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ON THE DRAWDOWN OF COMPLETELY ASYMMETRIC LÉVY PROCESSES

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Abstract. The *drawdown* process Y of a completely asymmetric Lévy process X is equal to X reflected at its running supremum $\overline{X}$: $Y = \overline{X} - X$. In this paper we explicitly express in terms of the scale function and the Lévy measure of X the law of the sextuple of the first-passage time of Y over the level $a > 0$, the time $\overline{G}_{\tau_a}$ of the last supremum of X prior to τ_a, the infimum $\underline{X}_{\tau_a}$ and supremum $\overline{X}_{\tau_a}$ of X at τ_a and the undershoot $a - Y_{\tau_a-}$ and overshoot $Y_{\tau_a} - a$ of Y at τ_a. As application we obtain explicit expressions for the laws of a number of functionals of drawdowns and rallies in a completely asymmetric exponential Lévy model.

1. Introduction

A completely asymmetric Lévy process is a real-valued stochastic process with càdlàg paths that has independent stationary increments whose jump sizes all have the same sign. Its *drawdown process*, also known as the reflected process, is the difference of its running supremum and its current value. Closely related is the *rally* process which is defined as the difference of the current value and the running infimum, and is equal to the drawdown process of the negative of the process. This paper is concerned with a *distributional* study of the drawdown process of a completely asymmetric Lévy process For such a Lévy process X the law is identified of the following sextuple concerning its drawdown process Y:

$$(\tau_a, \overline{G}_{\tau_a}, \overline{X}_{\tau_a}, \underline{X}_{\tau_a}, a - Y_{\tau_a-}, Y_{\tau_a} - a), \qquad \text{where} \qquad Y_t = \sup_{0\le s\le t} X_s - X_t, \quad t \ge 0. \tag{1.1}$$

Here the components of the vector are given by τ_a, the first-passage time of Y over a level $a > 0$, $\overline{G}_{\tau_a}$, the last time that X is at its supremum prior to τ_a, $\overline{X}_{\tau_a}$ and $\underline{X}_{\tau_a}$, the running supremum and infimum at τ_a, and $a - Y_{\tau_a-}$ and $Y_{\tau_a} - a$, the undershoot and the overshoot of Y at the epoch τ_a.

The drawdown process has been the object of considerable interest in various areas of applied probability. It has been studied for instance in queueing theory (*e.g.*, Asmussen [2]), risk theory and mathematical genetics. The drawdown process has also been employed in financial modelling, in the construction of tractable, path-dependent risk/performance measures. In the context of real estate portfolio optimisation, Hamelink & Hoesli [20] considered the running maximum of the drawdown

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process as an investment performance criterion. Checkhlov et al. [14] introduced a one-parameter family of portfolio risk measures that was called conditional drawdown and defined to be equal to the mean of a percentage of the worst portfolio drawdowns. Pospisil et al. [32] proposed the probability of a drawdown of a given size occurring before a rally of a given size as risk measure, and calculated this probability in the setting of one-dimensional diffusion models; the finite horizon case for Brownian motion was treated by Zhang & Hadjiliadis [37].

A number of papers has been devoted to a distributional study of functionals of the drawdown process. The joint Laplace of the time to a given drawdown and the running maximum of a Brownian motion with drift was derived by Taylor [36]; this joint law was obtained by Lehoczky [24] in the case of a general diffusion. An explicit expression for the expectation and the density of the maximum drawdown of Brownian motion was derived by Douady *et al.* [16]; the case of Brownian motion with drift was covered by Magdon *et al.* [26] where also the large time asymptotics of the expectation were derived.

The drawdown process also features in the solution of a number of optimal investment problems. Under the geometric Brownian motion model the optimal time to exercise the Russian option, which pays out the largest historical value of the stock at the moment of exercise, was shown by Shepp & Shiryaev [34] to be given by the first-passage of a drawdown process over a certain constant level. Such a first-passage time is also optimal when linear cost is included (Meilijson [27]), or under a spectrally negative Lévy model for the stock price (Avram *et al.* [3]).

Although still widely used as benchmark, mainly on account of its analytical tractability, it is by now well established that many features of Samuelson's classical geometric Brownian motion model for the price of a stock are not supported by empirical data. A class of tractable models that captures typical features of stock returns data such as fat tails, asymmetry and excess kurtosis is that of exponential Lévy processes. This class has received considerable attention in the literature —we refer to Cont & Tankov [15] and Boyarchenko & Levendorskii [8] for background and references. By restricting ourselves to Lévy processes with jumps of a single sign, we are able to draw on the fluctuation theory for this class of stochastic processes, which is considerably more explicit than in the case of general Lévy processes. Empirical support for a model from this class was given in Carr & Wu [9], where options on the S&P 500 index were studied. Carr & Wu [9] demonstrated that the finite-moment log-stable model, which is an exponential Lévy model driven by a spectrally negative stable process, provided a good fit to quoted option prices across maturities.

By way of application, we employ the sextuple law to obtain semi-analytical expressions for the expectations of a number of path-functionals of the drawdown process of an exponential Lévy process, which provide a description of different aspects of the riskiness of the model: (i) the probability that, on a given time horizon, a new minimum is attained (e.g. by a jump) at the first moment of a drawdown of a given size; (ii) the expected size of the drawdown process at the first moment that a drawdown of a given size occurs, given that this happens before a finite time-horizon

and (iii) the probability that, on a finite time-horizon, a drawdown of a given size occurs before a rally of a given size. These and related explicit expressions can form the basis for the analysis of optimal investment problems involving the drawdown process—in the interest of brevity, such investigations are left for future research.

In the literature two approaches have been successfully adopted to identify the distributions of path-functionals of the drawdown process: the *martingale* approach, exemplified in e.g. [10, 29], which rests on the identification of certain martingales, and the *excursion-theoretic* approach, employed in a.o. [3, 4, 11, 12, 23, 31], which is based on the theorem by Itô stating that the process of excursions of Y away from zero forms a Poisson point process. In this paper we will follow the latter approach.

The remainder of the paper is organized as follows. In Section 2 preliminaries are reviewed and the notation is set. In Sections 3.1 and 3.2 the law of the sextuple is derived for a spectrally negative and spectrally positive Lévy process, respectively. In Section 4 the sextuple law is employed to derive analytically explicit identities for a number of characteristics of drawdowns and rallies in an exponential Lévy model. The proofs of the main theorems are contained in Section 5.

2. Preliminaries

In this section we set the notation. For the background on the fluctuation theory of spectrally negative Lévy processes refer to Bertoin [6, Chapter VII] and Kyprianou [22].

Let $X = (X_t)_{t\geq 0}$ be a Lévy process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, a filtered probability space which satisfies the usual conditions. Assume throughout the paper that X is spectrally negative. Under $\mathbb{P}$ we have $X_0 = 0$ $\mathbb{P}$-a.s. For any $x \in \mathbb{R}$ we denote by $\mathbb{P}_x$ the law of the Lévy process started from x, i.e. the law of the process $x + X$ under $\mathbb{P}$. To avoid the case of trivial reflected processes we exclude X that has monotone paths, i.e. X is assumed to be neither the negative of a subordinator nor a deterministic drift upwards. Since the jumps of X are all non-positive, the moment generating function $\mathbb{E}[\mathrm{e}^{\theta X_t}]$ exists for all $\theta \geq 0$ and is given by $\psi(\theta) = t^{-1} \log \mathbb{E}[\mathrm{e}^{\theta X_t}]$ for some function $\psi(\theta)$. The function ψ which is well defined at least on the positive half-axis where it is strictly convex with the property that $\lim_{\theta\to\infty} \psi(\theta) = +\infty$. Let $\Phi(0)$ be the largest root of $\psi(\theta) = 0$. On $[\Phi(0), \infty)$ the function ψ is strictly increasing and we denote its right-inverse function by $\Phi : [0, \infty) \to [\Phi(0), \infty)$.

For $q \geq 0$, there exists a continuous increasing function $W^{(q)} : [0, \infty) \to [0, \infty)$, called the *q-scale function*, with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x)\mathrm{d}x = (\psi(\theta) - q)^{-1}, \qquad \theta > \Phi(q). \tag{2.1}$$

The function $W^{(q)}$ is extended to $x \in (-\infty, 0)$ by $W^{(q)}(x) = 0$. A related q-scale function $Z^{(q)}$ is defined by

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(z)\mathrm{d}z, \qquad \text{for } x \in \mathbb{R}. \tag{2.2}$$

The q-scale function $W^{(q)}$ is left- and right-differentiable on $(0,\infty)$ and we denote the right- and left-derivative of $W^{(q)}$ by $W_+^{(q)\prime}$ and $W_-^{(q)\prime}$ respectively for any $q \geq 0$. Furthermore if a Gaussian component is present then for any $q \geq 0$ we have $W^{(q)} \in C^2(0,\infty)$ (see [13]). In this case we denote by $W^{(q)\prime}$ and $W^{(q)\prime\prime}$ the first and the second derivative of the scale function $W^{(q)}$ respectively.

We next briefly review two-sided exit results which will be employed in the sequel. Let $u < v$ and $x \in [u,v]$ and define the first-passage times T_u^- and T_v^+ as

$$T_u^- = \inf\{t \geq 0 : X_t < u\} \quad \text{and} \quad T_v^+ = \inf\{t \geq 0 : X_t > v\}.$$

Let $T_{u,v} = T_u^- \wedge T_v^+$ be the first time X started at x enters the set $\mathbb{R} \setminus [u,v]$. The two sided first-passage results [7] read

$$\mathbb{E}_x\left[\mathrm{e}^{-qT_{u,v}} I_{\{X_{T_{u,v}}=v\}}\right] = \frac{W^{(q)}(x-u)}{W^{(q)}(v-u)}, \qquad x \in [u,v], \tag{2.3}$$

$$\mathbb{E}_x\left[\mathrm{e}^{-qT_{u,v}} I_{\{X_{T_{u,v}}\leq u\}}\right] = Z^{(q)}(x-u) - Z^{(q)}(v-u)\frac{W^{(q)}(x-u)}{W^{(q)}(v-u)}, \qquad x \in [u,v]. \tag{2.4}$$

A Lévy process started at zero *creeps downwards* (resp. *upwards*) over a level $x < 0$ (resp. $x > 0$) if, with positive probability, the first time that it enters the set $(-\infty, x)$ (resp. (x,∞)) this does not happen by a jump. It is an immediate consequence of the Wiener-Hopf factorization that a Lévy process creeps both upwards and downwards if and only if its Gaussian component is not zero. Therefore a spectrally negative Lévy process with non-monotone trajectories creeps downwards if and only if it has a positive Gaussian coefficient (e.g. [6, p. 175]). In this case the results of Millar [28] imply that

$$\mathbb{E}_x\left[\mathrm{e}^{-qT_u^-} I_{\{X_{T_u^-}=u\}}\right] = \frac{\sigma^2}{2}\left(W^{(q)\prime}(x-u) - \Phi(q)W^{(q)}(x-u)\right) \quad \text{for any} \quad u \leq x,\ q \geq 0. \tag{2.5}$$

Both sides of the equality in (2.5) are understood to be equal to zero if $\sigma = 0$. The formula for the probability that X leaves the interval $[u,v]$ by hitting u follows from (2.3), (2.5) and the strong Markov property and is given by the following expression:

$$\mathbb{E}_x\left[\mathrm{e}^{-qT_{u,v}} I_{\{X_{T_{u,v}}=u\}}\right] = \frac{\sigma^2}{2}\left(W^{(q)\prime}(x-u) - \frac{W^{(q)\prime}(v-u)}{W^{(q)}(v-u)}W^{(q)}(x-u)\right) \tag{2.6}$$

for $x \in [u,v]$. Again the expression is understood to be equal to 0 if $\sigma = 0$.

3. Reflected spectrally negative Lévy processes

Let $\overline{X}_t = \sup_{0\leq u\leq t}\{X_u\}$ and $\underline{X}_t = \inf_{0\leq u\leq t}\{X_u\}$ and define the reflected processes $Y = (Y_t)_{t\geq 0}$ and $\widehat{Y} = (\widehat{Y}_t)_{t\geq 0}$ by

$$Y_t = \overline{X}_t - X_t \qquad \text{and} \qquad \widehat{Y}_t = X_t - \underline{X}_t. \tag{3.1}$$

The focus of this paper are the first-passage times over a level $a > 0$ of the reflected processes Y and $\widehat{Y}$,

$$\tau_a := \inf\{t \geq 0 : Y_t > a\} \qquad \text{and} \qquad \widehat{\tau}_a := \inf\{t \geq 0 : \widehat{Y}_t > a\}. \tag{3.2}$$

It is well known that the stopping times τ_a and $\widehat{\tau}_a$ are finite $\mathbb{P}$-a.s. Furthermore, in [3, Theorem 1] the Laplace transform of τ_a was identified as

$$\mathbb{E}[\mathrm{e}^{-q\tau_a}] = Z^{(q)}(a) - q\frac{W^{(q)}(a)^2}{W_+^{(q)\prime}(a)}. \tag{3.3}$$

The last times before time t that X visits its running supremum and infimum are denoted by $\overline{G}_t$ and $\underline{G}_t$ respectively, where

$$\overline{G}_t = \sup\left\{s \leq t \,:\, X_s \text{ or } X_{s-} = \overline{X}_s\right\} \quad \text{and} \quad \underline{G}_t = \sup\{s \leq t : X_s \text{ or } X_{s-} = \underline{X}_s\}. \tag{3.4}$$

Note that $\overline{G}_t$ (resp. $\underline{G}_t$) can be viewed as the last time before time t that the reflected process Y (resp. $\widehat{Y}$) is equal to 0.

In Section 3.1 we characterise the joint law of the following sextuple of random variables:

τ_a	the first-passage time over a level a of the reflected process Y,
$\overline{G}_{\tau_a}$	the last time that X was at its running supremum prior to the first-passage time τ_a,
$\overline{X}_{\tau_a}$	the supremum of X at the first-passage time τ_a,
$\underline{X}_{\tau_a}$	the infimum of X at the first-passage time τ_a,
Y_{τ_a-}	the position of the reflected process just before it crosses the level a,
$Y_{\tau_a} - a$	the overshoot of the reflected process Y over the level a.

In Section 3.2 we give the joint law of the following quadruple of random variables

$\widehat{\tau}_a$	the first-passage time over a level a of the reflected process $\widehat{Y}$,
$\underline{G}_{\widehat{\tau}_a}$	the last time that X was at its running infimum prior to the first-passage time $\widehat{\tau}_a$,
$\overline{X}_{\widehat{\tau}_a}$	the supremum of X at the first-passage time $\widehat{\tau}_a$,
$\underline{X}_{\widehat{\tau}_a}$	the infimum of X at the first-passage time $\widehat{\tau}_a$.

Note that in this case, since X is assumed to be spectrally negative, the reflected process $\widehat{Y}$ can only jump down. Since $\widehat{Y}$ is right-continuous with left limits at the first-passage time $\widehat{\tau}_a$ we have $\widehat{Y}_{\widehat{\tau}_a-} = \widehat{Y}_{\widehat{\tau}_a} = a$ a.s.

3.1. **The sextuple law.** We now give the law of the sextuple $\big(\tau_a, \overline{G}_{\tau_a}, \overline{X}_{\tau_a}, \underline{X}_{\tau_a}, Y_{\tau_a-}, Y_{\tau_a} - a\big)$. Define for any $a > 0$ and $p, q \geq 0$ the map $F_{p,q,a} : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$F_{p,q,a}(y) = \lambda(a,q)\exp\left(-y\lambda(a,p)\right), \qquad y \in \mathbb{R}_+, \tag{3.5}$$

where $\lambda(a,q)$ is the ratio of the derivative of the q-scale function and the q-scale function at a,

$$\lambda(a,q) = \frac{W_+^{(q)\prime}(a)}{W^{(q)}(a)}. \tag{3.6}$$

Let Λ be the Lévy measure and $\sigma^2 \geq 0$ the Gaussian coefficient of X. Denote by $R_a^{(q)}(\mathrm{d}y) = \mathbb{E}\left[\int_0^{\tau_a} \mathrm{e}^{-qt} I_{\{Y_t \in \mathrm{d}y\}} \mathrm{d}t\right]$ the q-resolvent measure of Y killed upon first exit from $[0,a]$, which may be expressed in terms of the q-scale function $W^{(q)}$ by ([30, Theorem 1])

$$R_a^{(q)}(\mathrm{d}y) = \left[\lambda(a,q)^{-1} W^{(q)}(\mathrm{d}y) - W^{(q)}(y)\mathrm{d}y\right], \qquad y \in [0,a], \tag{3.7}$$

Furthermore, let $\Delta^{(q)}(a)$ denote the expression that is given in terms of the derivatives of $W^{(q)}$ by

$$\Delta^{(q)}(a) = \frac{\sigma^2}{2}\left[W^{(q)\prime}(a) - \lambda(a,q)^{-1} W^{(q)\prime\prime}(a)\right], \tag{3.8}$$

where the expression on the right-hand side of the equality in (3.8) is taken to be zero if $\sigma = 0$. As we will see below, the Laplace transform of τ_a on the event that Y creeps over the level a is equal to $\Delta^{(q)}(a) = \mathbb{E}[\mathrm{e}^{-q\tau_a} I_{\{Y_{\tau_a}=a\}}]$.

Theorem 1. *Let $X_0 = x \in \mathbb{R}$ and $a > 0$ and define events*

$$A_o = \left\{\underline{X}_{\tau_a} \geq u, \overline{X}_{\tau_a} \in \mathrm{d}v, Y_{\tau_a -} \in \mathrm{d}y, Y_{\tau_a} - a \in \mathrm{d}h\right\} \quad \text{and} \quad A_c = \left\{\underline{X}_{\tau_a} \geq u, \overline{X}_{\tau_a} \in \mathrm{d}v, Y_{\tau_a} = a\right\},$$

where u, v, y and h satisfy

$$u \leq x,\ y \in [0,a],\ v \geq x \vee (u+a) \quad \text{and} \qquad h \in (0, v-u-a]. \tag{3.9}$$

Then for any $q, r \geq 0$ the following identities hold true:

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_a - r\overline{G}_{\tau_a}} I_{A_o}\right] = \frac{W^{(q+r)}((x-u)\wedge a)}{W^{(q+r)}(a)} F_{q+r,q,a}\left(v - (x \vee (u+a))\right) R_a^{(q)}(\mathrm{d}y)\, \Lambda\left(y - a - \mathrm{d}h\right), \tag{3.10}$$

$$\mathbb{E}_x\left[\mathrm{e}^{-q\tau_a - r\overline{G}_{\tau_a}} I_{A_c}\right] = \frac{W^{(q+r)}((x-u)\wedge a)}{W^{(q+r)}(a)} F_{q+r,q,a}\left(v - (x \vee (u+a))\right) \Delta^{(q)}(a) \tag{3.11}$$

where $I_{\{\cdot\}}$ denotes the indicator of the set $\{\cdot\}$ and $c \wedge d = \min\{c,d\}$, $c \vee d = \max\{c,d\}$ for $c, d \in \mathbb{R}$.

Remarks. (i) The formulas in (3.10) and (3.11) determine the law of the sextuple $\left(\tau_a, \overline{G}_{\tau_a}, \overline{X}_{\tau_a}, \underline{X}_{\tau_a}, Y_{\tau_a -}, Y_{\tau_a} - a\right)$. Indeed, note that if the parameters u, v, y and h do not satisfy the restrictions in (3.9), then we have $\mathbb{P}_x[A_o] = \mathbb{P}_x[A_c] = 0$. In particular, if we take $u = -\infty$, then (3.9) places a restriction neither on the size $h \in (0,\infty)$ of the overshoot of the reflected process nor on the level $v \geq x$ of the supremum of X attained at τ_a. This is also transparent from Figure 1.

(ii) Note that on the event A_o we have $X_{\tau_a} = v - a - h$ $\mathbb{P}_x$-a.s. and hence $\underline{X}_{\tau_a} = (v-a-h) \wedge \underline{X}_{\tau_a -}$. Define the event

$$A_o^- = \left\{\underline{X}_{\tau_a -} \geq u, \overline{X}_{\tau_a} \in \mathrm{d}v, Y_{\tau_a -} \in \mathrm{d}y, Y_{\tau_a} - a \in \mathrm{d}h\right\}.$$

Then the following implications hold:

$$h \in (0, v-u-a] \implies A_o^- = A_o \quad \mathbb{P}_x\text{-a.s.}$$
$$h > v-u-a \implies \underline{X}_{\tau_a} = v-a-h \quad \mathbb{P}_x\text{-a.s. on } A_o^- \quad \text{and} \quad \mathbb{P}_x[A_o] = 0.$$

Furthermore, it is clear from step (2) of the proof of Theorem 1 (presented in Section 5) that $\mathbb{E}_x\left[\mathrm{e}^{-q\tau_a - r\overline{G}_{\tau_a}} I_{A_o^-}\right]$ is given by the formula in (3.10), which in this case holds without a restriction on the size of the overshoot h from (3.9).

(iii) Note that the reflected process Y creeps over the level a (i.e. $\mathbb{P}[Y_{\tau_a} = a] > 0$) if and only if X creeps downwards which is the case precisely if its Gaussian coefficient σ^2 is strictly positive. Hence the expression $\Delta^{(q)}(a)$ given in (3.8) is well-defined, as $W^{(q)}$ is twice differentiable at $a > 0$ if σ^2 is strictly positive.

(iv) Schematic representations of a typical path of the process X in the events A_o and A_c of Theorem 1 are given in the bottom and top pictures in Figure 1. Note that A_c contains the paths of the reflected process Y when it creeps over the level a and that $Y_{\tau_a -} = a$ $\mathbb{P}$-a.s. on A_c. Similarly A_o consists of the paths of Y which first enter (a, ∞) by a jump.

Explicit expressions, in terms of the 0-scale function, for the marginal distributions of $\overline{X}_{\tau_a}$ and $\underline{X}_{\tau_a -}$ under $\mathbb{P}_x$ can be obtained from Theorem 1.

Corollary 2. *For any $a > 0$ and $x \in \mathbb{R}$, the following hold true:*

$$\mathbb{P}_x(x - \underline{X}_{\tau_a -} \le z) = W(z \wedge a)/W(a), \qquad z \in \mathbb{R}_+, \tag{3.12}$$
$$\mathbb{P}_x(\overline{X}_{\tau_a} - x \ge z) = \exp\left(-zW'_+(a)/W(a)\right), \qquad z \in \mathbb{R}_+, \tag{3.13}$$

where $W = W^{(0)}$ denotes the 0-scale function.

Proof of Corollary 2 Setting q and r equal to 0 and letting $u = -\infty$ in (3.10)–(3.11), and integrating over $y \in [0, a]$, $h \in (0, \infty)$ and $v \in (x, \infty)$ the right-hand sides of the these expressions we obtain that $c(a) = 1$ where

$$c(a) := \int_0^\infty \int_0^a R_a^{(0)}(\mathrm{d}y)\Lambda(y - a - \mathrm{d}h) + \Delta^{(0)}(a). \tag{3.14}$$

Fixing x, a, u_0, v_0, such that $a > 0$, $u_0 + a \le v_0$ and $x \in [u_0, v_0]$, and again integrating the right-hand side of (3.10) over $y \in [0, a]$, $h \in (0, \infty)$ and $v \in (v_0, \infty)$ and the right-hand side of Eqns. (3.11) over $v \ge v_0$ we find

$$\mathbb{P}_x(\underline{X}_{\tau_a -} \ge u_0, \overline{X}_{\tau_a} \ge v_0) = c(a) \cdot \frac{W^{(0)}((x-u_0) \wedge a)}{W^{(0)}(a)} \, \mathrm{e}^{-(v_0 - x \vee (u_0+a))\lambda(a,0)}, \qquad x \in \mathbb{R}.$$

We obtain (3.12) and (3.13) from this display by setting $v_0 = x \vee (u_0 + a)$ and $u_0 = -\infty$, respectively, and recalling that $c(a) = 1$. □

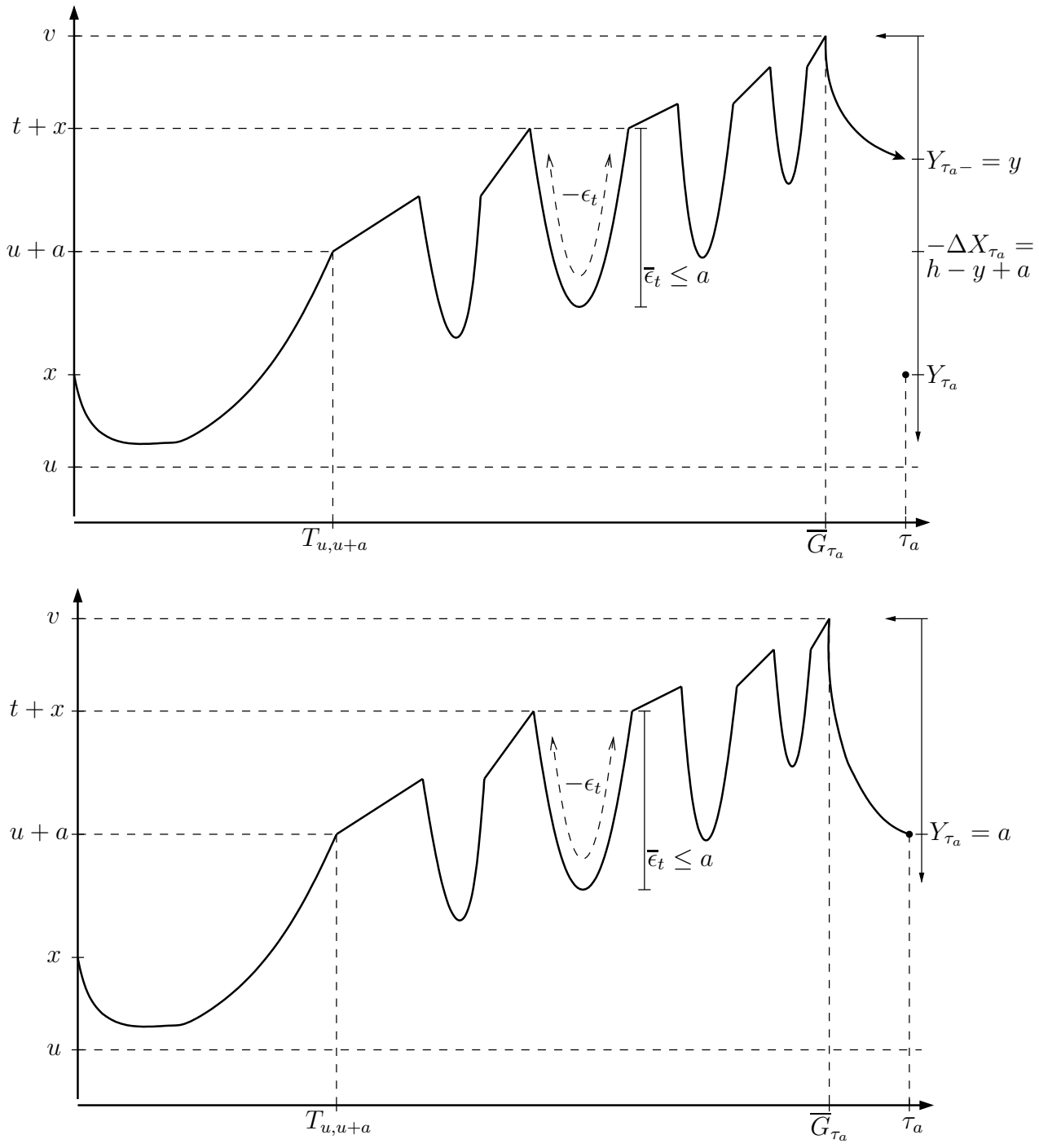


FIGURE 1. The top figure contains a schematic representation of a typical path of the process X in the event $A_o = \{\underline{X}_{\tau_a} \ge u, \overline{X}_{\tau_a} \in \mathrm{d}v, Y_{\tau_a-} \in \mathrm{d}y, Y_{\tau_a} - a \in \mathrm{d}h\}$. This figure illustrates the idea behind the proof of Theorem 1: the depicted path satisfies $T_{u,u+a} = T^+_{u+a}$ and $T^+_v = \overline{G}_{\tau_a}$. It is clear from the figure that the trajectory can be decomposed into the following three parts, each of which is analysed separately in the proof of Theorem 1: (1) the segment over the interval $[0, T_{u,u+a}]$, (2) the jump at time τ_a and (3) the segment that straddles $\overline{G}_{\tau_a}$ over the time interval $(T_{u,u+a}, \tau_a)$. The bottom figure depicts a schematic representation of a typical path of the process X in the event $A_c = \{\underline{X}_{\tau_a} \ge u, \overline{X}_{\tau_a} \in \mathrm{d}v, Y_{\tau_a} = a\}$.

Remarks. (i) If X is a Brownian motion with drift started at zero, i.e. $X_t = \mu t + \sigma W_t$, where $(W_t)_{t\geq 0}$ is a standard Brownian motion and $\mu \in \mathbb{R}, \sigma^2 > 0$, then the scale function takes the form

$$W(x) = \frac{1}{\mu}\left[1 - \mathrm{e}^{-\frac{2\mu}{\sigma^2}x}\right] \qquad \text{if} \quad \mu \neq 0, \tag{3.15}$$

$$W(x) = \frac{2}{\sigma^2}x \qquad \text{if} \quad \mu = 0. \tag{3.16}$$

Since X has no jumps, we have an almost sure equality $\overline{X}_{\tau_a} = a + X_{\tau_a}$. Therefore the random variable $a + X_{\tau_a}$ is by (3.13) exponentially distributed with parameter $(2\mu/\sigma^2)/(\mathrm{e}^{a2\mu/\sigma^2} - 1)$ (resp. $1/a$) if $\mu \neq 0$ (resp. $\mu = 0$). This fact, first observed by Lehoczky [24], is generalised by the formula in (3.13) to the class of spectrally negative Lévy processes.
(ii) Let X be a spectrally negative α-stable process with cumulant generating function $\psi(\theta) = (\sigma\theta)^\alpha$ for $\theta > 0$, where $\alpha \in (1, 2]$, $\sigma > 0$. Identity (2.1) and the definition of the Gamma function Γ imply that the scale function of X takes the form

$$W(x) = \frac{x^{\alpha-1}}{\sigma^\alpha \Gamma(\alpha)}, \qquad x \geq 0. \tag{3.17}$$

Corollary 2 implies that, under $\mathbb{P}_x$, the random variables $a^{-1}\left(x - \underline{X}_{\tau_a-}\right)$ and $a^{-1}\left(\overline{X}_{\tau_a} - x\right)$ follow Beta$(\alpha - 1, 1)$ and Exp$\left(\frac{\alpha-1}{a}\right)$ distributions with respective probability density functions

$$\mathbb{P}_x(x - \underline{X}_{\tau_a-} \in \mathrm{d}z) = I_{[0,a]}(z)\frac{\alpha-1}{a}\left(\frac{z}{a}\right)^{\alpha-2}\mathrm{d}z,$$

$$\mathbb{P}_x(\overline{X}_{\tau_a} - x \in \mathrm{d}z) = I_{(0,\infty)}(z)\frac{\alpha-1}{a}\,\mathrm{e}^{-z(\alpha-1)/a}\mathrm{d}z.$$

Note that these distributions do not depend on the value of σ. In the case of Brownian motion started at x (i.e. for $\alpha = 2$), the random variables $a^{-1}\left(x - \underline{X}_{\tau_a}\right)$ and $a^{-1}\left(\overline{X}_{\tau_a} - x\right)$ follow $U(0, 1)$ and Exp(1) distributions respectively.

3.2. **The quadruple law.** In this section we characterise the law of the quadruple $(\widehat{\tau}_a, \underline{G}_{\widehat{\tau}_a}, \underline{X}_{\widehat{\tau}_a}, \overline{X}_{\widehat{\tau}_a})$, where $\underline{G}_t$, defined in (3.4), is the last time the process X was at the infimum prior to time t. Before stating the result, we recall that $q \mapsto W^{(q)}(x)$ has a holomorphic extension to $\mathbb{C}$ for every $x \geq 0$ and that $(x, q) \mapsto Z^{(q)}(x)$ has a continuous extension to $[0, \infty) \times \mathbb{R}$, which is holomorphic in q for every $x \in \mathbb{R}$ (see [30], Lemma 2). Furthermore for any $u \geq 0$ let $\mathbb{P}^u$ be the exponentially tilted probability measure defined via the Esscher transform (see [2, Ch. XIII]). Then the scale functions $x \mapsto W_u^{(q-\psi(u))}(x)$ and $x \mapsto Z_u^{(q-\psi(u))}(x)$ satisfy the following identities for all $q \in \mathbb{C}, x \in \mathbb{R}$:

$$W_u^{(q-\psi(u))}(x) = \mathrm{e}^{-ux}W^{(q)}(x), \tag{3.18}$$

$$Z_u^{(q-\psi(u))}(x) = 1 + (q - \psi(u))\int_0^x \mathrm{e}^{-uz}W^{(q)}(z)\,\mathrm{d}z. \tag{3.19}$$

The identity in (3.18), for $q > \psi(u)$, follows by taking Laplace transforms on both sides and applying (2.1), and hence by analyticity for all $q \in \mathbb{C}$. The identity in (3.19) follows from (3.18) and the definition in (2.2).

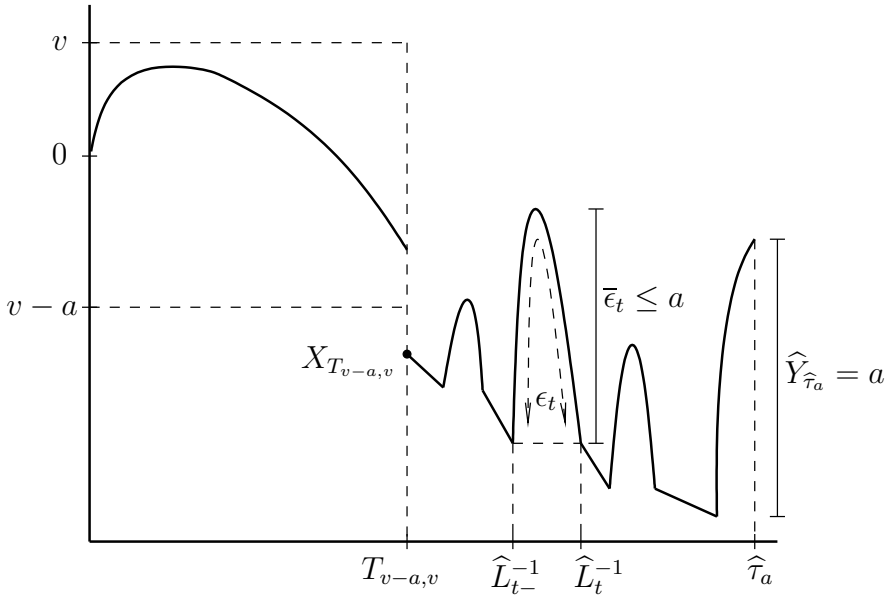


FIGURE 2. A schematic representation of a path which satisfies $T_{v-a,v} = T^-_{v-a}$ and is therefore in the event $\{\overline{X}_{\widehat{\tau}_a} < v\}$. At the moment of local time t the path in the figure has an excursion ϵ_t away from the infimum. The process $\widehat{L}^{-1}$ is the inverse local time at zero for the Markov process $\widehat{Y}$.

We now state the main result of this section.

Theorem 3. *Let $q, r, u, v \geq 0$. Denote $p = q + r - \psi(u)$. Then the following identity holds:*

$$\begin{aligned} &\mathbb{E}_x\left[\mathrm{e}^{-q\widehat{\tau}_a - r\underline{G}_{\widehat{\tau}_a} + u\underline{X}_{\widehat{\tau}_a}} I_{\{\overline{X}_{\widehat{\tau}_a} < v\}}\right] \\ &= \frac{W^{(q+r)}(a)}{W^{(q)}(a)}\left[\mathrm{e}^{-u(a-x)}\frac{Z^{(p)}_u(a+x-v)}{Z^{(p)}_u(a)} - \mathrm{e}^{-u(a-v)}\frac{W^{(q+r)}(a+x-v)}{W^{(q+r)}(a)}\right]. \end{aligned} \tag{3.20}$$

Remarks. (i) If $a < v$, then the definition of $\widehat{\tau}_a$ implies $\mathbb{P}\left[\overline{X}_{\widehat{\tau}_a} < v\right] = 1$. Therefore Theorem 3, together with equations (3.18) and (3.19) (recall that $W^{(q)}(y) = 0$ for all $q \geq 0$ and $y < 0$), implies the identities

$$\mathbb{E}\left[\mathrm{e}^{-q\widehat{\tau}_a + u\underline{X}_{\widehat{\tau}_a}}\right] = \mathrm{e}^{-au}/Z^{(q-\psi(u))}_u(a), \qquad q, u \geq 0, \tag{3.21}$$

$$\mathbb{E}[\mathrm{e}^{-q\widehat{\tau}_a - r\underline{G}_{\widehat{\tau}_a}}] = \frac{W^{(q+r)}(a)}{W^{(q)}(a)Z^{(q+r)}(a)}, \qquad q, r \geq 0. \tag{3.22}$$

The special case of formulae (3.21) and (3.22) for $u = 0$ and $r = 0$ respectively (i.e. the Laplace transform of $\widehat{\tau}_a$) is well-known (see [30], Proposition 2). Note that in (3.22) the random variable under the expectation does not depend on the starting point of the process X.
(ii) Since X is spectrally negative, we have $\widehat{Y}_{\widehat{\tau}_a} = a$ almost surely. Hence formula (3.20) also yields the joint law of the quadruple $(\widehat{\tau}_a, \underline{G}_{\widehat{\tau}_a} X_{\widehat{\tau}_a}, \overline{X}_{\widehat{\tau}_a})$.

(iii) Under the law $\mathbb{P}_x$, the cdf of the random variable $\overline{X}_{\widehat{\tau}_a} - x$ is given by

$$\mathbb{P}_x(\overline{X}_{\widehat{\tau}_a} - x \leq z) = 1 - \frac{W((a-z)\wedge a)}{W(a)}, \qquad z \in \mathbb{R}. \tag{3.23}$$

Note that the distribution of $\overline{X}_{\widehat{\tau}_a} - x$ under $\mathbb{P}_x$ is absolutely continuous with respect to the Lebesgue measure if and only if $W(0) = 0$. If $W(0) > 0$, this distribution has an atom only at a, which is of size $W(0)/W(a)$. Furthmore, the Laplace transform of $x - \underline{X}_{\widehat{\tau}_a}$ is given in terms of the scale function W of X by the formula

$$\mathbb{E}_x\left[\mathrm{e}^{-u\left(x-\underline{X}_{\widehat{\tau}_a}\right)}\right] = \mathrm{e}^{-ua} \Big/ \left[1 - \psi(u)\int_0^a \mathrm{e}^{-uz} W(z)\,\mathrm{d}z\right]. \tag{3.24}$$

(iv) In the special case when X is a spectrally negative α-stable process with cumulant generating function $\psi(\theta) = (\sigma\theta)^\alpha$, where $\alpha \in (1,2]$, $\sigma > 0$, the scale function takes the form $W(x) = x^{\alpha-1}/(\sigma^\alpha\Gamma(\alpha))$, for $x \geq 0$, by (3.17). Formula in (3.23) implies that under $\mathbb{P}_x$, for any $x \in \mathbb{R}$, the random variable $a^{-1}\left(\overline{X}_{\widehat{\tau}_a} - x\right)$ follows a Beta$(1, \alpha-1)$ distribution with probability distribution

$$\mathbb{P}_x(\overline{X}_{\widehat{\tau}_a} - x \in \mathrm{d}z) \;\; = \;\; I_{[0,a]}(z)\,\frac{\alpha-1}{a}\left(1 - \frac{z}{a}\right)^{\alpha-2}\mathrm{d}z.$$

The Laplace transform of the random variable $a^{-1}\left(x - \underline{X}_{\widehat{\tau}_a}\right)$ is, by (3.24), equal to

$$\mathbb{E}_x\left[\mathrm{e}^{-ua^{-1}\left(x-\underline{X}_{\widehat{\tau}_a}\right)}\right] = \mathrm{e}^{-u}\frac{\Gamma(\alpha)}{\Gamma(\alpha,u)}, \qquad \text{where} \quad \Gamma(z,y) = \int_y^\infty \mathrm{e}^{-s}s^{z-1}\mathrm{d}s \tag{3.25}$$

is the incomplete gamma function. Note that the laws of $a^{-1}\left(\overline{X}_{\widehat{\tau}_a} - x\right)$ and $a^{-1}\left(x - \underline{X}_{\widehat{\tau}_a}\right)$ do not depend on the value of σ. In particular, in the case of Brownian motion started at x (i.e. $\alpha = 2$), the random variables $a^{-1}\left(\overline{X}_{\widehat{\tau}_a} - x\right)$ and $a^{-1}\left(x - \underline{X}_{\widehat{\tau}_a}\right)$ follow $U(0,1)$ and Exp(1) distributions respectively.

4. Application: drawdowns and rallies in exponential Lévy models

Let the stochastic process $S = (S_t)_{t\geq 0}$ model the price of a stock or a foreign exchange rate. The (absolute) drawdown and rally processes of S are defined by the difference $\overline{S} - S$ of its running supremum $\overline{S}_t = \sup_{0\leq u\leq t}\{S_u\}$ and the current value, and the difference $S - \underline{S}$ of the current value and the running infimum $\underline{S}_t = \inf_{0\leq u\leq t}\{S_u\}$. Their relative counterparts, the *relative drawdown process* and *relative drawup* or *relative rally processes* of S are given by $\overline{S}/S$ and $S/\underline{S}$. Let D_α be the first time the price process S drops below its running supremum by at least $(100\alpha)\%$ with $\alpha \in (0,1)$, and denote by U_β be the first time the stock price S rallies above its running infimum by at least $(100\beta)\%$ with $\beta > 0$. In some trading strategies buy- and sell-signals for S are generated on the basis of its relative drawdown or drawup processes. For example, a commonly used strategy is to buy at the epoch U_β, and to sell at the epoch D_α (see e.g. Levich & Thomas [25] for an analysis of such a strategy in an foreign exchange setting). In this section we will consider the following four criteria that provide descriptions of different aspects of the risk associated to investing in S:

(i) The probability that S attains a new (all-time) minimum at the moment D_α of a first relative drawdown of size $(100\alpha)\%$, on the event that D_α is before T, where $T>0$ is a given time-horizon.
(ii) The probability that S attains a new maximum at the moment U_β of a first relative drawup of size $(100\beta)\%$, on the event that U_β is before T.
(iii) The expected (absolute) drawdown of S at the epoch D_α, on the event that D_α is before T.
(iv) The probability that a relative drawdown of size $(100\alpha)\%$ occurs before a relative drawup of size $(100\beta)\%$, on the event that D_α is before T.

Assume henceforth that the price process is modelled as $S=(S_t)_{t\geq 0}$ where

$$S_t = S_0\exp(X_t), \quad t\geq 0,$$

with $S_0>0$ and $X=(X_t)_{t\geq 0}$ a completely asymmetric Lévy process. Then, using the notation of Section 3, we have $\overline{S}=\mathrm{e}^{\overline{X}}$, $\underline{S}=\mathrm{e}^{\underline{X}}$, and therefore $\overline{S}/S=\mathrm{e}^{Y}$, $S/\underline{S}=\mathrm{e}^{\widehat{Y}}$, and find

$$D_\alpha = \inf\left\{t\geq 0 \,:\, \overline{S}_t/S_t \geq 1/(1-\alpha)\right\} = \tau_a, \qquad \text{where} \quad a=-\log(1-\alpha), \tag{4.1}$$

$$U_\beta = \inf\left\{t\geq 0 \,:\, S_t/\underline{S}_t \geq 1+\beta\right\} = \widehat{\tau}_b, \qquad \text{where} \quad b=\log(1+\beta). \tag{4.2}$$

Furthermore, the quantities in (i)–(iv) can be expressed in the notation of Section 3 as follows:

$$\begin{aligned}
\mathbb{P}[S_{D_\alpha}=\underline{S}_{D_\alpha}, D_\alpha<T] &= \mathbb{P}[\tau_a=\underline{G}_{\tau_a}, \tau_a<T],\\
\mathbb{P}[S_{U_\beta}=\overline{S}_{U_\beta}, U_\beta<T] &= \mathbb{P}[\widehat{\tau}_b=\overline{G}_{\widehat{\tau}_b}, \widehat{\tau}_b<T],\\
\mathbb{E}[(\overline{S}_{D_\alpha}-S_{D_\alpha})I_{\{D_\alpha<T\}}] &= S_0\mathbb{E}[(\mathrm{e}^{\overline{X}_{\tau_a}}-\mathrm{e}^{X_{\tau_a}})I_{\{\tau_a<T\}}],\\
\mathbb{P}[D_\alpha<U_\beta, D_\alpha<T] &= \mathbb{P}[\tau_a<\widehat{\tau}_b, \tau_a<T].
\end{aligned}$$

In the following sections these quantities are explicitly expressed in terms of scale functions, employing the sextuple and quadruple laws that were derived in Section 3.

4.1. **The probability of attaining a new minimum at the first moment of a drawdown.** In the next result the distributions of the two random vectors $(\tau_a, \widehat{Y}_{\tau_a})$ and $(\widehat{\tau}_a, Y_{\widehat{\tau}_a})$ are explicitly identified in terms of scale functions. Note that the process X is equal to its running minimum at the first moment of drawdown of size a if and only if $\widehat{Y}_{\tau_a}$ is equal to zero. Similarly, X is equal to its running minimum at the first moment of a drawup of size b if and only if $Y_{\widehat{\tau}_a}$ is equal to zero.

Corollary 4. *Let $a>0$. (i) For $b,q,\theta\geq 0$ the following identities hold true:*

$$\mathbb{E}\left[\mathrm{e}^{-q\tau_a}I_{\{\widehat{Y}_{\tau_a}>b\}}\right] = \mathrm{e}^{-b\lambda(a,q)}\left[1-\lambda(a,q)\int_0^a \frac{W^{(q)}(y)}{W^{(q)}(a)}\,\mathrm{d}y\right]M_{q,a}, \tag{4.3}$$

$$\mathbb{E}\left[\mathrm{e}^{-q\widehat{\tau}_a-\theta Y_{\widehat{\tau}_a}}\right] = 1-\left(\frac{(q-\psi(\theta))\mathrm{e}^{-a\theta}}{Z_\theta^{(q-\psi(\theta))}(a)}+\frac{\theta}{W^{(q)}(a)}\right)\int_0^a W^{(q)}(z)\,\mathrm{d}z, \tag{4.4}$$

where $\lambda(a,q) = W_+^{(q)\prime}(a)/W^{(q)}(a)$ *and* $M_{q,a} := \mathbb{E}[\mathrm{e}^{-\lambda(a,q)(Y_{\tau_a}-a)}]$ *is given by*

$$M_{q,a} = Z_\lambda^{(-\psi(\lambda))}(a) - W_\lambda^{(-\psi(\lambda))}(a)\frac{-\psi(\lambda)W_\lambda^{(-\psi(\lambda))}(a) + \lambda Z_\lambda^{(-\psi(\lambda))}(a)}{W_{\lambda+}^{(-\psi(\lambda))\prime}(a) + \lambda W_\lambda^{(-\psi(\lambda))}(a)} \quad \textit{with } \lambda = \lambda(a,q). \tag{4.5}$$

(ii) The Laplace transforms of $t \mapsto \mathbb{P}(\tau_a < t, \tau_a = \underline{G}_{\tau_a})$ *and* $t \mapsto \mathbb{P}(\widehat{\tau}_a < t, \widehat{\tau}_a = \overline{G}_{\widehat{\tau}_a})$ *are given by*

$$\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}[\tau_a < t, \tau_a = \underline{G}_{\tau_a}]\mathrm{d}t = \frac{1}{q}\left\{1 - \lambda(a,q)\int_0^a \frac{W^{(q)}(y)}{W^{(q)}(a)}\,\mathrm{d}y\right\} M_{q,a}, \tag{4.6}$$

$$\int_0^\infty \mathrm{e}^{-qt}\mathbb{P}[\widehat{\tau}_a < t, \widehat{\tau}_a = \overline{G}_{\widehat{\tau}_a}]\mathrm{d}t = \frac{1}{q}\left\{1 - \lambda(a,q)\int_0^a \frac{W^{(q)}(y)}{W^{(q)}(a)}\mathrm{d}y\right\}. \tag{4.7}$$

Remarks. (i) From the formulas in (4.6) and (4.7) it follows that the non-negative random variables $\widehat{Y}_{\tau_a}$ and $Y_{\widehat{\tau}_a}$ have atoms at zero of the sizes

$$\mathbb{P}\left[\widehat{Y}_{\tau_a} = 0\right] = 1 - \left(1 - \frac{W'_+(a)}{W(a)^2}\int_0^a W(y)\mathrm{d}y\right) M_{0,a}, \tag{4.8}$$

$$\mathbb{P}\left[Y_{\widehat{\tau}_a} = 0\right] = 1 - \frac{W'_+(a)}{W(a)^2}\int_0^a W(y)\,\mathrm{d}y. \tag{4.9}$$

Furthermore, the formula in (4.3), implies that conditional on the event $\left\{\widehat{Y}_{\tau_a} > 0\right\}$, $\widehat{Y}_{\tau_a}$ follows an exponential distribution with mean $W(a)/W'_+(a)$.

(ii) In the case X is Brownian motion with non-zero drift μ and Gaussian coefficient σ^2, the scale function is given by (3.15). Hence the atoms of $\widehat{Y}_{\tau_a}$ and $Y_{\widehat{\tau}_a}$ at zero are by (4.8) and (4.9) of the sizes

$$\mathbb{P}\left[\widehat{Y}_{\tau_a} = 0\right] = \frac{\mathrm{e}^{-a2\mu/\sigma^2} - 1 + a2\mu/\sigma^2}{(\mathrm{e}^{-a\mu/\sigma^2} - \mathrm{e}^{a\mu/\sigma^2})^2}. \qquad \mathbb{P}\left[Y_{\widehat{\tau}_a} = 0\right] = \frac{\mathrm{e}^{a2\mu/\sigma^2} - 1 - a2\mu/\sigma^2}{(\mathrm{e}^{-a\mu/\sigma^2} - \mathrm{e}^{a\mu/\sigma^2})^2}.$$

Furthermore, $\widehat{Y}_{\tau_a}$ and $Y_{\widehat{\tau}_a}$ conditioned to be stirctly positive follow exponential distributions:

$$Y_{\widehat{\tau}_a}\,|Y_{\widehat{\tau}_a} > 0 \sim \mathrm{Exp}\left(\frac{2\mu/\sigma^2}{\mathrm{e}^{2a\mu/\sigma^2} - 1}\right), \qquad \widehat{Y}_{\tau_a}\left|\widehat{Y}_{\tau_a} > 0 \sim \mathrm{Exp}\left(\frac{2\mu/\sigma^2}{1 - \mathrm{e}^{-2a\mu/\sigma^2}}\right).\right.$$

If $\mu = 0$, formula (4.8) implies $\mathbb{P}\left[\widehat{Y}_{\tau_a} = 0\right] = 1/2$. In this case formula (4.3) implies that $\mathbb{P}\left[\widehat{Y}_{\tau_a} \in \mathrm{d}b|\widehat{Y}_{\tau_a} > 0\right] = (\mathrm{e}^{-b/a}/a)\mathrm{d}b$ for $b > 0$. These results coincide with the findings of [19, Prop. 2.2, Cor. 2.6].

(i) For later reference we note that the formula in (4.4) can equivalently be expressed as

$$\mathbb{E}\left[\mathrm{e}^{-q\widehat{\tau}_a - \theta Y_{\widehat{\tau}_a}}\right] = 1 - \frac{\int_0^\infty \mathrm{e}^{-\theta z}W_+^{(q)\prime}(a+z)\,\mathrm{d}z}{\int_0^\infty \mathrm{e}^{-\theta z}W^{(q)}(a+z)\,\mathrm{d}z}\int_0^a \frac{W^{(q)}(z)}{W^{(q)}(a)}\,\mathrm{d}z \tag{4.10}$$

for any $q \geq 0$ and $\theta > \Phi(q)$.

Proof of Corollary 4 The identity $\widehat{Y}_{\tau_a} = \overline{X}_{\tau_a} - Y_{\tau_a} - \underline{X}_{\tau_a}$ implies that, for any $h > 0$, on the event

$$\left\{\widehat{Y}_{\tau_a} \leq b, Y_{\tau_a} - a \in \mathrm{d}h\right\} = \left\{\overline{X}_{\tau_a} \leq a + b + h + \underline{X}_{\tau_a}, Y_{\tau_a} - a \in \mathrm{d}h\right\}$$

the inequality $\overline{X}_{\tau_a} \leq a+b+h$ holds $\mathbb{P}_0$-a.s. The formula in (3.10) of Theorem 1 and the definition of $F_{q,q,a}$ given in (3.5) imply

$$\mathbb{E}\left[\mathrm{e}^{-q\tau_a} I_{\{\widehat{Y}_{\tau_a}\leq b, Y_{\tau_a}-a\in \mathrm{d}h\}}\right] = I_1(\mathrm{d}h) + I_2(\mathrm{d}h), \tag{4.11}$$

where I_1 and I_2 are measures on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ given by

$$\begin{aligned} I_1(\mathrm{d}h) &:= \mathbb{E}\left[\mathrm{e}^{-q\tau_a} I_{\{\widehat{Y}_{\tau_a}\leq b, Y_{\tau_a}-a\in \mathrm{d}h, \overline{X}_{\tau_a}\leq b+h\}}\right] = \nu^0_{q,a}(\mathrm{d}h) - \mathrm{e}^{-b\lambda(a,q)}\nu^1_{q,a}(\mathrm{d}h), \\ I_2(\mathrm{d}h) &:= \mathbb{E}\left[\mathrm{e}^{-q\tau_a} I_{\{\widehat{Y}_{\tau_a}\leq b, Y_{\tau_a}-a\in \mathrm{d}h, \overline{X}_{\tau_a}\in (b+h, a+b+h]\}}\right] = \lambda(a,q)\mathrm{e}^{-b\lambda(a,q)}\nu^1_{q,a}(\mathrm{d}h)\int_0^a \frac{W^{(q)}(y)}{W^{(q)}(a)}\,\mathrm{d}y. \end{aligned}$$

In these expressions $\lambda(a,q)$ is given in (3.6) and $\nu^j_{q,a}$, $j\in\{0,1\}$, are measures supported on $\mathbb{R}_+$ and defined by the following formula

$$\nu^j_{q,a}(\mathrm{d}h) = \mathrm{e}^{-hj\lambda(a,q)}\int_0^a R^{(q)}_a(\mathrm{d}y)\Lambda(y-a-\mathrm{d}h), \tag{4.12}$$

where $R^{(q)}_a$ is the resolvent measure given in (3.7) and Λ is the Lévy measure of X. An analogous argument, based on the formula in (3.11) of Theorem 1, yields the formula

$$\mathbb{E}\left[\mathrm{e}^{-q\tau_a} I_{\{\widehat{Y}_{\tau_a}\leq b, Y_{\tau_a}=a\}}\right] = \left[1-\mathrm{e}^{-b\lambda(a,q)} + \lambda(a,q)\mathrm{e}^{-b\lambda(a,q)}\int_0^a \frac{W^{(q)}(y)}{W^{(q)}(a)}\,\mathrm{d}y\right]\Delta^{(q)}(a), \tag{4.13}$$

where $\Delta^{(q)}(a)$ is given in (3.8). Observe that, on account of the compensation formula applied to $(\Delta X_t)_{t\geq 0}$, the following relations hold true:

$$\mathbb{E}[\mathrm{e}^{-q\tau_a}] = \nu^0_{q,a}(0,\infty) + \Delta^{(q)}(a), \tag{4.14}$$

$$\mathbb{E}[\mathrm{e}^{-\lambda(Y_{\tau_a}-a)}] = \nu^1_{q,a}(0,\infty) + \Delta^{(q)}(a), \qquad \lambda = \lambda(a,q), \tag{4.15}$$

where the expressions for $\mathbb{E}[\mathrm{e}^{-q\tau_a}]$ and $\mathbb{E}[\mathrm{e}^{-\lambda(Y_{\tau_a}-a)}]$ in terms of scale functions are given in (3.3) and (4.5) (see [3, Thm. 1]). By integrating (4.11) over $h\in(0,\infty)$, adding (4.13) and using (4.14), (4.15) and (3.3) we obtain the formula in (4.3).

It follows from Theorem 3 that the identity

$$\mathbb{E}\left[\mathrm{e}^{-q\widehat{\tau}_a + uX_{\widehat{\tau}_a}} I_{\{\overline{X}_{\widehat{\tau}_a}<v\}}\right] = \frac{Z^{(q-\psi(u))}_u(a-v)}{Z^{(q-\psi(u))}_u(a)} - \mathrm{e}^{uv}\frac{W^{(q)}(a-v)}{W^{(q)}(a)}. \tag{4.16}$$

holds for $q,u,v\geq 0$. The Laplace transform in v of the identity in (4.16) evaluated at $u=\theta$ implies the equality in (4.4). Since $\mathbb{E}\left[\mathrm{e}^{-q\widehat{\tau}_a} I_{\{Y_{\widehat{\tau}_a}=0\}}\right] = \lim_{\theta\to\infty}\mathbb{E}\left[\mathrm{e}^{-q\widehat{\tau}_a-\theta Y_{\widehat{\tau}_a}}\right]$, we obtain the formula in (4.7) by noting that $\{Y_{\widehat{\tau}_a}=0\} = \{\widehat{\tau}_a = \overline{G}_{\widehat{\tau}_a}\}$ and applying the identity

$$\lim_{\theta\to\infty}\frac{\theta\int_0^\infty \mathrm{e}^{-\theta z}W^{(q)\prime}_+(a+z)\,\mathrm{d}z}{\theta\int_0^\infty \mathrm{e}^{-\theta z}W^{(q)}(a+z)\,\mathrm{d}z} = \frac{W^{(q)\prime}_+(a)}{W^{(q)}(a)},$$

which follows from well-known properties of the scale functions. □

4.2. **The expected drawdown $\overline{S} - S$ at D_α.**

Corollary 5. *The Laplace transform of $T \mapsto \mathbb{E}[(\overline{S}_{D_\alpha} - S_{D_\alpha})\mathbf{1}_{\{D_\alpha<T\}}]$ is given by*

$$\begin{aligned}&\int_0^\infty \mathrm{e}^{-qT}\mathbb{E}[(\overline{S}_{D_\alpha} - S_{D_\alpha})\mathbf{1}_{\{D_\alpha<T\}}]\mathrm{d}T\\ &= \frac{S_0\lambda}{\lambda-1}\left[\int_0^a h_q(x)W^{(q)}(x)\mathrm{d}x\right] - \frac{S_0}{\lambda-1}\left[\int_0^a h_q(x)W^{(q)}(\mathrm{d}x) - \frac{1}{q}\right], \qquad q > \psi(1),\end{aligned} \tag{4.17}$$

where $a = -\log(1-\alpha)$, $\lambda = \lambda(a,q) = W_+^{(q)\prime}(a)/W^{(q)}(a)$ and $h_q(x) = 1 - (1 - q^{-1}\psi(1))\mathrm{e}^{-x}$.

Proof. Recall that $a = -\log(1-\alpha)$. In the notation of Section 3, the Laplace transform on the left-hand side of equation (4.17) is equal to

$$\frac{S_0}{q}\mathbb{E}[\mathrm{e}^{-q\tau_a}\{\mathrm{e}^{\overline{X}_{\tau_a}} - \mathrm{e}^{X_{\tau_a}}\}] = \frac{S_0}{q}\mathbb{E}[\mathrm{e}^{-q\overline{G}_{\tau_a}+\overline{X}_{\tau_a}}\{\mathrm{e}^{-q(\tau_a-\overline{G}_{\tau_a})}(1-\mathrm{e}^{-Y_{\tau_a}})\}].$$

Direct integration of the expressions in Theorem 1 with $u = -\infty$ and $x = 0$ shows that, for $q > \psi(1)$,

$$\begin{aligned}&\mathbb{E}[\mathrm{e}^{-q\tau_a}\{\mathrm{e}^{\overline{X}_{\tau_a}} - \mathrm{e}^{X_{\tau_a}}\}]\\ &= \mathbb{E}[\mathrm{e}^{-q\overline{G}_{\tau_a}+\overline{X}_{\tau_a}}]\{\mathbb{E}[\mathrm{e}^{-q(\tau_a-\overline{G}_{\tau_a})}] - \mathbb{E}[\mathrm{e}^{-q(\tau_a-\overline{G}_{\tau_a})-Y_{\tau_a}}]\},\end{aligned} \tag{4.18}$$

with

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{-q(\tau_a-\overline{G}_{\tau_a})-(Y_{\tau_a}-a)}] &= \frac{\lambda(a,q)}{\lambda(a,0)}\\ &\quad\cdot\left[Z_1^{(q-\psi(1))}(a) - W_1^{(q-\psi(1))}(a)\frac{(q-\psi(1))W_1^{(q-\psi(1))}(a) + Z_1^{(q-\psi(1))}(a)}{W_{1+}^{(q-\psi(1))\prime}(a) + W_1^{(q-\psi(1))}(a)}\right],\end{aligned} \tag{4.19}$$

$$\mathbb{E}[\mathrm{e}^{-q(\tau_a-\overline{G}_{\tau_a})}] = \frac{\lambda(a,q)}{\lambda(a,0)}\left[Z^{(q)}(a) - q\lambda(a,q)^{-1}W^{(q)}(a)\right], \tag{4.20}$$

$$\mathbb{E}[\mathrm{e}^{-q\overline{G}_{\tau_a}+\overline{X}_{\tau_a}}] = \frac{\lambda(a,0)}{\lambda(a,q)}\cdot\frac{\lambda(a,q)}{\lambda(a,q)-1}, \tag{4.21}$$

where the functions $Z_1^{(q-\psi(1))}(a)$ and $W_1^{(q-\psi(1))}$ are defined in equations (3.19) and (3.18). Inserting the expressions in equations (4.19) and (4.20) into equation (4.18) yields the expression in equation (4.17). □

4.3. **Probability of a large drawdown preceding or following a small rally.** In this section we identify explicit expressions for the probabilities of the events $\{D_\alpha < U_\beta\}$ and $\{D_\alpha > U_\beta\}$ in the case when the size $1/(1-\alpha)$ of the relative drawdown is larger or equal to the size $1+\beta$ of the relative drawup. Note that the case $1/(1-\alpha) \geq 1+\beta$ is the case of most interest in practice, since the event $\{D_\alpha < U_\beta\}$ corresponds to the risk held by an investor of losing more in an investment than they would gain if the security rallied. We refer to Hadjiliadis & Vecer [19] for further background on the use of this probability as risk-measure.

The results in this section are a consequence of Corollary 4 and the following key identity:

Lemma 6. *Pick any numbers $a \geq b > 0$. Then the following equality holds almost surely:*

$$\{\tau_a < \widehat{\tau}_b\} = \left\{a - b < Y_{\widehat{\tau}_b}\right\}. \tag{4.22}$$

Remark. In the case $b > a$ and the Lévy measure of X is non-zero, the equality analogous to (4.22) does not hold. The event $\{\tau_a < \widehat{\tau}_b\}$ is different from $\{b - a > \widehat{Y}_{\tau_a}\}$ as it depends in an essential way on the supremum $\overline{\widehat{Y}}_{\tau_a}$ of the reflected process $\widehat{Y}$ at τ_a. The identification of the law of the random variable $\overline{\widehat{Y}}_{\tau_a}$ is beyond the scope of the current paper.

Proof of Lemma 6. Let $\overline{Y}$ and $\overline{\widehat{Y}}$ denote the processes given by the running suprema of the reflected processes Y and $\widehat{Y}$ respectively. In Hadjiliadis & Vecer [19] the following model-free equalities were established:

$$Y + \widehat{Y} = \overline{Y} \vee \overline{\widehat{Y}} = \overline{\widehat{Y}} + \left[0 \vee \left(\overline{Y} - \overline{\widehat{Y}}\right)\right] \tag{4.23}$$

Since $\overline{\widehat{Y}}_{\widehat{\tau}_b} = \widehat{Y}_{\widehat{\tau}_b} = b$ a.s., we find $\{\tau_a < \widehat{\tau}_b\} = \{a < \overline{Y}_{\widehat{\tau}_b}\} = \left\{a - b < \overline{Y}_{\widehat{\tau}_b} - \overline{\widehat{Y}}_{\widehat{\tau}_b}\right\}$, and the lemma follows in view of (4.23). □

By combining Lemma 6 with Corollary 4 we obtain a semi-analytical expression for the probability of interest:

Corollary 7. *Let $a \geq b > 0$ and $t \geq 0$. Then we have $\mathbb{P}\left[\widehat{\tau}_b < \tau_a \wedge t\right] = \mathbb{P}\left[Y_{\widehat{\tau}_b} \leq a - b, \widehat{\tau}_b < t\right]$, where the joint Laplace transform of $(u,t) \mapsto \mathbb{P}\left[Y_{\widehat{\tau}_b} \leq u, \widehat{\tau}_b < t\right]$ is given by the formula*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \mathrm{e}^{-\theta u - qt} \mathbb{P}\left[Y_{\widehat{\tau}_b} \leq u, \widehat{\tau}_b < t\right] \mathrm{d}u = \\ &\frac{1}{q\theta}\left[1 + \left(\frac{(\psi(\theta) - q)\mathrm{e}^{-b\theta}}{Z_\theta^{(q-\psi(\theta))}(b)} - \frac{\theta}{W^{(q)}(b)}\right)\int_0^b W^{(q)}(y)\,\mathrm{d}y\right]. \end{aligned} \tag{4.24}$$

for any $\theta, q > 0$. Moreover for any $a > 0$ we have

$$\mathbb{P}\left[\tau_a < \widehat{\tau}_a\right] \;=\; \frac{W'_+(a)}{W(a)^2}\int_0^a W(z)\,\mathrm{d}z.$$

Remarks. (i) It is a consequence of the representation of the scale function $Z_\theta^{(-\psi(\theta))}$ given in (3.19), that the Laplace transform in (4.24) can be expressed as

$$\theta\int_0^\infty \mathrm{e}^{-\theta u}\mathbb{P}\left[Y_{\widehat{\tau}_b} \leq u\right]\mathrm{d}u = 1 + \left(1/\int_0^\infty \mathrm{e}^{-z\theta}W(b+z)\,\mathrm{d}z - \theta/W(b)\right)\int_0^b W(y)\,\mathrm{d}y.$$

(ii) The probability measure in Corollary 7 does not depend on the starting point of the process X, since the reflected processes $Y, \widehat{Y}$, and hence the stopping times $\tau_a, \widehat{\tau}_b$, are independent of X_0. Therefore equality in (4.24) remains valid if $\mathbb{P}[\cdot]$ is replaced by $\mathbb{P}_x[\cdot]$ for any starting point $x \in \mathbb{R}$.
(iii) Recall from (3.15) that in the case X is a Brownian motion with drift $\mu \in \mathbb{R} \setminus \{0\}$ and the

Gaussian coefficient $\sigma^2 > 0$, the scale function takes the form $W(x) = \left(1 - \mathrm{e}^{-x2\mu/\sigma^2}\right)/\mu$ for $x \geq 0$. The formula in Corollary 7 yields

$$\mathbb{P}\left[\tau_a < \widehat{\tau}_a\right] = \frac{\mathrm{e}^{-a2\mu/\sigma^2} - 1 + a2\mu/\sigma^2}{\left(\mathrm{e}^{a\mu/\sigma^2} - \mathrm{e}^{-a\mu/\sigma^2}\right)^2},$$

which coincides with the formula in (2.1) of [19]. In particular if $\mu = 0$ then Corollary 7 yields $\mathbb{P}\left[\tau_a < \widehat{\tau}_a\right] = 1/2$, which follows of course also directly by symmetry.

4.4. **Example: Carr & Wu model.** Carr & Wu [9] documents the persistence of the implied volatility skew across maturities in the exchange traded options on the S&P 500 index. The evidence presented in [9] suggests that the left tail of the risk-neutral distribution of returns of the index remains "fat" as the maturity increases. In order to capture this phenomenon Carr & Wu [9] model the risk-neutral evolution of the index level by the following stochastic differential equation:

$$\mathrm{d}S_t/S_{t-} = (r - q)\,\mathrm{d}t + \Sigma\,\mathrm{d}L_t^{\alpha,-1}, \qquad \alpha \in (1,2),\ \Sigma > 0, \tag{4.25}$$

where r and q denote the continuously compounded risk free rate and dividend yield. The driver $L^{\alpha,-1}$ is assumed to be an α-stable Lévy martingale with maximal negative skewness, i.e. $L^{\alpha,-1}$ is a spectrally negative α-stable process. In this model the decay of the left tail of the log return (i.e. of the quantity $\mathbb{P}\left[\log(S_T/S_t) < s\right]$ for any fixed $0 \leq t < T$) is asymptotically equal to $|s|^{-\alpha}$ as s tends to $-\infty$, while the right tail decays exponentially, so that the model (4.25) captures the observed phenomenon of the persistence of the implied volatility skew across maturities.

In order to understand the risk-neutral probabilities of drawdowns and rallies in this model for the S&P 500 index, note that the process S can be expressed as $S = \exp(X)$, where X is a spectrally negative α-stable process with drift. The cumulant generating function of X takes the form

$$\psi(\theta) = \mu\theta + (\sigma\theta)^\alpha \quad \text{for} \quad \theta > 0, \quad \text{where} \quad \alpha \in (1,2), \tag{4.26}$$

for some $\mu \in \mathbb{R}, \sigma > 0$ that depend on r, q, Σ and α. Note that in the limit case of $\alpha = 2$, the process X corresponds to Brownian motion with drift μ and variance $2\sigma^2$ at time one, and in this case the model (4.25) reduces to the Black-Scholes model.

As can be seen from Proposition 7, the probability of a drawdown preceding a rally in the model (4.25) is given in terms of scale functions of the process X, which is expressed in terms of the Mittag-Leffler function

$$E_{\beta,\gamma}(y) = \sum_{n=0}^{\infty} \frac{y^n}{\Gamma(n\beta + \gamma)}, \quad y \in \mathbb{R}.$$

Note that $E_{\beta,\gamma}$ is an entire function if β, γ are strictly positive and that it generalises the exponential $E_{1,1}(y) = \mathrm{e}^y$ and the hyperbolic cosine $E_{2,1}(y^2) = \cosh(y)$ functions. Furrer [18] identified the 0-scale function W of X as

$$W(x) = \frac{1}{\mu}\left[1 - E_{\alpha-1,1}\left(-\frac{\mu}{\sigma^\alpha}x^{\alpha-1}\right)\right], \qquad x \geq 0, \alpha \in (1,2]. \tag{4.27}$$

As a consequence, the q-scale function $W^{(q)}$ of X, which is related to W by the well-known expression $W^{(q)}(x) = \sum_{k\geq 0} q^k W^{\star(k+1)}(x)$ that is given in terms of the k-th convolution $W^{\star k}$, $k \in \mathbb{N}$, of W with itself, admits the following series representation:

$$W^{(q)}(x) = \frac{1}{\mu}\left[1 - E_{\alpha-1,1,1}\left(-\frac{\mu}{\sigma^\alpha}x^{\alpha-1}, -\frac{qx}{\mu}\right)\right], \qquad x, q \geq 0, \alpha \in (1,2], \tag{4.28}$$

where, for any nonnegative β, γ and δ, the function $E_{\beta,\gamma,\delta} : \mathbb{R}^2 \to \mathbb{R}$ is the function that is defined by

$$E_{\beta,\gamma,\delta}(y,z) := \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{s_{k,n}}{\Gamma(n\beta + k\delta + \gamma)} z^k y^n, \qquad y, z \in \mathbb{R},$$

where the coefficients $(s_{k,n}, n, k \in \mathbb{N}\cup\{0\})$ are given by

$$s_{k,n+1} = \left(\begin{array}{c} n \\ k \end{array}\right), \qquad k, n \in \mathbb{N}\cup\{0\}, \qquad s_{0,0} = 1.$$

Note that $s_{k,n} = \frac{(n-1)!}{k!(n-1-k)!}$ if $k = 0, 1, \ldots, n-1$, and $s_{k,n}$ is equal to zero otherwise.

Remarks. (i) In the limit as α approaches 2, the formula in (4.27) tends to the scale function of the Brownian motion with volatility $\sigma\sqrt{2}$, which is given in (3.15).
(ii) It follows from the formula in (4.27) that if the modulus of the drift μ tends to zero, then in the limit we obtain the scale function, of the spectrally negative α-stable process, which is given in (3.17).
(iii) It is assumed in (4.27) that $\alpha > 1$ and hence the series $E_{\alpha-1,1}$ converges uniformly on compact subsets. Therefore the derivative and the integral of the scale function can be obtained by performing these operations under the summation:

$$W'(x) = \frac{\alpha-1}{\sigma^\alpha}x^{\alpha-2}E'_{\alpha-1,1}\left(-\frac{\mu}{\sigma^\alpha}x^{\alpha-1}\right), \qquad x > 0. \tag{4.29}$$

$$\int_0^a W(z)\,\mathrm{d}z = \frac{a}{\mu}\left[1 - E_{\alpha-1,2}\left(-\frac{\mu}{\sigma^\alpha}a^{\alpha-1}\right)\right], \qquad a > 0. \tag{4.30}$$

We now present an analytically tractable expression for the probability of a large drawdown occurring before a small rally. Let $\mathcal{L}^{-1}$ denote the inverse Laplace transform.

Corollary 8. *Let $a = -\log(1-x)$, $x \in (0,1)$, and $b = \log(1+y)$, $y > 0$, and $a > b$. Then the probability of a drawdown of size $(100x)\%$, in the Carr & Wu model (4.25) occurs before a rally of size $(100y)\%$ is given by $\mathbb{P}[\tau_a < \widehat{\tau}_b] = \mathcal{L}^{-1}(F)(a-b)$ where*

$$F(\theta) = \frac{1}{\theta} - \left[\mu \mathrm{e}^{-b\theta} \Big/ \left(\sum_{n=1}^{\infty}\left(-\frac{\mu}{\sigma^\alpha\theta^{\alpha-1}}\right)^n \frac{\Gamma(n(\alpha-1)+1, b\theta)}{\Gamma(n(\alpha-1)+1)}\right) + 1/W(b)\right]\int_0^b W(z)\,\mathrm{d}z, \tag{4.31}$$

where $\Gamma(n(\alpha-1)+1, b\theta)$ denotes the incomplete gamma function (see (3.25)) and the scale function W and its anti-derivative are explicitly given in (4.27) and (4.30).

Remark. Standard Laplace inversion algorithms, such as the one given in Abate and Whitt [1], would typically evaluate the function in (4.31) about twenty times in order to compute the probability $\mathbb{P}\left[Y_{\widehat{\tau}_b} \leq a-b\right]$. Since the series in expression (4.31) is dominated by a geometric series, the error at each evaluation of the Laplace transform can be controlled easily.

In the case when the relative drawdown level $1/(1-x)$ is strictly larger than the size $1+y$ of the relative drawup (i.e. rally) of the index S, the probability of the market crash $\mathbb{P}\left[\tau_a < \widehat{\tau}_b\right]$, where $a = -\log(1-x)$ and $b = \log(1+y)$, is by Proposition (7) equal to $\mathbb{P}\left[Y_{\widehat{\tau}_b} \leq a-b\right]$. By Proposition 7 the Laplace transform of the function $u \mapsto \mathbb{P}\left[Y_{\widehat{\tau}_b} \leq u\right]$ is given by the formula in (4.31).

The previous results yield a closed-form expression for the probability of a drawdown occurring before a rally in the case these are of the same size in the log scale (cf. (4.1) and (4.2)):

Corollary 9. *The probability of a drawdown of size* $(100x)\%$*,* $x \in (0,1)$*, in the Carr & Wu model* (4.25) *occurs before a rally of size* $(100x/(1-x))\%$ *is given by*

$$\mathbb{P}\left[\tau_a < \widehat{\tau}_a\right] = (\alpha-1)AE'_{\alpha-1,1}\left[-A\right]\frac{1-E_{\alpha-1,2}\left[-A\right]}{\left(1-E_{\alpha-1,1}\left[-A\right]\right)^2}, \quad \textit{with} \quad A = \frac{\mu}{\sigma^\alpha}a^{\alpha-1}, \tag{4.32}$$

where $a = -\log(1-x)$.

Note that in the limiting case $\alpha = 2$, formula (4.32) yields the well-known expression for Brownian motion with drift μ and volatility $\sigma\sqrt{2}$. Formula (4.32) follows from Proposition 7 and the formulas (4.27), (4.29) and (4.30).

5. Proofs

5.1. **Proof of Theorem 1.** We now give a proof of Theorem 1 based on Itô's excursion theory.

Start by noting that since both sides in (3.10) and (3.11) are continuous in q, it suffices to prove the formulae in (3.10) and (3.11) for $q > 0$, which will be assumed without loss of generality. We prove the result in three steps. Step (1) deals with the segment of a path in

$$A_o = \left\{\underline{X}_{\tau_a} \geq u, \overline{X}_{\tau_a} \in \mathrm{d}v, Y_{\tau_a-} \in \mathrm{d}y, Y_{\tau_a} - a \in \mathrm{d}h\right\}$$

over the time interval $[0, T_{u,u+a}]$ by applying the strong Markov property at time $T_{u,u+a}$. Step (2) extracts the final jump ΔX_{τ_a} in the expectation of (3.10) by applying the compensation formula for the Poisson point process $(\Delta X_t)_{t\geq 0}$. Step (3) applies the Itô's excursion theory to the segment of a path in A_o over the interval $(T_{u,u+a}, \tau_a)$. In the case of the event A_c the structure of the proof is similar with steps (2) and (3) merged as there is no overshoot at time τ_a.

Step (1) Note that $\mathbb{P}_x\left[\{T_{u,u+a} = T_u^-\} \cap A_o\right] = 0$ (see Figure 1) and that on A_o we have $T_{u,u+a} < \overline{G}_{\tau_a}$ $\mathbb{P}_x$-a.s. We can therefore apply the strong Markov property at $T_{u,u+a}$ and identity (2.3) to the

expectation in (3.10):

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-q\tau_a - r\overline{G}_{\tau_a}} I_{A_o}\right] &= \mathbb{E}_x\left[\mathrm{e}^{-(q+r)T_{u,u+a}} I_{\{T_{u,u+a}=T^+_{u+a}\}} \mathrm{e}^{-q(\tau_a - T_{u,u+a}) - r\left(\overline{G}_{\tau_a} - T_{u,u+a}\right)} I_{A_o}\right] \\ &= \frac{W^{(q+r)}(a \wedge (x-u))}{W^{(q+r)}(a)}\, \mathbb{E}_{x\vee(u+a)}\left[\mathrm{e}^{-q\tau_a - r\overline{G}_{\tau_a}} I_{A_o}\right]. \qquad (5.1)\end{aligned}$$

Step (2) By (5.1) we may assume that the starting point x of the process X is greater or equal to $u+a$. Since $Y_{\tau_a -} \in [0,a]$ this implies that

$$\underline{X}_{\tau_a -} \geq u \quad \mathbb{P}_x\text{-a.s. and hence} \quad \left\{\underline{X}_{\tau_a} \geq u\right\} = \left\{X_{\tau_a -} + \Delta X_{\tau_a} \geq u\right\}\ \mathbb{P}_x\text{-a.s.} \qquad (5.2)$$

In order to deal with the jump at time τ_a we apply the compensation formula (see Section O.5 in [6]) to the Poisson point process $(\Delta X_t)_{t\geq 0}$ as follows:

$$\begin{aligned}\mathbb{E}_x\left[\mathrm{e}^{-q\tau_a - r\overline{G}_{\tau_a}} I_{A_o}\right] &= \mathbb{E}_x\left[\mathrm{e}^{-q\tau_a - r\overline{G}_{\tau_a}} I_{\left\{\overline{X}_{\tau_a}\in \mathrm{d}v, \underline{X}_{\tau_a}\geq u, Y_{\tau_a -}\in \mathrm{d}y, Y_{\tau_a}-a\in \mathrm{d}h\right\}}\right] \\ &= \mathbb{E}_x\left[\sum_{t\geq 0} \mathrm{e}^{-qt - r\overline{G}_t} I_{\left\{\sup_{s<t}\{Y_s\}\leq a, Y_t > a, \overline{X}_t \in \mathrm{d}v, Y_{t-}\in \mathrm{d}y\right\}} I_{\{y-a-\Delta X_t \in \mathrm{d}h, \Delta X_t \geq u+y-v\}}\right] \\ &= I_{\{h\in(0,u-v-a)\}} \Lambda\left(y-a-\mathrm{d}h\right) \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qt-r\overline{G}_t} I_{\left\{\overline{X}_t\in \mathrm{d}v, Y_{t-}\in \mathrm{d}y, \sup_{s<t}\{Y_s\}\leq a\right\}} \mathrm{d}t\right] \\ &= I_{\{h\in(0,u-v-a)\}} \Lambda\left(y-a-\mathrm{d}h\right) \left(\mathbb{E}_x \times E\right)\left[\mathrm{e}^{-r\overline{G}_\eta} I_{\left\{\overline{X}_\eta \in \mathrm{d}v, Y_\eta \in \mathrm{d}y, \eta<\tau_a\right\}}\right]/q, \qquad (5.3)\end{aligned}$$

where the sum runs over all $t \geq 0$ such that $\Delta X_t \neq 0$ and, as noted before, $q > 0$, and η is an exponential random variable defined on a probability space $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), P)$ such that $(\mathbb{P}\times P)[\eta > s|\mathcal{F}] = P(\eta > s) = \mathrm{e}^{-qs}$ for $s \in \mathbb{R}_+$. The second equality follows by (5.2). The third equality follows from the compensation formula for the Poisson point process of jumps of X and the fact that the inequality $\Delta X_t \geq u+y-v$ is equivalent to the restriction on h given in (3.9).

Step (3) The final task is to calculate the expectation in (5.3). This requires the Itô excursion theory for the reflected process Y. Since X is spectrally negative, the local time at zero of Y can be taken to be the running supremum of X, i.e. $L = \overline{X} - x$, under $\mathbb{P}_x$. Let $(t, \epsilon_t)_{t\geq 0}$ be the Poisson point process of excursions of X from its supremum $\overline{X}$, taking values in $[0,\infty)\times(\mathcal{E}\cup\{\partial\})$, where

$$\mathcal{E} = \left\{\varepsilon \in D(\mathbb{R}) : \exists \zeta \in (0,\infty] \text{ such that } \varepsilon(\zeta) = 0 \text{ if } \zeta < \infty,\ \varepsilon(0) \geq 0,\ \varepsilon(t) > 0\ \forall t \in (0,\zeta)\right\},$$

$D(\mathbb{R})$ is the Skorokhod space and ∂ is the graveyard state, with the intensity measure $\mathrm{d}t \times n(\mathrm{d}\varepsilon)$, where n denotes the Itô excursion measure. For each instance of local time $t \in (0, v-x]$, the inverse local time L^{-1} satisfies $L_{L_t^{-1}} = \overline{X}_{L_t^{-1}} - x = t$ since $L_t^{-1} = T^+_{x+t}$. Since the excursions are indexed by local time, for any $t \in (0, v-x]$ we have $\epsilon_t = \{X_{L^{-1}_{t-}} - X_{L^{-1}_{t-}+s} : 0 < s \leq L_t^{-1} - L_{t-}^{-1}\}$ if $L_{t-}^{-1} < L_t^{-1}$ and $\epsilon_t = \partial$ if $L_{t-}^{-1} = L_t^{-1}$. This implies that the excursions can be indexed by a subset of actual time that is given by the left-end points of excursion intervals. For any excursion $\varepsilon \in \mathcal{E}$ define $T_a(\varepsilon) = \inf\{s > 0 : \varepsilon(s) > a\}$ (with the convention $\inf\emptyset = \infty$) and let $\zeta(\varepsilon)$ be the life time of ε.

Define also the height of the excursion ε by $\overline{\varepsilon} = \sup\{\varepsilon(s) : 0 < s < \zeta(\varepsilon)\}$. We now obtain in the case $y > 0$

$$
\begin{aligned}
&(\mathbb{E}_x \times E)\left[\mathrm{e}^{-r\overline{G}_\eta} I_{\left\{\overline{X}_\eta \in \mathrm{d}v, Y_\eta \in \mathrm{d}y, \eta < \tau_a\right\}}\right] \\
&= (\mathbb{E}_x \times E)\left[\sum_{g\geq 0} \mathrm{e}^{-r\overline{G}_g} I_{\left\{\sup_{h<g}\{\overline{\epsilon}_h\}\leq a,\, \eta\in(g, g+(T_a\wedge\zeta)(\epsilon_g)),\, \overline{X}_g\in \mathrm{d}v\right\}} I_{\{\epsilon_g(\eta-g)\in \mathrm{d}y\}}\right] \\
&= (\mathbb{E}_x \times E)\left[\int_0^\infty \mathrm{d}\overline{X}_s \mathrm{e}^{-r\overline{G}_s} I_{\left\{\sup_{h<s}\{\overline{\epsilon}_h\}\leq a,\, \overline{X}_s\in \mathrm{d}v\right\}} I_{\{\eta>s\}}\, n\left[t' < (T_a\wedge\zeta)(\varepsilon),\, \varepsilon(t')\in \mathrm{d}y\right]\big|_{t'=\eta-s}\right] \\
&= \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qs-r\overline{G}_s} I_{\left\{\sup_{h<s}\{\overline{\epsilon}_h\}\leq a,\, \overline{X}_s\in \mathrm{d}v\right\}} \mathrm{d}\overline{X}_s\right] \\
&\cdot E\left[n\left(\eta < (T_a\wedge\zeta)(\varepsilon),\, \varepsilon(\eta)\in \mathrm{d}y\right)\right], \qquad (5.4)
\end{aligned}
$$

where the sum in the first equality is over all left-end points of excursion intervals. The second equality follows from the compensation formula of excursion theory (see [6, Cor. IV.11]) and the third from Fubini's theorem, the independence of X and η and the fact $P[\eta > s] = \mathrm{e}^{-qs}$. The second expectation in (5.4) is given by

$$
E\left[n\left(\eta < (T_a\wedge\zeta)(\varepsilon),\, \varepsilon(\eta)\in \mathrm{d}y\right)\right] = q\left[W_+^{(q)\prime}(y) - \frac{W_+^{(q)\prime}(a)}{W^{(q)}(a)} W^{(q)}(y)\right]\mathrm{d}y \qquad y\in(0,a). \qquad (5.5)
$$

The formula in (5.5) was proved in [30] (see equation (22)).

In order to obtain the first expectation in (5.4) first note that, since at the time L_t^{-1} the process X is at its supremum level $x + t$ (i.e. $L_t^{-1} = T_{x+t}^+$), we must have $\overline{G}_{L_t^{-1}} = L_t^{-1}$ $\mathbb{P}_x$-a.s for all $t > 0$. Furthermore the identity $\overline{X}_{L_t^{-1}} - x = t$ implies that if we reparametrise the integral under the first expectation in (5.4) using inverse local time we find

$$
\begin{aligned}
&\mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qs-r\overline{G}_s} I_{\left\{\sup_{h<s}\{\overline{\epsilon}_h\}\leq a,\, \overline{X}_s\in \mathrm{d}v\right\}} \mathrm{d}\overline{X}_s\right] \\
&= \mathbb{E}_x\left[\mathrm{e}^{-(q+r)L_{v-x}^{-1}} I_{\left\{\sup_{h<L_{v-x}^{-1}}\{\overline{\epsilon}_h\}\leq a\right\}}\right]\mathrm{d}v \\
&= \mathrm{e}^{-\Phi(q+r)(v-x)}\mathbb{E}_x\left[\mathrm{e}^{-(q+r)L_{v-x}^{-1}+\Phi(q+r)(v-x)} I_{\left\{\sup_{h<L_{v-x}^{-1}}\{\overline{\epsilon}_h\}\leq a\right\}}\right]\mathrm{d}v \\
&= \mathrm{e}^{-\Phi(q+r)(v-x)}\mathbb{P}_x^{\Phi(q+r)}\left[T_v^+ < \tau_a\right]\mathrm{d}v. \qquad (5.6)
\end{aligned}
$$

The third equality follows by an application of the Esscher change of measure formula and the equality in (5.6) is a consequence of the following

$$
\{T_v^+ < \tau_a\} = \{\forall t\in[0, v-x] \text{ with } \epsilon_t \neq \partial,\, \overline{\epsilon}_t \leq a\} \quad \mathbb{P}_x\text{ -a.s.}
$$

Since $N_t = \#\{u\in[0,t] : \overline{\epsilon}_u > a, \epsilon_u \neq \partial\}$ is a Poisson process with parameter $n^{\Phi(q+r)}(\overline{\varepsilon} > a)$ under the measure $\mathbb{P}_x^{\Phi(q+r)}$, where $n^{\Phi(q+r)}$ is the Itô excursion measure of X under the measure $\mathbb{P}_x^{\Phi(q+r)}$,

we obtain

$$\mathbb{P}_x^{\Phi(q+r)}\left[T_v^+ < \tau_a\right] = \mathrm{e}^{-(v-x)n^{\Phi(q+r)}(\overline{\varepsilon}>a)}, \qquad \text{where} \tag{5.7}$$

$$n^{\Phi(q+r)}\left(\overline{\varepsilon} > a\right) = \frac{W'_{\Phi(q+r)+}(a)}{W_{\Phi(q+r)}(a)} = \frac{W_+^{(q+r)\prime}(a)}{W^{(q+r)}(a)} - \Phi(q+r) \tag{5.8}$$

and $W_{\Phi(q+r)}(a)$ denotes the 0-scale function under the measure $\mathbb{P}_x^{\Phi(q+r)}$. The first equality in (5.8) is a well-known representation of the scale function in terms of the excursion measure and the second equality follows from the Esscher change of measure formula $W_{\Phi(q+r)}(x) = \mathrm{e}^{-x\Phi(q+r)}W^{(q+r)}(x)$.

Since $\overline{G}_\eta = \eta$ on the set $\{Y_\eta = 0\}$ we find in the case $y = 0$

$$\begin{aligned}
(\mathbb{E}_x \times E)[\mathrm{e}^{-rG_\eta} I_{\{\overline{X}_\eta > v, Y_\eta = 0, \eta < \tau_a\}}] &= (\mathbb{E}_x \times E)[\mathrm{e}^{-r\eta} I_{\{\overline{X}_\eta > v, Y_\eta = 0, \eta < \tau_a\}}] \\
&= \mathbb{E}_x\left[\int_0^\infty q\mathrm{e}^{-(r+q)t} I_{\{T_v^+ < t, Y_t = 0, t < \tau_a\}}\mathrm{d}t\right] \\
&= \frac{q}{r+q}\mathbb{E}_x[\mathrm{e}^{-(r+q)T_v^+} I_{\{T_v^+ < \tau_a\}}]\mathbb{E}_0\left[\int_0^{\tau_a}(r+q)\mathrm{e}^{-(q+r)t} I_{\{Y_t=0\}}\mathrm{d}t\right] \\
&= q\exp\left(-v\frac{W_+^{(q+r)\prime}(a)}{W^{(q+r)}(a)}\right)\cdot\frac{W^{(q+r)}(a)}{W_+^{(q+r)\prime}(a)}W^{(q+r)}(0),
\end{aligned}$$

where the first factor follows by combining (5.6), (5.7) and (5.8) and the second factor follows from [30, equation (22)]. Since $W^{(q+r)}(0) = W^{(q)}(0)$ we thus obtain

$$(\mathbb{E}_x \times E)[\mathrm{e}^{-rG_\eta} I_{\{\overline{X}_\eta \in \mathrm{d}v, Y_\eta = 0, \eta < \tau_a\}}] = q\exp\left(-v\frac{W^{(q+r)\prime}(a)}{W^{(q+r)}(a)}\right)W^{(q)}(0). \tag{5.9}$$

Identities (5.1), (5.3), (5.4), (5.5), (5.6) and (5.7) and (5.9) together imply the formula in (3.10). This concludes the proof in the case of the event A_o.

In the case of creeping, step (1) consists of the application of the strong Markov property and the first factor in (3.11) follows from an analogous calculation to (5.1) with A_o replaced by A_c. We therefore assume in what follows that $x \geq u + a$.

Step (2) does not feature in the context of A_c as there is no overshoot at τ_a. For the analogue of step (3) note that $x \geq u + a$ implies (see also Figure (1)) the inclusion

$$\{Y_{\tau_a} = a\} \subset \left\{\underline{X}_{\tau_a} \geq u\right\} \quad \mathbb{P}_x\text{ -a.s.}$$

The following excursion calculation, similar to (5.4), provides a key step:

$$\begin{aligned}
&\mathbb{E}_x\left[\mathrm{e}^{-q\tau_a - r\overline{G}_{\tau_a}} I_{\left\{\overline{X}_{\tau_a}\in\mathrm{d}v, Y_{\tau_a}=a\right\}}\right] \\
&= \mathbb{E}_x\left[\sum_{g\geq 0}\mathrm{e}^{-gq-r\overline{G}_g} I_{\left\{\sup_{h<g}\{\overline{\epsilon}_h\}\leq a,\, \overline{X}_{\tau_a}\in\mathrm{d}v\right\}}\mathrm{e}^{-qT_a(\epsilon_g)} I_{\{T_a(\epsilon_g)<\zeta(\epsilon_g), \epsilon_g(T_a(\epsilon_g))=a\}}\right] \qquad (5.10) \\
&= \mathbb{E}_x\left[\int_0^\infty \mathrm{e}^{-qs-r\overline{G}_s} I_{\left\{\sup_{h<s}\{\overline{\epsilon}_h\}\leq a,\, \overline{X}_s\in\mathrm{d}v\right\}}\mathrm{d}\overline{X}_s\right]\int_{\mathcal{E}}\mathrm{e}^{-qT_a(\varepsilon)} I_{\{T_a(\varepsilon)<\zeta(\varepsilon), \varepsilon(T_a(\varepsilon))=a\}}\, n(\mathrm{d}\varepsilon) \qquad (5.11)
\end{aligned}$$

The summation in (5.10) is over the left-end points of the excursion intervals and the equality in (5.11) follows from the compensation formula of Itô's excursion theory.

The expectation in the first factor in (5.11) is identical to (5.6). To find the second factor in (5.11) first note for any $x > 0$ the following identities hold as a consequence of the compensation formula:

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{-qT^-_{-a}} I_{\left\{X_{T^-_{-a}} = -a, T^-_{-a} < T^+_x\right\}}\right] \\
&= \mathbb{E}\left[\sum_{g\geq 0} \mathrm{e}^{-gq} I_{\left\{\sup_{h<g}\{\overline{\epsilon}_h - \overline{X}_h\} \leq a,\, \overline{X}_g \leq x\right\}} \mathrm{e}^{-qT_{a+\overline{X}_g}(\epsilon_g)} I_{\left\{T_{a+\overline{X}_g}(\epsilon_g) < \zeta(\epsilon_g), \epsilon_g(T_{a+\overline{X}_g}(\epsilon_g)) = a + \overline{X}_g\right\}}\right] \\
&= \mathbb{E}\left[\int_0^x \mathrm{e}^{-qL^{-1}_y} I_{\left\{\sup_{h<L^{-1}_y}\{\overline{\epsilon}_h - \overline{X}_h\} \leq a\right\}} \mathrm{d}y \int_{\mathcal{E}} \mathrm{e}^{-qT_{a+y}(\varepsilon)} I_{\{T_{a+y}(\varepsilon) < \zeta(\varepsilon), \varepsilon(T_{a+y}(\varepsilon)) = a+y\}} n(\mathrm{d}\varepsilon)\right].
\end{aligned}$$

The second equality follows by reparametrising the integral which arises in the compensation formula using inverse local time. Since $L^{-1}_0 = 0$ a.s. and the trajectories of X and L^{-1} are right-continuous, the following equality holds:

$$\int_{\mathcal{E}} \mathrm{e}^{-qT_a(\varepsilon)} I_{\{T_a(\varepsilon) < \zeta(\varepsilon), \varepsilon(T_a(\varepsilon)) = a\}}\, n(\mathrm{d}\varepsilon) \;=\; \lim_{x \searrow 0} x^{-1} \mathbb{E}\left[\mathrm{e}^{-qT^-_{-a}} I_{\left\{X_{T^-_{-a}} = -a, T^-_{-a} < T^+_x\right\}}\right]. \tag{5.12}$$

This limit can also be derived from Proposition 2 in [12]. Insert identity (2.6) into (5.12), apply L'Hopital's rule and the fact $W^{(q)\prime}(0) = \frac{2}{\sigma^2}$ to obtain a formula for the second factor in (5.11). This implies (3.11) and therefore concludes the proof of the theorem. □

5.2. **Proof of Theorem 3.** We start by noting that the spectral negativity of X implies the following identity $X_{\widehat{\tau}_a} - \underline{X}_{\widehat{\tau}_a} = \widehat{Y}_{\widehat{\tau}_a} = a$ a.s. We can therefore apply the Esscher transform to (3.20) to obtain

$$\mathbb{E}\left[\mathrm{e}^{-q\widehat{\tau}_a - r\underline{G}_{\widehat{\tau}_a} + u\underline{X}_{\widehat{\tau}_a}} I_{\left\{\overline{X}_{\widehat{\tau}_a} < v\right\}}\right] = \mathrm{e}^{-au} \mathbb{E}^u\left[\mathrm{e}^{-(q-\psi(u))\widehat{\tau}_a - r\underline{G}_{\widehat{\tau}_a}} I_{\left\{\overline{X}_{\widehat{\tau}_a} < v\right\}}\right]. \tag{5.13}$$

If $a < v$, then $\mathbb{P}\left[\overline{X}_{\widehat{\tau}_a} < v\right] = 1$ as remarked above (see also Figure 2). Note further that

$$\left\{\overline{X}_{\widehat{\tau}_a} < v\right\} = \left\{T^-_{v-a} < T^+_v\right\} \qquad \text{for any} \quad v \in [0, a].$$

Observe that $X_{T^-_{v-a}} \leq v - a$ $\mathbb{P}$-a.s. Hence the strong Markov property applied to the right-hand side of (5.13) and formula (2.4) yield

$$\begin{aligned}
\mathbb{E}^u\left[\mathrm{e}^{-(q-\psi(u))\widehat{\tau}_a - r\underline{G}_{\widehat{\tau}_a}} I_{\left\{\overline{X}_{\widehat{\tau}_a} < v\right\}}\right] &= \mathbb{E}^u\left[\mathrm{e}^{-(q+r-\psi(u))T^-_{v-a}} I_{\left\{T^-_{v-a} < T^+_v\right\}}\right] \mathbb{E}^u\left[\mathrm{e}^{-(q-\psi(u))\widehat{\tau}_a - r\underline{G}_{\widehat{\tau}_a}}\right] \\
&= \left[Z^{(p)}_u(a-v) - Z^{(p)}_u(a) \frac{W^{(p)}_u(a-v)}{W^{(p)}_u(a)}\right] \frac{W^{(q+r)}(a)}{W^{(q)}(a) Z^{(p)}_u(a)}
\end{aligned}$$

for all $v \geq 0$, where $p = q + r - \psi(u)$. To complete the proof we need to show that

$$\mathbb{E}^u\left[\mathrm{e}^{-(q-\psi(u))\widehat{\tau}_a - r\underline{G}_{\widehat{\tau}_a}}\right] = \frac{W^{(q+r)}(a)}{W^{(q)}(a) Z^{(p)}_u(a)} \quad \text{for} \quad q, r, u \geq 0. \tag{5.14}$$

This, together with (3.18)–(3.19), implies the formula in the theorem.

By the compensation formula applied to the Poisson point process of excursions of $\widehat{Y}$ away from zero,

$$\mathbb{E}^u[\mathrm{e}^{-(q-\psi(u))\widehat{\tau}_a - r\underline{G}_{\widehat{\tau}_a}}] = \mathbb{E}^u\left[\int_0^\infty \mathrm{e}^{-pt} I_{\{t<\underline{G}_{\widehat{\tau}_a}\}} \mathrm{d}\widehat{L}_t\right] \int_{\mathcal{E}} \mathrm{e}^{-(q-\psi(u))T_a(\varepsilon)} I_{\{T_a(\varepsilon)<\zeta(\varepsilon)\}} \widehat{n}^u(\mathrm{d}\varepsilon), \tag{5.15}$$

where $\widehat{n}^u$ is the corresponding excursion measure under the probability measure $\mathbb{P}^u$, $\widehat{L}$ a local time process of $\widehat{Y}$ at zero and T_a and ζ are as in the proof of Theorem 1 (see page 20). From Bertoin [6, Ch. VII, Prop. 15] it follows that for any $(\mathcal{F}_t)$-stopping time τ and $A \in \mathcal{F}_\tau$ we have

$$\widehat{n}^u(A \cap \{\tau < \zeta\}) = \lim_{x \searrow 0} W_u(x)^{-1} \mathbb{E}^u_x\left[I_{A\cap\{\tau<T_0^-\}}\right] \qquad \text{for any} \quad u \geq 0. \tag{5.16}$$

This limit can also be derived from [12, Proposition 2]. We now apply (5.16) to the stopping time $\tau = T_a^+$ to obtain

$$\begin{aligned}
\int_{\mathcal{E}} \mathrm{e}^{-(q-\psi(u))T_a(\varepsilon)} I_{\{T_a(\varepsilon)<\zeta(\varepsilon)\}} \widehat{n}^u(\mathrm{d}\varepsilon) &= \lim_{x\searrow 0} W_u(x)^{-1} \mathbb{E}^u_x\left[\mathrm{e}^{-(q-\psi(u))T_a^+} I_{\{T_a^+<T_0^-\}}\right] \\
&= W_u^{(q-\psi(u))}(a)^{-1} \lim_{x\searrow 0} \frac{W_u^{(q-\psi(u))}(x)}{W_u(x)} \\
&= W_u^{(q-\psi(u))}(a)^{-1} \qquad (5.17)
\end{aligned}$$

for any $q \geq \psi(u)$. The second equality follows from (2.3). The third equality is a consequence of the well-known identity $W^{(l)}(x) = \sum_{k=0}^\infty l^k W^{*(k+1)}(x)$, which holds for all $l \in \mathbb{C}$, $x \geq 0$, where W^{*k} denotes the k-th convolution of W with itself.

The identity in (5.15) for $r = 0$, the Laplace transform $\mathbb{E}^u\left[\mathrm{e}^{-(q-\psi(u))\widehat{\tau}_a}\right] = 1/Z_u^{(q-\psi(u))}(a)$, given in [30, Prop. 2], and identity (5.17) imply

$$\mathbb{E}^u\left[\int_0^\infty \mathrm{e}^{-qt} I_{\{t<\underline{G}_{\widehat{\tau}_a}\}} \mathrm{d}\widehat{L}_t\right] = \frac{W_u^{(q-\psi(u))}(a)}{Z_u^{(q-\psi(u))}(a)}$$

for $q \geq \psi(u)$. This identity, together with (5.15) and (5.17), yields (5.14) for large q. Analyticity of expressions in (5.14) on both sides of the identity imply (5.14) for all parameter values. This concludes the proof of the theorem. □

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