

**Imperial College
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**Interscale energy transfer in bypass
transition due to free-stream turbulence**

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I would like to dedicate this thesis to my loving parents ...

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements.

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Abstract

We consider the inter-scale energy transfer process in a boundary layer undergoing bypass transition using a generalised Karman-Howarth equation. The analysis is performed using a high-resolution direct numerical simulation (DNS) database. The bypass transition is triggered by homogeneous and isotropic free-stream turbulence satisfying a Von Karman spectrum.

We first consider the evolution of the second-order structure function (defined as the square of the difference of the streamwise velocity fluctuations between two points) at different locations in the transition region. We apply conditional sampling based on the local instantaneous intermittency and derived new analytic expressions that generalise existing decompositions of single-point statistics to two-point statistics. It is found that in the transition region, laminar streaky structures maintain their geometrical characteristics in the physical and scale spaces well inside the transition region, even after the initial break down to form turbulent spots. Further conditional analysis reveals that the outer mode is the dominant secondary instability mechanism and shows how turbulence spots penetrate the boundary layer and approach the wall. We also analyse the evolution of the two-point intermittency field, and find that the volume enclosed by an isosurface of a given value grows in both directions, with the growth in the streamwise direction being especially large.

In order to study the dynamics of the structure function and explore the production and interscale energy transfer in bypass transition, the Karman-Howarth-Monin-Hill (KMH) equation is then employed. This is the evolution equation of the scale energy, i.e. the energy contained within eddies of a specific length scale. This equation is very general and can be applied to inhomogeneous and anisotropic flows, like bypass transition. Maps of scale energy production and flux vectors are visualised on two-dimensional planes and three-dimensional hyperplanes that comprise both physical and separation spaces. In the transitional region, the maps show strong inverse cascade in the streamwise direction near the wall. The energy flux vectors emanate from a region of strong production and transfer energy to larger streamwise scales. The inverse cascade is mainly due to the non-linear interaction flux component, and this component competes with the one due to mean flow inhomogeneity. By superposing

the instantaneous velocity fields and the energy flux vectors, we relate the inverse cascade process to the growth of turbulent spots.

We also apply the conditional averaging framework that we developed for the structure function to the KMH equation and derive a conditionally-averaged KMH equation as well as expressions for the decomposition of energy flux vectors. Because conditional averaging does not commute with the spatial differentiation operation, the derivatives of conditionally-averaged two-point variables are calculated directly in the scale space. The conditionally-averaged KMH analysis shows that the strong inverse cascade in the streamwise direction is mainly due to the non-linear interaction terms across the upstream and downstream laminar/turbulent interfaces of the turbulent spots; we find that the contribution from the downstream one dominates. Many of the observed features are explained by a schematic cartoon that considers a propagating diamond-shaped turbulent spot. Finally, we compare the 3-dimensional energy flux map within the turbulent spots and within the fully turbulent region. A similar shape of the flux map is found, proving the dynamical similarities between turbulent spots and fully turbulent region in terms of interscale transfer.

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Chapter 1

Introduction

1.1 Bypass transition

Since the pipe flow experiments in 1883 (Reynolds, 1883), Reynolds observed the existence of two different flow regimes, laminar and turbulent. In his experiment, the process of laminar to turbulent transition was described for the first time. Transition (and subsequent turbulence) affects the performance of many engineering devices. For example, in aeronautical applications, it increases friction drag, thereby increasing fuel costs and the emission of harmful pollutants to the environment.

It is well-known that there exists a critical Reynolds number near 2000 in pipe flow, below which transition doesn't occur. Because the laminar profile is stable for all Reynolds number in pipe flow, to trigger transition, the Reynolds number need to be large enough and the the perturbation needs to be sufficient strong (Eckhardt et al., 2007). Similar conclusion can be drawn for Couette flow using linear stability analysis. Poiseuille flow is however subject to viscous linear instability whose critical Reynolds numbers is about 5772 (Schmid and Henningson, 2012).

Unlike internal flows, boundary layer transition (e.g., flat plate) usually depends on the external disturbance. The physical process that governs the initial growth of external disturbances that can lead to transition in a boundary layer is called receptivity (Morkovin, 1969, Reshotko, 1984). Natural transition (also known as orderly transition) is initiated by two-dimensional Tollmien-Schlichting instability waves, which grow exponentially in the streamwise direction with very slow growth rate; the unstable waves then experience secondary, three-dimensional, instability and eventually break down to turbulence. This transition scenario is applicable when external disturbances are small. When the disturbances are relatively strong, e.g. high turbulent intensity in the free-stream (larger than about

0.5 – 1%), roughness, acoustic waves, etc. the transition process bypasses the natural route of Tollmien-Schlichting waves; this is called bypass transition (Morkovin, 1969).

In this thesis, we consider bypass transition due to free-stream turbulence. Transition of this type has been studied extensively through experiments (Matsubara and Alfredsson, 2001) and numerical simulations (Zaki and Durbin, 2005).

The process of bypass transition mainly takes places in three steps,

(a) Penetration of the free-stream turbulent fluctuations inside the boundary layer

Receptivity explains how the vortical perturbations in the free-stream penetrate within the boundary layer. It is found that only low-frequency fluctuations can penetrate, while high-frequency ones remain in the free-stream, through a process called shear sheltering (Hunt and Durbin, 1999). Zaki and Saha (2009) explains this by considering the ratio of wall-normal diffusion time scale and free-stream convection timescale. The ratio can be described as the number of waves ($n \sim \frac{\tau_{diffusion}}{\tau_{convection}}$) in the free-stream passing a relatively moving observer inside the boundary layer (see figure 1.1),

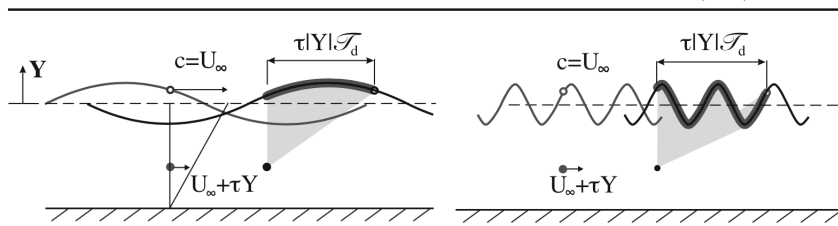


Figure 1.1 The penetration of low-frequency mode (left) and high-frequency mode (right) (Zaki, 2013).

If perturbations have relatively longer convection time scales (low-frequency modes), n is small, which means an observer inside boundary can resolve the disturbance; the opposite happens if the diffusion time scale is longer.

(b) Growth of the disturbances (streaks)

After the low-frequency vortical disturbances penetrate into the boundary layer, forward and backward streamwise streaks are formed (see figure 1.2), which are ubiquitously found in experiments (Klebanoff, 1971, Westin et al., 1994, Matsubara and Alfredsson, 2001). These long structures are also called Klebanoff modes by Kendall (1991), and are considered as transition precursors.

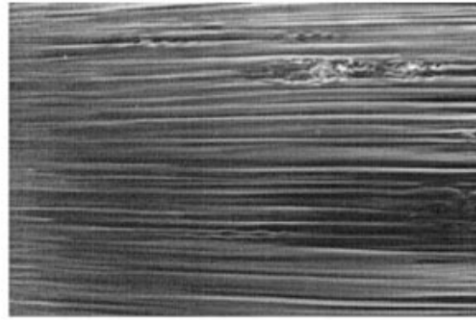


Figure 1.2 Streaky structures in the laminar region from the experiment of Matsubara and Alfredsson (2001).

The amplification of streamwise fluctuations can be explained by the lift-up effect. The simplest model is the case of homogeneous mean shear (inviscid flow) (Phillips, 1969). It can be shown analytically that the streamwise perturbation grows linearly in time (algebraic growth) and depends on shear and vertical momentum. The evolution equations for small perturbations (that take into account viscosity) are the Orr-sommerfeld (OS) and the Squire (SQ) equations (see Appendix A, equation A.11 and equation A.13), which are written in terms of perturbation wall-normal velocity and vorticity. When viscosity is present, the streamwise perturbation also in the beginning grows algebraically in time, but eventually decays exponentially (transient growth). The spacing between high and low-speed streaks in the spanwise direction has been investigated by the theory of optimal disturbances (Andersson et al., 1999, Luchini, 2000, Schmid and Henningson, 2012). It is found that for the boundary layer case, the length scale (averaging spacing) between two streaks is of the order of boundary layer thickness.

(c) Secondary instability (formation of turbulent spots)

At this stage, the streaks formed start to breakdown to turbulent spots due to secondary instability. It has been found that low-speed streaks lift up to the edge of the boundary layer, interacting with high-frequency modes in the free-stream (Jacobs and Durbin, 2001, Zaki and Durbin, 2005).

Experimental and visual studies explain the instability as the result of the meandering of the streaks (Asai et al., 2002, Mans et al., 2005, 2007, Mandal et al., 2010). Varicose and sinuous instability correspond to symmetric and antisymmetric deformation of the streaks respectively (see figure 1.3). These two types of instability can be caused by the inflectional velocity profile in the spanwise direction (Brandt and Henningson, 2002, Brandt et al., 2004)

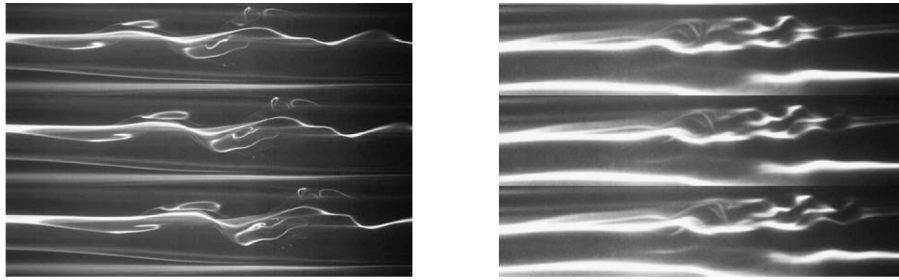


Figure 1.3 The sequence of the streaks breakdown through sinuous (left panel) and varicose (right panel) instability from the experiment of Mans et al. (2005).

Andersson et al. (2001) used inviscid stability theory and found that the sinuous instability is the dominant one and more often leads to streak breakdown. While Andersson et al. (2001) studied steady streaks, Vaughan and Zaki (2011) utilised Floquet analysis to investigate the stability of unsteady periodic streaks. Two types of secondary instability were identified, (1) an ‘outer’ type, which contains a ‘Kelvin-Helmholtz’-like instability that takes place near the edge of the boundary layer with a sinuous/antisymmetric mechanism and (2) a ‘inner’ type, at the overlapping region between the high- and low streaks near the wall with a varicose/symmetric mechanism. Using the same theory, Hack and Zaki (2014) successfully predicted the localized secondary instability of particular streaks under broadband free-stream forcing at given pressure gradient conditions. It was found that unstable streaks tend to be lifted away from the wall; this proves that the ‘outer’ type instability plays a significant role in bypass transition under free-stream turbulence.

(c) Turbulent spots merging and fully turbulent flow

Streaks breakdown to turbulent spots. These intermittent turbulent patches (see figure 1.4) were first described in the experimental work of Emmons (1951). Artificially generated turbulent spots were later studied experimentally and characteristics such as shape, growth rate, lateral spread angle, and substructure were investigated (Wynanski et al., 1976, Perry et al., 1981, Itsweire and Van Atta, 1984). However, the turbulent spots that appear during the bypass transition under free-stream turbulence look different compared to the artificial ones (Durbin and Wu, 2007). Anthony et al. (2005) studied spots in bypass transition and showed that they indeed differ from the conventional, Emmons-type spots. In the former case, turbulent spots form at a more upstream place and are like fast-moving streaks unlike the classic spreading arrowheads; also, some irregular and asymmetrical shapes are also found in Anthony’s heat flux pictures (Anthony et al., 2005).

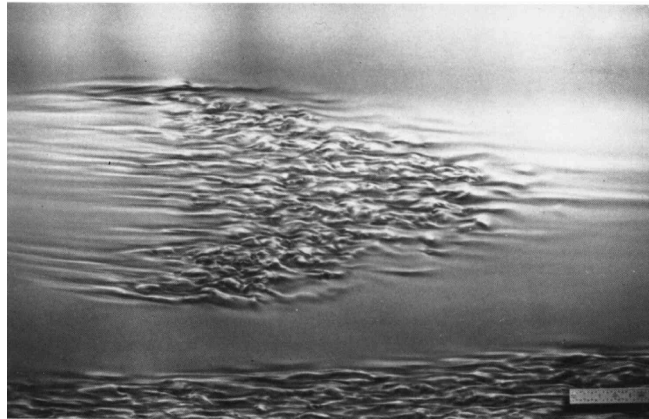


Figure 1.4 Emmons-type turbulent spot from the experiment of Emmons (1951).

Besides experiment, DNS of turbulent spot has been done by Singer (1996), where the general features of the turbulent spot and its individual flow structures were explored. The growth of turbulent spots in terms of the propagating rate and size under different external pressure gradients was studied by Nolan and Zaki (2013). In the simulation of Wu et al. (2017), hairpin structures were found upstream and within the transitional-turbulent spot. Some studies have focused on the similarity between the statistics within the turbulent spot and fully turbulent region. In the work of Park et al. (2012), an octant analysis based on the combined signs of the streamwise, wall-normal, and temperature fluctuations, exploring the momentum and heat fluxes is applied within turbulent spots. Wu et al. (2020) found that turbulent spots at the early stage of the transition have a similar volume-area fractal scaling comparing with high-Reynolds-number turbulence. Marxen and Zaki (2019) examined the intermittent turbulence characteristics in the transition region. In their research, conditional- and ensemble-average are used to explore the characteristics of turbulent spots. They found that in the core region of mature spots, the first- and second-order statistics are similar to fully turbulent flow.

The above-mentioned research is limited to the analysis of one-point statistics, e.g, mean velocity and root-mean-square quantities. This analysis cannot describe the dynamics (how energy transfers between different positions and length scales) of the transition process and this is the central question that we are trying to analyse in this thesis.

The standard approach to describe the dynamics of energy transfer is the turbulent kinetic energy (TKE) equation. This equation governs the evolution of TKE in physical space and contains the mechanisms that determine its evolution, such as production, dissipation, advection by the mean flow and turbulent fluctuations etc. However, this is a single-point equation, and does not contain information about the energy of different length scales. On the other hand, if the flow is homogeneous in all directions, one can employ Fourier series

and derive an evolution equation for the energy spectrum, see the books of Pope (2000) and Davidson (2015). This equation, formulated in spectral space, contains a term representing the kinetic energy transfer between scales (characterised by wavenumbers). This equation is also not suitable for our purposes, because transitional flows are strongly inhomogeneous (mainly in the wall-normal, but also in the streamwise directions), and spectral formulations do not apply. For the type of flows we are dealing with in this thesis, only the Karman-Howarth-Monin-Hill (KMH) equation is suitable, which will be introduced in the following section.

1.2 Karman-Howarth-Monin-Hill equation

The KMH equation can be used to analyse the interscale energy transfer process in very general flow settings. In this section, we trace the origin of the KMH equation, and review the building blocks that mathematically characterise the gorgeous world of turbulence.

Kolmogorov's hypotheses

A picture of energy cascade was introduced by Richardson (1922). In his description, energy is extracted from large-scale eddies, then transfer to smaller scale eddies, and finally dissipate due to viscous effects. Inspired by this picture, Kolmogorov made three hypotheses for sufficiently high Reynolds number turbulent flow (Kolmogorov, 1941b,a); (1) Small scale eddies are isotropic i.e. their statistics do not depend on the orientation. (2) There exists an universal equilibrium range, where statistics depend only on viscosity ν and dissipation rate ε . Through dimensional analysis, the Kolmogorov scales are found to be, length scale $\sim (\nu^3/\varepsilon)^{1/4}$, velocity scale $\sim (\nu\varepsilon)^{1/4}$, and time scale $\sim (\nu/\varepsilon)^{1/2}$. (3) There is an inertial subrange, where the statistics only depend on the dissipation rate ε , and not viscosity ν .

Structure function

The variable to verify Kolmogorov's hypotheses is the second-order structure function (Kolmogorov, 1941b,a, Pope, 2000), defined as the mean square of the difference of velocity between two points, x_i^+ and x_i^- , i.e., $\overline{du_i du_i} = \overline{(u_i^+ - u_i^-)^2}$. Under the assumption of Kolmogorov's hypothesis, the structure function at the inertial subrange depends on only the dissipation rate and the separation (r) between the two points. Using dimensional analysis the only possible expression is $\overline{du_i du_i} \sim (\varepsilon r)^{2/3}$. This so-called 2/3th law has been verified in many flows (Saddoughi and Veeravalli, 1994, Sreenivasan and Antonia, 1997). Besides the structure function, other properties can also be used to quantify the dynamics of turbulence, for example, the two-point velocity correlation tensor and the energy spectrum. For homogeneous isotropic turbulence (HIT) these properties are related to each other (Davidson, 2015).

For instance, velocity correlation function and energy spectrum are Fourier transform pairs. Velocity correlation appears in the expansion $\overline{du_i du_i} = \overline{(u_i^+ - u_i^-)^2} = \overline{(u_i^+)^2} + \overline{(u_i^-)^2} - 2\overline{u_i^+ u_i^-}$; the correlation, $\overline{u_i^+ u_i^-}$, tends 0 when the separation (r) between the + point and – point goes to infinity.

Both energy spectrum and second-order structure function contain information on the distribution of turbulent kinetic energy. The energy spectrum quantifies the energy distribution in wavenumber space. The structure function on the other hand represents the energy within an eddy with radius r (also called scale energy), and since it is defined in physical, rather than wavenumber space, it is easier to interpret physically.

Karman-Howarth equation

Karman and Howarth (1938) first derived a transport equation for the velocity correlation tensor for HIT. Later Kolmogorov (1941a) reformulated the equation in terms of structure function and proved the famous $-4/5$ 'th law that links the interscale energy flux, the separation between the two points, and the dissipation rate. Under HIT conditions, this flux is always negative, indicating forward cascade, i.e. energy is transferred from large to small scales. The advantage of this equation is that even for HIT, it is expressed in spatial variables, rather than wavenumbers. This feature allows for a generalisation to more complex flows, that can include the effect of anisotropy, shear, and convection due to mean flow (Monin and Yaglom, 1975, Antonia et al., 1997, 2000, Danaila et al., 2001, 2004, 2012).

Karman-Howarth-Monin-Hill equation

The most general form of the Karman-Howarth equation appears in Hill (2002a). For a weak formulation, which is more suitable for non-smooth, i.e. rough, velocity fields, see Duchon and Robert (2000), Dubrulle (2019). The equation is derived directly from the Navier-Stokes equations applied at points x_i^+ and x_i^- and contains all the mechanisms that determine the evolution of the second-order structure function i.e. how the energy at a particular scale is affected by production, dissipation and transfer in the physical X_i , and scale r_i spaces. It does not make the assumptions of homogeneity and/or isotropy, and is thus suitable for the analysis of inter-scale energy transfer in highly inhomogeneous flow (bypass transition in our case). We note that in the limit of very large separations r , the KMH equation tends to the TKE (more precisely, it tends to twice the sum of the TKE's at points x_i^+ and x_i^- , see more details in chapter 3).

This equation has been applied extensively to fully developed, wall-bounded turbulent flows. It was employed for the first time by Marati et al. (2004) to a low Reynolds number channel flow. The results showed that the well-known zonal decomposition of the flow in viscous layer, buffer region, log-layer and bulk region is also meaningful in the context of a scale-by-scale budget. Saikrishnan et al. (2012) studied the effect of Reynolds number in the range $Re_\tau = 180 - 934$ on the scale energy balance, while Cimorelli and De Angelis (2012) focused on $Re_\tau = 300$ and investigated in particular the inverse cascade for length scales larger than a cut-off value, and the associated difficulties for LES subgrid scale modelling if the filter size is larger than this value. Visualisation of the scale energy flux in a 3D hyper-plane (Cimorelli et al., 2013) revealed spiral-like paths that originate from two energy sources, one located close to the wall (inner region) and the other further away (outer region). In the work of Cimorelli et al. (2015), particular attention was paid to the dynamics of the outer source, while on Cimorelli et al. (2016) the role of attached eddies on sustaining a reverse cascade towards the bulk flow was studied. Gatti et al. (2019) developed a highly-efficient numerical algorithm to compute the terms of the KMH equation for the high-Reynolds number channel flow, where the homogeneous direction is calculated in Fourier space. More recently, Gatti et al. (2020) considered the balance equations for all components of the second order structure function tensor and the interactions via the pressure-strain term.

The preceding short review has focused on attached wall-bounded flows as more relevant to the subject of this thesis, but the equation has been applied to several other inhomogeneous and anisotropic flows, such as the near wake behind a square prism (Alves Portela et al., 2017, 2020), Von Karman flows (Dubrulle, 2019, Knutsen et al., 2020), planar jets (Cimorelli et al., 2021) and the recirculating flow behind a bump (Mollicone et al., 2018).

1.3 Motivation and objectives

Figure 1.5 shows contours of the streamwise velocity fluctuation at a plane parallel to the wall at distance $Y = 1.3L_0$ (L_0 will be defined later) during transition. We can see that the flow has a wide range of scales, from the upstream laminar to the downstream turbulent regions. There are high- and low-speed streaks (long structures in the laminar region), while a turbulent spot appears that is merging with the turbulent zone.

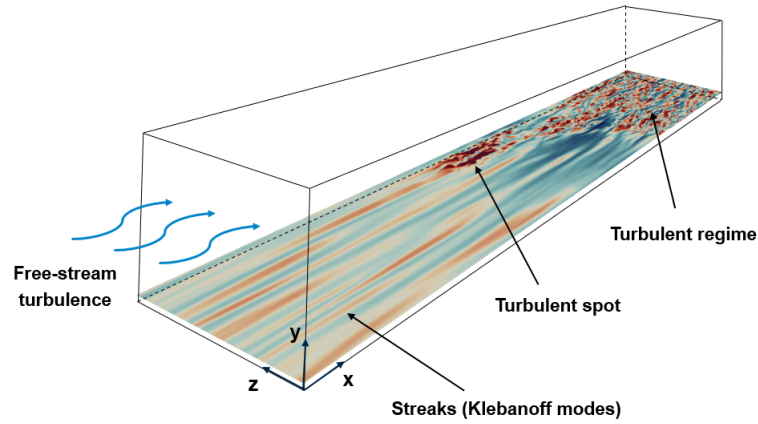


Figure 1.5 Contours of the instantaneous streamwise velocity fluctuations at $Y = 1.3L_0$.

Intuitively, the process of growth of a nascent turbulent patch to a larger one, and the subsequent merging of nearby patches, is reminiscent of an inverse energy cascade process, in which energy flows from small to large scales. For example, in two-dimensional turbulence, the inverse energy cascade results in the growth of the integral length scale (chapter 10 in Davidson (2015)). There are several questions around this notion that have not been considered so far in the literature: what is the mechanism that causes the growth of spots, especially in the streamwise direction? Is this indeed related to inverse energy cascade? How can this process be characterised quantitatively? More generally, how does energy transfer between different length scales in transitional flows differ from fully developed turbulent flows? These are the questions we will try to answer in this thesis.

The main objectives of this thesis are:

- (1) To quantify the inter-scale energy transfer process in the transitional region using the KMH equation.
- (2) To identify the mechanism through which this transfer process takes place i.e., the role of non-linear turbulent interactions, the interaction with the mean flow, the effect of pressure-velocity correlation and the viscous effect.

(3) To quantify, in this highly intermittent flow, the laminar and turbulent contributions on the dominant scale energy flux using a conditionally-averaged KMH equation.

(4) To compare the conditionally-averaged results sampled in the transitional region with the standard-averaged results sampled in the turbulent region.

1.4 Outline

The outline of the thesis is provided below:

Chapter (1) introduces bypass transition and the KMH equation.

Chapter (2) introduces the numerical method used to obtain the DNS database, the inlet conditions (special focus is given on the inlet turbulent fluctuations that satisfy a given spectrum), and validation. An intermittency field is also defined that will be used later to perform conditionally-averaged operations.

Chapter (3) introduces the KMH equation and its different forms in more detail.

In chapter (4), we derive a general expression of the conditionally-averaged decomposition for two-point statistics and analyse the evolution of the standard- and conditionally-averaged second-order structure functions at different streamwise locations. Part of the results of this chapter are published in Yao et al. (2020).

In chapter (5), we implement the standard-averaged KMH equation at different streamwise locations; also results from instantaneous analysis and PDFs of the non-linear scale energy flux are presented.

In chapter (6), we apply the framework developed in chapter (4) to derive a conditionally-averaged KMH equation in order to shed more light in the findings of chapter (5).

Chapter (7) closes this thesis with summarising the main conclusions and suggesting some lines of research for future work.

Chapter 2

Direct Numerical Simulation of bypass transition due to free-stream turbulence.

In this chapter, details about the Direct Numerical Simulation used to produce the DNS database are provided. The database contains 350 uncorrelated snapshots. For each time instant, the velocities in the streamwise, wall-normal and spanwise directions, pressure and a binary indicator function are stored. Special focus is given on the generation of the inlet velocity fluctuations that satisfy a given energy spectrum.

2.1 Governing equations and computational domain

The flow field is governed by the incompressible Navier-Stokes (NS) and continuity equations,

$$\begin{aligned}\rho \frac{\partial u_i}{\partial t} + \rho \frac{\partial u_i u_j}{\partial x_j} &= -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \\ \frac{\partial u_j}{\partial x_j} &= 0\end{aligned}\tag{2.1}$$

where ρ and μ are fluid density and viscosity respectively. In the following, the velocity components (u_i), Cartesian coordinates (x_i), time (t) and pressure (p) are nondimensionalised by free-stream velocity (U_∞), Blasius similarity variable (L_0), time scale ($T_0 = L_0/U_\infty$), and pressure (ρU_∞^2) respectively. The Blasius similarity variable is defined as $L_0 = \sqrt{\nu X_0/U_\infty}$, where X_0 is the distance of the domain inlet from the leading edge of the flat plate. The non-dimensional form of the governing equations takes form,

$$\begin{aligned}\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} &= -\frac{\partial p}{\partial x_i} + \frac{1}{Re} \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \\ \frac{\partial u_j}{\partial x_j} &= 0\end{aligned}\tag{2.2}$$

where Re is the Reynolds number based on L_0 , i.e., $Re = L_0 U_\infty / \nu$. The inlet Reynolds number is $Re_{L_0} = 160$. Based on the inlet momentum thickness (θ_0), the Reynolds number is $Re_{\theta_0} = 110$.

The computational domain size is $(3000 \times 200 \times 150)L_0$ in the streamwise (X), wall-normal (Y) and spanwise (Z) directions respectively (see figure 2.1). The number of cells in each direction are $2049 \times 192 \times 169$ respectively. The mesh is uniform in the streamwise and spanwise directions, while in the wall-normal direction the cells grow with expansion ratio 1.06 until $Y = 20L_0$; further away from the wall the mesh is uniform. The resulting spacings in wall units (indicated by $^+$) are $\Delta x_{max}^+ \sim 11.78$, $\Delta y_{max}^{+(w)} \sim 0.24$, and $\Delta z_{max}^+ \sim 7.14$. The subscript *max* denotes maximum values (found in the fully turbulent region), while $\Delta y_{max}^{+(w)}$ is evaluated at the centroid of the first cell close to the wall.

We let the flow develop for a sufficiently long time to reach statistically steady state and then collect statistics for 3 flow-through times. The time step is set to $0.2T_0$, which results in a maximum CFL number of around 0.35. Two-point statistics are calculated from 350 samples with time and spanwise averaging. The time separation between two samples is $20T_0$, i.e. we sample every 100 time steps. Convergence of the running average is reported in figure 2.2, where we show two examples of convergence of $\overline{du^2}$ with increasing number of samples. The location considered is at the fully turbulent region, $(X - X_0)/L_0 = 2415$, and the wall-normal distance is fixed to $Y/L_0 = 1.3$. The top panel is for a small streamwise separation $r_1/L_0 = 5$ and the bottom panel is for larger separation, $r_1/L_0 = 100$. Validation of single-point statistics with experiments is provided at section 2.5.

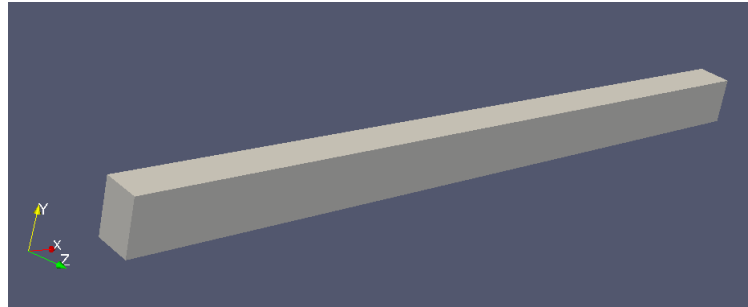


Figure 2.1 Computational domain with the size of $(3000 \times 200 \times 150)L_0$.

Figure 2.2

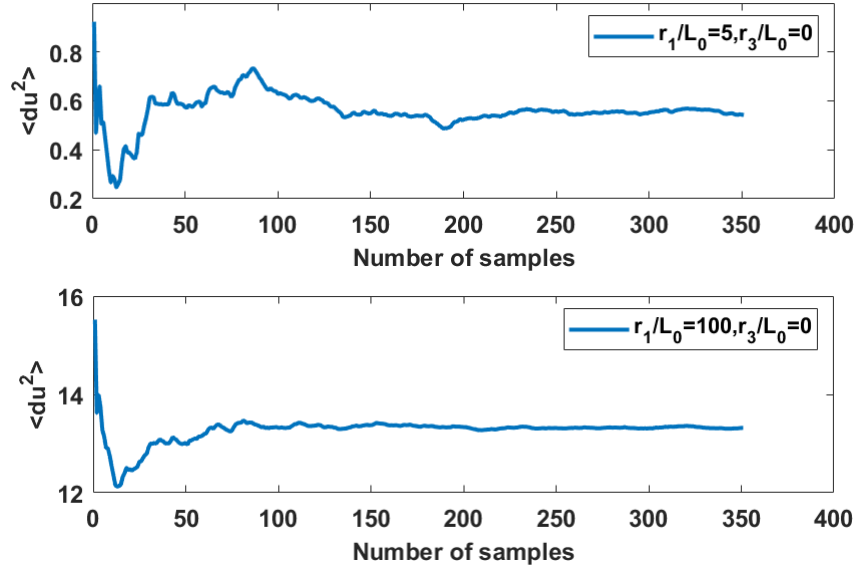


Figure 2.2 Running average of $\overline{du^2}$ versus the number of samples at a streamwise location inside the fully turbulent region, $(X - X_0)/L_0 = 2415$, at fixed height $Y/L_0 = 1.3$ and two different r_1 separations.

2.2 Numerical method

An in-house finite volume CFD solver, called Pantarhei, is used for the DNS simulation. The code has been used in the past to simulate a number of transitional and turbulent flows (Xiao and Papadakis, 2017, 2019, Alves Portela et al., 2017, 2020, Paul et al., 2018, Başbuğ et al., 2018). A fractional step method was implemented for the discretization of the equation set 2.2. A third-order backward difference scheme is used for the transient term and a third-order quadratic extrapolation is utilised to advance the convection terms in time (explicitly). The viscous terms are treated implicitly. The values (e.g, velocity and pressure) and their gradient at the finite volume surfaces are approximated using linear interpolation and central finite differencing schemes between adjacent cell centres respectively. The code is parallelised with the PETSC library (Balay et al., 2021) and that the linear systems of equations are solved using GMRES coupled with an algebraic multigrid preconditioner from the Hypre library (Falgout and Yang, 2002).

2.2.1 Fractional step method

In the incompressible flow solver, a standard incremental pressure-correction algorithm is used (Guermond et al., 2006), as explained in the following steps,

(1) Solve equation 2.3 for the predicted velocity $\hat{u}_i^{n+1}$ with pressure from the previous time step, n ,

$$\frac{\rho}{\Delta t} \left(\frac{11}{6} \hat{u}_i^{n+1} - 3u_i^n + \frac{3}{2} u_i^{n-1} - \frac{1}{3} u_i^{n-2} \right) - \frac{\partial}{\partial x_j} \left(\mu \frac{\partial \hat{u}_i^{n+1}}{\partial x_j} \right) = -\frac{\partial p^n}{\partial x_j} + f(t^{n+1}) \quad (2.3)$$

where $f(t^{n+1}) = -\frac{\partial}{\partial x_j} (3(\rho u_i u_j)^n - 3(\rho u_i u_j)^{n-1} + (\rho u_i u_j)^{n-2})$ represents the extrapolated convection term, which is moved to the right-hand side as source term. The viscous term is treated implicitly.

(2) Solve a Poisson equation for pressure correction, $\hat{\phi}^{n+1}$, which is obtained by taking the divergence of the equation 2.3 and using incompressibility condition,

$$\frac{\partial}{\partial x_i} \left(\frac{\partial \hat{\phi}^{n+1}}{\partial x_i} \right) = \frac{11\rho}{6\Delta t} \frac{\partial \hat{u}_i^{n+1}}{\partial x_i} \quad (2.4)$$

(3) Update the velocity and pressure field to obtain the values at the $n+1$ time step using equations 2.5 and 2.6.

$$\frac{\rho}{6\Delta t} (11u_i^{n+1} - 11\hat{u}_i^{n+1}) + \frac{\partial \hat{\phi}^{n+1}}{\partial x_i} = 0, \quad (2.5)$$

$$p^{n+1} = \hat{\phi}^{n+1} + p^n. \quad (2.6)$$

2.2.2 Boundary conditions

Boundary conditions are imposed to solve the differential equations of $\hat{\phi}^{n+1}$ and $\hat{u}_i^{n+1}$. At the wall, the no-slip ($\hat{u}_i^{n+1} = 0$) wall boundary condition is set (where n is the unit vector normal to the boundary); $\frac{\partial \hat{\phi}^{n+1}}{\partial n} = 0$ is implemented for the pressure. In the spanwise direction, cyclic boundary conditions are applied. For the free-stream boundary condition (top surface of the domain), a condition of zero normal and shear stress is applied, i.e, $v \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_1}{\partial x_2} \right) = 0$, $v \left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} \right) = 0$, and $v \left(-p + \frac{\partial u_2}{\partial x_2} \right) = 0$. At the outlet, a convective boundary condition, $\frac{\partial \hat{u}_i^{n+1}}{\partial t} + u_n \frac{\partial \hat{u}_i^{n+1}}{\partial n} = 0$, and $\frac{\partial \hat{\phi}^{n+1}}{\partial x_i} = 0$ are utilised where n is the direction normal to the outlet boundary surface.

At the inlet, $\frac{\partial \hat{\phi}^{n+1}}{\partial n} = 0$ is implemented for the pressure. The instantaneous velocity comprises the Blasius velocity profile to which vortical free-stream disturbances that satisfy a given spectrum are superimposed. More details are provided in the following sections.

2.2.3 Inlet free-stream turbulence

The superposition of the Blasius velocity profile and continuous modes of the OS and SQ equations (derived in appendix A) is widely used. Jacobs and Durbin (2001) for the first time used OS modes to generate the inlet conditions. A similar method was used by Brandt et al. (2004), and Hack and Zaki (2014), where the homogeneous SQ equation is also solved. The derivation of the OS and SQ equation and the corresponding boundary condition in the free-stream are provided in appendix A. Here we used a similar method; the detailed process is as follows.

(1) Step one

We start with selecting uniformly distributed wavenumbers k_1, k_2, k_3 on the surface of a spherical shell with radius $k = \sqrt{(k_1^2 + k_2^2 + k_3^2)}$, see in figure 2.3. The range of k depends on the mesh resolution and the size of the domain in the free-stream. In our case, above the boundary layer, a triply periodic domain with size $L_x \times L_y \times L_z = (150 \times 150 \times 150)L_0$ is defined. The range of the streamwise wavenumber is, $k_1 \in [\frac{2\pi}{L_x}, \frac{\pi}{\Delta x}]$, where Δx is the mesh spacing in the x direction. The size of the domain and the mesh spacing therefore define a minimum ($k_{1_{min}}$) and a maximum ($k_{1_{max}}$) wavenumber. The *min/max* values are the same in the other two directions, i.e. for k_2 and k_3 , therefore

$$|k| \in \left[\sqrt{k_{1_{min}}^2 + k_{2_{min}}^2 + k_{3_{min}}^2}, \sqrt{k_{1_{max}}^2 + k_{2_{max}}^2 + k_{3_{max}}^2} \right] \quad (2.7)$$

In the present case, the values are $|k| = [0.13, 1.03]$. An increment $dk = 0.1$ is chosen, which gives 10 shells in the wavenumber space. There are roughly 140 modes on each shell, and only positive k_2 is chosen. This is due to the fact that $+k_2$ and $-k_2$ modes already coexist in the solution of OS/SQ modes (Brandt et al., 2004).

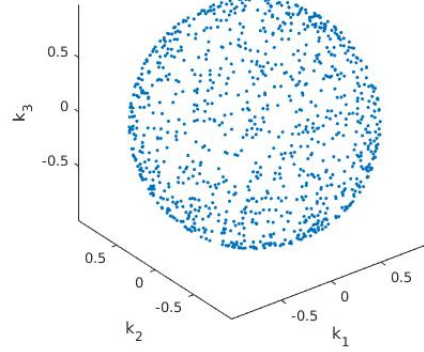


Figure 2.3 Uniformly distributed wavenumbers on the surface of a spherical shell with radius $k=1$.

(2) Step two

Using the mode wavenumbers and temporal frequency ($\omega = k_1 U_\infty$, and $U_\infty = 1$ in the freestream, then $\omega = k_1$), we solve the OS/SQ equations using a finite differencing method (see appendix A). Examples of the distribution of the real parts of the OS wall-normal modes ($\hat{v}(y)$) are shown in figure 2.4. It is clear that some modes penetrate inside the boundary layer ($y = 5$), but some do not; this is due to shear sheltering (Jacobs and Durbin, 1998). Away from the boundary layer, the modes have sinusoidal distribution. The SQ modes ($\hat{\eta}(y)$) have similar shape so that they are not shown here.

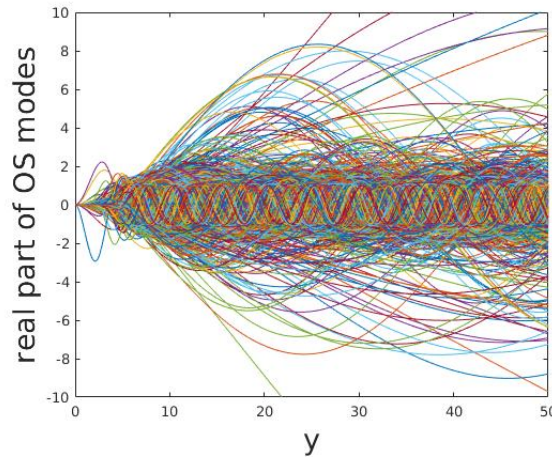


Figure 2.4 Real part of $\hat{v}(y)$ profile.

(3) Step three

Inlet free-stream turbulence satisfies a Von Karman energy spectrum,

$$E(k) = Tu_{FS}^2 \frac{1.196(kL_{11})^4}{[0.558 + (kL_{11})^2]^{17/6}} L_{11}, \quad Tu_{FS} = 3.4\%, \quad L_{11} = 5 \quad (2.8)$$

where Tu_{FS} is the turbulent intensity in the free-stream. Note that in the experiment, the free-stream turbulence intensity is set to 3.3%, however in the current simulation 3.4% is used in order to match the transition onset. The difference can be caused by other factors such as free-stream integral length scale, resolved energy spectrum etc. The free-stream integral length scale $L_{11} = 5$ (normalized by L_0) is chosen so that $L_{11}/\delta_0 = 1$, where δ_0 is the boundary layer thickness at the inlet.

Using the generation method for the HIT developed by Rogallo (1981), it is possible to compute the amplitude of the modes so that they reproduce the spectrum given above. Rogallo's method is based on the Craya-Herring Frame of Reference (Sagaut and Cambon, 2008), which is defined on a tangent plane normal to the wave number vector $\vec{k} = (k_1, k_2, k_3)$. The unit vectors, $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$, in this coordinate system are related to the unit vectors, $(\hat{i}, \hat{j}, \hat{k})$, of the Cartesian system with,

$$\vec{e}_1 = \vec{e}_2 \times \vec{e}_3 = \frac{k_1 k_2}{k k_{13}} \hat{i} - \frac{k_{13}}{k} \hat{j} + \frac{k_2 k_3}{k k_{13}} \hat{k}, \quad (2.9)$$

$$\vec{e}_2 = -(\vec{k} \times \vec{n})/|\vec{k} \times \vec{n}| = \frac{k_3}{k_{13}} \hat{i} - \frac{k_1}{k_{13}} \hat{k}, \quad (2.10)$$

$$\vec{e}_3 = \vec{k}/|k| = \frac{k_1}{|k|} \hat{i} + \frac{k_2}{|k|} \hat{j} + \frac{k_3}{|k|} \hat{k} \quad (2.11)$$

where a vertical unit vector is chosen $\vec{n} = (0, 1, 0)$, and $k = \sqrt{k_1^2 + k_2^2 + k_3^2}$, $k_{13} = \sqrt{k_1^2 + k_3^2}$. $\vec{e}_3$ is assumed to be along the $\vec{k}$ direction.

The velocity fluctuation can be written in this coordinate system as,

$$\vec{u} = A\vec{e}_1 + B\vec{e}_2 + C\vec{e}_3 \quad (2.12)$$

and taking the dot product with the wavenumber vector $\vec{k}$, we obtain,

$$\vec{u} \cdot \vec{k} = (A\vec{e}_1 + B\vec{e}_2 + C\vec{e}_3) \cdot \vec{k} \quad (2.13)$$

Due to divergence-free property of the velocity field (for the incompressible flow in Fourier space, the continuity equation satisfies $\vec{u} \cdot \vec{k} = 0$), the left hand side equals to 0. For the right hand side, $\vec{e}_1$ and $\vec{e}_2$ are normal to $\vec{e}_3$, so that coefficient C has to be 0.

Therefore, the divergence-free velocity field in Fourier space has only two components (along the directions $\vec{e}_1$ and $\vec{e}_2$) in the Craya-Herring frame. Converting the velocity from the $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ frame back to $(\hat{i}, \hat{j}, \hat{k})$ frame, we obtain,

$$\begin{aligned}\vec{u}(\vec{k}) &= A\vec{e}_1 + B\vec{e}_2 \\ &= \left(\frac{Ak_1k_2}{kk_{13}} + \frac{Bk_3}{k_{13}}\right)\hat{i} - \left(\frac{Ak_{13}}{k}\right)\hat{j} + \left(\frac{Ak_2k_3}{kk_{13}} - \frac{Bk_1}{k_{13}}\right)\hat{k} \\ &= \tilde{u}(\vec{k})\hat{i} + \tilde{v}(\vec{k})\hat{j} + \tilde{w}(\vec{k})\hat{k}\end{aligned}\quad (2.14)$$

It can be easily proved that $AA^* + BB^* = \tilde{u}\tilde{u}^* + \tilde{v}\tilde{v}^* + \tilde{w}\tilde{w}^*$ where $*$ denotes complex conjugate. The next step is to match the energy spectrum E_k . This is achieved by forming the random coefficients $A(k) = F(k)e^{i\theta_1}\cos\phi$ and $B(k) = F(k)e^{i\theta_2}\sin\phi$, where $F(k)$ can be obtained from the energy spectrum E_k and θ_1 , θ_2 and ϕ are random angles between 0 and 2π .

Similar to Brandt et al. (2004), the modes are integrated on each wavenumber shell (N modes on each shell), therefore,

$$TKE = \int_S E(k)dk \quad (2.15)$$

where

$$E(k)dk = \sum_N \frac{\tilde{u}\tilde{u}^* + \tilde{v}\tilde{v}^* + \tilde{w}\tilde{w}^*}{2} = \sum_N \frac{AA^* + BB^*}{2} = \sum_N \frac{F(k)^2}{2} \approx N \frac{F(k)^2}{2} \quad (2.16)$$

leading to,

$$F(k) = \sqrt{\frac{2E(k)dk}{N}} \quad (2.17)$$

(4) Step four

We can couple the solution of the homogeneous OS/SQ equation ($\hat{v}$ and $\hat{\eta}$) with the coefficients $A(k)$ and $B(k)$. Similar to Hack and Zaki (2014), we define,

$$\hat{v}_{scaled} = -\frac{Ak_{13}}{k} \frac{\hat{v}}{C_{os}}, \quad \hat{\eta}_{scaled} = iBk_{13} \frac{\hat{\eta}}{C_{sq}}, \quad (2.18)$$

where C_{os} and C_{sq} are the average magnitude of the complex numbers of $\hat{v}$ and $\hat{\eta}$ in the free-stream. The streamwise and spanwise components can be expressed by $\hat{v}_{scaled}$ and $\hat{\eta}_{scaled}$ using continuity equation and the definition of wall-normal vorticity,

$$\begin{aligned}
\hat{u}_{scaled} &= \frac{ik_1}{k_{13}^2} \frac{\partial \hat{v}_{scaled}}{\partial y} - \frac{ik_3}{k_{13}^2} \hat{\eta}_{scaled}, \\
\hat{v}_{scaled} &= \hat{v}_{scaled}, \\
\hat{w}_{scaled} &= \frac{ik_3}{k_{13}^2} \frac{\partial \hat{v}_{scaled}}{\partial y} + \frac{ik_1}{k_{13}^2} \hat{\eta}_{scaled}.
\end{aligned} \tag{2.19}$$

The real part of $\hat{v}_{scaled}$ along the wall-normal direction y is shown in the figure 2.5. It is evident that the amplitude of mode is decreased significantly in order to match the spectrum energy, while inside the boundary layer the characteristics of OS/SQ modes are maintained. The energy spectrum $E(k)$ is calculated through modes by $E(k) = \frac{1}{dk} \sum_N \frac{\tilde{u}_{scaled} \tilde{u}_{scaled}^* + \tilde{v}_{scaled} \tilde{v}_{scaled}^* + \tilde{w}_{scaled} \tilde{w}_{scaled}^*}{2}$ and is shown in figure 2.6.

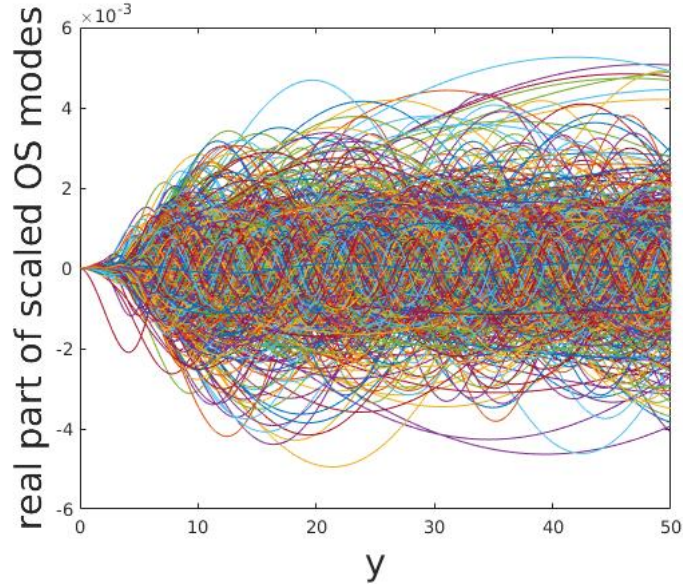


Figure 2.5 Real part of $\hat{v}_{scaled}$ profile

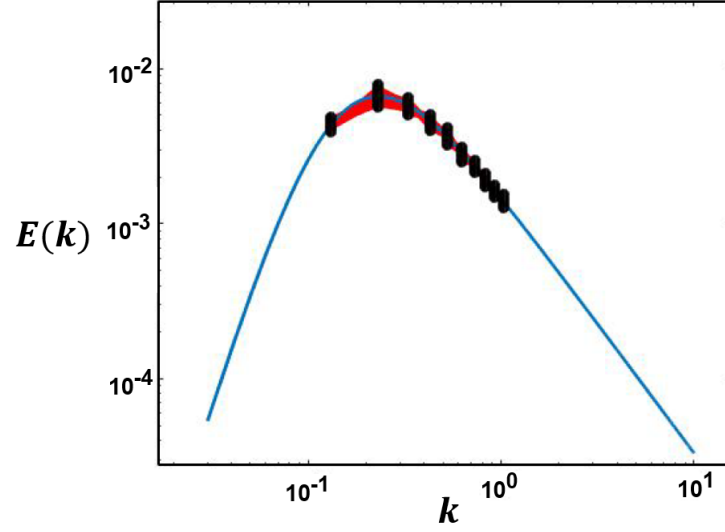


Figure 2.6 Resolved energy of the Von Karman spectrum. The black dots represent the 10 wavenumber shells with radius k .

Each black ‘cluster’ represents a wavenumber sphere. The blue line is the analytic spectrum as Eqn.2.8. It can be seen that the given spectrum is well reproduced.

(6) Final step

Evaluate inlet conditions from expressions below,

$$\begin{aligned} u_{inlet} &= U_{Blasius} + Re \left(\sum \hat{u}_{scaled}(y) e^{i(k_3 z - \omega t)} \right), \\ v_{inlet} &= V_{Blasius} + Re \left(\sum \hat{v}_{scaled}(y) e^{i(k_3 z - \omega t)} \right), \\ w_{inlet} &= 0 + Re \left(\sum \hat{w}_{scaled}(y) e^{i(k_3 z - \omega t)} \right). \end{aligned} \quad (2.20)$$

2.3 Visualisation of instantaneous turbulent spots

In figure 2.7, we use the instantaneous wall-normal fluctuation velocity, v' , to visualise turbulent structures (spots) at $Y/L_0 = 5$. A large and fully formed structure can be seen on the right of the top panel. Note the rapid change of the sign of v over the structure. A smaller structure, just forming, can be discerned in the area $(X - X_0)/L_0 \approx 1050 - 1100$ and $Z/L_0 \approx 25$. We track the growth of this turbulent structure at 4 time instants. The growth of the structure in the streamwise and spanwise directions is evident.

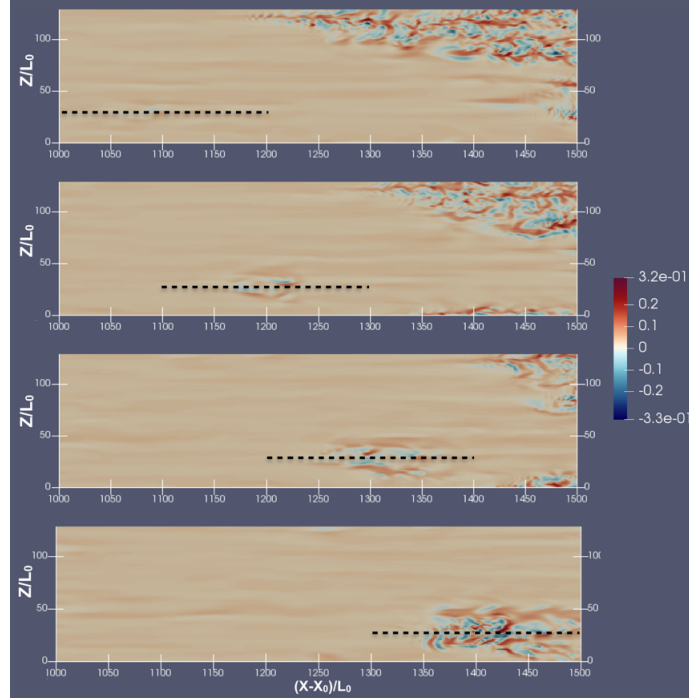


Figure 2.7 Flow snapshots tracking a spot from its incipient formation (top row) to fully developed form (bottom row); contour plots of the wall-normal velocity fluctuations at $Y = 5L_0$. The time interval between each snapshot is $190T_0$.

In figure 2.8, we show contours of the instantaneous streamwise velocity through the centre of the spot, at the same time instants. The horizontal domain is shifted in the streamwise direction in order to account for the translating motion of the structure; the domain shown corresponds to the horizontal dashed lines in the panels of figure 2.7. On the top panel, we see the lifting up of the low speed streak (negative fluctuating velocities). The velocity field at this time instant is smooth, but in the panel below the footprints of secondary instability start to appear at $Y \sim 12L_0$, $(X - X_0)/L_0 \sim 1180 - 1220$. The turbulent motion moves closer to the wall and fills the boundary layer below $10L_0$; a fully formed spot has been formed, as can be seen in the bottom row. The observed growth of spots cannot be captured with single-point statistics, but two-point statistics, especially ones conditioned on turbulent events, is a very effective statistical tool for this purpose.

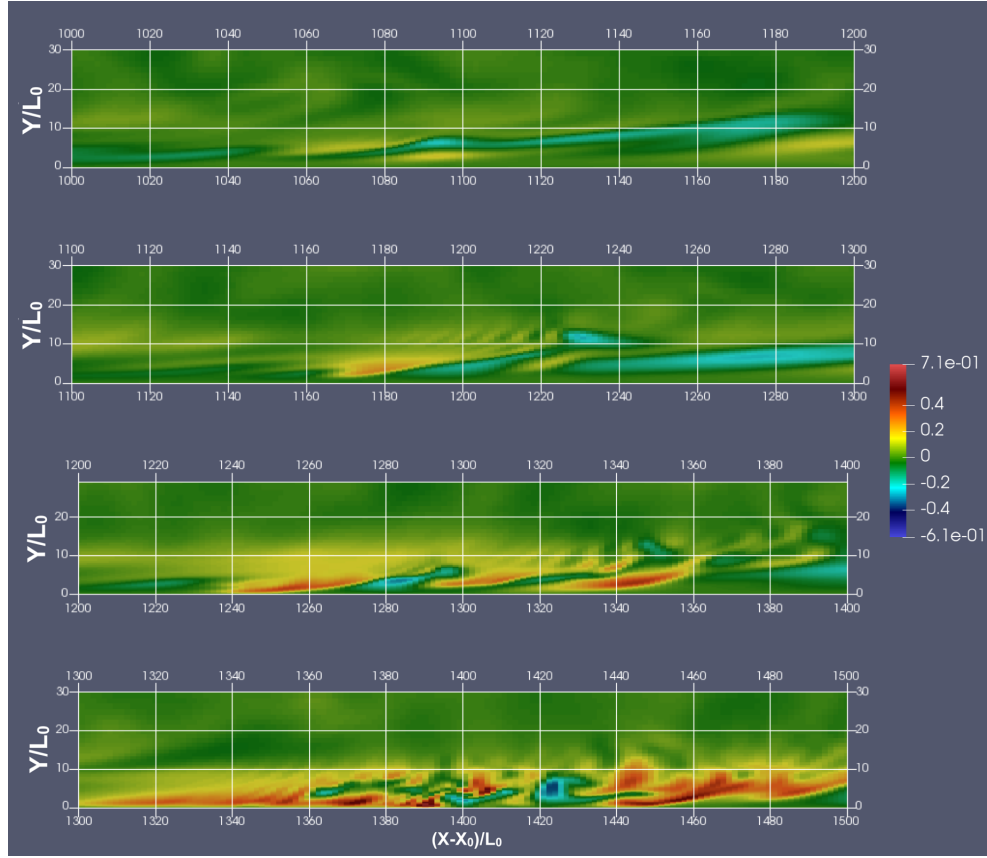


Figure 2.8 Contours of the instantaneous streamwise velocity through the centre of the spot. The translation of the domain visualised in the streamwise direction corresponds to the horizontal dashed lines in the figure 2.7

2.4 Binary intermittency field

We define an instantaneous binary indicator function, $\tau(X, Y, Z, t)$, equal to 1 if the flow is turbulent and 0 if it is laminar. The value of the binary indicator function at one point is computed based on the standard deviation of $D = |v'| + |w'|$ of the surrounding points at a fixed height, where v and w are the wall-normal and spanwise velocities fluctuations respectively. The surrounding points form a star-shaped pattern as shown below

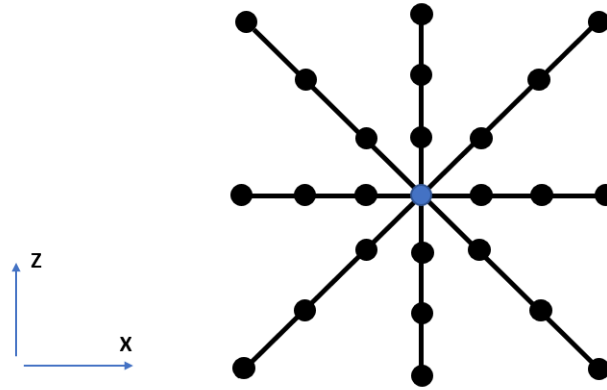


Figure 2.9 Sketch of the points chosen to compute the standard deviation. The blue point is the local center point.

We found that the number of the points used to define the star affects the resolution of the indicator function, for example if too few points are chosen, sporadic gaps with 0 values can be found within the fully turbulent region and within the turbulent spots. So we have empirically chosen 25 neighbouring points around a central point to calculate the local standard deviation (shown in figure 2.9). The optimal threshold value of D at one fixed height is determined using Otsu's method (Otsu, 1979). The method identifies turbulent and non-turbulent regions very well. Figure 2.10 shows one example of identifying the boundary of one turbulent spot. More recently, machine learning algorithms have been used to detect the laminar/turbulent interface (Wu et al., 2020).

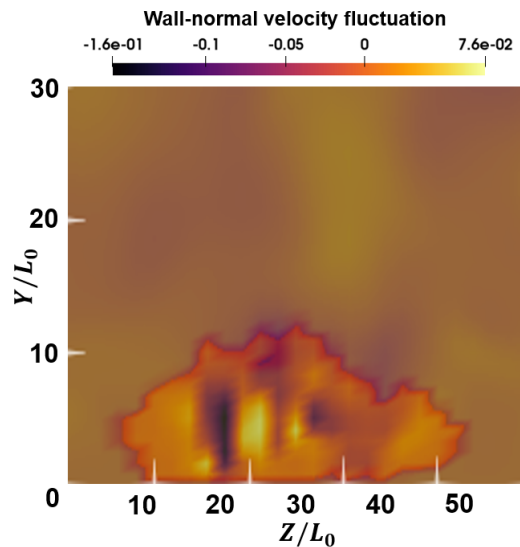


Figure 2.10 Identification of the laminar/turbulent interface (boundary of the one turbulent spot) in the (Y, Z) plane.

At each computational point (x, y, z) , we have stored velocities, pressure and the binary indicator function, i.e., (u, v, w, p, τ) . These 350 instantaneous flow snapshots make up the DNS database.

2.5 Validation

Our bypass transition DNS data base refers to the T3A flat plate experiment Roach (1990). In this section, mean properties (time- and spanwise averaged) and TKE budget are calculated to validate the DNS database. We also compare against the numerical results of Jacobs and Durbin (2001) who have simulated the same case.

2.5.1 Mean statistics

To check the implementation of the inlet conditions, in panel (a) of figure 2.11 we compare the decay rate of the free-stream turbulence intensity against the results of Jacobs and Durbin (2001); we notice that the two sets of results match well. The skin friction coefficient, C_f , against the momentum thickness, Re_θ , also matches well with the T3A experiments (Roach, 1990), as shown in panel (b) of figure 2.11. There is a small deviation for $250 < Re_\theta < 350$, where the current result is slightly lower than the experimental data. In Jacob's work, the C_f is slightly overestimated (refer to figure 4 in Jacobs and Durbin (2001)).

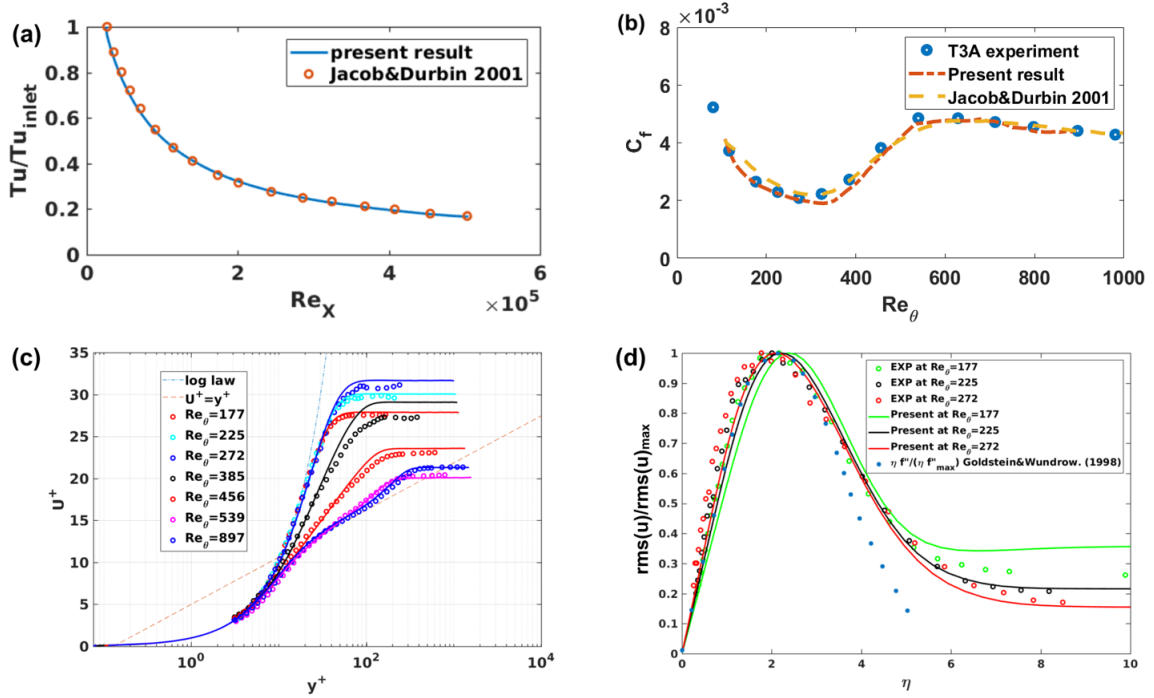


Figure 2.11 Panels (a) Free-stream turbulent intensity, Tu against Re_x and (b) Skin friction coefficient, C_f , against Re_θ are compared with DNS data of Jacobs and Durbin (2001); (c) Mean streamwise velocity, U^+ against y^+ (solid lines are the present DNS results, circles are the T3A experiment results (Roach, 1990)); (d) $rms(u)/rms(u)_{max}$ against the Blasius variable, η .

In panel (c) we compare the mean streamwise velocity (in wall units) against the T3A experiment (Roach, 1990). In the late transition and fully turbulent region the matching is good, however, the mean velocity in the transition region is more difficult to match. This is due to differences in the free-stream conditions that are very difficult to reproduce exactly. Panel (d) displays the profile of $rms(u)/rms(u)_{max}$ against the Blasius variable, η . The circles are calculated using the raw data from the T3A database (Roach, 1990). The profile matches well (up to $\eta \approx 4$) with the normalised shape of $\eta F''$, where F is the Blasius function. The latter formula was originally used by Taylor (1939) and Klebanoff (1971) for experimental data fitting. The streamwise velocity fluctuation profile clearly indicates the presence of the so-called Klebanoff modes.

2.5.2 Turbulent Kinetic Energy budget

In order to provide further evidence regarding the accuracy of the database, we compute the budgets of the TKE equation. The TKE equation takes form,

$$\underbrace{\overline{u'_i u'_j \frac{\partial U_i}{\partial x_j}}}_{\text{Production}(\mathcal{P}_k)} + \underbrace{\frac{1}{2} \overline{\frac{\partial U_j u'_i u'_i}{\partial x_j}}}_{\text{Convection}(\mathcal{A}_k)} + \underbrace{\frac{1}{2} \overline{\frac{\partial u'_i u'_i u'_j}{\partial x_j}}}_{\text{Turbulent diffusion}(\mathcal{T}_k)} = \underbrace{-\frac{1}{\rho} \overline{\frac{\partial p' u'_i}{\partial x_i}}}_{\text{Pressure}(\mathcal{P}_{p,k})} + \underbrace{\frac{\nu}{2} \overline{\frac{\partial}{\partial x_j} \left(\frac{\partial u'_i u'_i}{\partial x_j} \right)}}_{\text{Viscous diffusion}(\mathcal{D}_k)} - \underbrace{\nu \overline{\frac{\partial u'_i}{\partial x_j} \frac{\partial u'_i}{\partial x_j}}}_{\text{Dissipation}(\mathcal{E}_k)} \quad (2.21)$$

We calculate each term of the TKE equation at downstream fully turbulent region (at location $Re_\theta = 800$). Our DNS simulation matches very well with Spalart's DNS data (Spalart, 1988) (see figure. 2.12). Further validation regarding the variation of intermittency along the transition region is provided in chapter 5. We also provide evidence of balancing the KMH equation in chapter 3.

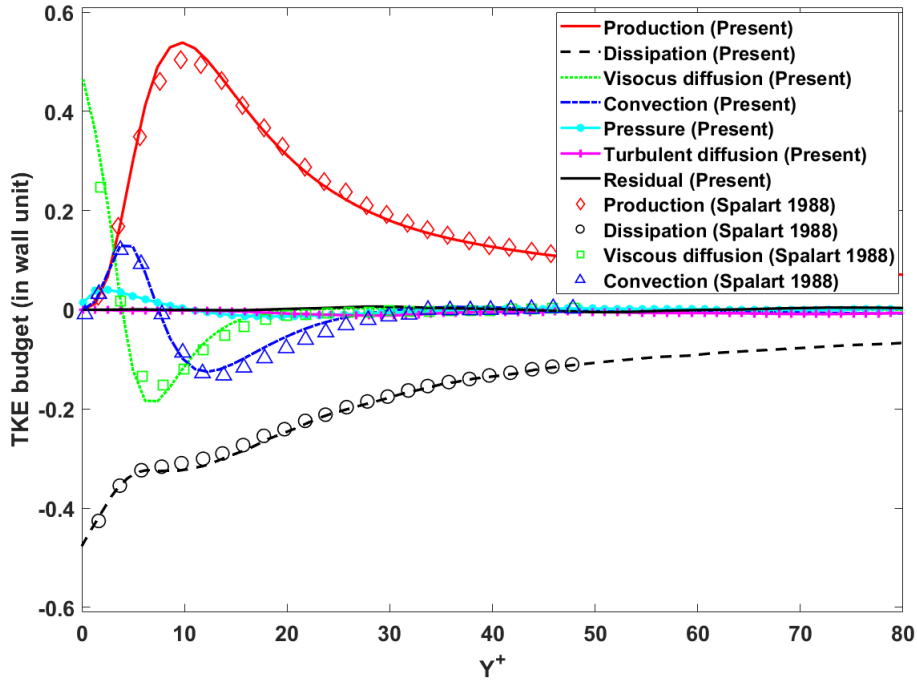


Figure 2.12 TKE budget at fully turbulent region. The lines are the results from the present database for $Re_\theta = 800$; the open symbols are from Spalart (1988) for $Re_\theta = 1410$.

Chapter 3

Karman-Howarth-Monin-Hill equation

3.1 Standard KMH equation

The KMH equation is a balance equation for the second-order structure function, dq^2 , defined as $dq^2 = (du'_i)^2 = (u'^+_i - u'^-_i)^2$, where u'^+_i and u'^-_i are the velocity fluctuations around the time-averages U_i^+ and U_i^- at two points x_i^+ and x_i^- , respectively (refer to schematic figure 3.1). Denoting the midpoint as $X_i = 0.5(x_i^+ + x_i^-)$ and the separation between the two points as $r_i = x_i^+ - x_i^-$, it is evident that for a general three-dimensional inhomogeneous and anisotropic flow, dq^2 is a function of 3 spatial and 3 separation (or scale) variables, i.e. $dq^2(X_i, r_i)$. The transport equation of $\overline{dq^2(X_i, r_i)}$ takes the form, refer to Hill (2002a).

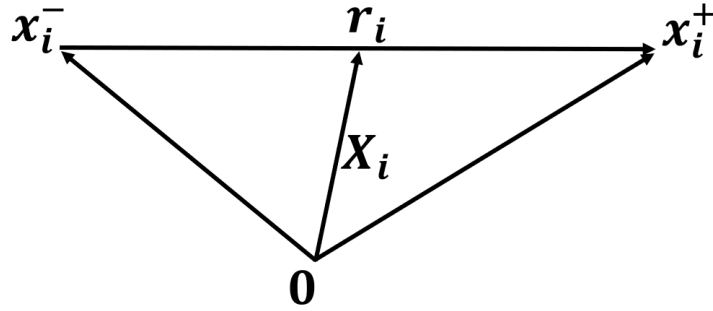


Figure 3.1 The relationship between x_i^+ , x_i^- and X_i , r_i .

$$\begin{aligned}
& \underbrace{\frac{\overline{\partial dq^2}}{\partial t}}_{\text{Transient term } (\mathcal{A}_t)} + \underbrace{\frac{\overline{\partial \frac{(U_j^+ + U_j^-)}{2} dq^2}}{\partial X_j}}_{\text{Mean flow advection } (\mathcal{A}_U)} + \underbrace{\frac{\overline{\partial du'_j dq^2}}{\partial r_j}}_{\text{Nonlinear interscale transfer } (\Pi)} + \underbrace{\frac{\overline{\partial dU_j dq^2}}{\partial r_j}}_{\text{Linear interscale transfer } (\Pi_U)} = \\
& \underbrace{-\frac{\overline{\partial \frac{(u_j'^+ + u_j'^-)}{2} dq^2}}{\partial X_j}}_{\text{Turbulent advection } (\mathcal{A}_u)} - \underbrace{2\frac{\overline{\partial du'_j dp'}}{\partial X_j}}_{\text{Pressure-velocity correlation } (\mathcal{T}_p)} + \underbrace{\frac{1}{2}\nu \frac{\overline{\partial^2 dq^2}}{\partial X_j \partial X_j}}_{\text{Diffusion in physical space } (\mathcal{D}_x)} + \underbrace{2\nu \frac{\overline{\partial^2 dq^2}}{\partial r_j \partial r_j}}_{\text{Diffusion in scale space } (\mathcal{D}_r)} \\
& \underbrace{-2\overline{du'_j du'_i} \frac{\partial dU_i}{\partial r_j} - \overline{(u_j'^+ + u_j'^-)} du'_i \frac{\partial dU_i}{\partial X_j}}_{\text{Production by mean flow } (\mathcal{P})} - \underbrace{4\nu \left(\frac{1}{4} \frac{\overline{\partial du'_i}}{\partial X_j} \frac{\overline{\partial du'_i}}{\partial X_j} + \frac{\overline{\partial du'_i}}{\partial r_j} \frac{\overline{\partial du'_i}}{\partial r_j} \right)}_{\text{Dissipation } (\varepsilon)}
\end{aligned} \tag{3.1}$$

The detailed derivation process is presented in appendix B. As noted in the introduction, $\overline{dq^2}$ can be interpreted as the energy of eddies with size smaller than $r = |r_i|$. The physical meaning of each term is given below:

- $\mathcal{A}_t = \frac{\overline{\partial dq^2}}{\partial t}$ represents the rate of change of $\overline{dq^2}$ in time.
- $\mathcal{A}_U = \frac{\overline{\partial \frac{(U_j^+ + U_j^-)}{2} dq^2}}{\partial X_j}$ is the advection term due to mean flow.
- $\Pi = \frac{\overline{\partial du'_j dq^2}}{\partial r_j}$ is the inter-scale energy transfer term due to non-linear interaction.
- $\Pi_U = \frac{\overline{\partial dU_j dq^2}}{\partial r_j}$ represents the interscale transfer due to mean flow field.
- $\mathcal{A}_u = -\frac{\overline{\partial \frac{(u_j'^+ + u_j'^-)}{2} dq^2}}{\partial X_j}$ represents the transport of scale energy due to turbulent fluctuations in the physical space.
- $\mathcal{T}_p = \frac{\overline{\partial \frac{(u_j'^+ + u_j'^-)}{2} dq^2}}{\partial X_j}$ is the pressure and velocity correlation. Note that if we define $p'_* = \frac{p'^+ + p'^-}{2}$ as the midpoint value of the fluctuating pressure, it is easy to prove that $\frac{\overline{\partial du'_j dp'}}{\partial X_j} = 8 \frac{\overline{\partial du'_j p'_*}}{\partial r_i}$ (see appendix C), so the pressure-velocity correlation term can be written in flux divergence form either in physical or scale space (here we write this term in physical space).
- $\mathcal{D}_x = \frac{1}{2}\nu \frac{\overline{\partial^2 dq^2}}{\partial X_j \partial X_j}$ and $\mathcal{D}_r = 2\nu \frac{\overline{\partial^2 dq^2}}{\partial r_j \partial r_j}$ are the viscous diffusion terms in physical and scale spaces.

- $\mathcal{P} = -2\overline{du'_j du'_i} \frac{\partial du_i}{\partial r_j} - (\overline{u'^+_{j*}} + \overline{u'^-_{j*}}) \overline{du'_i} \frac{\partial du_i}{\partial X_j}$ is the production term due to mean flow derivatives.
- $\varepsilon = 4\nu \left(\frac{1}{4} \frac{\partial \overline{du'_i}}{\partial X_j} \frac{\partial \overline{du'_i}}{\partial X_j} + \frac{\partial \overline{du'_i}}{\partial r_j} \frac{\partial \overline{du'_i}}{\partial r_j} \right)$ is the dissipation term.

3.2 Different forms the KMH equation and relation to the TKE equation

To provide a deeper understanding of the KMH equation, we present here, (1) the divergence form; (2) the link between the KMH equation and the classic Kolmogorov equation and how to get $-4/5$ th law; (3) the link between the KMH equation and the TKE equation.

3.2.1 Divergence form of the KMH equation

For statistically steady flows, as the one is considered in the present thesis, the transient term $\mathcal{A}_t = 0$. Defining $U_{j*} = \frac{U_j^+ + U_j^-}{2}$ and $u'_{j*} = \frac{u'^+_{j*} + u'^-_{j*}}{2}$ as the midpoint values of the mean and fluctuating velocities respectively, and grouping together the terms computed in physical (X_i) and scale (r_i) spaces, equation (3.1) can be rearranged as

$$\begin{aligned}
 & \underbrace{\frac{\partial [\overline{U_{i*} dq^2} + \overline{u'_{j*} dq^2} + 2\overline{du'_j dp'} - \frac{1}{2} \nu \frac{\partial dq^2}{\partial x_j}]}_{\text{Transport in physical space}} \frac{\partial}{\partial X_j} + \underbrace{\frac{\partial [\overline{du'_j dq^2} + \overline{dU_j dq^2} - 2\nu \frac{\partial dq^2}{\partial r_j}]}_{\text{Transport in scale space}} \frac{\partial}{\partial r_j} = \\
 & \underbrace{-2\overline{du'_i du'_j} \frac{\partial du_i}{\partial r_j} - 2\overline{du'_i u'_{j*}} \frac{\partial du_i}{\partial X_j}}_{\text{Production}} - \underbrace{4\nu \left(\frac{1}{4} \frac{\partial \overline{du'_i}}{\partial X_j} \frac{\partial \overline{du'_i}}{\partial X_j} + \frac{\partial \overline{du'_i}}{\partial r_j} \frac{\partial \overline{du'_i}}{\partial r_j} \right)}_{\text{Dissipation}}
 \end{aligned} \tag{3.2}$$

Equation 3.2 can be written as the divergence of fluxes in the two spaces as

$$\frac{\partial \phi_{sj}}{\partial X_j} + \frac{\partial \phi_{rj}}{\partial r_j} = \mathcal{P} - \varepsilon \equiv \mathcal{P}_{net}, \tag{3.3}$$

and define

$$\phi_{sj} = \underbrace{\overline{U_{j*} dq^2}}_{=\phi_{sj}^M} + \underbrace{\overline{u'_{j*} dq^2}}_{=\phi_{sj}^F} + \underbrace{\overline{2du'_j dp'}}_{=\phi_{sj}^P} - \underbrace{\frac{1}{2} \nu \frac{\partial dq^2}{\partial X_j}}_{=\phi_{sj}^V}, \tag{3.4}$$

as the flux vector in physical space and

$$\phi_{rj} = \underbrace{\overline{du'_j dq^2}}_{=\phi_{rj}^F} + \underbrace{\overline{dU_j dq^2}}_{=\phi_{rj}^M} - \underbrace{2v \frac{\partial dq^2}{\partial r_i}}_{=\phi_{rj}^V}, \quad (3.5)$$

as the flux vector in scale space. As before,

$$\mathcal{P} = -2\overline{du'_j du'_i} \frac{\partial dU_i}{\partial r_j} - 2\overline{u'_{j*} du'_i} \frac{\partial dU_i}{\partial X_j} \quad (3.6)$$

is the production term and

$$\varepsilon = 4v \left(\frac{1}{4} \frac{\partial \overline{du'_i}}{\partial X_j} \frac{\partial \overline{du'_i}}{\partial X_j} + \frac{\partial \overline{du'_i}}{\partial r_j} \frac{\partial \overline{du'_i}}{\partial r_j} \right) = 2(\varepsilon^+ + \varepsilon^-) = 4\varepsilon^* \quad (3.7)$$

is the dissipation term. In the above equation, $\varepsilon^+ = v \frac{\partial u'^+}{\partial x_j^+} \frac{\partial u'^+}{\partial x_j^+}$ and $\varepsilon^- = v \frac{\partial u'^-}{\partial x_j^-} \frac{\partial u'^-}{\partial x_j^-}$ are the local dissipation at points x_i^+ and x_i^- respectively, while $\varepsilon^* = \frac{\varepsilon^+ + \varepsilon^-}{2}$ is the midpoint value. Note that in (3.3) we have also defined the net production as $\mathcal{P}_{net} = \mathcal{P} - \varepsilon = \mathcal{P} - 4\varepsilon^*$.

In equations (3.4) and (3.5) we have decomposed the flux vectors into constituent components due to mean flow, turbulent fluctuations, pressure and viscous diffusion; these are denoted with superscripts M , F , P and V respectively, i.e. $\phi_s = \phi_s^M + \phi_s^F + \phi_s^V + \phi_s^P$ and $\phi_r = \phi_r^M + \phi_r^F + \phi_r^V$. In summary, we have 4 flux vector pairs arising from different mechanisms, (ϕ_s^M, ϕ_r^M) , (ϕ_s^F, ϕ_r^F) , (ϕ_s^V, ϕ_r^V) , $(\phi_s^P, 0)$.

This equation allows us to take a broader view of energy cascade here compared to Richardson's view as expressed in his poem, "Big whorls have little whorls Which feed on their velocity, and little whorls have lesser whorls And so on to viscosity". This means that large eddies transfer energy to the smaller ones and finally dissipate due to viscosity. However, energy cascade can be forward (as described by Richardson) or inverse depending on the local flow conditions. One of the issues is that there is no such strict definition of 'eddy'. However, using simple geometry, we may classify the eddies using six dimensions. The sketches presented in figure 3.2 demonstrate this idea.

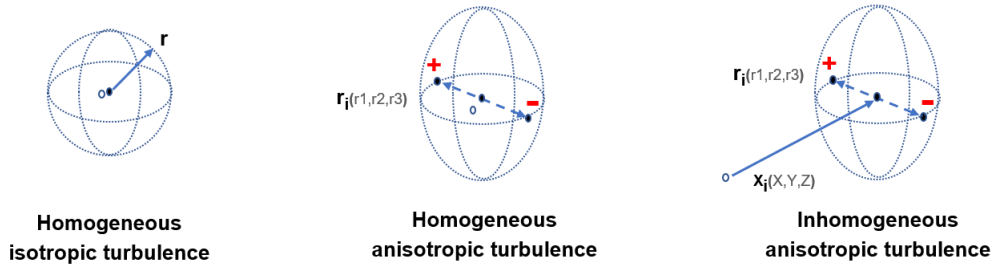


Figure 3.2 Sketches for different types of turbulent eddies.

For the HIT, the average eddy can be considered as a sphere. The statistics are independent of the orientation and physical location of the eddy, so only one parameter is needed to describe the eddy, its radius r (see the left plot in figure 3.2). For homogeneous but anisotropic turbulence, the orientations need to be considered because the statistics now depend on the direction. We define the shape of eddies using the separations r_1, r_2, r_3 in streamwise, wall-normal, spanwise directions (see the middle plot in figure 3.2). For the most general case, i.e, inhomogeneous and anisotropic turbulence, the spatial locations of the eddy are also important, which is indicated by the coordinates of the middle point between x_i^+ and x_i^- (see the right plot in figure 3.2).

The cascade process in inhomogeneous and anisotropic turbulence can be quantified as the energy transfer process between scales that are defined by the separation of the two points. For example, If $r_2 = 0$ is assumed, we consider a 'circle-like' eddy with the shape as shown in figure 3.3 left; if $r_2 = 0, r_3 = 0$, a 'streak-like' eddy is considered (see figure 3.3, right).

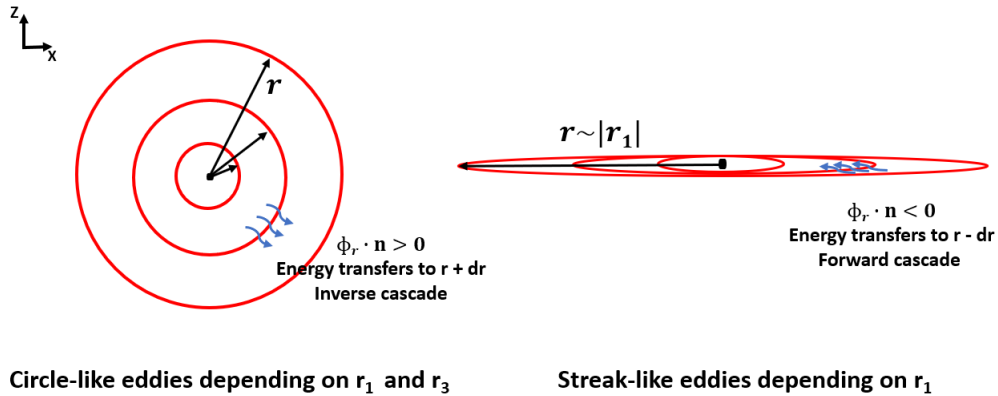


Figure 3.3 Sketches for the energy transfer between scales.

The forward or inverse cascades can be quantified by integrating the divergence of scale flux vector (ϕ_{r_i}) over a volume defined by the separations r_1, r_2, r_3 . By using the Gauss' divergence theorem,

$$\int_{r_i} \frac{\partial \phi_{ri}}{\partial r_i} dr_i = \int_{S_r} (\phi_r \cdot n) dS \quad (3.8)$$

where n is the unit vector normal to the volume boundary, we convert the volume integral to a surface integral. If $\phi_r \cdot n < 0$ is negative, the energy is transferred inwards, i.e, towards the smaller scale structures, representing a local forward cascade (in the direction n), if $\phi_r \cdot n > 0$, the energy is transferred outwards, i.e, towards the larger scale structures, representing a local inverse cascade. Note that the divergence of the energy flux vector also provides important quantitative information about energy gathering ($\frac{\partial \phi_{ri}}{\partial r_i} > 0$) or releasing ($\frac{\partial \phi_{ri}}{\partial r_i} < 0$) (Chiarini et al., 2022).

3.2.2 Relation to the $-4/5$ law

It is possible to write equation (3.3) in an even more compact form as,

$$\frac{\partial \phi_{\xi_i}}{\partial \xi_i} = \mathcal{P} - 4\varepsilon^*, \quad (3.9)$$

where $\xi = (x_1, x_2, x_3, r_1, r_2, r_3)$ is the composite position vector and $\phi_\xi(\xi) = (\phi_{s1}, \phi_{s2}, \phi_{s3}, \phi_{r1}, \phi_{r2}, \phi_{r3})$ is a 6-dimensional flux vector, with each component depending on ξ .

This equation is general and valid for inhomogeneous and anisotropic flows. Useful simplifications can be made for special flow cases. For example, if the flow is homogeneous in all three directions (but not necessarily isotropic), both the flux term in physical space $\frac{\partial \phi_{si}}{\partial x_i}$ and the production term $\mathcal{P}$ vanish, and therefore $\xi = (r_1, r_2, r_3)$ and $\phi_\xi(\xi) = (\phi_{r1}, \phi_{r2}, \phi_{r3})$. In this case, (3.9) simplifies to

$$\frac{\partial \phi_{ri}}{\partial r_i} = -4\varepsilon^*, \quad (3.10)$$

where $\phi_{ri} = \overline{du'_i dq^2} - 2\nu \frac{\partial \overline{dq^2}}{\partial r_i}$. Most importantly, using Taylor series expansions of ε^+ and ε^- around point X_i , Hill (2002b) proves that ε^* does not depend on the separation r_i , i.e. it is constant. Integrating over a sphere of radius r in scale space (with volume $V_r = \frac{4}{3}\pi r^3$ and surface $S_r = 4\pi r^2$), and using the Gauss' divergence theorem, $\int_{V_r} \frac{\partial \phi_{ri}}{\partial r_i} dV = \int_{S_r} (\phi_{ri} n_i) dS$, where $\phi_{ri} n_i$ is the flux normal to the spherical boundary, (3.10) becomes

$$\frac{1}{4\pi r^2} \int_{S_r} \overline{du'_n dq^2} - 2\nu \frac{\partial \overline{dq^2}}{\partial r_n} dS = -\frac{4}{3}\varepsilon^* r, \quad (3.11)$$

where $du'_n = du'_i n_i$. If in addition the flow is isotropic, the normal flux $\phi_{r_n} = \phi_r \cdot n = \overline{du'_n dq^2} - 2\nu \frac{\partial \overline{du'^2_n}}{\partial r_n}$ is independent of the orientation n . Using the isotropic relations for second and third order tensors together with the continuity equation, (3.11) is simplified to the well known $-4/5$ -th law

$$\overline{du'^3_n} - 6\nu \frac{\partial \overline{du'^2_n}}{\partial r_n} = -\frac{4}{5}\epsilon^* r, \quad (3.12)$$

refer to Hill (2002a) for the details. This equation represents the limiting form of (3.3) for homogeneous and isotropic turbulence. For large Reynolds number, the viscous term is small and $\overline{du'^3_n} < 0$, indicating forward energy cascade from large to small scales; this is the Richardson cascade.

3.2.3 Relation to the Turbulent Kinetic Energy equation

As the separation increases, i.e. $r = |r_i| \rightarrow \infty$, the velocity correlation $\overline{u'^+_i u'^-_i} \rightarrow 0$ and therefore $\overline{dq^2} \rightarrow 2(k^+ + k^-)$, where $k^+ = \frac{1}{2}\overline{u'^+_i u'^+_i}$, $k^- = \frac{1}{2}\overline{u'^-_i u'^-_i}$ are the TKE's at points x^+_i and x^-_i respectively. Thus for very large separations, the asymptotic form of (3.2) is twice the sum of the TKE transport equations (2.21) applied at each point. Both the nonlinear interscale transfer $\frac{\partial \overline{du'_j dq^2}}{\partial r_j}$ and the turbulent advection $\frac{\partial \overline{\frac{(u'^+_i + u'^-_i)}{2} dq^2}}{\partial X_j}$ tend to $\frac{1}{2} \left(\frac{\partial}{\partial x^+_j} \overline{u'^+_i u'^+_i u'^+_j} + \frac{\partial}{\partial x^-_j} \overline{u'^-_i u'^-_i u'^-_j} \right)$ as $|r_i| \rightarrow \infty$, thus both contribute equally to the turbulent diffusion term of the TKE equation (see (2.21)). The asymptotic forms of the scale energy, non-linear inter-scale energy transfer rate and turbulent advection at very large separations are presented in appendix D.

3.3 Balance of the KMH equation

In the flow considered in this thesis, only the spanwise direction x_3 is homogeneous, so averaging is performed in time and along x_3 . The composite position vector ξ and the flux vector $\phi_\xi(\xi)$ thus become 5-dimensional, i.e.

$$\xi = (x_1, x_2, r_1, r_2, r_3) \quad \text{and} \quad \phi_\xi(\xi) = (\phi_{s1}, \phi_{s2}, \phi_{r1}, \phi_{r2}, \phi_{r3}). \quad (3.13)$$

Due to this high dimensionality judicious choices must be made on the selection of the projection hyper-planes that will allow the extraction of the maximum amount of information. In the present thesis, we project the flux vector to the 2D planes $(r_3, x_2 = Y)$, $(r_1, x_2 = Y)$, (r_1, r_3) and the 3D hyper-plane $(r_1, r_3, x_2 = Y)$, i.e. we plot (ϕ_{r3}, ϕ_{s2}) , (ϕ_{r1}, ϕ_{s2}) , (ϕ_{r1}, ϕ_{r3})

and $(\phi_{r_1}, \phi_{r_3}, \phi_{s_2})$ at several streamwise locations x_1 . For all cases, we have assumed that the separation in the wall normal direction, $r_2 = 0$.

We provide evidence that the computed fields balance the KMH equation. The budgets for the KMH equation at two locations, $TR2$ and TU , are shown in figures 3.4 and 3.5 respectively. Two separation pairs are considered for each location, and all terms are plotted against the wall-normal direction. At small separations, the residual is less than 1% of maximum production. At large separations, it is less than 4 – 5%. Similar results have been found for other streamwise locations and separations.

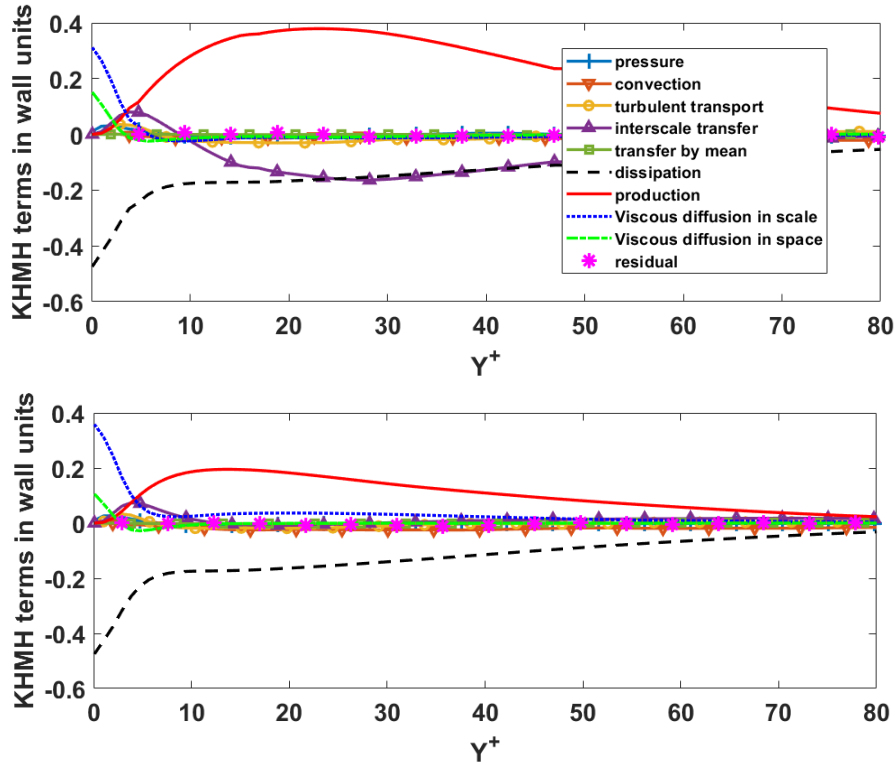


Figure 3.4 Budgets of the KMH equation at location ($TR2$) for $(r_3, r_1) = (0L_0, 75L_0)$ (top) and $(r_3, r_1) = (8L_0, 75L_0)$ (bottom).

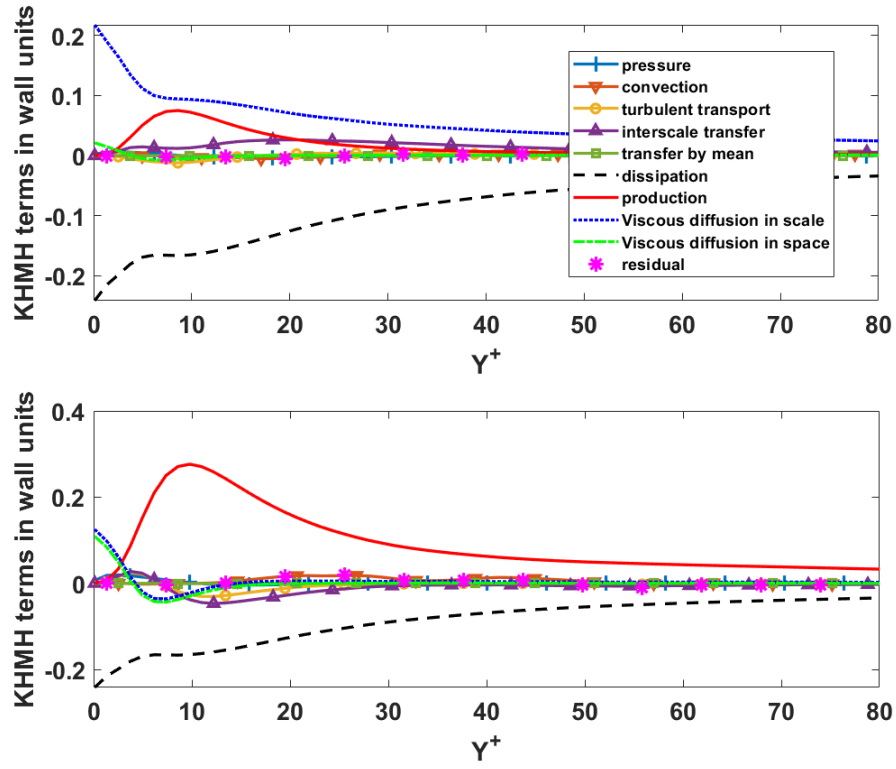


Figure 3.5 Budgets of the KMH equation at the fully turbulent region (TU) for $(r_3, r_1) = (0.1L_0, 0L_0)$ (top) and $(r_3, r_1) = (18L_0, 0L_0)$ (bottom).

Chapter 4

Evolution of the second-order structure function in a transitional boundary layer[†]

4.1 Introduction

As mentioned by Eckhardt et al. (1998), one of the challenging questions in transition is the emergence of the turbulent cascade. This provides the motivation to employ two-point statistics as explained in the previous chapters. In this chapter we consider the evolution of the second-order structure function in a transitional boundary layer; we consider the KMH equation in chapters 5 and 6.

Previous two-point analyses in bypass transition have employed the velocity correlation function (Matsubara and Alfredsson, 2001, Ovchinnikov et al., 2008, Fransson and Alfredsson, 2003, Westin et al., 1994). In this chapter, instead of using two-point velocity correlation functions, we employ the second-order structure function for the fluctuation of the streamwise velocity, u' , as an analysis tool. As defined earlier, the structure function is a two-point statistical quantity, defined as $\overline{du'^2} = \overline{(u'^+ - u'^-)^2}$, where the superscripts $^+$ and $^-$ indicate two points in space. The coordinates of the points are defined using the location of the mid-point $\vec{X} = (X, Y, Z)$ and the separation vector $\vec{r} = (r_1, r_2, r_3)$, i.e. $\vec{x}^\pm = \vec{X} \pm \vec{r}/2$. Similar to the velocity correlation, $\overline{du'^2}$ is also a six-dimensional quantity, $\overline{du'^2}(X, Y, Z, r_1, r_2, r_3)$ (when spanwise averaged, as in the present work, it becomes a five-dimensional quantity). As will be seen later, important physical characteristics, such as coherent structure location and energy density at a particular length scale, can be extracted from the distribution of $\overline{du'^2}$. We prefer however to work with the second-order structure function, as opposed to two-point correlation, for the following reasons:

[†]Part of the results of this chapter was published in Physical Review Fluids (Yao et al., 2020).

(1) There is no loss of information, because there is a direct relationship between these two-point statistical quantities, $\overline{du'^2} = \overline{(u'^+ - u'^-)^2} = \overline{(u'^+)^2} + \overline{(u'^-)^2} - 2\overline{u'^+ u'^-}$; the last term is the velocity correlation function. When the separation is along a homogeneous direction, $\overline{(u'^+)^2} = \overline{(u'^-)^2} = \overline{u'^2}$ and the above equation can be simplified to $\overline{du'^2} = 2 \left(\overline{u'^2} - \overline{u'^+ u'^-} \right)$.

(2) It has a more clear physical meaning, because $\overline{du'^2} + \overline{dv'^2} + \overline{dw'^2}$ can be regarded as the scale energy in physical space, i.e. it quantifies the energy contained within an eddy of size (length scale) $|\vec{r}|$ which is centred at point $\vec{X} = (X, Y, Z)$. Strictly speaking, the structure function also contains a term proportional to the enstrophy of scales greater than $|\vec{r}|$, however, enstrophy is large at the dissipative scales, and the enstrophy term can be ignored if the scales $|\vec{r}|$ are much larger than these scales; refer to Davidson (2015) and Marati et al. (2004) for more details about this interpretation and the caveats for small $|\vec{r}|$; refer also to the previous chapter 3. In the above sum, $\overline{du'^2}$ has the highest contribution, so this term is a good approximation of the scale energy. Note that since the structure function is formulated in physical space, it is applicable to general, spatially varying flows. This is in contrast to the more traditional view of the energy density in spectral (wavenumber) space, which is applicable to statistically homogeneous flows only.

(3) Its transport equation is the generalized Karman-Howarth equation (Hill, 2001, 2002a), which has been increasingly used as a statistical tool for the analysis of inter-scale energy transfer in channel flow (Marati et al., 2004, Cimorelli et al., 2013, 2015, 2016, Mollicone et al., 2018), homogeneous isotropic turbulence (Yasuda and Vassilicos, 2018), flow around a square prism (Alves Portela et al., 2017), and in the wake of a fractal grid (Gomes-Fernandes et al., 2015). The study of the structure function in this chapter paves the way to conduct further KMHM analysis in the following chapters.

Additionally, since we are studying a transitional flow, it is of interest to analyse the effect of intermittency on the statistics of $\overline{du'^2}$ via conditional sampling. Although this approach has long been applied to single-point statistics (Pope, 2000), we could not find a similar analysis for two-point statistics in the literature. We therefore derive the relevant equations and study the conditionally averaged fields. As will be shown, many interesting physical features of transition can be elucidated from the analysis of these fields.

4.2 Derivation of conditionally-averaged two-point statistics

4.2.1 Time- and spanwise intermittency

As mentioned in section 2.4, we define an instantaneous binary indicator function, $\tau(X, Y, Z, t)$, that takes the value 1 if the flow is turbulent and 0 if it is laminar. The time and spanwise average of the τ field

$$\gamma(X, Y) = \frac{1}{\Delta T L_z} \int_0^{\Delta T} \int_0^{L_z} \tau(X, Y, Z, t) dz dt \quad (4.1)$$

is known as intermittency, γ . A 2-dimensional contour plot is shown in figure (4.1); it is very similar to figure (5) in Marxen and Zaki (2019). In figure 4.1, vertical solid lines are marked as the calculation points for the structure function. The green line is located at laminar region ($\gamma = 0$); the red lines are located during the transition ($0 < \gamma < 1$) and the blue line is located at fully turbulent region ($\gamma = 1$). By arguing that turbulent spots are born with equal probability, Narashimha (1957) proposed a ‘universal’ distribution for intermittency, which matches well with the present result of $\gamma_{max}(X) = \max(\gamma(X, Y))$ (Narasimha, 1985), as shown in figure (4.2).

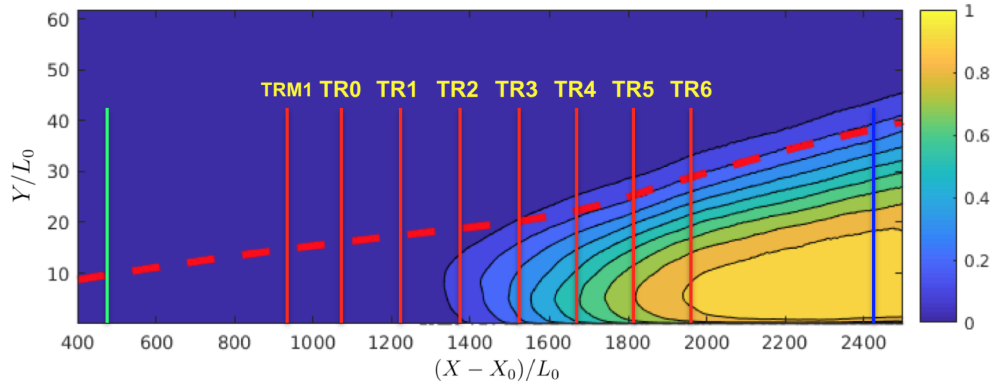


Figure 4.1 Contour plot of the intermittency field γ . The red dashed line represents the boundary layer thickness. The vertical solid lines in green, red and blue represent the streamwise locations where the second-order structure functions are calculated. The green line is in the laminar region ($(X - X_0)/L_0 = 540$) and the blue in the turbulent region ($(X - X_0)/L_0 = 2415$). The eight red lines, $TRM1$ and $TR0 - TR6$ are in the transition region and their coordinates are 940, 1065, 1215, 1365, 1515, 1665, 1815, 1965 respectively.

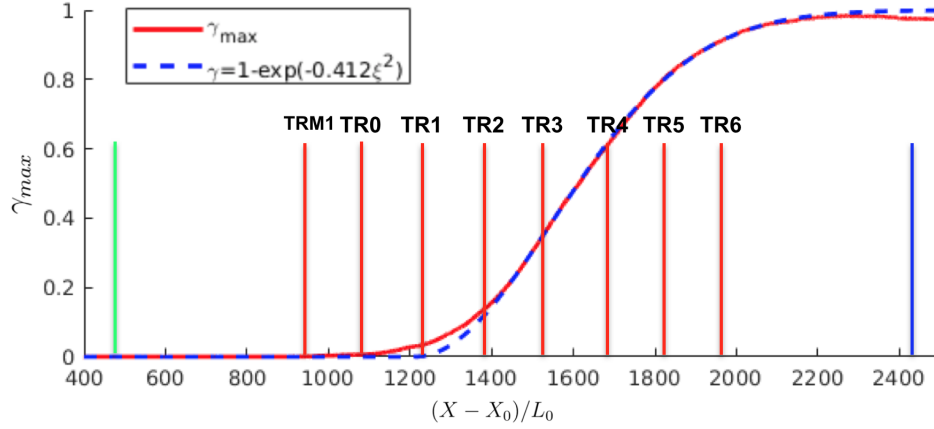


Figure 4.2 Variation of γ_{max} (solid line) in the streamwise direction and comparison with the formula of Narashima (dashed line), where $\xi = (X - X_s)/(X_{\gamma=0.75} - X_{\gamma=0.25})$, and X_s is the location transition starts, taken to be at $(X_s - X_0)/L_0 = 1100$.

The binary indicator function and the intermittency field allow us to define conditional averaged statistics, as explained in the next subsection.

4.2.2 Derivation of conditional two-point structure function

We start by first summarising existing expressions for single-point statistics. We use capital symbols, for example Φ , to denote instantaneous quantities and lower case symbols, u_i or ϕ , to denote fluctuations around a time and spanwise-averaged variable, $\bar{\Phi}$.

A conventionally averaged quantity $\bar{\Phi}$, can be decomposed into conditionally-averaged quantities using the intermittency γ as follows:

$$\bar{\Phi} = \gamma \bar{\Phi}^{(T)} + (1 - \gamma) \bar{\Phi}^{(L)}, \quad (4.2)$$

where the superscripts (T) and (L) indicate averages conditioned on turbulent and laminar events respectively. The conditionally averaged variables are defined as,

$$\bar{\Phi}^{(T)} = \frac{1}{\Delta T L_z \gamma} \int_0^{\Delta T} \int_0^{L_z} \tau \Phi dz dt \quad \bar{\Phi}^{(L)} = \frac{1}{\Delta T L_z (1 - \gamma)} \int_0^{\Delta T} \int_0^{L_z} (1 - \tau) \Phi dz dt \quad (4.3)$$

Using the Reynolds decomposition $\phi = \Phi - \bar{\Phi}$, the variance of fluctuations can be written as,

$$\overline{\phi^2} = \overline{\Phi^2} - \bar{\Phi}^2 \quad (4.4)$$

and similarly for the conditional variances,

$$\overline{\phi^2}^{(T)} = \overline{\Phi^2}^{(T)} - (\overline{\Phi}^{(T)})^2 \quad \overline{\phi^2}^{(L)} = \overline{\Phi^2}^{(L)} - (\overline{\Phi}^{(L)})^2 \quad (4.5)$$

The variance can also be written in terms of conditionally-averaged quantities as Pope (2000),

$$\overline{\phi^2} = \gamma \overline{\phi^2}^{(T)} + (1 - \gamma) \overline{\phi^2}^{(L)} + \gamma(1 - \gamma) (\overline{\Phi}^{(T)} - \overline{\Phi}^{(L)})^2 \quad (4.6)$$

The same decomposition can be employed for the time and spanwise-averaged Reynolds stresses. The latter can be written in terms of conditionally-averaged quantities as Pope (2000), Marxen and Zaki (2019),

$$\overline{u'_i u'_j} = \gamma \overline{u'_i u'_j}^{(T)} + (1 - \gamma) \overline{u'_i u'_j}^{(L)} + \gamma(1 - \gamma) \left(dU_i^{(T)} - dU_i^{(L)} \right) \left(dU_j^{(T)} - dU_j^{(L)} \right) \quad (4.7)$$

The one-point conditional decomposition can be extended to two-point statistics; the two points are indicated by the superscripts $+$ and $-$. The conditional averaging now includes three different states that are defined as,

$$\text{States} = \begin{cases} \text{Turbulent-Turbulent (TT)} & \tau^+ = 1, \tau^- = 1 \\ \text{Turbulent-Laminar (TL)} & \tau^+ = 1, \tau^- = 0 \text{ or } \tau^+ = 0, \tau^- = 1 \\ \text{Laminar-Laminar (LL)} & \tau^+ = 0, \tau^- = 0 \end{cases} \quad (4.8)$$

A schematic diagram for each state is shown in figure (4.3). we have to emphasis that in this chapter we combine TL and LT events together. This is because the square mean of the velocity fluctuation difference does not change if $+$ and $-$ switch off, i.e., $(du')^2 = (u'^+ - u'^-)^2 = (u'^- - u'^+)^2$. However this is not the case in chapter 6, which will be explained in that chapter.

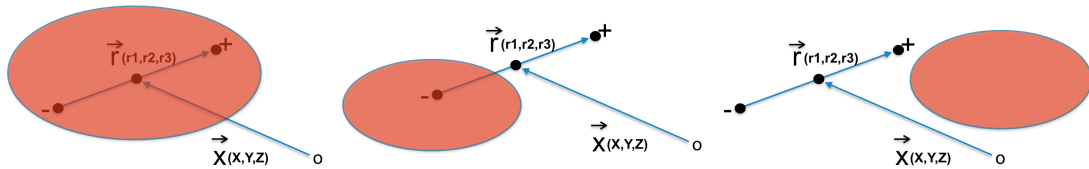


Figure 4.3 Sketches of the three states. Red ovals represent turbulent spots. Both points are within a turbulent spot (state TT, left), one point inside a turbulent spot, the other in laminar region (state TL, middle), both points in the laminar region (state LL, right).

Using the identity

$$\tau^+ \tau^- + (\tau^+ (1 - \tau^-) + (1 - \tau^+) \tau^-) + (1 - \tau^+) (1 - \tau^-) = 1, \quad (4.9)$$

the instantaneous velocity difference $du = u^+ - u^-$ can be written as,

$$du = \tau^+ \tau^- du + (\tau^+ (1 - \tau^-) + (1 - \tau^+) \tau^-) du + (1 - \tau^+) (1 - \tau^-) du \quad (4.10)$$

Taking the time and spanwise average of the above equation, we get the following decomposition,

$$dU = \gamma^{(TT)} dU^{(TT)} + \gamma^{(TL)} dU^{(TL)} + \gamma^{(LL)} dU^{(LL)} \quad (4.11)$$

where

$$\begin{aligned} dU^{(TT)} &= \frac{1}{\Delta T L_z \gamma^{(TT)}} \int_0^{\Delta T} \int_0^{L_z} \tau^+ \tau^- (u^+ - u^-) dz dt \\ dU^{(TL)} &= \frac{1}{\Delta T L_z \gamma^{(TL)}} \int_0^{\Delta T} \int_0^{L_z} (\tau^+ (1 - \tau^-) + (1 - \tau^+) \tau^-) (u^+ - u^-) dz dt \\ dU^{(LL)} &= \frac{1}{\Delta T L_z \gamma^{(LL)}} \int_0^{\Delta T} \int_0^{L_z} (1 - \tau^+) (1 - \tau^-) (u^+ - u^-) dz dt \end{aligned} \quad (4.12)$$

are the conditional averages and,

$$\begin{aligned} \gamma^{(TT)} &= \frac{1}{\Delta T L_z} \int_0^{\Delta T} \int_0^{L_z} \tau^+ \tau^- dz dt, \\ \gamma^{(TL)} &= \frac{1}{\Delta T L_z} \int_0^{\Delta T} \int_0^{L_z} (\tau^+ (1 - \tau^-) + (1 - \tau^+) \tau^-) dz dt, \\ \gamma^{(LL)} &= \frac{1}{\Delta T L_z} \int_0^{\Delta T} \int_0^{L_z} (1 - \tau^+) (1 - \tau^-) dz dt, \end{aligned} \quad (4.13)$$

are the two-point intermittencies; the latter generalise equation (4.1). Due to the identity (4.9), their sum is

$$\gamma^{(TT)} + \gamma^{(TL)} + \gamma^{(LL)} = 1. \quad (4.14)$$

The definition (4.13) indicates that $\gamma^{(TT)}, \gamma^{(TL)}, \gamma^{(LL)}$ can be physically interpreted as probabilities. For example, $\gamma^{(TT)}$ is the probability that both points $\vec{x}^\pm$ are located within turbulent patches; similar interpretations can be given to $\gamma^{(TL)}$ and $\gamma^{(LL)}$. The sum of the probabilities is equal to 1 (equation 4.14) and (4.11) is a weighted sum that involves the probabilities of the three states and the corresponding conditional averages.

Similar to the single-point case, the variance and conditional variance for the fluctuations difference can be written as,

$$\begin{aligned}\overline{du'^2} &= \overline{(du)^2} - (dU)^2 & \overline{du'^2}^{(TT)} &= \overline{(du)^2}^{(TT)} - (dU^{(TT)})^2 \\ \overline{du'^2}^{(TL)} &= \overline{(du)^2}^{(TL)} - (dU^{(TL)})^2 & \overline{du'^2}^{(LL)} &= \overline{(du)^2}^{(LL)} - (dU^{(LL)})^2\end{aligned}\quad (4.15)$$

Using equations (4.11) and (4.15) and after some algebra, we obtain the following decomposition of $\overline{du'^2}$

$$\overline{du'^2} = \gamma^{(TT)} \overline{du'^2}^{(TT)} + \gamma^{(TL)} \overline{du'^2}^{(TL)} + \gamma^{(LL)} \overline{du'^2}^{(LL)} + \Psi(dU^{(TT)}, dU^{(TL)}, dU^{(LL)}) \quad (4.16)$$

where,

$$\begin{aligned}\Psi(dU^{(TT)}, dU^{(TL)}, dU^{(LL)}) &= \gamma^{(TT)} (dU^{(TT)})^2 + \gamma^{(TL)} (dU^{(TL)})^2 + \gamma^{(LL)} (dU^{(LL)})^2 \\ &\quad - (\gamma^{(TT)} dU^{(TT)} + \gamma^{(TL)} dU^{(TL)} + \gamma^{(LL)} dU^{(LL)})^2\end{aligned}\quad (4.17)$$

The above equation generalises the existing decomposition of single-point statistics, equation (4.6), to two-point ones. We checked its validity by performing numerical experiments with synthetic data.

4.3 Two-point structure function in laminar and turbulent region

In this section, contour plots of the time- and spanwise- averaged streamwise structure functions are presented on (r_3, Y) and (r_1, Y) planes at the streamwise locations X_i shown as vertical solid lines in figure (4.1). For each plane, we consider that the other separations are 0. For example, in the (r_1, Y) plane we plot $\overline{du'^2}(r_1, Y; X_i, r_2 = 0, r_3 = 0)$, while for the (r_3, Y) plane we plot $\overline{du'^2}(r_3, Y; X_i, r_1 = 0, r_2 = 0)$. For all plots, the values are normalised by the square of the local friction velocity, $u_\tau^2(X_i)$.

The conventionally averaged $\overline{du'^2}$ is first plotted in the laminar and turbulent regions. This is followed by similar plots in the transitional region, and finally we present the maps of conditionally averaged quantities.

4.3.1 Streaky laminar region

The structure function in the streaky laminar region, corresponding to the green line in figure (4.1), is shown in figure (4.4). Note that the separations in the spanwise and streamwise directions as well as the wall-normal distance are normalised by two scales; one local (boundary layer thickness, δ) and one global (inlet Blasius variable, L_0).

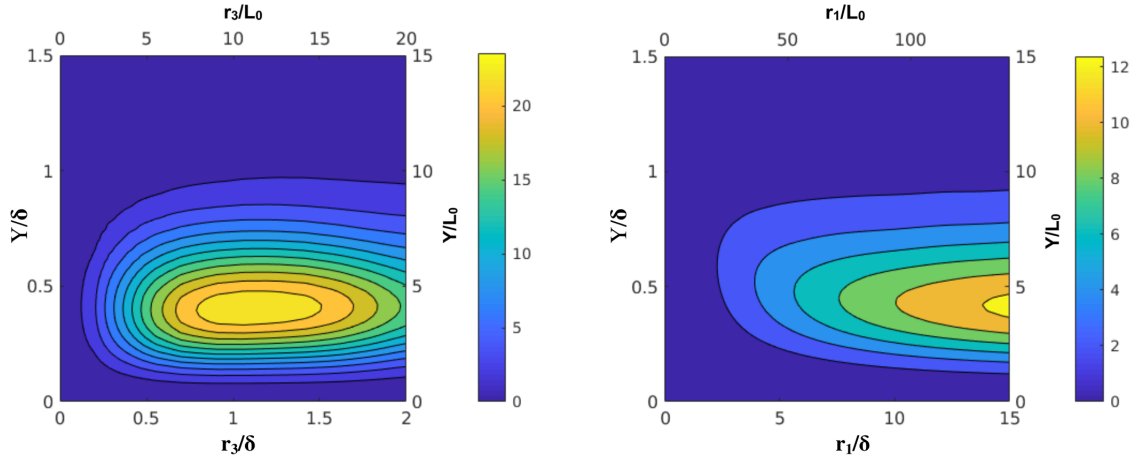


Figure 4.4 $\overline{du'^2}$ on (r_3, Y) (left) and (r_1, Y) (right) planes in the laminar region. Note that left and bottom axes are normalised by the local boundary layer thickness (δ), while top and right axes are normalised by the inlet Blasius similarity variable, L_0 .

A clear peak can be observed in the $\overline{du'^2}$ on (r_3, Y) plane as shown in the left panel of figure (4.4). The peak corresponds to $r_3 \approx 1.1\delta$ (or $r_3 \approx 3\delta^*$, or $r_3 \approx 10 - 11L_0$ in global units). As mentioned in the introduction, in the homogeneous direction, $\overline{du'^2} = 2(\overline{u'^2} - \overline{u'^+u'^-})$, and due to the negative sign before the correlation term, the minimum in the $\overline{u'^+u'^-}$ corresponds to a maximum of structure function. This distance is the average spacing between laminar high and low speed streaks. The value of $r_3 \approx 1.1\delta$ matches the location of the first minimum point of the streamwise two-point velocity correlation function (red spot in figure (4.5)). When normalised with the displacement thickness, the value of $\approx 3\delta^*$ is within the region reported in experimental and numerical theoretical studies (Matsubara and Alfredsson, 2001, Ovchinnikov et al., 2008), but smaller compared to $4.08\delta^*$, the value predicted by the optimal perturbation theory of Andersson et al. (1999) and Luchini (2000). This may be related to the fact that the peak in the optimal perturbation (figure 2 in Andersson et al. (1999)) is not very sharp, and the exact value selected is determined by the imposed length scale and the intensity of the free-stream turbulence at the inlet.

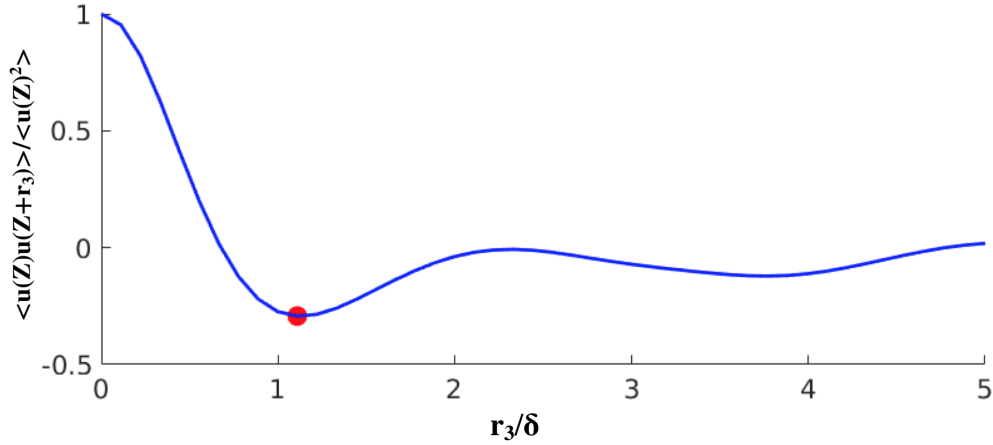


Figure 4.5 Second-order streamwise velocity correlation function with respect to spanwise separation at wall-normal distance $Y_0 \sim 4L_0$.

The scale energy peak in the wall-normal direction is located at $Y \approx 0.4\delta$ (or $Y \approx 4L_0$). This indicates the presence of Klebanoff modes, where the largest turbulent intensity of streamwise fluctuation is located around the middle of the laminar boundary layer (Andersson et al., 1999, Matsubara and Alfredsson, 2001). This is also consistent with the one-point statistics shown in figure 2.11(d), i.e. peak at $\eta \approx 2.2$ (with thickness $\eta \approx 5$).

The right panel of figure (4.4) shows the structure function on the (r_1, Y) plane. The largest values are also located at $Y \approx 0.4\delta$ as expected, but $\overline{du'^2}$ grows continuously with the streamwise separation r_1 , even until the maximum value considered, $r_1 = 15\delta$. We have examined also larger separations (plots not shown) and found that $\overline{du'^2}$ continues to increase, but at a slower rate. We expect that at the very large r_1 , the velocity fluctuations at the two points $\vec{x}^\pm$ will become uncorrelated and $\overline{du'^2}$ will tend to $\overline{(u'^+)^2} + \overline{(u'^-)^2}$. If the flow were homogeneous in the streamwise direction, then $\overline{du'^2}$ would tend to a constant value, equal to $2\overline{u'^2}$; this is nicely shown in figure 2 of Davidson et al. (2006). In our case however, the flow has strong spatial inhomogeneity and $\overline{du'^2}$ continues to increase for large r_1 , but at reduced rate.

4.3.2 Fully turbulent region

The structure function in the fully turbulent region corresponding to the blue line in figure (4.1) is shown in Figure (4.6). Again we normalise the separations and wall-normal distances using local wall units (bottom axes) as well as the global reference quantity L_0 (top axes). In the (r_3, Y) plane (left panel), a peak is found at $r_3^+ \approx 70$ and $Y^+ \approx 16$. This corresponds to the spanwise distance of the near-wall streaks and the energy peak region is located in the buffer layer. The results are very similar to the structure function calculated in channel

flow (Marati et al., 2004, Saikrishnan et al., 2012). In terms of global units, the maximum is located at $r_3 \approx 7 - 8L_0$, while in the laminar region it was at $r_3 \approx 10 - 11L_0$. It is also closer to the wall, because the mean velocity profile has become steeper (as can be seen from the C_f coefficient) and the production takes place closer to the wall.

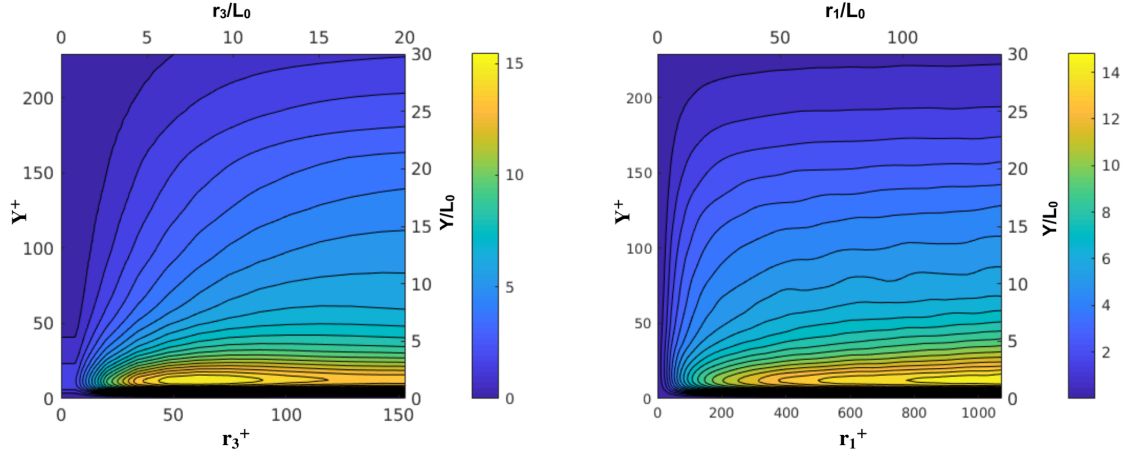


Figure 4.6 $\overline{du'^2}$ on (r_3, Y) (left) and (r_1, Y) (right) planes in the fully turbulent region. Left and bottom axes are in local wall units, while top and right axes are in global units of L_0 .

From the (r_1, Y) plane (right panel), we can see that the structure function grows over much larger separations in the streamwise direction compared to the spanwise one; this is indicative of very long streaky structures. Again the largest values in the wall-normal direction for different separations are located within the buffer layer, at $Y^+ \approx 16$, as expected.

The maps shown in figures 4.4 and 4.6 represent the two limits, the laminar and turbulent regimes respectively. In both regions, our results match previous experimental and theoretical work. In the following sections, we explore the evolution of the two-point statistics in the transitional regime and discuss the results with respect to the two limiting maps presented above.

4.4 Two-point structure function maps in transitional region

In this section, we plot maps of the standard and conditionally-averaged structure functions in the transition region on the (r_3, Y) and (r_1, Y) planes, at the locations of the red lines shown in figure (4.1).

4.4.1 Evolution of the standard structure function on $(r_3/r_1, Y)$ plane

From figure (4.7), we see the morphing of the structure functions on the (r_3, Y) plane from the laminar-like to the turbulent-like shape, compare with the left panels of Figures (4.4) and (4.6) respectively. In the streamwise locations $TR0 - TR3$, the characteristics of the laminar streaks are largely retained, i.e. the peak appears at $Y \approx 5L_0$ (slightly further from the wall compared to the distance of $4L_0$ in the fully laminar regime) and $r_3 \approx 10L_0$. At later stages, however, starting from $TR4$, and more evidently in $TR5$ and $TR6$, the shape changes significantly and the peak location approaches closer to the wall, $Y \approx 1.6L_0$. The spanwise location of the peak, however, is maintained up to $TR5$ and only in $TR6$ it is reduced to the turbulent value of $7 - 8L_0$.

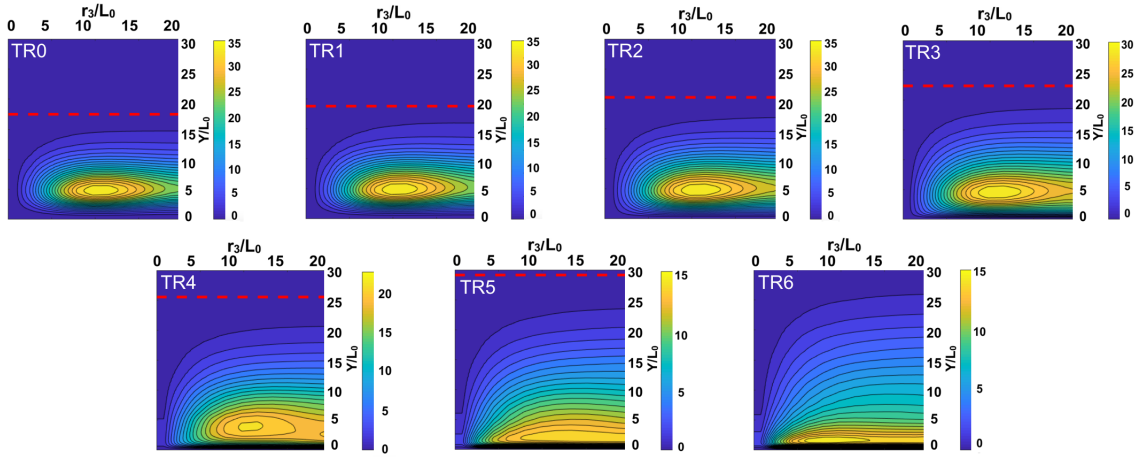


Figure 4.7 Evolution of $\overline{du'^2}$ on (r_3, Y) plane along the streamwise direction. The locations, denoted as $TR0 - TR6$, are defined in figure (4.1). In this and all subsequent figures, the red dash line represents the local boundary layer thickness.

In figure (4.8) we plot the location of the peak normalised with the displacement thickness, $r_3^{(min)}/\delta^*$ (solid red line). The values are between $2 - 3$ until the late stage of transition (location $TR4$, $(X - X_0)/L_0 \approx 1700$), and match with the numerical results of Ovchinnikov et al. (2008) (figure 12 in which $r_3^{(min)}$ tends to $2\delta^*$) and the experimental results of Matsubara and Alfredsson (2001) (figure 8 in which $r_3^{(min)}$ tends to $3\delta^*$). Differences might be due to the different inlet conditions of the free-stream turbulence. After $TR4$ the spacing decreases, approaching the value for the fully turbulent region.

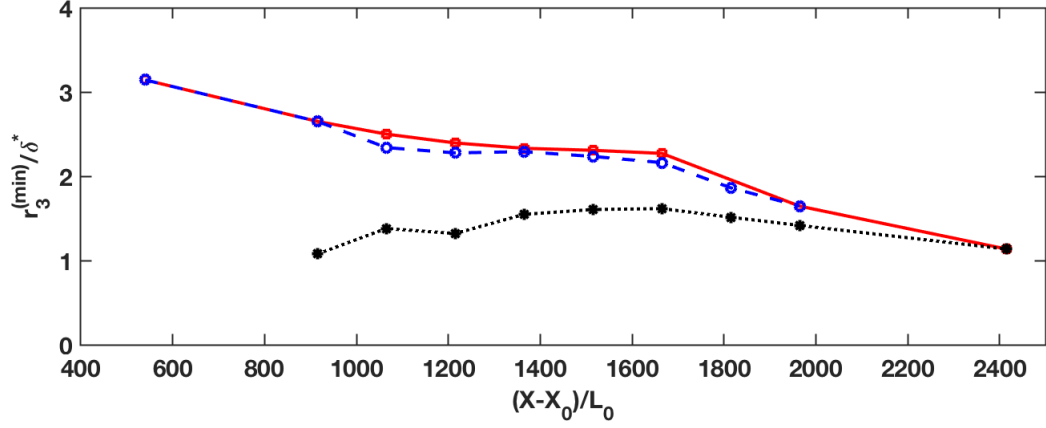


Figure 4.8 Variation of the $r_3^{(min)}/\delta^*$ along the streamwise direction. The solid red line refers to the maps of $\overline{du'^2}$ in figure (4.7), while the blue dash and the black dotted lines refer to the maps of $\gamma^{(LL)}\overline{du'^2}^{(LL)}$ in figure (4.10) and $\gamma^{(TT)}\overline{du'^2}^{(TT)}$ in figure (4.12) respectively. Circles mark the streamwise locations of figure (4.1).

Similar to the results on (r_3, y) plane, the structure function $\overline{du'^2}$ on (r_1, Y) plane, shown in figure (4.9), morphs from laminar-like to turbulent-like shape. The peak of the scale energy moves from $Y \approx 5L_0$ down to $Y \approx 1.6L_0$. It appears that at $TR3$, there is an extended peak region from $Y \approx 1 - 5L_0$. This is the streamwise location where the peaks amalgamate and further downstream the bottom peak starts to dominate.

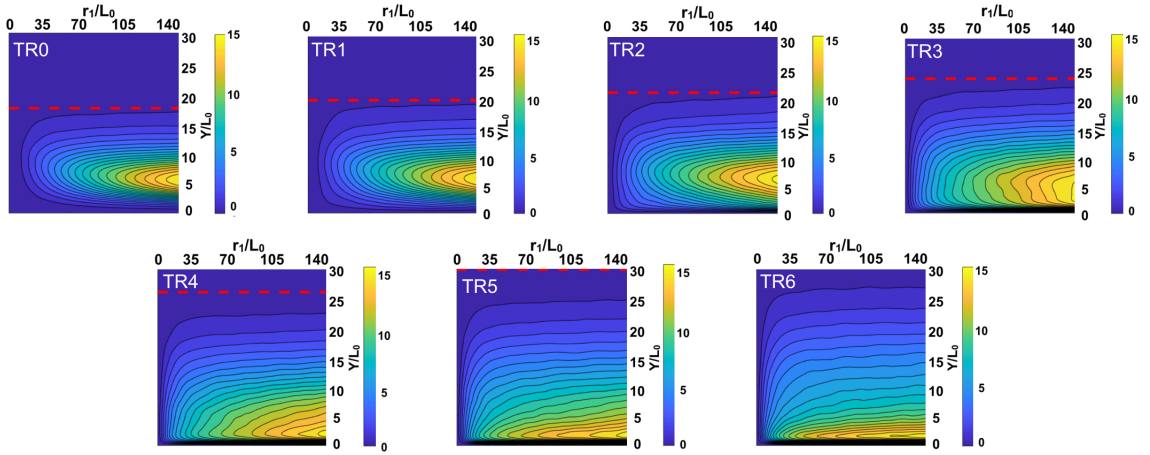


Figure 4.9 Evolution of $\overline{du'^2}$ on (r_1, Y) plane in the transitional region.

The structure functions presented above evolve smoothly between the two limiting shapes presented in the previous section 4.3. However standard time and spanwise averaging conceals important information. The next subsections present the evolution of conditionally

averaged structure functions. As will be seen, the latter provide richer information about the transition, especially when the conditioning is based on turbulent events.

4.4.2 Evolution of structure function on $(r_3/r_1, Y)$ plane conditioned on laminar events

The evolution of $\gamma^{(LL)} \overline{du'^2}^{(LL)}$ is shown in figure (4.10). Note that we chose to plot $\gamma^{(LL)} \overline{du'^2}^{(LL)}$, instead of $\overline{du'^2}^{(LL)}$, because the former appears in the decomposition (4.16). It is evident that the laminar streaks persist throughout the transition, retaining the same spanwise length scale and peak energy location, i.e. peak at $r_3 \approx 11L_0$ and $Y \approx 5L_0$. It is also interesting to observe that the peak energy stays at $Y \approx 5L_0$, even though the boundary layer is growing (the red dash lines represent the thickness of the boundary layer). Compare for example the bottom row of figure (4.10) with that of figure (4.7); the differences at locations *TR4* and *TR5* are striking. In the last plot for *TR6*, the laminar characteristics become less clear and the peak of $\gamma^{(LL)} \overline{du'^2}^{(LL)}$ decreases to $r_3 \approx 8L_0$. The plot is also less smooth compared to the other plots, because at the late stages of transition streaks start to merge with the turbulent boundary layer, thereby reducing the number of samples available for averaging conditioned on laminar events.

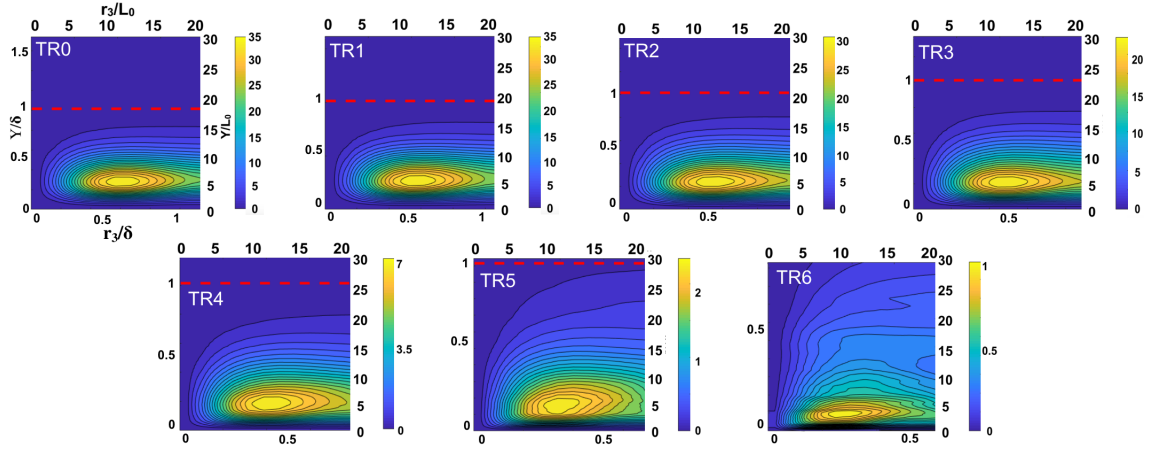


Figure 4.10 Evolution of $\gamma^{(LL)} \overline{du'^2}^{(LL)}$ on (r_3, Y) plane.

The variation of $r_3^{(min)}/\delta^*$ along X is also shown in figure (4.8) with dashed blue line. It is similar to the red line, i.e. the values slowly decrease to $2.3\delta^*$, remain nearly constant, and then decrease faster (as expected due to the rapid growth of δ^*).

Similar conclusions can be made for the (r_1, Y) maps, shown in Figure 4.11. The laminar streaky structures retain their characteristics deep into the transition region. Indeed, compared

with figure 4.9, we notice strong similarities for locations $TR0 - TR2$ as expected, but for $TR3$ and $TR4$ we observe significant differences; in figure 4.11 the shape of the latter two is much closer to the laminar shape. For $TR5$ and $TR6$ the shape becomes again similar to that shown in 4.9. Even though we condition on laminar events, the underlying average flow has been modified and its footprint is reflected in the $\gamma^{(LL)} \overline{du^2}^{(LL)}$ maps and the peak appears closer to the wall. As before, the plot for $TR6$ is not very smooth because of the reduced number of valid samples.

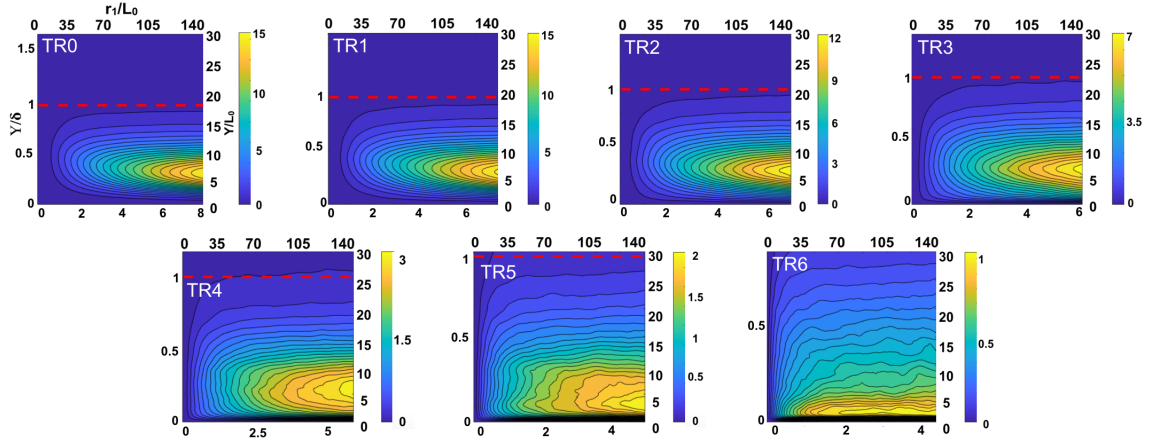


Figure 4.11 Evolution of $\gamma^{(LL)} \overline{du^2}^{(LL)}$ on (r_1, Y) plane

4.4.3 Evolution of structure function on $(r_3/r_1, Y)$ plane conditioned on turbulent events

The maps conditioned on turbulent events are shown in Figure (4.12). It is very interesting to notice that, at the earliest stage of transition at location $TR0$, turbulent motions exist near the middle of the boundary layer, with a spanwise length scale $\approx 7.5L_0$. These motions can be detected only when we apply conditional averaging; indeed they are absent when we apply standard time and spanwise averaging as evidenced in figure 4.7. This demonstrates that the proposed conditional sampling can reveal features that would not have been possible to bring to the fore otherwise.

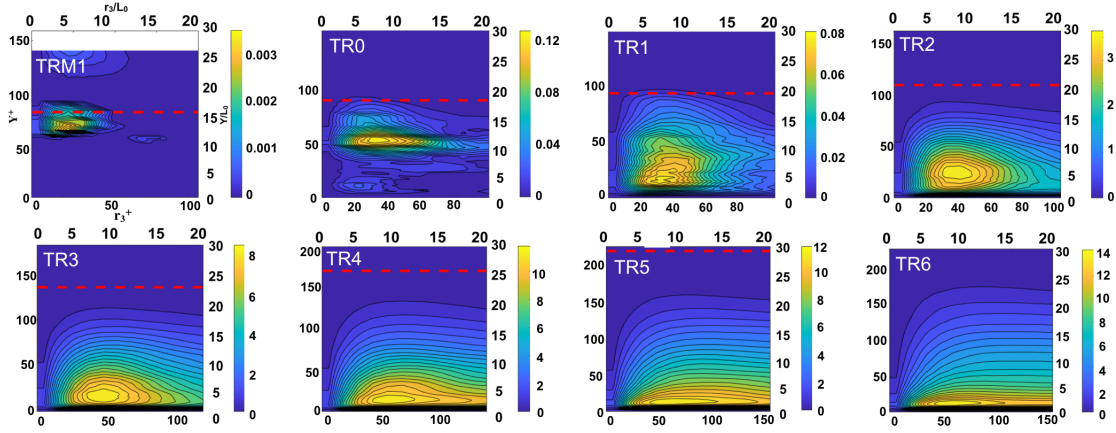


Figure 4.12 Evolution of $\gamma^{(TT)} \overline{du^2}^{(TT)}$ on (r_3, Y) plane; their corresponding locations are $TRM1, TR0 - TR6$ shown in figure (4.1)

The presence of turbulent motions in the middle of the boundary layer at the very early stages of transition is consistent with the outer type of the secondary instability mentioned in Vaughan and Zaki (2011). In order to shed more light into this feature, we have considered one more location, $TRM1$, located upstream of $TR0$, as shown in figure 4.1. At this location, we can clearly see that the turbulent activity starts near the edge of the boundary layer, at $Y \approx 12L_0$, entirely consistent with the instantaneous pattern shown in figure 2.8 (second row from the top). The spanwise length scale at this earlier location is $5L_0$, exactly equal to the length scale L_{11} that we impose on the energy spectrum at the inlet, equation (2.8). Note also that the turbulent motions outside the boundary layer, i.e. in the free stream, are also captured, and have the imposed length scale of $L_{11} = 5L_0$.

The succession of maps also demonstrates how the turbulent motions that were initiated at the top of the boundary layer, penetrate deep inside it and approach the wall quite early, even at $TR1$ and more clearly in $TR2$. These near-wall coherent structures (turbulent spots) become more visible at the late-stages of transition $TR4 - TR6$.

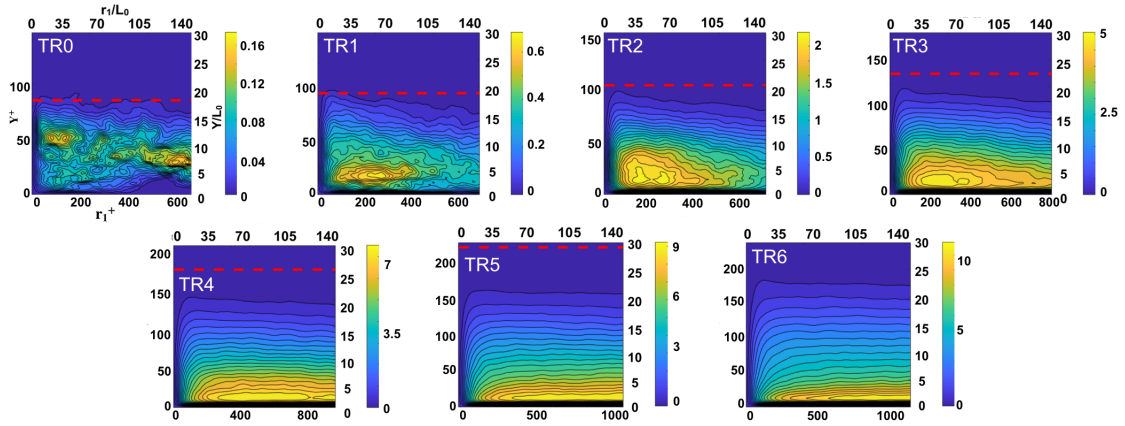
It is very instructive to compare figures 4.10 and 4.12 that show $\gamma^{(LL)} \overline{du^2}^{(LL)}$ and $\gamma^{(TT)} \overline{du^2}^{(TT)}$ respectively in the (r_3, Y) plane. When conditioning on laminar events, the peak is located at separation $r_3 \approx 10$ at all positions $TR0 - TR6$; as already mentioned, this is very close to the peak observed in the laminar state, see left panel of figure 4.4. When conditioning however on turbulent events, the location moves to $r_3 \approx 7 - 8$ at all streamwise positions, very close to the peak for the fully turbulent state, see right panel of 4.6. The two states, therefore, coexist during the whole transition process, and most importantly both retain their structural and geometric features that are determined at their inception. For the laminar state, the inception of Klebanoff modes is determined by the receptivity process

between the free-stream turbulence and the laminar boundary layer, while for the turbulent state by the secondary instability. Note that it is only through conditional averaging that we can elucidate these important characteristics.

We note that in the experiments of Fransson and Alfredsson (2003) with and without boundary layer suction, the spanwise scale of the streaky structures was found to be the same, despite a two-fold reduction in the thickness of the boundary layer during suction. This led the authors to conclude that the length scale was determined by the receptivity process that was the same for both cases. This is in accordance with our findings. The present results are also consistent with those of Marxen and Zaki (2019). Using conditional averaging, they demonstrated the similarity of the first-order statistics between spots and fully turbulent flow at the late stages of transition (not in the baby spots).

The variation of $r_3^{(min)}/\delta^*$ is also shown in figure (4.8) with a black dotted line. The first point at *TRM1* and the last one at the fully turbulent region have difference of 5%, but the two peaks are located at very different wall-normal distances. This figure also shows very clearly the coexistence of two length scales that correspond to the two different states of the flow.

Figure (4.13) shows similar behaviour in the (r_1, Y) plane, i.e. turbulent motions start near the top half of the boundary layer and then penetrate towards the wall; again this is consistent with the outer type secondary instability (Vaughan and Zaki, 2011). However the (r_1, Y) maps are distinctly different compared with the (r_3, Y) ones. In the latter, as we have just shown, the r_3 separation corresponding to the maximum value of $\gamma^{(TT)}\overline{du'^2}^{(TT)}$ was nearly constant throughout the transition process. From the (r_1, Y) maps however, it is evident that the separation r_1 corresponding to the maximum grows significantly, from $r_1 \approx 30L_0$ to $r_1 > 140L_0$. The peak area also becomes long and flat. There is a very clear physical explanation for this change of the length scale in the r_1 direction; it is due to the longitudinal growth of the structures within the turbulent spots. We explore this growth in the next section, by investigating the three-dimensional structure of the $\gamma^{(TT)}$ fields.

Figure 4.13 Evolution of $\gamma^{(TT)} \overline{du^2}^{(TT)}$ on (r_1, Y) plane

There is one final question that we consider in this section before we proceed with the analysis of $\gamma^{(TT)}$. Why we do not detect the spanwise growth of the spots from the $\gamma^{(TT)} \overline{du^2}^{(TT)}$ plots in the (r_3, Y) plane while we can detect the streamwise growth in the (r_1, Y) plane? The answer is the following. Since we are performing averaging conditioned on turbulent events, the length scale of the peak value in the (r_3, Y) plane corresponds to the average spanwise distance between the positive and negative streaks inside the spots. This is constant. However, the streamwise length scale of the streaks within the spot correlates much better with its length. For this reason, the r_1 location of the peak that quantifies the length of the streaks, grows significantly as the transition progresses.

4.5 Two-point intermittency field $\gamma^{(TT)}$

The intermittency $\gamma(X, Y)$ shown in figure (4.1) is a single-point, two dimensional, quantity and is not suitable for the analysis of the spot growth. On the other hand, the two-point intermittency field, $\gamma^{(TT)}$, contains a lot more information and, as we will see below, is a more appropriate quantity to use; for this reason, in this section, we focus on $\gamma^{(TT)}$.

As mentioned in section 4.2.2, $\gamma^{(TT)}(X, Y; r_1, r_2, r_3)$ has a clear physical interpretation as the probability that both points $\vec{x}^\pm = \vec{X} \pm \vec{r}/2$ are located in turbulent patches or spots (but not necessarily the same spots). This probability is of course equal to the percentage of time that both points spend inside turbulent spots (averaged in the spanwise direction). It is clear that setting the separation distance to 0, i.e. $r_1 = r_2 = r_3 = 0$, results in $\gamma^{(TT)}(X, Y; 0, 0, 0) = \gamma(X, Y)$, i.e. the intermittency shown in figure 4.1.

Let's now fix the physical location (X, Y) and consider an iso-surface of constant $\gamma^{(TT)}(r_1, r_2, r_3)$ and the volume of fluid enclosed by this surface. Based on the aforemen-

tioned interpretation, the fluid contained within this volume has a probability larger than or equal to the value of the iso-surface to be within one (or more) turbulent regions. This does not mean that this iso-surface indicates the shape of a turbulent spot. The proper way to identify the spot shape is to follow the tracking method described by Marxen and Zaki (2019). Nevertheless, we would expect the volume enclosed within an isosurface of constant value to grow as the transition process develops, in a similar way as a spot grows.

In figure (4.14) we plot $\gamma^{(TT)}$ in the (r_3, Y) at different streamwise locations, i.e. we assume $r_1 = 0, r_2 = 0$. The distribution of $\gamma^{(TT)}$ for $r_3 = 0$ at each map corresponds to the intermittency shown in figure 4.1. In essence then, this figure shows the spanwise extent of the turbulent region at each streamwise location. The pink dash lines indicate the isoline $\gamma^{(TT)} = 0.7\gamma_{max}^{(TT)}$, where $\gamma_{max}^{(TT)}$ is the maximum value for the particular X location. Note that the plots are symmetric with respect to $r_3 = 0$. Indeed, $\gamma^{(TT)}$ does not depend on the sign of r_3 ; a negative value will simply swap the points $\vec{x}^\pm$ without changing $\gamma^{(TT)}$. The same is valid for all separations, i.e. $\gamma^{(TT)}(X, Y; r_1, r_2, r_3) = \gamma^{(TT)}(X, Y; -r_1, -r_2, -r_3)$.

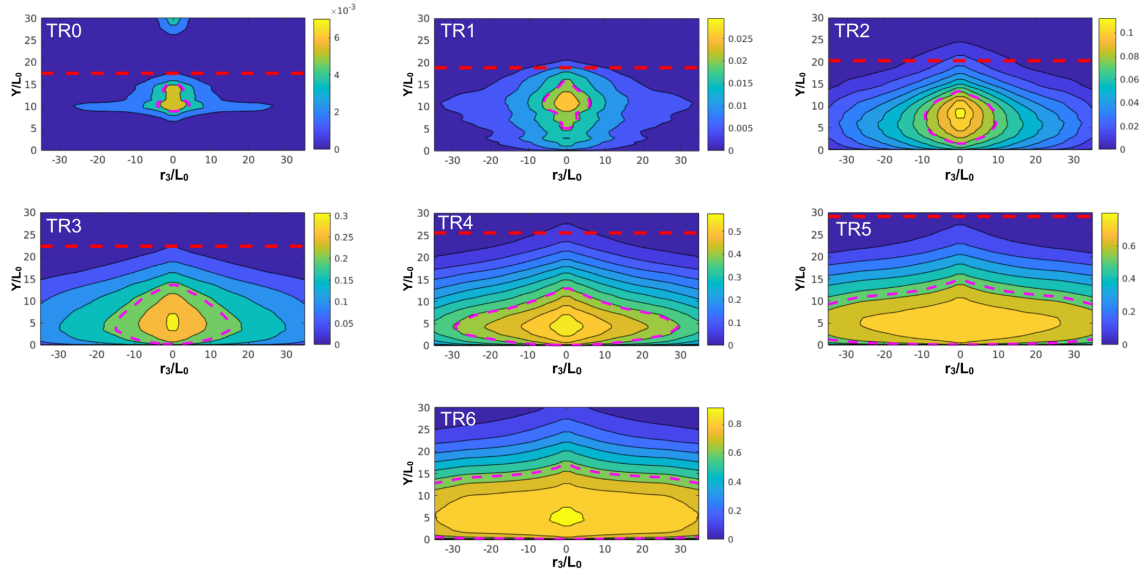


Figure 4.14 Evolution of $\gamma^{(TT)}$ on (r_3, Y) plane; pink isolines are defined as $\gamma^{(TT)} = 0.7\gamma_{max}^{(TT)}$

The subplots in figure (4.14) are consistent with the scenario of the previous section. At $TR0$ the peak of the $\gamma^{(TT)}$ is located at $Y \approx 10 - 12$ and occupies a narrow region. It then moves in the wall-normal direction and expands horizontally in the r_3 direction. It is evident that the spot is forming at the top half of the boundary layer and then gradually fills the lower part. This corresponds to the top-down mechanism due to the outer type instability (Nolan and Zaki, 2013).

This top-to-bottom process can also be seen from the plots in the (r_1, Y) plane, shown in figure (4.15); again notice the symmetry with respect to $r_1 = 0$. The difference with the previous plots in the (r_3, Y) plane is that the turbulent region is much longer in the streamwise direction. For example at $TR4$, the pink isoline extends up to around $150L_0$, while on the (r_3, Y) plane, the pink up to around $30L_0$.

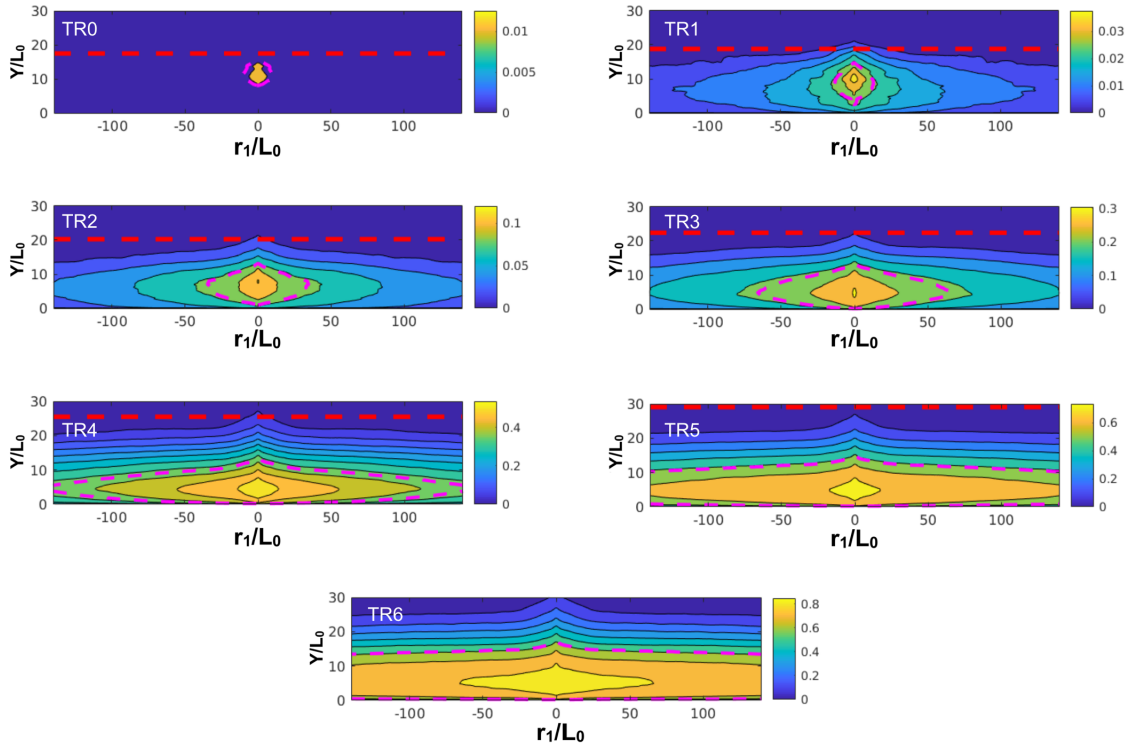


Figure 4.15 Evolution of $\gamma^{(TT)}$ on (r_1, Y) plane; pink isolines are defined as $\gamma^{(TT)} = 0.7\gamma_{max}^{(TT)}$

In figure (4.16) three-dimensional plots of $\gamma^{(TT)}$ in the (r_1, r_3, Y) domain are shown, for three streamwise locations $TR2, TR3, TR4$. Each plot was obtained by considering different values of r_3 and computing $\gamma^{(TT)}(r_1, Y; r_3)$ and stacking the results. For $TR2$ we considered the values $r_3 = 0, 3, 5, 6$, while for $TR3$ and $TR4$, $r_3 = 0, 5, 10, 12$ and $r_3 = 0, 5, 10, 12, 13, 15, 20, 21$ respectively. The black isolines are defined as $\gamma^{(TT)} = 0.7\gamma_{max}^{(TT)}$, where the $\gamma_{max}^{(TT)}$ is evaluated at $r_3 = 0$ (the black line on the bottom contour); the values of $\gamma_{max}^{(TT)}$ are 0.084, 0.21, 0.35 for $TR2, TR3, TR4$ respectively. In these plots, a long spot-like volume can be observed with long streamwise length and much shorter spanwise extent. The size of the volume increases along with the streamwise direction, as expected. This feature explains the growth of r_1 separation observed earlier in figure 4.13.

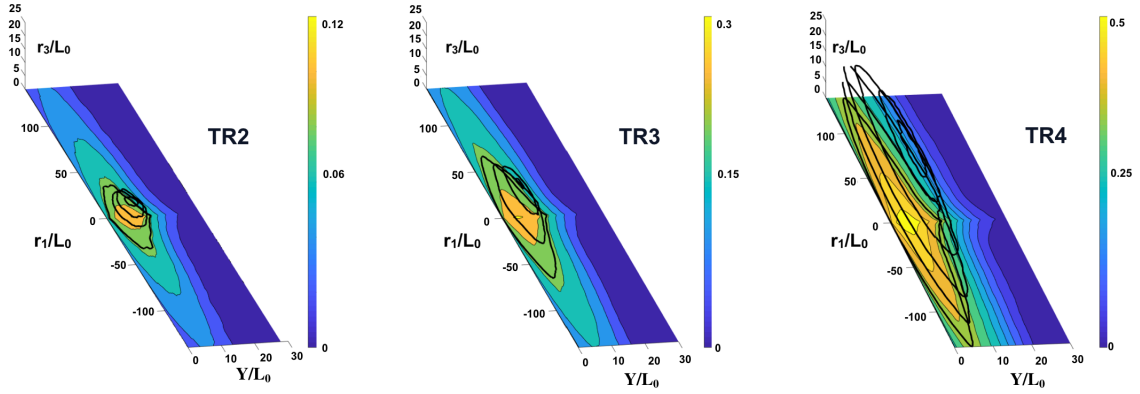


Figure 4.16 Three dimensional plots of $\gamma^{(TT)}$ at streamwise locations $TR2, TR3, TR4$ (from left to right). The black isolines denote $\gamma^{(TT)} = 0.7\gamma_{max}^{(TT)} (r_3 = 0)$.

4.6 Conclusions

We use the second-order structure function of the streamwise velocity as a statistical tool to explore bypass transition due to free-stream turbulence in a flat plate boundary layer. In order to account for the intermittent formation of turbulent spots during the transition process, we derive a new decomposition of the time and spanwise-averaged structure function in terms of conditionally-averaged quantities. The conditioning process is based on the instantaneous state of the flow at the two points that are used to define the structure function. Three states are defined (a) both points are in turbulent patches (TT state), (b) both in laminar (LL state) and (c) one in laminar, the other in turbulent patch (LT state). The derived expressions generalise existing expressions for one-point statistics to two-point statistics. Such analysis sheds light into the process of bypass transition from a new perspective.

The conditional structure functions are calculated along the streamwise direction, at 8 locations, from the laminar region to fully turbulent. The results conditioned on the laminar state, indicate that the laminar streaks maintain their physical features even after mixing with the intermittent turbulent spots. Results conditioned on turbulent state show that the initial turbulent motion starts near the boundary layer edge, consistent with the so-called outer secondary instability. Again the spanwise length scale of the turbulent spots is retained throughout the transition, but the streamwise increases significantly. It is only through the use of conditional statistics that these features can be clearly elucidated. The two-point intermittency $\gamma^{(TT)}$ was also examined. We noticed the growth in the volume of fluid enclosed by isosurfaces of this quantity along the both directions. The growth was observed mostly along the r_1 separation, indicative of the large increase of the length of the turbulent spots.

Recall that $\overline{du'^2} + \overline{dv'^2} + \overline{dw'^2}$ (where $\overline{du'^2}$ makes the largest contribution) represents the energy contained within an eddy of size $r = |\vec{r}|$ which is centred at point $\vec{X} = (X, Y, Z)$. Its evolution, as already explained in the previous chapter, is governed by the generalised Karman-Howarth equation, which is valid for general inhomogeneous flows, so it is applicable to the present case. Using the database generated, we can compute the rate of energy transfer between structures of different length scales, and use this information to draw the energy flow paths that determine the growth of turbulent spots. Furthermore, it is well known that spots merge to form larger spots. This implies an inverse energy transfer process, i.e. energy flowing from small to large scales, in contrast to the standard view of forward cascade (energy from large to small scales). Is there a competition between extracting energy from the mean flow (forward cascade) and from the inverse cascade process? What is the relative contribution of these two mechanisms? In the next chapter, these questions will be explored using the KMH equation.

Chapter 5

Analysis of interscale energy transfer using the standard time-averaged KHMH equation ^{††}

5.1 Introduction

The success of KHMH in describing the complex interscale dynamics of near-wall turbulence justifies the adoption of this framework to explore similar issues in bypass transition. As explained earlier, the equation is very general, and therefore applicable in laminar, transitional and fully turbulent regimes. In this chapter, the KHMH equation is employed to study the production and interscale energy transfer in a boundary layer undergoing bypass transition due to free-stream turbulence. The energy flux between different length scales is calculated at several streamwise locations covering the laminar, transitional, and turbulent regimes. Maps of scale energy production and flux vectors are visualised on two-dimensional planes and three-dimensional hyper-planes that comprise both physical and separation spaces. We will study the spatial evolution of the interscale energy fluxes at 6 locations, placed in the laminar (*LA*), transitional (*TR0*, *TR1*, *TR2*, *TR3*) and fully turbulent (*TA*) regions; these locations are referred to the skin friction coefficient and are shown in the right panel of figure 5.1. Note that in this chapter the numbering of the streamwise locations is different from chapter 4. We reduce the number of streamwise locations because we found some of the results are very similar, so only important locations are retained in this chapter.

^{††}Part of the results of this chapter have been submitted for publication in Journal of Fluid Mechanics (at the time of writing the revised version was still under review (Yao et al., 2021)).

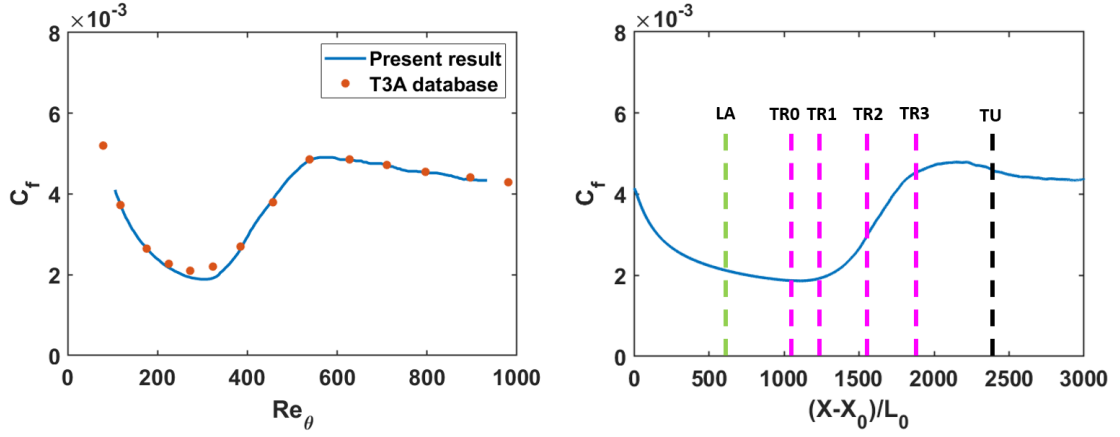


Figure 5.1 Skin friction coefficient, C_f , against Re_θ (left panel) and distance from the inlet of the domain, $(X - X_0)/L_0$ (right panel). The green and black vertical lines denote locations at the laminar (LA) and turbulent (TU) regions respectively. The purple vertical lines denote 4 stations $TR0, TR1, TR2, TR3$ within the transitional region. The coordinates are $(X - X_0)/L_0 = 540, 1065, 1215, 1515, 1815$, and 2415 for $LA, TR0, TR1, TR2, TR3$, and TU respectively.

5.2 Budgets of the TKE equation

Before we proceed to examine the evolution of two-point statistics using the KMH equation, it is instructive to examine first the budgets of the turbulent kinetic energy (TKE) equation at these 6 locations (as the TKE equation is the asymptotic form of the KMH equation at very large scale). Figure 5.2 shows the evolution of the budget terms plotted against the distance from the wall for all streamwise positions; variables are normalised using the local wall units. The meaning of the open symbols and lines can be referred to the TKE equation that was presented in section 2.12. The matching between the present results at the TU region (where $Re_\theta = 800$) and the reference data (where $Re_\theta = 1410$) is very good for all terms. This provides further evidence of the validity of the generated database. Comparison between the lines and symbols indicates the deviation from the fully turbulent state at each location.

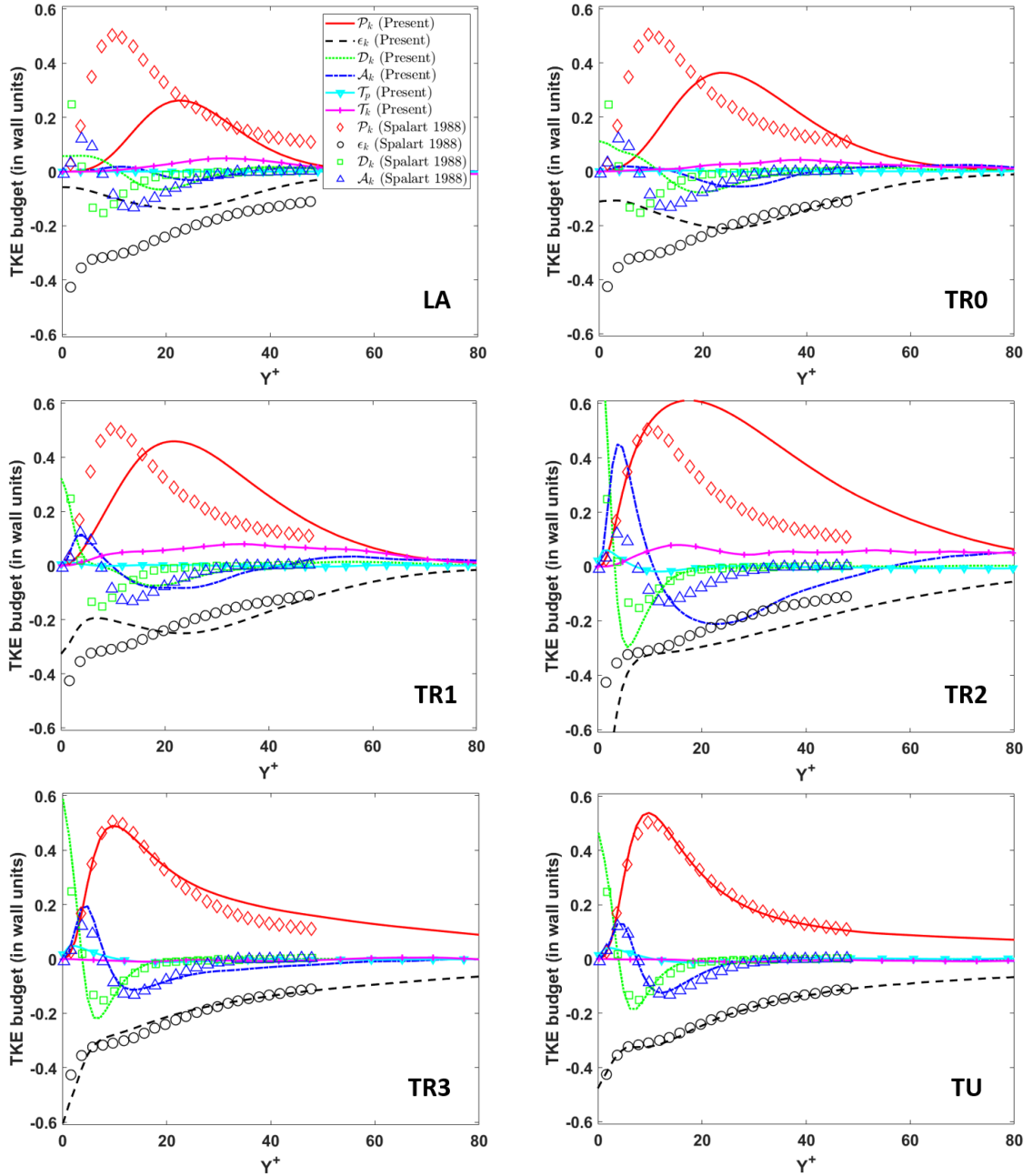


Figure 5.2 TKE budgets at locations LA , $TR0$, $TR1$, $TR2$, $TR3$, TU . The symbols represent data from Spalart (1988) for $Re_\theta = 1410$.

In the laminar regime (LA panel), where the boundary layer is distorted by Klebanoff modes, the production term has approximately symmetric shape, peaks at $Y^+ \approx 21$ and reaches a relatively small value. As the flow transitions, the production peak grows, moves closer to the wall and finally settles at $Y^+ \approx 16$ (panel TU), i.e. in the buffer layer. A production overshoot appears at $TR2$, which is in the middle of the transition region. At this

position the maximum intermittency is about 40% (refer to figure 4 of Yao et al. (2020)) and grows fast indicating the rapid formation and growth of turbulent spots. Nolan and Zaki (2013) report an overshoot in the streamwise rms velocity in exactly the same position, i.e. when the skin friction is midway between the laminar and turbulent values.

At LA , dissipation peaks within the Klebanoff mode core, but further downstream the maximum value grows, and beyond $TR1$ it settles at the wall, where it is balanced by viscous diffusion, as expected. Turbulent diffusion has small contribution at LA , $TR0$ and $TR1$, but becomes very significant at $TR2$, with a sharp positive peak close to the wall at $Y^+ \approx 5$ and a shallower negative trough at $Y^+ \approx 22$. At this location, three terms dominate for $Y^+ > 15$, production, dissipation and turbulent diffusion. Note also the very sharp rise of dissipation close to the wall, and the important contribution of turbulent diffusion to balance it. Further downstream, at $TR3$ and TU , all terms settle to smaller values.

The largest deviation from equilibrium, quantified by the difference between production and dissipation for $Y^+ > 15$, appears in $TR2$; this is the position with the largest growth of C_f as evidenced in figure 5.1, consistent with the rapid increase of intermittency, as mentioned earlier. At $TR3$, where C_f starts to stabilise marking the end of transition, the budgets are very similar to the fully turbulent state, TU . In the latter state turbulence is in equilibrium, with production balancing dissipation for $Y^+ > 25$. In all locations the convection term plays a minor role. It becomes more important in $TR2$ (and less so in $TR1$) for $Y^+ > 60$, where it balances production, dissipation and turbulent diffusion.

The strong wall-normal and streamwise inhomogeneity are more clearly demonstrated in the contour plots of TKE, production, turbulent transport, and dissipation shown in figure 5.3, where all terms are normalized by global variables (free-stream velocity U_∞ , and Blasius variable at the inlet L_0). Turbulent kinetic energy and turbulent transport peak at $(X - X_0)/L_0 \approx 1650$, slightly downstream of $TR2$. When normalised with global variables, the production term reaches maximum in the turbulent regime. The peak of TKE therefore at $(X - X_0)/L_0 \approx 1650$ arises from the strong positive value of turbulent transport close to the wall (yellow region in panel c). Dissipation grows sharply downstream of $(X - X_0)/L_0 \approx 1600$ and is maximised at the wall.

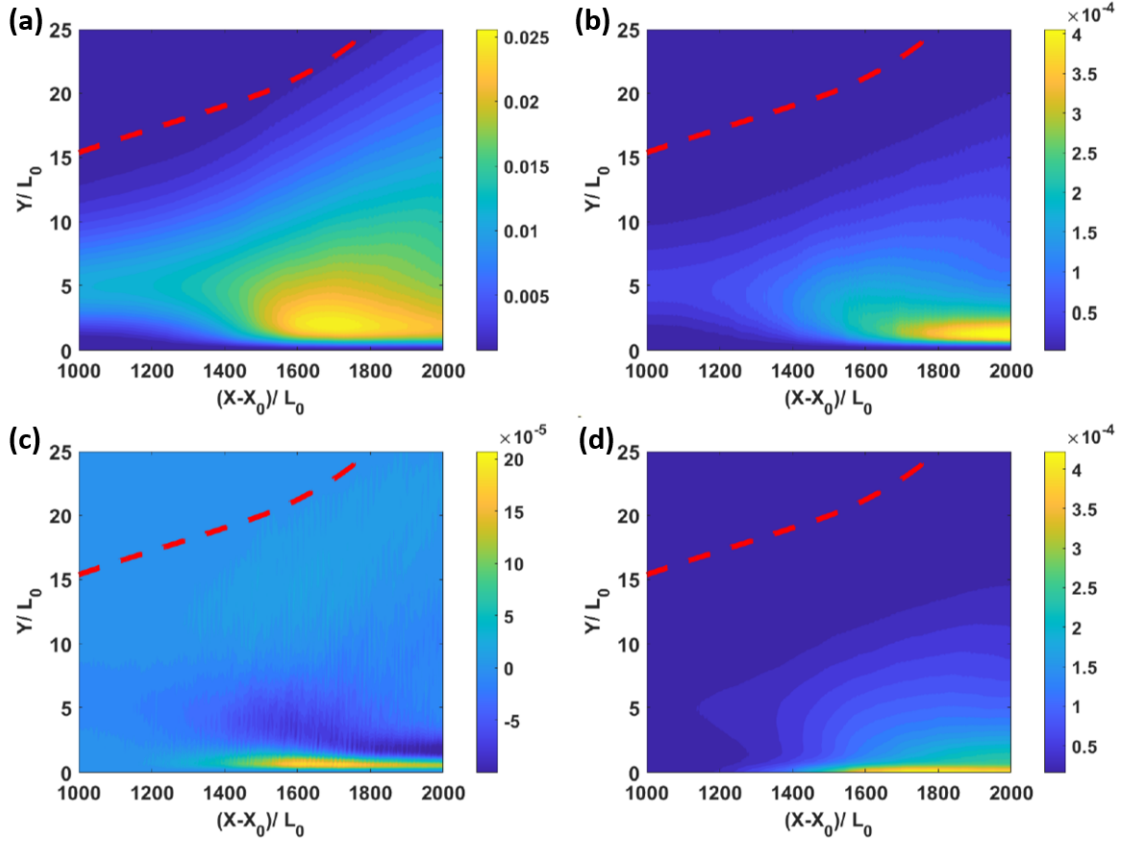


Figure 5.3 Contour plots of (a) turbulent kinetic energy, (b) production, (c) turbulent diffusion, and (d) dissipation. All variables normalised with global units. The red dashed line indicate the location of the boundary layer thickness.

The TKE is one-point equation and energy is transported only in physical space. In the next section we look at the transport of energy in both physical and scale spaces using the KMH equation.

5.3 Energy fluxes on the (r_3, Y) plane

5.3.1 General characteristics

Figure 5.4 shows the evolution of the flux vector components (ϕ_{r_3}, ϕ_{s_2}) in the (r_3, Y) plane (with $r_1 = 0$) from the laminar to turbulent region. Wall-normal distances and separations are normalized by the local wall variables (left and bottom axes) and global variables (top and right axes). The pink dashed line indicates the local boundary layer thickness. In order to facilitate the interpretation of these plots, we also superimpose coloured contours of the net production, $\mathcal{P}_{net} = \mathcal{P} - \varepsilon$.

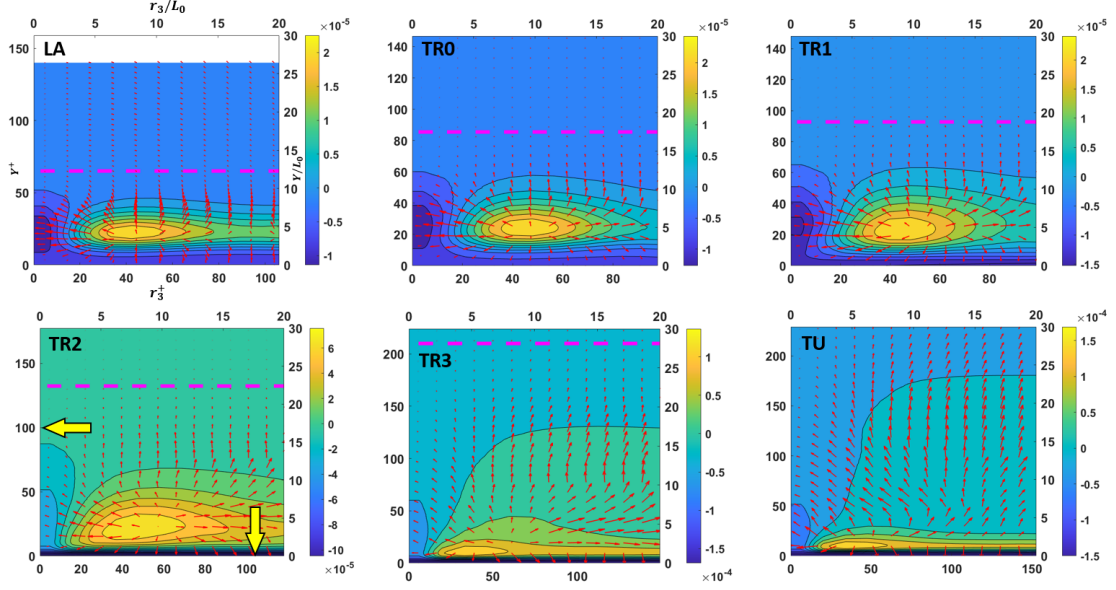


Figure 5.4 Evolution of energy flux vector (ϕ_{r_3}, ϕ_{s_2}) on (r_3, Y) plane (red arrows) from laminar to turbulent region. The coloured contour is the net production (normalised with global variables) and the pink dashed lines indicate the local boundary layer thickness. The yellow arrows at the TR2 plot point to the two sink regions, $r_3 = 0$ and $Y = 0$. The maximum vector magnitude (corresponding to the longest arrow on each plot) is $5 \times 10^{-5}, 7.6 \times 10^{-5}, 9.1 \times 10^{-5}, 1.7 \times 10^{-4}, 2.4 \times 10^{-4}, 2.2 \times 10^{-4}$ from LA to TU respectively.

At all spatial locations $\mathcal{P}_{net}$ peaks at particular r_3 separations and wall-normal distances Y (the peaks appear as concentrated yellow regions). The production $\mathcal{P}$ arises from the inhomogeneity of the time-average flow in the physical and scale spaces, as evidenced by the two terms in equation (3.6). We analysed the contribution of each term and found that the large production arises due to the inhomogeneity on scale space, and in particular from the component $-2\overline{du'_2 du'_1} \frac{\partial du_1}{\partial r_2} \Big|_{r_2=0}$. The contribution of the second term is exactly 0 (due to the spanwise homogeneity of the time-average flow). It is easier to understand the physical mechanism that leads to high production if we focus on the laminar region (top-left panel, denoted as LA). In this region Klebanoff modes (laminar streaks) are present that result in positive and negative streamwise velocity fluctuations, therefore du_1 is maximised when r_3 is equal to the distance between two streaks. The wall-normal fluctuation differences du_2 are also maximised at the same distance (refer to figure 24 in Xiao and Papadakis (2019)) resulting in large time and spanwise averages $\overline{du'_2 du'_1}$. The shear rate in scale space can be written as $\frac{\partial du_1}{\partial r_2} = \frac{1}{2}(\frac{\partial du_1}{\partial x_2^+} - \frac{\partial du_1}{\partial x_2^-}) = \frac{1}{2}(\frac{\partial U_1^+}{\partial x_2^+} + \frac{\partial U_1^-}{\partial x_2^-})$ and due to spanwise homogeneity of U_1 and the fact that $r_2 = 0$ (meaning that $x_2^+ = x_2^- = x_2 = Y$, thus $U_1^+ = U_1^- = U_1$) this can be simplified to $\frac{\partial du_1}{\partial r_2} \Big|_{r_2=0} = \frac{\partial U_1}{\partial x_2}$. The production term can therefore be written as

$\mathcal{P} = -2\overline{du'_2 du'_1} \frac{\partial U_1}{\partial x_2}$. The corresponding term for the turbulent kinetic energy equation is $\mathcal{P}_k \approx -\overline{u'_2 u'_1} \frac{\partial U_1}{\partial x_2}$, which is of course uniform in the spanwise direction, see figure 5.3(b). The peak production location of the KMH equation in the (r_3, Y) plane is therefore determined by the spanwise distance between the Klebanoff modes and the wall-normal distance at which the product of the second-order structure function $\overline{du'_2 du'_1}$ and the shear $\frac{\partial U_1}{\partial x_2}$ is maximum. The mean flow therefore injects energy at a specific r_3 separation, which is determined by the receptivity process arising from the interaction between free-stream turbulence forcing and shear during the early development of the boundary layer. This process controls the spacing between the Klebanoff modes, which is of the order of the boundary layer thickness close to the inlet.

As the flow evolves, the location of peak net production changes smoothly and approaches the wall. The largest energy production appears at the TU location, inside the buffer layer, at $Y^+ \approx 12$ and $r_3^+ \approx 40$, consistent with the results of Cimarelli et al. (2015) for channel flow. The large production at specific regions in the (r_3, Y) plane strongly affects the energy transfer in the physical and scale spaces, to which we now turn.

For all streamwise locations the energy fluxes originate from the $\mathcal{P}_{net}$ peak location and radiate outwards. In the laminar region (LA), most of the energy produced flows to smaller r_3 scales and is dissipated (forward cascade). There is weak inverse cascade to larger r_3 scales, while the energy flux in the wall-normal direction is negligible. At the turbulent region (TU) the fluxes show a different behaviour. The energy injected from the time-average flow radiates in all directions. There are two sinks indicated by yellow arrows in the TR2 plot of figure 5.4; one towards the wall (where it is dissipated) and the other towards small r_3 scales (indicating a forward cascade). An inverse energy cascade also appears, i.e. energy flows to larger r_3 scales to the right of the $\mathcal{P}_{net}$ before most of the flux vectors then change direction and bend towards the Y axis ($r_3 = 0$). This map is very similar to that of turbulent channel flow (Cimarelli et al., 2013, 2016, 2015).

In the transition region, the scale energy transfer pattern morphs progressively from laminar to turbulent shape. Downstream of TR0, the energy flux in the wall-normal direction (i.e towards the bulk of the flow) energises the region above the Klebanoff modes, increasing the thickness of the boundary layer. This indicates that the equilibrium boundary layer at TU is forming gradually during the transition process. Significant inverse cascade appears towards larger r_3 scales, but it is noteworthy that the bending of the energy flux vectors towards small scales does not appear in the transition region, but only at the TU region. It seems that strong inverse cascade is a central characteristic of transition; we will elaborate more on this important feature again later.

In order to shed more light into the patterns shown in figure 5.4, we turn our attention to the decompositions (3.4) and (3.5) and investigate the contributions of fluxes due to mean flow, non-linear interactions, viscosity, and pressure.

5.3.2 Decomposition of the energy fluxes

We focus in particular on the $TR2$ location, in the middle of the transition region. In figure 5.5 we plot the four pairs of fluxes $(\phi_{r_3}^F, \phi_{s_2}^F)$, $(\phi_{r_3}^V, \phi_{s_2}^V)$, $(\phi_{r_3}^M, \phi_{s_2}^M)$ and $(0, \phi_{s_2}^P)$ in the (r_3, Y) plane; these are denoted with the capital bold letters **F**, **V**, **M** and **P** respectively. We also plot the sum $(\phi_{r_3}^F + \phi_{r_3}^V, \phi_{s_2}^F + \phi_{s_2}^V)$, denoted as **F+V**, and the total flux (ϕ_{r_3}, ϕ_{s_2}) , which is of course identical to that shown in the lower left panel of figure 5.4. In all plots, the flux vectors are plotted as arrows to show the direction, but we also superimpose the flux magnitude as contours.

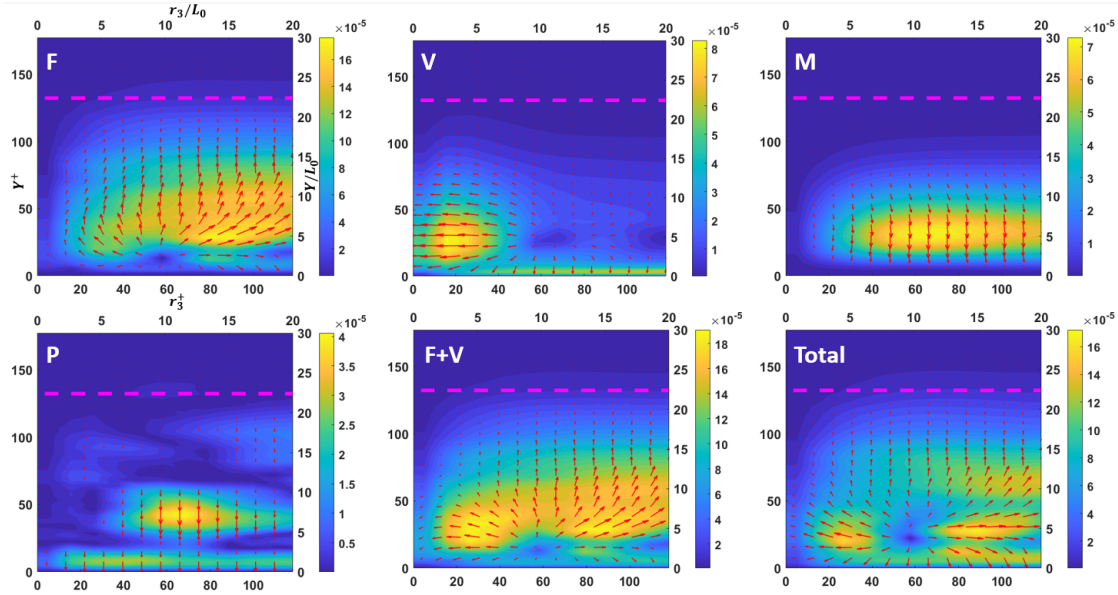


Figure 5.5 Maps of flux vectors due to nonlinear interaction between velocity fluctuations (F), viscosity (V), Mean velocity (M), pressure (P), combination ($F + V$), and total on (r_3, Y) plane at $TR2$ location. Contours represent the magnitude of the corresponding flux vectors.

We first note that the fluxes due to mean flow and pressure have components only in physical space, i.e. the wall-normal direction; this is because $\phi_{r_3}^P = 0$ and $\phi_{r_3}^M = \overline{dU_i dq^2} = 0$ due to homogeneity in the spanwise direction. Therefore in the r_3 direction the fluxes are due to the non-linear and viscous terms only, $\phi_{r_3}^F$ and $\phi_{r_3}^V$. The latter term results in forward cascade, while the former transfers energy to larger scales (inverse cascade) and to the bulk of the flow. The superposition of these two terms results in the **F+V** pattern, which is almost

identical to the total in the r_3 direction. Adding the contribution of pressure and mean flow terms, that both transfer energy towards the wall, readjusts the direction of the arrows, and completes the picture. The readjustment is especially evident at $Y \approx 5L_0$, and results in almost horizontal total inverse fluxes.

In figure 5.6 we plot the evolution of the two most important terms (viscous and non-linear) at all streamwise locations. For comparison purposes, we also include the total flux in the same plot (right column). We first notice that the viscous flux (middle column) results mainly in forward cascade (but there are some regions of very weak inverse cascade) and energy towards the wall. Note that the cascade processes due to viscous effect might be different from the conventional Kolmogorov cascade and associated with viscous diffusion. As transition progresses, the forward cascade acts at progressively smaller scales, while the energy transfer towards the wall becomes stronger. The latter is firmly established over a wide range of scales in the viscous sublayer when the turbulent region is reached. The viscous effect can be clearly seen also on the total energy flux plots, especially in the TR3 and TU locations. The energy transfer due to non-linear interaction term is more involved. This term mainly acts to transfer energy to larger scales (inverse cascade) and this is evident at all streamwise locations, even in the laminar (LA) region (but it is quite weak there). Starting from the TR0 location, it also acts to transfer energy to the bulk of the flow. The radial flux pattern that appears in the total flux plots (right column) is thus due to the superposition of the effects of the non-linear interaction and viscous terms.

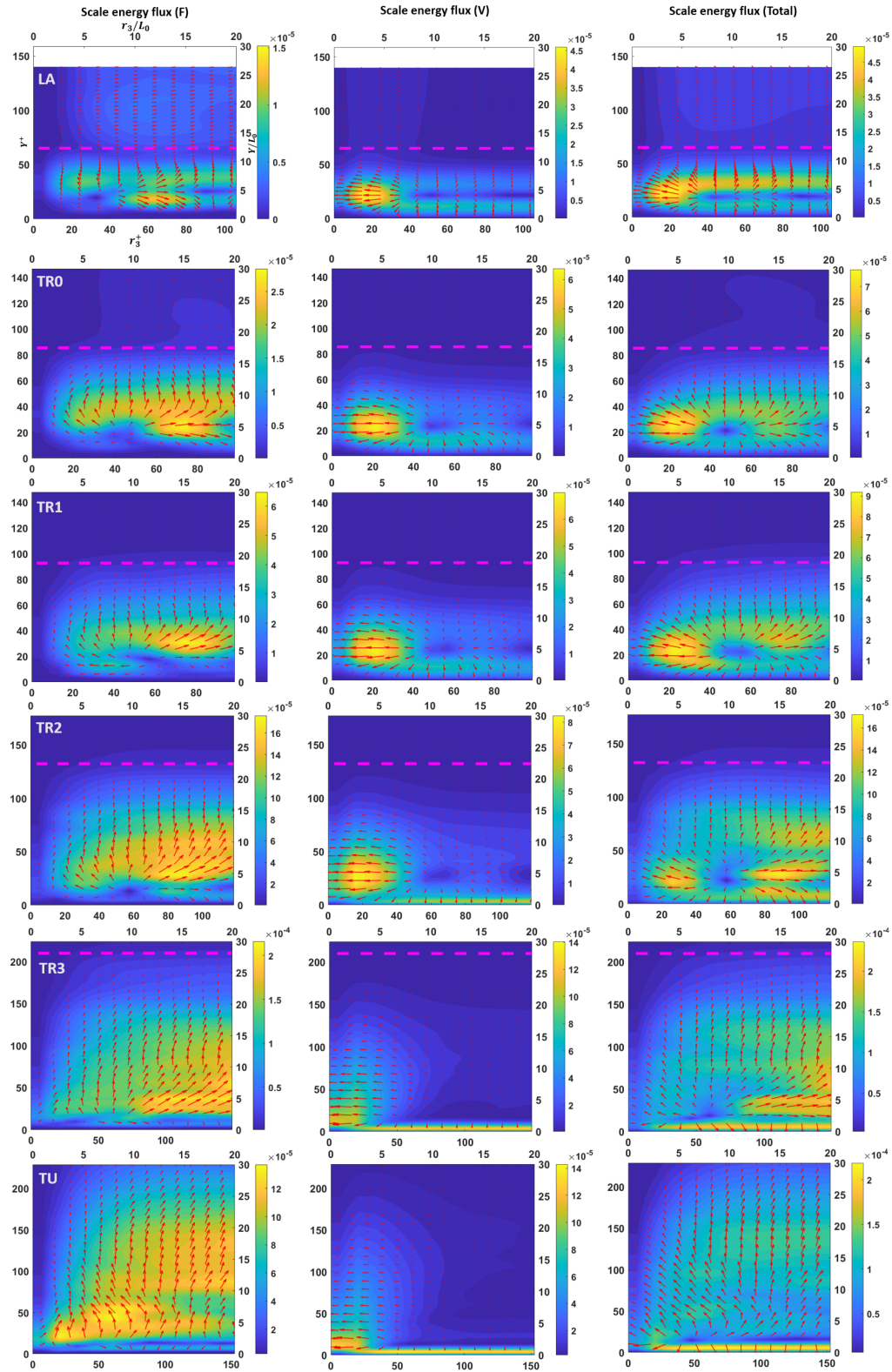


Figure 5.6 Evolution of energy fluxes due to the non-linear interaction term (left column) and the viscous term (middle column) in the (r_3, Y) plane. The total flux is shown in the right column. Contours represent the magnitude of the corresponding flux vectors. The pink dashed line is the local boundary layer thickness.

We now turn our attention to energy transfer in the (r_1, Y) plane.

5.4 Energy fluxes on the (r_1, Y) plane

For each streamwise location, we select the r_3 value that corresponds to the peak net production shown in Figure 5.4; the resulting separations are $r_3/L_0 = 8, 8, 8, 8, 5, 5$ at positions $LA, TR0, TR1, TR2, TR3, TU$ respectively. This allows us to investigate the behaviour of the energy fluxes in the particular r_3 separation at which the energy due the mean shear is injected.

5.4.1 General characteristics

Figure 5.7 shows the evolution of the energy flux vector (ϕ_{r_1}, ϕ_{s_2}) on this plane. A core region of strong net production, indicated by yellow colour, can be clearly seen; this region persists throughout transition. At the LA location, this region is concentrated at small r_1 values, $0 < r_1/L_0 < 75$, but as transition progresses, it moves closer to the wall and it is elongated. Gradually, it occupies the whole range of r_1 separations considered, $0 < r_1/L_0 < 150$. Note also that apart from the first two locations LA and $TR0$, net production can take negative values very close to the wall (because of strong dissipation).

From the high production region, energy flux vectors emanate with $\phi_{r_3} > 0$, indicating inverse cascade. At LA and $TR0$ locations this is barely noticeable, but at $TR1$ it is very strong in the region $0 < r_1/L_0 < 100$. It is also present at the downstream locations, $TR2, TR3$ but at larger separations, $75 < r_1/L_0 < 150$, and closer to the wall. It is very interesting to notice that once transition is complete, the inverse cascade is significantly subdued. Indeed, we can barely see it in the TU region (although the strong net production region can be still seen very clearly). In the next section we probe in more depth the origin of this inverse cascade, by examining the evolution of the different components of the energy flux.

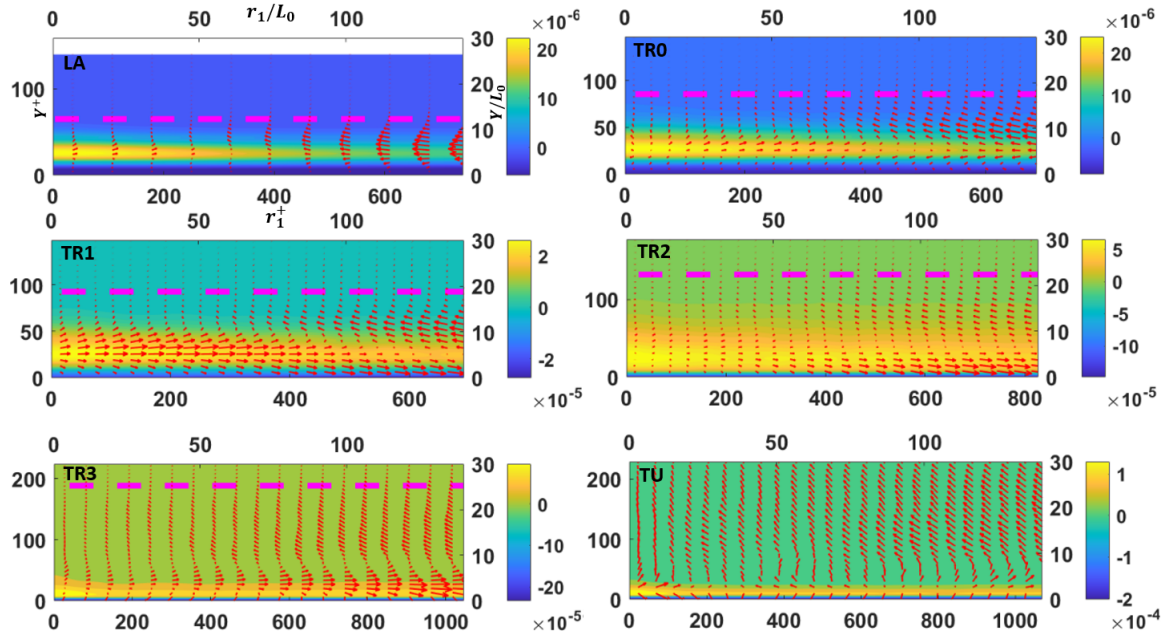


Figure 5.7 Evolution of the energy flux vector (ϕ_{r_1}, ϕ_{s_2}) on (r_1, Y) plane from laminar to turbulent region (red vectors). The contour values correspond to net production, and the pink dashed lines indicate the local boundary layer thickness. For $LA, TR0, TR1, TR2, TR3, TU$ locations, the separations are $r_3/L_0 = 8, 8, 8, 8, 5, 5$ respectively; these correspond to the peak net production shown in figure 5.4. The maximum vector magnitude (corresponding to the longest arrow in each plot) is $5 \times 10^{-4}, 2.5 \times 10^{-4}, 6 \times 10^{-4}, 1 \times 10^{-3}, 8 \times 10^{-4}, 3.5 \times 10^{-4}$ from LA to TU respectively.

5.4.2 Decomposition of the energy fluxes.

In the (r_3, Y) plane it was found that the non-linear interaction and viscous components of the energy flux were the most dominant. In the (r_1, Y) plane however, the viscous component is much less important, while energy flux due to mean flow, $\phi_{r_1}^M = \overline{dU_1 dq^2}$, assumes a more significant role. This is expected due to the strong streamwise inhomogeneity of the flow in the transition region. The sign of $\phi_{r_1}^M$ is determined by dU_1 . In order to facilitate the interpretation of the results, contours of U_1 are shown in figure 5.8 (top panel). For small r_1 , dU_1 has the same sign as the normal strain rate $\frac{\partial U_1}{\partial x_1}$, thus contours of the latter are also shown in the figure (bottom panel). In the transition region between $TR1$ and $TR3$, a zone of accelerating flow appears close to the wall, in agreement with the rise of C_f shown in figure 5.1. In the laminar region the flow decelerates, with the strongest negative strain around the centre of the boundary layer.

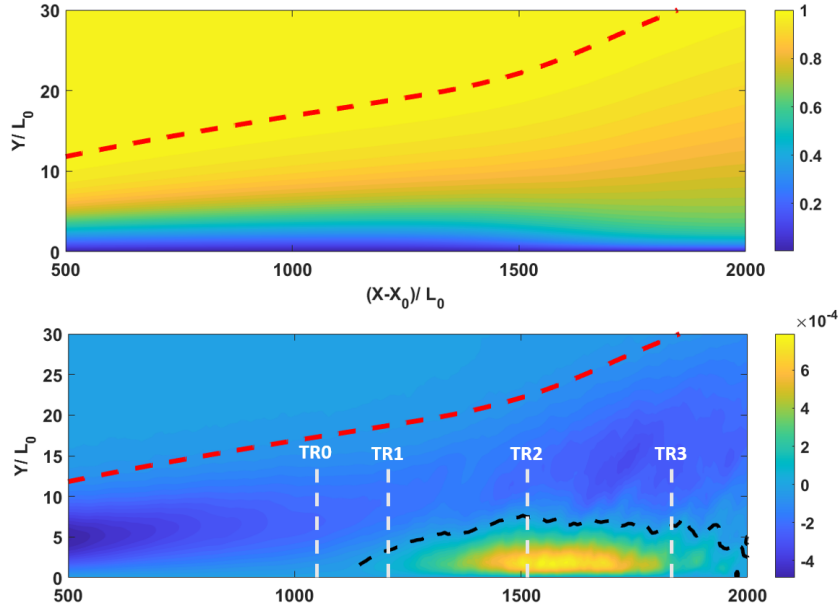


Figure 5.8 Contour plots of the mean streamwise velocity U_1 (top) and the normal strain rate $\frac{dU_1}{dX_1}$ (bottom). The black dashed curve denotes the $\frac{dU_1}{dX_1} = 0$ isoline. The red dashed line indicates the boundary layer thickness.

The evolution of the mean (M) and non-linear interaction (F) components of the energy flux are shown in figure 5.9. Note the competition between the two components, which is quite subtle. In the laminar region LA, $dU_1 < 0$ and the energy flux due to the mean flow leads to forward cascade; however the non-linear interaction leads to inverse cascade. The two contributions cancel each other out, and this explains why the total flux shown in the top-left panel of figure 5.7 is very weak for $0 < r_1/L_0 < 75$. At $TR0$ the C_f has a minimum and $dU_1 \approx 0$ for small r_1 , but for larger separations $dU_1 > 0$, consistent with the steeper rise of C_f at the start of transition. Again at this location the two components compete, with $\phi_{r_1}^M < 0$ (forward cascade) and $\phi_{r_1}^F > 0$ (inverse cascade). At $TR1$, dU_1 changes sign across the boundary layer ($dU_1 < 0$ away from the wall and $dU_1 \geq 0$ close to it, refer to figure 5.8), but $\phi_{r_1}^F > 0$ always, leading to overall strong inverse cascade. At $TR2$ and $TR3$, $dU_1 > 0$ close to the wall and very weak further away, while $\phi_{r_1}^F$ exhibits a mixture of forward and inverse cascade. Finally at TU, $dU_1 < 0$ away from the wall for large separations (due to the growth of the boundary layer thickness), while again $\phi_{r_1}^F$ does not exhibit a clear behaviour. It is evident therefore, that at least in the earlier stages of transition, a strong inverse cascade process exists, which is predominately due to the non-linear interaction term.

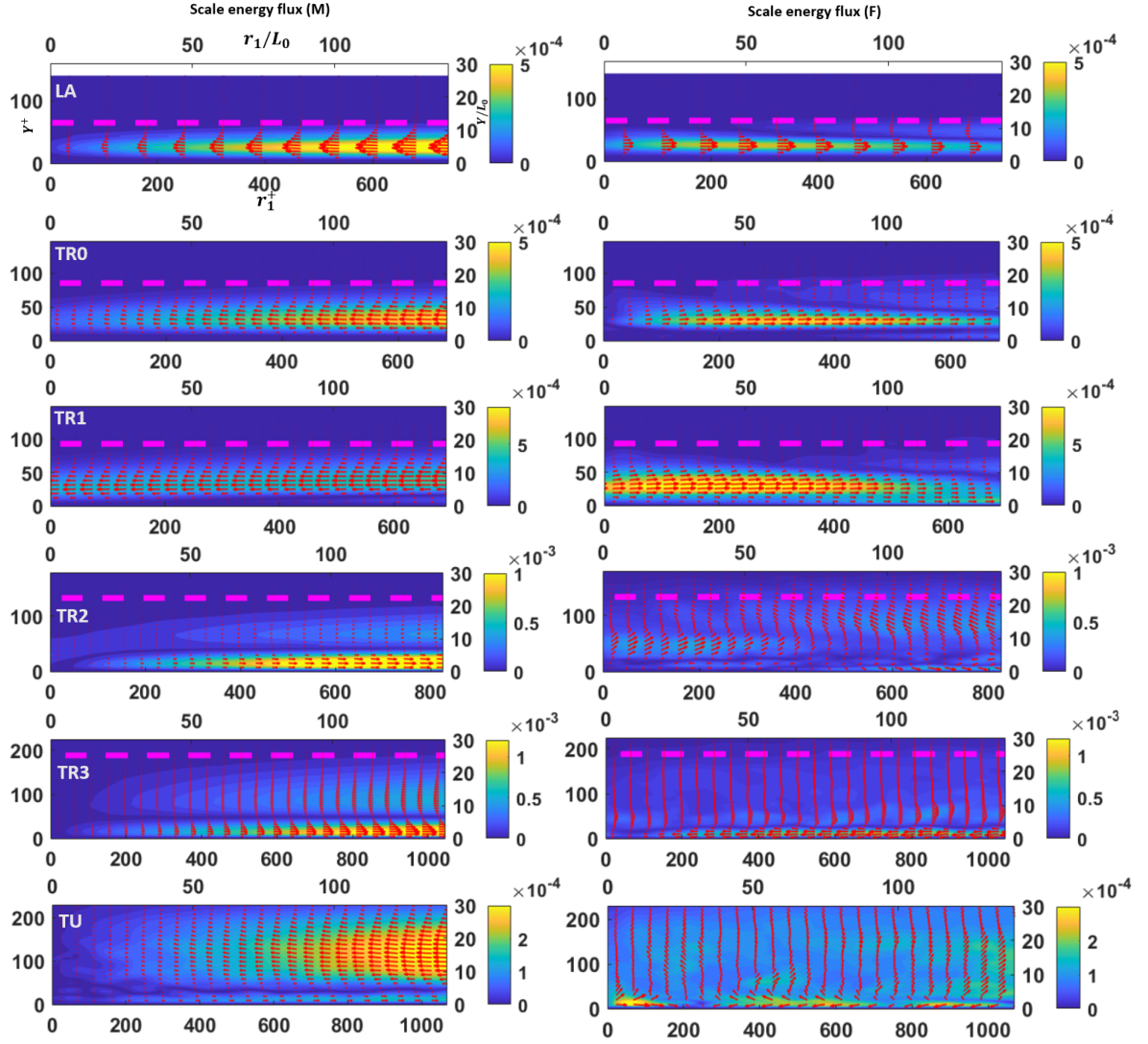


Figure 5.9 Evolution of the mean (M) and non-linear interaction (F) components of the energy flux (red vectors) on the (r_1, Y) plane. The contours are the magnitude of the corresponding flux vector. The spanwise separations are $r_3/L_0 = 8, 8, 8, 8, 5, 5$ from LA to TU respectively.

5.5 Energy fluxes in the (r_1, r_3) plane

In figure 5.10 maps of the scale energy flux vectors (ϕ_{r_1}, ϕ_{r_3}) in the (r_1, r_3) plane at 4 streamwise locations are shown; the wall normal distances correspond to the peak $\mathcal{P}_{net}$ of figure 5.4. Contours of the net energy production are also superimposed. The panel at location TR1 confirms the strong inverse cascade at regions of high $\mathcal{P}_{net}$, between $5 < r_3/L_0 < 15$. It is clear that the inverse cascade is mainly directed towards the r_1 direction; in the r_3 direction it appears only for $r_3/L_0 > 10$, in agreement with figure 5.4, but it is much weaker.

A recirculation pattern in the flow of energy is also evident, i.e. energy emanating from the large net production region flows to large r_1 and then recirculates back to small r_1 and r_3 scales. This recirculation look is much more clear in locations TR2 and TU. In TR3 it is not so evident, possibly because of the narrow range of the r_1 and r_3 separations considered.

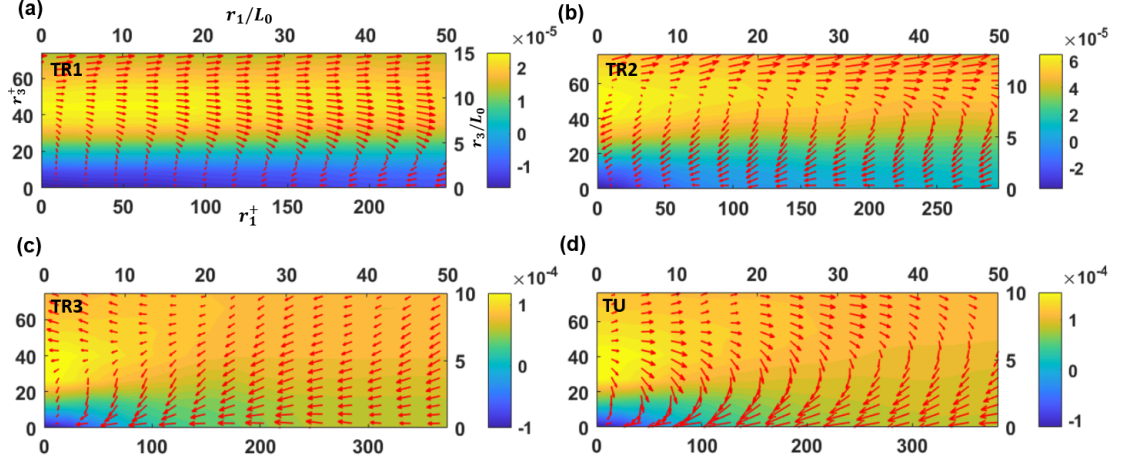


Figure 5.10 Scale energy flux vectors (ϕ_{r_1}, ϕ_{r_3}) on (r_1, r_3) plane at $Y/L_0 = 5$ for $TR1$ and $TR2$; $Y/L_0 = 1.3$ for $TR3$ and TU (red arrows). Contour plots of net production are superimposed. The maximum vector magnitude (corresponding to the longest arrow on each plot) is 4.5×10^{-4} , 1×10^{-3} , 6×10^{-4} , 7×10^{-4} from $TR1$ to TU respectively.

5.6 Maps of energy flux in the 3D (r_1, r_3, Y) hyper-plane

In the previous sections we examined the evolution of the energy fluxes on the 2D planes (r_3, Y) , (r_1, Y) and (r_1, r_3) . The most significant finding is that during the transition, a strong inverse cascade of energy exists that originates from the net production peak region and is mainly directed along the r_1 direction. However, previous results provide only limited understanding because they are visualised on 2D planes, so in this section, we map the energy paths on the 3D hyper-plane (r_1, r_3, Y) . These maps are shown in figure 5.11 for three streamwise locations LA , $TR2$ and TU , and are visualised from two different viewing angles. To aid understanding, we also superimpose the isosurface $0.7 \max(\mathcal{P}_{net})$ in yellow colour. For $TR2$ and TU we focus on different areas of the map; we do this by plotting the stream-tubes crossing small spheres placed at different r_1 separations (refer to the figure legend for more details).

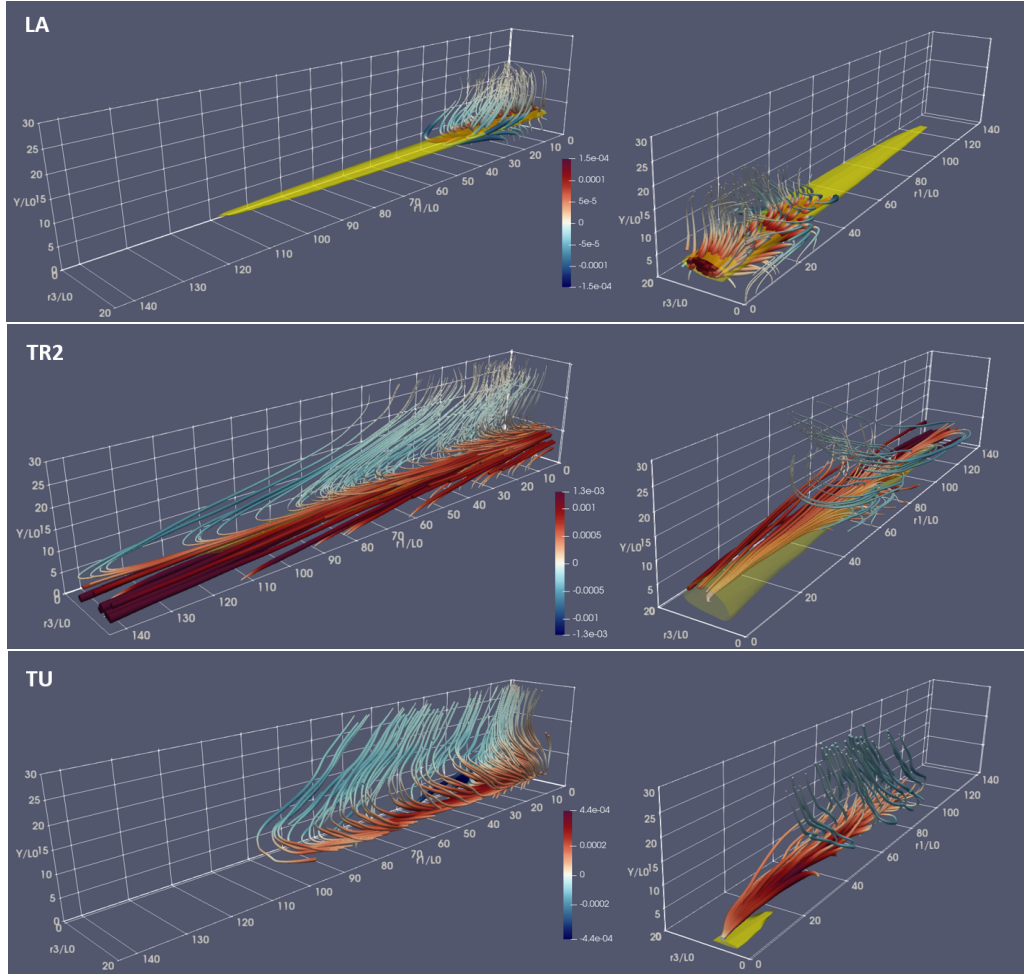


Figure 5.11 Maps of energy flux in the 3D (r_1, r_3, Y) hyper-plane in the form of stream-tubes for 3 streamwise locations. The yellow isosurface denotes $\max(\mathcal{P}_{net})$ factored by 0.7. The stream-tubes are coloured according to the sign of ϕ_{r_1} (red for positive, blue for negative, thus indicating inverse and forward cascade respectively). The colour bar refers to the value of ϕ_{r_1} . The radius of the tubes grows with increasing magnitude of the flux vector. Each plot was generated by placing a sphere at a specific location and tracing the stream-tubes crossing the sphere. For *LA*, a sphere of radius $15L_0$ is placed at $(r_1 = 5L_0, r_3 = 7.5L_0, Y = 5L_0)$ (left and right column show the same stream-tubes from different angles); For *TR2*, a sphere of radius $30L_0$ is placed at two points $(r_1 = 20L_0 \text{ and } 70L_0, r_3 = 7.5L_0, Y = 5L_0)$ (left column) and at $(r_1 = 70L_0, r_3 = 7.5L_0, Y = 5L_0)$ (right column). For *TU*, a sphere of radius $30L_0$ is placed at two points $(r_1 = 10L_0 \text{ and } 60L_0, r_3 = 7.5L_0, Y = 1.3L_0)$ (left column) and at $(r_1 = 60L_0, r_3 = 7.5L_0, Y = 1.3L_0)$ (right column).

In the laminar (*LA*) and transition regions (*TR2*), the net production region (in yellow colour) is centred at $r_3 = 10L_0, Y = 5L_0$ and is elongated in the r_1 direction. At *TU*, this region moves closer to the wall ($r_3 = 5L_0, Y = 1.3L_0$) and significantly shrinks in the streamwise direction (r_1). In all three locations, energy stream-tubes emanate from the production

region but their behaviour is different. In the laminar (*LA*) region, a weak inverse cascade along the r_1 direction is observed and then the flow energy pathlines bend upwards to the bulk of the flow and downwards to the wall region. At *TR2*, the energy flux is an order of magnitude larger compared to *LA*. Very strong inverse cascade is present in the r_1 direction, consistent with figures 5.7 and 5.10, but the energy paths then bend back to small r_1 and r_3 and towards the bulk of the flow; this explains the recirculating pattern noticed in figure 5.10. In the turbulent *TU* region, there is still inverse cascade in the r_1 direction, but it is weaker and confined to a smaller r_1 domain. Again, once energy has travelled for a sufficient distance in the r_1 direction, the stream-tubes bend to smaller r_1 and r_3 scales and to the bulk of the flow. The flux map at this location is very similar to that of channel flow, refer to Cimarelli et al. (2013).

5.7 Instantaneous interscale energy fluxes in physical space

The analysis presented in the previous sections identified a strong inverse cascade close to the wall, at wall-normal locations $Y < 10L_0$, which is mainly directed in the streamwise direction. In this section, we relate the energy transfer to the physical processes that take place during transition, and in particular to the growth of turbulent spots.

Figure 5.12 shows four snapshots of the instantaneous wall-normal velocity fluctuation at $Y = 1.3L_0$; the time delay between two snapshots is $dt = 500$ time units. In panel (a), two turbulent patches are starting to form at locations $(X - X_0)/L_0 \approx 1300$ and 1400. The two patches grow and propagate downstream as shown in panel (b), and finally merge to form a large turbulent patch, which is clearly visible in panel (c). This large patch then propagates as a single entity, see panel (d).

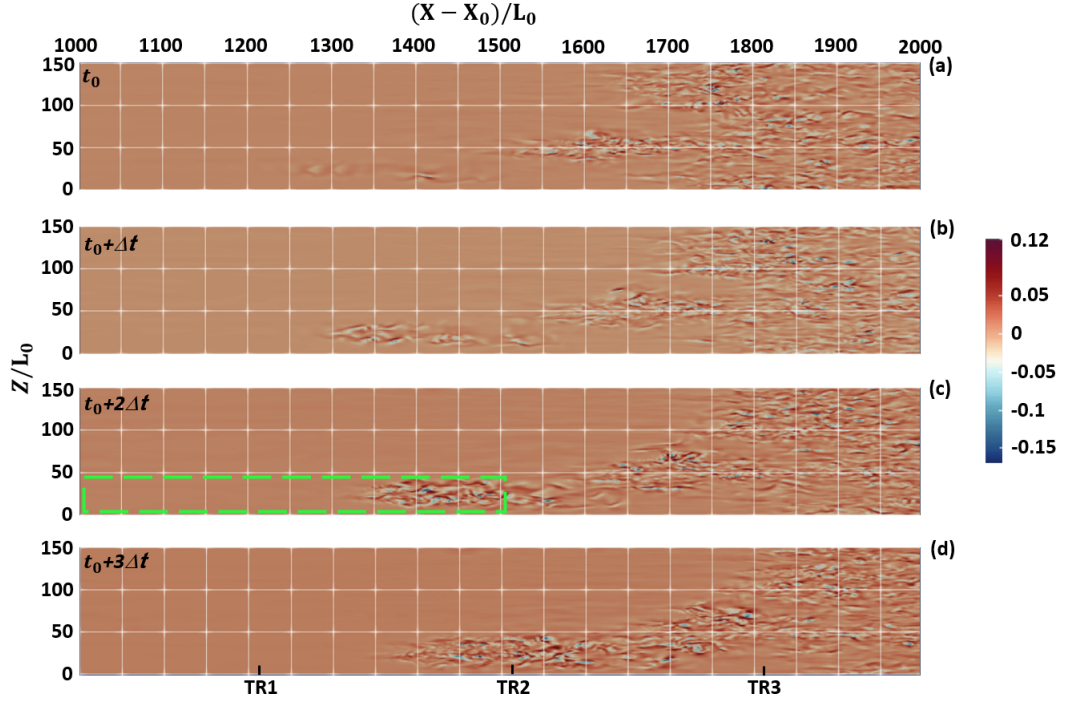


Figure 5.12 Contour plots of the instantaneous wall-normal velocity fluctuation at $Y = 1.3L_0$ at four time instances. The time separation Δt between two snapshots is 500 time units.

We now focus our attention to the rectangular area marked with a green dashed line in panel (c), which contains both laminar and turbulent regions. This area extends between $1002 - 1502$ and $0 - 50$ in the streamwise and spanwise directions respectively, and covers the region between locations $TR1$ and $TR2$ (see markings at the bottom of panel d). In figure 5.13 contour plots of the streamwise velocity fluctuations within the selected rectangular area are shown. We chose now two points, one red (at $(X - X_0)/L_0 = 1252$, within a laminar Klebanoff streak), and one green (at $(X - X_0)/L_0 = 1402$, within a streak that breaks down). We think of these two points as fixed middle points around which we consider separations r_1 up to $150L_0$ (consistent with figure 5.7) and r_3 up to $50L_0$. These separations define two regions, $[1177, 1327] \times [0, 50]$ and $[1327, 1477] \times [0, 50]$, that are marked with dashed yellow and black borders respectively in figure 5.13.

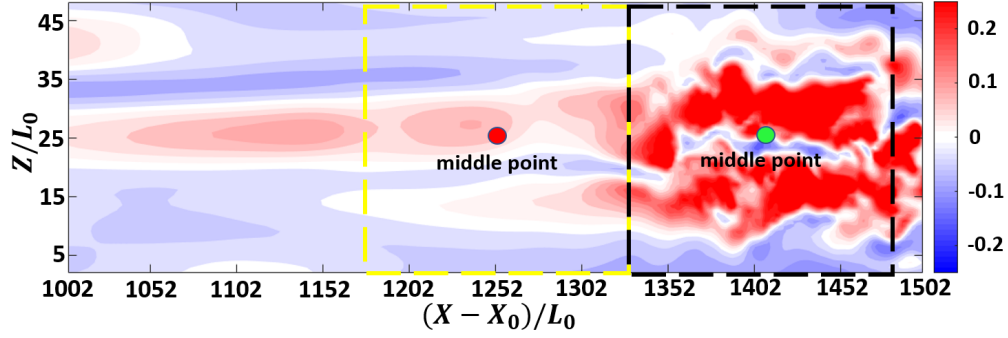


Figure 5.13 Contour plots of the instantaneous streamwise velocity fluctuations within the rectangular area marked with a green border in panel (c) of figure 5.12.

We can investigate the relation between the instantaneous flow and the scale fluxes within these two regions by plotting them in two separate figures. However the interpretation of the results is facilitated if the two figures are superimposed on a single plot. We do this on the left panel of figure 5.14 for the first region $[1177, 1327] \times [0, 50]$. In this panel, we superimpose the inter-scale energy flux vector (ϕ_{r_1}, ϕ_{r_3}) on the (r_1, r_3) plane (top and right axes) together with the instantaneous streamwise fluctuations in the (X, Z) plane (left and bottom axes). The process is as follows: we select an arbitrary point $x_i^+ = X_i + r_i/2$ to the right of the middle point X_i , and find the corresponding $x_i^- = X_i - r_i/2$, as shown in the figure. We then compute the energy flux vector using the fluctuations in these two points, and place the vector at x_i^+ (and since $x_i^+ = X_i + r_i/2$, the legends of the top and right axes are $r_1/(2L_0)$ and $r_3/(2L_0)$ respectively). In this way we can relate directly the instantaneous inter-scale energy fluxes with the dynamics of the streak breakdown and turbulent patch formation and growth. The right boundary of the area considered in figure 5.14 is just upstream of the streak breakdown, see also figure 5.13. Note the stronger streamwise velocity fluctuations when the two high speed streaks start to break down (at $Z/L_0 \approx 10$ and ≈ 30) and the associated inverse cascade. Forward cascade is observed instantaneously around the narrow low speed streak located in between, at $Z/L_0 \approx 20$. In the right panel, we present both the flux vectors and contours of the flux component ϕ_{r_1} ; the contours show that the inverse cascade is relatively weak at this time instant and location.

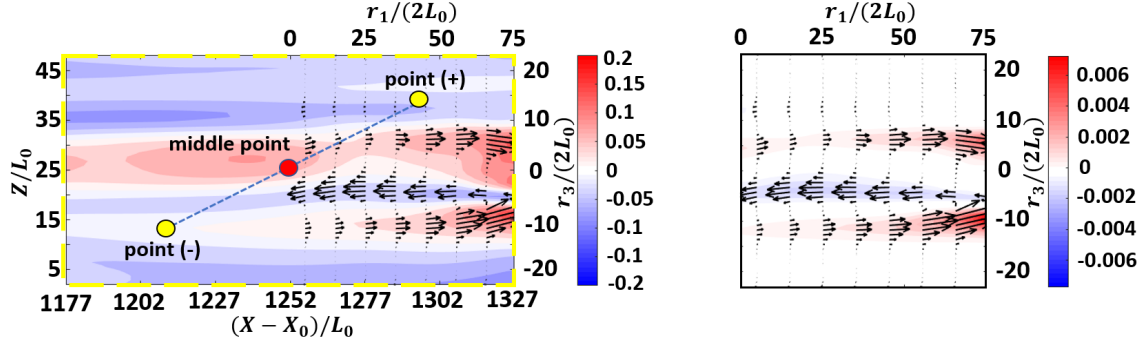


Figure 5.14 Contours of the streamwise velocity fluctuations (left) and of the energy flux component ϕ_{r_1} (right). In both panels, the superposed black arrows indicate the direction of the energy flux vector (ϕ_{r_1}, ϕ_{r_3}) . The middle point is located at $(X - X_0)/L_0 = 1252$.

We repeat the process for the region marked by the black border in figure 5.13. The middle point is now at $(X - X_0)/L_0 = 1402$ and is located firmly within the streak breakdown region; indeed it is surrounded by two high speed streaks that disintegrate to form a turbulent spot; the spot can be clearly seen in panel (c) of figure 5.12. The results, shown in figure 5.15, indicate that the large velocity fluctuations inside the streaks contribute to an inverse cascade around the middle point, and this corresponds to the growth of the high-speed streaks within the turbulent spot. Comparison of the streamwise flux components ϕ_{r_1} (right panels of figures 5.14 and 5.15) shows that the inverse cascade is much stronger when the streaks breakdown. This is because the inter-scale flux scales with the cube of velocity difference, so a small increase in du_1 results in much larger fluxes.

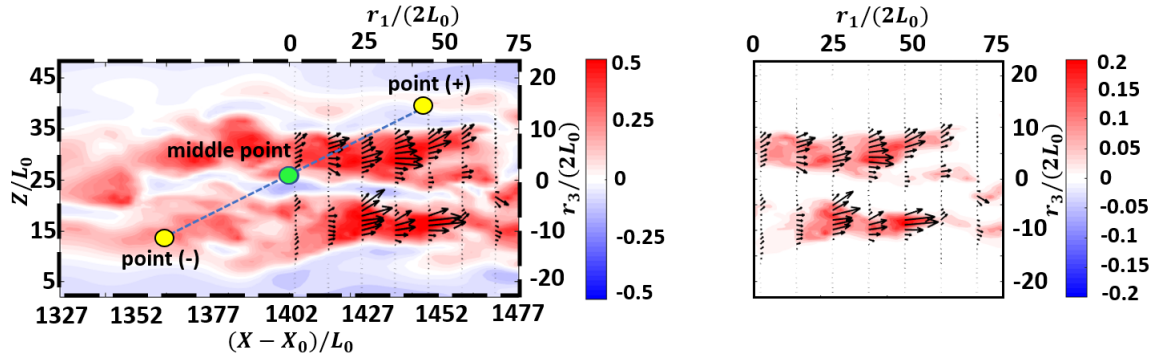


Figure 5.15 Contours of the streamwise velocity fluctuations (left) and of the energy flux component ϕ_{r_1} (right). In both panels, the superposed black arrows indicate the direction of the energy flux vector (ϕ_{r_1}, ϕ_{r_3}) . The middle point is located at $(X - X_0)/L_0 = 1402$.

In both regions considered, inverse cascade appears when the instantaneous flow accelerates between x_i^- to x_i^+ . This can also be seen in the average picture in figure 5.7.

Note that the interpretation of inverse cascade for fixed values of r_3 (for example for those values where $\mathcal{P}_{net}$ is maximised, as in figure 5.9) refers to the growth of streaks within spots. This is of course related to the growth of spots themselves, as the latter consist of high-speed streaks that grow individually.

The analysis of the growth of spots as single entities in the spanwise direction using the KMH formulation is possible, but it is more involved. It would require the identification of a spot using an indicator function as in Marxen and Zaki (2019) and Yao et al. (2020), positioning of the midpoint X_i at the centroid of the spot, and integration of the KMH equation in the scale space around the spot bounding surface. This is a generalisation of the integration over a sphere performed by Hill (2002a) and can elucidate very clearly the mechanisms for interscale energy transfer to the spot as a single entity. This is however beyond the scope of the present thesis.

The previous analysis demonstrated that the scale fluxes depend strongly on the distance from the wall and are highly intermittent. This motivates the investigation of their probability density function (PDF). Figure 5.16 shows the PDF of ϕ_{r_1} (top panel) at the streamwise location $TR1$ for fixed separation $(r_1, r_3) = (50L_0, 8L_0)$ and three different wall-normal distances. We also plot the PDFs of the components $\phi_{r_1}^M = dU_1 dq^2$ (mid panel) and $\phi_{r_1}^F = du'_1 dq^2$ (bottom panel). In total 55,000 time and spanwise samples were used to construct these PDFs. Although this is not ideal, because spanwise samples smooth out the fluxes around localised spots shown in figures 5.12, the PDFs still provide a more complete picture compared to time-averages.

At the separations considered there is strong inverse cascade, as shown in figure 5.7. At $Y = 7L_0$, where the inverse cascade is maximised, the shape of the PDF of ϕ_{r_1} is slightly asymmetric and skewed towards positive values. At the other two locations, the PDF is nearly symmetric resulting in very low time-averages. The PDFs of $\phi_{r_1}^M$ are one-sided; the sign depends on dU_1 and varies with wall-normal distance. This results in a competition between $\phi_{r_1}^F$ and $\phi_{r_1}^M$ in the core of the boundary layer. However, $\phi_{r_1}^M$ attains small values, and the overall effect on ϕ_{r_1} is small. Note the presence of rare (low probability) but very strong positive and negative fluxes of $\phi_{r_1}^F$ at $Y = 7L_0$. Similar behavior has been observed for the PDFs of fluxes in homogeneous isotropic turbulence, see figure 3 in Yasuda and Vassilicos (2018) (see also Ishihara et al. (2009) Aoyama et al. (2005)), but in this case the time-average is negative (standard forward cascade). Therefore the presence of turbulent spots in a transitional flow modifies the average picture, even though the results are spanwise-averaged and the localised spot effect is diluted; this indicates that it must be quite strong. The rare positive and negative events are still there, and it is not clear how the two forms of intermittency, one due to small-scale structures present in turbulent flows and the other

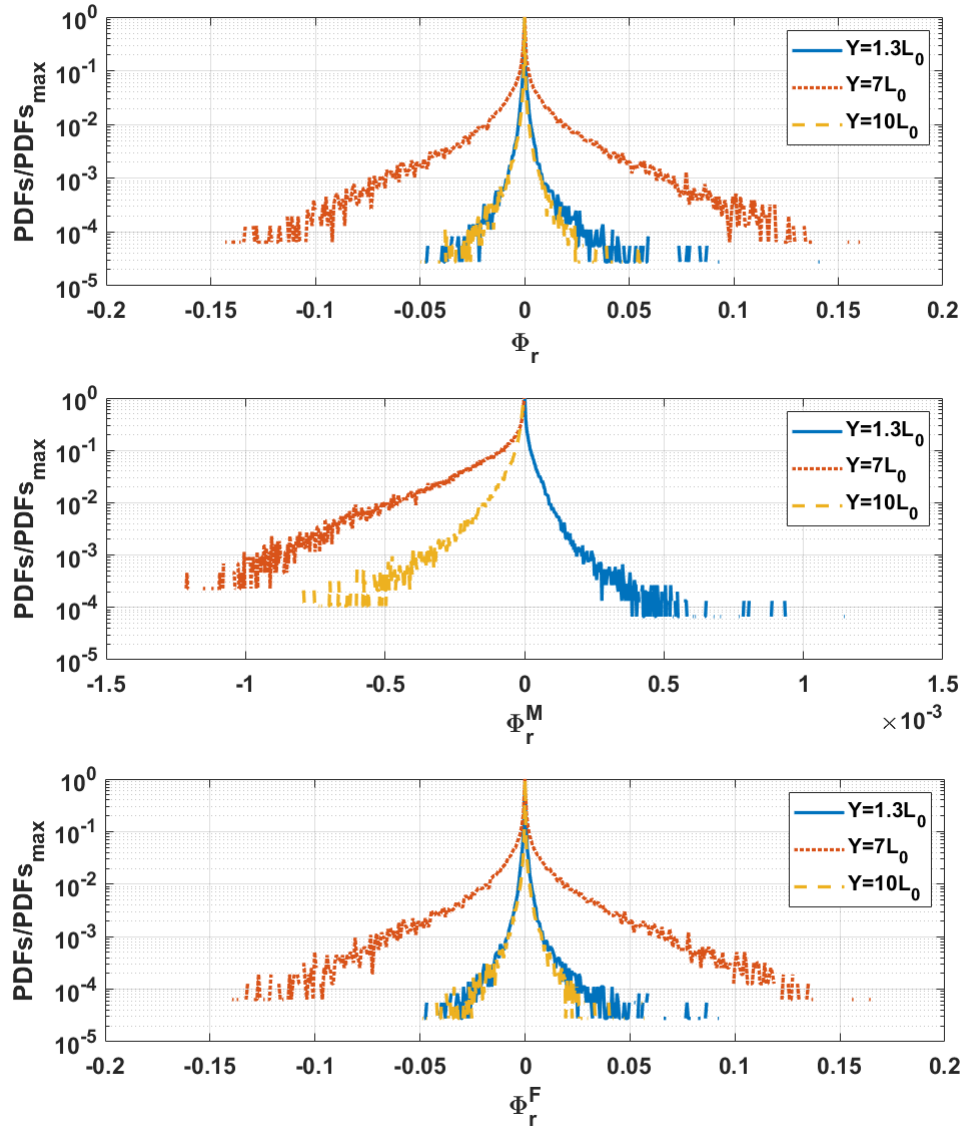


Figure 5.16 PDFs of ϕ_{r_1} (top) with the mean (middle) and fluctuating (bottom) components at $TR1$ location for $(r_1 = 50L_0, r_2 = 0, r_3 = 8L_0)$.

due to transition, interact. This motivates a more careful analysis of interscale fluxes, for example one in which the fluxes are conditioned on the flow state (laminar or turbulent) at points x_i^+ and x_i^- . Such conditional analysis has been performed by Yao et al. (2020) for the second-order structure functions and can be extended to interscale fluxes as well; we leave this for future work.

5.8 Conclusions

We apply the KMH equation to analyse the interscale energy flux in a flat plate boundary layer undergoing bypass transition due to forcing from free-stream turbulence. Energy from the mean flow is injected at a particular spanwise length scale and wall-normal position, both of which are determined by the interaction between the external forcing and the mean shear (linear receptivity process) that takes place in the early development of the boundary layer. From this production peak, energy fluxes radiate to different directions in physical and scale spaces. Two- and three-dimensional flux maps in the transitional regime show a very strong inverse cascade near the wall (i.e. energy flows from small to large scales) in the streamwise direction. Analysis of the constituent components of the total flux indicates that the inverse cascade is due to the non-linear interactions between velocity fluctuations. Other components, such as the one due to mean flow inhomogeneity, are non-negligible, but are not strong enough to affect the flux direction. As the transition process approaches completion, the maps morph to those encountered in other fully developed turbulent flows, such as channel flow, where there is weak inverse cascade in the streamwise direction, and then flux vectors bend towards small scales.

Superposition of the instantaneous flux vectors with the corresponding flow fields relates the cascade process to the growth of turbulent structures in the streamwise direction. The sign of the time and spanwise- average flux is positive across the boundary layer thickness and this is related to predominantly accelerating events that lead to the elongation of vortex structures within a spot, and consequently to the spot itself. The probability density function of the flux is asymmetric and skewed towards positive values. However, strong but rare forward and inverse cascade events are present, as in homogeneous isotropic turbulence.

The present results have painted a picture of interscale transfer in transitional flows that deviates significantly from that of fully developed, wall-bounded, turbulent flows. Spanwise averaging dilutes the transfer process within nascent and developing spots, as a large proportion of the flow remains laminar in the early stages of transition. More in depth investigations using conditional averaging, an operation that naturally accounts for intermittency effects due to transition, are certainly warranted. Such investigations will elucidate the contributions

to the average interscale transfer arising from different events, i.e. when the points x_i^+ and x_i^- are both located in laminar or turbulent patches, or one is placed in a laminar and the other in a turbulent region. The conditionally-averaged KMH analysis will be presented in the next chapter.

Chapter 6

Analysis of interscale energy transfer using the conditionally-averaged KHMH equation.

6.1 Introduction

Results from the application of the standard-averaged KHMH equation to the boundary layer bypass transition case were presented in chapter 5. In that chapter, a strong inverse cascade in the streamwise direction is found due to the non-linear interaction at the transitional locations, however, all the results were spanwise-averaged. Laminar streaks and turbulent patches however coexist during the transition and crucial information might be missed when averaging over the spanwise direction.

In this chapter, in order to separate the influence between laminar and turbulent events, we derived a conditionally-averaging KHMH equation and decompose scale energy flux vectors and the production term conditionally. The conditionally-averaging technique follows the process outlined in section 4.2.2, where a general expression of the conditionally-averaged decomposition for two-points properties is derived. This formula enables the analysis of the energy cascade process separately at the laminar and turbulent regions. The conditionally-averaged analysis is implemented at different streamwise locations from laminar (LA), transitional ($TR1, TR2, TR3$) and turbulent (TU), which are the same as in chapter 5. We re-plot the skin friction coefficient in figure 6.1 and in order to facilitate the interpretation of results we superimpose the variation of intermittency γ .

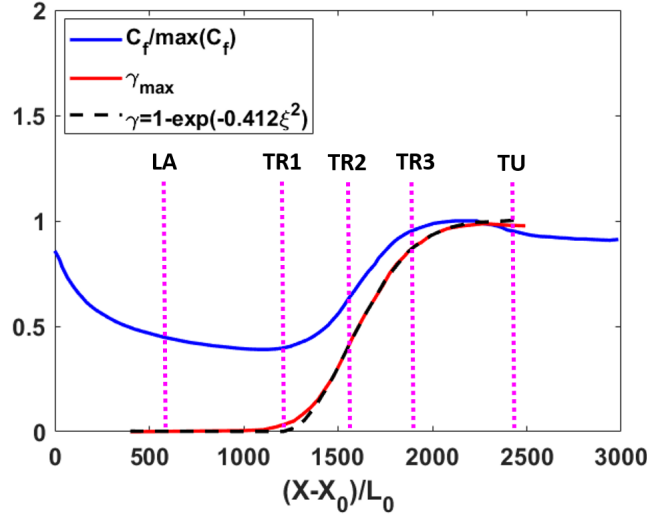


Figure 6.1 Skin friction coefficient, C_f (blue solid line), γ_{max} (red solid line), and γ from Narasimha (1985)'s formula (black dash line; $\xi = (X - X_s)/(X_{\gamma=0.75} - X_{\gamma=0.25})$; $(X_s - X_0) = 1100L_0$ where the transition starts) against $(X - X_0)/L_0$. The purple vertical lines denote one at laminar location $LA = 540L_0 + X_0$, three at transitional locations $TR1, TR2, TR3 = 1215, 1515, 1815L_0 + X_0$ and one at fully turbulent region $TU = 2415L_0 + X_0$.

6.2 Two-point conditionally-averaged operation

In order to derive the conditional time-averaged form of the KMH equation, we need first to define the conditions under which the averaging is performed. As already explained before, we employ the local binary function $\tau(X_i, t)$ to distinguish between instantaneous laminar and turbulent events (or states) at point X_i at time t . More specifically, $\tau(X_i, t)$ takes the value of 0 for the former, and 1 for the latter event. This binary function is computed using the standard deviation of $D = |v| + |w|$; refer to Marxen and Zaki (2019) and Yao et al. (2020) for more details.

Since the KMH equation involves two points, the conditions for the averaging operation should be defined using the states at both points, so we need to consider four combinations. For example, if both x_i^+ and x_i^- are located within a turbulent patch, this is denoted as a Turbulent-Turbulent (or TT) event, and it is defined by the condition $\tau^+ \tau^- = 1$, where $\tau^+ = \tau(x_i^+, t)$ and $\tau^- = \tau(x_i^-, t)$. Similarly, if both points are within the laminar region, this is a Laminar-Laminar (or LL) event, and it is defined by the condition $(1 - \tau^+)(1 - \tau^-) = 1$. If x_i^+, x_i^- are within a turbulent and a laminar patch respectively, this is a Turbulent-Laminar (or TL) event, defined by $\tau^+(1 - \tau^-) = 1$. Finally, if x_i^+, x_i^- are within a laminar and turbulent region respectively, this is a Laminar-Turbulent (or LT) event, defined by $(1 - \tau^+)\tau^- = 1$. In our notation, the first capital event letter denotes the state of point x_i^+ and the second the

state of x_i^- . We will also use the generic notation AA to refer to a general event, i.e. $AA = TT$ or LL or TL or LT . The four events are shown schematically in figure 6.2. Note that in this chapter for reasons to become clear later, we separate TL and LT events (but in the previous chapter we considered them together).

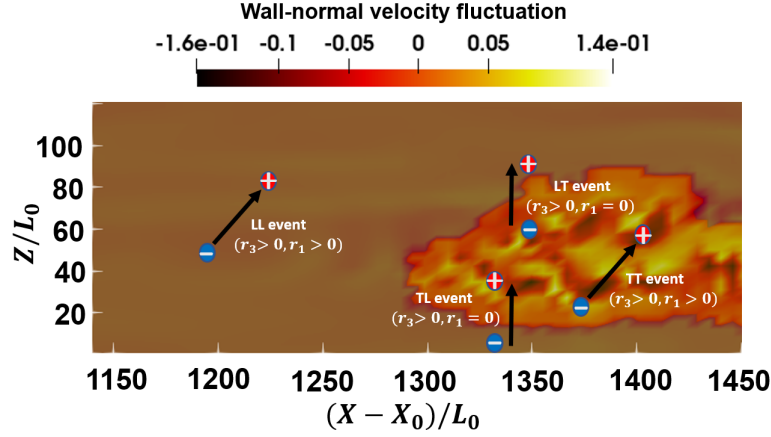


Figure 6.2 Contour plot of wall-normal velocity fluctuations in the transitional region. A turbulent spot is clearly visualised on the right half of the figure. Within the spot $\tau = 1$, while outside (grey region) $\tau = 0$. The four different two-point event types, LL , TT , TL and LT , are shown.

We can now define the time- and spanwise-average two-point intermittencies as,

$$\gamma^{(TT)} \equiv \frac{1}{\Delta T L_z} \int_0^{\Delta T} \int_0^{L_z} \tau^+ \tau^- dz dt \quad (6.1a)$$

$$\gamma^{(TL)} \equiv \frac{1}{\Delta T L_z} \int_0^{\Delta T} \int_0^{L_z} \tau^+ (1 - \tau^-) dz dt \quad (6.1b)$$

$$\gamma^{(LT)} \equiv \frac{1}{\Delta T L_z} \int_0^{\Delta T} \int_0^{L_z} (1 - \tau^+) \tau^- dz dt \quad (6.1c)$$

$$\gamma^{(LL)} \equiv \frac{1}{\Delta T L_z} \int_0^{\Delta T} \int_0^{L_z} (1 - \tau^+) (1 - \tau^-) dz dt \quad (6.1d)$$

Since $\tau^+ \tau^- + (1 - \tau^+) (1 - \tau^-) + \tau^+ (1 - \tau^-) + (1 - \tau^+) \tau^- = 1$, we have $\gamma^{(TT)} + \gamma^{(TL)} + \gamma^{(LT)} + \gamma^{(LL)} = 1$, thus the general intermittency $\gamma^{(AA)}$ is the probability that event AA will occur over a period of time ΔT along L_z .

The conditional time-averages of the general two-point variable, $dQ = Q(x_i^+) - Q(x_i^-)$, can be defined as,

$$\overline{dQ}^{(TT)}(X, Y; r_1, r_3) \equiv \frac{1}{\Delta T L_z \gamma^{(TT)}} \int_0^{\Delta T} \int_0^{L_z} \tau^+ \tau^- dQ dz dt \quad (6.2a)$$

$$\overline{dQ}^{(TL)}(X, Y; r_1, r_3) \equiv \frac{1}{\Delta T L_z \gamma^{(TL)}} \int_0^{\Delta T} \int_0^{L_z} \tau^+ (1 - \tau^-) dQ dz dt \quad (6.2b)$$

$$\overline{dQ}^{(LT)}(X, Y; r_1, r_3) \equiv \frac{1}{\Delta T L_z \gamma^{(LT)}} \int_0^{\Delta T} \int_0^{L_z} (1 - \tau^+) \tau^- dQ dz dt \quad (6.2c)$$

$$\overline{dQ}^{(LL)}(X, Y; r_1, r_3) \equiv \frac{1}{\Delta T L_z \gamma^{(LL)}} \int_0^{\Delta T} \int_0^{L_z} (1 - \tau^+) (1 - \tau^-) dQ dz dt \quad (6.2d)$$

This means that the standard time-average can be decomposed as

$$\begin{aligned} \overline{dQ}(X, Y; r_1, r_3) &= \frac{1}{\Delta T L_z} \int_0^{\Delta T} \int_0^{L_z} dQ dz dt \\ &= \gamma^{(TT)} \overline{dQ}^{(TT)} + \gamma^{(TL)} \overline{dQ}^{(TL)} + \gamma^{(LT)} \overline{dQ}^{(LT)} + \gamma^{(LL)} \overline{dQ}^{(LL)} \end{aligned} \quad (6.3)$$

The definitions are similar for the conditionally-average two-point variable $Q_* = 0.5 [Q(x_i^+) + Q(x_i^-)]$, for example,

$$\overline{Q_*}^{(TT)}(X, Y; r_1, r_3) \equiv \frac{1}{\Delta T L_z \gamma^{(TT)}} \int_0^{\Delta T} \int_0^{L_z} \tau^+ \tau^- Q_* dz dt \quad (6.4)$$

and similarly for the other events.

Referring back to figure 6.2, it is clear that $\overline{dQ}^{(TL)}$ and $\overline{dQ}^{(LT)}$ are two-point averages across the laminar/turbulent interface. For $r_1 = 0$ and $r_3 \neq 0$ (case shown in figure 6.2), these two conditional averages are taken across the interface in the spanwise direction, therefore they capture the properties of the interface in that direction. For $r_1 \neq 0$ and $r_3 = 0$ (case shown in figure 6.3), LT averaging is taken across the tail (i.e. the downstream end) of a turbulent spot, while TL is taken across the head of the spot (i.e. the upstream end). It is therefore possible to distinguish the different properties of the head or tail of a spot using the appropriate conditionally averaged quantity. This observation is important and facilitates the physical interpretation of the results presented in the following sections.

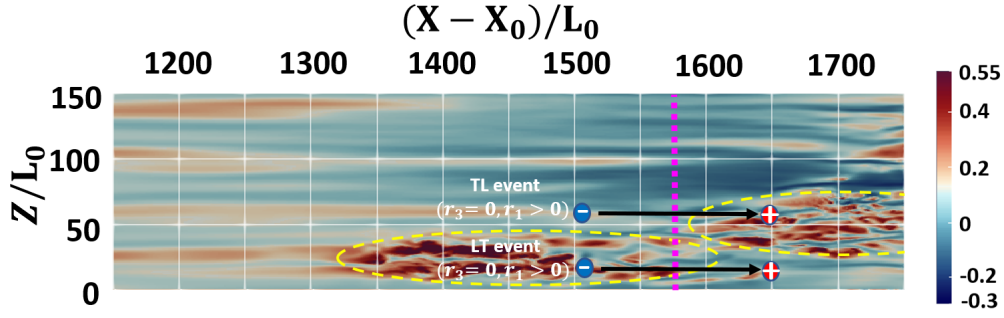


Figure 6.3 Contour plot of instantaneous streamwise velocity fluctuations. Yellow ovals demarcate two turbulent spots. The purple vertical dotted line represents a fixed streamwise location, X . Two event types, TL and LT , are shown with $r_1 \neq 0$ and $r_3 = 0$.

We now proceed to derive the conditionally-averaged KMH equation.

6.3 Derivation of Conditionally-averaged KMH equation

We start with the Navier-Stokes equations at two points x_i^+ and x_i^- ,

$$\frac{\partial u_i^+}{\partial t} + u_j^+ \frac{\partial u_i^+}{\partial x_j^+} = -\frac{\partial p^+}{\partial x_i^+} + \nu \frac{\partial^2 u_i^+}{\partial x_j^+ \partial x_j^+} \quad (6.5a)$$

$$\frac{\partial u_i^-}{\partial t} + u_j^- \frac{\partial u_i^-}{\partial x_j^-} = -\frac{\partial p^-}{\partial x_i^-} + \nu \frac{\partial^2 u_i^-}{\partial x_j^- \partial x_j^-} \quad (6.5b)$$

We now define the conditional velocity fluctuation difference as

$$du_i'^{(AA)} \equiv du_i - dU_i^{(AA)}, \quad (6.6)$$

where $AA = TT$ or LL or TL or LT as mentioned earlier, $du_i = u_i^+ - u_i^-$ and $dU_i^{(AA)} = \overline{du_i}^{(AA)}$ (from definition (6.2)). It is straightforward to prove that $\overline{du_i'^{(AA)}}^{(AA)} = 0$; this is the equivalent of $\overline{du_i'} = 0$ in standard averaging.

Similarly, the conditional average fluctuation velocity between the two points is defined as

$$u_{i*}'^{(AA)} \equiv u_{i*} - U_{i*}^{(AA)}, \quad (6.7)$$

where $u_{i*} = 0.5(u_i^+ + u_i^-)$ and $U_{i*}^{(AA)} = \overline{u_{i*}}^{(AA)}$ (from definition (6.4)), and again $\overline{u_{i*}'^{(AA)}}^{(AA)} = 0$.

Subtracting equation (6.5b) from (6.5a), multiplying each term by $2du_i'^{(AA)}$, and then applying the (AA) averaging operation as defined in equation (6.2) we obtain,

$$\begin{aligned}
 & \underbrace{\overline{2du_i'^{(AA)} \frac{\partial du_i}{\partial t}}^{(AA)}}_{\text{Transient term}} + \underbrace{\overline{2du_i'^{(AA)} u_j^+ \frac{\partial du_i}{\partial x_j^+} + 2du_i'^{(AA)} u_j^- \frac{\partial du_i}{\partial x_j^-}}^{(AA)}}_{\text{Non-linear term}} = \\
 & \underbrace{\overline{-2du_i'^{(AA)} \left(\frac{\partial dp}{\partial x_i^+} - \frac{\partial dp}{\partial x_i^-} \right)}^{(AA)}}_{\text{Pressure-velocity correlation}} + \underbrace{\overline{2vdu_i'^{(AA)} \frac{\partial^2 du_i}{\partial x_j^+ \partial x_j^+} + 2vdu_i'^{(AA)} \frac{\partial^2 du_i}{\partial x_j^- \partial x_j^-}}^{(AA)}}_{\text{Viscous term}}, \tag{6.8}
 \end{aligned}$$

where we have used $\frac{\partial u_i^+}{\partial x_j^-} = 0$ and $\frac{\partial u_i^-}{\partial x_j^+} = 0$ (because x_j^+ and x_j^- are independent variables).

We now define the two-point conditional energy as $dq^{2(AA)} = du_i'^{(AA)} du_i'^{(AA)}$ and seek its transport equation in the physical and scale spaces. Applying the variable transformation $X_i = 0.5(x_i^+ + x_i^-)$ and $r_i = x_i^+ - x_i^-$ and the definitions (6.6) and (6.7) into (6.8), after some algebra we get the following conditionally-averaged KMH equation for $dq^{2(AA)}$,

$$\begin{aligned}
 & \underbrace{\overline{\frac{\partial dq^{2(AA)}}{\partial t}}^{(AA)}}_{\text{Transient term}} + \underbrace{\overline{U_{j*}^{(AA)} \frac{\partial dq^{2(AA)}}{\partial X_j}}^{(AA)}}_{\text{Mean advection}} + \underbrace{\overline{u_{j*}'^{(AA)} \frac{\partial dq^{2(AA)}}{\partial X_j}}^{(AA)}}_{\text{Turbulent advection}} + \underbrace{\overline{dU_j^{(AA)} \frac{\partial dq^{2(AA)}}{\partial r_j}}^{(AA)}}_{\text{Linear transfer}} \\
 & + \underbrace{\overline{du_j'^{(AA)} \frac{\partial dq^{2(AA)}}{\partial r_j}}^{(AA)}}_{\text{Nonlinear transfer}} = \underbrace{\overline{-2du_i'^{(AA)} \frac{\partial dp'^{(AA)}}{\partial X_i}}^{(AA)}}_{\text{Pressure-velocity correlation}} + \underbrace{\overline{v \frac{1}{2} \frac{\partial^2 dq^{2(AA)}}{\partial X_j \partial X_j}}^{(AA)}}_{\text{Physical diffusion}} + \underbrace{\overline{2v \frac{\partial^2 dq^{2(AA)}}{\partial r_j \partial r_j}}^{(AA)}}_{\text{Scale diffusion}} \\
 & \underbrace{\overline{-2du_i'^{(AA)} u_{*j}'^{(AA)} \frac{\partial dU_i^{(AA)}}{\partial X_j} - 2du_i'^{(AA)} du_j'^{(AA)} \frac{\partial dU_i^{(AA)}}{\partial r_j}}^{(AA)}}_{\text{Production}} \\
 & - \underbrace{2v \left(\overline{\frac{\partial du_i'^{(AA)}}{\partial x_j^+} \frac{\partial du_i'^{(AA)}}{\partial x_j^+}} + \overline{\frac{\partial du_i'^{(AA)}}{\partial x_j^-} \frac{\partial du_i'^{(AA)}}{\partial x_j^-}} \right)^{(AA)}}_{\text{Dissipation}} \tag{6.9}
 \end{aligned}$$

The form of equation (6.9) is similar to the standard form (3.1). This is due to the appropriate definitions of the conditional fluctuating quantities in (6.6) and (6.7) that satisfy $\overline{du_i'^{(AA)}}^{(AA)} = 0$ and $\overline{u_{*i}'^{(AA)}}^{(AA)} = 0$. There is however an important difference. Standard

averaging commutes with the spatial differentiation operation, for example,

$$\frac{\partial \overline{dQ}}{\partial r_j} = \frac{1}{\Delta T L_z} \frac{\partial}{\partial r_j} \left(\int_0^{\Delta T} \int_0^{L_z} dQ dz dt \right) = \frac{1}{\Delta T L_z} \int_0^{\Delta T} \int_0^{L_z} \frac{\partial dQ}{\partial r_j} dz dt = \overline{\frac{\partial dQ}{\partial r_j}} \quad (6.10)$$

This is however not the case for conditionally-averaged quantities, for example

$$\frac{\partial \overline{dQ}^{(TT)}}{\partial r_j} = \frac{1}{\Delta T L_z} \frac{\partial}{\partial r_j} \left(\frac{1}{\gamma^{(TT)}} \int_0^{\Delta T} \int_0^{L_z} \tau^+ \tau^- dQ dz dt \right) \quad (6.11)$$

while

$$\frac{\partial \overline{dQ}^{(TT)}}{\partial r_j} = \frac{1}{\Delta T L_z} \left(\frac{1}{\gamma^{(TT)}} \int_0^{\Delta T} \int_0^{L_z} \tau^+ \tau^- \frac{\partial dQ}{\partial r_j} dz dt \right) \neq \overline{\frac{\partial dQ}{\partial r_j}}^{(TT)}, \quad (6.12)$$

because $\gamma^{(TT)}$ depends on r_j . This lack of commutation has several implications. First, equation (6.9) does not have a conservative form, i.e. one like (3.2). For example, for the non-linear inter-scale energy transfer term, we have

$$\overline{du_j'^{(AA)} \frac{\partial dq^{2(AA)}}{\partial r_j}}^{(AA)} = \overline{\frac{\partial dq^{2(AA)}}{\partial r_j} du_j'^{(AA)}}^{(AA)} \neq \overline{\frac{\partial dq^{2(AA)}}{\partial r_j} du_j'^{(AA)}}^{(AA)} \quad (6.13)$$

The first equality in the above equation is because $\frac{\partial du_j'^{(AA)}}{\partial r_j} = 0$ (this is equivalent to $\frac{\partial \overline{du_j'}}{\partial r_j} = 0$ for standard averaging). It is easy to prove; for example if $AA = TT$, we have

$$\begin{aligned} \frac{\partial du_j'^{(TT)}}{\partial r_j} &= \frac{\partial du_j}{\partial r_j} - \frac{\partial dU_j^{(TT)}}{\partial r_j} = \frac{\partial du_j}{\partial r_j} - \frac{\partial}{\partial r_j} \left(\frac{1}{\Delta T L_z \gamma^{(TT)}} \int_0^{\Delta T} \int_0^{L_z} \tau^+ \tau^- dU_j dz dt \right) = \\ &= \frac{\partial du_j}{\partial r_j} - \frac{\partial}{\partial r_j} \left(\frac{dU_j}{\Delta T L_z \gamma^{(TT)}} \int_0^{\Delta T} \int_0^{L_z} \tau^+ \tau^- dz dt \right) = \frac{\partial du_j}{\partial r_j} - \frac{\partial dU_j}{\partial r_j} = 0 - 0 = 0, \end{aligned} \quad (6.14)$$

because du_j and dU_j satisfy the continuity equation in the scale space. We also took into account that dU_j is constant with respect to z and t and used (6.1a). The inequality in (6.13) means that the non-linear inter-scale transfer term cannot be written in divergence form. The same issue appears in the conditionally-averaged TKE equation (Marxen and Zaki, 2019).

The second implication is computational. For standard averaging, all the terms in the KMH equation can be evaluated numerically by differentiating locally at points x_i^+ and x_i^- ,

for example

$$\begin{aligned} \frac{\partial dU_i}{\partial r_j} &= \frac{\partial \overline{du_i}}{\partial r_j} = \frac{\partial \overline{du_i}}{\partial r_j} = \frac{1}{2} \left(\frac{\partial (\overline{u_i^+ - u_i^-})}{\partial x_j^+} - \frac{\partial (\overline{u_i^+ - u_i^-})}{\partial x_j^-} \right) = \frac{1}{2} \left(\frac{\partial \overline{u_i^+}}{\partial x_j^+} + \frac{\partial \overline{u_i^-}}{\partial x_j^-} \right) = \\ &= \frac{1}{2} \left(\frac{\partial \overline{u_i^+}}{\partial x_j^+} + \frac{\partial \overline{u_i^-}}{\partial x_j^-} \right) = \frac{1}{2} \left(\frac{\partial U_i^+}{\partial x_j^+} + \frac{\partial U_i^-}{\partial x_j^-} \right) \end{aligned} \quad (6.15)$$

However, this is not the case for the conditionally-averaged velocity difference,

$$\begin{aligned} \frac{\partial dU_i^{(AA)}}{\partial r_j} &= \frac{\partial \overline{du_i}^{(AA)}}{\partial r_j} \neq \frac{\partial \overline{du_i}^{(AA)}}{\partial r_j} = \frac{1}{2} \left(\frac{\partial (\overline{u_i^+ - u_i^-})^{(AA)}}{\partial x_j^+} - \frac{\partial (\overline{u_i^+ - u_i^-})^{(AA)}}{\partial x_j^-} \right) \\ &= \frac{1}{2} \left(\frac{\partial \overline{u_i^+}^{(AA)}}{\partial x_j^+} + \frac{\partial \overline{u_i^-}^{(AA)}}{\partial x_j^-} \right) \end{aligned} \quad (6.16)$$

This means that the derivatives of the conditionally-averaged two-point variables must be calculated directly in scale space. The process and validation against standard averaging are presented in appendix F.

6.4 Conditional decomposition of non-linear interscale flux

In this section we decompose the non-linear energy flux in scale space $\overline{(du_i')^2 du_j'}$ that appears in equation (3.2) into the constituent conditionally-averaged components, i.e. we seek to derive an expression similar to (6.3). The decomposition for the second-order structure function (quadratic quantity) was derived in Yao et al. (2020). Here we extend the method to the energy flux (cubic quantity). We have

$$\begin{aligned} \phi^F &= \overline{(du_i')^2 du_j'} = \overline{(du_i - dU_i)^2 (du_j - dU_j)} \\ &= \overline{(du_i)^2 du_j} - \overline{(du_i)^2 dU_j} - 2\overline{du_i du_j dU_i} + 2\overline{dU_i dU_j dU_i} \\ &= \gamma^{(TT)} \overline{(du_i)^2 du_j}^{(TT)} + \gamma^{(TL)} \overline{(du_i)^2 du_j}^{(TL)} + \gamma^{(LT)} \overline{(du_i)^2 du_j}^{(LT)} + \\ &\quad \gamma^{(LL)} \overline{(du_i)^2 du_j}^{(LL)} - \overline{(du_i)^2 dU_j} - 2\overline{du_i du_j dU_i} + 2\overline{dU_i dU_j dU_i} \end{aligned} \quad (6.17)$$

where we applied (6.3) to $\overline{(du_i)^2 du_j}$.

We now write each of the terms $\overline{(du_i)^2 du_j}^{(AA)}$, where $AA = TT$ or LL or TL or LT , in terms of conditional fluctuations. To do this, we use equation (6.6), and express the

conditional non-linear energy flux $\phi^{F(AA)} = \overline{\left(du_i'^{(AA)}\right)^2 du_j'^{(AA)}}^{(AA)}$ as

$$\begin{aligned} \phi^{F(AA)} &= \overline{\left(du_i'^{(AA)}\right)^2 du_j'^{(AA)}}^{(AA)} = \overline{(du_i)^2 du_j}^{(AA)} - \overline{(du_i)^2}^{(AA)} dU_j^{(AA)} \\ &\quad - \overline{2du_i du_j}^{(AA)} dU_i^{(AA)} - 2dU_i^{(AA)} dU_j^{(AA)} dU_i^{(AA)} dU_j^{(AA)} \end{aligned} \quad (6.18)$$

Solving for $\overline{(du_i)^2 du_j}^{(AA)}$ and substituting into (6.17), we obtain the desired decomposition

$$\phi^F = \gamma^{(TT)} \phi^{F(TT)} + \gamma^{(TL)} \phi^{F(TL)} + \gamma^{(LT)} \phi^{F(LT)} + \gamma^{(LL)} \phi^{F(LL)} + \phi, \quad (6.19)$$

where the extra term ϕ is given by,

$$\begin{aligned} \phi &= \gamma^{(TT)} \left[\overline{(du_i)^2}^{(TT)} dU_j^{(TT)} + \overline{2du_i du_j}^{(TT)} dU_i^{(TT)} - 2dU_i^{(TT)} dU_i^{(TT)} dU_j^{(TT)} \right] \\ &+ \gamma^{(TL)} \left[\overline{(du_i)^2}^{(TL)} dU_j^{(TL)} + \overline{2du_i du_j}^{(TL)} dU_i^{(TL)} - 2dU_i^{(TL)} dU_i^{(TL)} dU_j^{(TL)} \right] \\ &+ \gamma^{(LT)} \left[\overline{(du_i)^2}^{(LT)} dU_j^{(LT)} + \overline{2du_i du_j}^{(LT)} dU_i^{(LT)} - 2dU_i^{(LT)} dU_i^{(LT)} dU_j^{(LT)} \right] \\ &+ \gamma^{(LL)} \left[\overline{(du_i)^2}^{(LL)} dU_j^{(LL)} + \overline{2du_i du_j}^{(LL)} dU_i^{(LL)} - 2dU_i^{(LL)} dU_i^{(LL)} dU_j^{(LL)} \right] \\ &- \overline{(du_i)^2} dU_j - \overline{2du_i du_j} dU_i + 2dU_i dU_i dU_j \end{aligned} \quad (6.20)$$

Figure 6.4 presents contours of all the terms appearing in (6.19) for the decomposition of the energy flux vector $(\phi_{r_3}^F, \phi_{s_2}^F)$ at location *TR2* on (r_3, Y) plane (the separations in the other two directions are equal to 0, i.e. $r_1 = 0, r_2 = 0$). In the figure, we combine the results of *TL* and *LT* together due to homogeneity in the spanwise direction. The upper left panel, denoted as LHS, depicts the results from the direct calculation of the flux vector $(\phi_{r_3}^F, \phi_{s_2}^F)$ using standard time-average, i.e. $\overline{(du_i')^2 du_j'}$. The lower right panel, denoted as RHS, presents the sum of all terms in the (6.19). It is clear that the results in the two panels are essentially identical, confirming the validity of the derived decomposition. The flux vector originates from the focal point $Y/L_0 \approx 2.5$ and $r_3 \approx 10$ and transfers energy radially to different directions. The strongest flux is located in the region $0 < Y/L_0 < 10, r_3/L_0 > 10$ and is found to be positive, indicating strong inverse cascade. This pattern was observed in Yao et al. (2021).

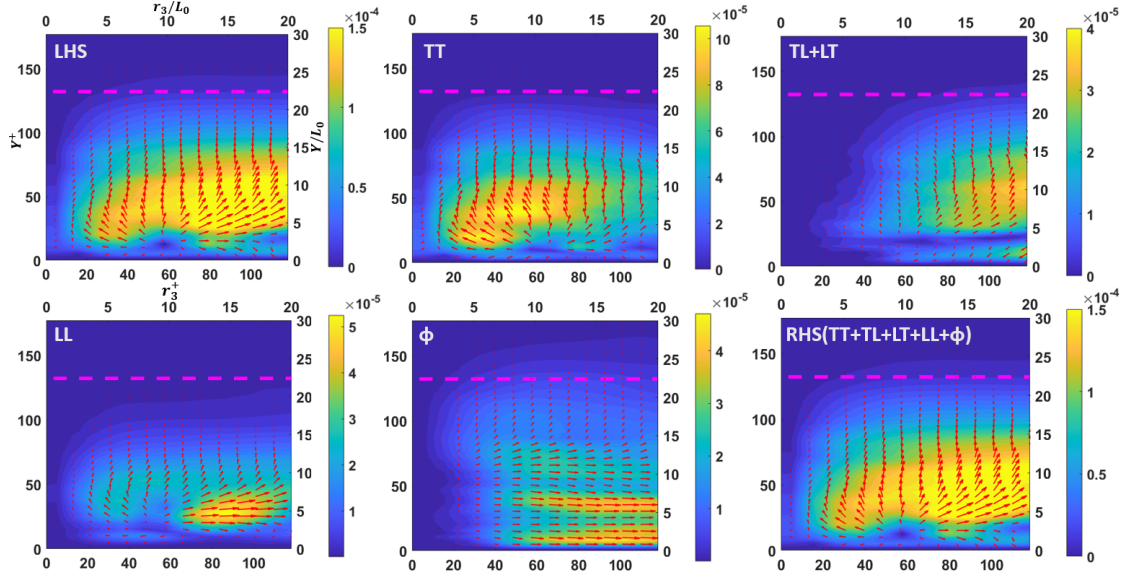


Figure 6.4 Interscale flux vector $\phi^F = \overline{(du'_i)^2 du'_j}$ (LHS) and the constituent components $\gamma^{(AA)} \phi^{F(AA)} = \gamma^{(AA)} \overline{(du'_i)^2 du'_j^{(AA)}}$ (where $AA=TT, TL+LT, LL$) and ϕ in the (r_3, Y) plane at location $TR2$. RHS denotes the sum of the terms in the right hand side of equation (6.19). Contours represent the magnitude of the flux vectors and the purple horizontal lines indicate the local boundary layer thickness.

Apart from verifying the decomposition (6.19), figure 6.4 provides important insight into the origin of the aforementioned energy flux pattern. It is interesting to observe that the inverse cascade arises from the $TL+LT$, LL , and ϕ components; all contribute to the strong positive flux in the region $0 < Y/L_0 < 10, r_3/L_0 > 10$ with values that are of the same order of magnitude. Outside this region, they have small values. On the other hand, the TT component contributes to the forward cascade in the region left of the focal point. The shape therefore of the energy flux vectors on the (r_3, Y) plane arises from the superposition of the contribution of TT term which is responsible for forward cascade, and all the other terms which are responsible for inverse cascade.

The constituent terms plotted in figure 6.4 are weighted by the corresponding two-point intermittencies, $\gamma^{(AA)}$, which can have very small values depending on the location examined, refer to figure 6.1. In the following section, the evolution of $\overline{(du'_i)^2 du'_j^{(AA)}}$ at different streamwise locations is examined.

6.5 Evolution of conditionally-averaged non-linear energy fluxes

6.5.1 Fluxes on (r_3, Y) plane

Figure 6.5 shows the conditionally averaged energy fluxes $(\phi_{r_3}^{F(AA)}, \phi_{s_2}^{F(AA)})$ where $AA = TT$ or $TL + LT$ or LL at locations $TR1, TR2, TR3$. The last row (marked as 'Total') depicts $(\phi_{r_3}^F, \phi_{s_2}^F)$. In all plots, $r_1 = 0$ and $r_2 = 0$.

At the early stages of transition, at location $TR1$, the flux vector $(\phi_{r_3}^{F(LL)}, \phi_{s_2}^{F(LL)})$ is dominant and is almost identical to the total flux. We can clearly see strong inverse cascade occurring to the right of the focal point, which is mainly due to LL events. The vectors $(\phi_{r_3}^{F(TT)}, \phi_{s_2}^{F(TT)})$ and $(\phi_{r_3}^{F(TL)}, \phi_{s_2}^{F(TL)}) + (\phi_{r_3}^{F(LT)}, \phi_{s_2}^{F(LT)})$ are very localised and their behaviour is difficult to interpret. This may be due to the small number of TT and TL/LT events at this early transition location. Notice however the high values of the magnitude of the TT vector, $(\phi_{r_3}^{F(TT)}, \phi_{s_2}^{F(TT)})$, compared to all the other vectors. However, its overall contribution to the total flux, $\gamma^{(TT)} \phi_{r_3}^{F(TT)}$, is small because the values of intermittency $\gamma^{(TT)}$ are negligible at this location (see figure 6.1).

At location $TR2$, TT and TL/LT events start to play a more significant role as expected, but they act at different areas of the map. Events of LL type maintain their dominant contribution to inverse cascade, while TL & LT events also amplify and contribute to the inverse cascade in the area $10L_0 < r_3 < 20L_0$ above $Y \approx 5L_0$. Notice again that TT -averaged flux vectors have a much higher magnitude compared to the total flux, and show a mixture of forward and inverse cascade to the left and right of the focal point respectively (with the forward cascade being slightly stronger). This picture is consistent with figure 6.4 where the conditionally-averaged fluxes are weighted by the two-point intermittency.

At location $TR3$, TT events assume the dominant role because $\gamma^{(TT)}$ approaches 1, LL events are localised (in the same way that TT events were localised at $TR1$ location), while TL & LT events again give rise to inverse cascade. Interestingly, as intermittency increases, TT events show stronger inverse cascade. For comparison, the flux vector $(\phi_{r_3}^F, \phi_{s_2}^F)$ at the fully turbulent location TU is plotted in figure 6.6. At this location the inverse cascade is weakened (but it is still visible) and energy flows in the wall-normal direction before looping back to small spanwise length scales.

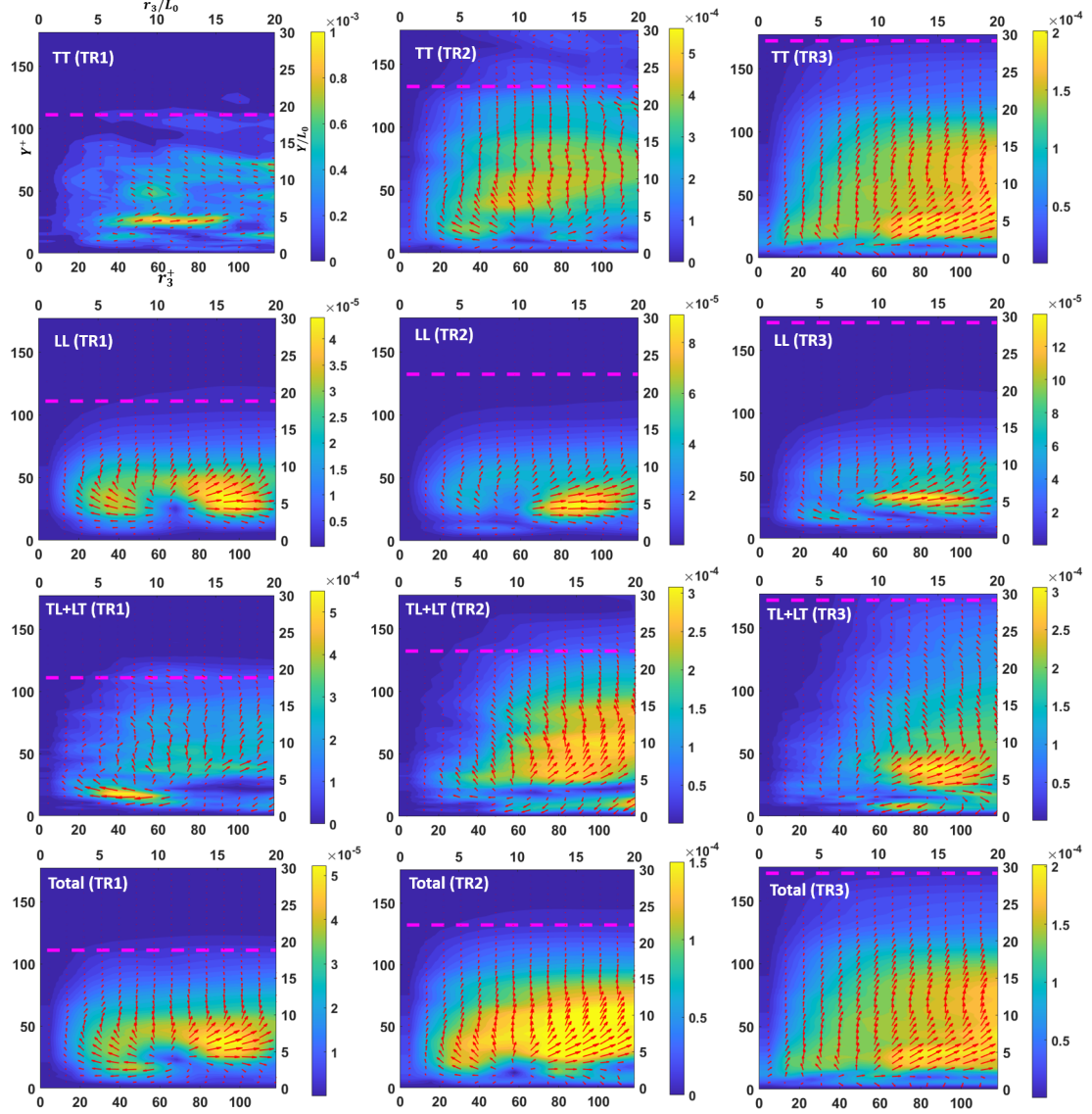


Figure 6.5 conditionally-averaged flux vectors $(\phi_{r_3}^{F(AA)}, \phi_{s_2}^{F(AA)})$ (first, second and third row respectively starting from the top) and standard-averaged flux vector $(\phi_{r_3}^F, \phi_{s_2}^F)$ (bottom row) at locations $TR1$ (left column), $TR2$ (middle column) and $TR3$ (right column). The contours represent the magnitude of the corresponding flux vectors and the purple horizontal lines indicate the local boundary layer thickness.

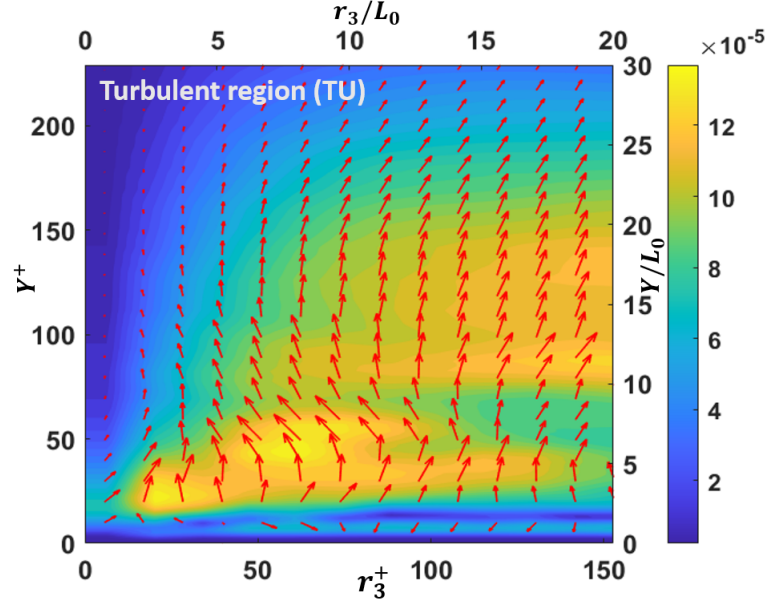


Figure 6.6 Standard-averaged flux vector $(\phi_{r_3}^F, \phi_{s_2}^F)$ at TU location. The contour represents the magnitude of the flux vectors.

The strong inverse cascade found at $TR2$ and $TR3$ locations is clearly not observed at the fully turbulent region. The origin of the inverse cascade arises mainly due to TL & LT events. Indeed, in the $TR2$ location their magnitude is 5-6 times larger compared to LL events (that also contribute to inverse cascade), while in $TR3$ about 2-3 times larger. The laminar/turbulent interface, therefore, plays a crucial role in the inverse cascade process. In the next section, we focus on the $TR2$ location and explore in more depth the cascade process in the three-dimensional (r_1, r_3, Y) hyperplane.

6.5.2 Fluxes on the (r_1, r_3, Y) hyperplane and (r_1, r_3) plane

We project the conditionally-averaged fluxes $\phi^{F(AA)}$ on the (r_1, r_3, Y) hyperplane, i.e. plot three-dimensional maps of the vector $(\phi_{r_1}^{F(AA)}, \phi_{r_3}^{F(AA)}, \phi_{s_2}^{F(AA)})$. We select location $TR2$, which is in the middle of the transition region, and has single-point intermittency about 0.4 (refer to figure 6.1).

Stream-tubes obtained from the total and the non-linear flux vectors (standard or conditionally-averaged) are shown in figure 6.7; see caption for details. Panel (a) shows the standard-averaged total energy flux vector $(\phi_{r_1}, \phi_{r_3}, \phi_{s_2})$ that includes the non-linear, linear, pressure, and viscous components, refer to equation (3.5). This plot is very similar to that of panel (b) that shows the non-linear component $(\phi_{r_1}^F, \phi_{r_3}^F, \phi_{s_2}^F)$, the latter being the dominant component, see also Yao et al. (2021) for a more detailed discussion on the other components. Both

plots depict a dense cluster of stream-tubes with energy flowing to larger r_1 scales (inverse cascade) with energy originating at $Y \approx 5L_0$ and $r_3 \approx 10L_0$. There is milder inverse cascade in larger r_3 scales. In another (smaller) cluster energy flux vectors rotate and bend towards the Y-axis ($r_1 = 0, r_3 = 0$).

The decomposition (6.19) allows us to probe in more detail the origin of the strong inverse cascade in the r_1 direction and identify the flow events that determine it. In the homogeneous spanwise direction TL and LT events can be combined together as $TL + LT$ (see figure 6.2). However, the streamwise direction is inhomogeneous and these events have to be considered separately. As mentioned earlier, for a fixed (X, Y) spatial location and r_3 separation, a TL event with $r_1 > 0$ takes place across the upstream laminar/turbulent interface of a spot. On the other hand, an LT event is taken across the downstream interface, refer to figure 6.3.

Panels (c) and (d) of figure 6.7 show the stream-tubes for the conditionally-averaged flux vectors $(\phi_{r_1}^{F(LT)}, \phi_{r_3}^{F(LT)}, \phi_{s_2}^{F(LT)})$ and $(\phi_{r_1}^{F(TL)}, \phi_{r_3}^{F(TL)}, \phi_{s_2}^{F(TL)})$ respectively. It is very clear that the LT flux vector contributes most strongly to the inverse cascade; this corresponds to the downstream laminar/turbulent interface of a spot. Indeed there is a cluster of stream-tubes whose direction indicates transfer of energy to larger streamwise scales, up to $r_1 \approx 50L_0$. The stream-tubes then bend towards smaller scales, $r_1, r_3 \rightarrow 0$. On the other hand, TL events that correspond to energy flux across the upstream interface of a turbulent spot, also contribute to inverse cascade, but over a shorter streamwise range, $0 < r_1 < 20L_0$. This clearly indicates that inter-scale energy transfer processes are different at the upstream and downstream interfaces of turbulent spots.

Panels (e) and (f) of figure 6.7 show the stream-tubes of the conditionally-averaged flux vectors $(\phi_{r_1}^{F(TT)}, \phi_{r_3}^{F(TT)}, \phi_{s_2}^{F(TT)})$ and $(\phi_{r_1}^{F(LL)}, \phi_{r_3}^{F(LL)}, \phi_{s_2}^{F(LL)})$. The former corresponds to interscale transfer within spots; there is weak inverse cascade and the streamtubes bend towards smaller scales. The latter corresponds to time instants where both points are located within laminar regions. There is strong forward cascade to small scales and then bending and energy transfer to larger scales away from the wall. This is a quite complicated energy flux pattern, which is difficult to interpret physically.

Plots in the 3D hyperplane (r_1, r_3, Y) visualise the main features of the energy flux paths, but can hide important detail. To uncover this detail, in figure 6.8 we plot the flux vectors in the (r_1, r_3) plane at the specific height $Y = 4.5L_0$. The total flux vector and the standard-averaged non-linear component (panels (a) and (b) respectively) are very similar and show a recirculating pattern with inverse cascade for $r_3 > 10$ over the range of r_1 examined and forward cascade for $r_3 < 10L_0$. It is very interesting to see that around $r_3 = 10L_0$ the energy flux is negligible; this cannot be easily observed from figure 6.7. The conditionally-averaged fluxes also show detail that cannot be discerned from the 3D plots. For example, the strong

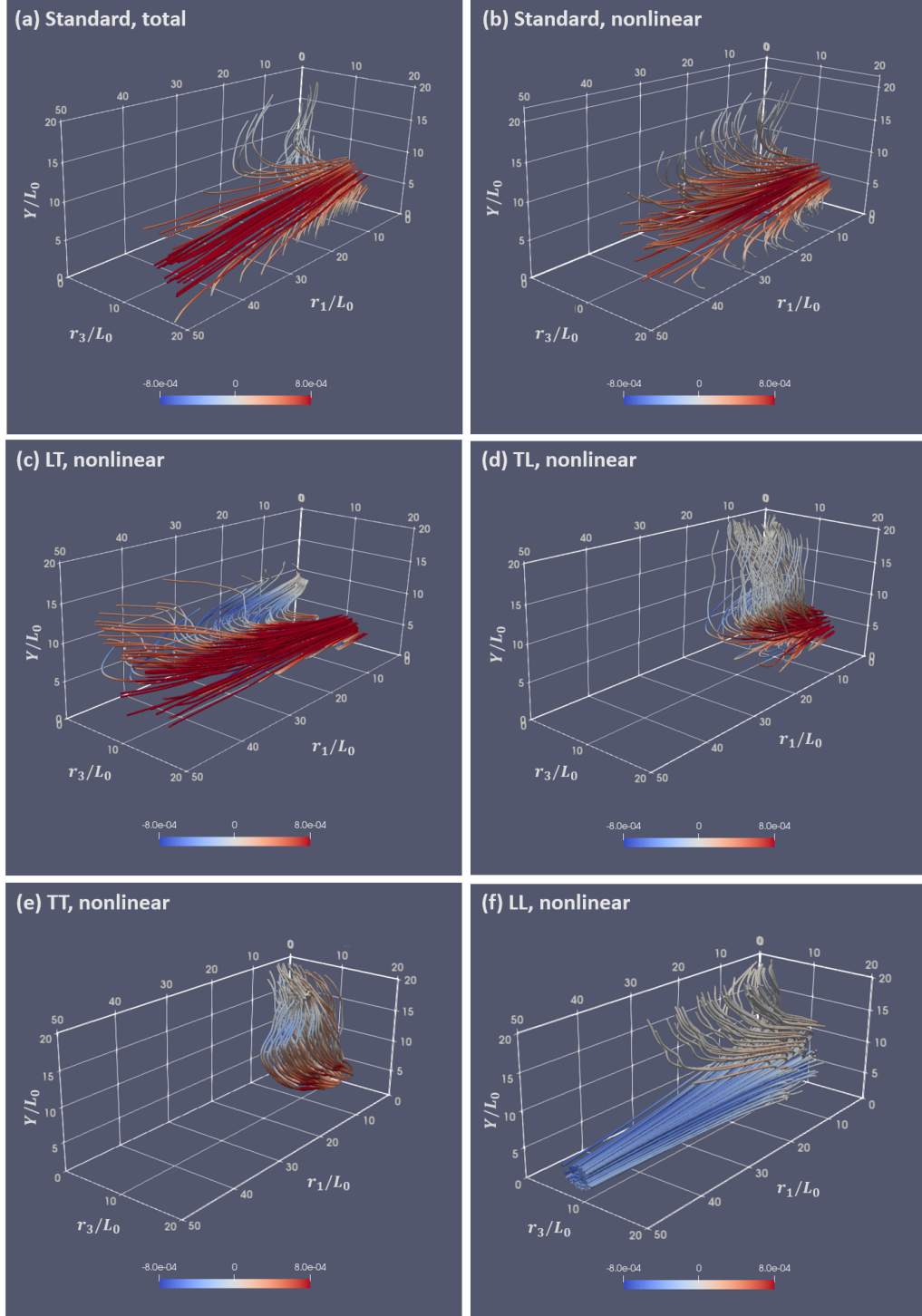


Figure 6.7 Stream-tubes of (a) the total flux vector $(\phi_{r_1}, \phi_{r_3}, \phi_{s_2})$, (b) the standard-averaged non-linear flux vector $(\phi_{r_1}^F, \phi_{r_3}^F, \phi_{s_2}^F)$, and the conditionally-averaged vectors (c) $(\phi_{r_1}^{F(LT)}, \phi_{r_3}^{F(LT)}, \phi_{s_2}^{F(LT)})$, (d) $(\phi_{r_1}^{F(TL)}, \phi_{r_3}^{F(TL)}, \phi_{s_2}^{F(TL)})$, (e) $(\phi_{r_1}^{F(TT)}, \phi_{r_3}^{F(TT)}, \phi_{s_2}^{F(TT)})$ and (f) $(\phi_{r_1}^{F(LL)}, \phi_{r_3}^{F(LL)}, \phi_{s_2}^{F(LL)})$. The plots were generated by placing a sphere of radius $5L_0$ at point $(r_1, r_3, Y) = (5L_0, 10L_0, 3L_0)$ and tracing the stream-tubes crossing the sphere. The stream-tubes are coloured according to the sign of the first component i.e. ϕ_{r_1} , $\phi_{r_1}^F$, $\phi_{r_1}^{F(LT)}$, $\phi_{r_1}^{F(TL)}$, $\phi_{r_1}^{F(TT)}$ and $\phi_{r_1}^{F(LL)}$ (red for positive, blue for negative, thus indicating inverse or forward cascade in the r_1 direction respectively). The colour bars also refer to the value of the first component (the min/max values are the same to facilitate comparison).

inverse cascade of $\phi_{r_1}^{F(LT)}$ extends over the whole range of r_3 and r_1 , while for $\phi_{r_1}^{F(TL)}$ it extends only in a specific range of separations, $r_3 \approx (5 - 15)L_0$, depending on r_1 , as can be seen from panels (c) and (d) respectively. The flux component $\phi_{r_1}^{F(LL)}$ clearly demonstrates forward cascade over the whole r_3 range examined (panel (e)), but $\phi_{r_1}^{F(TT)}$ is very small around $r_3 \approx 10$ and increases at the boundaries of the domain (panel (f)). These plots confirm that the strong inverse cascade in the standard-averaged flux is due to LT events at the downstream interface of the turbulent spots. Interestingly, the energy fluxes due to TL and LL events almost cancel out around $r_3 \approx 10L_0$, and this explains the very small fluxes in this area for $\phi_{r_1}^F$. Notice also that the TT events account for the forward cascade observed in $\phi_{r_1}^F$ and ϕ_{r_1} for small r_3 separations.

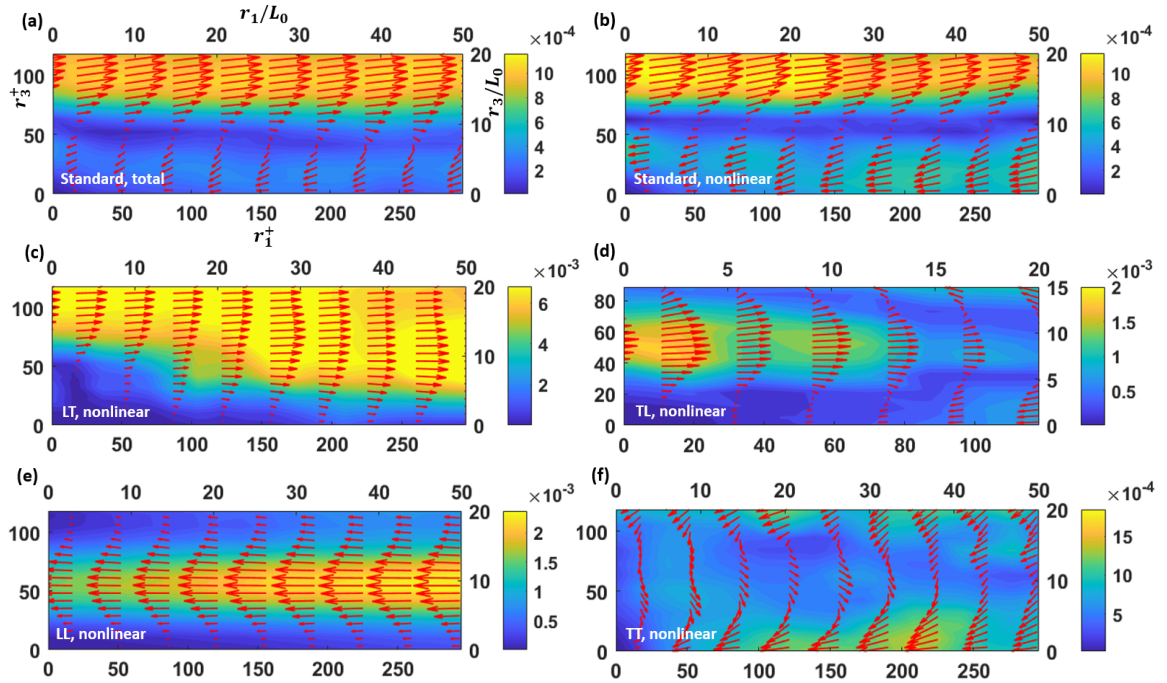


Figure 6.8 Flux vectors (a) (ϕ_{r_1}, ϕ_{r_3}) , (b) $(\phi_{r_1}^F, \phi_{r_3}^F)$, (c) $(\phi_{r_1}^{F(LT)}, \phi_{r_3}^{F(LT)})$, (d) $(\phi_{r_1}^{F(TL)}, \phi_{r_3}^{F(TL)})$, (e) $(\phi_{r_1}^{F(TT)}, \phi_{r_3}^{F(TT)})$ and (f) $(\phi_{r_1}^{F(LL)}, \phi_{r_3}^{F(LL)})$ on the (r_1, r_3) plane at wall-normal height $Y = 4.5L_0$ and location $TR2$. Contours represent the magnitude of the energy flux vectors in the plane.

The above figures have demonstrated the central role of the downstream laminar/turbulent interface on the interscale transfer and in particular the inverse cascade over a large range of scales. Additionally, TL events were localised in a smaller range of spanwise and streamwise separations. We now try to explain physically this behaviour with the aid of the cartoons shown in figure 6.9. More specifically, we consider a fixed X location (for figures 6.7 and

6.8 $X = X_{TR2}$) and follow a turbulent spot of diamond shape as it propagates to the right and crosses this location. The arrowhead apexes at the upstream and downstream ends are rather realistic approximations. This can be seen from figure 6.10 where we demarcate the boundaries of two turbulent spots. It is also consistent with experimental spot observations, refer to figures 12 and 14 in Anthony et al. (2005). The sharp corners of the spot around the maximum thickness are less realistic; the shape is more rounded in this region as can be seen from figure 18 of Marxen and Zaki (2019). However, analysis of this simplified shape can provide significant insight, as will be seen next.

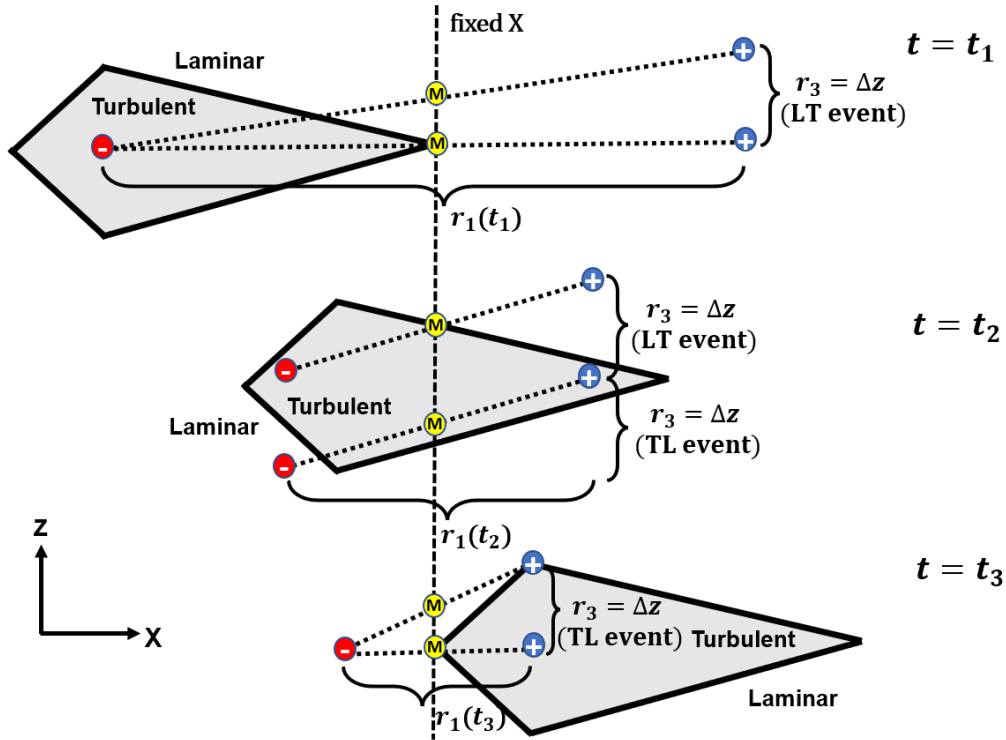


Figure 6.9 A propagating turbulent spot of diamond shape crossing a fixed X location; the view shown is in the (X, Z) plane. Red, yellow and blue dots represent the x_i^- , X_i , and x_i^+ points respectively. Only points with fixed $r_3 = \Delta z$ separation are shown.

In figure 6.9, snapshots of the propagating spot at three time instants t_1, t_2 and t_3 are shown. In all snapshots, we consider a fixed middle point (denoted with a yellow dot) located at the streamwise position, X . The blue and red dots represent the x_i^+ and x_i^- points respectively. Only points with a fixed spanwise separation, $r_3 = \Delta z$, are shown in the figure. At time $t = t_1$, the downstream apex of the spot lies exactly at the fixed X location. It can be seen that for an *LT* event (the only type of events possible at this time instant), the streamwise separation $r_1(t_1)$ is very long, of the order of the spot length. For $\Delta z = 0$, $r_1(t_1)$ attains a maximum value, equal to twice the spot length.

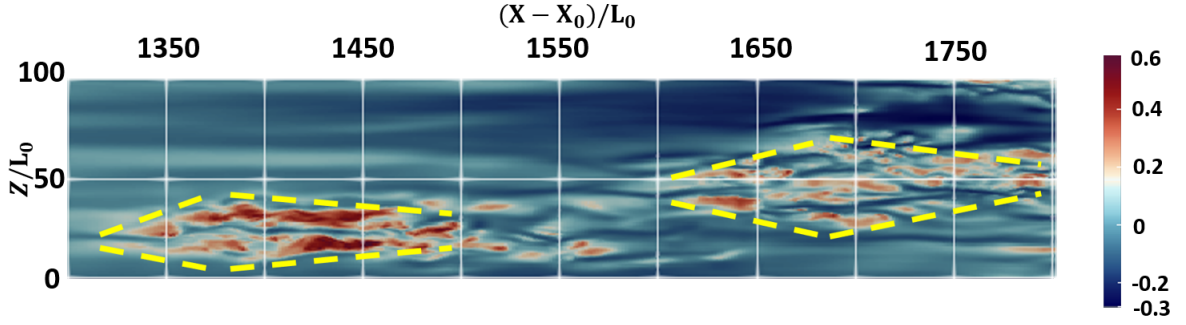


Figure 6.10 Contour plot of instantaneous streamwise velocity fluctuations. The boundaries of two turbulent spots in the transition region (marked as dashed yellow lines) indicate an approximately diamond spot shape.

At time instant $t = t_2$, approximately half of the spot has crossed X . The valid spanwise locations of the middle point are determined by the spreading angle of the front apex. Note that for fixed $r_3 = \Delta z$, TL and LT events co-exist, but it is clear that the $r_1(t_2)$ separation of an LT event is shorter to the one at $t = t_1$, i.e. $r_1(t_2) < r_1(t_1)$. At $t = t_3$, the whole spot has crossed the considered location, thus the rear apex is at X . At this time instant, only TL events are possible. It can be seen that only a narrow range of r_1 separations is admissible for a given r_3 . The actual range depends on the spreading angle of the rear apex. Thus on average, the valid r_1 separations corresponding to TL events at t_2 and t_3 is more narrow compared to LT events at $t = t_1$ and t_2 . This explains the inverse cascade over a wider range of separations for LT events shown in panel 6.8(c). We also conjecture that the largest admissible r_1 value mentioned earlier explains why the stream-tubes showed in figure 6.7(c) reach up to a maximum r_1 (the exact value depends on point of origin of the stream-tubes in the (r_3, Y) plane) and then bend backward towards small scales.

To provide further insight into the observed behaviour of the conditionally-averaged fluxes, we examine in more detail the flux vector component in the r_1 direction $\phi_{r_1}^{F(AA)} = \overline{\left(du_i'^{(AA)}\right)^2 du_1'^{(AA)}}^{(AA)}$.

In figure 6.11, the probability density functions (PDFs) of $\left(du_i'^{(AA)}\right)^2 du_1'^{(AA)}$ for $AA = LT$, TL , TT and LL are plotted for separations $r_1 = 30L_0$ and $r_3 = 10L_0$ at plane $Y = 4.5L_0$. The PDF of the instantaneous flux $\left(du_i'^{(LT)}\right)^2 du_1'^{(LT)}$ is asymmetric and skewed to positive values (implying inverse cascade after time-averaging). This is also the case but it is less evident for $\left(du_i'^{(TL)}\right)^2 du_1'^{(TL)}$, while $\left(du_i'^{(TT)}\right)^2 du_1'^{(TT)}$ is almost symmetric, and $\left(du_i'^{(LL)}\right)^2 du_1'^{(LL)}$ is skewed to the left. Note the large positive and negative fluctuations of the instantaneous fluxes compared to the time-average values reported in figure 6.8. This means that instantaneously

energy flows in either direction, and intense fluxes of relatively low probability tip the balance in one direction or another after time-averaging.

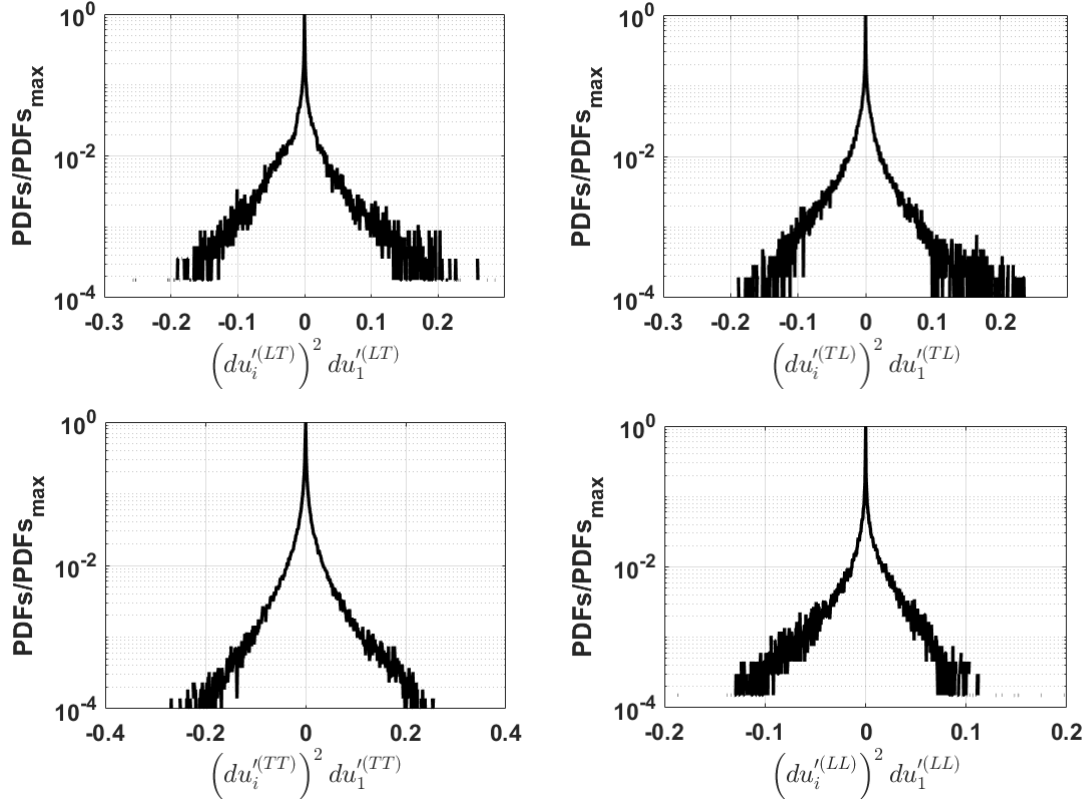


Figure 6.11 PDFs of $\left(du_i'^{(AA)}\right)^2 du_1'^{(AA)}$ for $AA = LT, TL, TT$ and LL for $r_1 = 30L_0$ and $r_3 = 10L_0$ at $Y = 4.5L_0$.

6.6 Conditionally-averaged scale energy production and transfer within a turbulent spot and comparison with fully developed turbulence

The conditional averaging operations defined in section 6.2 allow us to compute the scale energy production and interscale transfer within a turbulent spot and compare them with the corresponding quantities in the fully turbulent region. Similar work has been done for single-point statistics, for example, the turbulent kinetic energy (Park et al., 2012, Nolan and Zaki, 2013, Marxen and Zaki, 2019).

6.6.1 Conditional decomposition of production

We start by deriving a decomposition similar to (6.19) for the production term of the KMH equation (3.2). This term consists of two components, one due to inhomogeneity of the mean flow in scale space, $\mathcal{P}_r = -2\overline{du'_i du'_j} \frac{\partial dU_i}{\partial r_j}$, and the other due to inhomogeneity in physical space, $\mathcal{P}_s = -2\overline{du'_i u'_{j*}} \frac{\partial dU_i}{\partial x_j}$. Here we decompose the former component, $\mathcal{P}_r$, which is the dominant one; this is also discussed below. Applying a process similar to that described in section 6.4, we obtain

$$\begin{aligned} \mathcal{P}_r = & -2\overline{du'_i du'_j} \frac{\partial dU_i}{\partial r_j} = \\ & -2 \left(\gamma^{(TT)} \right)^2 \overline{du_i^{(TT)} du_j^{(TT)}} \frac{\partial dU_i^{(TT)}}{\partial r_j} - 2 \left(\gamma^{(TL)} \right)^2 \overline{du_i^{(TL)} du_j^{(TL)}} \frac{\partial dU_i^{(TL)}}{\partial r_j} \\ & - 2 \left(\gamma^{(LT)} \right)^2 \overline{du_i^{(LT)} du_j^{(LT)}} \frac{\partial dU_i^{(LT)}}{\partial r_j} - 2 \left(\gamma^{(LL)} \right)^2 \overline{du_i^{(LL)} du_j^{(LL)}} \frac{\partial dU_i^{(LL)}}{\partial r_j} \\ & - \phi \end{aligned} \tag{6.21}$$

where

$$\begin{aligned} \phi = & 2 \left(\gamma^{(TT)} \right)^2 dU_i^{(TT)} dU_j^{(TT)} \frac{\partial dU_i^{(TT)}}{\partial r_j} + 2 \left(\gamma^{(TL)} \right)^2 dU_i^{(TL)} dU_j^{(TL)} \frac{\partial dU_i^{(TL)}}{\partial r_j} \\ & + 2 \left(\gamma^{(LT)} \right)^2 dU_i^{(LT)} dU_j^{(LT)} \frac{\partial dU_i^{(LT)}}{\partial r_j} + 2 \left(\gamma^{(LL)} \right)^2 dU_i^{(LL)} dU_j^{(LL)} \frac{\partial dU_i^{(LL)}}{\partial r_j} \\ & + 2\gamma^{(TT)} \overline{du_i du_j}^{(TT)} \left[dU_i^{(TT)} \frac{\partial \gamma^{(TT)}}{\partial r_j} + \frac{\partial}{\partial r_j} \left(\gamma^{(LL)} dU_i^{(LL)} + \gamma^{(TL)} dU_i^{(TL)} + \gamma^{(LT)} dU_i^{(LT)} \right) \right] \\ & + 2\gamma^{(TL)} \overline{du_i du_j}^{(TL)} \left[dU_i^{(TL)} \frac{\partial \gamma^{(TL)}}{\partial r_j} + \frac{\partial}{\partial r_j} \left(\gamma^{(LL)} dU_i^{(LL)} + \gamma^{(TT)} dU_i^{(TT)} + \gamma^{(LT)} dU_i^{(LT)} \right) \right] \\ & + 2\gamma^{(LT)} \overline{du_i du_j}^{(LT)} \left[dU_i^{(LT)} \frac{\partial \gamma^{(LT)}}{\partial r_j} + \frac{\partial}{\partial r_j} \left(\gamma^{(LL)} dU_i^{(LL)} + \gamma^{(TL)} dU_i^{(TL)} + \gamma^{(TT)} dU_i^{(TT)} \right) \right] \\ & + 2\gamma^{(LL)} \overline{du_i du_j}^{(LL)} \left[dU_i^{(LL)} \frac{\partial \gamma^{(LL)}}{\partial r_j} + \frac{\partial}{\partial r_j} \left(\gamma^{(TT)} dU_i^{(TT)} + \gamma^{(TL)} dU_i^{(TL)} + \gamma^{(LT)} dU_i^{(LT)} \right) \right] \\ & - 2dU_i dU_j \frac{\partial dU_i}{\partial r_j} \end{aligned} \tag{6.22}$$

Note that the terms $\mathcal{P}_r^{(AA)} = -2\overline{du_i'^{(AA)} du_j'^{(AA)}}^{(AA)} \frac{\partial du_i'^{(AA)}}{\partial r_j}$ that appear in (6.21) are the production terms of the conditionally-averaged KMH equation (6.9). Due to the strong shear in the wall-normal direction, the dominant component of $\mathcal{P}_r = -2\overline{du_i' du_j'} \frac{\partial du_i}{\partial r_j}$ is $\mathcal{P}_{r(1,2)} = -2\overline{du_1' du_2'} \frac{\partial du_1}{\partial r_2} \Big|_{r_2=0}$. It can be easily proved (see Yao et al. (2021)) that for $r_1 = 0$ this component is equal to $-2\overline{du_1' du_2'} \frac{\partial du_1}{\partial x_2}$. The corresponding production component due to inhomogeneity of the mean flow in physical space, $\mathcal{P}_s = -2\overline{du_1' u_{2*}'} \frac{\partial du_1}{\partial x_2}$, is much smaller (and equal to 0 if $r_1 = 0$, in addition to $r_2 = 0$). Therefore in this section we consider only $\mathcal{P}_{r(1,2)} = -2\overline{du_1' du_2'} \frac{\partial du_1}{\partial r_2} \Big|_{r_2=0}$ and the corresponding conditionally-averaged (TT) component $\mathcal{P}_{r(1,2)}^{(TT)} = -2\overline{du_1'^{(TT)} du_2'^{(TT)}}^{(TT)} \frac{\partial du_1^{(TT)}}{\partial r_2} \Big|_{r_2=0}$ within a turbulent spot. We focus at location $TR2$ and compare the aforementioned component with $\mathcal{P}_{r(1,2)}$ evaluated at the fully turbulent region, TU .

We also extend the decomposition (6.19) to the total fluxes in physical and scales spaces ϕ_s and ϕ_r , defined in equations (3.4) and (3.5) respectively. The resulting expressions are

$$\phi_{sj} = \gamma^{(TT)} \phi_{sj}^{(TT)} + \gamma^{(TL)} \phi_{sj}^{(TL)} + \gamma^{(LT)} \phi_{sj}^{(LT)} + \gamma^{(LL)} \phi_{sj}^{(LL)} + \phi, \quad (6.23)$$

where

$$\begin{aligned} \phi_{sj}^{(AA)} = & \gamma^{(AA)} \overline{U_{j*}^{(AA)} dq^{2(AA)}}^{(AA)} + \overline{u_{j*}'^{(AA)} dq^{2(AA)}}^{(AA)} + 2\overline{du_j'^{(AA)} dp'^{(AA)}}^{(AA)} - \frac{1}{2} \nu \frac{\partial dq^{2(AA)}}{\partial x_j}^{(AA)}. \end{aligned} \quad (6.24)$$

The factor $\gamma^{(AA)}$ appearing in the first term is due to the separate decomposition of U_{*j} and $\overline{dq^2}$ in (3.4). Also

$$\phi_{rj} = \gamma^{(TT)} \phi_{rj}^{(TT)} + \gamma^{(TL)} \phi_{rj}^{(TL)} + \gamma^{(LT)} \phi_{rj}^{(LT)} + \gamma^{(LL)} \phi_{rj}^{(LL)} + \phi, \quad (6.25)$$

where

$$\phi_{rj}^{(AA)} = \overline{du_j'^{(AA)} dq^{2(AA)}}^{(AA)} + \gamma^{(AA)} \overline{du_j^{(AA)} dq^{2(AA)}}^{(AA)} - 2\nu \frac{\partial dq^{2(AA)}}{\partial r_j}^{(AA)}. \quad (6.26)$$

The full expressions, including the remainder terms, are provided in Appendix E.

6.6.2 Conditionally- and standard-averaged production and fluxes

In figure 6.12, contours of $\mathcal{P}_{r(1,2)}^{(TT)}$ at *TR2* and of $\mathcal{P}_{r(1,2)}$ at *TR2* and *TU* are plotted in the (r_3, Y) plane for $r_1 = 0$. It can be seen that the production peaks within the turbulent spot and the fully turbulent region are located at approximate the same spanwise separation and wall-normal height, $r_3 \approx 5L_0$ and $Y \approx 1.3L_0$ respectively. On the other hand, the peak of $\mathcal{P}_{r(1,2)}$ at *TR2* is found to be further away from the wall and at larger r_3 length scale, $Y \approx 4.5L_0$ and $r_3 \approx 7.5L_0$. We mark the spanwise scales where the peaks appear, i.e. $r_3 = 5L_0$ and $7.5L_0$, with vertical dotted lines in figure 6.12, and plot the variation of the three production terms along these lines in figure 6.13. Notice the very close matching of the conditionally-average production $\mathcal{P}_{r(1,2)}^{(TT)}$ at *TR2* (dashed line) and the standard-averaged production $\mathcal{P}_{r(1,2)}$ at *TU* (solid line) close to the wall (for $Y \leq 1.3L_0$) while further away the two sets deviate. On the other hand, $\mathcal{P}_{r(1,2)}$ at *TR2* shows significantly different behaviour even close to the wall, and of course peaks at a different distance.

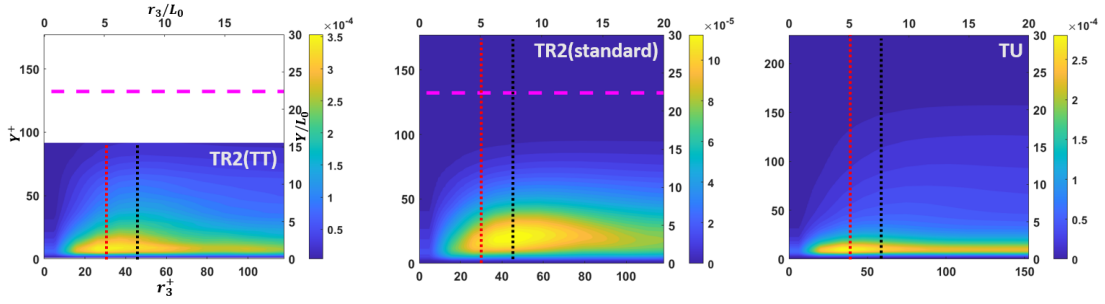


Figure 6.12 Contour plots of conditionally-averaged production $\mathcal{P}_{r(1,2)}^{(TT)}$ at *TR2* (left), standard-averaged production $\mathcal{P}_{r(1,2)}$ at *TR2* (middle) and *TU* (right) on the (r_3, Y) plane for $r_1 = 0$. The red and black vertical dotted lines are placed at $r_3 = 5L_0$ and $7.5L_0$. The horizontal purple line indicates the local boundary layer thickness.

The left panel of figure 6.14 shows stream-tubes in the (r_1, r_3, Y) hyperplane obtained from the conditionally-averaged total fluxes $\phi_s^{(TT)}$ and $\phi_r^{(TT)}$ at *TR2* together with an isosurface of the conditionally-averaged production. On the right panel we plot ϕ_s , ϕ_r and production in the fully turbulent region. The latter figure reflects the dynamics of near-wall turbulence; energy is extracted from the mean flow at the buffer layer where the production peak is located, then it is transferred away from the wall and towards larger r_1 scales before bending back to smaller scales (dissipation region). This behaviour is related to the self-sustained turbulence mechanism near the wall, see Cimarelli et al. (2013). A similar pattern can be discerned in the left panel, but the inverse cascade and flow of energy away from the wall is over a smaller range of r_1 separations (up to $r_1 \approx 10$); the stream-tubes again bend towards small scales. There are also some deviations between the two plots for larger r_3

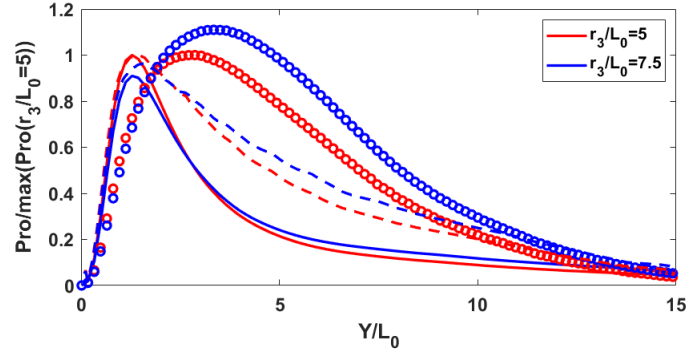


Figure 6.13 Variation of production terms along the wall-normal distance at $r_3/L_0 = 5.0$ and 7.5 . The variation is along the dotted vertical lines shown in figure 6.12 that pass through the corresponding production peaks. The plots are normalised by the value at $r_3/L_0 = 5.0$. The solid lines, dashed lines and circles represent $\mathcal{P}_{r(1,2)}$ at TU , $\mathcal{P}_{r(1,2)}^{(TT)}$ and $\mathcal{P}_{r(1,2)}$ at $TR2$ respectively.

separations. If the centre of the sphere (that is used to trace the stream-tubes) is placed at smaller r_3 and the radius is reduced, the similarity between the two panels is more evident, see figure 6.15 and caption for details.

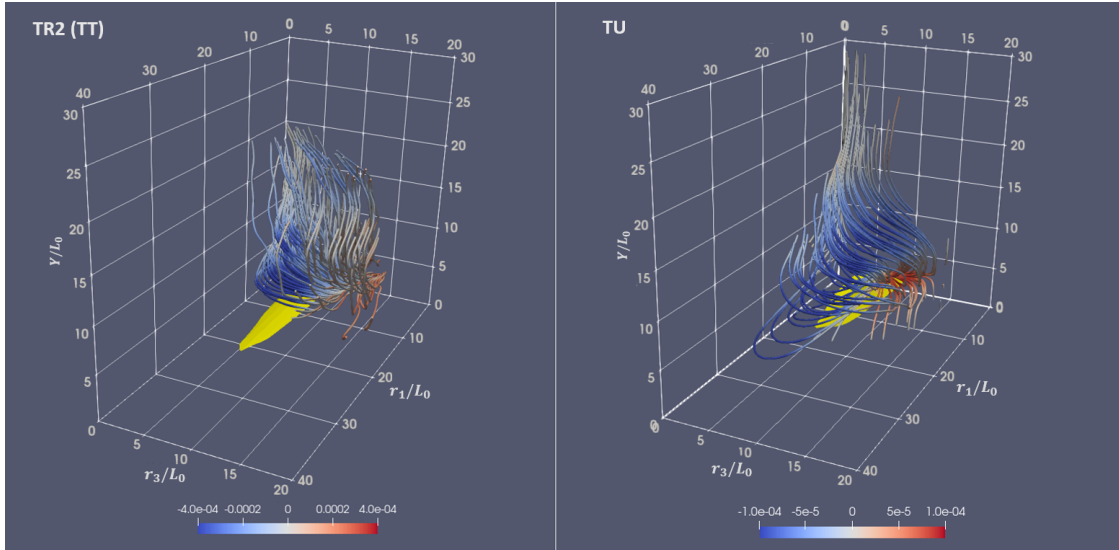


Figure 6.14 Stream-tubes of the conditionally-averaged total flux vector $(\phi_{r_1}^{(TT)}, \phi_{r_3}^{(TT)}, \phi_{s_2}^{(TT)})$ at $TR2$ (left) and of the standard-averaged vector $(\phi_{r_1}, \phi_{r_3}, \phi_{s_2})$ at TU in the 3D (r_1, r_3, Y) hyper-plane. The stream-tubes are colored according to the sign of ϕ_{r_3} (red for positive, blue for negative, thus indicating inverse and forward cascade respectively). The color bar refers to the value of ϕ_{r_3} . The tubes were generated by placing a sphere of radius $15L_0$ at $(r_1 = 5L_0, r_3 = 5L_0, Y = 1.3L_0)$ and tracing the stream tubes crossing the sphere.

The shorter range of inverse cascade in the $TR2$ location compared to the TU is probably because the spots are still developing, the merging is not yet complete, thus they have a smaller footprint in the streamwise direction. Note also the similarities of the left panels of figures 6.14 and 6.15 with the bottom left panel of figure 6.7 that shows the non-linear component of the flux vector only; this confirms that this is the most important component that determines the overall behaviour.

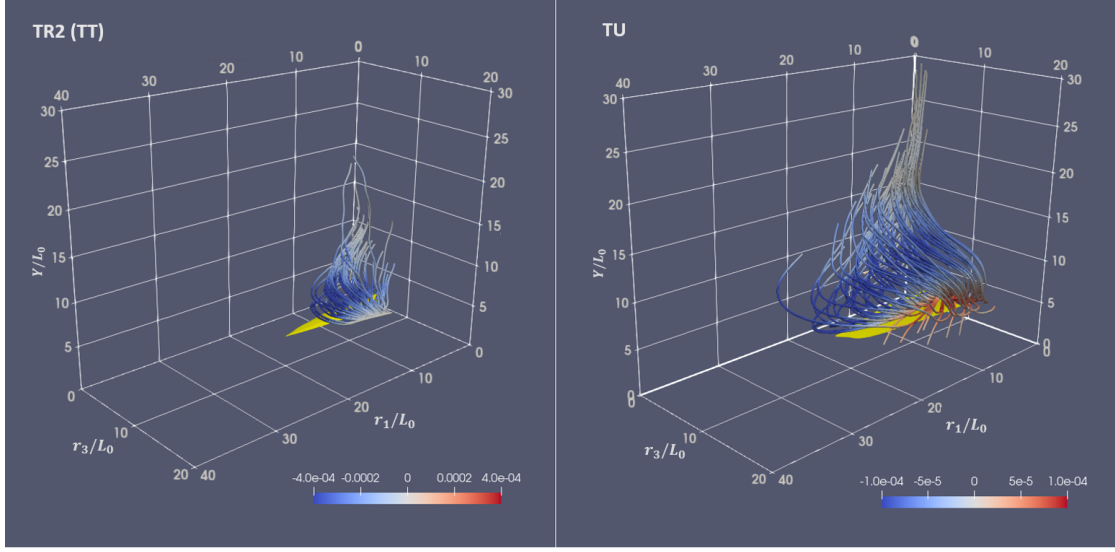


Figure 6.15 Same as figure 6.14, but the sphere is now placed at $(r_1 = 5L_0, r_3 = 2.5L_0, Y = 1.3L_0)$ and has smaller radius, $4L_0$.

6.7 Conclusion

In this work, we used the KMH equation to explore the inter-scale energy transfer of bypass transition. Due to the presence of intermittent turbulent spots during transition, we derived a conditionally-averaged KMH equation and decomposition of different flux terms that appear in this equation. The two-point derivative is evaluated directly in scale space because conditional-averaging does not commute with the spatial derivative operator.

We found a strong inverse cascade on the (r_3, Y) plane in the spanwise separation. This inverse cascade is due to the non-linear energy flux across the lateral laminar/turbulent interfaces ($TL + LT$ event) of the turbulent spots. In the (r_1, r_3, Y) hyper-plane, a strong inverse cascade is found in the r_1 direction due to non-linear flux across the downstream boundary (LT event) of the turbulent spots over a large range of r_1 separation and across the upstream boundary (TL event) over a small range of r_1 . A sketch that demonstrates the propagation of a diamond shape turbulent spot at a fixed streamwise location has helped the

physical interpretation of the results. It is the geometry and growth of the turbulent spot that causes the cascade difference at the upstream and downstream interfaces. Additionally, PDFs are calculated to prove the dominance of the positive non-linear energy flux.

In the last section, we plot a 3-dimensional production superimposed with the total energy flux within the turbulent spots at $TR2$ and fully turbulent region. There are both similarities and differences between these two regions. The similarity is that in both regions, a cluster of stream-tubes exists that originates from the production peak and transfers energy towards small scales and the near-wall region. The spiral shape of the energy flux is similar that its shape in channel flow Cimarelli et al. (2013). However, the spiral shape of the energy flux is compressed to a small size due to the thinner boundary layer thickness during the transition. Also, there exists a cluster of positive scale energy flux within the spot spreading in the r_1 and r_3 direction, which is not found in the fully turbulent region.

Chapter 7

Conclusions and future work

7.1 Conclusions

Most previous numerical or experimental studies of bypass transition focused on the analysis of single points statistics (mean properties) or two-point correlations (mainly to find the distance between Klebanoff modes). On the other hand, theoretical studies focused on linear or non-linear stability analysis. There has not been an analysis of the process of the transfer of energy between different length scales in bypass transition, similar to studies that have been conducted for other fully developed wall-bounded turbulent flows. This is the motivation for the work presented in this thesis. To this end, we set up a high-resolution DNS simulation and used the two-point KMH equation to analyze the inter-scale energy transfer process into the boundary layer bypass transition.

The bypass transition is triggered by free-stream homogeneous and isotropic turbulence, satisfying the Von Karman spectrum with turbulent intensity 3.4%. We store 350 uncorrelated snapshots in a database, which is validated against benchmarks experiments and simulations.

We first consider the evolution of the second-order function of the streamwise velocity fluctuation from the laminar to turbulent region. The results are presented on physical and scale spaces. The maximum value in the physical space provides information regarding the wall-normal location at which energy is extracted from the mean flow; while in the scale space, it represents the coherent length scale in the streamwise and spanwise directions. Particularly in the spanwise direction, it represents the average spacing between the laminar or turbulent streaks. In order to separate the contributions from laminar and turbulent events during transition, we apply conditional sampling based on the instantaneous local intermittency and derive a general conditional decomposition for two-point statistics. By doing so, the standard averaging can be decomposed to contributions arising from turbulent-turbulent (TT), turbulent-laminar/laminar-turbulent (TL/LT) and laminar-laminar (LL) events. It is

found from the LL conditionally-averaged results that the laminar streaky structures maintain well their geometrical characteristics in the physical and scale space, even after the initial breakdown to form turbulent spots. Analysis of the TT conditioned structure function confirms the top-to-down formation mechanism of the turbulent spots, consistent with the 'outer type' secondary instability mechanism. The growth of two-point intermittency in the streamwise direction indicates the growth of the turbulent spots along X .

To describe the dynamics of the structure function, the Karman-Howarth-Monin-Hill equation is then utilised. This is a generalised Karman-Howarth equation derived directly from the Navier-Stokes equations without any approximations. Using this equation, we can analyze the local processes of energy transfer in physical and scale spaces. To this end, the divergence form of the KMH equation is employed.

It is found that there exists a strong inverse cascade during the transition in the streamwise and spanwise directions. We decompose the scale energy flux into four contributions that account for the effect of non-linear and linear interactions, pressure-velocity correlations, and viscous effects. It is found that competition exists between the linear and non-linear effects in the streamwise direction, and between non-linear and viscous in the spanwise direction. The large inverse cascade is found to be dominated by the non-linear interaction flux component. We link this finding with the instantaneous flow field and find that it is related to the growth of the turbulent spots.

To further analyse the interscale transfer process, we derive a conditionally-averaged KMH equation, and decomposed the scale energy flux into 4 components, arising from TT (turbulent-turbulent), LL (laminar-laminar), TL (turbulent-laminar) and LT (laminar-turbulent) events. Conditional averaging doesn't commute with the spatial derivative operator, so we compute the conditionally-averaged two-point properties directly in the scale space. The validity of the algorithm is tested by comparing against derivatives of standard-averaged two-point quantities.

The conditionally-averaged scale energy flux unveils information hidden under standard averaging and further explains the origin of the strong inverse cascade. The TL conditional averaged scale energy flux reveals that the strong inverse cascade in the spanwise direction is mainly due to the non-linear scale energy flux across the lateral turbulent spots boundary. For the inverse cascade in the streamwise direction, it is the non-linear scale flux across the downstream laminar/turbulent interface of the turbulent spots that makes the strongest contribution. The flux across the downstream and upstream interfaces is different due to the inhomogeneity of the turbulent spots in the streamwise direction. Lastly, the TT -averaged fluxes show that there are both similarities and differences between the turbulent spots and the fully turbulent region. The difference is that there exists spanwise inverse cascade within

turbulent spots which is not found in the fully turbulent region. On the other hand, a very similar 3-dimensional flux map is found, which confirms the energy transfer dynamics is very similar within the turbulent spots and turbulent boundary layer.

7.2 Future work

(1) For the conditionally-averaged KMHM analysis (section 6), we mainly focused on the non-linear scale energy flux and production, which were found to be the dominant components. It will be also interesting to explore the linear, pressure, viscous scale/spatial energy fluxes, and dissipation at different streamwise locations.

(2) In the present KMHM analysis, only the streamwise r_1 and spanwise r_3 separations are considered. These were combined with the wall-normal direction Y and reduced the six-dimensional (in our case five-dimensional) KMHM equation to a three-dimensional space that can be visualised. However, the wall-normal separation r_2 is also important, because this direction is the very inhomogeneous for wall-bounded flows. This is worth exploring in the future.

(3) Using our conditionally-averaged framework to the structure function and the KMHM equation, we have recovered the top-to-down formation of the turbulent spot. This is the result of the 'outer-mode' secondary instability, a characteristic of the bypass transition due to free-stream turbulence. It would be also interesting to explore different boundary conditions (receptivity), such as adding leading-edge effect, considering different pressure gradients, and modifying the characteristics of the free-stream turbulence (intensity, length scale, etc.). It will be interesting to explore inter-scale energy transfer process for these cases as well.

(4) Further research can also focus on the applications of the conditionally-averaged framework that was developed in this thesis to other flows that also have a sharp turbulent/non-turbulent interface, for example jet flows. It would be very interesting to investigate the role of these interfaces on the interscale energy flux processes.

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Appendix A

The Orr-Sommerfeld and Squire equations

Using Reynolds decomposition, the velocity and pressure fields can be decomposed as,

$$\mathbf{u} = \mathbf{U} + \mathbf{u}', \quad p = P + p'. \quad (\text{A.1})$$

Substituting Eqn.A.1 into NS equation (Eqn. 2.1), and ignoring the nonlinear term, yields the linearized NS equation for the velocity fluctuations,

$$\begin{aligned} \nabla \cdot \mathbf{u}' &= 0 \\ \frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{u}' + (\mathbf{u}' \cdot \nabla) \mathbf{U} &= -\nabla p' + \frac{1}{Re} \nabla^2 \mathbf{u}' \end{aligned} \quad (\text{A.2})$$

Considering only parallel flows, in which $U = U(y), V = 0, W = 0$, the linearized N-S equation can then be simplified to,

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \quad (\text{A.3})$$

$$\frac{\partial u'}{\partial t} + U \frac{\partial u'}{\partial x} + v' \frac{\partial U}{\partial y} = -\frac{\partial p'}{\partial x} + \frac{1}{Re} \left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} + \frac{\partial^2 u'}{\partial z^2} \right) \quad (\text{A.4})$$

$$\frac{\partial v'}{\partial t} + U \frac{\partial v'}{\partial x} = -\frac{\partial p'}{\partial y} + \frac{1}{Re} \left(\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2} + \frac{\partial^2 v'}{\partial z^2} \right) \quad (\text{A.5})$$

$$\frac{\partial w'}{\partial t} + U \frac{\partial w'}{\partial x} = -\frac{\partial p'}{\partial z} + \frac{1}{Re} \left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2} \right) \quad (\text{A.6})$$

Taking the derivative with respect to x, y and z of equations A.4, A.5 and A.6 respectively, and adding together, we obtain Poisson's equation for pressure,

$$\nabla^2 p' = -2 \frac{\partial U}{\partial y} \frac{\partial v'}{\partial x} \quad (\text{A.7})$$

Taking Laplacian of equation A.5 and using equation A.7, we can obtain the wall-normal perturbation equation,

$$\left[\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \nabla^2 - U'' \frac{\partial}{\partial x} - \frac{1}{Re} \nabla^4 \right] v' = 0 \quad (\text{A.8})$$

Taking $\frac{\partial}{\partial z}$ of equation A.4, and $\frac{\partial}{\partial x}$ of equation A.6, adding them together, then utilizing the definition of vorticity, $\eta' = \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}$, we can obtain the equation for wall-normal vorticity.

$$\left[\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} - \frac{1}{Re} \nabla^2 \right] \eta' = -U' \frac{\partial v'}{\partial z} \quad (\text{A.9})$$

We now assume that the perturbations have wavelike expression,

$$\begin{aligned} v' &= \hat{v}(y) e^{i(k_1 x + k_3 z - \omega t)} \\ \eta' &= \hat{\eta}(y) e^{i(k_1 x + k_3 z - \omega t)} \end{aligned} \quad (\text{A.10})$$

where k_1, k_2 , and k_3 are the wavenumbers in streamwise, wall-normal and spanwise directions; ω is the disturbance frequency. Substituting equation A.10 into equation A.8, obtains the Orr-Sommerfeld equation,

$$\left[(-i\omega + ik_1 U)(D^2 - k_{13}^2) - ik_1 U'' - \frac{1}{Re}(D^2 - k_{13}^2)^2 \right] \hat{v} = 0 \quad (\text{A.11})$$

where $k_{13}^2 = k_1^2 + k_3^2$. Its boundary condition are, $\hat{v} = 0, \frac{\partial \hat{v}}{\partial y} = 0$ on the wall. In a semi-bounded domain (e.g boundary layer case), the boundary condition applied in the free-stream is given by Jacobs and Durbin (1998),

$$\frac{(\hat{v}'' + k_2^2 \hat{v})|_{y_1}}{(\hat{v}'' + k_2^2 \hat{v})|_{y_2}} = e^{k_{13}(y_2 - y_1)} \quad (\text{A.12})$$

where y_1 and y_2 are the two points below the topmost grid points.

Similarly, substituting Eqn.A.10 into Eqn.A.9, obtains the Squire equation,

$$\left[(-i\omega + ik_1 U) - \frac{1}{Re}(D^2 - k_{13}^2)^2 \right] \hat{\eta} = -ik_3 U' \hat{v} \quad (\text{A.13})$$

Equations A.11 and A.13 are regarded as a coupled eigenvalue problem. However, the homogeneous Squire equation (Zaki and Durbin, 2005) is used. This is because the OS/SQ equations are decoupled in the freestream. The inhomogeneous SQ solution can result in very high streamwise velocities (Hack and Zaki, 2014, Brandt et al., 2004). The boundary conditions of the homogeneous Squire condition are, $\hat{\eta} = 0|_{y=0}$ and $\hat{\eta} = 1|_{y \rightarrow \infty}$.

Appendix B

Derivation of the KMH equation

The KMH equation is a six-dimensional equation with independent variables the coordinates X_i and r_i , which represent ‘spatial/physical’ and ‘scale/separation’ spaces respectively. The relationship between them and x_i^+ , x_i^- are shown as below and figure B.1,

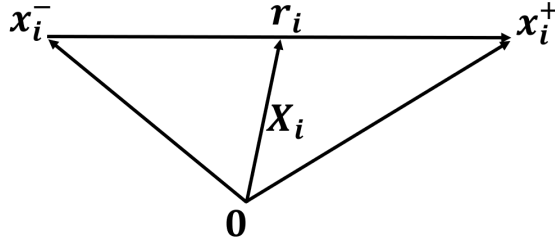


Figure B.1 The relationship between x_i^+ , x_i^- and X_i , r_i .

$$X_i \equiv (x_i^+ + x_i^-)/2 \quad r_i \equiv x_i^+ - x_i^- \quad (\text{B.1})$$

We transform the partial derivatives with respect to x_i^+ and x_i^- to the spaces of X_i , r_i , using chain rule,

$$\begin{aligned} \frac{\partial u_i^+(X_i, r_j)}{\partial x_i^+} &= \frac{\partial u_i^+}{\partial X_i} \frac{\partial X_i}{\partial x_i^+} + \frac{\partial u_i^+}{\partial r_i} \frac{\partial r_i}{\partial x_i^+} \\ \frac{\partial u_i^-(X_i, r_j)}{\partial x_i^-} &= \frac{\partial u_i^-}{\partial X_i} \frac{\partial X_i}{\partial x_i^-} + \frac{\partial u_i^-}{\partial r_i} \frac{\partial r_i}{\partial x_i^-} \end{aligned} \quad (\text{B.2})$$

Use equation. (B.1), we obtain,

$$\begin{aligned}\frac{\partial X_i}{\partial x_i^+} &= \frac{1}{2} + \frac{1}{2} \frac{\partial u_i^-}{\partial x_i^+} = \frac{1}{2} & \frac{\partial r_i}{\partial x_i^+} &= 1 - \frac{\partial u_i^-}{\partial x_i^+} = 1 \\ \frac{\partial X_i}{\partial x_i^-} &= \frac{1}{2} \frac{\partial u_i^+}{\partial x_i^-} + \frac{1}{2} = \frac{1}{2} & \frac{\partial r_i}{\partial x_i^-} &= \frac{\partial u_i^+}{\partial x_i^-} - 1 = -1\end{aligned}\quad (\text{B.3})$$

Substitute equation (B.3) into equation (B.2), we obtain,

$$\begin{aligned}\frac{\partial u_i^+(X_i, r_i)}{\partial x_i^+} &= \frac{1}{2} \frac{\partial u_i^+}{\partial X_i} + \frac{\partial u_i^+}{\partial r_i} \implies \frac{\partial}{\partial x_i^+} = \frac{1}{2} \frac{\partial}{\partial X_i} + \frac{\partial}{\partial r_i} \\ \frac{\partial u_i^-(X_i, r_i)}{\partial x_i^-} &= \frac{1}{2} \frac{\partial u_i^-}{\partial X_i} - \frac{\partial u_i^-}{\partial r_i} \implies \frac{\partial}{\partial x_i^-} = \frac{1}{2} \frac{\partial}{\partial X_i} - \frac{\partial}{\partial r_i}\end{aligned}\quad (\text{B.4})$$

Adding and subtracting the equations in (B.4), we get

$$\frac{\partial}{\partial X_i} = \frac{\partial}{\partial x_i^+} + \frac{\partial}{\partial x_i^-} \quad \frac{\partial}{\partial r_i} = \frac{1}{2} \left(\frac{\partial}{\partial x_i^+} - \frac{\partial}{\partial x_i^-} \right) \quad (\text{B.5})$$

In summery, we can obtain the relationships below between the partial derivatives with respect to x_i^+, x_i^-, X_i and r_i ,

$$\frac{\partial}{\partial x_i^+} = \frac{1}{2} \frac{\partial}{\partial X_i} + \frac{\partial}{\partial r_i}, \quad \frac{\partial}{\partial x_i^-} = \frac{1}{2} \frac{\partial}{\partial X_i} - \frac{\partial}{\partial r_i}, \quad \frac{\partial}{\partial r_i} = \frac{1}{2} \left(\frac{\partial}{\partial x_i^+} - \frac{\partial}{\partial x_i^-} \right), \quad \frac{\partial}{\partial X_i} = \frac{\partial}{\partial x_i^+} + \frac{\partial}{\partial x_i^-} \quad (\text{B.6})$$

We now writhe the NS equations and continuity equations at two independent points, x^+ and x^- ,

$$\frac{\partial u_i^+}{\partial x_i^+} = 0, \quad \frac{\partial u_i^+}{\partial t} + u_j^+ \frac{\partial u_i^+}{\partial x_j^+} = -\frac{\partial p^+}{\partial x_i^+} + \nu \frac{\partial^2 u_i^+}{\partial x_j^+ \partial x_j^+} \quad (\text{B.7})$$

$$\frac{\partial u_i^-}{\partial x_i^-} = 0, \quad \frac{\partial u_i^-}{\partial t} + u_j^- \frac{\partial u_i^-}{\partial x_j^-} = -\frac{\partial p^-}{\partial x_i^-} + \nu \frac{\partial^2 u_i^-}{\partial x_j^- \partial x_j^-} \quad (\text{B.8})$$

Importantly, since x^+ and x^- are independent, the velocity derivative with respect to different points is zero, i.e,

$$\frac{\partial u_i^+}{\partial x_i^-} = 0 \quad \frac{\partial u_i^-}{\partial x_i^+} = 0 \quad (\text{B.9})$$

We define the velocity fluctuations difference $du' \equiv u'^+ - u'^-$,

$$du' = du - \overline{du} \quad (\text{B.10})$$

and the average velocity fluctuation $u'_* \equiv (u'^+ + u'^-)/2$,

$$u'_* = u_* - \overline{u_*} \quad (\text{B.11})$$

It is important to note that the time average of the fluctuation properties is zero, i.e., $\overline{du'} = 0$ and $\overline{u'_*} = 0$. Then subtracting equation (B.8) from (B.7), using equation (B.10), and multiplying each term by $2du'_i$, and then implement time average operation (denoted as overline), we obtain

$$\begin{aligned} & \underbrace{\overline{2du'_i \frac{\partial du_i}{\partial t}}}_{\text{Transient (T)}} + \underbrace{\overline{2du'_i u_j^+ \frac{\partial du_i}{\partial x_j^+} + 2du'_i u_j^- \frac{\partial du_i}{\partial x_j^-}}}_{\text{Non-linear (N)}} = \\ & \underbrace{\overline{-2du'_i \left(\frac{\partial dp}{\partial x_i^+} + \frac{\partial dp}{\partial x_i^-} \right)}}_{\text{Pressure-velocity (P)}} + \underbrace{\overline{\nu 2du'_i \frac{\partial^2 du_i}{\partial x_j^+ \partial x_j^+} + \nu 2du'_i \frac{\partial^2 du_i}{\partial x_j^- \partial x_j^-}}}_{\text{Viscous (V)}} \end{aligned} \quad (\text{B.12})$$

(1) Transient term (T) is equal to 0 because we consider a statistically steady case,

$$T = \overline{2du'_i \frac{\partial (dU_i + du'_i)}{\partial t}} = \overline{2du'_i \frac{\partial dU_i}{\partial t}} + \overline{2du'_i \frac{\partial du'_i}{\partial t}} = \overline{2du'_i \frac{\partial du'_i}{\partial t}} \quad (\text{B.13})$$

(2) Pressure term (P)

$$\begin{aligned} P &= \overline{-2du'_i \left(\frac{\partial (dP + dp')}{\partial x_i^+} + \frac{\partial (dP + dp')}{\partial x_i^-} \right)} \\ &= \overline{-2du'_i \left(\frac{1}{2} \frac{\partial dP}{\partial X_i} + \frac{\partial dP}{\partial r_i} + \frac{1}{2} \frac{\partial dp'}{\partial X_i} + \frac{\partial dp'}{\partial r_i} + \frac{1}{2} \frac{\partial dP}{\partial X_i} - \frac{\partial dP}{\partial r_i} + \frac{1}{2} \frac{\partial dp'}{\partial X_i} - \frac{\partial dp'}{\partial r_i} \right)} \\ &= \overline{-2du'_i \left(\frac{\partial dP}{\partial X_i} + \frac{\partial dp'}{\partial X_i} \right)} \\ &= \overline{-2du'_i \frac{\partial dp'}{\partial X_i}} \end{aligned} \quad (\text{B.14})$$

(3) Viscous term (V)

$$\begin{aligned}
V &= \overline{2\nu du'_i \left(\frac{\partial^2 dU_i}{\partial x_j^+ \partial x_j^+} + \frac{\partial^2 du'_i}{\partial x_j^+ \partial x_j^+} + \frac{\partial^2 dU_i}{\partial x_j^- \partial x_j^-} + \frac{\partial^2 du'_i}{\partial x_j^- \partial x_j^-} \right)} \\
&= \overline{2\nu du'_i \left(\frac{\partial^2 du'_i}{\partial x_j^+ \partial x_j^+} + \frac{\partial^2 du'_i}{\partial x_j^- \partial x_j^-} \right)} \\
&= \overline{2\nu \left(\frac{\partial}{\partial x_j^+} \left(du'_i \frac{\partial du'_i}{\partial x_j^+} \right) - \frac{\partial du'_i}{\partial x_j^+} \frac{\partial du'_i}{\partial x_j^+} \right) + 2\nu \left(\frac{\partial}{\partial x_j^-} \left(du'_i \frac{\partial du'_i}{\partial x_j^-} \right) - \frac{\partial du'_i}{\partial x_j^-} \frac{\partial du'_i}{\partial x_j^-} \right)} \\
&= \overline{\nu \left(\frac{\partial}{\partial x_j^+} \left(\frac{\partial du'_i du'_i}{\partial x_j^+} \right) \right) - 2\nu \frac{\partial du'_i}{\partial x_j^+} \frac{\partial du'_i}{\partial x_j^+} + \nu \left(\frac{\partial}{\partial x_j^-} \left(\frac{\partial du'_i du'_i}{\partial x_j^-} \right) \right) - 2\nu \frac{\partial du'_i}{\partial x_j^-} \frac{\partial du'_i}{\partial x_j^-}} \\
&= \overline{\nu \left(\left(\frac{1}{2} \frac{\partial}{\partial X_j} + \frac{\partial}{\partial r_j} \right)^2 (du'_i du'_i) + \left(\frac{1}{2} \frac{\partial}{\partial X_j} - \frac{\partial}{\partial r_j} \right)^2 (du'_i du'_i) \right) - 2\nu \left(\frac{\partial du'_i}{\partial x_j^+} \frac{\partial du'_i}{\partial x_j^+} + \frac{\partial du'_i}{\partial x_j^-} \frac{\partial du'_i}{\partial x_j^-} \right)} \\
&= \overline{\nu \frac{1}{2} \frac{\partial^2 du'_i du'_i}{\partial X_j \partial X_j} + 2\nu \frac{\partial^2 du'_i du'_i}{\partial r_j \partial r_j} - 2\nu \left(\frac{\partial du'_i}{\partial x_j^+} \frac{\partial du'_i}{\partial x_j^+} + \frac{\partial du'_i}{\partial x_j^-} \frac{\partial du'_i}{\partial x_j^-} \right)}
\end{aligned} \tag{B.15}$$

(4) Nonlinear term (N)

$$\begin{aligned}
N &= \overline{2du'_i u_j^+ \frac{\partial du_i}{\partial x_j^+} + 2du'_i u_j^- \frac{\partial du_i}{\partial x_j^-}} \\
&= \overline{2du'_i u_j^+ \frac{\partial (dU_i + du'_i)}{\partial x_j^+} + 2du'_i u_j^- \frac{\partial (dU_i + du'_i)}{\partial x_j^-}} \\
&= \overline{2du'_i u_j^+ \left(\frac{1}{2} \frac{\partial dU_i}{\partial X_j} + \frac{\partial dU_i}{\partial r_j} \right) + 2du'_i u_j^+ \left(\frac{1}{2} \frac{\partial du'_i}{\partial X_j} + \frac{\partial du'_i}{\partial r_j} \right)} \\
&\quad + \overline{2du'_i u_j^- \left(\frac{1}{2} \frac{\partial dU_i}{\partial X_j} - \frac{\partial dU_i}{\partial r_j} \right) + 2du'_i u_j^- \left(\frac{1}{2} \frac{\partial du'_i}{\partial X_j} - \frac{\partial du'_i}{\partial r_j} \right)} \\
&= \overline{2du'_i u_{*j} \frac{\partial dU_i}{\partial X_j} + 2du'_i du_j \frac{\partial dU_i}{\partial r_j} + 2du'_i u_{*j} \frac{\partial du'_i}{\partial X_j} + 2du'_i du_j \frac{\partial du'_i}{\partial r_j}} \\
&= \overline{2du'_i (u'_{*j} + U_{*j}) \frac{\partial dU_i}{\partial X_j} + 2du'_i (du'_j + dU_j) \frac{\partial dU_i}{\partial r_j} + 2du'_i (u'_{*j} + U_{*j}) \frac{\partial du'_i}{\partial X_j} + 2du'_i (du'_j + dU_j) \frac{\partial du'_i}{\partial r_j}} \\
&= \overline{2du'_i U_{*j} \frac{\partial du'_i}{\partial X_j} + 2du'_i dU_j \frac{\partial du'_i}{\partial r_j} + 2du'_i u'_{*j} \frac{\partial dU_i}{\partial X_j} + 2du'_i du'_j \frac{\partial dU_i}{\partial r_j} + 2du'_i u'_{*j} \frac{\partial du'_i}{\partial X_j} + 2du'_i du'_j \frac{\partial du'_i}{\partial r_j}}
\end{aligned} \tag{B.16}$$

We add all terms together to obtain,

$$\begin{aligned}
& \overline{2du'_i U_{*j} \frac{\partial du'_i}{\partial X_j}} + \overline{2du'_i u'_{*j} \frac{\partial du'_i}{\partial X_j}} \\
& \overline{2du'_i dU_j \frac{\partial du'_i}{\partial r_j}} + \overline{2du'_i du'_j \frac{\partial du'_i}{\partial r_j}} = -\overline{2du'_i \frac{\partial dp'}{\partial X_i}} - \overline{2du'_i u'_{*j} \frac{\partial dU_i}{\partial X_j}} - \overline{2du'_i du'_j \frac{\partial dU_i}{\partial r_j}} \quad (\text{B.17}) \\
& + \overline{v \frac{1}{2} \frac{\partial^2 du'_i du'_i}{\partial X_j \partial X_j}} + \overline{2v \frac{\partial^2 du'_i du'_i}{\partial r_j \partial r_j}} - 2v \left(\frac{\partial du'_i}{\partial x_j^+} \frac{\partial du'_i}{\partial x_j^+} + \frac{\partial du'_i}{\partial x_j^-} \frac{\partial du'_i}{\partial x_j^-} \right)
\end{aligned}$$

Rearrange Eqn. (B.17) to combine du'_i and du'_i and define the two-point energy as $dq^2 = du'_i du'_i$. We then have the standard KMH equation.

$$\begin{aligned}
& \underbrace{\overline{\frac{\partial \frac{(U_j^+ + U_j^-)}{2} dq^2}{\partial X_j}}}_{\text{Mean flow advection } (\mathcal{A}_U)} + \underbrace{\overline{\frac{\partial du'_j dq^2}{\partial r_j}}}_{\text{Nonlinear interscale transfer } (\Pi)} + \underbrace{\overline{\frac{\partial dU_j dq^2}{\partial r_j}}}_{\text{Linear interscale transfer } (\Pi_U)} = \\
& \underbrace{\overline{\frac{\partial \frac{(u_j'^+ + u_j'^-)}{2} dq^2}{\partial X_j}}}_{\text{Turbulent advection } (\mathcal{A}_u)} - \underbrace{2 \overline{\frac{\partial du'_j dp'}{\partial X_j}}}_{\text{Pressure-velocity correlation } (\mathcal{T}_p)} + \underbrace{\frac{1}{2} v \overline{\frac{\partial^2 dq^2}{\partial X_j \partial X_j}}}_{\text{Diffusion in physical space } (\mathcal{D}_x)} + \underbrace{2v \overline{\frac{\partial^2 dq^2}{\partial r_j \partial r_j}}}_{\text{Diffusion in scale space } (\mathcal{D}_r)} \quad (\text{B.18}) \\
& \underbrace{-2 \overline{du'_j du'_i \frac{\partial dU_i}{\partial r_j}} - \overline{(u_j'^+ + u_j'^-) du'_i \frac{\partial dU_i}{\partial X_j}}}_{\text{Production by mean flow } (\mathcal{P})} - \underbrace{4v \left(\frac{1}{4} \overline{\frac{\partial du'_i}{\partial X_j} \frac{\partial du'_i}{\partial X_j}} + \overline{\frac{\partial du'_i}{\partial r_j} \frac{\partial du'_i}{\partial r_j}} \right)}_{\text{Dissipation } (\varepsilon)}
\end{aligned}$$

Appendix C

Different forms of the pressure-velocity correlation term

The pressure-velocity correlation term can be expressed in either scale or physical spaces. This is proved as below.

In physical space,

$$\begin{aligned} \overline{-2du'_i \frac{\partial p'^+}{\partial x_i^+} - 2du'_i \frac{\partial p'^-}{\partial x_i^-}} &= \overline{-2du'_i \frac{\partial(p'^+ - p'^-)}{\partial x_i^+} + 2du'_i \frac{\partial(p'^+ - p'^-)}{\partial x_i^-}} \\ &= \overline{-2du'_i \frac{\partial p'}{\partial x_i^+} + 2du'_i \frac{\partial p'}{\partial x_i^-}} \\ &= \overline{-2du'_i \left(\frac{1}{2} \frac{\partial p'}{\partial X_i} + \frac{\partial p'}{\partial r_i} \right) + 2du'_i \left(\frac{1}{2} \frac{\partial p'}{\partial X_i} - \frac{\partial p'}{\partial r_i} \right)} \\ &= \overline{-2du'_i \frac{\partial p'}{\partial X_i}} \end{aligned} \tag{C.1}$$

In scale space,

$$\begin{aligned} \overline{-2du'_i \frac{\partial p'^+}{\partial x_i^+} - 2du'_i \frac{\partial p'^-}{\partial x_i^-}} &= \overline{-2du'_i \frac{\partial(p'^+ + p'^-)}{\partial x_i^+} - 2du'_i \frac{\partial(p'^+ + p'^-)}{\partial x_i^-}} \\ &= \overline{-4du'_i \frac{\partial p'_*}{\partial x_i^+} - 4du'_i \frac{\partial p'_*}{\partial x_i^-}} \\ &= \overline{-4du'_i \left(\frac{\partial p'_*}{\partial x_i^+} - \frac{\partial p'_*}{\partial x_i^-} \right)} \\ &= \overline{-8du'_i \frac{\partial p'_*}{\partial r_i}} \end{aligned} \tag{C.2}$$

Appendix D

Asymptotic form of the KMH terms for very large separations

Asymptotic forms are derived using relationship,

$$\frac{\partial}{\partial r_i} = \frac{1}{2} \left(\frac{\partial}{\partial x_i^+} - \frac{\partial}{\partial x_i^-} \right), \quad \frac{\partial}{\partial X_i} = \frac{\partial}{\partial x_i^+} + \frac{\partial}{\partial x_i^-} \quad (\text{D.1})$$

and

$$\overline{u_i'^+ u_i'^-} = 0 \quad \text{at very large separations the correlation tends to 0.} \quad (\text{D.2})$$

The scale energy at large separation can be written as,

$$\overline{dq^2} = \overline{(du_i')^2} = \overline{(u_i'^+ - u_i'^-)^2} = \overline{(u_i'^+)^2 + (u_i'^-)^2 - 2u_i'^+ u_i'^-} = \overline{(u_i'^+)^2 + (u_i'^-)^2} = 2k^+ + 2k^- \quad (\text{D.3})$$

where

$$k^+ = \frac{1}{2} \overline{u_i'^+ u_i'^+} \quad k^- = \frac{1}{2} \overline{u_i'^- u_i'^-} \quad (\text{D.4})$$

The non-linear inter-scale energy transfer rate at large separations takes form,

$$\begin{aligned}
\frac{\partial \overline{du'_j dq^2}}{\partial r_j} &= \frac{\partial \overline{(u'_j{}^+ - u'_j{}^-)(u'_i{}^+ - u'_i{}^-)^2}}{\partial r_j} \\
&= \frac{\partial \overline{(u'_j{}^+ - u'_j{}^-)[(u'_i{}^+)^2 + (u'_i{}^-)^2]}}{\partial r_j} \\
&= \frac{\partial \overline{u'_j{}^+(u'_i{}^+)^2 - u'_j{}^-(u'_i{}^+)^2 + u'_j{}^+(u'_i{}^-)^2 - u'_j{}^-(u'_i{}^-)^2}}{\partial r_j} \\
&= \frac{\partial \overline{u'_j{}^+(u'_i{}^+)^2 - u'_j{}^-(u'_i{}^-)^2}}{\partial r_j} \tag{D.5} \\
&= \frac{1}{2} \left(\frac{\partial \overline{u'_j{}^+(u'_i{}^+)^2 - u'_j{}^-(u'_i{}^-)^2}}{\partial x_j^+} - \frac{\partial \overline{u'_j{}^+(u'_i{}^+)^2 - u'_j{}^-(u'_i{}^-)^2}}{\partial x_j^-} \right) \\
&= \frac{1}{2} \left(\frac{\partial \overline{u'_j{}^+(u'_i{}^+)^2}}{\partial x_j^+} + \frac{\partial \overline{u'_j{}^-(u'_i{}^-)^2}}{\partial x_j^-} \right)
\end{aligned}$$

The turbulent advection at large separation takes form,

$$\begin{aligned}
\frac{\partial \overline{\frac{(u'_j{}^+ + u'_j{}^-)}{2} dq^2}}{\partial X_j} &= \frac{1}{2} \frac{\partial \overline{(u'_j{}^+ + u'_j{}^-)(u'_i{}^+ - u'_i{}^-)^2}}{\partial X_j} \\
&= \frac{1}{2} \frac{\partial \overline{(u'_j{}^+ + u'_j{}^-)[(u'_i{}^+)^2 + (u'_i{}^-)^2]}}{\partial X_j} \\
&= \frac{1}{2} \frac{\partial \overline{u'_j{}^+(u'_i{}^+)^2 + u'_j{}^-(u'_i{}^+)^2 + u'_j{}^+(u'_i{}^-)^2 + u'_j{}^-(u'_i{}^-)^2}}{\partial X_j} \\
&= \frac{1}{2} \frac{\partial \overline{u'_j{}^+(u'_i{}^+)^2 + u'_j{}^-(u'_i{}^-)^2}}{\partial X_j} \tag{D.6} \\
&= \frac{1}{2} \left(\frac{\partial \overline{u'_j{}^+(u'_i{}^+)^2 + u'_j{}^-(u'_i{}^-)^2}}{\partial x_j^+} + \frac{\partial \overline{u'_j{}^+(u'_i{}^+)^2 + u'_j{}^-(u'_i{}^-)^2}}{\partial x_j^-} \right) \\
&= \frac{1}{2} \left(\frac{\partial \overline{u'_j{}^+(u'_i{}^+)^2}}{\partial x_j^+} + \frac{\partial \overline{u'_j{}^-(u'_i{}^-)^2}}{\partial x_j^-} \right)
\end{aligned}$$

Appendix E

Derivation of conditionally-averaged decompositions

E.1 Decomposition of the second-order structure function

$$\begin{aligned}\overline{(du')^2} &= \overline{(du)^2} - (dU)^2 \\ &= \gamma^{(TT)}\overline{(du)^2}^{(TT)} + \gamma^{(TL)}\overline{(du)^2}^{(TL)} + \gamma^{(LT)}\overline{(du)^2}^{(LT)} + \gamma^{(LL)}\overline{(du)^2}^{(LL)} \\ &\quad - (\gamma^{(TT)}dU^{(TT)} + \gamma^{(TL)}dU^{(TL)} + \gamma^{(LT)}dU^{(LT)} + \gamma^{(LL)}dU^{(LL)})^2 \\ &= \gamma^{(TT)}(\overline{(du')^2}^{(TT)} + (dU^{(TT)})^2) + \gamma^{(TL)}(\overline{(du')^2}^{(TL)} + (dU^{(TL)})^2) + \\ &\quad \gamma^{(LT)}(\overline{(du')^2}^{(LT)} + (dU^{(LT)})^2) + \gamma^{(LL)}(\overline{(du')^2}^{(LL)} + (dU^{(LL)})^2) \\ &\quad - (\gamma^{(TT)}dU^{(TT)} + \gamma^{(TL)}dU^{(TL)} + \gamma^{(LT)}dU^{(LT)} + \gamma^{(LL)}dU^{(LL)})^2 \\ &= \gamma^{(TT)}\overline{(du')^2}^{(TT)} + \gamma^{(TL)}\overline{(du')^2}^{(TL)} + \gamma^{(LT)}\overline{(du')^2}^{(LT)} + \gamma^{(LL)}\overline{(du')^2}^{(LL)} \\ &\quad + \gamma^{(TT)}(dU^{(TT)})^2 + \gamma^{(TL)}(dU^{(TL)})^2 + \gamma^{(LT)}(dU^{(LT)})^2 + \gamma^{(LL)}(dU^{(LL)})^2 \\ &\quad - (\gamma^{(TT)}dU^{(TT)} + \gamma^{(TL)}dU^{(TL)} + \gamma^{(LT)}dU^{(LT)} + \gamma^{(LL)}dU^{(LL)})^2 \\ &= \gamma^{(TT)}\overline{(du')^2}^{(TT)} + \gamma^{(TL)}\overline{(du')^2}^{(TL)} + \gamma^{(LT)}\overline{(du')^2}^{(LT)} + \gamma^{(LL)}\overline{(du')^2}^{(LL)} + \Phi\end{aligned}\tag{E.1}$$

where,

$$\begin{aligned}\Phi &= +\gamma^{(TT)}(dU^{(TT)})^2 + \gamma^{(TL)}(dU^{(TL)})^2 + \gamma^{(LT)}(dU^{(LT)})^2 + \gamma^{(LL)}(dU^{(LL)})^2 \\ &\quad - (\gamma^{(TT)}dU^{(TT)} + \gamma^{(TL)}dU^{(TL)} + \gamma^{(LT)}dU^{(LT)} + \gamma^{(LL)}dU^{(LL)})^2\end{aligned}\tag{E.2}$$

E.2 Decomposition of terms in scale space

Non-linear scale energy flux

$$\begin{aligned}
& \overline{(du'_i)^2 du'_j} = \overline{(du_i)^2 du_j} - \overline{(du_i)^2 dU_j} - \overline{2du_i du_j dU_i} + 2dU_i dU_j dU_j \\
& = \gamma^{(TT)} \overline{(du_i)^2 du_j}^{(TT)} + \gamma^{(TL)} \overline{(du_i)^2 du_j}^{(TL)} + \gamma^{(LT)} \overline{(du_i)^2 du_j}^{(LT)} + \gamma^{(LL)} \overline{(du_i)^2 du_j}^{(LL)} \\
& \quad - \overline{(du_i)^2 dU_j} - \overline{2du_i du_j dU_i} + 2dU_i dU_j dU_j \\
& = \gamma^{(TT)} \left(\overline{(du'_i)^2 du'_j}^{(TT)} + \overline{(du_i)^2}^{(TT)} dU_j^{(TT)} + \overline{2du_i du_j}^{(TT)} dU_i^{(TT)} - 2dU_i^{(TT)} dU_i^{(TT)} dU_j^{(TT)} \right) \\
& \quad + \gamma^{(TL)} \left(\overline{(du'_i)^2 du'_j}^{(TL)} + \overline{(du_i)^2}^{(TL)} dU_j^{(TL)} + \overline{2du_i du_j}^{(TL)} dU_i^{(TL)} - 2dU_i^{(TL)} dU_i^{(TL)} dU_j^{(TL)} \right) \\
& \quad + \gamma^{(LT)} \left(\overline{(du'_i)^2 du'_j}^{(LT)} + \overline{(du_i)^2}^{(LT)} dU_j^{(LT)} + \overline{2du_i du_j}^{(LT)} dU_i^{(LT)} - 2dU_i^{(LT)} dU_i^{(LT)} dU_j^{(LT)} \right) \\
& \quad + \gamma^{(LL)} \left(\overline{(du'_i)^2 du'_j}^{(LL)} + \overline{(du_i)^2}^{(LL)} dU_j^{(LL)} + \overline{2du_i du_j}^{(LL)} dU_i^{(LL)} - 2dU_i^{(LL)} dU_i^{(LL)} dU_j^{(LL)} \right) \\
& \quad - \overline{(du_i)^2 dU_j} - \overline{2du_i du_j dU_i} + 2dU_i dU_j dU_j \\
& = \gamma^{(TT)} \overline{(du'_i)^2 du'_j}^{(TT)} + \gamma^{(TL)} \overline{(du'_i)^2 du'_j}^{(TL)} + \gamma^{(LT)} \overline{(du'_i)^2 du'_j}^{(LT)} + \gamma^{(LL)} \overline{(du'_i)^2 du'_j}^{(LL)} + \phi
\end{aligned} \tag{E.3}$$

where,

$$\begin{aligned}
\phi & = \gamma^{(TT)} \left(\overline{(du_i)^2}^{(TT)} dU_j^{(TT)} + \overline{2du_i du_j}^{(TT)} dU_i^{(TT)} - 2dU_i^{(TT)} dU_i^{(TT)} dU_j^{(TT)} \right) \\
& \quad + \gamma^{(TL)} \left(\overline{(du_i)^2}^{(TL)} dU_j^{(TL)} + \overline{2du_i du_j}^{(TL)} dU_i^{(TL)} - 2dU_i^{(TL)} dU_i^{(TL)} dU_j^{(TL)} \right) \\
& \quad + \gamma^{(LT)} \left(\overline{(du_i)^2}^{(LT)} dU_j^{(LT)} + \overline{2du_i du_j}^{(LT)} dU_i^{(LT)} - 2dU_i^{(LT)} dU_i^{(LT)} dU_j^{(LT)} \right) \\
& \quad + \gamma^{(LL)} \left(\overline{(du_i)^2}^{(LL)} dU_j^{(LL)} + \overline{2du_i du_j}^{(LL)} dU_i^{(LL)} - 2dU_i^{(LL)} dU_i^{(LL)} dU_j^{(LL)} \right) \\
& \quad - \overline{(du_i)^2 dU_j} - \overline{2du_i du_j dU_i} + 2dU_i dU_j dU_j
\end{aligned} \tag{E.4}$$

Linear scale energy flux

$$\begin{aligned}
\overline{(du'_i)^2 dU_j} &= \overline{(du_i)^2 dU_j} - dU_i dU_i dU_j \\
&= \left(\gamma^{(TT)} \overline{(du_i)^2}^{(TT)} + \gamma^{(TL)} \overline{(du_i)^2}^{(TL)} + \gamma^{(LT)} \overline{(du_i)^2}^{(LT)} + \gamma^{(LL)} \overline{(du_i)^2}^{(LL)} \right) \\
&\quad \left(\gamma^{(TT)} dU_j^{(TT)} + \gamma^{(TL)} dU_j^{(TL)} + \gamma^{(LT)} dU_j^{(LT)} + \gamma^{(LL)} dU_j^{(LL)} \right) - dU_i dU_i dU_j \\
&= (\gamma^{(TT)})^2 \overline{(du_i)^2}^{(TT)} dU_j^{(TT)} + (\gamma^{(TL)})^2 \overline{(du_i)^2}^{(TL)} dU_j^{(TL)} \\
&\quad + (\gamma^{(LT)})^2 \overline{(du_i)^2}^{(LT)} dU_j^{(LT)} + (\gamma^{(LL)})^2 \overline{(du_i)^2}^{(LL)} dU_j^{(LL)} + \\
&\quad \gamma^{(TT)} \overline{(du_i)^2}^{(TT)} \left(\gamma^{(TL)} dU_j^{(TL)} + \gamma^{(LT)} dU_j^{(LT)} + \gamma^{(LL)} dU_j^{(LL)} \right) + \\
&\quad \gamma^{(TL)} \overline{(du_i)^2}^{(TL)} \left(\gamma^{(TT)} dU_j^{(TT)} + \gamma^{(LT)} dU_j^{(LT)} + \gamma^{(LL)} dU_j^{(LL)} \right) + \\
&\quad \gamma^{(LT)} \overline{(du_i)^2}^{(LT)} \left(\gamma^{(TT)} dU_j^{(TT)} + \gamma^{(TL)} dU_j^{(TL)} + \gamma^{(LL)} dU_j^{(LL)} \right) + \\
&\quad \gamma^{(LL)} \overline{(du_i)^2}^{(LL)} \left(\gamma^{(TT)} dU_j^{(TT)} + \gamma^{(TL)} dU_j^{(TL)} + \gamma^{(LT)} dU_j^{(LT)} \right) + \\
&\quad - dU_i dU_i dU_j \\
&= (\gamma^{(TT)})^2 \left(\overline{(du'_i)^2}^{(TT)} dU_j^{(TT)} + dU_i^{(TT)} dU_i^{(TT)} dU_j^{(TT)} \right) + \\
&\quad (\gamma^{(TL)})^2 \left(\overline{(du'_i)^2}^{(TL)} dU_j^{(TL)} + dU_i^{(TL)} dU_i^{(TL)} dU_j^{(TL)} \right) + \\
&\quad (\gamma^{(LT)})^2 \left(\overline{(du'_i)^2}^{(LT)} dU_j^{(LT)} + dU_i^{(LT)} dU_i^{(LT)} dU_j^{(LT)} \right) + \\
&\quad (\gamma^{(LL)})^2 \left(\overline{(du'_i)^2}^{(LL)} dU_j^{(LL)} + dU_i^{(LL)} dU_i^{(LL)} dU_j^{(LL)} \right) + \\
&\quad \gamma^{(TT)} \overline{(du_i)^2}^{(TT)} \left(\gamma^{(TL)} dU_j^{(TL)} + \gamma^{(LT)} dU_j^{(LT)} + \gamma^{(LL)} dU_j^{(LL)} \right) + \\
&\quad \gamma^{(TL)} \overline{(du_i)^2}^{(TL)} \left(\gamma^{(TT)} dU_j^{(TT)} + \gamma^{(LT)} dU_j^{(LT)} + \gamma^{(LL)} dU_j^{(LL)} \right) + \\
&\quad \gamma^{(LT)} \overline{(du_i)^2}^{(LT)} \left(\gamma^{(TT)} dU_j^{(TT)} + \gamma^{(TL)} dU_j^{(TL)} + \gamma^{(LL)} dU_j^{(LL)} \right) + \\
&\quad \gamma^{(LL)} \overline{(du_i)^2}^{(LL)} \left(\gamma^{(TT)} dU_j^{(TT)} + \gamma^{(TL)} dU_j^{(TL)} + \gamma^{(LT)} dU_j^{(LT)} \right) + \\
&\quad - dU_i dU_i dU_j \\
&= (\gamma^{(TT)})^2 \overline{(du'_i)^2}^{(TT)} dU_j^{(TT)} + (\gamma^{(TL)})^2 \overline{(du'_i)^2}^{(TL)} dU_j^{(TL)} + \\
&\quad (\gamma^{(LT)})^2 \overline{(du'_i)^2}^{(LT)} dU_j^{(LT)} + (\gamma^{(LL)})^2 \overline{(du'_i)^2}^{(LL)} dU_j^{(LL)} + \phi
\end{aligned}
\tag{E.5}$$

where

$$\begin{aligned}
\phi = & (\gamma^{(TT)})^2 dU_i^{(TT)} dU_i^{(TT)} dU_j^{(TT)} + (\gamma^{(TL)})^2 dU_i^{(TL)} dU_i^{(TL)} dU_j^{(TL)} \\
& + (\gamma^{(LT)})^2 dU_i^{(LT)} dU_i^{(LT)} dU_j^{(LT)} + (\gamma^{(LL)})^2 dU_i^{(LL)} dU_i^{(LL)} dU_j^{(LL)} + \\
& \gamma^{(TT)} \overline{(du_i)^2}^{(TT)} \left(\gamma^{(TL)} dU_j^{(TL)} + \gamma^{(LT)} dU_j^{(LT)} + \gamma^{(LL)} dU_j^{(LL)} \right) + \\
& \gamma^{(TL)} \overline{(du_i)^2}^{(TL)} \left(\gamma^{(TT)} dU_j^{(TT)} + \gamma^{(LT)} dU_j^{(LT)} + \gamma^{(LL)} dU_j^{(LL)} \right) + \\
& \gamma^{(LT)} \overline{(du_i)^2}^{(LT)} \left(\gamma^{(TT)} dU_j^{(TT)} + \gamma^{(TL)} dU_j^{(TL)} + \gamma^{(LL)} dU_j^{(LL)} \right) + \\
& \gamma^{(LL)} \overline{(du_i)^2}^{(LL)} \left(\gamma^{(TT)} dU_j^{(TT)} + \gamma^{(TL)} dU_j^{(TL)} + \gamma^{(LT)} dU_j^{(LT)} \right) + \\
& - dU_i dU_i dU_j
\end{aligned} \tag{E.6}$$

Viscous scale energy flux

$$\begin{aligned}
-2\nu \frac{\overline{\partial du_i^2}}{\partial r_j} &= -2\nu \left(\frac{\overline{\partial (du_i)^2}}{\partial r_j} - 2dU_i \frac{\partial dU_i}{\partial r_j} - 2dU_i \frac{\overline{\partial du_i}}{\partial r_j} + \frac{\partial (dU_i)^2}{\partial r_j} \right) \\
&= -2\nu \left(\gamma^{(TT)} \frac{\overline{\partial (du_i)^2}^{(TT)}}{\partial r_j} + \gamma^{(TL)} \frac{\overline{\partial (du_i)^2}^{(TL)}}{\partial r_j} + \gamma^{(LT)} \frac{\overline{\partial (du_i)^2}^{(LT)}}{\partial r_j} + \gamma^{(LL)} \frac{\overline{\partial (du_i)^2}^{(LL)}}{\partial r_j} \right. \\
&\quad \left. - 2dU_i \frac{\partial dU_i}{\partial r_j} - 2dU_i \frac{\overline{\partial du_i}}{\partial r_j} + \frac{\partial (dU_i)^2}{\partial r_j} \right) \\
&= -2\nu \left(\gamma^{(TT)} \left(\frac{\overline{\partial (du_i')^2}^{(TT)}}{\partial r_j} + 2dU_i^{(TT)} \frac{\partial dU_i^{(TT)}}{\partial r_j} + 2dU_i^{(TT)} \frac{\overline{\partial du_i}^{(TT)}}{\partial r_j} - \frac{\partial (dU_i^{(TT)})^2}{\partial r_j} \right) \right. \\
&\quad - 2\nu \left(\gamma^{(TL)} \left(\frac{\overline{\partial (du_i')^2}^{(TL)}}{\partial r_j} + 2dU_i^{(TL)} \frac{\partial dU_i^{(TL)}}{\partial r_j} + 2dU_i^{(TL)} \frac{\overline{\partial du_i}^{(TL)}}{\partial r_j} - \frac{\partial (dU_i^{(TL)})^2}{\partial r_j} \right) \right. \\
&\quad - 2\nu \left(\gamma^{(LT)} \left(\frac{\overline{\partial (du_i')^2}^{(LT)}}{\partial r_j} + 2dU_i^{(LT)} \frac{\partial dU_i^{(LT)}}{\partial r_j} + 2dU_i^{(LT)} \frac{\overline{\partial du_i}^{(LT)}}{\partial r_j} - \frac{\partial (dU_i^{(LT)})^2}{\partial r_j} \right) \right. \\
&\quad \left. \left. - 2\nu \left(\gamma^{(LL)} \left(\frac{\overline{\partial (du_i')^2}^{(LL)}}{\partial r_j} + 2dU_i^{(LL)} \frac{\partial dU_i^{(LL)}}{\partial r_j} + 2dU_i^{(LL)} \frac{\overline{\partial du_i}^{(LL)}}{\partial r_j} - \frac{\partial (dU_i^{(LL)})^2}{\partial r_j} \right) \right) \right. \right. \\
&\quad \left. \left. - 2\nu \left(-2dU_i \frac{\partial dU_i}{\partial r_j} - 2dU_i \frac{\overline{\partial du_i}}{\partial r_j} + \frac{\partial (dU_i)^2}{\partial r_j} \right) \right) \right. \\
&= -2\nu \left(\gamma^{(TT)} \frac{\overline{\partial (du_i')^2}^{(TT)}}{\partial r_j} + \gamma^{(TL)} \frac{\overline{\partial (du_i')^2}^{(TL)}}{\partial r_j} + \gamma^{(LT)} \frac{\overline{\partial (du_i')^2}^{(LT)}}{\partial r_j} + \gamma^{(LL)} \frac{\overline{\partial (du_i')^2}^{(LL)}}{\partial r_j} \right) + \phi
\end{aligned}$$

where

$$\begin{aligned}
\phi &= -2\nu \gamma^{(TT)} \left(2dU_i^{(TT)} \frac{\partial dU_i^{(TT)}}{\partial r_j} + 2dU_i^{(TT)} \frac{\overline{\partial du_i}^{(TT)}}{\partial r_j} - \frac{\partial (dU_i^{(TT)})^2}{\partial r_j} \right) \\
&\quad - 2\nu \gamma^{(TL)} \left(2dU_i^{(TL)} \frac{\partial dU_i^{(TL)}}{\partial r_j} + 2dU_i^{(TL)} \frac{\overline{\partial du_i}^{(TL)}}{\partial r_j} - \frac{\partial (dU_i^{(TL)})^2}{\partial r_j} \right) \\
&\quad - 2\nu \gamma^{(LT)} \left(2dU_i^{(LT)} \frac{\partial dU_i^{(LT)}}{\partial r_j} + 2dU_i^{(LT)} \frac{\overline{\partial du_i}^{(LT)}}{\partial r_j} - \frac{\partial (dU_i^{(LT)})^2}{\partial r_j} \right) \\
&\quad - 2\nu \gamma^{(LL)} \left(2dU_i^{(LL)} \frac{\partial dU_i^{(LL)}}{\partial r_j} + 2dU_i^{(LL)} \frac{\overline{\partial du_i}^{(LL)}}{\partial r_j} - \frac{\partial (dU_i^{(LL)})^2}{\partial r_j} \right) \\
&\quad - 2\nu \left(-2dU_i \frac{\partial dU_i}{\partial r_j} - 2dU_i \frac{\overline{\partial du_i}}{\partial r_j} + \frac{\partial (dU_i)^2}{\partial r_j} \right)
\end{aligned}$$

(E.7)

Production term in scale space

$$\begin{aligned}
2\overline{du'_i du'_j} \frac{\partial dU_i}{\partial r_j} &= 2\overline{du_i du_j} \frac{\partial dU_i}{\partial r_j} - 2dU_i dU_j \frac{\partial dU_i}{\partial r_j} \\
&= 2 \left(\gamma^{(TT)} \overline{du_i du_j}^{(TT)} + \gamma^{(TL)} \overline{du_i du_j}^{(TL)} + \gamma^{(LT)} \overline{du_i du_j}^{(LT)} + \gamma^{(TT)} \overline{du_i du_j}^{(TT)} \right) * \\
&\quad \left(\frac{\partial \gamma^{(TT)} dU_i^{(TT)} + \gamma^{(TL)} dU_i^{(TL)} + \gamma^{(LT)} dU_i^{(LT)} + \gamma^{(LL)} dU_i^{(LL)}}{\partial r_j} \right) - 2dU_i dU_j \frac{\partial dU_i}{\partial r_j} \\
&= 2 \left(\gamma^{(TT)} \overline{du_i du_j}^{(TT)} + \gamma^{(TL)} \overline{du_i du_j}^{(TL)} + \gamma^{(LT)} \overline{du_i du_j}^{(LT)} + \gamma^{(TT)} \overline{du_i du_j}^{(TT)} \right) * \\
&\quad \left(\gamma^{(TT)} \frac{\partial dU_i^{(TT)}}{\partial r_j} + dU_i^{(TT)} \frac{\partial \gamma^{(TT)}}{\partial r_j} + \gamma^{(TL)} \frac{\partial dU_i^{(TL)}}{\partial r_j} + dU_i^{(TL)} \frac{\partial \gamma^{(TL)}}{\partial r_j} + \right. \\
&\quad \left. \gamma^{(LT)} \frac{\partial dU_i^{(LT)}}{\partial r_j} + dU_i^{(LT)} \frac{\partial \gamma^{(LT)}}{\partial r_j} \right) + \gamma^{(LL)} \frac{\partial dU_i^{(LL)}}{\partial r_j} + dU_i^{(LL)} \frac{\partial \gamma^{(LL)}}{\partial r_j} \Bigg) - 2dU_i dU_j \frac{\partial dU_i}{\partial r_j} \\
&= 2(\gamma^{(TT)})^2 \overline{du_i du_j}^{(TT)} \frac{\partial dU_i^{(TT)}}{\partial r_j} \\
&\quad + 2\gamma^{(TT)} \overline{du_i du_j}^{(TT)} \left(dU_i^{(TT)} \frac{\partial \gamma^{(TT)}}{\partial r_j} + \frac{\partial (\gamma^{(TL)} dU_i^{(TL)} + \gamma^{(LT)} dU_i^{(LT)} + \gamma^{(LL)} dU_i^{(LL)})}{\partial r_j} \right) \\
&\quad + 2(\gamma^{(TL)})^2 \overline{du_i du_j}^{(TL)} \frac{\partial dU_i^{(TL)}}{\partial r_j} \\
&\quad + 2\gamma^{(TL)} \overline{du_i du_j}^{(TL)} \left(dU_i^{(TL)} \frac{\partial \gamma^{(TL)}}{\partial r_j} + \frac{\partial (\gamma^{(TT)} dU_i^{(TT)} + \gamma^{(LT)} dU_i^{(LT)} + \gamma^{(LL)} dU_i^{(LL)})}{\partial r_j} \right) \\
&\quad + 2(\gamma^{(LT)})^2 \overline{du_i du_j}^{(LT)} \frac{\partial dU_i^{(LT)}}{\partial r_j} \\
&\quad + 2\gamma^{(LT)} \overline{du_i du_j}^{(LT)} \left(dU_i^{(LT)} \frac{\partial \gamma^{(LT)}}{\partial r_j} + \frac{\partial (\gamma^{(TT)} dU_i^{(TT)} + \gamma^{(TL)} dU_i^{(TL)} + \gamma^{(LL)} dU_i^{(LL)})}{\partial r_j} \right) \\
&\quad + 2(\gamma^{(LL)})^2 \overline{du_i du_j}^{(LL)} \frac{\partial dU_i^{(LL)}}{\partial r_j} \\
&\quad + 2\gamma^{(LL)} \overline{du_i du_j}^{(LL)} \left(dU_i^{(LL)} \frac{\partial \gamma^{(LL)}}{\partial r_j} + \frac{\partial (\gamma^{(TT)} dU_i^{(TT)} + \gamma^{(TL)} dU_i^{(TL)} + \gamma^{(LT)} dU_i^{(LT)})}{\partial r_j} \right) \\
&\quad - 2dU_i dU_j \frac{\partial dU_i}{\partial r_j}
\end{aligned} \tag{E.8}$$

$$\begin{aligned}
&= 2(\gamma^{(TT)})^2 \left(\overline{du'_i du'_j}^{(TT)} \frac{\partial dU_i^{(TT)}}{\partial r_j} + dU_i^{(TT)} dU_j^{(TT)} \frac{\partial dU_i^{(TT)}}{\partial r_j} \right) \\
&+ 2\gamma^{(TT)} \overline{du_i du_j}^{(TT)} \left(dU_i^{(TT)} \frac{\partial \gamma^{(TT)}}{\partial r_j} + \frac{\partial(\gamma^{(LL)} dU_i^{(LL)} + \gamma^{(TL)} dU_i^{(TL)} + \gamma^{(LT)} dU_i^{(LT)})}{\partial r_j} \right) \\
&+ 2(\gamma^{(TL)})^2 \left(\overline{du'_i du'_j}^{(TL)} \frac{\partial dU_i^{(TL)}}{\partial r_j} + dU_i^{(TL)} dU_j^{(TL)} \frac{\partial dU_i^{(TL)}}{\partial r_j} \right) \\
&+ 2\gamma^{(TL)} \overline{du_i du_j}^{(TL)} \left(dU_i^{(TL)} \frac{\partial \gamma^{(TL)}}{\partial r_j} + \frac{\partial(\gamma^{(LL)} dU_i^{(LL)} + \gamma^{(TT)} dU_i^{(TT)} + \gamma^{(LT)} dU_i^{(LT)})}{\partial r_j} \right) \\
&+ 2(\gamma^{(LT)})^2 \left(\overline{du'_i du'_j}^{(LT)} \frac{\partial dU_i^{(LT)}}{\partial r_j} + dU_i^{(LT)} dU_j^{(LT)} \frac{\partial dU_i^{(LT)}}{\partial r_j} \right) \\
&+ 2\gamma^{(LT)} \overline{du_i du_j}^{(LT)} \left(dU_i^{(LT)} \frac{\partial \gamma^{(LT)}}{\partial r_j} + \frac{\partial(\gamma^{(LL)} dU_i^{(LL)} + \gamma^{(TT)} dU_i^{(TT)} + \gamma^{(TL)} dU_i^{(TL)})}{\partial r_j} \right) \\
&+ 2(\gamma^{(LL)})^2 \left(\overline{du'_i du'_j}^{(LL)} \frac{\partial dU_i^{(LL)}}{\partial r_j} + dU_i^{(LL)} dU_j^{(LL)} \frac{\partial dU_i^{(LL)}}{\partial r_j} \right) \\
&+ 2\gamma^{(TT)} \overline{du_i du_j}^{(TT)} \left(dU_i^{(TT)} \frac{\partial \gamma^{(TT)}}{\partial r_j} + \frac{\partial(\gamma^{(LL)} dU_i^{(LL)} + \gamma^{(TL)} dU_i^{(TL)} + \gamma^{(LT)} dU_i^{(LT)})}{\partial r_j} \right) \\
&- 2dU_i dU_j \frac{\partial dU_i}{\partial r_j} \\
&= 2(\gamma^{(TT)})^2 \overline{du'_i du'_j}^{(TT)} \frac{\partial dU_i^{(TT)}}{\partial r_j} + 2(\gamma^{(TL)})^2 \overline{du'_i du'_j}^{(TL)} \frac{\partial dU_i^{(TL)}}{\partial r_j} + \\
&\quad 2(\gamma^{(LT)})^2 \overline{du'_i du'_j}^{(LT)} \frac{\partial dU_i^{(LT)}}{\partial r_j} + 2(\gamma^{(LL)})^2 \overline{du'_i du'_j}^{(LL)} \frac{\partial dU_i^{(LL)}}{\partial r_j} + \phi
\end{aligned}$$

where

$$\begin{aligned}
\phi = & 2(\gamma^{(TT)})^2 dU_i^{(TT)} dU_j^{(TT)} \frac{\partial dU_i^{(TT)}}{\partial r_j} + 2(\gamma^{(TL)})^2 dU_i^{(TL)} dU_j^{(TL)} \frac{\partial dU_i^{(TL)}}{\partial r_j} \\
& + 2(\gamma^{(LT)})^2 dU_i^{(LT)} dU_j^{(LT)} \frac{\partial dU_i^{(LT)}}{\partial r_j} + 2(\gamma^{(LL)})^2 dU_i^{(LL)} dU_j^{(LL)} \frac{\partial dU_i^{(LL)}}{\partial r_j} + \\
& + 2\gamma^{(TT)} \overline{du_i du_j}^{(TT)} \left(dU_i^{(TT)} \frac{\partial \gamma^{(TT)}}{\partial r_j} + \left(\frac{\partial(\gamma^{(LL)} dU_i^{(LL)} + \gamma^{(TL)} dU_i^{(TL)} + \gamma^{(LT)} dU_i^{(LT)})}{\partial r_j} \right) \right) \\
& + 2\gamma^{(TL)} \overline{du_i du_j}^{(TL)} \left(dU_i^{(TL)} \frac{\partial \gamma^{(TL)}}{\partial r_j} + \left(\frac{\partial(\gamma^{(LL)} dU_i^{(LL)} + \gamma^{(TT)} dU_i^{(TT)} + \gamma^{(LT)} dU_i^{(LT)})}{\partial r_j} \right) \right) \\
& + 2\gamma^{(LT)} \overline{du_i du_j}^{(LT)} \left(dU_i^{(LT)} \frac{\partial \gamma^{(LT)}}{\partial r_j} + \left(\frac{\partial(\gamma^{(LL)} dU_i^{(LL)} + \gamma^{(TL)} dU_i^{(TL)} + \gamma^{(TT)} dU_i^{(TT)})}{\partial r_j} \right) \right) \\
& + 2\gamma^{(LL)} \overline{du_i du_j}^{(LL)} \left(dU_i^{(LL)} \frac{\partial \gamma^{(LL)}}{\partial r_j} + \left(\frac{\partial(\gamma^{(TT)} dU_i^{(TT)} + \gamma^{(TL)} dU_i^{(TL)} + \gamma^{(LT)} dU_i^{(LT)})}{\partial r_j} \right) \right) \\
& - 2dU_i dU_j \frac{\partial dU_i}{\partial r_j}
\end{aligned} \tag{E.9}$$

Dissipation in scale space

$$\begin{aligned}
4\nu \frac{\overline{\partial du'_i}}{\partial r_j} \frac{\overline{\partial du'_i}}{\partial r_j} &= 4\nu \left(\frac{\overline{\partial du_i}}{\partial r_j} \frac{\overline{\partial du_i}}{\partial r_j} - 2 \frac{\overline{\partial du_i}}{\partial r_j} \frac{\partial dU_i}{\partial r_j} + \frac{\partial dU_i}{\partial r_j} \frac{\partial dU_i}{\partial r_j} \right) \\
&= 4\nu \left(\gamma^{(TT)} \frac{\overline{\partial du_i}}{\partial r_j} \frac{\overline{\partial du_i}}{\partial r_j}^{(TT)} + \gamma^{(TL)} \frac{\overline{\partial du_i}}{\partial r_j} \frac{\overline{\partial du_i}}{\partial r_j}^{(TL)} + \gamma^{(LT)} \frac{\overline{\partial du_i}}{\partial r_j} \frac{\overline{\partial du_i}}{\partial r_j}^{(LT)} \right. \\
&\quad \left. + \gamma^{(LL)} \frac{\overline{\partial du_i}}{\partial r_j} \frac{\overline{\partial du_i}}{\partial r_j}^{(LL)} - 2 \frac{\overline{\partial du_i}}{\partial r_j} \frac{\partial dU_i}{\partial r_j} + \frac{\partial dU_i}{\partial r_j} \frac{\partial dU_i}{\partial r_j} \right) \\
&= 4\nu \gamma^{(TT)} \left(\frac{\overline{\partial du'_i}}{\partial r_j} \frac{\overline{\partial du'_i}}{\partial r_j}^{(TT)} + 2 \frac{\overline{\partial du_i}}{\partial r_j} \frac{\partial dU_i^{(TT)}}{\partial r_j} - \frac{\partial dU_i^{(TT)}}{\partial r_j} \frac{\partial dU_i^{(TT)}}{\partial r_j} \right) \\
&\quad + 4\nu \gamma^{(TL)} \left(\frac{\overline{\partial du'_i}}{\partial r_j} \frac{\overline{\partial du'_i}}{\partial r_j}^{(TL)} + 2 \frac{\overline{\partial du_i}}{\partial r_j} \frac{\partial dU_i^{(TL)}}{\partial r_j} - \frac{\partial dU_i^{(TL)}}{\partial r_j} \frac{\partial dU_i^{(TL)}}{\partial r_j} \right) \\
&\quad + 4\nu \gamma^{(LT)} \left(\frac{\overline{\partial du'_i}}{\partial r_j} \frac{\overline{\partial du'_i}}{\partial r_j}^{(LT)} + 2 \frac{\overline{\partial du_i}}{\partial r_j} \frac{\partial dU_i^{(LT)}}{\partial r_j} - \frac{\partial dU_i^{(LT)}}{\partial r_j} \frac{\partial dU_i^{(LT)}}{\partial r_j} \right) \\
&\quad + 4\nu \gamma^{(LL)} \left(\frac{\overline{\partial du'_i}}{\partial r_j} \frac{\overline{\partial du'_i}}{\partial r_j}^{(LL)} + 2 \frac{\overline{\partial du_i}}{\partial r_j} \frac{\partial dU_i^{(LL)}}{\partial r_j} - \frac{\partial dU_i^{(LL)}}{\partial r_j} \frac{\partial dU_i^{(LL)}}{\partial r_j} \right) \\
&\quad - 4\nu \left(2 \frac{\overline{\partial du_i}}{\partial r_j} \frac{\partial dU_i}{\partial r_j} + \frac{\partial dU_i}{\partial r_j} \frac{\partial dU_i}{\partial r_j} \right) \\
&= 4\nu \left(\gamma^{(TT)} \frac{\overline{\partial du'_i}}{\partial r_j} \frac{\overline{\partial du'_i}}{\partial r_j}^{(TT)} + \gamma^{(TL)} \frac{\overline{\partial du'_i}}{\partial r_j} \frac{\overline{\partial du'_i}}{\partial r_j}^{(TL)} + \gamma^{(LT)} \frac{\overline{\partial du'_i}}{\partial r_j} \frac{\overline{\partial du'_i}}{\partial r_j}^{(LT)} \right. \\
&\quad \left. + \gamma^{(LL)} \frac{\overline{\partial du'_i}}{\partial r_j} \frac{\overline{\partial du'_i}}{\partial r_j}^{(LL)} \right) + \phi
\end{aligned}
\tag{E.10}$$

where

$$\begin{aligned}
\phi = & 4\nu\gamma^{(TT)} \left(2\frac{\overline{\partial du_i}}{\partial r_j} \frac{\partial dU_i^{(TT)}}{\partial r_j} - \frac{\partial \overline{du_i}}{\partial r_j} \frac{\partial dU_i^{(TT)}}{\partial r_j} \right) \\
& + 4\nu\gamma^{(TL)} \left(2\frac{\overline{\partial du_i}}{\partial r_j} \frac{\partial dU_i^{(TL)}}{\partial r_j} - \frac{\partial \overline{du_i}}{\partial r_j} \frac{\partial dU_i^{(TL)}}{\partial r_j} \right) \\
& + 4\nu\gamma^{(LT)} \left(2\frac{\overline{\partial du_i}}{\partial r_j} \frac{\partial dU_i^{(LT)}}{\partial r_j} - \frac{\partial \overline{du_i}}{\partial r_j} \frac{\partial dU_i^{(LT)}}{\partial r_j} \right) \\
& + 4\nu\gamma^{(LL)} \left(2\frac{\overline{\partial du_i}}{\partial r_j} \frac{\partial dU_i^{(LL)}}{\partial r_j} - \frac{\partial \overline{du_i}}{\partial r_j} \frac{\partial dU_i^{(LL)}}{\partial r_j} \right) - 4\nu \left(2\frac{\overline{\partial du_i}}{\partial r_j} \frac{\partial dU_i}{\partial r_j} + \frac{\partial dU_i}{\partial r_j} \frac{\partial dU_i}{\partial r_j} \right)
\end{aligned} \tag{E.11}$$

E.3 Decomposition of terms in physical space

Non-linear physical energy flux

$$\overline{(du'_i)^2 u'_{j*}} = \gamma^{(TT)} \overline{(du'_i)^2 u'_{j*}}^{(TT)} + \gamma^{(TL)} \overline{(du'_i)^2 u'_{j*}}^{(TL)} + \gamma^{(LT)} \overline{(du'_i)^2 u'_{j*}}^{(LT)} + \gamma^{(LL)} \overline{(du'_i)^2 u'_{j*}}^{(LL)} + \phi \tag{E.12}$$

where,

$$\begin{aligned}
\phi = & \gamma^{(TT)} \left(\overline{(du_i)^2}^{(TT)} U_{j*}^{(TT)} + \overline{2du_i u_{j*}}^{(TT)} dU_i^{(TT)} - 2dU_i^{(TT)} dU_i^{(TT)} U_{j*}^{(TT)} \right) \\
& + \gamma^{(TL)} \left(\overline{(du_i)^2}^{(TL)} U_{j*}^{(TL)} + \overline{2du_i u_{j*}}^{(TL)} dU_i^{(TL)} - 2dU_i^{(TL)} dU_i^{(TL)} U_{j*}^{(TL)} \right) \\
& + \gamma^{(LT)} \left(\overline{(du_i)^2}^{(LT)} U_{j*}^{(LT)} + \overline{2du_i u_{j*}}^{(LT)} dU_i^{(LT)} - 2dU_i^{(LT)} dU_i^{(LT)} U_{j*}^{(LT)} \right) \\
& + \gamma^{(LL)} \left(\overline{(du_i)^2}^{(LL)} U_{j*}^{(LL)} + \overline{2du_i u_{j*}}^{(LL)} dU_i^{(LL)} - 2dU_i^{(LL)} dU_i^{(LL)} U_{j*}^{(LL)} \right) \\
& - \overline{(du_i)^2} U_{j*} - \overline{2du_i u_{j*}} dU_i + 2dU_i dU_i U_{j*}
\end{aligned} \tag{E.13}$$

Linear physical energy flux

$$\begin{aligned}
\overline{(du'_i)^2 U_{j*}} = & (\gamma^{(TT)})^2 \overline{(du'_i)^2}^{(TT)} U_{j*}^{(TT)} + (\gamma^{(TL)})^2 \overline{(du'_i)^2}^{(TL)} U_{j*}^{(TL)} + \\
& (\gamma^{(LT)})^2 \overline{(du'_i)^2}^{(LT)} U_{j*}^{(LT)} + (\gamma^{(LL)})^2 \overline{(du'_i)^2}^{(LL)} U_{j*}^{(LL)} + \phi
\end{aligned} \tag{E.14}$$

where

$$\begin{aligned}
\phi = & (\gamma^{(TT)})^2 dU_i^{(TT)} dU_i^{(TT)} U_{j*}^{(TT)} + (\gamma^{(TL)})^2 dU_i^{(TL)} dU_i^{(TL)} U_{j*}^{(TL)} \\
& + (\gamma^{(LT)})^2 dU_i^{(LT)} dU_i^{(LT)} U_{j*}^{(LT)} + (\gamma^{(LL)})^2 dU_i^{(LL)} dU_i^{(LL)} U_{j*}^{(LL)} + \\
& \gamma^{(TT)} \overline{(du_i)^2}^{(TT)} \left(\gamma^{(TL)} U_{j*}^{(TL)} + \gamma^{(LT)} U_{j*}^{(LT)} + \gamma^{(LL)} U_{j*}^{(LL)} \right) + \\
& \gamma^{(TL)} \overline{(du_i)^2}^{(TL)} \left(\gamma^{(TT)} U_{j*}^{(TT)} + \gamma^{(LT)} U_{j*}^{(LT)} + \gamma^{(LL)} U_{j*}^{(LL)} \right) + \\
& \gamma^{(LT)} \overline{(du_i)^2}^{(LT)} \left(\gamma^{(TT)} U_{j*}^{(TT)} + \gamma^{(TL)} U_{j*}^{(TL)} + \gamma^{(LL)} U_{j*}^{(LL)} \right) + \\
& \gamma^{(LL)} \overline{(du_i)^2}^{(LL)} \left(\gamma^{(TT)} U_{j*}^{(TT)} + \gamma^{(TL)} U_{j*}^{(TL)} + \gamma^{(LT)} U_{j*}^{(LT)} \right) + \\
& - dU_i dU_i U_{j*}
\end{aligned} \tag{E.15}$$

Viscous physical energy flux

$$\begin{aligned}
-\frac{1}{2} \mathbf{v} \frac{\partial \overline{du_i'^2}}{\partial X_j} = & -\frac{1}{2} \mathbf{v} \left(\gamma^{(TT)} \frac{\partial \overline{(du_i')^2}^{(TT)}}{\partial X_j} + \gamma^{(TL)} \frac{\partial \overline{(du_i')^2}^{(TL)}}{\partial X_j} \right. \\
& \left. + \gamma^{(LT)} \frac{\partial \overline{(du_i')^2}^{(LT)}}{\partial X_j} + \gamma^{(LL)} \frac{\partial \overline{(du_i')^2}^{(LL)}}{\partial X_j} \right) + \phi
\end{aligned}$$

where

$$\begin{aligned}
\phi = & -\frac{1}{2} \mathbf{v} \gamma^{(TT)} \left(2dU_i^{(TT)} \frac{\partial dU_i^{(TT)}}{\partial X_j} + 2dU_i^{(TT)} \frac{\partial \overline{du_i}^{(TT)}}{\partial X_j} - \frac{\partial (dU_i^{(TT)})^2}{\partial X_j} \right) \\
& -\frac{1}{2} \mathbf{v} \gamma^{(TL)} \left(2dU_i^{(TL)} \frac{\partial dU_i^{(TL)}}{\partial X_j} + 2dU_i^{(TL)} \frac{\partial \overline{du_i}^{(TL)}}{\partial X_j} - \frac{\partial (dU_i^{(TL)})^2}{\partial X_j} \right) \\
& -\frac{1}{2} \mathbf{v} \gamma^{(LT)} \left(2dU_i^{(LT)} \frac{\partial dU_i^{(LT)}}{\partial X_j} + 2dU_i^{(LT)} \frac{\partial \overline{du_i}^{(LT)}}{\partial X_j} - \frac{\partial (dU_i^{(LT)})^2}{\partial X_j} \right) \\
& -\frac{1}{2} \mathbf{v} \gamma^{(LL)} \left(2dU_i^{(LL)} \frac{\partial dU_i^{(LL)}}{\partial X_j} + 2dU_i^{(LL)} \frac{\partial \overline{du_i}^{(LL)}}{\partial X_j} - \frac{\partial (dU_i^{(LL)})^2}{\partial X_j} \right) \\
& -\frac{1}{2} \mathbf{v} \left(-2dU_i \frac{\partial dU_i}{\partial X_j} - 2dU_i \frac{\partial \overline{du_i}}{\partial X_j} + \frac{\partial (dU_i)^2}{\partial X_j} \right)
\end{aligned} \tag{E.16}$$

Pressure flux in physical space

$$2\overline{du'_i dp'} = 2 \left(\gamma^{(TT)} \overline{du'_i dp'}^{(TT)} + \gamma^{(TL)} \overline{du'_i dp'}^{(TL)} + \gamma^{(LT)} \overline{du'_i dp'}^{(LT)} + \gamma^{(LL)} \overline{du'_i dp'}^{(LL)} \right) + \phi \quad (\text{E.17})$$

where

$$\phi = 2(\gamma^{(TT)} dU_i^{(TT)} dP^{(TT)} + \gamma^{(TL)} dU_i^{(TL)} dP^{(TL)} + \gamma^{(LT)} dU_i^{(LT)} dP^{(LT)} + \gamma^{(LL)} dU_i^{(LL)} dP^{(LL)} - dU_i dP)$$

Production term in physical space

$$2\overline{du'_i u'_{*j}} \frac{\partial dU_i}{\partial X_j} = 2(\gamma^{(TT)})^2 \overline{du'_i u'_{*j}}^{(TT)} \frac{\partial dU_i^{(TT)}}{\partial X_j} + 2(\gamma^{(TL)})^2 \overline{du'_i u'_{*j}}^{(TL)} \frac{\partial dU_i^{(TL)}}{\partial X_j} + 2(\gamma^{(LT)})^2 \overline{du'_i u'_{*j}}^{(LT)} \frac{\partial dU_i^{(LT)}}{\partial X_j} + 2(\gamma^{(LL)})^2 \overline{du'_i u'_{*j}}^{(LL)} \frac{\partial dU_i^{(LL)}}{\partial X_j} + \phi \quad (\text{E.18})$$

where

$$\begin{aligned} \phi = & 2(\gamma^{(TT)})^2 dU_i^{(TT)} U_{j*}^{(TT)} \frac{\partial dU_i^{(TT)}}{\partial X_j} + 2(\gamma^{(TL)})^2 dU_i^{(TL)} U_{j*}^{(TL)} \frac{\partial dU_i^{(TL)}}{\partial X_j} \\ & + 2(\gamma^{(LT)})^2 dU_i^{(LT)} U_{j*}^{(LT)} \frac{\partial dU_i^{(LT)}}{\partial X_j} + 2(\gamma^{(LL)})^2 dU_i^{(LL)} U_{j*}^{(LL)} \frac{\partial dU_i^{(LL)}}{\partial X_j} + \\ & + 2\gamma^{(TT)} \overline{du_i u_{*j}}^{(TT)} \left(dU_i^{(TT)} \frac{\partial \gamma^{(TT)}}{\partial X_j} + \left(\frac{\partial(\gamma^{(LL)} dU_i^{(LL)} + \gamma^{(TL)} dU_i^{(TL)} + \gamma^{(LT)} dU_i^{(LT)})}{\partial X_j} \right) \right) \\ & + 2\gamma^{(TL)} \overline{du_i u_{*j}}^{(TL)} \left(dU_i^{(TL)} \frac{\partial \gamma^{(TL)}}{\partial X_j} + \left(\frac{\partial(\gamma^{(LL)} dU_i^{(LL)} + \gamma^{(TT)} dU_i^{(TT)} + \gamma^{(LT)} dU_i^{(LT)})}{\partial X_j} \right) \right) \\ & + 2\gamma^{(LT)} \overline{du_i u_{*j}}^{(LT)} \left(dU_i^{(LT)} \frac{\partial \gamma^{(LT)}}{\partial X_j} + \left(\frac{\partial(\gamma^{(LL)} dU_i^{(LL)} + \gamma^{(TL)} dU_i^{(TL)} + \gamma^{(TT)} dU_i^{(TT)})}{\partial X_j} \right) \right) \\ & + 2\gamma^{(LL)} \overline{du_i u_{*j}}^{(LL)} \left(dU_i^{(LL)} \frac{\partial \gamma^{(LL)}}{\partial X_j} + \left(\frac{\partial(\gamma^{(TT)} dU_i^{(TT)} + \gamma^{(TL)} dU_i^{(TL)} + \gamma^{(LT)} dU_i^{(LT)})}{\partial X_j} \right) \right) \\ & - 2dU_i U_{*j} \frac{\partial dU_i}{\partial X_j} \end{aligned} \quad (\text{E.19})$$

Dissipation in scale space

$$\begin{aligned} \nu \frac{\overline{\partial du'_i}}{\partial X_j} \frac{\overline{\partial du'_i}}{\partial X_j} = \nu \left(\gamma^{(TT)} \frac{\overline{\partial du'_i}}{\partial X_j} \frac{\overline{\partial du'_i}}{\partial X_j}^{(TT)} + \gamma^{(TL)} \frac{\overline{\partial du'_i}}{\partial X_j} \frac{\overline{\partial du'_i}}{\partial X_j}^{(TL)} + \gamma^{(LT)} \frac{\overline{\partial du'_i}}{\partial X_j} \frac{\overline{\partial du'_i}}{\partial X_j}^{(LT)} \right. \\ \left. + \gamma^{(LL)} \frac{\overline{\partial du'_i}}{\partial X_j} \frac{\overline{\partial du'_i}}{\partial X_j}^{(LL)} \right) + \phi \end{aligned} \quad (\text{E.20})$$

where

$$\begin{aligned} \phi = \nu \gamma^{(TT)} \left(2 \frac{\overline{\partial du_i}^{(TT)}}{\partial X_j} \frac{\partial dU_i^{(TT)}}{\partial X_j} - \frac{\partial dU_i^{(TT)}}{\partial X_j} \frac{\partial dU_i^{(TT)}}{\partial X_j} \right) \\ + \nu \gamma^{(TL)} \left(2 \frac{\overline{\partial du_i}^{(TL)}}{\partial X_j} \frac{\partial dU_i^{(TL)}}{\partial X_j} - \frac{\partial dU_i^{(TL)}}{\partial X_j} \frac{\partial dU_i^{(TL)}}{\partial X_j} \right) \\ + \nu \gamma^{(LT)} \left(2 \frac{\overline{\partial du_i}^{(LT)}}{\partial X_j} \frac{\partial dU_i^{(LT)}}{\partial X_j} - \frac{\partial dU_i^{(LT)}}{\partial X_j} \frac{\partial dU_i^{(LT)}}{\partial X_j} \right) \\ + \nu \gamma^{(LL)} \left(2 \frac{\overline{\partial du_i}^{(LL)}}{\partial X_j} \frac{\partial dU_i^{(LL)}}{\partial X_j} - \frac{\partial dU_i^{(LL)}}{\partial X_j} \frac{\partial dU_i^{(LL)}}{\partial X_j} \right) - \nu \left(2 \frac{\overline{\partial du_i}}{\partial X_j} \frac{\partial dU_i}{\partial X_j} + \frac{\partial dU_i}{\partial X_j} \frac{\partial dU_i}{\partial X_j} \right) \end{aligned} \quad (\text{E.21})$$

Appendix F

Calculation of derivatives of two-point quantities in scale space

In this Appendix we provide the steps for the numerical evaluation of derivatives of two-point directly in scale space.

F.1 Calculation of $\frac{\partial dU_i^{(AA)}}{\partial r_3}$

1. From the stored database, extract $U_i^{(AA)}$ at the desired X_0 location, at a fixed height Y_0 , along the Z direction.
2. Build the $dU_i^{(AA)}$ data structure in r_3 space from the discrete data in Z space; a schematic is shown in Figure F.1.
3. Calculate the gradient of $dU_i^{(AA)}$ with respect to r_3 using second-order central differences.
4. Repeat steps (i)-(iii) for different heights Y_0 .

This method is validated by comparing against the gradient evaluated at points x_i^+ and x_i^- , for example for the standard-averaged streamwise velocity dU_i this is,

$$\frac{\partial dU_i}{\partial r_3} = \frac{1}{2} \left(\frac{\partial(U_i^+ - U_i^-)}{\partial x_3^+} - \frac{\partial(U_i^+ - U_i^-)}{\partial x_3^-} \right) = \frac{1}{2} \left(\frac{\partial U_i^+}{\partial x_3^+} + \frac{\partial U_i^-}{\partial x_3^-} \right). \quad (\text{F.1})$$

Both approaches give identical results, refer to figure F.2 that shows contours of $\frac{\partial dU_1}{\partial r_3}$ in the (r_3, Y) plane. Note that the values are very small because in order to test our method, we averaged U_1 in time only.

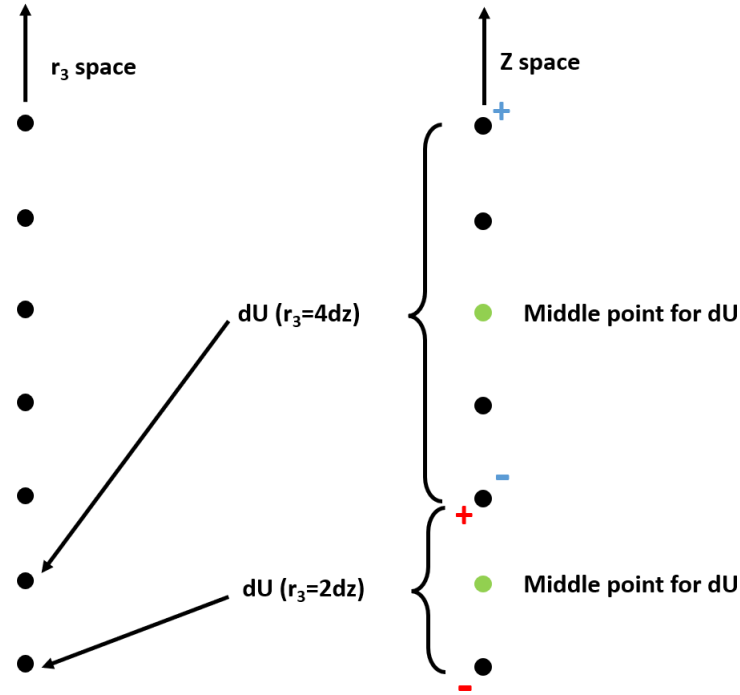


Figure F.1 Building of data structure in r_3 space from the data in Z space. The red and blue points marked with + and - represent two pairs of points with separation $r_3 = 2\Delta z$ and $r_3 = 4\Delta z$ respectively in Z space. The green circles are the middle points of the corresponding pairs.

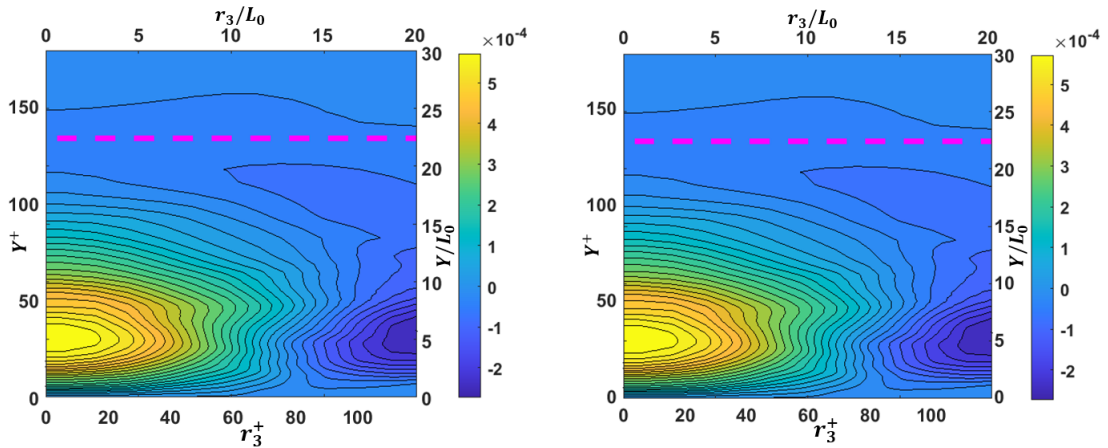


Figure F.2 Contours of $\frac{\partial dU_1}{\partial r_3}$ in the (r_3, Y) plane. Evaluation directly in scale space (left) and from equation (F.1) (right).

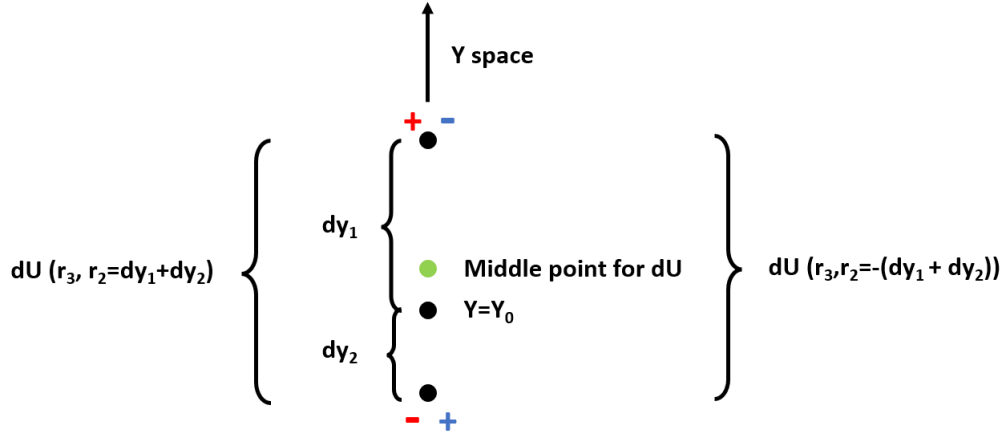


Figure F.3 Sketch for the calculation of $\left. \frac{\partial dU_i^{(AA)}}{\partial r_2} \right|_{r_2=0}$.

F.2 Calculation of $\left. \frac{\partial dU_i^{(AA)}}{\partial r_2} \right|_{r_2=0}$

1. Consider a cell with centroid at distance $Y = Y_0$ from the wall. The distances of the centroids of the cells located above and below are dy_1 and dy_2 respectively, refer to figure F.3. Recall that the grid is non-uniform in the wall-normal direction, so $dy_1 \neq dy_2$. Calculate $dU_i^{(AA)}(r_2 = dy_1 + dy_2) = U_i^{(AA)}(Y_0 + dy_1) - U_i^{(AA)}(Y_0 - dy_2)$ i.e. the velocity difference between the red $+/-$ markers in figure F.3.
2. Calculate $dU_i^{(AA)}(r_2 = -(dy_1 + dy_2)) = -dU_i^{(AA)}(r_2 = dy_1 + dy_2)$ i.e. the velocity difference between the blue $+/-$ markers.
3. Use central difference scheme to compute

$$\left. \frac{\partial dU_i^{(AA)}}{\partial r_2} \right|_{r_2=0} = \frac{dU_i^{(AA)}(r_2 = dy_1 + dy_2) - dU_i^{(AA)}(r_2 = -(dy_1 + dy_2))}{2(dy_1 + dy_2)} \quad (\text{F.2})$$

and store the value at the midpoint (denoted with a green dot in figure F.3).

4. Repeat steps (i)-(iii) for cell centroids at different heights Y_0 .
5. Because the mesh is non-uniform, the middle point is not located at Y_0 , so interpolate values of $\left. \frac{\partial dU_i^{(AA)}}{\partial r_2} \right|_{r_2=0}$ at midpoints to obtain value at Y_0 .

Comparison with evaluation at points x_i^+ and x_i^- for the standard-averaged streamwise velocity shows that the results are identical, refer to figure F.4.

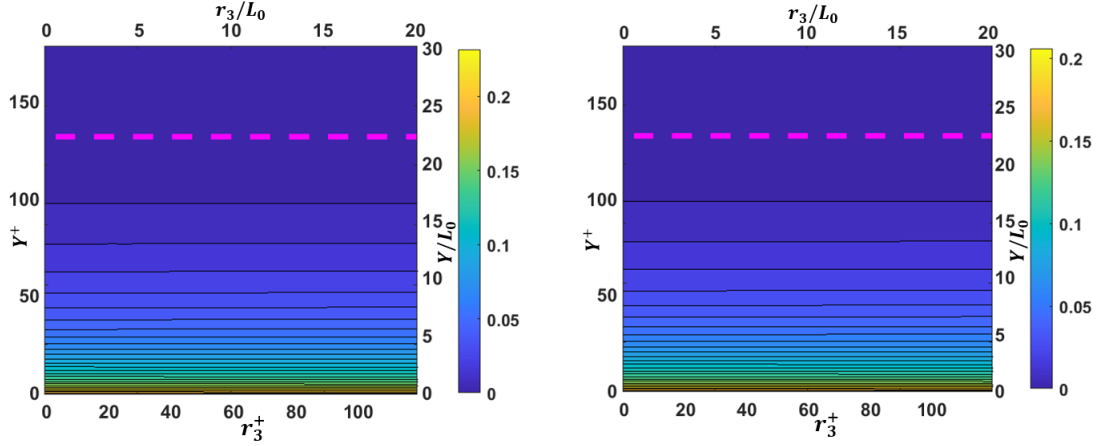


Figure F.4 Contour plot of $\left. \frac{\partial dU_1}{\partial r_2} \right|_{r_2=0}$ in the (r_3, Y) plane. Evaluation directly in scale space (left) and from $\frac{1}{2} \left(\frac{\partial U_1^+}{\partial x_2^-} + \frac{\partial U_1^-}{\partial x_2^-} \right)$ (right)

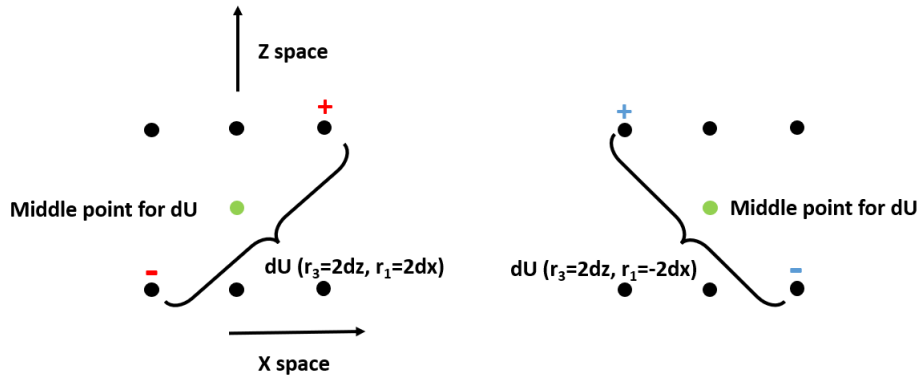


Figure F.5 Sketch for the calculation of $\left. \frac{\partial dU_i^{(AA)}}{\partial r_1} \right|_{r_1=0}$.

F.3 Calculation of $\left. \frac{\partial dU_i^{(AA)}}{\partial r_1} \right|_{r_1=0}$

1. Calculate $dU_i^{(AA)}(r_3, r_1 = 2\Delta x)$, i.e. the velocity difference between the red $+/-$ points in figure F.5, at a fixed height $y = Y_0$.
2. Calculate $dU_i^{(AA)}(r_3, r_1 = -2\Delta x)$, i.e. the velocity difference between the blue $+/-$ points in figure F.5, at a fixed height $y = Y_0$.

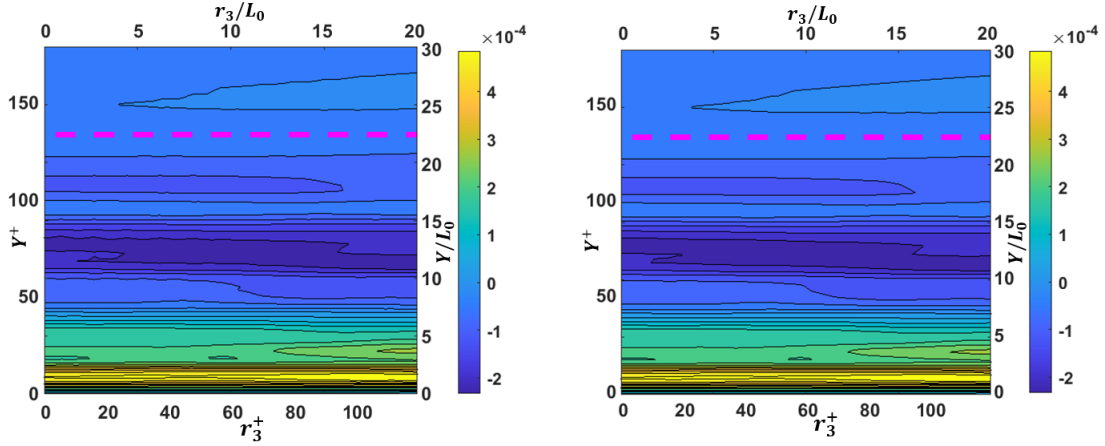


Figure F.6 Contour plot of $\left. \frac{\partial dU_1}{\partial r_1} \right|_{r_1=0}$ in the (r_3, Y) plane. Evaluation directly in scale space (left) and from $\frac{1}{2} \left(\frac{\partial U_1^+}{\partial x_1^+} + \frac{\partial U_1^-}{\partial x_1^-} \right)$ (right).

3. Use central difference scheme to compute

$$\left. \frac{\partial dU_i^{(AA)}}{\partial r_1} \right|_{r_1=0} = \frac{dU_i^{(AA)}(r_3, r_1 = 2\Delta x) - dU_i^{(AA)}(r_3, r_1 = -2\Delta x)}{2(2\Delta x)} \quad (\text{F.3})$$

and store the value at the middle point (marked with a green dot in figure F.5).

4. Repeat steps (i)-(iii) at different heights.

Schematic F.5 shows the four points involved for $\left. \frac{\partial dU_i^{(AA)}}{\partial r_1} \right|_{r_1=0}$. The process is similar for the evaluation of $\frac{\partial dU_i^{(AA)}}{\partial r_1}$ for different r_1 values.

Comparison with evaluation at points x_i^+ and x_i^- for the standard-averaged streamwise velocity shows identical results, see figure F.6.