

AUTOMORPHISMS OF FREE PLANES

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ABSTRACT

Free planes, analogous in certain respects to free groups, have been studied for some time. In this thesis, we make a contribution to the study of the automorphisms of these planes. Our first main result is that every free plane has a polarity. We then investigate some of the properties of these polarities. In particular, we prove that a polarity in a finitely generated free plane has only a finite number of absolute elements. We determine certain conditions satisfied by this number, and show that for every number satisfying these conditions, there is a polarity with this number of absolute elements. We also give a necessary and sufficient condition for polarities in a given free plane to be conjugate in the collineation group of that plane.

Our other main result is restricted to the free plane $\mathbb{F}_2$ generated by a quadrangle. The collineation group Γ_2 of the plane $\mathbb{F}_2$ has recently been shown to be generated by the symmetric group of degree 4 and a further element of Γ_2 of order 2. We determine the orders and the conjugacy classes of the elements of Γ_2 which have finite order, and prove that every non-trivial proper normal sub-group of Γ_2 is torsion-free. From this we are able to deduce that Γ_2 coincides with its commutator subgroup, and hence is insoluble. In our investigation of the properties of polarities in free planes, we prove that all the polarities in $\mathbb{F}_2$ are conjugate in Γ_2 . It follows

from this that the orthogonal groups of these polarities are also conjugate in Γ , and hence isomorphic. Our conjecture is that they are isomorphic to the cyclic group of order 2.

Also submitted as part of this thesis is a published note, which contains our first main result.

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1. BASIC CONCEPTS AND DEFINITIONS

1.1 Projective planes

Let $\mathcal{P}$ and $\mathcal{L}$ be two disjoint sets, whose elements will be called points and lines respectively. Let I be a symmetric relation (to be called incidence relation) defined on $\mathcal{P} \cup \mathcal{L}$ such that $x I y$ implies $x \in \mathcal{P}$ and $y \in \mathcal{L}$ or $x \in \mathcal{L}$ and $y \in \mathcal{P}$. We define a projective plane as a system consisting of the sets $\mathcal{P}$ and $\mathcal{L}$ and the incidence relation I , and satisfying the following axioms:

- A1: Given any two⁽¹⁾ points P, Q , there is exactly one line incident with both P and Q .
- A2: Given any two lines ℓ, m , there is at least one point incident with both ℓ and m .
- A3: There exist four points, no three of which are incident with the same line.

It follows trivially from A1 that any two lines ℓ, m are incident with at most one point. Taken together with A2, this implies

(1)

Throughout this thesis, when we speak of two objects x and y , we shall imply that they are distinct objects. When we speak simply of objects x and y , we then allow the possibility that $x = y$; that is, x and y may be different names for the same object. This convention will also apply to the use of the natural numbers three, four, ...

A2': Given any two lines ℓ, m , there is one and only one point incident with both ℓ and m .

For symmetry, A1, A2' and A3 are usually taken as the axioms defining a projective plane.

The line (point) incident with two points (lines) x and y will be denoted by xy .

A set of four points $P_i (i = 1, 2, 3, 4)$ as in A3, will be called a quadrangle, and denoted by $P_1 P_2 P_3 P_4$.

It is readily shown from the axioms that

A3': In a projective plane, there exist four lines, no three of which are incident with the same point.

Conversely, from A1, A2' and A3' we can derive A3. This result enables us to construct from any given projective plane $\mathcal{P}$ another projective plane $\mathcal{P}^*$, called the dual of $\mathcal{P}$: The points of $\mathcal{P}^*$ are the lines of $\mathcal{P}$, the lines of $\mathcal{P}^*$ are the points of $\mathcal{P}$ and the incidence relation I^* defined on $\mathcal{P}^*$ is identical with the incidence relation I on $\mathcal{P}$. That $\mathcal{P}^*$ is, in fact, a projective plane follows, as we have already stated, from A1, A2' and A3'.

The symmetrical role played by "points" and "lines" in the definitions of a projective plane $\mathcal{P}$ and its dual $\mathcal{P}^*$ gives rise to the Principle of Duality. This may be stated as follows: Let $\mathcal{R}$ be a theorem on projective planes, and let $\mathcal{R}^*$ be derived from $\mathcal{R}$ by interchanging the words "points" and "lines" in the enunciation of $\mathcal{R}$. Then $\mathcal{R}^*$ is also a theorem on projective planes. The truth of this principle follows from the fact that if a statement $\mathcal{R}$ is true of

a class of $\mathcal{C}$ of projective planes $\mathcal{P}$, then $\mathcal{P}^*$ is true of the class $\mathcal{C}^*$ of the duals $\mathcal{P}^*$ of $\mathcal{P}$, and $\mathcal{C}^*$, as shown above, is itself a class of projective planes. In view of this, we shall sometimes refer to lines as duals of points and vice versa.

An alternative definition of a projective plane regards lines as subsets of the set of points. In this case, we speak of a point belonging to a line, etc. The equivalence of the two points of view is easily derived.⁽²⁾ In this thesis, we do not restrict ourselves to either point of view, but use language and notation appropriate to one or the other as we find convenient. We also allow ourselves to use more descriptive expressions as "a point lies on a line", "a line passes through a point", and so on.

A system which satisfies axioms A1 and A2 but definitely violates A3 will be called a degenerate projective plane. The term "projective plane" will henceforth be used to include the degenerate case. Where necessary, we shall refer to projective planes, as defined earlier, as non-degenerate projective planes. The degenerate projective planes are readily listed (see Hall [5], page 232). The following lemma of Baer ([3], lemma 1) characterises them:

(2)

See, for example, Pickert [8] p.4, where this is done for the wider class of partial planes (definition on p. 9 of this thesis).

A projective plane $\mathcal{P}$, containing both points and lines, is degenerate if and only if it contains a point P and a line ℓ such that every point of $\mathcal{P}$, except possibly P , lies on ℓ , and every line of $\mathcal{P}$, except possibly ℓ , passes through P .

1.2 Partial planes

A generalisation of the concept of projective plane, introduced by Hall [5], is that of partial plane. This may be defined as a system consisting of two disjoint sets $\mathcal{P}$ and $\mathcal{L}$ of points and lines respectively and an incidence relation I between the points and the lines, which satisfies the axiom:

B1: Any two points P, Q are incident with at most one line.

As before it follows trivially from B1 that

B2: Any two lines ℓ, m are incident with at most one point.

We remark that the Principle of Duality applies to this more general concept as well.

Let $\mathcal{P}$ be a partial plane, and $\mathcal{Q}$ a subset of the set of points and lines of $\mathcal{P}$. Then we say that $\mathcal{Q}$ is a sub-partial plane of $\mathcal{P}$ and write $\mathcal{Q} \subseteq \mathcal{P}$ if $x, y \in \mathcal{Q}$ and $x I y$ in $\mathcal{Q}$ implies $x I y$ in $\mathcal{P}$. We shall also say that $\mathcal{P}$ is an extension of $\mathcal{Q}$. We may similarly define a projective sub-plane of a given projective plane or, more generally, of a given partial plane.

Let $\{\mathcal{Q}_j\}$ be a set of partial planes such that $\mathcal{Q}_j \subseteq \mathcal{Q}_{j+1}$ ($j = 0, 1, 2, \dots$). Let the elements of a system $\mathcal{P}$ be given by the union of the elements of the $\mathcal{Q}_j$ and the incidence relation in $\mathcal{P}$

be defined as follows: If $x, y \in \mathcal{P}$, then $x I y$ in $\mathcal{P}$ if there is an integer j such that $x, y \in \mathcal{Q}_j$ and $x I y$ in $\mathcal{Q}_j$. $\mathcal{P}$ can be shown to be a partial plane, which we call the union of the partial planes $\mathcal{Q}_j$.

A partial plane $\mathcal{Q}$ is said to be complete in its extension $\mathcal{P}$ if for any two points (lines) $x, y \in \mathcal{Q}$ such that the line (point) $xy \in \mathcal{P}$, we also have $xy \in \mathcal{Q}$.

A partial plane is said to be degenerate if it may be embedded⁽³⁾ as a sub-partial plane in some degenerate projective plane. Otherwise, it is non-degenerate.

Let $\mathcal{B}$ be a sub-partial plane of a projective plane $\mathcal{P}$. Then, it can be shown that the intersection of all the projective sub-planes of $\mathcal{P}$ (degenerate and non-degenerate) containing $\mathcal{B}$ is itself a projective sub-plane of $\mathcal{P}$, and is the unique least sub-plane⁽⁴⁾ of $\mathcal{P}$ containing $\mathcal{B}$. We call this sub-plane the sub-plane of $\mathcal{P}$ generated by $\mathcal{B}$ and denote it by $[\mathcal{B}]$. The sub-plane $[\mathcal{B}]$ is non-degenerate if and only if $\mathcal{B}$ is non-degenerate.

(3)

We use the expression "embedding a partial plane $\mathcal{Q}$ in a partial plane $\mathcal{P}$ " in the usual algebraic sense of an isomorphism of $\mathcal{Q}$ into $\mathcal{P}$. We give the definition of mappings of one partial plane into another on pp. 17 ff.

(4)

The abbreviation "plane" will sometimes be used to stand for "projective plane".

To be inserted after line 6 from below:

It is readily verified that if $\mathcal{P}$ is a projective plane and $\mathcal{B} \subseteq \mathcal{P}$, then the $\mathcal{B}$ -stage of elements of $[\mathcal{B}]$ has the properties (i) - (iii). One then deduces that $[\mathcal{B}]$ is generated by $\mathcal{B}$ in the sense just defined.

The projective sub-plane $[B]$ may also be defined in this way: Let Q be any sub-partial plane of a projective plane P . We denote by Q^+ the set of all points (lines) in P , but not in Q , which are incident in P with at least two lines (points) of Q . We now define a sequence of ascending sub-partial planes of P as follows:-

$$B_0 = B, \quad B_{j+1} = B_j \cup B_j^+ \quad (j = 0, 1, 2, \dots)$$

Then $[B]$ is the union of the partial planes B_j . The B -stage of an element $x \in [B]$, written $s_B(x)$, is defined to be the unique least non-negative integer j such that $x \in B_j$. Where there is no risk of ambiguity about the generating partial plane B , we shall refer simply to the stage of an element x and write $s(x)$.

In general, we shall say that a partial plane P is generated by its sub-partial plane B if to every element x in P we can assign a non-negative integer $s(x)$ as its stage such that:

- (i) precisely the elements of B have stage zero;
- (ii) if $s(x) > 0$, then there are at least two elements of lower stage incident with x ;
- (iii) if $s(x) > 0$, then there is at most one element of stage less than $s(x) - 1$ incident with x .

An element x in P , but not in B , will be said to be B -bound if it is incident with more than two elements of stage $\leq s(x)$. We note that, from (ii), there exist at least two such elements (of stage lower than $s(x)$), and hence x is B -bound if it is incident with a dual element of the same stage as itself.

The question naturally arises: 'Can an arbitrary partial plane be embedded in a projective plane? And if so, can this be done uniquely (up to isomorphism)?'

The answer to the first question is yes; in fact, the following construction, which we give only in outline, ⁽⁵⁾ shows that what appears to be an even stronger result holds - that any partial plane generates a projective plane. Given a partial plane $\mathcal{B}$, which is not a projective plane, we construct new elements $\mathcal{B}^+$ as follows: We divide the set of points in $\mathcal{B}$ into subsets such that

- (i) each subset contains only points no two of which are already incident with a line in $\mathcal{B}$;
- (ii) any two points not incident with a line in $\mathcal{B}$ belong to exactly one subset;
- (iii) each subset contains at least two points.

We now define each of these subsets to be a new line, incident with precisely the points in that subset. Similarly, we define new points from the set of lines of $\mathcal{B}$, and we denote the totality of new elements by $\mathcal{B}^+$. We now put

$$\mathcal{B}_0 = \mathcal{B}, \quad \mathcal{B}_{j+1} = \mathcal{B}_j \cup \mathcal{B}_j^+ \quad (j = 0, 1, 2, \dots).$$

It is then readily verified that the $\mathcal{B}_j$ form an ascending sequence of partial planes, whose union is a projective plane generated by $\mathcal{B}$.

(5)

For a fuller account, we refer to Pickert [8], pp. 14 - 15.

It will be evident from the above construction that the resulting projective plane is not, in general, unique. A simple example to show this is that the free plane of degree 2, the Fano configuration and the rational plane are three non-isomorphic projective planes each of which is generated by a quadrangle. (6)

There is, however, a canonical way of carrying out the construction. In defining $\mathcal{B}_j^+$ from $\mathcal{B}$ we shall require that every element of $\mathcal{B}_j^+$ be incident with exactly two dual elements of $\mathcal{B}_j$ ($j = 0, 1, 2, \dots$). The projective plane thus obtained is easily seen to be unique. We shall call it the free completion of $\mathcal{B}$ and denote it by $F(\mathcal{B})$. (7)

We observe that every element x in $F(\mathcal{B})$, but not in $\mathcal{B}$, (i.e. such that $s_{\mathcal{B}}(x) > 0$) is incident with exactly two dual elements of lower stage than x itself in $F(\mathcal{B})$. We shall refer to these dual elements as the elements defining x in $F(\mathcal{B})$.

(6)

"Free planes" are defined on p. 14 below. The Fano configuration is a finite projective plane with 7 points and 7 lines (see Skornyakov [10] p. 23) and the rational plane is a projective plane which may be co-ordinatized by the rational field (for the co-ordinatizing of projective planes, see the relevant sections of Skornyakov [10] or Pickert [8]).

(7)

We note that the free completion of a projective plane $\mathcal{P}$ may trivially be defined to be $\mathcal{P}$ itself.

The partial planes $\mathcal{B}_j$ ($j = 0, 1, 2, \dots$) contained in $F(\mathcal{B})$ will be called the complete free extensions of $\mathcal{B}$. Let $\mathcal{B}'$ be a partial plane such that $\mathcal{B} \subseteq \mathcal{B}' \subseteq F(\mathcal{B})$ and for every $x \in \mathcal{B}'$ with $s_{\mathcal{B}}(x) = j + 1$, $\mathcal{B}'$ also contains the two elements defining x in $F(\mathcal{B})$. Then we say that $\mathcal{B}'$ is a free extension of $\mathcal{B}$, and conversely that $\mathcal{B}$ is a free contraction of $\mathcal{B}'$. $\mathcal{B}'$ is then a free extension of $\mathcal{B}$ if and only if $\mathcal{B}$ generates $\mathcal{B}'$ and $\mathcal{B}'$ contains no $\mathcal{B}$ -bound elements. The $\mathcal{B}$ -stage of an element in $\mathcal{B}'$ is the same as its $\mathcal{B}$ -stage in $F(\mathcal{B})$. It is also evident that the complete free extensions $\mathcal{B}_j$ of $\mathcal{B}$ are themselves free extensions of $\mathcal{B}$.

It can be shown that if a partial plane $\mathcal{B}''$ is a free extension (contraction) of a partial plane $\mathcal{B}'$, which is itself a free extension (contraction) of a partial plane $\mathcal{B}$, then $\mathcal{B}''$ is a free extension (contraction) of $\mathcal{B}$. Also, a series of free extensions and free contractions may be carried out in any order. (Hall [5], Theorems 4.3 and 4.7. See also Pickert [8], Theorem 11, page 17.) Two partial planes are said to be freely equivalent if either may be obtained from the other by free extensions and free contractions. That free equivalence is a true equivalence relation is readily shown. Also, freely equivalent partial planes have the same free completion.

1.3 Free planes

The free completion of a partial plane consisting of $n (\geq 2)$ points on a line, the common line of incidence, and two points off

this line is called the free (projective) plane⁽⁸⁾ of degree n .⁽⁹⁾

We denote the free plane of degree n by $\mathcal{F}_n$, and the generating partial plane just defined by $\mathcal{B}_n^{(0)}$. The complete free extensions of $\mathcal{B}_n^{(0)}$ will be denoted by $\mathcal{B}_n^{(j)}$ ($j = 0, 1, 2, \dots$). We note that for $n = 2$, we may leave out the "common line of incidence", and we shall always consider $\mathcal{F}_2$ to be generated by a quadrangle.

A useful notion in the theory of free planes is that of rank.⁽¹⁰⁾ We shall say that a partial plane is finite if it contains a finite number of elements; it will be infinite otherwise. The rank $r(\mathcal{P})$ of a finite partial plane $\mathcal{P}$ is the number $2|\mathcal{P}| - |\mathcal{I}|$, where $|\mathcal{P}|$ is the number of elements in $\mathcal{P}$ and $|\mathcal{I}|$ the number of incidences in $\mathcal{P}$ given by the number of unordered pairs (x, y) with $x, y \in \mathcal{P}$ and $x \mathcal{I} y$.

(8)

For certain purposes, as in the theory of free products of projective planes, one considers the degenerate projective planes as free planes (Kopeikina [7]).

(9)

Though n need not be finite, we shall in the main, in this thesis, be concerned with free planes of finite degree. Hence, unless otherwise stated, n may be assumed to be finite.

(10)

The notion was introduced by Marshall Hall, but the name is due to Kopeikina.

It is easily shown that if $\mathcal{P}$ is a finite partial plane, then any finite free extension (contraction) of $\mathcal{P}$ has rank equal to that of $\mathcal{P}$. Hence, provided that the partial planes involved are all finite, rank is invariant under free equivalence. We now define the rank of the free completion $F(\mathcal{P})$ of a finite partial plane $\mathcal{P}$ to be $r(\mathcal{P})$. It is readily calculated that $r(\mathcal{F}_n) = n + 6$. An infinitely generated free plane will be said to have infinite rank, of cardinality equal to that of the set of collinear points.

Using the notion of rank, Hall [5] has shown that free planes $\mathcal{F}_n, \mathcal{F}_{n'}$ are non-isomorphic if $n \neq n'$.

By a configuration, we shall always understand a finite partial plane. A configuration $\mathcal{P}$ will be said to be confined if every element of $\mathcal{P}$ is incident in $\mathcal{P}$ with at least three dual elements. A partial plane is said to be open if it contains no confined configuration. (11)

1.4 Mappings

The image of an element x under a mapping α will be denoted by $x\alpha$. Consequently, the product $\alpha\beta$ of the mappings α and β , understood in the usual sense of the successive carrying out of α and

(11)

Observe that unlike "confined" we do not restrict the use of the term "open" to configurations. In fact, Dembowski [4] studies non-degenerate open (projective) planes, which are necessarily infinite. (See Theorem 20 of T.G. Ostrom: "Transitivities in projective planes". Canadian J. Math. 9(1957), 380-99.)

β , will be performed in that order. The mapping α will also be denoted by $\alpha: x \rightarrow x\alpha$. If $x\alpha = x$, we shall say that x is a fixed element of α . We shall say that the mapping α of a set $\mathcal{P}$ is an extension of the mapping β of $\mathcal{Q}$, if $\mathcal{Q} \subseteq \mathcal{P}$ and $x\alpha = x\beta$ for all $x \in \mathcal{Q}$.

A homomorphism α of a partial plane $\mathcal{P}$ into a partial plane $\mathcal{Q}$ is a mapping of the points of $\mathcal{P}$ into the points of $\mathcal{Q}$ and the lines of $\mathcal{P}$ into the lines of $\mathcal{Q}$ such that $x, y \in \mathcal{P}$ and $x I y$ in $\mathcal{P}$ implies $x\alpha I y\alpha$ in $\mathcal{Q}$. The homomorphic image of a projective plane is easily shown to be itself a projective plane. We note, however, that the homomorphic image may be degenerate even though the original plane is not (Skornyakov [10], Theorem 5).

A one-to-one mapping α of the points and lines of a partial plane $\mathcal{P}$ onto the points and lines of a partial plane $\mathcal{Q}$ such that α is a homomorphism of $\mathcal{P}$ onto $\mathcal{Q}$ and α^{-1} is a homomorphism of $\mathcal{Q}$ onto $\mathcal{P}$ is called an isomorphism of $\mathcal{P}$ onto $\mathcal{Q}$. If $\mathcal{P}$ (and hence $\mathcal{Q}$) is a non-degenerate projective plane, then one-oneness is sufficient for a homomorphism α to be an isomorphism.⁽¹²⁾ That this need not be the case with partial planes in general is shown by the following example:

(12)

In fact, this is true of the wider class of ~~partial~~ planes in which every element is incident with at least two dual elements.

Let $\mathcal{P}$ consist of a line ℓ and two points P, Q off ℓ , and $\mathcal{Q}$ consist of a line m and two points R, S on m . A one-one mapping α of $\mathcal{P}$ onto $\mathcal{Q}$ is defined by $P\alpha = R$, $Q\alpha = S$, $\ell\alpha = m$. α is a homomorphism of $\mathcal{P}$ onto $\mathcal{Q}$, but α^{-1} is not a homomorphism of $\mathcal{Q}$ onto $\mathcal{P}$. If we leave out Q from $\mathcal{P}$ and S from $\mathcal{Q}$, this example also shows that one-oneness is not sufficient if $\mathcal{P}$ is a degenerate projective plane.

If $\mathcal{Q} = \mathcal{P}$, an isomorphism of $\mathcal{P}$ onto $\mathcal{Q}$ is called a collineation. If $\mathcal{P}$ is a projective plane, and α a collineation of $\mathcal{P}$, then the system of fixed elements of α is a projective sub-plane of $\mathcal{P}$, which we call the fixed sub-plane of α .

A one-one mapping α of a partial plane $\mathcal{P}$ onto itself which takes points onto lines and lines onto points such that for $x, y \in \mathcal{P}$, $x I y$ if and only if $x\alpha I y\alpha$, is called a correlation. A partial plane $\mathcal{P}$ admitting of a correlation is then seen to be isomorphic to its dual $\mathcal{P}^*$. We shall say that $\mathcal{P}$ is self-dual. The product of two correlations is a collineation. In particular, a correlation θ such that $\theta^2 = 1$, the identity mapping, is called a polarity.

In a polarity θ , the line $P\theta$ is called the polar of the point P , and the point $\ell\theta$ is called the pole of the line ℓ . Since $\theta^2 = 1$, it is trivial that P is the pole of ℓ if and only if ℓ is the polar of P . An element x such that $x I x\theta$ in a polarity θ will be called an absolute element.

The collineations and correlations of a partial plane constitute its automorphisms. The set Π of all automorphisms of

a partial plane $\mathcal{P}$ form a group, the automorphism group of $\mathcal{P}$. The subset Γ of all the collineations of $\mathcal{P}$ forms a subgroup of Π . If the partial plane $\mathcal{P}$ is self-dual, then Γ is a normal subgroup of index 2 in Π . Otherwise, $\Gamma = \Pi$.

The automorphism group of $\mathcal{P}_n$ will be denoted by Π_n and the collineation group by Γ_n .

1.5 Notation and other definitions

We shall in the main use the following notation:

Roman capital letters will be used to denote points.

Roman lower case letters will be used to denote:

- (i) lines;
- (ii) elements of a partial plane, when we do not wish to specify whether these are points or lines. For this purpose we shall use letters towards the end of the alphabet;
- (iii) integers.

In polarities, corresponding pole and polar will be denoted by corresponding capital and lower case Roman letters.

Curly capital letters $\mathcal{P}, \mathcal{Q}, \dots$ will be used for partial planes, and in particular, for projective planes.

Greek capital letters will be used for groups of automorphisms of partial planes.

Greek lower case letters will be used for automorphisms of partial planes, except the identity mapping, which will be denoted by 1.

The following set-theoretical symbols, among others, will be used:

$\in$: is a member of

$\subseteq$: is a subset of

$\subset$: is a proper subset of

$P \setminus Q$: complement of Q in P . Here we shall not require that Q be a subset of P .

$\cup$: union of

$\cap$: intersection of

$\emptyset$: the empty set.

By a flag (x, y) will be meant a point-line pair with $x \perp y$. When it is not necessary to specify the elements, we shall speak simply of a flag.

If $x \in Q$ is incident in Q with k dual elements, we say that x has multiplicity k in Q . In particular we shall say that x is isolated, simple, double, ... if it has multiplicity 0, 1, 2, ...

Let Δ be a subgroup of the group Σ and $\alpha, \beta \in \Sigma$.

Then we shall say that α and β are conjugate in Δ if there is an element $\delta \in \Delta$ such that $\delta^{-1} \alpha \delta = \beta$.

2. INTRODUCTION TO THE THESIS

2.1 Historical

Free planes were introduced and first studied by Marshall Hall [5] in 1943. He derived a number of their properties, particularly those of the finitely generated free planes, and pointed out an analogy between free planes and free groups. This analogy was further developed by L. I. Kopeikina [7], who introduced the notion of free products of projective planes, parallel to that of free products of groups. Marshall Hall had shown that the finitely generated sub-planes of a free plane are themselves free; Kopeikina extended this result to all sub-planes of a free plane. Hall's result was a consequence of his theorem that a finitely generated non-degenerate open plane is free, but he remarked that this theorem might not be true if the restriction "finitely generated" were removed. Kopeikina gave an example of an infinitely generated open plane which is not free, thus proving Hall's remark to be right.

The next major advance in the theory of free planes was made by P. Dembowski [4]. He proved, among other things, that the collineation group Γ_n of the free plane $\mathcal{F}_n$ is not transitive on the points (lines) of $\mathcal{F}_n$, and that the transitive class under Γ_n of every element of $\mathcal{F}_n$ is infinite. (Here n may be infinite.) Using these results, he was able to give infinite classes of open planes which are not free. He showed, however, that the open planes are precisely those planes which may be described as locally free; that is, every finitely generated sub-plane of such a plane is free. Marshall Hall had shown

that every free plane contains an $\mathcal{F}_2$, and that, for finite n , every $\mathcal{F}_n$ contains an $\mathcal{F}_{n+1}$. Dembowski extended this result by showing that a free plane $\mathcal{F}_2$ contains an infinitely generated sub-plane, giving yet another instance of the analogy between free planes and free groups. ⁽¹⁾

However, not much else was known of the nature of the collineation groups $\mathcal{C}_n$ ⁽²⁾ until recently (1963), when R. Sandler [9] determined a system of three generators of the group $\mathcal{C}_2$. ⁽³⁾ For finite n , greater than 2, he determined a subgroup of $\mathcal{C}_n$, which he conjectures is the whole group $\mathcal{C}_n$. This still remains an open

(1)

The analogous result holds for free groups of rank greater than one, and also for other free systems as free nets, free loops, and free groupoids.

(2)

Since a free plane $\mathcal{F}_n$ contains no confined configurations, its collineation group $\mathcal{C}_n$ contains no perspectivities - the existence of a perspectivity in $\mathcal{C}_n$ would imply that of a Desargues configuration in $\mathcal{F}_n$, and this latter is a confined configuration. For the definitions of perspectivity and the Desargues configuration, and a systematic study of the connection between them, we refer to R. Baer: "Homogeneity of projective planes". American J. Math. 64 (1942) 137 - 52.

(3)

In a paper to be published, he gives a finite set of defining relations for $\mathcal{C}_2$. (Privately communicated.)

question.

2.2

In this thesis, we make some contribution to the development of the theory of free planes as outlined above. Dembowski [4] demonstrated that free planes are self-dual by defining a correlation on each $\mathcal{F}_n$ for arbitrary n . The correlations were all of infinite order and one immediately asks whether free planes have correlations of finite order and, in particular, whether they have polarities. That free planes have polarities we show in § 4. We further prove that the number j of absolute points in a polarity in $\mathcal{F}_n$ is finite and satisfies the conditions: $j \equiv n \pmod{2}$, and $0 \leq j \leq n$. We also demonstrate that for any number j satisfying these conditions, there is a polarity in $\mathcal{F}_n$ with precisely j absolute points. In § 5 we give a necessary and sufficient condition for polarities in $\mathcal{F}_n$ to be conjugate in Γ_n .

As stated earlier, Sandler [9] has determined three generators for Γ_2 . It was at first conjectured that Γ_2 is simple, but this has been shown to be false by Prof. N.S. Mendelsohn.⁽⁴⁾ However, we are able to show in § 6 that Γ_2 is insoluble. We determine completely the orders and the conjugacy classes of elements of finite order in Γ_2 and show that every proper normal subgroup of Γ_2 , distinct from

(4)

Privately communicated.

the identity, is torsion-free.

In the study of finite projective spaces, the orthogonal groups are of interest, since they lead to certain simple groups.⁽⁵⁾ The orthogonal group $\mathcal{O}(\theta)$ with respect to a given polarity θ in a plane $\mathcal{P}$ is defined to be the set of all collineations of $\mathcal{P}$ which commute with θ . That the set of collineations is a subgroup of the group of collineations of $\mathcal{P}$ is easily shown. We are naturally led to ask what the orthogonal groups for polarities in free planes are. Even if it were desirable to determine these groups in all cases, it would be a very difficult task, if not an impossible one, to perform. In §6.2, we consider the orthogonal groups in the case of polarities in $\mathcal{F}_2$. Now all the polarities in $\mathcal{F}_2$ are conjugate in $\mathcal{G}_2$; hence the orthogonal groups are also conjugate in $\mathcal{G}_2$ and so are isomorphic. We show that the problem of determining these groups is related to that of determining the stabiliser in $\mathcal{G}_2$ of three points belonging to a generating quadrangle of $\mathcal{F}_2$. This second problem we solve completely, but we are only able to make a conjecture as to what the orthogonal groups are.

2.3 Some remarks on the concept of a free plane

The question may rightly be asked whether the concept of

(5)

See for example, E. Artin: "Geometric Algebra". Interscience, New York (1957).

freeness properly applies to the class of projective planes which have been called free planes. Given a mathematical system, we say that a sub-system of it has the homomorphism property if every member of the system is a homomorphic image of some member of the sub-system. The homomorphism property is generally considered one of the most significant properties of a free sub-system, and such a sub-system is often effectively defined in terms of this property. (6)

This is one point of view. A second one is that expressed by Marshall Hall. In his own words, "the concept of a 'free' system, a system in which no relations hold save those implied by the axioms,

(6)

cf.: R.H. Bruck: "A Survey of Binary Systems". Springer, Berlin (1958) p. 5.

A.G. Kurosh: "General Algebra." Translated by K.A. Hirsch. Chelsea, New York (1963) p. 136.

H. Federer and B. Jonsson: "Some properties of free groups." Trans. American Math. Soc. 68 (1950) pp. 1 - 2.

C. Chevalley: "Fundamental Concepts of Algebra". Academic, New York (1956) p. 18.

is tenable even though no (homomorphism) theorem may hold."⁽⁷⁾

On either view, however, there is another class of planes which have as good a claim to be called free planes, namely the planes which Kopeikina [7] has called the completely free planes. A completely free plane is defined to be the free completion of a partial plane consisting of a number of isolated elements. This class of planes is actually a sub-class of the class of free planes (Kopeikina [7]).

Now a finitely generated completely free plane has even rank. We deduce immediately from Hall's result, that every $\mathcal{F}_n$ contains an $\mathcal{F}_{n+1}$, that a sub-plane of a completely free plane need not itself be completely free. On the other hand, every sub-plane of a

(7)

M. Hall, Jr.: "The Theory of Groups." Macmillan, New York (1959) p. 94.

It is also interesting to note that when Marshall Hall introduced the concept and name of free planes, he remarked that the analogy between these planes and free groups did not "extend to making every projective plane a homomorphic image of a free plane." Actually the homomorphism property holds for free planes. This follows from theorem 10 of Pickert [8], page 15, and the observation that a non-degenerate projective plane $\mathcal{P}$ may be generated by the points on any line ℓ in $\mathcal{P}$, the line ℓ and two points off ℓ .

free plane is free. The analogue of this result holds for most other free systems, and is again generally considered as the second most significant property of free systems. We are thus led to uphold the claim of the original class of planes to be called free planes.

3. PRELIMINARY RESULTS:

We list in this section a number of results which we shall require subsequently, but which we have not already mentioned. Most of them have been previously known and already published. We give the relevant reference(s) in such cases.

(A) Let $F(\mathcal{P})$ be the free completion of a partial plane $\mathcal{P}$. Then the confined configurations contained in $F(\mathcal{P})$ are precisely those already contained in $\mathcal{P}$.

(Pickert [8], theorem 12, p. 18; but see also concluding remarks to §4 of Hall [5].)

Corollary: A free plane is an open plane. (Hall [5], theorem 4.8).

(B) A non-degenerate open configuration is free equivalent to a configuration $\mathcal{C}_n^{(c)}$ for some $n \geq 2$.

(Hall [5], theorem 4.9)

Corollary 1: If $\mathcal{P}$ is a non-degenerate open configuration, then $r(\mathcal{P}) \geq 8$.

Corollary 2: Every finitely generated non-degenerate open plane is free.

(Hall [5], theorem 4.13).

(C) If the finite partial plane $\mathcal{P}$ is generated by the configuration $\mathcal{P}_0$, then $r(\mathcal{P}) \leq r(\mathcal{P}_0)$, and there is equality if and only if $\mathcal{P}$ is a free extension of $\mathcal{P}_0$.

(Dembowski [4], lemma 1.3).

Corollary: If $\mathcal{P}$ is a finite sub-partial plane of $\mathcal{T}_n$ such that

$r(\mathcal{P}) = n + 6$ and $\mathcal{P}$ generates all of $\mathcal{T}_n$, then $\mathcal{P}$ generates $\mathcal{T}_n$ freely.
(Sandler [9], theorem 3).

(D) Let α_0 be a homomorphism of a partial plane $\mathcal{B}$ into a projective plane $\mathcal{P}$. Then α_0 can be extended to a homomorphism α of $F(\mathcal{B})$ into $\mathcal{P}$. In general, the extension is not unique.

(The proof of (D) is essentially the same as that of theorem 10 of Pickert [8], p. 15).

(E) If φ is an isomorphism of a partial plane $\mathcal{B}$ onto a partial plane $\mathcal{B}'$, then φ extends uniquely to an isomorphism of $F(\mathcal{B})$ onto $F(\mathcal{B}')$.

Proof: The proof is by induction on the $\mathcal{B}$ - and $\mathcal{B}'$ -stages of elements in $F(\mathcal{B})$ and $F(\mathcal{B}')$.

Corollary: If each of the configurations $\mathcal{B}, \mathcal{B}'$ freely generates a given $\mathcal{T}_n$, and there is an isomorphism φ of $\mathcal{B}$ onto $\mathcal{B}'$ then φ extends (uniquely) to a collineation of $\mathcal{T}_n$.

Proof: The corollary follows directly from (E) and the fact that $F(\mathcal{B}) = \mathcal{T}_n = F(\mathcal{B}')$.

Remark: Unlike finitely generated free groups, however, it is not true that every homomorphism of a finitely generated free plane onto itself is an isomorphism. A counter-example is this:

Let $B_1 B_2 B_3 B_4$ be a generating quadrangle for an $\mathcal{T}_2$ and let $D_1 = (B_1 B_2) (B_3 B_4)$, $D_2 = (B_1 B_4) (B_2 B_3)$, $D_3 = (B_1 B_3) (B_2 B_4)$. The quadrangle $B_1 B_2 B_3 B_4$ is then a $\mathcal{B}_2^{(0)}$ for $\mathcal{T}_2$ and we define a homomorphism α_0 of $\mathcal{B}_2^{(0)}$ onto $\mathcal{B}_2^{(0)}$ as follows:-

$$B_1, B_2, B_3, B_4 \rightarrow B_1$$

$$B_1 B_2, B_3 B_4 \rightarrow B_1 B_2$$

$$B_1 B_3, B_2 B_4 \rightarrow B_1 B_3$$

$$B_1 B_4, B_2 B_3 \rightarrow B_1 B_4$$

$$D_1 \rightarrow B_2, D_2 \rightarrow B_4, D_3 \rightarrow B_3.$$

By the methods of result (D), α_0 may be extended to a homomorphism α of $F(\mathcal{B}_2^{(0)}) = \mathcal{F}_2$ onto $F(\mathcal{B}_2^{(0)}) = \mathcal{F}_2$. α is a homomorphism of $\mathcal{F}_2$ onto itself but it is clearly not an isomorphism.

(F) If α_0 is an automorphism of a partial plane $\mathcal{B}$, then α_0 extends uniquely to an automorphism α of $F(\mathcal{B})$. Moreover, the order of α_0 and the order of α are the same.

Proof: The proofs of both parts of (F) are by induction on the $\mathcal{B}$ -stage of elements of $F(\mathcal{B})$. (See also Dembowski [4], page 419.)

Definition: An automorphism of $F(\mathcal{B})$ which maps $\mathcal{B}$ onto itself will be called a $\mathcal{B}$ -stationary automorphism of $F(\mathcal{B})$.

Definition: Let $F(\mathcal{B})$ be the free completion of a partial plane $\mathcal{B}$ and $\mathcal{L}_0$ a sub-partial plane of $F(\mathcal{B})$. Then we shall say that $\mathcal{L}_0$ has the property N with respect to $\mathcal{B}$ if $s_{\mathcal{B}}(x) < s_{\mathcal{B}}(y)$ for all $x \in \mathcal{L}_0$, $y \in [\mathcal{B}] \setminus \mathcal{L}_0$, and $x I y$.

(G) If the partial plane $\mathcal{L}_0$ has the property N with respect to $\mathcal{B}$, then $[\mathcal{L}_0] = F(\mathcal{L}_0)$.

Proof: We have to show that $[\mathcal{L}_0]$ contains no $\mathcal{L}_0$ -bound elements. The proof is by induction on the $\mathcal{L}_0$ -stage j of elements of $[\mathcal{L}_0]$.

$j = 1$: Let $z \in [l]$ be of l -stage 1. Then z is incident in $[l]$ with at least two elements $y_1, y_2 \in l$. By property N $s_B(z) > s_B(y_i) > 0$, ($i = 1, 2$) and hence $z \notin B$. The element z is then defined in $F(B)$ by exactly two elements of lower l -stage, and using property N again, we conclude that these will be exactly the elements of lower l -stage in $[l]$ incident with z . Suppose now that, for some positive integer j , every element of $[l] \setminus l$ of l -stage $\leq j$ has been defined by precisely the two elements defining it in $F(B)$, but that an element z of l -stage $j + 1$ is l -bound; that is, we have $z I y_i$, $s_l(y_i) \leq j + 1$, ($i = 1, 2, 3$). Since z is incident in $[l]$ with at least one element x of l -stage j , which by the induction hypothesis is defined in $[l]$ by the elements defining it in $F(B)$, we deduce that $s_B(z) > s_B(x) > 0$, and so $z \notin B$. Hence one of the y_i 's, y_3 say, must be such that $s_B(y_3) > s_B(z) > 0$. If $s_l(y_3) \leq j$, then we deduce from this last set of inequalities and property N that $y_3 \notin l$; so that by the induction hypothesis we have $s_l(z) < s_l(y_3) \leq j$. This contradicts the assumption that $s_l(z) = j + 1$. On the other hand, if $s_l(y_3) = j + 1$, then y_3 must be incident with at least two elements w_1, w_2 say, of l -stage $\leq j$. Now either a $w_i \in l$ or by the induction hypothesis is already defined in $[l]$ by the two elements defining it in $F(B)$; hence it follows that $s_B(y_3) > s_B(w_i)$ ($i = 1, 2$). Since $z \neq w_1, w_2$, this implies that y_3 is a B -bound element, which is again a contradiction. Result (G) is thus proved.

Corollary: If $F(B)$ is the free completion of the partial plane B and l is complete in B , then $[l] = F(l)$.

Proof: It is readily verified that in this case $\mathcal{L}$ has the property $\mathcal{N}$ with respect to $\mathcal{B}$.

We now state two elementary results on polarities, which we shall use often in the sequel without explicit mention.

(H) In a given polarity, a point P lies on the polar of a second point Q if and only if Q lies on the polar of P , and dually.

(J) An absolute line contains exactly one absolute point, and dually.

Result (H) is proved directly from the definition of polarity and (J) follows from (H) and axiom A1.

Definition: We shall say that a configuration $\mathcal{L} \subseteq \mathcal{F}_n$ is of stage j if j is the maximum value of the stages of the elements of $\mathcal{L}$.

(K) Let $\mathcal{B}_2^{(e)}$ be defined as usual and $P_1 P_2 P_3 P_4$ be a generating quadrangle of the plane $\mathcal{F}_2$ of $\mathcal{B}_2^{(e)}$ - stage $j \geq 4$. If $D_1 = (P_1 P_2)(P_3 P_4)$, $D_2 = (P_1 P_4)(P_2 P_3)$, $D_3 = (P_1 P_3)(P_2 P_4)$, then exactly one of the P 's is of stage j , one of the D 's is of stage $j + 2$, and the other two D 's are of stage $< j$.

Proof: The proof of (K) is contained in the proof of theorem 6 of Sandler [9].

4. POLARITIES IN FREE PLANES

4.1 Existence of polarities in $\mathcal{F}_n$

We find it convenient to introduce the following definition:
By a figure $\mathcal{A}$ will be understood a configuration consisting of two points P, Q and two lines ℓ, m with $P, Q \in \ell$ and $P \in m$. We shall also designate a figure $\mathcal{A}$ by (P, Q, ℓ, m) , specifying its elements in that order.

Theorem 1: A free plane $\mathcal{F}_n$ of finite degree has a polarity.

Proof: A polarity is defined on a flag (A, a) if we make A and a pole and polar respectively of each other. Again, a polarity is defined on a figure $\mathcal{A}$ given by (A, P, a, p) if we make A and P the poles of a and p respectively.

Let $\mathcal{B}$ be a configuration consisting of x flags and y figure $\mathcal{A}$'s, where x and y are certain non-negative integers. A polarity ϑ is defined on $\mathcal{B}$ if, for each flag and each figure $\mathcal{A}$, we take the correspondences defined above. If x and y are sufficiently large, $\mathcal{B}$ will be non-degenerate. Also $\mathcal{B}$ is open, and hence, $\mathcal{B}$ will be a free generating configuration for some free plane $\mathcal{F}_n$ (result (B)).

The rank of a flag is 3, and the rank of a figure $\mathcal{A}$ is 5. Since rank is here clearly additive, $r(\mathcal{B}) = 3x + 5y$. We also recall that $r(\mathcal{F}_n) = n + 6$.

We now consider the equation

$$3x + 5y = n + 6 \quad \dots\dots\dots (1)$$

Since 3 and 5 are co-prime, it follows from a well-known result in number-theory that the equation (1) is soluble in non-negative integers

for $n \geq 9$. By direct verification, we also find that it is soluble in non-negative integers for $n \geq 2$. For $\mathcal{B}$ to be non-degenerate, it is sufficient that one of the following should hold:

- (i) $x \geq 0$ and $y \geq 0$;
- (ii) $x = 0$ and $y \geq 1$;
- (iii) $y = 0$ and $x \geq 2$;

It is again readily verified that the solutions we get for equation (1) all satisfy this condition for $n \geq 2$.

Hence, for any given finite n , we may, by choosing suitable values of x and y , obtain a free generating configuration $\mathcal{B}$ for a free plane $\mathcal{F}_n$. A polarity $\mathcal{O}$ has been defined on $\mathcal{B}$, and by result (F), this can be extended (uniquely) to a polarity in the entire plane $\mathcal{F}_n$.

Note: For the existence of polarities in free planes of infinite degree, we refer to Remark (c) of §6 of the note submitted with this thesis, Abbiw-Jackson [1].

4.2 Absolute elements in a polarity in $\mathcal{F}_n$

Let $\mathcal{R}$ be a property which partial planes may have. Then, following usual usage we shall say that a partial plane $\mathcal{P}$ is minimal $\mathcal{R}$ if it has the property $\mathcal{R}$ but no proper sub-partial plane of $\mathcal{P}$ has the property $\mathcal{R}$. In particular, a minimal free generating configuration $\mathcal{M}$ for $\mathcal{F}_n$ is a configuration $\mathcal{M}$ which

- (i) freely generates $\mathcal{F}_n$,
- (ii) $\mathcal{M}' \subseteq \mathcal{M}$ freely generates $\mathcal{F}_n$ implies that $\mathcal{M}' = \mathcal{M}$.

This is equivalent to saying that $\mathfrak{M}$ freely generates $\mathfrak{F}_n$, and $\mathfrak{M}$ cannot be freely contracted.

Definition: A partial plane $\mathcal{P}$ will be said to be invariant under an automorphism α (assumed to be defined in some partial plane containing $\mathcal{P}$) if $\mathcal{P}\alpha = \mathcal{P}$; that is, if α fixes $\mathcal{P}$ as a partial plane, not necessarily elementwise. If α is a polarity, we shall simply say that $\mathcal{P}$ is self-polar under α .

Though we do not need our next lemma until §6, we give it here as its proof substantially contains that of Lemma 2, which we require immediately.

Lemma 1: Let α be a collineation of $\mathfrak{F}_n$. Then if α has finite order k , there exists a minimal free generating configuration $\mathfrak{M}$ for $\mathfrak{F}_n$ which is invariant under α .

Proof: Let $\mathcal{B}^{(0)}_n$ be the free generating configuration for $\mathfrak{F}_n$ defined in §1.3. We shall first show that there exists in $\mathfrak{F}_n$ a configuration $\mathcal{L}_0$ which freely generates $\mathfrak{F}_n$ and is invariant under α . We consider the configuration $\mathcal{D} = \bigcup_{i=0}^{k-1} \mathcal{B}^{(0)}_n \alpha^i$. $\mathcal{D}$ generates $\mathfrak{F}_n$ (since $\mathcal{D} \supseteq \mathcal{B}^{(0)}_n$), and $\mathcal{D}\alpha = \mathcal{D}$; hence if $r(\mathcal{D}) = n + 6$, then by the corollary to result (C) we may put $\mathcal{L}_0 = \mathcal{D}$. Generally, however, $r(\mathcal{D})$ will be greater than $n + 6$. We then choose the least integer j such that the complete free extension $\mathcal{B}^{(j)}_n$ contains $\mathcal{D}$. We write $\mathcal{P}_0$ for $\mathcal{B}^{(j)}_n$. If $\mathcal{P}_0\alpha = \mathcal{P}_0$, we may put $\mathcal{L}_0 = \mathcal{P}_0$. Otherwise, there exists in $\mathcal{P}_0$ a non-empty (finite) set $\mathcal{Q}_1$ of elements defined by $\mathcal{Q}_1 = \{x \mid x \in \mathcal{P}_0, x\alpha \notin \mathcal{P}_0\}$. We shall call such elements exceptional elements in $\mathcal{P}_0$. We choose an exceptional element

x in $\mathcal{P}_0$ of least $\mathcal{B}_n^{(0)}$ -stage in $\mathcal{F}_n$. Clearly $s(x) \geq 1$. Let y, z be the two dual elements defining x in $\mathcal{F}_n$. Since $s(x)$ is minimal, it follows that the two elements $y\alpha, z\alpha \in \mathcal{P}_0$ (in fact, at least one of them must be of stage j). We freely extend $\mathcal{P}_0$ by adjoining the intersection or join of $y\alpha$ and $z\alpha$. We repeat the process on the elements of $\mathcal{Q}_1 \setminus \{x\}$. Since $\mathcal{Q}_1$ is finite, we obtain eventually a finite free extension $\mathcal{P}_1$ of $\mathcal{P}_0$ such that $\mathcal{P}_1 = \mathcal{P}_0 \cup \mathcal{P}_0\alpha$.

We next consider the (finite) set $\mathcal{Q}_2$ of exceptional elements in $\mathcal{P}_1$. If $\mathcal{Q}_2 \neq \emptyset$, we apply the above procedure to the elements of $\mathcal{Q}_2$. We thus obtain a finite free extension $\mathcal{P}_2$ of $\mathcal{P}_1$ (and hence a finite free extension of $\mathcal{P}_0$), where $\mathcal{P}_2 = \mathcal{P}_1 \cup \mathcal{P}_1\alpha = \mathcal{P}_0 \cup \mathcal{P}_0\alpha \cup \mathcal{P}_0\alpha^2$. If $\mathcal{Q}_2 = \emptyset$, we put $\mathcal{P}_2 = \mathcal{P}_1$. After $k-1$ applications of this procedure, we obtain a finite free extension $\mathcal{P}_{k-1}$ of $\mathcal{P}_0$ which contains no exceptional elements, since $\mathcal{P}_{k-1} = \bigcup_{i=0}^{k-1} \mathcal{P}_0\alpha^i$ and hence $\mathcal{P}_{k-1}\alpha = \mathcal{P}_{k-1}$ ($\alpha^k = 1$). Also each $\mathcal{P}_i$, being a free extension of $\mathcal{P}_0$ and hence of $\mathcal{B}_n^{(0)}$, freely generates $\mathcal{F}_n$. The partial plane $\mathcal{P}_{k-1}$ is the configuration $\mathcal{L}$ we require. (We observe trivially that $\mathcal{L}$ may be put equal to any $\mathcal{P}_i$, $0 \leq i < k-1$, if it happens that, for some such i $\mathcal{P}_i\alpha = \mathcal{P}_i$.)

Since $\mathcal{L}$ is finite, it contains a configuration $\mathcal{M}$ which is minimal with respect to the following two conditions (taken together):

- (i) freely generating $\mathcal{F}_n$,
- (ii) being invariant under α . (*)

We now show that $\mathcal{M}$ is also minimal with respect to condition (i) only. Suppose this is not the case. Then $\mathcal{M}$ contains a double

element x , say. Let x be incident in $\mathcal{M}$ with the two dual elements y, z . From (ii) it follows that $\mathcal{M}$ also contains the elements $x\alpha, x\alpha^2, \dots, x\alpha^{k-1}$, incident respectively with exactly the pairs of dual elements $(y\alpha, z\alpha), (y\alpha^2, z\alpha^2), \dots, (y\alpha^{k-1}, z\alpha^{k-1})$. Since the x 's are dual to the y 's and z 's, the removal of an x does not affect the multiplicity in $\mathcal{M}$ of the other x 's. Hence we may freely contract $\mathcal{M}$ by removing the set $\mathcal{X} = \{x\alpha^i \mid i = 0, 1, \dots, k-1\}$. The configuration $\mathcal{M}' = \mathcal{M} \setminus \mathcal{X}$ then satisfies both conditions (i) and (ii). Since $\mathcal{M}' \subset \mathcal{M}$ this contradicts the choice of $\mathcal{M}$ as minimal. The proof of the lemma is now complete.

Lemma 2: Let $\mathcal{O}$ be a polarity in $\mathcal{F}_n$. Then there exists a minimal self-polar (under $\mathcal{O}$) free generating configuration λ for $\mathcal{F}_n$.

Proof: The proof of lemma 1 up to the point marked with the asterisk (*) still holds if, instead of a collineation α (of finite order), we take a correlation α (also of finite order). In particular, this is true for a polarity $\mathcal{O}$, proving lemma 2.

Notation: If $\mathcal{O}$ is a polarity in a given free plane $\mathcal{F}_n$, we shall denote by $\lambda(\mathcal{O})$ a minimal self-polar (under $\mathcal{O}$) free generating configuration for $\mathcal{F}_n$.

Remark 1: The result obtained in the last paragraph of the proof of lemma 1 does not necessarily hold if α is a correlation. A counter-example is this: For $\mathcal{F}_2$, an $\lambda(\mathcal{O})$ is given by the configuration consisting of a flag $(A_1, A_1\mathcal{O})$ and a figure $\mathcal{A}$ defined by $(A_2, P, A_2\mathcal{O}, P\mathcal{O})$. This configuration is, however, not a minimal free generating configuration for $\mathcal{F}_2$.

We note, however, that in any $\lambda(\vartheta)$, a double element must necessarily be an absolute element; otherwise the final argument in the proof of lemma 1 will then essentially apply.

Remark 2: For a given ϑ , the minimal self-polar free generating configuration $\lambda(\vartheta)$ constructed in lemma 2 is not unique. We have given an example to show this in our note (Abbiw-Jackson [1], page 32). That example further shows that two such configurations need not even be isomorphic.

We now prove a lemma which will characterise certain isomorphic minimal self-polar free generating configurations.

Lemma 3: Let ϑ, ϑ' be polarities in a given free plane $\mathcal{F}_n$ and $\lambda(\vartheta), \lambda(\vartheta')$ be as defined above. Then there is an isomorphism φ of $\lambda(\vartheta)$ onto $\lambda(\vartheta')$ such that $x\varphi\vartheta' = x\vartheta\varphi$ for all $x \in \lambda(\vartheta)$ if and only if ϑ and ϑ' are conjugate in $\mathcal{F}_n$.

Proof: "Only if" - Since $F(\lambda(\vartheta)) = \mathcal{F}_n = F(\lambda(\vartheta'))$, by the corollary to result (E), φ extends uniquely to a collineation α of $\mathcal{F}_n$. Now, for all $x \in \lambda(\vartheta)$,

$$x\alpha\vartheta'\alpha^{-1} = x\varphi\vartheta'\varphi^{-1} = x\vartheta$$

Hence the polarities $\alpha\vartheta'\alpha^{-1}$ and ϑ coincide on $\lambda(\vartheta)$. By result (F), this implies that $\vartheta = \alpha\vartheta'\alpha^{-1}$, proving this part of the lemma.

"If" - By lemma 2, ϑ determines an $\lambda(\vartheta)$. For the corresponding $\lambda(\vartheta')$, we take $(\lambda(\vartheta))\alpha$, where α is a collineation of $\mathcal{F}_n$ such that $\vartheta' = \alpha^{-1}\vartheta\alpha$. For,

$$[(\lambda(\vartheta))\alpha]\vartheta' = [(\lambda(\vartheta))\alpha](\alpha^{-1}\vartheta\alpha) = (\lambda(\vartheta))\vartheta = \lambda(\vartheta).$$

Hence $(\lambda(\mathcal{B}))_{\alpha}$ is self-polar under $\mathcal{B}'$. That $(\lambda(\mathcal{B}))_{\alpha}$ is an $\lambda(\mathcal{B}')$ follows from this and the fact that $\lambda(\mathcal{B})$ is a minimal self-polar (under $\mathcal{B}$) free generating configuration. The required isomorphism ϕ from $\lambda(\mathcal{B})$ onto $(\lambda(\mathcal{B}))_{\alpha}$ is obtained simply by restricting α to $\lambda(\mathcal{B})$.

Theorem 2: Let $\mathcal{P}$ be a free extension of a partial plane $\mathcal{B}$ and $\mathcal{B}$ a polarity defined on $\mathcal{P}$ such that $\mathcal{B}\mathcal{B} = \mathcal{B}$. Then all the absolute elements in $\mathcal{P}$ are contained in $\mathcal{B}$.

Proof: We recall that we may assign to each element in $\mathcal{P}$ a unique non-negative integer, which is its $\mathcal{B}$ -stage. Every element of $\mathcal{P}$, not in $\mathcal{B}$, is then uniquely defined by two dual elements of lower stage.

We first prove that $s(x\mathcal{B}) = s(x)$ for all elements x in $\mathcal{P}$. We do this by induction on the stage j of an element in $\mathcal{P}$. Since $\mathcal{B}$ is self-polar, this is true for $j = 0$. Suppose we have shown that $s(x\mathcal{B}) = s(x)$ for all elements x of stage $\leq j$, for some non-negative integer j . We now consider an element x of stage $j + 1$. Let y, z be the two dual elements (of lower stage) defining x in $\mathcal{P}$. At least one of these two elements, y , say, must be of stage j . By the induction hypothesis $s(y\mathcal{B}) = s(y) = j$ and $s(z\mathcal{B}) = s(z) \leq j$. Now $y\mathcal{B}$ and $z\mathcal{B}$ are distinct elements, each incident with $x\mathcal{B}$. Moreover, they define $x\mathcal{B}$ in $\mathcal{P}$; otherwise, $s(x\mathcal{B}) \leq \max\{s(y\mathcal{B}), s(z\mathcal{B})\} = j$. Hence, from $x = (x\mathcal{B})\mathcal{B}$ and the induction hypothesis, $s(x) = s((x\mathcal{B})\mathcal{B}) = s(x\mathcal{B}) \leq j$. This contradicts the initial assumption that $s(x) = j + 1$. Since $s(z\mathcal{B}) \leq j$ and $s(y\mathcal{B}) = j$, it

follows that $s(x^\vartheta) = j + 1$, proving our assertion.

Now, if x is an absolute element, then $x \perp x^\vartheta$. Since $s(x^\vartheta) = s(x)$, this would make x a $\mathcal{B}$ -bound element if $x \in \mathcal{P} \setminus \mathcal{B}$. But this is impossible, since $\mathcal{P}$ is a free extension of $\mathcal{B}$. Any absolute elements in $\mathcal{P}$ must, therefore, be contained in $\mathcal{B}$.

Corollary: The number of absolute elements in a polarity ϑ in $\mathcal{F}_n$ is finite.

Proof: From lemma 2, there exists in $\mathcal{F}_n$ a self-polar (under ϑ) finite partial plane $\lambda(\vartheta)$, of which $\mathcal{F}_n$ is a free extension. The corollary then follows directly from theorem 2.

Theorem 3: If j is the number of absolute points (and hence of absolute lines) in a polarity in $\mathcal{F}_n$, then

- (i) $j \equiv n \pmod{2}$;
- (ii) $0 \leq j \leq n$;
- (iii) for $n = 2$, every polarity has $j = 2$ and further, all polarities are conjugate in $\mathcal{F}_2$;
- (iv) for $n > 2$, there is a polarity in $\mathcal{F}_n$ for each j satisfying conditions (i) and (ii).

Proof (i): From considerations of symmetry, we deduce that the rank of a self-polar configuration with no absolute elements must be even.

Suppose now that a polarity ϑ in $\mathcal{F}_n$ has absolute points A_i with polars a_i ($i = 1, \dots, j$). By lemma 2, ϑ determines a configuration $\lambda(\vartheta)$, which, by theorem 2, contains all the absolute elements. Suppose now that the point A_i is incident in $\lambda(\vartheta)$

with exactly $x_i + 1$ ($x_i \geq 0$) lines. (One of these lines will be a_i and the remaining x_i will be non-absolute.) The line a_i will then be incident with exactly $x_i + 1$ points (including A_i). If we now remove the pair (A_i, a_i) , we decrease $|\lambda(\vartheta)|$ by 2 and the number of incidences by $2x_i + 1$, thus increasing the rank by $2x_i - 3$. We note that the remaining configuration is still self-polar under ϑ . We now remove in turn $(A_2, a_2), (A_3, a_3) \dots (A_j, a_j)$. The final configuration, λ' say, is self-polar and has no absolute elements. Hence $r(\lambda')$ is even ($= 2y$, say) and since $r(\lambda(\vartheta)) = n + 6$, we arrive at the equation

$$n + 6 + 2 \sum_{i=1}^j x_i - 3j = 2y.$$

It is immediate from this equation that $j \equiv n \pmod{2}$.

(iii): To prove part (iii) we shall show that every $\lambda(\vartheta)$ for a polarity ϑ in $\mathcal{F}_2$ consists of a flag and a figure $\mathcal{A}$ (with the obvious correspondences for poles and polars).

Let ℓ be an $\lambda(\vartheta)$ for $\mathcal{F}_2$ (with respect to some polarity ϑ). Since ℓ is open, it contains an element x of multiplicity less than three in ℓ .

Suppose x is an isolated element in ℓ . Then its corresponding element x^ϑ is also isolated, and the configuration $\ell' = \ell \setminus \{x, x^\vartheta\}$ is self-polar and has rank 4. The only possibility is for ℓ' to consist of a pair of corresponding isolated elements y, y^ϑ . ℓ then consists of two isolated points and two isolated lines, which is a degenerate configuration. Hence ℓ cannot contain an isolated element.

We next assume that ℓ contains a simple element x . Here there

are two possibilities: x may be absolute or non-absolute. If x is absolute, then $\mathcal{L}$ contains the flag (x, x^θ) , and $\mathcal{L}' = \mathcal{L} \setminus \{x, x^\theta\}$ has rank 5. Again, one verifies that $\mathcal{L}'$ can only be a figure $\mathcal{A}$, (Y, Z, y, z) say. If x is a non-absolute element, then $\mathcal{L}' = \mathcal{L} \setminus \{x, x^\theta\}$ has rank 6. Direct verification once again shows that $\mathcal{L}'$ must consist of two flags $(y, y^\theta), (z, z^\theta)$ say. x is incident with an element of one of these flags and x^θ with the other element of the same flag. Hence in both cases, $\mathcal{L}$ consists of a flag and a figure $\mathcal{A}$, as asserted.

Finally, if $\mathcal{L}$ contains a double element x , then x must be absolute (see Remark 1 after lemma 2). The configuration $\mathcal{L}' = \mathcal{L} \setminus \{x, x^\theta\}$ now has rank 7. Again, $\mathcal{L}'$ can only consist of a flag (y, y^θ) say, and a corresponding pair of isolated elements z and z^θ . The element x will then be incident with one of z and z^θ , and x^θ incident with the other (by result (J), x cannot be incident with y or y^θ). Since also $x \perp x^\theta$, our assertion is again proved. This completes the proof of our statement that every $\lambda(\mathcal{C})$ consists of a flag and a figure $\mathcal{A}$.

If $\mathcal{C}, \mathcal{C}'$ are polarities in $\mathcal{F}_2$ with minimal self-polar free generating configurations ~~for~~ $\lambda(\mathcal{C}), \lambda(\mathcal{C}')$, we may define an isomorphism between $\lambda(\mathcal{C})$ and $\lambda(\mathcal{C}')$ by mapping the flags onto each other and the figure $\mathcal{A}$'s onto each other in a canonical way. The second part of (iii), now follows from lemma 3.

(ii): We have shown by (iii) that (ii) is true for $n = 2$ (the least positive integer in which we are interested). The proof

of (ii) now proceeds by induction on n . The method of proof is similar to that used in the proof of the first part of (iii). From a given $\lambda(\mathcal{C})$, we obtain a self-polar sub-partial plane λ' of lower rank by removing suitable elements from $\lambda(\mathcal{C})$. If λ' is non-degenerate, it can be shown that the result follows directly from the induction hypothesis. If λ' is degenerate, then we investigate what the various possible configurations are and show that the result holds in each case. The entire proof is given in our note, Abbiw-Jackson [1], pages 34 - 6.

(iv): To prove the final part of the theorem, it is sufficient to display a minimal self-polar free generating configuration for each relevant j . This also we have done in our note, Abbiw-Jackson [1], pages 36 - 7.

5. CONJUGATE POLARITIES IN A FREE PLANE

5.1

Throughout this section (as indeed was the case in the previous one), by conjugate polarities in a given free plane $\mathcal{F}_n$ will always be understood polarities which are conjugate in the collineation group $\mathcal{G}_n$ of $\mathcal{F}_n$.

In lemma 3, we characterised such polarities. An immediate consequence of that lemma is that conjugate polarities have the same number of absolute elements. A simple example shows that the converse of this is not true. In a given $\mathcal{F}_3$, we may define a polarity ϑ , with three absolute points, whose minimal self-polar free generating configuration $\lambda(\vartheta)$ consists of three flags $(x, x\vartheta)$, $(y, y\vartheta)$ and $(z, z\vartheta)$, without further incidences. A second polarity ϑ' , also with three absolute points, may have $\lambda(\vartheta')$ given by the configuration in fig. 1.

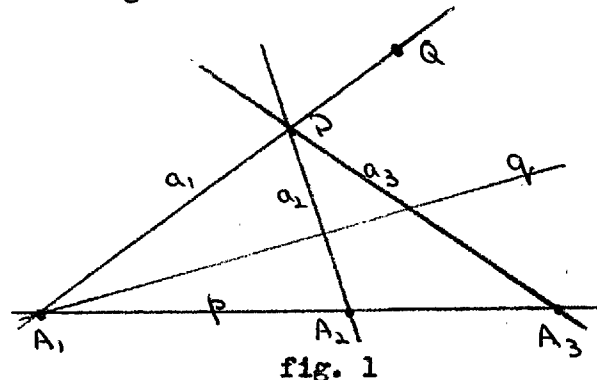


fig. 1

(Respective poles and polars are given by corresponding capital and lower case letters).

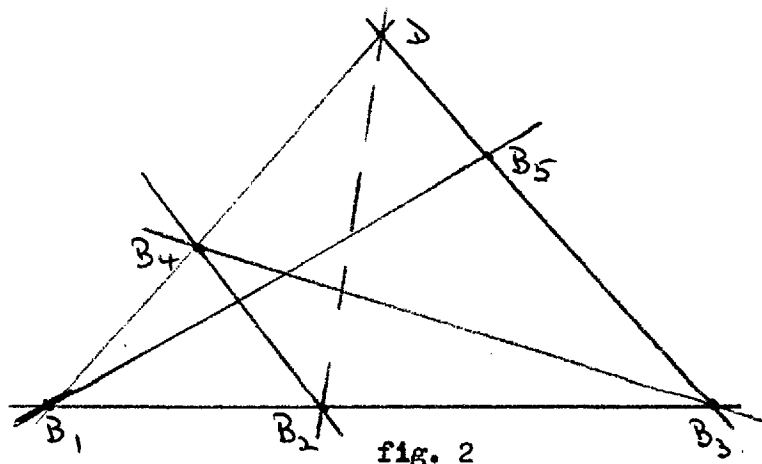


fig. 2

We show in fig. 2 the configurations $\lambda(\theta)$ and $\lambda(\theta')$ as sub-partial planes of $\mathcal{F}_3$, which is freely generated by the collinear points B_1, B_2, B_3 , the line B_1B_2 and the points B_4, B_5 . The polarity θ is defined by:

$$B_1 \rightarrow B_1B_4, \quad B_2 \rightarrow B_2B_4, \quad B_3 \rightarrow B_3B_5$$

and θ' by:

$$B_i \rightarrow B_iB_4 \quad (i = 1, 2, 3)$$

$$D = (B_1B_4)(B_3B_5) \rightarrow B_1B_5$$

Clearly, θ, θ' cannot be conjugate, since the absolute points of one are collinear but those of the other are not.

In this section, we shall give a sufficient condition for polarities θ, θ' to be conjugate (theorem 8). From lemma 3, this condition is also seen to be necessary. We shall, moreover, show that for θ, θ' to be conjugate, it is not sufficient to assume that their configurations of absolute elements λ, λ' are isomorphic and that the sub-planes $[\lambda], [\lambda']$ which they generate are also isomorphic.

5.2 Relatively confined configurations

A generalisation of the notion of a confined configuration is that of one configuration confined relative to another.⁽¹⁾

Definition: We shall say that a configuration $\mathcal{Q}$ is confined relative to a configuration $\mathcal{P}$, if every element of $\mathcal{Q}$, not in $\mathcal{P}$, has multiplicity at least three in $\mathcal{Q}$.

We see then that $\mathcal{Q}$ is confined in the sense defined on page 16 if $\mathcal{P}$ is the empty set $\emptyset$. Also, trivially, every configuration is confined relative to itself.

Lemma 4: (A) If $\mathcal{Q}$, $\mathcal{R}$ are confined relative to $\mathcal{P}$, then $\mathcal{Q} \cup \mathcal{R}$ is confined relative to $\mathcal{P}$.

(B) If $\mathcal{R}$ is confined relative to $\mathcal{Q}$ and $\mathcal{Q}$ confined relative to $\mathcal{P}$, and if $\mathcal{R} \supseteq \mathcal{Q}$, then $\mathcal{R}$ is confined relative to $\mathcal{P}$.

(C) If $F(\mathcal{B})$ is the free completion of $\mathcal{B}$ and $\mathcal{Q} \subseteq F(\mathcal{B})$ is confined relative to $\mathcal{P} \subseteq \mathcal{B}$, then $\mathcal{Q} \subseteq \mathcal{B}$.

Proof: Parts (A) and (B) are readily proved by set-theoretical arguments. We give only the proof of (B) to illustrate the method. We have to show that every element $x \in \mathcal{R} \setminus \mathcal{Q}$ has multiplicity at least three in $\mathcal{R}$. Now if $x \in \mathcal{R} \setminus \mathcal{P}$, then $x \in \mathcal{R} \setminus \mathcal{Q}$ or $x \in \mathcal{Q} \setminus \mathcal{P}$.

(1)

This generalisation was first made by Kopeikina [7].

In the first case, x has multiplicity at least three in $\mathcal{R}$, since $\mathcal{R}$ is confined relative to $\mathcal{Q}$. In the second case, x has multiplicity at least three in $\mathcal{Q}$ ($\mathcal{Q}$ is confined relative to $\mathcal{P}$), and since $\mathcal{R} \supseteq \mathcal{Q}$, x has the required multiplicity in $\mathcal{R}$ as well. Thus (B) is proved.

The proof of (C) is essentially the same as that of result (A).

Lemma 5: Let $\mathcal{A}, \mathcal{B}$ be configurations such that $\mathcal{A} \subseteq \mathcal{B}$.

Let $\mathcal{S}$ be the set of all configurations in $F(\mathcal{B})$ confined relative to $\mathcal{A}$. Then $\mathcal{S}$ contains a unique maximal element $\mathcal{K}$.

Proof: The set $\mathcal{S}$ is not empty, since $\mathcal{A} \in \mathcal{S}$. By lemma 4(C) if $\mathcal{P} \in \mathcal{S}$, then $\mathcal{P} \subseteq \mathcal{B}$. Since $\mathcal{B}$ is finite, $\mathcal{S}$ contains a maximal element $\mathcal{K}$, say. By lemma 4A, $\mathcal{K}$ must be unique.

Definition: Given $\mathcal{A}, \mathcal{B}$ and $\mathcal{K}$ as in lemma 5, we shall call $\mathcal{K}$ the closure of $\mathcal{A}$ in $F(\mathcal{B})$.

We note that since $\mathcal{A}$ is confined relative to itself, it follows that $\mathcal{K} \supseteq \mathcal{A}$.

Lemma 6: If $\mathcal{K}$ is the closure of $\mathcal{A}$ in $F(\mathcal{B})$ and the configuration $\mathcal{K}'$ is such that $\mathcal{K} \subset \mathcal{K}' \subseteq F(\mathcal{B})$, then $\mathcal{K}' \setminus \mathcal{K}$ contains an element of multiplicity less than three in $\mathcal{K}'$.

Proof: Suppose the lemma is false. Then $\mathcal{K}'$ is confined relative to $\mathcal{K}$, which implies by lemma 4B, that $\mathcal{K}'$ is confined relative to $\mathcal{A}$. This contradicts the definition of $\mathcal{K}$ as the maximal configuration confined relative to $\mathcal{A}$.

Theorem 6: In a given $\mathcal{I}_n$, let $\mathcal{A}$ be a non-empty configuration consisting of j flags, where $j \equiv n \pmod{2}$ and let $\mathcal{K}$ be the closure

of $\mathcal{A}$ in $\mathcal{F}_n$. If a polarity ϑ can be defined on $\mathcal{K}$ such that $\mathcal{A}$ is the system of absolute elements of ϑ , then ϑ can be extended to a polarity Θ in $\mathcal{F}_n$. The extension of ϑ to Θ is not, in general, unique.

Proof: We choose a minimal free generating configuration for $\mathcal{F}_n$ such that $\mathcal{M} \supset \mathcal{A}$. Then, by lemma 4C, $\mathcal{M} \supset \mathcal{K}$. We shall show that we can always reduce $\mathcal{M}$ to a "standard" form. The extensions and contractions to be carried out below will all be free.

(i) $\mathcal{M} = \mathcal{K}$: By result (F), ϑ can be extended (uniquely) to a polarity Θ in $\mathcal{F}_n$. Since any $\lambda(\Theta)$ contains $\mathcal{A}$ and hence $\mathcal{K}$ and $\mathcal{K}$ is itself an $\lambda(\Theta)$ in this case, it follows that here $\mathcal{K}$ is the unique $\lambda(\Theta)$.

(ii) $\mathcal{M} \supset \mathcal{K}$: We choose a flag (A_1, a_1) in $\mathcal{A}$. If $j > 1$, we choose a second flag (A_2, a_2) . By lemma 6, and the minimality of $\mathcal{M}$, there is an isolated or simple element in $\mathcal{M} \setminus \mathcal{K}$. Let P be a simple point in $\mathcal{M} \setminus \mathcal{K}$, not incident with a line through A_1 . We adjoin PA_1 , and then remove P , thus transforming P into a simple line through A_1 . Similarly, a simple point on a line not incident with A_2 may be transformed into a line through A_2 , and dually a simple line not through a point on a_1 (a_2) may be transformed into a simple point on a_1 (a_2). If P is a simple point on A_1A_2 , we may transform it into a simple line through a_1a_2 by adding the join of P and a_1a_2 , and removing P ; and conversely we may transform a simple line through a_1a_2 into a simple point on A_1A_2 . Since $\mathcal{M}$ is non-degenerate, there

is an element x in $\mathcal{M}$ such that $x \neq A, A_1, a, a_1$ and x is not incident with either A, A_1 or a, a_1 . We may transfer a simple element on A, A_1 or a, a_1 into a simple element on x . By suitable combinations of these transformations, we can transfer any simple element in $\mathcal{M} \setminus \mathcal{K}$ into a simple element on A , or a . Moreover, providing only that the resulting configuration will not be degenerate, we are free to choose whether to transform any given simple element into a line on A , or a point on a .

An isolated point P may be replaced with two simple lines by adjoining PA , and PA_1 , and removing P . An isolated line may be similarly replaced. Combining with the previous transformations, we may replace an isolated element in $\mathcal{M} \setminus \mathcal{K}$ with simple elements on A , or a .

After thus transforming an isolated or simple element, we remove any double elements, not in $\mathcal{K}$, which may occur, thus obtaining a minimal free generating configuration containing $\mathcal{K}$.

We now assert that the above procedures can be repeated until all the elements of $\mathcal{M}$, not in $\mathcal{K}$, are simple elements on A , or a . For otherwise, let $\mathcal{M}'$ be a minimal free generating configuration properly containing $\mathcal{K}$, not of the form described, and to which we cannot apply the above transformations. Let $\mathcal{M}'' = \mathcal{M}' \setminus \mathcal{X}$, where $\mathcal{X}$ is the set of all simple elements on A , or a , (except possibly A , and a , themselves). Since $\mathcal{K} \cap \mathcal{X} = \emptyset$, it follows that $\mathcal{M}'' \supset \mathcal{K}$. $\mathcal{M}'' \setminus \mathcal{K}$ must then contain only elements of multiplicity greater than two, whence by lemma 4B, $\mathcal{M}''$ is confined relative to $\mathcal{A}$. This

contradicts the definition of $\mathcal{K}$. When $\mathcal{M}$ is such that all the elements of $\mathcal{M} \setminus \mathcal{K}$ are simple elements on A_1 or a_1 , we shall say that $\mathcal{M}$ is of standard form.

Now, by an argument similar to that used in the proof of theorem 3(i), we have $r(\mathcal{K}) \equiv j \pmod{2}$, and hence $r(\mathcal{M}) \equiv r(\mathcal{K}) \pmod{2}$. Also, here $r(\mathcal{M} \setminus \mathcal{K}) = r(\mathcal{M}) - r(\mathcal{K}) > 0$. Hence $r(\mathcal{M} \setminus \mathcal{K})$ is some positive even number, which is also the number of simple elements in $\mathcal{M} \setminus \mathcal{K}$, when $\mathcal{M}$ is reduced to a standard form. We take exactly half of such elements as simple lines through A_1 and the other half as simple points on a_1 . The resulting standard form is clearly not degenerate.

The polarity ϑ in $\mathcal{K}$ can now be extended to a polarity ϑ' in $\mathcal{M}$ by assigning to each point in $\mathcal{M} \setminus \mathcal{K}$ exactly one line in $\mathcal{M} \setminus \mathcal{K}$ as its polar. By result (F), ϑ' can then be extended (uniquely) to a polarity ϑ'' in $\mathcal{F}_n$. It must also be obvious from the foregoing that ϑ' and hence ϑ'' is, in general, not unique.

If $j = 1$, but $\mathcal{K} \supset \mathcal{A}$, then we can choose a point and a line in $\mathcal{K} \setminus \mathcal{A}$ to play the roles of A_1 and a_1 . (Note that it is not necessary to our argument that A_1 and a_1 be an incident pair.) The above reduction to a standard form then applies in exactly the same way.

If $j = 1$ and $\mathcal{K} = \mathcal{A}$, we may, by essentially similar methods, reduce a minimal free generating configuration $\mathcal{M} \supset \mathcal{A}$ into a standard form given by a flag (A_1, a_1) , an isolated point P and an isolated line p , and equal numbers of simple elements on A_1 and a_1 . The

extension of ϑ to a polarity ϑ' in $\mathfrak{M}$ is made as above and by result (F), ϑ' can be extended to a polarity ϑ'' in $\mathfrak{F}_n$.

This completes the proof of the theorem.

Remark: If $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{K}$ are as defined in lemma 5, then from the reduction of a minimal free generating configurations to a standard form, we can deduce that $\mathcal{K} = \bigcap \mathcal{L}$.

$$\{\mathcal{L} \mid \mathcal{L} \supseteq \mathcal{A}; F(\mathcal{L}) = F(\mathcal{B})\}.$$

Lemma 7: If $\mathcal{A}$ is the system of absolute elements of a polarity ϑ in $\mathfrak{F}_n$, and $\mathcal{K}$ the closure of $\mathcal{A}$ in $\mathfrak{F}_n$, then $\mathcal{K}\vartheta = \mathcal{K}$.

Proof: Let $\lambda(\vartheta)$ have the usual meaning. If $\mathcal{A} = \emptyset$, then $\mathcal{K} = \emptyset$ and there is nothing to prove. Suppose now that $\mathcal{A} \neq \emptyset$. Then we may reduce $\lambda(\vartheta)$ to a standard form with the elements $\mathcal{K}$ not in $\mathcal{K}$ as simple points on a line Q , and their polars through a point A , where $a\vartheta = A$ and $A\perp a$. If we remove the elements $\mathcal{K}$, we are still left with a self-polar configuration, which is $\mathcal{K}$.

Theorem 7: Let n be even and ϑ a polarity in $\mathfrak{F}_n$ without absolute elements. Then any $\lambda(\vartheta)$ may be reduced to one of the following two configurations:

- (i) for $n \equiv 0 \pmod{4}$, $\lambda(\vartheta)$ consists of a pair of flags $(P, Q\vartheta)$, $(Q, P\vartheta)$, and a certain number of isolated points and their polars;
- (ii) for $n \equiv 2 \pmod{4}$, $\lambda(\vartheta)$ consists of isolated points and their polars.

Proof: The proof is by induction on n , using methods similar to those of theorem 3(ii). The relevant least positive integers are $n = 4$ for (i) and $n = 6$ for (ii). For $n = 4$, the methods referred to

Theorem 8:

(1) The last sentence of the statement of the theorem should read:

Then θ, θ' are conjugate if they have the same (finite) number of absolute elements and if there exists a collineation α of $\mathcal{F}_n$ such that $x\alpha\theta' = x\theta\alpha$ for all $x \in \mathcal{K}$.

(2) The first sentence of the proof should be replaced by:

Let x be an absolute element of θ . Then $x \perp x\theta$ and hence $x\alpha \perp (x\theta)\alpha$. Since $\mathcal{A} \subseteq \mathcal{K}$, $x \in \mathcal{K}$ and it follows that $x\theta\alpha = x\alpha\theta'$. Hence $x\alpha \perp (x\alpha)\theta'$; that is, $x\alpha$ is an absolute element of θ' . Thus $\mathcal{A}\alpha \subseteq \mathcal{A}'$; that $\mathcal{A}\alpha = \mathcal{A}'$ now follows from the fact that $\mathcal{A}$ and $\mathcal{A}'$ consist of the same finite number of flags.

lead directly to the configuration stated in the theorem. For $n = 6$, we either arrive at the stated configuration or one consisting of four flags (P_1, p_1) , (P_2, p_1) , (P_3, p_4) and (P_4, p_3) , where $P_i \mathcal{Q} = p_i$ ($i = 1, 2, 3, 4$). We now freely extend this configuration by adjoining $P_5 = P_1 P_3$ and $P_6 = p_2 p_3$, and freely contract by removing P_2, P_3, p_2, p_3 . The remaining configuration is then in the required form. The proof of the theorem is now completed by induction.

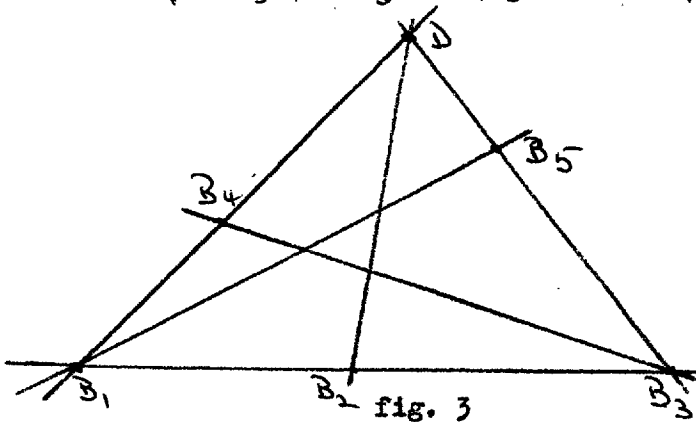
Theorem 8: Let $\mathcal{A}, \mathcal{A}'$ be the configurations of absolute elements of the polarities $\mathcal{Q}, \mathcal{Q}'$ in a free plane $\mathcal{F}_n$, and $\mathcal{K}, \mathcal{K}'$ the closures of $\mathcal{A}, \mathcal{A}'$ in $\mathcal{F}_n$. Then $\mathcal{Q}$ and $\mathcal{Q}'$ are conjugate if there exists a collineation α of $\mathcal{F}_n$ such that $x \alpha \mathcal{Q}' = x \mathcal{Q} \alpha$ for all $x \in \mathcal{K}$.

Proof: Since $\mathcal{A} \subseteq \mathcal{K}$, it follows from the condition $x \alpha \mathcal{Q}' = x \mathcal{Q} \alpha$, all $x \in \mathcal{K}$ that α maps $\mathcal{A}$ onto $\mathcal{A}'$. Also automorphisms of $\mathcal{F}_n$ clearly preserve the closures of configurations, and hence $\mathcal{K}' = \mathcal{K} \alpha$. Now $\mathcal{A}$ and $\mathcal{A}'$ are either both empty or both non-empty. If $\mathcal{A} = \mathcal{A}' = \mathcal{K} = \mathcal{K}' = \emptyset$, the condition of the theorem is vacuously satisfied; the theorem in this case follows from theorem 7 and lemma 3.

If $\mathcal{A}$ is non-empty, then by lemma 7, $\mathcal{K}$ is self-polar under $\mathcal{Q}$; and since $\mathcal{A}'$ is then non-empty, $\mathcal{K}'$ is also self-polar under $\mathcal{Q}'$. We can now take $\lambda(\mathcal{Q})$ and $\lambda(\mathcal{Q}')$ in the standard forms defined in theorem 6. That $\mathcal{Q}$ and $\mathcal{Q}'$ are conjugate follows from lemma 3, since we may now extend the condition on $\mathcal{K}$ to $\lambda(\mathcal{Q})$.

Remark: We now show that for ϑ, ϑ' to be conjugate, it is not sufficient to assume that $\mathcal{A}, \mathcal{A}'$ are isomorphic and that they generate isomorphic sub-planes $[\mathcal{A}], [\mathcal{A}']$. We define a polarity ϑ with $\mathcal{A} = \mathcal{K} = \lambda(\vartheta)$ consisting of three flags as in § 5.1. A second polarity ϑ' is defined as follows (the elements are as in fig. 2):

$$\begin{aligned} B_1 &\rightarrow B_3 B_5, & B_2 &\rightarrow B_2 D, & B_3 &\rightarrow B_1 B_4 \\ B_4 &\rightarrow B_3 B_4, & B_5 &\rightarrow B_1 B_5, & D &\rightarrow B_1 B_3. \end{aligned}$$



Now $\mathcal{A}'$ consists of three flags $(B_1, B_1\vartheta')$ $(B_4, B_4\vartheta')$, $(B_5, B_5\vartheta')$, and hence $\mathcal{A}, \mathcal{A}'$ are isomorphic (qua configurations). Also $\mathcal{K}' = \lambda(\vartheta')$ is the configuration shown in fig 3. Since $\mathcal{A}'$ is complete in $\lambda(\vartheta')$ it follows from result (G) that $[\mathcal{A}'] = F(\mathcal{A}')$, and hence $[\mathcal{A}']$ is an $\mathbb{F}_3$. Since $\mathcal{A}$ generates the entire plane $\mathbb{F}_3$, it follows that $[\mathcal{A}]$ and $[\mathcal{A}']$ are isomorphic. Now $\lambda(\vartheta)$ and $\lambda(\vartheta')$ are both unique, and since they are obviously non-isomorphic, it follows from lemma 3 that ϑ and ϑ' are not conjugate.

6. SOME PROPERTIES OF THE COLLINEATION GROUP OF THE FREE PLANE GENERATED BY A QUADRANGLE

6.1: We have mentioned already that Sandler [9] has determined a set of three generators for Γ_2 and also a finite set of defining relations. In this sub-section, we shall determine the conjugacy classes of elements of finite order in Γ_2 , and show that if $\Delta \neq 1$ is a proper normal sub-group of Γ_2 , then Δ is torsion-free. We derive from this that Γ_2 coincides with its commutator-subgroup, and hence is insoluble.

Our methods are essentially geometrical, making heavy use of the minimal free generating configurations for $\mathcal{F}_n$. The lemmas leading to our main theorems are not restricted to $\mathcal{F}_2$ and may be of some interest in themselves.

We proved in lemma 1 that if α is a collineation of finite order in $\mathcal{F}_n$, then there is a minimal free generating configuration $\mathcal{M}$ for $\mathcal{F}_n$ such that $\mathcal{M}\alpha = \mathcal{M}$. We shall denote such a configuration by $\mathcal{M}(\alpha)$. Two such $\mathcal{M}(\alpha)$'s will be said to be α -free equivalent.

Lemma 8: Let α, α' be collineations of finite order in a given free plane $\mathcal{F}_n$, and $\mathcal{M}(\alpha), \mathcal{M}(\alpha')$ as defined above. Then, there is an isomorphism φ of $\mathcal{M}(\alpha)$ onto $\mathcal{M}(\alpha')$ such that $x\varphi\alpha' = x\alpha\varphi$ for all $x \in \mathcal{M}(\alpha)$ if and only if α and α' are conjugate in Γ_n .

Proof: The proof is essentially the same as that of lemma 3, which similarly characterises polarities conjugate in Γ_n .

To be inserted at the end of the proof of lemma 9:

That $\mathcal{M}'$ is non-degenerate in both cases is readily shown, using Baer's characterisation of degenerate planes given on page 9.

Configurations $\mathcal{M}(\alpha)$, $\mathcal{M}(\alpha')$ as in lemma 8 will be said to be $\mathcal{F}_n$ -collineation isomorphic.

Lemma 9: Let $\mathcal{M}$ be a non-degenerate open configuration which

- (i) cannot be freely contracted;
- (ii) contains an element with multiplicity greater than 2.

Then $r(\mathcal{M}) \geq 9$.

Proof: Suppose the element x in $\mathcal{M}$ is incident with the dual elements $y_1, y_2, \dots, y_t$ ($t \geq 3$). Then if one of the y_i 's is a simple element, we remove it. The remaining configuration $\mathcal{M}'$ is open and non-degenerate; hence by corollary 1 of result (B), $r(\mathcal{M}') \geq 8$, and $r(\mathcal{M}) = r(\mathcal{M}') + 1 \geq 9$.

If no y_i is a simple element, they must all have multiplicity at least three. Since $\mathcal{M}$ is open, it must contain an element z of multiplicity 0 or 1 (there can be no double elements in view of (i)). We remove some such element z . The configuration $\mathcal{M}' = \mathcal{M} \setminus \{z\}$ is again open and non-degenerate, and the lemma follows as before.

Corollary: For $\mathcal{F}_2$, the minimal free generating configurations are exhausted by the following types:

$\mathcal{J}$ (i): a quadrangle; or its dual, a quadrilateral.⁽¹⁾

$\mathcal{J}$ (ii): three isolated points and an isolated line; the dual of this.

(1)

The quadrilateral determined by the lines l_i ($i = 1, 2, 3, 4$) will be denoted by $l_1 l_2 l_3 l_4$

J (iii): two flags and an isolated point (line).

Proof: A minimal free generating configuration for $\mathcal{F}_2$ must, in view of lemma 9, contain only isolated or simple elements. It can then be directly verified that the above exhaust the possible configurations.

Lemma 10: Let $F(\mathcal{B})$ be the free completion of the partial plane $\mathcal{B}$ and α a $\mathcal{B}$ -stationary collineation. Let $\mathcal{L}$ be the sub-partial plane of $\mathcal{B}$ which is fixed elementwise by α . Then, if α is of odd order, the fixed subplane of α is precisely $[\mathcal{L}] = F(\mathcal{L})$. ($\mathcal{L}$ and hence $F(\mathcal{L})$ may be degenerate.)

Proof: We denote the fixed subplane of α by $\mathcal{F}(\alpha)$. Then since the join or intersection of two fixed elements is also a fixed element, it follows that $[\mathcal{L}] \subseteq \mathcal{F}(\alpha)$.

Suppose now that the lemma is false and let $x \in \mathcal{F}(\alpha)$ be an element of least $\mathcal{B}$ -stage in $F(\mathcal{B})$ such that $x \notin [\mathcal{L}]$. Then, obviously $s(x) > 0$. Let y_1, y_2 be the two dual elements (of lower stage) defining x in $F(\mathcal{B})$. Since α clearly preserves the stages of elements in $F(\mathcal{B})$ and also incidences, we must have either (i) $y_i \alpha = y_i$ ($i = 1, 2$), or (ii) $y_1 \alpha = y_2$ and $y_2 \alpha = y_1$. In the first case, both y_1 and y_2 are fixed elements and, from the minimality of $s(x)$, must belong to $[\mathcal{L}]$. But this implies that $x \in [\mathcal{L}]$, giving a contradiction. In the other case, y_1 and y_2 form a transitive class (under α) of order 2, and hence the order of α must be even or infinite. This contradicts our assumption that α is of odd order. Hence $x \in \mathcal{F}(\alpha)$ implies $x \in [\mathcal{L}]$ and so $[\mathcal{L}] = \mathcal{F}(\alpha)$.

From our earlier observation about the join or intersection of two fixed elements being itself a fixed element, it follows that $\mathcal{L}$ is complete in $\mathcal{G}$, whence by the corollary to result (G), $[\mathcal{L}] = F(\mathcal{L})$.

Theorem 9: The elements of Γ_2 of finite order have orders 2, 3, or 4. The elements of order 2 form one conjugacy class, and so do the elements of order 4. The elements of order 3 split into two conjugacy classes.

Proof: From lemma 1 and the corollary to lemma 9, we deduce immediately that the only possible orders for elements of Γ_2 of finite order are 2, 3 or 4. That collineations with these orders actually occur in Γ_2 follows from the fact that the stationary collineations (with respect to a generating quadrangle in $\mathcal{F}_2$) form a subgroup of Γ_2 isomorphic with Σ^4 , the symmetric group on four letters. Again, using the corollary to lemma 9, we can deduce the following results:-

(a) α a collineation of order 4: Then $\mathcal{M}(\alpha)$ can only be of type $\mathcal{I}(1)$; that is, a quadrangle or a quadrilateral. If a quadrangle $B_1 B_2 B_3 B_4$ is an $\mathcal{M}(\alpha)$ such that $B_i \alpha = B_{i+1}$ ($i = 1, 2, 3, 4$; $B_5 = B_1$), then we may obtain a second $\mathcal{M}(\alpha)$ by freely adjoining the four lines $B_i B_{i+1}$ and freely removing the four points B_i ($i = 1, 2, 3, 4$). The second configuration is a quadrilateral and also an $\mathcal{M}(\alpha)$. From lemma 8, we deduce that all elements of order 4 in Γ_2 form one conjugacy class in Γ_2 .

(b) α a collineation of order 2: Here $\mathcal{M}(\alpha)$ can assume any one of eight configurations no two of which are $\mathcal{F}_n$ -collineation

isomorphic. There are four of $\mathcal{J}(1)$ and two each of $\mathcal{J}(ii)$ and $\mathcal{J}(iii)$.

We now show that all the eight configurations are α -free equivalent:

Let $B_1 B_2 B_3 B_4$ be a generating quadrangle for $\mathcal{F}_2$ such that

$B_1 \alpha = B_2, B_2 \alpha = B_3, B_3 \alpha = B_4, B_4 \alpha = B_1$. Also let

$D_1 = (B_1 B_3)(B_2 B_4)$ and $D_2 = (B_1 B_4)(B_2 B_3)$. Then $D_i \alpha = D_i (i = 1, 2)$.

The quadrangles (1): $B_1 B_2 B_3 B_4$ and (2): $D_1 D_2 B_1 B_2$, and the

quadrilaterals (3): $(B_1 B_3)(B_2 B_4)(B_1 B_4)(B_2 B_3)$, and (4):

$(B_1 B_2)(B_3 B_4)(B_1 B_3)(B_2 B_4)$ are four $\mathcal{M}(\alpha)$'s of type $\mathcal{J}(1)$; the

configurations (5): $D_1, B_3, B_4, B_1 B_2$ and (6): $D_1, B_1 B_2, B_1 B_4, B_2 B_3$,

are two $\mathcal{M}(\alpha)$'s of type $\mathcal{J}(ii)$; and, finally, the configurations

(7): D_1 together with the two flags $(B_1, B_1 B_4)$ and $(B_2, B_2 B_3)$, and

(8): $B_1 B_2$ together with the two flags $(B_3, B_2 B_3)$ and $(B_4, B_1 B_4)$ are

two $\mathcal{M}(\alpha)$'s of type $\mathcal{J}(iii)$. Again, by lemma 8, this shows that all

collineations of order 2 form one conjugacy class in Γ .

(c) Collineations of order 3 fall into two conjugacy classes, characterised by two invariant minimal free generating configurations of type $\mathcal{J}(i)$:

(i) A quadrangle $B_1 B_2 B_3 B_4$ such that $B_1 \alpha = B_2, B_2 \alpha = B_3,$

$B_3 \alpha = B_4, B_4 \alpha = B_1$,

(ii) A quadrilateral $l_1 l_2 l_3 l_4$ such that $l_1 \alpha = l_2$

$l_2 \alpha = l_3, l_3 \alpha = l_4, l_4 \alpha = l_1$.

(These are easily shown to be α -free equivalent to two configurations

of type $\mathcal{J}(ii)$). That the two configurations are not α -free

equivalent follows from lemma 10, since the fixed subplane of one

consists of an isolated point, and that of the other is an isolated line.

(We observe, however, that collineations of order 3 form one conjugacy class in the automorphism group $\overline{\Gamma}_2$ of $\mathcal{F}_2$.)

Theorem 10: If Δ is a normal sub-group of Γ_2 such that $1 \subset \Delta \subset \Gamma_2$, then Δ is torsion-free.

Proof: Sandler [9] has shown that Γ_2 is generated by Σ^4 , the symmetric group of degree 4, and a further collineation τ of order 2. Since Σ^4 is itself generated by its transpositions (of order 2), it follows that Γ_2 is generated by the conjugacy class of elements of order 2. Hence if a normal subgroup Δ contains an element of order 2, then $\Delta = \Gamma_2$. Trivially this is also true if Δ contains an element α of order 4, since α^2 is then of order 2.

If Δ contains an element α of order 3, let $\mathcal{M}(\alpha)$ be given by the quadrangle $B_1 B_2 B_3 B_4$ with $B_1 \alpha = B_2$, $B_2 \alpha = B_3$, $B_3 \alpha = B_4$, $B_4 \alpha = B_1$. We define a second collineation β on $\mathcal{F}_2$ as follows: $B_1 \rightarrow B_2$, $B_2 \rightarrow B_4$, $B_4 \rightarrow B_1$, $B_3 \rightarrow B_3$. By lemma 8, β is conjugate to α in Γ_2 . Also $\alpha\beta$ is given by

$$B_1 \rightarrow B_4, \quad B_4 \rightarrow B_1, \quad B_2 \rightarrow B_3, \quad B_3 \rightarrow B_2,$$

from which it follows that $\alpha\beta$ is of order 2. We can similarly construct an element $\alpha\beta$ of order 2, if $\mathcal{M}(\alpha)$ is of the other type. Hence if Δ contains an element of order 3, it also contains an element of order 2 and thus $\Delta = \Gamma_2$. This completes the proof.

Corollary: Γ_2 coincides with its commutator-subgroup Γ_2' .

Proof: Since Γ_2' is normal in Γ_2 , it is sufficient, by theorem 10, to show that Γ_2' contains an element of finite order, distinct

from the identity. Since $\Sigma^4 \subset \Sigma$, it follows that $(\Sigma^4)' \subseteq \Sigma'$.

But $(\Sigma^4)' = A^4$, the alternating group of degree 4. The corollary is thus proved, since A^4 is finite and non-trivial.

A remark on lemma 9: It would have been desirable to prove a much stronger result than lemma 9; that is, one which gives the lower bound for the rank of a configuration (M) satisfying condition (i) of that lemma as a non-trivial function of the maximum value of the multiplicities of elements of (M) . That we cannot, in fact, improve on the lemma in this direction is shown by the following example:

Dembowski [4], pages 431-2, has constructed a sequence $\mathbb{Z}$ of collinear points $z_1, z_2, \dots$, each z_i being the vertex of an almost-confined configuration $\mathcal{P}_i$, say.⁽²⁾ It is readily shown that $r(\mathcal{P}_i) = 8$, all i . For each i , we define an extension $\mathcal{Q}_i$ of $\mathcal{P}_i$ by adding a simple line y_i incident with z_i . Now $r(\mathcal{Q}_i) = 9$, and all the $\mathcal{Q}_i$ satisfy condition (i) of lemma 9. But by choosing i sufficiently large, we get a $\mathcal{Q}_i$ with a line zz_i of arbitrarily large multiplicity.

(2)

For the definitions of "vertex" and "almost-confined configuration" ("Scheittelelement" and "fastgeschlossene Konfiguration" in [4]), see page 427 of [4].

6.2

In § 4, we showed that if ϑ is a polarity in $\mathbb{T}_2$, a minimal self-polar (under ϑ) free generating configuration $\lambda(\vartheta)$ always consists of a flag and a figure A . We may, without loss of generality, represent $\lambda(\vartheta)$ as in fig. 4, where $B_1 B_2 B_3 B_4$ is a generating quadrangle $\mathcal{B}_2^{(c)}$ for $\mathbb{T}_2$, $D = (B_1 B_3)(B_2 B_4)$, and ϑ is defined by:

$$B_1 \rightarrow B_1 B_3, \quad B_2 \rightarrow B_2 B_3, \quad D \rightarrow B_1 B_4.$$

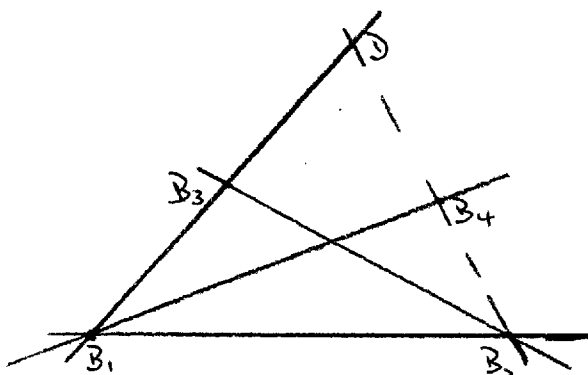


fig. 4

Lemma 11: If α belongs to the orthogonal group $\mathcal{T}(\vartheta)$ of a polarity ϑ , and A is the configuration of absolute elements of ϑ , then $A\alpha = A$.

Proof: Let x be an absolute element, and write

$$x\alpha = y, \quad (x\vartheta)\alpha = z. \quad \text{Then}$$

$$y\vartheta = x\alpha\vartheta = x\vartheta\alpha = z,$$

and also $x \perp x\vartheta$ implies $x\alpha \perp (x\vartheta)\alpha$ or $y \perp z$. Hence y is an absolute element, and $A\alpha \subseteq A$. Since α is a collineation, we must have $A\alpha = A$.

Corollary: If $\mathcal{Q}$ is a polarity in $\mathcal{T}_2$, with $\mathcal{L}(\mathcal{Q})$ as given in fig. 4, and if $\alpha \in \mathcal{T}(\mathcal{Q})$, then either

$$(i) \quad B_i \alpha = B_i \quad (i = 1, 2, 3)$$

$$\text{or (ii) } B_1 \alpha = B_2, \quad B_2 \alpha = B_1, \quad B_3 \alpha = B_3.$$

Proof: Since $\mathcal{Q}$ has only two absolute points B_1, B_2 , it follows from the lemma that α either fixes each of B_1, B_2 or it interchanges them. Correspondingly, α fixes the absolute lines $B_1 B_2, B_1 B_3$ or it interchanges them. In either case, it is trivial that $B_3 = (B_1 B_3)(B_2 B_3)$ is fixed by α .

From the corollary, we see that the problem of determining $\mathcal{T}(\mathcal{Q})$ for a polarity in $\mathcal{T}_2$ is related to that of the stabiliser of three points which belong to a generating quadrangle of $\mathcal{T}_2$.

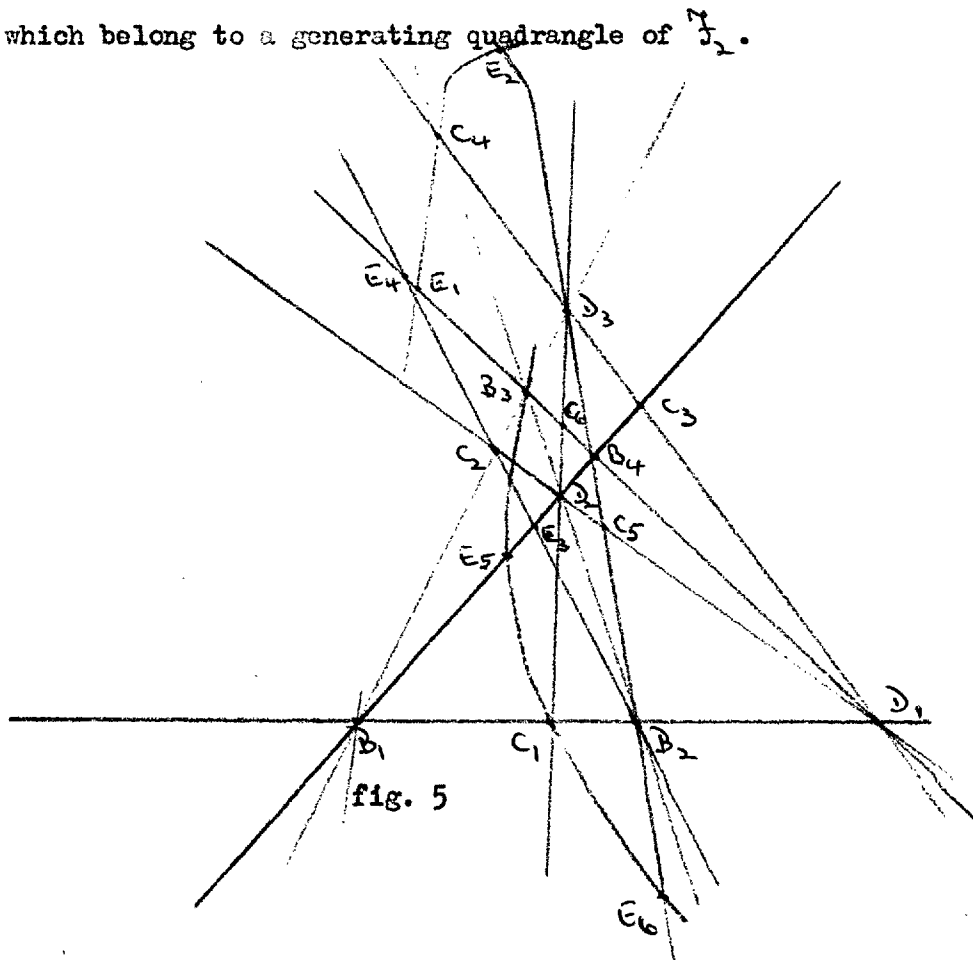


fig. 5

Sandler ([9], Theorem 7) has shown that Γ_2 is generated by the three collineations (the elements are as shown in fig. 5).

$$\begin{aligned} \varphi_1 &: B_1 \rightarrow B_2, & B_2 \rightarrow B_1, & B_3 \rightarrow B_3, & B_4 \rightarrow B_4; \\ \varphi_2 &: B_1 \rightarrow B_2, & B_2 \rightarrow B_3, & B_3 \rightarrow B_4, & B_4 \rightarrow B_1; \\ \tau &: B_1 \rightarrow B_1, & B_2 \rightarrow D_1, & B_3 \rightarrow B'_2, & B_4 \rightarrow D_2. \end{aligned}$$

Theorem 11: The stabiliser Σ of the three points B_1, B_2, B_3 is the group generated by the conjugates of $(\varphi, \tau)^3$ by elements of the symmetric group of degree three, which is the subgroup of $\mathcal{B}_2^{(e)}$ - stationary collineations fixing B_4 .

Proof: We shall say that a collineation α of Γ_2 is of stage j if it maps $\mathcal{B}_2^{(e)}$ onto a quadrangle of stage j .

We observe that a collineation in Σ is completely determined when we know the quadrangle onto which $\mathcal{B}_2^{(e)}$ is mapped. We verify directly that the identity is the only collineation of stage 0 belonging to Σ , and that there are no such collineations of stage 2 or 4. Also the six quadrangles $B_1 B_2 B_3 E_i$ ($1 \leq i \leq 6$) are the only quadrangles of stage 6 which generate all of $\mathcal{T}_2$. These quadrangles are the images of $\mathcal{B}_2^{(e)}$ under the six collineations, which are the conjugates of $(\varphi, \tau)^3$ referred to in the theorem. We denote the respective collineations by σ_i ($1 \leq i \leq 6$), and also observe that the σ_i comprise three collineations and their inverses.

The proof now proceeds by induction on the stage j , of a collineation $\sigma \in \Sigma$. Suppose that for some integer j , every collineation of stage j is such that σ is a product of the σ_i . We have shown that $j \geq 6$. Let $\alpha \in \Sigma$ map $\mathcal{B}_2^{(e)}$ onto a quadrangle $B_1 B_2 B_3 P = \mathcal{B}_2^{(e)}$.

say where $s_{\mathcal{B}_2^{(0)}}(P) = j$. We use fig. 5 and replace B_+ by P , and E_i by Q_i ($1 \leq i \leq 6$). We now make repeated use of result (K) and the fact that if $x \in \mathcal{T}_2$ such that $s(x) > 0$, then at least one of the two elements defining x in $\mathcal{T}_2$ is of stage $s(x) - 1$. From result (K), one of the D 's, D_3 say, must be of stage $j + 2$, and D_1, D_2 of stage $< j$. The two lines B_1P, B_3P then define P in $\mathcal{T}_2$ and we may take $s(B_3P) = j - 1$, whence $s(D_1) = j - 2$. Since $B_1B_3D_1D_2$ is also a generating quadrangle of $\mathcal{T}_2$, we deduce that $s(D_2) < j - 2$, and $s(C_2) < j - 2$. The two lines C_1D_2, B_1B_2 must then define D_1 in $\mathcal{T}_2$, whence $s(C_2D_2) = j - 3$. Arguing in this way, we conclude finally that those of the quadrangles $B_1B_2B_3Q_i$ which are of $\mathcal{B}_2^{(0)}$ -stage $> j$ are of stage $j + 6$. Let $B_1B_2B_3Q$ be one such quadrangle and β the collineation mapping $\mathcal{B}_2^{(0)}$ onto $B_1B_2B_3Q$, which is of $\mathcal{B}_2^{(0)}$ -stage 6. Now the collineation $\gamma = \alpha^{-1}\beta$ maps $B_1B_2B_3P = \mathcal{B}_2^{(0)}$ onto $B_1B_2B_3Q$. Hence the conjugate $\alpha\gamma\alpha^{-1}$ of γ will map $\{(B_1B_2B_3P)\alpha^{-1}\} = \mathcal{B}_2^{(0)}$ onto a quadrangle of $\mathcal{B}_2^{(0)}$ -stage 6, i.e. onto one of the quadrangles $B_1B_2B_3E_i$. Hence

$$\alpha\gamma\alpha^{-1} = \sigma_j \quad \text{for some } j \ (j = 1, 2, \dots, 6),$$

and putting $\beta = \alpha\gamma$, we get $\beta = \sigma_j\alpha$. By the induction hypothesis, α is a product of the σ_i and hence β is also such a product. Since from theorem 6 of Sandler [9], every $\beta \in \Sigma$ may be obtained from a collineation α of lower stage as above, the theorem is proved.

Conjecture: The orthogonal group $\mathcal{T}(\mathcal{B})$ of a polarity $\mathcal{B}$ in $\mathcal{T}_2$ is isomorphic to the cyclic group of order 2.

Given $\lambda(\theta)$ as in fig. 4, we verify that each of θq_1 , and $q_1\theta$ is defined by:

$$B_1 \rightarrow B_2 B_3, \quad B_2 \rightarrow B_1 B_3, \quad B_3 \rightarrow B_1 B_2$$

$$B_4 \rightarrow D_2 D_3.$$

Hence $q_1 \in T(\theta)$. We also verify directly that none of the σ_i belong to $T(\theta)$. We have not been able to prove this latter to be the case for a general σ belonging to Σ , but the relations between poles and polars in the verifications for the σ_i make this very likely.

REFERENCES

1. ABBIW-JACKSON, D.K.: "Polarities in free planes". Proc. London Math. Soc. 3(15) (1965) 26 - 38.
2. BAER, R.: "Polarities in finite projective planes". Bull. American Math. Soc. 52 (1946) 77 - 93.
3. BAER, R.: "Projectivities with fixed points on every line". ibid. 52 (1946) 273 - 86.
4. DEMBOWSKI, P: "Freie und offene projektive Ebenen". Math. Zeitschrift 72 (1960) 410 - 38.
5. HALL, M. JR.: "Projective planes". Trans. American Math. Soc. 54 (1943) 229 - 77.
6. HALL, M. JR.: "Projective planes and related topics." Calif. Inst. Tech. 1954.
7. KOPEIKINA, L.I.: "Free decompositions of projective planes". Izv. Akad. Nauk. SSSR Ser. Math. 9 (1945) 495 - 526.
(Russian). (German translation available.)
8. PICKERT, G.: "Projektive Ebenen". Springer, Berlin. 1955.
9. SANDLER, R.: "Collineation groups of free planes". Trans. American Math. Soc. 107 (1963) 129 - 39.

10. SKORNYAKOV, L.A.: "Projective planes". Translation 99
American Math. Soc. Providence. 1953.

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POLARITIES IN FREE PLANES

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1. Introduction

The concept of free planes was introduced by Marshall Hall (4) in 1943, but, until recently, rather little work has been done on the theory. Dembowski (3) has demonstrated that free planes possess correlations, but as far as is known to the author it has not been previously shown that they have polarities. In §3 we prove the existence of polarities in all free planes of finite degree.

We next show, in §4, that all polarities have only a finite number of absolute elements. Finally, in §5, we prove that, if j is the number of absolute points (and hence of absolute lines) in a polarity of a free plane of finite degree n , then $j \equiv n \pmod{2}$ and $0 \leq j \leq n$. Moreover, except for $n = 2$, there is a polarity with j absolute points for every j satisfying these conditions. When $n = 2$, there is only one type of polarity (with $j = 2$), and all the polarities are conjugate in the group of automorphisms of the free plane of degree 2.

In view of our opening remark, we give in the next section a fairly comprehensive account of some of the concepts and notions in the theory of free planes, and also some elementary results.

2. Definitions and elementary results

A collection of objects of two kinds, called points and lines, on which a binary relation (incidence relation) is defined, is said to form a *projective plane* if

- (A) any two distinct points are incident with exactly one line;
- (B) any two distinct lines are incident with exactly one point;
- (C) there exist at least four points, no three of which are incident with the same line.

A collection of points and lines together with an incidence relation, which satisfies (A) and (B) but definitely violates (C), is called a *degenerate projective plane*. By contrast, projective planes are sometimes referred to as *non-degenerate (or proper) projective planes*.

We now introduce the concept of *partial plane*. This is defined to be a collection of points and lines, together with an incidence relation, satisfying

- (a) any two distinct points are incident with at most one line;
- (b) any two distinct lines are incident with at most one point.

We note that projective planes are instances of partial planes. Also a partial plane with a finite number of elements (i.e. points and lines) is referred to as a *configuration*.

A partial plane is said to be *degenerate* if it can be embedded as a sub-partial plane in some degenerate projective plane.

Let $\mathfrak{P}'$ be a sub-partial plane of a partial plane $\mathfrak{P}$. Then we say that $\mathfrak{P}'$ is *closed in $\mathfrak{P}$* if

- (i) every line of $\mathfrak{P}$ which is incident with two distinct points of $\mathfrak{P}'$ is also in $\mathfrak{P}'$,
- (ii) every point of $\mathfrak{P}$ which is incident with two distinct lines of $\mathfrak{P}'$ is also in $\mathfrak{P}'$.

A partial plane $\mathfrak{P}$ is *generated* by its sub-partial plane $\mathfrak{Q}$ if $\mathfrak{Q} \subseteq \mathfrak{P}' \subseteq \mathfrak{P}$ and $\mathfrak{P}'$ closed in $\mathfrak{P}$ implies that $\mathfrak{P}' = \mathfrak{P}$.

It is clear that if $\mathfrak{P}'$ is a projective sub-plane of a projective plane $\mathfrak{P}$, then $\mathfrak{P}'$ is closed in $\mathfrak{P}$. Hence, if a projective plane $\mathfrak{P}$ is generated by its sub-partial plane $\mathfrak{Q}$, then no proper sub-partial plane of $\mathfrak{P}$ which contains $\mathfrak{Q}$ can be a projective plane.

Given a partial plane $\mathfrak{P}$ which is not a projective plane, we may, subject to certain obvious restrictions, form subsets of the set of points, such that every subset contains only points not already incident with a line, and define a new line to be incident with each subset. For details of such a construction we refer the reader to Hall (4). We call the partial plane consisting of $\mathfrak{P}$ and the new lines an *extension* of $\mathfrak{P}$. A canonical way of constructing the new lines is to define each subset to consist of exactly two points which are not already incident with a line and to include every such point-pair as a subset. More precisely:

Let $\mathfrak{P}_0$ be a partial plane, which is not a projective plane. Define $\mathfrak{P}_1$ as follows. For every pair of points P, Q not incident with a line of $\mathfrak{P}_0$, we define a new line l to be incident with P and Q and with only these points. Put $\mathfrak{P}_1 = \mathfrak{P}_0 \cup \Sigma l$, with obvious incidences; clearly, $\mathfrak{P}_1$ is also a partial plane. Define $\mathfrak{P}_2$ in dual fashion from $\mathfrak{P}_1$, and similarly $\mathfrak{P}_3, \mathfrak{P}_4, \dots$. The partial plane $\mathfrak{P} = \bigcup_{m=1}^{\infty} \mathfrak{P}_m$ is easily seen to be a projective plane, which we call the *free completion* of $\mathfrak{P}_0$. (There is essentially only one way of extending $\mathfrak{P}_0$ to its free completion.) We also say that $\mathfrak{P}_m$ is a *free extension* of $\mathfrak{P}_{m-1}$ ($m = 1, 2, \dots$). More generally, we say that a partial

† Hall uses the term 'complete' where we have 'closed'. We have adopted the terminology of Grace E. Bates. (She treats the analogous situation of half-nets in her paper, 'Free loops and nets and their generalisations', *American J. Math.* 69 (1947) 501.) We also adopt her viewpoint in the definition of the concept of 'generation of a partial plane by its sub-partial plane', and in part of the definitions of homomorphic mappings, which we give later on.

plane $\mathfrak{P}'$ is a *free extension* of another partial plane $\mathfrak{P}$ if when we put $\mathfrak{P} = \mathfrak{P}_0$ in the above construction we have $\mathfrak{P}_0 \subseteq \mathfrak{P}' \subseteq \mathfrak{P}_m$ for some $m \geq 0$. Conversely, $\mathfrak{P}$ is a *free contraction* of $\mathfrak{P}'$ if $\mathfrak{P}'$ is a free extension of $\mathfrak{P}$. Two partial planes are said to be *free equivalent* if either may be obtained from the other by a succession of free extensions or free contractions or both. It is easily verified that the free completion $\mathfrak{P}$ (as well as each $\mathfrak{P}_m$) is generated by $\mathfrak{P}_0$. We say that $\mathfrak{P}_0$ *freely generates* $\mathfrak{P}$ (and also any free extension of $\mathfrak{P}_0$).

The free completion of the partial plane consisting of n (≥ 2) collinear points, the common line of incidence, and two points off this line, is called the *free plane of degree n* (n need not be finite). We shall designate it by $\mathfrak{P}^n$, and the partial planes constructed above by $\mathfrak{P}_m^n$.

In generating $\mathfrak{P}$ from $\mathfrak{P}_0$, we say that an element P is of stage m if $P \in \mathfrak{P}_m$ but $P \notin \mathfrak{P}_{m-1}$ ($m = 1, 2, \dots$), and that P is of stage zero if $P \in \mathfrak{P}_0$. We write $s(P) = m$. A result we shall make use of later is that, if $\mathfrak{P}$ is freely generated by $\mathfrak{P}_0$, then every element of stage greater than zero is incident with exactly two elements of earlier stage. We shall say that the two latter elements define the first.

A useful concept in the development of the theory of free planes is that of *rank*[†] (also due to Marshall Hall (4), though the name is due to Kopeikina (5)). For a finite partial plane, this is defined to be the number $r(\mathfrak{P}) = 2(P + L) - I$, where P is the number of points, L the number of lines, and I the number of incidences given by the number of ordered pairs (P_s, l_i) such that the point P_s is incident with the line l_i . That rank is invariant under free equivalence is easily proved ((4) Theorem 4.10).

The rank of the free completion of a finite partial plane $\mathfrak{P}_0$ is defined to be the rank of $\mathfrak{P}_0$. In particular, the rank of a free plane of finite degree n is $n + 6$. An infinitely generated free plane is said to have infinite rank of cardinality equal to that of the collinear points.

A partial plane is said to be a *confined configuration* if

- (i) it has a finite number of elements,
- (ii) every point is on at least three lines, and every line contains at least three points.

An *open configuration (plane)* is one which contains no confined configuration. It can be shown that the free completion of a partial plane $\mathfrak{P}_0$ contains precisely those confined configurations that are already contained in $\mathfrak{P}_0$, whence it follows that free planes are open. Hall (4) has

[†] We observe that the notion of rank is essentially the same as the more familiar one of 'degree of freedom'. In a plane, a point or a line by itself has 2 degrees of freedom. If a point is constrained to lie on r lines, its degree of freedom is $2 - r$.

proved that finitely generated open planes are free,[†] but Kopeikina (5) has shown that this is not necessarily so if the planes are infinitely generated.

A *homomorphism* α of a partial plane $\mathfrak{P}$ into a partial plane $\mathfrak{Q}$ is a (single-valued) mapping of the points of $\mathfrak{P}$ into the points of $\mathfrak{Q}$ and the lines of $\mathfrak{P}$ into the lines of $\mathfrak{Q}$ such that incidence is preserved.

The homomorphic image of a projective plane is easily shown to be a projective plane.

A homomorphism α of a partial plane $\mathfrak{P}$ onto a partial plane $\mathfrak{Q}$ which is one-to-one and such that α^{-1} is a homomorphism of $\mathfrak{Q}$ onto $\mathfrak{P}$ is called an *isomorphism* of $\mathfrak{P}$ onto $\mathfrak{Q}$. (If $\mathfrak{P}$ (and hence $\mathfrak{Q}$) is a projective plane, then one-to-oneness is sufficient for a homomorphism to be an isomorphism. It is easily shown that this is not the case if $\mathfrak{P}$ is a proper partial plane.) If $\mathfrak{Q} = \mathfrak{P}$, then an isomorphism of $\mathfrak{P}$ onto $\mathfrak{Q}$ is called a *collineation*.

A one-to-one mapping of a partial plane onto itself which takes points onto lines and lines onto points, and which preserves incidence, is called a *correlation*. Thus the product of two correlations is a collineation. In particular, a correlation θ such that $\theta^2 = 1$, the identity mapping, is called a *polarity*.

If θ is a polarity and P a point of the partial plane $\mathfrak{P}$, we call the line $p = P\theta$ the *polar* of P ; then $P = p\theta$, and P is called the *pole* of p .[‡]

Elementary results

LEMMA 2.1. *The point P lies on the polar of a second point Q if and only if Q lies on the polar of P , and dually.*

LEMMA 2.2. *An absolute line contains exactly one absolute point, and dually.*

These are easily proved (see Baer (I), for instance).

In the next two lemmas, $\mathfrak{P}$ is the free completion of the partial plane $\mathfrak{P}_0$.

LEMMA 2.3. *Every homomorphism α_0 of $\mathfrak{P}_0$ into a projective plane $\mathfrak{Q}$ can be extended to a homomorphism α of $\mathfrak{P}$ into $\mathfrak{Q}$.*

In general α is not unique, and again if α_0 is an isomorphism it does not follow that α is also an isomorphism.

LEMMA 2.4. *If α_0 is a collineation (correlation) of $\mathfrak{P}_0$, then α_0 extends in a unique way to a collineation (correlation) α of $\mathfrak{P}$. Moreover, the order of α is the same as the order of α_0 ; so that, in particular, polarities extend to polarities.*

[†] We refer to this result below by (H).

[‡] We adopt this convention throughout this paper—a point is denoted by a capital Roman letter and its polar by the corresponding small letter.

Lemmas 2.3 and 2.4 are easily proved by induction on the partial planes $\mathfrak{P}_m$ defined above.

Also a collineation (correlation) α of $\mathfrak{P}^n$ is completely determined by its action on any sub-partial plane of $\mathfrak{P}^n$ which freely generates the whole of $\mathfrak{P}^n$.

LEMMA 2.5. *If $\mathfrak{P}$ is an extension of a finite partial plane $\mathfrak{P}_0$, then $r(\mathfrak{P}) \leq r(\mathfrak{P}_0)$, and there is equality if and only if $\mathfrak{P}$ is a free extension of $\mathfrak{P}_0$.*

From this may be deduced Theorem 3 of Sandler (6), that if $\mathfrak{P}_0$ is a finite sub-partial plane of $\mathfrak{P}^n$ which has rank $n+6$ and which generates all of $\mathfrak{P}^n$, then $\mathfrak{P}_0$ generates $\mathfrak{P}^n$ freely.

Lemma 2.5 is trivially false if the rank is infinite.

3. Existence of polarities in $\mathfrak{P}^n$

THEOREM 3.1. *Every free plane of finite degree possesses polarities.*

Proof. In Figs. 1 and 2 below we have, clearly, defined *polarities* in the respective partial planes.

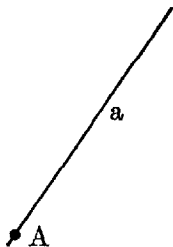


FIG. 1

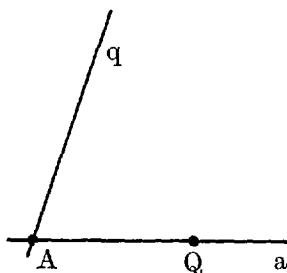


FIG. 2

If we take a large enough finite number of (1) or (2) or both, with no further incidences, then we obtain a non-degenerate open configuration, and hence, by (H), we have a generating configuration for some $\mathfrak{P}^n$.

Now the rank of (1) is 3 and the rank of (2) is 5. Also, rank is clearly additive if we take the set-theoretic union of a number of disjoint finite partial planes. We consider the equation

$$3x + 5y = n + 6.$$

It is solvable in non-negative integers for all $n \geq 2$. This follows for $n \geq 9$ from the well-known result that if a and b are positive integers and $(a, b) = 1$, then the equation

$$ax + by = c$$

has solutions in non-negative integers for all $c \geq ab$; and for $2 \leq n < 9$ by direct verification. Hence for every $\mathfrak{P}^n$ we can, by taking appropriate

numbers of (1) and (2), obtain a generating configuration on which a polarity θ is defined, and, by Lemma 2.4, θ can be extended (uniquely) to a polarity in the whole of $\mathfrak{P}^n$.

4. Minimal self-polar generating configurations

A finite sub-partial plane $\mathfrak{C}$ of $\mathfrak{P}^n$ is called a *minimal self-polar generating configuration* if it is self-polar and freely generates the whole of $\mathfrak{P}^n$ but is such that no proper sub-partial plane of $\mathfrak{C}$ is both self-polar and free equivalent to $\mathfrak{C}$.

We remark that a minimal self-polar generating configuration is not necessarily a *minimal generating configuration*, defined as a configuration $\mathfrak{M}$ which freely generates $\mathfrak{P}^n$ but is such that no proper sub-partial plane of $\mathfrak{M}$ is free equivalent to $\mathfrak{M}$.

We now show that such minimal self-polar generating configurations exist for every polarity θ of $\mathfrak{P}^n$. Let $\mathfrak{P}_0$ be the generating configuration of $\mathfrak{P}^n$ defined in § 2, and let θ be a polarity in $\mathfrak{P}^n$. Put $\mathfrak{P}'_0 = \mathfrak{P}_0\theta$ and $\mathfrak{Q} = \mathfrak{P}_0 \cup \mathfrak{P}'_0$. Then $\mathfrak{Q}$ is self-polar but, in general, will have rank greater than $n+6$. We shall now construct a self-polar configuration $\mathfrak{S}$ which has rank $n+6$. If $\mathfrak{Q}$ has rank $n+6$, put $\mathfrak{Q} = \mathfrak{S}$. Otherwise, let $\mathfrak{P}_m^n$ be the smallest partial plane containing $\mathfrak{Q}$. If $\mathfrak{P}_m^n$ is self-polar, put $\mathfrak{P}_m^n = \mathfrak{S}$. If not, let P in $\mathfrak{P}_m^n$ be an element of least stage such that $P\theta \notin \mathfrak{P}_m^n$. For the sake of definiteness, we shall take P to be a point. Then the two (unique) lines l, m (say) defining P in $\mathfrak{P}^n$ are such that the distinct points $l\theta, m\theta \in \mathfrak{P}_m^n$ (in fact, at least one of these must be of the m th stage). We extend $\mathfrak{P}_m^n$ by freely adjoining the line $P\theta = l\theta \cup m\theta$. Since the number of elements in $\mathfrak{P}_m^n$ whose poles or polars are not in $\mathfrak{P}_m^n$ is necessarily finite, we obtain, after a finite number of repetitions of this process, a free extension $\mathfrak{S}$ of $\mathfrak{P}_m^n$ which is self-polar. Since $\mathfrak{P}_m^n$ has rank $n+6$, $\mathfrak{S}$ is also of rank $n+6$ and generates the whole of $\mathfrak{P}^n$; hence it generates $\mathfrak{P}^n$ freely (by Lemma 2.5).

We now freely contract $\mathfrak{S}$ but so that the remaining configuration is always self-polar; that is, we remove an element and its pole or polar simultaneously. That we can do this is shown by the fact that, if Q in $\mathfrak{S}$ is a non-absolute point, then we may freely remove Q if and only if Q is on exactly two lines, l, m (say). From Lemma 2.2 it follows that q ($\neq l, m$), the polar of Q , contains exactly the points L, M ($\neq Q$), the poles of l, m respectively. Hence we may freely remove both Q and q together. We carry out this free contraction until we obtain a configuration $\mathfrak{C}$ which cannot be thus further contracted (this situation will obviously be reached after a finite number of steps). $\mathfrak{C}$ is the required minimal self-polar generating configuration.

That $\mathfrak{C}$ is not unique is shown by the following example.

The configuration given in Fig. 3 is self-polar and freely generates $\mathbb{P}^3$. (In the above notation, it is an instance of a $\mathfrak{Q}$ which is also an $\mathfrak{S}$.) We may successively remove the corresponding elements (Q_3, q_3) , (Q_2, q_2) , and (Q_5, q_5) , thus obtaining as a minimal self-polar generating configuration $\mathfrak{C}$ the sub-partial plane defined by the points A, Q_1, Q_4 and their respective polars a, q_1, q_4 . But we could also have obtained a second minimal

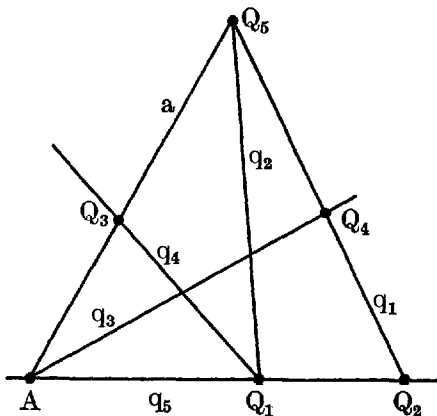


FIG. 3

self-polar generating configuration $\mathfrak{C}'$ by removing in turn (Q_4, q_4) , (Q_2, q_2) , and (Q_5, q_5) , so that $\mathfrak{C}'$ is the sub-partial plane defined by the points A, Q_1, Q_3 and their respective polars. Now $\mathfrak{C}$ and $\mathfrak{C}'$ are not even isomorphic.

The following proposition, however, holds.

THEOREM 4.1. *Let θ, θ' be two polarities of $\mathbb{P}^n$, and let $\mathfrak{C}, \mathfrak{C}'$ be respective minimal self-polar generating configurations. Suppose also that there is an isomorphism φ of $\mathfrak{C}$ onto $\mathfrak{C}'$ such that, if P is any element of $\mathfrak{C}$ and $P' = P\varphi$, then $p' = p\varphi$, where p, p' are respectively the polars (or poles) of P, P' under θ, θ' .† Then θ and θ' are conjugate (by a collineation) in the group of automorphisms of $\mathbb{P}^n$.*

Proof. The isomorphism φ maps $\mathfrak{C} \subset \mathbb{P}^n$ onto $\mathfrak{C}' \subset \mathbb{P}^n$. Hence, by Lemma 2.3, φ extends to a homomorphism α of $\mathbb{P}^n$ onto a projective plane $\mathbb{P}^n \alpha$ containing $\mathfrak{C}'$. Since $\mathfrak{C}'$ generates $\mathbb{P}^n$, it follows that $\mathbb{P}^n \alpha = \mathbb{P}^n$. Hence α is a homomorphism of $\mathbb{P}^n$ onto itself. By induction on the finite free extensions of $\mathfrak{C}$ and $\mathfrak{C}'$ (corresponding to the $\mathbb{P}_m$'s

† That is, the isomorphism φ preserves the polarity θ .

defined in §2), it can easily be shown that α is one-to-one, and hence a collineation of $\mathfrak{P}^n$.†

Now let P, P', p, p' be as defined above. Then

$$\begin{aligned} P\alpha\theta'\alpha^{-1} &= P\phi\theta'\alpha^{-1} \\ &= P'\theta'\alpha^{-1} = p'\alpha^{-1} = p'\phi^{-1} = p. \end{aligned}$$

Hence the polarities θ and $\alpha\theta'\alpha^{-1}$ of $\mathfrak{P}^n$ coincide on $\mathfrak{C}$. Since a correlation of $\mathfrak{C}$ extends uniquely to a correlation of $\mathfrak{P}^n$ (Lemma 2.4), it follows that $\theta = \alpha\theta'\alpha^{-1}$. This proves the theorem.

THEOREM 4.2. *Let θ be a polarity of a partial plane $\mathfrak{P}$, which is a (finite or infinite) free extension of its self-polar (under θ) sub-partial plane $\mathfrak{S}$. Then all the absolute elements of $\mathfrak{P}$ are contained in $\mathfrak{S}$.*

COROLLARY 4.3. *The number of absolute elements in any polarity θ of $\mathfrak{P}^n$ is finite.*

Proof. Suppose the theorem is false, and let A be an absolute element of least stage in the free generation of $\mathfrak{P}$ from $\mathfrak{S}$ such that $A \in \mathfrak{P}$ but $A \notin \mathfrak{S}$. We shall take A to be a point, and its stage to be m (≥ 1). Let A be defined by the lines p, q whose poles are P, Q respectively. Then the polar a of A is different from p, q . (If a were equal to p or q , this would imply for $m = 1$ that $a \in \mathfrak{S}$ and, since $\mathfrak{S}$ is self-polar, that $A \in \mathfrak{S}$; for $m > 1$ it would contradict the choice of A as having minimal stage.) Hence $s(a) > s(A)$, and A has been used in defining a . Since the point A lies on p and q , its polar a also contains the points P and Q (Lemma 2.1). It follows then that at least one of P, Q (say P) has stage greater than $s(a)$. Now let P_1, P_2 define the line p . Their polars p_1, p_2 then contain P , and since we already have $s(a) < s(P)$, at least one of p_1, p_2 (say p_1) has its stage greater than $s(P)$. We also note that $s(P_1) < s(p)$. We thus obtain a sequence of elements,

$$A, p, P_1, p_1, \dots$$

of strictly decreasing stage, whose poles or polars

$$a, P, p_1, P_2, \dots$$

have strictly increasing stages.

Since $s(A) = m$ is finite, the first sequence eventually terminates in an element L (say) of $\mathfrak{S}$. However, the pole or polar l of L is such that $s(l) > s(a) > m$. This contradicts our initial assumption that $\mathfrak{S}$ is self-polar, and hence the theorem is proved.

^{But not}
† ~~In fact,~~ every homomorphism of a finitely generated free plane onto itself is an isomorphism.

To deduce the corollary, we simply observe that $\mathfrak{P}^n$ is a free extension of the minimal self-polar generating configuration $\mathfrak{C}$ (constructed above), and that $\mathfrak{C}$ is finite.

5. Main results

From considerations of symmetry we deduce that a self-polar configuration with no absolute elements has even rank.

Suppose now that there are j (≥ 1) absolute points in a polarity. Let A_i ($i = 1, 2, \dots, j$) be an absolute point and a_i its polar. Suppose also that A_i lies on $x_i + 1$ ($x_i \geq 0$) lines (one of these will be the line a_i , and the remaining x_i lines will all be non-absolute (Lemma 2.2)). Then by Lemma 2.1, a_i contains exactly $x_i + 1$ points (one of which is A_i). If we now remove A_i and a_i , we decrease the number of points by 1, the number of lines by 1, and the number of incidences by $2x_i + 1$, thus increasing the rank by $2x_i - 3$. If we remove all the j absolute points and their polars from the configuration $\mathfrak{C}$ (defined as above), we obtain the equation

$$n + 6 + 2 \sum_{i=1}^j x_i - 3j = 2y,$$

for some integer y . From this equation follows the first part of

THEOREM 5.1. *If j is the number of absolute points in a polarity in $\mathfrak{P}^n$ then*

- (i) $j \equiv n \pmod{2}$,
- (ii) $0 \leq j \leq n$,
- (iii) *if $n > 2$ there is a polarity in $\mathfrak{P}^n$ for every j satisfying (i) and (ii) (this shows that our results are the best possible); all polarities in $\mathfrak{P}^2$ have $j = 2$ and, moreover, they are all conjugate in the automorphism group of $\mathfrak{P}^2$.*

Proof of (ii). We proceed by induction on the rank of the minimal self-polar generating configuration $\mathfrak{C}$ constructed in §4. We note that $\mathfrak{C}$ is non-degenerate and open.

Suppose that the result is true for all such configurations of rank less than $n + 6$. We now take $r(\mathfrak{C}) = n + 6$ and consider various cases:

- (a) If $\mathfrak{C}$ contains no absolute points, the result is immediate.
- (b) Suppose that $\mathfrak{C}$ has an absolute point A , incident only with its own polar a . We remove the pair (A, a) . Let $\mathfrak{C}'$ be the remaining configuration. $\mathfrak{C}'$ is also self-polar, and $r(\mathfrak{C}') = n + 3$, $j(\mathfrak{C}') = j(\mathfrak{C}) - 1$. We now consider the following cases:

$\mathfrak{C}'$ non-degenerate. From the inductive hypothesis, we obtain

$$j(\mathfrak{C}) - 1 \leq n - 3,$$

whence $j < n$.

$\mathfrak{C}'$ degenerate. In any case, $r(\mathfrak{C}') \geq 5$, and hence $\mathfrak{C}'$ contains at least 2 points (and 2 lines). If $\mathfrak{C}'$ contains exactly 2 points and 2 lines, we have the following possibilities:

$r(\mathfrak{C}') = 5$. $\mathfrak{C}'$ is the configuration in Fig. 2. Thus $j(\mathfrak{C}) = n = 2$ and (ii) is proved.

$r(\mathfrak{C}') = 6$. $\mathfrak{C}'$ consists of two independent flags. (A flag is a single point on a single line.) Then either the points lie on their respective polars, in which case $j(\mathfrak{C}) = 3$, or each point lies on the polar of the other and $j(\mathfrak{C}) = 1$. Condition (ii) is again satisfied, since $n = 3$.

$r(\mathfrak{C}') = 7$. $\mathfrak{C}'$ consists of a flag (which gives a corresponding pair of absolute elements) and an isolated point and its polar. Here $j(\mathfrak{C}) = 2 < n = 4$.

$r(\mathfrak{C}') = 8$. $\mathfrak{C}'$ contains exactly two isolated points and their polars. Thus $j(\mathfrak{C}) = 1 < n = 5$, and (ii) is again proved.

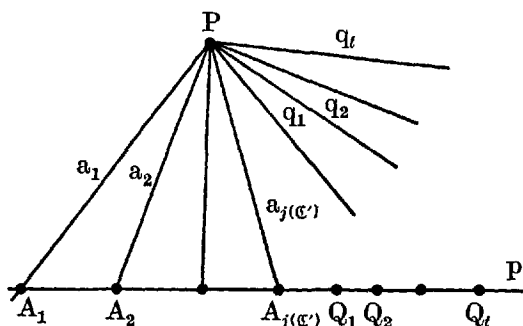


FIG. 4

If $\mathfrak{C}'$ contains more than 2 points (and 2 lines), then there is a point P in $\mathfrak{C}'$ such that all lines except possibly the polar p of P contain P and dually ((2) Lemma 1).

If P is an absolute point (so that P lies on p), then, by Lemma 2.2, $\mathfrak{C}'$ contains no other absolute elements and $j(\mathfrak{C}) = 2 < n$, n being at least 4.

If P is non-absolute, then $\mathfrak{C}'$ is as shown in Fig. 4.

We have the equation

$$n + 3 = r(\mathfrak{C}') = 4 + 3t + j(\mathfrak{C}'),$$

with $t \geq 0$. Hence $j(\mathfrak{C}') \leq n - 1$, and $j(\mathfrak{C}) \leq n$.

(c) Suppose now that every absolute point is incident with some non-absolute line. If there is an absolute point A which is incident with exactly one non-absolute line, then we remove A and its polar a . With $\mathfrak{C}'$ as defined in (b), we have $r(\mathfrak{C}') = n + 5$.

$\mathfrak{C}'$ non-degenerate. By the inductive hypothesis,

$$j-1 \leq n-1,$$

$$j \leq n.$$

$\mathfrak{C}'$ degenerate. It is easily verified that the only possibility is $r(\mathfrak{C}') = 8$, and $\mathfrak{C}'$ then consists of two isolated points and their polars. Thus

$$j(\mathfrak{C}) = 1 < n = 3.$$

(d) Finally, if every absolute point is incident with at least two non-absolute lines (and hence with at least three lines) then, since $\mathfrak{C}$ is open, $\mathfrak{C}$ contains at least one non-absolute point P which is incident with at most one line. ($\mathfrak{C}$, being minimal, does not contain non-absolute points incident with exactly two lines.) We remove one such point P and its polar p , thus reducing the rank of $\mathfrak{C}$ by 4 if P is an isolated point and by 2 if P is incident with one line.

If $\mathfrak{C}'$ is non-degenerate, then (ii) follows immediately from the inductive hypothesis.

If $\mathfrak{C}'$ is degenerate, there is only one possibility: $\mathfrak{C}'$ consists of an absolute line a containing its pole A and at least two non-absolute points, whose polars then contain A . $j(\mathfrak{C}) = 1 < n$, n being not less than 5. This completes the proof of (ii).

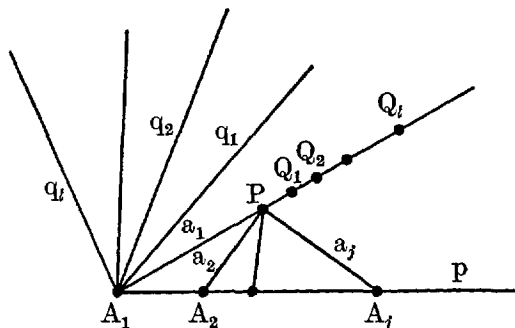


FIG. 5

Proof of (iii). For the final part of the theorem, we give examples of minimal self-polar generating configurations with the required number of absolute points. From the results of §§ 2 and 4, we know that such a configuration determines a polarity of $\mathbb{P}^n$ with the same absolute points. If $j \geq 2$, we take the configuration in Fig. 5 with $t = \frac{1}{2}(n-j+2)$. (If $j = 2$, we leave out P and p .)

The configuration in Fig. 6 satisfies the case when $j = 1$, with $t = \frac{1}{2}(n-1)$.

If $j = 0$ then, as in Theorem 3.1, the equation

$$4x + 6y = n + 6$$

has non-negative integral solutions for $n \equiv 0 \pmod{2}$ and $n \geq 2$. Now a configuration consisting of an isolated point and its polar has rank 4, and one consisting of two independent flags has rank 6. (In the second configuration we make each point lie on the polar of the other.) Hence, by taking appropriate numbers of the two configurations, we obtain a (minimal) self-polar configuration which freely generates $\mathfrak{P}^n$ for any even n (> 2), and has $j = 0$.

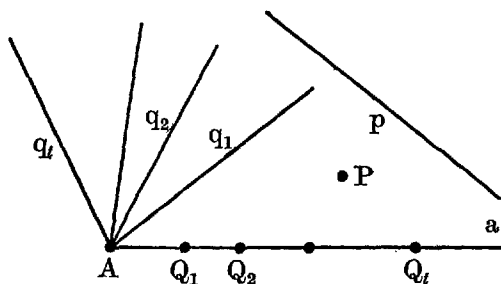


FIG. 6

For $n = 2$, it is not hard to verify that the only possible minimal self-polar generating configuration is the partial plane which is the set-theoretic union of the configurations given in Figs. 1 and 2. Hence all such configurations satisfy the conditions of Theorem 4.1, from which it follows that all polarities in $\mathfrak{P}^2$ are conjugate (by a collineation) in the automorphism group of $\mathfrak{P}^2$.

6. Remarks

(a) It is easily shown that polarities are not completely determined (up to conjugacy—by a collineation—in the automorphism group) by the number of absolute elements. For instance, if $j = n = 3$ we may have for a self-polar generating configuration $\mathfrak{C}$ three independent flags (each point lying on its own polar), or we may have the configuration in Fig. 5 with $t = 1$, and the two polarities are clearly not conjugate in the automorphism group of $\mathfrak{P}^3$.

The question of polarities with isomorphic structures of absolute elements will be discussed in a subsequent note.

(b) It may be of some interest to compare our results with some of those obtained by Baer (1) for polarities in finite planes. Baer shows that the number j of absolute points in a polarity in a finite plane of order n (i.e. $n + 1$ points on every line) satisfies

- (i) $j \equiv n + 1 \pmod{2}$,
- (ii) $j \geq n + 1$.

(c) The partial plane in Fig. 5 can easily be extended to give a polarity in $\mathfrak{P}^n$ when n is infinite. There does not seem to be much of interest that can be said in this case.

(d) Finally, the methods used in §4 can also be used to obtain information about collineations of $\mathfrak{P}^n$ which have finite order. I hope to publish later some results I have obtained in this respect.

I wish to express my sincere gratitude to my supervisor, Dr A. Wagner, for his encouragement and guidance in the preparation of this paper. My thanks are also due to the referee whose suggestions have led to a considerable improvement in the presentation of this paper, especially of §4.

REFERENCES

1. R. BAER, 'Polarities in finite projective planes', *Bull. American Math. Soc.* 52 (1946) 77–93.
2. R. BAER, 'Projectivities with fixed points on every line', *ibid.* 52 (1946) 273–80.
3. P. DEMBOWSKI, 'Freie und offene projektive Ebenen', *Math. Zeitschrift* 72 (1960) 410–38.
4. M. HALL, JR., 'Projective planes', *Trans. American Math. Soc.* 54 (1943) 229–77.
5. L. KOPEIKINA, 'Decompositions of projective planes', *Bull. Acad. Sci. URSS Ser. Math. (Izvestija Akad. Nauk SSSR)* 9 (1945) 495–526 (Russian). See also *Math. Reviews* 8 (1947) 167.
6. R. SANDLER, 'Collineation groups of free planes', *Trans. American Math. Soc.* 107 (1963) 129–39.

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