UNIQUENESS AND SHORT TIME REGULARITY OF THE WEAK KÄHLER-RICCI FLOW

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Abstract. Let X be a compact Kähler manifold. We prove that the Kähler-Ricci flow starting from arbitrary closed positive $(1,1)$-currents is smooth outside some analytic subset. This regularity result is optimal, meaning that the flow has positive Lelong numbers for short time if the initial current has. We also prove that the flow is unique when starting from currents with zero Lelong numbers.

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1. Introduction

Over the last few decades, the Ricci flow, introduced by Hamilton in [23],

$$\frac{dg}{dt} = -2\mathrm{Ric}(g)$$

has found spectacular applications in Riemannian geometry [27], [28], [29]. As observed by Bando, starting from a Kähler metric, the Kähler property is preserved by the Ricci flow and the resulting flow is called the Kähler-Ricci flow. Its long time-existence is now well known thanks to the work of Cao [6], Tsuji [39] and Tian-Zhang [38]. Convergence of the flow at infinity is also of great interest, as shown in [6], as this effect can be used to find Kähler-Einstein metrics if they exist. Since then, the Kähler-Ricci flow, as well as its twisted version, has been studied intensively.

Of particular interest is the attempt to run the Kähler-Ricci flow from singular data (see [7], [8], [34], [37], [22]), and this can be considered as an alternative way to regularize currents. Another motivation for studying the weak Kähler-Ricci flow comes from an analytic analogue of Mori's Minimal Model Program. It is conjectured (see [34], [35], [36]) that the minimal model of a projective algebraic variety can be obtained as the eventual limit of the Kähler-Ricci flow after passing through singularities. At each singular time, the flow

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develops singularities along some analytic subset of X, and so to continue through these singularities it is necessary to restart the flow from singular objects.

In [34], Song-Tian succeeded in starting the flow from continuous initial data. Remarkable progress has recently been made by Guedj and Zeriahi [22] which allows us to define a unique maximal flow starting from any closed positive $(1,1)$-current. Moreover, as shown in [22], the flow slowly smoothes out the initial current and the speed of the regularizing effect depends on the value of the Lelong number of the initial current.

The scalar parabolic Monge-Ampère equation. Before going further and stating the main results of the paper, let us fix some notations. Let X be a compact Kähler manifold of dimension n and $\alpha_0 \in H^{1,1}(X,\mathbb{R})$ a Kähler class. Fix η a smooth closed $(1,1)$-form on X. The η-twisted Kähler-Ricci flow, introduced in [9], is the following:

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) + \eta \ , \ \omega_t|_{t=0} = T_0, \tag{1.1}$$

where T_0 is a fixed closed positive $(1,1)$-current in α_0. When T_0 is a Kähler form, as mentioned above, the flow admits a unique smooth solution on a maximal interval $[0, T_{\max})$, where

$$T_{\max} := \sup\{t \geq 0 \,|\, tK_X + t\{\eta\} + \alpha_0 \ \text{ is nef}\}.$$

It is standard (and more convenient) to rewrite the flow as a scalar parabolic Monge-Ampère equation. Fix ω a Kähler form in α_0. Let φ_0 be a global potential of T_0, i.e. $T_0 := \omega + dd^c\varphi_0$. Set

$$\chi := \eta - \mathrm{Ric}(\omega), \ \theta_t := \omega + t\chi,$$

and consider the following equation

$$\frac{\partial \varphi_t}{\partial t} = \log\left[\frac{(\theta_t + dd^c\varphi_t)^n}{\omega^n}\right], \ \varphi_t \to \varphi_0 \text{ as } t \to 0. \tag{1.2}$$

If φ_t solves (1.2) then a straightforward computation shows that $\omega_t := \theta_t + dd^c\varphi_t$ solves the flow (1.1). Conversely, if ω_t solves the flow (1.1) then it follows from the dd^c-lemma that we can write

$$\omega_t = \theta_t + dd^c\varphi_t,$$

where φ_t solves the parabolic Monge-Ampère equation (1.2).

The maximal Kähler-Ricci flow. Fix $T_0 = \omega + dd^c\varphi_0$ a closed positive $(1,1)$ current in the class α_0.

One of the natural tools to measure local singularities of a quasi-plurisubharmonic functions is the Lelong number (the definition will be recalled in Section 2). Sometimes it is more convenient to work with the integrability index instead of the Lelong numbers. The integrability index of T_0 (or φ_0) at a point $x \in X$ is defined by

$$c(T_0, x) = c(\varphi_0, x) := \sup\left\{\lambda > 0 \ \Big|\ e^{-2\lambda\varphi_0} \in L^1(V_x)\right\},$$

where V_x is some neighborhood around x. We denote by $c(T_0) = c(\varphi_0)$ the infimum of $c(T_0, x)$ when x runs over X. As X is compact it is well known that $c(T_0) > 0$.

It follows from Skoda's integrability theorem that one has the following "optimal" comparison between the Lelong number of φ_0 at a point $x_0 \in X$ and the local integrability index of φ_0 at x_0:

$$\frac{1}{\nu(\varphi_0, x_0)} \leq c(\varphi_0, x_0) \leq \frac{n}{\nu(\varphi_0, x_0)}. \tag{1.3}$$

In particular $c(\varphi_0) = +\infty$ if and only if $\nu(\varphi_0, x) = 0$ for all $x \in X$ (see [33] for Skoda's theorem and [41] for a uniform version).

Let $\varphi_{0,j}$ be a sequence of smooth ω_0-psh functions decreasing to φ_0. Such sequences exist thanks to [11] (see also [3]). Let $\varphi_{t,j}$ be the unique solution of the parabolic equation (1.2) with initial data $\varphi_{0,j}$. If $1/2c(T_0) < T_{\max}$, as shown by Guedj and Zeriahi in [22], when $j \to +\infty$ the sequence $\varphi_{t,j}$ decreases to φ_t which satisfies the following:

- For each $t > 0$, φ_t is a θ_t-psh function. *Moreover, if $t > 1/2c(T_0)$, φ_t is smooth on X and solves (1.2) in the classical sense.*
- φ_t converges in capacity to φ_0 as $t \to 0$.

The flow constructed by approximation as above was also shown to be maximal, meaning that it dominates any "weak" solution of (1.2). The regularity of this flow was obtained for t not too small and nothing was known when $t < 1/2c(T_0)$. Example 6.4 in [22] suggests that there might be no regularity at all due to the presence of positive Lelong numbers. However, as in Demailly's regularization theorem [10, 11], one can expect that the regularizing effect happens outside some analytic subset. Our first result shows that it is indeed the case.

Theorem 1.1. *Assume that $1/2c(T_0) < T_{\max}$. Then the maximal Kähler-Ricci flow starting from T_0 is smooth in a Zariski open subset of X.*

More precisely, the Zariski open subset in Theorem 1.1 is described by the complement of Lelong superlevel sets of T_0. For each $s > 0$ we define

$$D_s := \{x \in X \mid \nu(T_0, x) \geq s\}.$$

The precise statement of Theorem 1.1 says that for each $\varepsilon > 0$ there exists an analytic subset $Y_\varepsilon := D_{\kappa(\varepsilon)}$ such that ω_t is smooth in $X \setminus Y_\varepsilon$ for every $t > \varepsilon$. Here the constant $\kappa(\varepsilon)$ depends only on ε, $c(\varphi_0)$ and $T_{\max}$ in such a way that it decreases to 0 as ε goes to 0. This constant $\kappa(\varepsilon)$ is approximately ε, up to an error term in Demailly's regularizing process. An interesting particular case is when ω represents the first Chern class of an ample holomorphic line bundle over X. In this case the error term is zero and $\kappa(\varepsilon)$ can be taken to be $b\varepsilon$, for some positive constant b depending only on $c(\varphi_0)$ and $T_{\max}$. Siu's theorem [31] guarantees that Y_ε is an analytic subset of X.

Remark. Our regularity result is similar to the classical result on regularization of quasi-plurisubharmonic functions due to Demailly [10, 11]. In both results one can approximate singular ω-plurisubharmonic (psh for short) functions by quasi-psh functions (where we gain positivity by changing the Kähler class) which are smooth in Zariski open subsets. The new feature in our theorem is that along the flow we keep information on the Monge-Ampère measures given by the equation (1.2). The approximants in Demailly's theorem all have analytic singularities. It is expected that our flow in Theorem 1.1 also has analytic singularities for positive times. However, with our techniques in this paper we are not able to confirm this belief (see [14, Question 43] for a conjecture in this direction).

We also show that our result in Theorem 1.1 is optimal in the sense that any maximal Kähler-Ricci flow starting from currents having positive Lelong numbers has positive Lelong numbers in short-time (see Theorem 4.5). Moreover, we show that there is a propagation of singularities of the flow.

The assumption $1/2c(\varphi_0) < T_{\max}$ is necessary to insure that the maximal scalar solution φ_t is well-defined (i.e. not identically $-\infty$). It is automatically satisfied if $T_{\max} = +\infty$ or if the Lelong numbers of the initial potential φ_0 are not very big (thanks to (1.3)). In Example 4.1 we have a simple situation where the assumption $1/2c(T_0) < T_{\max}$ does not hold and the limiting potential is identically $-\infty$. Although the potentials converge to $-\infty$, the currents are still well-defined. In this example the flow is stationary and singular, so there is no regularization effect at all. In our main result, when the condition $1/2c(T_0) < T_{\max}$ is satisfied the singularities of the flow will be decreasing in time and when t reaches $1/2c(T_0)$ they disappear and the flow is smooth for $t > 1/2c(T_0)$ as shown in [22, Theorem 6.1]. We believe that without the condition $1/2c(T_0) < T_{\max}$ the flow (of currents) is still well-defined but there is probably no regularization effect.

Our previous analysis gives rise to the following definition.

Definition 1.2. A function $\varphi : (0, T_{\max}) \times X \to \mathbb{R}$ is called a weak solution of the equation (1.2) starting from φ_0 if the following conditions are satisfied:

(i) For each $t > 0$, the function φ_t is θ_t-psh on X.
(ii) The function $t \mapsto \varphi_t$ is continuous as a map from $[0, T_{\max})$ to $L^1(X)$.
(iii) For each $\varepsilon > 0$ there exists an analytic subset $D_\varepsilon \subset X$ such that the function

$$(t, x) \mapsto \varphi_t(x)$$

is smooth on $[\varepsilon, T_{\max}) \times (X \setminus D_\varepsilon)$ where the equation (1.2) is satisfied in the classical sense. Moreover $D_\varepsilon = \emptyset$ if $\varepsilon > 1/(2c(\varphi_0))$.

A family (ω_t) of closed positive $(1,1)$ currents is called a weak Kähler Ricci flow (1.1) starting from $T_0 = \omega + dd^c\varphi_0$ if $\omega_t = \theta_t + dd^c\varphi_t$ and φ_t is a weak solution of the equation (1.2).

The second goal of this paper is to study the stability and the uniqueness of the weak Kähler-Ricci flow. As mentioned above, the flow constructed in [22] is maximal among weak solutions of the parabolic equation (1.2). However it is not clear whether this maximal solution is the unique weak-Kähler-Ricci flow. When the inital current has zero Lelong numbers, we prove that the uniqueness holds:

Theorem 1.3. *Assume that $c(T_0) = +\infty$. Then the following holds:*

- **Uniqueness:** *Any weak Kähler-Ricci flow starting from T_0 is maximal. In other words the flow is unique.*
- **Stability:** *Assume that $T_{0,j}$ is a sequence of positive closed $(1,1)$-currents converging to T_0 in the L^1 topology. Then the corresponding maximal Kähler Ricci flow $\omega_{t,j}$ converges to ω_t in the following sense: for each $t \in (0, T_{\max})$, $\omega_{t,j}$ converges in $\mathcal{C}^\infty(X)$ to ω_t as $j \to +\infty$.*

The case when T_0 has positive Lelong number is more delicate as the maximal flow has unbounded potential in short time. The usual method using the classical maximum principle and the pluripotential comparison principle, which are the main tools in studying parabolic equations, breaks down. However, we also have a partial result:

Theorem 1.4. *Assume that $c(T_0)$ is finite and $1/(2c(T_0)) < T_{\max}$. Then any weak flow having the same singularities as the maximal flow is also maximal*

Here, we say that two flows ω_t, Ω_t have the same singularity if their potentials u_t, v_t satisfy

$$u_t - C \leq v_t \leq u_t + C, \ \forall t$$

for some constant C.

When the initial current T_0 has some local regularity properties it is natural to ask for stronger convergence of the flow when the time goes to zero. If T_0 has continuous potential it was shown in [34] that the solution φ_t converges uniformly to φ_0. When T_0 has finite energy the convergence is in energy as shown in [22]. In the last section we prove the following:

Theorem 1.5. *Let $\varphi_0 \in \mathrm{PSH}(X,\omega)$ be such that $1/2c(\varphi_0) < T_{\max}$ and φ_t be the maximal solution of (1.2) starting from φ_0.*

- *If φ_0 is continuous in an open subset $U \subset X$ then φ_t converges locally uniformly to φ_0 in U.*
- *If $e^{\gamma\varphi_0} \in \mathcal{C}^\infty(X)$ for some positive constant γ and φ_0 is strictly ω-plurisubharmonic on X, then φ_t converges in $\mathcal{C}^\infty_{\mathrm{loc}}(\Omega)$ to φ_0, where $\Omega = \{\varphi_0 > -\infty\}$.*

The first statement generalizes the uniform convergence of Song-Tian [34]. The second one covers the particular cases of currents having analytic singularities.

Organization of the paper. The organization of the paper is as follows. In section 2 we recall basic pluripotential theory that will be needed later on. In Section 3 we establish various a priori estimates. The proof of Theorem 1.1 will be given in Section 4 while the proof of Theorem 1.3 and Theorem 1.4 will be given in Section 5. In Section 6 we investigate fine convergence of the flow at time $t = 0$ when more regularity on the initial current is assumed, proving Theorem 1.5 (see Theorem 6.1 and Theorem 6.2).

2. Preliminaries

In this section we recall recent results in pluripotential theory that will be needed in the sequel. We also review basic techniques in parabolic complex Monge-Ampère equations.

2.1. **Recap on pluripotential theory.** Let (X,ω) be a compact Kähler manifold of complex dimension n. The set $\mathrm{PSH}(X,\omega)$ contains all ω-plurisubharmonic functions on X (ω-psh for short). A function u is called ω-psh on X if it is upper semicontinuous and integrable on X, and $\omega + dd^c u \geq 0$ in the weak sense of currents.

For a bounded ω-psh function u, it follows from [1] that the Monge-Ampère operator $(\omega + dd^c u)^n$ is well-defined as a non-negative Borel measure on X. If $u \in \mathrm{PSH}(X,\omega)$ is unbounded, we can still define the non-pluripolar Monge-Ampère measure of u. It follows from the local property in the plurifine topology of the Monge-Ampère operator that the following sequence of Borel measures

$$\mathbf{1}_{\{u>-j\}}(\omega + dd^c \max(u,-j))^n$$

is non-decreasing in j. Its limit as $j \to +\infty$ is called the non-pluripolar Monge-Ampère measure of u, denoted by

$$\mathrm{MA}(u) = (\omega + dd^c u)^n := \lim_{j\to+\infty} (\omega + dd^c \max(u, -j))^n.$$

As shown in [21], $\mathrm{MA}(u)$ vanishes on pluripolar sets. Its total mass takes values in $[0, \int_X \omega^n]$. As defined in [21], u belongs to $\mathcal{E}(X,\omega)$ if the total mass of its non-pluripolar Monge-Ampère measure is maximal, i.e. $\int_X \mathrm{MA}(u) = \int_X \omega^n$.

Recall that the Lelong number of $u \in \mathrm{PSH}(X,\omega)$ at some point $x_0 \in X$ is defined as the supremum over all $\gamma \geq 0$ such that

$$u(x) \leq \gamma \log|z| + O(1), \forall x \in V_{x_0},$$

where (V_{x_0}, z) is a local holomorphic coordinate centered at x_0.

As shown in [21], functions in $\mathcal{E}(X,\omega)$ have mild singularities: if $u \in \mathcal{E}(X,\omega)$, it has zero Lelong number at any point in X. This together with Skoda's theorem [33] yield

$$\int_X e^{-Au} dV < +\infty, \ \forall A > 0.$$

Observe that the integrability index $c(u) = +\infty$ if and only if $\nu(u,x) = 0$ for any $x \in X$.

2.2. **The generalized Monge-Ampère capacity.** In [15,16] we introduce and study the so called generalized Monge-Ampère capacity defined as

$$\mathrm{Cap}_\psi(E) := \sup\left\{\int_E \mathrm{MA}(u) \,|\, u \in \mathrm{PSH}(X,\omega), \ \ \psi - 1 \leq u \leq \psi\right\}, \qquad E \subset X$$

where ψ is a possibly singular $\delta\omega$-psh function, $\delta \in (0,1)$, and $\mathrm{MA}(u)$ denotes the non-pluripolar Monge-Ampère measure of u. Observe that if $\psi \equiv C$, then we recover the classical Monge-Ampère capacity introduced in [1], [25], [20]. Such generalized capacities are the key ingredient to study singularities of solutions of degenerate complex Monge-Ampère equations and they enjoy several properties. Among those, it can be proven that when $\psi \in \mathcal{E}(X,\omega)$ then the ψ-capacity characterizes pluripolar sets, namely

$$\mathrm{Cap}^*_\psi(E) = 0 \iff \text{ E is a pluripolar subset of } X,$$

where Cap^*_ψ is the outer ψ-capacity defined for any subset $E \subset X$ by

$$\mathrm{Cap}^*_\psi(E) := \inf\left\{\mathrm{Cap}_\psi(U) \ \middle|\ U \text{ is an open subset of } X, \ E \subset U\right\}.$$

We refer the reader to [16] for more details.

Here below we recall a classical lemma (which goes back to [26]) that it is used in the proof of Lemma 3.3.

Lemma 2.1. *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing, right continuous function such that $g(+\infty) = 0$. Assume that there exists $C > 0$ such that g satisfies*

$$s\, g(t+s) \leq C\, g(t)^2 \qquad \forall s \in [0,1],\ t > 0. \tag{2.1}$$

Let $t_0 > 0$ be such that $g(t_0) \leq (2C)^{-1}$. Then we have

$$g(t) = 0, \quad \forall t \geq 2 + t_0.$$

We refer the reader to [26] (or [17]) for a proof of the above lemma.

2.3. **The comparison principle.** One of the main tools in studying the Kähler-Ricci flow is the comparison principle. The following version is classical:

Proposition 2.2. *Let $u, v : [0, T_{\max}) \times X \to \mathbb{R}$ be smooth functions such that u_t, v_t are strictly θ_t-psh functions and*

$$\dot{u_t} \leq \log \left[\frac{(\theta_t + dd^c u_t)^n}{\omega^n} \right] \quad \text{and} \quad \dot{v_t} \geq \log \left[\frac{(\theta_t + dd^c v_t)^n}{\omega^n} \right].$$

Then $u_t - v_t \leq \sup_X (u_0 - v_0)$. In particular if φ_t, ψ_t are solutions of (1.2) and $\varphi_0 \geq \psi_0$ then $\varphi_t \geq \psi_t$ for all $t > 0$.

We refer to [22, Corollary 2.2] for a proof.

In dealing with the weak Kähler-Ricci flow it is often more convenient to use the following generalized comparison principle, which allows the subsolution to have singularities:

Proposition 2.3. *Assume that $\varphi : [0, T_{\max}) \times X \to \mathbb{R}$ is a smooth solution of the parabolic equation (1.2). Assume that ψ is a weak subsolution of (1.2) with initial data $\psi_0 \leq \varphi_0$ and $c(\psi_0) = +\infty$. Then $\psi_t \leq \varphi_t, \ \forall t \in [0, T_{\max})$.*

Here a weak subsolution with initial data ψ_0 is by definition a smooth function $\psi : (0, T_{\max}) \times X \to \mathbb{R}$ such that

$$\dot{\psi_t} \leq \log \left[\frac{(\theta_t + dd^c \psi_t)^n}{\omega^n} \right], \ \forall t \in (0, T_{\max}), \quad \psi_t \to \psi_0 \ \text{ in } \ L^1(X).$$

In Proposition 2.2, the functions φ, ψ are both smooth in $[0, T_{\max}) \times X$ while in the above statement there is no regularity assumption on ψ. We also stress that the result still holds without the assumption that $c(\varphi_0) = +\infty$ (see Section 4 and Section 5). The proof without this condition is however more involved and we establish it later.

Proof. Fix $\varepsilon > 0$ and note that $\psi - \varphi$ is a smooth function on $[\varepsilon, T] \times X$, for any $T < T_{\max}$. It then follows from the maximum principle that $\psi - \varphi$ attains its maximum on $[\varepsilon, T] \times X$ at a point $(\varepsilon, x_\varepsilon)$, thus

$$\psi_t - \varphi_t \leq \sup_X (\psi_\varepsilon - \varphi_\varepsilon).$$

Now, the function $(\varepsilon, x) \to \varphi_\varepsilon(x)$ is smooth by assumption and in particular continuous. It then follows from Hartogs' lemma and from the fact that ψ_t converges to ψ_0 in L^1 when $t \to 0$ that

$$\sup_X (\psi_\varepsilon - \varphi_\varepsilon) \to \sup_X (\psi_0 - \varphi_0) \leq 0 \quad \text{as } \varepsilon \to 0.$$

It then follows that $\psi_t \leq \varphi_t$ for all $t > 0$. □

Remark 2.4. One can also prove Proposition 2.3 by using the following observation: any weak solution of the parabolic equation (1.2) is upper semi-continuous in two variables (t, x). The latter fact follows from Hartogs' lemma as the following lemma shows.

Lemma 2.5. *Let $T > 0$ be a constant and let $\varphi : [0, T] \to \mathrm{PSH}(X, \omega)$ be a continuous map where the latter space is equipped with the L^1 topology. Then $\varphi(t, x)$ as a function of two variables (t, x) is upper semicontinuous on $[0, T] \times X$.*

We thank Vincent Guedj for the following proof:

Proof. Let (φ_t) be a weak solution of the equation (1.2) and assume by contradiction that there exists a sequence (t_j, x_j) converging to (t, x) such that

$$\limsup_{j\to+\infty} \varphi_{t_j}(x_j) > \varphi_t(x).$$

Since the function $t \mapsto \varphi_t$ is continuous as a map from $[0, T_{\max})$ to $L^1(X)$, it follows that φ_{t_j} converges in $L^1(X)$ to φ_t. We can assume that $\varphi_t(x) > -\infty$ (the other case follows similarly). Take $\varepsilon > 0$ small enough such that

$$\limsup_{j\to+\infty} \varphi_{t_j}(x_j) > \varphi_t(x) + \varepsilon.$$

It follows from the semicontinuity of φ_t that there exists a small closed ball B around x such that

$$\varphi_t(y) < \varphi_t(x) + \varepsilon/2, \ \forall y \in B.$$

Then for j big enough we will have $x_j \in B$. Thus

$$\limsup \varphi_{t_j}(x_j) \leq \limsup \sup_B \varphi_{t_j} \leq \sup_B \varphi_t < \varphi_t(x) - \varepsilon/2,$$

as follows from Hartogs' lemma. This yields a contradiction. □

2.4. **Equisingular approximation.** Let φ be a ω-psh function on X. We are interested in approximating φ by quasi-psh functions which are less singular than φ and such that their singularities are somehow comparable to those of φ. In this paper we will make use of Demailly's equisingular approximation. For each $c > 0$, define

$$D_c := \left\{ z \in X \ \middle| \ \nu(\varphi, z) \geq c \right\}.$$

It follows from a result of Y.T. Siu [31] that each D_c is an analytic subset of X (see [24] or [12] for alternative proofs). If the Lelong number of φ is positive at some point on X it was conjectured by Demailly and Kollár in [13] that $e^{-\alpha\varphi}$ is not integrable at the critical value α. This openness conjecture was recently proved by Berndtsson [2] (see also [30], [19] for alternative proof) while the two dimensional case was proven some years ago by Favre and Jonsson [18].

Demailly's equisingular approximation gives us a sequence of quasi-psh functions (φ_j) having analytic singularities converging to φ. Moreover, the sequence (φ_j) also keeps track on the Lelong numbers of φ.

Theorem 2.6 (Demailly's equisingular approximation)**.** *[10,12] Let φ be a ω-plurisubharmonic function on X. There exists a sequence of quasi-plurisubharmonic functions (φ_m) such that*

(i) *$\varphi \leq \varphi_m$ and φ_m converges pointwise and in $L^1(X)$ to φ as $m \to +\infty$.*
(ii) *φ_m has logarithmic singularities, $e^{2m\varphi_m}$ is smooth on X and vanishes on the set of points x such that $e^{-2m\varphi}$ is not locally integrable near x.*
(iii) *$\omega + dd^c\varphi_m \geq -\varepsilon_m\omega$, where $\varepsilon_m > 0$ converges to 0 as $m \to +\infty$.*
(iv) *$\int_X e^{2m(\varphi_m-\varphi)}dV < +\infty$.*

Lemma 2.7. *Fix $\varphi \in \mathrm{PSH}(X, \omega)$. Then for each $\varepsilon > 0$ there exists $\kappa(\varepsilon) > 0$ and $\psi_\varepsilon \in PSH(X, \omega) \cap \mathcal{C}^\infty(X \setminus D_{\kappa(\varepsilon)})$ such that*

$$\int_X e^{2(\psi_\varepsilon-\varphi)/\varepsilon}dV < +\infty.$$

Here, $\kappa(\varepsilon) > 0$ decreases to 0 as $\varepsilon \searrow 0$.

Proof. Fix $\varepsilon > 0$ and let $c(\varphi)$ be the integrability index of φ, i.e.

$$c(\varphi) := \sup\left\{t > 0 \;\middle|\; e^{-2t\varphi} \in L^1(X)\right\}.$$

We can assume that $c(\varphi) < +\infty$ (otherwise the conclusion is straightforward). Let (φ_m) be Demailly's approximating sequence of φ (see Theorem 2.6). Note that φ_m is smooth outside the analytic subset $D_{1/m} = \{z \in X \;\big|\; \nu(\varphi, z) \geq \frac{1}{m}\}$ and

$$\omega + dd^c\varphi_m \geq -\varepsilon_m\omega,$$

and the constant ε_m decreases to 0 as m goes to $+\infty$. We let $m(\varepsilon) >> 1$ denote the smallest integer m such that

$$m > \frac{2}{\varepsilon(1+\varepsilon_m)} \quad \text{and} \quad \frac{2\varepsilon_m}{\varepsilon(1+\varepsilon_m)} < c(\varphi).$$

Hölder's inequality shows that the function

$$\psi_\varepsilon := \frac{\varphi_m}{1+\varepsilon_m},$$

is smooth outside $D_{\kappa(\varepsilon)}$ and satisfies our requirements with $\kappa(\varepsilon) = m(\varepsilon)^{-1}$. □

Remark 2.8. The constants ε_m come from the gluing process in Demailly's proof [10]. More specifically, the functions φ_m are constructed by gluing the local approximants obtained via the Bergman kernel. In order to glue these local functions a small loss of positivity due to the change of holomorphic coordinates is necessary; this is where the errors ε_m comes from. We refer to [10] for more details. In a particular case when ω represents the first Chern class of an ample holomorphic line bundle over X, one can argue globally and avoid the gluing process (see e.g. [20, Theorem 8.1] for a proof). Hence ε_m can be taken to be zero in this case and we can choose $k(\varepsilon) = \varepsilon$.

3. A priori estimates

Let (X, ω) be a compact Kähler manifold of dimension n. The goal is to run the flow from a singular initial data $\varphi_0 \in \mathrm{PSH}(X, \omega)$ having positive Lelong numbers. The strategy is to pick $(\varphi_{0,j})$ a smooth sequence of strictly ω-psh functions decreasing to φ_0 as $j \to +\infty$ and to consider $(\varphi_{t,j})$ the smooth flow running from $\varphi_{0,j}$. The goal is to establish uniform estimates that will allow us to pass to the limit. Thus, in the sequel we work with $\varphi_{t,j}$ (that we will denote by φ_t for simplicity) but to get uniform estimates we should always take into account the singular behavior of the initial data φ_0.

3.1. **Notations.** From this point on we fix $\varepsilon_0 > 0$ and a function $\psi \in \mathrm{PSH}(X, \omega) \cap \mathcal{C}^\infty(X \setminus D)$ for some analytic subset $D \subset X$. Let E_1, E_2 denote the following quantities

$$E_1 := \int_X \exp\left(\frac{2\psi - 2\varphi_0}{\varepsilon_0}\right) dV < +\infty,$$

$$E_2 := \int_X \exp\left(\frac{-\psi}{T}\right)\omega^n < +\infty,$$

where $1/(2c(\varphi_0)) < T < T_{\max}$. Observe that such a function ψ exists thanks to Lemma 2.7 and E_2 is finite since $T > 1/2c(\varphi_0)$ and by construction ψ is less singular than φ_0. We also underline that D is a Lelong number super-level set related to the function φ_0. Since we

approximate φ_0 from above by a smooth sequence $(\varphi_{0,j})$, the corresponding constants E_1^j are uniform in j and we can pass to the limit when $j \to +\infty$.

In the remaining part of this section φ_0 will be a smooth strictly ω-psh function on X. One should keep in mind that φ_0 in this a priori estimate section plays the role of the approximating sequence $\varphi_{0,j}$.

We set, just for simplicity,

$$\theta_t := \omega + t\eta - t\mathrm{Ric}(\omega).$$

Let φ_t be the solution of the parabolic Monge-Ampère equation:

$$\frac{\partial \varphi_t}{\partial t} = \log\left[\frac{(\theta_t + dd^c\varphi_t)^n}{\omega^n}\right],\ \varphi_t|_{t=0} = \varphi_0,$$

on a maximal interval $[0, T_{\max})$. Recall that $T_{\max}$ is defined by

$$T_{\max} := \sup\{t \geq 0 \,|\, tK_X + t\{\eta\} + \alpha_0 \ \text{ is nef}\}.$$

The existence of the solution on this maximal interval follows from [6], [39], [38]. We fix T and S such that

$$\frac{1}{2c(\varphi_0)} < T < S < T_{\max}.$$

Since our purpose is to prove regularity of the maximal solution of the equation (1.2) on $(0, T]$, for simplicity we assume that

$$\frac{2\omega}{3} \leq \theta_t \leq 2\omega, \forall t \in [0, S].$$

Thus for each $t \in [0, S]$ we have

$$\theta_t = \frac{t\theta_S}{S} + \frac{S-t}{S}\omega \geq \left(1 - \frac{t}{3S}\right)\omega.$$

3.2. **The $\mathcal{C}^0$ estimate.** We recall the following upper bound for φ_t whose proof using the standard parabolic maximum principle can be found, for example, in [22].

Lemma 3.1. *The following estimate holds*

$$\varphi_t \leq \sup_X \varphi_0 + (n\log 2)t, \quad \forall t \in [0, T].$$

As we will see in the next sections, if we start the flow from a current with positive Lelong numbers there is no hope for the flow to be smooth. Thus a "reasonable" $\mathcal{C}^0$ estimate must involve some singular quasi-plurisubharmonic function. In our recent works [15] and [16] we have developed the theory of generalized capacities to deal with these singular estimates. In particular we proved that if $\mathrm{MA}(\varphi) \leq Ce^{-\phi}\omega^n$ for some quasi-plurisubharmoinc function ϕ then for any $a > 0$ small enough (i.e. $a\phi \in \mathrm{PSH}(X, \omega/2)$) there exists a uniform constant $A > 0$ such that

$$\varphi \geq a\phi - A.$$

We use the same ideas to establish the following $\mathcal{C}^0$ estimate for the complex parabolic Monge-Ampère equation (1.2).

Theorem 3.2. *Fix $\varepsilon > \varepsilon_0$. For $t \in [\varepsilon, T]$ the following estimate holds*

$$\varphi_t \geq \left(1 - \frac{t}{2T}\right)\psi - C,$$

where C is a positive constant depending on ε, our fixed parameters and an upper bound for E_1 and E_2.

Proof. Fix $t \in [\varepsilon, T]$. It follows from [22, Proposition 3.1] that

$$\dot{\varphi}_t \leq \frac{\varphi_t - \varphi_0}{t} + n, \tag{3.1}$$

and so

$$(\theta_t + dd^c\varphi_t)^n = e^{\dot{\varphi}_t} \leq e^{n+(\varphi_t-\varphi_0)/t}\omega^n.$$

If φ_0 has zero lelong numbers then, as shown in [22], the estimate (3.1) is enough (and crucial) to get a $\mathcal{C}^0$ estimate for φ_t using Kołodziej's theorem [26]. However, for the general case, namely when φ_0 has positive Lelong numbers, this method breaks down since the bound on $\mathrm{MA}(\varphi_t)$ is no longer integrable. This is why we need to use the generalized capacities that we introduced in [16].

Now set

$$\psi_t = \left(1 - \frac{t}{2T}\right)\psi.$$

Observe that

$$\theta_t \geq \left(1 - \frac{t}{3S}\right)\omega \quad \text{and} \quad dd^c\psi_t + \left(1 - \frac{t}{2T}\right)\omega \geq 0.$$

It then follows that ψ_t is $\delta\theta_t$-plurisubharmonic with $\delta \in (0,1)$ depending on ε_0, T, S, namely

$$\delta = \frac{6TS - 3S\varepsilon_0}{6TS - 2T\varepsilon_0} < 1.$$

At this point we apply Lemma 3.3 below with $A = 1/t$ and $u = \varphi_0/t$ to get the desired estimate. In order to do that, we also need to bound the following quantity

$$\int_X \exp\left(\frac{q(\psi_t - \varphi_0)}{t}\right)\omega^n \tag{3.2}$$

for some $q > 1$. Indeed, by fixing $p > 2$, $2/q$ being its conjugate such that

$$\frac{pq}{\varepsilon} < \frac{2}{\varepsilon_0}$$

and using Hölder's inequality we obtain

$$\int_X e^{\frac{q(\psi_t-\varphi_0)}{t}}\omega^n = \int_X e^{\frac{q(\psi-\varphi_0)}{t}} e^{\frac{-q\psi}{2T}}\omega^n \leq \left(\int_X e^{\frac{pq(\psi-\varphi_0)}{t}}\omega^n\right)^{1/p}\left(\int_X e^{\frac{-\psi}{T}}\omega^n\right)^{q/2}.$$

The right-hand side of the above inequality is finite as follows from the construction of ψ in Lemma 2.7:

$$\int_X e^{\frac{pq(\psi-\varphi_0)}{t}}\omega^n \leq \int_X e^{\frac{2(\psi-\varphi_0)}{\varepsilon_0}}\omega^n < +\infty.$$

Thus we can bound the term in (3.2) in terms of E_1 and E_2. Therefore, our $\mathcal{C}^0$ estimate only depends on the fixed parameters $\varepsilon, \varepsilon_0, T, S$ and on upper bounds for E_1, E_2. The proof is thus complete. □

Lemma 3.3. *Assume that $\varphi \in \mathcal{E}(X,\omega)$ and $(\omega + dd^c\varphi)^n \leq e^{A\varphi - u}\omega^n$, where $A > 0$, u is some measurable function such that $e^{A\psi-u} \in L^q(X)$ for $\psi \in \mathrm{PSH}(X,\delta\omega)$ with $\delta \in (0,1)$ and for some $q > 1$. Then we have the following estimate*

$$\varphi \geq \psi - C,$$

where C is a positive constant depending only on A, δ, ω, p (where p is the conjugate exponent of q) and on an upper bound for $\int_X e^{q(A\psi-u)}\omega^n$.

Proof. The proof follows the arguments in [15]. The goal is to prove that the function

$$H(t) := \left[\mathrm{Cap}_\psi(\varphi < \psi - t)\right]^{1/n}$$

satisfies the inequality (2.1). Since $H(t)$ is right continuous and $H(t)$ goes to zero as t goes to $+\infty$ (see [15, Lemma 2.6]) we can conclude using Lemma 2.1 above.

Fix $s \in [0,1]$, $t > 0$, then by [15, Proposition 2.8] we get

$$s^n \mathrm{Cap}_\psi(\varphi < \psi - t - s) \leq \int_{\{\varphi<\psi-t\}} \mathrm{MA}(\varphi).$$

Using the assumption and Hölder's inequality we then have

$$\begin{aligned}
\int_{\{\varphi<\psi-t\}} \mathrm{MA}(\varphi) &\leq \int_{\{\varphi<\psi-t\}} e^{A\varphi-u}\,\omega^n \\
&= \int_{\{\varphi<\psi-t\}} e^{A(\varphi-\psi)} e^{A\psi-u}\,\omega^n \\
&\leq e^{-At}\mathrm{Vol}(\{\varphi<\psi-t\})^{1/p} \left(\int_{\{\varphi<\psi-t\}} e^{q(A\psi-u)}\,\omega^n\right)^{1/q}.
\end{aligned}$$

It follows from [20] that

$$\mathrm{Vol}(\{\varphi<\psi-t\})^{1/p} \leq C(p,\omega,\delta,n)\left[\mathrm{Cap}_{(1-\delta)\omega}(\varphi<\psi-t)\right]^2.$$

Thanks to [15, Lemma 2.7] we also have

$$\mathrm{Cap}_{(1-\delta)\omega} \leq \mathrm{Cap}_\psi.$$

Taking into account all of the above yields

$$sH(t+s) \leq C^{1/n}e^{-At/n}H(t)^2 \leq BH(t)^2,$$

where $B = C^{1/n}$ and C depends on ω, p, δ, A and an upper bound for $\|e^{A\psi-u}\|_{L^q(X)}$. Now, choose $t_\infty > 0$ such that

$$2C^{2/n}e^{-At_\infty/n}\mathrm{Vol}(X)^{2/n} < 1.$$

We then have $H(t_\infty + 1) < (2B)^{-1}$. It then follows from Lemma 2.1 that

$$\varphi \geq \psi - t_\infty - 3.$$

□

3.3. **The $\mathcal{C}^2$ estimate.** We recall the following well-known result (see [40] for a proof).

Lemma 3.4. *Let α, β be positive $(1,1)$-forms. Then*

$$n\left(\frac{\alpha^n}{\beta^n}\right)^{\frac{1}{n}} \leq \mathrm{tr}_\beta(\alpha) \leq n\left(\frac{\alpha^n}{\beta^n}\right)\left(\mathrm{tr}_\alpha(\beta)\right)^{n-1}.$$

Recall that $(\varphi_t)_{0\leq t<T_{\max}}$ is a smooth solution of the parabolic Monge-Ampère equation (1.2) starting from a smooth strictly ω-psh function φ_0.

We use Δ_t to denote the Laplacian with respect to the Kähler metric ω_t which is defined by

$$\omega_t := \theta_t + dd^c\varphi_t.$$

In order to establish the $\mathcal{C}^2$ estimates we need a lower bound for $\dot{\varphi}_t$. Let us stress that during the proof of our estimates, the constants $C_1, C_2...$ stand for various positive constants which are under control.

Proposition 3.5. *For all $(t,x) \in (\varepsilon, T] \times X$,*

$$\dot{\varphi}_t(x) \geq n\log(t-\varepsilon) + A(\Psi_t - \varphi_t)(x) - C,$$

where A, C are positive constants depending on ε, on an upper bound for E_1, E_2 and on our fixed parameters. Here, Ψ_t is defined by

$$\Psi_t = \left(1 - \frac{t}{2S}\right)\psi.$$

Proof. Consider the following function G defined on $(\varepsilon, T] \times X$ by

$$G(t,x) := \dot{\varphi}_t(x) + A(\varphi_t - \Psi_t) - n\log(t-\varepsilon), \quad \Psi_t := \left(1 - \frac{t}{2S}\right)\psi,$$

where A is a positive constant to be specified later. Observe that G is smooth on $(\varepsilon, T] \times X_0$, where

$$X_0 := \{x \in X \mid \psi(x) > -\infty\}.$$

In the sequel, all computations take place in $(\varepsilon, T] \times X_0$. We compute

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) G = A\dot{\varphi}_t - \frac{n}{t-\varepsilon} + A\frac{\psi}{2S} - nA + A\mathrm{tr}_{\omega_t}(\theta_t + dd^c\Psi_t) + \mathrm{tr}_{\omega_t}(\chi).$$

It follows from

$$\theta_t \geq \left(1 - \frac{t}{3S}\right)\omega \quad \text{and} \quad dd^c\Psi_t + \left(1 - \frac{t}{2S}\right)\omega \geq 0$$

that

$$\theta_t + dd^c\Psi_t \geq \frac{t\omega}{6S} \geq \frac{\varepsilon\omega}{6S}.$$

We choose $A > 0$ big enough so that

$$\frac{A\varepsilon\omega}{6S} + \chi \geq \omega.$$

Thus we have

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) G \geq A\dot{\varphi}_t - \frac{n}{t-\varepsilon} - nA + A\frac{\psi}{2S} + \mathrm{tr}_{\omega_t}(\omega). \tag{3.3}$$

It follows from Lemma 3.4 that

$$\mathrm{tr}_{\omega_t}(\omega) \geq n\left(\frac{\omega^n}{\omega_t^n}\right)^{1/n} = n\exp\left(\frac{-\dot{\varphi}_t}{n}\right). \tag{3.4}$$

It is elementary that for each small constant $b>0$ and each big constant $B>0$ there exists a constant B' depending only on b and B such that

$$be^x - Bx \geq -B', \quad \forall x \in \mathbb{R}.$$

Using this observation, we can find positive constants C_1, C_2 under control such that

$$A\dot{\varphi}_t + n\exp\left(\frac{-\dot{\varphi}_t}{n}\right) \geq \exp\left(\frac{-\dot{\varphi}_t}{n} - C_1\right) - C_2. \tag{3.5}$$

From (3.3), (3.4) and (3.5) we deduce that

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) G \geq \exp\left(\frac{-\dot{\varphi}_t}{n} - C_1\right) + A\frac{\psi}{2S} - \frac{n}{t-\varepsilon} - C_2 - An. \tag{3.6}$$

Notice that the function G is smooth on $(\varepsilon, T]\times X_0$ and it goes to $+\infty$ on the boundary. Thus we see that G attains its minimum on $(\varepsilon, T]\times X_0$ at some point $(t_0, x_0)\in(\varepsilon, T]\times X_0$. It follows from the minimum principle and (3.6) that, at (t_0, x_0), we have

$$\dot{\varphi}_t \geq -n\log\left(C_2 + \frac{n}{t_0-\varepsilon} - A\frac{\psi}{2S} + An\right) - nC_1.$$

We then get

$$\begin{aligned} G(t_0,x_0) &\geq -C_3 - n\log\left[C_5(t_0-\varepsilon) + n - \frac{A(t_0-\varepsilon)\psi(x_0)}{2S}\right] + A(\varphi_{t_0} - \Psi_{t_0}) \\ &\geq -C_4 - n\log\left[C_5(t_0-\varepsilon) + n - \frac{A(t_0-\varepsilon)\psi(x_0)}{2S}\right] - \delta\psi(x_0), \end{aligned}$$

where $\delta = \frac{At_0(S-T)}{2TS} > 0$ and the last inequality follows from Theorem 3.2. We then obtain a uniform lower bound for $G(t_0, x_0)$, and hence the result follows. □

Theorem 3.6. *Fix $\varepsilon > \varepsilon_0$. For all $x\in X$ and $t\in[\varepsilon, T]$ one has*

$$0 \leq (t-\varepsilon)\log \mathrm{tr}_\omega(\omega_t) \leq -A\psi + C,$$

where A, C are positive constants depending on ε, an upper bound for E_1, E_2 and our fixed parameters.

Proof. Consider

$$H := (t-\varepsilon)\log \mathrm{tr}_\omega(\omega_t) - A(\varphi_t - \Psi_t), t\in[\varepsilon, T], x\in X,$$

where A is a positive constant to be specified later and the function Ψ_t is defined by

$$\Psi_t := \left(1 - \frac{t}{2S}\right)\psi.$$

Notice that H is smooth on $[\varepsilon, T]\times X_0$ and H goes to $-\infty$ on the boundary of X_0, where

$$X_0 := \{x\in X \mid \psi(x) > -\infty\}.$$

Set $u = \mathrm{tr}_\omega(\omega_t)$. We compute

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) H = \log u + \frac{t-\varepsilon}{u}\frac{\partial u}{\partial t} - A\dot{\varphi}_t - A\frac{\psi}{2S} - (t-\varepsilon)\Delta_t \log u + A\Delta_t(\varphi_t - \Psi_t).$$

As in the proof of Proposition 3.5 we have

$$\mathrm{tr}_{\omega_t}(\theta_t + dd^c\Psi_t) \geq \frac{\varepsilon}{6S}\mathrm{tr}_{\omega_t}(\omega).$$

Thus we obtain

$$\Delta_t(\varphi_t - \Psi_t) \leq n - \frac{\varepsilon}{6S}\mathrm{tr}_{\omega_t}(\omega).$$

Thanks to [5, Lemma 2.2] (which is an improvement of [32]) we also have

$$\Delta_t \log u \geq \frac{\mathrm{tr}_\omega(dd^c\dot{\varphi}_t)}{\mathrm{tr}_\omega(\omega_t)} - B\,\mathrm{tr}_{\omega_t}(\omega),$$

where B depends only on a lower bound for the holomorphic bisectional curvature of ω. Furthermore,

$$\frac{t-\varepsilon}{u}\frac{\partial u}{\partial t} = \frac{t-\varepsilon}{u}\left(\mathrm{tr}_\omega(\chi) + \mathrm{tr}_\omega(dd^c\dot{\varphi}_t)\right) \leq \frac{t-\varepsilon}{u}\mathrm{tr}_\omega(dd^c\dot{\varphi}_t) + C_1(t-\varepsilon)\,\mathrm{tr}_{\omega_t}(\omega)$$

where in the last inequality we use the fact that $\mathrm{tr}_\omega(\chi)$ is bounded from above together with the trivial inequality $n \leq \mathrm{tr}_{\omega_t}(\omega)\mathrm{tr}_\omega(\omega_t)$. Thus

$$\frac{t-\varepsilon}{u}\frac{\partial u}{\partial t} - (t-\varepsilon)\Delta_t \log u \leq (B + C_1)(t-\varepsilon)\,\mathrm{tr}_{\omega_t}(\omega).$$

It follows from Lemma 3.4 applied to $\alpha = \omega_t$ and $\beta = \omega$ that

$$\mathrm{tr}_\omega(\omega_t) \leq n\, e^{\dot{\varphi}_t}\left(\mathrm{tr}_{\omega_t}(\omega)\right)^{n-1}.$$

Using the inequality $(n-1)\log x < x + C_n$, we obtain

$$\log u \leq \log n + \dot{\varphi}_t + (n-1)\log \mathrm{tr}_{\omega_t}(\omega) \leq \dot{\varphi}_t + \mathrm{tr}_{\omega_t}(\omega) + C_2.$$

Thus we have

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) H \leq -(A-1)\dot{\varphi}_t + \left[(B + C_1)(t-\varepsilon) + 1 - \frac{A\varepsilon}{6S}\right]\mathrm{tr}_{\omega_t}(\omega) - \frac{A\psi}{2S} + C_3.$$

We will choose $A > 0$ big enough such that

$$(B + C_1)T + 1 - \frac{A\varepsilon}{6S} \leq -1.$$

Assume now that H achieves its maximum at some point $(t_0, x_0) \in [\varepsilon, T] \times X_0$. If $t_0 = \varepsilon$ then we are done. Assume that $t_0 > \varepsilon$. From now on our computations take place at this maximum point. By the maximum principle we have

$$0 \leq \left(\frac{\partial}{\partial t} - \Delta_t\right) H \leq C_3 - (A-1)\dot{\varphi}_t - \frac{A\psi}{2S} - \mathrm{tr}_{\omega_t}(\omega),$$

or, equivalently

$$\mathrm{tr}_{\omega_t}(\omega) \leq C_3 - (A-1)\dot{\varphi}_t - \frac{A\psi}{2S}. \tag{3.7}$$

In particular, since $\mathrm{tr}_{\omega_t}(\omega) \geq 0$, we have the following upper bound for $\dot{\varphi}_t$:

$$\dot{\varphi}_t \leq \frac{C_3}{A-1} - \frac{A\psi}{2S(A-1)}. \tag{3.8}$$

It follows from Lemma 3.4 that

$$\mathrm{tr}_{\omega_t}(\omega) \geq n \exp\left(\frac{-\dot{\varphi}_t}{n}\right).$$

Plugging this into (3.7) we obtain

$$\mathrm{tr}_{\omega_t}(\omega) \leq C_4 - \frac{A\psi}{4S}. \tag{3.9}$$

Thus applying again Lemma 3.4 yields

$$\log u \leq \log n + \dot{\varphi}_t + (n-1)\log\left[C_4 - \frac{A\psi}{4S}\right].$$

From this and from (3.8) we obtain

$$H \leq C_5 - A\left[\varphi_t - \left(1 - \frac{t}{2S} - \frac{t-\varepsilon}{2(A-1)S}\right)\psi\right] + (n-1)(t-\varepsilon)\log\left[C_4 - \frac{A\psi}{4S}\right].$$

Now, we increase A a little bit, if necessary, so that

$$\delta := \frac{\varepsilon}{2T} - \frac{\varepsilon}{2S} - \frac{T}{2(A-1)S} > 0.$$

Thus since $\psi \leq 0$ we obtain

$$H \leq C_5 - A\left[\varphi_t - \left(1 - \frac{t}{2T}\right)\psi\right] + A\delta\psi + (n-1)(t-\varepsilon)\log\left[C_4 - \frac{A\psi}{4S}\right].$$

The second term is uniformly bounded from above thanks to Theorem 3.2. Now, using the obvious inequality $-bx + \log x \leq C$ we obtain a uniform upper bound for H at the point (t_0, x_0) and the result follows. □

3.4. **More estimates.** In this subsection we derive some other a priori estimates for the solutions of the parabolic equation (1.2). These estimates will be used later to prove stronger convergence of the flow at time zero when the initial data has some regularity properties (Theorems 6.2 and 6.3). The setting and the notations will be the same as those in the previous subsection.

Proposition 3.7. *Assume that there exist smooth ω-plurisubharmonic functions ϕ_1, ϕ_2 and positive constants $0 < \delta < 1$, $C_1 > 0$ such that*

$$\dot{\varphi}_0 \geq C_1\phi_1 \quad \text{and} \quad \varphi_0 \geq \delta\phi_2.$$

Fix $T_1 \in (0, T_{\max})$ such that $\theta_{T_1} > \frac{\delta+1}{2}\omega$. Then there exists a uniform constant C_2 depending on C_1, T_1 and δ such that

$$\dot{\varphi}_t \geq C_2(\phi_2 + 1) + C_1\phi_1, \quad \forall t \in [0, T_1].$$

Remark 3.8. When dealing with the convergence of the flow at time zero the condition on θ_{T_1} is satisfied for T_1 small enough.

Proof. Consider the following function

$$H(t,x) := \dot{\varphi}_t - C_1\phi_1 + A(\varphi_t - \delta\phi_2),$$

where $A > 0$ is a big constant to be specified later. We use the same arguments as in Proposition 3.5. Since H is smooth on $[0,T_1] \times X$ it attains a minimum at some point (t_0, x_0). If $t_0 = 0$ then we are done since $H(0,.) \geq 0$ by our construction.

Assume now that $t_0 > 0$. We do our computation below at this minimum point (t_0, x_0). By the minimum principle we have

$$0 \geq \left(\frac{\partial}{\partial t} - \Delta_t\right) H \geq -An + A\dot{\varphi}_t + \left(\frac{(1-\delta)A}{2} - C_1 - A_1\right) \mathrm{tr}_{\omega_t}(\omega).$$

Here we use the fact that $\theta_t > \frac{\delta+1}{2}\omega$ and $\chi \geq -A_1\omega$, for some constant A_1 under control.

Now, we choose A big enough such that

$$\frac{(1-\delta)A}{2} - C_1 - A_1 > 1.$$

Using Lemma 3.4 combined with the inequality $e^x \geq Bx - C_B$, we obtain a uniform lower bound for $\dot{\varphi}_{t_0}(x_0)$. The lower bound for $H(t_0, x_0)$ then follows observing that, by [22, Lemma 2.9], $\varphi_t \geq \varphi_0 - c(t)$, and so $\varphi_t - \delta\phi_2 \geq -c(t)$ where $c(t) \to 0$ as $t \to 0$.

□

Proposition 3.9. *Assume that there exist smooth ω-plurisubharmonic function ϕ_1, ϕ_2 and a positive constant $0 < \delta < 1$ such that*

$$\Delta_\omega \varphi_0 \leq e^{-C_1\phi_1} \quad \text{and} \quad \varphi_0 \geq \delta\phi_2.$$

Fix $T_1 \in (0, T_{\max})$ such that $\theta_{T_1} > \frac{\delta+1}{2}\omega$. Then there exist uniform positive constants C_2, C_3 depending on C_1, T_1 and δ such that

$$\mathrm{tr}_\omega \omega_t \leq C_3 e^{-C_1\phi_1 - C_2\phi_2}, \ \forall t \in [0, T_1].$$

Proof. Consider the function

$$H(t,x) = \log \mathrm{tr}_\omega(\omega_t) + C_1\phi_1 - A(\varphi_t - \delta\phi_2),$$

where $A > 0$ is a big constant (under control) to be specified later. This is a smooth function on $[0,T_1] \times X$ and hence it attains its maximum at some point (t_0, x_0). If $t_0 = 0$ we are done since by our construction $H(0,.) \leq \log n$. Assume now that $t_0 > 0$. From now on our computations take place at this maximum point (t_0, x_0). By the maximum principle and by the same arguments as in Theorem 3.6 we eventually get

$$0 \leq \left(\frac{\partial}{\partial t} - \Delta_t\right) H \leq A_1 - A\dot{\varphi}_t - \left(\frac{(1-\delta)A}{2} - B - C_1 - A_2\right) \mathrm{tr}_{\omega_t}(\omega), \tag{3.10}$$

where $-B < 0$ is a lower bound for the holomorphic bisectional curvature of ω, and A_2 is a constant under control. Here we use the fact that $\theta_t \geq \frac{1+\delta}{2}\omega$ for all $t \in [0, T_1]$ and

$$-\mathrm{tr}_{\omega_t}(dd^c\phi_1) \leq \mathrm{tr}_{\omega_t}(\omega), \quad \mathrm{tr}_{\omega_t}(\chi) \geq -A_2 \mathrm{tr}_{\omega_t}(\omega).$$

It follows from Lemma 3.4 that

$$\mathrm{tr}_{\omega_t}\omega \geq n \exp\left(\frac{-\dot{\varphi}_t}{n}\right).$$

Plugging this into (3.10), choosing A big enough and using the inequality $-\varepsilon x + \log x \leq C_\varepsilon$ with $0 < \varepsilon < (An)^{-1}$, we obtain an upper bound for $\mathrm{tr}_{\omega_t}(\omega)$ at (t_0, x_0). We infer that there exists a uniform constant $C > 0$ such that $H(t_0, x_0) \leq C$. Thus $H(t, x) \leq C$ and so

$$\log \mathrm{tr}_\omega(\omega_t) \leq C - C_1\phi_1 - A\delta\phi_2$$

or equivalently

$$\mathrm{tr}_\omega(\omega_t) \leq C' e^{-C_1\phi_1 - A\delta\phi_2}.$$

□

4. Regularity of weak solutions

In this section we study short time regularity of the Kähler-Ricci flow (1.1) starting from a current having positive Lelong numbers. In particular, we give a proof of Theorem 1.1 and show that the result is optimal.

In the whole section we assume that φ_0 is a ω-psh function on X such that

$$\frac{1}{2c(\varphi_0)} < T_{\max}.$$

This condition guarantees that the approximating smooth solutions will not converge uniformly to $-\infty$ (see [22, Lemma 2.9]).

4.1. **Construction of the maximal weak solution.** For convenience, let us recall the construction of the maximal weak solution due to Guedj and Zeriahi [22]. Let $(\varphi_{0,j})$ be a decreasing sequence of smooth strictly ω-plurisubharmonic functions which converges to φ_0. Let $(\varphi_{t,j})$ be the sequence of smooth solutions of the equation (1.2) starting from $\varphi_{0,j}$. When j goes to $+\infty$, for each $t > 0$ fixed, the sequence $(\varphi_{t,j})$ decreases to a well-defined θ_t-psh function φ_t (not identically $-\infty$ thanks to [22, Lemma 2.9]).

Example 4.1. Let us stress that the condition $1/2c(\varphi_0) < T_{\max}$ is necessary to insure that the maximal flow is well-defined as the following example shows.

Consider $X = \mathbb{P}^1$ and $\omega = 4\omega_{FS}$ whose potential is given in homogeneous coordinate by

$$\rho = 2\log(|z|^2 + |w|^2).$$

It is well-known that $\mathrm{Ric}(\omega) = \omega$. Consider the Kähler-Ricci flow

$$\dot{\omega}_t = -\mathrm{Ric}\ \omega_t$$

and observe that in this case we have $T_{\max} = 1$. Consider the following sequence of functions

$$\varphi_{0,j} := 2\log(|z|^2 + |w|^2/j) - \rho.$$

The sequence $(\varphi_{0,j})_j$ decreases to a ω-psh function φ_0 with $c(\varphi_0) = 1/4$. Thus $1/2c(\varphi_0) = 2 > T_{\max}$. A straightforward computation shows that

$$\omega + dd^c\varphi_{0,j} = e^{-\varphi_{0,j} - \log j}\omega.$$

Let us define

$$\varphi_{t,j} := (1-t)\varphi_{0,j} - t\log j - t + (t-1)\log(1-t), \quad t \in (0,1)$$

and note that $\varphi_{t,j}$ solves the following parabolic Monge-Ampère equation

$$\omega - t\mathrm{Ric}(\omega) + dd^c u_t = e^{\dot{u}_t}\omega,$$

with initial data $\varphi_{0,j}$. We can also check that $\varphi_{t,j}$ decreases to $-\infty$ if $t > 0$. Hence the maximal solution constructed in [22] is not well-defined in this case. Note, however, that the flow $\omega_{t,j}$ converges to $\omega_t = (1-t)\omega_0$.

Lemma 4.2. *The limit φ_t obtained as above does not depend on the choice of the approximating sequence $(\varphi_{0,j})$.*

Proof. Let $\psi_{0,j}$ be another decreasing sequence of smooth strictly ω-psh functions which converges to φ_0. Let $(\psi_{t,j})$ be the corresponding smooth solutions of (1.2) starting from $\psi_{0,j}$. Fix $k \in \mathbb{N}$ and $\varepsilon > 0$. By Hartogs' lemma there exists $j_k \in \mathbb{N}$ such that

$$\psi_{0,j} \leq \varphi_{0,k} + \varepsilon, \quad \forall j \geq j_k.$$

It follows from the comparison principle (Proposition 2.2) that

$$\psi_{t,j} \leq \varphi_{t,k} + \varepsilon, \quad \forall j \geq j_k.$$

Now, letting $k \to +\infty$ and then $\varepsilon \to 0$ we get one inequality. The result follows since we can inverse the role of φ_t and ψ_t in the above arguments. □

Remark 4.3. The proof given above also shows that if φ_t and ψ_t are two maximal solutions of (1.2) with $\varphi_0 \leq \psi_0$ then $\varphi_t \leq \psi_t$.

4.2. **Short time singularity.** It was shown by Guedj-Zeriahi in [22, Theorem 6.1] that when $t > 1/2c(\varphi_0)$, $\omega_t := \theta_t + dd^c\varphi_t$ is a Kähler form satisfying the twisted Kähler-Ricci flow equation (1.1). In particular they proved that for $t > 1/2c(\varphi_0)$ φ_t is a smooth and verifies equation (1.2).

In what follows we investigate the behavior of the maximal solution in short time. We show that the maximal solution of the equation (1.2) starting from a current with positive Lelong numbers is unbounded for short time. This relies on a quite simple observation that we will explain in the next lines before entering into the details of the proof of Lemma 4.4.

Assume that $\varphi_0 \in \mathrm{PSH}(X, \omega)$ has positive Lelong number at some $x_0 \in X$, then

$$\varphi_0 \leq \gamma \log|z| + C$$

for some positive constants γ, C. Without loss of generality we can assume that $\varphi_0 = \phi(z)\log|z|^2$, where z is a local coordinate in a neighborhood V around $z = 0$ and ϕ is a cut-off function which vanishes outside V and $\phi \equiv 1$ near $z = 0$. We can also assume that $\varphi_0 \in \mathrm{PSH}(X, \omega/2)$ and $T_{\max} > 1$. Let φ_t be the maximal solution of the equation (1.2) with initial data φ_0. Fix $\varepsilon > 0$ small enough. Consider the following family

$$u_{t,\varepsilon} := [1 - (n+1)t]\,\phi(z)\log(|z|^2 + \varepsilon^2) + Ct.$$

We can choose $C > 0$ large enough (which does not depend on ε) such that $u_{t,\varepsilon}$ is a supersolution of the equation (1.2). Since $u_{0,\varepsilon} \geq \varphi_0$ it follows from the comparison principle that $u_{t,\varepsilon} \geq \varphi_t$, $\forall t \in (0, 1/(n+1))$. By letting $\varepsilon \searrow 0$ we can conclude that φ_t is unbounded and has positive Lelong number for very short time.

In the Lemma below we prove that the solution is singular for all $0 < t < 1/2nc(\varphi_0)$. When $n = 1$ this coupled with [22, Theorem 6.1] gives a complete picture of the regularity properties of the Kähler-Ricci flow.

Lemma 4.4. *Let $\phi \in \mathrm{PSH}(X,\omega)$ be such that $e^{\gamma\phi} \in \mathcal{C}^\infty(X)$ for some positive constant γ. Assume that $\varphi_0 \leq \phi \leq 0$. Then there exists a positive constant C depending on an upper bound for $dd^c e^{\gamma\phi}$ such that*

$$\varphi_t \leq (1-n\gamma t)\phi + Ct, \ \forall t \in [0, 1/n\gamma].$$

Proof. Notice that in the statement of the lemma we impose implicitly that $1/n\gamma < T_{\max}$. For simplicity we also assume that

$$\theta_t \leq \omega, \ \forall t \in [0, 1/n\gamma].$$

In practice γ will be very large and these conditions are irrelevant. By replacing ϕ by the function $\gamma^{-1}\log(e^{\gamma\phi}+\varepsilon)$ and letting $\varepsilon \to 0$ we can assume that ϕ is actually smooth on X. Exploiting the same arguments in the proof of Lemma 4.2 we can also assume that φ_0 is smooth.

Observe that there is a uniform constant $C > 0$ depending only on an upper bound for $dd^c e^{\gamma\phi}$ such that

$$dd^c\phi \leq Ce^{-\gamma\phi}\omega.$$

Set

$$\phi_t := (1-n\gamma t)\phi + t\log(C^n 2^n).$$

Since $\phi \leq 0$ and $dd^c\phi \leq Ce^{-\gamma\phi}\omega$ we see that

$$0 \leq (\omega + dd^c\phi_t) \leq (1+Ce^{-\gamma\phi})\omega \leq 2Ce^{-\gamma\phi}\omega.$$

It then follows that

$$(\omega + dd^c\phi_t)^n \leq 2^n C^n e^{-n\gamma\phi}\omega^n = e^{\dot{\phi}_t}\omega^n.$$

Then ϕ_t is a supersolution of the following parabolic equation

$$(\omega + dd^c u_t)^n = e^{\dot{u}_t}\omega^n,$$

while φ_t is a subsolution since we have assumed that $\theta_t \leq \omega$. The result follows by the classical maximum principle. □

Note that the above result shows that for short time we have propagation of sigularities (as it is highlighted in Theorem 4.5 below) and that our C^0 estimate in Theorem 3.2 is optimal.

Theorem 4.5. *Assume that φ_0 has positive Lelong numbers. Then we have the following relation between the Lelong number superlevel sets of φ_t and φ_0:*

$$D_c(\varphi_0) \subset D_{c(t)}(\varphi_t), \quad c(t) = c - 2nt.$$

In particular, the maximal solution φ_t has positive Lelong numbers for any $t < 1/2n c(\varphi_0)$.

Proof. Let $c > 0$ and $x_0 \in D_c(\varphi_0)$. It follows from the very definition of the Lelong number that we can find a function $\phi \in \mathrm{PSH}(X, A\omega)$ for some positive constant A such that

$$e^{2\phi/c} \in \mathcal{C}^\infty(X) \ \text{ and } \ \phi \geq \varphi_0.$$

In fact, the function ϕ can be written locally as

$$\phi(z) = c\log|z|,$$

where z is a local coordinate centered at x_0. Applying Lemma 4.4 we get

$$\varphi_t \leq (1-2nt/c)\phi + Ct.$$

It follows that $\nu(\varphi_t, x_0) \geq c - 2nt$, hence $x_0 \in D_{c(t)}(\varphi_t)$.

In particular, if $t < 1/2nc(\varphi_0)$ then for a constant c such that $2nt < c < 1/c(\varphi_0)$ the set $D_c(\varphi_0)$ is non-empty as follows from Skoda's integrability theorem since $e^{-\frac{2}{c}\varphi_0} \notin L^1$. It follows from the first part of the theorem that $D_{c(t)}(\varphi_t)$ is non-empty for t small, where $c(t) = c - 2nt$. □

Our previous analysis describes various singular situations. It also sheds some light on the behavior of the maximal solution in short time. In fact, it strongly suggests that the maximal solution is smooth outside analytic subsets of X. We next prove Theorem 1.1, confirming this suggestion by employing our previous a priori estimates.

4.3. **Proof of Theorem 1.1.** Let $(\varphi_{0,j})$ be a decreasing sequence of smooth strictly ω-psh functions converging to φ_0. This is always possible thanks to Demailly's regularization (see [11], or [3]). Let $(\varphi_{t,j})$ be smooth solutions of the parabolic equation (1.2) starting from $\varphi_{0,j}$. It was proven by Guedj-Zeriahi in [22] that, as $j \to +\infty$, the family $\varphi_{t,j}$ decreases to φ_t, which is a θ_t-psh function on X. They also proved that for each $t > (2c(\varphi_0))^{-1}$, the function φ_t is smooth on X and satisfies (1.2) pointwise on X. Moreover, $\varphi_t \to \varphi_0$ in capacity as $t \to 0$.

We investigate the regularity of φ_t for small t. Let us fix $\varepsilon_0 > 0$ and $\varepsilon > \varepsilon_0$. Let T, S be as in Section 3. Thanks to Lemma 2.7 we fix $\psi \in \mathrm{PSH}(X, \omega) \cap \mathcal{C}^\infty_{\mathrm{loc}}(X \setminus D_\kappa)$ (the constant $\kappa = \kappa(\varepsilon_0)$ is defined by Lemma 2.7) such that

$$\int_X \exp\left(\frac{2\psi - 2\varphi_0}{\varepsilon_0}\right) dV < +\infty. \tag{4.1}$$

We can always assume that $\psi \leq 0$. Recall that by the choice of T and since ψ is less singular than φ_0, we always have

$$\int_X \exp\left(\frac{-\psi}{T}\right) \omega^n < +\infty. \tag{4.2}$$

We will prove that φ_t is smooth on $X \setminus D_\kappa$ for each $t > \varepsilon$. Fix a compact subset $K \Subset X \setminus D_\kappa$. We apply Lemma 3.1 and Theorem 3.2 for the flow $\varphi_{t,j}$ (and use the function ψ above) and note that, thanks to (4.1) and (4.2) the constants in these theorems do not depend on j. Thus

$$\sup_{t \in [\varepsilon, T]} \sup_K |\varphi_{t,j}| \leq C(\varepsilon, K).$$

It follows from Theorem 3.6 that we also have a uniform bound on $\Delta\varphi_{t,j}$:

$$\sup_{t \in [\varepsilon, T]} \sup_K |\Delta_\omega(\varphi_{t,j})| \leq C(\varepsilon, K).$$

Now, from Proposition 3.5 and from the fact that the functions $\varphi_{t,j}$ satisfy the equation (1.2) we obtain a locally uniform bound on $\dot{\varphi}_{t,j}$:

$$\sup_{t \in [\varepsilon, T]} \sup_K \left| \frac{d\varphi_{t,j}}{dt} \right| \leq C(\varepsilon, K).$$

Then using the complex parabolic Evans-Krylov theory together with Schauder's estimate (see [4, Theorem 4.1.4]), we obtain higher order estimates on $[\varepsilon, T] \times K$ just as in [22]. This justifies the smoothness of φ_t on $X \setminus D_\kappa$.

Remark 4.6. In the proof of Theorem 1.1 we have assumed (for the sake of the presentation) that we can choose T, S such that

$$0 < \frac{1}{2c(\varphi_0)} < T < S < T_{\max}, \quad \text{and} \quad \theta_S \geq \frac{2\omega}{3}.$$

In general, the number $2/3$ above should be replaced by some positive constant in $(0,1)$ depending on the number $2c(\varphi_0)T_{\max} > 1$. The constants $\kappa(\varepsilon)$ should also be changed to $b\kappa(\varepsilon)$ for some positive constant b.

5. Uniqueness and stability

5.1. **Currents with zero Lelong numbers.** In this subsection we study the uniqueness and stability of the Kähler-Ricci flow starting from currents having zero Lelong numbers. The estimate (5.1) below was proven in [22] for the maximal flow. The ideas in [22] are the prove such estimates (in a uniform way) for the smooth approximating flows. We prove in the lemma below that, in the case of zero Lelong numbers, the same estimate holds for any weak solution. The idea of our proof is quite simple: we follow the same strategy of Guedj-Zeriahi, but instead of dealing with the flow φ_t itself we deal with the flow $\varphi_{t+\varepsilon}$ (for $\varepsilon > 0$) and then we let $\varepsilon \to 0$. The key point is that the new flow is smooth *up to zero* and the classical comparison principle can be applied.

Lemma 5.1. *Assume that (φ_t) is a weak solution of the parabolic equation (1.2) with initial data φ_0. Assume also that $c(\varphi_0) = +\infty$. Let β be a positive constant such that $2\beta T_{\max} > 1$. Then*

$$\varphi_t \geq (1 - 2\beta t)\varphi_0 - C(t), \quad 0 < t < (2\beta)^{-1}. \tag{5.1}$$

where $C(t)$ depends only on t and $C(t) \searrow 0$ as $t \searrow 0$.

Proof. Fix a positive constant α such that $2\beta - \alpha > 1/T_{\max}$. Take a small constant $\varepsilon > 0$ such that

$$(2\beta - \alpha)\omega + (1 + 2\beta\varepsilon)\chi > 0.$$

Set

$$u_t := \varphi_{t+\varepsilon}, \ t \in [0, T_{\max} - \varepsilon).$$

Since $c(\varphi_0) = +\infty$, by definition u is smooth on $X \times [0,T]$. Applying [22, Lemma 2.9] we see that for any $x \in X$ and any $t \in [0, (2\beta)^{-1} - \varepsilon)$,

$$(1 - 2\beta t)u_0(x) + \alpha t v_\varepsilon(x) + n(t\log t - t) \leq u_t(x), \tag{5.2}$$

where v_ε is the unique continuous ω-psh function satisfying

$$\alpha^n(\omega + dd^c v_\varepsilon)^n = e^{\alpha v_\varepsilon - 2\beta\varphi_\varepsilon}\omega^n.$$

Since φ_0 has zero Lelong numbers everywhere and since φ_ε converges to φ_0 in L^1 as $\varepsilon \searrow 0$ it follows from [41, Theorem 3.1] that

$$\int_X e^{-4\beta\varphi_\varepsilon}\omega^n \leq C,$$

for a uniform constant $C > 0$. Therefore, it follows from Kołodziej's $\mathcal{C}^0$ estimate (see [26]) that v_ε is uniformly bounded. Thus from (5.2) we get

$$(1 - 2\beta t)\varphi_\varepsilon(x) - C(t) \leq \varphi_{t+\varepsilon}(x).$$

Letting $\varepsilon \to 0$ we obtain the result. □

From Lemma 5.1 we immediately get the following convergence result.

Corollary 5.2. *Assume that $c(\varphi_0) = +\infty$ and φ_t is a weak solution of the parabolic equation (1.2). Then φ_t converges to φ_0 in capacity as t goes to zero.*

We are now in the position to prove the following comparison principle.

Theorem 5.3. *Let $\varphi_0, \psi_0 \in \mathrm{PSH}(X,\omega)$ be such that $c(\varphi_0) = +\infty$. Let $(\varphi_t), (\psi_t)$ be weak solutions of the equation (1.2) starting from φ_0 and ψ_0 respectively. Then*

$$\varphi_0 \leq \psi_0 \Longrightarrow \varphi_t \leq \psi_t, \forall t > 0.$$

In particular, if $c(\varphi_0) = +\infty$ then there is a unique weak solution of the equation (1.2) with initial data φ_0.

Proof. We divide the proof into two steps.
Step 1. We assume that $\chi \geq 0$. Fix $\varepsilon > 0$ and consider the following function

$$v_t := \psi_{t+\varepsilon} + C(\varepsilon),$$

where $C(\varepsilon)$ is the constant defined in Lemma 5.1. Accordingly we have that $v_0 \geq \psi_0$. We compute

$$\dot{v}_t = \dot{\psi}_{t+\varepsilon} = \log \left[\frac{(\omega + (t+\varepsilon)\chi + dd^c v_t)^n}{\omega^n} \right].$$

Since $\chi \geq 0$ and φ_t is a weak solution of (1.2) it follows that

$$\dot{\varphi}_t = \log \left[\frac{(\omega + t\chi + dd^c \varphi_t)^n}{\omega^n} \right] \leq \log \left[\frac{(\omega + (t+\varepsilon)\chi + dd^c \varphi_t)^n}{\omega^n} \right].$$

Our computations above show that v_t is a smooth solution of the following parabolic equation

$$\dot{v}_t = \log \left[\frac{(\omega + (t+\varepsilon)\chi + dd^c v_t)^n}{\omega^n} \right],$$

while φ_t is a weak subsolution. Since $\varphi_0 \leq \psi_0 \leq v_0$ it follows from the generalized comparison principle (Proposition 2.3) that $\varphi_t \leq v_t$. Since $C(\varepsilon) \searrow 0$ as $\varepsilon \searrow 0$ the result follows.

Step 2. We remove the non-negative assumption on χ. The idea is to make a change of variable to get rid of the sign of χ. Without loss of generality we can assume that $T_{\max} > 1$ and hence $\chi > -\omega$. We make the following change of variable

$$u_t := e^t \varphi_{1-e^{-t}}.$$

Then u_t satisfies the following equation

$$\dot{u}_t = u_t - nt + \log \left[\frac{(\omega + (e^t - 1)\chi' + dd^c u_t)^n}{\omega^n} \right],$$

where $\chi' = \omega + \chi > 0$. Thus this change of variable gives us another equation with the twisted form $\chi' > 0$. We then can apply our techniques in the first step. For the reader's convenience we explain the proof in full details.

Fix $\varepsilon > 0$ small enough and consider the following families:

$$u_t := e^t \varphi_{1-e^{-t}} \ , \ v_t := e^{t+\varepsilon} \psi_{1-e^{-t-\varepsilon}} + n\varepsilon(e^t - 1) + e^{t+\varepsilon} C(1 - e^{-\varepsilon}),$$

where $C(s)$ is the constant defined in Lemma 5.1. Observe that v is smooth on $[0,t_0]\times X$, where t_0 is a fixed positive constant such that $1-e^{-\varepsilon-t_0}<T_{\max}$. We can assume that φ_t and ψ_t are negative for small t. Then it follows from Lemma 5.1 (choose $2\beta=1$) that

$$v_0\geq\varphi_0=u_0.$$

Since $\chi':=\chi+\omega\geq 0$ it follows that

$$\dot{u_t}\leq u_t-nt+\log\left[\frac{(\omega+(e^{t+\varepsilon}-1)\chi'+dd^cu_t)^n}{\omega^n}\right],$$

and

$$\dot{v_t}=v_t-nt+\log\left[\frac{(\omega+(e^{t+\varepsilon}-1)\chi'+dd^cv_t)^n}{\omega^n}\right].$$

It thus follows from the comparison principle (Proposition 2.3) that $u_t\leq v_t,\forall t\in[0,t_0]$. Letting $\varepsilon\searrow 0$ we obtain $\varphi_t\leq\psi_t$ for any $t\in(0,t_0)$. Then we can conclude that $\varphi_t\leq\psi_t$ for any $t\in(0,T_{\max})$. □

The following stability result follows from the uniqueness of the flow.

Theorem 5.4. *Let φ_0 be a ω-psh function and $\varphi_{0,j}$ a sequence of ω-psh functions such that $\varphi_{0,j}\to\varphi_0$ in L^1. Assume also that $c(\varphi_0)=+\infty$. Denote by $\varphi_{t,j}$ and φ_t the maximal solutions of (1.2) with initial data $\varphi_{0,j}$ and φ_0 respectively. Then for each $t>0$,*

$$\varphi_{t,j}\to\varphi_t\quad \textit{in } \mathcal{C}^\infty(X) \textit{ as }\quad j\to+\infty.$$

Proof. Assume first that the sequence $(\varphi_{0,j})$ is decreasing. Then we can find a decreasing sequence $(\psi_{0,j})$ of smooth strictly ω-psh functions which converges to φ_0 and satisfies

$$\varphi_0\leq\varphi_{0,j}\leq\psi_{0,j},\quad\forall j\in\mathbb{N}.$$

Let $(\varphi_{t,j})$ and $(\psi_{t,j})$ be the corresponding weak solutions of the equation (1.2) with initial data $\varphi_{0,j}$ and $\psi_{0,j}$ respectively. It follows from the comparison principle (Theorem 5.3) that

$$\varphi_t\leq\varphi_{t,j}\leq\psi_{t,j},\quad\forall j\in\mathbb{N}.$$

By Lemma 4.2 we know that $\psi_{t,j}\searrow\varphi_t$, hence $\varphi_{t,j}\searrow\varphi_t$. Moreover it follows from [22] that for each $\varepsilon>0$, $T<T_{\max}$, $k\in\mathbb{N}$ we have

$$\|\varphi_j\|_{\mathcal{C}^k([\varepsilon,T]\times X)}\leq C(\varepsilon,T,k),$$

for $j>j(\varepsilon)$ large enough. Then using Arzelà-Ascoli theorem, we know that, for each $t>0$, $\varphi_{t,j}$ converges to φ_t in $\mathcal{C}^\infty(X)$ as j goes to $+\infty$. Hence the result follows.

Now, let us prove the general statement. We can assume that φ_0 is negative. Recall that, for each j fixed, $\varphi_{t,j}$ converges in $L^1(X)$ to $\varphi_{0,j}$ as $t\searrow 0$. The arguments in the previous step also apply in this case yielding that for each $t>0$, $\varphi_{t,j}$ converges to some ψ_t in $\mathcal{C}^\infty(X)$ as j goes to $+\infty$. Note that, by construction, ψ_t is such that

$$\frac{\partial\psi_t}{\partial t}=\log\left[\frac{(\theta_t+dd^c\psi_t)^n}{\omega^n}\right],\ \forall t>0.$$

It follows from Lemma 5.1 that

$$\varphi_{t,j}\geq(1-2\beta t)\varphi_{0,j}-C(t),\forall t\in(0,(2\beta)^{-1}).$$

Letting $j \to +\infty$, then $t \searrow 0$ we obtain $\liminf_{t\to 0} \psi_t \geq \varphi_0$. To see the reverse inequality we set

$$u_{0,j} := \left(\sup_{k\geq j} \varphi_{0,k}\right)^* \searrow \varphi_0.$$

Let $u_{t,j}$ be the maximal solution of (1.2) with initial data $u_{0,j}$. By the comparison principle we have

$$u_{t,j} \geq \varphi_{t,j}.$$

It follows from the previous step that $(u_{t,j})$ decreases to φ_t, hence $\psi_t \leq \varphi_t$. Since φ_t converges in $L^1(X)$ to φ_0, we infer that ψ_t converges in $L^1(X)$ to φ_0. From the comparison principle (Theorem 5.3) we obtain $\varphi_t \equiv \psi_t$. □

5.2. **Currents with positive Lelong numbers.** In this section we try and prove the uniqueness result for the general case: initial currents with positive Lelong numbers. We fix the following notations. Assume that φ_0 is a ω-psh function on X, η is a smooth closed $(1,1)$-form on X. Assume also that

$$\frac{1}{2c(\varphi_0)} < T_{\max} := \sup\left\{t > 0 \;\middle|\; \{\omega\} + t\eta - tc_1(X) > 0\right\}.$$

Fix two positive constants α, β such that

$$\frac{1}{2c(\varphi_0)} < \frac{1}{2\beta} < \frac{1}{2\beta - \alpha} < T_{\max}.$$

Recall that the twisted Kähler-Ricci flow reads in the level of potentials as:

$$\dot{\varphi}_t = \log\left[\frac{(\theta_t + dd^c\varphi_t)^n}{\omega^n}\right], \tag{5.3}$$

where $\theta_t := \omega + t\eta - t\mathrm{Ric}(\omega)$. We recall the definition of weak solutions of the parabolic equation (5.3).

Definition 5.5. A function $\varphi : (0, T_{\max}) \times X \to \mathbb{R}$ is called a weak solution of the equation (5.3) starting from φ_0 if the following conditions are satisfied:

(i) For each $t > 0$, the function φ_t is θ_t-psh on X. Moreover the function $t \mapsto \varphi_t$ is continuous as a map from $[0, T_{\max})$ to $L^1(X)$.

(ii) For each $\varepsilon > 0$, there exists an analytic subset $D_\varepsilon \subset X$ such that the function

$$(t, x) \mapsto \varphi_t(x)$$

is smooth in $[\varepsilon, T_{\max}) \times (X \setminus D_\varepsilon)$ and satisfies the equation (5.3) in the classical sense. Moreover, D_ε is empty if $\varepsilon > 1/(2c(\varphi_0))$.

We first show that the maximal solution constructed in [22] is indeed "maximal", meaning that it dominates any weak solution (in the sense of Definition 5.5).

Lemma 5.6. *Let φ_t be the maximal solution of (5.3) starting from φ_0 and let ψ_t be a weak solution with initial data ψ_0. Then*

$$\psi_0 \leq \varphi_0 \Longrightarrow \psi_t \leq \varphi_t, \quad \forall t \in (0, T_{\max}).$$

Proof. Let $\varphi_{0,j}$ be the smooth sequence decreasing to φ_0 and $\varphi_{t,j}$ be the smooth sequence of approximants running from $\varphi_{0,j}$ and decreasing pointwise to φ_t. It thus suffices to show that $\psi_t \leq \varphi_{t,j}$ for any fixed j. To simplify the notation, in the remaining part of the proof we drop the index j. Fix $0 < T < T_{\max}$ and $\varepsilon > \delta > 0$. Denote by Y_ε the analytic set outside of which the weak solution ψ_t is smooth for $t \geq \varepsilon$. Consider

$$H(t,x) := \psi_{t+\varepsilon} - \varphi_{t+\varepsilon+\delta} + \delta\phi \qquad t \in [0, T-\varepsilon],\ x \in X_\varepsilon$$

where $\phi \leq 0$ is a ω-psh function that is smooth on $X_\varepsilon := X \setminus Y_\varepsilon$ and goes to $-\infty$ on Y_ε. Observe that by construction H is upper semicontinuos on $[0, T-\varepsilon] \times X_\varepsilon$ and $\phi = -\infty$ on Y_ε. This implies that H achieves its maximum at some point $(t_0, x_0) \in [0, T-\varepsilon] \times X_\varepsilon$. We claim that $t_0 = 0$. Indeed, arguing by contradiction we get that at (t_0, x_0)

$$\begin{aligned} 0 \leq \frac{\partial}{\partial t} H &= \dot{\psi}_{t+\varepsilon} - \dot{\varphi}_{t+\varepsilon+\delta} \\ &= \log\left[\frac{(\theta_t + \varepsilon\chi + dd^c\psi_{t+\varepsilon})^n}{(\theta_t + (\varepsilon+\delta)\chi + dd^c\varphi_{t+\varepsilon+\delta})^n}\right] \\ &\leq \log\left[\frac{(\theta_t + dd^c\varphi_{t+\varepsilon+\delta} + \varepsilon\chi - \delta dd^c\phi)^n}{(\theta_t + (\varepsilon+\delta)\chi + dd^c\varphi_{t+\varepsilon+\delta})^n}\right] < 0 \end{aligned}$$

where in the last inequality we assume $\chi \geq \omega$. Observe that this last assumption costs no generality since otherwise we can perform a change of variables and argue as in Section 5.1. Thus we have, by letting $\delta \searrow 0$,

$$\psi_{t+\varepsilon} - \varphi_{t+\varepsilon} \leq H(0, x_0) \leq \sup_X(\psi_\varepsilon - \varphi_\varepsilon).$$

Furthermore, since $(t,x) \to \varphi_t(x)$ is smooth, and in particular continuous, it follows from Hartogs' lemma that

$$\sup_X(\psi_\varepsilon - \varphi_\varepsilon) \to \sup_X(\psi_0 - \varphi_0) \leq 0.$$

The conclusion follows when we let ε go to zero. □

In the remaining part of this subsection we prove a "partial" uniqueness result in the case of positive Lelong numbers.

Lemma 5.7. *Assume that φ_0 is as above. Let (φ_t) be a weak solution of the equation (5.3) starting from φ_0 and assume that φ_t and the maximal solution have the same singularities. Then*

$$(1 - 2\beta t)\varphi_0 - C(t) \leq \varphi_t, \quad \forall 0 < t < (2\beta)^{-1},$$

where $C(t)$ is a uniform constant depending only on t, and converges to zero as t goes to zero.

Proof. Fix two small positive constants ε, δ such that

$$(2\beta - \alpha - \delta)\omega + (1 + 2\beta\varepsilon)\chi > 0. \tag{5.4}$$

Consider the following function

$$u_t = (1 - 2\beta t)\varphi_\varepsilon + \alpha t v_\varepsilon + n(t\log t - t),$$

where v_ε is the unique bounded function in $\mathrm{PSH}(X,\omega)$ such that

$$\alpha^n(\omega + dd^c v_\varepsilon)^n = e^{\alpha v_\varepsilon - 2\beta\varphi_\varepsilon}\omega^n. \tag{5.5}$$

Observe also that v_ε is uniformly bounded by a constant independent of ε. This fact follows since we have chosen β such that $\beta < c(\varphi_0)$.

Let Y_ε be the analytic subset outside of which the maximal solution is smooth for $t \geq \varepsilon$. Since φ_t has the same singularities as the maximal solution, it follows that

$$\sup_{t\in[0,(2\beta)^{-1}]} \sup_{X_\varepsilon}(u_t - \varphi_{t+\varepsilon}) < +\infty.$$

Here $X_\varepsilon := X \setminus Y_\varepsilon$. Let ϕ be a negative ω-plurisubharmonic function on X which is smooth on X_ε and goes to $-\infty$ on Y_ε. By our construction the following function

$$H(t,x) = u_t(x) - \varphi_{t+\varepsilon}(x) + \delta t\phi, \quad t \in [0,(2\beta)^{-1}],\ x \in X_\varepsilon,$$

is smooth on $[0,(2\beta)^{-1}] \times X_\varepsilon$. Moreover $H(t,x)$ converges to $-\infty$ uniformly in t as x approaches Y_ε. It then follows that H attains a maximum at some $(t_0,x_0) \in [0,(2\beta)^{-1}] \times X_\varepsilon$. By the choice of δ and ε and by the construction of u_t it is easy to see that $t_0 = 0$. If this was not the case then by the maximum principle we would have, at this point,

$$0 \leq \frac{\partial}{\partial t}H \leq \alpha v_\varepsilon - 2\beta\varphi_\varepsilon + n\log t - \log\left[\frac{(\omega + (t+\varepsilon)\chi + dd^c\varphi_{t+\varepsilon})^n}{\omega^n}\right]. \tag{5.6}$$

On the other hand, by (5.4) and by the construction of u_t we have

$$\alpha t(\omega + dd^c v_\varepsilon) < \omega + (t+\varepsilon)\chi + \delta t dd^c\phi + dd^c u_t.$$

At the point (t_0,x_0) where H attains its maximum we have

$$\alpha t(\omega + dd^c v_\varepsilon) < \omega + (t+\varepsilon)\chi + dd^c\varphi_{t+\varepsilon}. \tag{5.7}$$

Now, from (5.5), (5.6) and (5.7) we get a contradiction. Letting δ go to 0 we obtain $u_t \leq \varphi_{t+\varepsilon}$. We finally let $\varepsilon \to 0$ to conclude. □

We will prove the following uniqueness result.

Theorem 5.8. *Assume that φ_0 is as above. Let (φ_t) be a weak solution of the equation (5.3) starting from φ_0 and assume that φ_t has the same singularities as the maximal solution. Then φ_t is also maximal.*

Proof. It costs no generality to assume that $\varphi_t \leq 0$ for small t. We first treat the case when $\chi > 0$. Let (ψ_t) be the maximal solution of the equation (5.3) starting from φ_0. We prove that $\varphi_t \equiv \psi_t$. Since ψ_t is maximal, thanks to Lemma 5.6 it suffices to prove that $\psi_t \leq \varphi_t$.

Fix $T \in (0, T_{\max})$, $\varepsilon > 0$ and consider the following function

$$u_t := \varphi_{t+\varepsilon} + C(\varepsilon),\ t \in [0, T-\varepsilon],\ x \in X_\varepsilon,$$

where $C(\varepsilon)$ and X_ε are defined as in Lemma 5.7. We remark that by this construction and by Lemma 5.7 it holds that

$$u_0 = \varphi_\varepsilon + C(\varepsilon) \geq \varphi_0.$$

Fix a small positive constant $\delta < \varepsilon$ such that $\delta\omega < \varepsilon\chi$. Consider now the following function

$$G(t,x) := \psi_t(x) - u_t(x) + \delta\phi, \quad t \in [0, T-\varepsilon],\ x \in X_\varepsilon,$$

where ϕ is also defined in Lemma 5.7. Since (φ_t) and (ψ_t) have the same singularity, meaning that the difference is uniformly bounded in t, one can apply the arguments in Lemma 5.7 to see that G attains a maximum at some point (t_0,x_0). Again, t_0 is forced to be 0 by the

maximum principle. Indeed, if t_0 were not 0, by the maximum principle we would have, at (t_0, x_0),

$$0 \leq \frac{\partial}{\partial t} G = \log \left[\frac{(\theta_t + dd^c \psi_t)^n}{(\theta_t + \varepsilon\chi + dd^c u_t)^n} \right] < \log \left[\frac{(\theta_t + \varepsilon\chi + \delta dd^c \phi + dd^c \psi_t)^n}{(\theta_t + \varepsilon\chi + dd^c u_t)^n} \right] \leq 0,$$

which is impossible.

Change of variables: In the previous step we have assumed that $\chi \geq 0$. To remove this assumption we perform a change of variables exactly as in the case of zero Lelong numbers. It is harmless to assume that $\chi' := \omega + \chi > 0$. Now, let (φ_t) be a weak solution and consider

$$u_t := e^t \varphi_{1-e^{-t}}, t \in [0, +\infty).$$

We compute

$$\dot{u}_t = u_t - nt + \log \left[\frac{\omega + (e^t - 1)\chi' + dd^c u_t)^n}{\omega^n} \right]. \tag{5.8}$$

Thus, any weak (maximal) solution of the equation (5.3) is in one-to-one correspondence with a weak (maximal) solution of the equation (5.8) by a change of variables. Now, we combine the previous step with the proof of Theorem 5.3 to conclude. □

5.3. **Proof of Theorem 1.3 and Theorem 1.4.** Theorem 1.3 follows from Theorem 5.3 and Theorem 5.4. Theorem 1.4 follows from Theorem 5.8.

6. Convergence at time zero

In this section we investigate the convergence at zero of weak solutions of the Kähler-Ricci flow. Let φ_0 be a ω-plurisubharmonic function on X and assume that

$$\frac{1}{2c(\varphi_0)} < T_{\max}.$$

Let (φ_t) be the maximal solution of the equation (5.3) starting from φ_0. If φ_0 is continuous Song-Tian [34] proved that φ_t converges uniformly to φ_0. When φ_0 has finite energy, it was shown in [22] that φ_t converges in energy (the convergence in capacity always holds). If φ_0 is continuous in some open subset $U \subset X$ we prove in the following that the convergence is locally uniform in U. This generalizes the uniform convergence of [34] when $U = X$.

Theorem 6.1. *Assume that φ_0 is continuous in an open subset U of X. Then φ_t converges in $L^\infty_{\rm loc}(U)$ to φ_0.*

Proof. It costs no generality to assume that $\varphi_t \leq 0$ for small $t > 0$. It follows from [22, Lemma 2.9] that there exist a uniform constant C and a small time t_0 such that

$$\varphi_t \geq \varphi_s + n(t-s)\left[\log(t-s) - C\right], \ \forall 0 \leq s \leq t \leq t_0.$$

Set $\psi_t := \varphi_t + nCt - nt \log t$. Using the convexity of the function $x \log x$ we obtain

$$\psi_t \geq \psi_s - nt \log 2.$$

Now, set $u_t = \psi_t + nt \log 4$ and observe that

$$u_t \geq u_{t/2}.$$

Thus for each $t > 0$ the sequence $u_{t2^{-j}}$ decreasingly converges to $u_0 = \varphi_0$ as j goes to $+\infty$. By Dini's theorem the convergence is locally uniform in U. □

When the initial current has analytic singularities we obtain the following expected convergence in its regular locus:

Theorem 6.2. *Assume that $e^{\gamma\varphi_0}$ is smooth on X for some positive constant γ and φ_0 is strictly ω-psh in X. Then the maximal solution φ_t of the equation (5.3) converges in $\mathcal{C}^\infty_{\rm loc}(\Omega)$ to φ_0 as t goes to 0. Here, $\Omega := \{\varphi_0 > -\infty\}$.*

Proof. First observe that, since $e^{\gamma\varphi_0}$ is smooth on X, $\Omega = \{\varphi_0 > -\infty\}$ is an open set. Thanks to Lemma 4.2 the maximal solution does not depend on the choice of the approximating sequence. Thus we can take

$$\varphi_{0,j} := \frac{1}{\gamma}\log\left(e^{\gamma\varphi_0} + \frac{1}{j}\right)$$

as the smooth sequence decreasing to φ_0. To simplify the notation we can assume that $\gamma = 1$. For each j, let $\varphi_{t,j}$ be the smooth solution of (5.3) starting from $\varphi_{0,j}$. Denote by $\omega_t := \theta_t + dd^c\varphi_{t,j}$. We want to prove that there exists a positive constant C such that

$$\frac{1}{C}\omega \leq \omega_t \leq C\omega$$

on each compact subset $K \subset \Omega$. The result will then follow from [4, Theorem 3.2.17]. We start proving the right-hand side inequality. Since, in K, ω_t and ω are both strictly positive forms, it suffices to prove that $\mathrm{tr}_\omega(\omega_t) \leq C$ in K. Note that φ_0 is a $\delta\omega$-psh function for some $\delta \in (0,1)$. Furthermore, φ_0 is smooth on Ω and by construction we have $\varphi_{0,j} \geq \varphi_0$. Moreover

$$\Delta\varphi_{0,j} \leq Be^{-\varphi_0},$$

for some uniform constant $B > 0$. Indeed by a simple computation we get

$$dd^c e^{\varphi_{0,j}} = e^{\varphi_{0,j}} dd^c\varphi_{0,j} + e^{\varphi_{0,j}} d\varphi_{0,j} \wedge d^c\varphi_{0,j} \leq B\omega$$

where the last inequality follows from the fact that e^{φ_0} is smooth on X. For $t > 0$ small enough we have $\theta_t \geq \frac{\delta+1}{2}\omega$. Thus we can apply Proposition 3.9 to get the uniform estimate on $\mathrm{tr}_\omega(\omega_t)$. It remains to prove that $C^{-1}\omega \leq \omega_t$ on each $K \subset \Omega$. By construction and from the fact that φ_0 is strictly ω-psh we get

$$c\omega^n < (\omega + dd^c\varphi_{0,j})^n = e^{\dot\varphi_{0,j}}\omega^n$$

where c is a uniform positive constant. This easily implies $\dot\varphi_{0,j} > \log c$. Applying Proposition 3.7, we infer that there exists a uniform constant $C_1 > 0$ such that

$$\dot\varphi_{t,j} \geq C_1(\varphi_0 + 1).$$

By construction $\omega_t^n = e^{\dot\varphi_{t,j}}\omega^n$ and so, thanks to the above estimate, we infer that there exists a uniform constant $C_2 = C(K)$ such that $\omega_t^n \geq C_2^{-1}\omega^n$ on each $K \subset \Omega$. Let now $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of the matrix associated to ω_t with respect to ω. Then the latter inequality means that $\lambda_1 \cdots \lambda_n \geq \frac{1}{C_2}$. This combined with the previous estimate (i.e. $\max_i \lambda_i \leq C$) gives that $\min_i \lambda_i \geq \frac{1}{C_3}$ for some uniform constant C_3. Hence the conclusion. □

The same arguments as above can also be applied in other contexts. Assume that $0 < f \in L^1(X)$ is smooth in the complement of some closed subset $D \subset X$. We say that f satisfies the condition $\mathcal{H}_f$ if

$$f = e^{\psi^+ - \psi^-},\ \psi^\pm \in \mathrm{PSH}(X, C\omega),\ \psi^- \in L^\infty_{\mathrm{loc}}(X \setminus D),$$

for some positive constant C. Assume also that f satisfies the compatibility condition $\int_X f\omega^n = \int_X \omega^n$. Let $\varphi_0 \in \mathcal{E}(X, \omega)$ be the unique normalized solution of the complex Monge-Ampère equation

$$(\omega + dd^c\varphi_0)^n = f\omega^n.$$

We are going to prove the following convergence result:

Theorem 6.3. *Assume the above setting and let φ_t be the unique weak solution of the equation (5.3) with initial data φ_0. Then φ_t converges to φ_0 in $\mathcal{C}^\infty_{\mathrm{loc}}(X \setminus D)$.*

Proof. We approximate $\psi^\pm$ by using Demailly's technique (see [11]). Let $\psi^\pm_j$ denote these approximants. We then use Yau's work [40] to find $\varphi_{0,j} \in \mathrm{PSH}(X, \omega) \cap \mathcal{C}^\infty(X)$ such that

$$(\omega + dd^c\varphi_{0,j})^n = c_j e^{\psi^+_j - \psi^-_j}\omega^n,$$

where c_j are normalization constants and $\varphi_{0,j}$ are normalized by $\sup_X \varphi_{0,j} = 0$. It was proved in [15] that c_j converges to 1 as j goes to $+\infty$ and the following estimates hold:

- For each $\varepsilon > 0$ there exists $C_1 = C(\varepsilon) > 0$ such that
$$\varphi_{0,j} \geq \varepsilon\psi^-_j - C_1.$$
- There exists a uniform constant $C_2 > 0$ such that
$$\Delta_\omega(\varphi_{0,j}) \leq e^{-C_2\psi^-_j}.$$

Now, for each j let $\varphi_{t,j}$ be the unique smooth solution of the parabolic equation (5.3) with initial data $\varphi_{0,j}$. It follows from Theorem 5.4 that $\varphi_{t,j}$ converges to φ_t the unique weak solution of (5.3). Using Proposition 3.9 we get a uniform estimate for $\mathrm{tr}_\omega(\omega_{t,j})$ on each compact subset $K \Subset X \setminus D$. This together with Proposition 3.7 and [4, Theorem 3.2.17] yield locally uniform bounds on all derivatives of φ_t. This explains the convergence in $\mathcal{C}^\infty_{\mathrm{loc}}(X \setminus D)$ of φ_t to φ_0. □

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