

Scattering constructions for nonlinear wave equations on Kerr-AdS spacetimes

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Statement of originality

I confirm that the contents of this thesis are my own work and that all contributions from other sources are referenced appropriately.

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Abstract

By means of a backwards scattering construction, existence of a class of exponentially decaying solutions of the nonlinear massive wave equation $\square_g \psi + \alpha \psi = \mathcal{F}(\psi, \partial \psi)$ on the Kerr-Anti-de Sitter black hole exterior region is proven. Scattering data characterising the exponential decay is posed on the future event horizon, paired with Dirichlet boundary conditions at null infinity. The resulting solutions form a large class, exhibiting all functional degrees of freedom associated with the problem. However, given that (at best) inverse logarithmic decay has been established for even general solutions of the linear ($\mathcal{F} = 0$) forward problem [37], they are decidedly non-generic. The *Hawking-Reall bound* on the angular momentum of the spacetime is not imposed. In this setting, it is known that the forward problem admits exponentially growing mode solutions [15]. The prescribed exponential decay of the horizon data is utilised in overcoming the *blueshift effect* one encounters when solving backwards, as is familiar from the related result in the asymptotically flat setting [8].

In order to establish key techniques for the scattering construction on the Kerr-Anti-de Sitter exterior, similar constructions on the asymptotically flat Minkowski space and Schwarzschild black hole exterior are presented. These are followed by a nonlinear local existence result on pure Anti-de Sitter space, serving as an introduction to asymptotically Anti-de Sitter spacetimes and nonlinear techniques.

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Chapter 1

Introduction

1.1 The black hole stability problem

The *black hole stability problem* is a central question of mathematical general relativity. The Einstein vacuum equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{1.1.1}$$

for a spacetime¹ $(\mathcal{M}, g)$ with cosmological constant Λ admit a handful of known explicit black hole solutions, of which one would like to determine asymptotic stability or instability under small perturbations. More precisely, given appropriate initial data sufficiently close to data for an explicit reference black hole solution of (1.1.1), one asks whether the maximal development under (1.1.1) satisfies the following criteria for asymptotic stability:

1. The maximal development contains a black hole region,
2. The maximal development remains globally sufficiently close to the reference black hole spacetime.
3. The maximal development asymptotically approaches the reference black hole in time.²

In the $\Lambda \geq 0$ case, great strides have been made. Namely, full nonlinear asymptotic stability of the slowly rotating Kerr [26, 43–45, 53] and Kerr-de Sitter [20, 32] spacetimes. There have been many other important contributions in this direction, including, but not limited to [1, 7, 9–11]. In the $\Lambda < 0$ setting, the picture is not so clear. However, one has the following conjecture.

¹Throughout this thesis, $\mathcal{M}$ is assumed to be $(3 + 1)$ -dimensional.

²In reality, one does not ask for the resulting solution to asymptotically approach a black hole with exactly the same parameters as the reference, but one in the same “family” of black hole spacetimes (with possibly different mass or angular momentum, for example). One can see, for instance, [10] for an in-depth discussion of the precise formulation of asymptotic stability.

Conjecture 1.1.1 (Asymptotic (in)stability of Anti-de Sitter space [36, 38]). Anti-de Sitter (AdS) space is asymptotically unstable (respectively, stable) as a solution of (1.1.1) with Dirichlet (respectively, optimally dissipative) boundary conditions at future null infinity.

The conjectured stability of AdS under optimally dissipative boundary conditions will be proven in upcoming work of Holzegel-Luk-Smulevici-Warnick [39]. With regards to the instability portion of Conjecture 1.1.1, Moschidis proved instability of AdS in spherical symmetry coupled to two matter models, Einstein null dust [47] and Einstein massless Vlasov [48] equations, with Dirichlet boundary conditions. Bizòn and Rostworowski derived instability numerically for the spherically symmetric Einstein massless scalar field equation with Dirichlet boundary conditions [2].

The Holzegel-Smulevici instability conjecture was, in fact, stated for the full asymptotically-AdS Kerr-AdS family of black hole spacetimes [36] (of which AdS itself is a special case). Nonetheless, based on the key result (Theorem 1.3.2) of this thesis, a large class of perturbations of Kerr-AdS spacetimes which decay sufficiently fast will be constructed in subsequent work, in the manner of Dafermos-Holzegel-Rodnianski [8]. Before addressing the problem in the setting of full gravity, it is prudent to consider the so-called *poor man’s linearisation* of (1.1.1), the scalar wave equation³

$$\square_g \psi + \alpha \psi = \mathcal{F}(\psi, D\psi) \quad (1.1.2)$$

on a fixed spacetime background with metric g , mass $\alpha = 0$ for $(\mathcal{M}, g)$ with $\Lambda = 0$ and more general masses in the $\Lambda < 0$ case. As such, we will explore this setting, first focusing on the more familiar $\Lambda = 0$ backgrounds which admit a well-posed initial value problem for (1.1.2). After developing an appropriate scattering construction for decaying scalar waves in this setting, we will move to the asymptotically AdS backgrounds on which the pertinent problem is an initial *boundary* value problem with Dirichlet boundary conditions at infinity. An in-depth discussion of the eventual goal of a scattering construction for a class of fast-decaying perturbations of asymptotically Anti-de Sitter black hole spacetimes under the full Einstein vacuum system (1.1.1) is postponed until Chapter 4.

1.2 The “poor man’s linearisation” on asymptotically flat backgrounds

In the globally hyperbolic asymptotically flat setting, the problem of interest is the initial value problem

$$\begin{cases} \square_g \psi = \mathcal{F}(\psi, D\psi) = 0 \\ (\psi, \partial_t \psi)|_{\{t=\tau\}} = (\psi_0, \psi_1), \end{cases} \quad (1.2.1)$$

for a massless scalar wave. We restrict our attention to the linear ($\mathcal{F} = 0$) problem, as including nonlinear terms involves additional complications (namely, the null condition)

³Here, $\square_g$ is the d’Alembertian $\square_g = \frac{1}{\sqrt{|\det g|}} \partial_\mu (\sqrt{|\det g|} g^{\mu\nu} \partial_\nu \psi)$ with respect to the metric g .

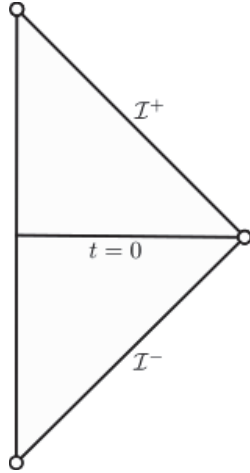


Figure 1.1: The Penrose diagram of Minkowski space. Here, $\mathcal{I}^\pm$ are future (respectively, past) null infinity.

which will not arise when we come to consider the nonlinear problem in the asymptotically AdS setting.

We will begin by discussing problem (1.2.1) where we fix $g = \eta$, the Minkowski metric, from the perspective of scattering and Friedlander's *radiation field*.

1.2.1 Scattering on Minkowski space: Friedlander's *radiation field*

Minkowski space $(\mathcal{M}_M, \eta)$ is the maximally symmetric $\Lambda = 0$ solution of (1.1.1). The Minkowski metric in standard (t, r, θ, ϕ) coordinates is given by

$$\eta = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1.2.2)$$

For simplicity, the discussion of this section will concern problem (1.2.1) with $g = \eta$ in these coordinates. However, in practice, we will supplement these with double-null coordinates (u, v, θ, ϕ) which are better suited to the geometry of the scattering problem on asymptotically flat spacetimes.

Decay for solutions ψ of (1.2.1) on Minkowski space in the regime of compactly supported initial data is well understood. The Strong Huygen's Principle ensures that the solution is compactly supported in time at each fixed radius, whilst the work of Friedlander [21–23] precisely describes the radial asymptotics of the solution. We collect two key results of Friedlander here, before later proving a version of them using the modern language of the energy method in Section 2.2. Firstly, Friedlander showed that the radial asymptotics of the solution ψ of (1.2.1) are uniquely defined.

Theorem 1.2.1. (Existence and uniqueness of the radiation field of an initially compactly supported wave on Minkowski space [21]) Let ψ be a smooth solution of (1.2.1) arising from

compactly supported initial data (ψ_0, ψ_1) at $t = \tau \geq 0$. Then ψ admits a unique, uniform radiation field $h_{\mathcal{I}^+}$:

$$\lim_{r \rightarrow \infty} r\psi(t, r, \theta, \phi) = h_{\mathcal{I}^+}(t - r, \theta, \phi).$$

This result is paired with a second which gives the reverse implication: that prescribed radial asymptotics uniquely determine a solution ψ of

$$\square_\eta \psi = 0. \tag{1.2.3}$$

Theorem 1.2.2. (A scattering construction for waves on Minkowski space from prescribed compactly supported radiation fields [21]) Let $h_{\mathcal{I}^+}(t - r, \theta, \phi)$ be a sufficiently smooth function of compact support in $t - r$. Then there exists a unique smooth solution ψ of (1.2.3) on $t = \tau \geq 0$ attaining $h_{\mathcal{I}^+}$ as its radiation field:

$$\lim_{r \rightarrow \infty} r\psi(t, r, \theta, \phi) = h_{\mathcal{I}^+}(t - r, \theta, \phi).$$

The proof of (the existence part of) Theorem 1.2.2 given in this thesis (see Sections 2.2.5-2.2.7) acts as a first, relatively straightforward construction via the scattering problem

$$\begin{cases} \square_\eta \psi = 0, \\ r\psi|_{\mathcal{I}^+} = h_{\mathcal{I}^+}. \end{cases} \tag{1.2.4}$$

This will lay the groundwork for the more involved constructions for scalar waves on black hole spacetimes.

1.2.2 Scattering on asymptotically flat black hole exteriors

Let us now discuss the appropriate analogue of Theorem 1.2.2 on $\Lambda = 0$ black hole exteriors, starting with those which are spherically symmetric.

The Schwarzschild spacetimes are a one-parameter family of spherically symmetric $\Lambda = 0$ solutions $(\mathcal{M}_{S(M)}, g_{S(M)})$ of (1.1.1). For each fixed choice of mass parameter $M > 0$, the corresponding maximally extended spacetime contains a black hole region. The Schwarzschild spacetimes are a subfamily of the two-parameter (mass and angular momentum) axially symmetric Kerr family.

In (t^*, r, θ, ϕ) coordinates, the spacetime metric on the *exterior* region $r \geq 2M$ of the Schwarzschild black hole (whose event horizon $\mathcal{H}^\pm$ lies at $\{r = 2M\}$) is given by

$$g_{S(M)} = -\left(1 - \frac{2M}{r}\right)(dt^*)^2 + \frac{4M}{r}dt^*dr + \left(1 + \frac{2M}{r}\right)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1.2.5}$$

From now on, we shall drop the M -dependence from subscripts for brevity of notation. When considering problem (1.2.1) in the setting $g = g_S$, one must take several new facets into account. First of all, one may define a radiation field h of the solution ψ arising from

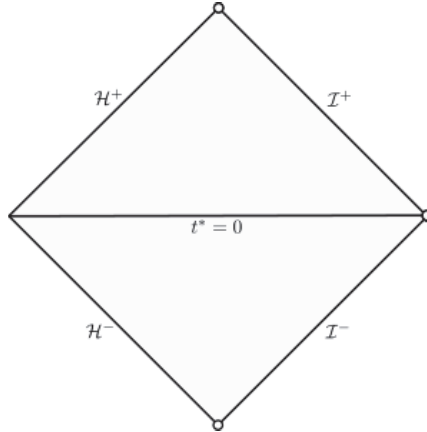


Figure 1.2: The Penrose diagram of the Schwarzschild black hole exterior region. Here, $\mathcal{I}^\pm$ are future (respectively, past) null infinity and $\mathcal{H}^\pm$ are the future (respectively, past) event horizons, which lie at $r = 2M$.

compactly supported initial data as before, obtaining the analogue of Theorem 1.2.1 [13]. However, energy of the solution initialised at $t^* \geq 0$ is now lost via both future null infinity $\mathcal{I}^+$ and the future event horizon $\mathcal{H}^+$, meaning that h is bifurcate. As such, the appropriate scattering problem (analogous to (1.2.4)) is

$$\begin{cases} \square_{g_S} = 0, \\ (\psi|_{\mathcal{H}^+}, \psi|_{\mathcal{I}^+}) = h = (h_{\mathcal{H}^+}, h_{\mathcal{I}^+}). \end{cases} \quad (1.2.6)$$

Secondly, the Strong Huygens' Principle no longer holds. One instead finds that compactly supported (as well as more general) initial data gives rise to a solution decaying at the (sharp) rate t^{-3} [16] (in time $t = t^* - 2M \log(r - 2M)$). Motivated by Theorem 1.2.2 on Minkowski space, one may resultingly like to prescribe a polynomially decaying radiation field at infinity as scattering data for (1.2.6) in the hope of obtaining a unique solution ψ . However, in doing so, one encounters a geometric obstruction: the blueshift effect, which drives exponential growth along the event horizon. As such, to close estimates and derive a solution ψ of (1.2.6), one prescribes an exponentially decaying radiation field on the event horizon. See Section 2.1.2.1 for a simplified explanation of this mechanism. Thankfully, by assuming exponential decay of the radiation field, one avoids a further geometric feature, trapped null geodesics, of the Schwarzschild exterior which manifests in degeneration of energy estimates for the forward problem (1.2.1) at the *photonsphere*, $r = 3M$. One thus obtains the following result (see Sections 2.3.3-2.3.5).

Theorem 1.2.3. (A scattering construction for linear waves on the Schwarzschild exterior from prescribed exponentially decaying radiation fields) Let $h = (h_{\mathcal{H}^+}(t^*, \theta, \phi), h_{\mathcal{I}^+}(t^*, \theta, \phi))$ be sufficiently smooth, exponentially decaying and appropriately matched bifurcate scattering data on the event horizon and future null infinity. Then there exists a smooth solution

ψ of (1.2.3) on $t^* \geq t_0^* > 0$ attaining h as its radiation field. More precisely,

$$\lim_{r \rightarrow \infty} r\psi(t^*, r, \theta, \phi) = h_{\mathcal{I}^+}(t^*, \theta, \phi), \quad \psi|_{\mathcal{H}^+} = h_{\mathcal{H}^+}.$$

Amazingly, due to the assumed exponential decay of the scattering data, generalising Theorem 1.2.3 to include the full family of (sub)extremal axially symmetric Kerr exteriors involves no additional conceptual difficulties. The construction is entirely insensitive to the geometric complication of *superradiance* which one must confront in other settings. The analogues of Theorems 1.2.1 and 1.2.3 for solutions ψ of (1.2.1) on the Kerr exterior follow from a result of Dafermos, Rodnianski and Shlapentokh-Rothman [13]. These results will not be proven here. However, the analogous (to Theorem 1.2.3) construction on Kerr-AdS will be given in detail in Chapter 4. There one sees the (relatively minor) equivalent difficulties arising from moving away from spherical symmetry of the background.

Given this basis of scattering constructions for decaying scalar waves on asymptotically flat backgrounds, we now shift our attention to the asymptotically AdS setting.

1.3 The “poor man’s linearisation” on asymptotically AdS backgrounds

On an asymptotically AdS spacetime with metric g , the appropriate *poor man’s linearisation* of (1.1.1) is a *massive*⁴ scalar wave (or Klein-Gordon) equation

$$\square_g \psi + \alpha \psi = \mathcal{F}(\psi, D\psi). \quad (1.3.1)$$

The mass α will be assumed to satisfy the *Breitenlohner-Freedman bound* $\alpha < \frac{9}{4}$ which ensures a positive $T = \partial_t$ energy, as first demonstrated by Breitenlohner and Freedman on AdS itself [4]. The range $\alpha > 0$ is sometimes referred to as the range of “negative mass”.

Due to the non-globally hyperbolic nature of these spacetimes, one must study (1.3.1) in the context of an initial *boundary* value problem. Since we are motivated by the setting where Conjecture 1.1.1 predicts instability, we focus on the Dirichlet problem:

$$\begin{cases} \square_g \psi + \alpha \psi = \mathcal{F}(\psi, \partial \psi), \\ (\psi, \partial_t \psi)|_{\{t=\tau\}} = (\psi_0, \psi_1), \quad r^{\frac{3}{2}-s} \psi|_{\mathcal{I}} = 0, \quad s = \sqrt{\frac{9}{4} - \alpha}. \end{cases} \quad (1.3.2)$$

1.3.1 Anti-de Sitter space

The maximally symmetric $\Lambda = -\frac{3}{\ell^2}$ ($\ell > 0$) solution of (1.1.1) is the anti-de Sitter (AdS) spacetime $(\mathcal{M}_{AdS}, g_{AdS})$. In standard (t, r, θ, ϕ) coordinates, for each fixed cosmological

⁴For the specific choice of conformal mass $\alpha = 2$, the support of the Green’s function of (1.3.1) with $g = g_{AdS}$ and $\mathcal{F} = 0$ is restricted to the light cone. In this sense, (1.3.1) with $\alpha = 2$, $\mathcal{F} = 0$ is the analogue of the zero mass equation on asymptotically flat space [34].

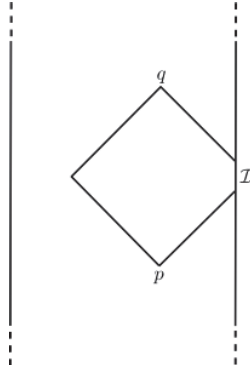


Figure 1.3: A non-compact causal diamond on the Penrose diagram of the non-globally hyperbolic Anti-de Sitter space. Here, $\mathcal{I}$ is null infinity.

constant, the metric reads

$$g_{AdS} = -\left(1 + \frac{r^2}{\ell^2}\right)dt^2 + \frac{1}{1 + \frac{r^2}{\ell^2}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1.3.3)$$

On AdS, the “radiation field” at infinity of solutions of (1.3.2) is determined by the boundary conditions. Dirichlet boundary conditions enforce that the solution loses no energy through $\mathcal{I}$, implying that there exist no decaying solutions of (1.3.2) with $g = g_{AdS}$. As such, there is no meaningful analogue of Theorem 1.2.2 in this setting. Our consideration of AdS itself will be limited to a proof of local existence for the nonlinear problem (1.3.2), which follows from linear well-posedness on general asymptotically-AdS spacetimes [33]. This appears in Chapter 3. In order to define an appropriate notion of scattering, we next consider asymptotically-AdS black hole spacetimes, where we will see (Theorem 1.3.1) that some amount of decay occurs.

As in the $\Lambda = 0$ setting, there is no additional conceptual difficulty arising from generalising the analogue of Theorem 1.2.3 from spherical to axial symmetry. As such, we next discuss problem (1.3.2) on Kerr-AdS exteriors, without first considering Schwarzschild-AdS.

1.3.2 Scattering on the Kerr-AdS black hole exterior

For each fixed negative cosmological constant $\Lambda = -\frac{3}{\ell^2}$ ($\ell > 0$), the Kerr-AdS spacetimes $(\mathcal{M}_{KAdS}, g_{KAdS}(M, a))$ are a two-parameter family of axially symmetric solutions of (1.1.1). Given a fixed choice of mass $M > 0$, event horizon radius r_+ and (*sub*)*extremal* angular momentum⁵ a

$$|a| \leq \sqrt{\frac{M\ell^2}{r_+} - \ell^2 - 2r_+^2} \quad \text{and} \quad |a| < \ell. \quad (1.3.4)$$

⁵The second requirement ensures that expression (1.3.5) defines a Lorentzian metric (one can check via direct calculation that $\det g$ is undefined when $|a| = \ell$, due to the vanishing of Ξ and when $|a| > \ell$, there exists $\theta = \theta'$ at which $\det g|_{\theta'} > 0$). The first bound ensures that (1.3.5) describes a black hole spacetime (*extremal* in the case of equality). See [52].

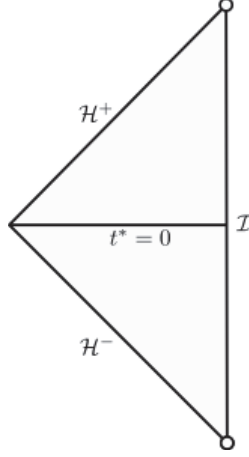


Figure 1.4: The Penrose diagram of the Kerr-Anti-de Sitter black hole exterior. Here $\mathcal{I}$ is null infinity and $\mathcal{H}^\pm$ are the future (respectively, past) event horizons.

one has a black hole spacetime with metric (in familiar Boyer-Lindquist $(t, r, \theta, \tilde{\phi})$ coordinates)

$$g_{KAdS} = -\frac{\Delta_- - \Delta_\theta a^2 \sin^2 \theta}{\Sigma} dt^2 + \frac{\Sigma}{\Delta_-} dr^2 + \frac{\Sigma}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta(r^2 + a^2)^2 - \Delta_- a^2 \sin^2 \theta}{\Xi^2 \Sigma} \sin^2 \theta d\tilde{\phi}^2 - 2 \frac{\Delta_\theta(r^2 + a^2)^2 - \Delta_-}{\Xi \Sigma} a \sin^2 \theta d\tilde{\phi} dt, \quad (1.3.5)$$

which extends to a regular metric (4.2.1) on the entire exterior region $r \geq r_+$. The quantities $\Delta_\pm$, Δ_θ , Σ and Ξ are defined in (4.2.2).

The asymptotic-in-time behaviour of general solutions of the *linear* ($\mathcal{F} = 0$) problem (1.3.2) with $g = g_{KAdS}(M, a)$ for a satisfying the *Hawking-Reall bound*

$$|a|\ell < r_+^2 \quad (1.3.6)$$

is captured in the following Theorem of Holzegel-Smulevici.

Theorem 1.3.1. (Generic logarithmic decay for the linear massive wave equation with Dirichlet boundary conditions on the Kerr-AdS exterior [36]) Set $\mathcal{F} = 0$ and consider (1.3.2) on the exterior of a fixed Kerr-AdS black hole satisfying the Hawking-Reall bound (1.3.6). Solutions ψ arising from generic initial data (ψ_0, ψ_1) decay logarithmically in time and no faster.

In the setting where one allows the Hawking-Reall bound (1.3.6) to be violated, one encounters even “worse” decay behaviour. This was demonstrated by Dold, who derived exponentially growing mode solutions of the linear problem [15].

It is precisely the generically slow decay of solutions of the linear initial boundary value problem (1.3.2) which motivates studying a (backwards) scattering problem in order to

extract exceptional fast-decaying solutions. With that being said, the problem with which we shall work is the global towards the future Dirichlet scattering problem

$$\begin{cases} \square_{KAdS}\psi + \alpha\psi = \mathcal{F}, & \alpha < \frac{9}{4} \\ \psi|_{\mathcal{H}^+} = h_{\mathcal{H}^+}, & r^{\frac{3}{2}-s}\psi|_{\mathcal{I}} = 0, \quad s = \sqrt{\frac{9}{4} - \alpha}, \end{cases} \quad (1.3.7)$$

with smooth scattering data $h_{\mathcal{H}^+}$, capturing the desired fast-decay, prescribed on the black hole event horizon $\mathcal{H}^+$. As was the case on the Schwarzschild background (Theorem 1.2.3), we must require that $h_{\mathcal{H}^+}$ decays exponentially in time as a result of the blueshift effect at the horizon. In Chapter 4 we prove the following key result which is stated informally below.

Theorem 1.3.2. (A scattering construction for exponentially decaying nonlinear waves on the Kerr-AdS exterior) Under appropriate assumptions on $\mathcal{F}$ (see Section 4.2.3.2), there exist a large class of solutions ψ of (1.3.7) on the region $t^* \geq t_0^* > 0$ of the (sub)extremal⁶ (1.3.4) Kerr-AdS exterior which decay exponentially in time towards the future.

We note that the exponentially decaying solutions of (1.3.7) which we derive on the Kerr-AdS exterior are significantly more exceptional than those on the asymptotically flat Schwarzschild exterior. This is due to the aforementioned polynomial decay of general solutions of the initial value problem in that setting, compared with the general slow decay here (Theorem 1.3.1).

1.4 Outline

The body of the thesis is divided into three chapters, whose contents are as follows. Each chapter begins with its own introduction, which is more in-depth than what has been presented so far.

Chapter 2 In this Chapter, the “poor man’s linearisation” is discussed with respect to asymptotically flat backgrounds. Firstly, the appropriate versions of each of Friedlander’s results on Minkowski space (Theorem 1.2.1 and Theorem 1.2.2) are stated formally and proven. This is followed by the statement and proof of Theorem 1.2.3 on the Schwarzschild black hole exterior.

Chapter 3 Here, we begin our study of asymptotically Anti-de Sitter backgrounds with a statement and proof of local existence for the nonlinear problem (1.3.2) on Anti-de Sitter space itself.

⁶Note that this result does not assume the Hawking-Reall bound, in contrast to Theorem 1.3.1

Chapter 4 This chapter contains an adapted version of the upcoming paper *A scattering construction for nonlinear wave equations on Kerr-Anti-de Sitter spacetimes* [35], whose main result is Theorem 1.3.2. Therein, a detailed discussion of moving from the “poor man’s linearisation” to an analogous result for the Einstein vacuum equations (1.1.1) is also provided.

After the body of the thesis, several appendices are included. Appendix A contains definitions associated with the *energy method*, as well as a general energy identity. Several important inequalities appear in Appendix B, including some Sobolev embeddings and *Grönwall’s inequality*. In Appendix C, the *Arzelà-Ascoli Theorem* is stated, alongside a related, useful result. Finally, Appendix D contains a discussion of regular double-null coordinates on the Schwarzschild black hole exterior.

1.5 Notation

Throughout this thesis, the following notational conventions will be used.

- By ∂_{x^i} , we shall denote the standard partial derivative with respect to a given coordinate system $\{x^i\}_{i=0}^3$.
- ∂ (with no subscript) is used to denote an arbitrary partial derivative.
- By $\sigma = (\sigma_1, \dots, \sigma_n)$ (in places, other lowercase Greek letters are used), we shall denote a multi-index with entries σ_i , $i \in \mathbb{N}$ and length $|\sigma| = \sum_i \sigma_i$. Given a collection of vector fields X_ℓ , for instance $\{\partial_{x^\ell}\}_{\ell=0}^3$, we shall denote a general concatenation of them by $X^\sigma = X_0^{\sigma_1} X_1^{\sigma_2} \dots$.
- By $\mathcal{g}$, we shall denote the metric induced on the sphere at fixed time and radius (in the relevant coordinates) by the background metric of a given spacetime.
- In coordinates (θ, ϕ) on said sphere, we shall then denote, by ∇ , Δ the associated gradient and Laplace operators

$$\nabla f = \mathcal{g}^{AB} \nabla_A f \cdot \nabla_B, \quad \Delta = \mathcal{g}^{AB} \nabla_A \nabla_B,$$

(here, f is a sufficiently smooth function),

- by $|\nabla \dots \nabla f|^2 = \mathcal{g}^{AA'} \dots \mathcal{g}^{BB'} \nabla_A \dots \nabla_B f \cdot \nabla_{A'} \dots \nabla_{B'} f$, the norm of the gradient with respect to the induced metric on the sphere,
- and, by $d\omega$, the associated volume form:

$$d\omega = \sin \theta d\theta d\phi.$$

- Finally, by $\{\Omega_j\}_{j=1}^3$ we shall denote the *angular momentum operators*, a set of vector fields which form a basis for the action of $SO(3)$ on the spheres.

Chapter 2

The “poor man’s linearisation” on asymptotically flat spacetimes

2.1 Introduction

In this chapter, we study the “poor man’s linearisation” of the Einstein vacuum equation on asymptotically flat spacetimes, the linear wave equation

$$\square_g \psi = 0,$$

focusing specifically on Minkowski space and the Schwarzschild black hole exterior. We first consider the forward problem (1.2.1) on Minkowski space, followed by the scattering problem (1.2.4). We prove Theorems 1.2.1 and 1.2.2 in this setting, the latter of which serves as a first demonstration of the scattering construction which will be key for the discussion of the Kerr-AdS black hole exterior in Chapter 4. Next, on the Schwarzschild black hole exterior we prove Theorem 1.2.6 via a further scattering construction. This will showcase a scattering problem with bifurcate data (on the horizon and at infinity), as well as the key complication of working with spacetimes possessing an event horizon in the setting of backwards-in-time evolution, the *blueshift effect*. Both features will also play a role when it comes to the scattering construction on Kerr-AdS in Chapter 4.

Here, we provide a brief outline of the content appearing in this chapter.

2.1.1 Minkowski space

In the following sections, we shall restate informal versions of Theorem 1.2.1 and Theorem 1.2.2 and discuss the key elements of their proofs.

2.1.1.1 The forward problem: Existence and uniqueness of Friedlander's *radiation field*

We begin our study of equation (1.2.3) on Minkowski space by studying the *forward problem*

$$\begin{cases} \square_{\eta}\psi = 0 \\ (\psi, \partial_t\psi)|_{t=\tau} = (\psi_0, \psi_1) \end{cases} \quad (2.1.1)$$

with smooth and compactly supported initial data (ψ_0, ψ_1) . For the purposes of this problem, it is most convenient to write the equation in *double-null coordinates* ($u = t - r, v = t + r$),

$$\frac{1}{r}\partial_u\partial_v(r\psi) = \Delta\psi, \quad (2.1.2)$$

and integrate (2.1.2) directly in u, v .

In these coordinates, the Minkowski metric η (1.2.2) reads

$$\eta = -dudv + \frac{1}{4}(u-v)^2(d\theta^2 + \sin^2\theta d\phi^2).$$

We shall prove the following result (a null-coordinate restatement of Theorem 1.2.1).

Theorem 2.1.1. (Existence and uniqueness of the radiation field of an initially compactly supported wave on Minkowski space [21]¹) Let ψ be a smooth solution of (2.1.1) arising from initial data (ψ_0, ψ_1) at time $t = \tau \geq 0$ supported on $r \leq R_{supp} < \infty$ (Figure 2.1). Then there exists a unique, v -independent function $h_{\mathcal{I}^+}(u, \theta, \phi)$ (the *radiation field* of ψ) for which

$$r\psi(u, v, \theta, \phi) = h_{\mathcal{I}^+}(u, \theta, \phi) + \mathcal{O}\left(\frac{1}{r}\right).$$

Let us discuss Theorem 2.1.1, firstly in terms of the toy problem for spherically symmetric waves, before outlining key elements of the proof in full generality.

Spherically symmetric solutions of problem (2.1.1) Before discussing the proof of Theorem 2.1.1 in the general case, let us first consider the case where we restrict to spherically symmetric solutions $\psi(u, v)$ of (2.1.1). Such solutions satisfy equation (2.1.2) with $\Delta\psi = 0$ and, therefore, satisfy the system

$$\partial_u(r\psi) = H(u), \quad \partial_v(r\psi) = F(v), \quad (2.1.3)$$

where the functions H and F are yet to be determined. Integrating equations (2.1.3) in u and v , respectively, yields the expression

$$r\psi = h(u) + C_1(v) = f(v) + C_2(u). \quad (2.1.4)$$

¹The result itself is originally due to Friedlander [21], however, the proof given in this thesis is original.

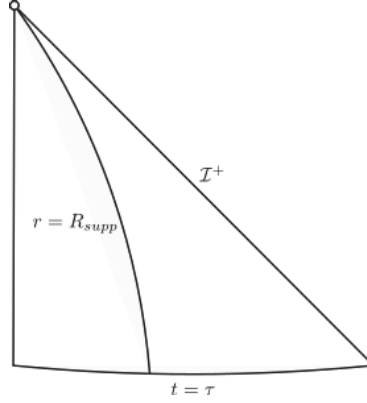


Figure 2.1

Here, h and f are, respectively, the u - and v -antiderivatives of H and F , whilst C_1 and C_2 are unknown functions. By (2.1.4), one may express the solution as

$$r\psi = h(u) + f(v). \quad (2.1.5)$$

Now, since ψ is initially compactly supported, the function f must also vanish at all points $(t = \tau, r)$ where $r > R_{supp}$. Furthermore, since f is u -independent, f must in fact vanish at all $v > \tau + R_{supp}$. This immediately yields Theorem 2.1.1 under the restriction to spherically symmetric ψ .

General solutions of problem (2.1.1) Outside of spherical symmetry, we will take the approach of estimating the (now non-vanishing) right-hand side of equation (2.1.2) as an error term and directly integrating the left-hand side. The key ingredient we will exploit in order to estimate $\Delta\psi$ is a so-called r^p estimate (see Section 2.2.2). This gives uniform control of ψ itself. Commuting the equation with second-order concatenations of the *angular momentum operators* $\{\Omega_j\}_{j=1}^3$ then gives a uniform bound for $\Delta\psi$ by the initial data. The result easily follows.

Let us now discuss the backwards problem.

2.1.1.2 The backwards problem: A scattering construction for linear waves from their radiation fields

Given existence and uniqueness of the radiation field $h_{\mathcal{I}^+}$, we now consider the *backwards problem*

$$\begin{cases} \square_\eta \psi = 0 \\ r\psi|_{\mathcal{I}^+} = h_{\mathcal{I}^+}(u, \theta, \phi) \end{cases} \quad (2.1.6)$$

with smooth, v -independent and compactly supported (in u) scattering data $h_{\mathcal{I}^+}$. In this setting, we will prove the following result (a null-coordinate restatement of Theorem 1.2.2).

Theorem 2.1.2. (A scattering construction for linear waves on Minkowski space from prescribed compactly supported radiation fields [21]) Let $h_{\mathcal{I}^+}(u, \theta, \phi)$ be a smooth function supported on a compact interval $[u_0, u_1]$ of u (Figure 2.2). Then there exists a smooth solution ψ of problem (2.1.6) on $t \geq \tau \geq 0$.

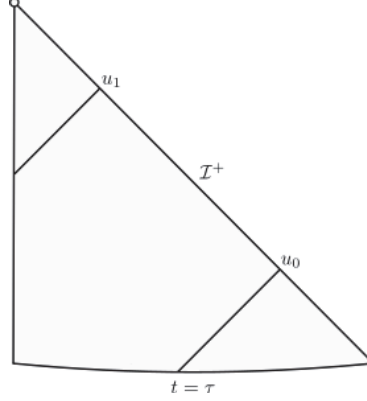


Figure 2.2

The proof of Theorem 2.1.2 employs the aforementioned r^p estimates as part of an iteration scheme, consisting of a sequence of finite-in-time problems

$$\begin{cases} \square_\eta \psi_i = 0 \\ r\psi_i|_{v=v_i} = h_{\mathcal{I}^+}(u, \theta, \phi), \quad \psi_i|_{u=u_1} = 0 \end{cases} \quad (2.1.7)$$

with scattering data at finite $v = v_i$ (see Figure 2.3). It will be shown in Section 2.2.6 that the finite problems (2.1.7) approximate the full problem (2.1.6) in the limit $v_i \rightarrow \infty$. This is achieved by utilising the *energy method*, which is very widely applicable and will be used throughout this thesis. The method involves multiplying the equation by an appropriate globally timelike (in this case, $T = \partial_t$ will suffice) vector field applied to ψ

$$\begin{aligned} & T\psi \cdot \square_\eta \psi = 0 \\ \Rightarrow & -\partial_t \psi \cdot \partial_t^2 \psi + \partial_t \psi \cdot \partial_r^2 \psi + \partial_t \psi \cdot \Delta \psi + \frac{2}{r} \partial_t \psi \cdot \partial_r \psi + \frac{r^2 \cos \theta}{\sin \theta} \partial_t \psi \cdot \partial_\theta \psi = 0 \end{aligned}$$

and integrating by parts over a well-chosen spacetime region.² Here, this will be a cut-off null cone (see Section 2.2.6 and Figure 2.9). This allows one to estimate H^1 -norms of the solutions ψ_i of the finite problems (2.1.7) on past spacelike slices of constant time by the approximate scattering data on $v = v_i$ as:

$$E^T[\psi_i](t) \leq C \int_{\{v=v_i\}} \left[\left(\frac{h}{1+r} \right)^2 + (\partial_u h)^2 + \frac{1}{1+r^2} |\nabla h|^2 \right] r^2 du d\omega, \quad (2.1.8)$$

²This method can also be carried out more abstractly using the framework of the *energy-momentum tensor* and *currents* (see Appendix A). In doing so, one integrates the energy identity (A.0.1) over the spacetime region via the Divergence Theorem. This approach will be used in the discussion of Schwarzschild, Anti-de Sitter space and Kerr-Anti-de Sitter space.

where

$$E^T[\psi_i](t) = \int_{\Sigma_t} \left[(\partial_t \psi_i)^2 + (\partial_r \psi_i)^2 + |\nabla \psi_i|^2 \right] r^2 dr d\omega. \quad (2.1.9)$$

Commuting with $T = \partial_t$ and $\{\Omega_j\}_{j=1}^3$ sufficiently many times gives analogous estimates for their H^k norms. By Sobolev embedding in the spacelike slices, one can then obtain L^∞ estimates for the ψ_i and their derivatives. This allows application of the *Arzelà-Ascoli Theorem* (see Appendix C) to deduce a convergent subsequence of the ψ_i , tending to our target solution ψ of the full problem (2.1.6).

We will now discuss the analogous scattering result and its proof on the Schwarzschild black hole exterior.

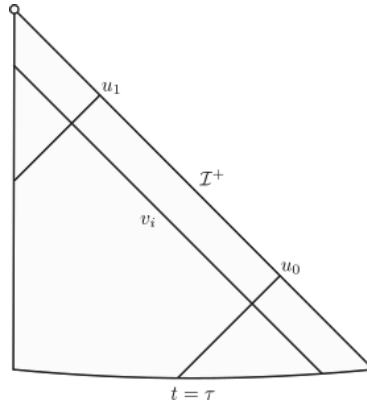


Figure 2.3

2.1.2 The Schwarzschild black hole exterior

As discussed in Chapter 1, we would like to prove Theorem 1.2.3 on the Schwarzschild exterior. In the following, we set up a null-coordinate statement of the result and discuss the key elements of the proof, highlighting the differences to the Minkowski case along the way.

2.1.2.1 A scattering construction for linear waves on the Schwarzschild black hole exterior

The relevant scattering problem on the Schwarzschild exterior is

$$\begin{cases} \square_{g_S} \psi = 0, \\ \psi|_{\mathcal{H}^+} = h_{\mathcal{H}^+}, \quad r\psi|_{\mathcal{I}^+} = h_{\mathcal{I}^+}, \end{cases} \quad (2.1.10)$$

with bifurcate characteristic scattering data on the event horizon and future null infinity.

We will work with the equation itself in (t^*, r, θ, ϕ) coordinates (in which the metric is (1.2.5)):

$$\begin{aligned} \square_{g_S} \psi = & - \left(1 + \frac{2M}{r}\right) \partial_{t^*}^2 \psi + \left(1 - \frac{2M}{r}\right) \partial_r^2 \psi + \frac{4M}{r} \partial_{t^*} \partial_r \psi + \Delta \psi \\ & + \frac{6M}{r^2} \partial_{t^*} \psi + \frac{2}{r} \left(1 - \frac{M}{r}\right) \partial_r \psi + \frac{\cos \theta}{r^2 \sin \theta} \partial_\theta \psi = 0, \end{aligned} \quad (2.1.11)$$

however, it is convenient to introduce double-null coordinates, which we will use to define the scattering data, among other things. To do so, we fix an $R > 3M$ sufficiently large³ and define the Regge-Wheeler radial coordinate

$$r^* = r + 2M \log(r - 2M) - R, \quad (2.1.12)$$

and corresponding null coordinates $(u = t - r^*, v = t + r^*)$. In (u, v, θ, ϕ) coordinates, the spacetime metric reads

$$g_S(M) = -4 \left(1 - \frac{2M}{r}\right) du dv + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.1.13)$$

Note that this metric (2.1.13) is only regular on $r > 2M$, away from the event horizon $\mathcal{H}^+$. However, one can define (U, V) coordinates which are regular across $\mathcal{H}^+$ (see Appendix D). Given this, we proceed in (u, v, θ, ϕ) coordinates, with the caveat that whenever a function is evaluated $u = \infty$, we are really doing so at the corresponding finite value in the regular (U, V) coordinates.

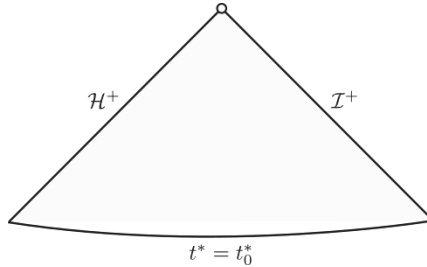


Figure 2.4

We will assume (and it will become clear why in the subsequent discussion) that the smooth scattering data $(h_{\mathcal{H}^+}, h_{\mathcal{I}^+})$ we prescribe for problem (2.1.10) satisfies

$$F_{\mathcal{H}^+}[h_{\mathcal{H}^+}] \leq C \exp(-B \cdot v), \quad F_{\mathcal{I}^+}[h_{\mathcal{I}^+}] \leq C \exp(-B \cdot u) \quad (2.1.14)$$

for all $t = \frac{1}{2}(u + v) > 0$, some $C > 0$ and some constant $B > 0$ sufficiently large, depending on the surface gravity κ .⁴ Here, $F_{\mathcal{H}^+}$, $F_{\mathcal{I}^+}$ are H^1 -norms of $h_{\mathcal{H}^+}$, $h_{\mathcal{I}^+}$, respectively. Truncated

³Largeness will be required when we apply the r^2 -weighted estimates (see Section 2.3.2) in Section 2.3.5.

⁴The surface gravity of a Killing horizon is the constant κ which satisfies $\nabla^a(T^b T_b) = -2\kappa T^a$, where T is the Killing field which generates the horizon.

forms of these will appear as the horizon (respectively, $v = v_i$) flux term in a subsequent energy inequality (2.1.17). In practice, one must also assume higher-order analogues of (2.1.14).

We will prove the following result (a null coordinate restatement of Theorem 1.2.3).

Theorem 2.1.3. (A scattering construction for linear waves on the Schwarzschild exterior from prescribed exponentially decaying radiation fields⁵) Let $h_{\mathcal{H}^+}(v, \theta, \phi)$ and $h_{\mathcal{I}^+}(u, \theta, \phi)$ be sufficiently smooth, exponentially decaying (in v and u respectively, as in (2.1.14)) functions. Then there exists a smooth solution ψ of problem (2.1.10) on $t^* \geq t_0^* > 0$.

The proof of Theorem 2.1.3 employs an iteration, similar to that used for the analogous result on Minkowski space, on the finite-in-time problems (see Figure 2.5)

$$\begin{cases} \square_{g_S} \psi_i = 0, \\ \psi|_{\mathcal{H}^+} = h_{\mathcal{H}^+}, \quad r\psi|_{v=v_i} = h_{\mathcal{I}^+}. \end{cases} \quad (2.1.15)$$

Here, $h_{\mathcal{H}^+}, h_{\mathcal{I}^+}$ are appropriately truncated scattering data (see Section 2.3.4). That problems (2.1.15) approximate the full problem (2.1.10) in the limit $v_i \rightarrow \infty$ will be shown in section 2.3.4.

The presence of a black hole region in the spacetime introduces some new difficulties, of which we now provide an overview.

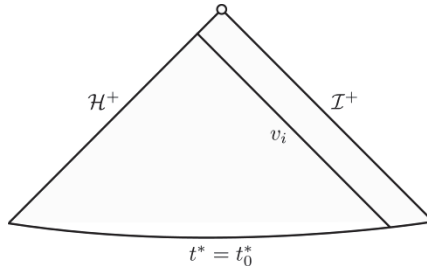


Figure 2.5

Degeneration of the T energy As on Minkowski space, we would like to apply an appropriate vector field multiplier to the equation and integrate over a suitable spacetime region to estimate all derivatives of the solution ψ_i by the scattering data (in a manner analogous to (2.1.8)). However, in doing so, using the Killing vector field $T = \partial_{t^*}$ now produces an energy (the analogue of (2.1.9))

$$E^T[\psi](t^*) = \int_{\Sigma_{t^*}} \left[\left(1 + \frac{2M}{r}\right) (\partial_{t^*} \psi)^2 + \left(1 - \frac{2M}{r}\right) (\partial_r \psi)^2 + |\nabla \psi|^2 \right] r^2 dr d\omega, \quad (2.1.16)$$

⁵This result follows from [13], however, a different argument is used there.

which degenerates at the event horizon $r = 2M$. This is a manifestation of the fact that on the Schwarzschild spacetime (in fact, all black hole spacetimes with $\kappa > 0$), T is not globally uniformly timelike on the entire exterior region - it becomes *null* on the horizon.⁶ In order to remedy this and derive an estimate which controls *all* derivatives of the solution non-degenerately (allowing us to apply the Arzelà-Ascoli Theorem to obtain a solution of the full problem (2.1.10)), we must employ an alternative, necessarily non-Killing, vector field multiplier. This leads us to the next difficulty.

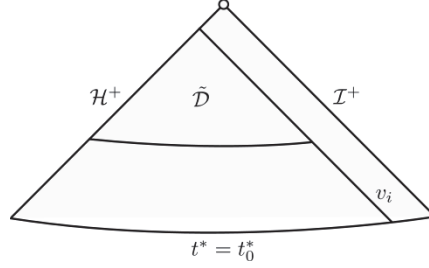


Figure 2.6

Gravitational blueshift In the pursuit of a non-degenerate energy estimate analogous to (2.1.8), we use a globally uniformly timelike vector field N as our multiplier:

$$N\psi \cdot \square_{g_S}\psi = 0.$$

Integrating over the spacetime region $\tilde{\mathcal{D}}$ (depicted in Figure 2.6) bounded by a constant t^* slice⁷ and the null hypersurfaces ($v = v_i$, $\mathcal{H}^+$) on which the truncated scattering data is posed in the finite problem (2.1.15) then gives rise to the estimate

$$E^N[\psi_i](t^*) \leq F_{\mathcal{H}^+ \cap [t^*, T]} + F_{\{v=v_i\} \cap [t^*, T]} + C_\kappa \int_{t^*}^T E^N[\psi_i](t^{*'}) dt^{*'}, \quad (2.1.17)$$

where the constant $C_\kappa > 0$ depends linearly on κ . Here, $F_{\mathcal{H}^+ \cap [t^*, T]}$ and $F_{\{v=v_i\} \cap [t^*, T]}$ are first-order flux terms through the horizon and $v = v_i$ (the bifurcate data analogue of the boundary term on $v = v_i$ in (2.1.8)). Thus, in order to control the energy purely in terms of the scattering data, we must treat the (κ dependent) bulk term. This is easily achieved, thanks to the assumption of exponential decay (2.1.14) on the scattering data, as follows.

By applying the assumed decay (2.1.14) of the scattering data, (2.1.17) gives

$$E^N[\psi_i](t^*) \leq C \exp(-B \cdot t^{*'}) + C_\kappa \int_{t^*}^T E^N[\psi_i](t^{*'}) dt^{*'}. \quad (2.1.18)$$

⁶In the case of non-zero angular momentum (for example, on Kerr or Kerr-AdS), T is in fact non-timelike on a region (the *ergoregion*) extending beyond the event horizon, and may even be spacelike at the horizon.

⁷In practice, we will work with mixed spacelike-null slices Σ_{t^*} (see Section 2.3.1).

Via a Grönwall inequality (see Appendix B), (2.1.18) implies

$$\begin{aligned}
E^N[\psi_i](\tau) &\leq C \exp(-B \cdot t^{*'}) + \int_{t^*}^T C_k C \exp(-B \cdot t^{*'}) \exp\left(\int_{t^*}^{t^{*'}} C_\kappa ds\right) dt^{*'} \\
&= C \exp(-B \cdot t^{*'}) + \int_{t^*}^T C_k C \exp\left((C_\kappa - B)t^{*'}\right) dt^{*'} \\
&\leq C \left(1 + \frac{C_\kappa}{B - C_\kappa}\right) \exp(-B \cdot t^*) \\
&= \tilde{C} \exp(-B \cdot t^*),
\end{aligned} \tag{2.1.19}$$

with

$$\tilde{C} = C \left(1 + \frac{C_\kappa}{B - C_\kappa}\right) > 0,$$

provided that the rate $B > 0$ of decay (2.1.14) along the horizon is strictly greater than the constant C_κ .

As on Minkowski space, commuting with appropriate vector fields produces analogous estimates for the H^k norms of the solutions ψ_i of the finite problems (2.1.15). However, due to the aforementioned degeneration of the $T = \partial_{t^*}$ energy (2.1.16), one must also commute with N to control all higher-order derivatives non-degenerately. Each commutation produces an additional bulk term, meaning that the assumed rate of decay $B > 0$ of the scattering data (2.1.14) must be chosen sufficiently large, depending on the number of commutations, as well as C_κ . One may then proceed precisely as before, Sobolev embedding in the mixed null-spacelike slices to obtain L^∞ estimates for the ψ_i and their derivatives, before applying the *Arzelà-Ascoli Theorem* (see Appendix C) to obtain a solution of the full problem (2.1.10).

2.2 Minkowski space

We begin our discussion of Minkowski space with some preliminary definitions (Section 2.2.1). These are followed by the statements of some key estimates we will require later (Section 2.2.2). With these in hand, we proceed with tackling the forward problem (2.1.1) in Sections 2.2.3-2.2.4. Finally, in Sections 2.2.5-2.2.7, we discuss the backwards problem (2.1.6).

2.2.1 Preliminaries

Here, we state some basic definitions.

Vector fields In order to generate energy estimates to sufficiently high order, we will commute the equation (2.1.2) with the (Killing) *commuting* vector fields:

$$\Gamma_M \in \left\{ T = \partial_t = \frac{1}{2}(\partial_u + \partial_v), \{\Omega_j\}_{j=1}^3 \right\}. \tag{2.2.1}$$

The energies We recall the definitions of the standard Sobolev spaces on Minkowski space. In Section 2.2.2 we demonstrate that these arise when using the vector field $T = \partial_t$ as a multiplier and commuting with T , $\{\Omega_j\}_{j=1}^3$.

Definition 2.2.1. (Sobolev norms on spacelike slices of Minkowski space) Let t be a fixed time. Then for each $k \in \mathbb{N}$, we define the Sobolev space

$$H_M^k(\Sigma_t) = \{\psi : [0, \infty) \times S^2 \rightarrow \mathbb{R} \mid \|\psi\|_{H_M^k(\Sigma_t)} < \infty\}$$

of scalar fields ψ on Σ_t associated with the energy

$$\|\psi\|_{H_M^1(\Sigma_t)}^2 = \int_{\Sigma_t} \left[(\partial_t \psi)^2 + (\partial_r \psi)^2 + |\nabla \psi|^2 + \psi^2 \right] r^2 dr d\omega, \quad (2.2.2)$$

and, for $k \geq 2$,

$$\|\psi\|_{H_M^k(\Sigma_t)}^2 = \int_{\Sigma_t} \sum_{\substack{|\sigma| \leq k-1, \\ |\rho| \leq k}} \left[|\nabla \Omega^{\sigma_1} \partial_t^{\sigma_2} \partial_r^{\sigma_3} \psi|^2 + (\Omega^{\rho_1} \partial_t^{\rho_2} \partial_r^{\rho_3} \psi)^2 \right] r^2 dr d\omega. \quad (2.2.3)$$

By $\|\cdot\|_{\hat{H}_M^k}$, we denote the above norms, minus their zeroth-order terms.

We also recall the following standard definitions of L^p spaces on Minkowski space.

Definition 2.2.2 (L^p -norms on spacelike slices of Minkowski space). Let $t \in \mathbb{R}$ be a fixed time. Then for $p \in \{2, \infty\}$ we define the spaces

$$L_M^p(\Sigma_t) = \{\psi : [0, \infty) \times S^2 \rightarrow \mathbb{R} \mid \|\psi\|_{L_M^p(\Sigma_t)}^2 < \infty\}$$

of scalar fields ψ on Σ_t associated with the norms

$$\|\psi\|_{L_M^2(\Sigma_t)}^2 = \int_{\Sigma_t} |\psi|^2 r^2 dr d\omega, \quad \|\psi\|_{L_M^\infty(\Sigma_t)} = \sup_{\Sigma_t} |\psi|.$$

2.2.2 Key estimates

In this section, we collect some estimates which will be used to prove Theorems 2.1.1 and 2.1.2. Firstly, we state energy conservation and local energy boundedness results on the spacelike slices Σ_t . These are followed by some r^p -weighted estimates of Dafermos-Rodnianski [12] on slices of constant u .

Conservation of energy Although we will not directly use the following result, it is included here as an interesting comparison to the stronger localised estimate (Proposition 2.2.5) which appears below, as well as a first application of the energy method.

Proposition 2.2.3. (Conservation of energy on Minkowski space) Suppose ψ is a smooth solution of problem (2.1.1) arising from smooth compactly supported initial data (ψ_0, ψ_1) . Then ψ satisfies the energy conservation law

$$\int_{\Sigma_t} \left[(\partial_t \psi)^2 + (\partial_r \psi)^2 + |\nabla \psi|^2 \right] r^2 dr d\omega = \int_{\Sigma_\tau} \left[(\partial_t \psi)^2 + (\partial_r \psi)^2 + |\nabla \psi|^2 \right] r^2 dr d\omega, \quad (2.2.4)$$

at each $t \geq \tau$.

This follows easily from applying $\partial_t \psi$ to the equation as a multiplier, integrating over the spacetime slab $\mathcal{D} = \mathcal{M}_M \cap [\tau, t]$ and dropping boundary terms as a result of compact support.

Local energy boundedness By localising to the spacetime region of the following definition, one can recover the stronger coercive estimate of Proposition 2.2.5 below.

Definition 2.2.4. (A cut-off cone in Minkowski space) Fix $t_2 > t_1$, $\bar{R} > 0$ and let $B_r \subset \mathbb{R}^3$ be the ball of radius $r > 0$ centered at the origin in $\mathbb{R}^3$ (whose boundary we will label S_r). We will denote by C the “cut-off cone”

$$C = \bigcup_{\tau \in [t_1, t_2]} \{t = \tau\} \times B_{\bar{R}+t_2-\tau},$$

and by ∂C its boundary (see Figure 2.7).

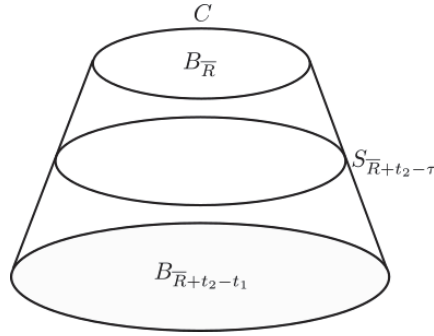


Figure 2.7: The “cut-off cone” of Definition 2.2.4.

Applying $\partial_t \psi$ to the equation as a multiplier once more and integrating over C , we obtain the following local energy boundedness result. In Section 2.2.6, we will apply a similar estimate for a particular choice of $\bar{R} > 0$ in the backward direction to control the solution on a spacelike slice by the scattering data. Higher order analogues follow from commuting the equation with the commuting vector fields Γ_M .

Proposition 2.2.5. (Local energy boundedness on Minkowski space) Fix $t_2 > t_1$ and $\bar{R} > 0$. Suppose ψ is a smooth solution of problem (2.1.1) arising from smooth compactly supported initial data (ψ_0, ψ_1) . Then ψ satisfies

$$\begin{aligned} & \int_{\{t=t_2\} \times B_{\bar{R}}} \left[(\partial_t \psi)^2 + (\partial_r \psi)^2 + |\nabla \psi|^2 \right] r^2 dr d\omega \\ & \leq \int_{\{t=t_1\} \times B_{\bar{R}+t_2-t_1}} \left[(\partial_t \psi)^2 + (\partial_r \psi)^2 + |\nabla \psi|^2 \right] r^2 dr d\omega. \end{aligned}$$

r^2 -weighted energy estimates Here we state the r^p -weighted energy estimates for the forward and backwards problems (2.1.1), (2.1.6) on Minkowski space, with $p = 2$. It is possible to obtain similar estimates for $p > 2$ [12] (up to non-positive-definite bulk terms), however, these will not be required here. These results follow from applying $r^p \partial_v \psi$ as a multiplier to the equation in double-null coordinates (2.1.2) and integrating by parts over an appropriate spacetime region ($D_{t_1-R}^{t_2-R}$ below). One can see [12] for further details.

In the following, given fixed $t_2 > t_1$, $R > 0$, we will denote by

$$D_{t_1-R}^{t_2-R} = \mathcal{M}_M \cap \{r \geq R\} \cap \{u \in [t_1 - R, t_2 - R]\},$$

the null partial spacetime strip, and by $\tilde{\Sigma}_{t_1}$ the mixed spacelike-null slice

$$\tilde{\Sigma}_{t'} = \left\{ \{t = t'\} \cap \{r \leq R\} \right\} \cup \left\{ \{u = t' - R\} \cap \{v \geq t' + R\} \right\}$$

(see Figure 2.8).

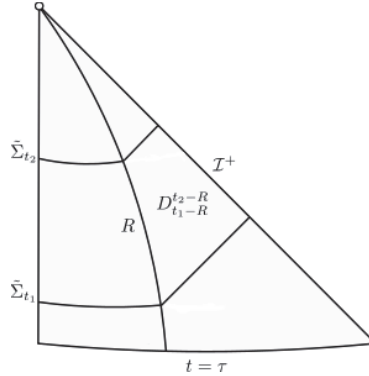


Figure 2.8

Proposition 2.2.6. (Estimate for r^2 -weighted energies for the forwards problem on Minkowski space [12]) Fix $t_2 > t_1 \geq \tau$ and $R > 0$. Suppose ψ is a smooth solution of problem (2.1.1) arising from smooth compactly supported initial data. Then ψ satisfies the r^2 -weighted estimate

$$\int_{u=t_2-R, v \geq t_2+R} r^2 (\Gamma_M^\alpha \partial_v (r\psi))^2 dv d\omega + \int_{D_{t_1-R}^{t_2-R}} 2r (\Gamma_M^\alpha \partial_v (r\psi))^2 du dv d\omega$$

$$\leq \int_{u=t_1-R, v \geq t_1+R} r^2 (\Gamma_M^\alpha \partial_v(r\psi))^2 dv d\omega + C \int_{\tilde{\Sigma}_{t_1}} \left[(\Gamma_M^\alpha \partial_t \psi)^2 + (\Gamma_M^\alpha \partial_r \psi)^2 + |\nabla \Gamma_M^\alpha \psi|^2 \right] dr d\omega,$$

for all $|\alpha| \geq 0$, some $C > 0$.

Proposition 2.2.6 follows immediately from the $|\alpha|=0$ result appearing in [12] after commuting with Γ_M .

The following result for the backwards problem is not proven in [12], however, it follows from the same argument used to prove the (forward) (p-WE-Mink) estimate appearing therein, adapted to the backwards setting.

Proposition 2.2.7. (Estimate for r^2 -weighted energies for the backwards problem on Minkowski space [12]) Fix $t_2 > t_1 \geq 0$ and $R > 0$. Suppose ψ is a smooth solution of problem (2.1.6) arising from smooth compactly supported scattering data. Then ψ satisfies the r^2 -weighted estimate

$$\begin{aligned} & \int_{u=t_1-R, v \geq t_1+R} r^2 (\Gamma_M^\alpha \partial_v(r\psi))^2 dv d\omega + \int_{D_{t_1-R}^{t_2-R}} 2r (\Gamma_M^\alpha \partial_v(r\psi))^2 du dv d\omega \\ & \leq \int_{u=t_2-R, v \geq t_2+R} r^2 (\Gamma_M^\alpha \partial_v(r\psi))^2 dv d\omega + C \int_{\tilde{\Sigma}_{t_2}} \left[(\Gamma_M^\alpha \partial_t \psi)^2 + (\Gamma_M^\alpha \partial_r \psi)^2 + |\nabla \Gamma_M^\alpha \psi|^2 \right] dr d\omega, \end{aligned}$$

for all $|\alpha| \geq 0$, some $C > 0$.

With these estimates in hand, we can now proceed to formally stating and proving Theorem 2.1.1.

2.2.3 The forward problem: Existence and uniqueness of Friedlander's *radiation field*

We begin our study of the forward problem (2.1.1) on Minkowski space by defining the following initial data quantities.

Definition 2.2.8. (Initial data quantities $\tilde{\mathfrak{D}}_M^m$ for the forward problem on Minkowski space) Given sufficiently regular initial data $(\psi_0, \psi_1) : \mathbb{R}^3 \rightarrow \mathbb{R}$ on Σ_τ , we define the quantity

$$\tilde{\mathfrak{D}}_M^m = \int_{\Sigma_\tau} \sum_{|\alpha| \leq m-1} \left[\psi_0^2 + (\Gamma^\alpha \partial_t \psi_0)^2 + (\Gamma^\alpha \partial_r \psi_0)^2 + |\nabla \Gamma^\alpha \psi_0|^2 \right] r^2 dr d\omega$$

for each $m \geq 1$.

In terms of these initial data quantities, Theorem 2.1.1 is restated formally as follows.

Theorem 2.2.9. (Existence and uniqueness of the radiation field of an initially compactly supported wave on Minkowski space [21]) Let ψ be a smooth solution of problem (2.1.1) arising from smooth initial data (ψ_0, ψ_1) at time $t = \tau \geq 0$ supported on $r \leq R_{supp} < \infty$

and satisfying $\tilde{\mathfrak{D}}_M^5 < \infty$. Then there exists a unique, v -independent function $h_{\mathcal{I}^+}(u, \theta, \phi) : \mathbb{R} \times S^2 \rightarrow \mathbb{R}$ (the *radiation field* of ψ) such that

$$\left| \psi(u, v, \theta, \phi) - \frac{h_{\mathcal{I}^+}(u, \theta, \phi)}{r} \right| \leq C \tilde{\mathfrak{D}}_M^5 \frac{1}{1 + r(u, v)^2},$$

for some $C > 0$.

The proof of Theorem 2.2.9 involves integrating equation (2.1.2) in the null directions and using the r^2 -weighted estimate (Proposition 2.2.6), combined with Sobolev embedding (Theorem B.1.2 and Corollary B.1.3) to control the right-hand side. This is carried out as follows.

2.2.4 Proving existence and uniqueness of the radiation field

In order to prove Theorem 2.2.9, we require an estimate for the right-hand side of equation (2.1.2). In the proof below, this will be applied to allow the left-hand side of the equation (2.1.2) to be directly integrated in u and v .

Proposition 2.2.10. (An estimate for $\Delta\psi$ on Minkowski space) Let ψ be as in Theorem 2.2.9. Then

$$|\Delta\psi| \leq C \tilde{\mathfrak{D}}_M^5 \frac{1}{1 + r^3},$$

for some $C > 0$.

Proof. First, suppose that (u, v) is such that $t(u, v) > \tau$ and $r(u, v) > R_{supp}$. Then choose v_2 such that the sphere (u, v_2) lies in a finite time slice Σ_{τ_2} with $\tau_2 \geq \tau$, at a finite radius $R > R_{supp}$. By the Fundamental Theorem of Calculus, one has

$$|(r\psi)(u, v, \theta, \phi)| \leq \int_{v_2}^v |\partial_v(r\psi)(u, v', \theta, \phi)| dv' + |(r\psi)(u, v_2, \theta, \phi)|. \quad (2.2.5)$$

Applying the Sobolev inequality on S^2 (Corollary B.1.3) to the first term on the right-hand side of (2.2.5) yields

$$\begin{aligned} & \int_{v_2}^v |\partial_v(r\psi)(u, v', \theta, \phi)| dv' \\ & \leq C \int_{v \geq v_2} \int_{S^2} \left[(\partial_v(r\psi))^2 + |\nabla \partial_v(r\psi)|^2 + |\nabla^2 \partial_v(r\psi)|^2 \right] r^2 dv d\omega \\ & \leq C \sum_{|\sigma| \leq 2} \int_{v \geq v_2} \int_{S^2} (\Omega^\sigma \partial_v(r\psi))^2 r^2 dv d\omega \\ & \leq C \sum_{|\sigma| \leq 2} \int_{\tilde{\Sigma}_{t_1}} \left[(\Omega^\sigma \partial_t \psi)^2 + (\Omega^\sigma \partial_r \psi)^2 + |\nabla \Omega^\sigma \psi|^2 \right] dr d\omega \end{aligned} \quad (2.2.6)$$

$$\begin{aligned}
&= C \sum_{|\sigma| \leq 2} \int_{\Sigma_{t_1}} \left[(\Omega^\sigma \partial_t \psi)^2 + (\Omega^\sigma \partial_r \psi)^2 + |\nabla \Omega^\sigma \psi|^2 \right] dr d\omega \\
&\leq C \tilde{\mathfrak{D}}_M^3,
\end{aligned} \tag{2.2.7}$$

for some $C > 0$. Here, (2.2.6) follows from the r^2 -weighted estimates (Proposition 2.2.6 with $|\alpha| \leq 2$) by choosing $t_1 = \tau$ and R as above to ensure the boundary term on u_1 vanishes. Applying (2.2.7) to (2.2.5) then gives

$$|(r\psi)(u, v, \theta, \phi)| \leq C \tilde{\mathfrak{D}}_M^3 + |(r\psi)(u, v_2, \theta, \phi)|. \tag{2.2.8}$$

We can estimate the second term on the right-hand side of (2.2.8) as follows. For each u and its corresponding fixed v_2 , choosing r_1, r_2 so that $0 < r_1 < R < r_2 < \infty$, we have that

$$\begin{aligned}
&|(r\psi)(u, v_2, \theta, \phi)| \\
&= |R\psi(u, v_2, \theta, \phi)| \\
&\leq C \int_{\Sigma_\tau \cap [r_1, r_2]} \left[\psi^2 + (\partial_r \psi)^2 + |\nabla \psi|^2 + (\partial_r^2 \psi)^2 + |\nabla^2 \psi|^2 + |\nabla \partial_r \psi|^2 \right] r^2 dr d\omega
\end{aligned} \tag{2.2.9}$$

$$\leq C \left(\int_{\Sigma_\tau \cap [r_1, r_2]} \left[\psi^2 + \sum_{|\sigma|=1} (\Omega^\sigma \psi)^2 \right] r^2 dr d\omega + \tilde{\mathfrak{D}}_M^1 \right) \tag{2.2.10}$$

$$\leq C \left(\int_{\Sigma_\tau \cap [r_1, r_2]} \left[(\partial_r \psi)^2 + |\nabla \psi|^2 \right] r^2 dr d\omega + \tilde{\mathfrak{D}}_M^2 \right) \tag{2.2.11}$$

$$\leq C \tilde{\mathfrak{D}}_M^2, \tag{2.2.12}$$

for some $C > 0$. Here, (2.2.9) follows from Sobolev embedding in Σ_τ (Theorem B.1.2), (2.2.10) from Proposition 2.2.5, and (2.2.11) from Hardy's inequality (Theorem B.2.1) and Proposition 2.2.5. The final inequality (2.2.12) follows from Proposition 2.2.5.

Now, combining estimates (2.2.8) and (2.2.12) gives

$$|(r\psi)(u, v, \theta, \phi)| \leq C \tilde{\mathfrak{D}}_M^3. \tag{2.2.13}$$

for some $C > 0$.

Next, suppose instead that (u, v) is such that $t = t(u, v) > \tau$ and $r(u, v) \leq R_{supp}$. Then

$$\begin{aligned}
|(r\psi)(u, v, \theta, \phi)| &\leq C \int_{\Sigma_t \cap \{r \leq R_{supp}\}} \psi^2 r^2 dr d\omega \\
&\leq C \int_{\Sigma_t \cap \{r \leq R_{supp} + t - \tau\}} \left[(\partial_r \psi)^2 + |\nabla \psi|^2 \right] r^2 dr d\omega
\end{aligned} \tag{2.2.14}$$

$$\leq C \int_{\Sigma_\tau \cap \{r \leq R_{supp}\}} \left[(\partial_r \psi)^2 + |\nabla \psi|^2 \right] r^2 dr d\omega \tag{2.2.15}$$

$$\leq C \tilde{\mathfrak{D}}_M^1, \tag{2.2.16}$$

for some $C > 0$, where (2.2.14) follows from Hardy's inequality (Theorem B.2.1) and (2.2.15) from local energy boundedness (Proposition 2.2.5).

Combining estimates (2.2.13) and (2.2.16) gives the pointwise bound

$$|\psi| \leq C \tilde{\mathfrak{D}}_M^3 \frac{1}{1+r}, \quad (2.2.17)$$

for some $C > 0$. By commuting the equation with Ω^σ for $|\sigma| \leq 2$ and Sobolev embedding in $\mathbb{R}^3$ (Theorem B.1.2) one obtains

$$r^2 |\Delta \psi| \leq r^2 |\nabla^2 \psi| \leq C \sum_{|\alpha|=2} |\Omega^\alpha \psi| \leq C \tilde{\mathfrak{D}}_M^5 \frac{1}{1+r},$$

as required. $\square$

Given Proposition 2.2.10, we may proceed with proving Theorem 2.2.9 by integrating equation (2.1.2) directly.

Proof of Theorem 2.2.9. By Proposition 2.2.10 and the equation (2.1.2), we have that

$$|\partial_u \partial_v(r\psi)| = r |\Delta \psi| \leq C \tilde{\mathfrak{D}}_M^5 \frac{1}{1+r^2}, \quad (2.2.18)$$

for some $C > 0$.

Fix $u_0 = \tau + R$ with $R > R_{supp}$. Then, by integrating (2.2.18) in u , one obtains

$$\begin{aligned} |\partial_v(r\psi)(u, v, \theta, \phi)| &\leq \int_{u_0}^u |\partial_u \partial_v(r\psi)(u', v, \theta, \phi)| du' \\ &\leq \int_{u_0}^u C \tilde{\mathfrak{D}}_M^5 \frac{1}{1+r(u', v)^2} du' \\ &\leq C \tilde{\mathfrak{D}}_M^5 (u - u_0) \frac{1}{1+r^2}, \end{aligned} \quad (2.2.19)$$

for some $C > 0$. Now fixing $\tilde{v}$ and integrating (2.2.19) in v gives

$$\begin{aligned} \left| (r\psi)(u, v, \theta, \phi) - (r\psi)(u, \tilde{v}, \theta, \phi) \right| &\leq \int_v^{\tilde{v}} |\partial_v(r\psi)(u, v', \theta, \phi)| dv' \\ &\leq \int_v^{\tilde{v}} C \tilde{\mathfrak{D}}_M^5 (u - u_0) \frac{1}{1+r(u, v')^2} dv' \\ &\leq C \tilde{\mathfrak{D}}_M^5 \frac{1}{1+r(u, v)}, \end{aligned} \quad (2.2.20)$$

for some $C > 0$. Since the right-hand side of (2.2.20) vanishes in the limit $r \rightarrow \infty$, independently of $\tilde{v}$, the result follows. $\square$

This completes our consideration of the forwards problem.

2.2.5 The backwards problem: A scattering construction for linear waves from their radiation fields

We begin our discussion of the backwards problem (2.1.6) on Minkowski space with the following definition of scattering data quantities.

Definition 2.2.11. (Data quantities $\mathfrak{D}_M^k$ for the scattering problem on Minkowski space) Given sufficiently regular scattering data $h_{\mathcal{I}^+} : \mathbb{R} \times S^2 \rightarrow \mathbb{R}$ supported on $[u_0, u_1]$ with $u_0 < u_1$, we define the quantity

$$\mathfrak{D}_M^k = \int_{u_0}^{u_1} \int_{S^2} \sum_{|\alpha| \leq k-1} \left[h^2 + (\Gamma_M^\alpha \partial_u h)^2 + |\nabla \Gamma_M^\alpha h|^2 \right] r^2 du d\omega.$$

In terms of these data quantities, we can now formally state the result we wish to prove for the backwards problem on Minkowski space.

Theorem 2.2.12. (A scattering construction for linear waves on Minkowski space from prescribed compactly supported radiation fields [21]) Let $h_{\mathcal{I}^+}(u, \theta, \phi) : \mathbb{R} \times S^2$ be smooth scattering data supported on $[u_0, u_1]$ with $u_0 < u_1 < \infty$, $u_0(r = \infty) = \tau \geq 0$, such that $\mathfrak{D}_M^5 < \infty$. Then there exists a smooth solution ψ of problem (2.1.6) attaining $h_{\mathcal{I}^+}$ as its radiation field, i.e.

$$\left| \psi(u, v, \theta, \phi) - \frac{h(u, \theta, \phi)}{1 + r(u, v)} \right| \leq \frac{C \mathfrak{D}_M^5}{(1 + r(u, v))^2}. \quad (2.2.21)$$

In order to prove Theorem 2.2.12, we derive uniform estimates for solutions ψ_i of a sequence of finite (in v) problems (2.1.7) approximating the full problem (2.1.6). This is carried out in the following section. These estimates will subsequently allow an application of the Arzelà-Ascoli Theorem to give the required result.

2.2.6 The iteration scheme

Fix a sequence $\{v_i\}_{i=1}^\infty$ of v -values such that $r(u_1, v_1) > 0$. We will study the finite in v problems

$$\begin{cases} \frac{1}{r} \partial_u \partial_v (r \psi_i) = \Delta \psi_i \\ r \psi_i|_{v=v_i} = h_{\mathcal{I}^+}(u, \theta, \phi), \quad \psi_i|_{u=u_1} = 0, \end{cases} \quad (2.1.7)$$

with truncated scattering data posited on $v = v_i$, satisfying

$$h_{\mathcal{I}^+}(u, v = v_i, \theta, \phi) = \varpi_i(u) h_{\mathcal{I}^+}(u, \theta, \phi),$$

where

$$\varpi_i(u) = \begin{cases} 1 & \text{for } u \leq u_1 - 0.2 \\ 0 & \text{for } u > u_1 - 0.1. \end{cases}$$

The key element of the proof of Theorem 2.2.12 is the following uniform boundedness result for their solutions.

Theorem 2.2.13. (Uniformly bounded solutions of the finite problems on Minkowski space) Let i be fixed. The corresponding smooth solution ψ_i in of problem (2.1.7) satisfies

$$\|\psi_i\|_{H_M^4(\Sigma_t)} \leq C\mathfrak{D}_M^5, \quad (2.2.22)$$

$$\|D^\sigma \psi_i\|_{L_M^\infty(\Sigma_t)} \leq C\mathfrak{D}_M^{\sigma+3}, \quad (2.2.23)$$

at all times $t \geq \tau$, for some $C > 0$ and all $|\sigma| \leq 2$.

In the proof of Theorem 2.2.13, we will require the following proposition.

Proposition 2.2.14. (Estimates for derivatives non-tangential to $\{v = v_i\}$ on Minkowski space) Fix i and let ψ_i be the corresponding smooth solution of problem (2.1.7). Then one has the following estimates for selected v -derivatives of ψ_i :

$$\int_{\{v=v_i\}} \left[(\partial_u \partial_v \psi_i)^2 + |\nabla \partial_v \psi_i|^2 \right] r^2 du d\omega \leq C(\mathfrak{D}_M^2)^2 \quad (2.2.24)$$

Proof. Let us begin with the first term on the left-hand side of (2.2.24).

By the equation (2.1.2), we have that

$$\begin{aligned} r\Delta \psi_i &= \partial_u \partial_v (r\psi_i) = \partial_u \psi_i - \partial_v \psi_i + r\partial_u \partial_v \psi_i \\ \Rightarrow \quad \partial_u \partial_v \psi_i &= \Delta \psi_i - \frac{1}{r} \partial_u \psi_i + \frac{1}{r} \partial_v \psi_i \end{aligned} \quad (2.2.25)$$

$$\Rightarrow \quad (\partial_u \partial_v \psi_i)^2 \leq C \left((\Delta \psi_i)^2 + \frac{1}{r^2} (\partial_u \psi_i)^2 + \frac{1}{r^2} (\partial_v \psi_i)^2 \right), \quad (2.2.26)$$

for some $C > 0$. Integrating (2.2.26) and using the scattering data to control derivatives tangential to $\{v = v_i\}$ gives

$$\begin{aligned} \int_{\{v=v_i\}} (\partial_u \partial_v \psi_i)^2 r^2 du d\omega &\leq C \int_{\{v=v_i\}} \left[|\nabla^2 h|^2 + \frac{1}{r^2} \left(\partial_u \left(\frac{h}{r} \right) \right)^2 + \frac{1}{r^2} (\partial_v \psi_i)^2 \right] r^2 du d\omega \\ &\leq C(\mathfrak{D}_M^2)^2 + C \int_{\{v=v_i\}} \frac{1}{r^2} (\partial_v \psi_i)^2 r^2 du d\omega, \end{aligned} \quad (2.2.27)$$

for some $C > 0$. To deal with the remaining non-tangential derivative in (2.2.27), we use that

$$r\partial_v \psi_i = \partial_v (r\psi_i) - \psi_i. \quad (2.2.28)$$

Since ψ is already controlled by the scattering data on $\{v = v_i\}$, we estimate $\partial_v (r\psi_i)$. At each fixed $u > u_0$, the Fundamental Theorem of Calculus gives

$$|\partial_v (r\psi_i)(u, v_i, \theta, \phi)| = \int_{u_0}^u |\partial_u \partial_v (r\psi_i)(u', v_i, \theta, \phi)| du'$$

$$= \int_{u_0}^u |\Delta h(u', \theta, \phi)| du' \quad (2.2.29)$$

$$\leq (u - u_0)^{\frac{1}{2}} \left(\int_{u_0}^u |\Delta h(u', \theta, \phi)|^2 du' \right)^{\frac{1}{2}} \quad (2.2.30)$$

where (2.2.29) follows from the equation (2.1.2) and (2.2.30) from the Cauchy-Schwarz inequality. The boundary term at u_0 vanishes by definition of the support of h . Now, integrating (2.2.28) gives

$$\begin{aligned} \int_{\{v=v_i\}} |\partial_v \psi_i|^2 r^2 du d\omega &= \int_{\{v=v_i\}} |\partial_v(r\psi_i) - \psi_i|^2 r^2 du d\omega \\ &\leq \int_{\{v=v_i\}} \left[|\partial_v(r\psi_i)|^2 + |\psi_i|^2 \right] r^2 du d\omega \\ &\leq \int_{\{v=v_i\}} \left[C \int_{u_0}^u |\Delta h(u', \theta, \phi)|^2 du' + |\psi_i|^2 \right] r^2 du d\omega \quad (2.2.31) \\ &\leq \int_{\{v=v_i\}} \left[C \int_{u_0}^u |\nabla^2 h(u', \theta, \phi)|^2 du' + |\psi_i|^2 \right] r^2 du d\omega \\ &= \int_{\{v=v_i\}} \left(\frac{h}{r} \right)^2 r^2 du d\omega + C \int_{u_0}^u \left[\int_{\{v=v_i\}} |\nabla^2 h|^2 r^2 du d\omega \right] du' \\ &\leq \int_{\{v=v_i\}} \left(\frac{h}{r} \right)^2 du d\omega + C \int_{u_0}^u \mathfrak{D}_M^2 du' \\ &\leq C(\mathfrak{D}_M^2)^2 \quad (2.2.32) \end{aligned}$$

for some $C > 0$. Here, (2.2.31) follows from (2.2.30). Applying (2.2.32) to (2.2.27) gives the estimate for the $\partial_u \partial_v \psi_i$ derivative appearing in (2.2.24).

To control the remaining $\nabla \partial_v$ term appearing on the left-hand side of (2.2.24), we simply commute estimate (2.2.32) with the angular momentum operators $\{\Omega_j\}_{j=1}^3$, giving

$$\int_{\{v=v_i\}} |\nabla \partial_v \psi_i|^2 r^2 du d\omega \leq C \sum_{j=1}^3 \int_{\{v=v_i\}} (\Omega_j \partial_v \psi_i)^2 r^2 du d\omega \leq C(\mathfrak{D}_M^2)^2.$$

□

With this estimate in hand, we can now prove Theorem 2.2.13.

Proof of Theorem 2.2.13. Fix i and let C_{v_i} be the cut-off cone (see Definition 2.2.4) with t_2 chosen such that Σ_{t_2} does not intersect the region on which the solution ψ_i is supported, $\tau \leq t_1 < t_2$, and $\bar{R} > 0$ chosen such that the null boundary of the cone lies on $\{v = v_i\}$ (see Figure 2.9). Integrating the energy identity (A.0.1) with $X = T$ over C_{v_i} yields

$$\int_{\Sigma_{t_1} \cap \{v \leq v_i\}} \left[(\partial_t \psi_i)^2 + (\partial_r \psi_i)^2 + |\nabla \psi_i|^2 \right] r^2 dr d\omega$$

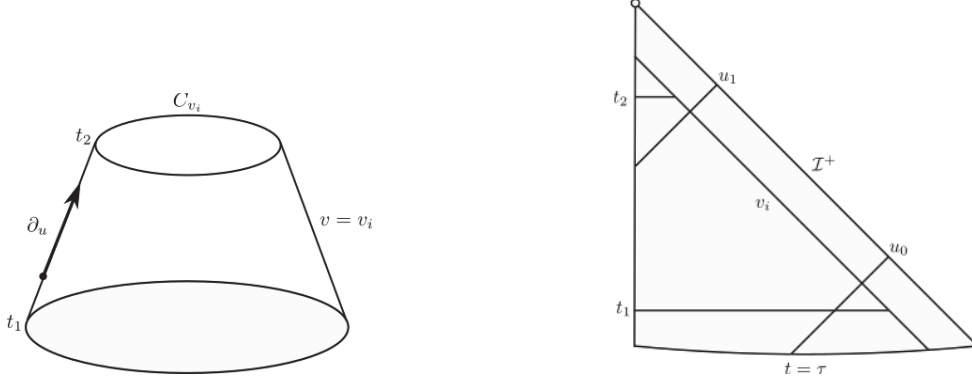


Figure 2.9

$$= \int_{\{v=v_i\} \cap \{t \in [t_1, t_2]\}} \frac{1}{2} \left[(\partial_t \psi_i - \partial_r \psi_i)^2 + |\nabla \psi_i|^2 \right] r^2 du d\omega, \quad (2.2.33)$$

since the boundary term on t_2 vanishes by construction. Let us consider the terms on the right-hand side of (2.2.33). We have that

$$\begin{aligned} & \int_{\{v=v_i\} \cap \{t \in [t_1, t_2]\}} \frac{1}{2} \left[(\partial_t \psi_i - \partial_r \psi_i)^2 + |\nabla \psi_i|^2 \right] r^2 du d\omega \\ &= \int_{\{v=v_i\} \cap \{t \in [t_1, t_2]\}} \frac{1}{2} \left[(\partial_u \psi_i)^2 + |\nabla \psi_i|^2 \right] r^2 du d\omega \\ &= \int_{\{v=v_i\} \cap \{t \in [t_1, t_2]\}} \frac{1}{2} \left[\left(\partial_u \left(\frac{h}{r} \right) \right)^2 + \frac{1}{r^2} |\nabla h|^2 \right] r^2 du d\omega \end{aligned} \quad (2.2.34)$$

$$\begin{aligned} &= \int_{\{v=v_i\} \cap \{t \in [t_1, t_2]\}} \frac{1}{2} \left[\frac{1}{r^2} (\partial_u h)^2 + \frac{h^2}{r^4} + \frac{2}{r^3} h \partial_u h + \frac{1}{r^2} |\nabla h|^2 \right] r^2 du d\omega \\ &\leq C \int_{\{v=v_i\} \cap \{t \in [t_1, t_2]\}} \left[\left(\frac{h}{r} \right)^2 + (\partial_u h)^2 + \frac{1}{r^2} |\nabla h|^2 \right] r^2 du d\omega \end{aligned} \quad (2.2.35)$$

$$\leq C \mathfrak{D}_M^1, \quad (2.2.36)$$

for some $C > 0$. Here, (2.2.34) follows from the fact that the derivatives ∂_u, ∇ are tangent to the boundary of the cone (which lies on $\{v = v_i\}$, where $\psi = \frac{h}{r}$). Inequality (2.2.35) follows from the Cauchy-Schwarz inequality. Since h is v -independent, estimate (2.2.36) is independent of i . Applying (2.2.36) to (2.2.33) gives

$$\int_{\Sigma_{t_1} \cap \{v \leq v_i\}} \left[(\partial_t \psi_i)^2 + (\partial_r \psi_i)^2 + |\nabla \psi_i|^2 \right] r^2 dr d\omega \leq C \mathfrak{D}_M^1. \quad (2.2.37)$$

In order to obtain higher-order analogues of (2.2.37), we commute estimate (2.2.37) with the Γ_M commuting vector fields (2.2.1). Let us demonstrate how this is carried out for the

second order estimate, as the process at higher-order is similar. After commuting once with T , one obtains the estimate

$$\begin{aligned}
& \int_{\Sigma_{t_1} \cap \{v \leq v_i\}} \left[(\partial_t^2 \psi_i)^2 + (\partial_t \partial_r \psi_i)^2 + |\nabla \partial_t \psi_i|^2 \right] r^2 dr d\omega \\
& \leq C \int_{\{v=v_i\} \cap \{t \in [t_1, t_2]\}} \left[(\partial_u \partial_v \psi_i)^2 + (\partial_u^2 \psi_i)^2 + |\nabla \partial_u \psi_i|^2 + |\nabla \partial_v \psi_i|^2 \right] r^2 du d\omega \\
& \leq C \int_{\{v=v_i\} \cap \{t \in [t_1, t_2]\}} \left[(\partial_u \partial_v \psi_i)^2 + \left(\partial_u^2 \frac{h}{r} \right)^2 + \left| \nabla \partial_u \left(\frac{h}{r} \right) \right|^2 + |\nabla \partial_v \psi_i|^2 \right] r^2 du d\omega \\
& \leq C (\mathfrak{D}_M^2)^2,
\end{aligned} \tag{2.2.38}$$

for some $C > 0$. Here, (2.2.38) follows from Proposition 2.2.14.

To control the remaining second order derivatives (which are purely spatial), we first commute (2.2.37) twice with the angular momentum operators $\{\Omega_i\}_{i=1}^3$. This gives the estimate

$$\begin{aligned}
& \int_{\Sigma_{t_1} \cap \{v \leq v_i\}} \left[|\nabla \partial_t \psi|^2 + |\nabla \partial_r \psi_i|^2 + |\nabla^2 \psi_i|^2 \right] r^2 dr d\omega \\
& \leq C \int_{\{v=v_i\} \cap \{t \in [t_1, t_2]\}} \sum_{j,k=1}^3 \left[(\Omega_j \partial_u \psi_i)^2 + (\Omega_j \Omega_k \partial_u \psi_i)^2 + |\nabla \Omega_j \psi_i|^2 + |\nabla \Omega_j \Omega_k \psi_i|^2 \right] r^2 du d\omega \\
& \leq C (\mathfrak{D}_M^3)^2,
\end{aligned} \tag{2.2.39}$$

for some $C > 0$. The final inequality (2.2.39) follows as a result of all derivatives appearing on the right-hand side of the previous inequality being tangential to $\{v = v_i\}$.

Estimates (2.2.38) and (2.2.39) control all second order derivatives on ψ_i , except for the second-order radial derivative. This is easily controlled via the equation after applying (2.2.37), (2.2.38) and (2.2.39) to control all other terms.

After commuting sufficiently many times and repeating these steps, one recovers the estimate

$$\|\psi_i\|_{\dot{H}_M^4(\Sigma_t)} \leq C \mathfrak{D}_M^5$$

for some $C > 0$. To prove estimate (2.2.22), we also require an L^2 estimate for ψ_i . By Hardy's inequality (Theorem B.2.1),

$$\|\psi_i\|_{L^2(\Sigma_t \cap \{v \leq v_i\})}^2 \leq C \int_{\Sigma_t \cap \{v \leq v_i\}} \left[(\partial_r \psi_i)^2 + |\nabla \psi_i|^2 \right] r^2 dr d\omega \leq C \mathfrak{D}_M^1,$$

for some $C > 0$, where the final inequality follows from (2.2.37).

Finally, An application of Theorem B.1.2 then yields estimate (2.2.23). This completes the proof. $\square$

We now have the ingredients required to prove Theorem 2.2.12.

2.2.7 Convergence of the iteration scheme and attainment of the radiation field

Proof of 2.2.12. By Theorem 2.2.13, we have uniform boundedness of the solutions ψ_i to the finite problems (2.1.7) and their first-order derivatives $D\psi_i$. Since we have set $r\psi_i|_{v=v_i} = h_{\mathcal{I}^+}(u, \theta, \phi)$ to be v -independent, ψ_i extends to the function $r^{-1}h_{\mathcal{I}^+}$ on $[v_i, \infty)$. Furthermore, since $\psi_i|_{u=u_1} = 0$, ψ_i extends to a function which is identically zero on $[u_1, \infty)$. Thus, we may apply the *Arzèla-Ascoli Theorem* on the domain $[\tau, \infty) \times [0, \infty) \times S^2$ to obtain a uniformly convergent subsequence $\psi_{i_j} \rightarrow \psi$. We may also apply *Arzèla-Ascoli* to the derivatives $\{D\psi_{i_j}, D^2\psi_{i_j}, \dots\}$ up to sufficiently high order to obtain uniformly convergent subsequences $D\psi_{i_j} \rightarrow D\psi$, $D^2\psi_{i_j} \rightarrow D^2\psi$. Thus, the limiting function ψ solves (2.1.2). It remains to show estimate (2.2.21).

If the sphere (u, v) is such that $r(u, v) = 0$, then clearly

$$\begin{aligned} & |r\psi(u, v, \theta, \phi) - h(u, \theta, \phi)| \\ &= |h(u, \theta, \phi)| \\ &\leq C \int_{u_0}^{u_1} \int_{S^2} \left[h^2 + (\partial_u h)^2 + |\nabla h|^2 + (\partial_u^2 h)^2 + |\nabla \partial_u h|^2 + |\nabla^2 h|^2 \right] r^2 du d\omega \\ &\leq C \mathfrak{D}_M^2, \end{aligned}$$

for some $C > 0$, by Sobolev embedding (Corollary B.1.3).

Now consider (u, v) such that $r(u, v) = R > 0$. By the triangle inequality and scattering data for ψ_i ,

$$\begin{aligned} & |r\psi(u, v, \theta, \phi) - h(u, \theta, \phi)| \\ &= |r\psi(u, v, \theta, \phi) - r\psi_{i_k}(u, v_{i_k}, \theta, \phi)| \\ &\leq |r\psi(u, v, \theta, \phi) - r\psi_{i_k}(u, v, \theta, \phi)| \\ &\quad + |r\psi_{i_k}(u, v, \theta, \phi) - r\psi_{i_k}(u, v_{i_k}, \theta, \phi)| \\ &= |r\psi(u, v, \theta, \phi) - r\psi_{i_k}(u, v, \theta, \phi)| + \left| \int_v^{v_{i_k}} (\partial_v(r\psi_{i_k}))(u, v', \theta, \phi) dv' \right|. \end{aligned} \quad (2.2.40)$$

Inequality (2.2.40) follows from an application of the Fundamental Theorem of Calculus. In order to control the second term on the right-hand side of (2.2.40) independently of i_k , we apply the Fundamental Theorem of Calculus once again, giving

$$|r\psi_{i_k}(u, v, \theta, \phi)| \leq \left| \int_v^{v_{i_k}} (\partial_v(r\psi_{i_k}))(u, v', \theta, \phi) dv' \right| + |r\psi_{i_k}(u, v_{i_k}, \theta, \phi)|. \quad (2.2.41)$$

Consider the second term on the right-hand side of (2.2.41). By Sobolev embedding on S^2 (Corollary B.1.3), we have that

$$\left| \int_v^{v_{i_k}} (\partial_v(r\psi_{i_k}))(u, v', \theta, \phi) dv' \right|$$

$$\begin{aligned}
&\leq C \int_{S^2} \int_v^{v_{ik}} \left[(\partial_v(r\psi_{ik})) + |\nabla \partial_v(r\psi_{ik})|^2 + |\nabla^2 \partial_v(r\psi_{ik})|^2 \right] r^2 dv' d\omega \\
&\leq C \sum_{|\sigma| \leq 2} \int_{S^2} \int_v^{v_{ik}} (\Omega^\sigma \partial_v(r\psi_{ik})) r^2 dv' d\omega
\end{aligned} \tag{2.2.42}$$

Fixing $R > 0$ as above and choosing t_2 sufficiently large so that $v = t_2 + R > v_{ik}$ and applying the r^2 -weighted estimate (Proposition 2.2.7) with $|\alpha| \leq 2$ to the right-hand side of (2.2.42) gives

$$\begin{aligned}
&\sum_{|\sigma| \leq 2} \int_{S^2} \int_v^{v_{ik}} (\Omega^\sigma \partial_v(r\psi_{ik})) r^2 dv' d\omega \\
&\leq C \sum_{|\sigma| \leq 2} \int_{\tilde{\Sigma}_{t_2}} \left[(\Omega^\sigma \partial_t \psi_{ik})^2 + (\Omega^\sigma \partial_r \psi_{ik})^2 + |\nabla \Omega^\sigma \psi_{ik}|^2 \right] r^2 dr d\omega \\
&= C \sum_{|\sigma| \leq 2} \int_{\Sigma_{t_2}} \left[(\Omega^\sigma \partial_t \psi_{ik})^2 + (\Omega^\sigma \partial_r \psi_{ik})^2 + |\nabla \Omega^\sigma \psi_{ik}|^2 \right] r^2 dr d\omega \\
&\leq C \mathfrak{D}_M^3
\end{aligned} \tag{2.2.43}$$

for some $C > 0$, where (2.2.43) follows from Theorem 2.2.13.

For the second term on the right-hand side of (2.2.41), we simply use the scattering data and Sobolev embedding (Corollary B.1.3) as before, to obtain

$$|r\psi_{ik}(u, v_{ik}, \theta, \phi)| \leq C \mathfrak{D}_M^2,$$

for some $C > 0$. Thus,

$$|\psi_{ik}| \leq C \frac{\mathfrak{D}_M^3}{1+r}, \tag{2.2.44}$$

for some $C > 0$. Via the equation (2.1.2) and the appropriate backwards analogue of Proposition 2.2.10, (2.2.44) then gives

$$|\partial_u \partial_v(r\psi_{ik})| = |r \Delta \psi_{ik}| \leq \frac{C \mathfrak{D}_M^5}{1+r^2}. \tag{2.2.45}$$

for some $C > 0$. Fix $u < u_1$. Then, by the Fundamental Theorem of Calculus, the assumption $\psi_i(u_1) = 0$ and estimate (2.2.45)

$$\begin{aligned}
|\partial_v(r\psi_{ik})(u, v, \theta, \phi)| &= \int_u^{u_1} |\partial_u \partial_v(r\psi_{ik})(u', v, \theta, \phi)| du' \\
&\leq \int_u^{u_1} \frac{C \mathfrak{D}_M^5}{1+r(u', v, \theta, \phi)^2} du' \\
&\leq \left(\int_u^{u_1} \frac{1}{1+r(u', v, \theta, \phi)^4} du' \right)^{\frac{1}{2}} \cdot \left(\int_u^{u_1} C^2 (\mathfrak{D}_M^5)^2 du' \right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq \frac{C\mathfrak{D}_M^5}{1+r(u_1, v)^2}$$

for some $C > 0$. Now, by virtue of $|u_1 - u| < \infty$, we have that

$$\frac{1+r(u, v)}{1+r(u_1, v)} = \frac{1+v-u}{1+v-u_1} = 1 + \frac{u_1-u}{1+v-u_1} \in [1, \infty),$$

we have that

$$\frac{1}{1+r(u_1, v)^2} \leq \frac{C}{1+r(u, v)^2}$$

for some $C > 0$. Hence,

$$|\partial_v(r\psi_{i_k})(u, v, \theta, \phi)| \leq \frac{C\mathfrak{D}_M^5}{1+r(u, v)^2}. \quad (2.2.46)$$

Finally, integrating (2.2.46) in v gives the following estimate for the second term on the right-hand side of (2.2.40).

$$\begin{aligned} \left| \int_v^{v_{i_k}} \partial_v(r\psi_{i_k})(u, v, \theta, \phi) dv \right| &\leq \int_v^{v_{i_k}} |\partial_v(r\psi_{i_k})(u, v, \theta, \phi)| dv \\ &\leq \int_v^{v_{i_k}} \frac{C\mathfrak{D}_M^5}{1+r(u, v)^2} dv \\ &\leq \frac{C\mathfrak{D}_M^5}{1+r(u, v)^2} \end{aligned}$$

for some $C > 0$. Now, as $k \rightarrow \infty$, (2.2.40) gives:

$$\left| \psi(u, v, \theta, \phi) - \frac{h(u, \theta, \phi)}{r(u, v)} \right| \leq \frac{C\mathfrak{D}_M^5}{1+r(u, v)^2}$$

for some $C > 0$, as required. □

2.3 The Schwarzschild black hole exterior

We begin our discussion of the Schwarzschild black hole exterior in Section 2.3.1 with some basic definitions, before stating some key estimates we will use (Section 2.3.2). In Sections 2.3.3-2.3.5 we will then discuss the backwards problem (2.1.10).

2.3.1 Preliminaries

Let us start by stating some definitions we will require.

Vector fields We define the globally uniformly timelike vector field

$$N_S = \partial_v + f(r) \frac{1}{1 - \frac{2M}{r}} \partial_u, \quad \text{where} \quad f(r) = \begin{cases} 1 & \text{for } r < R - \frac{1}{10}, \\ 1 - \frac{2M}{r} & \text{for } r \geq R. \end{cases} \quad (2.3.1)$$

Here R is as in (2.1.12) and f interpolates smoothly between its two cases. N_S will be used as both a multiplier and commutator, in order to obtain coercive energy estimates. As discussed in Section 2.1.2, the analogous estimates using only $T = \partial_{t^*}$ degenerate at $\mathcal{H}^+$, meaning that they do not control derivatives transverse to the horizon.

The *commuting vector fields* we shall use are

$$\Gamma_S \in \left\{ T = \partial_{t^*}, N_S, \{\Omega_j\}_{j=1}^3 \right\},$$

and by $\tilde{\Gamma}_S$ we will denote the subset of Killing commuting vector fields:

$$\tilde{\Gamma}_S \in \left\{ T, \{\Omega_j\}_{j=1}^3 \right\}.$$

A mixed null-spacelike foliation It is convenient to define the following foliation of the Schwarzschild exterior region.

Let $t^* = t + 2M \log(r - 2M)$ and R be as in (2.1.12). Then, for each fixed t^* we shall denote by Σ_{t^*} the leaf in the foliation satisfying

$$\Sigma_{t^*} = \left(\mathcal{M}_S \cap \left\{ (t^*, r, \theta, \phi) \mid r < R \right\} \right) \cup \left(\mathcal{M}_S \cap \left\{ (t^*, r, \theta, \phi) \mid u = \tau, r \geq R \right\} \right). \quad (2.3.2)$$

An example of such a hypersurface is depicted in Figure 2.10.

The unit normal to the spacelike part of each leaf Σ_{t^*} is given by

$$n_{t^*} = \sqrt{-g^{t^*t^*}} \partial_{t^*} - \frac{g^{t^*r}}{\sqrt{-g^{t^*t^*}}} \partial_r.$$

The energies We define the following Sobolev spaces on the mixed null-spacelike slices of the Schwarzschild black hole exterior.

Definition 2.3.1. (Sobolev norms on mixed null-spacelike slices of Schwarzschild) Let $t^* = \tau$ be a fixed time. Then for each $k \in \mathbb{N}$, we define the Sobolev space

$$H_S^k(\Sigma_\tau) = \{ \psi : [2M, \infty) \times S^2 \rightarrow \mathbb{R} \mid \|\psi\|_{H_S^k(\Sigma_\tau)} < \infty \}$$

of scalar fields ψ on Σ_τ associated with the energy

$$\begin{aligned} \|\psi\|_{H_S^1(\Sigma_\tau)}^2 &= \int_{\Sigma_\tau \cap \{r < R\}} \left[(\partial_{t^*} \psi)^2 + (\partial_r \psi)^2 + |\nabla \psi|^2 + \psi^2 \right] r^2 dr d\omega \\ &\quad + \int_{\Sigma_\tau \cap \{r \geq R\}} \left[(\partial_v \psi)^2 + |\nabla \psi|^2 + \psi^2 \right] r^2 dv d\omega \end{aligned} \quad (2.3.3)$$

and, for $k \geq 2$,

$$\begin{aligned} \|\psi\|_{H_S^k(\Sigma_\tau)}^2 = & \sum_{\substack{|\sigma| \leq k-1, \\ |\rho| \leq k}} \left(\int_{\Sigma_\tau \cap \{r < R\}} \left[|\nabla \Omega^{\sigma_1} \partial_{t^*}^{\sigma_2} \partial_r^{\sigma_3} \psi|^2 + (\Omega^{\rho_1} \partial_{t^*}^{\rho_2} \partial_r^{\rho_3} \psi)^2 \right] r^2 dr d\omega \right. \\ & \left. + \int_{\Sigma_\tau \cap \{r \geq R\}} \left[|\nabla \Omega^{\sigma_1} \partial_v^{\sigma_2} \psi|^2 + (\Omega^{\rho_1} \partial_v^{\rho_2} \psi)^2 \right] r^2 dv d\omega \right). \end{aligned} \quad (2.3.4)$$

By $\|\cdot\|_{\dot{H}_S^k}$, we denote the above norms, minus their zeroth-order terms.

We also define the following standard definitions of L^p spaces on the slices of the Schwarzschild exterior.

Definition 2.3.2 (L^p -norms on mixed null-spacelike slices of Schwarzschild). Let $t^* = \tau \in \mathbb{R}$ be a fixed time. Then for $p \in \{2, \infty\}$ we define the spaces

$$L_S^p(\Sigma_\tau) = \{\psi : [2M, \infty) \times S^2 \rightarrow \mathbb{R} \mid \|\psi\|_{L_S^p(\Sigma_\tau)}^2 < \infty\}$$

of scalar fields ψ on Σ_t associated with the norms

$$\|\psi\|_{L_S^2(\Sigma_\tau)}^2 = \int_{\Sigma_\tau \cap \{r < R\}} |\psi|^2 r^2 dr d\omega + \int_{\Sigma_\tau \cap \{r \geq R\}} |\psi|^2 r^2 dv d\omega, \quad \|\psi\|_{L_S^\infty(\Sigma_\tau)} = \sup_{\Sigma_\tau} |\psi|.$$

2.3.2 Key estimates

Here we collect two key results that we require to prove Theorem 2.1.3 - r^2 -weighted estimates (Proposition 2.3.3) and a Sobolev embedding in the mixed null-spacelike slices (Theorem 2.3.4).

r^2 -weighted energy estimates We will require r^2 -weighted estimates (analogous to Proposition 2.2.7) in our analysis on Schwarzschild. These follow (after commuting with Γ_S) from the same argument appearing in [12], adapted to the backwards problem (2.1.10). The key element is applying $r^p \partial_v(r\psi)$ as a multiplier to the equation (2.1.11), after rewriting it in double-null coordinates. Here Σ_{t^*} is the mixed null-spacelike slice (2.3.2) defined above and, given fixed $t_2^* > t_1^* \geq 0$, $R > 0$, $D_{t_1^*-R}^{t_2^*-R}$ is the partial null spacetime strip (see Figure 2.10)

$$D_{t_1^*-R}^{t_2^*-R} = \mathcal{M}_S \cap \{r \geq R\} \cap \{u \in [t_1^* - R, t_2^* - R]\}.$$

Proposition 2.3.3. (Estimate for r^2 -weighted energies for the backwards problem on the Schwarzschild black hole exterior [12]) Fix $t_2^* > t_1^* \geq 0$ and $R > 0$ sufficiently large.⁸ Suppose

⁸Largeness is required due to the appearance of an additional bulk term of bad sign for small r which one does not see in the analogous estimate on Minkowski space.

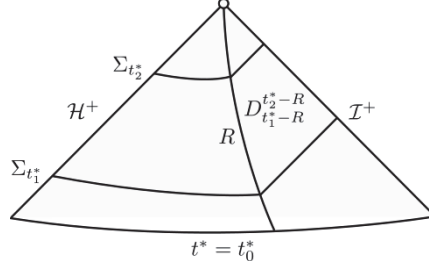


Figure 2.10

ψ is a smooth solution of problem (2.1.10) arising from smooth scattering data. Then ψ satisfies the r^2 -weighted estimate

$$\begin{aligned} & \int_{u=t_1^*-R, v \geq t_1^*+R} r^2 (\tilde{\Gamma}_S^\alpha \partial_v(r\psi))^2 dv d\omega + \int_{D_{t_1^*-R}^{t_2^*-R}} 2r (\tilde{\Gamma}_S^\alpha \partial_v(r\psi))^2 du dv d\omega \\ & \leq \int_{u=t_2^*-R, v \geq t_2^*+R} r^2 (\tilde{\Gamma}_S^\alpha \partial_v(r\psi))^2 dv d\omega + C \left(\int_{\Sigma_{t_2^*} \cap \{r < R\}} \left[(\tilde{\Gamma}_S^\alpha \partial_t \psi)^2 + (\tilde{\Gamma}_S^\alpha \partial_r \psi)^2 + |\nabla \tilde{\Gamma}_S^\alpha \psi|^2 \right] dr d\omega \right. \\ & \quad \left. + \int_{\Sigma_{t_2^*} \cap \{r \geq R\}} \left[(\tilde{\Gamma}_S^\alpha \partial_v \psi)^2 + |\nabla \tilde{\Gamma}_S^\alpha \psi|^2 \right] dv d\omega \right), \end{aligned}$$

for all $|\alpha| \geq 0$, some $C > 0$.

Sobolev embedding on null-spacelike slices We also require the following Sobolev embedding result. This follows from the Sobolev embedding theorem for functions on bounded domains of $\mathbb{R}^n$ (Corollary B.1.3), as well as that for functions on $\mathbb{R}^n$ (and unbounded domains) (Theorem B.1.2).

Theorem 2.3.4. (L^∞ Sobolev embedding on mixed null-spacelike slices Σ_τ in Schwarzschild) Let $\psi : ([2M, R) \times S^2) \cup ([R, \infty) \times [t^*, \infty) \times S^2) \rightarrow \mathbb{R}$ be a smooth function. Then ψ satisfies the Sobolev inequality

$$\|\psi\|_{L^\infty(\Sigma_{t^*})} \leq C \|\psi\|_{H_S^2(\Sigma_{t^*})}.$$

2.3.3 The backwards problem: A scattering construction for linear waves from their radiation fields

Let us define the following scattering data quantities, in terms of which we will state the main result of this section.

Definition 2.3.5. (Data quantities $\mathfrak{D}_S^k$ for the scattering problem on the Schwarzschild black hole exterior) Given sufficiently regular bifurcate scattering data $(h_{\mathcal{H}^+}, h_{\mathcal{I}^+}) : \mathbb{R} \times S^2 \rightarrow \mathbb{R}$,

we define the quantities

$$\begin{aligned}\mathfrak{D}_{S\mathcal{I}^+}^k &= \int_0^\infty \int_{S^2} \sum_{|\alpha| \leq k-1} \exp(2\Upsilon_k \cdot u) \left[h_{\mathcal{I}^+}^2 + \left(\partial_u \Gamma_S^\alpha h_{\mathcal{I}^+} \right)^2 + |\nabla \Gamma_S^\alpha h_{\mathcal{I}^+}|^2 \right] r^2 du d\omega, \\ \mathfrak{D}_{S\mathcal{H}^+}^k &= \int_0^\infty \int_{S^2} \sum_{|\alpha| \leq k-1} \exp(2\Upsilon_k \cdot v) \left[h_{\mathcal{H}^+}^2 + \left(\partial_v \Gamma_S^\alpha h_{\mathcal{H}^+} \right)^2 + |\nabla \Gamma_S^\alpha h_{\mathcal{H}^+}|^2 \right] r^2 dv d\omega,\end{aligned}$$

where Υ_k is a constant depending on k . We shall denote by $\mathfrak{D}_S^k$ the sum of the data quantites

$$\mathfrak{D}_S^k = \mathfrak{D}_{S\mathcal{I}^+}^k + \mathfrak{D}_{S\mathcal{H}^+}^k.$$

Given this definition, we can now formally restate the result we will prove for the backwards problem (2.1.10) on the Schwarzschild black hole exterior.

Theorem 2.3.6. (A scattering construction for exponentially decaying linear waves on the Schwarzschild black hole exterior from prescribed radiation fields) Let $t_0^* > 0$ be fixed and $(h_{\mathcal{H}^+}(v, \theta, \phi), h_{\mathcal{I}^+}(u, \theta, \phi)) : \mathbb{R} \times S^2 \rightarrow \mathbb{R}$ be smooth scattering data satisfying $\mathfrak{D}_S^5 < \infty$ for $2\Upsilon_{\kappa,5} > a_{\kappa,5}$.⁹ Then there exists a smooth solution $\psi : \{t^* \geq t_0^*\} \times \{r \geq r_+\} \times S^2 \rightarrow \mathbb{R}$ of problem (2.1.10) attaining $h_{\mathcal{I}^+}$ as its radiation field, i.e.

$$\left| \psi(u, v, \theta, \phi) - \frac{h_{\mathcal{I}^+}(u, \theta, \phi)}{r} \right| \leq \frac{C\mathfrak{D}_S^5}{r(u, v)^2}. \quad (2.3.5)$$

In order to prove Theorem 2.3.6, we proceed in a similar manner as on Minkowski space by deriving uniform estimates for solutions ψ_i of a sequence of finite (in v) problems (2.1.15) approximating the full problem (2.1.10). This is carried out in the following section. These estimates will subsequently allow an application of the Arzelà-Ascoli Theorem, as before, to give the required result.

2.3.4 The iteration scheme

For the finite problems (2.1.15), we will take truncated scattering data $(h_{\mathcal{H}^+i}, h_{\mathcal{I}^+i})$ satisfying

$$\begin{aligned}h_{\mathcal{H}^+i}(u = \infty, v, \theta, \phi) &= \begin{cases} h_{\mathcal{H}^+} & \text{for } v < v_i - \vartheta_i, \quad \vartheta_i = \frac{1}{10}v_i \\ 0 & \text{for } v \geq v_i, \end{cases} \\ h_{\mathcal{I}^+i}(u, v = v_i, \theta, \phi) &= \varpi_i(u) h_{\mathcal{I}^+}(u, \theta, \phi),\end{aligned}$$

where

$$\varpi_i(u) = \begin{cases} 0 & \text{for } r(u, v_i) < 2.5M \\ 1 & \text{for } r(u, v_i) \geq 2.5M + 0.1. \end{cases}$$

Here, $h_{\mathcal{H}^+i}$ and ϖ_i are assumed to interpolate smoothly between their respective cases.

We prove the following estimates for the solutions ψ_i of the finite problems (2.1.15), which will be applied to prove Theorem 2.3.6.

⁹This is a constant $a_{\kappa,k=5}$, depending on the surface gravity κ , which grows with k . One can see where this constant arises in the proof of Proposition 2.3.8.

Theorem 2.3.7. (Uniformly bounded solutions of the finite problems on the Schwarzschild black hole exterior) Let $t_0^* > 0$, i be fixed and $(h_{\mathcal{H}^+i}, h_{\mathcal{I}^+i})$ be truncated scattering data satisfying $\mathfrak{D}_S^5 < \infty$ for $2\Upsilon_{\kappa,5} > a_{\kappa,5}$. The corresponding smooth solution ψ_i of problem (2.1.15) satisfies

$$\|\psi_i\|_{H_S^4(\Sigma_{t^*})}^2 \leq C \exp(-2\Upsilon_{\kappa,5} \cdot t^*) \mathfrak{D}_S^5, \quad (2.3.6)$$

$$\|D^\sigma \psi_i\|_{L_S^\infty(\Sigma_{t^*})}^2 \leq C \exp(-2\Upsilon_{\kappa,5} \cdot t^*) \mathfrak{D}_S^5, \quad (2.3.7)$$

at all $t^* \geq t_0^*$, for some $C > 0$ and all $|\sigma| \leq 2$.

This will follow from the following Proposition, via an application of Theorem 2.3.4.

Proposition 2.3.8. Let i, ψ_i be as in Theorem 2.3.7. Then

$$\|\psi_i\|_{H_S^4(\Sigma_{t^*})}^2 \leq C \exp(-2\Upsilon_{\kappa,5} \cdot t^*) \mathfrak{D}_S^5$$

at all $t^* \geq t_0^*$, for some $C > 0$.

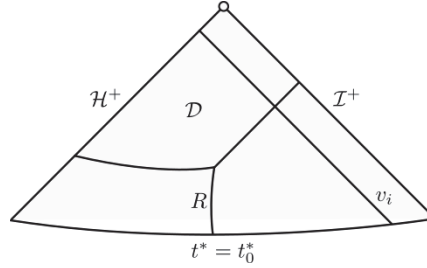


Figure 2.11

Proof of Proposition 2.3.8. Integrating the energy identity (A.0.1) for the vector field N_S over the spacetime region $\mathcal{D}$ depicted in Figure 2.11 gives (via an application of the Divergence Theorem)

$$\begin{aligned} & \int_{\Sigma_{t^*} \cap \{r < R\}} J_\alpha^N[\psi_i] \cdot n_{t^*}^\alpha \cdot r^2 dr d\omega + \int_{\Sigma_{t^*} \cap \{r \geq R\}} J_v^N[\psi_i] \cdot r^2 dv d\omega \\ &= \int_{\{v=v_i\}} J_u^N[\psi_i] \cdot r^2 du d\omega + \int_{\mathcal{H}^+} J_v^N[\psi_i] \cdot r^2 dv d\omega + \int_{\mathcal{D} \cap \{r < R\}} K^N[\psi_i] \cdot r^2 du dv d\omega \\ &\leq C \left[\exp(-2\Upsilon_{\kappa,1} \cdot t^*) \mathfrak{D}_S^1 + \int_{t^*}^T \int_{\Sigma_{t^*} \cap \{r < R\}} a_{\kappa,1} J_\alpha^N[\psi_i] \cdot n_{t^*}^{\alpha 2} dr d\omega dt^* \right], \end{aligned} \quad (2.3.8)$$

where (2.3.8) follows from the assumption $\mathfrak{D}_S^5 < \infty$ on the scattering data. By direct calculation, one finds that $a_{\kappa,1} = 4M$ is the reciprocal of the surface gravity κ . The last term on the right-hand side of (2.3.8) is of positive sign, hence, it cannot simply be dropped if

we wish to control the left-hand side by scattering data. Instead, we will use a Grönwall inequality (Theorem B.3.1). Accordingly, we define the functions

$$\begin{aligned} E_{\psi_i}(t^*) &= \int_{\Sigma_{t^*} \cap \{r < R\}} J_\alpha^N[\psi_i] \cdot n_{t^*}^\alpha \cdot r^2 dr d\omega + \int_{\Sigma_{t^*} \cap \{r \geq R\}} J_v^N[\psi_i] \cdot r^2 dv d\omega \\ b_{\psi_i}(t^{*'}) &= C \exp(-2\Upsilon_{\kappa,1} \cdot t^{*'}) \mathfrak{D}_S^1 \end{aligned}$$

and rewrite the energy identity (2.3.8) as an inequality in $E_{\psi_i}(t^*)$ and $b_{\psi_i}(t^{*'})$:

$$E_{\psi_i}(t^*) \leq b_{\psi_i}(t^{*'}) + \int_{t^*}^T a_{\kappa,1} \cdot E_{\psi_i}(t^{*'}) dt^{*'}.$$

Applying Theorem B.3.1 then yields

$$\begin{aligned} E_{\psi_i}(t^*) &\leq b_{\psi_i}(t^{*'}) + \int_{t^*}^T a_{\kappa,1} b_{\psi_i}(t^{*'}) \exp\left(\int_{t^*}^{t^{*'}} a_{\kappa,1} ds\right) dt^{*'} \\ &= b_{\psi_i}(t^{*'}) + \int_{t^*}^T a_{\kappa,1} b_{\psi_i}(t^{*'}) \exp\left((t^{*'} - t^*) a_{\kappa,1}\right) dt^{*'} \\ &= b_{\psi_i}(t^{*'}) + C \exp(-a_{\kappa,1} \cdot t^*) \int_{t^*}^T a_{\kappa,1} \mathfrak{D}_S^1 \exp\left((a_{\kappa,1} - 2\Upsilon_{\kappa,1}) t^{*'}\right) dt^{*'} \\ &= b_{\psi_i}(t^{*'}) - C \exp(-a_{\kappa,1} \cdot t^*) \int_{t^*}^T \frac{a_{\kappa,1} \mathfrak{D}_S^1}{2\Upsilon_{\kappa,1} - a_{\kappa,1}} \partial_{t^{*'}} \left(\exp\left((a_{\kappa,1} - 2\Upsilon_{\kappa,1}) t^{*'}\right) \right) dt^{*'} \\ &\leq b_{\psi_i}(t^{*'}) + C \exp(-a_{\kappa,1} \cdot t^*) \frac{a_{\kappa,1} \mathfrak{D}_S^1}{2\Upsilon_{\kappa,1} - a_{\kappa,1}} \exp\left((a_{\kappa,1} - 2\Upsilon_{\kappa,1}) t^*\right) \\ &\leq C \exp(-2\Upsilon_{\kappa,1} \cdot t^*) \mathfrak{D}_S^1 \left(1 + \frac{a_{\kappa,1}}{2\Upsilon_{\kappa,1} - a_{\kappa,1}}\right) \\ &\leq C \exp(-2\Upsilon_{\kappa,1} \cdot t^*) \mathfrak{D}_S^1, \end{aligned}$$

for some $C > 0$, provided $2\Upsilon_{\kappa,1} > a_{\kappa,1}$. Hence,

$$\|\psi_i\|_{\dot{H}^1(\Sigma_{t^*} \cap \{v \leq v_i\})}^2 \leq C E_{\psi_i}(t^*) \leq C \exp(-2\Upsilon_{\kappa,1} \cdot t^*) \mathfrak{D}_S^1,$$

for some $C > 0$. In order to derive bounds for higher-order derivatives, we apply concatenations of the commuting vector fields Γ_S to sufficiently high order. Commutations with T and the Ω_j generate no additional terms, so we need only worry about bulk terms generated by commutations with N_S . The bulk after one commutation with N_S is given by

$$K^N(N\psi_i) = T_{\alpha\beta}(N\psi_i)^{(N)}\pi^{\alpha\beta} + \square(N\psi_i) \cdot N\psi_i. \quad (2.3.9)$$

After expanding the second term on the right-hand side of (2.3.9) as

$$\square(N\psi_i) \cdot N\psi_i = \left[2^{(N)}\pi^{\alpha\beta} \partial_\alpha \partial_\beta \psi_i + \left(2(\nabla^\alpha{}^{(N)}\pi_{\alpha\mu}) - (\nabla_\mu^{(N)}\pi_\alpha^\alpha) \right) \nabla^\mu \psi_i \right] \cdot N\psi_i, \quad (2.3.10)$$

one sees that the bulk (2.3.9) involves only derivatives of ψ_i up to second order. Moreover, by choice of N_S (2.3.1), the bulk contribution vanishes outside of a compact ($r < R$) region of r , meaning that the corresponding term in the commuted energy estimate,

$$\begin{aligned} & \int_{\Sigma_{t^*} \cap \{r < R\}} J_\alpha^N[N\psi_i] \cdot n_{t^*}^\alpha \cdot r^2 dr d\omega + \int_{\Sigma_{t^*} \cap \{r \geq R\}} J_v^N[N\psi_i] \cdot r^2 dv d\omega \\ &= \int_{\{v=v_i\}} J_u^N[N\psi_i] \cdot r^2 du d\omega + \int_{\mathcal{H}^+} J_v^N[N\psi_i] \cdot r^2 dv d\omega \\ &+ \int_{t^*}^T \int_{\Sigma_{t^*} \cap \{r < R\}} \left(T_{\alpha\beta}(N\psi_i)^{(N)} \pi^{\alpha\beta} + \square(N\psi_i) \cdot N\psi_i \right) r^2 dr d\omega dt^*, \end{aligned}$$

is easily rewritten

$$E_{N\psi_i}(t^*) \leq b_{N\psi_i}(t^{*'}) + \int_{t^*}^T a_{\kappa,2} \cdot E_{N\psi_i}(t^{*'}) dt^{*'}$$

as before, with $a_{\kappa,2} > a_{\kappa,1}$. Applying Theorem B.3.1 as before gives

$$\begin{aligned} E_{N\psi_i}(t^*) &\leq C \exp(-2\Upsilon_{\kappa,2} \cdot t^*) \mathfrak{D}_S^2 \left(1 + \frac{a_{\kappa,2}}{2\Upsilon_{\kappa,2} - a_{\kappa,2}} \right) \\ &\leq C \exp(-2\Upsilon_{\kappa,2} \cdot t^*) \mathfrak{D}_S^2, \end{aligned}$$

for some $C > 0$, provided $2\Upsilon_{\kappa,2} > a_{\kappa,2}$. Thus,

$$\begin{aligned} \|\psi_i\|_{\dot{H}^2(\Sigma_{t^*} \cap \{v \leq v_i\})}^2 &\leq C \left(E_{\psi_i}(t^*) + E_{T\psi_i}(t^*) + E_{N\psi_i}(t^*) + \sum_{1 \leq j, k \leq 3} \left[E_{\Omega_j \psi_i}(t^*) + E_{\Omega_k \psi_i}(t^*) \right] \right) \\ &\leq C \exp(-2\Upsilon_{\kappa,3} \cdot t^*) \mathfrak{D}_S^3, \end{aligned}$$

for some $C > 0$. Repeating sufficiently many times, one obtains

$$\|\psi_i\|_{\dot{H}_S^4(\Sigma_{t^*} \cap \{v \leq v_i\})}^2 \leq C \exp(-2\Upsilon_{\kappa,5} \cdot t^*) \mathfrak{D}_S^5,$$

provided $2\Upsilon_{\kappa,5} > a_{\kappa,5}$, where $a_{\kappa,5}$ is the relevant (strictly larger) analogue of $a_{\kappa,1}$, $a_{\kappa,2}$.

Finally, an application of Hardy's inequality (Theorem B.2.1) on each segment $\Sigma_\tau \cap \{r < R\} \cap \{v \leq v_i\}$, $\Sigma_\tau \cap \{r \geq R\} \cap \{v \leq v_i\}$ of the slice Σ_τ gives

$$\|\psi_i\|_{L_S^2(\Sigma_{t^*} \cap \{v \leq v_i\})}^2 \leq C \exp(-2\Upsilon_{\kappa,1} \cdot t^*) \mathfrak{D}_S^1 \leq C \exp(-2\Upsilon_{\kappa,5} \cdot t^*) \mathfrak{D}_S^5,$$

for some $C > 0$. This completes the proof. $\square$

Proof of Theorem 2.3.7. Estimate (2.3.6) follows immediately from Proposition 2.3.8, whilst an application of Theorem 2.3.4 gives (2.3.7). $\square$

We can now complete the proof of the main result (Theorem 2.3.6).

2.3.5 Convergence of the iteration scheme and attainment of the scattering data

Proof. By an application of the Arzelà-Ascoli Theorem (familiar from the proof of the analogous result, Theorem 2.2.12, on Minkowski space), Theorem 2.3.7 gives a solution of the full problem 2.1.10. The uniform estimate (2.3.5) then follows from the r^p estimates (Proposition 2.3.3) (with R chosen to be that used to define the foliation, as in (2.1.12), (2.3.2)) and Theorem 2.3.7 in the same manner as the proof of Theorem 2.2.12. $\square$

Chapter 3

Local existence for nonlinear wave equations on Anti-de Sitter space

3.1 Introduction

The key goal of this chapter is to introduce important elements required in the following chapter (which contains a scattering construction for nonlinear waves on the Kerr-AdS black hole exterior). As mentioned previously, there is no such meaningful scattering construction on pure AdS. For this reason, our discussion of AdS itself is limited to proving a local existence result for nonlinear wave equations. This serves to illustrate the procedure one uses to prove the analogous (backwards) result for nonlinear wave equations on Kerr-AdS, which is utilised (without explicit proof) in the following chapter. The proof of local existence on AdS also serves as a first demonstration of a *bootstrap argument* which plays a key role in the subsequent discussion of Kerr-AdS.

3.1.1 Anti-de Sitter space

Since AdS itself has no horizon and possesses a timelike future null infinity, it is most convenient to work with standard (t, r, θ, ϕ) coordinates and spacelike hypersurfaces. As such, the AdS metric we will work with is as stated earlier:

$$g_{AdS} = -\left(1 + \frac{r^2}{\ell^2}\right)dt^2 + \frac{1}{1 + \frac{r^2}{\ell^2}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1.3.3)$$

We shall refer to the spacetime manifold itself as $\mathcal{M}_{AdS}$. In these coordinates, we will consider the equation

$$\square_{g_{AdS}}\psi + \alpha\psi = \mathcal{F}(\psi, D\psi), \quad (3.1.1)$$

where

$$\square_{g_{AdS}}\psi = -\frac{1}{1 + \frac{r^2}{\ell^2}}\partial_t^2\psi + \left(1 + \frac{r^2}{\ell^2}\right)\partial_r^2\psi + \Delta\psi + \frac{2}{r}\left(1 + \frac{r^2}{\ell^2}\right)\partial_r\psi + \frac{\cos\theta}{r^2\sin\theta}\partial_\theta\psi. \quad (3.1.2)$$

In lengthy expressions, the subscript of the $\square$ operator will be dropped. We will prove a local existence result for the corresponding initial-value Dirichlet problem:¹

$$\begin{cases} \square_{g_{AdS}} \psi + \alpha \psi = \mathcal{F}(\psi, \partial \psi) \\ (\psi, \partial_t \psi)|_{\{t=0\}} = (\psi_0, \psi_1), \quad r^{\frac{3}{2}-s} \psi|_{\mathcal{I}} = 0, \quad s = \sqrt{\frac{9}{4} - \alpha}, \end{cases} \quad (3.1.3)$$

with sufficiently regular initial data (ψ_0, ψ_1) . Throughout, we will restrict our attention to a nonlinearity $\mathcal{F}(\psi, \partial \psi)$ which is quadratic in the (appropriately weighted) first-order derivatives of ψ (see Section 3.2.3). We postpone an in-depth discussion of this until the scattering construction on the Kerr-AdS exterior is presented. However, one can have in mind that this choice of $\mathcal{F}$ is designed to mimic the nonlinearities which appear when the Einstein vacuum equation (1.1.1) is written in harmonic gauge.

3.1.2 Local existence of nonlinear waves on Anti-de Sitter space

The (informally stated, for now) result we will prove is the following.

Theorem 3.1.1 (Local existence for nonlinear wave equations on Anti-de Sitter space). Let (ψ_0, ψ_1) be sufficiently regular initial data at time $t = 0$. Then there exists some time $T > 0$ up until which problem (3.1.3) admits a regular solution.

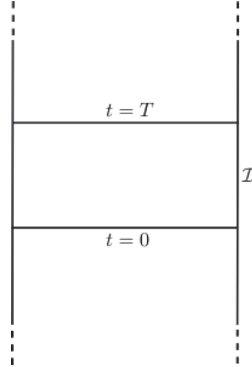


Figure 3.1

In order to prove Theorem 3.1.1, the key step is to utilise an iteration scheme, of a different spirit than those appearing in the earlier discussion of scattering constructions. In this case, the interval of time is the same at each step in the iteration (we are proving a purely local result), whilst the right-hand side of the equation changes. At each step in the iteration, we work with a purely *linear* inhomogeneous problem:

$$\begin{cases} \square_{g_{AdS}} \psi_i + \alpha \psi_i = \mathcal{F}_{i-1} \stackrel{def}{=} \mathcal{F}(\psi_{i-1}, \partial \psi_{i-1}) \\ (\psi_i, \partial_t \psi_i)|_{\Sigma_0} = (\psi_0, \psi_1), \quad r^{\frac{3}{2}-s} \psi_i|_{\mathcal{I}} = 0, \quad s = \sqrt{\frac{9}{4} - \alpha}, \end{cases} \quad (3.1.4)$$

¹By $r^{\frac{3}{2}-s} \psi|_{\mathcal{I}}$, we denote the limit $\lim_{r \rightarrow \infty} r^{\frac{3}{2}-s} \psi(t, r, \theta, \phi)$ for fixed (t, θ, ϕ) .

for $i \geq 2$. Via induction on i and a bootstrap argument, we obtain uniform boundedness of each ψ_i . This allows a familiar application of the Arzelà-Ascoli Theorem to derive a solution ψ of the full nonlinear problem (3.1.3).

3.1.3 Outline

In Section 3.2 we provide some basic definitions, as well as an explicit definition of the nonlinearity $\mathcal{F}$ we will allow on the right-hand side of equation (3.1.1). This is followed in Section 3.3 by a collection of estimates which will be called upon in proving Theorem 3.1.1. Finally, in Section 3.4, we will formally restate and prove the main result (Theorem 3.4.1).

3.2 Preliminaries

3.2.1 Geometric objects

We begin by defining some key geometric objects in AdS space.

In the standard (t, r, θ, ϕ) coordinates (corresponding to the metric (1.3.3)), the spacelike hypersurfaces Σ_t of constant time have unit normal and induced volume form

$$n_{\Sigma_t} = \frac{1}{\sqrt{1 + \frac{r^2}{\ell^2}}} \partial_t, \quad \text{vol}_{\Sigma_t} = \frac{1}{\sqrt{1 + \frac{r^2}{\ell^2}}} r^2 dr d\omega.$$

We also define the renormalised coordinate derivative operators (which will be utilised in the definition of the nonlinearity, among other things):

$$D \in \left\{ \frac{1}{1+r} \partial_t, r \partial_r, \frac{1}{1+r} \partial_\theta \psi, \frac{1}{1+r} \partial_\phi \psi \right\}, \quad \overline{D} \in \left\{ \partial_t, r \partial_r, \frac{1}{1+r} \partial_\theta \psi, \frac{1}{1+r} \partial_\phi \psi \right\}. \quad (3.2.1)$$

In order to derive energy estimates to sufficiently high order, we apply the (Killing) *commuting vector fields*

$$\Gamma_{AdS} \in \left\{ T, \{\Omega_j\}_{j=1}^3 \right\} \quad (3.2.2)$$

to the equation. The subscript of Γ will be dropped where necessary for compactness of expressions.

3.2.2 The energies

We define the following r -weighted energies on AdS space, which naturally arise when using T as a multiplier (see Section 3.3.1).

Definition 3.2.1 (Sobolev norms on spacelike slices in AdS space). Let t be a fixed time. Then, for each $k \in \mathbb{N}$ we define the Sobolev space

$$H_{AdS}^k(\Sigma_t) = \{ \psi : [0, \infty) \times S_{t,r}^2 \rightarrow \mathbb{R} \mid \|\psi\|_{H_{AdS}^k(\Sigma_t)}^2 < \infty \}$$

of scalar fields ψ on Σ_t associated with the energy

$$\|\psi\|_{H_{AdS}^1(\Sigma_t)}^2 = \int_{\Sigma_t} \left[\frac{1}{1+r^2} (\partial_t \psi)^2 + r^2 (\partial_r \psi)^2 + |\nabla \psi|^2 + \psi^2 \right] r^2 dr d\omega \quad (3.2.3)$$

and, for $k \geq 2$,

$$\begin{aligned} \|\psi\|_{H_{AdS}^k(\Sigma_t)}^2 = & \int_{\Sigma_t} \left[\sum_{\substack{|\sigma| \leq k-1, \\ |\rho| \leq k, \rho_2 \leq k-1}} \left(r^{2\sigma_3} |\nabla \Omega^{\sigma_1} \partial_t^{\sigma_2} \partial_r^{\sigma_3} \psi|^2 + r^{2\rho_3} (\Omega^{\rho_1} \partial_t^{\rho_2} \partial_r^{\rho_3} \psi)^2 \right) \right. \\ & \left. + \frac{1}{1+r^2} (\partial_t^k \psi)^2 \right] r^2 dr d\omega. \end{aligned} \quad (3.2.4)$$

On the spacelike slices of constant time, we also define the following L^p spaces.

Definition 3.2.2 (L^p -norms on spacelike slices of AdS space). Let $t \in \mathbb{R}$ be a fixed time. Then for $p \in \{2, \infty\}$ we define the spaces

$$L_{AdS}^p(\Sigma_t) = \{\psi : [0, \infty) \times S_{t,r}^2 \rightarrow \mathbb{R} \mid \|\psi\|_{L_{AdS}^p(\Sigma_t)} < \infty\}$$

of scalar fields ψ on Σ_t associated with the norms

$$\|\psi\|_{L_{AdS}^2(\Sigma_t)}^2 = \int_{\Sigma_t} |\psi|^2 r^2 dr d\omega, \quad \|\psi\|_{L_{AdS}^\infty(\Sigma_t)} = \sup_{\Sigma_t} |\psi|.$$

Finally, we define the function spaces in which the solutions we seek lie.

Definition 3.2.3 (Function spaces of the solution on AdS space). By CH_{AdS}^k , we denote the function space

$$CH_{AdS}^k = \left\{ \psi : \mathcal{M}_{AdS} \rightarrow \mathbb{R} \mid \psi \in \bigcap_{q=0}^{k-1} C^q(\mathbb{R}_t; H_{AdS}^{k-q}(\Sigma_t)) \text{ and } \frac{1}{1+r} \psi \in C^k(\mathbb{R}_t; L_{AdS}^2(\Sigma_t)) \right\}$$

of scalar fields ψ on $\mathcal{M}_{AdS}$.

3.2.3 A class of nonlinear wave equations

We postpone an in-depth consideration of the nature of and motivation for the nonlinearity we choose to study here until the discussion on Kerr-AdS (see Section 4.2.3). However, we state the condition which $\mathcal{F}$ must satisfy in order to close the estimates, before explicitly defining the $\mathcal{F}$ to which we restrict.

Condition 3.2.4 (The required estimate for $\mathcal{F}$ on AdS space). The right-hand side $\mathcal{F}$ of equation (3.1.1) must satisfy

$$\begin{aligned} \sum_{|\sigma| \leq k-1} \|T^{\sigma_1} \Omega^{\sigma_2} \mathcal{F}\|_{L_{AdS}^2(\Sigma_t)} & \leq C_k \|\psi\|_{H_{AdS}^k(\Sigma_t)} \left(\|\psi\|_{H_{AdS}^{k-1}(\Sigma_t)} \right)^n, \\ \sum_{|\gamma| \leq \lceil \frac{k-2}{2} \rceil} \|\square_{g_{AdS}}^\gamma \mathcal{F}\|_{L_{AdS}^2(\Sigma_t)}^2 & \leq \tilde{C}_k \|\psi\|_{H_{AdS}^k(\Sigma_t)} \left(\|\psi\|_{H_{AdS}^{k-1}(\Sigma_t)} \right)^p \end{aligned} \quad (3.2.5)$$

for some $k \geq 9$, $C_k, \tilde{C}_k > 0$ and $n, p \geq 0$.

This condition is satisfied by a very general class of $\mathcal{F}$ but, for simplicity, we shall focus on the following specific choice of (quadratic in $D\psi$ (3.2.1)) nonlinearity. Considering the frame

$$e_0 = \frac{1}{1+r}\partial_t, \quad e_1 = r\partial_r, \quad e_2 = \frac{1}{1+r}\partial_\theta, \quad e_3 = \frac{1}{1+r}\partial_\phi, \quad (3.2.6)$$

define

$$\mathcal{F} = F^{\mu\nu}(t, r, \theta, \phi) \cdot e_\mu \psi e_\nu \psi. \quad (3.2.7)$$

Here, $F : \mathcal{M}_{AdS} \rightarrow T\mathcal{M}_{AdS} \otimes T\mathcal{M}_{AdS}$ is a $(0, 2)$ -tensor with components $F_{\mu\nu} : \mathcal{M}_{AdS} \rightarrow \mathbb{R}$ which are smooth, bounded (with respect to the frame e_i) functions² with uniformly bounded T , Ω and e_1 derivatives up to order $k-1$. As we shall show in Proposition 3.3.9 below, this choice of $\mathcal{F}$ (3.2.7) satisfies Condition 3.2.4 with $n = 1$, $p = 3$.

3.3 Key estimates

In this section, we collect several estimates for solutions of problem (3.1.3), which will be required in the proof of Theorem 3.1.1.

3.3.1 Coercivity of the energy

After direct calculation, one finds that the T -energy

$$\int_{\Sigma_t} J_t^T[\psi] \cdot n_{\Sigma_t}^t \cdot \text{vol}_{\Sigma_t} = \frac{1}{2} \int_{\Sigma_t} \left[\frac{1}{1 + \frac{r^2}{\ell^2}} (\partial_t \psi)^2 + \left(1 + \frac{r^2}{\ell^2} \right) (\partial_r \psi)^2 + |\nabla \psi|^2 - \alpha \psi^2 \right] r^2 dr d\omega, \quad (3.3.1)$$

involves a zeroth-order term of bad sign for $\alpha > 0$. As such, some additional work is required to obtain an appropriate coercive energy with which we can work in the manner familiar from the previous chapter. It will become apparent in the proof of the following coercivity result that this is precisely the reason for requiring the Breitenlohner-Freedman bound $\alpha < \frac{9}{4}$ on the mass.

Proposition 3.3.1 (A coercive estimate for the T -energy on AdS). Let t be fixed and ψ be a solution of (3.1.3) in $H_{AdS}^k(\Sigma_t)$. Then the T -energy (3.3.1) of ψ satisfies the coercive estimate

$$\int_{\Sigma_t} J_t^T[\psi] \cdot n_{\Sigma_t}^t \cdot \text{vol}_{\Sigma_t} \geq \|\psi\|_{H_{AdS}^1(\Sigma_t)}^2$$

²This is modulo the standard degeneration of the frame $\frac{1}{1+r}\partial_\theta, \frac{1}{1+r}\partial_\phi$ on S^2 .

Note that the reverse implication

$$\|\psi\|_{H_{AdS}^1(\Sigma_t)}^2 \geq C \int_{\Sigma_t} J_t^T[\psi] \cdot n_{\Sigma_t}^t \cdot \text{vol}_{\Sigma_t}$$

is trivially true for some $C > 0$.

Proof. Via integration by parts in r , the Dirichlet boundary conditions and the Cauchy-Schwarz inequality, we have that

$$\begin{aligned} \int_{\Sigma_t} \psi^2 r^2 dr d\omega &\leq \int_{\Sigma_t} \frac{1}{3} \psi^2 \partial_r(r^3) dr d\omega \\ &= - \int_{\Sigma_t} \frac{1}{3} \partial_r(\psi^2) r^3 dr d\omega \\ &= - \int_{\Sigma_t} \frac{2}{3} \psi \partial_r \psi \cdot r^3 dr d\omega \\ &\leq \left(\int_{\Sigma_t} \frac{4}{9} \psi^2 r^2 dr d\omega \right)^{\frac{1}{2}} \left(\int_{\Sigma_t} (\partial_r \psi)^2 r^4 dr d\omega \right)^{\frac{1}{2}}. \end{aligned}$$

This gives

$$\int_{\Sigma_t} \frac{9}{4} \psi^2 r^2 dr d\omega \leq \int_{\Sigma_t} (\partial_r \psi)^2 r^4 dr d\omega < \int_{\Sigma_t} \left(1 + \frac{r^2}{\ell^2} \right) (\partial_r \psi)^2 r^2 dr d\omega,$$

ensuring that the zeroth-order term in the T -energy (3.3.1) can be absorbed by the $\partial_r \psi$ term, without degeneracy, for all $\alpha < \frac{9}{4}$. $\square$

3.3.2 Estimates for higher order radial derivatives

We will require the following estimates for radial and mixed radial derivatives of general order.

Proposition 3.3.2 (An estimate for the second-order radial derivative of ψ on AdS space). Let ψ be a function in $H_{AdS}^2(\Sigma_t)$ satisfying Dirichlet boundary conditions $r^{\frac{3}{2}-s}\psi|_{\mathcal{I}} = 0$. Then we have that

$$\int_{\Sigma_t} (\partial_r^2 \psi)^2 r^6 dr d\omega \leq C \left(\|\psi\|_{H_{AdS}^1(\Sigma_t)}^2 + \|T\psi\|_{H_{AdS}^1(\Sigma_t)}^2 + \int_{\Sigma_t} (\square_g \psi + \alpha \psi)^2 r^2 dr d\omega \right),$$

for some $C > 0$.

Proposition 3.3.2 follows from a standard elliptic estimates argument (see [33]), as a result of the fact that T is globally timelike on AdS space. A similar estimate (Proposition 4.3.6) will be utilised on the “far” region of Kerr-AdS where T is timelike, however, there one must employ a more complex argument to cover the entire exterior region.

By commuting with Γ_{AdS} , one obtains the following higher-order estimate.

Proposition 3.3.3 (An estimate for higher-order derivatives of at most second order in the radial direction on AdS space). Let ψ be a function in $H_{KAdS}^k(\Sigma_t)$ for $k \geq 2$ satisfying Dirichlet boundary conditions $r^{\frac{3}{2}-s}\psi|_{\mathcal{I}} = 0$. Then,

$$\int_{\Sigma_t} \sum_{|\sigma|=k-2} (\Gamma^\sigma \partial_r^2 \psi)^2 r^6 dr d\omega \leq C \sum_{\substack{|\alpha| \leq k-1, \\ |\beta| \leq k-2}} \left[\|\Gamma^\alpha \psi\|_{H_{AdS}^1(\Sigma_t)}^2 + \int_{\Sigma_t} \left(\Gamma^\beta (\square_g \psi + \alpha \psi) \right)^2 r^2 dr d\omega \right],$$

for some $C > 0$.

Finally, we have the following result for general derivatives.

Proposition 3.3.4 (A k^{th} -order estimate for general derivatives on AdS space). Let ψ be a function in $H_{AdS}^k(\Sigma_t)$ for $k \geq 2$ satisfying Dirichlet boundary conditions $r^{\frac{3}{2}-s}\psi|_{\mathcal{I}} = 0$. Then,

$$\begin{aligned} & \sum_{\substack{|\sigma|=k, \\ \sigma_2 \geq 2}} \int_{\Sigma_t} r^{2\sigma_2} (\Gamma^{\sigma_1} \partial_r^{\sigma_2} \psi)^2 r^2 dr d\omega \\ & \leq C \sum_{\substack{|\alpha| \leq k-1, \\ |\beta| \leq k-2, \\ |\gamma| \leq \lceil \frac{k-2}{2} \rceil}} \left[\|\Gamma^\alpha \psi\|_{H_{AdS}^1(\Sigma_t)}^2 + \int_{\Sigma_t} \left(\left(\Gamma^\beta (\square_g \psi + \alpha \psi) \right)^2 + \left(\square_g^\gamma (\square_g \psi + \alpha \psi) \right)^2 \right) r^2 dr d\omega \right], \end{aligned}$$

for some $C > 0$.

Proposition 3.3.3 follows from applying $\square$ to the equation (3.1.1) sufficiently many times, multiplying by $\Delta^p \psi$ for p sufficiently large, integrating by parts and using Proposition 3.3.2 and Proposition 3.3.3 to control terms where necessary. One can see Section 4.3.2 for similar arguments.

3.3.3 Sobolev embedding in spacelike slices

We will also require the following Sobolev embedding results.

Theorem 3.3.5 (L^∞ Sobolev embedding on spacelike slices Σ_t of AdS space). Fix t and let ψ be a function in $H_{AdS}^k(\Sigma_t)$ for $k \geq 3$ satisfying Dirichlet boundary conditions $r^{\frac{3}{2}-s}\psi|_{\mathcal{I}} = 0$. Then one can bound ψ and its D derivatives (3.2.1) up to order $|\sigma| \leq k-3$ uniformly as

$$\begin{aligned} & \|D^\sigma \psi\|_{L^\infty(\Sigma_t)} \\ & \leq \frac{C}{1+r^{\frac{3}{2}}} \left[\sum_{\substack{|\alpha| \leq |\sigma|+2, \\ |\beta| \leq |\sigma|+1, \\ |\gamma| \leq \lceil \frac{|\sigma|+1}{2} \rceil}} \left[\|\Gamma^\alpha \psi\|_{H_{AdS}^1(\Sigma_t)}^2 + \int_{\Sigma_t} \left(\left(\Gamma^\beta (\square_g \psi + \alpha \psi) \right)^2 \right. \right. \right. \end{aligned}$$

$$+ \left(\square_g^\gamma (\square_g \psi + \alpha \psi) \right)^2 \Big)^{\frac{1}{2}} r^2 dr d\omega \Big] \Big]^{\frac{1}{2}} \quad (3.3.2)$$

for some $C > 0$.³ Up to order $|\sigma| = k - 4$, one has estimate (3.3.2) for the $\overline{D}$ derivatives (3.2.1).

Proof. Let $|\sigma| \leq k - 3$. For any fixed r , as a result of the Dirichlet boundary conditions, we have that

$$\begin{aligned} |D^\sigma \psi(t, r, \theta, \phi)| &\leq \int_r^\infty |\partial_r D^\sigma \psi| dr \\ &\leq \left(\int_r^\infty \frac{1}{1+r^4} dr \right)^{\frac{1}{2}} \left(\int_r^\infty |\partial_r D^\sigma \psi|^2 (1+r^4) dr \right)^{\frac{1}{2}} \\ &\leq \frac{C}{1+r^{\frac{3}{2}}} \left(\int_r^\infty |\partial_r D^\sigma \psi|^2 r^4 dr \right)^{\frac{1}{2}}, \end{aligned} \quad (3.3.3)$$

for some $C > 0$. Moreover, by Theorem B.1.1,

$$\begin{aligned} &\int_r^\infty |\partial_r D^\sigma \psi|^2 r^4 dr \\ &\leq \tilde{C} \sum_{j=0}^2 \int_{\Sigma_t} |\nabla^j \partial_r D^\sigma \psi|^2 r^6 dr d\omega \\ &\leq C \sum_{|\rho| \leq 2} \int_{\Sigma_t} \frac{1}{1+r^2} |\Omega^\rho \partial_r D^\sigma \psi|^2 r^6 dr d\omega \\ &\leq C \sum_{|\rho| \leq 2} \int_{\Sigma_t} |\Omega^\rho \partial_r D^\sigma \psi|^2 r^4 dr d\omega \\ &\leq C \sum_{\substack{|\alpha| \leq |\sigma|+2, \\ |\beta| \leq |\sigma|+1, \\ |\gamma| \leq \lceil \frac{|\sigma|+1}{2} \rceil}} \left[\|\Gamma^\alpha \psi\|_{H_{AdS}^1(\Sigma_{t^*})}^2 + \int_{\Sigma_{t^*}} \left(\left(\Gamma^\beta (\square_g \psi + \alpha \psi) \right)^2 + \left(\square_g^\gamma (\square_g \psi + \alpha \psi) \right)^2 \right) r^2 dr d\omega \right] \end{aligned} \quad (3.3.4)$$

for some $C > 0$, by Proposition 3.3.4. Combining (3.3.3) and (3.3.4) gives the desired result. For $|\sigma| \leq k - 4$, one reproduces (3.3.2) for the $\overline{D}$ derivatives by controlling the top-order ∂_t derivative by the zeroth-order term of (3.2.3) after $|\sigma|$ T commutations. $\square$

In fact, via elliptic estimates, we can prove the following Sobolev embedding with stronger (α -dependent) weights. Since we will not use this result in the proof of local existence (and

³Note that the same estimate (as well as those appearing in Theorem 3.3.6 and Theorem 3.3.7) holds for Ω^σ derivatives, controlling angular derivatives $\partial_\theta, \partial_\phi$ with a stronger radial weight.

for simplicity), we state it here for solutions ψ of the linear wave equation $\square_{g_{AdS}}\psi + \alpha\psi = 0$. However, one can also prove an analogous estimate for $\square_{g_{AdS}}\psi + \alpha\psi = \mathcal{F} \neq 0$ with appropriate $\mathcal{F}$.

Theorem 3.3.6 (An improved L^∞ Sobolev embedding on spacelike slices Σ_t in AdS space [33, Section 7.3]). Fix t and let ψ be a function in $H_{AdS}^k(\Sigma_t)$ (with $\square_{g_{AdS}}\psi + \alpha\psi = 0$) for $k \geq 3$. Then one can bound ψ and its D derivatives (3.2.1) up to order $|\sigma| \leq k - 3$ uniformly as

$$\|D^\sigma \psi\|_{L^\infty(\Sigma_t)} \leq \frac{C}{1 + r^{\frac{3}{2} + \delta}} \sum_{|\alpha| \leq |\sigma| + 2} \|\Gamma^\alpha \psi\|_{H_{AdS}^1(\Sigma_t)} \quad (3.3.5)$$

for some $C > 0$, all $\delta < s = \sqrt{\frac{9}{4} - \alpha}$. Up to order $|\sigma| = k - 4$, one has estimate (3.3.2) for the $\bar{D}$ derivatives (3.2.1).

Theorem 3.3.6 follows by using the equation (3.1.1) to control, via expression (3.1.2), the second-order t -derivative $\partial_t^2 \psi$ by purely spatial derivatives which appear with stronger r -weights in the energy (4.2.9). One can then use a higher-order T -commuted energy to control the integrand in (3.3.3), with additional room for the desired stronger weight in r .

We will also make use of the following embedding for derivatives where only 1 is arbitrary.

Theorem 3.3.7 (L^∞ Sobolev embedding for derivatives where only 1 is arbitrary). Fix t and let ψ be a function in $H_{AdS}^k(\Sigma_t)$ for $k \geq 3$ satisfying Dirichlet boundary conditions $r^{\frac{3}{2} - s} \psi|_{\mathcal{I}}$. Then, for each $|\sigma| \leq k - 4$, one has

$$\begin{aligned} & \|\Gamma^\sigma D\psi\|_{L^\infty(\Sigma_t)} \\ & \leq \frac{C}{1 + r^{\frac{3}{2}}} \left[\sum_{\substack{|\alpha| \leq |\sigma| + 3, \\ |\beta| \leq |\sigma| + 2, \\ |\gamma| \leq \lceil \frac{|\sigma| + 2}{2} \rceil}} \left[\|\Gamma^\alpha \psi\|_{H_{AdS}^1(\Sigma_t)}^2 + \int_{\Sigma_t} \left(\left(\Gamma^\beta (\square_g \psi + \alpha\psi) \right)^2 \right. \right. \right. \\ & \quad \left. \left. \left. + \left(\square_g^\gamma (\square_g \psi + \alpha\psi) \right)^2 \right) r^2 dr d\omega \right] \right]^{\frac{1}{2}}. \quad (3.3.6) \end{aligned}$$

3.3.4 Energy estimates for the linear inhomogeneous problem

We now prove general n^{th} -order energy estimates for solutions ψ of problem (3.1.3). This is first carried out with respect to a general inhomogeneous term $\tilde{\mathcal{F}}$. Estimates for our chosen nonlinear term $\mathcal{F}$ (3.2.7) are then derived in the subsequent section.

Proposition 3.3.8. Suppose ψ is a CH_{KAdS}^k solution of (3.1.3), with inhomogeneity $\mathcal{F} = \tilde{\mathcal{F}}$ satisfying

$$\sum_{|\sigma| \leq n-1} \|T^{\sigma_1} \Omega^{\sigma_2} \tilde{\mathcal{F}}\|_{L^2(\Sigma_t)} < \infty$$

for all $n \leq k$, t in $[0, T]$. Then ψ satisfies the general n^{th} order energy estimate

$$\begin{aligned} & \sup_{t \in [0, T]} \|\psi\|_{H_{AdS}^n(\Sigma_t)}^2 \\ & \leq C \left[\|\psi\|_{H_{AdS}^n(\Sigma_0)}^2 + \left(\int_0^T \sum_{|\sigma| \leq n-1} \|T^{\sigma_1} \Omega^{\sigma_2} \tilde{\mathcal{F}}\|_{L^2(\Sigma_t)} dt \right)^2 \right. \\ & \quad \left. + \sum_{\substack{|\alpha| \leq n-1, \\ |\beta| \leq n-2, \\ |\gamma| \leq \lceil \frac{n-2}{2} \rceil}} \left(\|\Gamma_{AdS}^\alpha \psi\|_{H_{AdS}^1(\Sigma_t)}^2 + \int_{\Sigma_t} \left(\left(\Gamma_{AdS}^\beta \tilde{\mathcal{F}} \right)^2 + \left(\square_{g_{AdS}}^\gamma \tilde{\mathcal{F}} \right)^2 \right) r^2 dr d\omega \right) \right], \quad (3.3.7) \end{aligned}$$

for some $C > 0$ and all $n \leq k$.

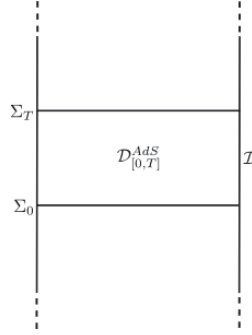


Figure 3.2

Proof. We apply $T\psi$ to the equation (3.1.1) as a multiplier and integrate by parts over the region $\mathcal{D}_{[0,T]}^{AdS} = \mathcal{M}_{AdS} \cap [0, T]$ (see Figure 3.2) to obtain

$$\begin{aligned} \int_{\mathcal{D}_{[0,T]}^{AdS}} |T\psi \cdot \tilde{\mathcal{F}}| r^2 dt dr d\omega &= \int_{\mathcal{D}_{[0,T]}^{AdS}} |T\psi (\square_{AdS} \psi + \alpha \psi)| r^2 dt dr d\omega \\ &\geq C \int_{\partial \mathcal{D}_{[0,T]}^{AdS}} J_\mu^T[\psi] \cdot n_{\Sigma_t}^\mu \cdot \text{vol}_{\Sigma_t}. \end{aligned}$$

Thus, after dropping the boundary term at infinity by virtue of the Dirichlet boundary condition, we have that

$$\int_{\Sigma_T} J_t^T[\psi] \cdot n_{\Sigma_T}^t \cdot \text{vol}_{\Sigma_T} \leq \int_{\Sigma_0} J_t^T[\psi] \cdot n_{\Sigma_0}^t \cdot \text{vol}_{\Sigma_0} + C \int_{\mathcal{D}_{[0,T]}^{AdS}} |T\psi \cdot \tilde{\mathcal{F}}| r^2 dr d\omega dt.$$

Applying Proposition 3.3.1 gives

$$\|\psi\|_{H_{AdS}^1(\Sigma_T)}^2 \leq C \left(\|\psi\|_{H_{AdS}^1(\Sigma_0)}^2 + \int_{\mathcal{D}_{[0,T]}^{AdS}} |T\psi \cdot \tilde{\mathcal{F}}| r^2 dr d\omega dt \right). \quad (3.3.8)$$

Since we can perform estimate (3.3.8) over smaller time intervals, we in fact have that

$$\sup_{t \in [0, T]} \|\psi\|_{H^1_{AdS}(\Sigma_t)}^2 \leq C \left(\|\psi\|_{H^1_{AdS}(\Sigma_0)}^2 + \int_{\mathcal{D}_{[0, T]}^{AdS}} |T\psi \cdot \tilde{\mathcal{F}}| r^2 dr d\omega dt \right).$$

In order to generate higher-order estimates, we commute with the commuting (Killing) vector fields Γ_{AdS} :

$$\sum_{|\sigma| \leq n-1} \sup_{t \in [0, T]} \|\Gamma_{AdS}^\sigma \psi\|_{H^1_{AdS}(\Sigma_t)}^2 \leq C \left(\|\psi\|_{H^1_{AdS}(\Sigma_0)}^2 + \sum_{|\sigma| \leq n-1} \int_{\mathcal{D}_{[0, T]}^{AdS}} |\Gamma_{AdS}^\sigma (T\psi \cdot \tilde{\mathcal{F}})| r^2 dr d\omega dt \right). \quad (3.3.9)$$

Using Proposition 3.3.4 to control the missing radial and mixed radial derivatives, we have that

$$\begin{aligned} & \sup_{t \in [0, T]} \|\psi\|_{H^n_{AdS}(\Sigma_t)}^2 \\ & \leq C \left[\|\psi\|_{H^n_{AdS}(\Sigma_0)}^2 + \sum_{|\sigma| \leq n-1} \int_{\mathcal{D}_{[0, T]}^{AdS}} |T^{\sigma_1} \Omega^{\sigma_2} (T\psi \cdot \tilde{\mathcal{F}})| r^2 dr d\omega dt \right. \\ & \quad \left. + \sum_{\substack{|\alpha| \leq n-1, \\ |\beta| \leq n-2, \\ |\gamma| \leq \lceil \frac{n-2}{2} \rceil}} \left(\|\Gamma_{AdS}^\alpha \psi\|_{H^1_{AdS}(\Sigma_t)}^2 + \int_{\Sigma_t} \left(\left(\Gamma_{AdS}^\beta \tilde{\mathcal{F}} \right)^2 + \left(\square_{g_{AdS}}^\gamma \tilde{\mathcal{F}} \right)^2 \right) r^2 dr d\omega \right) \right]. \quad (3.3.10) \end{aligned}$$

Expanding the bulk term on the right-hand side of (3.3.10) arising from the inhomogeneity $\tilde{\mathcal{F}}$, we find that

$$\begin{aligned} & \int_0^T \int_{\Sigma_t} \sum_{|\sigma| \leq n-1} |T^{\sigma_1} \Omega^{\sigma_2} (T\psi \cdot \tilde{\mathcal{F}})| r^2 dr d\omega dt \\ & \leq \int_0^T \int_{\Sigma_t} \sum_{|\tilde{\sigma}| \leq |\sigma| \leq n-1} |T^{\sigma_1 - \tilde{\sigma}_1 + 1} \Omega^{\sigma_2 - \tilde{\sigma}_2} \psi \cdot T^{\tilde{\sigma}_1} \Omega^{\tilde{\sigma}_2} \tilde{\mathcal{F}}| r^2 dr d\omega dt \\ & \leq \int_0^T \sum_{\substack{|\tilde{\sigma}| \leq n, \\ |\sigma| \leq n-1}} \|T^{\tilde{\sigma}_1} \Omega^{\tilde{\sigma}_2} \psi\|_{L^2(\Sigma_t)} \|T^{\sigma_1} \Omega^{\sigma_2} \tilde{\mathcal{F}}\|_{L^2(\Sigma_t)} dt \\ & \leq \frac{\varepsilon}{2} \sup_{t \in [0, T]} \|\psi\|_{H^n_{AdS}(\Sigma_t)}^2 + \frac{1}{2\varepsilon} \left(\int_0^T \sum_{|\sigma| \leq n-1} \|T^{\sigma_1} \Omega^{\sigma_2} \tilde{\mathcal{F}}\|_{L^2(\Sigma_t)}^2 dt \right). \quad (3.3.11) \end{aligned}$$

Here $\varepsilon > 0$ is chosen sufficiently small to allow the first term on the right-hand side of (3.3.11) to be absorbed on the left-hand side of (3.3.10), yielding

$$\sup_{t \in [0, T]} \|\psi\|_{H^n_{AdS}(\Sigma_t)}^2$$

$$\begin{aligned}
&\leq C \left[\|\psi\|_{H_{AdS}^n(\Sigma_0)}^2 + \left(\int_0^T \sum_{|\sigma| \leq n-1} \|T^{\sigma_1} \Omega^{\sigma_2} \tilde{\mathcal{F}}\|_{L^2(\Sigma_t)} dt \right)^2 \right. \\
&\quad \left. + \sum_{\substack{|\alpha| \leq n-1, \\ |\beta| \leq n-2, \\ |\gamma| \leq \lceil \frac{n-2}{2} \rceil}} \left(\|\Gamma_{AdS}^\alpha \psi\|_{H_{AdS}^1(\Sigma_t)}^2 + \int_{\Sigma_t} \left(\left(\Gamma_{AdS}^\beta \tilde{\mathcal{F}} \right)^2 + \left(\square_{g_{AdS}}^\gamma \tilde{\mathcal{F}} \right)^2 \right) r^2 dr d\omega \right) \right],
\end{aligned}$$

for some $C > 0$, as required. $\square$

3.3.5 Estimating the nonlinearity

We now turn our attention to estimating the $\tilde{\mathcal{F}}$ contribution in (3.3.7) for our particular choice of nonlinearity $\mathcal{F}$ (3.2.7). We first prove the following bound on the second and final terms on the right-hand side of (3.3.7). This is followed by an explanation of how the remaining (strictly lower-order) $\mathcal{F}$ -dependent terms are treated.

Proposition 3.3.9. Let ψ be as in Proposition 3.3.8 with $k \geq 9$ and suppose $\mathcal{F}$ is a nonlinearity of the form (3.2.7). Then $\mathcal{F}$ satisfies the estimates

$$\begin{aligned}
\sum_{|\sigma| \leq k-1} \|T^{\sigma_1} \Omega^{\sigma_2} \mathcal{F}\|_{L_{AdS}^2(\Sigma_t)} &\leq C \|\psi\|_{H_{AdS}^k(\Sigma_t)} \|\psi\|_{H_{AdS}^{k-1}(\Sigma_t)}, \\
\sum_{|\gamma| \leq \lceil \frac{k-2}{2} \rceil} \|\square_{g_{AdS}}^\gamma \mathcal{F}\|_{L_{AdS}^2(\Sigma_t)}^2 &\leq \tilde{C} \|\psi\|_{H_{AdS}^{k-1}(\Sigma_t)}^4
\end{aligned} \tag{3.3.12}$$

for some $C, \tilde{C} > 0$.

Proof. By the chain rule and assumption (3.2.7) on $\mathcal{F}$, we have that

$$\begin{aligned}
\sum_{|\sigma| \leq k-1} \|T^{\sigma_1} \Omega^{\sigma_2} \mathcal{F}\|_{L^2(\Sigma_t)} &= \sum_{|\sigma| \leq k-1} \|T^{\sigma_1} \Omega^{\sigma_2} (F^{\mu\nu}(t, x) \cdot e_\mu \psi e_\nu \psi)\|_{L^2(\Sigma_t)} \\
&\leq C_F \sum_{\substack{|\sigma| \leq k-1 \\ \delta \leq \sigma_1, \gamma \leq \sigma_2 \\ 0 \leq \mu, \nu \leq 3}} \|T^{\sigma_1 - \delta} \Omega^{\sigma_2 - \gamma} e_\mu \psi \cdot T^\delta \Omega^\gamma e_\nu \psi\|_{L^2(\Sigma_t)},
\end{aligned} \tag{3.3.13}$$

where C_F is a constant depending on the functions $F^{\mu\nu}(t, x)$ appearing in $\mathcal{F}$ (3.2.7). Assuming, without loss of generality, that $\delta + \gamma \leq \frac{k-1}{2}$ yields

$$\sum_{\substack{|\sigma| \leq k-1 \\ \delta \leq \sigma_1, \gamma \leq \sigma_2 \\ 0 \leq \mu, \nu \leq 3}} \|T^{\sigma_1 - \delta} \Omega^{\sigma_2 - \gamma} e_\mu \psi \cdot T^\delta \Omega^\gamma e_\nu \psi\|_{L^2(\Sigma_t)} \tag{3.3.14}$$

$$\leq \sum_{\substack{|\sigma| \leq k-1 \\ |\delta| \leq \frac{k-1}{2}, \\ 0 \leq \mu, \nu \leq 3}} \|T^{\sigma_1} \Omega^{\sigma_2} e_\mu \psi\|_{L^2(\Sigma_t)} \|T^{\delta_1} \Omega^{\delta_2} e_\nu \psi\|_{L^\infty(\Sigma_t)} \tag{3.3.15}$$

$$\leq C \|\psi\|_{H_{AdS}^k(\Sigma_t)} \left[\|\psi\|_{H_{AdS}^{k-1}(\Sigma_t)} + \sum_{|\gamma| \leq \lceil \frac{|\delta|+2}{2} \rceil} \|\square_{g_{AdS}}^\gamma \mathcal{F}\|_{L^2(\Sigma_t)} \right]. \quad (3.3.16)$$

Here, the last inequality follows from Theorem 3.3.7 and the assumption $k \geq 9$. By the assumption $k \geq 9$, the $\square$ terms on the right-hand side of (3.3.16) are of lower-order than those appearing in (3.3.12), so we demonstrate how to treat those first, as follows.

We treat the second-order terms in each $\square$ explicitly. All other terms can be dealt with similarly. We have that

$$\begin{aligned} & \sum_{|\gamma| \leq \lceil \frac{k-2}{2} \rceil} \|\square_{g_{AdS}}^\gamma \mathcal{F}\|_{L^2(\Sigma_t)}^2 \\ & \leq C \sum_{|\gamma| \leq \lceil \frac{k-2}{2} \rceil} \left\| (r^2 \partial_r^2)^{\gamma_1} \left(\frac{1}{r^2} \partial_t^2 \right)^{\gamma_2} \left(\frac{1}{r^2} \partial_\theta^2 \right)^{\gamma_3} \left(\frac{1}{r^2} \partial_\phi^2 \right)^{\gamma_4} (F^{\mu\nu}(t, x) \cdot e_\mu \psi e_\nu \psi) \right\|_{L^2(\Sigma_t)}^2 + \dots \\ & \leq C_F \sum_{\substack{|\gamma| \leq \lceil \frac{k-2}{2} \rceil, \\ \delta \leq \gamma_1, \rho \leq \gamma_2, \sigma \leq \gamma_3, \beta \leq \gamma_4 \\ 0 \leq \mu, \nu \leq 3}} \left\| (r^2 \partial_r^2)^{\gamma_1 - \delta} \left(\frac{1}{r^2} \partial_t^2 \right)^{\gamma_2 - \rho} \dots e_\mu \psi \cdot (r^2 \partial_r^2)^\delta \left(\frac{1}{r^2} \partial_t^2 \right)^\rho \dots e_\nu \psi \right\|_{L^2(\Sigma_t)}^2, \end{aligned}$$

where C_F is a constant depending on the functions $F^{\mu\nu}(t, x)$ appearing in $\mathcal{F}$ (3.2.7). Without loss of generality, we assume that $\delta + \rho + \sigma + \beta \leq \frac{1}{2} \lceil \frac{k-2}{2} \rceil \leq \frac{k-2}{4}$, yielding

$$\begin{aligned} & \sum_{\substack{|\gamma| \leq \lceil \frac{k-2}{2} \rceil, \\ \delta \leq \gamma_1, \rho \leq \gamma_2, \sigma \leq \gamma_3, \beta \leq \gamma_4 \\ 0 \leq \mu, \nu \leq 3}} \left\| (r^2 \partial_r^2)^{\gamma_1 - \delta} \left(\frac{1}{r^2} \partial_t^2 \right)^{\gamma_2 - \rho} \dots e_\mu \psi \cdot (r^2 \partial_r^2)^\delta \left(\frac{1}{r^2} \partial_t^2 \right)^\rho \dots e_\nu \psi \right\|_{L^2(\Sigma_t)}^2 \\ & \leq \sum_{\substack{|\gamma| \leq \lceil \frac{k-2}{2} \rceil, \\ |\sigma| \leq \frac{k-2}{4}, \\ 0 \leq \mu, \nu \leq 3}} \left\| (r^2 \partial_r^2)^{\gamma_1} \left(\frac{1}{r^2} \partial_t^2 \right)^{\gamma_2} \dots e_\mu \psi \right\|_{L^2(\Sigma_t)}^2 \left\| (r^2 \partial_r^2)^{\sigma_1} \left(\frac{1}{r^2} \partial_t^2 \right)^{\sigma_2} \dots e_\nu \psi \right\|_{L^\infty(\Sigma_t)}^2 \\ & \leq C \|\psi\|_{H_{AdS}^{k-1}(\Sigma_t)}^4, \end{aligned}$$

for some $C > 0$. Here, we have applied Theorem 3.3.5 (treating lower order terms on the right-hand side of (3.3.2) via the same argument) and the assumption $k \geq 9$.

Applying the same argument to the $\square$ terms on the right-hand side of (3.3.16), we have that

$$\sum_{|\sigma| \leq k-1} \|T^{\sigma_1} \Omega^{\sigma_2} \mathcal{F}\|_{L^2(\Sigma_{t^*})} \leq C \|\psi\|_{H_{AdS}^k(\Sigma_t)} \|\psi\|_{H_{AdS}^{k-1}(\Sigma_t)}.$$

□

We can treat the other terms (of strictly lower order) on the right-hand side of (3.3.7) similarly. The Γ commuted terms are immediately controlled by the above. This gives

$$\begin{aligned} & \sum_{\substack{|\beta| \leq k-2, \\ |\gamma| \leq \lceil \frac{k-2}{2} \rceil}} \int_{\Sigma_t} (\Gamma_{AdS}^\beta \mathcal{F})^2 r^2 dr d\omega + \int_{\tau}^{\tau'} \sum_{|\beta| \leq k-3} \|\Gamma_{AdS}^\beta \mathcal{F}\|_{L^2(\Sigma_t)}^2 dt \\ & \leq C \int_{\tau}^{\tau'} \|\psi\|_{H_{AdS}^{k-2}(\Sigma_t)}^2 \|\psi\|_{H_{AdS}^{k-3}(\Sigma_t)}^2 dt. \end{aligned} \quad (3.3.17)$$

It is worth noting that the estimate of Proposition 3.3.9 leaves room for additional r -weights in the nonlinearity $\mathcal{F}$, as a result of the $r^{\frac{3}{2}}$ weight in Theorem 3.3.5. In fact, if one instead applies the stronger result of Theorem 3.3.6, even more room is gained. As we shall see in the discussion of Kerr-AdS, one in fact only requires a Sobolev embedding with an $r^{\frac{1}{2}}$ -weight for the scattering construction (Theorem 4.3.14).

With all necessary estimates in hand, we proceed with proving local existence.

3.4 Local existence for nonlinear wave equations on Anti-de Sitter space

We will apply the estimates appearing in Section 3.3 in order to prove the following nonlinear local existence result on Anti-de Sitter space.

Theorem 3.4.1 (Local existence for nonlinear wave equations on AdS space). Let $\psi_0 \in H_{AdS}^k(\Sigma_0)$ for $k \geq 9$, $\psi_1 = 0$ and $\mathcal{F}$ be as in (3.2.7). Then there exists a time $T > 0$ up to which the nonlinear initial value Dirichlet problem (3.1.3) admits a solution $\psi : \{0 \leq t \leq T\} \times \{r \geq 0\} \times S_{t,r}^2 \rightarrow \mathbb{R} \in CH_{AdS}^k$ satisfying

$$\sup_{t \in [0, T]} \|\psi\|_{H_{AdS}^k(\Sigma_t)}^2 \leq A. \quad (3.4.1)$$

for some $A > 0$, depending on ψ_0 . Moreover, the solution satisfies

$$\begin{aligned} \sum_{|\sigma| \leq k-3} \|r^{\frac{3}{2}} D^\sigma \psi\|_{L_{AdS}^\infty(\Sigma_t)}^2 & \leq A, \\ \sum_{|\sigma| \leq k-4} \|r^{\frac{3}{2}} \overline{D}^\sigma \psi\|_{L_{AdS}^\infty(\Sigma_t)}^2 & \leq A. \end{aligned} \quad (3.4.2)$$

In order to prove Theorem 3.4.1, we adapt Luk's proof of the analogous result on Minkowski space [46]. This involves constructing an appropriate sequence of purely *linear* problems approximating the full nonlinear problem (3.1.3). Via an inductive argument on the iterates and a familiar application of the *Arzela-Ascoli Theorem*, we obtain a limiting solution of problem (3.1.3).

Remark (Uniqueness of solutions). Uniqueness of the solution given by Theorem 3.4.1 is not proven here. However, said solution is expectedly unique in the energy class CH_{AdS}^k . For a discussion of uniqueness for solutions of the *linear* equation on more general asymptotically AdS spacetimes, one can see [33].

3.4.1 An approximating sequence of linear waves

Consider the sequence of problems

$$\begin{cases} \square\psi_i + \alpha\psi_i = \mathcal{F}_{i-1} \stackrel{def}{=} \mathcal{F}(\psi_{i-1}, \partial\psi_{i-1}) \\ (\psi_i, \partial_t\psi_i)|_{\Sigma_0} = (\psi_0, \psi_1), \quad r\psi_i|_{\mathcal{I}} = 0 \end{cases} \quad (3.1.4)$$

for $i \geq 2$, whose solutions $\{\psi_i\}_{i=2}^\infty \in CH_{AdS}^k$ are to be determined recursively from the initial data $\psi_0 \in H_{AdS}^k(\Sigma_0)$, $\psi_1 = 0$.

For each fixed i , the equation in (3.1.4) is strictly *linear*. The right-hand side $\mathcal{F}_{i-1}$ is a linear inhomogeneity, depending on the previous element ψ_{i-1} of the sequence of solutions. As such, linear local well-posedness [33] ensures that such a sequence of unique solutions $\{\psi_i\}_{i=2}^\infty$ exists. We will prove the following Theorem.

Theorem 3.4.2 (Uniform boundedness of solutions of the approximating linear problems on AdS space). Let $\psi_0 \in H_{AdS}^k(\Sigma_0)$ and $\psi_1 = 0$. Then there exists $A, T > 0$ (depending on ψ_0) such that for each i , the corresponding solution $\psi_i \in CH_{AdS}^k$ of problem (3.1.4) satisfies

$$\sup_{t \in [0, T]} \|\psi_i\|_{H_{AdS}^k(\Sigma_t)} \leq A.$$

Theorem 3.4.2 follows immediately from Proposition 3.4.3.

Proposition 3.4.3 (Induction on i on AdS space). Let (ψ_0, ψ_1) be as in Theorem 3.4.2. Then the following statements hold.

1. The solution $\psi_2 \in CH_{AdS}^k$ of problem (3.1.4) with $i = 2$ satisfies

$$\sup_{t \in [0, T]} \|\psi_2\|_{H_{AdS}^k(\Sigma_t)} \leq A,$$

for some $T > 0$ sufficiently small and $A > 0$ sufficiently large (both depending on ψ_0).

2. Let $i > 2$ be fixed. Suppose that the solution $\psi_{i-1} \in CH_{AdS}^k$ of problem (3.1.4) satisfies the inductive hypothesis

$$\sup_{t \in [0, T]} \|\psi_{i-1}\|_{H_{AdS}^k(\Sigma_t)} \leq A. \quad (3.4.3)$$

Then ψ_i also satisfies

$$\sup_{t \in [0, T]} \|\psi_i\|_{H_{AdS}^k(\Sigma_t)} \leq A, \quad (3.4.4)$$

as well as

$$\begin{aligned} \sum_{|\sigma| \leq k-3} \|r^{\frac{3}{2}} D^\sigma \psi_i\|_{L^\infty_{AdS}(\Sigma_t)}^2 &\leq A, \\ \sum_{|\sigma| \leq k-4} \|r^{\frac{3}{2}} \bar{D}^\sigma \psi_i\|_{L^\infty_{AdS}(\Sigma_t)}^2 &\leq A. \end{aligned} \quad (3.4.5)$$

In order to prove Proposition 3.4.3, we employ a bootstrap argument which closes as a result of the following.

Proposition 3.4.4 (Improving the bootstrap assumption on AdS space). Let ψ_0 , ψ_1 and ψ_{i-1} be as in Proposition 3.4.3. Furthermore, suppose that ψ_i satisfies the bootstrap assumption

$$\|\psi_i\|_{H^k_{AdS}(\Sigma_t)} \leq 2A \quad (3.4.6)$$

at some $t \in [0, T]$. Then the bootstrap constant can be improved, so that

$$\|\psi_i\|_{H^k_{AdS}(\Sigma_t)} \leq A.$$

Proof of Proposition 3.4.4. By Proposition 3.3.8 and Proposition 3.3.9, we have that

$$\begin{aligned} &\sup_{t \in [0, T]} \|\psi_i\|_{H^k_{AdS}(\Sigma_t)}^2 \\ &\leq C \left[\|\psi_i\|_{H^k_{AdS}(\Sigma_0)}^2 + \|\psi_i\|_{H^k_{AdS}(\Sigma_t)}^2 + \left(\int_0^T \|\psi_{i-1}\|_{H^k_{AdS}(\Sigma_t)} \|\psi_{i-1}\|_{H^{k-1}_{AdS}(\Sigma_t)} dt \right)^2 \right. \\ &\quad \left. + \|\psi_{i-1}\|_{H^{k-1}_{AdS}(\Sigma_t)}^4 + \int_0^T \|\psi_{i-1}\|_{H^{k-2}_{AdS}(\Sigma_t)}^2 \|\psi_{i-1}\|_{H^{k-3}_{AdS}(\Sigma_t)}^2 dt \right] \\ &\leq C \left[\|\psi_i\|_{H^k_{AdS}(\Sigma_0)}^2 + \left(\int_0^T \|\psi_{i-1}\|_{H^k_{AdS}(\Sigma_t)} \|\psi_{i-1}\|_{H^{k-1}_{AdS}(\Sigma_t)} dt \right)^2 \right. \\ &\quad \left. + \int_0^T \left(\|\psi_{i-1}\|_{H^{k-1}_{AdS}(\Sigma_t)}^4 + \|\psi_i\|_{H^k_{AdS}(\Sigma_t)}^2 \right) dt \right], \end{aligned} \quad (3.4.7)$$

for some $C > 0$. Then, by assumptions (3.4.3) and (3.4.6),

$$\|\psi_i\|_{H^k_{AdS}(\Sigma_t)}^2 \leq C \left(\frac{A}{2C} + T^2 A^4 + T A^4 + 4T A^2 \right) \leq A.$$

as desired. Here, the final inequality follows for $A(\psi_0)$ sufficiently large and T sufficiently small, so that $T^2 A^4 + T A^4 + 4T A^2 \leq \frac{A}{2C}$. $\square$

We can now prove Proposition 3.4.3 as follows.

Proof of Proposition 3.4.3. We prove each statement in turn, beginning with the base case.

1. Fix $i = 2$. The function ψ_2 solves problem (3.1.4) with trivial right-hand side $\mathcal{F}_1 = 0$, which by Proposition 3.3.8 and Proposition 3.3.9 gives

$$\sup_{t \in [0, T]} \|\psi_2\|_{H_{AdS}^k(\Sigma_t)}^2 \leq C \|\psi_0\|_{H_{AdS}^k(\Sigma_0)}^2 \leq A,$$

for some $C > 0$. Here, the first inequality follows from the initial condition on Σ_0 and the second by choosing $A(\psi_0) > 0$ sufficiently large.

2. We shall show that the set

$$\mathcal{B} = \left\{ t \in [0, T] \mid \psi_i \text{ satisfies } \|\psi_i\|_{H_{AdS}^k(\Sigma_\tau)} \leq A \ \forall \ \tau \in [0, t] \right\}$$

is non-empty, open and closed, proving $\mathcal{B} = [0, T]$.

- Since $t = 0 \in \mathcal{B}$, $\mathcal{B}$ is clearly non-empty.
- Let t_j be a sequence of times in $\mathcal{B}$. Then by continuity of ψ_i and the assumed bound on ψ_i , t_j has a limit in $\mathcal{B}$. Thus $\mathcal{B}$ is closed.
- Suppose $t > 0 \in \mathcal{B}$. Then, by continuity of ψ_i there exists a small radius $\varepsilon > 0$ such that, for all τ in the open ball $\mathbf{B}_\varepsilon(t) = (t - \varepsilon, t + \varepsilon)$, ψ_i satisfies

$$\|\psi_i\|_{H_{AdS}^k(\Sigma_\tau)} \leq 2A.$$

By Proposition 3.4.4, this implies

$$\|\psi_i\|_{H_{AdS}^k(\Sigma_\tau)} \leq A,$$

so that $\mathbf{B}_\varepsilon(t) \subset \mathcal{B}$ and $\mathcal{B}$ is open.

Thus $\mathcal{B} = [0, T]$. Since $[0, T]$ is a closed interval and ψ_i is continuous, (3.4.4) follows. Estimates (3.4.5) follow from an application of Theorem 3.3.5.

□

We can now complete the proof of Theorem 3.4.1.

3.4.2 A limit of the approximating sequence

Proof of Proposition 3.4.1. By Proposition 3.4.3, we have uniform boundedness of ψ_i and its derivatives for each fixed i . We can now perform a familiar application of the Arzelà-Ascoli Theorem to obtain our target solution ψ , satisfying the required estimates. □

Chapter 4

A scattering construction for nonlinear wave equations on Kerr-Anti-de Sitter spacetimes

4.1 Introduction

This chapter concerns solutions of the massive scalar wave equation

$$\square_g \psi + \alpha \psi = \mathcal{F}(\psi, \partial \psi), \quad (1.3.1)$$

on the exterior ($r \geq r_+$) of a fixed $(3+1)$ -dimensional Kerr-Anti-de Sitter (Kerr-AdS) background¹ $(\mathcal{M}_{KAdS}, g_{KAdS})$ with cosmological constant $\Lambda = -\frac{3}{\ell^2}$ ($\ell > 0$), mass $M > 0$ and future event horizon ($\mathcal{H}^+$) radius r_+ whose angular momentum a satisfies²

$$|a| \leq \sqrt{\frac{M\ell^2}{r_+} - \ell^2 - 2r_+^2} \quad \text{and} \quad |a| < \ell. \quad (1.3.4)$$

Recall that, in *Boyer-Lindquist* coordinates, the metric takes the familiar form

$$\begin{aligned} g_{KAdS} = & -\frac{\Delta_- - \Delta_\theta a^2 \sin^2 \theta}{\Sigma} dt^2 + \frac{\Sigma}{\Delta_-} dr^2 + \frac{\Sigma}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta(r^2 + a^2)^2 - \Delta_- a^2 \sin^2 \theta}{\Xi^2 \Sigma} \sin^2 \theta d\tilde{\phi}^2 \\ & - 2 \frac{\Delta_\theta(r^2 + a^2)^2 - \Delta_-}{\Xi \Sigma} a \sin^2 \theta d\tilde{\phi} dt, \end{aligned} \quad (1.3.5)$$

¹We will usually drop the ‘KAdS’ subscript (from g , $\square_g$ and other relevant objects), in order to maintain compactness of mathematical expressions.

²Recall that the second requirement ensures that expression (1.3.5) defines a Lorentzian metric (one can check via direct calculation that $\det g$ is undefined when $|a| = \ell$, due to the vanishing of Ξ and when $|a| > \ell$, there exists $\theta = \theta'$ at which $\det g|_{\theta'} > 0$). The first bound ensures that (1.3.5) describes a black hole spacetime (*extremal* in the case of equality). See [52].

where Δ_- , Δ_θ , Σ and Ξ are functions of r and θ , defined below in (4.2.2). In practice, alternative (t^*, r, θ, ϕ) coordinates will be used, as these remedy that Boyer-Lindquist coordinates cease to be regular at the event horizon. The spacetime metric in these coordinates, as well as the relation between the two coordinate systems is stated in Section 4.2.1. One may consult Figure 4.1 for examples of hypersurfaces of constant t and t^* time respectively.

Recall that, in (1.3.1), the mass α is assumed to satisfy the *Breitenlohner-Freedman bound*, $\alpha < \frac{9}{4}$, within which positivity of the energy was first derived in the “pure” Anti-de Sitter (AdS) context [4]. The range $\alpha > 0$ corresponds to “negative mass.” The class of nonlinearity $\mathcal{F}$ treated here will be defined in Section 4.2.3. For now, one should think of $\mathcal{F}$ as polynomial in first-order “unit” derivatives $D\psi$ (see (4.2.7)) of the solution and consider $\mathcal{F} = \frac{1}{r^2}(\partial_{t^*}\psi)^2$ as one explicit example. In particular, in contrast to the asymptotically flat case, no algebraic condition on $\mathcal{F}$ is required. Specifically, one need not impose the *null condition* which is crucial in the asymptotically flat case [5, 41, 42]. This is due to the fact that derivatives in the null directions decay at the same rate in r , meaning that there are no “good” and “bad” derivatives. However, some general r -weighted condition (4.2.12), analogous to Condition 3.2.4 on AdS, on $\mathcal{F}$ is required. Note already that condition (4.2.12) allows a broad class of nonlinearities. In particular, we permit terms of the form $g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi$ which mimic the first-order terms appearing when one writes the Einstein equation in harmonic gauge. This is discussed further in Section 4.2.3.2.

4.1.1 The main result

Since asymptotically-AdS spacetimes are not globally hyperbolic, the pertinent setting in which to study (1.3.1) is an *initial boundary value problem*. Here, we restrict our attention to the Dirichlet problem

$$\begin{cases} \square_{g_{KAdS}}\psi + \alpha\psi = \mathcal{F}(\psi, \partial\psi), \\ (\psi, \partial_{t^*}\psi)|_{\{t^*=\tau\}} = (\psi_0, \psi_1), \quad r^{\frac{3}{2}-s}\psi|_{\mathcal{I}} = 0, \quad s = \sqrt{\frac{9}{4} - \alpha}, \end{cases} \quad (4.1.1)$$

with initial conditions prescribed on a spacelike hypersurface $\{t^* = \tau\}$ and boundary conditions at the timelike conformal boundary $\mathcal{I}$, which can formally be viewed as the limiting hypersurface $\{r = \infty\}$ (see Figure 4.1). However, one may easily extend the discussion that follows to Neumann or Robin boundary conditions in the manner of Holzegel-Warnick [40].

The objective of this chapter is to construct and prove quantitative estimates for an expectedly non-generic, but nonetheless large, class of exponentially decaying solutions of (4.1.1). No symmetry assumptions will be made on the class of solutions which, moreover, exhibit the full functional degrees of freedom of the problem (4.1.1). The main result can be stated informally as follows (a restatement of Theorem 1.3.2):

Theorem 4.1.1 (Exponentially decaying nonlinear waves on the Kerr-AdS exterior). Under appropriate assumptions on $\mathcal{F}$ (see Section 4.2.3.2), there exist a large class of (classical) solutions ψ of (1.3.1) with vanishing Dirichlet boundary conditions at $\mathcal{I}$, parameterised by

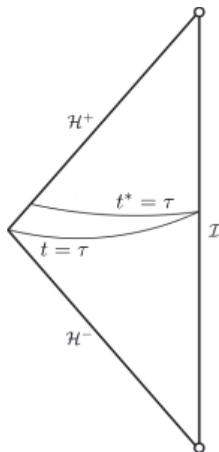


Figure 4.1: The Penrose diagram of the Kerr-AdS *exterior* region $\{r \geq r_+\}$ depicting examples of constant t and t^* spacelike hypersurfaces.

appropriate *scattering data* on $\mathcal{H}^+$, on the (sub)extremal³ (1.3.4) Kerr-AdS exterior defined globally to the future of some constant t^* hypersurface. These solutions, moreover, decay exponentially in time towards the future.

Theorem 4.1.1 follows from recasting the initial boundary value problem (4.1.1) as a *backwards* problem with exponentially decaying scattering data on the horizon, and constructing solutions via an iteration scheme, in a manner analogous to those employed on Minkowski space and the Schwarzschild black hole exterior. It is the exponentially decaying data which makes the solutions non-generic. However, it seems an extremely intricate problem to a priori characterise the initial data giving rise to such solutions as a set in the space of all Cauchy data for (4.1.1).

Remark (Solutions outside of the span of quasinormal modes). As one immediate application of the construction of Theorem 4.1.1, it follows that, even for the linear ($\mathcal{F} = 0$) equation (1.3.1), there are solutions which do not lie in the span of so-called quasinormal modes. Indeed, one can prescribe the scattering data to vanish after finite time to construct non-trivial solutions which vanish to the future of some late $t^* = \text{constant}$ slice. These solutions do not lie in the span of quasinormal modes. See [55] where such solutions and their relation to quasinormal modes are discussed in the Schwarzschild-de Sitter case.

Before discussing our construction to establish Theorem 4.1.1 in more detail, the following section provides a brief review of what is known about the *forward* problem (4.1.1).

4.1.2 Comparison with the forward problem

Understanding the behaviour of general solutions of the nonlinear initial boundary value problem (4.1.1) is key to the wider *black hole stability problem* (see the discussion of Section

³Note that this result does not assume the Hawking-Reall bound (1.3.6), outside of which exponentially *growing* mode solutions are known to exist (see Theorem 4.1.3 below).

4.1.4) and is a difficult open question. Holzegel and Smulevici [36] derived sharp logarithmic decay of general solutions of (4.1.1) for the class of *linear* ($\mathcal{F} = 0$) problems, in the setting where the *Hawking-Reall bound*

$$|a|\ell < r_+^2 \quad (1.3.6)$$

is satisfied. Informally, their result can be stated as follows.

Theorem 4.1.2 (Logarithmic decay of general linear waves on the Kerr-AdS exterior [36]). Consider a fixed Kerr-AdS background $(\mathcal{M}, g)$ satisfying the Hawking-Reall bound (1.3.6). General solutions of the initial boundary value problem for

$$\square_{g_{KAdS}} \psi + \alpha \psi = 0 \quad (4.1.2)$$

with Dirichlet boundary conditions on the subextremal Kerr-AdS exterior decay logarithmically and no faster.

See also the work of Gannot who constructs quasinormal mode solutions of (4.1.2) on Kerr-AdS [24, 25].

When the Hawking-Reall bound is violated, behaviour of solutions of (1.3.1) can generally be much worse. This was exhibited by Dold [15] who produced solutions of (4.1.2) which grow exponentially in time, for parameters violating (1.3.6).

Theorem 4.1.3. (Exponentially growing mode solutions of (4.1.2) [15]) For any ℓ , there is a Kerr-AdS exterior (violating the Hawking-Reall bound (1.3.6)) for which equation (4.1.2) with Dirichlet boundary conditions at $\mathcal{I}$ admits an exponentially growing mode solution.

The proof of Theorem 4.1.2 relies on frequency space analysis of solutions of the forwards problem (4.1.1). This allows the authors to deal with geometric features of the Kerr black-hole exterior, such as superradiance and stable trapping, which cannot be fully understood in physical space. On the other hand, as we shall see, solving backwards from (sufficiently fast) *exponentially decaying* scattering data for (1.3.1) not only makes it possible to treat the more general *nonlinear* ($\mathcal{F} \neq 0$) equations of Theorem 4.1.1, but also allows one to use purely physical space methods since the exponential decay estimates render the procedure insensitive to both trapping and superradiance.

One may expect that, for slower than exponentially decaying scattering data on $\mathcal{H}^+$, the construction of Theorem 4.1.1 can be generalised to again construct future global solutions on the exterior. Such solutions would now generically be singular at the event horizon, however. Characterising the scattering data which give rise to regular solutions at $\mathcal{H}^+$, which thus arise as solutions of the forward problem (4.1.1) with regular Cauchy data, is an interesting and difficult open problem.

4.1.3 Outline of the proof

As noted, Theorem 4.1.1 is proven by way of a scattering construction, inspired by that of Dafermos, Holzegel and Rodnianski [8] in the $\Lambda = 0$ setting. This involves prescribing

the exponentially fast decay of the solution to be constructed as data $h_{\mathcal{H}^+}$ along the event horizon of the Kerr-AdS black hole and evolving backwards to obtain a global towards the future solution of (1.3.1).

It is worth noting that (as is the case in the asymptotically flat setting), conceptually, the construction on Kerr-AdS is not significantly more difficult than that on the spherically symmetric Schwarzschild-AdS spacetime.

As is familiar from the discussion of Schwarzschild in Chapter 2, the key difficulty to overcome, the *blueshift effect* (see Section 4.1.3.1 below), already arises at the level of treating a *linear* problem (4.1.3). To extend to the full nonlinear problem, one must control an additional bulk term in the energy estimates. This is straightforward, provided $\mathcal{F}$ satisfies an appropriate assumption (4.2.12) (similar to Condition 3.2.4 on AdS).

The main step in the proof consists of obtaining a priori energy estimates for a sequence of approximating problems. We give an outline via a discussion of the linear and nonlinear aspects of the problem.

4.1.3.1 Linear aspects: The redshift effect

As is familiar from the discussion of the Schwarzschild black hole exterior, due to the lack of a globally uniformly timelike Killing vector field on Kerr-AdS, the gravitational blueshift means that *any* energy estimate generated by a timelike vector field will produce a non-vanishing bulk term which could, potentially, drive exponential growth (see Section 2.1.2). This should be contrasted with forward-in-time propagation where a (uniformly timelike) *redshift vector field* N can be carefully constructed to ensure all derivatives in the energy estimate appear with a good sign near the horizon. In order to close energy estimates and prove Theorem 4.1.1, we impose a rate of exponential decay on the scattering data which is stronger than the blueshift-generated exponential growth. Consequently, the methods used in this chapter do not allow for scattering data which decay any slower. It is unclear how one might address backwards evolution of (1.3.1) in the setting of polynomially decaying data, for instance⁴. However, in the *extremal* setting (the setting of equality in the first condition appearing in (1.3.4)), the redshift effect degenerates ($\kappa = 0$), and so one may be able to close estimates under weaker decay assumptions.

To see how this works in practice, let us consider a backwards, finite-in-time, linear toy scattering problem which should be thought of as approximating, in the limit $t^* \rightarrow \infty$, the full problem

$$\begin{cases} \square_{g_{KAdS}} \psi + \alpha \psi = 0, & \alpha < \frac{9}{4} \\ \psi|_{\mathcal{H}^+} = h_{\mathcal{H}^+}, & r^{\frac{3}{2}-s} \psi|_{\mathcal{I}} = 0, \quad s = \sqrt{\frac{9}{4} - \alpha}. \end{cases} \quad (4.1.3)$$

In (4.1.3), the smooth scattering data $h_{\mathcal{H}^+}$ satisfies

$$F_{\mathcal{H}^+}[h_{\mathcal{H}^+}] \leq C \exp(-B \cdot \tau), \quad (4.1.4)$$

⁴In the $\Lambda = 0$ setting, it is conjectured that polynomially decaying perturbations of the Kerr event horizon give rise to *weak null singularities* [8]. One can expect similar behaviour in the $\Lambda < 0$ setting.

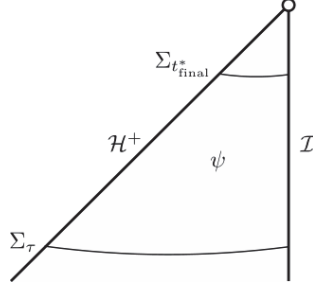


Figure 4.2: The region of integration for estimate (4.1.6).

for some $C > 0$ and some constant $B > 0$ sufficiently large, depending on the surface gravity κ . Here, $F_{\mathcal{H}^+}$ is an H^1 -norm of $h_{\mathcal{H}^+}$, a truncated form of which will appear as the horizon flux term in a subsequent energy inequality (4.1.6). One must also assume higher-order analogues of (4.1.4) but, for now, the estimate will only be demonstrated to first order.

The toy problem we consider is one with mixed spacelike and null scattering data:

$$\begin{cases} \square_{g_{KAdS}} \psi + \alpha \psi = 0, & \alpha < \frac{9}{4} \\ \psi|_{\mathcal{H}^+ \cap \{t^* \leq t^*_{\text{final}}\}} = \tilde{\chi}(t^*) h_{\mathcal{H}^+}, & (\psi, \partial_{t^*} \psi)|_{\Sigma_{t^*_{\text{final}}}} = 0, \quad r^{\frac{3}{2}-s} \psi|_{\mathcal{I}} = 0, \quad s = \sqrt{\frac{9}{4} - \alpha}. \end{cases} \quad (4.1.5)$$

Here, the scattering data on the horizon is truncated via an appropriate cut-off function $\tilde{\chi}$ which vanishes with all derivatives at t^*_{final} . Note that the energy $F_{\mathcal{H}^+ \cap [\tau, t^*_{\text{final}}]} = F_{\mathcal{H}^+}[\tilde{\chi}(t^*) h_{\mathcal{H}^+}]$ of the cut-off scattering data is controlled by that without the cut-off (and in turn controlled by (4.1.4)). Local well-posedness of problem (4.1.5) follows from a characteristic data analogue of [33]. That problem (4.1.5) indeed approximates (4.1.3) follows from the iteration procedure in Section 4.5 with $\mathcal{F} = 0$.

We define the non-degenerate energy (expressed in regular coordinates (t^*, r, θ, ϕ))

$$E[\psi](t^*) = \int_{\Sigma_{t^*}} \left[\frac{1}{r^2} (\partial_{t^*} \psi)^2 + r^2 (\partial_r \psi)^2 + |\nabla \psi|^2 + \psi^2 \right] \text{vol}_{\Sigma_{t^*}}.$$

Multiplying the equation in (4.1.5) by $N\psi$, where N is a uniformly timelike vector field equal to the stationary Killing field $T = \partial_{t^*}$ at infinity, and integrating by parts over the spacetime strip $\mathcal{D}_{[\tau, t^*_{\text{final}}]}^{KAdS} = \mathcal{M}_{KAdS} \cap [\tau, t^*_{\text{final}}]$ yields the energy estimate⁵

$$E[\psi](\tau) \leq F_{\mathcal{H}^+ \cap [\tau, t^*_{\text{final}}]} + F_{\mathcal{I} \cap [\tau, t^*_{\text{final}}]} + C_\kappa \int_\tau^{t^*_{\text{final}}} E[\psi](t^*) dt^*, \quad (4.1.6)$$

which is the analogue of estimate (2.1.17) on Schwarzschild. Here, $F_{\mathcal{H}^+}$ and $F_{\mathcal{I}}$ are the flux terms on the event horizon and future null infinity, respectively. The constant $C_\kappa > 0$ depends linearly on the surface gravity κ .

⁵In fact, the zeroth-order term arising from using N (or T) as a multiplier appears with a bad sign and, as such, some additional work is required to show that this multiplier controls $E[\psi]$. This is carried out in Section 4.3.1.

Applying the Dirichlet boundary conditions and assumed decay (4.1.4) along the horizon, (4.1.6) becomes

$$E[\psi](\tau) \leq C \exp(-B \cdot t^*) + C_\kappa \int_\tau^{t_{\text{final}}^*} E[\psi](t^{*'}) dt^{*'} \quad (4.1.7)$$

Via a Grönwall inequality, in the same manner as on Schwarzschild (Section 2.1.2), (4.1.7) implies

$$\begin{aligned} E[\psi](\tau) &\leq C \exp(-B \cdot \tau) + \int_\tau^{t_{\text{final}}^*} C_k C \exp(-B \cdot t^*) \exp\left(\int_\tau^{t^*} C_\kappa ds\right) dt^* \\ &= C \exp(-B \cdot \tau) + \int_\tau^{t_{\text{final}}^*} C_k C \exp\left((C_\kappa - B)t^*\right) dt^* \\ &\leq C \left(1 + \frac{C_\kappa}{B - C_\kappa}\right) \exp(-B \cdot \tau) \\ &= \tilde{C} \exp(-B \cdot \tau), \end{aligned} \quad (4.1.8)$$

with

$$\tilde{C} = C \left(1 + \frac{C_\kappa}{B - C_\kappa}\right) > 0,$$

provided that the rate $B > 0$ of decay (4.1.4) along the horizon is strictly greater than the constant C_κ .

4.1.3.2 Linear aspects: Controlling higher-order derivatives

In order to prove Theorem 4.1.1, one requires estimates analogous to (4.1.8) for higher order energies. Commuting with T , N and $\Phi = \partial_\phi$ (the Killing vector field associated with the axial symmetry of the background) gives non-degenerate control of higher order time (and mixed time) derivatives. This is carried out in Section 4.3.4, for a linear inhomogeneous problem. Commuting with T and Φ is trivial, however, commuting with N generates non-vanishing commutator terms. On Schwarzschild-AdS (indeed, as was done on AdS in Chapter 3), one could also commute with the angular momentum operators (without generating further additional bulk terms) to control the missing angular derivatives. However, outwith spherical symmetry, one must take more care. There are many ways to resolve this (and the terms generated by commuting with N) but, for simplicity, here this is achieved in Section 4.3.2 by constructing estimates from the equation (1.3.1). Note that the estimates obtained in this manner are not optimal. In particular the L^∞ Sobolev embedding (Theorem 4.3.14) one can prove using these estimates gives only $r^{-\frac{1}{2}}$ decay of ψ , $|\nabla\psi|$. One could improve these estimates by commuting with the angular momentum operators and treating the resulting bulk contributions as error terms, yielding $r^{-\frac{3}{2}}$ decay of ψ (as in Theorem 3.3.5 on AdS space) and angular derivatives with a stronger weight $|r\nabla\psi|$, plus further improvements,

depending on the Breitenlohner-Freedman bound. However, it will become clear that the weaker decay is sufficient to close the argument, even in the nonlinear case.

To control all missing spatial derivatives at general order, we require a four level hierarchy of L^2 estimates, which appear in Section 4.3.2. The first level (I) in the hierarchy controls general derivatives $D^\sigma \psi$ on the “far” region (where T is uniformly timelike). These elliptic estimates are stated in Section 4.3.2.1. Level (II) controls derivatives of the form $N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} D_x^2 \psi$, i.e., those which involve only two derivatives in general spatial directions, on the entire exterior region. These estimates are derived in Section 4.3.2.2. Level (III) contains the “near” region analogues of those estimates at level (I), and are derived in Section 4.3.2.3. Finally, the fourth level (IV) of L^2 estimates, appearing in Section 4.3.2.4, control general derivatives $D^\sigma \psi$ on the entire exterior region.

I Estimating $D^\sigma \psi$ on the “far” region

To construct the elliptic estimate of Proposition 4.3.7 (appearing in Section 4.3.2.1), we begin by estimating all second order spatial derivatives on the “far” region (captured by Proposition 4.3.6). To do so, we rewrite the equation (1.3.1) (in regular (t^*, r, θ, ϕ) coordinates) in the form

$$(\Delta + g^{rr} \partial_r^2 + 2g^{\phi r} \partial_\phi \partial_r) \psi = \square_g \psi + \alpha \psi + \dots, \quad (4.1.9)$$

where Δ denotes the Laplace-Beltrami operator of the spheres of constant (t^*, r) . The $\dots$ represent terms controlled by the first-order energy and the energy after one commutation with the Killing vector field T . On the “far” region, where T is timelike, the operator in brackets on the left-hand side of (4.1.9) is elliptic. Via a standard argument, this allows one to control all spatial derivatives on the “far” region by commuting with T and an estimate for $\mathcal{F}$ (see Proposition 4.3.6):

$$\begin{aligned} & \int_{\Sigma_{t^*}} \tilde{\xi}(r) \left(|\nabla^2 \psi|^2 + r^2 |\nabla \partial_r \psi|^2 + r^4 (\partial_r^2 \psi)^2 \right) r^2 dr d\omega \\ & \leq C \left(E[\psi](t^*) + E[T\psi](t^*) + \int_{\Sigma_{t^*}} (\square_g \psi + \alpha \psi)^2 r^2 dr d\omega \right), \end{aligned}$$

for some $C > 0$. Here, $\tilde{\xi}(r)$ is supported on the “far” region where T is timelike. Successively applying $\square_g$ to (4.1.12) and commuting sufficiently many times with T then allows one to control all derivatives on the “far” region in this manner (Proposition 4.3.7):

$$\begin{aligned} & \sum_{|\sigma|=k} \int_{\Sigma_{t^*}} \tilde{\xi}(r) r^{2\sigma_2} |\nabla^{\sigma_1} \partial_{t^*}^{\sigma_2} \partial_r^{\sigma_3} \psi|^2 r^2 dr d\omega \\ & \leq C \sum_{\substack{|\alpha| \leq k-1, \\ |\beta| \leq k-2, \\ |\gamma| \leq \lceil \frac{k-2}{2} \rceil}} \left[E[T^\alpha \psi](t^*) + \int_{\Sigma_{t^*}} \left(\left(T^\beta (\square_g \psi + \alpha \psi) \right)^2 + \left(\square_g^\gamma (\square_g \psi + \alpha \psi) \right)^2 \right) r^2 dr d\omega \right], \end{aligned} \quad (4.1.10)$$

for some $C > 0$, $k \geq 2$. The last two terms on the right-hand side of (4.1.10) will eventually be controlled by an estimate for $\mathcal{F}$.

II(A) Estimating $D_x^2\psi$ on the entire exterior

To construct the estimate of Proposition 4.3.9 (appearing in Section 4.3.2.2) for derivatives $N^{\sigma_1}T^{\sigma_2}\Phi^{\sigma_3}D_x^2\psi$, we begin by estimating all second order spatial derivatives $D_x^2\psi$. To do so, we first note that the second term on the left-hand side of (4.1.9) degenerates at the event horizon (where $g^{rr} = 0$). Furthermore, the third term carries a potentially bad sign which plays a role outside of the large- r regime. To deal with the latter, we rewrite (4.1.9) as

$$\Delta\psi + g^{rr}\partial_r^2\psi = \square_g\psi + \alpha\psi + \dots, \quad (4.1.11)$$

where the $\dots$ represent terms controlled by the first-order energy and the energy after one commutation with each of the Killing vector fields T , Φ . Away from the horizon (where $g^{rr} \neq 0$), (4.1.11) allows one to estimate all second order spatial derivatives by estimates for the nonlinearity $\mathcal{F}$ and the energy after one commutation with each of T , Φ . To counteract the degeneration at the horizon, we rewrite the $T^2\psi$ term on the right-hand side of (4.1.11) in terms of the redshift vector field N , which (close to the horizon) carries a non-degenerate ∂_r term:

$$N = \left(1 + \sqrt{-g^{t^*t^*}}\right)\partial_{t^*} - \frac{g^{t^*r}}{\sqrt{-g^{t^*t^*}}}\partial_r - \frac{g^{t^*\phi}}{\sqrt{-g^{t^*t^*}}}\partial_\phi.$$

This gives

$$\Delta\psi + \left(g^{rr} + \xi(r)\right)\partial_r^2\psi = \square_g\psi + \alpha\psi + \mathcal{O}\left(\frac{1}{r^2}\right)N^2\psi - \dots, \quad (4.1.12)$$

where $\xi(r)$ is a cut-off function supported on the horizon and up to a finite radius. The $\dots$, as before, represent terms controlled by the first-order energy and the energy after one commutation with each of T , Φ . Multiplying (4.1.12) by $\Delta\psi$, integrating by parts on the left and applying Cauchy-Schwarz on the right allows one to obtain control of $\nabla^2\psi$, $\nabla\partial_r\psi$, provided care is taken with the r -weights which appear. The first two terms on the right-hand side of (4.1.12) are controlled by estimates for the nonlinearity $\mathcal{F}$. This gives

$$\begin{aligned} & \int_{\Sigma_{t^*}} \left((\Delta\psi)^2 + r^2 |\nabla\partial_r\psi|^2 \right) r^2 dr d\omega \\ & \leq C \left[E[\psi](t^*) + E[T\psi](t^*) + E[N\psi](t^*) + E[\Phi\psi](t^*) + \int_{\Sigma_{t^*}} (\square_g\psi + \alpha\psi)^2 r^2 dr d\omega \right], \end{aligned} \quad (4.1.13)$$

for some $C > 0$. The integrand of the final term on the right-hand side of (4.1.13) will eventually be replaced by $\mathcal{F}^2$ and estimated. Estimate (4.1.13) does not close immediately due to the presence of the $E[N\psi]$ term. However, this is eventually controlled by estimates

for the N -commuted energy, which involve commutator terms supported on a compact r region (see Proposition 4.3.16). We have that

$$\begin{aligned}
& E[\psi](\tau) + E[T\psi](\tau) + E[N\psi](\tau) + E[\Phi\psi](\tau) \\
& \leq F_{\mathcal{H}^+ \cap [\tau, t_{\text{final}}^*]} + F_{\mathcal{I} \cap [\tau, t_{\text{final}}^*]} \\
& \quad + \int_{\tau}^{t_{\text{final}}^*} \left[C_k \left(E[\psi](t^{*'}) + E[T\psi](t^{*'}) + E[N\psi](t^{*'}) + E[\Phi\psi](t^{*'}) \right) \right. \\
& \quad \left. + \int_{\Sigma_{t^{*}}} \left(|[\square, N](\psi)| + \sum_{|\sigma| \leq 2} |N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} (N\psi \cdot \mathcal{F})| \right) r^2 dr d\omega \right] dt^{*'}, \quad (4.1.14)
\end{aligned}$$

where $E[\psi]$, $F_{\mathcal{H}^+ \cap [\tau, t_{\text{final}}^*]}$ and $F_{\mathcal{I} \cap [\tau, t_{\text{final}}^*]}$ are as above. Estimate (4.1.13) is then inserted on the right-hand side of (4.1.14) to control the commutator term $[\square, N](\psi)$ (which consists of general derivatives up to second order). An appropriate estimate for $\mathcal{F}$ is applied to the final term on the right-hand side of (4.1.14), and a Grönwall inequality is used (see the proof of Theorem 4.5.2) in the manner of (4.1.8), giving

$$E[\psi](\tau) + E[T\psi](\tau) + E[N\psi](\tau) + E[\Phi\psi](\tau) \leq \tilde{C}' \exp(-B \cdot \tau), \quad (4.1.15)$$

where $\tilde{C}'$ is strictly greater than the constant $\tilde{C}$ appearing in inequality (4.1.8). Estimate (4.1.15) now allows the second-order elliptic estimate (4.1.13) to close.

The second order radial derivative can then be controlled via the equation (4.1.12) itself, by virtue of the fact that we now control all other derivatives which appear therein. This yields (see Proposition 4.3.8)

$$\begin{aligned}
& \int_{\Sigma_{t^*}} \left((\Delta \psi)^2 + r^2 |\nabla \partial_r \psi|^2 + r^2 (\partial_r^2 \psi)^2 \right) r^2 dr d\omega \\
& \leq C \left[E[\psi](t^*) + E[T\psi](t^*) + E[N\psi](t^*) + E[\Phi\psi](t^*) + \int_{\Sigma_{t^*}} (\square_g \psi + \alpha \psi)^2 r^2 dr d\omega \right], \\
& \leq C \left[\exp(-B \cdot \tau) + \int_{\Sigma_{t^*}} (\square_g \psi + \alpha \psi)^2 r^2 dr d\omega \right], \quad (4.1.16)
\end{aligned}$$

for some $C > 0$.

II(B) Estimating $N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} D_x^2 \psi$ on the entire exterior

To produce the higher order estimates in this level of the hierarchy (Proposition 4.3.9), we successively commute (4.1.12) with N , T and Φ and repeat the same procedure. Since N does not commute with $\square_{g_{KAdS}}$, doing so generates additional terms at each step. However, these terms are always of lower order in N and involve at most two general derivatives, so are immediately controlled by lower order estimates. This gives (see Proposition 4.3.9)

$$\sum_{|\sigma|=k-2} \int_{\Sigma_{t^*}} \left((\Delta \Gamma^\sigma \psi)^2 + r^2 |\nabla \partial_r \Gamma^\sigma \psi|^2 + r^2 (\partial_r^2 \Gamma^\sigma \psi)^2 \right) r^2 dr d\omega$$

$$\leq C \sum_{\substack{|\alpha| \leq k-1, \\ |\beta| \leq k-2}} \left[E[\Gamma^\alpha \psi](t^*) + \int_{\Sigma_{t^*}} \left(\Gamma^\beta (\square_g \psi + \alpha \psi) \right)^2 r^2 dr d\omega \right], \quad (4.1.17)$$

for some $C > 0$, $k \geq 2$, where $\Gamma^\sigma = N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3}$.

III(A) Estimating $N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} \partial_\theta^2 \psi$ on the “near” region

Before estimating all general derivatives at level **III** (Theorem 4.3.10), as is carried out in Section 4.3.2.3, we note that (4.1.17) implies the estimate

$$\begin{aligned} & \int_{\Sigma_{t^*}} \sum_{|\sigma|=k-2} \xi(r) (\partial_{t^*}^{\sigma_1} \partial_r^{\sigma_2} \partial_\phi^{\sigma_3} \partial_\theta^2 \psi)^2 r^2 dr d\omega \\ & \leq C \sum_{\substack{|\alpha| \leq k-1, \\ |\beta| \leq k-2}} \left[E[\Gamma^\alpha \psi](t^*) + \int_{\Sigma_{t^*}} \left(\Gamma^\beta (\square_g \psi + \alpha \psi) \right)^2 r^2 dr d\omega \right], \end{aligned} \quad (4.1.18)$$

for derivatives of second order in ∂_θ . Here, $\xi(r)$ (as above) is supported on the “near” region, where N has a non-vanishing r -component. As such, it only remains to control derivatives $N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} \partial_\theta^{\sigma_4} \psi$ for $\sigma_4 > 2$.

III(B) Estimating $D^\sigma \psi$ on the “near” region

Let us demonstrate how to estimate $\partial_\theta^3 \psi$, $\partial_\theta^4 \psi$.

Rearranging (4.1.9) so that only terms involving ∂_θ derivatives remain on the left-hand side, one has

$$\frac{g^{\theta\theta}}{\sqrt{|\det g|}} \partial_\theta (\sqrt{|\det g|}) \partial_\theta \psi + \partial_\theta (g^{\theta\theta}) \partial_\theta \psi + \mathbb{A} \psi = \square_g \psi + \alpha \psi + \dots, \quad (4.1.19)$$

where $\dots$ represent terms controlled on the “near” region by the first-order energy and estimate (4.1.18). Applying $\mathbb{A}$ to (4.1.19) gives

$$\mathbb{A} \left(\frac{g^{\theta\theta}}{\sqrt{|\det g|}} \partial_\theta (\sqrt{|\det g|}) \partial_\theta \psi + \partial_\theta (g^{\theta\theta}) \partial_\theta \psi + \mathbb{A} \psi \right) = \mathbb{A} (\square_g \psi + \alpha \psi) + \dots \quad (4.1.20)$$

The $\dots$ on the right-hand side of (4.1.20) involve terms with derivatives of at most second order in θ , and so are controlled by (4.1.18). The first term on the right-hand side of (4.1.20) will eventually be controlled via an estimate for $\mathcal{F}$. Multiplying (4.1.20) by $\mathbb{A} \psi$, applying the Cauchy-Schwarz inequality to the right-hand side and integrating the $\mathbb{A}^2 \psi \mathbb{A} \psi$ term on the left-hand side by parts in the angular directions gives control of the $\partial_\theta^3 \psi$ derivative. The other terms on the left-hand side of (4.1.20) can be absorbed via ε -Cauchy-Schwarz. This gives the estimate

$$\int_{\Sigma_{t^*}} \xi(r) |\nabla^3 \psi|^2 r^2 dr d\omega$$

$$\leq C \sum_{\substack{|\alpha| \leq 3, \\ |\beta| \leq 2}} \left[E[\Gamma^\alpha \psi](t^*) + \int_{\Sigma_{t^*}} \left(\left(\Gamma^\beta(\square_g \psi + \alpha \psi) \right)^2 + \left(\Delta(\square_g \psi + \alpha \psi) \right)^2 \right) r^2 dr d\omega \right],$$

for some $C > 0$.

Repeating with the higher order multiplier $\Delta^2 \psi$ yields control of $\partial_\theta^4 \psi$:

$$\begin{aligned} & \int_{\Sigma_{t^*}} \xi(r) |\nabla^4 \psi|^2 r^2 dr d\omega \\ & \leq C \sum_{\substack{|\alpha| \leq 3, \\ |\beta| \leq 2}} \left[E[\Gamma^\alpha \psi](t^*) + \int_{\Sigma_{t^*}} \left(\left(\Gamma^\beta(\square_g \psi + \alpha \psi) \right)^2 + \left(\Delta(\square_g \psi + \alpha \psi) \right)^2 \right) r^2 dr d\omega \right], \end{aligned}$$

for some $C > 0$.

By successively applying Δ and higher-order $\Delta \psi$ multipliers, one controls all ∂_θ^σ derivatives. Commuting with T , N and Φ then gives (Theorem 4.3.10)

$$\begin{aligned} & \sum_{|\sigma|=k} \int_{\Sigma_{t^*}} \xi(r) r^{2\sigma_2} |\nabla^{\sigma_1} \partial_{t^*}^{\sigma_2} \partial_r^{\sigma_3} \psi|^2 r^2 dr d\omega \\ & \leq C \sum_{\substack{|\alpha| \leq k-1, \\ |\beta| \leq k-2, \\ |\gamma| \leq \lceil \frac{k-2}{2} \rceil}} \left[E[\Gamma^\alpha \psi](t^*) + \int_{\Sigma_{t^*}} \left(\left(\Gamma^\beta(\square_g \psi + \alpha \psi) \right)^2 + \left(\Delta^\gamma(\square_g \psi + \alpha \psi) \right)^2 \right) r^2 dr d\omega \right] \end{aligned} \tag{4.1.21}$$

for some $C > 0$, $k \geq 2$.

IV Estimating $D^\sigma \psi$ on the entire exterior

Finally, combining estimates (4.1.10) and (4.1.21) gives control of general derivatives on the entire exterior, as appears in Section 4.3.2.4 (Theorem 4.3.12):

$$\begin{aligned} & \sum_{|\sigma|=k} \int_{\Sigma_{t^*}} r^{2\sigma_2} |\nabla^{\sigma_1} \partial_{t^*}^{\sigma_2} \partial_r^{\sigma_3} \psi|^2 r^2 dr d\omega \\ & \leq C \sum_{\substack{|\alpha| \leq k-1, \\ |\beta| \leq k-2, \\ |\gamma| \leq \lceil \frac{k-2}{2} \rceil}} \left[E[\Gamma^\alpha \psi](t^*) + \int_{\Sigma_{t^*}} \left(\left(\Gamma^\beta(\square_g \psi + \alpha \psi) \right)^2 + \left(\square_g^\gamma(\square_g \psi + \alpha \psi) \right)^2 \right) r^2 dr d\omega \right], \end{aligned}$$

for some $C > 0$, $k \geq 2$.

4.1.3.3 The nonlinear problem

Let us now turn to the full nonlinear problem. We once again consider a finite-in-time approximation of the full nonlinear problem, analogous to (4.1.5) with a non-zero nonlinearity

$\mathcal{F}$ on the right-hand side of the equation. This forms part of an iteration scheme (see Section 4.5.1) whose limit will provide the desired result for the full problem. Local well-posedness for the finite-in-time nonlinear problems will not be proven here. However, it follows from an adaptation of the linear well-posedness result [33], or an argument similar to that used to prove Theorem 3.4.1 on AdS space.

The key ingredient of the proof of Theorem 4.1.1 is an energy estimate, in the spirit of (4.1.6), but now with an additional term arising from the nonlinearity $\mathcal{F}$:

$$E[\psi](\tau) \leq F_{\mathcal{H}^+ \cap [\tau, t_{\text{final}}^*]} + \int_{\tau}^{t_{\text{final}}^*} \left(C_{\kappa} E[\psi](t^{*'}) + \int_{\Sigma_{t^{*}}} |\mathcal{F}(\psi, \partial\psi) \cdot N\psi| \text{vol}_{\Sigma_{t^{*}}} \right) dt^{*'}. \quad (4.1.22)$$

In order to proceed as before, using the assumed exponential decay and a Grönwall inequality to derive exponential decay of $E[\psi](\tau)$, one must be able to bound this new term appropriately. Given that no null condition is required, this is easily insured, provided the general Condition 4.2.4 is satisfied. In particular, one must ensure that an appropriately r -weighted spatial L^2 norm of $\mathcal{F}$ and its derivatives is well-controlled by the energies (4.2.12). For simplicity it is assumed (assumption (4.2.14)) that $\mathcal{F}$ is quadratic in the unit derivatives $D\psi$ (4.2.7). It is shown in Proposition 4.3.17 that this class of $\mathcal{F}$ indeed satisfies Condition 4.2.4. The restriction on $\mathcal{F}$ could be weakened to allow for additional growth in r by commuting with angular momentum operators to deduce stronger radial decay, as discussed earlier. However, the class satisfying (4.2.14) can be readily adapted without additional decay to the key application of interest, the Einstein vacuum equation in an appropriate gauge. This is discussed further in Section 4.2.3.2. Condition 4.2.4 easily allows for more general nonlinearities than those of the assumed form (4.2.14), including higher order polynomials of $D\psi$, as well as additional linear terms.

Provided $\mathcal{F}$ satisfies appropriate assumptions (see Condition 4.2.4), one may then proceed as in the linear setting, commuting (4.1.22) with T and the redshift vector field N (Section 4.3.4) and applying a Grönwall inequality to derive exponential decay of the energies $E[T^{\alpha} N^{\beta} \psi]$ to sufficiently high order (Section 4.5.3). As was the case in the linear setting, one must employ elliptic estimates and the equation to control the “missing” angular and radial derivatives (as is done in Section 4.3.2).

4.1.4 Relation to the Einstein equation

It remains to present the problem in its wider context. The decay behaviour of solutions of (4.1.1) is closely related to the question of stability of the Kerr-AdS spacetimes as solutions of the *Einstein vacuum equation* (EVE)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \quad (4.1.23)$$

with negative cosmological constant $\Lambda = -\frac{3}{\ell^2}$ and the *black hole stability problem*, more generally. In an appropriate choice of coordinates, (4.1.23) can be written as a system of wave equations. As such, it is expected that Theorem 4.1.1 will play a central role in

deducing existence of a class of exponentially decaying gravitational perturbations of the Kerr-AdS black hole exterior. The technical details of applying Theorem 4.1.1 to (4.1.23) will be carried out in subsequent work. In order to move from Theorem 4.1.1 to an analogous statement in full gravity, one must deal with two concerns - firstly, the *quasilinear* nature of (4.1.23); secondly, its *gauge-invariance*. The first concern is straightforward to address - Theorem 4.1.1 is written as a statement for *semilinear* equations, purely for the purpose of simplicity. One may easily extend to quasilinear equations mimicking the structure of (4.1.23), as one need not capture a null condition in the nonlinearity. The insensitivity of the scattering construction to trapping means, moreover, that no additional difficulties arise from treating quasilinear nonlinearities. The key difficulty of working with (4.1.23) is rather the second concern - that of fixing a gauge. There are many possible choices of gauge in which to work with (4.1.23), *harmonic* and *double-null gauge* being two commonly used in the literature. Let us address each of these in turn, highlighting their advantages and disadvantages, and the degree to which they translate (4.1.23) into a system of equations treatable via the methods appearing in this work.

In *harmonic gauge*, (4.1.23) takes the form of a system of quasilinear massive wave equations, as derived by Enciso and Kamran [18] in their well-posedness result for (4.1.23) in the asymptotically AdS setting. There, one sees that the system carries little in the way of geometric structure and admits multiple different masses α . Each of these masses lies in the range to which Theorem 4.1.1 applies (see the discussion in Section 4.2.3.2).⁶ A drawback of harmonic gauge particular to the scattering problem is that it is not well-adapted to characteristic hypersurfaces on which one would like to pose data.

On the other hand, a *double-null gauge* is, by design, suited to characteristic hypersurfaces. Under this choice of gauge, (4.1.23) becomes a system of wave and transport equations for spacetime null curvature components. Although the system of equations in double-null gauge is considerably more cumbersome than in the harmonic gauge case, one has the aesthetic advantage of wave equations admitting only the conformal mass $\alpha = 2$ in $(3 + 1)$ -dimensions. In linearised gravity with double-null gauge, two gauge-independent curvature components, $\tilde{\alpha}^{[s]}$ for $s = \pm 2$, satisfy the Teukolsky equations⁷ - wave equations of the form (1.3.1) with masses $\alpha = 2$.

Given that one expects, via an appropriate choice of gauge, that Theorem 4.1.1 can be applied to (4.1.23) to obtain a class of exponentially decaying gravitational perturbations of Kerr-AdS spacetimes, let us now address the wider context of the black hole stability problem. Moschidis [47, 48] showed instability of pure AdS in the context of (4.1.23) in spherical symmetry coupled to appropriate matter models. Previously, Bizón and Rostworowski [2] derived an instability mechanism based on resonant frequencies and found instability for the spherically symmetric scalar field using numerical methods. Moreover, Theorem 4.1.2 suggests that general perturbations of Kerr-AdS spacetimes decay insufficiently fast to allow one to prove nonlinear stability. Thus the authors of [36] conjecture nonlinear instability⁸

⁶Furthermore, note that the Sobolev spaces used herein are mass independent, so treating multiple masses is straightforward.

⁷One may see [27], for example, for work in this direction in the Kerr-AdS setting.

⁸Note, however, that *linear* stability of the Schwarzschild-AdS subfamily has been proven [28–30].

of the Kerr-AdS exterior as a solution of (4.1.23) (see also, however, the discussion in [14]). As such, the class of exponentially decaying perturbations suggested by Theorem 4.1.1 are expectedly non-generic.

4.1.5 Outline

This chapter has the following structure. Section 4.2 introduces the Kerr-AdS spacetime metric and associated quantities, as well as some vector fields which will be key to proving Theorem 4.1.1. The precise class of wave equations (1.3.1) considered and associated boundary conditions are then specified. Finally, the energy spaces in which the analysis is carried out are defined, and some important energy quantities recalled. Section 4.3 presents several physical space estimates which will be key for the proof. In Section 4.4, the main result (formulated informally in Theorem 4.1.1) is presented rigorously as Theorem 4.4.2. Section 4.5 is concerned with the proof of Theorem 4.4.2.

4.2 Preliminaries

Before presenting the main result, the set up and key quantities are noted here. Firstly, in Section 4.2.1 the Kerr-AdS metric and relevant spacetime vector fields are introduced. The equation, associated boundary conditions and scattering problem formulation follow in Section 4.2.3. In particular, the class of nonlinearity $\mathcal{F}$ in (1.3.1) is specified at this stage. Section 4.2.2 presents the energy spaces in which solutions of the scattering problem will be analysed.

4.2.1 The Kerr-AdS family of spacetimes

4.2.1.1 The spacetime manifold and metric

For fixed negative cosmological constant $\Lambda = -\frac{3}{\ell^2}$, the *Kerr-Anti-de Sitter* (Kerr-AdS) spacetimes are a two-parameter family of solutions $(\mathcal{M}', g')$ of the Einstein vacuum equation (4.1.23). Each fixed choice of mass $M > 0$ and each angular momentum a such that (1.3.4) holds, where r_+ (the largest real root of Δ_- , see (4.2.2) below) is the event horizon radius, give a stationary, axisymmetric, rotating black hole with metric

$$\begin{aligned}
g = & -\frac{\Delta_- - \Delta_\theta a^2 \sin^2 \theta}{\Sigma} dt^{*2} + \frac{1}{\Sigma(1 + \frac{r^2}{\ell^2})^2} \left(\Delta_+ - a^2 \sin^2 \theta \left(\left(1 + \frac{r^2}{\ell^2}\right) + \frac{\Sigma}{\ell^2} \right) \right) dr^2 + \frac{\Sigma}{\Delta_\theta} d\theta^2 \\
& + \frac{\Delta_\theta(r^2 + a^2)^2 - \Delta_- a^2 \sin^2 \theta}{\Xi^2 \Sigma} \sin^2 \theta d\phi^2 - 2 \frac{\Delta_\theta(r^2 + a^2)^2 - \Delta_-}{\Xi \Sigma} a \sin^2 \theta d\phi dt^* \\
& + \frac{2}{(1 + \frac{r^2}{\ell^2})} \left(\frac{2Mr}{\Sigma} - \frac{a^2}{\ell^2} \sin^2 \theta \right) dt^* dr - \frac{2a \sin^2 \theta}{(1 + \frac{r^2}{\ell^2})} \left(1 + \frac{2Mr}{\Xi \Sigma} \right) d\phi dr
\end{aligned} \tag{4.2.1}$$

in coordinates (t^*, r, θ, ϕ) on the *exterior region* $\mathcal{M} = \mathbb{R} \times [r_+, \infty) \times S^2$. Here,

$$\begin{aligned}\Delta_{\pm} &= (r^2 + a^2) \left(1 + \frac{r^2}{\ell^2} \right) \pm 2Mr & \Sigma &= r^2 + a^2 \cos^2 \theta \\ \Delta_{\theta} &= 1 - \frac{a^2}{\ell^2} \cos^2 \theta & \Xi &= 1 - \frac{a^2}{\ell^2},\end{aligned}\tag{4.2.2}$$

One can check that *Anti-de Sitter* space and the one-parameter *Schwarzschild-Anti-de Sitter* family arise as special cases for the choices $a = M = 0$, or $a = 0$, $M > 0$, respectively.

We shall denote by $S_{t^*r}^2$ the sphere at a given time t^* and radius r , with induced metric

$$g = \frac{\Sigma}{\Delta_{\theta}} d\theta^2 + \frac{\Delta_{\theta}(r^2 + a^2)^2 - \Delta_- a^2 \sin^2 \theta}{\Xi^2 \Sigma} \sin^2 \theta d\phi^2.$$

The spacelike hypersurfaces Σ_{t^*} of constant t^* foliate the exterior region $\{r \geq r_+\}$, depicted in Figure 4.1. We shall denote by $n_{\Sigma_{t^*}}$ their future-directed unit normal vector

$$n_{\Sigma_{t^*}} = \sqrt{-g^{t^*t^*}} \partial_{t^*} - \frac{g^{t^*r}}{\sqrt{-g^{t^*t^*}}} \partial_r - \frac{g^{t^*\phi}}{\sqrt{-g^{t^*t^*}}} \partial_{\phi}\tag{4.2.3}$$

and by $\text{vol}_{\Sigma_{t^*}}$ their induced volume form

$$\text{vol}_{\Sigma_{t^*}} = \frac{\Sigma}{\Xi} \sin \theta \sqrt{-g^{t^*t^*}} dr d\theta d\phi.$$

We collect here the asymptotic (in r) behaviour of the inverse metric components $g^{\mu\nu}$

$$\begin{aligned}g^{t^*t^*} &= \mathcal{O}\left(\frac{1}{r^2}\right), & g^{rr} &= \mathcal{O}(r^2), & g^{\theta\theta} &= \mathcal{O}\left(\frac{1}{r^2}\right), & g^{\phi\phi} &= \mathcal{O}\left(\frac{1}{r^2}\right) \\ g^{t^*r} &= \mathcal{O}\left(\frac{1}{r^3}\right), & g^{t^*\phi} &= \mathcal{O}\left(\frac{1}{r^2}\right), & g^{r\phi} &= \mathcal{O}\left(\frac{1}{r^2}\right)\end{aligned}\tag{4.2.4}$$

and determinant

$$\det g = \mathcal{O}(r^4)\tag{4.2.5}$$

which will be applied in later estimates where it is key that r -weights do not become too large at infinity.

One may alternatively employ *Boyer-Lindquist* coordinates $(t, r, \theta, \tilde{\phi})$, defined implicitly by the relations

$$\frac{dt^*}{dr} = \frac{2Mr}{\Delta_-(1 + \frac{r^2}{\ell^2})} \quad \frac{d\phi}{dr} = \frac{a\Xi}{\Delta_-} \quad \frac{dt^*}{dt} = 1 \quad \frac{d\phi}{d\tilde{\phi}} = 1.$$

Then the metric (4.2.1) takes its, perhaps, more familiar form (1.3.5). However, since Boyer-Lindquist coordinates become irregular at the event horizon $\mathcal{H}^+$, we shall work in (t^*, r, θ, ϕ) from this point onwards.

4.2.1.2 Vector fields

One immediately obtains from (4.2.1) that the coordinate vector fields $T = \partial_{t^*}$, $\Phi = \partial_\phi$ are Killing. However, T is not uniformly timelike in the black hole exterior. As such, we will use a uniformly timelike vector field

$$N_{KAdS} = T + \xi(r)Y, \quad \xi(r) = \begin{cases} 1 & \text{on } r \leq r_0 \\ 0 & \text{on } r \geq r_1 \end{cases}, \quad r_+ < r_0 < r_1 < \infty \quad (4.2.6)$$

to obtain coercive energy estimates. Otherwise, control of derivatives transverse to the horizon degenerates. Throughout, r_0 is a fixed radius such that T is uniformly timelike on $r \geq r_0$, $r_1 > r_0$ is a sufficiently large (depending on a) fixed radius (see Lemma 4.3.2) and $\xi(r)$ is a positive function, interpolating smoothly between its two cases. For simplicity, we choose Y to be the future-directed unit normal (4.2.3) to the spacelike hypersurfaces Σ_{t^*} .

Let us now define the set of *commuting vector fields*

$$\Gamma_{KAdS} \in \{N_{KAdS}, T, \Phi\}.$$

From now on, we drop the ‘KAdS’ subscript. We shall denote by $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ a multi-index with entries σ_i , $i \in \mathbb{N}$, and length $|\sigma| = \sum_i \sigma_i$. A general concatenation of the Γ -vector fields will be written as

$$\Gamma^\sigma = N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3}.$$

These will be applied to the equation (1.3.1) in order to generate estimates for the solution ψ to sufficiently high-order.

Moreover, define the sets of normalised spacetime derivatives

$$D \in \left\{ \frac{1}{r} \partial_{t^*}, r \partial_r, \frac{1}{r} \partial_\theta \psi, \frac{1}{r} \partial_\phi \psi \right\}, \quad \bar{D} \in \left\{ \partial_{t^*}, r \partial_r, \frac{1}{r} \partial_\theta \psi, \frac{1}{r} \partial_\phi \psi \right\} \quad (4.2.7)$$

in terms of which the definition of the class of nonlinearity (Section 4.2.3.2), the Sobolev embedding Theorem 4.3.14 and Theorems 4.4.2-4.5.2 will be stated. General concatenations of these derivatives will be denoted in a similar manner as for the commuting vector fields, with multi-index $\sigma = (\sigma_0, \dots, \sigma_3)$. For example,

$$D^\sigma = \left(\frac{1}{r} \partial_{t^*} \right)^{\sigma_0} (r \partial_r)^{\sigma_1} \left(\frac{1}{r} \partial_\theta \right)^{\sigma_2} \left(\frac{1}{r} \partial_\phi \right)^{\sigma_3}.$$

We denote the purely spatial normalised derivatives $r \partial_r$, $\frac{1}{r} \partial_\theta$ and $\frac{1}{r} \partial_\phi$ by D_x .

4.2.2 The energies

We define the following weighted, non-degenerate energies on Kerr-AdS.⁹ It will become clear that these arise when integrating the energy identity (A.0.1) with $X = N$ and its commutations over an appropriate spacetime region.

⁹These are equivalent to the Cartesian “twisted” energies appearing in [18] when one restricts to Dirichlet boundary conditions.

Definition 4.2.1 (Sobolev norms on spacelike slices of KAdS). Let t^* be a fixed time. Then, for each $k \in \mathbb{N}$ we define the Sobolev space

$$H_{KAdS}^k(\Sigma_{t^*}) = \{\psi : [r_+, \infty) \times S_{t^*,r}^2 \rightarrow \mathbb{R} \mid \|\psi\|_{H_{KAdS}^k(\Sigma_{t^*})}^2 < \infty\}$$

of scalar fields ψ on Σ_{t^*} associated with the energy

$$\|\psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 = \int_{\Sigma_{t^*}} \left[\frac{1}{r^2} (\partial_{t^*} \psi)^2 + r^2 (\partial_r \psi)^2 + |\nabla \psi|^2 + \psi^2 \right] r^2 dr d\omega \quad (4.2.8)$$

and, for $k \geq 2$,

$$\|\psi\|_{H_{KAdS}^k(\Sigma_{t^*})}^2 = \int_{\Sigma_{t^*}} \left[\sum_{\substack{|\sigma| \leq k-1, \\ |\rho| \leq k, \rho_1 \leq k-1}} \left(r^{2\sigma_3} |\nabla^{1+\sigma_1} \partial_{t^*}^{\sigma_2} \partial_r^{\sigma_3} \psi|^2 + r^{2\rho_2} (\partial_{t^*}^{\rho_1} \partial_r^{\rho_2} \psi)^2 \right) + \frac{1}{r^2} (\partial_{t^*}^k \psi)^2 \right] r^2 dr d\omega. \quad (4.2.9)$$

We also define the following L^p -norms on spacelike slices of Kerr-AdS.

Definition 4.2.2 (L^p -norms on spacelike slices of KAdS). Let $t^* \in \mathbb{R}^+$ be a fixed time. Then for $p \in \{2, \infty\}$ we define the spaces

$$L_{KAdS}^p(\Sigma_{t^*}) = \{\psi : [r_+, \infty) \times S_{t^*,r}^2 \rightarrow \mathbb{R} \mid \|\psi\|_{L_{KAdS}^p(\Sigma_{t^*})}^2 < \infty\}$$

of scalar fields ψ on Σ_{t^*} associated with the norms

$$\|\psi\|_{L_{KAdS}^2(\Sigma_{t^*})}^2 = \int_{\Sigma_{t^*}} |\psi|^2 r^2 dr d\omega, \quad \|\psi\|_{L_{KAdS}^\infty(\Sigma_{t^*})} = \sup_{\Sigma_{t^*}} |\psi|.$$

In terms of these Sobolev and L^p spaces, we define the following classes of functions of time and space.

Definition 4.2.3 (Function spaces of the solution on KAdS). By CH_{KAdS}^k , we denote the function space

$$CH_{KAdS}^k = \left\{ \psi : \mathcal{M} \rightarrow \mathbb{R} \mid \psi \in \bigcap_{q=0}^{k-1} C^q(\mathbb{R}_{t^*}; H_{KAdS}^{k-q}(\Sigma_{t^*})) \text{ and } \frac{1}{r} \psi \in C^k(\mathbb{R}_{t^*}; L_{KAdS}^2(\Sigma_{t^*})) \right\}$$

of scalar fields ψ on $\mathcal{M}$.

4.2.3 A class of nonlinear wave equations

In this chapter, we study the nonlinear wave equation¹⁰

$$\square_g \psi + \alpha \psi = \mathcal{F}, \quad \alpha < \frac{9}{4}, \quad (4.2.10)$$

¹⁰The restriction $\alpha < \frac{9}{4}$ is known as the *Breitenlohner-Freedman* bound. This assumption ensures that (4.2.10) is well-posed, given appropriate initial and boundary conditions, in the general asymptotically-AdS setting [33].

for $\psi \in CH_{KAdS}^k(\Sigma_{t^*})$ with fixed $k \geq 9$ where $\mathcal{F}$ is a smooth function of the spacetime coordinates, the solution ψ and its normalised first-order derivatives $D\psi$ (4.2.7). In the following sections, we introduce appropriate boundary conditions before discussing the assumptions $\mathcal{F}$ satisfies and how this relates to the Einstein vacuum equation in harmonic gauge. We then state the scattering problem to be studied.

4.2.3.1 Boundary conditions at future null infinity

Given that the Kerr-AdS spacetime is not globally hyperbolic, equation (4.2.10) must be paired with an appropriate boundary condition along the timelike conformal boundary $\mathcal{I}$. In this chapter, we restrict our attention to a particular choice in the reflective category, Dirichlet boundary conditions:¹¹

$$r^{\frac{3}{2}-s}\psi|_{\mathcal{I}} = 0, \quad s = \sqrt{\frac{9}{4} - \alpha}, \quad (4.2.11)$$

where we write $r^{\frac{3}{2}-s}\psi|_{\mathcal{I}}$ to mean the limit

$$\lim_{r \rightarrow \infty} r^{\frac{3}{2}-s}\psi(t^*, r, \theta, \phi)$$

for each fixed (t^*, θ, ϕ) .

The methods used herein easily extend to include Neumann or Robin boundary conditions. However, one must use the formalism of *twisted* derivatives introduced by Warnick [54]. Dirichlet, Neumann and Robin boundary conditions were treated in this manner in the linear setting by Holzegel and Warnick [40].

It is worth noting that, since we do not commute with angular momentum operators (and therefore only show $r^{-\frac{1}{2}}$ decay of ψ , not the $r^{-\frac{3}{2}-s}$ decay of Theorem 3.3.6), we do not explicitly prove here that ψ attains this boundary condition. However, this is known from the well-posedness theory where angular momentum operators are utilised [34].

4.2.3.2 A general nonlinearity involving derivatives to first-order

In order to close the estimates appearing in this chapter, we require a general condition on the nonlinearity $\mathcal{F}$ (similar to Condition 3.2.4) on AdS space). This is encoded in the following estimates.

Condition 4.2.4 (The required estimates for $\mathcal{F}$ on KAdS). The right-hand side $\mathcal{F}$ of equation (4.2.10) must satisfy

$$\begin{aligned} \sum_{|\sigma| \leq k-1} \|N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} \mathcal{F}\|_{L^2_{KAdS}(\Sigma_{t^*})} &\leq C_k \|\psi\|_{H^k_{KAdS}(\Sigma_{t^*})} \left(\|\psi\|_{H^{k-1}_{KAdS}(\Sigma_{t^*})} \right)^n, \\ \sum_{|\gamma| \leq \lceil \frac{k-2}{2} \rceil} \|\square_g^\gamma \mathcal{F}\|_{L^2_{KAdS}(\Sigma_{t^*})}^2 &\leq \tilde{C}_k \|\psi\|_{H^k_{KAdS}(\Sigma_{t^*})} \left(\|\psi\|_{H^{k-1}_{KAdS}(\Sigma_{t^*})} \right)^p \end{aligned} \quad (4.2.12)$$

¹¹Note, the boundary condition is only really necessary in the range $\frac{5}{4} < \alpha < \frac{9}{4}$ where it excludes the Neumann branch of the solution. Otherwise, an assumption of finite energy suffices.

for some $k \geq 9$, $C_k, \tilde{C}_k > 0$ and $n, p \geq 0$.

Condition 4.2.4 allows for a very general class of $\mathcal{F}$. However, for simplicity, we shall restrict our attention to a nonlinearity which is quadratic in the first order normalised derivatives $D\psi$ (4.2.7) of the solution. In terms of the frame

$$e_0 = \frac{1}{r}\partial_{t^*}, \quad e_1 = r\partial_r, \quad e_2 = \frac{1}{r}\partial_\theta, \quad e_3 = \frac{1}{r}\partial_\phi, \quad (4.2.13)$$

we shall write

$$\mathcal{F} = F^{\mu\nu}(t^*, r, \theta, \phi) \cdot e_\mu \psi e_\nu \psi. \quad (4.2.14)$$

Here, $F : \mathcal{M} \rightarrow T\mathcal{M} \otimes T\mathcal{M}$ is a $(0, 2)$ -tensor with components $F_{\mu\nu} : \mathcal{M} \rightarrow \mathbb{R}$ which are smooth, bounded (with respect to the frame e_i) functions¹² with uniformly bounded T , N , Φ , e_1 and e_2 derivatives up to order $k - 1$. By Proposition 4.3.17 below, the nonlinearity (4.2.14) satisfies Condition 4.2.4 with $n = 1$, $p = 3$.

It is worth noting that the restriction to nonlinearities of the form (4.2.14) is not necessary to close the argument of this chapter. One can readily extend to include quasilinear equations, as well as more general $\mathcal{F}$ satisfying the obvious analogue of Condition 4.2.4. This includes, for example, higher order polynomials in $D\psi$, as well as additional linear terms with suitable asymptotics. Furthermore, one can treat systems

$$\square_g \psi_j + \alpha_j \psi_j = \mathcal{F}_j, \quad \alpha_j < \frac{9}{4}, \quad j = 1, \dots, \ell$$

of equations with the $\mathcal{F}_j$ satisfying the obvious analogue of Condition 4.2.4. Notably, one can also allow for $\mathcal{F}$ to have some additional growth in r by bootstrapping stronger r -weighted norms of ψ in the proof. This can be done (although we will not do this here) via commuting with angular momentum operators and performing an additional round of elliptic estimates.

The Einstein equations in harmonic gauge The generality of Condition 4.2.4 will, in particular, allow for treatment of the Einstein vacuum equation in later work. To see that this is possible, in principle, let us consider the Einstein equation in harmonic gauge. In [18], the Einstein vacuum equation in harmonic gauge is written symbolically, in (t, x, θ, ϕ) coordinates, in terms of a conformally rescaled metric $\bar{g} = x^2 g$ ($\{x = 0\}$ corresponds to $\mathcal{I}$) as

$$Q(g) = \bar{g}^{-1} \partial^2 \bar{g} + a_0(\bar{g}) \partial \bar{g} \partial \bar{g} + \frac{a_1(\bar{g})}{x} \partial \bar{g} + \frac{a_2(\bar{g})}{x^2} = 0.$$

Here, the $a_i(\bar{g})$ are smooth functions of x and $\bar{g}$. Due to the blow-up of some terms in the above towards the conformal boundary, the authors perform a technical process of renormalisation to obtain a system of nonlinear wave equations which can be treated via an earlier result for scalar waves [17]. The solution is written as a sum

$$g = \gamma + h, \quad h = x^{\frac{3}{2}} u$$

¹²This is modulo the standard degeneration of the frame $\frac{1}{r}\partial_\theta, \frac{1}{r}\partial_\phi$ on S^2 .

of terms γ and h which contain the most (respectively least) singular behaviour at the boundary. The authors construct the approximate solution γ of $Q(\gamma) = 0$ algebraically from the boundary data, iteratively constructing γ_ℓ such that $Q(\gamma_\ell) = \mathcal{O}(x^{\ell-1})$ and setting $\gamma = \gamma_\ell$ for ℓ sufficiently large. After fixing γ , they solve a tensorial equation

$$P_g u = -\frac{1}{x^{\frac{7}{2}}} Q(\gamma) + \frac{F(\bar{g})u}{x} + \int_0^1 x^{\frac{3}{2}} B(u, x\partial u) d\sigma \quad (4.2.15)$$

for u . Here, $F(\bar{g})$ is smooth and B is quadratic, with smooth coefficients depending on $x^2\gamma + \sigma x^{\frac{7}{2}}u$. The principal part of P_g is the wave operator $\square_{\bar{g}}$ of the rescaled metric. Having fixed $\gamma = \gamma_\ell$ for large enough ℓ , the highly singular weight in the first term is counteracted. This problem reduces to first ignoring the tensorial nature of equation (4.2.15) and solving a system of scalar equations

$$L_{g,\alpha_j} u_j = \mathcal{N}_j, \quad j = 0, \dots, 3. \quad (4.2.16)$$

Here, the left-hand side $L_{g,\alpha_j} u_j$ is related to a scalar wave equation with mass¹³ $\alpha_j \leq 0$ via

$$\square_g \psi_j + \alpha_j \psi_j = [1 + \mathcal{O}(x^2)] x^{\frac{7}{2}} L_{g,\alpha_j} u_j, \quad \psi_j = x^{\frac{3}{2}} u_j \quad (4.2.17)$$

and the right-hand side $\mathcal{N}_j$ schematically looks like

$$\mathcal{N}_j = x^{\frac{3}{2}} u_j^2 + x^{\frac{5}{2}} u_j \partial u_j + x^{\frac{7}{2}} (\partial u_j)^2. \quad (4.2.18)$$

This gives, via relation (4.2.17), a system of scalar wave equations

$$\square_g \psi_j + \alpha_j \psi_j = x^2 \psi_j^2 + x^3 \psi_j \partial \psi_j + x^4 (\partial \psi_j)^2 + \text{l.o.t.} \quad (4.2.19)$$

for ψ_j .

Local well-posedness of scalar wave equations of the form (4.2.10) with quadratic nonlinearities $\mathcal{F}$ akin to the quadratic (in ψ_j) term on the right-hand side of (4.2.19) was proven in an earlier paper of the authors [17]. In particular, in that setting, equation (4.2.10) with

$$\mathcal{F} = F^{\mu\nu}(t^*, x, \theta, \phi) \cdot \tilde{e}_\mu \psi \tilde{e}_\nu \psi, \quad F^{\mu\nu} = x^q \tilde{F}^{\mu\nu}(t^*, x, \theta, \phi), \quad q \geq 2 + \sqrt{\frac{9}{4} - \alpha}, \quad (4.2.20)$$

was considered. Here $\tilde{F}^{\mu\nu} : \mathcal{M} \rightarrow \mathbb{R}$ are smooth, bounded functions and $\tilde{e}_i$ is the frame analogous to (4.2.13) in (t^*, x, θ, ϕ) coordinates. Since the masses α_j of the system of equations (4.2.16) satisfy $\alpha_j \leq 0$, the relevant nonlinearity $\mathcal{F}$ of the form (4.2.20) has weight x^q with $q \geq \frac{7}{2}$, as appears in the final term on the right-hand side of (4.2.18).

We note that the class (4.2.20) of nonlinearity considered in [17] is strictly smaller than that (4.2.14) treated herein, due to the requirement of weights which decay in r . Most importantly, one can show that Condition 4.2.4 applies to all terms appearing on the right-hand side of (4.2.19).

¹³It is worth noting that equations (4.2.10) with “positive mass” $\alpha \leq 0$ are easier to treat than the more general equations treated herein. In particular, the corresponding energies are automatically coercive, meaning that the additional work appearing in Section 4.3.1 is not required if one restricts to these masses.

4.2.3.3 The scattering problem for nonlinear waves

All that remains is to supplement (4.2.10), (4.2.11) with appropriate scattering data $h_{\mathcal{H}^+}$ on the event horizon $\mathcal{H}^+$. It will become clear in subsequent sections that this scattering construction requires that $h_{\mathcal{H}^+}$ decay at a sufficiently fast exponential rate in t^* . That being said, the scattering problem we will study in this chapter is

$$\begin{cases} \square_g \psi + \alpha \psi = \mathcal{F} \\ \psi|_{\mathcal{H}^+} = h_{\mathcal{H}^+}, \quad r^{\frac{3}{2}-s} \psi|_{\mathcal{I}^-} = 0, \quad s = \sqrt{\frac{9}{4} - \alpha} \end{cases} \quad (4.2.21)$$

where we seek a solution $\psi \in CH_{KAdS}^k$ for $k \geq 9$. From Proposition 4.3.17 onwards, we will restrict our attention to $\mathcal{F}$ satisfying assumption (4.2.14).

4.3 Key estimates

Here, we collect a number of estimates which will be key in proving the main result. Firstly, in Section 4.3.1, coercivity of the energies is derived. This is followed in Section 4.3.2 by elliptic estimates for spatial derivatives. In Section 4.3.3, these elliptic estimates are applied to derive an L^∞ Sobolev embedding on the spacelike slices Σ_{t^*} . Finally, in sections 4.3.4 and 4.3.5, the previous estimates are appealed to in proving the k^{th} -order energy estimates which are central to the proof of Theorem 4.1.1.

4.3.1 Coercivity of the energy

In order to prove Theorem 4.1.1, one must be able to prove coercive estimates. By direct calculation, one finds that the (degenerate at $\mathcal{H}^+$) T -energy

$$\begin{aligned} & \int_{\Sigma_{t^*}} J_\mu^T[\psi] \cdot n_{\Sigma_{t^*}}^\mu \cdot \text{vol}_{\Sigma_{t^*}} \\ &= \frac{1}{2} \int_{\Sigma_{t^*}} \left[-g^{t^*t^*} (\partial_{t^*} \psi)^2 + \frac{\Delta_-}{\Sigma} (\partial_r \psi)^2 + |\nabla \psi|^2 + 2g^{r\phi} \partial_r \psi \partial_\phi \psi - \frac{\alpha}{\ell^2} \psi^2 \right] \frac{\Sigma}{\Xi} dr d\omega \end{aligned} \quad (4.3.1)$$

carries a zeroth-order term of bad sign.¹⁴ As was the case on Anti-de Sitter space (see Proposition 3.3.1), it will become clear in the proof of Lemma 4.3.2 that ensuring positivity of this energy requires stipulating the Breitenlohner-Freedman bound $\alpha < \frac{9}{4}$. An analogous zeroth-order term also appears in the non-degenerate N -energy

$$\int_{\Sigma_{t^*}} J_\mu^N[\psi] \cdot n_{\Sigma_{t^*}}^\mu \cdot \text{vol}_{\Sigma_{t^*}}, \quad (4.3.2)$$

¹⁴One must also deal with the $\partial_r \psi \partial_\phi \psi$ term, however, this can be absorbed by other terms via an application of Cauchy-Schwarz away from the horizon.

with which we will work throughout this chapter, where now the coefficient is a different function $-C_\alpha$ (with $C_\alpha \equiv \frac{\alpha}{\ell^2}$ for $r > r_1$). Trivially, for some $C > 0$ one has the estimate

$$\begin{aligned} & \int_{\Sigma_\tau} J_\mu^N[\psi] \cdot n_{\Sigma_\tau}^\mu \cdot \text{vol}_{\Sigma_\tau} + C \left[\int_{\Sigma_\tau \cap \{r \geq r_1\}} \frac{\alpha}{\ell^2} \psi^2 \frac{\Sigma}{\Xi} \text{d}r \text{d}\omega + \int_{\Sigma_\tau \cap \{r < r_1\}} C_\alpha \psi^2 \frac{\Sigma}{\Xi} \text{d}r \text{d}\omega \right] \\ & \geq \|\psi\|_{H_{KAdS}^1(\Sigma_\tau)}^2, \end{aligned}$$

however, we will show in Lemma 4.3.2 that the term on the “far region” $r \geq r_1$ can be dropped. In Lemma 4.3.3, it will be shown that the term on the “near region” $r < r_1$ can be exchanged for a spacetime term on a compact in r region.

We prove the following coercivity statement for the N -energy.

Proposition 4.3.1 (A coercive estimate for the non-degenerate N -energy on KAdS). Let ψ be a solution of (4.2.21) in CH_{KAdS}^1 with first time derivative in $C([\tau, \tau']; L^2(\Sigma_{t^*}))$. Then the N -energy (4.3.2) of ψ satisfies the coercive estimate

$$\begin{aligned} & \int_{\Sigma_\tau} J_\mu^N[\psi] \cdot n_{\Sigma_{t^*}}^\mu \cdot \text{vol}_{\Sigma_{t^*}} \\ & + \tilde{C}_\alpha \left[\int_{\Sigma_{\tau'} \cap \{r < r_1 + \delta\}} \psi^2 \frac{\Sigma}{\Xi} \text{d}r \text{d}\omega + \int_{\mathcal{D}_{[\tau, \tau']} \cap \{r < r_1 + \delta\}} \left(\psi^2 + (\partial_{t^*} \psi)^2 \right) \frac{\Sigma}{\Xi} \text{d}r \text{d}\omega \text{d}t^* \right] \\ & \geq \|\psi\|_{H_{KAdS}^1(\Sigma_\tau)}^2 \end{aligned} \tag{4.3.3}$$

for an appropriately chosen constant $\tilde{C}_\alpha$ depending on the mass α .

Note that the implication

$$\|\psi\|_{H_{KAdS}^1}^2 \geq C \int_{\Sigma_\tau} J_\mu^N[\psi] \cdot n_{\Sigma_{t^*}}^\mu \cdot \text{vol}_{\Sigma_{t^*}}$$

in the other direction is trivially true for some positive constant C .

Proposition 4.3.1 will be applied to the finite-in-time problems (4.5.1) in Section 4.5. In that setting, the integral on $\Sigma_{\tau'}$ appearing in (4.3.3) vanishes and so Proposition 4.3.1 shows that the N -current $J^N[\psi]$ controls the H_{KAdS}^1 norm of ψ up to a spacetime term:

$$\int_{\tau}^{\tau'} \|\mathbf{1}_{\{r < r_1 + \delta\}} \psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 \text{d}t^*,$$

supported away from infinity. This term can then be dealt with via a Grönwall inequality.

Proposition 4.3.1 follows from two estimates: one on the “far” region $r \geq r_1$ where $N = T$, and one on the “near” region. The former is a Hardy inequality, adapted from [34, Section 5.1], the proof of which demonstrates the importance of the aforementioned Breitenlohner-Freedman bound $\alpha < \frac{9}{4}$. This result is as follows.

Lemma 4.3.2 (A “far” region Hardy inequality on KAdS). Let $\chi(r)$ be a smooth, positive cut-off function satisfying

$$\chi(r) = \begin{cases} 0 & \text{on } r < r_1, \\ 1 & \text{on } r \geq r_1 + \delta, \end{cases}$$

for some $\delta > 0$, and increasing monotonically on $[r_1, r_1 + \delta]$, where r_1 is sufficiently large with respect to a . Then for a solution ψ of (4.2.21) in $H_{KAdS}^1(\Sigma_{t^*})$, one has that

$$\int_{\Sigma_{t^*}} \chi(r) \left[\frac{\Delta_-}{\Sigma} (\partial_r \psi)^2 - \frac{\alpha}{\ell^2 \Xi} \psi^2 \right] \frac{\Sigma}{\Xi} dr d\omega \geq 0$$

On the “near” region, the zeroth order term of the N -current $J^N[\psi]$ is estimated via a spacetime term in order to derive the following coercivity statement.

Lemma 4.3.3 (A “near” region estimate for the non-degenerate N -energy on KAdS). Let ψ be a solution of (4.2.21) in CH_{KAdS}^1 with first time derivative in $C([\tau, \tau']; L^2(\Sigma_{t^*}))$. Then

$$\int_{\Sigma_\tau \cap \{r < r_1 + \delta\}} \psi^2 \frac{\Sigma}{\Xi} dr d\omega \leq \int_{\Sigma_{\tau'} \cap \{r < r_1 + \delta\}} \psi^2 \frac{\Sigma}{\Xi} dr d\omega + C \int_{\mathcal{D}_{[\tau, \tau'] \cap \{r < r_1 + \delta\}}} \left[\psi^2 + (\partial_{t^*} \psi)^2 \right] \frac{\Sigma}{\Xi} dr d\omega dt^*, \quad (4.3.4)$$

for some $C > 0$.

We begin by proving Lemma 4.3.2.

Proof of Lemma 4.3.2. We adapt the proof of coercivity of the T -energy from [34, Lemma 4.1] as follows.

Firstly, we have that

$$\frac{\alpha}{\ell^2 \Xi} \int_{\Sigma_{t^*}} \chi(r) \psi^2 \Sigma dr d\omega \leq \frac{\alpha}{\ell^2 \Xi} \int_{\Sigma_{t^*}} \chi(r) \psi^2 (r^2 + a^2) dr d\omega. \quad (4.3.5)$$

Integrating the right-hand side of (4.3.5) by parts and applying the Cauchy-Schwarz inequality, one obtains

$$\begin{aligned} & \int_{\Sigma_{t^*}} \chi(r) \psi^2 (r^2 + a^2) dr d\omega \\ &= \int_{\Sigma_{t^*}} \frac{1}{3} \chi(r) \psi^2 \partial_r \left(r^3 - r_+^3 + 3a^2(r - r_+) \right) dr d\omega \\ &= - \int_{\Sigma_{t^*}} \frac{2}{3} \left(r^3 - r_+^3 + 3a^2(r - r_+) \right) \left[\chi(r) \psi \partial_r \psi + \frac{1}{2} \partial_r (\chi(r)) \psi^2 \right] dr d\omega \\ &\leq \left(\int_{\Sigma_{t^*}} \frac{4}{9} \chi(r) \psi^2 (r^2 + a^2) dr d\omega \right)^{\frac{1}{2}} \left(\int_{\Sigma_{t^*}} (\partial_r \psi)^2 \chi(r) \frac{[r^3 - r_+^3 + 3a^2(r - r_+)]^2}{r^2 + a^2} dr d\omega \right)^{\frac{1}{2}} \end{aligned} \quad (4.3.6)$$

$$\begin{aligned}
& - \int_{\Sigma_{t^*}} \frac{1}{3} \left(r^3 - r_+^3 + 3a^2(r - r_+) \right) \partial_r(\chi(r)) \psi^2 dr d\omega \\
& \leq \left(\int_{\Sigma_{t^*}} \frac{4}{9} \chi(r) \psi^2 (r^2 + a^2) dr d\omega \right)^{\frac{1}{2}} \left(\int_{\Sigma_{t^*}} (\partial_r \psi)^2 \chi(r) \frac{[r^3 - r_+^3 + 3a^2(r - r_+)]^2}{r^2 + a^2} dr d\omega \right)^{\frac{1}{2}},
\end{aligned} \tag{4.3.7}$$

where the final inequality follows from the fact that the second term of (4.3.7) is negative.

Thus,

$$\int_{\Sigma_{t^*}} \frac{9}{4} \chi(r) \psi^2 (r^2 + a^2) dr d\omega \leq \int_{\Sigma_{t^*}} (\partial_r \psi)^2 \chi(r) \frac{[r^3 - r_+^3 + 3a^2(r - r_+)]^2}{r^2 + a^2} dr d\omega.$$

Now, provided that the cut-off radius r_1 is chosen sufficiently large (depending on a), we have that

$$\left[\Delta_- - \frac{[r^3 - r_+^3 + 3a^2(r - r_+)]^2}{(r^2 + a^2)\ell^2} \right] \Big|_{\chi(r) \neq 0} > 0$$

implying

$$\int_{\Sigma_{t^*}} \chi(r) \left[\frac{\Delta_-}{\Sigma} (\partial_r \psi)^2 - \frac{\alpha}{\ell^2} \psi^2 \right] \frac{\Sigma}{\Xi} dr d\omega \geq 0 \quad \Longleftrightarrow \quad \alpha < \frac{9}{4},$$

as required. $\square$

Next, we prove Lemma 4.3.3 which covers the remaining “near” region, $r < r_1 + \delta$.

Proof of Lemma 4.3.3. On the “near” region $r < r_1 + \delta$, we estimate the zeroth-order term of (4.3.2) by a spacetime term via the Fundamental Theorem of Calculus:

$$\begin{aligned}
& \int_{\Sigma_\tau \cap \{r < r_1 + \delta\}} \psi^2 \frac{\Sigma}{\Xi} dr d\omega \\
& = \int_{\Sigma_{\tau'} \cap \{r < r_1 + \delta\}} \psi^2 \frac{\Sigma}{\Xi} dr d\omega - \int_\tau^{\tau'} \partial_{t^*} \left[\int_{\Sigma_{t^*} \cap \{r < r_1 + \delta\}} \psi^2 \frac{\Sigma}{\Xi} dr d\omega \right] dt^* \\
& = \int_{\Sigma_{\tau'} \cap \{r < r_1 + \delta\}} \psi^2 \frac{\Sigma}{\Xi} dr d\omega - \int_{\mathcal{D}_{[\tau, \tau']} \cap \{r < r_1 + \delta\}} \partial_{t^*}(\psi^2) \frac{\Sigma}{\Xi} dr d\omega dt^* \\
& = \int_{\Sigma_{\tau'} \cap \{r < r_1 + \delta\}} \psi^2 \frac{\Sigma}{\Xi} dr d\omega - \int_{\mathcal{D}_{[\tau, \tau']} \cap \{r < r_1 + \delta\}} 2\psi \cdot \partial_{t^*} \psi \frac{\Sigma}{\Xi} dr d\omega dt^* \\
& \leq \int_{\Sigma_{\tau'} \cap \{r < r_1 + \delta\}} \psi^2 \frac{\Sigma}{\Xi} dr d\omega + \int_{\mathcal{D}_{[\tau, \tau']} \cap \{r < r_1 + \delta\}} \left[2\psi^2 + \frac{1}{2}(\partial_{t^*} \psi)^2 \right] \frac{\Sigma}{\Xi} dr d\omega dt^* \\
& \leq \int_{\Sigma_{\tau'} \cap \{r < r_1 + \delta\}} \psi^2 \frac{\Sigma}{\Xi} dr d\omega + C \int_{\mathcal{D}_{[\tau, \tau']} \cap \{r < r_1 + \delta\}} \left[\psi^2 + (\partial_{t^*} \psi)^2 \right] \frac{\Sigma}{\Xi} dr d\omega dt^*,
\end{aligned} \tag{4.3.8}$$

for some $C > 0$. Here, (4.3.8) follows from the Cauchy-Schwarz inequality. $\square$

Proposition 4.3.1 follows as a straight forward corollary of Lemmas 4.3.2 and 4.3.3.

Proof of Proposition 4.3.1. By Lemma 4.3.2, the N -energy is coercive on the “far” region $r \geq r_1 + \delta$. Then, by adding an appropriate multiple

$$\tilde{C}_\alpha \left[\int_{\Sigma_{t^*} \cap \{r < r_1 + \delta\}} \psi^2 \frac{\Sigma}{\Xi} dr d\omega + \int_{\mathcal{D}_{[r, r']} \cap \{r < r_1 + \delta\}} \left[\psi^2 + (\partial_{t^*} \psi)^2 \right] \frac{\Sigma}{\Xi} dr d\omega dt^* \right]$$

of the right-hand side of the estimate appearing in Lemma 4.3.3 (choose, say, $\tilde{C}_\alpha = 2C_\alpha$, where C_α is the coefficient of the zeroth-order term in (4.3.2)), one obtains the desired result. $\square$

By commuting (4.3.3) with Γ^σ and applying the elliptic estimates appearing in the next section, one obtains analogous higher order coercive estimates for ψ sufficiently regular.

4.3.2 L^2 estimates for derivatives of order > 2 in space

The estimates appearing in this section will allow us to control higher-order angular and radial derivatives of the solution ψ . This allows one to avoid bulk terms generated by commuting with the angular momentum operators, as we did with no issue in the spherically symmetric Anti-de Sitter setting. We will prove a four-level hierarchy of estimates - the first for general derivatives on the “far” region, the second for derivatives of the form $\Gamma^\sigma D_x^2 \psi$ (where only two derivatives are in general directions) on the entire exterior, the third for general derivatives on the “near” region, and the fourth for general derivatives on the entire exterior.

We begin with preliminary second order elliptic estimates for angular derivatives, first on the spheres $S_{t^*, r}^2$ and then on the spacelike slices Σ_{t^*} . The first is given on the Kerr-AdS spheres $S_{t^*, r}^2$ at fixed time and radial distance, but holds for more general topological 2-spheres.

Lemma 4.3.4 (An elliptic estimate on the topological 2-spheres $S_{t^*, r}^2$ of KAdS [9, Prop 4.4.3]). Let ψ be a sufficiently smooth function on $S_{t^*, r}^2$ for $t^*, r \geq r^+$ fixed. Then

$$\int_{S_{t^*, r}^2} (\Delta \psi)^2 d\omega \geq c \int_{S_{t^*, r}^2} \left(|\nabla^2 \psi|^2 + 2K |\nabla \psi|^2 \right) d\omega, \quad (4.3.9)$$

for some $c > 0$, where K is the Gauss curvature of $S_{t^*, r}^2$.

Note that the K term in (4.3.9) can be negative, however, when it comes to applying Lemma 4.3.4 to prove Proposition 4.3.8 it will be treated as a lower-order term. As such, the sign does not matter.

Under assumption of sufficient regularity of ψ on the entire spacelike hypersurface Σ_{t^*} , one immediately obtains the following result as a consequence of Lemma 4.3.4.

Corollary 4.3.5 (An elliptic estimate for second-order angular derivatives on spacelike slices of KAdS). Let ψ be a function in $H_{KAdS}^2(\Sigma_{t^*})$. Then

$$\int_{\Sigma_{t^*}} (\Delta\psi)^2 r^2 dr d\omega \geq c \int_{\Sigma_{t^*}} \left(|\nabla^2 \psi|^2 + 2K |\nabla \psi|^2 \right) r^2 dr d\omega, \quad (4.3.10)$$

for some $c > 0$, where K is the Gauss curvature of $S_{t^*,r}^2$.

Given Corollary 4.3.5, we next recall some elliptic estimates on the “far” region.

4.3.2.1 Elliptic estimates for general derivatives on the “far” region (I)

We first recall the standard (see, for example, [33]) second-order elliptic estimate one can prove on the “far” region (outside of the ergoregion) on which T is everywhere timelike. This requires appealing to Corollary 4.3.5 and commuting once with T .

Proposition 4.3.6 (A “far” region second-order elliptic estimate on KAdS). Let ψ be a function in $H_{KAdS}^2(\Sigma_{t^*})$ satisfying Dirichlet boundary conditions (4.2.11) and $\tilde{\xi}(r)$ be a smooth, positive cut-off function satisfying

$$\tilde{\xi}(r) = \begin{cases} 0 & \text{on } r < r_0, \\ 1 & \text{on } r \geq r_0 + \tilde{\delta} \end{cases}$$

for some $\tilde{\delta} > 0$ such that $r_0 + \tilde{\delta} < r_1$, and increasing monotonically on $[r_0, r_0 + \tilde{\delta}]$. Here, r_0, r_1 are as in Section 4.2.1.2. Then we have that

$$\begin{aligned} & \int_{\Sigma_{t^*}} \tilde{\xi}(r) \left(|\nabla^2 \psi|^2 + r^2 |\nabla \partial_r \psi|^2 + r^4 (\partial_r^2 \psi)^2 \right) r^2 dr d\omega \\ & \leq C \left(\|\psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \|T\psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \int_{\Sigma_{t^*}} (\square_g \psi + \alpha \psi)^2 r^2 dr d\omega \right), \end{aligned} \quad (4.3.11)$$

for some $C > 0$.

Proposition 4.3.6 is a result of the fact that one can write the wave equation (1.3.1) as

$$\begin{aligned} & \frac{1}{\sqrt{|\det g|}} \partial_i (g^{ij} \sqrt{|\det g|} \partial_j \psi) \\ & = \square \psi - g^{t^* t^*} \partial_{t^*}^2 \psi - 2g^{t^* i} \partial_{t^*} \partial_i \psi - \frac{1}{\sqrt{|\det g|}} \partial_r (g^{t^* r} \sqrt{|\det g|}) \partial_t \psi, \end{aligned} \quad (4.3.12)$$

for $i = r, \theta, \phi$. The left-hand side of (4.3.12) is an elliptic operator on the region where $g_{\Sigma_{t^*}}$ (the induced metric on the slice Σ_{t^*}) is positive definite. This is precisely where one has

$$\det g_{\Sigma_{t^*}} = \frac{\Xi^2}{\Sigma^3} \left[\frac{\Delta_-}{\sin^2 \theta} - a^2 \Delta_\theta \right] > 0,$$

the same region on which T is timelike:

$$g_{t^*t^*} = -\frac{\sin^2 \theta}{\Sigma} \left[\frac{\Delta_-}{\sin^2 \theta} - a^2 \Delta_\theta \right] < 0.$$

Integrating (4.3.12) by parts gives control of $\nabla^2 \psi$, $\nabla \partial_r \psi$. The remaining $\partial_r^2 \psi$ can then be controlled directly by (4.3.12), given that one controls all other terms.

By successively applying $\square_g$ to (4.3.12) and repeating these steps, one obtains the following higher-order result.

Proposition 4.3.7 (A “far” region k^{th} -order elliptic estimate on KAdS). Let ψ be a function in $H_{KAdS}^k(\Sigma_{t^*})$ for $k \geq 2$ satisfying Dirichlet boundary conditions (4.2.11) and $\tilde{\xi}(r)$ be as in Proposition 4.3.6. Then we have that

$$\begin{aligned} & \sum_{|\sigma|=k} \int_{\Sigma_{t^*}} \tilde{\xi}(r) r^{2\sigma_2} |\nabla^{\sigma_1} \partial_{t^*}^{\sigma_2} \partial_r^{\sigma_3} \psi|^2 r^2 dr d\omega \\ & \leq C \sum_{\substack{|\alpha| \leq k-1, \\ |\beta| \leq k-2, \\ |\gamma| \leq \lceil \frac{k-2}{2} \rceil}} \left[\|T^\alpha \psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \int_{\Sigma_{t^*}} \left(\left(T^\beta (\square_g \psi + \alpha \psi) \right)^2 + \left(\square_g^\gamma (\square_g \psi + \alpha \psi) \right)^2 \right) r^2 dr d\omega \right], \end{aligned}$$

for some $C > 0$.

We can now proceed to estimate derivatives $\Gamma^\sigma D_x^2 \psi$ on the entire exterior, which will subsequently allow us to construct estimates for general derivatives on the “near” region. These will eventually be combined with Proposition 4.3.7 to control general derivatives on the entire exterior region.

4.3.2.2 Estimating spatial derivatives of at most second order in general directions on the entire exterior region (II)

We first derive an elliptic estimate for second order spatial derivatives on the spacelike slices Σ_{t^*} . This requires appealing to Corollary 4.3.5, commuting once with T and commuting once with the redshift vector field N in order to counteract the degenerate (at the horizon) g^{rr} weights appearing in the estimate involving T alone. One commutation with Φ is also employed, in order to deal treat a mixed angular-radial term of potentially bad sign. This result will then be generalised to a statement for higher order spatial derivatives of the form $\Gamma^\sigma D_x^2 \psi$ via successive commutation with the commuting vector fields Γ .

Proposition 4.3.8 (A non-degenerate second-order elliptic estimate on spacelike slices of KAdS). Let ψ be a function in $H_{KAdS}^2(\Sigma_{t^*})$ satisfying Dirichlet boundary conditions (4.2.11). Then we have that

$$\int_{\Sigma_{t^*}} \left(|\nabla^2 \psi|^2 + r^2 |\nabla \partial_r \psi|^2 + r^4 (\partial_r^2 \psi)^2 \right) r^2 dr d\omega$$

$$\leq C \left(\|\psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \|T\psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \|N\psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \|\Phi\psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 \right. \\ \left. + \int_{\Sigma_{t^*}} (\square_g \psi + \alpha \psi)^2 r^2 dr d\omega \right) \quad (4.3.13)$$

for some $C > 0$.

Proof. Let us consider the massive wave operator

$$\square_g \psi + \alpha \psi$$

appearing on the left-hand side of (4.2.10) in regular (t^*, r, θ, ϕ) -coordinates:

$$\square_g \psi + \alpha \psi = g^{t^* t^*} \partial_{t^*}^2 \psi + g^{rr} \partial_r^2 \psi + 2g^{t^* \phi} \partial_{t^*} \partial_\phi \psi + 2g^{t^* r} \partial_{t^*} \partial_r \psi \\ + \partial_r(g^{rr}) \partial_r \psi + \frac{g^{rr}}{\sqrt{|\det g|}} \partial_r(\sqrt{|\det g|}) \partial_r \psi + \frac{g^{\theta\theta}}{\sqrt{|\det g|}} \partial_\theta(\sqrt{|\det g|}) \partial_\theta \psi \\ + \partial_\theta(g^{\theta\theta}) \partial_\theta \psi + 2g^{\phi r} \partial_\phi \partial_r \psi + \partial_r(g^{t^* r}) \partial_{t^*} \psi + \frac{g^{t^* r}}{\sqrt{|\det g|}} \partial_r(\sqrt{|\det g|}) \partial_{t^*} \psi \\ + \frac{g^{r\phi}}{\sqrt{|\det g|}} \partial_r(\sqrt{|\det g|}) \partial_\phi \psi + \partial_r(g^{r\phi}) \partial_\phi \psi + \Delta \psi + \alpha \psi. \quad (4.3.14)$$

By rewriting the $\partial_{t^*}^2 \psi$ term in (4.3.14) in terms of $N^2 \psi$, rearranging, multiplying by $\Delta \psi$ and integrating over a spacelike slice Σ_{t^*} , one obtains

$$\left| \int_{\Sigma_{t^*}} \left[\frac{1}{r^2} N^2 \psi + \left(\frac{1}{r^2} + \xi \right) \partial_{t^*} \partial_\phi \psi + \left(\frac{1}{r^3} + \xi \right) \partial_{t^*} \partial_r \psi + \left(\frac{1}{r^4} + \xi \right) \partial_{t^*} \psi + (r + \xi) \partial_r \psi \right. \right. \\ \left. \left. + \frac{1}{r^2} \partial_\theta \psi + \left(\frac{1}{r^3} + \xi \right) \partial_\phi \psi + \square_g \psi + \alpha \psi \right] \Delta \psi \cdot r^2 dr d\omega \right| \\ \geq C \left| \int_{\Sigma_{t^*}} \left[\left(g^{rr} + \xi^2 (g^{t^* r})^2 \right) \partial_r^2 \psi + \xi^2 (g^{t^* \phi})^2 \partial_\phi^2 \psi + 2 \left(g^{\phi r} + \xi^2 g^{t^* r} g^{t^* \phi} \right) \partial_\phi \partial_r \psi \right. \right. \\ \left. \left. + \alpha \psi + \Delta \psi \right] \Delta \psi \cdot r^2 dr d\omega \right|, \quad (4.3.15)$$

for some $C > 0$. Note that by using the redshift vector field, the degenerately weighted $(g^{rr}|_{r+=0})$ second-order r -derivative has been supplemented with a non-degenerate near-horizon term.

We estimate all terms on the left-hand side of (4.3.15), beginning with the first:

$$\left| \int_{\Sigma_{t^*}} \frac{1}{r^2} N^2 \psi \cdot \Delta \psi \cdot r^2 dr d\omega \right| \leq \left(\frac{1}{\varepsilon} \int_{\Sigma_{t^*}} (N^2 \psi)^2 dr d\omega \right)^{\frac{1}{2}} \left(\varepsilon \int_{\Sigma_{t^*}} (\Delta \psi)^2 dr d\omega \right)^{\frac{1}{2}} \\ \leq \frac{1}{2\varepsilon} \int_{\Sigma_{t^*}} (N^2 \psi)^2 dr d\omega + \frac{\varepsilon}{2} \int_{\Sigma_{t^*}} (\Delta \psi)^2 dr d\omega$$

$$\leq \frac{1}{2\varepsilon} \|N\psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \frac{\varepsilon}{2} \int_{\Sigma_{t^*}} (\Delta\psi)^2 dr d\omega.$$

Here, $\varepsilon > 0$ is chosen sufficiently small to allow for the ε weighted second term to be absorbed by the $(\Delta\psi)^2$ term on the right-hand side of (4.3.15).

The terms on the left-hand side of (4.3.15) involving second-order t^* -derivatives and first-order derivatives respectively are dealt with similarly, yielding

$$\begin{aligned} & \left| \int_{\Sigma_{t^*}} \left[\left(\frac{1}{r^2} + \xi \right) \partial_{t^*} \partial_\phi \psi \cdot \Delta\psi + \left(\frac{1}{r^3} + \xi \right) \partial_{t^*} \partial_r \psi \cdot \Delta\psi \right] r^2 dr d\omega \right| \\ & \leq \frac{1}{2\varepsilon} \int_{\Sigma_{t^*}} \left[|\nabla \partial_{t^*} \psi|^2 + r^2 (\partial_r \partial_{t^*} \psi)^2 \right] r^2 dr d\omega + \varepsilon \int_{\Sigma_{t^*}} (\Delta\psi)^2 r^2 dr d\omega \\ & \leq \frac{1}{2\varepsilon} \|T\psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \varepsilon \int_{\Sigma_{t^*}} (\Delta\psi)^2 r^2 dr d\omega \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{\Sigma_{t^*}} \left[\left(\frac{1}{r^4} + \xi \right) \partial_{t^*} \psi \cdot \Delta\psi + (r + \xi) \partial_r \psi \cdot \Delta\psi + \frac{1}{r^2} \partial_\theta \psi \cdot \Delta\psi + \left(\frac{1}{r^3} + \xi \right) \partial_\phi \psi \cdot \Delta\psi \right] r^2 dr d\omega \right| \\ & \leq \frac{1}{2\varepsilon} \int_{\Sigma_{t^*}} \left[\frac{1}{r^2} (\partial_{t^*} \psi)^2 + r^2 (\partial_r \psi)^2 + |\nabla \psi|^2 \right] r^2 dr d\omega + 2\varepsilon \int_{\Sigma_{t^*}} (\Delta\psi)^2 r^2 dr d\omega \\ & \leq \frac{1}{2\varepsilon} \|\psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + 2\varepsilon \int_{\Sigma_{t^*}} (\Delta\psi)^2 r^2 dr d\omega. \end{aligned}$$

Analogously,

$$\left| \int_{\Sigma_{t^*}} \left[(\square_g \psi + \alpha\psi) \Delta\psi \cdot r^2 dr d\omega \right] \right| \leq \frac{1}{2\varepsilon} \int_{\Sigma_{t^*}} (\square_g \psi + \alpha\psi)^2 r^2 dr d\omega + \frac{\varepsilon}{2} \int_{\Sigma_{t^*}} (\Delta\psi)^2 r^2 dr d\omega.$$

Next, we determine what the right-hand side of (4.3.15) controls. Integrating the r -derivative terms by parts and dropping boundary terms at infinity as a result of the Dirichlet boundary conditions (4.2.11) gives

$$\begin{aligned} & \left| \int_{\Sigma_{t^*}} \left(g^{rr} + \xi^2 (g^{t^*r})^2 \right) \partial_r^2 \psi \cdot \Delta\psi \cdot r^2 dr d\omega \right| \\ & = \left| \int_{\Sigma_{t^*}} \left(g^{rr} + \xi^2 (g^{t^*r})^2 \right) \left[\partial_r (\partial_r \psi \cdot \Delta\psi) - \partial_r \psi \cdot \partial_r \Delta\psi \right] r^2 dr d\omega \right| \\ & = \left| \int_{\Sigma_{t^*}} \left[-\partial_r \left(\left(g^{rr} + \xi^2 (g^{t^*r})^2 \right) r^2 \right) \partial_r \psi \cdot \Delta\psi \right. \right. \\ & \quad \left. \left. - \left(g^{rr} + \xi^2 (g^{t^*r})^2 \right) \left(\partial_r \psi \cdot \Delta \partial_r \psi + \partial_r (g^{\theta\theta}) \partial_r \psi \cdot \partial_\theta^2 \psi + \partial_r (g^{\phi\phi}) \partial_r \psi \cdot \partial_\phi^2 \psi \right) r^2 \right] dr d\omega \right. \\ & \quad \left. + \int_{S_{t^*,r_+}^2} \xi^2 (g^{t^*r})^2 \partial_r \psi \cdot \Delta\psi \cdot r^2 d\omega \right| \end{aligned}$$

$$\geq C \left| \int_{\Sigma_{t^*}} \left[r^4 |\nabla \partial_r \psi|^2 + \left| \nabla \left((g^{rr} + \xi^2 (g^{t^*r})^2) r^2 \right) \partial_r \psi \cdot \nabla \partial_r \psi \right| - r^3 |\partial_r \psi \cdot \Delta \psi| - \frac{1}{2\varepsilon} |\nabla \psi|^2 r^2 \right] dr d\omega \right| \quad (4.3.16)$$

$$\geq C \left(\int_{\Sigma_{t^*}} \left[r^4 |\nabla \partial_r \psi|^2 - \frac{\varepsilon}{2} r^2 (\Delta \psi)^2 \right] dr d\omega - \frac{1}{2\varepsilon} \|\psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 \right) \quad (4.3.17)$$

for some $C > 0$. Here, as before, $\varepsilon > 0$ is chosen sufficiently small. Inequality (4.3.16) follows from applying the asymptotic behaviour of the inverse metric components (4.2.1.1) and absorbing the small contribution of the boundary term on r_+ by other terms. Line (4.3.17) follows from the Cauchy-Schwarz inequality. The second term on the right-hand side of (4.3.17) can be absorbed on the left-hand side of (4.3.15), whilst the final term can be absorbed by the $(\Delta \psi)^2$ term on the right-hand side of (4.3.15).

The mass term appearing on the right-hand side of (4.3.15) satisfies

$$\begin{aligned} \left| \int_{\Sigma_{t^*}} \alpha \psi \cdot \Delta \psi \cdot r^2 dr d\omega \right| &\leq \frac{1}{2\varepsilon} \int_{\Sigma_{t^*}} (\alpha \psi)^2 r^2 dr d\omega + \frac{\varepsilon}{2} \int_{\Sigma_{t^*}} (\Delta \psi)^2 r^2 dr d\omega \\ &\leq \frac{\alpha^2}{2\varepsilon} \|\psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \frac{\varepsilon}{2} \int_{\Sigma_{t^*}} (\Delta \psi)^2 r^2 dr d\omega, \end{aligned}$$

and so can be absorbed by other terms.

This leaves one terms of potentially bad sign on the right-hand side of (4.3.15). This satisfies

$$\begin{aligned} &\left| \int_{\Sigma_{t^*}} 2 \left(g^{\phi r} + \xi^2 g^{t^*r} g^{t^*\phi} \right) \partial_\phi \partial_r \psi \cdot \Delta \psi \cdot r^2 dr d\omega \right| \\ &\leq C \left[\int_{\Sigma_{t^*}} \frac{\varepsilon}{2} (\Delta \psi)^2 r^2 dr d\omega + \frac{1}{2\varepsilon} \|\Phi \psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 \right], \end{aligned} \quad (4.3.18)$$

for some $C > 0$. The first term on the right-hand side of (4.3.18) can be absorbed by the $(\Delta \psi)^2$ term on the right-hand side of (4.3.15), whilst the second can be moved to the left-hand side of (4.3.15).

For the Laplacian term on the right-hand side of (4.3.15), we appeal to Corollary 4.3.5. The Gauss curvature term of bad sign on the right-hand side of (4.3.10) is of first-order, so easily controlled via an application of the Cauchy-Schwarz inequality by $\|\psi\|_{H_{KAdS}^1(\Sigma_{t^*})}$ and absorbed by the left-hand side of (4.3.15). The remaining $\partial_\phi^2 \psi$ term on the right-hand side of (4.3.15) is of good sign. Finally, the second-order r -derivative of ψ is controlled via the equation, yielding estimate (4.3.13). $\square$

In order to generalise the estimate of Proposition 4.3.8 to higher order derivatives $\Gamma^\sigma D_x^2 \psi$, one must commute with N sufficiently many times and apply Δ^m as a multiplier for $m > 0$ sufficiently large. By doing so, one recovers the following result.

Proposition 4.3.9 (A non-degenerate k^{th} -order elliptic estimate on KAdS for derivatives where only 2 of k are arbitrary). Let ψ be a function in $H_{KAdS}^k(\Sigma_{t^*})$ for $k \geq 2$ satisfying Dirichlet boundary conditions (4.2.11). Then,

$$\begin{aligned} & \int_{\Sigma_{t^*}} \sum_{|\sigma|=k-2} \left[|\nabla^2 \Gamma^\sigma \psi|^2 + r^2 |\nabla \Gamma^\sigma \partial_r \psi|^2 + r^4 (\Gamma^\sigma \partial_r^2 \psi)^2 \right] r^2 dr d\omega \\ & \leq C \sum_{\substack{|\alpha| \leq k-1, \\ |\beta| \leq k-2}} \left[\|\Gamma^\alpha \psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \int_{\Sigma_{t^*}} \left(\Gamma^\beta (\square_g \psi + \alpha \psi) \right)^2 r^2 dr d\omega \right] \end{aligned} \quad (4.3.19)$$

for some $C > 0$.

Proof. In order to control third order derivatives of the form $\Gamma D_x^2 \psi$, we successively commute (4.3.14) with T , N and Φ . The process of proving the second order estimate can then be repeated, applying higher-order multipliers Δ^σ as required to successfully integrate by parts. Commuting with T and Φ generates no additional terms, so the $T D_x^2 \psi$, $\Phi D_x^2 \psi$ estimates are trivial. Commutator terms are generated in the $N D_x^2 \psi$ estimate, however, these are of at most second order (in general derivatives) and vanishing near infinity so controlled by the estimate of Proposition 4.3.8:

$$\begin{aligned} N \square_g \psi &= \square_g N \psi + N \left(\frac{1}{\sqrt{|\det g|}} \right) \partial_\mu (g^{\mu\nu} \sqrt{|\det g|} \partial_\nu \psi) + \frac{1}{\sqrt{|\det g|}} N \left(\partial_\mu (g^{\mu\nu} \sqrt{|\det g|}) \right) \partial_\nu \psi \\ &+ \frac{1}{\sqrt{|\det g|}} N (g^{\mu\nu} \sqrt{|\det g|}) \partial_\mu \partial_\nu \psi. \end{aligned} \quad (4.3.20)$$

Further commutations with N generate additional commutator terms which are also of at most second order in general derivatives and of lower order in N . These can always be controlled by estimates with fewer commutations. $\square$

Given Proposition 4.3.9, we can next estimate general derivatives on the “near” region.

4.3.2.3 Estimating general derivatives on the “near” region (III)

We would like to prove the following estimate, which gives control of general k^{th} -order derivatives of ψ on the “near” region. This, combined with the analogous “far” region result (Proposition 4.3.7) will allow us to control general k^{th} -order derivatives of ψ on the entire exterior region, non-degenerately.

Theorem 4.3.10 (A non-degenerate k^{th} -order “near” region estimate on KAdS). Let ψ be a function in $H_{KAdS}^k(\Sigma_{t^*})$ for $k \geq 2$ satisfying Dirichlet boundary conditions (4.2.11) and let $\xi(r)$ be as in the definition of N (4.2.6). Then,

$$\sum_{\substack{|\sigma|=k, \\ \sigma_2 \leq k-1}} \int_{\Sigma_{t^*}} \xi(r) r^{2\sigma_2} |\nabla^{\sigma_1} \partial_{t^*}^{\sigma_2} \partial_r^{\sigma_3} \psi|^2 r^2 dr d\omega$$

$$\leq C \sum_{\substack{|\alpha| \leq k-1, \\ |\beta| \leq k-2, \\ |\gamma| \leq \lceil \frac{k-2}{2} \rceil}} \left[\|\Gamma^\alpha \psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \int_{\Sigma_{t^*}} \left(\left(\Gamma^\beta (\square_g \psi + \alpha \psi) \right)^2 + \left(\Delta^\gamma (\square_g \psi + \alpha \psi) \right)^2 \right) r^2 dr d\omega \right]$$

for some $C > 0$.

In order to prove Theorem 4.3.10, we require the following result which is immediately implied by Proposition 4.3.9.

Proposition 4.3.11 (A non-degenerate k^{th} -order “near” region estimate on KAdS for derivatives where at most 2 of k are ∂_θ). Let ψ be a function in $H_{KAdS}^k(\Sigma_{t^*})$ for $k \geq 2$ satisfying Dirichlet boundary conditions (4.2.11) and let $\xi(r)$ be as in the definition of N (4.2.6). Then,

$$\begin{aligned} & \int_{\Sigma_{t^*}} \sum_{\substack{|\sigma|=k-2, \\ \sigma_4 \leq 2}} \xi(r) (\partial_{t^*}^{\sigma_1} \partial_r^{\sigma_2} \partial_\phi^{\sigma_3} \partial_\theta^{\sigma_4} \psi)^2 r^2 dr d\omega \\ & \leq C \sum_{\substack{|\alpha| \leq k-1, \\ |\beta| \leq k-2}} \left[\|\Gamma^\alpha \psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \int_{\Sigma_{t^*}} \left(\Gamma^\beta (\square_g \psi + \alpha \psi) \right)^2 r^2 dr d\omega \right] \end{aligned} \quad (4.3.21)$$

for some $C > 0$.

We can now prove Theorem 4.3.10. We demonstrate how to obtain control of the third and fourth-order ∂_θ derivatives. The general higher-order statement follows by iterating the same steps sufficiently many times and commuting with Γ .

Proof. Rearranging (4.3.14) so that only terms involving ∂_θ derivatives of ψ appear on the right-hand side gives

$$\begin{aligned} & \square_g \psi + \alpha \psi - g^{t^* t^*} \partial_{t^*}^2 \psi - g^{rr} \partial_r^2 \psi - 2g^{t^* \phi} \partial_{t^*} \partial_\phi \psi - 2g^{t^* r} \partial_{t^*} \partial_r \psi - \partial_r(g^{rr}) \partial_r \psi \\ & - \frac{g^{rr}}{\sqrt{|\det g|}} \partial_r(\sqrt{|\det g|}) \partial_r \psi - 2g^{\phi r} \partial_\phi \partial_r \psi - \partial_r(g^{t^* r}) \partial_{t^*} \psi - \frac{g^{t^* r}}{\sqrt{|\det g|}} \partial_r(\sqrt{|\det g|}) \partial_{t^*} \psi \\ & - \frac{g^{r\phi}}{\sqrt{|\det g|}} \partial_r(\sqrt{|\det g|}) \partial_\phi \psi - \partial_r(g^{r\phi}) \partial_\phi \psi - \alpha \psi \\ & = \frac{g^{\theta\theta}}{\sqrt{|\det g|}} \partial_\theta(\sqrt{|\det g|}) \partial_\theta \psi + \partial_\theta(g^{\theta\theta}) \partial_\theta \psi + \Delta \psi. \end{aligned} \quad (4.3.22)$$

Applying the Δ operator to both sides of expression (4.3.22), multiplying by $\xi(r) \Delta^p \psi$ for $p = 1, 2$ and integrating over Σ_{t^*} then yields

$$\left| \int_{\Sigma_{t^*}} \sum_{\substack{|\sigma| \leq 4, \\ \sigma_4 \leq 2}} \left[\Delta(\square_g \psi + \alpha \psi) + \partial_{t^*}^{\sigma_1} \partial_r^{\sigma_2} \partial_\phi^{\sigma_3} \partial_\theta^{\sigma_4} \psi \right] \xi(r) \Delta^p \psi \cdot r^2 dr d\omega \right|$$

$$\geq C \left| \int_{\Sigma_{t^*}} \left[\frac{g^{\theta\theta}}{\sqrt{|\det g|}} \partial_\theta(\sqrt{|\det g|}) \mathbb{A} \partial_\theta \psi + \partial_\theta(g^{\theta\theta}) \mathbb{A} \partial_\theta \psi + \mathbb{A}^2 \psi \right] \xi(r) \mathbb{A}^p \psi \cdot r^2 dr d\omega \right|, \quad (4.3.23)$$

for some $C > 0$.

Set $p = 1$. By Cauchy-Schwarz and Proposition 4.3.11, this implies

$$\begin{aligned} & \sum_{\substack{|\alpha| \leq 3, \\ |\beta| \leq 2}} \left[\|\Gamma^\alpha \psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \int_{\Sigma_{t^*}} \left(\left(\Gamma^\beta(\square_g \psi + \alpha \psi) \right)^2 + \left(\mathbb{A}(\square_g \psi + \alpha \psi) \right)^2 \right) r^2 dr d\omega \right] \\ & \geq C \int_{\Sigma_{t^*}} \left[\frac{g^{\theta\theta}}{\sqrt{|\det g|}} \partial_\theta(\sqrt{|\det g|}) \mathbb{A} \partial_\theta \psi + \partial_\theta(g^{\theta\theta}) \mathbb{A} \partial_\theta \psi + \mathbb{A}^2 \psi \right] \xi(r) \mathbb{A} \psi \cdot r^2 dr d\omega, \end{aligned} \quad (4.3.24)$$

for some $C > 0$. Control of third-order angular derivatives (in particular, the third-order ∂_θ derivative) arises from integrating the final term on the right-hand side of (4.3.24) by parts once in the angular directions:

$$\left| \int_{\Sigma_{t^*}} \xi(r) \cdot \mathbb{A}^2 \psi \cdot \mathbb{A} \psi \cdot r^2 dr d\omega \right| \geq \int_{\Sigma_{t^*}} \xi(r) |\nabla^3 \psi|^2 r^2 dr d\omega. \quad (4.3.25)$$

The remaining terms of potentially bad sign are controlled via Cauchy-Schwarz,

$$\begin{aligned} & \left| \int_{\Sigma_{t^*}} \left[\frac{g^{\theta\theta}}{\sqrt{|\det g|}} \partial_\theta(\sqrt{|\det g|}) \mathbb{A} \partial_\theta \psi \cdot \mathbb{A} \psi + \partial_\theta(g^{\theta\theta}) \mathbb{A} \partial_\theta \psi \cdot \mathbb{A} \psi \right] r^2 dr d\omega \right| \\ & \leq C \int_{\Sigma_{t^*}} \left[\varepsilon |\nabla^3 \psi|^2 + \frac{1}{\varepsilon} (\mathbb{A} \psi)^2 \right] r^2 dr d\omega, \end{aligned} \quad (4.3.26)$$

for some $C > 0$, where $\varepsilon > 0$ is chosen to be sufficiently small. The second term on the right-hand side of (4.3.26) can be absorbed on the left-hand side of (4.3.24).

Combining (4.3.25), (4.3.26) and (4.3.24) (with $p = 1$) gives

$$\begin{aligned} & \sum_{\substack{|\alpha| \leq 3, \\ |\beta| \leq 2}} \left[\|\Gamma^\alpha \psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \int_{\Sigma_{t^*}} \left(\left(\Gamma^\beta(\square_g \psi + \alpha \psi) \right)^2 + \left(\mathbb{A}(\square_g \psi + \alpha \psi) \right)^2 \right) r^2 dr d\omega \right] \\ & \geq C \int_{\Sigma_{t^*}} \xi(r) |\nabla^3 \psi|^2 r^2 dr d\omega, \end{aligned} \quad (4.3.27)$$

for some $C > 0$.

Now, set $p = 2$. Applying Cauchy-Schwarz with a sufficiently small $\varepsilon > 0$ weight to the left-hand side of (4.3.23) and applying Proposition 4.3.11 gives

$$\begin{aligned} & \sum_{\substack{|\alpha| \leq 3, \\ |\beta| \leq 2}} \left[\|\Gamma^\alpha \psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \int_{\Sigma_{t^*}} \left(\left(\Gamma^\beta(\square_g \psi + \alpha \psi) \right)^2 + \left(\mathbb{A}(\square_g \psi + \alpha \psi) \right)^2 \right) r^2 dr d\omega \right] \\ & \geq C \left| \int_{\Sigma_{t^*}} \left[\frac{g^{\theta\theta}}{\sqrt{|\det g|}} \partial_\theta(\sqrt{|\det g|}) \mathbb{A} \partial_\theta \psi + \partial_\theta(g^{\theta\theta}) \mathbb{A} \partial_\theta \psi + \mathbb{A}^2 \psi \right] \right| \end{aligned}$$

$$\left| -\varepsilon(\mathbb{A}^2\psi)^2 \right] \xi(r) \mathbb{A}^2\psi \cdot r^2 dr d\omega \Big| \quad (4.3.28)$$

The first two terms on the right-hand side of (4.3.28) are controlled by (4.3.27). The third term controls $\nabla^4\psi$ via an application of Corollary 4.3.5, after absorbing the Gauss curvature term of bad sign as before. This gives

$$\begin{aligned} & \sum_{\substack{|\alpha|\leq 3, \\ |\beta|\leq 2}} \left[\|\Gamma^\alpha\psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \int_{\Sigma_{t^*}} \left(\left(\Gamma^\beta(\square_g\psi + \alpha\psi) \right)^2 + \left(\mathbb{A}(\square_g\psi + \alpha\psi) \right)^2 \right) r^2 dr d\omega \right] \\ & \geq C \int_{\Sigma_{t^*}} \xi(r) \left[|\nabla^3\psi|^2 + |\nabla^4\psi|^2 \right] r^2 dr d\omega, \end{aligned}$$

for some $C > 0$. □

4.3.2.4 Estimating general derivatives on the entire exterior region (IV)

Given Proposition 4.3.7 and Theorem 4.3.10, we now have the following estimate for general k^{th} -order spatial derivatives on the entire slice Σ_{t^*} .

Theorem 4.3.12 (A non-degenerate k^{th} -order L^2 estimate on spacelike slices of KAdS). Let ψ be a function in $H_{KAdS}^k(\Sigma_{t^*})$ for $k \geq 2$ satisfying Dirichlet boundary conditions (4.2.11). Then,

$$\begin{aligned} & \sum_{\substack{|\sigma|=k, \\ \sigma_2 \leq k-1}} \int_{\Sigma_{t^*}} r^{2\sigma_2} |\nabla^{\sigma_1} \partial_{t^*}^{\sigma_2} \partial_r^{\sigma_3} \psi|^2 r^2 dr d\omega \\ & \leq C \sum_{\substack{|\alpha|\leq k-1, \\ |\beta|\leq k-2, \\ |\gamma|\leq \lceil \frac{k-2}{2} \rceil}} \left[\|\Gamma^\alpha\psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \int_{\Sigma_{t^*}} \left(\left(\Gamma^\beta(\square_g\psi + \alpha\psi) \right)^2 + \left(\square_g^\gamma(\square_g\psi + \alpha\psi) \right)^2 \right) r^2 dr d\omega \right] \end{aligned} \quad (4.3.29)$$

for some $C > 0$.

4.3.3 Sobolev embedding in spacelike slices

We recall the Sobolev embedding result for a general complete Riemannian manifold with positive injectivity radius and Ricci curvature bounded below [31], stating it for the spheres $S_{t^*,r}^2 \subset \mathcal{M}$. Theorem 4.3.14, an embedding on spacelike slices Σ_{t^*} will follow as a corollary.

Theorem 4.3.13 (L^∞ Sobolev embedding on spheres $S_{t^*,r}^2$ of KAdS [31, Thm 3.4]). Fix (t^*, r) and consider the Riemannian 2-submanifold $S_{t^*,r}^2$ of $\mathcal{M}$ equipped with the induced metric. Let $u : S_{t^*,r}^2 \rightarrow \mathbb{R}$ be a function in the Sobolev space

$$H_{S_{t^*,r}^2}^2 = \left\{ f : S_{t^*,r}^2 \rightarrow \mathbb{R} \mid \int_{S_{t^*,r}^2} \sum_{j=0}^2 |\nabla^j f|^2 r^2 d\omega < \infty \right\}.$$

Then one can bound u uniformly,

$$\sup_{S_{t^*,r}^2} |u|^2 \leq \tilde{c} \int_{S_{t^*,r}^2} \sum_{j=0}^2 |\nabla^j u|^2 r^2 d\omega,$$

for some $\tilde{c} > 0$.

Applying Theorem 4.3.12 and Theorem 4.3.13 allows us to derive the following embeddings.

Theorem 4.3.14 (L^∞ Sobolev embedding for normalised derivatives on spacelike slices Σ_{t^*} of KAdS). Fix t^* and let ψ be a function in $H_{KAdS}^k(\Sigma_{t^*})$ for $k \geq 3$ satisfying Dirichlet boundary conditions (4.2.11). Then one can bound ψ and its D derivatives (4.2.7) up to order $|\sigma| \leq k - 3$ uniformly as

$$\begin{aligned} & \|D^\sigma \psi\|_{L^\infty(\Sigma_{t^*})} \\ & \leq \frac{C}{r^{\frac{1}{2}}} \left[\sum_{\substack{|\alpha| \leq |\sigma|+2, \\ |\beta| \leq |\sigma|+1, \\ |\gamma| \leq \lceil \frac{|\sigma|+1}{2} \rceil}} \left[\|\Gamma^\alpha \psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \int_{\Sigma_{t^*}} \left(\left(\Gamma^\beta (\square_g \psi + \alpha \psi) \right)^2 + \left(\square_g^\gamma (\square_g \psi + \alpha \psi) \right)^2 \right) r^2 dr d\omega \right] \right]^{\frac{1}{2}} \end{aligned} \quad (4.3.30)$$

for some $C > 0$. Up to order $|\sigma| = k - 4$, one has estimate (4.3.30) for the $\overline{D}$ derivatives (4.2.7) of ψ .

Proof. Given that $\psi|_{\mathcal{I}^-} = 0$, we can apply the Fundamental Theorem of Calculus to obtain

$$\begin{aligned} |D^\sigma \psi| & \leq \int_r^\infty |\partial_r D^\sigma \psi| dr \\ & \leq \left(\int_r^\infty \frac{1}{r^2} dr \right)^{\frac{1}{2}} \left(\int_r^\infty |\partial_r D^\sigma \psi|^2 r^2 dr \right)^{\frac{1}{2}} \\ & \leq \frac{1}{r^{\frac{1}{2}}} \left(\int_r^\infty |\partial_r D^\sigma \psi|^2 r^2 dr \right)^{\frac{1}{2}}. \end{aligned} \quad (4.3.31)$$

Then, by Theorem 4.3.13, we have that

$$\begin{aligned} & \int_r^\infty |\partial_r D^\sigma \psi|^2 r^2 dr \\ & \leq \tilde{c} \sum_{j=0}^2 \int_{\Sigma_{t^*}} |\nabla^j \partial_r D^\sigma \psi|^2 r^4 dr d\omega \\ & \leq C \sum_{\substack{|\alpha| \leq |\sigma|+2, \\ |\beta| \leq |\sigma|+1, \\ |\gamma| \leq \lceil \frac{|\sigma|+1}{2} \rceil}} \left[\|\Gamma^\alpha \psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \int_{\Sigma_{t^*}} \left(\left(\Gamma^\beta (\square_g \psi + \alpha \psi) \right)^2 + \left(\square_g^\gamma (\square_g \psi + \alpha \psi) \right)^2 \right) r^2 dr d\omega \right] \end{aligned} \quad (4.3.32)$$

for some $C > 0$, where (4.3.32) follows from Theorem 4.3.12. Thus,

$$|D^\sigma \psi| \leq \frac{C}{r^{\frac{1}{2}}} \left[\sum_{\substack{|\alpha| \leq |\sigma|+2, \\ |\beta| \leq |\sigma|+1, \\ |\gamma| \leq \lceil \frac{|\sigma|+1}{2} \rceil}} \left[\|\Gamma^\alpha \psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \int_{\Sigma_{t^*}} \left(\left(\Gamma^\beta (\square_g \psi + \alpha \psi) \right)^2 + \left(\square_g^\gamma (\square_g \psi + \alpha \psi) \right)^2 \right) r^2 dr d\omega \right] \right]^{\frac{1}{2}}$$

for some $C > 0$, as required. The estimate for $\overline{D}^\sigma \psi$ with $|\sigma| \leq k - 4$ follows similarly, where the top-order T derivative is controlled with a stronger weight by the zeroth-order term of (4.2.9) after commuting $|\sigma|$ times with T . $\square$

We also note the following Sobolev embedding for derivatives where only 1 is arbitrary.

Theorem 4.3.15 (L^∞ Sobolev embedding on spacelike slices of KAdS for derivatives where only 1 is arbitrary). Fix t^* and let ψ be a function in $H_{KAdS}^k(\Sigma_{t^*})$ for $k \geq 3$ satisfying Dirichlet boundary conditions (4.2.11). Then, for each $|\sigma| \leq k - 4$, one has

$$\begin{aligned} & \|N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} D\psi\|_{L^\infty(\Sigma_{t^*})} \\ & \leq \frac{C}{r^{\frac{1}{2}}} \left[\sum_{\substack{|\alpha| \leq |\sigma|+3, \\ |\beta| \leq |\sigma|+2, \\ |\gamma| \leq \lceil \frac{|\sigma|+2}{2} \rceil}} \left[\|\Gamma^\alpha \psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \int_{\Sigma_{t^*}} \left(\left(\Gamma^\beta (\square_g \psi + \alpha \psi) \right)^2 + \left(\square_g^\gamma (\square_g \psi + \alpha \psi) \right)^2 \right) r^2 dr d\omega \right] \right]^{\frac{1}{2}} \end{aligned} \quad (4.3.33)$$

4.3.4 Energy estimates for the linear inhomogeneous problem

We now prove general n^{th} -order energy estimates for a solution ψ of (4.2.21). We begin with a result for the problem (4.2.21) with general inhomogeneity $\tilde{\mathcal{F}}$.

Proposition 4.3.16 (An n^{th} -order energy estimate for solutions of the linear inhomogeneous problem on KAdS). Suppose ψ is a CH_{KAdS}^k solution of (4.2.21), with inhomogeneity $\tilde{\mathcal{F}}$ satisfying

$$\begin{aligned} & \sum_{|\sigma| \leq n-1} \|N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} \tilde{\mathcal{F}}\|_{L^2(\Sigma_{t^*})} < \infty, \\ & \sum_{\substack{|\beta| \leq n-2, \\ |\gamma| \leq \lceil \frac{n-2}{2} \rceil}} \int_{\Sigma_{t^*}} \left((\Gamma^\beta \tilde{\mathcal{F}})^2 + (\square_g^\gamma \tilde{\mathcal{F}})^2 \right) r^2 dr d\omega < \infty \end{aligned}$$

for all $n \leq k$, t^* in $[\tau, \tau']$. Then ψ satisfies the general n^{th} order energy estimate

$$\|\psi\|_{H_{KAdS}^n(\Sigma_{t^*})}^2$$

$$\begin{aligned}
&\leq C \left[F_{\mathcal{H}^+ \cap [\tau, \tau']}^n[\psi] + \|\psi\|_{H_{KAdS}^n(\Sigma_{\tau'})}^2 + \sum_{\substack{|\beta| \leq n-2, \\ |\gamma| \leq \lceil \frac{n-2}{2} \rceil}} \int_{\Sigma_{t^*}} \left((\Gamma^\beta \tilde{\mathcal{F}})^2 + (\Box_g^\gamma \tilde{\mathcal{F}})^2 \right) r^2 dr d\omega \right. \\
&\quad + \int_{\tau}^{\tau'} \left(C_{\kappa, n} \|\psi\|_{H^n(\Sigma_{t^*})}^2 + \sum_{|\beta| \leq n-3} \|\Gamma^\beta \tilde{\mathcal{F}}\|_{L^2(\Sigma_{t^*})}^2 \right) dt^* \\
&\quad \left. + \left(\int_{\tau}^{\tau'} \sum_{|\sigma| \leq n-1} \|N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} \tilde{\mathcal{F}}\|_{L^2(\Sigma_{t^*})}^2 dt^* \right)^2 \right], \tag{4.3.34}
\end{aligned}$$

for some $C > 0$ and all $n \leq k$ where $F_{\mathcal{H}^+ \cap [\tau, \tau']}^n$ is a flux term through the event horizon, involving derivatives of the scattering data $h_{\mathcal{H}^+}$ of up to n^{th} -order. The constant $C_{\kappa, n}$ depends on the spacetime surface gravity κ and n .

Proof. We apply the vector field N (4.2.6) as a multiplier and integrate the equation over the spacetime region $\mathcal{D}_{[\tau, \tau']} = \mathcal{M} \cap [\tau, \tau']$. By direct calculation and an application of the Divergence Theorem, one obtains that

$$\begin{aligned}
\int_{\mathcal{D}_{[\tau, \tau']}} |N\psi \cdot \tilde{\mathcal{F}}| r^2 dt^* dr d\omega &= \int_{\mathcal{D}_{[\tau, \tau']}} |N\psi (\Box_g \psi + \alpha \psi)| r^2 dt^* dr d\omega \\
&\geq C \int_{\partial \mathcal{D}_{[\tau, \tau']}} J_\mu^N[\psi] \cdot n_{\Sigma_{t^*}}^\mu \cdot \text{vol}_{\Sigma_{t^*}}
\end{aligned}$$

for some $C > 0$. Hence, by integrating (A.0.1) over $\mathcal{D}$ and applying Proposition 4.3.1, it is clear that ψ satisfies

$$\begin{aligned}
&\int_{\Sigma_\tau} J_\mu^N[\psi] \cdot n_{\Sigma_\tau}^\mu \cdot \text{vol}_{\Sigma_\tau} + \tilde{C}_\alpha \left[\int_{\Sigma_{\tau'} \cap \{r < r_1\}} \psi^2 \frac{\Sigma}{\Xi} dr d\omega + \int_{\mathcal{D}_{[\tau, \tau']} \cap \{r < r_1\}} \left(\psi^2 + (\partial_{t^*} \psi)^2 \right) \frac{\Sigma}{\Xi} dr d\omega dt^* \right] \\
&\leq C \left(\int_{\mathcal{H}^+ \cap [\tau, \tau']} J_\mu^N[\psi] \cdot n_{\mathcal{H}^+}^\mu \cdot \text{vol}_{\mathcal{H}^+} + \int_{\Sigma_{\tau'}} J_\mu^N[\psi] \cdot n_{\Sigma_{\tau'}}^\mu \cdot \text{vol}_{\Sigma_{\tau'}} \right. \\
&\quad + \int_{\mathcal{D}_{[\tau, \tau']}} \left(|N\psi \cdot \tilde{\mathcal{F}}| r^2 dr d\omega + |K^N[\psi]| \text{vol}_{\Sigma_{t^*}} \right) dt^* \\
&\quad \left. + \tilde{C}_\alpha \left[\int_{\Sigma_{\tau'} \cap \{r < r_1\}} \psi^2 \frac{\Sigma}{\Xi} dr d\omega + \int_{\mathcal{D}_{[\tau, \tau']} \cap \{r < r_1\}} \left(\psi^2 + (\partial_{t^*} \psi)^2 \right) \frac{\Sigma}{\Xi} dr d\omega dt^* \right] \right).
\end{aligned}$$

where we have used the Dirichlet boundary condition to drop the boundary term on $\mathcal{I}$. Applying Proposition 4.3.1 to the integrals on the spacelike slices gives the energy estimate

$$\begin{aligned}
\|\psi\|_{H_{KAdS}^1(\Sigma_\tau)}^2 &\leq C \left[F_{\mathcal{H}^+ \cap [\tau, \tau']}^1[\psi] + \|\psi\|_{H_{KAdS}^1(\Sigma_{\tau'})}^2 \right. \\
&\quad \left. + \int_{\mathcal{D}_{[\tau, \tau']}} \left(\|\psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \|\partial_{t^*} \psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + |N\psi \cdot \tilde{\mathcal{F}}| r^2 dr d\omega \right) dt^* \right]
\end{aligned}$$

$$+ |K^N[\psi]| \text{vol}_{\Sigma_{t^*}}) dt^*],$$

where

$$F_{\mathcal{H}^+ \cap [\tau, \tau']}^1[\psi] = \int_{\mathcal{H}^+ \cap [\tau, \tau']} J_\mu^N[\psi] \cdot n_{\mathcal{H}^+}^\mu \cdot \text{vol}_{\mathcal{H}^+}.$$

We shall refer to the higher-order analogues of this horizon flux term as

$$F_{\mathcal{H}^+ \cap [\tau, \tau']}^m[\psi] = \sum_{|\sigma| \leq m-1} \int_{\mathcal{H}^+ \cap [\tau, \tau']} J_\mu^N[\Gamma^\sigma \psi] \cdot n_{\mathcal{H}^+}^\mu \cdot \text{vol}_{\mathcal{H}^+}.$$

Since we can perform the same estimate over smaller time intervals, we in fact have the following L^∞ in time estimate:

$$\begin{aligned} & \sup_{t^* \in [\tau, \tau']} \|\psi\|_{H_{KAdS(\Sigma_{t^*})}^1}^2 \\ & \leq C \left[F_{\mathcal{H}^+ \cap [\tau, \tau']}^1[\psi] + \|\psi\|_{H_{KAdS(\Sigma_{\tau'})}^1}^2 + \int_{\mathcal{D}[\tau, \tau']} \left(\|\psi\|_{H_{KAdS(\Sigma_{t^*})}^1}^2 + \|\partial_{t^*} \psi\|_{H_{KAdS(\Sigma_{t^*})}^1}^2 \right. \right. \\ & \quad \left. \left. + |N\psi \cdot \tilde{\mathcal{F}}| r^2 d\text{rd}\omega + |K^N[\psi]| \text{vol}_{\Sigma_{t^*}} \right) dt^* \right], \end{aligned}$$

In order to generate the general k^{th} -order estimate appearing in the Theorem, we commute the equation with Γ^σ :

$$\begin{aligned} & \sum_{|\sigma| \leq n-1} \sup_{t^* \in [\tau, \tau']} \|\Gamma^\sigma \psi\|_{H_{KAdS(\Sigma_{t^*})}^1}^2 \\ & \leq C \left[F_{\mathcal{H}^+ \cap [\tau, \tau']}^n[\psi] + \|\psi\|_{H_{KAdS(\Sigma_{\tau'})}^n}^2 + \int_{\mathcal{D}[\tau, \tau']} \sum_{|\sigma| \leq n-1} \left(\|\psi\|_{H_{KAdS(\Sigma_{t^*})}^n}^2 + |K^N[N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} \psi]| \text{vol}_{\Sigma_{t^*}} \right. \right. \\ & \quad \left. \left. + \left(|N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} (N\psi \cdot \tilde{\mathcal{F}})| + |[\square_g, N^{\sigma_1}](\psi)| \right) r^2 d\text{rd}\omega \right) dt^* \right]. \end{aligned} \tag{4.3.35}$$

Expanding the term in (4.3.35) arising from the inhomogeneity $\tilde{\mathcal{F}}$ via the product rule yields

$$\begin{aligned} & \int_\tau^{\tau'} \int_{\Sigma_{t^*}} \sum_{|\sigma| \leq n-1} |N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} (N\psi \cdot \tilde{\mathcal{F}})| r^2 d\text{rd}\omega dt^* \\ & \leq \int_\tau^{\tau'} \int_{\Sigma_{t^*}} \sum_{|\tilde{\sigma}| \leq |\sigma| \leq n-1} |N^{\sigma_1 - \tilde{\sigma}_1 + 1} T^{\sigma_2 - \tilde{\sigma}_2} \Phi^{\sigma_3 - \tilde{\sigma}_3} \psi \cdot N^{\tilde{\sigma}_1} T^{\tilde{\sigma}_2} \Phi^{\tilde{\sigma}_3} \tilde{\mathcal{F}}| r^2 d\text{rd}\omega dt^* \\ & \leq \int_\tau^{\tau'} \sum_{\substack{|\tilde{\sigma}| \leq n, \\ |\sigma| \leq n-1}} \|N^{\tilde{\sigma}_1} T^{\tilde{\sigma}_2} \Phi^{\tilde{\sigma}_3} \psi\|_{L^2(\Sigma_{t^*})} \|N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} \tilde{\mathcal{F}}\|_{L^2(\Sigma_{t^*})} dt^* \end{aligned} \tag{4.3.36}$$

$$\leq \frac{\varepsilon}{2} \sup_{t^* \in [\tau, \tau']} \|\psi\|_{H_{KAdS}^n(\Sigma_{t^*})}^2 + \frac{1}{2\varepsilon} \left(\int_{\tau}^{\tau'} \sum_{|\sigma| \leq n-1} \|N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} \tilde{\mathcal{F}}\|_{L^2(\Sigma_{t^*})} dt^* \right)^2 \quad (4.3.37)$$

Estimate (4.3.36) follows from relabelling indices and an application of the Cauchy-Schwarz inequality. Finally, (4.3.37) follows from the definition of the n^{th} -order energy of ψ . We choose $\varepsilon > 0$ sufficiently small to allow the first term of (4.3.37) to eventually be absorbed by the left-hand side of estimate (4.3.35).

Returning to (4.3.35), we use that the redshift vector field N is Killing on $r \geq r_1$, so that bulk terms generated by commuting with N are non-vanishing only on the compact r -region $r_+ \leq r < r_1$. This ensures no large- r bulk contribution, giving the bound

$$\begin{aligned} \int_{\Sigma_{t^*}} \sum_{|\sigma| \leq n-1} |K^N[N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} \psi]| \text{vol}_{\Sigma_{t^*}} &\leq C_{\kappa, n} \int_{\Sigma_{t^*}} \sum_{|\sigma| \leq n-1} J_{\mu}^N[N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} \psi] \cdot n_{\Sigma_{t^*}}^{\mu} \cdot \text{vol}_{\Sigma_{t^*}} \\ &\leq C_{\kappa, n} \|\psi\|_{H_{KAdS}^n(\Sigma_{t^*})}^2, \end{aligned} \quad (4.3.38)$$

where $C_{\kappa, n}$ depends on the spacetime surface gravity κ and the number of commutations. This leaves only the commutator term on the right-hand side of (4.3.35) to be treated. By direct calculation (for example, at first-order (4.3.20)) one finds that $[\square_g, N^{\sigma_1}]$ is an operator of order σ_1 . Clearly, its terms involve at most two general derivatives. Moreover, since $N = T$ near infinity (and $[\square_g, T] = 0$), one need not worry about r -weights. As such, Proposition 4.3.9 gives

$$\begin{aligned} \int_{\Sigma_{t^*}} \sum_{\sigma_1 \leq n-1} |[\square_g, N^{\sigma_1}](\psi)| &\leq C \int_{\Sigma_{t^*}} \sum_{|\sigma| \leq n-3} \left[|\Gamma^{\sigma} \nabla^2 \psi|^2 + r^2 |\Gamma^{\sigma} \nabla \partial_r \psi|^2 + r^4 (\Gamma^{\sigma} \partial_r^2 \psi)^2 \right] r^2 dr d\omega \\ &\leq C \sum_{\substack{|\alpha| \leq n-2, \\ |\beta| \leq n-3}} \left[\|\Gamma^{\alpha} \psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \int_{\Sigma_{t^*}} \left(\Gamma^{\beta} (\square_g \psi + \alpha \psi) \right)^2 r^2 dr d\omega \right] \\ &= C \sum_{\substack{|\alpha| \leq n-2, \\ |\beta| \leq n-3}} \left[\|\Gamma^{\alpha} \psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \int_{\Sigma_{t^*}} (\Gamma^{\beta} \tilde{\mathcal{F}})^2 r^2 dr d\omega \right]. \end{aligned} \quad (4.3.39)$$

Combining estimates (4.3.35), (4.3.37), (4.3.38) and (4.3.39) gives

$$\begin{aligned} &\sum_{|\sigma| \leq n-1} \sup_{t^* \in [\tau, \tau']} \|\Gamma^{\sigma} \psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 \\ &\leq C \left[F_{\mathcal{H}^+ \cap [\tau, \tau']}^n[\psi] + \|\psi\|_{H_{KAdS}^n(\Sigma_{\tau'})}^2 + \left(\int_{\tau}^{\tau'} \sum_{|\sigma| \leq n-1} \|N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} \tilde{\mathcal{F}}\|_{L^2(\Sigma_{t^*})} dt^* \right)^2 \right. \\ &\quad \left. + \int_{\tau}^{\tau'} \left(C_{\kappa, n} \|\psi\|_{H^n(\Sigma_{t^*})}^2 + \sum_{|\beta| \leq n-3} \|\Gamma^{\beta} \tilde{\mathcal{F}}\|_{L^2(\Sigma_{t^*})}^2 \right) dt^* \right], \end{aligned} \quad (4.3.40)$$

for some $C > 0$, where the first term on the right-hand side of (4.3.39) has been absorbed by the higher order $C_{\kappa, n}$ -weighted bulk term. The left-hand side of estimate (4.3.40) does not

control all n^{th} -order derivatives of ψ . In particular, the purely angular, angular-radial and purely radial derivatives of order $n \geq 2$ are missing.

By Theorem 4.3.12, we have that

$$\begin{aligned}
& \sum_{|\sigma| \leq n} \int_{\Sigma_{t^*}} r^{2\sigma_2} |\nabla^{\sigma_1} \partial_r^{\sigma_2} \psi|^2 r^2 dr d\omega \\
& \leq C \sum_{\substack{|\alpha| \leq n-1, \\ |\beta| \leq n-2, \\ |\gamma| \leq \lceil \frac{n-2}{2} \rceil}} \left[\|\Gamma^\alpha \psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \int_{\Sigma_{t^*}} \left(\left(\Gamma^\beta (\square_g \psi + \alpha \psi) \right)^2 + \left(\square_g^\gamma (\square_g \psi + \alpha \psi) \right)^2 \right) r^2 dr d\omega \right] \\
& = C \sum_{\substack{|\alpha| \leq n-1, \\ |\beta| \leq n-2, \\ |\gamma| \leq \lceil \frac{n-2}{2} \rceil}} \left[\|\Gamma^\alpha \psi\|_{H_{KAdS}^1(\Sigma_{t^*})}^2 + \int_{\Sigma_{t^*}} \left((\Gamma^\beta \tilde{\mathcal{F}})^2 + (\square_g^\gamma \tilde{\mathcal{F}})^2 \right) r^2 dr d\omega \right], \tag{4.3.41}
\end{aligned}$$

Finally, combining estimates (4.3.40) and (4.3.41) produces estimate (4.3.34) for all derivatives of ψ up to n^{th} -order. $\square$

4.3.5 Estimating the nonlinearity

Next, we estimate the nonlinearity $\mathcal{F}$ appearing in problem (4.2.21) of the form (4.2.14). This will eventually allow us to treat the terms involving $\mathcal{F}$ on the right-hand side of estimate (4.3.34). We begin with the following proposition which provides an estimate for the top-order $\mathcal{F}$ terms on the right-hand side of (4.3.34). Then we will comment on treating the other (lower-order) $\mathcal{F}$ -dependent terms on the right-hand side of (4.3.34). We note that Proposition 4.3.17 confirms that the nonlinearity (4.2.14) indeed satisfies Condition 4.2.4.

Proposition 4.3.17 (Estimates for the nonlinearity on KAdS). *Let ψ be as in Proposition 4.3.16 with $k \geq 9$ and suppose $\mathcal{F}$ is a nonlinearity of the form (4.2.14). Then $\mathcal{F}$ satisfies the estimates*

$$\begin{aligned}
& \sum_{|\sigma| \leq k-1} \|N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} \mathcal{F}\|_{L^2(\Sigma_{t^*})} \leq C \|\psi\|_{H_{KAdS}^k(\Sigma_{t^*})} \|\psi\|_{H_{KAdS}^{k-1}(\Sigma_{t^*})} \\
& \sum_{|\gamma| \leq \lceil \frac{k-2}{2} \rceil} \|\square_g^\gamma \mathcal{F}\|_{L^2(\Sigma_{t^*})}^2 \leq \tilde{C} \|\psi\|_{H_{KAdS}^{k-1}(\Sigma_{t^*})}^4
\end{aligned} \tag{4.3.42}$$

for some $C, \tilde{C} > 0$.

Proof. By the chain rule and assumption (4.2.14) on $\mathcal{F}$, we have that

$$\begin{aligned}
\sum_{|\sigma| \leq k-1} \|N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} \mathcal{F}\|_{L^2(\Sigma_{t^*})} &= \sum_{|\sigma| \leq k-1} \|N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} (F^{\mu\nu}(t^*, x) \cdot e_\mu \psi e_\nu \psi)\|_{L^2(\Sigma_{t^*})} \\
&\leq C_F \sum_{\substack{|\sigma| \leq k-1 \\ \delta \leq \sigma_1, \gamma \leq \sigma_2, \rho \leq \sigma_3 \\ 0 \leq \mu, \nu \leq 3}} \|N^{\sigma_1 - \delta} T^{\sigma_2 - \gamma} \Phi^{\sigma_3 - \rho} e_\mu \psi \cdot N^\delta T^\gamma \Phi^\rho e_\nu \psi\|_{L^2(\Sigma_{t^*})},
\end{aligned} \tag{4.3.43}$$

where C_F is a constant depending on the functions $F^{\mu\nu}(t^*, x)$ appearing in $\mathcal{F}$ (4.2.14). Assuming, without loss of generality, that $\delta + \gamma \leq \frac{k-1}{2}$ yields

$$\begin{aligned}
& \sum_{\substack{|\sigma| \leq k-1 \\ \delta \leq \sigma_1, \gamma \leq \sigma_2, \rho \leq \sigma_3 \\ 0 \leq \mu, \nu \leq 3}} \|N^{\sigma_1 - \delta} T^{\sigma_2 - \gamma} \Phi^{\sigma_3 - \rho} e_\mu \psi \cdot N^\delta T^\gamma \Phi^\rho e_\nu \psi\|_{L^2(\Sigma_{t^*})} \\
& \leq \sum_{\substack{|\sigma| \leq k-1 \\ |\delta| \leq \frac{k-1}{2}, \\ 0 \leq \mu, \nu \leq 3}} \|N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} e_\mu \psi\|_{L^2(\Sigma_{t^*})} \|N^{\delta_1} T^{\delta_2} \Phi^{\delta_3} e_\nu \psi\|_{L^\infty(\Sigma_{t^*})} \\
& \leq C \|\psi\|_{H_{KAdS}^k(\Sigma_{t^*})} \left[\|\psi\|_{H_{KAdS}^{k-1}(\Sigma_{t^*})} + \sum_{|\gamma| \leq \lceil \frac{|\delta|+2}{2} \rceil} \|\square_g^\gamma \mathcal{F}\|_{L^2(\Sigma_{t^*})} \right]. \tag{4.3.44}
\end{aligned}$$

Here, the last inequality follows from Theorem 4.3.15 and the assumption $k \geq 9$. By the assumption $k \geq 9$, the $\square$ terms on the right-hand side of (4.3.44) are of lower-order than those appearing in (4.3.42), so we demonstrate how to treat those first, as follows.

We expand the $\square_g$ term as follows, treating the second-order terms in each $\square$ explicitly. The other terms are dealt with similarly.

$$\begin{aligned}
& \sum_{|\gamma| \leq \lceil \frac{k-2}{2} \rceil} \|\square_g^\gamma \mathcal{F}\|_{L^2(\Sigma_{t^*})}^2 \\
& \leq C \sum_{|\gamma| \leq \lceil \frac{k-2}{2} \rceil} \left\| (r^2 \partial_r^2)^{\gamma_1} \left(\frac{1}{r^2} \partial_{t^*}^2 \right)^{\gamma_2} \left(\frac{1}{r^2} \partial_\theta^2 \right)^{\gamma_3} \left(\frac{1}{r^2} \partial_\phi^2 \right)^{\gamma_4} (F^{\mu\nu}(t^*, x) \cdot e_\mu \psi e_\nu \psi) \right\|_{L^2(\Sigma_{t^*})}^2 + \dots \\
& \leq C_F \sum_{\substack{|\gamma| \leq \lceil \frac{k-2}{2} \rceil, \\ \delta \leq \gamma_1, \rho \leq \gamma_2, \sigma \leq \gamma_3, \beta \leq \gamma_4 \\ 0 \leq \mu, \nu \leq 3}} \left\| (r^2 \partial_r^2)^{\gamma_1 - \delta} \left(\frac{1}{r^2} \partial_{t^*}^2 \right)^{\gamma_2 - \rho} \dots e_\mu \psi \cdot (r^2 \partial_r^2)^\delta \left(\frac{1}{r^2} \partial_{t^*}^2 \right)^\rho \dots e_\nu \psi \right\|_{L^2(\Sigma_{t^*})}^2,
\end{aligned}$$

where C_F is a constant depending on the functions $F^{\mu\nu}(t^*, x)$ appearing in $\mathcal{F}$ (4.2.14). Assuming, without loss of generality, that $\delta + \rho + \sigma + \beta \leq \frac{1}{2} \lceil \frac{k-2}{2} \rceil \leq \frac{k-2}{4}$ yields

$$\begin{aligned}
& \sum_{\substack{|\gamma| \leq \lceil \frac{k-2}{2} \rceil, \\ \delta \leq \gamma_1, \rho \leq \gamma_2, \sigma \leq \gamma_3, \beta \leq \gamma_4 \\ 0 \leq \mu, \nu \leq 3}} \left\| (r^2 \partial_r^2)^{\gamma_1 - \delta} \left(\frac{1}{r^2} \partial_{t^*}^2 \right)^{\gamma_2 - \rho} \dots e_\mu \psi \cdot (r^2 \partial_r^2)^\delta \left(\frac{1}{r^2} \partial_{t^*}^2 \right)^\rho \dots e_\nu \psi \right\|_{L^2(\Sigma_{t^*})}^2 \\
& \leq \sum_{\substack{|\gamma| \leq \lceil \frac{k-2}{2} \rceil, \\ |\sigma| \leq \frac{k-2}{4}, \\ 0 \leq \mu, \nu \leq 3}} \left\| (r^2 \partial_r^2)^{\gamma_1} \left(\frac{1}{r^2} \partial_{t^*}^2 \right)^{\gamma_2} \dots e_\mu \psi \right\|_{L^2(\Sigma_{t^*})}^2 \left\| (r^2 \partial_r^2)^{\sigma_1} \left(\frac{1}{r^2} \partial_{t^*}^2 \right)^{\sigma_2} \dots e_\nu \psi \right\|_{L^\infty(\Sigma_{t^*})}^2 \\
& \leq C \|\psi\|_{H_{KAdS}^{k-1}(\Sigma_{t^*})}^4,
\end{aligned}$$

for some $C > 0$. Here, we have applied Theorem 4.3.14 (treating lower order terms on the right-hand side of (4.3.30) via the same argument) and the assumption $k \geq 9$.

Applying the same argument to the $\square$ terms on the right-hand side of (4.3.44), we have that

$$\sum_{|\sigma| \leq k-1} \|N^{\sigma_1} T^{\sigma_2} \Phi^{\sigma_3} \mathcal{F}\|_{L^2(\Sigma_{t^*})} \leq C \|\psi\|_{H_{KAdS}^k(\Sigma_{t^*})} \|\psi\|_{H_{KAdS}^{k-1}(\Sigma_{t^*})}.$$

□

By the same method, we treat the other terms (of strictly lower order) on the right-hand side of (4.3.34):

$$\begin{aligned} & \sum_{\substack{|\beta| \leq k-2, \\ |\gamma| \leq \lceil \frac{k-2}{2} \rceil}} \int_{\Sigma_{t^*}} (\Gamma^\beta \mathcal{F})^2 r^2 dr d\omega + \int_{\tau}^{\tau'} \sum_{|\beta| \leq k-3} \|\Gamma^\beta \mathcal{F}\|_{L^2(\Sigma_{t^*})}^2 dt^* \\ & \leq C \int_{\tau}^{\tau'} \|\psi\|_{H_{KAdS}^{k-2}(\Sigma_{t^*})}^2 \|\psi\|_{H_{KAdS}^{k-3}(\Sigma_{t^*})}^2 dt^*, \end{aligned} \quad (4.3.45)$$

for some $C > 0$. We now have all the ingredients required to proceed to stating and proving the main result.

4.4 The main result

In order to precisely formulate the main result, we define the following k^{th} -order quantities $\mathfrak{D}_k$ which depend on the scattering data $h_{\mathcal{H}^+}$.

Definition 4.4.1 (Weighted k^{th} -order data quantities on KAdS). Given scattering data $h_{\mathcal{H}^+} : \{t^* \in [0, \infty)\} \times S_{t^*, r_+}^2 \rightarrow \mathbb{R}$ on $\mathcal{H}^+$, we define the quantity $\mathfrak{D}_k$ (which depends on derivatives of the scattering data $h_{\mathcal{H}}$ up to k^{th} -order) as

$$\mathfrak{D}_k = \sum_{|\sigma| \leq k} \int_{S_{t^*, r_+}^2} \int_0^\infty |\exp(\Upsilon_k \cdot t^*) \nabla^{\sigma_1} \partial_{t^*}^{\sigma_2} h_{\mathcal{H}^+}|^2 dt^* d\omega,$$

where Υ_k is a constant depending on k .

Given this, we now state the main result.

Theorem 4.4.2 (Exponentially decaying nonlinear waves on Kerr-AdS). Let $t_0^* > 0$ be fixed, $k \geq 9$ and let $h_{\mathcal{H}^+} : \mathbb{R}^+ \times S_{t^*, r_+}^2 \rightarrow \mathbb{R}$ be sufficiently smooth scattering data with

$$\mathfrak{D}_k < \infty \quad (4.4.1)$$

for associated constant (see Definition 4.4.1) $\Upsilon_{\kappa,k} \geq B_k$, with $B_k > 0$ sufficiently large depending on k and the spacetime surface gravity κ . Then there exists a solution $\psi : \{t^* \geq t_0^*\} \times \{r > r_+\} \times S_{t^*,r}^2 \rightarrow \mathbb{R}$ in CH_{KAdS}^k of problem (4.2.21) satisfying

$$\|\psi\|_{H_{KAdS}^k(\Sigma_{t^*})}^2 \leq \overline{C} \exp(-B_k \cdot t^*) \mathfrak{D}_k. \quad (4.4.2)$$

for some $\overline{C} > 0$. Moreover, the solution satisfies

$$\begin{aligned} \sum_{|\sigma| \leq k-3} \|r^{\frac{1}{2}} D^\sigma \psi\|_{L^\infty(\Sigma_{t^*})}^2 &\leq C \exp(-B_k \cdot t^*) \mathfrak{D}_k, \\ \sum_{|\sigma| \leq k-4} \|r^{\frac{1}{2}} \overline{D}^\sigma \psi\|_{L^\infty(\Sigma_{t^*})}^2 &\leq C \exp(-B_k \cdot t^*) \mathfrak{D}_k. \end{aligned} \quad (4.4.3)$$

for some $C > 0$.

Remark (Uniqueness of solutions). Note that Theorem 4.4.2 makes no claims regarding uniqueness of the solution ψ . However, by applying the estimates appearing in sections 4.3.4 and 4.3.5 to the difference of two solutions ψ'_1, ψ'_2 , one can derive uniqueness in the class of exponentially decaying solutions. One can see [8] for an analogous discussion of the problem of uniqueness in the setting of a scattering problem for exponentially decaying perturbations of the asymptotically flat Kerr black hole exterior.

4.5 The proof

This section is concerned with the proof of Theorem 4.4.2. Firstly, a sequence of finite-in-time problems approximating the full problem are constructed (Section 4.5.1). The bootstrap argument of Section 4.5.2 is used to show existence of uniformly bounded, exponentially decaying solutions of these finite problems via the bootstrap improvement argument appearing in Section 4.5.3. Finally, a standard argument is used to ensure the sequence of solutions of the finite problems converges to the target solution of the global towards the future nonlinear problem. This appears in Section 4.5.4.

4.5.1 An approximating sequence of nonlinear waves

Given a sequence $t_i^* \rightarrow \infty$ of times to the future of a fixed $t_0^* > 0$, we consider the finite-in-time problems

$$\begin{cases} \square_g \psi_i + \alpha \psi_i = \mathcal{F}_i = \mathcal{F}(D\psi_i, D\psi_i), \\ \psi_i|_{\mathcal{H}^+} = h_{\mathcal{H}^+ i}, \quad \psi_i|_{\Sigma_{t_i^*}} = 0, \quad r^{\frac{3}{2}-s} \psi_i|_{\mathcal{I}} = 0, \quad s = \sqrt{\frac{9}{4} - \alpha} \end{cases} \quad (4.5.1)$$

for $\psi_i \in CH_{KAdS}^k$. Here $h_{\mathcal{H}^+ i}$ is the appropriately truncated scattering data, satisfying

$$h_{\mathcal{H}^+ i}(t^*, r = r_+, \theta, \phi) = \begin{cases} h_{\mathcal{H}^+}(t^*, \theta, \phi) & \text{for } t^* \leq t_i^* - \vartheta_i, \quad \vartheta_i = \frac{1}{10} t_i^* \\ 0 & \text{for } t^* = t_i^* \end{cases} \quad (4.5.2)$$

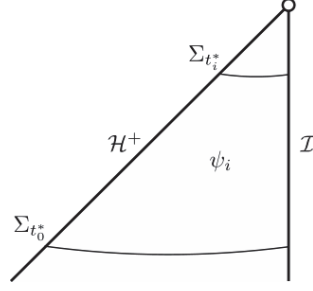


Figure 4.3: The spacetime region on which the finite-in-time problems are constructed.

and interpolating smoothly between its two cases.

We prove the following estimates on the ψ_i and their derivatives.

Theorem 4.5.1 (Uniformly bounded solutions of the finite problems on KAdS). Let $t_0^* > 0$ be fixed, $k \geq 9$ and let $h_{\mathcal{H}^+} : \mathbb{R}^+ \times S_{t^*, r_+}^2 \rightarrow \mathbb{R}$ be sufficiently smooth truncated scattering data with

$$\mathfrak{D}_k < \infty \quad (4.5.3)$$

for associated constant (see Definition 4.4.1) $\Upsilon_{\kappa, k} \geq B_k$, with $B_k > 0$ sufficiently large depending on k and the spacetime surface gravity κ . Then there exists a solution $\psi_i : \{t^* \in [t_0^*, t_i^*]\} \times \{r \geq r_+\} \times S_{t^*, r}^2 \rightarrow \mathbb{R} \in CH_{KAdS}^k$ of the nonlinear scattering problem (4.5.1) satisfying

$$\|\psi_i\|_{H_{KAdS}^k(\Sigma_{t^*})}^2 \leq \overline{C} \exp(-B_k \cdot t^*) \mathfrak{D}_k \quad (4.5.4)$$

for some $\overline{C} > 0$. Moreover, the solution satisfies

$$\begin{aligned} \sum_{|\sigma| \leq k-3} \|r^{\frac{1}{2}} D^\sigma \psi_i\|_{L^\infty(\Sigma_{t^*})}^2 &\leq C \exp(-B_k \cdot t^*) \mathfrak{D}_k, \\ \sum_{|\sigma| \leq k-4} \|r^{\frac{1}{2}} \overline{D}^\sigma \psi_i\|_{L^\infty(\Sigma_{t^*})}^2 &\leq C \exp(-B_k \cdot t^*) \mathfrak{D}_k. \end{aligned} \quad (4.5.5)$$

for some $C > 0$.

4.5.2 The bootstrap argument

We prove Theorem 4.5.1 via the following bootstrap argument. Openness of the bootstrap set $\mathcal{B}$ is given by Theorem 4.5.2, which is stated and proven in Section 4.5.3. For the duration of this section and Section 4.5.3, i is fixed.

Proof of Theorem 4.5.1. Consider the set

$$\mathcal{B} = \left\{ t^* \in [t_0^*, t_i^*] \mid \exists \text{ a solution } \psi_i \text{ in } CH_{KAdS}^k \text{ of (4.5.1) on } [t^*, t_i^*] \text{ satisfying (4.5.4) } \forall \tau \in [t^*, t_i^*] \right\}. \quad (4.5.6)$$

We will show that $\mathcal{B} = [t_0^*, t_i^*]$.

- Firstly, by local existence of solutions of (4.5.1), $\mathcal{B}$ is non-empty.
- Furthermore, by local existence, continuity and the bootstrap assumption (4.5.4), any sequence of times $\tilde{\tau}_j$ in $\mathcal{B}$ converges to a limit $\tilde{\tau}$ in $\mathcal{B}$. As such, $\mathcal{B}$ is closed.
- Now, let $t^* \in \mathcal{B}$. Then, by continuity of the solution ψ_i and a further application of local existence, there exists a small open ball $\mathbf{B}_\epsilon(t^*) = (t^* - \epsilon, t^* + \epsilon)$ of radius $\epsilon > 0$ on which ψ_i solves (4.5.1) and satisfies

$$\|\psi_i(\tau)\|_{H_{\text{KAdS}}^k(\Sigma_\tau)}^2 \leq 2\overline{C} \exp(-B_k \cdot \tau) \mathfrak{D}_k. \quad (4.5.7)$$

Theorem 4.5.2 below implies that the bound (4.5.7) can be improved

$$\|\psi_i(\tau)\|_{H_{\text{KAdS}}^k(\Sigma_\tau)}^2 \leq \overline{C} \exp(-B_k \cdot \tau) \mathfrak{D}_k,$$

so that $\mathbf{B}_\epsilon(t^*) \subset \mathcal{B}$ and $\mathcal{B}$ is open.

Hence, $\mathcal{B}$ is open, closed and non-empty, implying $\mathcal{B} = [t_0^*, t_i^*]$.

The uniform bound (4.5.9) follows from Theorem 4.5.2, completing the proof. $\square$

4.5.3 Improving the bootstrap assumption

The following bootstrap assumption improvement result gives openness of the bootstrap set $\mathcal{B}$ and the uniform bound (4.5.9). These are required for the proof of Theorem 4.5.1 appearing in the previous section. The proof follows from the energy estimate for the linear inhomogeneous problem (Proposition 4.3.16) and that for the nonlinearity (Proposition 4.3.17).

Theorem 4.5.2 (Improving the bootstrap assumption on KAdS). Let $t_0^* > 0$ be fixed, $k \geq 9$ and let $h_{\mathcal{H}^+}$ be as in Theorem 4.5.1. Furthermore, let $t_0^* < T < t_i^*$ and suppose there exists a solution $\psi_i \in CH_{\text{KAdS}}^k$ of problem (4.5.1) satisfying (4.5.4) for some $\overline{C} > 0$ sufficiently large at $\tau \in [T, t_i^*]$. Then ψ_i decays in t^* with an improved constant

$$\|\psi_i\|_{H_{\text{KAdS}}^k(\Sigma_\tau)}^2 \leq \frac{\overline{C}}{2} \exp(-B_k \cdot \tau) \mathfrak{D}_k. \quad (4.5.8)$$

and satisfies

$$\begin{aligned} \sum_{|\sigma| \leq k-3} \|r^{\frac{1}{2}} D^\sigma \psi_i\|_{L^\infty(\Sigma_{t^*})}^2 &\leq C \exp(-B_k \cdot t^*) \mathfrak{D}_k, \\ \sum_{|\sigma| \leq k-4} \|r^{\frac{1}{2}} \overline{D}^\sigma \psi_i\|_{L^\infty(\Sigma_{t^*})}^2 &\leq C \exp(-B_k \cdot t^*) \mathfrak{D}_k. \end{aligned} \quad (4.5.9)$$

for $B_n = C_{\kappa, n}$ and some $C > 0$.

Proof. By Propositions 4.3.16 and 4.3.17, as well as estimate (4.3.45), ψ_i satisfies

$$\begin{aligned}
& \|\psi_i\|_{H_{KAdS}^k(\Sigma_\tau)}^2 \\
& \leq C \left[F_{\mathcal{H}^+ \cap [\tau, t_i^*]}^k[\psi_i] + \|\psi_i\|_{H_{KAdS}^k(\Sigma_{t_i^*})}^2 + \|\psi_i\|_{H_{KAdS}^{k-1}(\Sigma_\tau)}^4 + \int_\tau^{t_i^*} \|\psi_i\|_{H_{KAdS}^{k-2}(\Sigma_{t^*})}^2 \|\psi_i\|_{H_{KAdS}^{k-3}(\Sigma_{t^*})}^2 dt^* \right. \\
& \quad \left. + \int_\tau^{t_i^*} C_{\kappa,k} \|\psi_i\|_{H^k(\Sigma_{t^*})}^2 dt^* + \left(\int_\tau^{t_i^*} \|\psi_i\|_{H_{KAdS}^k(\Sigma_{t^*})} \|\psi_i\|_{H_{KAdS}^{k-1}(\Sigma_{t^*})} dt^* \right)^2 \right] \\
& \leq C \left[F_{\mathcal{H}^+ \cap [\tau, t_i^*]}^k[\psi_i] + \|\psi_i\|_{H_{KAdS}^k(\Sigma_{t_i^*})}^2 + \left(\int_\tau^{t_i^*} \|\psi_i\|_{H_{KAdS}^k(\Sigma_{t^*})} \|\psi_i\|_{H_{KAdS}^{k-1}(\Sigma_{t^*})} dt^* \right)^2 \right. \\
& \quad \left. + \int_\tau^{t_i^*} \left(\|\psi_i\|_{H_{KAdS}^{k-1}(\Sigma_{t^*})}^2 + \|\psi_i\|_{H_{KAdS}^{k-3}(\Sigma_{t^*})}^2 + C_{\kappa,k} \right) \|\psi_i\|_{H^k(\Sigma_{t^*})}^2 dt^* \right] \tag{4.5.10}
\end{aligned}$$

where

$$F_{\mathcal{H}^+ \cap [\tau, t_i^*]}^k[\psi_i] \leq C \exp(-2\Upsilon_{\kappa,k} \cdot t^*) \mathfrak{D}_k$$

for some $C > 0$, by truncation of the scattering data (4.5.2).

By the decay assumption (4.5.3) on the approximate scattering data $h_{\mathcal{H}^+}$, the assumption that $\psi_i|_{\Sigma_{t_i^*}} = 0$, and applying the bootstrap assumption (4.5.4) to the lower order norm in the final term on the right-hand side, (4.5.10) becomes

$$\begin{aligned}
\|\psi_i\|_{H_{KAdS}^k(\Sigma_\tau)}^2 & \leq C \left[\exp(-2\Upsilon_{\kappa,k} \cdot t^*) \mathfrak{D}_k + \int_\tau^{t_i^*} \left(2\overline{C} \exp(-B_k \cdot \tau) \mathfrak{D}_k + C_{\kappa,k} \right) \|\psi_i\|_{H^k(\Sigma_{t^*})}^2 dt^* \right. \\
& \quad \left. + \left(\int_\tau^{t_i^*} \|\psi_i\|_{H_{KAdS}^k(\Sigma_{t^*})} \left(\overline{C} \exp(-B_k \cdot t^*) \mathfrak{D}_k \right)^{\frac{1}{2}} dt^* \right)^2 \right] \\
& \leq C \left[\exp(-2\Upsilon_{\kappa,k} \cdot t^*) \mathfrak{D}_k + \int_\tau^{t_i^*} \left(3\overline{C} \exp(-B_k \cdot \tau) \mathfrak{D}_k + C_{\kappa,k} \right) \|\psi_i\|_{H^k(\Sigma_{t^*})}^2 dt^* \right] \tag{4.5.11}
\end{aligned}$$

Let

$$\begin{aligned}
a(C_{\kappa,k}) & = C \left(3\overline{C} \exp(-B_k \cdot \tau) \mathfrak{D}_k + C_{\kappa,k} \right), \\
b(t^*) & = C \exp(-2\Upsilon_{\kappa,k} \cdot t^*) \mathfrak{D}_k.
\end{aligned}$$

Then a Grönwall inequality [51, Proposition 2.6] yields

$$\begin{aligned}
& \|\psi_i\|_{H_{KAdS}^k(\Sigma_\tau)}^2 \\
& \leq b(t^*) + \int_\tau^{t_i^*} a(C_{\kappa,k}) b(t^*) \exp \left(\int_\tau^{t^*} a(C_{\kappa,k}) ds \right) dt^* \\
& = b(t^*) + \int_\tau^{t_i^*} a(C_{\kappa,k}) b(t^*) \exp \left((t^* - \tau) a(C_{\kappa,k}) \right) dt^*
\end{aligned}$$

$$\begin{aligned}
&= b(t^*) + C \exp(-a(C_{\kappa,k}) \cdot \tau) \int_{\tau}^{t_i^*} a(C_{\kappa,k}) \mathfrak{D}_k \exp\left((a(C_{\kappa,k}) - 2\Upsilon_{\kappa,k})t^*\right) dt^* \\
&= b(t^*) - C \exp(-a(C_{\kappa,k}) \cdot \tau) \int_{\tau}^{t_i^*} \frac{a(C_{\kappa,k}) \mathfrak{D}_k}{2\Upsilon_{\kappa,k} - a(C_{\kappa,k})} \partial_{t^*} \left(\exp\left((a(C_{\kappa,k}) - 2\Upsilon_{\kappa,k})t^*\right) \right) dt^* \\
&\leq b(t^*) + C \exp(-a(C_{\kappa,k}) \cdot \tau) \frac{a(C_{\kappa,k}) \mathfrak{D}_k}{2\Upsilon_{\kappa,k} - a(C_{\kappa,k})} \exp\left((a(C_{\kappa,k}) - 2\Upsilon_{\kappa,k})\tau\right) \\
&\leq C \exp(-2\Upsilon_{\kappa,k} \cdot \tau) \mathfrak{D}_k \left(1 + \frac{a(C_{\kappa,k})}{2\Upsilon_{\kappa,k} - a(C_{\kappa,k})} \right)
\end{aligned} \tag{4.5.12}$$

The desired improved estimate

$$||\psi_i||_{H_{KAdS(\Sigma_\tau)}^k}^2 \leq \frac{\overline{C}}{2} \exp(-B_k \cdot \tau) \mathfrak{D}_k \tag{4.5.13}$$

follows, provided $2\Upsilon_{\kappa,k} \geq B_k$ and

$$\frac{\overline{C}}{2} \left[\frac{2C}{\overline{C}} + \frac{2Ca(C_{\kappa,k})}{\overline{C}(2\Upsilon_{\kappa,k} - a(C_{\kappa,k}))} \right] \leq \frac{\overline{C}}{2}. \tag{4.5.14}$$

Inequality (4.5.14) holds, provided $2\Upsilon_{\kappa,k} > a(C_{\kappa,k})$ and $\overline{C}$ and B_k are sufficiently large, depending on $C_{\kappa,k}$, C and $\mathfrak{D}_k$.

To obtain the uniform bounds (4.5.9), we apply Theorem 4.3.14. For each $|\sigma| < k - 3$, this gives

$$\begin{aligned}
&||\overline{D}^\sigma \psi_i||_{L^\infty(\Sigma_\tau)} \\
&\leq \frac{C}{r^{\frac{1}{2}}} \left[\sum_{\substack{|\alpha| \leq |\sigma|+2, \\ |\beta| \leq |\sigma|+1, \\ |\gamma| \leq \lceil \frac{|\sigma|+1}{2} \rceil}} \left[||\Gamma^\alpha \psi||_{H_{KAdS(\Sigma_\tau)}^1}^2 + \int_{\Sigma_\tau} \left(\left(\Gamma^\beta (\square_g \psi + \alpha \psi) \right)^2 + \left(\square_g^\gamma (\square_g \psi + \alpha \psi) \right)^2 \right) r^2 dr d\omega \right] \right]^{\frac{1}{2}} \\
&= \frac{C}{r^{\frac{1}{2}}} \left[\sum_{\substack{|\alpha| \leq |\sigma|+2, \\ |\beta| \leq |\sigma|+1, \\ |\gamma| \leq \lceil \frac{|\sigma|+1}{2} \rceil}} \left[||\Gamma^\alpha \psi||_{H_{KAdS(\Sigma_\tau)}^1}^2 + \int_{\Sigma_\tau} \left((\Gamma^\beta \mathcal{F})^2 + (\square_g^\gamma \mathcal{F})^2 \right) r^2 dr d\omega \right] \right]^{\frac{1}{2}} \\
&\leq \frac{C}{r^{\frac{1}{2}}} \left[||\psi||_{H_{KAdS(\Sigma_\tau)}^{|\sigma|+3}}^2 + ||\psi||_{H_{KAdS(\Sigma_\tau)}^{|\sigma|+2}}^2 ||\psi||_{H_{KAdS(\Sigma_\tau)}^{|\sigma|+1}}^2 \right]^{\frac{1}{2}} \\
&\leq \frac{C}{r^{\frac{1}{2}}} \left(\frac{\overline{C}}{2} \right)^{\frac{1}{2}} \exp\left(-\frac{B_k}{2} \cdot \tau\right) \mathfrak{D}_{|\sigma|+3}^{\frac{1}{2}},
\end{aligned}$$

where the third inequality follows from estimate (4.3.45) and the last from (4.5.13). For $|\sigma| = k - 3$, one obtains the same estimate for D^σ . $\square$

4.5.4 Convergence of the solution

All that remains is to prove Theorem 4.4.2 as a consequence of Theorem 4.5.1.

Proof of Theorem 4.4.2. By Theorem 4.5.1, we have uniform boundedness of the solutions ψ_i to the finite problems (4.5.1) and their first-order derivatives $D\psi_i$. Since we have set $\psi_i|_{\Sigma_{t_i^*}} = 0$, each solution ψ_i trivially extends to a function which is identically zero on $[t_i^*, \infty)$. Thus, we may apply the *Arzèla-Ascoli Theorem* on the domain $[t_0^*, \infty) \times \{r \geq r_{\mathcal{H}}\} \times S_{t^*, r}^2$ to obtain a uniformly convergent subsequence $\psi_{i_j} \rightarrow \psi$. We may also apply *Arzèla-Ascoli* to the derivatives $\{D\psi_{i_j}, D^2\psi_{i_{j_k}}, \dots\}$ up to sufficiently high order to obtain uniformly convergent subsequences $D\psi_{i_j} \rightarrow D\psi$, $D^2\psi_{i_{j_k}} \rightarrow D^2\psi$. Thus, the limiting function ψ solves (4.2.10). Finally, the uniform bound (4.5.5) gives the estimates (4.4.3). $\square$

Appendices

Appendix A

The energy momentum tensor

We recall [34] the *energy-momentum tensor* of a solution ψ of (1.1.2)

$$T_{\mu\nu} \stackrel{def}{=} \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} (g^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi - \alpha \psi^2).$$

The *current* of ψ along a given vector field X is

$$J_\mu^{(X)}[\psi] \stackrel{def}{=} T_{\mu\nu}[\psi] X^\nu,$$

and the associated bulk is

$$K^X[\psi] \stackrel{def}{=} T_{\mu\nu}[\psi]^{(X)} \pi^{\mu\nu}.$$

Here, $^{(X)}\pi_{\mu\nu}$ denotes the *deformation tensor*¹ of the vector field X :

$$^{(X)}\pi_{\mu\nu} \stackrel{def}{=} \frac{1}{2} \nabla_\mu X_\nu + \nabla_\nu X_\mu.$$

Recall the relation [34]

$$\nabla^\mu (J_\mu^X[\psi]) = K^X[\psi]. \tag{A.0.1}$$

¹N.B., if X is a Killing field on $(\mathcal{M}, g)$, its deformation tensor is trivial and the resulting bulk vanishes.

Appendix B

Inequalities

B.1 Sobolev inequalities

Theorem B.1.1 (L^∞ Sobolev embedding on smooth Riemannian 2-manifolds [31, Thm 3.4]). Let $(\mathcal{M}, h)$ be a smooth Riemannian 2-manifold with Ricci curvature bounded below and positive injectivity radius. Suppose $u : \mathcal{M} \rightarrow \mathbb{R}$ is a function in the Sobolev space

$$H_{\mathcal{M}}^2 = \left\{ f : \mathcal{M} \rightarrow \mathbb{R} \mid \int_{\mathcal{M}} \sum_{j=0}^2 |\nabla^j f|^2 \text{vol}_h < \infty \right\}.$$

Then one can bound u uniformly,

$$\sup_{\mathcal{M}} |u|^2 \leq \tilde{c} \int_{\mathcal{M}} \sum_{j=0}^2 |\nabla^j u|^2 \text{vol}_h,$$

for some $\tilde{c} > 0$.

Theorem B.1.2 (L^∞ Sobolev embedding on $\mathbb{R}^3$). Let $f : \mathbb{R}^3 \rightarrow \mathbb{R} \in H^2(\mathbb{R}^3)$. Then f satisfies the Sobolev inequality:

$$\|f\|_{L^\infty(\mathbb{R}^3)} \leq C \|f\|_{H^2(\mathbb{R}^3)}, \tag{B.1.1}$$

for some $C > 0$.

Proof. Let $\hat{f}$ be the Fourier transform of f . Then for all $\mathbf{x} \in \mathbb{R}^3$,

$$\begin{aligned} f(\mathbf{x}) &= \int_{\mathbb{R}^3} \hat{f}(\xi) e^{i(\mathbf{x} \cdot \xi)} d\xi \\ \Rightarrow |f(\mathbf{x})| &\leq \int_{\mathbb{R}^3} |\hat{f}(\xi)| d\xi. \end{aligned}$$

Now, by the Cauchy-Schwarz inequality and the *Fourier Inversion Theorem* [50, Thm 7.7, p186-187], we have

$$|f(x)| \leq \left(\int_{\mathbb{R}^3} \left[(1 + |\xi|^2) |\hat{f}(\xi)| \right]^2 d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} (1 + |\xi|^2)^{-2} d\xi \right)^{\frac{1}{2}} \\ \leq C \|f\|_{H^2(\mathbb{R}^3)}$$

for some $C > 0$. □

Corollary B.1.3 (L^∞ Sobolev embedding on a bounded, open subset of $\mathbb{R}^3$). Let $\Omega \subseteq \mathbb{R}^3$ be a bounded, open set and $f : \Omega \rightarrow \mathbb{R}$ a function in $H^2(\Omega)$. Then f satisfies the Sobolev inequality

$$\|f\|_{L^\infty(\Omega)} \leq C \|f\|_{H^2(\Omega)}. \quad (\text{B.1.2})$$

Proof. By the Extension Theorem [19, 5.4, Thm 1], f extends to a compactly supported function f_{ext} on $\mathbb{R}^3$ satisfying

$$\|f_{\text{ext}}\|_{H^1(\mathbb{R}^3)} \leq C \|f\|_{H^1(\Omega)},$$

for some $C > 0$. Since f_{ext} is compactly supported, we may apply Theorem B.1.2 to obtain

$$\|f\|_{L^\infty(\Omega)} \leq \|f_{\text{ext}}\|_{L^\infty(\mathbb{R}^3)} \leq C \|f_{\text{ext}}\|_{H^2(\mathbb{R}^3)},$$

for some $C > 0$. By Remark (i) [19, 5.4, Remarks], we also have that

$$\|f_{\text{ext}}\|_{H^2(\mathbb{R}^3)} \leq C \|f\|_{H^2(\Omega)},$$

for some $C > 0$. Hence,

$$\|f\|_{L^\infty(\Omega)} \leq C \|f\|_{H^2(\Omega)},$$

for some $C > 0$, as desired. □

B.2 Hardy's inequality

Theorem B.2.1 (Hardy's inequality [49, p.72]). Let $f \in H^1(B_r(0))$, where $r > 0$ and $B_r(0)$ is the open ball of radius r centered at the origin in $\mathbb{R}^3$. Then

$$\left\| \frac{f}{|x|} \right\|_{L^2(B_r(0))} \leq 2 \|\nabla f\|_{L^2(B_r(0))},$$

where ∇ is the gradient operator.

B.3 Grönwall's inequality

Theorem B.3.1 (Grönwall's inequality [51, Proposition 2.6]). Suppose $0 < T < T_f$ and $v : [T, T_f] \rightarrow \mathbb{R}$ satisfies

$$v(t) \leq b(t) + \int_t^{T_f} a(s)v(s)ds$$

for all $T \leq t \leq T_f$, for some $a, b : [T, T_f] \rightarrow [0, \infty)$. Then

$$v(t) \leq b(t) + \int_t^{T_f} a(s)b(s) \exp\left(\int_s^t a(s')ds'\right)ds$$

for all $T \leq t \leq T_f$.

Appendix C

The Arzelà-Ascoli Theorem

In proving several results appearing in this thesis, we employ the Arzelà-Ascoli Theorem (stated below as Theorem C.0.3). We first include here the following relevant definition of equicontinuity of a sequence of functions. This is followed by Proposition C.0.2, which we shall rely on to ensure that the sequences of functions (solutions ψ_i of finite problems, for example (2.1.7)) and their derivatives) with which we work indeed satisfy the assumption of equicontinuity, required to apply Theorem C.0.3 which follows.

Definition C.0.1 (Uniform equicontinuity). We say that a sequence of functions $\{f_n\}_{n \in \mathbb{N}}$, $f_n : \mathbb{R}^m \rightarrow \mathbb{R}$ is uniformly equicontinuous if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - y| < \delta$ then $|f_n(x) - f_n(y)| < \epsilon$ for all $n \in \mathbb{N}$.

Proposition C.0.2 (Uniform equicontinuity from uniformly bounded first-order derivatives). Let $\{f_i\}_{i=1}^\infty$ be a sequence of continuous functions $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ whose first-order derivatives are everywhere well-defined and uniformly bounded. Then the sequence f_i is uniformly equicontinuous.

Proof. Let the f_i have first-order derivatives uniformly bounded by a constant M , so that for all $c \in \mathbb{R}^n$ and all multi-indices $\alpha \in \mathbb{N}^n$ with $|\alpha| = 1$

$$|\partial^\alpha f_i(c)| \leq M.$$

The f_i are continuous, so for $x_1 < x_2$, we may apply the *Mean Value Theorem* to derive existence of a $c \in \mathbb{R}^n$ satisfying

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Hence, for all $x_1 < x_2 \in \mathbb{R}^n$,

$$|f(x_2) - f(x_1)| \leq M|x_2 - x_1|$$

and so the f_i are Lipschitz continuous. It easily follows that for all $\epsilon > 0$, if $|x - y| < \frac{\epsilon}{M}$, then

$$|f_n(x) - f_n(y)| \leq M|x - y| < \epsilon$$

for all $n \in \mathbb{N}$. □

Theorem C.0.3 (Arzelà-Ascoli¹). Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of uniformly bounded and uniformly equicontinuous functions $f_n : \mathbb{R}^m \rightarrow \mathbb{R}$. Then there exists a subsequence $\{f_{n_k}\}_{n,k \in \mathbb{N}}$ which converges pointwise to a function $f : \mathbb{R}^m \rightarrow \mathbb{R}$. Moreover, the convergence is uniform on compact sets.

¹See, for example, the Corollary of Theorem A5 in [50, Appendix A, p411].

Appendix D

Regular double-null coordinates on the Schwarzschild black hole exterior

The Schwarzschild metric in double-null coordinates (u, v, θ, ϕ)

$$g_S(M) = -4 \left(1 - \frac{2M}{r} \right) du dv + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.1.13)$$

is only regular on $r > 2M$, away from the event horizon $\mathcal{H}^+$. However, the renormalised vector field

$$\frac{1}{1 - \frac{2M}{r}} \partial_u \quad (D.0.1)$$

extends regularly across $\mathcal{H}^+$. One can see this by defining new null coordinates

$$U = -\exp\left(-\frac{u}{4M}\right), \quad V = \exp\left(\frac{v}{4M}\right),$$

in terms of which the standard radial coordinate r is given implicitly by

$$-UV = \exp\left(\frac{r}{4M}\right) \frac{r}{2M} \left(1 - \frac{2M}{r}\right). \quad (D.0.2)$$

The coordinate vector fields of (u, v) and (U, V) are related by

$$\partial_u = -\frac{1}{4M \exp(\frac{u}{4M})} \partial_U, \quad \partial_v = \frac{\exp(\frac{v}{4M})}{4M} \partial_V. \quad (D.0.3)$$

Via the relations (D.0.2) and (D.0.3), one ensures regularity of the renormalised vector field (D.0.1)

$$\frac{1}{1 - \frac{2M}{r}} \partial_u = \exp\left(\frac{r}{4M}\right) \frac{r}{8M^2} \frac{1}{V} \partial_U$$

across $r = 2M$.

Bibliography

- [1] L. Andersson, T. Bäckdahl, P. Blue, and S. Ma, *Stability for linearized gravity on the Kerr spacetime*, [arXiv:1903.03859 \[gr-qc\]](#) (2019).
- [2] P. Bizón and A. Rostworowski, *Weakly turbulent instability of anti-de Sitter spacetime*, Physical Review Letters **107** (2011), no. 3, 0331102:1–4.
- [3] R.H. Boyer and R.W. Lindquist, *Maximal analytic extension of the Kerr metric*, Journal of Mathematical Physics **8** (1967), no. 2, 265–281.
- [4] P. Breitenlohner and D. Z. Freedman, *Positive energy in anti-de Sitter backgrounds and gauged extended supergravity*, Physics Letters B **115** (1982), no. 3, 197–201.
- [5] D. Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, Communications in Pure and Applied Mathematics **39** (1986), 267–282.
- [6] ———, *The formation of black holes in general relativity*, [arXiv:0805.3880 \[gr-qc\]](#) (2008).
- [7] D. Christodoulou and S. Klainerman, *The global nonlinear stability of the Minkowski space*, Princeton University Press, Princeton, 1994.
- [8] M. Dafermos, G. Holzegel, and I. Rodnianski, *A scattering theory construction of dynamical vacuum black holes*, [arXiv:1306.5364 \[gr-qc\]](#) (2013).
- [9] ———, *The linear stability of the Schwarzschild solution to gravitational perturbations*, Acta Mathematica **222** (2019), 1–214.
- [10] M. Dafermos, G. Holzegel, I. Rodnianski, and M. Taylor, *The non-linear stability of the Schwarzschild family of black holes*, [arXiv:2104.08222 \[gr-qc\]](#) (2021), 1–513.
- [11] ———, *Quasilinear wave equations on asymptotically flat spacetimes with applications to Kerr black holes*, [arXiv:2212.14093 \[gr-qc\]](#) (2022), 1–98.
- [12] M. Dafermos and I. Rodnianski, *A new physical-space approach to decay for the wave equation with applications to black hole spacetimes*, XVIth International Congress on Mathematical Physics (2010), 421–432.
- [13] M. Dafermos, I. Rodnianski, and Y. Shlapentokh-Rothman, *A scattering theory for the wave equation on Kerr black hole exteriors*, [arXiv:1412.8379 \[gr-qc\]](#) (2014).
- [14] O. J. Dias, G. T. Horowitz, D. Marolf, and J. E. Santos, *On the nonlinear stability of asymptotically anti-de Sitter solutions*, Classical and Quantum Gravity **29** (2012), no. 23, 235019:1–24.
- [15] D. Dold, *Unstable mode solutions to the Klein-Gordon equation in Kerr-anti-de Sitter spacetimes*, Communications in Mathematical Physics **350** (2017), 639–697.
- [16] R. Donniger, W. Schlag, and A. Soffer, *On pointwise decay of linear waves on a Schwarzschild black hole background*, Communications in Mathematical Physics **309** (2012), 51–86.

- [17] A. Enciso and N. Kamran, *A singular initial-boundary value problem for nonlinear wave equations and holography in asymptotically anti-de Sitter spaces*, Journal de Mathématiques Pures et Appliquées **103** (2015), no. 4, 1053–1091.
- [18] ———, *Lorentzian Einstein metrics with prescribed conformal infinity*, Journal of Differential Geometry **112** (2019), no. 3, 505–554.
- [19] L.C. Evans, *Partial Differential Equations*, 2nd ed., American Mathematical Society, Providence, Rhode Island, 2010.
- [20] A.J. Fang, *Nonlinear stability of the slowly-rotating Kerr-de Sitter family*, [arXiv:2112.07183 \[math.AP\]](#) (2021).
- [21] F.G. Friedlander, *On the radiation field of pulse solutions of the wave equation*, Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences **269** (1962), no. 1336, 53–65.
- [22] ———, *On the radiation field of pulse solutions of the wave equation. II*, Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences **279** (1964), no. 1378, 386–394.
- [23] ———, *On the radiation field of pulse solutions of the wave equation. III*, Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences **299** (1967), no. 1457, 264–278.
- [24] O. Gannot, *Quasinormal modes for Schwarzschild–AdS black holes: exponential convergence to the real axis*, Communications in Mathematical Physics **330** (2014), 771–799.
- [25] ———, *Existence of quasinormal modes for Kerr-AdS black holes*, Annales Henri Poincaré **18** (2017), 2757–2788.
- [26] E. Giorgi, S. Klainerman, and J. Szeftel, *Wave equations estimates and the nonlinear stability of slowly rotating Kerr black holes*, [arXiv:2205.14808 \[math.AP\]](#) (2022).
- [27] O. Graf and G. Holzegel, *Mode stability results for the Teukolsky equations on Kerr-anti-de Sitter spacetimes*, Classical and Quantum Gravity **40** (2023), no. 045003, 1–43.
- [28] ———, *Linear Stability of Schwarzschild-Anti-de Sitter spacetimes I: The system of gravitational perturbations*, [arXiv:2408.02251 \[gr-qc\]](#) (2024).
- [29] ———, *Linear Stability of Schwarzschild-Anti-de Sitter spacetimes II: Logarithmic decay of solutions to the Teukolsky system*, [arXiv:2408.02252 \[gr-qc\]](#) (2024).
- [30] ———, *Linear Stability of Schwarzschild-Anti-de Sitter spacetimes III: Quasimodes and sharp decay of gravitational perturbations*, [arXiv:2410.21994 \[gr-qc\]](#) (2024).
- [31] E. Hebey, *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities*, Courant Lecture Notes in Mathematics, vol. 5, New York: Courant Institute, 1999.
- [32] P. Hintz and A. Vasy, *The global non-linear stability of the Kerr-de Sitter family of black holes*, Acta Mathematica **220** (2018), no. 1, 1–206.
- [33] G. Holzegel, *Well-posedness for the massive wave equation on asymptotically anti-de Sitter spacetimes*, Journal of Hyperbolic Differential Equations **9** (2012), no. 2, 239–261.
- [34] ———, *On the massive wave equation on slowly rotating Kerr-AdS spacetimes*, Communications in Mathematical Physics **294** (2010), no. 1, 169–197.
- [35] G. Hood, *A scattering construction for nonlinear wave equations on Kerr-Anti-de Sitter spacetimes (in preparation)*.
- [36] G. Holzegel and J. Smulevici, *Decay properties of Klein-Gordon fields on Kerr-AdS spacetimes*, Communications on Pure and Applied Mathematics **66** (2013), no. 11, 1751–1802.

- [37] ———, *Quasimodes and a lower bound on the uniform energy decay rate for Kerr-AdS spacetimes*, Analysis & PDE **7** (2014), no. 5, 1057–1090.
- [38] G. Holzegel, J. Luk, J. Smulevici, and C. Warnick, *Asymptotic properties of linear field equations in Anti-de Sitter space*, Communications in Mathematical Physics **374** (2020), no. 2, 1125–1178.
- [39] ———, *The global nonlinear stability of the anti-de Sitter space with dissipative boundary conditions (in preparation)*.
- [40] G. Holzegel and C. Warnick, *Boundedness and growth for the massive wave equation on asymptotically anti-de Sitter black holes*, Journal of Functional Analysis **266** (2014), no. 4, 2436–2485.
- [41] F. John, *Blow-up for quasi-linear wave equations in three space dimensions*, Fritz John: Collected Papers **1** (1985), 566–588.
- [42] S. Klainerman, *The null condition and global existence to nonlinear wave equations*. Nonlinear Systems of Partial Differential Equations in Applied Mathematics Part I (Santa Fe, NM, 1984), Lectures in Applied Mathematics **23** (1986), 293–326.
- [43] S. Klainerman and J. Szeftel, *Kerr stability for small angular momentum*, [arXiv:2104.11857 \[math.AP\]](#) (2021).
- [44] ———, *Construction of GCM spheres in perturbations of Kerr*, Annals of PDE **8** (2022), no. 2, 1–153. Article 17.
- [45] ———, *Effective results on uniformization and intrinsic GCM spheres in perturbations of Kerr*, Annals of PDE **8** (2022), no. 2, 1–89. Article 18.
- [46] J. Luk, *Introduction to nonlinear wave equations*, Stanford University, Department of Mathematics.
- [47] G. Moschidis, *A proof of the instability of AdS for the Einstein null-dust system with an inner mirror*, Analysis and PDE **13** (2020), no. 6, 1671–1754.
- [48] ———, *A proof of the instability of AdS for the Einstein-massless Vlasov system*, Inventiones Mathematicae **231** (2023), 467–672.
- [49] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, Singapore, 1987.
- [50] ———, *Functional Analysis*, 2nd ed., McGraw-Hill, Singapore, 1991.
- [51] V. Schlue and M. Taylor, *Inverse modified scattering and polyhomogeneous expansions for the Vlasov–Poisson system*, [arXiv:2404.15885 \[math.AP\]](#) (2024).
- [52] J. Veselý and M. Žofka, *The Kerr-Newman-(anti-)de Sitter spacetime: Extremal configurations and electrogeodesics*, General Relativity and Gravitation **51** (2019), no. 11, 156:1–22.
- [53] D. Shen, *Construction of GCM hypersurfaces in perturbations of Kerr*, Annals of PDE **9** (2023), no. 1, 1–112. Article 11.
- [54] C. Warnick, *The massive wave equation in asymptotically AdS spacetimes*, Communications in Mathematical Physics **321** (2013), no. 1, 85–111.
- [55] ———, *(In)completeness of quasinormal modes*, Acta Physica Polonica B, Proceedings Supplement **15** (2022), no. 1-A13.