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An adaptive selective frequency damping method

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The selective frequency damping (SFD) method is an alternative to classical Newton's method to obtain unstable steady-state solutions of dynamical systems. However, this method has two main limitations: it does not converge for arbitrary control parameters, and when it does converge, the time necessary to reach a steady-state solution may be very long. In this paper, we present an adaptive algorithm to address these two issues. We show that by evaluating the dominant eigenvalue of a "partially converged" steady flow, we can select a control coefficient and a filter width that ensure an optimum convergence of the SFD method. We apply this adaptive method to several classical test cases of computational fluid dynamics and we show that a steady-state solution can be obtained with a very limited (or without any) *a priori* knowledge of the flow stability properties. © 2015 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4932107>]

I. INTRODUCTION

To compute stability analysis numerically with high accuracy, it is crucial to carefully choose the base flow around which the governing equations will be linearised. The steady-state solution is mathematically appropriate because it is a solution of the system considered. In computational fluid dynamics, if a flow is linearly stable, obtaining a steady-state solution is trivial, we only have to execute the code and wait long enough until the flow becomes constant in time. However for unstable flows, obtaining a steady-state solution is a concrete challenge. The problem of finding an unstable fixed point of a non-linear system has to be addressed.

The selective frequency damping (SFD) method¹ appears to be an efficient alternative to classical Newton's method to solve this problem in the field of fluid dynamics. It is based on the filtering of unstable temporal frequencies. The convergence of the SFD method is governed by two parameters, the control coefficient χ and the filter width Δ . For arbitrary choice of these parameters, the method may not be able to control the evolution of the instabilities within the flow. Hence, the method does not always converge. Even when a steady-state solution can be found, convergence may be very slow. The selection of the parameters χ and Δ is central for users of the SFD method. We intend to address this issue in this study.

We present an adaptive procedure that couples the SFD method and a global stability analysis method. The idea is to approximate the dominant eigenvalue of the flow studied during the execution of the solver implementing the SFD method. This approximation is used to tune χ and Δ by using a simple one-dimensional model. The strength of this procedure is that it requires only very limited (or in some cases not any) knowledge of the flow behaviour before executing the code.

In Sec. II, we present how optimum parameters of the SFD method can be selected when the dominant eigenvalue of the system studied is known. In Sec. III, we propose an adaptive procedure to automatically select parameters that ensure an optimum convergence of the SFD method. Finally in Sec. IV, we show that this procedure can be successfully applied to obtain unstable steady-state solutions of several classical test cases of computational fluid dynamics.

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II. EVALUATION OF OPTIMUM PARAMETERS

Global stability analysis aims to determine if the largest eigenvalue of a linearised dynamical system has a modulus larger than one or not. When the SFD method is applied to this system, it aims to control the evolution of the least stable eigenvalue. Hence, determining if the SFD method can control a dynamical system may be narrowed down to: can the SFD method control the evolution of the dominant instability of this system?

The idea of analysing the influence of the SFD method on the eigenvalues of a linearised system was introduced by Åkervik *et al.*¹ However here, we only focus our attention on the least stable mode of the system studied.

First we recall the formulation of the SFD method as it was originally introduced. We consider the dynamical system $\dot{\mathbf{q}} = \mathbf{F}(\mathbf{q})$, where $\mathbf{q}$ represents the problem unknown(s) and $\mathbf{F}$ is an operator (which can be nonlinear). A fixed point of $\mathbf{F}$ can be found by solving the augmented system

$$\begin{cases} \dot{\mathbf{q}} = \mathbf{F}(\mathbf{q}) - \chi(\mathbf{q} - \bar{\mathbf{q}}), \\ \dot{\bar{\mathbf{q}}} = \frac{\mathbf{q} - \bar{\mathbf{q}}}{\Delta}, \end{cases} \quad (1)$$

where χ is the control coefficient (real and positive), $\bar{\mathbf{q}}$ is a filtered version of $\mathbf{q}$, and Δ is the filter width of a first-order low-pass time filter (real and strictly positive). The steady-state solution is reached when $\mathbf{q} = \bar{\mathbf{q}}$.

We now assume the dominant eigenvalue (denoted λ_D) of the unstable dynamical system studied is known. Then we introduce the one-dimensional model

$$u^{n+1} = \lambda_D u^n. \quad (2)$$

For a given control coefficient χ and filter width Δ , if the SFD method is able to force (2) to evolve towards its steady-state (which is $u = \bar{u} = 0$), it means that it is able to control the instabilities related to λ_D . Then the same parameters can be used in the SFD method applied to an unstable fluid system which has λ_D for dominant eigenvalue.

We define $\mathcal{L}$ as being the linear operator modelling the actions of the feedback control and the low-pass time filter of the SFD method such that

$$\mathcal{L} = \begin{pmatrix} -\chi & \chi \\ 1/\Delta & -1/\Delta \end{pmatrix}. \quad (3)$$

Then the discrete encapsulated formulation of the SFD method² applied to one-dimensional model (2) is defined by

$$\begin{pmatrix} u^{n+1} \\ \bar{u}^{n+1} \end{pmatrix} = e^{\mathcal{L}} \begin{pmatrix} \lambda_D u^n \\ \bar{u}^n \end{pmatrix} = \underbrace{e^{\mathcal{L}} \begin{pmatrix} \lambda_D & 0 \\ 0 & 1 \end{pmatrix}}_{\mathcal{M}} \begin{pmatrix} u^n \\ \bar{u}^n \end{pmatrix}, \quad (4)$$

where $\mathcal{M}$ is the iteration matrix transforming $(u^n, \bar{u}^n)$ into $(u^{n+1}, \bar{u}^{n+1})$. This matrix depends on the dominant eigenvalue λ_D , the control coefficient χ , and the filter width Δ .

Convergence of (4) is only governed by the two eigenvalues of $\mathcal{M}$. If they both have a modulus strictly smaller than one, then system (4) is stable and converges towards its steady-state solution. Otherwise, the system is unstable and the steady-state cannot be reached.

As $\mathcal{M}$ is a 2×2 matrix, evaluating the modulus of its eigenvalues is simple. For a fixed λ_D , we can manually select a control coefficient χ and a filter width Δ that ensure both eigenvalues of $\mathcal{M}$ to be smaller than one. Then the instabilities are controlled by the SFD method and convergence of (4) towards its steady-state is guaranteed. However, this convergence may be slow.

The fastest convergence of (4) is achieved when the modulus of the dominant eigenvalue of $\mathcal{M}$ is minimum. As we can evaluate the eigenvalues of $\mathcal{M}$ for every $\chi \geq 0$ and every $\Delta > 0$ (for a fixed λ_D), a basic line search algorithm³ can be implemented to obtain the optimum parameters χ_{opt} and Δ_{opt} . Then a fast convergence of the SFD method applied to a flow which has λ_D for dominant eigenvalue can be achieved if χ_{opt} and Δ_{opt} are used.

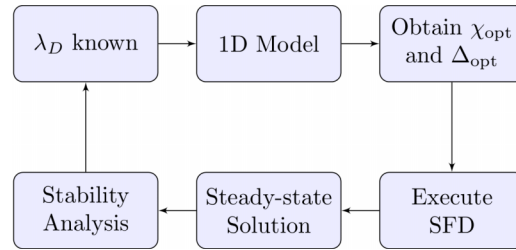


FIG. 1. Illustration of the implicit problem of the unadapted SFD method. λ_D is the dominant eigenvalue of the flow; χ_{opt} and Δ_{opt} are the parameters that ensure an optimum convergence of the SFD method applied to (2).

The issue with this feature is that to be able to adjust the parameters of the SFD method, the dominant eigenvalue λ_D of the system is required. But this eigenvalue is obtained only after computing stability analysis, and stability analysis requires to linearise the governing equation around a steady-state solution, which is what we intend to reach with the SFD method. In other words, to ensure convergence of the SFD method towards a steady-state solution, we need to know this steady-state solution. Fig. 1 illustrates this implicit problem. The closed loop shows that analysing the stability of one-dimensional model (2) is useful only if the dominant eigenvalue λ_D is known *a priori*.

In Sec. III, we present an adaptive algorithm to get around this issue.

III. ADAPTIVE ALGORITHM

In this section, we propose a procedure to address the issue of selecting appropriate parameters for the SFD method. Our algorithm links together the SFD method and a flow stability analysis method (namely, a modified Arnoldi iteration method). The idea is to use a “partially converged” steady base flow for computing stability analysis and obtain an approximation of the dominant eigenvalue (denoted $\tilde{\lambda}$). Then we use the procedure described in Sec. II to obtain the control coefficient $\tilde{\chi}$ and the filter width $\tilde{\Delta}$ that ensure the fastest convergence of the SFD method applied to $u^{n+1} = \tilde{\lambda}u^n$. These parameters are an approximation of the optimum parameters χ_{opt} and Δ_{opt} . The SFD method is then executed using the parameters $\tilde{\chi}$ and $\tilde{\Delta}$, so the instabilities can be damped more efficiently.

The main aspect of the adaptive procedure proposed here is the selection of the “partially converged” steady-state. In this study the choice made was to execute the SFD method for a given number T of time units before computing stability analysis of this “partial” base flow. The user has to define this time interval considering that it has to be large enough to allow the flow to evolve but not too large to avoid wasting computational time executing the SFD method with badly suited parameters.

After defining the “partially converged” steady-state, the stability analysis method has to be computed. Note that here the base flow used is not exactly a steady-state solution of the governing equations. The role of the SFD method is to reduce the temporal frequencies within the flow. Hence as the problem is not converged yet, the base flow selected is only an approximation of the steady-state solution. However, our experiments suggested that this analysis gives a good approximation of the dominant eigenvalue of the flow studied.

At this stage, the one-dimensional model presented in Sec. II is used to approximate the optimum parameters. The line search algorithm returns the control coefficient and filter width that minimize the eigenvalues of $\mathcal{M}$ (see Eq. (4)). Hence, these parameters ensure the fastest convergence of the SFD method applied to $u^{n+1} = \tilde{\lambda}u^n$. After updating the parameters $\tilde{\chi}$ and $\tilde{\Delta}$, the SFD method is executed again for T time units.

This adaptive process is iterated until the norm $\|\mathbf{q} - \bar{\mathbf{q}}\|_{\text{inf}}$ becomes smaller than a desired tolerance. When this becomes true, we say that the approximation of the steady base flow is good enough; hence, $\tilde{\chi}$ and $\tilde{\Delta}$ are fixed until convergence. If this tolerance is too large, then the flow will not have enough time to evolve, and the “partially converged” flow will not be a good approximation of the steady-state. If this tolerance is too small, stability analysis will be computed repeatedly using base flows close to each other which will waste computational resources. Our experiments suggested that a tolerance of 10^{-2} seems to be a good compromise.

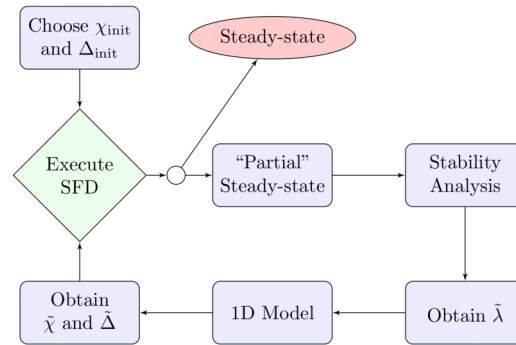


FIG. 2. Illustration of the adaptive SFD method. χ_{init} and Δ_{init} are the initial parameters of the SFD method; $\tilde{\lambda}$ is an approximation of the dominant eigenvalue of the flow; $\tilde{\chi}$ and $\tilde{\Delta}$ are the parameters that ensure an optimum convergence of the SFD method applied to $u^{n+1} = \tilde{\lambda}u^n$. Note that the circle (right of “execute SFD”) represents a group of criteria to determine if the stability analysis method has to be computed or if the SFD parameters are fixed until convergence is reached.

The behaviour of this adaptive method is illustrated in Fig. 2. Note that when stability analysis has to be computed more than once (i.e., the loop of Fig. 2 is executed several times), the initial condition of the modified Arnoldi iteration method is the final solution of the previous computation. This allows us to speed up convergence of the stability analysis solver.

As we only seek an approximation of the dominant eigenvalue, the flow stability analysis method does not need to be very well converged. We fixed the tolerance of the modified Arnoldi iteration method to be 10^{-3} . A more accurate evaluation of the eigenvalue of an “approximated” steady-state state would result in a waste of computational resources.

Also, we recall that the SFD method is based on the damping of unstable temporal frequencies. Then it can only control unstable problems with oscillatory growth of the instabilities (i.e., the imaginary part of the dominant eigenvalue is non-zero). Otherwise there is no frequency to be damped by the method. Hence, all the $\tilde{\lambda}$ successively computed for this algorithm must have a non-zero imaginary part.

All users of the SFD method (in any formulation) must define an initial control coefficient and filter width. To execute the adaptive SFD method, the time T between two consecutive execution, of the stability analysis method is an additional parameter that has to be defined.

IV. NUMERICAL SIMULATIONS

In this section, we present the application of the adaptive algorithm detailed in Sec. III to some classical examples of computational fluid dynamics. The aim is to show that the method is able to reach unstable steady-state solutions even if very little (or not any) information about the flow properties is available.

The algorithm has been implemented into the Nektar++ spectral/*hp* element framework.⁴ It couples two methods that act as wrappers around a flow solver that are the encapsulated SFD method² and a modified Arnoldi iteration method.^{5,6} The incompressible Navier-Stokes solver used throughout this section implements an unstabilised continuous Galerkin method⁷ to discretize the problem in space and a second order *velocity-correction* scheme⁸ for time-integration.

In our implementation, the user has to define the initial parameters χ_{init} and Δ_{init} and the time T for which the SFD method is executed before computing stability analysis. We define the length of one Arnoldi iteration to be equal to one time unit. Hence if the stability analysis solver converges in n iterations, the computational time will be approximately the same as executing the Navier-Stokes solver for n time units.

The adaptive algorithm is executed until $\|\mathbf{q} - \bar{\mathbf{q}}\|_{\text{inf}} < 10^{-2}$. Once this mark is reached, we consider that the SFD parameters are a good approximation of the optimum ones. Then the control coefficient and the filter width are fixed until convergence (i.e., when $\|\mathbf{q} - \bar{\mathbf{q}}\|_{\text{inf}} < 10^{-8}$).

We now present the behaviour of the adaptive SFD method applied to the incompressible flow past a cylinder at $Re = 100$. We show that this procedure is able to automatically select parameters

that allow optimum convergence even if the initial control coefficient and filter width are poorly chosen.

It has been shown² that the parameters' pair $\chi = 1$ and $\Delta = 0.5$ does not allow the SFD method to control the instabilities within the flow past a cylinder at $Re = 100$. Hence by keeping these parameters fixed along the execution of the SFD method, a steady-state solution cannot be found.

To show that the adaptive procedure presented in Sec. III allows flexibility concerning the initial parameters chosen, we use $\chi_{\text{init}} = 1$ and $\Delta_{\text{init}} = 0.5$. We define $T = 50$, which means that the SFD method is executed for 50 time units between each execution of the modified Arnoldi method (until $\|\mathbf{q} - \bar{\mathbf{q}}\|_{\text{inf}} < 10^{-2}$). The simulation is initialised with a vanishing flow field (i.e., $(u_0, v_0) = (0, 0)$) in order to illustrate the fact that convergence of the SFD method does not depend on the choice of an initial condition close to the targeted steady-state.

A snapshot of the first “partially converged” base flow obtained (at $t = 50$) with these settings is presented in Fig. 3(a). This flow configuration is not symmetric but the oscillations are reduced in comparison with the vortex shedding of the uncontrolled case. At this stage, $\|\mathbf{q} - \bar{\mathbf{q}}\|_{\text{inf}} = 0.089$, which means that the current flow only approximate coarsely the steady-state solution. However, this flow field is used as base flow to compute the modified Arnoldi method for the first time. The stability analysis solver converges (i.e., the residual becomes smaller than 10^{-3}) in 56 iterations. The dominant eigenvalue (denoted $\tilde{\lambda}$) returned has a growth rate of 0.104 and a frequency of 0.821. Then the procedure presented in Sec. II is used to determine the pair of parameters that would enable the fastest convergence of the SFD method applied to the one-dimensional model $u^{n+1} = \tilde{\lambda}u^n = \exp(0.104 \pm 0.821 i)u^n$. We obtained that the optimum parameters for this problem are $\tilde{\chi} = 0.481$ and $\tilde{\Delta} = 2.696$.

The SFD method is then executed for another 50 time units with $\tilde{\chi}$ and $\tilde{\Delta}$, which are approximations of the optimum parameters. At $t = 100$, $\|\mathbf{q} - \bar{\mathbf{q}}\|_{\text{inf}} = 0.016$ and the modified Arnoldi method is executed a second time using a second “partially converged” base flow. Convergence is reached in 45 iterations and the dominant eigenvalue returned has a growth rate of 0.134 and a frequency of 0.769. The corresponding approximated optimum parameters are $\tilde{\chi} = 0.471$ and $\tilde{\Delta} = 3.028$.

The SFD method is executed a third time with the updated parameters. However this time, it is executed for less than 50 time units because $\|\mathbf{q} - \bar{\mathbf{q}}\|_{\text{inf}}$ becomes smaller than 10^{-2} at $t = 109$. Then we consider that the current flow field is a good enough approximation of the steady-state solution. Hence, we compute a stability analysis one last time. The modified Arnoldi method converges in 32 iterations and the dominant eigenvalue returned has a growth rate of 0.132 and a frequency of 0.764.

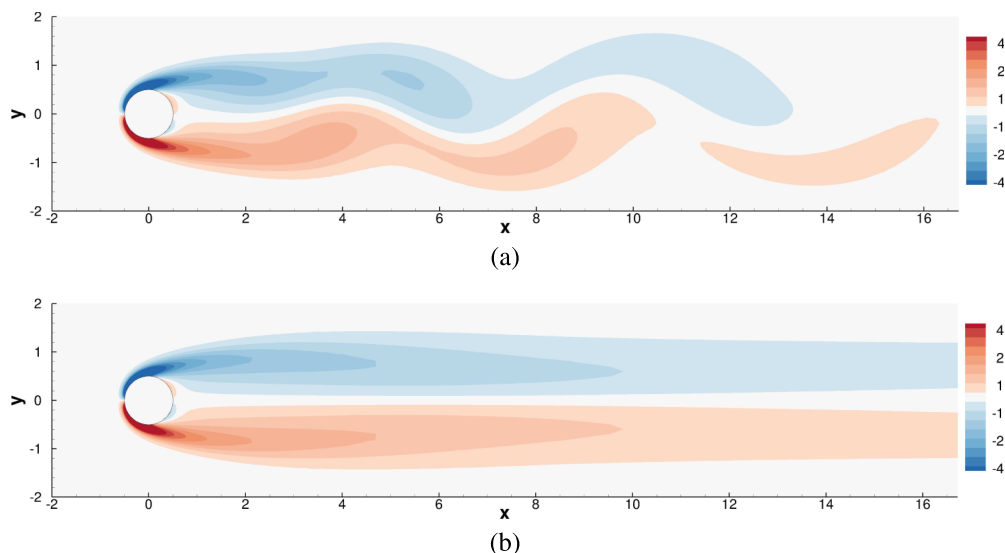


FIG. 3. Vorticity of the incompressible flow past a cylinder at $Re = 100$. “Partially converged” base flow used to compute the first instance of the modified Arnoldi iteration method (a); steady-state solution obtained at the end of the execution of the adaptive SFD method (b).

The corresponding approximated optimum parameters are $\tilde{\chi} = 0.467$ and $\tilde{\Delta} = 3.050$. These parameters are then fixed for the remaining of the solver execution. The converged steady-state solution obtained is shown in Fig. 3(b).

The convergence history of the adaptive algorithm is presented in Fig. 4. The three horizontal plateaux of the solid (red) curve represent the time spent computing stability analysis of the “partially converged” base flows. This allows us to visualise the impact of the additional computational work required by the adaptive algorithm. In Fig. 4, we also report the time evolution of $\|\mathbf{q} - \bar{\mathbf{q}}\|_{\text{inf}}$ when the SFD parameters are fixed to be $\chi = 1$ and $\Delta = 0.5$. This curve (the blue dotted one) is such that it decreases for a certain time, then increases abruptly, and eventually oscillates around a fixed value. This represents the fact that no steady-state can be found by keeping a control coefficient of 1 and a filter width of 0.5 fixed along the execution of the SFD method.

We computed an *a posteriori* stability analysis using the converged steady-state solution as base flow. The growth rate found was 0.127 and the frequency 0.741. Note that the three successive executions of the Arnoldi method returned approximations of the dominant eigenvalues that were gradually closer to the converged one. Knowing this dominant eigenvalue, we can evaluate the parameters that allow us to reach the optimum convergence rate of the unadapted SFD method. The optimum control coefficient is $\chi_{\text{opt}} = 0.451$ and the optimum filter width is $\Delta_{\text{opt}} = 3.144$. We executed the unadapted SFD method with these optimum parameters and the convergence history of this simulation is also reported in Fig. 4. We can notice that the adaptive SFD method converges at the same exponential rate as the unadapted one executed with the optimum parameters. Hence, even if the adaptive algorithm does not improve the convergence rate *per se*, it assists the user in order to reach optimum convergence speed. Note that for this case, *a priori* knowledge of the flow behaviour was not necessary. No assumption was made in order to define appropriate values of χ_{init} and Δ_{init} . We actually did the opposite because we used $\chi_{\text{init}} = 1$ and $\Delta_{\text{init}} = 0.5$ whereas we knew that these parameters were not suitable for the unadapted SFD method.

We realised other numerical simulations to test the robustness of our adaptive algorithm by considering different bluff body flows. The extension of the incompressible flow past a cylinder at higher Reynolds numbers was straightforward. We considered the (two-dimensional) case $Re = 300$ with $\chi_{\text{init}} = 1$, $\Delta_{\text{init}} = 2$, and the Arnoldi method executed every 100 time units. A steady-state solution was found and the stability analysis computed using the converged base flow returned that the dominant eigenvalue of this flow has a growth rate of 0.163 and a frequency of 0.470 (which corresponds to the optimum parameters $\chi_{\text{opt}} = 0.334$ and $\Delta_{\text{opt}} = 5.951$).

One may argue that if the geometry studied contains axial symmetry (like for the flow past a cylinder), using the SFD method, or any method that provides a steady-state solution of the Navier-Stokes equations, is unnecessary. For example, Mao and Blackburn⁹ presented a stability analysis (global and local) of the incompressible flow past a square cylinder (up to $Re = 300$) using a base flow obtained on a semi-domain with symmetric boundary conditions on the horizontal axis. We applied the adaptive SFD method to the incompressible flow past an ellipse with an angle of attack which is a case that cannot be studied with the help of symmetry planes. The ellipse’s major axis is 1, its minor axis is 0.4, the angle of attack is 30° , and the flow is studied at $Re = 150$ (based on the major axis length).

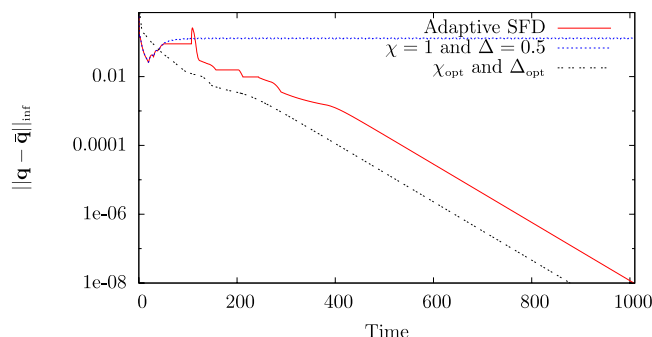


FIG. 4. Convergence history of the adaptive SFD method for $\chi_{\text{init}} = 1$ and $\Delta_{\text{init}} = 0.5$. The cases of the unadapted SFD method for $\chi = 1$, $\Delta = 0.5$, and for the optimum parameters $\chi_{\text{opt}} = 0.451$, $\Delta_{\text{opt}} = 3.144$ are also reported.

No assumption was made about this flow behaviour and the settings of our simulations were such that $\chi_{\text{init}} = 1$, $\Delta_{\text{init}} = 1$, and the Arnoldi method executed every 50 time units. A steady-state solution was found and the stability analysis computed using the converged base flow returned that the dominant eigenvalue of this flow has a growth rate of 0.168 and a frequency of 1.283 (which corresponds to the optimum parameters $\chi_{\text{opt}} = 0.792$ and $\Delta_{\text{opt}} = 1.673$).

The incompressible flow past a rotating cylinder is another case that cannot be studied with the help of symmetry planes. The rotation of the cylinder impacts the stability of the flow. At $Re = 100$, Pralits *et al.*¹⁰ have shown that for a rotation rate $0 \leq \alpha \leq 1.8$ the flow is unstable and von Kármán vortex streets are present. They become weaker as α increases. This instability is called shedding mode I. If the rotation rate is in the range $1.8 \leq \alpha \leq 4.85$, the flow becomes stable. If the rotation rate is increased again, a second unstable mode appears (called shedding mode II) for a range $4.85 \leq \alpha \leq 5.17$. And eventually, for rotation rates above 5.17, the flow is stable. Note that this behaviour is only true in two-dimensions. In three-dimensions the presence of shedding mode I and the range of shedding mode II depend on the spanwise wave number.¹¹ To test our adaptive algorithm, we considered the problem of finding a steady-state solution of the unstable mode II. Then we studied the two-dimensional flow past a rotating cylinder at $Re = 100$ with a rotation rate $\alpha = 5$.

As suggested by Åkervik *et al.*,¹ we defined the initial SFD parameters of this test case based on observations of the nonlinear unstabilized case. Direct numerical simulations showed that the frequency of the shedding mode II is very low. Hence, the flow past a cylinder at the unstable shedding mode II is challenging for the SFD method because this method is well suited to obtain steady-state solutions of flows with high unstable frequencies. We initially took little care in the definition of the initial parameters of the adaptive SFD method and failed in forcing the flow to evolve towards a steady-state. The low frequency of the shedding mode II suggests that the filter width should be chosen to be relatively high. Larger filter widths enable us to control instabilities that arise on a larger time scale even if they may require an impractically long time to converge. We eventually found that $\chi_{\text{init}} = 1$, $\Delta_{\text{init}} = 5$, and executing the Arnoldi method every 300 time units are appropriate settings. For this test case, the initial filter width is carefully chosen (based on the behaviour of the unstabilized DNS) but nothing is assumed about the dominant eigenvalue of the flow. Our adaptive algorithm executes twice the stability analysis on “partially converged” base flows and eventually converges towards an unstable steady-state solution. This steady-state is shown in Fig. 5. We computed a stability analysis using the converged base flow and the output was such that the dominant eigenvalue of this flow has a growth rate of 0.036 and a frequency of 0.241 (which corresponds to the optimum parameters $\chi_{\text{opt}} = 0.140$ and $\Delta_{\text{opt}} = 9.610$). This dominant eigenvalue corresponds to the one reported by Pralits *et al.*¹⁰

To conclude this section on numerical simulations we provide some guidelines to use the adaptive algorithm detailed in Sec. III. If information about the frequency of the flow field is available, it could be used to define the initial value of the filter width. A large Δ_{init} (e.g., about 5) should be used for cases with a low frequency and a Δ_{init} close to 1 should be used for cases with a relatively high frequency. As large filter widths render the system evolution slow, the time between two consecutive execution of the stability analysis method should also be large in order to give the flow sufficient time to evolve. In our simulations we used $T \approx 50 \times \Delta_{\text{init}}$ and were always able to converge towards a steady state solution. If little or no information is available about the frequency of the unstabilized flow field, or if

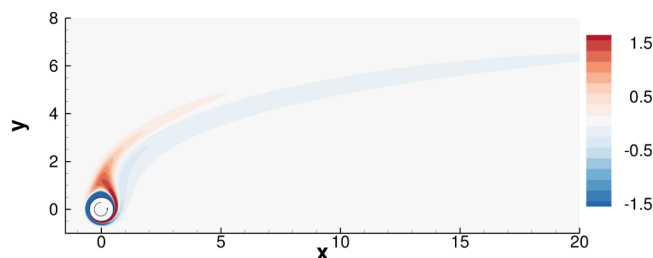


FIG. 5. Vorticity of the steady-state of the incompressible flow past a two-dimensional rotating cylinder at $Re = 100$ with a rotation rate $\alpha = 5$.

it would be too expensive to compute, we suggest to use $\Delta_{\text{init}} = 2$ and to compute the stability analysis method every 100 time units. For our adaptive algorithm, we found that always using $\chi_{\text{init}} = 1$ was appropriate. We also suggest to fix the SFD parameters when $\|\mathbf{q} - \bar{\mathbf{q}}\|_{\text{inf}}$ becomes smaller than 10^{-2} and to set up a tolerance of 10^{-3} for the execution of the Arnoldi method using “partially converged” base flows.

The simulations realised suggest that if the adaptive SFD method fails to converge with a given set of parameters, it should be restarted with a larger time interval T between consecutive executions of the stability analysis solver. We found out that a too small T does not give enough time to the flow to properly evolve which may result in a “partially converged” base flow which is a too coarse approximation of the converged steady-state. Restarting the simulation with a larger T may solve the convergence issue. Otherwise, one should increase the initial filter width. By combining these two aspects, we were able to find unstable steady-state solutions of all the flows considered.

V. CONCLUSION

An adaptive procedure to address the issue of selecting appropriate parameters for the SFD method is presented. This algorithm links together a SFD method, a stability analysis method and a procedure that determines the optimum coefficients of the SFD method applied to a simple one dimensional model. This adaptive method is based on several successive computations of the stability analysis method using “partially converged” base flows. This approximation is then used to tune the parameters of the SFD method to ensure an optimum convergence towards the steady-state solution.

This adaptive method was successfully applied to obtain unstable steady-states of several flows. The steady-state of the flow past a cylinder was obtained up to $Re = 300$. We also obtained steady-state solutions of flows that cannot be studied with the help of symmetry planes such as the flow past an ellipse with an angle of attack or past a rotating cylinder. For the later case we were interested in controlling the instabilities related to the shedding mode II. This was fairly challenging for our algorithm but a careful definition of the initial filter width Δ_{init} enables optimum convergence towards a steady-state. This set of test cases validates our adaptive method. Our next challenge is to study stability of more challenging flows such as fully three-dimensional vortex dominated flows for which no information is available about the instability frequency.

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