v_t^*) = -\tilde{\theta}v_t^* = -\bar{\kappa}(\bar{\kappa} - \kappa_{2,t}). \quad (3.69)$$

C. Proof of Proposition 9

Under type 2 agent's beliefs and her marginal valuation for the stock given by (3.52)

$$\frac{\tilde{J}_0}{D_0} = E_t^2 \left[\int_0^T \frac{\Lambda_{2,s}}{\Lambda_{2,0}} \frac{D_s}{D_0} ds \right] = E_t^1 \left[\int_0^T \frac{\xi_t \lambda_t^{-1}}{\xi_0 \lambda_0^{-1}} \frac{\Lambda_{1,s}}{\Lambda_{1,0}} \frac{D_s}{D_0} ds \right] \leq E_t^1 \left[\int_0^T \frac{\Lambda_{1,s}}{\Lambda_{1,0}} \frac{D_s}{D_0} ds \right] = \frac{J_0}{D_0} \quad (3.70)$$

because that the stochastic process $\frac{\xi_t \lambda_t^{-1}}{\xi_0 \lambda_0^{-1}}$ is a positive martingale and that the covariance of $\frac{\xi_t \lambda_t^{-1}}{\xi_0 \lambda_0^{-1}}$ and $\frac{\Lambda_{1,s}}{\Lambda_{1,0}} \frac{D_s}{D_0}$ is negative when the LTV constraint binds.

D. Proof of Proposition 11

Without difference in beliefs, under the representative agent's marginal valuation for the stock given by

$$\frac{J_0}{D_0} = E_t^1 \left[\int_0^T \frac{U_C(Y_s, \lambda_s)}{U_C(Y_0, \lambda_0)} \frac{D_s}{D_0} ds \right] = E_t^1 \left[\int_0^T e^{-\rho s} \frac{1 + \lambda_{1,s}}{1 + \lambda_{1,0}} ds \right] \leq E_t^1 \left[\int_0^T e^{-\rho s} ds \right] = \frac{\bar{J}_0}{D_0}. \quad (3.71)$$

The inequality is due to the fact that λ_t is a positive martingale and therefore a decreasing process for which the long run decrease rate is $\left(\hat{\mu}_{A,t}^1 - \hat{\mu}_{A,t}^2 \right)^2$.

E. Proof of Proposition 10

Apply Ito's lemma to each agent's first order conditions

$$e^{-\rho t} \frac{\partial U_i}{\partial C_{i,t}} = y_i \Lambda_{i,t} \quad (3.72)$$

and match the diffusion term,

$$-\frac{\frac{\partial^2 U_1}{\partial C_{1,t}^2}}{\frac{\partial U_1}{\partial C_{1,t}}} \sigma_{C_1} = \sigma_{C_1} = \kappa_{1,t} \quad (3.73)$$

$$-\frac{\frac{\partial^2 U_2}{\partial C_{2,t}^2}}{\frac{\partial U_2}{\partial C_{2,t}}} \sigma_{C_2} = \sigma_{C_2} = \kappa_{2,t} + \frac{v_t^*}{\sigma_{J,t}} \quad (3.74)$$

where $C_{i,t}$ satisfies

$$dC_{i,t} = \mu_{C_i} dt + \sigma_{C_i} dZ_{A,t}^i. \quad (3.75)$$

Market clearing condition in consumption goods implies that

$$C_{1,t} \sigma_{C_1} + C_{2,t} \sigma_{C_2} = \sigma_{Y,t} Y_t, \quad (3.76)$$

which implies (3.54) and (3.55) in the unconstrained region. In the constrained region, LTV constraint, $\tilde{\theta} = \frac{\bar{\kappa}}{\sigma_{J,t}}$ allows

$$\sigma_{C_2} = \bar{\kappa}, \quad (3.77)$$

combine with the equation above, we obtain the constrained part of (3.54) and (3.55).

The risk free rate follows from applying Ito's lemma to the state-price density for each agent, changing to the true probability measure, matching the deterministic terms and rearranging. Risk premium can be calculated using the consumption share weighted risk prices times the stock volatility. And the $\mu_{Y,t}^1$ and $\mu_{Y,t}^2$ comes from applying Ito's lemma to $Y_t = A_t N_t$ with respect to Agent 1 and Agent 2's probability measure.

F. Dynamics of the Output

The output depends on the technological development as well as the aggregate labor supply. The individual optimization yields the optimal consumption and leisure

$$C_{i,t} = \frac{e^{-\rho t}}{y^i \Lambda_{i,t}}, \quad (3.78)$$

$$L_{i,t}^\eta = \frac{e^{-\rho t}}{y_i \Lambda_{i,t} \frac{W_t}{P_t}}. \quad (3.79)$$

Consumption goods market clear:

$$\bar{N} - \sum_{i=1,2} N_{i,t} = \left(\frac{W_t}{P_t} \right)^{-\frac{1}{\eta}} \sum_{i=1,2} C_{i,t}^{\frac{1}{\eta}} \quad (3.80)$$

$$\bar{N} - N_t = \left(\frac{W_t}{P_t} \right)^{-\frac{1}{\eta}} Y_t^{\frac{1}{\eta}} \sum_{i=1,2} \nu_{i,t}^{\frac{1}{\eta}} = \left(\frac{W_t}{A_t P_t} \right)^{-\frac{1}{\eta}} N_t^{\frac{1}{\eta}} \sum_{i=1,2} \nu_{i,t}^{\frac{1}{\eta}}. \quad (3.81)$$

By Ito's lemma, the diffusion of Y_t follows

$$\sigma_Y = \sigma_A - \frac{\lambda_t}{N_t} \sigma_\lambda \frac{\partial N_t}{\partial \lambda_t}. \quad (3.82)$$

As

$$\sigma_\lambda = \kappa_1 - \kappa_2 - \frac{v_t^*}{\sigma_{J,t}} = \begin{cases} \frac{1}{\nu_{1,t}} \sigma_{Y,t} - \left(1 + \frac{\nu_{2,t}}{\nu_{1,t}} \right) \bar{\kappa} & \text{if the constraint binds,} \\ \bar{\mu}_t, & \text{otherwise.} \end{cases} \quad (3.83)$$

hence

$$\sigma_Y = \begin{cases} \frac{\sigma_A + \frac{\lambda_t}{N_t} \frac{\partial N_t}{\partial \lambda_t} \left(1 + \frac{\nu_{2,t}}{\nu_{1,t}} \right) \bar{\kappa}}{1 + \frac{\lambda_t}{N_t} \frac{\partial N_t}{\partial \lambda_t} \frac{1}{\nu_{1,t}}}, & \text{if the constraint binds,} \\ \sigma_A - \frac{1}{N_t} \lambda_t \bar{\mu}_t \frac{\partial N_t}{\partial \lambda_t}, & \text{otherwise.} \end{cases} \quad (3.84)$$

Assume that an equilibrium exists. The SDFs in the fictitious economy for the two agents are

$$\Lambda_{1,t} = e^{-\rho t} \frac{U_c(Y_t, \lambda_t)}{U_c(Y_0, \lambda_0)}, \quad \Lambda_{2,t} = \frac{\lambda_t}{\lambda_0} \Lambda_{1,t}, \quad (3.85)$$

where $U_c(Y_t, \lambda_t)$ is the marginal utility of a social planner which is defined as

$$U(Y_t, \lambda_t) = \max_{C_{1,t} + C_{2,t} \leq A_t, (\bar{N} - L_{1,t} - L_{2,t}) = Y_t} \log C_{1,t} + \frac{L_{1,t}^{1-\eta}}{1-\eta} + \lambda_t \left(\log C_{2,t} + \frac{L_{2,t}^{1-\eta}}{1-\eta} \right). \quad (3.86)$$

The stock price J_t is computed from the unconstrained agent's probability measure

$$J_t = E_t^1 \left[\int_t^T e^{-\rho s} \frac{U_c(Y_s, X_s, \lambda_s)}{U_c(Y_0, X_0, \lambda_0)} D_s ds \right]. \quad (3.87)$$

G. Proof of Corollary 11

I use Clark-Ocone Theorem - [Nualart \(2006\)](#) to prove Corollary 11. I consider the equity pricing equation respect to probability measure of the Type 1 Agent:

$$J_t = \frac{1}{U_C(C_t, \lambda_t)} E^1 \left[\int_s^T e^{-\rho s} U_C(C_t, \lambda_t) \frac{Y_s}{\epsilon} ds | \mathcal{F}_t \right]. \quad (3.88)$$

Define the $\mathcal{L}^2(\mathcal{P}^1)$ martingale M_t

$$M_t = E_t^1 \left[\int_t^T e^{-\rho s} U_C(C_s, \lambda_s) Y_s ds \right]. \quad (3.89)$$

By the martingale representation theorem, there exist a process $\phi(t)$ such that $E_t^1 \left[\int_t^T |\phi(t)|^2 ds \right] < \infty$ and $M_t = M_0 + \int_0^T \phi(t) dZ_{A,t}^1$.

The value of the stock can be rewritten as

$$J_t = \frac{1}{\epsilon U_C(C_t, \lambda_t)} \left[M_t - \int_0^t e^{-\rho s} U_C(C_s, \lambda_s) Y_s ds \right] \quad (3.90)$$

Applying Ito's lemma to equation above and comparing the diffusion term earns the volatility

$$\sigma_{J,t} = -\frac{U_{CC}(C_t, \lambda_t)}{U_C(C_t, \lambda_t)} Y_t \sigma_Y + \frac{U_{C\lambda}(C_t, \lambda_t)}{U_C(C_t, \lambda_t)} \lambda_t \sigma_\lambda + \frac{1}{E_t^1 \left[\int_t^T e^{-\rho s} U_C(C_s, \lambda_s) Y_s ds \right]} \phi(t) \quad (3.91)$$

$$= \sigma_Y + \left(\frac{\lambda_t}{1 + \lambda_t} - \frac{\lambda_t}{N_t} \frac{\partial N_t}{\partial \lambda_t} \right) \sigma_\lambda + \frac{1}{E_t^1 \left[\int_t^T e^{-\rho s} (1 + \lambda_s) ds \right]} \phi(t) \quad (3.92)$$

To complete the proof we need to use Malliavin derivatives to calculate the the response of M_t to

fundamental shocks. By Clark-Ocone Theorem, the diffusion term of dynamics of M_t is

$$\phi(t) = E_t^1 \left[\int_t^T e^{-\rho s} \mathcal{D}_t^1 \lambda_s ds \right] \quad (3.93)$$

where the Malliavin derivatives are

$$\frac{\mathcal{D}_t^1 \lambda_s}{\lambda_s} = -\sigma_\lambda(t) - \int_t^s \mathcal{D}_t^1 \sigma_\lambda(z) [\sigma_\lambda(z) dz + dZ_{A,z}^1] \quad (3.94)$$

$$\mathcal{D}_t^1 \sigma_\lambda(z) = \begin{cases} \mathcal{D}_t^1 \bar{\mu}(z) & \text{if } \sigma_\lambda = \bar{\mu}(z) \\ \frac{\partial \kappa_{1,z}}{\partial \lambda_z} \mathcal{D}_t^1 \lambda_z & \text{if } \sigma_\lambda = \kappa_{1,z} - \bar{\kappa} \end{cases} \quad (3.95)$$

$$\mathcal{D}_t^1 \bar{\mu}(z) = e^{-\int_t^z \frac{\bar{v}_u^2}{\sigma_A^2} du} \frac{1}{\sigma_A^2} (\bar{v}_t^1 - \bar{v}_t^2) \quad (3.96)$$

H. Extension: Speculation and Risk Sharing

To rationalize the risk sharing motive, we assume two types of households differs in their preference besides the priors. Specifically, we assume the utility of the more confident agent follows Instantaneous utility of type i representative agent is given by the following function that depends on the consumption $C_{i,u}$ and leisure $L_{i,t}$:

$$U(C_{1,t}, L_{1,t}) \equiv \frac{C_{1,t}^{1-\gamma}}{1-\gamma} + \log L_{1,t}, \quad (3.97)$$

$$U(C_{2,t}, L_{2,t}) \equiv \log C_{2,t} + \log L_{2,t}. \quad (3.98)$$

Apart from the heterogeneity in risk aversion γ to induce risk sharing, I keeps all other settings identical to the baseline model¹².

Both agents optimize the consumption, leisure and portfolio to maximize their expected utility, which is given by

$$V_{i,t} = \max_{C_{i,u}, N_{i,u}, \theta_{i,u} \in \Theta} E_t^i \left[\int_t^\infty e^{-\rho(u-t)} U_i(C_{i,u}, L_{i,u}) du \right]$$

subject to a dynamic budge constraint

$$dG_{i,t} = r_t G_{i,t} dt + \theta_{i,t} (dJ_t - r_t J_t + D_t dt) + \frac{W_t}{P_t} (\bar{N}_i - L_{i,t}) dt - C_{i,t} dt, \quad (3.100)$$

and the dynamics of consumption share

$$\frac{d\nu_{1,t}}{\nu_{1,t}} = \mu_{\nu_1} dt + \sigma_{\nu_1} dZ_t. \quad (3.101)$$

¹²I also consider an alternative utility function for Type 1 agent:

$$U(C_{1,t}, L_{1,t}) \equiv \frac{C_{1,t}^{1-\gamma}}{1-\gamma} + \log L_{1,t} \quad (3.99)$$

which assigns more weight to leisure smoothing. The derivations for asset prices and labor supply are similar to this proof

The FOC of Agent 1's optimization problem gives the optimal consumption and leisure choice:

$$\hat{C}_{1,t} = \left(\frac{e^{-\rho t}}{y_1 \Lambda_{1,t}} \right)^{\frac{1}{\gamma}}, \quad \hat{C}_{2,t} = \frac{e^{-\rho t}}{y_1 \Lambda_{2,t}} \quad (3.102)$$

$$\hat{L}_{1,t} = \frac{e^{-\rho t}}{y_1 \Lambda_{1,t} \frac{W_t}{P_t}}, \quad \hat{L}_{2,t} = \frac{e^{-\rho t}}{y_2 \Lambda_{2,t} \frac{W_t}{P_t}} \quad (3.103)$$

y_i is the Lagrange multiplier from the static budget constraint and solves the following equations

$$E_0^1 \left[\int_0^T e^{-\frac{\rho}{\gamma} t} \Lambda_{1,t}^{1-\frac{1}{\gamma}} dt \right] y_1^{-\frac{1}{\gamma}} + \frac{1}{\rho} (1 - e^{-\rho T}) y_1^{-1} = E_0^1 \left[\int_0^T \Lambda_{1,t} \frac{W_t}{P_t} \bar{N}_1 dt \right] + x_1 J_0, \quad (3.104)$$

$$y_2 = \frac{\frac{2}{\rho} (1 - e^{-\rho T})}{E_0^2 \left[\int_0^T \Lambda_{2,t} \frac{W_t}{P_t} \bar{N}_2 dt \right] + x_2 J_0}. \quad (3.105)$$

The labor market clearing condition leads to

$$A_t (\bar{N} - L_{1,t} - L_{2,t}) = C_{1,t} + C_{2,t} \quad (3.106)$$

$$A_t (\bar{N} - (1 + \lambda_t) L_{1,t}) = C_{1,t} + \lambda_t C_{1,t}^\gamma \quad (3.107)$$

$$(\bar{N} - (1 + \lambda_t) L_{1,t}) = \left(\frac{W_t}{A_t P_t} \right)^{\frac{1}{\gamma}} A_t^{\frac{1}{\gamma}-1} L_{1,t}^{\frac{1}{\gamma}} + \lambda_t \frac{W_t}{A_t P_t} L_{1,t} \quad (3.108)$$

The following proposition characterizes the equilibrium in this new setting.

Corollary 13. *Assume that an equilibrium exists. The SDFs in the fictitious for two types of households are*

$$\Lambda_{1,t} = e^{-\rho t} \frac{U_c(Y_t, \lambda_t)}{U_c(Y_0, \lambda_0)}, \quad \Lambda_{2,t} = \frac{\lambda_t}{\lambda_0} \Lambda_{1,t}, \quad (3.109)$$

where $U_c(Y_t, \lambda_t)$ is the marginal utility of a social planner which is defined as

$$U(Y_t, \lambda_t) = \max_{C_{1,t} + C_{2,t} \leq A_t (\bar{N} - L_{1,t} - L_{2,t}) = Y_t} U(C_{1,t}, L_{1,t}) + \lambda_t U(C_{2,t}, L_{2,t}). \quad (3.110)$$

The stock price J_0 is computed from the more confident agent's probability measure

$$J_0 = \frac{1}{\epsilon} E_0^1 \left[\int_0^T e^{-\rho s} \frac{U_c(Y_s, \lambda_s)}{U_c(Y_0, \lambda_0)} Y_s ds \right]. \quad (3.111)$$

The equilibrium consumption allocations are

$$C_{1,t} = \left(\frac{1}{\epsilon U_c(Y_t, \lambda_t)} \right)^{\frac{1}{\gamma}}, \quad C_{2,t} = \frac{\lambda_t}{\epsilon U_c(Y_t, \lambda_t)}. \quad (3.112)$$

The stochastic weighting process λ_t has dynamics given by (3.36) where λ_0 clears the markets at time $t = 0$, namely

$$C_1(\lambda_0, A_0) + C_2(\lambda_0, A_0) = A_0 (\bar{N} - L_1(\lambda_0, A_0) - L_2(\lambda_0, A_0)) \quad (3.113)$$

Conversely, if there exists processes $\Lambda_{1,t}$, $\Lambda_{2,t}$, λ_t and J_t satisfying equations (3.109) - (3.113), with $\sigma_Y > 0$, the associated optimal consumption portfolio policies clear all markets.

Proof: Similar to A.

Corollary 14. *The market price of risk for two agents are*

$$\kappa_{1,t} = \begin{cases} \frac{\gamma}{\nu_{1,t}} (\sigma_{Y,t} - \nu_{2,t}\bar{\kappa}) & \text{if the constraint binds,} \\ \mathbf{R}_t \sigma_{Y,t} + \nu_{2,t} \mathbf{R}_t \bar{\mu}, & \text{otherwise,} \end{cases} \quad (3.114)$$

$$\kappa_{2,t} = \begin{cases} \frac{\gamma}{\nu_{1,t}} (\sigma_{Y,t} - \nu_{2,t}\bar{\kappa}) - \bar{\mu}_t & \text{if the constraint binds,} \\ \mathbf{R}_t \sigma_{Y,t} - \frac{1}{\gamma} \nu_{1,t} \mathbf{R}_t \bar{\mu}, & \text{otherwise,} \end{cases} \quad (3.115)$$

where $\mathbf{R}_t$ is the aggregate risk aversion defined by

$$\mathbf{R}_t = \left(\frac{\nu_{1,t}}{\gamma} + \frac{\nu_{2,t}}{1} \right)^{-1}. \quad (3.116)$$

By Ito's lemma, the diffusion of Y_t

$$\sigma_Y = \sigma_A - \frac{\lambda_t}{N_t} \sigma_\lambda \frac{\partial N_t}{\partial \lambda_t} + \frac{A_t}{N_t} \sigma_A \frac{\partial N_t}{\partial A_t} \quad (3.117)$$

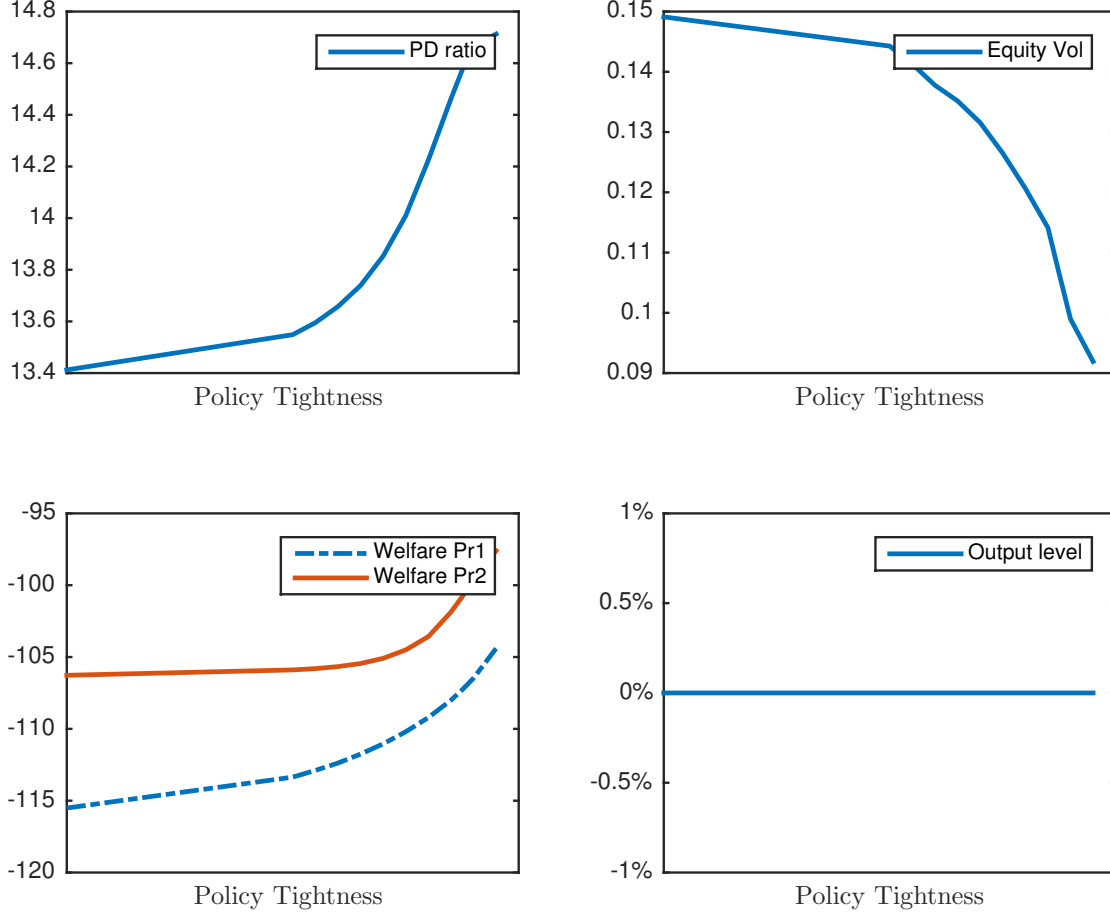
As

$$\sigma_\lambda = \kappa_1 - \kappa_2 - \frac{v_t^*}{\sigma_{J,t}} = \begin{cases} \frac{\gamma}{\nu_{1,t}} (\sigma_{Y,t} - \nu_{2,t}\bar{\kappa}) - \bar{\kappa} & \text{if the constraint binds,} \\ \bar{\mu}_t, & \text{otherwise.} \end{cases} \quad (3.118)$$

hence

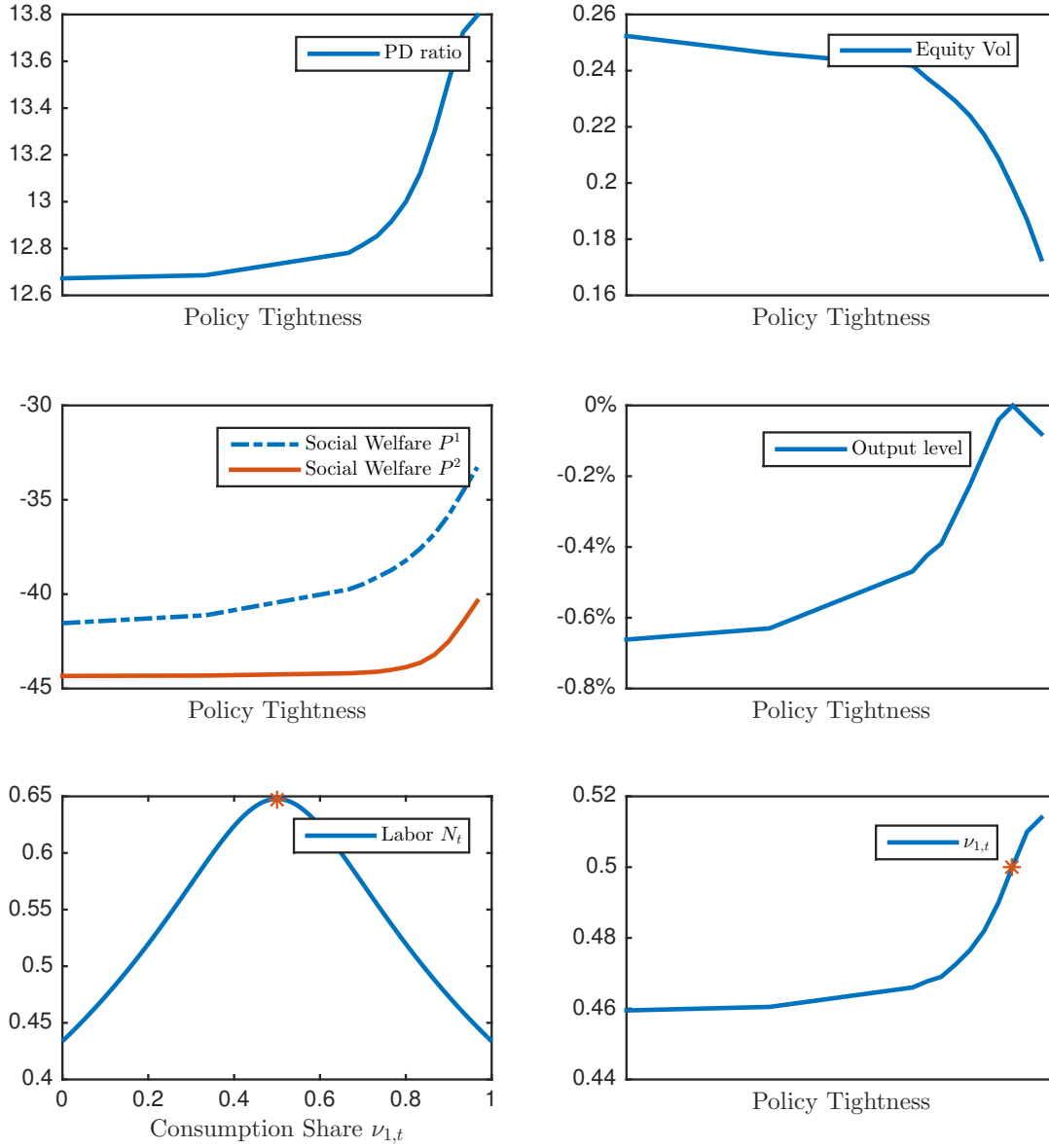
$$\sigma_Y = \begin{cases} \frac{\sigma_A + \frac{\lambda_t}{N_t} \frac{\partial N_t}{\partial \lambda_t} \left(1 + \gamma \frac{\nu_{2,t}}{\nu_{1,t}} \right) \bar{\kappa} + \frac{A_t}{N_t} \sigma_A \frac{\partial N_t}{\partial A_t}}{1 + \frac{\lambda_t}{N_t} \frac{\partial N_t}{\partial \lambda_t} \frac{\gamma}{\nu_{1,t}}}, & \text{if the constraint binds,} \\ \sigma_A - \frac{1}{N_t} \lambda_t \bar{\mu}_t \frac{\partial N_t}{\partial \lambda_t} + \frac{A_t}{N_t} \sigma_A \frac{\partial N_t}{\partial A_t}, & \text{otherwise.} \end{cases} \quad (3.119)$$

X. Appendix: Figures



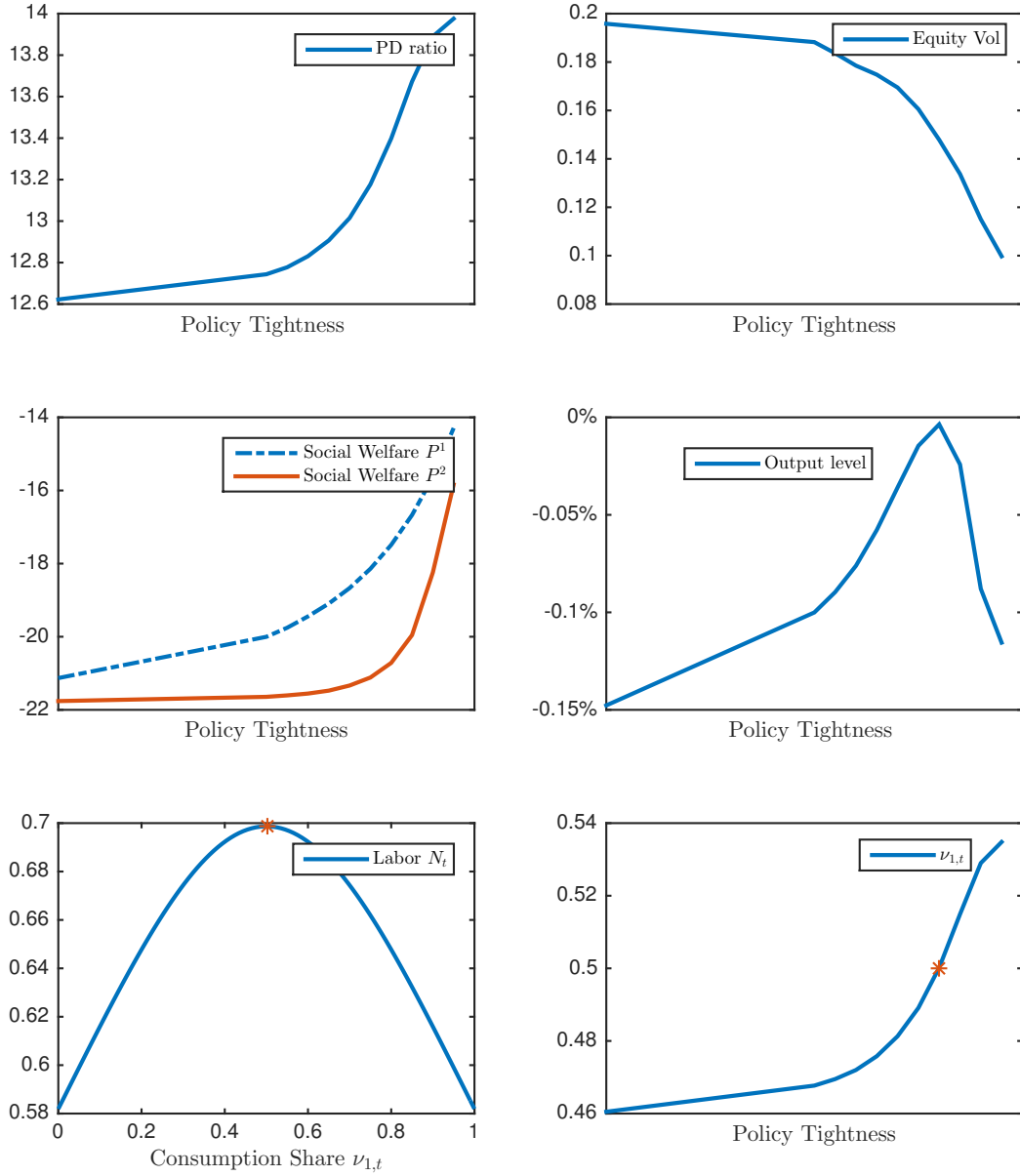
This figure plots the aggregate welfare, Price-dividend ratio and volatility of equity, and output level against different tightness of macro-prudential policy ($\bar{\kappa}$ is smaller, the policy is tighter). In this baseline calibration, $\eta = 1$, so the individual utility depends on the logarithmic of consumption and leisure. I set $g_t = \hat{\mu}_{A,t}^1 - \hat{\mu}_{A,t}^2 = 0.02$. The figures show that tighter policy increases the asset prices and reduces the equity volatility. The left bottom panel plots the two boundaries of social welfare under any probability measures, showing that tighter policy increases the social welfare even without knowing the true probability measure. The output gap is 0 as the income effect offsets the substitution effect fully with log preference on consumption and leisure.

Figure 3.1. A Special Parametrization: Discount Rate Channel



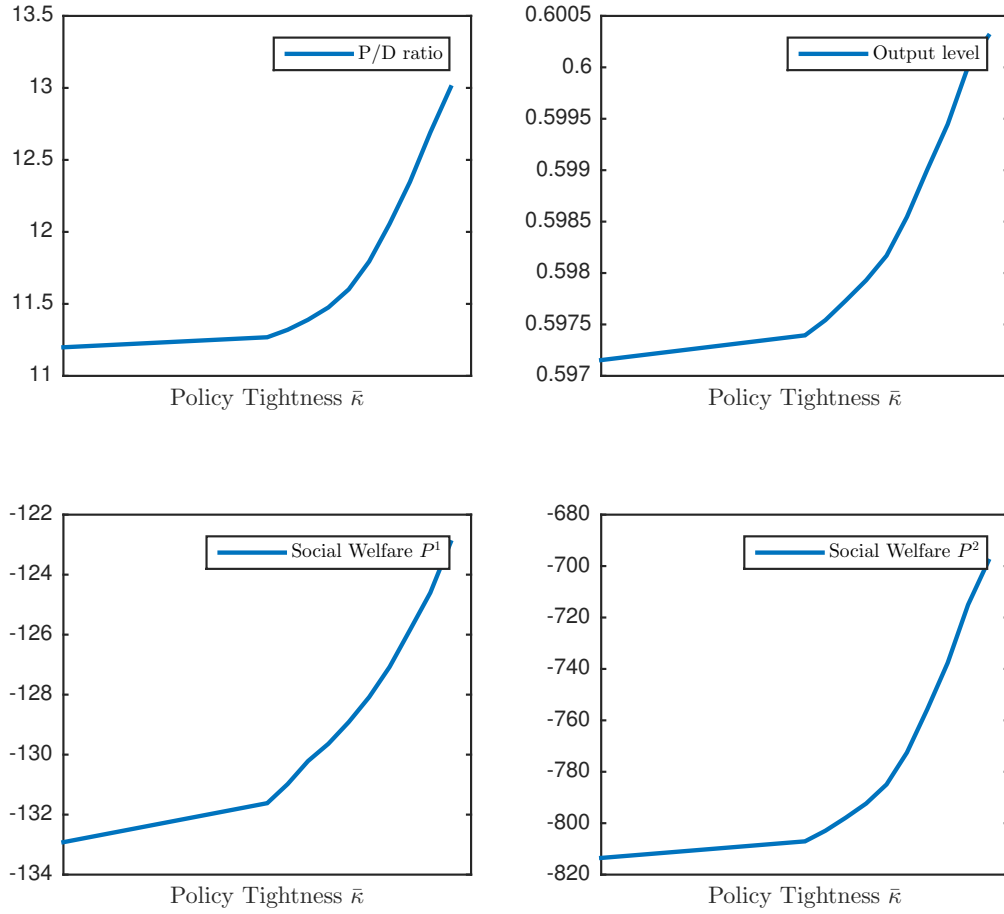
This figure plots the aggregate welfare, Price-dividend ratio and volatility of equity, and output level against different tightness of macro-prudential policy ($\bar{\kappa}$ is smaller, the policy is tighter). I set $\eta = 0.25$ and $g_t = \hat{\mu}_{A,t}^1 - \hat{\mu}_{A,t}^2 = 0.02$. The figures show that tighter policy increases the asset prices and reduces the equity volatility. Macro-prudential policy reduces the output gap, unlike the special parameterization which is to focus on the discount rate channel.

Figure 3.2. Impacts of Macro-prudential Policy: Baseline Calibration



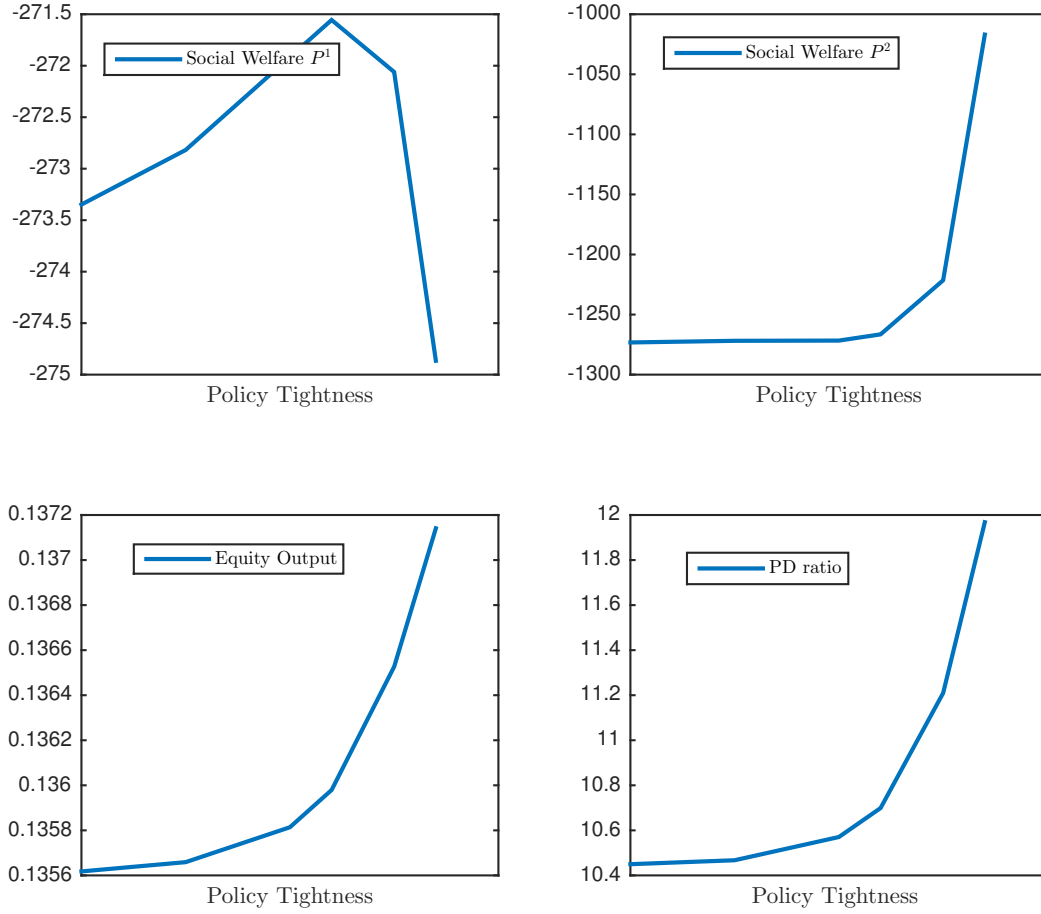
This figure plots the results with an alternative parameter choice for robustness purpose. The only difference is $\gamma = 2$ in this case. The results are similar to my baseline calibration.

Figure 3.3. Impacts of Macro-prudential Policy: An Alternative Calibration



This figure plots the results with the numerical in my extension model which allows risk sharing as well as speculation. The more risk averse agent's utility function is $\frac{C^{1-\gamma}}{1-\gamma} + \log L$. γ equals 2.

Figure 3.4. Model Extension: Risk Sharing and Speculation



This figure plots the results with the numerical in my extension model which allows risk sharing as well as speculation. The more risk averse agent's utility function is $\frac{C^{1-\gamma}}{1-\gamma} + \frac{L^{1-\gamma}}{1-\gamma}$, and I set $\gamma = 2$. I set the initial technology level $A_0 = 0.3$. The figure shows that moderate policy increases the social welfare unambiguously.

Figure 3.5. Risk Sharing and Speculation: An Alternative Utility Function

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