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**Hypergeometric functions and
new mirrors of Fano varieties**

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Declaration of Originality

I, Giulia Gugiatti, confirm that the work presented in this thesis is my own and that any collaboration has been explicitly acknowledged. Where information has been derived from the work of others, this has been indicated in the text.

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Abstract

In this thesis we outline a strategy to write the hypergeometric function of a Fano weighted complete intersection $X = X_{d_1, \dots, d_c} \subset \mathbb{P}(a_1, \dots, a_m)$ of dimension n as a period of a pencil of $(n - 1)$ -dimensional varieties. Conjecturally this function is a specialisation of the quantum period of X , thus the ideas in this thesis can be viewed as the basis of a new method to find Landau–Ginzburg mirrors for all Fano weighted complete intersections. Our strategy builds upon the work of Beukers, Cohen, and Mellit on finite hypergeometric functions, suggesting that the hypergeometric variation attached to X is linked to the variation of intermediate cohomology of a pencil of $(n - 1 + 2c)$ -dimensional varieties. This pencil can be viewed as a mirror of X of the wrong dimension.

We focus on anticanonical del Pezzo weighted complete intersections. The feature which makes these surfaces especially interesting is that, if the ambient space has all weights ≥ 2 , they have empty anticanonical linear system, and therefore fall out of the range of the known mirror constructions.

For each positive integer k we exhibit the hypergeometric function of the family of surfaces $X_{8k+4} \subset \mathbb{P}(2, 2k + 1, 2k + 1, 4k + 1)$ as a period of a pencil of curves of genus $3k + 1$. This series is the main piece of the Johnson–Kollár classification of anticanonical del Pezzo weighted hypersurfaces. For all k our pencil is a mirror of the family. We build a candidate mirror for the family of surfaces $X_{15} \subset \mathbb{P}(3, 3, 5, 5)$ by identifying the variation of H^3 of its wrong dimension mirror with a quotient of the variation of H^1 of a pencil of genus-six curves. We also provide evidence that our constructions are compatible with the work of Hori–Vafa.

We conclude by discussing our ideas to prove the conjectural Hodge–theoretic translation of the work of Beukers, Cohen, and Mellit.

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Chapter 1

Introduction

In this Chapter we give an overview of the type of problems considered in this thesis and a preliminary description of the main results obtained. We postpone until later chapters the precise statement of these results and a detailed literature review.

Motivation and Objectives

In this thesis we propose a method to approach the following:

Question 1 Let $X = X_{d_1, \dots, d_c} \subset \mathbb{P}(a_1, \dots, a_m)$ be a Fano quasismooth and wellformed weighted complete intersection of dimension n . Let $\lambda = \sum a_i - \sum d_i$. Consider the hypergeometric function:

$$\tilde{I}_X(\alpha) = \sum_{j=0}^{\infty} \frac{(d_1 j)! \dots (d_c j)! (\lambda j)!}{(a_1 j)! \dots (a_m j)!} \alpha^j \quad (1.1)$$

Is $\tilde{I}_X$ a period of a pencil $w: Y \rightarrow \mathbb{C}^\times$ of $(n-1)$ -dimensional varieties?

Our motivation to study this question is twofold.

On the one hand, the general shape of the Fano/Landau–Ginzburg (LG) correspondence suggests an identification of the regularised quantum period of an n -dimensional Fano variety X with a period of an n -dimensional LG model. This for us is simply a pair (Y, w) , where Y is a complex manifold of dimension n and $w: Y \rightarrow \mathbb{C}^\times$ is a complex function. Upon natural conjectural generalisations of the Quantum Lefschetz Principle to orbifold target spaces, for a Fano complete intersection X in a weighted projective space (a specialisation of) the regularised quantum period of X is given by the hypergeometric function $\tilde{I}_X$ in Equation (1.1). Thus our Question 1 is equivalent to the following:

Question 2 Does X admit a LG mirror (Y, w) ?

The existence of a LG mirror of X may be concretely understood whenever X admits a toric degeneration – which conjecturally is equivalent to assuming that X has non-empty anticanonical linear system – but otherwise is not known a priori. This motivates our Question 1 from the point of view of *mirror symmetry*, and the ideas we present in this thesis can be seen as the basis of a new method to find LG mirrors *for all Fano weighted complete intersections*.

On the other hand, the work of Katz [Kat90] shows that any irreducible hypergeometric local system supports a canonical variation of Hodge structure (VHS) of geometric origin; in other words, such local systems arise as the variation of the (co)homology groups of the fibres of a morphism. Given that the local system $\mathbb{H}_X$ of solutions to the irreducible differential equation annihilating $\tilde{I}_X$ has supposedly Hodge weight $n - 1$, it is natural to ask – even as there is a priori no reason to expect it – the following:

Question 3 Is $\mathbb{H}_X$ a subquotient of the variation of intermediate homology of a pencil of $(n - 1)$ -dimensional varieties?

This motivates our Question 1 – which really is just a more precise way of rephrasing this – from the point of view of *hypergeometric motives*.

Our approach

The source of ideas for our investigations is the arithmetic of hypergeometric functions. The work [BCM15] by Beukers, Cohen, and Mellit shows that finite hypergeometric functions correspond to point counts in positive characteristic on explicit pencils of algebraic varieties. This strongly suggests¹ that the corresponding analytic hypergeometric functions are periods of the same pencils now regarded over $\mathbb{C}$.

Specialising this to the hypergeometric function $\tilde{I}_X$ (1.1) of a Fano weighted complete intersection X of dimension n , one expects an identity of VHS

$$\mathbb{H}_X = \mathrm{gr}_{d-1}^W R^{d-1} v_! \mathbb{Q}(c) \quad (1.2)$$

away from the critical locus of v , where $v: W \rightarrow \mathbb{C}^\times$ is an explicit map on a manifold W of dimension $d = n + 2c$ depending only on $d_1, \dots, d_c$ and $a_1, \dots, a_m$. The LG model

¹Chapter 7 contains the seeds of a proof, in full generality, of our Hodge–theoretic interpretation of [BCM15]. The results in Chapter 5 and 6 provide strong evidence for this interpretation.

(W, v) can be interpreted as a mirror of X *of the wrong dimension*. Then it makes sense to ask if there is a morphism $w: Y \rightarrow \mathbb{C}^\times$, where Y is a n -dimensional manifold, such that $\mathrm{gr}_{d-1}^W R^{d-1} v_! \mathbb{C}$ identifies with a subquotient of $\mathrm{gr}_{n-1}^W R^{n-1} w_! \mathbb{C}$ away from the critical locus of v . The LG model (Y, w) would be a mirror *of the right dimension*.

In Chapter 3 we outline a strategy, divided into three main steps, to study the Hodge structure

$$\mathrm{gr}_{d-1}^W H^{d-1}(W_\alpha, \mathbb{Q})$$

where $W_\alpha = v^{-1}(\alpha)$ is the general fibre of v . The second step of this strategy builds upon a combinatorial iterative algorithm, based on the methods of the toric minimal model program (MMP), to find a simple compactification of a hypersurface in a torus. Our algorithm constructs a birational model for the natural compactification of the hypersurface in the toric variety associated to the Newton polytope of its equation by means of resolving and ‘shrinking’ the polytope. We present the algorithm, and some examples of its application in practice, in Chapter 4.

It would be interesting to show, in substantial generality, that the method proposed in this thesis extends the Hori–Vafa [HV00] construction. The results we present in Section 6.1 provide evidence for this.

Main results

We focus on hypergeometric variations of Hodge weight one and, in particular, on the hypergeometric functions $\tilde{I}_X$ of del Pezzo weighted complete intersections X which are anticanonical, that is $-K_X = \mathcal{O}_X(1)$. Anticanonical weighted complete intersections of arbitrary dimension living in projective spaces with all weights ≥ 2 are especially interesting in mirror symmetry since they have empty anticanonical linear system, and therefore fall out of the range of any known mirror construction.

Below we sketch the main results discussed in this thesis.

The work [CG19] with A. Corti, presented in Chapter 5, gives an affirmative answer to Question 1 for the series of surfaces

$$X_{8k+4} \subset \mathbb{P}(2, 2k+1, 2k+1, 4k+1) \tag{1.3}$$

where k is a positive integer. This series forms the main piece of the classification of anticanonical del Pezzo weighted hypersurfaces by Johnson–Kollár [JK01].

We prove:

Theorem A (Theorem 5.3). For all $k > 0$ integer, the hypergeometric function of the surface X_{8k+4} (1.3) equals a period of the pencil $w_k: Y_k \rightarrow \mathbb{C}^\times$ given by the surface

$$Y_k = (\alpha y^2 - h_k(\alpha, t_0, t_1) = 0) \subset \mathbb{P}(1, 1, 3k+2) \times \mathbb{C}^\times \quad (1.4)$$

where t_0, t_1, y are coordinates on $\mathbb{P}(1, 1, 3k+2)$, α is a coordinate on $\mathbb{C}^\times$ and

$$h_k(\alpha, t_0, t_1) = (4t_1^{2k+1} + \alpha t_0^{2k+1})(-64t_1^{4k+3} + t_0 t_1^{4k+2} - 32\alpha t_0^{2k+1} t_1^{2k+2} - 4\alpha^2 t_0^{4k+2} t_1)$$

together with the projection w_k on the second factor.

The curves $Y_{k,\alpha} = w_k^{-1}(\alpha)$ are hyperelliptic curves of genus $3k+1$. Since the local system $\mathbb{H}_k$ of solutions to the irreducible ODE annihilating the hypergeometric function of X_{8k+4} (1.3) has rank $6k+2$, an immediate consequence of Theorem A is that $\mathbb{H}_k = R^1 w_{k!} \mathbb{C}$ away from the critical locus of w_k .

For all $k > 0$, our LG model (Y_k, w_k) is a mirror for the surface $X = X_{8k+4}$. What makes Theorem A especially interesting is that, since the anticanonical linear system of the surfaces (1.3) is empty, these are entirely new mirrors.

Proving Theorem A is elementary: we expand the period of the LG model (Y_k, w_k) in power series with the help of the residue theorem. The main difficulty is to find the equation of the surface Y_k (1.4) (and the integration cycles on the curves $Y_{k,\alpha}$). We extract the equation of Y_k from the 4-dimensional LG model (W_k, v_k) given by [BCM15]:

Theorem B (Theorem 5.10). For all $k > 0$ integer, there is an identity:

$$\mathrm{gr}_3^W R^3 v_{k!} \mathbb{Q}(1) = R^1 w_{k*} \mathbb{Q}$$

away from the critical locus of v_k and w_k .

Our constructions in Section 5.2.2 show how the curve $Y_{k,\alpha}$ arises naturally from the geometry of the 3-fold $W_{k,\alpha} = v_k^{-1}(\alpha)$. Our MMP algorithm constructs a partial compactification $W_{k,\alpha} \subset \widehat{W}_{k,\alpha}$ supporting a del Pezzo fibration $\phi_{k,\alpha}: \widehat{W}_{k,\alpha} \rightarrow \mathbb{C}^\times$. Once we have the map $\phi_{k,\alpha}$, the proof of Theorem B is based on mixed Hodge theory. The key point is to notice that the quasismooth fibres of $\phi_{k,\alpha}$ are del Pezzo surfaces of Picard rank 2, and the curve $Y_{k,\alpha}$ is a 2:1 branched cover of $\mathbb{P}^1$ which singles out two distinguished generators of the Picard group.

The fact that the variation $\mathrm{gr}_3^W H^3(W_{k,\alpha}, \mathbb{Q})$ is the variation of H^1 of a pencil of curves is not trivial. There is no reason to expect it. Looking for an explanation, one naturally wonders whether the 3-folds $W_{k,\alpha}$ are rational: there are indeed very few examples of irrational 3-folds with this property. Our computations in Section 5.2.3 show that this is probably not the case.

In Chapter 6 we present the results of our investigations on some of the sporadic cases of the Johnson–Kollár list [JK01].

The first three sporadic cases of the list are the surfaces:

$$X_3 \subset \mathbb{P}^3 \quad X_4 \subset \mathbb{P}(1, 1, 1, 2) \quad X_6 \subset \mathbb{P}(1, 1, 2, 3) \quad (1.5)$$

These are the only smooth anticanonical del Pezzo weighted hypersurfaces.

For each of the surfaces (1.5), we show that the mirror which our method extracts out of the 4-dimensional mirror (W, v) coincides with the Hori–Vafa mirror (Y, w) . In particular we prove that:

Theorem C (Theorem 6.2). The 3-fold $W_\alpha = v^{-1}(\alpha)$ compactifies to the blow-up of $\mathbb{P}^3$ along the closure $\overline{Y}_\alpha \subset \mathbb{P}^2$ of the curve $Y_\alpha = w^{-1}(\alpha)$.

Theorem C immediately implies that $\mathrm{gr}_3^W H_c^3(W_\alpha, \mathbb{Q})(1) = \mathrm{gr}_1^W H_c^1(Y_\alpha, \mathbb{Q})$. In particular, in this case, the fibres W_α of the LG model (W, v) are rational 3-folds.

The second part of Chapter 6 is devoted to the study of the surface $X_{15} \subset \mathbb{P}(3, 3, 5, 5)$. This surface is the first case of [JK01] living in a projective space with all weights ≥ 3 . In Section 5.3 we propose a conjectural explanation for the existence of mirrors of anticanonical surfaces $X_d \subset \mathbb{P}(2, a_2, a_3, a_4)$ inspired by the works [Bor97, CKV20, Dol96, Voi93] on mirror Calabi–Yau pairs with dual involutions. The explanation is not applicable to anticanonical surfaces in projective spaces with weights ≥ 3 , and this makes the case of the surface $X = X_{15}$ particularly interesting.

Let (W, v) be the 4-dimensional LG model given by [BCM15]. We prove that:

Theorem D (Theorem 6.10). Away from the critical locus of the map v , $\mathrm{gr}_3^W R^3 v_! \mathbb{Q}(1)$ identifies with an explicit quotient of $\mathrm{gr}_1^W R^1 w_! \mathbb{Q}$, where $w: Y \rightarrow \mathbb{C}^\times$ is the pencil given by the surface

$$Y = (x^3 - t = -27ty^3 + 27\alpha t^2 y + \alpha(27 - t) = 0) \subset \mathbb{A}^3 \times \mathbb{C}^\times \quad (1.6)$$

where t, x, y are coordinates on $\mathbb{A}^3$ and α is a coordinate on $\mathbb{C}^\times$, together with the second projection w .

By identity (1.2), the LG model (Y, w) can be viewed as a candidate LG mirror of the surface X . The fibres $Y_\alpha = w^{-1}(\alpha)$ compactify to curves of genus 6.

Once again, the curves Y_α arise naturally from the 3-folds $W_\alpha = v^{-1}(\alpha)$. We construct a partial compactification $W_\alpha \subset \widehat{W}_\alpha$ which is a pencil $\phi_\alpha: \widehat{W}_\alpha \rightarrow \mathbb{C}^\times$ of cubic surfaces. Once we have the map ϕ_α , we use mixed Hodge theory to prove Theorem D. The key point is to notice that the 27 lines on the smooth fibres of ϕ contain a distinguished subset of 9 lines. The curve Y_α is the 9:1 branched cover of $\mathbb{C}^\times$ parametrizing these lines, and, in fact, $H^1(\mathbb{C}^\times, R^2\phi_{\alpha*}\mathbb{Q}) = H^1(\mathbb{C}^\times, \mathbb{V})$, where $\mathbb{V} \subset R^2\phi_{\alpha*}\mathbb{Q}$ is the sub-local system generated by the 9 lines.

Future directions

F. Rodriguez Villegas has proposed a generalisation of the work [BCM15] to finite multivariate hypergeometric functions. It would be extremely attractive to use the ideas in his work to eliminate the Hori–Vafa assumptions and find mirrors for all toric complete intersections.

It would be interesting to see that our new LG mirrors – as well as the wrong dimension mirrors (W, v) – satisfy the Homological Mirror Symmetry Conjecture.

The work [Wel87], combined with [BCM15], suggests that for all hypergeometric variations of Hodge weight one the answer to Question 3 is positive. It would be interesting to classify the hypergeometric local systems of Hodge weight one which are exactly the variation of H^1 of a pencil of curves. Sections 3.4 and 4.4 contain a preliminary discussion of this problem.

Outline

The main body of the thesis is structured into six Chapters. Naturally, the thesis partially coincides in content and in writing with the paper [CG19].

In Chapter 2 we provide the background material on hypergeometries and mirror symmetry required for the thesis. In Chapter 3 we summarize the work [BCM15] and we expand on our strategy to approach Question 1. We focus our attention on hypergeometric variations of Hodge weight one and, in particular, on anticanonical del Pezzo weighted complete intersections. In Chapter 4 we describe our MMP algorithm for hypersurfaces in tori. We compute the output of the algorithm in three examples. In Chapter 5 we give the

details of our work on the Johnson–Kollár series and we conjecture an explanation for the existence of mirrors of anticanonical surfaces $X_d \subset \mathbb{P}(2, a_2, a_3, a_4)$. Chapter 6 contains the exact statements and the proofs of Theorems C and D. In Chapter 7 we discuss our ideas to prove the conjectural Hodge-theoretic translation of [BCM15]. Appendix A contains the two lists [JK01] and [KP15] of anticanonical del Pezzo weighted surfaces.

Chapter 2

Hypergeometrics and LG mirrors

In this Chapter we provide the background material on the theory of hypergeometrics and on mirror symmetry for Fano varieties required for this thesis. The link between the two theories is that hypergeometric functions arise as generating functions for genus-zero Gromov–Witten invariants of certain, very special Fano varieties.

2.1 The notion of period

In this Section we state what we mean by periods of a morphism.

Definition 2.1. Let Y be a smooth algebraic manifold of dimension n and let $w: Y \rightarrow \mathbb{C}^\times$ be a quasi-projective morphism. For $\alpha \in \mathbb{C}^\times$ denote by $Y_\alpha = w^{-1}(\alpha)$ the fibre. Denote by $U \subset \mathbb{C}^\times$ the set of regular values of w , and by $w_U: Y_U = w^{-1}(U) \rightarrow U$ the restriction. Choose a pair (ω, γ) where:

- (i) $\omega \in \Gamma(U, w_U^* \Omega_{Y_U/U}^{n-1})$. For $\alpha \in U$, denote by $\omega_\alpha \in H^0(Y_\alpha, \Omega_{Y_\alpha}^{n-1})$ the corresponding form;
- (ii) $\gamma \in \Gamma(\Delta^\times, R^{n-1}w_U! \mathbb{Q}_{Y_U})$ where $0 \in \Delta \subset \mathbb{C}$ is a small disk and $\Delta^\times = \Delta \setminus \{0\}$. For $\alpha \in \Delta^\times$ denote by $\gamma_\alpha \in H_{n-1}(Y_\alpha, \mathbb{Q})$ the corresponding cycle.

The *period* of $w: Y \rightarrow \mathbb{C}^\times$ relative to (ω, γ) is the function

$$\pi(\alpha) = \int_{\gamma_\alpha} \omega_\alpha \quad (\alpha \in \Delta^\times) \quad (2.1)$$

In what follows we denote by $\mathcal{D}$ the sheaf of differential operators on $\mathbb{C}^\times$. The ring $\mathcal{D}(\mathbb{C}^\times)$ is $\mathbb{C}[\alpha^{\pm 1}, D]$ where $D = \alpha d/d\alpha$. We generally write elements of $\mathcal{D}$ in the form $\sum \alpha^j P_j(D)$, where the $P_j(\alpha)$ are polynomials in $\mathbb{C}[\alpha]$.

Remark 2.2. Any period π of $w: Y \rightarrow \mathbb{C}^\times$ satisfies polynomial ordinary differential equations, and the local system of solutions to the annihilator ideal of π is a subquotient of $\mathrm{gr}_{n-1}^W R^{n-1} w_U! \mathbb{C}$.

To see this, write

$$\mathcal{E} = (R^{n-1} w_U \star \mathbb{Q}) \otimes \mathcal{O}_U$$

The sheaf $\mathcal{E}$ carries an integrable algebraic connection $\nabla: \mathcal{E} \rightarrow \Omega_U^1 \otimes \mathcal{E}$ (*the Gauss–Manin connection*) by means of which we can regard it as a $\mathcal{D}_U$ -module, and the local systems

$$R^{n-1} w_U \star \mathbb{C} = \underline{\mathrm{Hom}}_{\mathcal{D}_U}(\mathcal{O}_U, \mathcal{E}), \quad \text{and} \quad R^{n-1} w_U! \mathbb{C} = \underline{\mathrm{Hom}}_{\mathcal{D}_U}(\mathcal{E}, \mathcal{O}_U)$$

are the local systems of flat sections and of solutions of $\mathcal{E}$. Indeed, if η is a section of $\mathcal{E}$, and $\eta = \sum \sigma_i \eta^i$ in a basis σ_i of a local trivialisation, the action of $D = \alpha \frac{d}{d\alpha}$ on η is given by

$$D \cdot \eta = \nabla_{\alpha \frac{d}{d\alpha}} \eta$$

where by definition $\nabla \eta = \sum \sigma_i \otimes d\eta^i$; thus any section η satisfies linear relations of the form:

$$\sum_{k=0}^h p_k(\alpha) D^k \eta = 0 \quad p_k(\alpha) \in \mathbb{C}[\alpha] \quad (2.2)$$

(In particular, for any η there is a unique – up to multiplication by a constant – minimal relation of the form (2.2), that is, where $h \leq \dim_{\mathbb{C}} H^{n-1}(Y_\alpha, \mathbb{C})$ is as small as possible, and $\deg p_h$ is as small as possible).

Now $w_U \star \Omega_{Y_U/U}^{n-1} \subset \mathcal{E}$ as the last piece of the Hodge filtration and we regard ω as a section of $\mathcal{E}$ by means of this inclusion. On the other hand we regard γ as a solution of $\mathcal{E}$ and we recover the period as $\pi = \gamma(\omega)$. We have an inclusion and a surjection of $\mathcal{D}_U$ -modules:

$$\mathcal{D}_U \cdot \omega \subset \mathcal{E}, \quad \text{and} \quad \gamma: \mathcal{D}_U \cdot \omega \rightarrow \mathcal{D}_U \cdot \pi$$

So we have $\underline{\mathrm{Hom}}_{\mathcal{D}_U}(\mathcal{D}_U \cdot \pi, \mathcal{O}_U) \subset \underline{\mathrm{Hom}}_{\mathcal{D}_U}(\mathcal{D}_U \cdot \omega, \mathcal{O}_U)$ and $R^{n-1} w_U! \mathbb{C} = \underline{\mathrm{Hom}}_{\mathcal{D}_U}(\mathcal{E}, \mathcal{O}_U) \twoheadrightarrow \underline{\mathrm{Hom}}_{\mathcal{D}_U}(\mathcal{D}_U \cdot \omega, \mathcal{O}_U)$.

Definition 2.3. [CCG⁺13, Definition 3.3] The *Picard–Fuchs operator* $L = L_\pi$ of the period π is the operator $L = \sum_{k=0}^h p_k(\alpha) D^k$ (uniquely defined up to multiplication by a constant) such that $L \cdot \pi = 0$, where h is taken as small as possible, and $\deg p_h$ is taken as small as possible. We say that the *order* of L is h .

Remark 2.4. Recall that a formal series $\sum c_n \alpha^n$ is killed by a polynomial differential operator $\sum_j \alpha^j P_j(D)$ if and only if its coefficients c_n satisfy *the linear recursion relation*:

$$\forall n \quad \sum_{j \leq n} P_j(n-j) c_{n-j} = 0 \quad (2.3)$$

In practice, one reconstructs the Picard–Fuchs operator of π by computing the first terms of the power series expansion of π (*the period sequence*) and trying to guess the recursion relation.

Remark 2.5. Observe that, if the local system of solutions to the Picard Fuchs operator of π is irreducible, then it coincides with the local system of solutions to the annihilator ideal of π and is therefore a subquotient of $\mathrm{gr}_{n-1}^W R^{n-1} w_U! \mathbb{C}$.

2.2 Hypergeometrics

Hypergeometric functions have a rich mathematical tradition going back to the 17th century. In this thesis we deal with hypergeometric functions in one complex variable. What is special about these functions is that they are solutions to ordinary differential equations *of geometric origin*.

2.2.1 Hypergeometric functions

Definition 2.6. Let $m \in \mathbb{N}_{>0}$ and let $\mathbf{a} = (a_1, \dots, a_m)$, $\mathbf{b} = (b_1, \dots, b_m)$ be two multisets of complex numbers. Assume that none of the a_i, b_j is in $\mathbb{Z}_{<0}$. The (generalized) hypergeometric function with parameter sets $\mathbf{a}, \mathbf{b}$ is the analytic function defined by the complex power series in α :

$$F(\mathbf{a}, \mathbf{b} | \alpha) = \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_m)_j}{(b_1)_j \cdots (b_m)_j} \alpha^j \quad (2.4)$$

where $(x)_n$ denotes the rising factorial (or Pochhammer symbol):

$$\begin{cases} x(x+1) \cdots (x+n-1) & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases}$$

The series (2.4) converges on the open disk $|\alpha| < 1$, and outside of $|\alpha| < 1$ the hypergeometric function is defined by analytic continuation with respect to α .

Remark 2.7. Using the Γ -function one can rewrite the series (2.4) as:

$$\sum_{j=0}^{\infty} \prod_{i=1}^m \frac{\Gamma(a_i + j) \Gamma(1 - b_i - j)}{\Gamma(a_i) \Gamma(1 - b_i)} (-1)^{mj} \alpha^j$$

In this thesis we focus on hypergeometric functions defined over $\mathbb{Q}$.

Definition 2.8. The function $F(\mathbf{a}, \mathbf{b}|\cdot)$ is defined over $\mathbb{Q}$ if $\forall i, j$ $a_i, b_j \in \mathbb{Q} \cap (0, 1]$, $a_i \neq b_j$, and the polynomials $\prod_{j=1}^m (x - e^{2\pi i a_j})$ and $\prod_{j=1}^m (x - e^{2\pi i b_j})$ are products of cyclotomic polynomials.

There is a connection between hypergeometric functions defined over $\mathbb{Q}$ and generating functions of certain factorial ratios (see [Bob09, Lemma 4.1]).

Let $\mathbf{p} = (p_1, \dots, p_r)$ and $\mathbf{q} = (q_1, \dots, q_s)$ be two lists of positive integers such that $\sum p_i = \sum q_j$. The function:

$$G(\mathbf{p}, \mathbf{q}; \alpha) = \sum_{j=0}^{\infty} \frac{(p_1 j)! \dots (p_r j)!}{(q_1 j)! \dots (q_s j)!} \alpha^j \quad (2.5)$$

is (after rescaling α) a hypergeometric series defined over $\mathbb{Q}$. Indeed, there exist two multisets $\mathbf{a}, \mathbf{b}$ as in Definition 2.8 such that

$$\frac{\prod_{j=1}^r x^{p_j} - 1}{\prod_{j=1}^s x^{q_j} - 1} = \prod_{j=1}^m \frac{x - e^{2\pi i a_j}}{x - e^{2\pi i b_j}}$$

and one can show that $G(\mathbf{p}, \mathbf{q}; \alpha/M) = F(\mathbf{a}, \mathbf{b}|\alpha)$, where $M = \frac{p_1^{p_1} \dots p_r^{p_r}}{q_1^{q_1} \dots q_s^{q_s}}$. Explicitly, setting

$$A = \bigcup_{i=1}^r \left\{ \frac{k}{p_i} \mid k = 1, \dots, p_i \right\} \quad B = \bigcup_{i=1}^s \left\{ \frac{k}{q_i} \mid k = 1, \dots, q_i \right\}$$

the multisets $\mathbf{a}$ and $\mathbf{b}$ are given by: $\mathbf{a} = A \setminus A \cap B$ and $\mathbf{b} = B \setminus A \cap B$.

Conversely, if $F(\mathbf{a}, \mathbf{b}|\cdot)$ is an hypergeometric function defined over $\mathbb{Q}$, there exists two lists $\mathbf{p}, \mathbf{q}$ as above such that $F(\mathbf{a}, \mathbf{b}|M\alpha) = G(\mathbf{p}, \mathbf{q}; \alpha)$. Thus any hypergeometric function over $\mathbb{Q}$ is (up to rescaling α) a generating function of factorial ratios of the form (2.5).

2.2.2 Hypergeometric operators

Consider two nonzero polynomials $P, Q \in \mathbb{C}[\alpha]$ of degree m . Following Katz [Kat90], we define the *hypergeometric operator* $H(P, Q)$ in $\mathcal{D}$ as

$$H(P, Q) = P(D) - \alpha Q(D) \quad (2.6)$$

and the *hypergeometric $\mathcal{D}$ -module* $\mathcal{H}(P, Q)$ on $\mathbb{C}^\times$ as $\mathcal{H}(P, Q) = \mathcal{D}/\mathcal{D}H(P, Q)$. Of course the $\mathcal{D}$ -module does not change by multiplying $H(P, Q)$ by a $\mathbb{C}^\times$ factor.

We also introduce a notation which makes explicit the factorization of P and Q . Assume that $P(\alpha) = p \prod_{j=1}^m (\alpha - v_j)$, $Q(\alpha) = q \prod_{j=1}^m (\alpha - w_j)$, and write $\lambda = p/q$. We

define:

$$H_\lambda(\mathbf{v}, \mathbf{w}) = \lambda \prod_{j=1}^m (D - v_i) - \alpha \prod_{j=1}^m (D - w_j) \quad (2.7)$$

and $\mathcal{H}_\lambda(\mathbf{v}, \mathbf{w}) = \mathcal{D}/\mathcal{D}H_\lambda(\mathbf{v}, \mathbf{w})$. Thus

$$H_\lambda(\mathbf{v}, \mathbf{w}) = 1/q H(P, Q) \quad \text{and} \quad \mathcal{H}_\lambda(\mathbf{v}, \mathbf{w}) = \mathcal{H}(P, Q)$$

The hypergeometric operator $H_\lambda(\mathbf{v}, \mathbf{w})$ has order m and a regular singularity at λ . We define the *hypergeometric local system* $\mathbb{H}_\lambda(\mathbf{v}, \mathbf{w})$ as the complex local system of rank m on $U = \mathbb{C}^\times \setminus \{\lambda\}$ whose sections are the solutions of the differential equation $H_\lambda(\mathbf{v}, \mathbf{w})\varphi = 0$. In the language of $\mathcal{D}$ -modules, the local system $\mathbb{H}_\lambda(\mathbf{v}, \mathbf{w})$ is given by the sheaf of homomorphisms $\underline{\text{Hom}}_{\mathcal{D}_U}(j^*\mathcal{H}_\lambda(\mathbf{v}, \mathbf{w}), \mathcal{O}_U)$, where $j: U \rightarrow \mathbb{C}^\times$ denotes inclusion.

The following result characterizes irreducible hypergeometric $\mathcal{D}$ -modules/local systems up to isomorphism (see [Kat90, Proposition 3.2, Corollary 3.2.1, Lemma 3.3] for details).

Proposition 2.9. *Let $\mathcal{H}_\lambda(\mathbf{v}, \mathbf{w})$ be a hypergeometric $\mathcal{D}$ -module on $\mathbb{C}^\times$. Then:*

- (1) $\mathcal{H}_\lambda(\mathbf{v}, \mathbf{w})$ is a irreducible $\mathcal{D}$ -module if and only if $\forall i, j \ v_i - w_j \notin \mathbb{Z}$.
- (2) Its isomorphism class determines the v_i and the $w_j \mod \mathbb{Z}$, and the scalar $\lambda \in \mathbb{C}^\times$.

Remark 2.10. Statement (2) can be rephrased by saying that a hypergeometric local system $\mathbb{H}_\lambda(\mathbf{v}, \mathbf{w})$ is *physically rigid* [Fed18, Proposition 2.4].

Remark 2.11. By Remark 2.4 a formal series $\sum c_n \alpha^n$ is killed by the hypergeometric operator $H(P, Q)$ (2.6) if and only if its coefficients c_n satisfy the two-term recursion:

$$P(n)c_n = Q(n-1)c_{n-1} \quad (2.8)$$

It follows that the function $G(\mathbf{p}, \mathbf{q}; \cdot)$ in Equation (2.5) is killed by the irreducible hypergeometric operator $H(P, Q)$ given by:

$$P = \prod q_i^{q_i} \prod (D - v_i) \quad Q = \prod p_i^{p_i} \prod (D - w_j)$$

where $\mathbf{v} = V \setminus V \cap W$, $\mathbf{w} = W \setminus V \cap W$ and

$$V = \bigcup_{i=1}^s \left\{ \frac{k}{q_i} \mid k = 0, \dots, q_i - 1 \right\} \quad W = \bigcup_{i=1}^r \left\{ \frac{k}{p_i} - 1 \mid k = 0, \dots, p_i - 1 \right\}$$

which shows that $F(\mathbf{a}, \mathbf{b}; \cdot)$ is killed by $H_1((1 - b_1, \dots, 1 - b_m), (-a_1, \dots, -a_m))$.

2.2.3 Motivic interpretation of hypergeometrics

The work [Kat90] shows that hypergeometric local systems $\mathbb{H}_\lambda(\mathbf{v}, \mathbf{w})$, where $\mathbf{v}, \mathbf{w}$ are multisets of rational numbers, admit a motivic interpretation.

For any $\lambda \in \mathbb{C}^\times$ and any $2m$ rational numbers v_i, w_j such that for all i, j $v_i \neq w_j \pmod{\mathbb{Z}}$, Katz constructs a smooth m -dimensional algebraic manifold Z , depending only on λ and v_i, w_j , together with a map $\pi: Z \rightarrow \mathbb{C}^\times$, such that the $\mathcal{D}$ -module $\mathcal{H}_\lambda(\mathbf{v}, \mathbf{w})$ is isomorphic to an irreducible component of the relative De Rham cohomology sheaf $H_{\text{DR}}^{m-1}(Z/\mathbb{C}^\times) = R^{m-1}\pi_*\Omega_{Z/\mathbb{C}^\times}^\bullet$ (see [Kat90, Theorem 5.4.4, Section 5.4] for the precise statement of this result and for its proof).

It follows that any irreducible hypergeometric local system $\mathbb{H}_\lambda(\mathbf{v}, \mathbf{w})$ with rational parameters naturally supports a canonical rational variation of Hodge structure (VHS)¹ — and the work of Katz provides an explicit geometric realisation of this VHS. In other words, by the discussion in Section 2.1, hypergeometric functions must admit integral representations as periods of one-parameter families of algebraic varieties.

Example 2.12 (The Legendre family). The hypergeometric function $F((\frac{1}{2}, \frac{1}{2}), (1, 1)|\alpha)$ equals (up to a constant $\mathbf{i}$) the period

$$\pi(\alpha) = \frac{1}{2\pi\mathbf{i}} \oint \frac{1}{\sqrt{x(x-1)(x-\alpha)}} dx$$

of the family of cubic curves $E_\alpha \subset \mathbb{P}^2$ with the familiar affine equation $y^2 = x(x-1)(x-\alpha)$. This is easy to show by applying the residue theorem around $\alpha = 0$.

Now suppose that the local system $\mathbb{H}_\lambda(\mathbf{v}, \mathbf{w})$ satisfies the additional assumption that the two sets $\{\exp 2\pi\mathbf{i}v_i\}$, $\{\exp 2\pi\mathbf{i}w_j\}$ are stable under complex conjugation. Then the Hodge numbers of the VHS supported by $\mathbb{H}_\lambda(\mathbf{v}, \mathbf{w})$ are computable by the formula proposed by Corti and Golyshev in [CG11] and later proved by Fedorov in [Fed18, Theorem 2].

Theorem 2.13. [CG11, Conjecture 1.4] For $\lambda \in \mathbb{C}^\times$, and $0 \leq v_1 \leq \dots \leq v_m < 1$ and $0 \leq w_1 \leq \dots \leq w_m < 1$ such that $\forall i, j$ $v_i \neq w_j$ and the sets $\{\exp 2\pi\mathbf{i}v_i\}$, $\{\exp 2\pi\mathbf{i}w_j\}$ are stable under complex conjugation, consider the hypergeometric operator $H = H_\lambda(\mathbf{v}, \mathbf{w})$. Write

$$p(k) = \text{card}\{j \mid v_j < w_k\} - k$$

¹The work [dSKP16, Proposition 2.1] shows that an irreducible rigid local system supports at most one rational VHS.

and set $p_+ = \max\{p(k)\}$, $p_- = \min\{p(k)\}$. The local system $\mathbb{H}$ of solutions to the ordinary differential equation $H\varphi = 0$ supports a VHS of weight $p_+ - p_-$ and Hodge numbers

$$h^{j-p_-, -j+p_+} = \text{card } p^{-1}(j)$$

Remark 2.14. If the Hodge weight of $\mathbb{H}$ is n , it is natural to ask — even as there is a priori no reason to expect it — if $\mathbb{H}$ is (a subquotient of) the variation of H_n of a one-parameter family of n -dimensional varieties.

2.3 The Fano/Landau–Ginzburg correspondence

The Mirror Symmetry Conjecture for Fano varieties takes the shape of the Fano/Landau–Ginzburg correspondence: on the one hand we have a Fano orbifold of dimension n , on the other hand we have a Landau–Ginzburg (LG) model of the same dimension. Given the rich structures involved on the two sides, the correspondence can be stated at many different levels. The Hodge–theoretic formulation, which is the approach we take here, states that there is an identity between two cohomological invariants: the regularised quantum period of the Fano orbifold, and a period of the LG model. Equivalently, the local system of solutions to the regularised quantum differential operator is of geometric origin. This formulation is explored, for instance, in the paper [Gol07].

While significant progress has been made towards developing a mirror symmetry theory in substantial generality, together with intrinsic constructions of mirror pairs, there are some general open questions requiring a deeper understanding:

- How general is the picture? Do all Fano orbifolds have a LG mirror?
- How precise is it? Can one define two sets $\mathcal{A}$ and $\mathcal{B}$ and a 1:1 correspondence $\mathcal{A} \rightarrow \mathcal{B}$?
- How concrete is it? Can we make a directory of Fano/LG models?

The correspondence may be concretely understood by considering only Fano varieties with a toric degeneration. The Gross–Siebert program [GS18], started in 2001, proposes a systematic construction of mirror pairs by deforming away from a maximally unipotent degeneration. Recent conjectures by Coates, Corti, Galkin, Golyshev, Kasprzyk, et al., building on the perspective in [CCG⁺13], suggest that Fano varieties with a toric degeneration correspond under mirror symmetry to LG models which restrict to (certain, very special) Laurent polynomials. Conjecturally and experimentally, such torus charts of the LG model are in bijection with toric degenerations of the given Fano variety.

Beyond the context of Fano varieties with toric degenerations, however, the Fano/LG correspondence is wide open, even for the simplest cases of weighted and toric complete intersections: the existence of LG mirrors is not known a priori, nor is a systematic mirror construction available. Fano varieties without a toric degeneration include (and conjecturally [CH17, Section 1] coincide with) Fano varieties whose anticanonical linear system is empty: indeed for $\mathbb{Q}$ -Gorenstein deformations the $h^0(X, -K_X)$ of the fibres is locally constant on the base, *see* [ACC⁺16], and for the toric central fibre $h^0(X_0, -K_{X_0}) \neq 0$.

In Chapter 3 we propose a method to study mirror symmetry *for any Fano weighted complete intersection*. The method will build on the fact that in this case the regularised quantum period is encoded by a hypergeometric series.

Below we recall the definition of quantum period of a Fano variety and we expand on the notion of mirror symmetry for Fano weighted complete intersections.

2.3.1 The quantum period

We recall the notion of quantum period of a Fano manifold and its generalisation to a Fano variety with quotient singularities. Roughly speaking, the quantum period of a Fano variety X is a generating function for some Gromov–Witten invariants of X counting the number of genus-zero, degree- d orbifold curves in X satisfying certain tangency conditions.

Fano manifolds Let X be a Fano manifold. The *quantum period* of X , denoted by G_X , is defined as the complex power series

$$G_X(\alpha) = 1 + \sum_{j \geq 2} \sum_{\substack{d \in H_2(X, \mathbb{Z}) \\ \langle d, -K_X \rangle = j}} \langle \phi_{\text{vol}} \cdot \psi^{j-2} \rangle_{0,1,d}^X \alpha^j \quad (2.9)$$

Here $\phi_{\text{vol}} \in H^{2\dim X}(X, \mathbb{Z})$ is the cohomology class of a point and $\psi \in H^2(X_{0,1,d}, \mathbb{Q})$ is the first Chern class of the universal cotangent line bundle on the moduli stack $X_{0,1,d}$ of degree- d stable maps to X from genus-zero curves with one marked point. For the definition of the Gromov–Witten invariant $\langle \phi_{\text{vol}} \cdot \psi^{j-2} \rangle_{0,1,d}^X$ we refer the reader to [CCG⁺13, CCGK16].

We recall how G_X arises from the genus-zero Gromov–Witten theory of X . The genus-zero Gromov–Witten invariants of X define an extension of the ordinary cup product in cohomology, the *small quantum product*, which describes how submanifolds of X intersect in a quantum way: roughly speaking, they intersect if they are connected by

pseudoholomorphic curves. This product is equivalent to an integrable algebraic connection (*the Dubrovin connection*) on the trivial bundle with fibre $H^{\text{ev}}(X, \mathbb{C}) \subset H^\bullet(X, \mathbb{C})$ over the torus $\mathbb{T} = \text{Spec} \mathbb{C}[H_2(X, \mathbb{Z})]$, by means of which we can regard the bundle as a $\mathcal{D}$ -module $\mathcal{M}$, where $\mathcal{D}$ is the sheaf of differential operators on $\mathbb{T}$. A solution of $\mathcal{M}$ is Givental’s *J-function*, defined in [Giv96]. The *J-function* is cohomology-valued, but we can regard its identity component J_X^0 as a $\mathbb{C}$ -valued function. The quantum period of X is defined as the restriction $G_X(\alpha) = J_X^0 \circ \kappa(\alpha)$, where $\kappa: \mathbb{C}^\times \rightarrow \mathbb{T}$ is the line of the anticanonical class $-K_X \in H_2(X, \mathbb{Z})$. By construction, G_X satisfies an algebraic ODE.

Fano orbifolds The notion of quantum period generalises to Fano varieties with quotient singularities. Let X be a Fano variety with quotient singularities and denote by $\mathcal{X}$ the corresponding well-formed orbifold. Gromov–Witten theory for orbifold target spaces was constructed by Chen–Ruan [CR04] in symplectic geometry and by Abramovic–Graber–Vistoli [AGV02] in algebraic geometry. The *quantum period* of X is a generating function for some genus-zero Gromov–Witten invariants of $\mathcal{X}$.

More precisely, let $\tilde{H}_{\text{CR}}^{<2}(\mathcal{X}, \mathbb{Q}) = \mathbb{Q}\mathbf{1}_{b_1} \oplus \cdots \oplus \mathbb{Q}\mathbf{1}_{b_r}$ be the subspace of the even part $H_{\text{CR}}^\bullet(\mathcal{X})$ of the Chen–Ruan cohomology of $\mathcal{X}$ generated by the identity classes $\mathbf{1}_{b_j}$ of the twisted sectors of $\mathcal{X}$ with age < 1 , and write

$$\delta_{d, i_1 \dots i_n} = -K_X \cdot d + \sum_{j=1}^n 1 - \text{age}(b_{i_j})$$

Then G_X is the function:

$$G_X(x_1, \dots, x_r, \alpha) = 1 + \sum_{d \in \text{Eff}(\mathcal{X})} \sum_{n \in \mathbb{N}} \sum_{1 \leq i_1, \dots, i_n \leq r} \text{GW}_{d, i_1, \dots, i_n} \frac{x_{i_1} \cdots x_{i_n}}{n!} \alpha^{\delta_{d, i_1 \dots i_n}} \quad (2.10)$$

We refer the reader to [OP18] for the definition of the Gromov–Witten invariant

$$\text{GW}_{d, i_1, \dots, i_n} = \langle \mathbf{1}_{b_{i_1}}, \dots, \mathbf{1}_{b_{i_n}}, \phi_{\text{vol}} \psi^{\delta_{d, i_1 \dots i_n} - 2} \rangle_{0, n+1, d}$$

[OP18, Proposition 3.4] shows that $G_X \in \mathbb{Q}[x_1, \dots, x_r][[\alpha]]$ and that the coefficient of G_X in α is zero.

The way in which G_X arises from the genus-zero Gromov–Witten theory of $\mathcal{X}$ extends the situation in the smooth case. The genus-zero Gromov–Witten invariants of $\mathcal{X}$ are encoded in a Lagrangian cone $\mathcal{L}_{\mathcal{X}}$, called the *Givental cone* of $\mathcal{X}$, determining and determined by the *J-function*: this generalises the *J-function* of a smooth variety and now takes as input classes $\gamma \in H_{\text{CR}}^\bullet(\mathcal{X})$ (see [CCIT15, Section 2.3]). The quantum period

$G_{\mathcal{X}}$ is obtained from the component of J_X along $\mathbf{1}_0 \in H^0(X, \mathbb{Q}) \subset H_{\text{CR}}^0(\mathcal{X})$ by restricting along the line of $-K_X$ and by setting

$$\gamma = \alpha^{1-\text{age}(b_1)} x_1 \mathbf{1}_{b_1} + \cdots + \alpha^{1-\text{age}(b_r)} x_1 \mathbf{1}_{b_r} \quad (2.11)$$

The regularised quantum period If $G_X(\alpha) = \sum_{\delta \in \mathbb{N}} c_{\delta} \alpha^{\delta}$ is the quantum period of X , the *regularised quantum period* of X is the Fourier–Laplace transform:

$$\widehat{G}_X(\alpha) = \frac{1}{\alpha} \int_0^{\infty} G(t) e^{-t/\alpha} dt = \sum_{\delta \in \mathbb{N}} \delta! c_{\delta} \alpha^{\delta} \quad (2.12)$$

Note that if X is smooth G_X is simply a power series in α , thus so is $\widehat{G}_X$. We refer the reader to [GGI16, Lemma 3.7.4, Lemma 3.7.6] for results about the convergence radius of the regularised quantum period.

Regularising the quantum period is necessary to compare it with the periods of a morphism. Indeed, the differential equation of lowest order, as in Definition 2.3, satisfied by G_X is not regular. The differential operator of lowest order, as in Definition 2.3, annihilating $\widehat{G}_X$ is called the *regularised quantum differential operator*.

2.3.2 Fano weighted complete intersections and Mirror Symmetry

We denote by $\mathbb{P}(a_1, \dots, a_m)$ the weighted projective space with weights $a_1 \leq \cdots \leq a_m$. We simply write $\mathbb{P}$ when m and the weights are clear from the context.

For $X = X_{d_1, \dots, d_c}$ a quasismooth [CR00, 6.3] and wellformed [CR00, 6.10] complete intersection in $\mathbb{P}(a_1, \dots, a_m)$, the adjunction formula for the canonical sheaf K_X [CR00, 6.14] states that

$$K_X = \mathcal{O}_X \left(\sum_{i=1}^c d_i - \sum_{i=1}^m a_i \right)$$

By definition X is Fano if and only if $-K_X$ is ample, i.e., if and only if $\sum_i d_i < \sum_i a_i$.

We call a Fano surface a del Pezzo surface.

Our notion of mirror symmetry

We state what we mean by mirror of a quasismooth wellformed Fano weighted complete intersection $X = X_{d_1, \dots, d_c} \subset \mathbb{P}(a_1, \dots, a_m)$ of dimension $n = m - 1 - c$. Let $\lambda = \sum a_i - \sum d_i > 0$.

Definition 2.15. The *regularised I-function* of X is defined as the complex power series

$$\widehat{I}_X(\alpha) = \sum_{j=0}^{\infty} \frac{(d_1 j)! \cdots (d_c j)! (\lambda j)!}{(a_1 j)! \cdots (a_m j)!} \alpha^{\lambda j} \quad (\alpha \in \mathbb{C}) \quad (2.13)$$

Definition 2.16. A *Landau–Ginzburg (LG) model* of dimension n is a pair (Y^n, w) as in Definition 2.1. The period in Equation (2.1) is called the *period of the LG model* relative to the pair (ω, γ) .

Definition 2.17. Let $X = X_{d_1, \dots, d_c} \subset \mathbb{P}(a_1, \dots, a_m)$ a quasismooth wellformed Fano weighted complete intersection of dimension n . A LG model (Y^n, w) is mirror of X if there exists a period π of (Y, w) such that for all α

$$\widehat{I}(\alpha) = \pi(\alpha) \quad (2.14)$$

Remark 2.18. Our notion of mirror symmetry is consistent with the general shape of the Fano/LG correspondence. Indeed our function $\widehat{I}_X$ is conjecturally a shift of the specialization at a point $\gamma \in H_{\text{CR}}^{<2}(\mathcal{X})$ (2.11) of the regularised quantum period of X (see [OP18]). More precisely, $\widehat{I}_X$ is obtained from specializing the (non-extended) $\mathcal{E}$ -twisted I -function of $\mathbb{P}$ [CCIT19, Section 4], where $\mathcal{E} = \bigoplus_{i=1}^c \mathcal{O}_{\mathbb{P}}(d_i)$. Conjecturally, for a Fano orbifold toric complete intersection $X \subset Y$ defined by a regular section of $\mathcal{E} = \bigoplus_{i=1}^c \mathcal{E}_i$, where $\mathcal{E}_1, \dots, \mathcal{E}_c$ are line bundles on Y , the $\mathcal{E}$ -twisted I -function of Y determines the J -function of X .

The work [CCIT19, Theorem 25] proves this statement when $\mathcal{E}$ is convex. The paper [Wan19] shows this for all positive hypersurfaces in proper toric Deligne–Mumford stacks, which include all positive hypersurfaces in weighted projective spaces.

Remark 2.19. Consider the hypergeometric series:

$$\widetilde{I}_X(\alpha) = \sum_{j=0}^{\infty} \frac{(d_1 j)! \dots (d_c j)! (\lambda j)!}{(a_1 j)! \dots (a_m j)!} \alpha^j \quad (2.15)$$

We refer to $\widetilde{I}_X$ as *the hypergeometric function of X* . By Remark 2.11 $\widetilde{I}_X$ is annihilated by the irreducible differential operator $H_X = H_{\alpha_0}(\mathbf{v}, \mathbf{w})$, where $\alpha_0 = \frac{\prod a_i^{a_i}}{\lambda^\lambda \prod d_i^{d_i}}$, $\mathbf{v} = V \setminus V \cap W$, $\mathbf{w} = W \setminus V \cap W$ and

$$V = \bigcup_{i=1}^m \left\{ \frac{k}{a_i} \mid k = 0, \dots, a_i - 1 \right\}$$

$$W = \bigcup_{i=1}^c \left\{ \frac{k}{d_i} - 1 \mid k = 0, \dots, d_i - 1 \right\} \cup \left\{ \frac{k}{\lambda} - 1 \mid k = 0, \dots, \lambda - 1 \right\}$$

Since $\widetilde{I}_X(\alpha^\lambda) = \widehat{I}(\alpha)$, it follows that $\widehat{I}(\alpha)$ is annihilated by the irreducible differential operator:

$$L_{\widehat{I}} = \alpha_0 \prod_i (D - \lambda v_i) - \alpha^\lambda \prod_i (D - \lambda w_i) \quad (2.16)$$

In other words, $\mathcal{D}/\mathcal{D}L_{\widehat{I}} \simeq \rho_\lambda^* \mathcal{H}_{\alpha_0}(\mathbf{v}, \mathbf{w})$, where $\rho_\lambda: \alpha \mapsto \tau = \alpha^\lambda$ is the Kummer pullback.

Remark 2.20. It follows directly from Definition 2.17 that $L_{\widehat{\Gamma}} = L_{\pi}$, where L_{π} is the Picard–Fuchs operator of π ; this could be taken as an equivalent formulation of mirror symmetry.

Remark 2.21. Periods of LG models are commonly defined as follows. Consider an element $\omega \in H^0(Y, \Omega_Y^n)$ and a cycle $\gamma \in H_n(Y, \mathbb{Q})$ such that $\int_{\gamma} \omega = 1$ and $\forall |\alpha| \ll 1 \ \gamma \in H_n(Y \setminus Y_{\alpha}, \mathbb{Q})$. The period of the LG model relative to (ω, γ) is the function:

$$\pi(\alpha) = \int_{\gamma} \frac{1}{1 - \alpha/w} \omega \quad (2.17)$$

This definition is stronger than our Definition (2.1). Indeed, by the residue theorem:

$$\pi(\alpha) = \int_{\gamma} \frac{1}{1 - \alpha/w} \omega = \int_{\gamma_{\alpha}} \omega_{\alpha}$$

where $\gamma_{\alpha} \in H_{n-1}(Y_{\alpha}, \mathbb{Q})$ is such that $\tau(\gamma_{\alpha}) = \gamma$, the map $\tau: H_{n-1}(Y_{\alpha}, \mathbb{Q}) \rightarrow H_n(Y, \mathbb{Q})$ being the Griffiths tube map, and

$$\omega_{\alpha} = \text{Res}_{Y_{\alpha}} \left(\frac{\omega}{1 - \alpha/w} \right) \in H^0(Y_{\alpha}, \Omega_{Y_{\alpha}}^{n-1})$$

Hori–Vafa mirrors

The work of Givental/Hori–Vafa [Giv98, HV00] constructs mirrors for all Fano toric complete intersections² $X \subset Y$ defined by regular sections of line bundles $L_1, \dots, L_c$ satisfying suitable compatibility conditions with respect to the torus-invariant divisors of the ambient space Y . We recall this construction for weighted complete intersections.

Let $X = X_{d_1, \dots, d_c} \subset \mathbb{P}(a_1, \dots, a_m)$ be a quasismooth wellformed Fano weighted complete intersection of dimension n . Assume that:

Hori–Vafa condition $\forall k \in \{1, \dots, c\} \exists S_k \subset \{1, \dots, m\}$ such that

$$S_i \cap S_j = \emptyset \ \forall i, j \quad \text{and} \quad d_k = \sum_{i \in S_k} a_i \quad (2.18)$$

Choose³ $S_1, \dots, S_c$ satisfying (2.18). Let $x_1, \dots, x_m$ be affine coordinates on $(\mathbb{C}^{\times})^m$. The *Hori–Vafa mirror* of X is defined as the LG model (Y, w) :

$$Y = \left(\begin{array}{l} \sum_{i \in S_k} x_i = 1, \ k \in \{1, \dots, c\} \\ \prod_{i=1}^m x_i^{a_i} = 1 \end{array} \right) \subset (\mathbb{C}^{\times})^m \quad w = \frac{1}{\sum_{i \notin \cup_k S_k} x_i} \quad (2.19)$$

²The papers [Giv98, HV00] consider smooth complete intersections in toric manifolds, but the mirror construction can be extended to complete intersections in toric Deligne–Mumford stacks.

³There could be more than one choice of subsets $S_1, \dots, S_c$ satisfying the condition (2.18).

One can check that the regularised I -function of X equals a period of (Y, w) . Indeed, by repeatedly applying the residue theorem in one complex variable, one sees that:

$$\widehat{I}_X(\alpha) = \int_{\Gamma} \frac{1}{1 - \alpha(\sum_{i \notin \cup_k S_k} x_i)} \frac{1}{1 - x_1^{-a_1} \dots x_m^{-a_m}} \prod_{k=1}^c \frac{1}{1 - \sum_{i \in S_k} x_i} \Omega \quad (2.20)$$

where

$$\Omega = \left(\frac{1}{2\pi i} \right)^m \frac{dx_1 \dots dx_m}{x_1 \dots x_m} \quad (2.21)$$

and $\Gamma \in H_{n+1}(\mathbb{C}^\times)^m, \mathbb{Z})$ is an explicit cycle. The left hand side can be rewritten as a period of the form (2.17) by residuating $c + 1$ times.

Remark 2.22. Note that, if $X = X_{d_1, \dots, d_c} \subset \mathbb{P}(a_1, \dots, a_m)$ satisfies the Hori–Vafa condition, then $|-K_X| \neq \emptyset$, hence conjecturally X admits a toric degeneration. The work [Prz11] gives a method to extract a Laurent polynomial out of the map w .

Example 2.23. Consider the cubic surface $X_3 \subset \mathbb{P}^3$. Pick $S_1 = \{2, 3, 4\}$. The Hori–Vafa mirror of X is $Y = (x_2 + x_3 + x_4 = 1, x_1 x_2 x_3 x_4 = 1) \subset (\mathbb{C}^\times)^4$, together with $w = 1/x_1$. It is easy to see that:

$$\widehat{I}_X(\alpha) = \left(\frac{1}{2\pi i} \right)^4 \int_{\Gamma} \frac{1}{1 - \alpha x_1} \frac{1}{1 - (x_1 x_2 x_3 x_4)^{-1}} \frac{1}{1 - x_2 - x_3 - x_4} \frac{dx_1 dx_2 dx_3 dx_4}{x_1 x_2 x_3 x_4}$$

where $\Gamma = \{|x_1| = R, |x_2| = |x_3| = |x_4| = \varepsilon \mid 0 < \varepsilon < 1/3, R \cdot \varepsilon^3 > 1\} \subset (\mathbb{C}^\times)^4$.

Now we extract a Laurent polynomial out of w . Solve the second equation defining Y for x_1 , and let x, y be affine coordinates on $(\mathbb{C}^\times)^2$. Set:

$$x_2 = \frac{1}{1 + x + y} \quad x_3 = \frac{x}{1 + x + y} \quad x_4 = \frac{y}{1 + x + y}$$

Then $Y \simeq (\mathbb{C}^\times)^2$ and $x_1 = \frac{(1+x+y)^3}{xy}$. It is easy to see that $\widehat{I}_X$ equals the period:

$$\pi(\alpha) = \left(\frac{1}{2\pi i} \right)^2 \oint \frac{1}{1 - \alpha/w} \frac{dx dy}{xy} = \sum_{j=0}^{\infty} \text{coeff}_1 \left(\frac{(1+x+y)^3}{xy} \right) \alpha^j$$

Example 2.24. Consider the surface $X = X_{6,6} \subset \mathbb{P}(2, 2, 3, 3, 3)$, which is a quasismooth and wellformed orbifold with $4 \times 1/3(1, 1)$ points and $-K_X = \mathcal{O}(1)$ (the surface X is the 13th sporadic case in Table A.2). Clearly X does not satisfy the Hori–Vafa condition, nor has a toric degeneration since $|-K_X| = \emptyset$. However, X admits a LG mirror. This builds upon computations of Don Zagier and it is discussed in [CK16].

Remark 2.25. In (2.19) switch the last equation defining the manifold Y with w , to obtain the LG model (Y, w) :

$$Y = \left(\begin{array}{l} \sum_{i \in S_k} x_i = 1, \quad k \in \{1, \dots, c\} \\ \sum_{i \notin \cup_k S_k} x_i = 1 \end{array} \right) \subset (\mathbb{C}^\times)^m \quad w = \prod_{i=1}^m x_i^{a_i} \quad (2.22)$$

The hypergeometric function of X is a period of this LG model. Indeed:

$$\tilde{I}_X(\alpha) = \int_{\Gamma} \frac{1}{1 - \alpha x_1^{-a_1} \dots x_m^{-a_m}} \prod_{k=1}^c \frac{1}{1 - \sum_{i \in S_k} x_i} \frac{1}{1 - \sum_{i \notin \cup_k S_k} x_i} \Omega \quad (2.23)$$

where Ω is the form in (2.21) and Γ is an explicit cycle in $H_m((\mathbb{C}^\times)^m, \mathbb{Z})$. Throughout this thesis, we will refer to the LG model (Y, w) in (2.22) as the Hori–Vafa mirror of X .

Chapter 3

A new method from the arithmetic of hypergeometric functions

In this Chapter we outline a strategy to determine if the hypergeometric function $\tilde{I}_X$ of a Fano weighted complete intersection X of dimension n is a period of a pencil of varieties of dimension $n - 1$. By Section 2.3.2 our strategy can be viewed as the basis of a new method to build mirrors *for all* Fano weighted complete intersections.

The source of ideas for our investigation is the arithmetic of hypergeometric functions: the work of Beukers, Cohen, and Mellit [BCM15] on finite hypergeometric series suggests that any hypergeometric function over $\mathbb{Q}$ is a period of a pencil of algebraic varieties which depends only on the hypergeometric parameters. The pencil attached to the function $\tilde{I}_X$ by [BCM15] can be interpreted as a LG mirror of X *of the wrong dimension*.

We focus on hypergeometric functions of Hodge weight one, and, in particular, on the hypergeometric functions of anticanonical del Pezzo weighted complete intersections.

3.1 Finite hypergeometric functions and point counting

There is a finite analogue of the analytical hypergeometric function $F(\mathbf{a}, \mathbf{b}|\cdot)$ which was introduced independently by Katz [Kat90] and by Greene [Gre87] in the late 1980's.

For a prime power q and two multisets $\mathbf{a}, \mathbf{b}$ of m rational numbers each, disjoint modulo $\mathbb{Z}$ and such that $\forall i, j \ (q-1)a_i, (q-1)b_j \in \mathbb{Z}$, the *finite hypergeometric function* $H_q(\mathbf{a}, \mathbf{b}|\cdot)$ is the complex-valued function:

$$H_q(\mathbf{a}, \mathbf{b}|\alpha) = -\frac{1}{q^\times} \sum_{j=0}^{q^\times-1} \frac{g(j + a_i q^\times)g(-j - b_i q^\times)}{g(a_i q^\times)g(-b_i q^\times)} \omega((-1)^m \alpha)^j \quad (3.1)$$

where $q^\times = q - 1$, $\omega: \mathbb{F}_q^\times \rightarrow \mathbb{C}^\times$ is a generator of the character group on $\mathbb{F}_q^\times$ and $g(\chi)$ is a Gauss sum (see [BCM15, Definition 1.1]).

Remark 3.1. Note the similarities with (2.7): Gauss sums are finite analogues of Γ -functions. In the finite case the assumption $a_i \neq b_j \bmod \mathbb{Z}$ is made to avoid certain degeneracies.

Finite hypergeometric functions appear to be closely related to point counting on one-parameter families of algebraic varieties defined over finite fields. From the motivic point of view, this relation suggests that the corresponding analytic hypergeometric functions can be represented as periods of the same families now regarded over $\mathbb{C}$. In other words, the trace of the q^{th} Frobenius, acting on the cohomology of the fibres of the family, is expected to be linked to the Picard–Fuchs operators of the family.¹

As an example we mention the case of the Legendre family, which appeared in 2.12:

Example 3.2. (K.Ono, 1998) Let q be an odd prime power and let $\alpha \in \mathbb{F}_q$ with $\alpha \neq 0, 1$. Let $E_\alpha \in \mathbb{P}^2$ the elliptic curve with affine equation $y^2 = x(x-1)(x-\alpha)$ and write $E_\alpha(\mathbb{F}_q)$ for the set of $\mathbb{F}_q$ -rational points. Then:

$$|E_\alpha(\mathbb{F}_q)| = H_q((1/2, 1/2), (1, 1)|\alpha)$$

up to factors depending only on q .

3.2 The work of Beukers, Cohen and Mellit

In their work [BCM15] Beukers, Cohen and Mellit show that finite hypergeometric functions defined over $\mathbb{Q}$, that is, which satisfy the assumptions of Definition 2.8, correspond to point counting on explicit families of varieties over $\mathbb{F}_q$.

They rewrite ([BCM15, Theorem 1.3]) a finite hypergeometric function over $\mathbb{Q}$ as $H(\mathbf{a}, \mathbf{b}|\alpha) = L(\mathbf{p}, \mathbf{q}; \alpha/M)$, where $L(\mathbf{p}, \mathbf{q}; \cdot)$ is a finite analogue of the function $G(\mathbf{p}, \mathbf{q}; \cdot)$ in Equation (2.5), and $M = \prod p_i^{p_i} / \prod q_j^{q_j}$ as in Section 2.2.1.

Then they prove the following:

¹The link between point counts via hypergeometrics and Picard–Fuchs hypergeometric operators has been extensively explored, see for instance [Igu58, Dwo69, CdlORV03, DKS⁺20]. In [Cle80] Clemens refers to it as “Manin’s Unity of Mathematics”. However, at the moment, there is no general theorem known to us for this link: these connections are typically done by hand. Our ideas in Chapter 7 are meant to be a contribution in this direction.

Theorem 3.3. [BCM15, Theorem 1.5] For all $\alpha \in \mathbb{F}_q^\times$, consider affine variety W_α given by the homogeneous equations:

$$\begin{cases} y_1 + \cdots + y_r - x_1 - \cdots - x_s = 0 \\ \alpha \cdot y_1^{p_1} \cdots y_r^{p_r} = x_1^{q_1} \cdots x_s^{q_s} \end{cases} \quad (\text{for all } j, x_j, y_j \neq 0) \quad (3.2)$$

If $\gcd(p_1, \dots, p_r, q_1, \dots, q_s) = 1$ and $M\alpha \neq 1$, there exists a suitable nonsingular completion $\overline{W}_\alpha$ of W_α such that $|\overline{W}_\alpha(\mathbb{F}_q)| = L(\mathbf{p}, \mathbf{q}; \alpha)$ up to factors depending only on r, s , and the prime power q .

Remark 3.4. One easily checks that the condition $\alpha \neq 0, 1/M$ corresponds to the non-singularity of the variety W_α . We write $U = \mathbb{C}^\times \setminus \{1/M\}$.

By the discussion in Section 3.1, Theorem 3.3 suggests that the hypergeometric function $G(\mathbf{p}, \mathbf{q}; \cdot)$ in Equation (2.5) is a period of the family of varieties W_α in Equation (3.2). In other words, if $\mathbb{H}$ is the local system of solutions to the irreducible hypergeometric equation annihilating $G(\mathbf{p}, \mathbf{q}; \cdot)$ and $v: W \rightarrow \mathbb{C}^\times$ is the map on

$$W = (y_1 + \cdots + y_r - x_1 - \cdots - x_s = 0) \subset \mathbb{T}^{r+s-1} \quad (3.3)$$

sending $(x_1 : \cdots : x_s : y_1 : \cdots : y_r) \mapsto \prod x_i^{q_i} / \prod y_i^{p_i}$, where $\mathbb{T}^{r+s-1} \subset \mathbb{P}^{r+s-1}$ is the torus, there shall be an identity:

$$\mathbb{H} = \mathrm{gr}_{d-1}^W R^{d-1} v_U! \mathbb{C} \quad (3.4)$$

where we write $d = r + s - 2$ for the complex dimension of W and $v_U: W_U = v^{-1}(U) \rightarrow U$ for the restriction.

Remark 3.5. We discuss this conjectural Hodge-theoretic translation of Theorem 3.3 in Chapter 7. In his work [Rod19] Rodriguez Villegas suggests it may be possible to verify explicitly that the formula 2.13 for the Hodge numbers of general hypergeometric variations is equivalent to the algorithm in [DK86] to compute the weight $d - 1$ mixed Hodge numbers of the middle cohomology of W_α for a generic $\alpha \in \mathbb{C}^\times$. This will be investigated further.

3.3 A program for weighted complete intersections

We specialise the above discussion to the hypergeometric function $\tilde{I}_X$ of a Fano quasismooth and wellformed weighted complete intersection $X = X_{d_1, \dots, d_c} \subset \mathbb{P}(a_1, \dots, a_m)$.

Write $n = \dim_{\mathbb{C}} X$ and $\lambda = \sum a_i - \sum d_j$ as in Section 2.3.2, and assume that $d_1 \leq \dots \leq d_c$. Let

$$\alpha_0 = \frac{a_1^{a_1} \dots a_m^{a_m}}{d_1^{d_1} \dots d_c^{d_c} \lambda^\lambda} \quad (3.5)$$

set $U = \mathbb{C}^\times \setminus \{\alpha_0\}$, and denote by $\mathbb{H}$ the local system of solutions to the irreducible differential equation satisfied by $\tilde{I}_X$. The general shape of the Fano/LG correspondence suggests that $\mathbb{H}$ has Hodge weight $n - 1$.

Consider the manifold W of dimension $d = n + 2c$ given by

$$W = (x_1 + \dots + x_m = y_1 + \dots + y_{c+1}) \subset \mathbb{T}^{m+c} \quad (3.6)$$

where $\mathbb{T}^{m+c} \subset \mathbb{P}^{m+c}$ is the torus and x_i, y_j are homogeneous coordinates on $\mathbb{P}^{m+c}$, together with the map $v: W \rightarrow \mathbb{C}^\times$ sending

$$(x_1 : \dots : x_s : y_1 : \dots : y_r) \mapsto \frac{x_1^{a_1} \dots x_m^{a_m}}{y_1^{d_1} \dots y_c^{d_c} y_{c+1}^\lambda} \quad (3.7)$$

By Remark 3.4 the fibre $W_\alpha = v^{-1}(\alpha)$ is singular if and only if $\alpha = \alpha_0$; we denote by $v_U: W_U = v^{-1}(U) \rightarrow U$ the restriction.

The above discussion suggests there is an identity of VHS:

$$\mathbb{H} = \mathrm{gr}_{d-1}^W R^{d-1} v_{U!} \mathbb{Q}(c) \quad (3.8)$$

where (k) denotes the Tate twist at the level of sheaves. Equivalently, the function $\tilde{I}_X$ should be a period π of $v: W \rightarrow \mathbb{C}^\times$.

Since the regularised I -function of X is $\hat{I}_X = \tilde{I}_X \circ \rho_\lambda$, where $\rho_\lambda: \alpha \mapsto \alpha^\lambda$ as in Remark 2.19, by Definition 2.17 the pair (W, v) can be interpreted as a LG mirror of X *of the wrong dimension*.

It makes sense to ask if there is a manifold Y of dimension n with a morphism $w: Y \rightarrow \mathbb{C}^\times$ such that $\mathrm{gr}_{d-1}^W R^{d-1} v_{U!} \mathbb{C}$ identifies with a subquotient of $\mathrm{gr}_{n-1} R^{n-1} w_{U!} \mathbb{C}$, where $w_U: Y_U = w^{-1}(U) \rightarrow U$ is the restriction. The pair (Y, w) would provide a LG mirror *of the right dimension*.

Below we describe our strategy to study, for all $\alpha \in U$, the pure Hodge structure

$$\mathrm{gr}_{d-1}^W H_c^{d-1}(W_\alpha, \mathbb{Q})$$

and to determine if this Hodge structure arises as a subquotient of the middle cohomology of a variety Y_α of dimension $n - 1$. In the affirmative, the pencil of varieties Y_α provides the required morphism $w: Y \rightarrow \mathbb{C}^\times$.

Step 1. We manipulate the equations of W_α to write it a hypersurface in a torus $\mathbb{T}$ of dimension d .

This is easily done by solving Equation (3.6) for one of the variables and plugging this into $\alpha y_1^{d_1} \dots y_c^{d_c} y_{c+1}^\lambda = x_1^{a_1} \dots x_m^{a_m}$.

For instance, by solving (3.6) by y_{c+1} and setting $y_i = x_{m+i}$ for $i = 1, \dots, c$, we obtain:

$$\alpha \prod_{i=m+1}^{m+c} x_i^{d_i} \left(\sum_{i=1}^m x_i - \sum_{i=m+1}^{m+c} x_i \right)^\lambda = \prod_{i=1}^m x_i^{a_i}$$

Then, in the chart $(x_{m+c} \neq 0)$, with coordinates $u_i = x_i/x_{m+c}$, W_α identifies with:

$$\left(\alpha \left(\sum_{i=1}^m u_i - \sum_{i=m+1}^{m+c-1} u_i - 1 \right)^\lambda - \frac{\prod_{i=1}^m u_i^{a_i}}{\prod_{i=1}^{c-1} u_{m+i}^{d_i}} = 0 \right) \subset (\mathbb{C}^\times)^d \quad (3.9)$$

Step 2. We construct a simple (and possibly partial) compactification $W_\alpha \subset \widehat{W}_\alpha$ in a toric variety with at worst canonical singularities such that

$$\mathrm{gr}_{d-1}^W H_c^{d-1}(W_\alpha, \mathbb{Q}) = \mathrm{gr}_{d-1}^W H_c^{d-1}(\widehat{W}_\alpha, \mathbb{Q})$$

The variety $\widehat{W}_\alpha$ we construct has **either** $K_{\widehat{W}_\alpha}$ nef, **or** there is a fibration $\widehat{W}_\alpha \rightarrow Z$ to a lower dimensional variety Z with $-K_{\widehat{W}_\alpha}$ relatively ample.

We construct $\widehat{W}_\alpha$ by means of a combinatorial algorithm, based on a variant of the toric minimal model program (MMP), for a hypersurface in a torus. We present the algorithm in Chapter 4.

When necessary, we combine the algorithm with the methods of variations of GIT quotients.

Step 3. We study the geometry of $\widehat{W}_\alpha$ to determine if the top weight piece in its middle cohomology is the subquotient of the top weight piece in the H_c^{n-1} of a $(n-1)$ -dimensional variety.

The hope is that $\widehat{W}_\alpha$ supports a Fano fibration $\phi_\alpha: \widehat{W}_\alpha \rightarrow \mathbb{P}^1$ (or $\mathbb{C}^\times$). Then one can use the Leray spectral sequence of the morphism ϕ_α with second page

$$E_2^{p,q} = H^p(\mathbb{P}^1, R^q \phi_{\alpha*} \mathbb{Q}) \implies H_c^{p+q}(\widehat{W}_\alpha, \mathbb{Q})$$

which is known to degenerate at E_2 , and analyse the sheaves $R^j \phi_{\alpha*} \mathbb{Q}$ for $j = d-1, d-2$ – whose stalk at t is the H^j of the Fano variety $\widehat{W}_{\alpha,t} = \phi_\alpha^{-1}(t)$ – to try to reconstruct a morphism $p_\alpha: Y_\alpha^{n-1} \rightarrow \mathbb{P}^1$ such that $H^{n-1}(Y_\alpha, \mathbb{Q})$ has the required property.

For instance, if $\widehat{W}_\alpha$ is a quasismooth 3-fold with a del Pezzo fibration $\phi_\alpha: \widehat{W}_\alpha \rightarrow \mathbb{C}^\times$ (whose singular fibres are mildly singular), then the only contribution to $\mathrm{gr}_3^W H_c^3(\widehat{W}_\alpha, \mathbb{Q})$ is given by $\mathrm{gr}_1^W H^1(\mathbb{C}^\times, R^2\phi_{\alpha*}\mathbb{Q})(-1)$; thus it is enough to construct a morphism $p_a: Y_a \rightarrow \mathbb{C}^\times$ such that

$$\mathrm{gr}_1^W H^1(Y_\alpha, \mathbb{Q}) = \mathrm{gr}_1^W H^1(\mathbb{C}^\times, p_{\alpha*}\mathbb{Q}) \rightarrow \mathrm{gr}_1^W H^1(\mathbb{C}^\times, R^2\phi_{\alpha*}\mathbb{Q})$$

This is what we do in the cases discussed in Chapter 5 and in Chapter 6.

Remark 3.6. If X is the weighted projective space $\mathbb{P} = \mathbb{P}(a_1, \dots, a_m)$, then

$$\tilde{I}_X(\alpha) = \sum_{j=0}^{\infty} \frac{(a_1 + \dots + a_m)j}{(a_1 j)! \dots (a_m j)!} \alpha^{(a_1 + \dots + a_m)j}$$

In this case the pair (W, v) defined by (3.6) and (3.7) has dimension $m - 1 = \dim_{\mathbb{C}} \mathbb{P}$ and coincides with the Hori–Vafa mirror of $\mathbb{P}$ (2.22).

In general, for a weighted complete intersection $X = X_{d_1, \dots, d_c} \subset \mathbb{P}(a_1, \dots, a_m)$ of dimension n , the pair (W, v) has wrong dimension $n + 2c$. It would be very interesting to show that, whenever X admits a Hori–Vafa mirror (Y, w) , the LG pair that our program extracts out of (W, v) coincides with (Y, w) . It would also be very interesting to establish if, in these cases, there exists a recurrent geometric relation between the LG pairs (Y, w) and (W, v) . This will be discussed further in the first Section of Chapter 6.

3.4 Hypergeometric local systems of weight one

Let $\mathbb{H}$ be an irreducible hypergeometric local system of Hodge weight one and let (W^d, v) be the pair associated to $\mathbb{H}$ by the work of Beukers, Cohen, and Mellit. Assuming identity (3.4), the work [Wel87], applied to the intermediate Jacobian of the Hodge structure $\mathrm{gr}_{d-1}^W H^{d-1}(W_\alpha, \mathbb{Q})$, ensures that there is a morphism $w: Y^2 \rightarrow \mathbb{C}^\times$ such that $\mathbb{H}$ identifies with a subquotient of $\mathrm{gr}_1^W R^1 w_* \mathbb{C}$ away from the critical locus of w . Indeed, [Wel87] shows that any principally polarized abelian variety is isomorphic to an abelian subvariety, called *Prym–Tjurin variety*, of the Jacobian of a curve (see [Wel87, Fact (1.19)]).

This fact is interesting for two reasons.

Firstly, from the point of view of mirror symmetry, it implies that, if identity (3.8) holds then any weighted complete intersection del Pezzo surface X admits a LG mirror (Y^2, w) . In particular, if X is a del Pezzo weighted hypersurface, the pair (W, v) associated to $\tilde{I}_X$ is a pencil of 3-folds. The papers [Bea77, Bel85, Tju72, BM79, Kan89] describe the presentation of the intermediate Jacobian of 3-folds with Kodaira dimension $\kappa = -\infty$

(Fano threefolds, conic bundles and pencils of del Pezzo surfaces) as Prym–Tjurin variety. We will mention how our results in Chapters 5 and 6 relate to these works.

Secondly, given that every irreducible hypergeometric local system of Hodge weight one arises as a subquotient of the variation of H^1 of a pencil of curves, it is interesting to consider the more restrictive question:

Question 3.7. Let $\mathbb{H}$ be an irreducible hypergeometric local system $\mathbb{H}$ of Hodge weight one. Is $\mathbb{H}$ isomorphic to the $R^1w_!\mathbb{C}$ of a pencil $w: Y \rightarrow \mathbb{C}^\times$ of curves?

Observe that, even as Question 3.7 is very natural from a Hodge–theoretic point of view, there a priori no reason to expect an affirmative answer: there are well-known examples of Hodge structures of weight one which are not the H^1 of a curve. For instance, the H^3 of the generic cubic 3-fold $X = X_3 \subset \mathbb{P}^4$ has Hodge weight one, and the work by Clemens–Griffiths [CG72] shows that the intermediate Jacobian of X is not isomorphic to the Jacobian of a curve (which eventually implies that X is not rational, see [CG72, Corollary 3.26]).

Building upon the formula [CG11] we have implemented a code to classify all hypergeometric functions defined over $\mathbb{Q}$ whose attached variation has Hodge weight one. Rodriguez Villegas [RV19] has determined that, assuming identity (3.4), the hypergeometric functions with parameters $\mathbf{p}, \mathbf{q}$ given by

$\mathbf{p}$	$\mathbf{q}$
(11,2)	(1,3,4,5)
(15,2)	(1,4,5,7)
(16,4)	(2,3,7,8)
(18,8)	(4,6,7,9)
(24,6)	(3,7,8,12)

correspond to the variation of H^3 of a pencil of cubic 3-folds; thus these variations cannot be isomorphic to the variation of H^1 of a pencil of curves. In Section 4.4 we show how to see this for the case $\mathbf{p}, \mathbf{q} = (11,2), (1,3,4,5)$ by applying our MMP algorithm for hypersurfaces in tori to the corresponding 3-folds W_α by Beukers, Cohen, and Mellit.

3.5 Anticanonical del Pezzo weighted complete intersections

We recall the notion of anticanonical Fano weighted complete intersection.

Definition 3.8. A quasismooth and wellformed Fano weighted complete intersection $X = X_{d_1, \dots, d_c} \subset \mathbb{P}(a_1, \dots, a_m)$ is *anticanonical* if

$$\sum_{j=1}^c d_j = \sum_{i=1}^m a_i - 1 \quad \text{that is} \quad -K_X = \mathcal{O}_X(1) \quad (3.10)$$

Remark 3.9. The regularised I -function $\widehat{I}_X$ of an anticanonical Fano weighted complete intersection X coincides with the hypergeometric function $\widetilde{I}_X$.

Anticanonical Fano weighted complete intersections are especially interesting in mirror symmetry since most of them are not covered by any known mirror construction.

Indeed, if $X = X_{d_1, \dots, d_c} \subset \mathbb{P}(a_1, \dots, a_m)$ is anticanonical, the anticanonical linear system $|-K_X| = |O_X(1)| \neq \emptyset$ if and only if $a_1 = 1$. Thus, by the discussion in Section 2.3 all anticanonical Fano complete intersections in weighted projective spaces with weights ≥ 2 fall out of the range Givental/Hori–Vafa [HV00, Giv98] and of the Gross–Siebert program [GS18].

Our work so far has focused on *anticanonical del Pezzo* weighted complete intersections. By [CCC11, Theorem 6] a weighted complete intersection del Pezzo surface which is quasi-smooth, well-formed, and is not the intersection with a linear cone has codimension $c \leq 2$.

Codimension one In their work [JK01] Johnson and Kollár provide the complete list of anticanonical hypersurfaces $X_d \subset \mathbb{P}(a_1, a_2, a_3, a_4)$ in weighted projective 3-spaces. The list consists of an infinite series and 22 sporadic cases (see Table A.1).

For an anticanonical del Pezzo surface $X_d \subset \mathbb{P}(a_1, a_2, a_3, a_4)$, the pair (W, v) defined by Equations (3.6) and (3.7) is given by 4-dimensional manifold:

$$W = (y_1 + y_2 = x_1 + x_2 + x_3 + x_4) \subset \mathbb{T}^5 \quad (3.11)$$

where $\mathbb{T}^5 \subset \mathbb{P}^5$ is the torus and x_i, y_j are homogeneous coordinates on $\mathbb{P}^5$, together with the map $v: W \rightarrow \mathbb{C}^\times$ sending

$$(x_1 : x_2 : x_3 : x_4 : y_1 : y_2) \mapsto \frac{x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4}}{y_1^{d_1} y_2^\lambda} \quad (3.12)$$

Remark 3.10. Recall that, for a positive weighted hypersurface X , the fact the hypergeometric function of X coincides (up to a shift) with a specialisation of the regularised quantum period of X is not conjectural but has been proven in the paper [Wan19].

Codimension two In their work [KP15] Kim and Park classify all anticanonical del Pezzo surfaces $X_{d_1, d_2} \subset \mathbb{P}(a_1, a_2, a_3, a_4, a_5)$ in weighted projective 4-spaces. The list consists of three infinite series and 38 sporadic cases (see Table A.2).

For an anticanonical del Pezzo surface $X_{d_1, d_2} \subset \mathbb{P}(a_1, a_2, a_3, a_4, a_5)$, the pair (W, v) defined by Equations (3.6) and (3.7) is given by 6-dimensional manifold:

$$W = (y_1 + y_2 + y_3 = x_1 + x_2 + x_3 + x_4 + x_5) \subset \mathbb{T}^7 \quad (3.13)$$

where $\mathbb{T}^7 \subset \mathbb{P}^7$ is the torus and x_i, y_j are homogeneous coordinates on $\mathbb{P}^7$, together with the map $v: W \rightarrow \mathbb{C}^\times$ sending

$$(x_1 : x_2 : x_3 : x_4 : x_5 : y_1 : y_2 : y_3) \mapsto \frac{x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4} x_5^{a_5}}{y_1^{d_1} y_2^{d_2} y_3^\lambda} \quad (3.14)$$

The Johnson–Kollár series and the third series constructed by Kim–Park are especially interesting from the point of view of mirror symmetry. Indeed they provide infinite lists of del Pezzo surfaces whose LG mirrors are not known a priori.

In Chapter 5 we use the strategy sketched in Section 3.3 to construct LG mirrors for the surfaces of the Johnson–Kollár series. The study of the third series by Kim–Park is ongoing work. In Chapter 4 we show how to use our MMP algorithm to construct a good compactification of the fibre of the wrong dimension mirror of this series.

In Chapter 6 we give evidence that our strategy is compatible with Hori–Vafa [HV00] and we construct a candidate LG mirror for the surface $X_{15} \subset \mathbb{P}(3, 3, 5, 5)$ in Table A.1. As we explain in Chapter 5, the existence of a LG mirror of this surface is particularly interesting as it is not naturally connected to any other mirror duality.

Chapter 4

An MMP algorithm for hypersurfaces in tori

In this Chapter we present a combinatorial iterative algorithm, based on the methods of the minimal model program, to find a good compactification of a hypersurface in a torus. We describe how to apply the algorithm to the fibres of the wrong dimension mirrors of anticanonical del Pezzo weighted complete intersections, and, as an example, we compute the output of the algorithm for the Johnson–Kollar series and the third series by Kim–Park. We conclude by showing how to use the algorithm to prove that, assuming identity (3.4), the hypergeometric local system attached to the series $G((11, 2), (1, 3, 4, 5); \cdot)$ cannot be a curve variation.

4.1 Notation and basic facts about toric varieties

We set up our notation for toric varieties and we recall some well-known facts which we use repeatedly.

For a lattice L , we denote by $L_{\mathbb{R}} = L \otimes_{\mathbb{Z}} \mathbb{R}$ the associated real vector space. For a torus $\mathbb{T}$, we denote by $M = \text{Hom}(\mathbb{T}, \mathbb{C}^{\times})$ the character lattice and we write $N = \text{Hom}(M, \mathbb{Z})$.

For a fan $\Sigma \subset N_{\mathbb{R}}$, we denote by F_{Σ} the associated toric variety. We denote by ρ_i the primitive generators of the 1-dimensional cones (or *rays*) of Σ , and by $D_i \subset F_{\Sigma}$ the corresponding divisors. We write $\Sigma(r)$ for the r -dimensional cones of Σ .

If F is a proper toric variety and $D = \sum a_i D_i$ is a Weil divisor on F , then we denote by

$$P_D = \bigcap_i \{m : \langle \rho_i, m \rangle \geq -a_i\} \subset M_{\mathbb{R}} \quad (4.1)$$

the polytope of D . It is well known that $P_D \cap M$ is a basis of $H^0(F, D)$.

Given a rational polytope $P \subset M_{\mathbb{R}}$, we denote by Σ_P the normal fan of P . If P is not full dimensional then Σ_P is a generalised fan, that is, all the cones of Σ_P contain a fixed vector subspace σ_0 with associated lattice $N_0 = N \cap \sigma_0$. We denote by F_P the toric variety for the fan $\Sigma_P/\sigma_0 \subset (N/N_0)_{\mathbb{R}}$.

Fact 1 There is a 1-to-1 correspondence between:

- (I) the set of polarised toric varieties (F, D) , that is, pairs of a proper toric variety F and (torus invariant) ample divisor D , and
- (II) the set of full-dimensional rational polytopes $P \subset M_{\mathbb{R}}$.

Given a pair (F, D) , the corresponding polyhedron is $P = P_D$. Conversely, given P , we let $F = F_P$ and $D = \sum b_i D_i$ where $P = \bigcap_i \{m : \langle \rho_i, m \rangle \geq -b_i\}$ is the unique facet presentation of P .

Fact 2 Suppose that D is a nef divisor on a proper toric variety F . Let $P = P_D$ be the polytope of D and (F_P, D_P) the corresponding polarised pair. Then (F_P, D_P) is an *ample model* of (F, D) , in other words:

- (i) there is a proper toric morphism $f : F \rightarrow F_P$, and
- (ii) $D = f^* D_P$.

Given a complete fan Σ in $N_{\mathbb{R}}$, if n is the dimension of N and k is the number of rays of Σ , we denote by $\rho : \mathbb{Z}^k \rightarrow N$ the linear map sending the i^{th} element of the standard basis of $\mathbb{Z}^k$ to ρ_i and we write $\mathbb{L} = \ker \rho$. The exact sequence:

$$0 \rightarrow \mathbb{L} \rightarrow \mathbb{Z}^k \xrightarrow{\rho} N$$

is called the fan sequence. Since $\text{coker } \rho$ is finite, the dual map $\rho^* : M \rightarrow \mathbb{Z}^k$ is injective. We write $\mathbb{L}^{\vee} = \text{coker } \rho^*$ and we denote by $\mathcal{W} : \mathbb{Z}^k \rightarrow \mathbb{L}^{\vee}$ the quotient map. The short exact sequence:

$$0 \rightarrow M \xrightarrow{\rho^*} \mathbb{Z}^k \xrightarrow{\mathcal{W}} \mathbb{L}^{\vee} \rightarrow 0 \quad (4.2)$$

is called the divisor sequence. It is well known that the group $\mathbb{Z}^k$ in (4.2) identifies with the group $\text{Div}_{\mathbb{T}}(F_{\Sigma})$ of torus-invariant Weil divisors of F_{Σ} , and $\mathbb{L}^{\vee}$ is isomorphic to the class group $\text{Cl}(F_{\Sigma})$: the image $\mathcal{W}(e_i)$ of the i^{th} element of the standard basis of $\mathbb{Z}^k$ is the class of the divisor D_i of F_{Σ} .

Applying $\text{Hom}_{\mathbb{Z}}(\cdot, \mathbb{C}^\times)$ to (4.2) gives an inclusion $\mathbb{G} = \text{Hom}_{\mathbb{Z}}(\mathbb{L}^\vee, \mathbb{C}^\times) \subset (\mathbb{C}^\times)^k$, hence an induced action of $\mathbb{G}$ on $\mathbb{C}^k$. We write $\mathcal{C} \subset \mathbb{L}_{\mathbb{R}}^\vee$ for the strictly convex cone spanned by the columns of $\mathcal{W}$, c_i for the i^{th} column of $\mathcal{W}$, and $\langle c_{i_1}, \dots, c_{i_r} \rangle$ for the $\mathbb{R}_{\geq 0}$ -span of $c_{i_1}, \dots, c_{i_r}$. Given ω in $\mathcal{C}$, we denote by F_ω the (almost geometric) quotient

$$\mathbb{C}^k //_{\omega} \mathbb{G} = \left(\mathbb{C}^k \setminus V(I_\omega) \right) // \mathbb{G}$$

where I_ω denotes the irrelevant ideal of ω :

$$I_\omega = (x_{i_1} \dots x_{i_r} \mid \omega \in \langle c_{i_1}, \dots, c_{i_r} \rangle) \subset \mathbb{C}[x_1, \dots, x_k]$$

We refer to $\mathcal{W}$ as the weight data for F_ω and to ω as a stability condition for F_ω . The fan determined by the decomposition of $\mathcal{C}$ obtained by taking all possible cones over the columns c_i is called the secondary fan. It is well known that if σ is the smallest cone of the secondary fan containing ω then for any $\omega' \in \text{Int}(\sigma)$ F_ω and $F_{\omega'}$ coincide. Moreover F_ω is isomorphic to the toric variety F_{P_D} , where $D = \sum a_i D_i$ and $\mathcal{W}((a_1, \dots, a_k)) = \omega$.

4.2 Compactifying a hypersurface in a torus

Let $f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$ be a Laurent polynomial whose Newton polytope $P = \text{Newt}(f)$ is d -dimensional. Let W be the hypersurface $(f = 0) \subset \mathbb{T}$, where $\mathbb{T} \simeq (\mathbb{C}^\times)^d$ is the torus with coordinates $x_1, \dots, x_d$. A naive way to compactify W is to consider the closure of W in the toric variety $F = F_P$ whose fan is the normal fan of P . However both F and the compactification of W in F can be highly singular. Here we describe a combinatorial iterative algorithm, based on the methods of the minimal model program, to construct a simpler compactification of W . The algorithm:

- has **input** the Newton polytope $P = \text{Newt}(f)$;
- has **output** a d -dimensional polytope $\overline{P}$ such that the toric variety $\overline{F} = F_{\overline{P}}$ has canonical singularities, and the compactification $W \subset \overline{W}$ in $\overline{F}$ has either canonical divisor $K_{\overline{W}}$ nef, or supports a fibration to a lower dimensional variety with $-K_{\overline{W}}$ relatively ample.

In what follows we explain the main ingredients of the algorithm: in Section 4.2.1 we construct the toric variety $F = F_P$, in Section 4.2.2 we build $\mathbb{Q}$ -Gorenstein partial resolutions of F , in Section 4.2.3 we present a combinatorial variant of the MMP for the divisor $D = D_P$ of the polytope P . Section 4.2.4 contains the description of the algorithm.

4.2.1 The toric variety $F = F_P$

Let $M = \text{Hom}(\mathbb{T}, \mathbb{Z})$ be the character lattice and let N be its dual lattice. The facet presentation of the polytope $P = \text{Newt}(f)$ is of the form:

$$P = \bigcap_{i=1}^k \{m \in M_{\mathbb{R}} : \langle \rho_i, m \rangle \geq -a_i\} \subset M_{\mathbb{R}}$$

for some $(\rho_i, a_i) \in N \times \mathbb{Z}$, and some $k \geq d + 1$. Hence the fan $\Sigma = \Sigma_P$ has k rays generated by the vectors ρ_i and $F = F_P$ is a compact d -dimensional toric variety. The compactification of W in F corresponds to an element of the linear system $|O_F(D)|$, where $D = \sum_{i=1}^k a_i D_i$ is the ample divisor of F associated to P (as in Fact 1).

Notation We set up the notation we use below. The variety $F = F_P$ may not be $\mathbb{Q}$ -Gorenstein. Let $G \rightarrow F$ be any small morphism such that K_G is $\mathbb{Q}$ -Cartier. If G does not have canonical singularities, write Σ_1 for the fan of G and set $S_1 = \{\rho_1, \dots, \rho_k\}$. For $i > 1$, let $S_i = \{\rho_1, \dots, \rho_k, \dots, \rho_{l_i}\}$ be the minimal set of vectors in N needed to obtain a refinement Σ_i of Σ_{i-1} with canonical i -dimensional cones, and write

$$\Theta_i = S_i \setminus S_{i-1} \tag{4.3}$$

Let $d_{i_1} < d_{i_2} < \dots < d_{i_{r_i}}$ be the discrepancies of the valuations corresponding to the elements of Θ_i , and let p_{i_j} be the number of elements of Θ_i whose associated valuation has discrepancy d_{i_j} .

We choose an order of the elements of Θ_i such that the p_{i_j} elements with discrepancy d_{i_j} precede the $p_{i_{j+1}}$ with discrepancy $d_{i_{j+1}}$. Moreover, if $\rho_h = \sum_i b_{hi} \rho_i$, we set

$$a_h = \sum_i b_{hi} a_i \tag{4.4}$$

4.2.2 $\mathbb{Q}$ -Gorenstein partial resolutions of F

Fix¹ $h \geq k$. For $\varepsilon \geq 0$ consider the polytope:

$$P(\varepsilon) = \bigcap_{i=1}^h \{m \in M_{\mathbb{R}} : \langle \rho_i, m \rangle \geq -a_i + \varepsilon\} \subset M_{\mathbb{R}} \tag{4.5}$$

¹The objects defined in this Section obviously depend on the choice of h , but, for the sake of keeping the notation light, we decided to omit the reference to h in our notation. The algorithm we describe below will iterate over h .

The polytope $P(\varepsilon)$ is a ‘shrinking’ of the polytope P . Indeed, for $\varepsilon = 0$ we have that:

$$P(0) = \bigcap_{i=1}^h \{m \in M_{\mathbb{R}} : \langle \rho_i, m_i \rangle \geq -a_i\} = \bigcap_{i=1}^k \{m \in M_{\mathbb{R}} : \langle \rho_i, m_i \rangle \geq -a_i\} = P$$

where the second equality follows from our choice (4.4) of the a_i .

For $0 < \varepsilon \ll 1$, $P(\varepsilon)$ is d -dimensional, the presentation in Equation (4.5) is the facet presentation of $P(\varepsilon)$, and the normal fan of $P(\varepsilon)$ is independent of ε ; denote it by $\tilde{\Sigma}$ and let $\tilde{F} = F_{P(\varepsilon)}$ be the corresponding toric variety. Let $\tilde{D} = \sum_{i=1}^h a_i D_i$; by Fact 1 above the divisor

$$\tilde{D}(\varepsilon) = \sum_{i=1}^h (-\varepsilon + a_i) D_i = \varepsilon K_{\tilde{F}} + \tilde{D} \quad (4.6)$$

is ample on $\tilde{F}$.

Lemma 4.1. *The following hold:*

- (a) $\tilde{F}$ is $\mathbb{Q}$ -Gorenstein and the map $f: \tilde{F} \dashrightarrow F$ is a morphism;
- (b) $\tilde{D} = f^*(D)$.

Sketch of proof. Note that $\tilde{D}$ is nef and $\tilde{D} = \tilde{D}(0)$. Moreover we have seen that for $\varepsilon = 0$ $P(0) = P$; thus by Fact 2 above (F, D) is the ample model of $(\tilde{F}, \tilde{D})$: in other words there is a morphism $f: \tilde{F} \rightarrow F$ and $\tilde{D} = f^*(D_P)$. Finally, one can check explicitly that the fan $\tilde{\Sigma}$ is a refinement of Σ such that for any $\sigma \in \tilde{\Sigma}(d)$, the minimal generators of the rays of σ lie on a hyperplane $H_\sigma \subset N$. In other words, $\tilde{F}$ is $\mathbb{Q}$ -Gorenstein. $\square$

Remark 4.2. Note that, when $h = k$ in Equation (4.5), the map $\tilde{F} \rightarrow F$ is a small morphism such that $K_{\tilde{F}}$ is $\mathbb{Q}$ -Cartier. Thus we can compute the set Θ_2 in (4.3) by refining the fan $\tilde{\Sigma}$ of $\tilde{F}$. Similarly, for $i > 2$, we can compute the set Θ_i by refining the fan $\tilde{\Sigma}$ of the variety $\tilde{F}$ obtained for $h = k + \text{card}(\Theta_{i-1})$.

4.2.3 A combinatorial minimal model program

For $0 < \varepsilon \ll 1$ the divisors $\tilde{D}(\varepsilon)$ are ample on $\tilde{F}$. We recover $\tilde{F}$ as the toric variety whose spanning fan is the normal fan of the polytopes $P(\varepsilon)$ for small values of ε .

The minimal model program with scaling

Our minimal model program with scaling consists of an inductively defined finite sequence of toric varieties F_j , divisors X_j , and a strictly increasing sequence of rational numbers

ε_j , and birational maps:

$$F_0 \dashrightarrow \cdots \dashrightarrow F_j \xrightarrow{t_j} F_{j+1} \dashrightarrow \cdots \dashrightarrow F_r$$

where:

1. $F_0 = \tilde{F}$, $X_0 = \tilde{D}$, and $\varepsilon_0 = \max\{\varepsilon > 0 \mid \varepsilon K_{\tilde{F}} + \tilde{D} \text{ is nef on } \tilde{F}\}$;
2. For $j \geq 0$, $t_j: F_j \dashrightarrow F_{j+1}$ is the divisorial contraction or flip of the face $R_j \subset \text{NE}(F_j)$ with $(\varepsilon_j K_j + X_j)|_{R_j} = 0$ and $(K_j + X_j)|_{R_j} < 0$;
3. For $j > 0$ X_j is the proper transform of X_{j-1} on F_j , $K_j = K_{F_j}$, and

$$\varepsilon_j = \max\{\varepsilon > \varepsilon_{j-1} \mid \varepsilon K_j + X_j \text{ is nef on } F_j\}$$

4. The program ends at F_r where **either**:

- (i) *The pair (F_r, X_r) is a minimal model*, that is, $\varepsilon_r = 1$ and $K_{F_r} + X_r$ is nef; **or**
- (ii) *The pair (F_r, X_r) is a Mori fibre space (Mfs)*, that is, $\varepsilon_r < 1$, the contraction $\psi: F_r \rightarrow S$ of R_r has relative dimension > 0 and $-(K_{F_r} + X_r)$ is ψ -ample. In this case ε_r is the quasi-effective threshold of the pair $(\tilde{F}, \tilde{D})$:

$$\varepsilon_r = \sup\{t \mid tK_{\tilde{F}} + \tilde{D} \in \overline{\text{Eff}}(\tilde{F})\}$$

Our construction of the varieties F_j

We recover the toric varieties F_j by means of the polytopes $P(\varepsilon)$ in (4.5). Indeed, by Fact 1, F_j is the toric variety whose spanning fan is the normal fan of the polytopes $P(\varepsilon)$ for $\varepsilon \in (\varepsilon_{j-1}, \varepsilon_j)$, since the divisors $\varepsilon K_j + X_j$ are ample on F_j . The threshold values $\cdots < \varepsilon_j < \varepsilon_{j+1} < \cdots$ are those where the polytope $P(\varepsilon)$ changes shape.²

Our strategy to compute the threshold values is the following. Consider the $(d+1)$ -dimensional lattice $N' = N \oplus \mathbb{Z}$ and the dual lattice M' , whose elements are pairs (m, r) with $m \in M$ and $r \in \mathbb{Z}$. Denote by $q: M' \rightarrow \mathbb{Z}$ the projection to the second factor. Let $\rho'_i = (\rho_i, -1) \in N'$, $i = 1, \dots, h$ and let P' be the $(d+1)$ -dimensional polyhedron:

$$P' = \bigcap_{i=1}^h \{m' \in (M')_{\mathbb{R}} : \langle m', \rho'_i \rangle \geq -a_i\} \subset (M')_{\mathbb{R}} \quad (4.7)$$

Then $\forall \varepsilon$ $P(\varepsilon) = q_{\mathbb{R}}^{-1}(\varepsilon) \cap P'$, and the threshold values occur at the vertices of P' .

²The paper [Ish99] describes a similar method to constructs a minimal model for a Σ -regular toric divisor with $\kappa(X) \geq 0$: by means of ‘puffing up’ of the polytope of the adjoint divisor.

4.2.4 Our algorithm

The algorithm consists of iterating over the index h the following two basic steps:

- (1) build the polyhedron P' in Equation (4.7), and
- (2) build the polytope $P(\varepsilon) = q_{\mathbb{R}}^{-1}(\varepsilon) \cap P'$, where $\varepsilon_{r-1} < \varepsilon < \varepsilon_r$ and $\varepsilon_{r-1}, \varepsilon_r$ are the two largest values attained by the last coordinates of the vertices of P' ,

until the toric variety $F_r = F_{P(\varepsilon)}$ has at worst canonical singularities. When this is true, one sets $\bar{P} = P(\varepsilon)$ and $\bar{F} = F_{\bar{P}}$.

More precisely, we proceed as follows. We build $P = \text{Newt}(f)$, we set $h = k$ in Equation (4.7), and we perform (1) and (2). Note that this corresponds to find the output of the MMP described in 4.2.3 for the pair $(\tilde{F}, \tilde{D})$ in Lemma 4.1, where $\tilde{F}$ is $\mathbb{Q}$ -Gorenstein and $\tilde{F} \rightarrow F$ is a small morphism. If the variety $F_r = F_{P(\varepsilon)}$ obtained out of Step (2) does not have canonical singularities, we compute the set Θ_2 (4.3) of vectors needed to make the 2-dimensional cones of $\tilde{\Sigma}$ canonical, and the discrepancies $d_{2_1} < \dots < d_{2_{r_2}}$. We set $h = h + p_{2_1}$, where p_{2_1} is the number of elements of Θ_2 with discrepancy d_{2_1} , and we perform (1) and (2). This corresponds to find the output of the MMP described in 4.2.3 for the pair $(\tilde{F}, \tilde{D})$ in Lemma 4.1, where now $\tilde{F}$ is a $\mathbb{Q}$ -Gorenstein partial resolution of F . If the variety F_r obtained out of Step (2) does not have canonical singularities, we set $h = h + p_{2_2}$, and we repeat the same procedure. If at $h = k + \text{card}(\Theta_2)$ the output F_r of Step (2) does not have canonical singularities, we compute the set of vectors Θ_3 (4.3) needed to make the 3-dimensional cones of $\tilde{\Sigma}$ canonical, and we continue as we did for Θ_2 .

The algorithm terminates after a finite number of steps. By construction $\bar{F} = F_{\bar{P}}$ has canonical singularities, and the closure $W \subset \bar{W}$ in $\bar{F}$ has either canonical divisor $K_{\bar{W}}$ nef, or supports a fibration to a lower dimensional variety with $-K_{\bar{W}}$ relatively ample.

Remark 4.3. Our choice of iterating over the index h as described above maximises the efficiency of the algorithm. Indeed, it may not be necessary to compute all the sets Θ_i (4.3) (which corresponds to build a canonical resolution of F), or to pick all the vectors in a set Θ_i , to obtain a variety F_r with canonical singularities out of the MMP. This will be visible in the examples of Sections 4.3 and 4.4. The reason for this is that, at any step $F_j \xrightarrow{-t_j} F_{j+1}$ of the MMP, for any geometric valuation ν , there is an inequality $d_\nu(K_{F_j}) \leq d_\nu(K_{F_{j+1}})$, where $d_\nu(K_{F_i})$ denotes the discrepancy at ν .

Remark 4.4. The algorithm can be implemented in Magma and upgraded to terminal singularities in the natural way.

4.3 Application to mirrors of del Pezzo surfaces

We describe how to apply the algorithm to the fibres of the wrong dimension mirrors of anticanonical del Pezzo weighted complete intersections.

4.3.1 Hypersurfaces

In Equations (3.11) and (3.12) we have defined the 4-dimensional pair (W, v) associated to the $\widehat{I}$ -function of an anticanonical del Pezzo surface $X = X_d \subset \mathbb{P}(a_1, \dots, a_4)$. Following the discussion in Section 3.3, we rewrite the fibre $W_\alpha = v^{-1}(\alpha)$ as the hypersurface:

$$(\alpha(u_1 + u_2 + u_3 + u_4 - 1) - u_1^{a_1} u_2^{a_2} u_3^{a_3} u_4^{a_4} = 0) \subset \mathbb{T}^4 \quad (4.8)$$

where $\mathbb{T}^4 \simeq (\mathbb{C}^\times)^4$ is the 4-dimensional torus with coordinates u_1, u_2, u_3, u_4 .

Let $M = \text{Hom}(\mathbb{T}^4, \mathbb{C}^\times)$ be the character lattice and $N = M^\vee$. Denote by $\{e_1, e_2, e_3, e_4\}$ the basis of N dual to the basis of M consisting of the coordinate functions u_i on $\mathbb{T}^4$. We study the Newton polytope $P \subset M_\mathbb{R}$ of the polynomial in Equation (4.8), defining the 3-fold W_α .

The polytope P is a 4-dimensional lattice polytope with 6 vertices $u_0 = \underline{0}$, u_i , $i \in \{1, 2, 3, 4\}$, and $u_5 = (a_1, a_2, a_3, a_4)$, and facet presentation:

$$P = \bigcap_{i=1}^8 \{m \in M_\mathbb{R} : \langle \rho_i, m \rangle \geq -l_i\} \subset M_\mathbb{R} \quad (4.9)$$

where $(\rho_i, l_i) \in N \times \mathbb{Z}$ are:

$$\left\{ \begin{array}{ll} (\rho_i, l_i) = (e_i, 0) & \text{for } i = 1, 2, 3, 4 \\ (\rho_5, l_5) = ((a_2 + a_3 + a_4 - 1)b_1/a_1, -b_1, -b_1, -b_1), b_1 \\ (\rho_6, l_6) = ((-b_2, (a_1 + a_3 + a_4 - 1)b_2/a_2, -b_2, -b_2), b_2 \\ (\rho_7, l_7) = ((-b_3, -b_3, (a_1 + a_2 + a_4 - 1)b_3/a_3, -b_3), b_3 \\ (\rho_8, l_8) = ((-b_4, -b_4, -b_4, (a_1 + a_2 + a_3 - 1)b_4/a_4), b_4 \end{array} \right. \quad (4.10)$$

and, for $i = 1, 2, 3, 4$, b_i is the smallest integer such that $(\sum_{j \neq i} a_j - 1)b_i/a_i \in \mathbb{N}$. Hence the fan $\Sigma = \Sigma_P$ is a complete fan with six 4-dimensional cones and eight rays generated by the vectors ρ_i , and F is a compact 4-dimensional toric variety. Note that P does not depend on α and so neither do Σ and F . The compactification of W_α in F corresponds to an element of the linear system $|O_F(D)|$, where $D = \sum l_i D_i$ is the ample divisor of F associated to P .

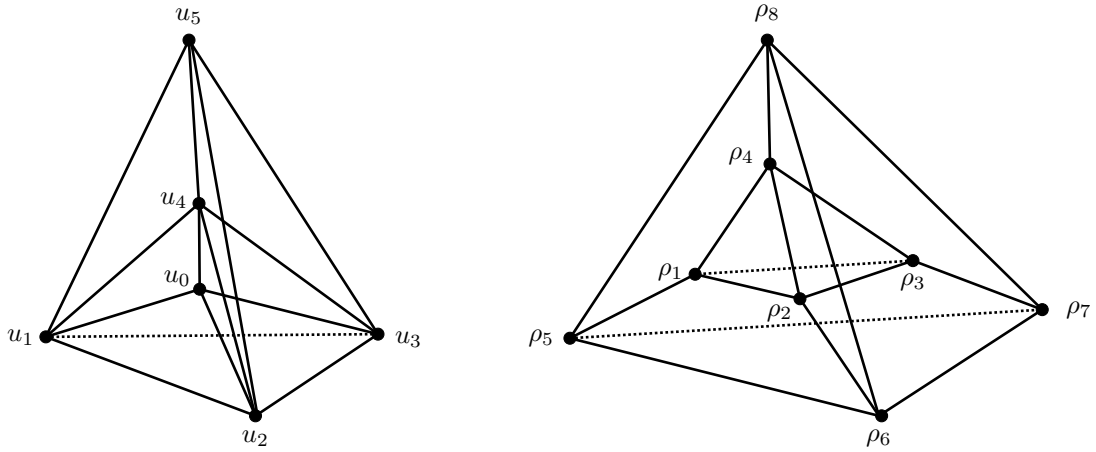


Figure 4.1: A picture of the polytope $P \subset M_{\mathbb{R}}$ and of the normal fan $\Sigma \subset N_{\mathbb{R}}$.

The Kollár–Johnson series $X_{8k+4} \subset \mathbb{P}(2, 2k+1, 2k+1, 4k+1)$

Fix $k > 0$ integer. Let $W_{k,\alpha} = W_{\alpha}$ be the 3-fold in Equation (4.8) where $(a_1, a_2, a_3, a_4) = (2, 2k+1, 2k+1, 4k+1)$. Then the Newton polytope P of the polynomial defining W_{α} has facet presentation given by Equations (4.9) and (4.10). More precisely:

$$\left\{ \begin{array}{ll} (\rho_i, l_i) = (e_i, 0) & \text{for } i = 1, 2, 3, 4 \\ (\rho_5, l_5) = ((4k+1, -1, -1, -1), 1) \\ (\rho_6, l_6) = ((-1, 3, -1, -1), 1) \\ (\rho_7, l_7) = ((-1, -1, 3, -1), 1) \\ (\rho_8, l_8) = ((-(4k+1), -(4k+1), -(4k+1), 4k+3), (4k+1)) \end{array} \right.$$

The variety $F = F_P$ is not $\mathbb{Q}$ -Gorenstein, and the $\mathbb{Q}$ -Gorenstein variety $\tilde{F} = F_{P(\varepsilon)}$ associated to the polytope P shrunk by $\varepsilon \ll 1$ (see Section 4.2.2) has noncanonical singularities. In particular, looking at the 2-dimensional cones of the fan of $\tilde{F}$, the cones $\tau_1 = \langle \rho_5, \rho_8 \rangle_+$ and $\tau_2 = \langle \rho_4, \rho_8 \rangle_+$ correspond to the affine open subsets

$$\tilde{F}_{\tau_1} \cong (\mathbb{C}^\times)^2 \times \frac{1}{4k+2}(1, 1) \quad \text{and} \quad \tilde{F}_{\tau_2} \cong (\mathbb{C}^\times)^2 \times \frac{1}{4k+1}(1, 2k)$$

and the surface quotient singularities $\frac{1}{4k+2}(1, 1)$ and $\frac{1}{4k+1}(1, 2k)$ are not canonical: the vectors $\frac{1}{4k+2}(\rho_5 + \rho_8)$ and $\frac{1}{4k+1}(\rho_4 + 2k\rho_8)$ correspond to valuations with discrepancies $-\frac{4k}{4k+2}$ and $-\frac{2k}{4k+1}$.

Since $-\frac{4k}{4k+2} < -\frac{2k}{4k+1}$, following the discussion in Section 4.2.4, we set:

$$\rho_9 = \frac{1}{4k+2}(\rho_5 + \rho_8) = (0, -1, -1, 1) \quad \text{and} \quad l_9 = 1$$

We build the 5-dimensional polyhedron P' in Equation (4.7), where $h = 9$ and $(\rho_i, l_i) \in N \times \mathbb{Z}$ are as defined above, and we find the sequence of threshold values:

$$\frac{1}{2}, \frac{2k+1}{3k+2}, \frac{2}{3}$$

Thus the minimal model program ends with a Mfs.

For all $\varepsilon \in \left(\frac{2k+1}{3k+2}, \frac{2}{3}\right)$, the polytope $P(\varepsilon)$ in Equation (4.5) is a 4-dimensional polytope with 9 vertices and 7 facets, whose normals are generated by $\rho_1, \rho_2, \rho_3, \rho_5, \rho_6, \rho_7, \rho_9$; thus its normal fan $\bar{\Sigma}$ has 7 rays and 9 maximal cones, which are listed in Figure 4.2. One can check that the 4-dimensional toric variety $\bar{F}$ is $\mathbb{Q}$ -Gorenstein and has canonical singularities.

The polytope $P(2/3)$ is the segment:

$$P\left(\frac{2}{3}\right) = \left[\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 1 \right), \left(\frac{2}{3}, \frac{2k+1}{3}, \frac{2k+1}{3}, \frac{4k+1}{3} \right) \right]$$

thus $S = \mathbb{P}^1$. More precisely, the normal fan of $P(2/3)$ has two maximal cones

$$C_1 = \{n \in N_{\mathbb{R}} : n_2 + n_3 + 2n_4 \geq 0\} \quad \text{and} \quad C_2 = \{n \in N_{\mathbb{R}} : n_2 + n_3 + 2n_4 \leq 0\}$$

intersecting along the hyperplane $C_0 = \{n \in N_{\mathbb{R}} : n_2 + n_3 + 2n_4 = 0\} \subset N_{\mathbb{R}}$.

The toric morphism $\psi: \bar{F} \rightarrow \mathbb{P}^1$ is induced by the quotient map $N \rightarrow N/N_0$, where $N_0 = N \cap C_0$, sending

$$(n_1, n_2, n_3, n_4) \mapsto n_2 + n_3 + 2n_4$$

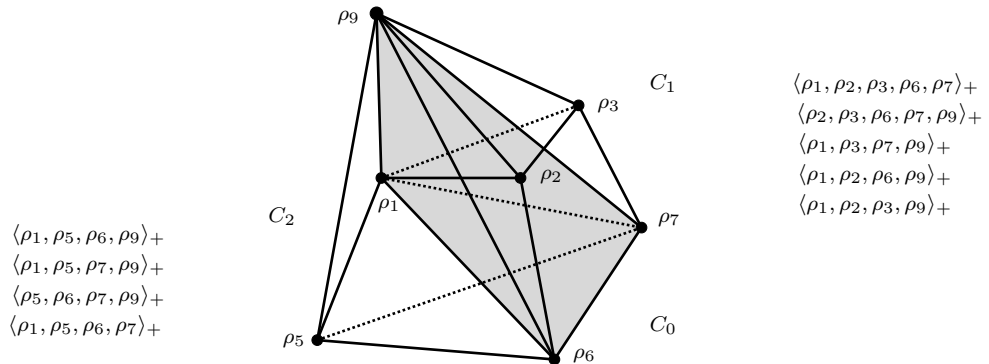


Figure 4.2: A picture of the fan $\bar{\Sigma} \subset N_{\mathbb{R}}$ and of the cones $C_1 \ni \rho_2, \rho_3$ and $C_2 \ni \rho_5$. The vectors $\rho_1, \rho_6, \rho_7, \rho_9$ span the hyperplane $C_0 \subset N_{\mathbb{R}}$, represented by the area coloured in gray. The cone C_1 is the union of the 5 maximal cones of $\bar{\Sigma}$ containing ρ_2 or/and ρ_3 , listed on the right of the picture. The cone C_2 is the union of the 4 maximal cones of $\bar{\Sigma}$ containing ρ_5 , listed on the left.

The preimage via $\psi: \bar{F} \rightarrow \mathbb{P}^1$ of the torus $\mathbb{C}^\times \subset \mathbb{P}^1$ is the non-compact toric variety G associated to the subfan $\Delta \subset \bar{\Sigma}$ given by the cones of $\bar{\Sigma}$ contained in C_0 . The fan Δ has four 3-dimensional cones (see Figure 4.2):

$$\langle \rho_6, \rho_1, \rho_9 \rangle_+ \quad \langle \rho_7, \rho_1, \rho_9 \rangle_+ \quad \langle \rho_6, \rho_7, \rho_1 \rangle_+ \quad \langle \rho_6, \rho_7, \rho_9 \rangle_+$$

Since the quotient N/N_0 is torsion-free, we may choose a splitting $N = N_0 \oplus N_1$ and regard Δ as a fan Δ_0 in N_0 ; we have $\Delta = \Delta_0 \oplus \{0\}$, thus $G = G_0 \times \mathbb{C}^\times$, where G_0 is the 3-dimensional toric variety with spanning fan $\Delta_0 \subset N_0$. To determine G_0 , note that the vectors $\rho_1, \rho_6, \rho_7, \rho_9$ spanning the hyperplane C_0 satisfy the linear relation $\rho_6 + \rho_7 + 2\rho_1 + 2\rho_9 = 0$. However G_0 is not the weighted projective space $\mathbb{P}(1, 1, 2, 2)$. Write $N_{00} = \mathbb{Z}\rho_6 + \mathbb{Z}\rho_1 + \mathbb{Z}\rho_9$. Then:

$$N_0 = N_{00} + 1/2 \cdot (1, 1, 1)\mathbb{Z}$$

We have $M_{00} \supset M_0 = \{m \in M_{00} : m_1 + m_2 + m_3 = 0 \bmod 2\}$ and $\mathbb{C}[M_0]$ is the ring of invariants $\mathbb{C}[M_{00}]^{\mu_2}$, where the group where μ_2 acts on $\mathbb{C}[M_{00}]$ by $\xi \cdot \chi^m = \xi^{m_1+m_2+m_3} \chi^m$. Equivalently G_0 is the quotient variety

$$G_0 = \mathbb{P}(1, 1, 2, 2)/\mu_2$$

where μ_2 acts with weights $(0, 1, 1, 1)$ on $\mathbb{P}(1, 1, 2, 2)$. Denoting by x_1, x_2, y_1, y_2 the weighted homogeneous coordinates of $\mathbb{P}(1, 1, 2, 2)$, the μ_2 action has weights $1/2(1, 1, 1)$ on the two charts $\{x_1 = 1\}$ and $\{x_2 = 1\}$, and $1/4(1, -1, 0)$ on the two charts $\{y_1 = 1\}$ and $\{y_2 = 1\}$.

To determine the equation of $W_\alpha \subset \mathbb{T}^4$ in $G = \mathbb{P}(1, 1, 2, 2)/\mu_2 \times \mathbb{C}^\times$, we choose the basis of $N_\mathbb{R}$ given by the four vectors $\rho_6, \rho_1, \rho_9, e_2$ and denote the dual basis by x, y, z, t . The change of coordinate matrix is:

$$\begin{bmatrix} -1 & 3 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

hence

$$u_1 = x^{-1}y \quad u_2 = x^3z^{-1}t \quad u_3 = x^{-1}z^{-1} \quad u_4 = x^{-1}z$$

Thus W_α identifies with the 3-fold:

$$\left(-at^{2k+1}y^2 + yz + z^2 + 1 - xz + tx^4 = 0 \right) / \mu_2 \subset \mathbb{T}_G \quad (4.11)$$

where $a = 1/\alpha$. The closure of $W_\alpha \subset \widehat{W}_\alpha$ in G is given by the weighted homogeneous equation:

$$\left(-at^{2k+1}y_1^2 + y_1y_2 + y_2^2 + x_1^4 - x_1x_2y_2 + tx_2^4 = 0\right) / \mu_2 \subset \mathbb{P}(1, 1, 2, 2) / \mu_2 \times \mathbb{C}^\times \quad (4.12)$$

The 3-fold $\widehat{W}_\alpha$ is a partial compactification of W_α . The second projection $\widehat{W}_\alpha \rightarrow \mathbb{C}^\times$ is the restriction of the toric morphism $\varphi: \overline{F} \rightarrow \mathbb{P}^1$. We will see in Chapter 5 that this projection is a degree 2 del Pezzo fibration.

4.3.2 Codimension two surfaces

In Equations (3.13) and (3.14) we have defined the 6-dimensional pair (W, v) associated to the $\widehat{I}$ -function of an anticanonical surface $X = X_{d_1, d_2} \subset \mathbb{P}(a_1, \dots, a_5)$. Following the discussion in Section 3.3, we rewrite the fibre $W_\alpha = v^{-1}(\alpha)$ as the hypersurface:

$$\left(\alpha(u_1 + u_2 + u_3 + u_4 + u_5 - u_6 - 1) - u_1^{a_1} u_2^{a_2} u_3^{a_3} u_4^{a_4} u_5^{a_5} u_6^{-d_1} = 0\right) \subset \mathbb{T}^6 \quad (4.13)$$

where $\mathbb{T}^6 \simeq (\mathbb{C}^\times)^6$ is the 6-dimensional torus with coordinates u_i .

Let $M = \text{Hom}(\mathbb{T}^6, \mathbb{C}^\times)$ be the character lattice and $N = M^\vee$. Denote the basis of N dual to the basis of M consisting of the coordinate functions u_i on $\mathbb{T}^6$ by $\{e_1, e_2, e_3, e_4, e_5, e_6\}$. We study the Newton polytope $P \subset M_\mathbb{R}$ of the polynomial in Equation (4.13), defining the 5-fold W_α .

The polytope P is a 6-dimensional lattice polytope with 8 vertices $u_0 = \underline{0}$, u_i , $i \in \{1, \dots, 6\}$, and $u_7 = (a_1, a_2, a_3, a_4, a_5, -d_1)$, and facet presentation

$$P = \bigcap_{j=1}^{15} \{m \in M_\mathbb{R} : \langle \rho_j, m \rangle \geq -l_j\} \subset M_\mathbb{R} \quad (4.14)$$

where $(\rho_j, l_j) \in N \times \mathbb{Z}$ are given by:

$$\begin{cases} (\rho_i, l_i) &= (e_i, 0) \\ (\rho_{i+5}, l_{i+5}) &= ((0, 0, \dots, d_1 b_i / a_i, \dots, 0, b_i), 0) \\ (\rho_{i+10}, l_{i+10}) &= ((-c_i, -c_i, \dots, (\sum_{j \neq i} a_j - d_1 - 1) c_i / a_i, \dots, -c_i), c_i) \end{cases} \quad (4.15)$$

for $i = 1, 2, 3, 4, 5$. Here b_i is the smallest integer such that $d_1 b_i / a_i \in \mathbb{N}$, c_i is the smallest integer such that $(\sum_{j \neq i} a_j - d_1 - 1) c_i / a_i \in \mathbb{N}$, and $d_1 b_i / a_i$ and $(\sum_{j \neq i} a_j - d_1 - 1) c_i / a_i$ are in the i^{th} position. Hence the fan $\Sigma = \Sigma_P$ is a complete fan with 8 6-dimensional cones and 15 rays generated by the vectors ρ_i . Note that P does not depend on α so neither do Σ and F . The compactification of W_α in F corresponds to an element of the linear system $|\mathcal{O}_F(D)|$, where $D = \sum l_j D_j$ is the ample divisor of F associated to P .

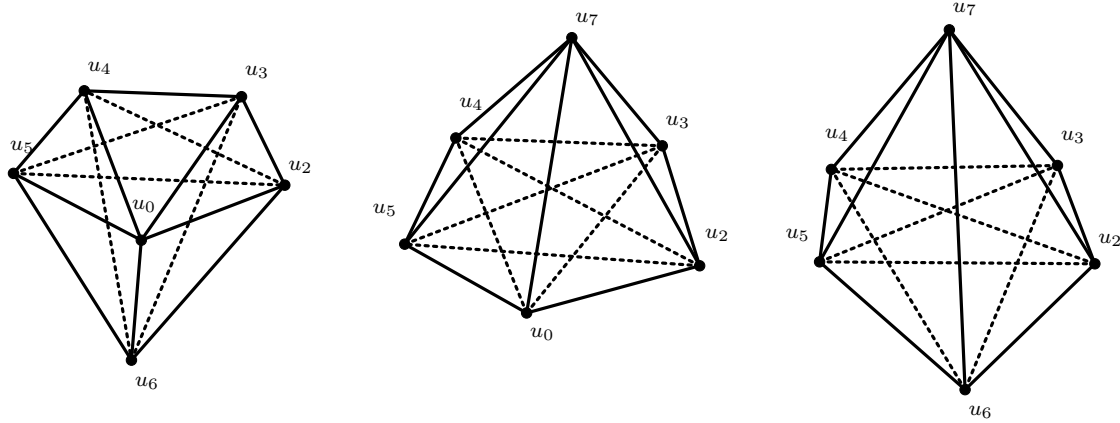


Figure 4.3: A picture of the facets of P with normals $\rho_1, \rho_6, \rho_{11}$.

The series $X_{6k+3,8k+2} \subset \mathbb{P}(2, 2k+1, 2k+1, 4k+1, 6k+1)$

Fix $k > 0$ integer. Let $W_{k,\alpha} = W_\alpha$ be the 5-fold in described by Equation (4.13) where $(a_1, a_2, a_3, a_4, a_5, d_1) = (2, 2k+1, 2k+1, 4k+1, 6k+1, 6k+3)$. Then the Newton polytope P of the polynomial defining W_α has facet presentation given by Equations (4.9) and 4.15. More precisely, $(\rho_i, l_i) = (e_i, 0)$ for $i = 1, 2, 3, 4, 5$, and

$$\left\{ \begin{array}{ll} (\rho_6, l_6) = & ((6k+3, 0, 0, 0, 0, 2), 0) \\ (\rho_7, l_7) = & ((0, 3, 0, 0, 0, 1), 0) \\ (\rho_8, l_8) = & ((0, 0, 3, 0, 0, 1), 0) \\ (\rho_9, l_9) = & ((0, 0, 0, 6k+3, 0, 4k+1), 0) \\ (\rho_{10}, l_{10}) = & ((0, 0, 0, 0, 6k+3, 6k+1), 0) \\ (\rho_{11}, l_{11}) = & ((4k, -1, -1, -1, -1, -1), 1) \\ (\rho_{12}, l_{12}) = & ((-(2k+1), 6k+1, -(2k+1), -(2k+1), -(2k+1), -(2k+1)), (2k+1)) \\ (\rho_{13}, l_{13}) = & ((-(2k+1), -(2k+1), 6k+1, -(2k+1), -(2k+1), -(2k+1)), (2k+1)) \\ (\rho_{14}, l_{14}) = & ((-1, -1, -1, 1, -1, -1), 1) \\ (\rho_{15}, l_{15}) = & ((-(6k+1), -(6k+1), -(6k+1), -(6k+1), 2k+1, -(6k+1), 6k+1)) \end{array} \right.$$

The variety $F = F_P$ is not $\mathbb{Q}$ -Gorenstein, and the $\mathbb{Q}$ -Gorenstein variety $\tilde{F} = F_{P(\varepsilon)}$ associated to the polytope P shrunk by $\varepsilon \ll 1$ has noncanonical singularities. The non-canonical 2-dimensional cones of the fan of $\tilde{F} = \tilde{F}_k$ vary with k ; however, the cones

$$\begin{aligned} & \langle \rho_4, \rho_9 \rangle_+ \quad \langle \rho_5, \rho_{10} \rangle_+ \quad \langle \rho_5, \rho_{15} \rangle_+ \quad \langle \rho_6, \rho_7 \rangle_+ \quad \langle \rho_6, \rho_8 \rangle_+ \quad \langle \rho_6, \rho_9 \rangle_+ \quad \langle \rho_6, \rho_{10} \rangle_+ \\ & \langle \rho_{10}, \rho_{15} \rangle_+ \quad \langle \rho_{11}, \rho_{12} \rangle_+ \quad \langle \rho_{11}, \rho_{13} \rangle_+ \quad \langle \rho_{11}, \rho_{15} \rangle_+ \quad \langle \rho_{12}, \rho_{15} \rangle_+ \quad \langle \rho_{13}, \rho_{15} \rangle_+ \end{aligned}$$

are noncanonical for all k . The ones corresponding to the worst singularities are $\langle \rho_{12}, \rho_{15} \rangle_+$, $\langle \rho_{13}, \rho_{15} \rangle_+$ and $\langle \rho_6, \rho_{10} \rangle_+$. The vectors $\frac{1}{2(4k+1)}(\rho_{12} + \rho_{15})$, $\frac{1}{2(4k+1)}(\rho_{12} + \rho_{15})$ correspond to valuations with discrepancy $-\frac{8k}{8k+2}$, and the vector $\frac{1}{6k+3}(\rho_6 + \rho_{10})$ corresponds to a valuation with discrepancy $-\frac{6k+1}{6k+3}$. Then we set:

$$\begin{cases} \rho_{16} = \frac{1}{2(4k+1)}(\rho_{12} + \rho_{15}) = (-1, 0, -1, -1, 0, -1) & l_{16} = 1 \\ \rho_{17} = \frac{1}{2(4k+1)}(\rho_{12} + \rho_{15}) = (-1, -1, 0, -1, 0, -1) & l_{17} = 1 \\ \rho_{18} = \frac{1}{6k+3}(\rho_6 + \rho_{10}) = (1, 0, 0, 0, 1, 1) & l_{18} = 0 \end{cases}$$

We build the 7-dimensional polyhedron P' in Equation (4.7), where $h = 18$. The last two threshold values are:

$$\frac{2k+1}{5k+3}, \frac{2}{5}$$

For all $\varepsilon \in (\frac{2k+1}{5k+3}, \frac{2}{5})$ the polytope $P(\varepsilon)$ is a 6-dimesional polytope with 14 vertices and 10 facets, whose normals are generated by the vectors $\rho_1, \rho_2, \rho_3, \rho_7, \rho_8, \rho_{11}, \rho_{14}, \rho_{16}, \rho_{17}, \rho_{18}$. Thus its normal fan $\bar{\Sigma}$ has 10 rays and 14 maximal cones. The toric variety $\bar{F}$ is $\mathbb{Q}$ -Gorenstein and has canonical singularities.

The polytope $P(2/5)$ is a segment, thus $S = \mathbb{P}^1$. More precisely, the fan of $P(2/5)$ has two maximal cones intersecting along the hyperplane

$$C_0 = \{n \in N_{\mathbb{R}} : n_2 + n_3 + 2n_4 + 3n_5 - 3n_6 = 0\}$$

The toric morphism $\varphi: \bar{F} \rightarrow \mathbb{P}^1$ is induced by the quotient map $N \rightarrow N/N_0$, where $N_0 = N \cap C_0$, sending $n \rightarrow n_2 + n_3 + 2n_4 + 3n_5 - 3n_6$.

The non-compact toric variety $G = \varphi^{-1}(\mathbb{C}^\times)$ is the variety associated to the subfan $\Delta \subset \bar{\Sigma}$ given by cones over the rays in C_0 . Choosing a splitting $N = N_0 \oplus N_1$, we can regard Δ as a fan Δ_0 in N_0 ; then $G \simeq G_0 \times \mathbb{C}^\times$, where G_0 is the 5-dimensional variety with fan Δ_0 . The rays of $\bar{\Sigma}$ in C_0 are the seven rays generated by $\rho_1, \rho_7, \rho_8, \rho_{14}, \rho_{16}, \rho_{17}, \rho_{18}$. Write these vectors as vectors in N_0 by forgetting the coordinate n_2 and let $M_0 = N_0^\vee$. Consider the divisor sequence

$$0 \rightarrow M_0 \rightarrow \mathbb{Z}^7 \rightarrow \mathbb{L}^\vee$$

A weight matrix for G_0 is the 2×7 matrix:

$$\mathcal{W} = \begin{bmatrix} 3 & 2 & 1 & 3 & 0 & 3 & 3 \\ -1 & -1 & 0 & -1 & 1 & -2 & -1 \end{bmatrix}$$

It is evident that the chamber of the secondary fan between the columns c_3 and c_5 of $\mathcal{W}$ corresponds to the weighted projective space $\mathbb{P} = \mathbb{P}(1, 2, 3, 3, 3, 3)$.

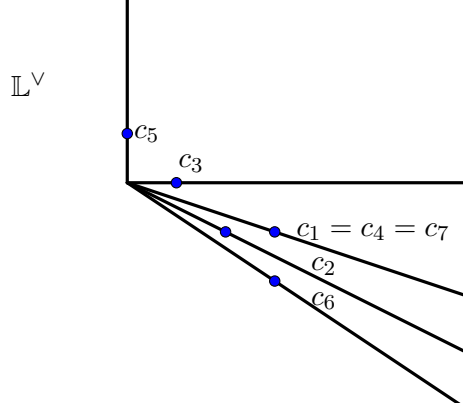


Figure 4.4: A picture of the columns c_i of $\mathcal{W}$ defining the secondary fan.

We rewrite the equation of $W_\alpha \subset \mathbb{T}^6$ in $G' = \mathbb{P} \times \mathbb{C}^\times$.³ Choose the basis of $N = N_0 \oplus N_1$ given by the vectors $\rho_7, \rho_1, \rho_{14}, \rho_{17}, \rho_{18}, \rho_2$ and denote by $v_1, v_2, v_3, v_4, v_5, t$ the dual basis. The change of coordinate matrix is:

$$\begin{bmatrix} 0 & 3 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 & -1 & -1 \\ -1 & -1 & 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

hence

$$u_1 = \frac{v_2 v_5}{v_3 v_4} \quad u_2 = \frac{t v_1^3}{v_3 v_4} \quad u_3 = \frac{1}{v_3} \quad u_4 = \frac{v_3}{v_4} \quad u_5 = \frac{v_5}{v_3} \quad u_6 = \frac{v_1 v_5}{v_3 v_4}$$

Thus W_α identifies with the 5-fold of Equation:

$$\left(v_2 v_5 + t v_1^3 + v_4 + v_3^2 + v_4 v_5 - v_1 v_5 - v_3 v_4 - a t^{2k+1} v_2^2 = 0 \right) \subset (\mathbb{C}^\times)^5 \times \mathbb{C}^\times$$

where $a = 1/\alpha$. The closure of $W_\alpha \subset \widehat{W}_\alpha$ in $\mathbb{P} \times \mathbb{C}^\times$ is given by the weighted homogeneous equation:

$$x_2 x_5 + t x_1^3 + x_0^3 x_4 + x_3^2 + x_4 x_5 - x_0 x_1 x_5 - x_3 x_4 - a t^{2k+1} x_2^2 = 0$$

where $x_0, \dots, x_5$ are the homogeneous coordinates on $\mathbb{P}(1, 2, 3, 3, 3, 3)$. The second projection $\widehat{W}_\alpha \rightarrow \mathbb{C}^\times$ is a Fano fibration, with fibres Fano 4-folds of degree 6.

³The toric variety $G' = \mathbb{P} \times \mathbb{C}^\times$ is a better ambient space where to study W_α than $G = G_0 \times \mathbb{C}^\times$. Indeed G is not $\mathbb{Q}$ -factorial, and G' is the output of an MMP for $K + D$ relative to the morphism $G \rightarrow \mathbb{C}^\times$, where $K = K_G$ and the closure of W_α in G is in $|\mathcal{O}_G(D)|$. This consists of a small resolution of G followed by the divisorial contraction of D_{16} , as it is easy to guess from Figure 4.4: G corresponds to $(6, -2) = 2c_1 \in L^\vee$, and $-(K + D)$ corresponds to $(9, -3) = 3c_1 \in L^\vee$.

4.4 Cubic threefolds

Consider the hypergeometric series G associated to the lists $\mathbf{p}, \mathbf{q} = (11, 2), (1, 3, 4, 5)$, and let $\mathbb{H}$ be the local system of solutions to the irreducible ODE annihilating G . By the discussion in Section 3.2, we expect an identity

$$\mathbb{H} = \mathrm{gr}_3^W R^3 v_U! \mathbb{Q}(1) \quad (4.16)$$

where $v: W \rightarrow \mathbb{C}^\times$ is the map on

$$W = (y_1 + y_2 = x_1 + x_2 + x_3 + x_4) \subset \mathbb{T}^5 \quad (4.17)$$

sending $(x_1 : \cdots : x_4 : y_1 : y_2) \mapsto \frac{x_1 x_2^3 x_3^4 x_4^5}{y_1^4 y_2^2}$, $U = \mathbb{C}^\times \setminus \{\frac{3^3 4^4 5^5}{11^{11} 2^2}\}$ and $v_U: W_U \rightarrow U$ is the restriction.

Solving Equation (4.17) for x_1 and working in the chart $(y_1 \neq 0)$, we can rewrite the fibre $W_\alpha = v^{-1}(\alpha)$ as:

$$(\alpha u_1^2 u_2^{-3} u_3^{-4} u_4^{-5} - (1 + u_1 - u_2 - u_3 - u_4) = 0) \subset \mathbb{T}^4 \quad (4.18)$$

where $\mathbb{T}^4 \simeq \mathbb{C}^\times$ is the 4-dimensional torus with coordinates u_i .

Below we show via our MMP algorithm that for all $\alpha \in U$ the 3-fold W_α compactifies to a smooth cubic 3-fold $\overline{W}_\alpha$ in $\mathbb{P}^4$. Assuming identity (4.16), this fact implies⁴ that the VHS supported by $\mathbb{H}$ cannot be a curve variation (see Section 3.4).

Let $M = \mathrm{Hom}(\mathbb{T}^4, \mathbb{C}^\times)$ and $N = M^\vee$. Denote by e_i the basis of N dual to the basis of M consisting of the coordinate functions u_i . The Newton polytope $P \subset M_\mathbb{R}$ of the polynomial in Equation (4.18) is a 4-dimensional lattice polytope with 6 vertices $u_0 = \mathbf{0}$, u_i , $i = 1, 2, 3, 4$, and $u_5 = (2, -3, -4, -5)$, and 8 facets determined by the pairs $(\rho_i, l_i) \in N \times \mathbb{Z}$:

$$\left\{ \begin{array}{ll} (\rho_1, l_1) = ((1, 0, 0, 0), 0) & (\rho_5, l_5) = ((-1, -1, -1, 1), 1) \\ (\rho_2, l_2) = ((2, 0, 1, 0), 0) & (\rho_6, l_6) = ((-3, 8, -3, -3), 3) \\ (\rho_3, l_3) = ((3, 2, 0, 0), 0) & (\rho_7, l_7) = ((-4, -4, 7, -4), 4) \\ (\rho_4, l_4) = ((5, 0, 0, 2), 0) & (\rho_8, l_8) = ((-5, -5, -5, 6), 5) \end{array} \right. \quad (4.19)$$

The compactification of W_α in $F = F_P$ corresponds to an element of the linear system $|O_F(D)|$, where $D = D_5 + 3D_6 + 4D_7 + 5D_8$.

⁴One can show that $\forall \alpha \in U \ \mathrm{gr}_3^W H_c^3(W_\alpha, \mathbb{Q}) = H_c^3(\overline{W}_\alpha, \mathbb{Q})$, as we do in Lemma 5.14.

The toric variety $F = F_P$ is not $\mathbb{Q}$ -Gorenstein; the fan of the $\mathbb{Q}$ -Gorenstein toric variety $\tilde{F} = F_{P(\varepsilon)}$, where $\varepsilon \ll 1$, has seven 2-dimensional noncanonical cones:

$$\langle \rho_5, \rho_8 \rangle_+ \quad \langle \rho_6, \rho_7 \rangle_+ \quad \langle \rho_6, \rho_8 \rangle_+ \quad \langle \rho_4, \rho_8 \rangle_+ \quad \langle \rho_5, \rho_6 \rangle_+ \quad \langle \rho_5, \rho_7 \rangle_+ \quad \langle \rho_7, \rho_8 \rangle_+$$

The vectors $\frac{1}{11}(\rho_5 + 2\rho_8)$, $\frac{1}{11}(\rho_6 + 2\rho_7)$, $\frac{1}{11}(2\rho_6 + \rho_8)$ correspond to valuations with discrepancy $-\frac{8}{11}$, the vector $\frac{1}{5}(2\rho_4 + \rho_8)$ corresponds to a valuation with discrepancy $-\frac{2}{5}$ and the vectors $\frac{1}{11}(\rho_5 + 7\rho_6)$, $\frac{1}{11}(7\rho_5 + \rho_7)$ and $\frac{1}{11}(7\rho_7 + \rho_8)$ correspond to valuations with discrepancies $-\frac{3}{11}$. Then we set

$$\begin{cases} \rho_9 &= \frac{1}{11}(\rho_5 + 2\rho_8) = (-1, -1, -1, 1) & a_9 = 1 \\ \rho_{10} &= \frac{1}{11}(\rho_6 + 2\rho_7) = (-1, 0, 1, -1) & a_{10} = 1 \\ \rho_{11} &= \frac{1}{11}(2\rho_6 + \rho_8) = (-1, 1, -1, 0) & a_{11} = 1 \end{cases}$$

We build the 5-dimensional polyhedron P' in Equation (4.7), where $h = 11$, and we find the sequence of threshold values:

$$\frac{11}{25}, \frac{1}{2}, \frac{3}{5}$$

For all $\varepsilon \in (1/2, 3/5)$ the polytope $P(\varepsilon)$ is a 4-simplex, whose normals are generated by the vectors $\rho_1, \rho_2, \rho_9, \rho_{10}, \rho_{11}$; these vectors satisfy the linear relation $\rho_9 + \rho_1 + \rho_2 + \rho_{10} + \rho_{11} = 0$, and any four of them form a basis of N . Thus $\bar{F} = F_{P(\varepsilon)}$ is the projective space $\mathbb{P}^4$. The polytope $P(3/5)$ is a point.

Denote by x_0, x_1, x_2, x_3, x_4 the homogeneous coordinates of $\mathbb{P}^4$. To determine the equation of the compactification $W_\alpha \subset \bar{W}_\alpha$ in $\mathbb{P}^4$, we choose $\rho_1, \rho_2, \rho_{10}, \rho_{11}$ as a basis of N , and we denote by y_1, y_2, y_3, y_4 the dual basis. The change of coordinates matrix is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 0 & 1 & -1 \\ -1 & 1 & -1 & 0 \end{bmatrix}$$

hence

$$u_1 = \frac{y_1 y_2^2}{y_3 y_4} \quad u_2 = y_4 \quad u_3 = \frac{y_1 y_2}{y_4} \quad u_4 = \frac{1}{y_3}$$

Thus W_α identifies with the 3-fold of Equation:

$$(\alpha y_1^2 = y_3 y_4 + y_1 y_2^2 - y_3 y_4^2 - y_2 y_3^2 - y_4) \subset \mathbb{T}^4$$

The closure $W_\alpha \subset \bar{W}_\alpha$ in $\mathbb{P}^4$ is described by the homogeneous equation:

$$(\alpha x_0 x_1^2 = x_0 x_3 x_4 + x_1 x_2^2 - x_3 x_4^2 - x_2 x_3^2 - x_0^2 x_4) \subset \mathbb{P}^4$$

This is a smooth cubic 3-fold in $\mathbb{P}^4$. □

Chapter 5

The Johnson–Kollár series

The work presented in this Chapter is joint with A. Corti and first appeared as [CG19].

The main piece of the classification of anticanonical quasismooth and wellformed two-dimensional weighted hypersurfaces [JK01] is given by the infinite series:

$$X_{8k+4} \subset \mathbb{P}(2, 2k+1, 2k+1, 4k+1) \quad k \in \mathbb{N}_{>0} \quad (5.1)$$

In this Chapter for all $k > 0$ integer we exhibit the regularised I -function of the surface $X = X_{8k+4}$ as a period of a pencil of hyperelliptic curves, and we show how this pencil arises from the pencil of 3-folds attached to the regularised I -function by the work of Beukers, Cohen, and Mellit. The pencil we construct is a LG mirror of the surface X . The feature which makes our construction especially interesting is that the anticanonical linear system of the surfaces (5.1) is empty, thus our LG mirrors fall out of the range of any known mirror construction.

In the last Section of the Chapter we discuss a conjectural explanation for the existence of mirrors of the surfaces (5.1) in terms of mirror symmetry of certain Calabi–Yau pairs.

5.1 Main result

5.1.1 Hyperelliptic integrals

For all $k > 0$ integer, the regularised I -function of the surface $X = X_{8k+4}$ is the hypergeometric series:

$$\widehat{I}_k(\alpha) = \sum_{j=0}^{\infty} \frac{((8k+4)j)!j!}{(2j)!((2k+1)j)!^2((4k+1)j)!} \alpha^j \quad (5.2)$$

This function is annihilated by the irreducible hypergeometric operator of order $6k + 2$:

$$H_k = \alpha_{k,0} \prod_{j=0}^{2k} \left(D - \frac{j}{2k+1} \right) \prod_{j=0}^{4k} \left(D - \frac{j}{4k+1} \right) - \alpha \prod_{\substack{j=1 \\ \gcd(j, 8k+4)=1}}^{8k+3} \left(D + \frac{j}{8k+4} \right)$$

where

$$\alpha_{k,0} = \frac{(4k+1)^{4k+1}}{4^{8k+3}(2k+1)^{2(2k+1)}} \quad (5.3)$$

Thus the local system $\mathbb{H}_k$ of solutions to the operator H_k is a irreducible complex local system of rank $6k + 2$ on the set $U_k = \mathbb{P}^1 \setminus \{0, \infty, \alpha_{k,0}\}$. The formula 2.13 shows that $\mathbb{H}_k$ has Hodge weight one.

Our pencil of curves Consider the pencil of hyperelliptic curves:

$$Y_k = (\alpha y^2 - h_k(\alpha, t_0, t_1) = 0) \subset \mathbb{P}(1, 1, 3k+2) \times \mathbb{C}^\times \quad (5.4)$$

where t_0, t_1, y are coordinates on $\mathbb{P}(1, 1, 3k+2)$, α is a coordinate on $\mathbb{C}^\times$ and

$$h_k(\alpha, t_0, t_1) = (4t_1^{2k+1} + \alpha t_0^{2k+1})(-64t_1^{4k+3} + t_0 t_1^{4k+2} - 32\alpha t_0^{2k+1} t_1^{2k+2} - 4\alpha^2 t_0^{4k+2} t_1)$$

together with $w_k: Y_k \rightarrow \mathbb{C}^\times$ the projection on the second factor.

Remark 5.1. For all $\alpha \neq \alpha_{k,0}$ the fibre $Y_{k,\alpha} = w_k^{-1}(\alpha)$ is a smooth hyperelliptic curve of genus $3k + 1$.

The period integral We define a pair (ω_k, γ_k) as follows. For all $\alpha \neq \alpha_{k,0}$:

- $\omega_{k,\alpha} = \frac{1}{2\pi i} \frac{t^{2k} dt}{y}$, where $t = t_1/t_0$;
- $\gamma_{k,\alpha}$ is the cycle that we describe in Section (5.1.2).

From this data we construct the period integral:

$$\pi_k(\alpha) = \frac{1}{2\pi i} \oint_{\gamma_{k,\alpha}} \frac{t^{2k} dt}{y} \quad (5.5)$$

Remark 5.2. The pair (Y_k, w_k) defines a LG model of dimension 2, according to Definition 2.16, and the period (5.5) is the period of the LG model (Y_k, w_k) relative to (ω_k, γ_k) .

Our primary result is:

Theorem 5.3. For all $k > 0$ integer, $\widehat{I}_k(\alpha) = \pi_k(\alpha)$, where

(A) $\widehat{I}_k(\alpha)$ is the hypergeometric function of Equation (5.2) above, and

(B) $\pi_k(\alpha)$ is the period of the LG model (Y_k, w_k) of Equation (5.5).

Remark 5.4. Theorem 5.3 can be interpreted as stating that LG model (Y_k, w_k) is a mirror of the surface X_{8k+4} (5.1).

Write $Y_{U_k} = w_k^{-1}(U_k)$ and denote by $w_{U_k}: Y_{U_k} \rightarrow U_k$ the restriction. By Theorem 5.3 and Remark 2.5, $\mathbb{H}_k$ is a subquotient of the local system $R^1 w_{U_k} \star \mathbb{C}$. Since the two local systems have both rank $6k + 2$, we obtain:

Corollary 5.5. *For all $k > 0$ integer, $\mathbb{H}_k$ is isomorphic to $R^1 w_{U_k} \star \mathbb{C}$.*

Remark 5.6. Corollary 5.5 shows that for the local system $\mathbb{H}_k$ the answer to Question 3.7 is positive.

5.1.2 The integration cycles

Denote by $p_{k,\alpha}: Y_{k,\alpha} \rightarrow \mathbb{P}^1$ the 2:1 cover. The cover $p_{k,\alpha}$ is branched at the $6k + 4$ roots of the polynomial:

$$h_{k,\alpha}(t) = t(4t^{2k+1} + \alpha)(-64t^{4k+2} + t^{4k+1} - 32\alpha t^{2k+1} - 4\alpha^2)$$

Lemma 5.7. *For $|\alpha| \ll 1$ the polynomial $h_{k,\alpha}$ has:*

- A root at $t = 0$;
- $4k + 1$ roots of norm $\sim |\alpha|^{\frac{2}{4k+1}}$;
- $2k + 1$ roots of norm $\sim |\alpha|^{\frac{1}{2k+1}}$;
- A root of norm $\sim \frac{1}{64}$.

Proof. Clearly $h_{k,\alpha}$ has a root at $t = 0$ and $2k + 1$ roots of norm $|\alpha/4|^{\frac{1}{2k+1}}$. Consider the polynomial $g_{k,\alpha}(t) = -64t^{4k+2} + t^{4k+1} - 32\alpha t^{2k+1} - 4\alpha^2$; then as $\alpha \rightarrow 0$ $g_{k,\alpha}(t)$ has a root $t_\alpha \sim \frac{1}{64}$ and $4k + 1$ roots t_α of norm $\sim |\alpha|^{\frac{2}{4k+1}}$. Indeed

$$\lim_{\alpha \rightarrow 0} g_{k,\alpha}(t) = t^{4k+1}(-64t + 1)$$

hence for $|\alpha| \ll 1$ $g_{k,\alpha}$ has a root that tends to $\frac{1}{64}$ and $4k + 1$ roots that tend to 0. Now t_α is a root if and only if $t_\alpha^{4k+1} = 4(4t_\alpha^{2k+1} + \alpha)^2$ and if $t_\alpha \rightarrow 0$, then $|t_\alpha|^{4k+1} \sim |\alpha|^2$. $\square$

Choose a continuous function $\rho_k: (0, 1) \rightarrow \mathbb{R}$ such that for $r \ll 1$:

$$r^{\frac{2}{4k+1}} \ll \rho_k(r) \ll r^{\frac{1}{2k+1}}$$

For $|\alpha| \ll 1$ let $\bar{\gamma}_{k,\alpha}$ be the circle of radius $\rho_k(|\alpha|)$ around the origin in $\mathbb{C}$ starting at $t_0 = \rho_k(|\alpha|)$. By Lemma 5.7 this circle divides $\mathbb{P}^1$ into two regions each containing an even number of branch points of $p_{k,\alpha}$, see Figure 5.1. Hence $p_{k,\alpha}^{-1}(\bar{\gamma}_{k,\alpha}) \subset Y_{k,\alpha}$ consists of two disjoint circles. We take $\gamma_{k,\alpha}$ to be the lift along which the power series expansion in Equation (5.6) is valid¹, see Figure 5.2.

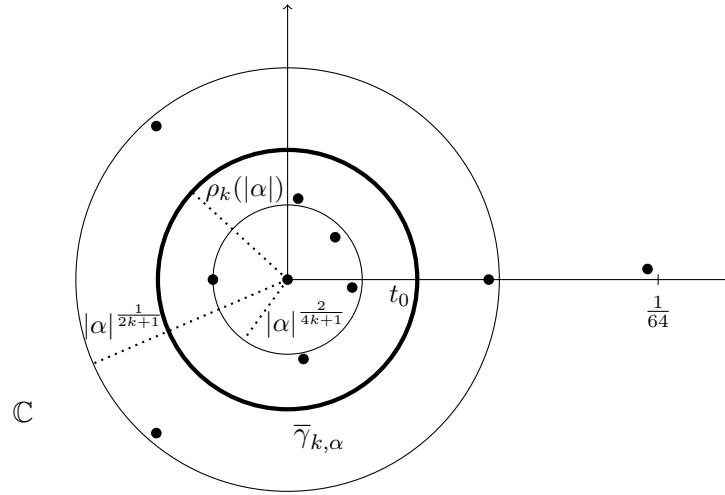


Figure 5.1: The circle $\bar{\gamma}_{k,\alpha} \subset \mathbb{C}$. The 10 marked points represent the roots of $h_{k,\alpha}$ for $k = 1$.

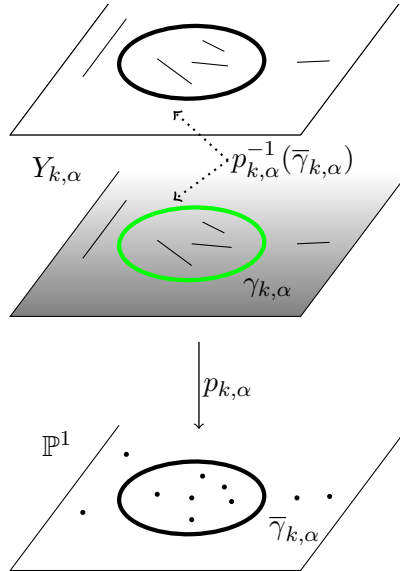


Figure 5.2: The circle $\gamma_{k,\alpha}$ on $Y_{k,\alpha}$. The gray sheet of the cover represents the one where (5.6) is valid. The 5 line segments on each sheet indicate a choice of branch cuts for $k = 1$.

¹Alternatively we choose the base point for our lift such that $y > 0$.

5.1.3 Proof of Theorem 5.3

The proof of Theorem 5.3 is an immediate consequence of the next two lemmas. In Lemma 5.8 we expand the period π_k (5.5) in power series. In Lemma 5.9 we show that the power series we obtain coincides with the hypergeometric series $\widehat{I}_k$ (5.2).

Lemma 5.8. *Let $\pi_k(\alpha)$ be the period integral described in Equation (5.5). Then, setting $m = (2k + 1)j$:*

$$\pi_k(\alpha) = \sum_{j=0}^{\infty} \binom{-1/2}{m} (-1)^m 4^{3m-j} \sum_{p=j}^{2m} \binom{-1/2}{p-j} \binom{2m}{p} \alpha^j \quad (5.6)$$

Proof. By a small manipulation we write the period as:

$$\pi_k(\alpha) = \frac{1}{2\pi i} \oint_{\bar{\gamma}_{k,\alpha}} \frac{1}{\sqrt{\left(1 + \frac{4t^{2k+1}}{\alpha}\right)}} \frac{1}{\sqrt{\left(1 - t \left(8 + \frac{2\alpha}{t^{2k+1}}\right)^2\right)}} \frac{dt}{t}. \quad (5.7)$$

By the defining inequalities of the function ρ_k , and our choice of $\bar{\gamma}_{k,\alpha}$, both the following power series expansions hold:

$$\begin{aligned} \left(1 + \frac{4t^{2k+1}}{\alpha}\right)^{-\frac{1}{2}} &= \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{4^n t^{(2k+1)n}}{\alpha^n} \\ \left(1 - t \left(8 + \frac{2\alpha}{t^{2k+1}}\right)^2\right)^{-\frac{1}{2}} &= \sum_{m=0}^{\infty} \binom{-1/2}{m} (-1)^m 4^{3m} t^m \left(1 + \frac{\alpha}{4t^{2k+1}}\right)^{2m} \end{aligned}$$

Plugging the two power series in (5.7), switching the series and the integral signs and using the binomial theorem we obtain:

$$\pi_k(\alpha) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{2m} \binom{-1/2}{n} \binom{-1/2}{m} \binom{2m}{p} (-1)^m 4^{3m+n-p} \alpha^{p-n} \frac{1}{2\pi i} \oint_{\bar{\gamma}_{k,\alpha}} \frac{dt}{t} t^{m-(2k+1)(p-n)}$$

Finally, applying the residue theorem around $t = 0$ and setting $j = p - n$, we obtain the result. $\square$

Lemma 5.9. *For all $k > 0$ and $j \geq 0$ integers, setting $m = (2k + 1)j$:*

$$\binom{-1/2}{m} (-1)^m 4^{3m-j} \sum_{p=j}^{2m} \binom{-1/2}{p-j} \binom{2m}{p} = \frac{((8k + 4)j)! j!}{(2j)! ((2k + 1)j)!^2 ((4k + 1)j)!} \quad (5.8)$$

Proof. Clearly (5.8) holds when $j = 0$, thus we assume $j \geq 1$. In what follows we repeatedly use the identity:

$$(2l - 1)!! = \frac{(2l)!}{2^l \cdot l!} \quad (l > 0 \text{ integer}) \quad (5.9)$$

We have:

$$\binom{-1/2}{m} = \frac{(-1)^m (2m-1)!!}{2^m m!} = \frac{(-1)^m}{4^m} \binom{2m}{m} \quad (5.10)$$

We set $i = p - j$ and we write the finite sum on the left hand side of (5.8) as:

$$\sum_{p=j}^{2m} \binom{-1/2}{p-j} \binom{2m}{p} = \sum_{i=0}^{2m-j} \binom{-1/2}{i} \binom{2m}{j+i} = \sum_{i=0}^{2m-j} \binom{-1/2}{i} \binom{2m}{2m-j-i} = \binom{2m-1/2}{2m-j}$$

where the last equality follows from the Chu–Vandermonde formula for generalised binomial coefficients:

$$\sum_{i=0}^n \binom{\beta}{i} \binom{\alpha}{n-i} = \binom{\beta+\alpha}{n} \quad (\alpha, \beta \in \mathbb{C}, n \in \mathbb{N})$$

Thus we can rewrite the left hand side of (5.8) as the product:

$$\binom{2m}{m} 4^{2m-j} \binom{2m-1/2}{2m-j}$$

Since $m = (2k+1)j$, this product is:

$$\binom{(4k+2)j}{(2k+1)j} 4^{(4k+1)j} \binom{(4k+2)j-1/2}{(4k+1)j} \quad (5.11)$$

Now we note that

$$\binom{(4k+2)j-1/2}{(4k+1)j} = \frac{((8k+4)j-1)!!}{(2j-1)!! 2^{(4k+1)j} ((4k+1)j)!}$$

Plugging this equality in the product (5.11) and using Equation (5.9) for $l = j$ and $l = (4k+2)j$, we obtain the right hand side of Equation (5.8). $\square$

5.2 Relation to the work of Beukers, Cohen, and Mellit

In this Section we explain how to find the equation of the curves $Y_{k,\alpha}$ starting from the 3-folds associated to $\widehat{I}_k$ by the work of Beukers, Cohen, and Mellit.

5.2.1 An isomorphism of local systems

By Section 3.5, for all $k > 0$ integer the pair (W_k, v_k) attached to the hypergeometric function $\widehat{I}_k$ by [BCM15] is given by the manifold:

$$W_k = (x_1 + x_2 + x_3 + x_4 - y_1 - y_2 = 0) \subset \mathbb{T}^5 \quad (5.12)$$

where $\mathbb{T}^5 \subset \mathbb{P}^5$ is the torus and the x_i, y_j are homogeneous coordinates on $\mathbb{P}^5$, together with the map $v_k: W_k \rightarrow \mathbb{C}^\times$ sending:

$$(x_1 : x_2 : x_3 : x_4 : y_1 : y_2) \mapsto \frac{x_1^2 (x_2 x_3)^{2k+1} x_4^{4k+1}}{y_1^{8k+4} y_2} \quad (5.13)$$

Let $\alpha_{k,0}$, U_k , and $w_{U_k}: Y_{U_k} \rightarrow U_k$ be as in Section 5.1.1.

The 3-fold $W_{k,\alpha} = v_k^{-1}(\alpha)$ is singular if and only if $\alpha = \alpha_{k,0}$. Write $W_{U_k} = v_k^{-1}(U_k)$ and denote by $v_{U_k}: W_{U_k} \rightarrow U_k$ the restriction. We expect an identity of VHS:

$$\mathbb{H}_k = \mathrm{gr}_3^W R^3 v_{U_k}! \mathbb{Q}(1) \quad (5.14)$$

We prove that:

Theorem 5.10. *For all $k > 0$ integer, there is an isomorphism:*

$$\mathrm{gr}_3^W R^3 v_{U_k}! \mathbb{Q}(1) = R^1 w_{U_k} \star \mathbb{Q} \quad (5.15)$$

Remark 5.11. This provides evidence for our Hodge–theoretic interpretation of the work [BCM15]. Indeed Theorem 5.10 and Corollary 5.5 imply identity (5.14) a posteriori.

Remark 5.12. The fact that the variation $\mathrm{gr}_3^W H^3(W_{k,\alpha}, \mathbb{Q})$ is the variation of H^1 of a pencil of curves is not trivial. As discussed in Section 3.4, there is no reason to expect it. To explain this, one naturally wonders whether the 3-folds $W_{k,\alpha}$ are rational: there are indeed very few examples of irrational 3-folds whose intermediate Jacobian is the Jacobian of a curve, see [Sar82, Sar80]. Our analysis in Section 5.2.3 shows this is probably not the case. This makes our Theorem 5.10 even more interesting.

5.2.2 Proof of Theorem 5.10

We prove that for all $\alpha \in U_k$ there is an isomorphism of pure Hodge structures:

$$\mathrm{gr}_3^W H_c^3(W_{k,\alpha}, \mathbb{Q})(1) = H^1(Y_{k,\alpha}, \mathbb{Q}) \quad (5.16)$$

For the rest of this Section we fix $k > 0$ and $\alpha \neq \alpha_{k,0}$ and we omit all reference to k and α .

Notation Following the discussion in Section 3.3 and setting $a = 1/\alpha$ for convenience, we rewrite the 3-fold $W = v^{-1}(\alpha)$ as:

$$\left(u_1 + u_2 + u_3 + u_4 - 1 - au_1^2 u_2^{2k+1} u_3^{2k+1} u_4^{4k+1} = 0 \right) \subset \mathbb{T}^4 \quad (5.17)$$

where $\mathbb{T}^4 \simeq (\mathbb{C}^\times)^4$ is the torus with coordinates u_i .

Similarly, we rewrite $Y = w^{-1}(\alpha)$ as

$$\left(a^2 y^2 - (4at^{2k+1} + 1)(a^2 t^{4k+2} - 4t(4at^{2k+1} + 1)^2) = 0 \right) \subset \mathbb{P}(1, 1, 3k+2) \quad (5.18)$$

Finally, we write $p_{\mathbb{C}^\times}: Y_{\mathbb{C}^\times} = p^{-1}(\mathbb{C}^\times) \rightarrow \mathbb{C}^\times$ for the restriction of the 2:1 branched cover $p: Y \rightarrow \mathbb{P}^1$.

The partial compactification In Section 4.3.1 we have constructed a partial compactification $W \subset \widehat{W}$ in the (non compact) toric variety

$$G = \mathbb{P}(1, 1, 2, 2) / \mu_2 \times \mathbb{C}^\times$$

where μ acts with weights $(0, 1, 1, 1)$ on $\mathbb{P}(1, 1, 2, 2)$. The 3-fold W identifies with the 3-fold of Equation (4.11) in the torus $\mathbb{T}_G \subset G$, and the closure $W \subset \widehat{W}$ in G is described by Equation (4.12).

Lemma 5.14 below states that $\mathrm{gr}_3^W H_c^3(W, \mathbb{Q}) = \mathrm{gr}_3^W H_c^3(\widehat{W}, \mathbb{Q})$ hence for our purpose we might as well work with $\widehat{W}$ in place of W . The advantage of working with $\widehat{W}$ is that, as stated in Lemma 5.15, the second projection $\phi: \widehat{W} \rightarrow \mathbb{C}^\times$ is a fibration with fibres del Pezzo surfaces. The point of our MMP algorithm is precisely to make this structure manifest. The crucial result is Lemma 5.18, which constructs an algebraic cycle $Z \subset \mathrm{CH}_2(Y \times_{\mathbb{C}^\times} \widehat{W})$ inducing an isomorphism $p_* \mathbb{Q}_Y \rightarrow R^2 \phi_* \mathbb{Q}_{\widehat{W}}(1)$.

Remark 5.13. The 3-fold $\widehat{W}$ is quasismooth and we think at it as a smooth orbifold. More precisely $\widehat{W}$ has nonisolated quotient singularities: although it is singular, it is a rational homology manifold. Because of this, for the purpose of cohomological computations, we can pretend that $\widehat{W}$ is smooth. Below we take the convention that the set of regular values of the map $\phi: \widehat{W} \rightarrow \mathbb{C}^\times$ is the set of values $t \in \mathbb{C}^\times$ such that the corresponding fibre $\widehat{W}_t$ is quasismooth.

Lemma 5.14. *There is an identity of pure Hodge structures:*

$$\mathrm{gr}_3^W H_c^3(W, \mathbb{Q}) = \mathrm{gr}_3^W H_c^3(\widehat{W}, \mathbb{Q}) \quad (5.19)$$

Proof. Note first that the 3-fold W is nonsingular but noncompact, thus $H_c^3(W, \mathbb{Q})$ is a mixed Hodge structure with weights ≤ 3 , and so is $H_c^3(\widehat{W}, \mathbb{Q})$. Consider the divisor $D = \widehat{W} \setminus W$, and denote by $i: D \hookrightarrow \widehat{W}$ and $j: W \hookrightarrow \widehat{W}$ the inclusions. We have a long exact sequence of mixed Hodge structures

$$\cdots \rightarrow H_c^2(D, \mathbb{Q}) \rightarrow H_c^3(W, \mathbb{Q}) \rightarrow H_c^3(\widehat{W}, \mathbb{Q}) \rightarrow H_c^3(D, \mathbb{Q}) \rightarrow \cdots$$

To prove (5.19), we check that $\mathrm{gr}_3^W H_c^2(D, \mathbb{Q}) = \mathrm{gr}_3^W H_c^3(D, \mathbb{Q}) = (0)$. To this end, we study the geometry of the surface D ; D is the union $D = \bigcup_{i=1}^4 D_i$, where:

$$D_1 = \widehat{W} \cap (x_1 = 0), \quad D_2 = \widehat{W} \cap (x_2 = 0), \quad D_3 = \widehat{W} \cap (y_1 = 0), \quad D_4 = \widehat{W} \cap (y_2 = 0)$$

One can check that $D \subset \widehat{W}$ is (locally the quotient of) a simple normal crossing divisor with no 0-dimensional stata. By setting

$$D^{[1]} = \bigsqcup_i D_i \quad \text{and} \quad D^{[2]} = \bigsqcup_{i < j} D_{ij}, \quad \text{with } D_{ij} = D_i \cap D_j$$

we get a strict simplicial resolution $D^{[2]} \rightrightarrows D^{[1]} \rightarrow D$ and the long exact sequence:

$$\cdots \rightarrow \bigoplus_i H_c^{m-1}(D_i, \mathbb{Q}) \rightarrow \bigoplus_{i < j} H_c^{m-1}(D_{ij}, \mathbb{Q}) \rightarrow H_c^m(D, \mathbb{Q}) \rightarrow \bigoplus_i H_c^m(D_i, \mathbb{Q}) \rightarrow \cdots$$

It follows that $H_c^2(D, \mathbb{Q})$ has weights ≤ 2 . Now choose $m = 3$ and examine $H_c^3(D, \mathbb{Q})$. On the left hand side, $\bigoplus_{i < j} H_c^2(D_{ij}, \mathbb{Q})$ has weights ≤ 2 . On the right hand side, $\bigoplus_i H_c^3(D_i, \mathbb{Q})$ a priori has weights ≤ 3 . To conclude, we next show that, in fact, for $i = 1, 2, 3, 4$, $H_c^3(D_i, \mathbb{Q})$ has weights < 3 . Consider first the surface D_1 , given by

$$\left(-at^{2k+1}y_1^2 + y_2^2 + y_1y_2 + tx_2^4 = 0 \right) / \mu_2 \subset \mathbb{P}(1, 2, 2) / \mu_2 \times \mathbb{C}^\times$$

This is the same as the surface

$$\left(-at^{2k+1}y_1^2 + y_2^2 + y_1y_2 + tz_2^2 = 0 \right) / \mu_2 \subset \mathbb{P}^2 / \mu_2 \times \mathbb{C}^\times \quad (5.20)$$

where z_2, y_1, y_2 are homogeneous coordinates of $\mathbb{P}^2$ and μ_2 acts as $(y_1, y_2) \mapsto (-y_1, -y_2)$ on the affine piece ($z_2 = 1$). Note that the quotient of $\mathbb{P}^2$ by the μ_2 -action is the weighted projective space $\mathbb{P}(1, 1, 2)$ with homogeneous coordinates $y_1, y_2, w_2 = z_2^2$. In $\mathbb{P}(1, 1, 2)$ (5.20) becomes

$$\left(at^{2k+1}y_1^2 - y_2^2 - y_1y_2 = tw_2 \right) \subset \mathbb{P}(1, 1, 2)$$

thus, since $t \neq 0$, we conclude that $D_1 \simeq \mathbb{P}^1 \times \mathbb{C}^\times$ and then $H_c^3(D_1, \mathbb{Q})$ is a pure Hodge structure of weight 2. An almost identical argument holds for D_2 . The surface D_4 is given by

$$\left(-at^{2k+1}y_1^2 + x_1^4 + tx_2^4 = 0 \right) / \mu_2 \subset \mathbb{P}(1, 1, 2) / \mu_2 \times \mathbb{C}^\times$$

which is the same as the surface

$$\left(-at^{2k+1}w_1 + z_1 + tz_2 = 0 \right) \subset \mathbb{P}^2 \times \mathbb{C}^\times$$

where z_1, z_2, w_1 are homogeneous coordinates on $\mathbb{P}^2$, thus also $D_4 \simeq \mathbb{P}^1 \times \mathbb{C}^\times$ and $H_c^3(D_4, \mathbb{Q})$ is a pure Hodge structure of weight 2. To conclude, consider the surface D_3 , given by

$$(y_2^2 + x_1^4 - x_1x_2y_2 + tx_2^4 = 0) / \mu_2 \subset \mathbb{P}(1, 1, 2) / \mu_2 \times \mathbb{C}^\times$$

By means of the substitution $y_2 \mapsto y_2 - x_1x_2/2$ write this as

$$\left(y_2^2 + x_1^4 - \frac{x_1^2x_2^2}{4} + tx_2^4 = 0\right)/\mu_2 \subset \mathbb{P}(1, 1, 2)/\mu_2 \times \mathbb{C}^\times$$

and note that this is the same as the surface

$$\left(w_2^2 = z_1z_2(-z_1^2 + \frac{z_1z_2}{4} - tz_2^2)\right) \subset \mathbb{P}(1, 1, 2) \times \mathbb{C}^\times \quad (5.21)$$

where $z_1 = x_1^2, z_2 = x_2^2, w_2 = y_2^2$ are homogeneous coordinates on $\mathbb{P}(1, 1, 2)$. A natural compactification of (5.21) is the surface $\overline{D}_3$ given by

$$\left(w_2^2 = z_1z_2(-t_0^2z_1^2 + t_0^2\frac{z_1z_2}{4} - t_0t_1z_2^2)\right) \subset \mathbb{P}(1, 1, 2) \times \mathbb{P}^1$$

Note that $\overline{D}_3$ is a 2:1 cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along a divisor in $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4, 2)|$, thus $\overline{D}_3$ is a rational surface and hence $H^3(\overline{D}_3, \mathbb{Q}) = (0)$. Then $H^3(D_3, \mathbb{Q})$ has weights ≤ 2 , since, by setting $\Gamma = \overline{D}_3 \setminus D_3$, we have the long exact sequence of mixed Hodge structures

$$\cdots \rightarrow H_c^2(\Gamma, \mathbb{Q}) \rightarrow H_c^3(D_3, \mathbb{Q}) \rightarrow H^3(\overline{D}_3, \mathbb{Q}) \rightarrow \cdots$$

and $\mathrm{gr}_3^W H^2(\Gamma, \mathbb{Q}) = (0)$. □

Lemma 5.15. *Let $\phi: \widehat{W} \rightarrow \mathbb{C}^\times$ be the projection onto the second factor. Denote by Δ the set of critical values of ϕ and write $\Omega = \mathbb{C}^\times \setminus \Delta$. Let $j: \Omega \hookrightarrow \mathbb{C}^\times$ be the natural inclusion and denote by $\phi_\Omega: \widehat{W}_\Omega = \phi^{-1}(\Omega) \rightarrow \Omega$ the induced morphism. Let $\delta_1, \delta_2, \delta \in \mathbb{C}[t]$ be the polynomials:*

$$\delta_1(t) = 4at^{2k+1} + 1 \quad \delta_2(t) = a^2t^{4k+2} - 4t \left(4at^{2k+1} + 1\right)^2 \quad \delta = \delta_1 \cdot \delta_2 \quad (5.22)$$

Write $\Omega_1 = \mathbb{C}^\times \setminus \{\delta_1 = 0\}$, $\widehat{W}_{\Omega_1} = \phi^{-1}(\Omega_1)$. Denote by K the function field $k(\mathbb{C}^\times) = \mathbb{C}(t)$. Then:

(1) $\Delta = \{\delta = 0\}$. If $\delta_1(t) = 0$, $\widehat{W}_t$ has a unique non quasismooth point $p_t = (0 : 0 : -2 : 1)$.

If $\delta_2(t) = 0$, $\widehat{W}_t$ has a unique non quasismooth point

$$q_t = \left(1 : \sqrt{\frac{2\delta_1(t)}{at^{2k+1}}} : \frac{1}{\delta_1(t)} : \frac{2at^{2k+1}}{\delta_1(t)}\right) \quad (5.23)$$

In both cases the non quasismooth point is an ordinary double point.

(2) After the change of coordinates:

$$y_1 \mapsto \frac{y_1}{2} - \frac{x_1x_2}{2(4at^{2k+1} + 1)} \quad \text{and} \quad y_2 \mapsto \frac{y_1}{2} + y_2 - \frac{x_1x_2}{2} \quad (5.24)$$

the equation of the fibre $\widehat{W}_t$ over $t \in \Omega_1$ is:

$$\left(-(4at^{2k+1} + 1)y_1^2 + y_2^2 + x_1^4 + tx_2^4 - \frac{at^{2k+1}}{4at^{2k+1} + 1}(x_1x_2)^2 = 0\right)/\mu_2 \subset G \quad (5.25)$$

- (3) For all $t \in \Omega$ the fibre $\widehat{W}_t$ is a quasismooth del Pezzo surface of degree 2 with two $1/4 \cdot (1, -1)$ points $p^\pm$ on $(x_1 = x_2 = 0)$, and intersects $(y_1 = y_2 = 0)$ in two points $q^\pm$. In the coordinates of Equation (5.25),

$$p_t^\pm = \left(0 : 0 : 1 : \pm \frac{\sqrt{\delta_1(t)}}{2} \right) \quad \text{and} \quad q_t^\pm = \left(\sqrt{\frac{at^{2k+1} \pm \sqrt{\delta_2(t)}}{2 \cdot \delta_1(t)}} : 1 : 0 : 0 \right) \quad (5.26)$$

- (4) For all $t \in \Omega$ the fibre $\widehat{W}_t$ has Picard rank $r = h^2(\widehat{W}_t, \mathbb{Q}) = 2$. More specifically, $\widehat{W}_t$ contains a configuration of lines as pictured in Figure 5.3, and a basis of $\text{Pic}(\widehat{W}_t)_\mathbb{Q}$ is given the classes of the curves $C_{t,1} \sim C_{t,4}$ and $C_{t,2} \sim C_{t,3}$.

- (5) In the coordinates of Equation (5.25), the map $\phi_1: Y_1 = \widehat{W} \cap (x_1 = x_2 = 0) \rightarrow \mathbb{C}^\times$ is a 2:1 cover branched along $(\delta_1 = 0)$, and the map $\phi_2: Y_2 = \widehat{W}_{\Omega_1} \cap (y_1 = y_2 = 0) \rightarrow \Omega_1$ is a 2:1 cover branched along $(\delta_2 = 0)$. In particular,

$$\phi_1^{-1}(t) = \begin{cases} p_t^\pm & \text{if } \delta_1(t) \neq 0 \\ p_t & \text{if } \delta_1(t) = 0 \end{cases} \quad \text{and} \quad \phi_2^{-1}(t) = \begin{cases} q_t^\pm & \text{if } \delta_2(t) \neq 0 \\ q_t & \text{if } \delta_2(t) = 0 \end{cases} \quad (5.27)$$

- (6) The Picard rank of $\widehat{W}_K$ is $\rho = 1$.

- (7) $R^1\phi_*\mathbb{Q} = R^3\phi_*\mathbb{Q} = (0)$.

- (8) The natural homomorphism $R^2\phi_*\mathbb{Q} \rightarrow j_*j^*R^2\phi_*\mathbb{Q} = j_*R^2\phi_{\Omega*}\mathbb{Q}$ is an isomorphism.

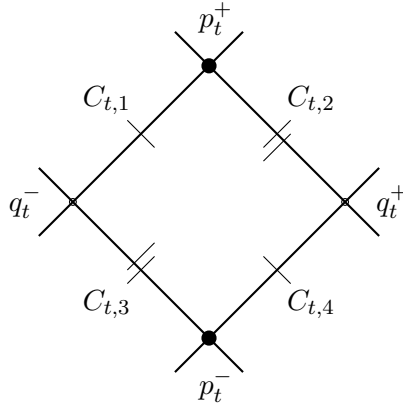


Figure 5.3: A quasismooth fibre $\widehat{W}_t$ and the four points $p_t^\pm, q_t^\pm$ in $\widehat{W}_t$. Numerically equivalent curves on $\widehat{W}_t$ are marked with the same symbol.

Proof. To prove (1), fix $t \in \mathbb{C}^\times$ and compute the Jacobian of the polynomial in Equation (4.12). If $t \in \Delta$ the non quasismooth point of $\widehat{W}_t$ is an ordinary double point since the Hessian of (4.12) at that point is invertible.

Assertion (2) is a simple substitution.

To prove (3), fix $t \in \Omega$ and consider the quotient map $\sigma: \mathbb{P}(1, 1, 2, 2) \rightarrow \mathbb{P}(1, 1, 2, 2)/\mu_2$. Let $\widehat{V}_t = \sigma^{-1}(\widehat{W}_t)$ be the preimage. Note that:

$$-K_{\widehat{V}_t} = \sigma^* \left(-K_{\widehat{W}_t} \right) \quad \text{and} \quad K_{\widehat{V}_t}^2 = 2 \cdot K_{\widehat{W}_t}^2$$

since σ is 2:1 and étale in codimension 1. We have $-K_{\widehat{V}_t} = \mathcal{O}(2)$ and $-K_{\widehat{V}_t}^2 = 4$, thus $\widehat{V}_t \subset \mathbb{P}(1, 1, 2, 2)$ is a del Pezzo surface of degree 4. The orbifold points of $\widehat{V}_t$, of type $1/2 \cdot (1, 1)$, are the two points $P_t^\pm$ of $\widehat{V}_t$ on the line $(x_1 = x_2 = 0)$. In the coordinates of Equation (5.25),

$$P_t^\pm = \left(0 : 0 : 1 : \pm \frac{\sqrt{\delta_1(t)}}{2} \right)$$

Note that on the affine piece $(y_1 = 1) \subset \mathbb{P}(1, 1, 2, 2)$ the group μ_2 acts by

$$(x_1, x_2, y_2) \rightarrow (ix_1, -ix_2, y_2)$$

Hence $\widehat{W}_t \subset \mathbb{P}(1, 1, 2, 2)/\mu_2$ is a del Pezzo surface of degree 2 with two $1/4 \cdot (1, -1)$ points $p_t^\pm = \sigma(P_t^\pm)$. These are the only orbifold points of $\widehat{W}_t$ since $(1 : 0 : 0 : 0)$ and $(0 : 1 : 0 : 0)$ do not satisfy Equation (5.25). Setting $y_1 = y_2 = 0$ in Equation (5.25), we find that $\widehat{W}_t \cap (y_1 = y_2 = 0)$ is given by the two points $q_t^\pm$ in Equation (5.26).

To prove (4), note that the crepant resolution $\widetilde{W}_t$ of $\widehat{W}_t$ is a smooth weak del Pezzo surface of degree 2. By Demazure's Theorem this is the blow-up of $\mathbb{P}^2$ in 7 nongeneral points, thus it has Picard rank $r = 8$. Then the surface $\widehat{W}_t$, obtained by blowing down 6 exceptional curves on $\widetilde{W}_t$, has Picard rank $r = 2$.

We explain how the geometry of $\widehat{W}_t$ singles out a distinguished basis of generators of $\text{Pic}(W_t)_{\mathbb{Q}}$. Setting

$$\nu_{\pm}(t) = \sqrt{\frac{at^{2k+1} \pm \sqrt{\delta_2(t)}}{2 \cdot \delta_1(t)}}$$

on $\mathbb{P}(1, 1, 2, 2)$ Equation (5.25) factors as:

$$(y_2 - \sqrt{\delta_1}y_1)(y_2 + \sqrt{\delta_1}y_1) + (x_1 - \nu_+x_2)(x_1 + \nu_+x_2)(x_1 - \nu_-x_2)(x_1 + \nu_-x_2) = 0$$

This exhibits eight lines on $\widehat{V}_t$, four of which passing through P_t^+ , the other four passing thorough P_t^- , as pictured on the left of Figure 5.4. The four lines on $\widehat{V}_t$ through P_t^+ correspond to two orbits under the μ_2 action, and the same holds for the four lines through P_t^- . Namely, in $\mathbb{P}(1, 1, 2, 2)/\mu_2$ Equation (5.25) can only be factored as:

$$(y_2 - \sqrt{\delta_1}y_1)(y_2 + \sqrt{\delta_1}y_1) + (x_1^2 - \nu_+^2x_2^2)(x_1^2 - \nu_-^2x_2^2) = 0$$

This exhibits four lines on the surface $\widehat{W}_t$, as pictured in the middle of Figure 5.4, two of which passing through p_t^+ :

$$C_{t,1}: (y_2 - \sqrt{\delta_1(t)}y_1 = x_1^2 - \nu_+^2(t)x_2^2 = 0) \quad C_{t,2}: (y_2 - \sqrt{\delta_1(t)}y_1 = x_1^2 - \nu_-^2(t)x_2^2 = 0) \quad (5.28)$$

the other two passing through p_t^- :

$$C_{t,3}: (y_2 + \sqrt{\delta_1(t)}y_1 = x_1^2 - \nu_+^2(t)x_2^2 = 0) \quad C_{t,4}: (y_2 + \sqrt{\delta_1(t)}y_1 = x_1^2 - \nu_-^2(t)x_2^2 = 0) \quad (5.29)$$

To see that $C_{t,1} \sim C_{t,4}$ note that, on $\widetilde{W}_t$, the union of the strict transform $\widetilde{C}_{t,1}$ and the three exceptional curves above p_t^+ , as pictured on the right of Figure 5.4, supports a fibre of a conic bundle where another fibre is supported on the union of the strict transform $\widetilde{C}_{t,4}$ and the three exceptional curves above p_t^- . Taking direct image this implies indeed that $C_{t,1} \sim C_{t,4}$. Similarly, $C_{t,2} \sim C_{t,3}$. Then the two curves $C_{t,1}$ and $C_{t,2}$ form a basis of $\text{Pic}(\widehat{W}_t)_{\mathbb{Q}}$.

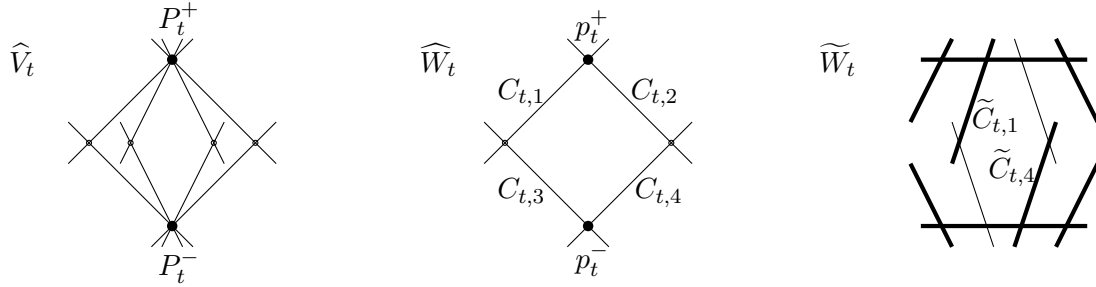


Figure 5.4: The surfaces $\widehat{V}_t$, $\widehat{W}_t$ and $\widetilde{W}_t$. The eight lines on $\widehat{V}_t$ descend to four lines on $\widehat{W}_t$. The two sets of bold lines on $\widetilde{W}_t$ correspond to two distinct fibers of a conic bundle over $\mathbb{P}^1$.

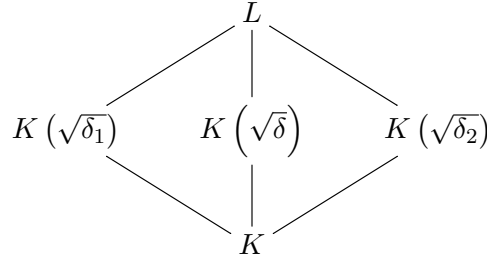
Statement (5) follows immediately from Equation (5.25), together with (1) and (3). In particular, to prove (5.27) one checks that, in the coordinates of Equation (4.12), if $\delta_1(t) = 0$ then $\phi_1^{-1}(t) = (0 : 0 : 2 : -1) = p_t$, and if $\delta_2(t) = 0$ then $\phi_2^{-1}(t)$ is given by

$$\left(\sqrt{\frac{at^{2k+1}}{2\delta_1(t)}} : 1 : \sqrt{\frac{at^{2k+1}}{2\delta_1^3(t)}} : \sqrt{\frac{2(at^{2k+1})^3}{\delta_1^3(t)}} \right)$$

which is the point q_t in (5.23).

We prove (6) as follows. Note that the function fields of Y_1 and Y_2 are $k(Y_1) = K(\sqrt{\delta_1})$ and $k(Y_2) = K(\sqrt{\delta_2})$. Set $L = K(\sqrt{\delta_1}, \sqrt{\delta_2})$ and consider the lattice of Galois field

extensions:



The Galois group $\text{Gal}(L/K) \simeq C_2 \times C_2$ is generated by the two automorphisms σ_1 and σ_2 , where:

$$\sigma_1|_{K(\sqrt{\delta_1})} = \text{id} \quad \sigma_1(\sqrt{\delta_2}) = -\sqrt{\delta_2} \quad \text{and} \quad \sigma_2|_{K(\sqrt{\delta_2})} = \text{id} \quad \sigma_2(\sqrt{\delta_1}) = -\sqrt{\delta_1}$$

By (4), $\text{Pic}(\widehat{W}_L)$ is generated by the classes of the curves C_1 and C_2 , of Equation (5.28). Then by Galois descent $\text{Pic}(\widehat{W}_K)$ has only one generator, given by the class of the curve $C_1 + C_2 + C_3 + C_4 \subset \widehat{W}_K$, since $C_1 + C_2 + C_3 + C_4$ is the only curve in $\widehat{W}_L$ which is invariant with respect to the induced action of $\text{Gal}(L/K)$.

Assertion (7) follows from the fact that, by (1) and (4), $\forall t \in \mathbb{C}^\times$ $H^1(\widehat{W}_t, \mathbb{Q}) = H^3(\widehat{W}_t, \mathbb{Q}) = (0)$.

To prove (8), consider a singular value $s \in \Delta$ and the fibre $\widehat{W}_s$. To check that $R^2\phi_*\mathbb{Q} \rightarrow j_*R^2\phi_{\Omega*}\mathbb{Q}$ is an isomorphism in a neighbourhood of s , it is enough to show that the natural homomorphism

$$H^2(\widehat{W}_s, \mathbb{Q}) \rightarrow H^2(\widehat{W}_t, \mathbb{Q})^{T_s}$$

is an isomorphism, where $t \in \mathbb{C}^\times$ is near s , $T_s: H^2(\widehat{W}_t, \mathbb{Q}) \rightarrow H^2(\widehat{W}_t, \mathbb{Q})$ is the local monodromy operator at s , and $H^2(\widehat{W}_t, \mathbb{Q})^{T_s}$ denotes the group of monodromy invariants. Indeed on the one hand, by the proper base change theorem, $H^2(\widehat{W}_s, \mathbb{Q})$ is the fibre at s of $R^2\phi_*\mathbb{Q}$, and on the other hand $H^2(\widehat{W}_t, \mathbb{Q})^{T_s}$ is the fibre at s of $j_*R^2\phi_{\Omega*}\mathbb{Q}$.

We have an exact triangle of constructible complexes on $\widehat{W}_s$:

$$\mathbb{Q} \rightarrow \psi\mathbb{Q} \rightarrow \varphi\mathbb{Q} \xrightarrow{+1}$$

where ψ and φ are the nearby and vanishing cycle functors [GRR72, Exposé I]. Since $\widehat{W}_s$ has isolated hyperquotient singularities — in fact by (4) it only has one non quasismooth point — $\varphi\mathbb{Q}$ is supported at the non quasismooth point of $\widehat{W}_s$ and is concentrated in degree 2 [Mil68]. Thus from the exact sequence:

$$(0) = H^1(\widehat{W}_s, \varphi\mathbb{Q}) \rightarrow H^2(\widehat{W}_s, \mathbb{Q}) \rightarrow H^2(\widehat{W}_s, \psi\mathbb{Q}) = H^2(\widehat{W}_t, \mathbb{Q}) \rightarrow \dots$$

we conclude that the natural homomorphism $H^2(\widehat{W}_s, \mathbb{Q}) \rightarrow H^2(\widehat{W}_t, \mathbb{Q})$ is injective. By the local invariant cycle Theorem [dC17, Theorem 1.4.1] then $H^2(\widehat{W}_s, \mathbb{Q})$ is the group of monodromy invariant cycles in $H^2(\widehat{W}_t, \mathbb{Q})$. $\square$

Remark 5.16. Combining Equations (5.18) and (5.22), note that the curve Y is given by:

$$(a^2y^2 - \delta(t) = 0) \subset \mathbb{P}(1, 1, 3k + 2)$$

Lemma 5.17. *There is an identity of mixed Hodge structures:*

$$H_c^3(\widehat{W}, \mathbb{Q}) = H_c^1(\mathbb{C}^\times, R^2\phi_*\mathbb{Q}) \quad (5.30)$$

Proof. Note that the two functors ϕ_* and $\phi_!$ coincide, since the map ϕ is proper. Consider the Leray spectral sequence of ϕ with second page $E_2^{p,q} = H_c^p(R^q\phi_*\mathbb{Q}) \implies H_c^{p+q}(\widehat{W}, \mathbb{Q})$. For reasons of space and since $R^3\phi_*\mathbb{Q} = R^1\phi_*\mathbb{Q} = (0)$ by Lemma 5.15(7), the spectral sequence degenerates at the second page. Indeed,

$$E_2 = \begin{bmatrix} H_c^0(\mathbb{C}^\times, \mathbb{Q}) & (0) & H_c^0(\mathbb{C}^\times, R^2\phi_*\mathbb{Q}) & (0) & H_c^0(\mathbb{C}^\times, R^4\phi_*\mathbb{Q}) \\ H_c^1(\mathbb{C}^\times, \mathbb{Q}) & (0) & H_c^1(\mathbb{C}^\times, R^2\phi_*\mathbb{Q}) & (0) & H_c^1(\mathbb{C}^\times, R^4\phi_*\mathbb{Q}) \\ H_c^2(\mathbb{C}^\times, \mathbb{Q}) & (0) & H_c^2(\mathbb{C}^\times, R^2\phi_*\mathbb{Q}) & (0) & H_c^2(\mathbb{C}^\times, R^4\phi_*\mathbb{Q}) \end{bmatrix}$$

thus $d_2: E_2^{p,q} \rightarrow E_2^{p+2,q-1}$ is zero, and there is no more cohomology for a nonzero d_k when $k > 2$. Then the groups contributing to $H_c^3(\widehat{W}, \mathbb{Q})$ are $H_c^0(\mathbb{C}^\times, R^3\phi_*\mathbb{Q}) = (0)$, $H_c^1(\mathbb{C}^\times, R^2\phi_*\mathbb{Q})$, and $H_c^2(\mathbb{C}^\times, R^1\phi_*\mathbb{Q}) = (0)$. $\square$

Lemma 5.18. *There is an isomorphism of mixed sheaves on $\mathbb{C}^\times$:*

$$p_{\mathbb{C}^\times \star} \mathbb{Q}_{Y_{\mathbb{C}^\times}} \rightarrow R^2\phi_* \mathbb{Q}_{\widehat{W}}(1) \quad (5.31)$$

Proof. The sketch of the proof is as follow: let $\Omega \subset \mathbb{C}^\times$ and $j: \Omega \hookrightarrow \mathbb{C}^\times$ be as in Lemma 5.15 and let $p_\Omega: Y_\Omega = p^{-1}(\Omega) \rightarrow \Omega$ be the induced morphism; we first construct a homomorphism

$$z: p_{\Omega \star} \mathbb{Q}_{Y_\Omega} \rightarrow R^2\phi_* \mathbb{Q}_{\widehat{W}_\Omega}(1) \quad (5.32)$$

and then we show that it is an isomorphism. This concludes the proof since on the one hand it is obvious that $j_* p_{\Omega \star} \mathbb{Q}_{Y_\Omega} = p_{\mathbb{C}^\times \star} \mathbb{Q}_{Y_{\mathbb{C}^\times}}$, on the other hand by Lemma 5.15(8) $j_* R^2\phi_{\Omega \star} \mathbb{Q}_{\widehat{W}_\Omega} = R^2\phi_* \mathbb{Q}_{\widehat{W}}$.

Consider the diagram

$$\begin{array}{ccc}
 & Y_\Omega \times_\Omega \widehat{W}_\Omega & \\
 \phi'_\Omega \swarrow & & \searrow p'_\Omega \\
 Y_\Omega & & \widehat{W}_\Omega \\
 p_\Omega \searrow & & \swarrow \phi_\Omega \\
 & \Omega &
 \end{array}$$

Below we construct a cycle

$$Z \in \text{CH}_2 \left(Y_\Omega \times_\Omega \widehat{W}_\Omega \right)$$

inducing the homomorphism z via the natural maps:

$$\begin{aligned}
 \text{CH}_2 \left(Y_\Omega \times_\Omega \widehat{W}_\Omega \right) &\rightarrow H_4^{BM} \left(Y_\Omega \times_\Omega \widehat{W}_\Omega \right) = \\
 &= \text{Hom}_{D_{cc}^b} \left(R p_{\Omega*} \mathbb{Q}_{Y_\Omega}, R \phi_{\Omega*} \mathbb{Q}_{\widehat{W}_\Omega}[2](1) \right) \rightarrow \text{Hom} \left(p_{\Omega*} \mathbb{Q}_{Y_\Omega}, R^2 \phi_{\Omega*} \mathbb{Q}_{\widehat{W}_\Omega}(1) \right)
 \end{aligned}$$

Namely, if s is a section of $p_{\Omega*} \mathbb{Q}_{Y_\Omega}$, then $z(s) = p'_{\Omega*} (Z \cap \phi'_\Omega{}^* s)$.

Let $\overline{Z} \subset \widehat{W}_\Omega$ be the union over all $t \in \Omega$ of the four curves $C_{t,1}, \dots, C_{t,4}$ in the fibre $\widehat{W}_t$ described in Lemma 5.15, that is:

$$\overline{Z} = \left(-(4at^{2k+1} + 1)y_1^2 + y_2^2 = x_1^4 - \frac{at^{2k+1}}{4at^{2k+1} + 1}(x_1x_2)^2 + tx_2^4 = 0 \right) / \mu_2 \subset \widehat{W}_\Omega$$

We have that:

$$Y_\Omega \times_\Omega \overline{Z} = Z_1 + Z_2 \subset Y_\Omega \times_\Omega \widehat{W}_\Omega$$

is the sum of two irreducible components and we take Z to be one of these components. In order to see this, let $K = \mathbb{C}(t)$ and $L = K(\sqrt{\delta_1}, \sqrt{\delta_2})$, as in the proof of Lemma 5.15, and consider the field extensions $K \subset K(\sqrt{\delta}) \subset L$, where $\delta = \delta_1 \delta_2$. The key point is to notice that Y_Ω is a 2:1 cover of Ω with function field $k(Y_\Omega) = K(\sqrt{\delta})$. Indeed, as observed in Remark 5.16, the curve Y is a 2:1 cover of $\mathbb{P}^1$ branched at the $6k + 4$ roots of the polynomial δ . Then, since the generator of the Galois group $\text{Gal}(L/K(\sqrt{\delta}))$ exchanges C_1 with C_4 and C_2 with C_3 on $\widehat{W}_L$, by Galois descent the cycle $C_1 + C_2 + C_3 + C_4$ on $\widehat{W}_{k(Y_\Omega)}$ splits into two components $C_1 + C_4$ and $C_2 + C_3$, each defined over $k(Y_\Omega)$ and corresponding to the two irreducible components Z_1 and Z_2 .

It remains to show that the induced homomorphism z is an isomorphism, which can be checked at the generic point η , or indeed at any point $t \in \Omega$. This is true because, as proved in Lemma 5.15(4), the set $\{C_1 + C_4, C_2 + C_3\}$ is a basis of $\text{Pic}(\widehat{W}_t)_\mathbb{Q} = H^2(\widehat{W}_t, \mathbb{Q})$. $\square$

Proof of Theorem 5.10 Let $W \subset \widehat{W}$ be the partial compactification of Equation (4.12) and $\phi: \widehat{W} \rightarrow \mathbb{C}^\times$ the projective degree 2 del Pezzo fibration of Lemma 5.15. Applying in sequence Lemmas 5.14, 5.17, 5.18 we see:

$$\mathrm{gr}_3^W H_c^3(W, \mathbb{Q}) = \mathrm{gr}_3^W H_c^3(\widehat{W}, \mathbb{Q}) = \mathrm{gr}_3^W H_c^1(\mathbb{C}^\times, R^2\phi_*\mathbb{Q}) = (\mathrm{gr}_1^W H_c^1(\mathbb{C}^\times, p_{\mathbb{C}^\times}\mathbb{Q}))(-1)$$

and the latter group is $H^1(Y, \mathbb{Q})(-1)$. $\square$

5.2.3 A conic bundle structure

The del Pezzo fibration $\phi: \widehat{W} \rightarrow \mathbb{C}^\times$ is not a Mori fibre space in the Mori category: $\widehat{W}$ has strictly canonical singularities along two sections of ϕ corresponding to the orbifold points $p_t^\pm \in \widehat{W}_t$ (all t) of Lemma 5.15(3).

In this Section we construct a birational model W' with terminal singularities and a Mori fibration $\psi: W' \rightarrow \mathbb{F}_1$ in the Mori category.

In light of the discussion in Section 3.4, it is natural to ask if our computation of the cohomology of W in terms of the fibration $\phi: \widehat{W} \rightarrow \mathbb{C}^\times$ really is the easiest way to find the group $H^1(Y, \mathbb{Q})$. We think that it is. The fibration $\psi: W' \rightarrow \mathbb{F}_1$ gives a richer but more complicated way to understand the cohomology of W .

Construction Consider the toric variety G' with weight matrix²:

t_1	t_2	v_1	v_2	z_1	z_2	z
1	1	0	-1	0	$-k-1$	$-k$
0	0	1	1	0	0	-2
0	0	0	0	1	1	1

and irrelevant ideal $(t_1, t_2)(v_1, v_2)(z, z_1, z_2)$. Note the morphism $G' \rightarrow \mathbb{F}_1$. Set

$$\widetilde{\delta}_1(t_1, t_2) = 4at_1^{2k+1} + t_2^{2k+1}$$

and consider the hypersurface $W' \subset G'$ given by

$$-z_1^2 + t_2\widetilde{\delta}_1z_2^2 + v_1v_2z^2 \left(\widetilde{\delta}_1v_1^2 - at_1^{2k+1}t_2v_1v_2 + t_1t_2\widetilde{\delta}_1v_2^2 \right) = 0 \quad (5.33)$$

Theorem 5.19. (A) W' has terminal singularities and the morphism $\psi: W' \rightarrow \mathbb{F}_1$ is a conic bundle Mfs;

²This matrix defines an action of $\mathbb{G}_m^3$ on $\mathbb{A}^7$ and G' is a GIT quotient.

(B) There is a commutative diagram:

$$\begin{array}{ccc} \widehat{W} & \dashrightarrow & W' \\ \phi \downarrow & & \downarrow \psi \\ \mathbb{C}^\times & \dashleftarrow & \mathbb{F}_1 \end{array}$$

Proof. (A) It is possible, and not too hard, to verify explicitly that W' has isolated cA singularities.

(B) We start by exhibiting an explicit birational map from a chart of G to a chart of G' .

Consider the chart $(t_2 = v_2 = z = 1) \subset G'$; this is isomorphic to $\mathbb{C}^4$ with coordinate functions t_1, v_1, z_1, z_2 . In this chart, W' is given by the equation:

$$-z_1^2 + \delta_1 z_2^2 + v_1 \left(\delta_1 v_1^2 - a t_1^{2k+1} v_1 + t_1 \delta_1 \right) = 0$$

Next consider the chart $(x_2 = 1) \subset G$; this is isomorphic to $\mathbb{G}_m \times \frac{1}{2}(1, 1, 1)$ with coordinate ring

$$\mathbb{C}[t] \otimes \mathbb{C}[x_1, y_1, y_2]^{\mu_2}$$

generated by the functions $t, x_1^2, x_1 y_1$, &c. In this chart, $\widehat{W}$ is given by the equation:

$$-\delta_1 y_1^2 + y_2^2 + x_1^4 - \frac{a t^{2k+1}}{\delta_1} x_1^2 + t = 0$$

We define a rational map from the chart in G to the chart in G' by:

$$v_1, z_1, z_2 \mapsto x_1^2, \delta_1 x_1 y_1, x_1 y_2$$

This map is in fact birational with inverse given by:

$$x_1^2, x_1 y_1, y_1^2, x_1 y_2, y_1 y_2, y_2^2 \mapsto v_1, \frac{z_1}{\delta_1}, \frac{z_1^2}{v_1 \delta_1^2}, z_2, \frac{z_1 z_2}{v_1 \delta_1}, \frac{z_2^2}{v_1}$$

It is easy to see that the equation for $\widehat{W}$ is transformed into the equation for W' . □

Remark 5.20. The variety W' has many singular points (more precisely, above $\widetilde{\delta}_1 = v_j = 0$, $t_1 = v_1 = 0$, $t_2 = v_j = 0$, $j = 1, 2$); thus the conic bundle $\psi: W' \rightarrow \mathbb{F}_1$, although it is a Mori fibre space, is not standard (in the birational geometry literature, a 3-fold Mfs $f: X \rightarrow S$ is a *standard conic bundle* if $\dim S = 2$ and X is nonsingular, which implies that S is also nonsingular, all fibres are conics, and the discriminant is normal crossing). Hence the conjectural rationality criteria [Isk96, Isk91b, Isk91a, Isk87] do not directly apply, though they suggest that W' is nonrational (and possibly even birationally rigid). It would be interesting to study the geometry of W' further.

Remark 5.21. The conic bundle $\psi: W' \rightarrow \mathbb{F}_1$ is an analog of a construction that in the singularity theory and mirror symmetry literature is called a “double suspension.” It is possible, of course, to find $H^1(Y, \mathbb{Q})$ in $H^3(W', \mathbb{Q})$, but it is easier to find it in $H^3(\widehat{W}, \mathbb{Q})$.

5.3 Mirror log Calabi-Yau pairs

In this Section we briefly discuss a conjectural explanation for the existence of LG mirrors of the surfaces (5.1) inspired to the works [Bor97, CKV20, Dol96, Voi93] on mirror Calabi-Yau pairs with dual involutions.

For all $k > 0$ integer, the surface $X = X_{8k+4}$ is the zero locus of a regular polynomial $p(x, y_1, y_2, z)$ of degree $8k + 4$ in the weighted homogeneous coordinates of the space $\mathbb{P}(2, 2k + 1, 2k + 1, 4k + 1)$.

Set $x = u^2$ and consider the surface:

$$\tilde{X} = \tilde{X}_{8k+4} \subset \mathbb{P}(1, 2k + 1, 2k + 1, 4k + 1) \quad (5.34)$$

defined by $p(u^2, y_1, y_2, z)$. The map $\tilde{X} \rightarrow X$ sending $u \mapsto x = u^2$ is a natural 2:1 cover of X , branched along the divisor $B = (x = 0) \subset \mathcal{O}_X(2)$. Moreover $-K_{\tilde{X}} = \mathcal{O}_{\tilde{X}}$. However, $\tilde{X}$ is not a K3 surface: it is easy to see that $\tilde{X}$ does not have du Val singularities. Therefore the results of the papers [Bor97, CKV20, Dol96, Voi93] are not directly applicable.

Nevertheless, one can consider the minimal resolution $\varphi: Z \rightarrow \tilde{X}$. The anticanonical divisor $-K_Z$ is a effective $\mathbb{Q}$ -Cartier divisor on Z . Then one can construct a log Calabi-Yau pair (Z, D) out of Z , and apply the intrinsic mirror symmetry program [GS18] to build a mirror log Calabi-Yau pair (V, E) . We believe that the mirror duality between (Z, D) and (V, E) could motivate the existence of a LG mirror of X .

Remark 5.22. It is natural to wonder whether there is a relation between the mirror (V, E) of (Z, D) and our LG mirror $(Y, w) = (Y_k, w_k)$ (5.4) of X .

Consider a surface $\tilde{Y}$ described in an affine patch by the equations

$$(v^2 - \delta_1(\alpha, t) = w^2 - \delta_2(\alpha, t) = 0) \subset \mathbb{A}^3 \times \mathbb{C}^\times$$

where δ_1 and δ_2 are the polynomials in Equation (5.22). The map $\tilde{Y} \rightarrow Y$ sending $(t, v, w) \mapsto (t, y = vw)$ is a natural 2:1 cover of Y , branched along the curve $C \subset Y$ determined by the $6k + 4$ ramification points of the curves $Y_\alpha = Y_{k, \alpha}$. In the spirit of [Bor97, CKV20, Dol96, Voi93], it would be very interesting to show that the surface $\tilde{Y}$ has trivial anticanonical bundle and that (V, E) arises from resolving $\tilde{Y}$.

Remark 5.23. The same argument above could be repeated for any anticanonical del Pezzo surface

$$X = X_d \subset \mathbb{P}(a_1, a_2, a_3, a_4) \quad \text{where} \quad a_1 = 2$$

Indeed the surface $\tilde{X} = \tilde{X}_d \subset \mathbb{P}(1, a_2, a_3, a_4)$ defined by $p(u^2, y_1, y_2, z)$, where p is the polynomial defining X in $\mathbb{P}(2, a_2, a_3, a_4)$ and u is the first coordinate of $\mathbb{P}(1, a_2, a_3, a_4)$, is a natural 2:1 cover of X , and has $-K_{\tilde{X}} = \mathcal{O}_{\tilde{X}}$. Instead, the argument is not applicable to anticanonical del Pezzo surfaces $X = X_d \subset \mathbb{P}(a_1, a_2, a_3, a_4)$, where $a_1 = n \geq 3$: the surface $\tilde{X} = \tilde{X}_d \subset \mathbb{P}(1, a_2, a_3, a_4)$ defined by $p(u^n, y_1, y_2, z)$, which is a natural $n : 1$ cover of X , is a surface of general type. Therefore, understanding mirror symmetry for these surfaces is especially interesting.

Chapter 6

Some sporadic cases in Table A.1

In this Chapter we study some of the sporadic cases of the Johnson–Kollár list.

In Section 6.1, for the three smooth cases in Table A.1, we prove that the Hori–Vafa mirror coincides with the LG model that the method outlined in Section 3.3 extracts out of the wrong dimension mirror; more precisely, we show that the fibres of the latter arise by blowing up $\mathbb{P}^3$ along the fibres of the Hori–Vafa mirror.

In Section 6.2, for the surface $X = X_{15} \subset \mathbb{P}(3, 3, 5, 5)$, we show that the variation of $\mathrm{gr}_3^W H_c^3(W_\alpha, \mathbb{Q})$ of the pencil of 3-folds associated to $\widehat{I}_X$ by [BCM15] is a subquotient of the variation of $\mathrm{gr}_1 H_c^1$ of a pencil of genus-six curves. The pencil is a candidate LG mirror for the surface X . This result is especially interesting in light of Remark 5.23: indeed X does not admit a natural cover $\widetilde{X} \rightarrow X$ with trivial anticanonical bundle.

6.1 The smooth cases

The first three sporadic cases of the Johnson–Kollár list are the surfaces:

$$X_3 \subset \mathbb{P}^3 \quad X_4 \subset \mathbb{P}(1, 1, 1, 4) \quad X_6 \subset \mathbb{P}(1, 1, 2, 3) \quad (6.1)$$

It is easy to see that these are the only smooth surfaces in Table A.1. Clearly each of these surfaces admits a Hori–Vafa mirror.

Here and in what follows, for X one of the surfaces in (6.1), we denote by (Y, w) and by (W, v) the Hori–Vafa (2.22) and the wrong dimension mirror (3.11) of X , and by $U \subset \mathbb{C}^\times$ the set of regular values of the maps v and w . We write $Y_\alpha = w^{-1}(\alpha)$ and $W_\alpha = v^{-1}(\alpha)$ for the fibres over $\alpha \in U$, and $\overline{Y}_\alpha \subset \mathbb{P}^2$ for the natural compactification of Y_α . The curve $\overline{Y}_\alpha \subset \mathbb{P}^2$ is an elliptic curve.

Remark 6.1. Denote by $\mathbb{H}_X$ the local system of solutions to the irreducible differential equation annihilating the regularised I -function of X . It is easy to check that $\mathbb{H}_X$ has rank 2, and the formula 2.13 shows that $\mathbb{H}_X$ has Hodge weight one. Therefore $\mathbb{H}_X$ coincides with the variation $\mathrm{gr}_1 H_c^1(Y_\alpha, \mathbb{Q}) = H^1(\overline{Y}_\alpha, \mathbb{Q})$.

We prove that:

Theorem 6.2. *For all $\alpha \in U$, W_α compactifies to the blow-up of $\mathbb{P}^3$ along a curve whose components are only rational curves and the elliptic curve $\overline{Y}_\alpha \subset \mathbb{P}^2$.*

An almost immediate consequence of Theorem 6.2 ([Voi07, Section 7.3.3]) is:

Corollary 6.3. *For all $\alpha \in U$, there is an identity:*

$$\mathrm{gr}_3^W H_c^3(W_\alpha, \mathbb{Q})(1) = H^1(\overline{Y}_\alpha, \mathbb{Q}) \quad (6.2)$$

Remark 6.4. Theorem 6.2 strongly suggests that the method outlined in Section 3.3 agrees with Hori–Vafa [HV00], and, more in general, as Theorem 5.10, provides strong evidence for our Hodge–theoretic interpretation of the work [BCM15]. Indeed an immediate consequence of Corollary 6.3 and Remark 6.1 is that $\mathbb{H}_X$ coincides with the variation $\mathrm{gr}_3^W H_c^3(W_\alpha, \mathbb{Q})(1)$.

Remark 6.5. It follows directly by Theorem 6.2 that the 3-fold W_α is rational. This is the most common situation in which the H^3 of a 3-fold equals the H^1 of a curve, and, as mentioned in Remark 5.12, is probably not the case of the fibres W_α of the wrong dimension mirrors of the surfaces (5.1).

In what follows we prove Theorem 6.2 for each of the surfaces (6.1).

6.1.1 Proof of Theorem 6.2 for $X_3 \subset \mathbb{P}^3$

The Hori–Vafa mirror of the cubic surface is the LG model given by the surface:

$$Y = (x_1 = x_2 + x_3 + x_4 = 1) \subset (\mathbb{C}^\times)^4$$

where x_i are coordinates on $(\mathbb{C}^\times)^4$, with the map $w = x_1 x_2 x_3 x_4$. The closure in $\mathbb{P}^2$ of the fibre $Y_\alpha = w^{-1}(\alpha)$ is the elliptic curve :

$$\overline{Y}_\alpha = (\alpha x^3 - (x - y - z)yz = 0) \subset \mathbb{P}^2 \quad (6.3)$$

where x, y, z are homogeneous coordinates on $\mathbb{P}^2$.

The wrong dimension mirror of the cubic surface is the LG model given by the 4-fold:

$$W = (x_1 + x_2 + x_3 + x_4 = y_1 + y_2) \subset \mathbb{T}^5$$

where $\mathbb{T}^5 \subset \mathbb{P}^5$ is the torus and x_i, y_j are homogeneous coordinates on $\mathbb{P}^5$, with the map $v = \frac{x_1 x_2 x_3 x_4}{y_1^3 y_2}$. Following the discussion in Section 3.3, we rewrite $W_\alpha = v^{-1}(\alpha)$ as:

$$W_\alpha = (\alpha(u_1 + u_2 + u_3 + u_4 - 1) - u_1 u_2 u_3 u_4 = 0) \subset \mathbb{T}^4 \quad (6.4)$$

where $\mathbb{T}^4$ is the torus with coordinates u_i .

Let $M = \text{Hom}(\mathbb{T}^4, \mathbb{C}^\times)$. The output of the algorithm applied to the Newton polytope P of the polynomial in Equation (6.4) is the polytope P itself. Indeed the toric variety $F = F_P$ is a (non $\mathbb{Q}$ -factorial) Fano variety with canonical singularities, and $-K_F - D$ is ample on F , where D is the divisor of P .

We study the variation of GIT of the toric variety $F = F_P$. The polytope P has six vertices and eight facets as in (4.10); more precisely,

$$\begin{cases} (\rho_5, l_5) = ((2, -1, -1, -1), 1) & (\rho_6, l_6) = ((-1, 2, -1, -1), 1) \\ (\rho_7, l_7) = ((-1, -1, 2, -1), 1) & (\rho_8, l_8) = ((-1, -1, -1, 2), 1) \end{cases}$$

Consider the divisor sequence

$$0 \rightarrow M \xrightarrow{\rho^*} \mathbb{Z}^8 \xrightarrow{\mathcal{W}} \mathbb{L}^\vee \rightarrow 0$$

A weight matrix for F is:

$$\mathcal{W} = \begin{bmatrix} -2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & -2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & -2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The vector $v = (1, 1, 1, 1) \in \mathbb{L}^\vee$ is a stability condition for F since $v = \mathcal{W}(D)$. It is easy to check that v is a ray of the secondary fan, and there are 24 chambers around v . These are given by the interiors of the cones:

$$\langle c_i, c_i + c_j, c_i + c_j + c_k, v \rangle_+$$

where c_i is the i^{th} column of $\mathcal{W}$, and $i, j, k \in \{5, 6, 7, 8\}$ are pairwise distinct.

Consider the toric variety F_{c_5} . The polytope P_{D_5} is a segment, thus $F_{c_5} = \mathbb{P}^1$. More precisely, the normal fan of P_{D_5} has two maximal cones intersecting along the hyperplane

$$C_0 = \{n \in N \mid n_2 + n_3 + n_4 = 0\}$$

Then let $F' = F_{v'}$, where $v' = (v+c_5)/2$. The fan Σ' of F' is a refinement of the normal fans of P and P_{D_5} . Thus, on the one hand, there is a nontrivial toric morphism $\varphi_1: F' \rightarrow F$. On the other hand, the quotient map $N \rightarrow N/N_0$, where $N_0 = N \cap C_0$, induces a toric morphism $\varphi_2: F' \rightarrow \mathbb{P}^1$.

We determine the non compact toric variety $G = \varphi_2^{-1}(\mathbb{C}^\times)$. The cones of Σ' contained in C_0 are:

$$\langle \rho_1, \rho_6, \rho_7 \rangle_+ \quad \langle \rho_1, \rho_6, \rho_8 \rangle_+ \quad \langle \rho_1, \rho_7, \rho_8 \rangle_+ \quad \langle \rho_6, \rho_7, \rho_8 \rangle_+$$

and $\rho_6, \rho_7, \rho_8, \rho_1$ satisfy the linear relation $\rho_6 + \rho_7 + \rho_8 + 3\rho_1 = 0$. Denote by N_{00} the 3-dimensional lattice spanned by ρ_6, ρ_7, ρ_8 and ρ_1 , and note that $N_0 = N_{00} + 1/3(0, 2, 1, 0)\mathbb{Z}$. Then, choosing a splitting $N = N_0 \oplus N_1$, we find that $G = \mathbb{P}(1, 1, 1, 3)/_{\mu_3} \times \mathbb{C}^\times$, where μ_3 acts with weights $(0, 2, 1, 0)$ on $\mathbb{P}(1, 1, 1, 3)$.

We determine the equation of $W_\alpha \subset \mathbb{T}^4$ in terms of the coordinates v_1, v_2, v_3, t dual to the basis of $N = N_0 \oplus N_1$ given by the four vectors $e_{0,1}, e_{0,2}, e_{0,3}, \rho_2$, where

$$e_{0,1} = (1, 0, 0, 0) \quad e_{0,2} = (0, -1, 1, 0) \quad e_{0,3} = (0, -1, 0, 1)$$

is a basis of N_0 . The coordinate change matrix is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

thus $u_1 = v_1, v_2 = \frac{t}{v_2 v_3}, u_3 = v_2, u_4 = v_3$. Then $W_\alpha \subset \mathbb{T}^4$ identifies with the 3-fold:

$$(\alpha(v_1 v_2 v_3 + t + v_2^2 v_3 + v_2 v_3^2 - v_2 v_3) - t v_1 v_2 v_3 = 0) \subset \mathbb{T}_G$$

The closure $W_\alpha \subset \overline{W}_\alpha$ in $\mathbb{P}^3 \times \mathbb{P}^1$ is given by:

$$(\alpha(t_0 x_1 x_2 x_3 + t_1 x_0^3 + t_0 x_2^2 x_3 + t_0 x_2 x_3^2 - t_0 x_0 x_2 x_3) - t_1 x_1 x_2 x_3 = 0) \subset \mathbb{P}^3 \times \mathbb{P}^1 \quad (6.5)$$

where the x_i are homogeneous coordinates on $\mathbb{P}^3$ and t_0, t_1 are homogeneous coordinates on $\mathbb{P}^1$. Rewriting the polynomial in Equation (6.5) as $t_0 f_3 + t_1 g_3$, where

$$f_3 = \alpha x_0^3 - x_1 x_2 x_3 \quad g_3 = \alpha x_2 x_3 (x_1 + x_2 + x_3 - x_0)$$

it is evident that $\overline{W}_\alpha$ is the blow up of $\mathbb{P}^3$ along the locus $(f_3 = g_3 = 0)$. This locus is given by the rational curves $(x_0^3 = x_3 = 0) \subset \mathbb{P}^3$, $(x_0^3 = x_2 = 0) \subset \mathbb{P}^3$, and the elliptic curve:

$$(\alpha x_0^3 - x_1 x_2 x_3 = 0, x_1 + x_2 + x_3 - x_0 = 0) \subset \mathbb{P}^3$$

This is the same as the elliptic curve $\overline{Y}_\alpha \subset \mathbb{P}^2$ in Equation (6.3). $\square$

6.1.2 Proof of Theorem 6.2 for $X_4 \subset \mathbb{P}(1, 1, 1, 2)$

The Hori–Vafa mirror of $X_4 \subset \mathbb{P}(1, 1, 1, 2)$ is the LG model given by the surface:

$$Y = (x_1 = x_2 + x_3 + x_4 = 1) \subset (\mathbb{C}^\times)^4$$

where x_i are coordinates on $(\mathbb{C}^\times)^4$, with the map $w = x_1 x_2 x_3 x_4^2$. The closure in $\mathbb{P}^2$ of the fibre $Y_\alpha = w^{-1}(\alpha)$ is the elliptic curve :

$$\overline{Y}_\alpha = (\alpha x^4 - (x - y - z)yz^2 = 0) \subset \mathbb{P}^2 \quad (6.6)$$

where x, y, z are homogeneous coordinates on $\mathbb{P}^2$.

The wrong dimension mirror of the cubic surface is the LG model given by the 4-fold:

$$W = (x_1 + x_2 + x_3 + x_4 = y_1 + y_2) \subset \mathbb{T}^5$$

where $\mathbb{T}^5 \subset \mathbb{P}^5$ is the torus and x_i, y_j are homogeneous coordinates on $\mathbb{P}^5$, with the map $v = \frac{x_1 x_2 x_3 x_4^2}{y_1^4 y_2}$. Following the discussion in Section 3.3, we rewrite $W_\alpha = v^{-1}(\alpha)$ as:

$$W_\alpha = (\alpha(u_1 + u_2 + u_3 + u_4 - 1) - u_1 u_2 u_3 u_4^2 = 0) \subset \mathbb{T}^4 \quad (6.7)$$

where $\mathbb{T}^4$ is the torus with coordinates u_i .

Let $M = \text{Hom}(\mathbb{T}^4, \mathbb{C}^\times)$ and $N = M^\vee$. The output of the algorithm applied to the Newton polytope P of the polynomial in Equation (6.7) is a polytope $\overline{P}$ such that the toric variety $\overline{F} = F_{\overline{P}}$ is a (non $\mathbb{Q}$ -factorial) Fano variety with canonical singularities. More precisely, $\overline{P}$ has six vertices and seven facets determined by the pairs $(\rho_i, l_i) \in N \times \mathbb{Z}$:

$$\left\{ \begin{array}{ll} (\rho_i, l_i) = (e_i, 0) & \text{for } i = 1, 2, 3 \\ (\rho_4, l_4) = ((3, -1, -1, -1), 1) \\ (\rho_5, l_5) = ((-1, 3, -1, -1), 1) \\ (\rho_6, l_6) = ((-1, -1, 3, -1), 1) \\ (\rho_7, l_7) = ((-1, -1, -1, 1), 1) \end{array} \right.$$

Consider the divisor sequence

$$0 \rightarrow M \xrightarrow{\rho^*} \mathbb{Z}^7 \xrightarrow{\mathcal{W}} \mathbb{L}^\vee \rightarrow 0$$

A weight matrix for $\overline{F}$ is:

$$\mathcal{W} = \begin{bmatrix} -2 & 2 & 2 & 1 & 0 & 0 & 1 \\ 2 & -2 & 2 & 0 & 1 & 0 & 1 \\ 2 & 2 & -2 & 0 & 0 & 1 & 1 \end{bmatrix}$$

The toric variety $\overline{F}$ coincides with F_{c_7} . Indeed, $\overline{F}$ is Fano, and $\mathcal{W}(-K_{\overline{F}}) = (4, 4, 4)$. The toric variety $F_{c_4} = \mathbb{P}^1$, since the polytope P_{D_4} is a segment. More precisely, the normal fan of P_{D_4} has two maximal cones intersecting along the hyperplane

$$C_0 = \{n \in N \mid n_2 + n_3 + 2n_4 = 0\}$$

Then let $F' = F_{v'}$, where $v' = (c_7 + c_4)/2$. The variety F' supports a non trivial toric morphism $\varphi_1: F' \rightarrow F$, and a toric morphism $\varphi_2: F' \rightarrow \mathbb{P}^1$ induced by the quotient map $N \rightarrow N_0$, where $N_0 = N \cap C_0$.

The preimage $G = \varphi_2^{-1}(\mathbb{C}^\times)$ is the noncompact toric variety $G = \mathbb{P}(1, 1, 2, 4)/\mu_2 \times \mathbb{C}^\times$, where μ_2 acts with weights $(0, 1, 1, 0)$ on $\mathbb{P}$. Indeed, the cones of Σ' contained in C_0 are:

$$\langle \rho_1, \rho_5, \rho_6 \rangle_+ \quad \langle \rho_1, \rho_5, \rho_7 \rangle_+ \quad \langle \rho_1, \rho_6, \rho_7 \rangle_+ \quad \langle \rho_5, \rho_6, \rho_7 \rangle_+$$

and $\rho_5, \rho_6, \rho_7, \rho_1$ satisfy the linear relation $\rho_5 + \rho_6 + 2\rho_7 + 4\rho_1 = 0$. Moreover, writing N_{00} for the 3-dimensional lattice spanned by $\rho_5, \rho_6, \rho_7, \rho_1$, one has $N_0 = N_{00} + 1/2(0, 1, 1, 0)\mathbb{Z}$.

We determine the equation of $W_\alpha \subset \mathbb{T}^4$ in terms of the coordinates v_1, v_2, v_3, t dual to the basis of N given by the four vectors $e_{0,1}, e_{0,2}, e_{0,3}, \rho_2$, where:

$$e_{0,1} = (1, 0, 0, 0) \quad e_{0,2} = (0, -1, 1, 0) \quad e_{0,3} = (0, -2, 0, 1)$$

is a basis of N_0 . The coordinate change matrix is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

thus $u_1 = v_1$, $v_2 = \frac{t}{v_2 v_3^2}$, $u_3 = v_2$, $u_4 = v_3$. Then $W_\alpha \subset \mathbb{T}^4$ identifies with the 3-fold:

$$(\alpha(v_1 v_2 v_3^2 + t + v_2^2 v_3^2 + v_2 v_3^3 - v_2 v_3^2) - t v_1 v_2 v_3^2 = 0) \subset \mathbb{T}_G$$

The closure $\overline{W}_\alpha$ of W_α in $\mathbb{P}^3 \times \mathbb{P}^1$ is given by:

$$(\alpha(t_0 x_1 x_2 x_3^2 + t_1 x_0^4 + t_0 x_2^2 x_3^2 + t_0 x_2 x_3^3 - t_0 x_0 x_2 x_3^2) - t_1 x_1 x_2 x_3^2 = 0) \subset \mathbb{P}^3 \times \mathbb{P}^1 \quad (6.8)$$

where the x_i are homogeneous coordinates on $\mathbb{P}^3$ and t_0, t_1 are homogeneous coordinates on $\mathbb{P}^1$. Rewriting the polynomial in Equation (6.8) as $t_0 f_4 + t_1 g_4$, where

$$f_4 = \alpha x_0^4 - x_1 x_2 x_3^2 \quad g_4 = \alpha x_2 x_3^2 (x_1 + x_2 + x_3 - x_0)$$

we see that $\overline{W}_\alpha$ is the blow up of $\mathbb{P}^3$ along the locus $(f_4 = g_4 = 0)$. This is given by the rational curves $(x_0^4 = x_3^2 = 0) \subset \mathbb{P}^3$, $(x_0^4 = x_2 = 0) \subset \mathbb{P}^3$, and the elliptic curve:

$$(\alpha x_0^4 - x_1 x_2 x_3^2 = 0, x_1 + x_2 + x_3 - x_0 = 0) \subset \mathbb{P}^3$$

This is the same as the elliptic curve $\overline{Y}_\alpha \subset \mathbb{P}^2$ of Equation (6.6). $\square$

6.1.3 Proof of Theorem 6.2 for $X_6 \subset \mathbb{P}(1, 1, 2, 3)$

The Hori–Vafa mirror of $X_6 \subset \mathbb{P}(1, 1, 2, 3)$ is the LG model given by the surface:

$$Y = (x_1 = x_2 + x_3 + x_4 = 1) \subset (\mathbb{C}^\times)^4$$

where x_i are coordinates on $(\mathbb{C}^\times)^4$, with the map $w = x_1 x_2 x_3^2 x_4^3$. The closure in $\mathbb{P}^2$ of the fibre $Y_\alpha = w^{-1}(\alpha)$ is the elliptic curve :

$$\overline{Y}_\alpha = (\alpha x^6 - (x - y - z)y^2 z^3 = 0) \subset \mathbb{P}^2 \quad (6.9)$$

where x, y, z are homogeneous coordinates on $\mathbb{P}^2$.

The wrong dimension mirror of the cubic surface is the LG model given by the 4-fold:

$$W = (x_1 + x_2 + x_3 + x_4 = y_1 + y_2) \subset \mathbb{T}^5$$

where $\mathbb{T}^5 \subset \mathbb{P}^5$ is the torus and x_i, y_j are homogeneous coordinates on $\mathbb{P}^5$, with the map $v = \frac{x_1 x_2 x_3^2 x_4^3}{y_1^6 y_2}$. Following the discussion in Section 3.3, we rewrite the fibre $W_\alpha = v^{-1}(\alpha)$ as:

$$W_\alpha = (\alpha(u_1 + u_2 + u_3 + u_4 - 1) - u_1 u_2 u_3^2 u_4^3 = 0) \subset \mathbb{T}^4 \quad (6.10)$$

where $\mathbb{T}^4$ is the torus with coordinates u_i .

Let $M = \text{Hom}(\mathbb{T}^4, \mathbb{C}^\times)$ and $N = M^\vee$. The output of the algorithm applied to the Newton polytope P of the polynomial in Equation (6.10) is a polytope $\overline{P}$ such that the toric variety $\overline{F} = F_{\overline{P}}$ is a (non $\mathbb{Q}$ -factorial) Fano variety with canonical singularities. More precisely, $\overline{P}$ has six vertices and six facets determined by the pairs $(\rho_i, l_i) \in N \times \mathbb{Z}$:

$$\left\{ \begin{array}{ll} (\rho_1, l_1) = ((1, 0, 0, 0), 0) & (\rho_4, l_4) = ((-1, 5, -1, -1), 1) \\ (\rho_2, l_2) = ((0, 1, 0, 0), 1) & (\rho_5, l_5) = ((-1, -1, 2, -1), 1) \\ (\rho_3, l_3) = ((5, -1, -1, -1), 1) & (\rho_6, l_6) = ((-1, -1, -1, 1), 1) \end{array} \right.$$

Consider the divisor sequence $0 \rightarrow M \xrightarrow{\rho^*} \mathbb{Z}^6 \xrightarrow{\mathcal{W}} \mathbb{L}^\vee \rightarrow 0$. A weight matrix for $\overline{F}$ is:

$$\mathcal{W} = \begin{bmatrix} 0 & 6 & 1 & 0 & 2 & 3 \\ 6 & 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

The vector $v = (1, 1) \in \mathbb{L}^\vee$ is a stability condition for $\overline{F}$, since $\mathcal{W}(-K_F) = (12, 12)$. The toric variety $F_{c_3} = \mathbb{P}^1$, since the polytope P_{D_3} is a segment. More precisely, the normal fan of P_{D_3} has two maximal cones intersecting along the hyperplane

$$C_0 = \{n \in N \mid n_2 + 2n_3 + 3n_4 = 0\}$$

Then, let $F' = F_{v'}$, where $v' = (v + c_3)/2$. The variety F' supports a non trivial toric morphism $\varphi_1: F' \rightarrow F$, and a toric morphism $\varphi_2: F' \rightarrow \mathbb{P}^1$ induced by the quotient map $N \rightarrow N_0$, where $N_0 = N \cap C_0$.

The preimage $G = \varphi_2^{-1}(\mathbb{C}^\times)$ is the noncompact toric variety $G = \mathbb{P}(1, 2, 3, 6) \times \mathbb{C}^\times$. Indeed, the cones of Σ' contained in C_0 are:

$$\langle \rho_1, \rho_4, \rho_5 \rangle_+ \quad \langle \rho_1, \rho_4, \rho_6 \rangle_+ \quad \langle \rho_1, \rho_5, \rho_6 \rangle_+ \quad \langle \rho_4, \rho_5, \rho_6 \rangle_+$$

The vectors $\rho_4, \rho_5, \rho_6, \rho_1$ satisfy the linear relation $\rho_4 + 2\rho_5 + 3\rho_6 + 6\rho_1 = 0$ and span N_0 .

We determine the equation of $W_\alpha \subset \mathbb{T}^4$ in terms of the coordinates v_1, v_2, v_3, t dual to the basis of N given by the four vectors $e_{0,1}, e_{0,2}, e_{0,3}, \rho_2$, where:

$$e_{0,1} = (1, 0, 0, 0) \quad e_{0,2} = (0, -2, 1, 0) \quad e_{0,3} = (0, -3, 0, 1)$$

is a basis of N_0 . The coordinate change matrix is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -3 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

thus $u_1 = v_1, v_2 = \frac{t}{v_2^2 v_3^3}, u_3 = v_2, u_4 = v_3$. Then $W_\alpha \subset \mathbb{T}^4$ identifies with the 3-fold:

$$(\alpha(v_1 v_2^2 v_3^3 + t + v_2^3 v_3^3 + v_2^2 v_3^4 - v_2^2 v_3^3) - t v_1 v_2^2 v_3^3 = 0) \subset \mathbb{T}_G$$

The closure $\overline{W}_\alpha$ of W_α in $\mathbb{P}^3 \times \mathbb{P}^1$ is given by:

$$(\alpha(t_0 x_1 x_2^2 x_3^3 + t_1 x_0^6 + t_0 x_2^3 x_3^3 + t_0 x_2^2 x_3^4 - t_0 x_0 x_2^2 x_3^3) - t_1 x_1 x_2^2 x_3^3 = 0) \subset \mathbb{P}^3 \times \mathbb{P}^1 \quad (6.11)$$

where the x_i are homogeneous coordinates on $\mathbb{P}^3$ and t_0, t_1 are homogeneous coordinates on $\mathbb{P}^1$. Rewriting the polynomial in Equation (6.11) as $t_0 f_6 + t_1 g_6$, where

$$f_6 = \alpha x_0^6 - x_1 x_2^2 x_3^3 \quad g_6 = \alpha x_2^2 x_3^3 (x_1 + x_2 + x_3 - x_0)$$

we see that $\overline{W}_a$ is the blow up of $\mathbb{P}^3$ along the locus $(f_6 = g_6 = 0)$. This locus is given by the rational curves $(x_0^6 = x_3^3 = 0) \subset \mathbb{P}^3$, $(x_0^6 = x_2^2 = 0) \subset \mathbb{P}^3$, and the elliptic curve:

$$(\alpha x_0^6 - x_1 x_2^2 x_3^3 = 0, x_1 + x_2 + x_3 - x_0 = 0) \subset \mathbb{P}^3$$

This is the same as the elliptic curve $\overline{Y}_\alpha \subset \mathbb{P}^2$ in Equation (6.9). $\square$

Remark 6.6. Observe that there is a recurrent link between the weights of the projective space $\mathbb{P} \supset X$ and the equation of the hyperplane C_0 .

Remark 6.7. For the non-smooth cases $X_d \subset \mathbb{P}(1, a_2, a_3, a_4)$ in Table A.1, the relation between the wrong dimension mirror and the Hori–Vafa mirror is not so straightforward. For instance, for the surface $X_{10} \subset \mathbb{P}(1, 2, 3, 5)$, the compactification $W_\alpha \subset \overline{W}_\alpha$ of the fibre of the wrong dimension mirror produced by our algorithm is:

$$\left(x_2 x_4 + x_0 x_1^5 + x_2^3 + x_3^2 - x_1 x_2 x_3 - \frac{1}{\alpha} x_0^2 x_4 = 0 \right) \subset \mathbb{P}(1, 1, 2, 3, 4)$$

where the x_i are weighted homogeneous coordinates on $\mathbb{P}(1, 1, 2, 3, 4)$. Extracting the fibre Y_α of the Hori–Vafa mirror out of the 3-fold $\overline{W}_\alpha$ is work in progress.

6.2 The surface $X_{15} \subset \mathbb{P}(3, 3, 5, 5)$

In this Section we study the regularised I -function of the surface $X = X_{15} \subset \mathbb{P}(3, 3, 5, 5)$. This is the first surface of Table A.1 living in a weighted projective space with all weights ≥ 3 . By Remark 5.23, this makes the surface X especially interesting.

6.2.1 A candidate LG mirror

The regularised I -function of the surface X is the hypergeometric series

$$\widehat{I}_X(\alpha) = \sum_{j=0}^{\infty} \frac{(15j)!j!}{(3j!)^2(5j)!^2} \alpha^j \quad (6.12)$$

Remark 6.8. This series is annihilated by the hypergeometric operator:

$$H_X = \prod_{i=0}^2 (D - i/3) \prod_{i=0}^4 (D - i/5) - 3^9 5^5 \alpha \prod_{\substack{i=1 \\ \gcd(i,15)=1}}^{14} (D + i/15)$$

Thus the local system $\mathbb{H}_X$ of solutions to the differential equation $H_X \cdot \varphi = 0$ is an irreducible complex local system of rank 8 on $U = \mathbb{C}^\times \setminus \{\alpha_0\}$, where $\alpha_0 = 1/3^9 5^5$. The formula 2.13 shows that $\mathbb{H}_X$ has Hodge weight one.

By the discussion in Section 3.5, the pair (W, v) associated to $\widehat{I}_X$ is given by the manifold

$$W = (y_1 + y_2 - x_1 - x_2 - x_3 - x_4 = 0) \subset \mathbb{T}^5 \quad (6.13)$$

where $\mathbb{T}^5 \subset \mathbb{P}^5$ is the torus and x_i, y_j are homogeneous coordinates on $\mathbb{P}^5$, together with the map $v: W \rightarrow \mathbb{C}^\times$ sending $(x_1 : x_2 : x_3 : x_4 : y_1 : y_2) \mapsto \frac{x_1^3 x_2^3 x_3^5 x_4^5}{y_1^{15} y_2}$. Write $W_U = v^{-1}(U)$ and denote by $v_U: W_U \rightarrow U$ the restriction. We expect an identity of VHS:

$$\mathbb{H}_X = \mathrm{gr}_3^W R^3 v_U! \mathbb{Q}(1) \quad (6.14)$$

Now consider the smooth surface:

$$Y = (x^3 - t = -27ty^3 + 27\alpha t^2 y + \alpha(27 - t) = 0) \subset \mathbb{A}^3 \times \mathbb{C}^\times \quad (6.15)$$

where t, x, y are coordinates on $\mathbb{A}^3$ and α is a coordinate on $\mathbb{C}^\times$, together with the second projection $w: Y \rightarrow \mathbb{C}^\times$. Let $P \subset \mathbb{A}^2 \times \mathbb{C}^\times$ be the surface cut out of $\mathbb{A}^2 \times \mathbb{C}^\times$ by the second equation in (6.15), namely:

$$P = (-27ty^3 + 27\alpha t^2 y + \alpha(27 - t) = 0) \subset \mathbb{A}^2 \times \mathbb{C}^\times \quad (6.16)$$

and write $z: P \rightarrow \mathbb{C}^\times$ for the second projection.

Remark 6.9. The fibres $Y_\alpha = w^{-1}(\alpha)$ and $P_\alpha = z^{-1}(\alpha)$ are singular if and only if $\alpha = \alpha_0$.

Let $Y_U = w^{-1}(U)$, $P_U = z^{-1}(U)$, and write $w_U: Y_U \rightarrow U$ and $z_U: P_U \rightarrow U$ for the restrictions. We prove that:

Theorem 6.10. *There is a short exact sequence:*

$$0 \longrightarrow \mathrm{gr}_1^W R^1 z_U! \mathbb{Q} \longrightarrow \mathrm{gr}_1^W R^1 w_U! \mathbb{Q} \longrightarrow \mathrm{gr}_3^W R^3 v_U! \mathbb{Q}(1) \longrightarrow 0 \quad (6.17)$$

Observe that, if identity (6.14) holds, Theorem 6.10 implies the following

Expectation: The local system $\mathbb{H}_X$ identifies with the quotient of $\mathrm{gr}_1^W R^1 w_U! \mathbb{Q}$ described by the short exact sequence (6.17).

Remark 6.11. The statement in the Expectation can be interpreted as stating that the LG model (Y, w) is a mirror of $X = X_{15}$. Finding a period of (Y, w) whose power series expansion coincides with $\widehat{I}_X$ is work in progress.

Remark 6.12. By the work [DK86] the rank of $\mathrm{gr}_3^W R^3 v_U! \mathbb{Q}$ is 8 (see Lemma 6.13), thus matches with the rank of $\mathbb{H}_X$. Then, since $\mathbb{H}_X$ is physically rigid, a way to prove identity (6.14) could be to show that the local monodromies of $\mathbb{H}_X$ and $\mathrm{gr}_3^W R^3 v_U! \mathbb{Q}$ are conjugate, see [Fed18, Proposition 2.4].

6.2.2 Proof of Theorem 6.10

We prove that for all $\alpha \in U$ there is a short exact sequence:

$$0 \longrightarrow \mathrm{gr}_1^W H_c^1(P_\alpha, \mathbb{Q}) \longrightarrow \mathrm{gr}_1^W H_c^1(Y_\alpha, \mathbb{Q}) \longrightarrow \mathrm{gr}_3^W H_c^3(W_\alpha, \mathbb{Q}) \xrightarrow{(1)} 0 \quad (6.18)$$

For the rest of this Section we fix $\alpha \in U$ and we omit all the reference to α .

Notation Following the discussion in Section 3.3 and setting $a = 1/\alpha$ for convenience, we rewrite the 3-fold $W = v^{-1}(\alpha)$ as:

$$W = (u_1 + u_2 + u_3 + u_4 - 1 - au_1^3 u_2^3 u_3^5 u_4^5 = 0) \subset \mathbb{T}^4 \quad (6.19)$$

where $\mathbb{T}^4 \simeq (\mathbb{C}^\times)^4$ is the torus with coordinates u_i .

Similarly, we rewrite $P = z^{-1}(\alpha)$ as:

$$P = (27(-aty^3 + t^2y + 1) - t = 0) \subset \mathbb{A}^2 \quad (6.20)$$

thus

$$Y = (x^3 - t = 27(-aty^3 + t^2y + 1) - t = 0) \subset \mathbb{A}^3 \quad (6.21)$$

The compactification In what follows we construct a suitable compactification W' of the 3-fold W in Equation (6.19).

Let $M = \mathrm{Hom}(\mathbb{T}^4, \mathbb{C}^\times)$ and $N = M^\vee$. The output the MMP algorithm applied to the Newton polytope of the polynomial in Equation (6.19) is a polytope $\overline{P}$ such that the toric variety $\overline{F} = F_{\overline{P}}$ is a (non $\mathbb{Q}$ -factorial) Fano variety with canonical singularities. More precisely, $\overline{P}$ has six vertices and six facets determined by the pairs $(\rho_i, l_i) \in N \times \mathbb{Z}$:

$$\left\{ \begin{array}{ll} (\rho_1, l_1) = ((1, 0, 0, 0), 0) & (\rho_4, l_4) = ((1, -4, -1, -1), 1) \\ (\rho_2, l_2) = ((0, 1, 0, 0), 0) & (\rho_5, l_5) = ((-1, -1, 2, -1), 1) \\ (\rho_3, l_3) = ((4, -1, -1, -1), 1) & (\rho_6, l_6) = (-1, -1, -1, 2), 1) \end{array} \right.$$

Consider the divisor sequence: $0 \rightarrow M \xrightarrow{\rho^*} \mathbb{Z}^6 \xrightarrow{\mathcal{W}} \mathbb{L}^\vee \rightarrow 0$. A weight matrix for $\overline{F}$ is

$$\mathcal{W} = \begin{bmatrix} -5 & 5 & 1 & -1 & 0 & 0 \\ 3 & -2 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Write c_i for the columns of $\mathcal{W}$. We have that $\mathcal{W}(-K_{\overline{F}}) = (0, 4)$, thus $\overline{F} = F_{c_5}$. Consider the chamber of the secondary fan between the rays generated by c_2 and c_3 . For all ω in this chamber, F_ω is the toric variety:

$$F' = \mathbb{P}(2, 3, 5, 5, 5)/\mu_3$$

where μ_3 acts with weights $(1, 0, 2, 0, 1)$. To see this, choose for instance the stability condition $\omega = (c_2 + c_3)/2 = (3, -1)$; then $F_\omega = F_{P_D}$, where $D \in \mathcal{W}^{-1}(\omega)$. The polytope P_D is a 4-simplex whose facet normals are generated by the vectors $\rho_1, \rho_3, \rho_4, \rho_5, \rho_6$. These vectors satisfy the relation $2\rho_3 + 3\rho_4 + 3\rho_5 + 5\rho_6 + 5\rho_1 = 0$, and, denoting by N' the lattice spanned by $\rho_3, \rho_4, \rho_5, \rho_6, \rho_1$, we have $N = N' + \frac{1}{3}(1, 0, 2, 0, 1)\mathbb{Z}$.

We write the equation of $W \subset \mathbb{T}^4$ in F' . Choose $\rho_3, \rho_5, \rho_6, \rho_1$ as a basis of $N_{\mathbb{R}}$ and write x, y, z, w for the dual basis. The change of coordinate matrix is:

$$\begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & 2 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Hence

$$u_1 = \frac{x^4 w}{yz} \quad u_2 = \frac{1}{xyz} \quad u_3 = \frac{y^2}{xz} \quad u_4 = \frac{z^2}{xy}$$

Thus W identifies with the 3-fold:

$$(x^5 w + 1 + y^3 + z^3 - xyz - aw^3 = 0) / \mu_3 \times \mu_3 \subset \mathbb{T}_{F'}$$

where $\mu_3 \times \mu_3$ acts with weights $(1, 1, 1, 1), (1, 2, 0, 1)$. The closure $W \subset W'$ in F' is given by the equation:

$$(x_1^5 y_3 + x_2^5 + y_1^3 + y_2^3 - x_1 x_2 y_1 y_2 - ay_3^3 = 0) / \mu_3 \subset F' \quad (6.22)$$

where x_1, x_2, y_1, y_2, y_3 are weighted homogeneous coordinates on F' .

Lemma 6.13. *The rank of $\mathrm{gr}_3^W H_c^3(W, \mathbb{Q})$ is 8.*

Proof. For $(p, q) = (3, 0)$ and $(2, 1)$ we compute the numbers

$$e^{p,q}(W) = \sum_k (-1)^k h^{p,q}(H_c^k(W, \mathbb{Q}))$$

where $h^{p,q}(H_c^k(W, \mathbb{Q})) = \dim H^{p,q}(\mathrm{gr}_{p+q}^W H_c^k(W, \mathbb{Q}))$. The Newton polytope P of the polynomial in Equation (6.19) does not have interior points. Thus by [DK86, Proposition 5.8] $e^{3,0} = 0$. Moreover, since the facets of P do not have interior points and $2P$ has four interior points, by [DK86, Corollary 5.9] $e^{2,1} = -4$.

It remains to check that for $k \neq 3$ $\mathrm{gr}_3^W H_c^k(W, \mathbb{Q}) = (0)$. This is obvious for $k < 3$: indeed W is nonsingular but noncompact, thus $H_c^k(W, \mathbb{Q})$ has weights $\leq k$. On the other hand, with the same techniques of Lemma 5.14, one can show that for $k > 3$ $\mathrm{gr}_3^W H_c^k(W, \mathbb{Q}) = \mathrm{gr}_3^W H_c^k(W', \mathbb{Q})$, and this is trivial since W' is a quasismooth compact 3-fold. $\square$

A pencil of cubic surfaces By manipulating the equation of $W' \subset F'$ we show that the 3-fold W can be rewritten as the intersection of the 3-fold $\widehat{W}$:

$$(t^2 u^2 z + u^3 + t^2 x^3 + y^3 - tuxy - atz^3 = 0) \subset \mathbb{P}^3 \times \mathbb{C}^\times \quad (6.23)$$

with $\mathbb{T}^3 \times \mathbb{C}^\times$, where $\mathbb{T}^3 \subset \mathbb{P}^3$ is the 3-dimensional torus. Here u, x, y, z are homogeneous coordinates on $\mathbb{P}^3$ and t is a coordinate on $\mathbb{C}^\times$.

Let x_1, x_2, y_1, y_2, y_3 be as in Equation (6.22). We perform a weighted blow-up of $\mathbb{P}(2, 3, 5, 5, 5)$ along the surface $(x_1 = 0, x_2 = 0)$ by setting $x_1 = sy_0^2$ and $x_2 = ry_0^3$. The polynomial in Equation (6.22) becomes:

$$s^5 y_0^{10} y_3 + r^5 y_0^{15} + y_1^3 + y_2^3 - sry_0^5 y_1 y_2 - ay_3^3 = 0$$

Then, on $(r \neq 0)$, the strict transform of W' is isomorphic to the 3-fold given by:

$$(s^5 y_0^2 y_3 + y_0^3 + y_1^3 + y_2^3 - sy_0 y_1 y_2 - ay_3^3 = 0) / \mu_3 \subset \mathbb{P}^3 \times \mathbb{C} / \mu_3 \quad (6.24)$$

where y_0, y_1, y_2, y_3 are homogeneous coordinates on $\mathbb{P}^3$, s is an affine coordinate, and μ_3 acts on $\mathbb{P}^3$ with weights $(0, 2, 0, 1)$ and on $\mathbb{C}$ with weight 1. For all $s \neq 0$ write $(u, x, y, z) = (sy_0, y_1/s, sy_2, y_3)$ and let $t = s^3$. Then the intersection of 3-fold of Equation (6.24) with $\mathbb{P}^3 \times \mathbb{C}^\times / \mu_3$ identifies with the 3-fold $\widehat{W} \subset \mathbb{P}^3 \times \mathbb{C}^\times$ of Equation (6.23).

The 3-fold $\widehat{W}$ of Equation (6.23) is a partial compactification of W . With the same techniques used in the proof of Lemma 5.14, one can show that:

Lemma 6.14. *There is an identity of pure Hodge structures:*

$$\mathrm{gr}_3^W H_c^3(W, \mathbb{Q}) = \mathrm{gr}_3^W H_c^3(\widehat{W}, \mathbb{Q})$$

Then, for the purpose of our computations, we can work with $\widehat{W}$ in place of W . The advantage of working with $\widehat{W}$ is that the second projection $\phi: \widehat{W} \rightarrow \mathbb{C}^\times$ is a pencil of cubic surfaces. The following Lemma describes the geometry of this pencil.

Lemma 6.15. *Let $\phi: \widehat{W} \rightarrow \mathbb{C}^\times$ be the projection to the second factor. Denote by Δ the set of critical values of ϕ and write $\Omega = \mathbb{C}^\times \setminus \Delta$ for the set of regular values. Let $j: \Omega \hookrightarrow \mathbb{C}^\times$ be the inclusion and denote by $\phi_\Omega: \widehat{W}_\Omega = \phi^{-1}(\Omega) \rightarrow \Omega$ the restriction. Let ω be a 3rd primitive root of 1. Then:*

1. The set $\Delta = (\delta = 0)$, where $\delta = \delta_1 \cdot \delta_2$ and:

$$\delta_1(t) = 27a - 4t^5 \quad \delta_2(t) = a(t - 27)^2 - 108t^5 \quad (6.25)$$

If t is a root of δ_1 , the fibre $\widehat{W}_t$ is singular at the point $(-2/3t^2 : 0 : 0 : 1)$. If t is a root of δ_2 , the fibre $\widehat{W}_t$ is singular at the three points

$$\left(3\omega^i t^{1/3} : 1 : \omega^{2i} t^{2/3} : \omega^i(-27+t)/(6t^{5/3})\right) \quad i = 1, 2, 3 \quad (6.26)$$

In both cases, the singular point is an ordinary double point.

2. For all $t \in \Omega$ the fibre $\widehat{W}_t$ is a smooth cubic surface in $\mathbb{P}^3$. In particular, $\widehat{W}_t$ contains 27 lines, and $H^2(\widehat{W}_t, \mathbb{Z}) = \text{Pic}(\widehat{W}_t) \simeq \mathbb{Z}^7$.

3. For all $t \in \Omega$ the fibre $\widehat{W}_t$ contains the nine lines¹ $l_{i,j} = P_i \cap E_j$, where, for $i, j = 1, 2, 3$, P_i and E_j are the planes:

$$P_i = (z - \lambda_i u = 0) \quad E_j = \left(y + \omega^j t^{2/3} x + \omega^{2j} \frac{t^{1/3}}{3} u = 0\right) \quad (6.27)$$

and $\lambda_1, \lambda_2, \lambda_3$ are the roots of the polynomial:

$$27(-at y^3 + t y^2 + 1) - t \quad (6.28)$$

In particular, for $j = 1, 2, 3$, the three lines $l_{1,j}, l_{2,j}, l_{3,j}$ meet at the Eckardt point $e_j = (0 : 1 : -\omega^j t^{2/3} : 0)$ (see Figure 6.1).

4. The union over $t \in \Omega$ of the lines $l_{i,j}$ spans a local system $\mathbb{V} \subset R^2 \phi_{\Omega*} \mathbb{Q}$ of rank 5.

5. $R^1 \phi_{\star} \mathbb{Q} = R^3 \phi_{\star} \mathbb{Q} = (0)$.

Proof. To prove (1), fix $t \in \mathbb{C}^\times$ and compute the Jacobian of the polynomial in Equation (6.23). For all $t \in \Delta$ the Hessian of the polynomial at t is invertible.

Assertion (2) is obvious (see Equation (6.23)).

To prove (3), intersect the cubic surface $\widehat{W}_t$ with the plane $(z = \lambda u)$ to find the planar cubic:

$$C_\lambda = ((-at\lambda^3 + t^2\lambda + 1)u^3 + t^2x^3 + y^3 - tuxy = 0) \subset (z = \lambda u) \quad (6.29)$$

There are two cases in which C_λ is singular. If $-at\lambda^3 + t^2\lambda + 1 = 0$, then C_λ is a nodal cubic (with a node at $(1 : 0 : 0 : \lambda)$). If $-at\lambda^3 + t^2\lambda + 1 \neq 0$, one easily sees that the Jacobian of the polynomial in Equation (6.29) vanishes if and only if $-at\lambda^3 + t^2\lambda + 1 - t/27 = 0$; this is the polynomial in Equation (6.28). If λ is a solution to this polynomial, then Equation (6.29) factors as:

$$\left(y + \omega t^{2/3} x + \omega^2 \frac{t^{1/3}}{3} u\right) \left(y + \omega^2 t^{2/3} x + \omega \frac{t^{1/3}}{3} u\right) \left(y + t^{2/3} x + \frac{t^{1/3}}{3} u\right) = 0$$

¹For convenience, we decided to omit the reference to t in our notation for the lines $l_{i,j}$.

These factors describe the three planes E_1, E_2, E_3 in Equation (6.27). The last part of Statement (3) is obvious.

To prove (4), look at the relations between the $l_{i,j}$. We have six triangles :

$$\begin{aligned} i) \quad & l_{1,1} + l_{1,2} + l_{1,3} = -K & iv) \quad & l_{1,1} + l_{2,1} + l_{3,1} = -K \\ ii) \quad & l_{2,1} + l_{2,2} + l_{2,3} = -K & v) \quad & l_{1,2} + l_{2,2} + l_{3,2} = -K \\ iii) \quad & l_{3,1} + l_{3,2} + l_{3,3} = -K & vi) \quad & l_{1,3} + l_{2,3} + l_{3,3} = -K \end{aligned}$$

where $-K = -K_{\widehat{W}_t}$. Identity (vi) is implied by the previous identities. It follows that the $l_{i,j}$ generate a local system $\mathbb{V} \subset R^2\phi_{\Omega} \star \mathbb{Q}$ of rank 5.

Statement (5) follows immediately from (1) and (2). $\square$

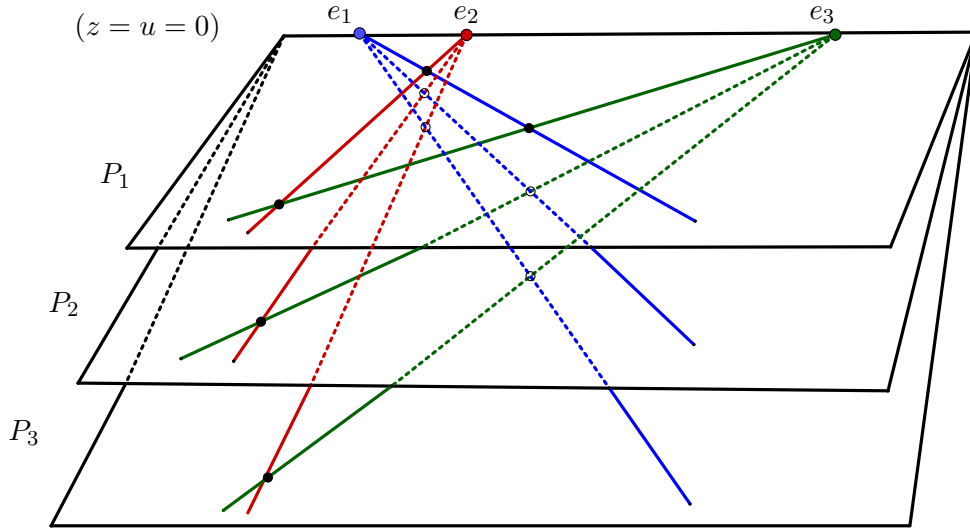


Figure 6.1: A picture of the nine lines $l_{i,j}$ on the smooth cubic surface $\widehat{W}_t$. The three planes P_1, P_2, P_3 meet along the line $(z = u = 0)$, containing the Eckardt points e_1, e_2, e_3 .

Remark 6.16. Note that the curve P in Equation (6.20) is cut out of $\mathbb{A}^2$ by the polynomial in Equation (6.28). It is easy to see that the discriminant of this polynomial (as polynomial in $\mathbb{C}(a, t)[y]$) is $t^2 \cdot \delta_2(t)$; $\delta_2(t)$ has double roots if and only if $\alpha = \alpha_0$ (see Remark 6.9).

Remark 6.17. The configuration of the remaining 18 lines on $\widehat{W}_t$ is more complicated. One way to determine their equations is to intersect $\widehat{W}_t$ with the quadric surfaces passing through three skew lines among the $l_{i,j}$. Here we omit this analysis since it is not relevant to the discussion.

Lemma 6.18. *Let $\widehat{W}_\Omega$ be as in Lemma 6.15. There is an identity of pure Hodge structures:*

$$\mathrm{gr}_3^W H_c^3(\widehat{W}, \mathbb{Q}) = \mathrm{gr}_3^W H_c^3(\widehat{W}_\Omega, \mathbb{Q}) \quad (6.30)$$

Proof. Consider the divisor $\widehat{W}_\Delta = \widehat{W} \setminus \widehat{W}_\Omega = \phi^{-1}(\Delta)$, where Δ is the set of critical values of ϕ , as in Lemma 6.15. We have a long exact sequence of mixed Hodge structures

$$\cdots \rightarrow H_c^2(\widehat{W}_\Delta, \mathbb{Q}) \rightarrow H_c^3(\widehat{W}_\Omega, \mathbb{Q}) \rightarrow H_c^3(\widehat{W}, \mathbb{Q}) \rightarrow H_c^3(\widehat{W}_\Delta, \mathbb{Q}) \rightarrow \cdots$$

To prove (6.30), we check that $\mathrm{gr}_3^W H_c^2(\widehat{W}_\Delta, \mathbb{Q}) = \mathrm{gr}_3^W H_c^3(\widehat{W}_\Delta, \mathbb{Q}) = (0)$. Observe that

$$\widehat{W}_\Delta = \bigcup_{\delta(t)=0} \widehat{W}_t$$

which implies that $\forall i \ H_c^i(\widehat{W}_\Delta, \mathbb{Q}) = \bigoplus_{\delta(t)=0} H_c^i(\widehat{W}_t, \mathbb{Q})$. Since by Lemma 6.15(1) for all roots t of δ the singularities of $\widehat{W}_t$ are ordinary double points, $H^3(\widehat{W}_t, \mathbb{Q}) = (0)$ and $H^2(\widehat{W}_t, \mathbb{Q})$ is a pure Hodge structure of weight two. This concludes the proof. $\square$

Lemma 6.19. *Let $\phi_\Omega: \widehat{W}_\Omega \rightarrow \Omega$ be as in Lemma 6.15. There is an identity of mixed Hodge structures:*

$$H_c^3(\widehat{W}_\Omega, \mathbb{Q}) = H_c^1(\Omega, R^2\phi_{\Omega*}\mathbb{Q}) \quad (6.31)$$

Proof. The statement follows immediately from the Leray spectral sequence for the smooth morphism ϕ_Ω , which is known to degenerate at the second page, combined with Lemma 6.15(5). $\square$

Lemma 6.20. *Let Y and P be the curves in (6.21) and (6.20). Let $\phi_\Omega: \widehat{W}_\Omega \rightarrow \Omega$ be as in Lemma 6.15 and write $\mathbb{W}$ for the rank-4 local system $\mathbb{V}/\langle -K \rangle$, where $\mathbb{V}$ is as in Lemma 6.15 and $\langle -K \rangle \subset R^2\phi_{\Omega*}\mathbb{Q}$ is the local system spanned by $-K = -K_{\widehat{W}_t}$. There is a short exact sequence:*

$$0 \longrightarrow \mathrm{gr}_1^W H_c^1(P, \mathbb{Q}) \longrightarrow \mathrm{gr}_1^W H_c^1(Y, \mathbb{Q}) \longrightarrow \mathrm{gr}_1^W H_c^1(\Omega, \mathbb{W})(1) \longrightarrow 0 \quad (6.32)$$

Moreover, $\mathrm{gr}_1^W H_c^1(\Omega, \mathbb{W})$ has rank 8.

Proof. Let $j: \Omega \hookrightarrow \mathbb{C}^\times$ be as in Lemma 6.15 and consider the two maps $f: Y \rightarrow \mathbb{C}^\times$ and $p: P \rightarrow \mathbb{C}^\times$ given by the projection onto the coordinate t . Let $Y_\Omega = f^{-1}(\Omega)$, $P_\Omega = p^{-1}(\Omega)$ and write $f_\Omega: Y_\Omega \rightarrow \Omega$, $p_\Omega: P_\Omega \rightarrow \Omega$ for the restrictions. Let $e: E \rightarrow \mathbb{C}^\times$ be the 3:1 cover of $\mathbb{C}^\times$ defined by $x^3 = t$. Let $E_\Omega = e^{-1}(\Omega)$, and write $e_\Omega: E_\Omega \rightarrow \Omega$ for the restriction. We have that $Y_\Omega = P_\Omega \times_\Omega E_\Omega$.

Consider the commutative diagram

$$\begin{array}{ccccc}
 & & Y_\Omega & & \\
 & \swarrow & \downarrow & \searrow & \\
 P_\Omega & & \begin{array}{c} 9:1 \\ \downarrow \end{array} & & E_\Omega \\
 & \searrow & & \swarrow & \\
 & & \Omega & &
 \end{array}$$

(The arrows from P_Ω and E_Ω to Ω are labeled 3:1.)

The key point is to observe that $f_\Omega: Y_\Omega \rightarrow \Omega$ is the 9:1 cover parametrizing the nine lines $l_{i,j}$ on the fibres $\widehat{W}_t$ of $\phi_\Omega: \widehat{W}_\Omega \rightarrow \Omega$, and $p_\Omega: P_\Omega \rightarrow \Omega$ and $e_\Omega: E_\Omega \rightarrow \Omega$ are the 3:1 covers parametrizing the triangles of lines on the planes P_i and E_j in Equation (6.27). The maps $Y_\Omega \rightarrow P_\Omega$ and $Y_\Omega \rightarrow E_\Omega$ can be thought as the map sending $l_{i,j} \mapsto P_i$ and $l_{i,j} \mapsto E_j$.

Then we have an exact sequence:

$$0 \longrightarrow \mathbb{Q}_\Omega \longrightarrow p_\Omega \star \mathbb{Q} \oplus e_\Omega \star \mathbb{Q} \longrightarrow f_\Omega \star \mathbb{Q} \longrightarrow \mathbb{W}(1) \longrightarrow 0$$

By splitting this sequence into the two short exact sequences:

$$0 \longrightarrow \mathbb{Q}_\Omega \longrightarrow p_\Omega \star \mathbb{Q} \oplus e_\Omega \star \mathbb{Q} \longrightarrow \mathbb{L} \longrightarrow 0 \quad (6.33)$$

$$0 \longrightarrow \mathbb{L} \longrightarrow f_\Omega \star \mathbb{Q} \longrightarrow \mathbb{W}(1) \longrightarrow 0 \quad (6.34)$$

we obtain the short exact sequence of pure Hodge structures in (6.32). Indeed, the long exact sequence of compactly supported cohomology groups associated to (6.33) is:

$$\begin{aligned}
 \cdots \rightarrow H_c^1(\Omega, \mathbb{Q}_\Omega) &\rightarrow H_c^1(P_\Omega, \mathbb{Q}) \oplus H_c^1(E_\Omega, \mathbb{Q}) \rightarrow H_c^1(\Omega, \mathbb{L}) \rightarrow \\
 &\mathbb{Q}(-1) \hookrightarrow \mathbb{Q}(-1) \oplus \mathbb{Q}(-1) \rightarrow H_c^2(\Omega, \mathbb{L}) \rightarrow (0)
 \end{aligned}$$

thus $H_c^2(\Omega, \mathbb{L}) = \mathbb{Q}(-1)$ and, since $\mathrm{gr}_1^W H_c^1(\Omega, \mathbb{Q}_\Omega) = \mathrm{gr}_1^W H_c^1(E_\Omega, \mathbb{Q}) = H^1(\mathbb{P}^1, \mathbb{Q}) = (0)$, we have

$$\mathrm{gr}_1^W H_c^1(P_\Omega, \mathbb{Q}) \simeq \mathrm{gr}_1^W H_c^1(\Omega, \mathbb{L}) \quad (6.35)$$

The long exact sequence of compactly supported cohomology groups associated to (6.34) is:

$$(0) = H_c^0(\Omega, \mathbb{W})(1) \rightarrow H_c^1(\Omega, \mathbb{L}) \rightarrow H_c^1(Y_\Omega, \mathbb{Q}) \rightarrow H_c^1(\Omega, \mathbb{W})(1) \rightarrow \mathbb{Q}(-1) \rightarrow \cdots \quad (6.36)$$

where the first equality is due to the fact that $H^0(\Omega, R^2\phi_\Omega \star \mathbb{Q}) = H^0(\Omega, \langle -K \rangle)$ as the map $\phi: \widehat{W} \rightarrow \mathbb{C}^\times$ is a Mori fibre space. Now, since $\mathrm{gr}_1^W H_c^1(P_\Omega, \mathbb{Q}) = \mathrm{gr}_1^W H_c^1(P, \mathbb{Q})$ and the same holds for $Y_\Omega \subset Y$, (6.36) combined with (6.35) gives rise to the short exact sequence in (6.32).

We compute the rank of $\mathrm{gr}_1^W H_c^1(P, \mathbb{Q})$ by considering the compactification $P \subset \bar{P}$:

$$\bar{P} = (-27asty^3 + 27t^2yw^2 + (27s - t)sw^3 = 0) \subset \mathbb{P}^1 \times \mathbb{P}^1$$

where s, t, y, w are homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$. It is easy to see that the first projection $\bar{p}: \bar{P} \rightarrow \mathbb{P}^1$ has six ramification points of multiplicity 2 (five over $\delta_2(t) = 0$ and one over $(0 : 1)$) and one ramification point of multiplicity 3 (over $(1 : 0)$). Then by the Hurwitz's formula $\bar{P}$ has genus 2. This shows that $\mathrm{gr}_1^W H_c^1(P, \mathbb{Q})$ has rank 4. Analogously, one can show that Y compactifies to a curve of genus 6, thus $\mathrm{gr}_1^W H_c^1(Y, \mathbb{Q})$ has rank 12. It follows from (6.32) that $\mathrm{gr}_1^W H_c^1(\Omega, \mathbb{W})$ has rank 8. $\square$

Proof of Theorem 6.10 Let $W \subset \widehat{W}$, $\phi_\Omega: \widehat{W}_\Omega \rightarrow \Omega$, and $\mathbb{W}$ be as above. By Lemmas 6.13, 6.14, 6.18, and 6.19

$$\mathrm{gr}_3^W H_c^3(W, \mathbb{Q}) = \mathrm{gr}_3^W H_c^3(\widehat{W}, \mathbb{Q}) = \mathrm{gr}_3^W H_c^3(\widehat{W}_\Omega, \mathbb{Q}) = \mathrm{gr}_1^W H_c^1(\Omega, R^2\phi_{\Omega*}\mathbb{Q})$$

has rank 8. Since by Lemma 6.20 also $\mathrm{gr}_1^W H_c^1(\Omega, \mathbb{W})$ has rank 8, we obtain that:

$$\mathrm{gr}_1^W H_c^1(\Omega, R^2\phi_{\Omega*}\mathbb{Q}) = \mathrm{gr}_1^W H_c^1(\Omega, \mathbb{W})$$

This, combined with (6.32), gives the short exact sequence in (6.18). $\square$

Remark 6.21. It is possible to study the intermediate Jacobian of $\widehat{W}$ by the methods of the paper [Kan89], but we think this would be more complicated.

Chapter 7

Towards a Hodge–theoretic translation of Theorem 3.3

In Chapter 3 we discussed the natural Hodge–theoretic translation of the result [BCM15] by Beukers, Cohen, and Mellit, which we summarize below:

Expectation Given two multisets $\mathbf{p} = (p_1, \dots, p_r)$, $\mathbf{q} = (q_1, \dots, q_s)$ of integers such that $\sum p_i = \sum q_j$, consider the pair (W, v) given by the manifold of dimension $d = r + s - 2$:

$$W = (y_1 + \dots + y_r = x_1 + \dots + x_s) \subset \mathbb{T}^{r+s-1} \quad (7.1)$$

where $\mathbb{T}^{r+s-1} \subset \mathbb{P}^{r+s-1}$ is the torus and x_i, y_j are homogeneous coordinates, together with the map $v: W \rightarrow \mathbb{C}^\times$ sending

$$(x_1 : \dots : x_s : y_1 : \dots : y_r) \mapsto \frac{x_1^{q_1} \dots x_s^{q_s}}{y_1^{p_1} \dots y_r^{p_r}} \quad (7.2)$$

Then:

- (I) the hypergeometric series $G = G(\mathbf{p}, \mathbf{q}; \cdot)$ is a period of $v: W \rightarrow \mathbb{C}^\times$, or, equivalently,
- (II) the local system $\mathbb{H}$ of solutions to the irreducible ODE annihilating G is a subquotient of $\mathrm{gr}_{d-1}^W R^{d-1} v_U! \mathbb{Q}(1)$ where $U = \mathbb{C}^\times \setminus \left\{ \frac{q_1^{q_1} \dots q_s^{q_s}}{p_1^{p_1} \dots p_r^{p_r}} \right\}$ and $v_U: W_U = v^{-1}(U) \rightarrow U$ is the restriction.

In this Chapter we present our ideas to prove (I) and (II). Turning these ideas into complete proofs is ongoing work.

7.1 Background

We start by recalling some properties of the Hadamard product of power series and of the convolution of hypergeometric $\mathcal{D}$ -modules. Our ideas to prove (I) and (II) will build upon these properties.

7.1.1 Hadamard product of power series

Definition 7.1. If $A = \sum_{n \geq 0} a_n t^n$ and $B = \sum_{n \geq 0} b_n t^n$ are power series with radius of convergence r_a and r_b , the *Hadamard product* C of A and B , denoted by $C = A \star B$, is the power series defined as

$$C = \sum_{n \geq 0} c_n t^n \quad \text{where} \quad c_n = a_n b_n \quad (\text{and radius of convergence } r_c > r_a r_b) \quad (7.3)$$

It is easy to show that for $|t| < r_a r_b$ the Hadamard product $C = A \star B$ is the (multiplicative) *convolution* of A and B :

$$C(t) = \frac{1}{2\pi i} \int_{|u|=\rho} A\left(\frac{t}{u}\right) B(u) \frac{du}{u} \quad (7.4)$$

where we choose $\frac{|t|}{r_a} < \rho < r_b$. Indeed, $C(t)$ converges uniformly for $|t| < r_a r_b$, and the left hand side of (7.4) is:

$$\frac{1}{2\pi i} \int_{|u|=\rho} \sum a_n \left(\frac{t}{u}\right)^n \sum_m b_m u^m \frac{du}{u}$$

Since the two series in the expression above converge uniformly on $(|u| = \rho)$, this is:

$$\frac{1}{2\pi i} \int_{|u|=\rho} \sum_n a_n b_n t^n \frac{du}{u} = \sum_n a_n b_n t^n$$

The power series $E = \sum t^n = \frac{1}{1-t}$ is the identity element for the Hadamard product.

Definition 7.2. Let $A = \sum a_n t^n$ be a formal series. The *Hadamard inverse* of A is defined as

$$A^\star = \sum \frac{1}{a_n} t^n \quad (7.5)$$

The set of formal series with the sum and the Hadamard product forms an algebra.

We use identity (7.4) to show that if A and B are periods (in the sense of Remark 2.21) then so is their Hadamard product C .

Let $f: X \rightarrow \mathbb{C}^\times$ and $g: Y \rightarrow \mathbb{C}^\times$ be two morphisms. Their ‘product morphism’ $fg: X \times Y \rightarrow \mathbb{C}^\times$ sends $(x, y) \mapsto f(x)g(y)$.

Proposition 7.3. *If A and B are periods of (X, f) and (Y, g) , then $C = A \star B$ is a period of $(X \times Y, fg)$. More precisely, if*

$$A(t) = \int_{\gamma_X} \frac{1}{1-t/f} \omega_X \quad B(t) = \int_{\gamma_Y} \frac{1}{1-t/g} \omega_Y \quad (7.6)$$

where $\omega_X, \gamma_X, \omega_Y, \gamma_Y$ are as in Remark 2.21, then

$$C(t) = \int_{\gamma_X \times \gamma_Y} \frac{1}{1-t/fg} \omega_X \wedge \omega_Y \quad (7.7)$$

Proof. For $|t| < r_a r_b$, by combining identity (7.4) and (7.6) we write:

$$C(t) = \frac{1}{2\pi i} \int_{\gamma_X \times \gamma_Y} \left(\int_{|u|=\rho} \frac{1}{1-t/uf} \frac{1}{1-u/g} \frac{du}{u} \right) \omega_X \times \omega_Y$$

Let m_f and m_g be the minimum of f along γ_X and the minimum of g along γ_Y . For $|t|/\rho < m_f$ and $\rho < m_g$ we can develop the two factors

$$\frac{1}{1-t/uf} \quad \frac{1}{1-u/g}$$

in power series. Plugging the power series in the internal integral above, we get:

$$\sum_{i,j} \frac{t^i}{f^i g^j} \int_{|u|=\rho} u^{j-i-1} du = 2\pi i \sum_k \frac{t^k}{(fg)^k} = 2\pi i \frac{1}{1-t/fg}$$

where the second equality follows from the residue theorem around $u = 0$ and the last equality holds if $|t| < m_f m_g$. $\square$

Note that the morphism fg is the the same as the composition

$$fg: X \times Y \xrightarrow{f \times g} \mathbb{C}^\times \times \mathbb{C}^\times \xrightarrow{\mu} \mathbb{C}^\times$$

where μ denotes the product map. Then it is natural to wonder if, when A is a period of (X, f) , its Hadamard inverse $A^\star$ is a period of

$$1/f: X \xrightarrow{f} \mathbb{C}^\times \xrightarrow{i} \mathbb{C}^\times$$

where i denotes the inversion map. More precisely, writing L for the Picard–Fuchs operator of A , one may wonder whether $A^\star$ is a solution to $i_\star L = i^\star L$. In general there is no reason to expect this, as it is not possible to deduce a linear recursion relation for $A^\star$ from the linear recursion relation for A , see Remark 2.11. However, if A is a hypergeometric series, we can prove the following:

Proposition 7.4. *If $A = \sum a_n t^n$ is annihilated by the hypergeometric operator $H(P, Q)$, then A^* is annihilated by the hypergeometric operator $H(R, S)$ given by*

$$H(R, S) = H(Q(D-1), P(D+1)) \quad (7.8)$$

and

$$\mathcal{H}(R, S) \simeq \mathcal{D}/\mathcal{D} \ i^* H(P, Q) = i^* \mathcal{H}(P, Q) \quad (7.9)$$

Proof. Rewrite the linear recursion relation for A as:

$$Q(n-1) \frac{1}{a_n} - P(n) \frac{1}{a_{n-1}} = 0$$

It follows that A^* is killed by the hypergeometric operator $H(R, S)$ in Equation (7.8).

Explicitly, if $H(P, Q) = p \prod (D - v_i) - t q \prod (D - w_j)$, then

$$H(R, S) = q \prod (D - 1 - w_i) - t p \prod (D + 1 - v_i)$$

On the other hand,

$$i^* H(P, Q) = P(-D) - \frac{1}{t} Q(-D) = -\frac{1}{t} (Q(-D) - t P(-D))$$

hence $(-t)i^* H(P, Q) = H(Q(-D), P(-D))$. Explicitly,

$$H(Q(-D), P(-D)) = (-1)^m \left(q \prod (D + w_i) - t p \prod (D + v_i) \right)$$

where $m = \deg P = \deg Q$. By Corollary 2.9 this concludes the proof. $\square$

Remark 7.5. Proposition 7.4 shows that, if A is a solution to the operator $H(P, Q)$, then A^* is a section of a local system isomorphic to the local system of solutions to the operator $i^* H(P, Q)$. Thus, if A is a period of (X, f) , A^* should be a period of $(X, 1/f)$. However, it is not clear how to explicitly represent A^* as a period of $(X, 1/f)$ by knowing the representation of A as a period of (X, f) .

7.1.2 Convolution of hypergeometrics

We recall the definition of convolution of $\mathcal{D}$ -modules in [Kat90, Chapter 5].

Given a smooth scheme X over $\mathbb{C}$, denote by $\mathcal{D}_X$ the sheaf of differential operators on X , by $D(X, \mathcal{D})$ the derived category of left $\mathcal{D}_X$ -modules on X and by $D^{\text{b,hol}}(X)$ the subcategory of $D(X, \mathcal{D})$ corresponding to holonomic $\mathcal{D}$ -modules.

Definition 7.6. Given a smooth $\mathbb{C}$ -groupscheme $G/\mathbb{C}$, the convolution of two objects in $D^{\text{b,hol}}(G)$ is defined as

$$K \star L = R\mu_*(K \boxtimes L) \quad (7.10)$$

where $\mu: G \times_{\mathbb{C}} G \rightarrow G$ is the product map and $K \boxtimes L$ denotes the external tensor product of K and L .

Now consider two hypergeometric operators $H(P, Q)$ and $H(R, S)$ of degree m . If $A = \sum a_n t^n$ and $B = \sum b_n t^n$ are hypergeometric series annihilated by $H(P, Q)$ and $H(R, S)$, then their coefficients satisfy the two-term recursion relations:

$$P(n)a_n = Q(n-1)a_{n-1} \quad R(n)b_n = S(n-1)b_{n-1}$$

Then clearly the Hadamard product $C = A \star B$ satisfies the two-term recursion relation:

$$P(n)R(n)a_n b_n = Q(n-1)S(n-1)a_{n-1}b_{n-1}$$

hence it is a solution of $H(PR, QS)$. This suggests there should be an isomorphism of $\mathcal{D}$ -modules:

$$\mathcal{H}(P, Q) \star \mathcal{H}(R, S) \simeq \mathcal{H}(PR, QS) \quad (7.11)$$

In fact, this isomorphism holds whenever PR and QS do not have common zeroes mod $\mathbb{Z}$, see [Kat90, Theorem 5.3.1]. The explicit version of (7.11) is:

$$\mathcal{H}_\lambda(v_i, w_j) \star \mathcal{H}_\mu(u_i, z_j) \simeq \mathcal{H}_{\lambda\mu}(v_i u_i, w_j z_j) \quad (7.12)$$

provided that $\mathcal{H}_{\lambda\mu}(v_i u_i, w_j z_j)$ is irreducible.

7.2 Our approach

7.2.1 Expectation (I) and the inverse of a multinomial

Observe that, according to Expectation (I), the series $G(\mathbf{p}, \mathbf{q}; \cdot)$ is a period of the pair (W, v) defined by Equations (7.1) and (7.2), and the series $G(\mathbf{q}, \mathbf{p}; \cdot)$ is a period of the pair $(W, 1/v)$. This is consistent with the considerations in Section 7.1. Indeed, $G(\mathbf{q}, \mathbf{p}; \cdot)$ is the Hadamard inverse of $G(\mathbf{p}, \mathbf{q}; \cdot)$.

In particular, for multinomial coefficients

$$C_j = \binom{(c_1 + \cdots + c_k)j}{c_1 j, \dots, c_k j}$$

Expectation (I) states that:

(a) the series $\sum C_j \alpha^j$ is a period of (W, v) , where

$$W = (y_1 = x_1 + \cdots + x_k) \subset \mathbb{T}^k \quad v(x_1 : \cdots : x_k : y_1) = \frac{x_1^{c_1} \cdots x_k^{c_k}}{y_1^{c_1 + \cdots + c_k}}$$

(b) the series $\sum \frac{1}{C_j} \alpha^j$ is a period of $(W, 1/v)$.

We already know that Statement (a) holds, see Remark 3.6. Indeed $(W, v) = (Y, w)$, where

$$Y = (1 = x_1 + \cdots + x_k) \subset (\mathbb{C}^\times)^k \quad w(x_1, \dots, x_k) = x_1^{c_1} \cdots x_k^{c_k}$$

and we have seen that

$$\sum C_j \alpha^j = \int_\gamma \frac{1}{1 - \alpha/w} \omega \quad (7.13)$$

for an explicit choice of γ and ω on Y , see Equation (2.23). If we could prove Statement (b), or, more precisely, that:

$$\sum \frac{1}{C_j} \alpha^j = \int_{\gamma'} \frac{1}{1 - \alpha w} \omega' \quad (7.14)$$

for some choice of γ' and ω' on Y , then we could prove Expectation (I) in full generality. We explain this below.

Idea Let $\lambda = \sum q_i = \sum p_i$. Rewrite $G = G(\mathbf{p}, \mathbf{q}; \cdot)$ as the Hadamard product $G = Q \star P^\star$, where $Q = \sum Q_j \alpha^j$, $P = \sum P_j \alpha^j$, and

$$Q_j = \binom{\lambda j}{q_{1j}, \dots, q_{sj}} \quad P_j = \binom{\lambda j}{p_{1j}, \dots, p_{rj}} \quad (7.15)$$

Now, Q is a period of (X, f) of the form $\int_{\gamma_X} \frac{1}{1 - \alpha/f} \omega_X$, where

$$X = (x_1 + \cdots + x_s = 1) \subset (\mathbb{C}^\times)^s \quad f(x_1, \dots, x_s) = x_1^{q_1} \cdots x_s^{q_s} \quad (7.16)$$

If $P^\star$ is a period of $(Y, 1/g)$ of the form $\int_{\gamma_Y} \frac{1}{1 - \alpha g} \omega_Y$, where

$$Y = (y_1 + \cdots + y_r = 1) \subset (\mathbb{C}^\times)^r \quad g(y_1, \dots, y_r) = y_1^{p_1} \cdots y_r^{p_r} \quad (7.17)$$

then by Proposition 7.3 we obtain that G is a period of the pair $(X \times Y, f/g)$. Since the map $I: X \times Y \rightarrow \mathbb{P}^{r+s-1}$ sending

$$(x_1, \dots, x_s, y_1, \dots, y_r) \mapsto (x_1 : \cdots : x_s : y_1 : \cdots : y_r)$$

identifies $(X \times Y, f/g)$ with $((x_1 + \cdots + x_s \neq 0) \cap W, v)$, with W and v as in Equations (7.1) and (7.2), we also obtain that G is a period of (W, v) . $\square$

Remark 7.7. Proving an identity of the form (7.14) seems hard. Roughly speaking, it would require to transport the cycle γ in (7.13) from $\alpha = 0$ to $\alpha = \infty$. Note that the pair (Y, w) — that is, the Hori–Vafa mirror of $\mathbb{P}(c_1, \dots, c_k)$ — is (after a fibrewise compactification) a pencil of Calabi–Yau varieties

$$\left((x_1 - x_2 - \dots - x_k)^{c_1} x_2^{c_2} \dots x_k^{c_k} = \alpha \cdot x_1^{\sum c_i} \right) \subset \mathbb{P}^k$$

While the fibre at $\alpha = 0$ is well-behaved, the geometry at $\alpha = \infty$ is very different, and there is a priori no canonical choice of a cycle of integration.

7.2.2 Towards a proof of Expectation (II)

We believe that proving Expectation (II) is easier than proving Expectation (I). Our ideas for a proof of (II) involve four steps which we describe below. Steps 1 and 2 rely on the theory of hypergeometrics, while the analysis in Steps 3 and 4 involves only algebraic geometry.

Notation As in (II), denote by $\mathbb{H}$ the irreducible hypergeometric local system on

$$U = \mathbb{C}^\times \setminus \left\{ \frac{q_1^{q_1} \dots q_s^{q_s}}{p_1^{p_1} \dots p_r^{p_r}} \right\}$$

attached to the series $G = G(\mathbf{p}, \mathbf{q}; \cdot)$. Let $Q = \sum Q_j \alpha^j$ and $P = \sum P_j \alpha^j$ be the hypergeometric series defined by Equation (7.15). Denote by $\mathbb{H}_Q, \mathbb{H}_{P^*}$ the irreducible hypergeometric local systems on

$$U' = \mathbb{C}^\times \setminus \left\{ \frac{q_1^{q_1} \dots q_s^{q_s}}{\lambda^\lambda} \right\} \quad \text{and} \quad U'' = \mathbb{C}^\times \setminus \left\{ \frac{\lambda^\lambda}{p_1^{p_1} \dots p_r^{p_r}} \right\}$$

attached to Q and P^* . The external tensor product $\mathbb{H}_Q \boxtimes \mathbb{H}_{P^*}$ is a local system on $U' \times U''$. Let $\mu: \mathbb{C}^\times \times \mathbb{C}^\times \rightarrow \mathbb{C}^\times$ and $i: \mathbb{C}^\times \rightarrow \mathbb{C}^\times$ be the product map and the inversion map, as in Section 7.1.1. Let (X, f) and (Y, g) be the pairs defined by Equations (7.16) and (7.17). Denote by $n = s - 2$ and $m = r - 2$ the relative dimension of f and g . The relative dimension of $f/g: X \times Y \rightarrow \mathbb{C}^\times$ is $n + m + 1 = r + s - 3 = d - 1$, where d is the dimension of the manifold W in Equation (7.1).

Step 1 Show that Katz’s Convolution Theorem (see Section 7.1.2) implies that there is a surjective morphism:

$$R^1 \mu_V! \left(\mathbb{H}_Q \boxtimes \mathbb{H}_{P^*}|_{\mu^{-1}(U)} \right) \twoheadrightarrow \mathbb{H} \tag{7.18}$$

where $V = \mu \left(U' \times U'' \cap \mu^{-1}(U) \right)$ and $\mu_V: \mu^{-1}(V) \rightarrow V$ is the restriction.

Remark 7.8. In light of Katz's result, the morphism (7.18) is something natural to expect. However, Katz's theorem is a statement about perverse sheaves, and needs to be translated in terms of the local system $\mathbb{H}$ on U .

Step 2 We know how to describe $\mathbb{H}_Q$ and $\mathbb{H}_{P^\star}$ in terms of the morphisms f and g . Indeed, since Q is a period of (X, f) , the local system $\mathbb{H}_Q$ is a subquotient of

$$\mathrm{gr}_n^W R^n f_{U'} ! \mathbb{Q}$$

where $f_{U'}: X_{U'} = f^{-1}(U') \rightarrow U'$ is the restriction. On the other hand, by Proposition 7.4 the local system $\mathbb{H}_{P^\star} \simeq i^* \mathbb{H}_P$, where $\mathbb{H}_P$ is the irreducible hypergeometric local system attached to the series P ; since P is a period of (Y, g) , it follows that $\mathbb{H}_{P^\star}$ is a subquotient of

$$\mathrm{gr}_m^W R^m (1/g)_{U''} ! \mathbb{Q}$$

where $(1/g)_{U''}: Y_{U''} = f^{-1}(U'') \rightarrow U''$ is the restriction.

Step 3 Show that:

$$gr_{n+m+1}^W R^{n+m+1} \mu_V ! (\mathrm{gr}_n^W R^n f_{U'} ! \mathbb{Q} \boxtimes \mathrm{gr}_m^W R^m (1/g)_{U''} ! \mathbb{Q})|_{\mu^{-1}(U)} \quad (7.19)$$

is a subquotient of

$$\mathrm{gr}_{n+m+1}^W R^{n+m+1} (f/g)_U ! \mathbb{Q} \quad (7.20)$$

where $(f/g)_U: (X \times Y)_U = (f/g)^{-1}(U) \rightarrow U$ is the restriction.

Remark 7.9. Roughly speaking, we know how this is supposed to go. The ingredients are the following:

- the Leray spectral sequence for the composite map $f/g = \mu \circ f \times 1/g$ shows that $R^1 \mu_! R^{n+m}(f \times 1/g)_! \mathbb{Q}$ is a subquotient of $R^{n+m+1}(f/g)_! \mathbb{Q}$;
- the product $R^n f_! \mathbb{Q} \boxtimes R^m (1/g)_! \mathbb{Q}$ is a Künneth component of $R^{n+m}(f \times 1/g)_!$;
- if Σ is a k -dimensional variety, $i: \Delta \hookrightarrow \Sigma$ is the closed embedding of a subvariety and $j: \Omega = \Sigma \setminus \Delta \rightarrow \Sigma$ is the open inclusion of the complement, via the exact triangle

$$Rj_! \mathbb{Q}_\Omega \rightarrow \mathbb{Q}_\Sigma \rightarrow i_* \mathbb{Q}_\Delta \xrightarrow{+1} \quad (7.21)$$

one can often show that $\mathrm{gr}_k^W H_c^k(\Omega, \mathbb{Q}) = \mathrm{gr}_k^W H_c^k(\Sigma, \mathbb{Q})$.

Step 4 Show that:

$$\mathrm{gr}_{n+m+1}^W R^{n+m+1}(f/g)_{U!}\mathbb{Q} \simeq \mathrm{gr}_{d-1}^W R^{d-1}v_{U!}\mathbb{Q} \quad (7.22)$$

Remark 7.10. We have seen in Section 7.2.1 that $X \times Y \hookrightarrow W$ as the intersection $Z = (x_1 + \cdots + x_s \neq 0) \cap W$ and that the map f/g reads as the map v in Equation (7.2) on Z . Then write $D = (x_1 + \cdots + x_s = 0) \cap W$, and, for all $\alpha \in U$ let $W_\alpha = v^{-1}(\alpha)$, $D_\alpha = D \cap W_\alpha$, $Z_\alpha = Z \cap W_\alpha$. By the triangle (7.21) with $(\Sigma, \Omega, \Delta) = (W_\alpha, Z_\alpha, D_\alpha)$, to prove the isomorphism in (7.22) it is enough to show that

$$\mathrm{gr}_{d-1}^W H_c^{d-1}(D_\alpha, \mathbb{Q}) = 0$$

Once proven (7.18), (7.19), (7.22), Expectation (II) follows by applying in sequence the four steps.

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Appendix A

Anticanonical del Pezzo surfaces

This Appendix contains the lists of anticanonical del Pezzo weighted surfaces constructed in [JK01] and [KP15].

	Weights	Degree
$k > 0$ integer:	$(2, 2k+1, 2k+1, 4k+1)$	$8k+4$
Sporadic:	$(1, 1, 1, 1)$	3
	$(1, 1, 1, 2)$	4
	$(1, 1, 2, 3)$	6
	$(1, 2, 3, 5)$	10
	$(1, 3, 5, 7)$	15
	$(1, 3, 5, 8)$	16
	$(2, 3, 5, 9)$	18
	$(3, 3, 5, 5)$	15
	$(3, 5, 7, 11)$	25
	$(3, 5, 7, 14)$	28
	$(3, 5, 11, 18)$	36
	$(5, 14, 17, 21)$	56
	$(5, 19, 27, 31)$	81
	$(5, 19, 27, 50)$	100
	$(7, 11, 27, 37)$	81
	$(7, 11, 27, 44)$	88
	$(9, 15, 17, 20)$	60
	$(9, 15, 23, 23)$	69
	$(11, 29, 39, 49)$	127
	$(11, 49, 69, 128)$	256
	$(13, 23, 35, 57)$	127
	$(13, 35, 81, 128)$	256

Table A.1: The Johnson–Kollár list.

	Weights	Multidegree
$k > 0$ integer:	(1,1, $k+1$, $k+1$, $2k+1$)	($2k+2$, $2k+2$)
	(1,2, $2k+1$, $2k+1$, $4k+1$)	($4k+2$, $4k+3$)
	(2, $2k+1$, $2k+1$, $4k+1$, $6k+1$)	($6k+3$, $8k+2$)
Sporadic:	(1,1,1,1,1)	(2,2)
	(1,2,2,3,3)	(4,6)
	(1,2,3,4,5)	(6,8)
	(1,3,3,5,5)	(6,10)
	(1,4,5,7,11)	(12,15)
	(1,4,7,10,13)	(14,20)
	(1,5,8,12,19)	(20,24)
	(1,5,9,13,17)	(18,26)
	(1,7,11,17,27)	(28,34)
	(1,7,12,17,23)	(24,35)
	(1,8,13,19,31)	(32,39)
	(1,9,15,23,23)	(24,46)
	(2,2,3,3,3)	(6,6)
	(2,3,4,5,5)	(8,10)
	(2,3,5,6,7)	(10,12)
	(3,3,5,5,7)	(10,12)
	(3,5,6,8,13)	(16,18)
	(3,5,7,9,11)	(16,18)
	(4,5,7,10,13)	(18,20)
	(5,7,10,14,23)	(28,30)
	(5,9,12,20,31)	(36,40)
	(5,14,17,21,37)	(42,51)
	(6,7,9,11,14)	(18,28)
	(6,8,9,11,13)	(22,24)
	(9,15,23,23,31)	(46,54)
	(9,15,23,23,37)	(46,60)
	(9,23,30,38,67)	(76,90)
	(10,17,25,34,43)	(60,68)
	(11,18,27,44,61)	(72,88)
	(11, 27,36,62,97)	(108,124)
	(11,29, 39,49,59)	(88,98)
	(11,29,39,49,67)	(78,116)
	(11,29,38,48,85)	(96,114)
	(13,22,55,76,97)	(110,152)
	(13,23,34,56,89)	(102,112)
	(13,23,35,47,57)	(70,104)
	(13,23,35,57,79)	(92,114)
	(14,19,25,32,45)	(64,70)

Table A.2: The Kim–Park list.