

A constructive method for plane-wave representations of special functions

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Abstract

A general constructive scheme for the derivation of plane-wave representations of special functions is proposed. Illustrative examples of the construction are given. As one case study, new integral representations of the elliptic Weierstrass $\wp$ function are derived; these complement, and generalize, similar new plane-wave integral representations of the same function recently found by Dienstfrey & Huang [*J. Math. Anal. Appl.*, **316**, 142–160, (2006)] using other techniques. Our approach is inspired by recent developments in the so-called Fokas transform for the solution of boundary value problems for partial differential equations.

Keywords: plane wave representation, elliptic function, transform method, special functions

1. Introduction

The present work has been inspired by a recent paper of Dienstfrey & Huang [7] in which some new integral representations of the classical elliptic functions, and some other related functions, are derived. Among other results, those authors derive the following integral representation of the Weierstrass $\wp$ function:

$$\wp(z) = \frac{1}{z^2} + 8 \int_0^\infty \lambda \left[e^{-\lambda} \sinh^2 \left(\frac{z\lambda}{2} \right) f_1(\lambda, \tau) + e^{i\tau\lambda} \sin^2 \left(\frac{z\lambda}{2} \right) f_2(\lambda, \tau) \right] d\lambda, \quad (1)$$

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where

$$f_1(\tau, \lambda) = \frac{\cosh^2(\tau\lambda/2)}{1 - 2e^{-\lambda} \cosh(\tau\lambda) + e^{-2\lambda}}, \quad f_2(\tau, \lambda) = \frac{\cos^2(\lambda/2)}{1 - 2e^{i\tau\lambda} \cos(\lambda) + e^{2i\tau\lambda}}, \quad (2)$$

and τ is the lattice ratio. (1) is not valid for all $z \in \mathbb{C}$; it has a finite domain of validity as a consequence of the particular derivation given in [7]. Indeed, for (1) to be valid it is necessary that $z \in D(\tau)$ defined by

$$D(\tau) = \{z | \operatorname{Re}[-1 \pm z \pm \tau] < 0, \operatorname{Im}(\tau \pm z) > 0\}. \quad (3)$$

Unfortunately, this region may not even include the entire fundamental polygon of the doubly periodic function. Dienstfrey & Huang [7] discuss the possibility of deriving fast numerical routines based on their new integral representations but they acknowledge the inconvenience of having such cumbersome restrictions on the domains of validity. Expression (1) is derived using Laplace-Mellin representations of multipoles together with results on the summation of two-dimensional geometric series.

The purpose of this paper is to find *other* – also apparently new – plane-wave integral representations for the Weierstrass $\wp$ function that have strong similarities to (1). But the results are different and, in particular, they enjoy the important distinction that they are valid throughout the *entire* fundamental polygon and with no cumbersome restrictions on various sub-domains of validity. More significantly, the construction here is very general and we use the Weierstrass $\wp$ function as an important example to showcase the possibilities of the method for other special functions.

Dienstfrey & Huang [7] were motivated in their enquiries by the fact that so-called “plane-wave representations” have proven to be a valuable tool in the development of recent embodiments of fast multipole methods [13, 14]. In particular, an elementary but significant observation made by Hrycak & Rokhlin [14], and which underlies a significant improvement in the numerical implementation of fast multipole methods, is that if z and z' are complex numbers with

$\text{Re}(z - z') > 0$ then

$$\frac{1}{z - z'} = \int_0^\infty e^{-x(z - z')} dx. \quad (4)$$

This simple mathematical fact is stated as Lemma 2.8 of [14]. But precisely the *same* mathematical observation has recently been used by the author [4] as the basis for a new and elementary derivation of a transform pair for analytic functions defined over simply connected convex polygons of general shape. Those transform pairs were first written down by Fokas & Kapaev [11] who derived them using completely different methods based on the spectral analysis of a Lax pair and use of Riemann-Hilbert methods. The author's new derivation in [4] leads the way to extensions to find generalized transform pairs for *circular domains*, that is, domains whose boundaries are made up of circular arcs and straight lines [4, 5] (this even includes multiply connected domains [5]).

Inspection of (1) reveals that its integrands are of the form where the only z dependence appears in purely exponential factors of the form $e^{i\lambda z}$ (see (95) of §6 to see this explicitly). While referred to as “plane-wave representations” they are also known as *Ehrenpreis-type* integral representations [10] and the significance of the latter has resurfaced in recent years as a consequence of investigations into a new transform approach to both linear and nonlinear PDEs pioneered in recent decades by A. S. Fokas and collaborators [9, 10, 2].

One aim of the present paper is to indicate connections between all the mathematical ideas just described: plane wave representations, elliptic and other special functions, Ehrenpreis integral representations and the Fokas transform method.

A key result here is to derive the following new plane-wave integral representation for the Weierstrass $\wp$ function with periods $2l$ and $2hi$:

$$\begin{aligned} \wp(z) = & \frac{1}{z^2} + \frac{1}{2\pi} \int_{L_1} \rho_1(k) [e^{ikz} + e^{-ikz} - 2] dk \\ & + \frac{1}{2\pi} \int_{L_2} \left[\rho_2(k) + \frac{k}{1 - e^{2ikl}} \right] [e^{ikz} + e^{-ikz} - 2] dk, \end{aligned} \quad (5)$$

where $\rho_1(k)$ and $\rho_2(k)$ are entire functions given explicitly by the rapidly con-

vergent sums

$$\begin{aligned}\rho_1(k) &= -2\pi^2 e^{-kh} \sin(kl) \sum_{m \geq 1} \frac{(-1)^m m \rho^m}{k^2 l^2 - m^2 \pi^2} \left\{ \frac{1 + \rho^{2m}}{1 - \rho^{2m}} k + \frac{m\pi}{l} \right\}, \\ \rho_2(k) &= i\pi^2 e^{-kh} e^{-ikl} \sum_{m \geq 1} \frac{(-1)^m m \rho^m}{k^2 l^2 - m^2 \pi^2} \left[\frac{1 + \rho^{2m}}{1 - \rho^{2m}} k (1 - e^{2kh}) + \frac{m\pi}{l} (1 + e^{2kh}) \right],\end{aligned}\tag{6}$$

and where $\rho = e^{-\pi h/l} < 1$ is set by the lattice ratio. The integral representation (5) is valid *everywhere* in the fundamental polygon. It is different to (1) found
40 by Dienstfrey & Huang [7].

The idea of our constructive method is general and can be applied to find analogous plane-wave integral representations of other special functions. It is believed that our new understanding of the results of [7] might well have further mathematical ramifications, not least for other functions closely related to
45 elliptic function theory such as Eisenstein series and the Riemann zeta function (also discussed in [7]). Indeed our work here follows in the same spirit as recent work by Fokas & Glasser [12] who study a boundary value problem for Laplace's equation in a special domain (the exterior to the so-called Hankel contour) and also make use of mathematical ideas derived from the Fokas transform approach,
50 together with some conformal mapping ideas. In this way those authors find some novel identities for certain special functions including the gamma function and the hypergeometric functions. They are also able to point out an intriguing mathematical connection between a Neumann boundary value problem for Laplace's equation and the Riemann hypothesis.

55 2. A transform method for polygons

Suppose P is an N -sided simply connected, convex polygon as shown in Figure 1 for the case $N = 4$. The main result to be used in our construction of plane-wave integral representations is the following representation for a function

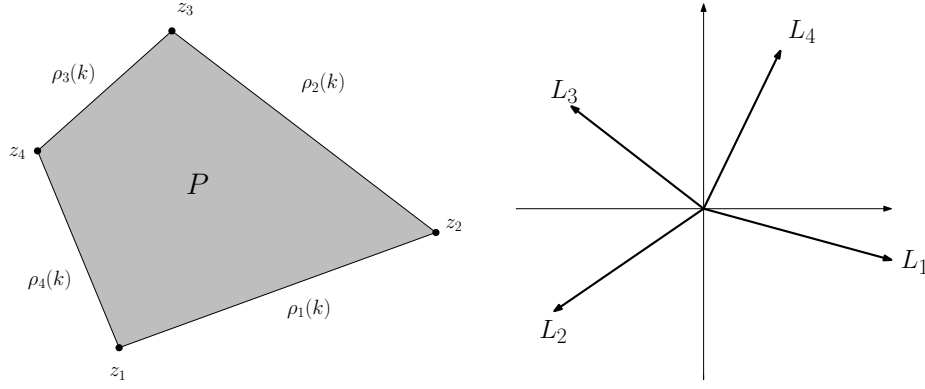


Figure 1: A simply connected convex polygon P and spectral rays $\{L_j | j = 1, \dots, N\}$.

$f(z)$ analytic in some such P :

$$f(z) = \frac{1}{2\pi} \sum_{j=1}^N \int_{L_j} \rho_j(k) e^{ikz} dk, \quad (7)$$

$$\rho_j(k) = \int_{\text{edge } j} f(z) e^{-ikz} dz,$$

where L_j is the ray in the complex k -plane having argument that is minus the angle made by side j to the real axis in the z -plane. The so-called *spectral functions* $\{\rho_j(k)\}$ satisfy the following *global relation*:

$$\sum_{j=1}^N \rho_j(k) = 0, \quad k \in \mathbb{C}. \quad (8)$$

Fokas & Kapaev [11] first derived this result using a Lax pair formulation together with Riemann-Hilbert methods.

A more elementary rederivation of the transform pair (7) has been given recently by the author [4]. We now reproduce this derivation for two reasons: first, it is brief and instructive; second, it is based on the following elementary geometrical observation which turns out to be precisely the simple result (4) of Lemma 2.8 of Hrycak and Rokhlin [14] and used there to convert multipole expansions into exponential ones for use in a modified fast multipole scheme. In [4] the same simple mathematical fact is stated in the following form. Suppose a point z' lies on some finite length slit on the real axis and z is in the upper

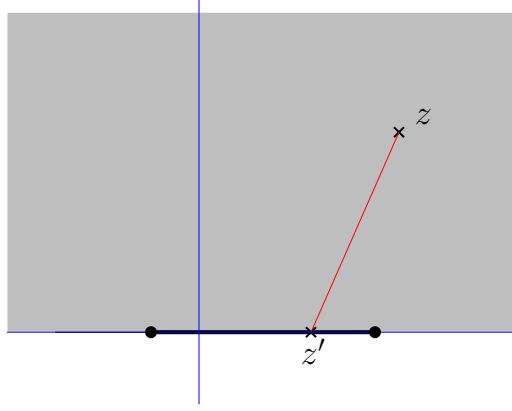


Figure 2: Geometrical positioning of z and z' for the validity of (11).

half plane – see Figure 2 – then clearly

$$0 < \arg[z - z'] < \pi. \quad (9)$$

It follows trivially that

$$\int_0^\infty e^{ik(z-z')} dk = \left[\frac{e^{ik(z-z')}}{i(z-z')} \right]_0^\infty = \frac{1}{i(z'-z)} \quad (10)$$

or,

$$\frac{1}{z'-z} = i \int_0^\infty e^{ik(z-z')} dk, \quad 0 < \arg[z - z'] < \pi. \quad (11)$$

It is easy to check that the contribution from the upper limit of integration vanishes for the particular choices of z' and z to which we have restricted consideration.

On the other hand, suppose z' lies on some other finite length slit making angle χ with the positive real axis and suppose that z is in the half plane shown in Figure 3 (the half-plane “to the left” of the slit as one follows its tangent with uniform inclination angle χ). Now the affine map

$$z' \mapsto e^{-i\chi}(z' - \alpha), \quad z \mapsto e^{-i\chi}(z - \alpha), \quad (12)$$

for example, where the (unimportant) constant α is shown in Figure 3, takes the slit to the real axis, and z to the upper-half plane, and

$$0 < \arg[e^{-i\chi}(z - \alpha) - e^{-i\chi}(z' - \alpha)] < \pi. \quad (13)$$

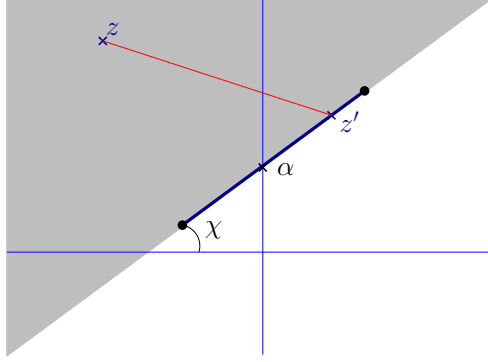


Figure 3: Geometrical positioning of z and z' for the validity of (15).

Hence, on use of (11) with the substitutions (12), we can write

$$\frac{1}{e^{-i\chi}(z' - \alpha) - e^{-i\chi}(z - \alpha)} = i \int_0^\infty e^{ik(e^{-i\chi}(z - \alpha) - e^{-i\chi}(z' - \alpha))} dk, \quad (14)$$

or, on cancellation of α and rearrangement,

$$\frac{1}{z' - z} = i \int_0^\infty e^{ie^{-i\chi}k(z - z')} e^{-i\chi} dk. \quad (15)$$

This expression is valid uniformly for all z and z' having the geometrical positioning depicted in Figure 3.

Now consider a bounded convex polygon P with N sides $\{S_j | j = 1, \dots, N\}$. Figure 4 shows an example with $N = 3$. For a function $f(z)$ analytic in P , Cauchy's integral formula provides that for $z \in P$,

$$f(z) = \frac{1}{2\pi i} \oint_{\partial P} \frac{f(z') dz'}{z' - z}. \quad (16)$$

We can separate the boundary integral into a sum over the N sides:

$$f(z) = \frac{1}{2\pi i} \sum_{j=1}^N \int_{S_j} f(z') \frac{1}{(z' - z)} dz'. \quad (17)$$

But if side S_j has inclination χ_j then (15) can be used, with $\chi \mapsto \chi_j$, to re-express the Cauchy kernel, that is $1/(z' - z)$, uniformly for all $z \in P$ and z' on the respective sides:

$$f(z) = \frac{1}{2\pi i} \sum_{j=1}^N \int_{S_j} f(z') \left\{ i \int_0^\infty e^{ie^{-i\chi_j}k(z - z')} e^{-i\chi_j} dk \right\} dz'. \quad (18)$$

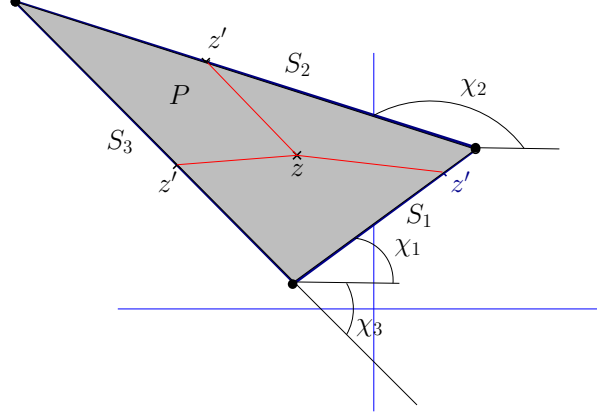


Figure 4: A convex polygon P as an intersection of $N = 3$ half-planes with N angles $\{\chi_j | j = 1, 2, 3\}$. Formula (15) can be used in the Cauchy integral formula with $\chi = \chi_j$ when z' is on side S_j (for $j = 1, 2, 3$).

On reversing the order of integration we can write

$$f(z) = \frac{1}{2\pi} \sum_{j=1}^N \int_{\mathcal{L}} \rho_{jj}(k) e^{-i\chi_j} e^{ie^{-i\chi_j} kz} dk, \quad (19)$$

where, for integers m, n between 1 and N , we define the *spectral matrix* [4] to be

$$\rho_{mn}(k) \equiv \int_{S_n} f(z') e^{-ie^{-i\chi_m} kz'} dz', \quad (20)$$

and where $\mathcal{L} = [0, \infty)$ is the *fundamental contour* [4] for straight line edges.

Thus we arrive at the following transform pair:

$$\begin{aligned} f(z) &= \frac{1}{2\pi} \sum_{j=1}^N \int_{\mathcal{L}} \rho_{jj}(k) e^{-i\chi_j} e^{ie^{-i\chi_j} kz} dk, \\ \rho_{mn}(k) &= \int_{S_n} f(z') e^{-ie^{-i\chi_m} kz'} dz'. \end{aligned} \quad (21)$$

Notice that only the diagonal terms of the spectral matrix appear in the integral representation of $f(z)$.

The spectral matrix elements have their own analytical structure. Observe that, for any $k \in \mathbb{C}$, and for any $m = 1, \dots, N$,

$$\sum_{n=1}^N \rho_{mn}(k) = \sum_{n=1}^N \int_{S_n} f(z') e^{-ie^{-i\chi_m} kz'} dz' = \int_{\partial P} f(z') e^{-ie^{-i\chi_m} kz'} dz' = 0, \quad (22)$$

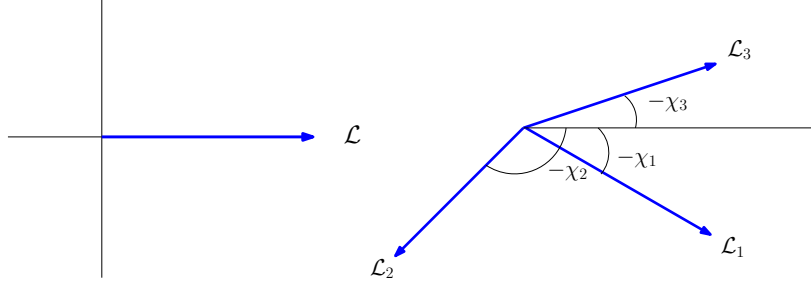


Figure 5: The fundamental contour $\mathcal{L}$ for straight line edges (left). The spectral rays $\{\mathcal{L}_j|j = 1, 2, 3\}$ from Fokas & Kapaev [11] for the $N = 3$ polygon P shown in Figure 4 (right). The latter rays are images of $\mathcal{L}$ under the N transformations (23).

65 where we have used Cauchy's theorem and the fact that $f(z')e^{-ie^{-i\chi_m}kz'}$ (for $m = 1, \dots, N$) is analytic inside P . We refer to (22) as *global relations* relating different elements of the spectral matrix. There are N such relations but each is an equivalent statement of the analyticity of $f(z)$ in the domain P .

The formulation above differs from that given in [11] where the notions of spectral matrix and fundamental contour are not introduced. But it is easy to arrive at the form (7) as given by [11] by now defining, for each $j = 1, \dots, N$, the change of spectral variable given by

$$\lambda_j = e^{-i\chi_j} k \quad (23)$$

70 in both the spectral functions *and* the integral representation in (21) then, rather than a sum of N contributions over the single fundamental contour in the spectral plane, it is easy to check that (21) reproduces (7).

Transform pairs for unbounded polygons, such as strips and semi-strips, can also be derived [11] with minor modifications to the derivation above. The only difference is that the global relations are now valid only in restricted parts of the spectral k -plane. For example, if P is the infinite strip $-l < \text{Re}[z] < l$ of Figure 6 then for any $f(z)$ analytic in this strip we have the transform pair

$$f(z) = \frac{1}{2\pi} \int_{\tilde{L}_1} \rho_1(k) e^{ikz} dk + \frac{1}{2\pi} \int_{\tilde{L}_2} \rho_2(k) e^{ikz} dk, \quad (24)$$

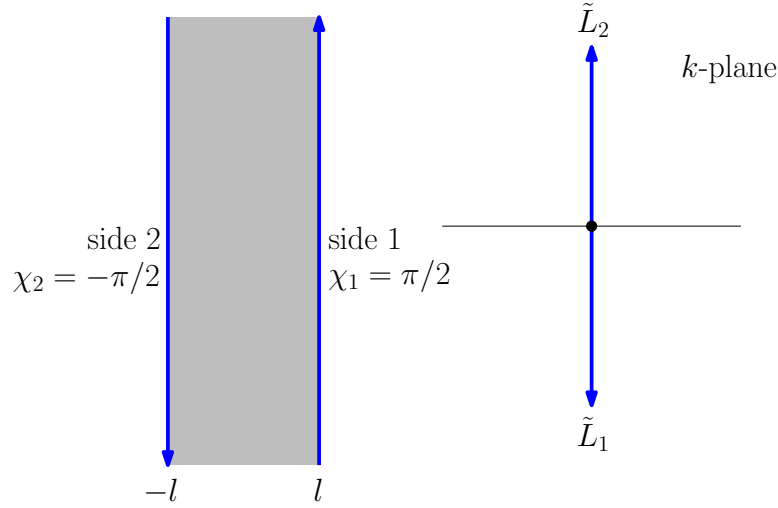


Figure 6: The infinite strip $-l < \text{Re}[z] < l$ with its two sides; the angles χ_1 and χ_2 are shown. The corresponding rays of integration $\tilde{L}_1$ and $\tilde{L}_2$ in the k -plane are also shown; $\tilde{L}_j$ makes angle $-\chi_j$ with the positive k -axis.

where the rays $\tilde{L}_1$ and $\tilde{L}_2$ are also shown in Figure 6 and the spectral functions are

$$\rho_1(k) = \int_{l-i\infty}^{l+i\infty} f(z)e^{-ikz} dz, \quad \rho_2(k) = \int_{-l+i\infty}^{-l-i\infty} f(z)e^{-ikz} dz. \quad (25)$$

The global relation in this case is

$$\rho_1(k) + \rho_2(k) = 0, \quad k \in i\mathbb{R}. \quad (26)$$

which, it is noted, is only valid for purely imaginary k values.

3. The cotangent function

As a first example suppose we seek a periodic function, with period $2l$,

$$f(z) = f(z - 2l) \quad (27)$$

and having simple poles at $z = 2ln$ for $n \in \mathbb{Z}$. By the periodicity it is enough
75 to restrict consideration to the single period window $-l < \text{Re}[z] < l$ where
the function has a single simple pole at $z = 0$. This period window is clearly

the same two-sided, unbounded polygon of Figure 6 and the transform method described in §2 can be applied.

Let the two sides of the period window be labelled as sides 1 and 2 as shown in Figure 6. We will write

$$f(z) = f_s(z) + \hat{f}(z), \quad f_s(z) = \frac{1}{z} \quad (28)$$

and seek an integral representation of the function $\hat{f}(z)$ which is analytic in this period window (and decaying in the far-field). By the defining properties of $f(z)$ we realize that it is, in fact, the cotangent function,

$$f(z) = \frac{\pi}{2l} \cot\left(\frac{\pi z}{2l}\right), \quad (29)$$

but this observation will not be needed in the treatment to follow and will be
80 invoked only at the end.

Define the spectral functions

$$\rho_1(k) = \int_{l-i\infty}^{l+i\infty} \hat{f}(z) e^{-ikz} dz, \quad \rho_2(k) = \int_{-l+i\infty}^{-l-i\infty} \hat{f}(z) e^{-ikz} dz. \quad (30)$$

An integral representation for $\hat{f}(z)$ is

$$\hat{f}(z) = \frac{1}{2\pi} \int_{\tilde{L}_1} \rho_1(k) e^{ikz} dk + \frac{1}{2\pi} \int_{\tilde{L}_2} \rho_2(k) e^{ikz} dk, \quad (31)$$

where the rays $\tilde{L}_1$ and $\tilde{L}_2$ are shown in Figure 6. The global relation [11] in this case is (26).

We now show how to find the spectral functions. On multiplication of (27) by e^{-ikz} and integration along side 1 we find

$$\int_{l-i\infty}^{l+i\infty} \hat{f}(z) e^{-ikz} dz - \int_{l-i\infty}^{l+i\infty} \hat{f}(z-2l) e^{-ikz} dz = S(k), \quad (32)$$

where

$$S(k) \equiv \int_{l-i\infty}^{l+i\infty} e^{-ikz} \left[\frac{1}{z-2l} - \frac{1}{z} \right] dz = \begin{cases} -2\pi i e^{-2ikl}, & \text{Im}[k] < 0, \\ -2\pi i, & \text{Im}[k] > 0. \end{cases} \quad (33)$$

On performing a change of integration variable in the second integral on the left hand side of (32) we find

$$\rho_1(k) + e^{-2ikl} \rho_2(k) = S(k). \quad (34)$$

The global relation (26) then implies that

$$\rho_1(k)(1 - e^{-2ikl}) = S(k), \quad k \in i\mathbb{R}. \quad (35)$$

Hence

$$\rho_1(k) = \begin{cases} -\frac{2\pi i e^{-2ikl}}{1 - e^{-2ikl}}, & \text{Im}[k] < 0, \\ -\frac{2\pi i}{1 - e^{-2ikl}}, & \text{Im}[k] > 0. \end{cases} \quad (36)$$

It follows from (31) and (26) that

$$\hat{f}(z) = \frac{1}{2\pi} \int_0^{-i\infty} -\frac{2\pi i e^{-2ikl}}{1 - e^{-2ikl}} e^{ikz} dk + \frac{1}{2\pi} \int_0^{i\infty} \frac{2\pi i}{1 - e^{-2ikl}} e^{ikz} dk. \quad (37)$$

On rearrangement, and after a change of integration variable $k \mapsto -k$ in the second integral in (37), we find

$$f(z) = \frac{1}{z} + i \int_0^{-i\infty} \frac{e^{ikz} - e^{-ikz}}{1 - e^{2ikl}} dk. \quad (38)$$

By the earlier observation (29) we have derived the following plane wave, or Ehrenpreis-type, integral representation of the cotangent function:

$$\frac{\pi}{2l} \cot\left(\frac{\pi z}{2l}\right) = \frac{1}{z} + i \int_0^{-i\infty} \frac{e^{ikz} - e^{-ikz}}{1 - e^{2ikl}} dk. \quad (39)$$

Remark: It is interesting to remark that this integral representation of such a standard trigonometric function is difficult to find in standard texts, even though the exponential decay of the integrand for z in the period strip renders it numerically more efficient to compute the Fourier-type integral in (39) than to add up contributions from the infinite periodic array of simple poles along the real z -axis. Of course, this idea of replacing a “lattice sum” in the physical z -plane with an integral in a spectral k -plane is at the heart of Ewald summation techniques (although the construction there is very different).

Remark: We will need to make use of the result (39) later in §5 which is why this particular example was chosen.

4. The gamma function

Consider now, as a second example, a function $f(z)$ satisfying the functional equation

$$f(z+1) = zf(z) \quad (40)$$

analytic in the strip $0 < \text{Re}[z] < 1$ and satisfying the normalization

$$f(1) = 1. \quad (41)$$

This function is not periodic, but once a representation for $f(z)$ in the strip
 95 $0 < \text{Re}[z] < 1$ is found, the functional equation (40) can be used to find the analytic continuation of the function outside this strip.

Let P be the strip of analyticity $0 < \text{Re}[z] < 1$; the geometry is identical to that of Figure 6 except that the width of the strip is now unity. For z in P we know we can write

$$f(z) = \frac{1}{2\pi} \int_{\tilde{L}_1} \rho_1(k) e^{ikz} dk + \frac{1}{2\pi} \int_{\tilde{L}_2} \rho_2(k) e^{ikz} dk, \quad (42)$$

with spectral functions

$$\rho_1(k) = \int_{1-i\infty}^{1+i\infty} e^{-ikz} F(z) dz, \quad \rho_2(k) = \int_{-l+i\infty}^{-l-i\infty} e^{-ikz} F(z) dz. \quad (43)$$

The global relation is again (26). It follows that (42) can be written as

$$f(z) = \frac{1}{2\pi} \int_{-\infty}^{i\infty} \rho_2(k) e^{ikz} dk. \quad (44)$$

Now multiply (40) by e^{-ikz} and integrate over side 2:

$$\int_{i\infty}^{-i\infty} e^{-ikz} f(z+1) dz = \int_{i\infty}^{-i\infty} e^{-ikz} z f(z) dz. \quad (45)$$

On making the change of variable $\eta = z+1$ in the integral on the left hand side, and on noticing that the integral on the right hand side is $i d\rho_2(k)/dk$ we find

$$-e^{ik} \rho_1(k) = i \frac{d\rho_2(k)}{dk}. \quad (46)$$

A second invocation of the global relation (26) leads to the first-order linear ordinary differential equation:

$$\frac{d\rho_2(k)}{dk} = -ie^{ik}\rho_2(k), \quad (47)$$

which has the solution

$$\rho_2(k) = A \exp(-e^{ik}) \quad (48)$$

for some constant A . On substitution into (44),

$$f(z) = \frac{A}{2\pi} \int_{-i\infty}^{i\infty} \exp(-e^{ik}) e^{ikz} dk. \quad (49)$$

This is the required plane-wave integral representation. Even so, it is instructive now to introduce the change of integration variable

$$t = e^{ik}, \quad dt = itdk \quad (50)$$

so that

$$f(z) = -\frac{A}{2\pi i} \int_0^\infty \exp(-t) t^{z-1} dt. \quad (51)$$

The requirement (41) yields $A = -2\pi i$ so that

$$f(z) = \int_0^\infty \exp(-t) t^{z-1} dt, \quad z \in P. \quad (52)$$

Of course, we know that the function analytic in the strip $0 < \text{Re}[z] < 1$ and satisfying the functional relation (40) is the classical gamma function $\Gamma(z)$ and, by dint of our construction, in (52) we have just derived the well-known integral representation of it [1].

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5. The Weierstrass $\wp$ function

We now turn our attention to the Weierstrass $\wp$ function considered by Dienstfrey & Huang [7]. In contrast to the two examples already given, we have not found the following integral representations reported elsewhere in the literature.

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The defining properties of the Weierstrass $\wp$ function are that it is an even, doubly periodic, meromorphic function of z having a second order pole at $z = 0$ and at all its periodic images under the transformations

$$z \mapsto z + 2l, \quad z \mapsto z + 2ih, \quad (53)$$

where l and h are positive constants. It is now natural to consider P to be the rectangular polygon

$$-l < x < l, \quad -h < y < h \quad (54)$$

as shown in Figure 7. We seek a doubly-periodic function $f(z)$ for which this rectangular polygon is a fundamental region and where $f(z)$ has a second-order pole with Laurent expansion

$$f(z) = \frac{1}{z^2} + \mathcal{O}(z^2) \quad (55)$$

in a neighbourhood of $z = 0$.

For our construction we let

$$f(z) = f_s(z) + \hat{f}(z), \quad (56)$$

where

$$f_s(z) = c \operatorname{cosec}^2\left(\frac{\pi z}{2l}\right), \quad c = \left(\frac{\pi}{2l}\right)^2. \quad (57)$$

$f_s(z)$ has a second order pole at $z = 0$ and no other singularities in the fundamental polygon. It is also even in z , i.e., $f_s(-z) = f_s(z)$. We seek a correction term $\hat{f}(z)$ such that $f(z)$ has the required functional properties:

$$f(-z) = f(z) \quad (58)$$

with

$$f(z + 2l) = f(z) \quad (59)$$

and

$$f(z + 2ih) = f(z). \quad (60)$$

These conditions will determine $\hat{f}(z)$ up to a constant; the latter is determined by ensuring that the Laurent series of $f(z)$ about $z = 0$ has vanishing constant term.

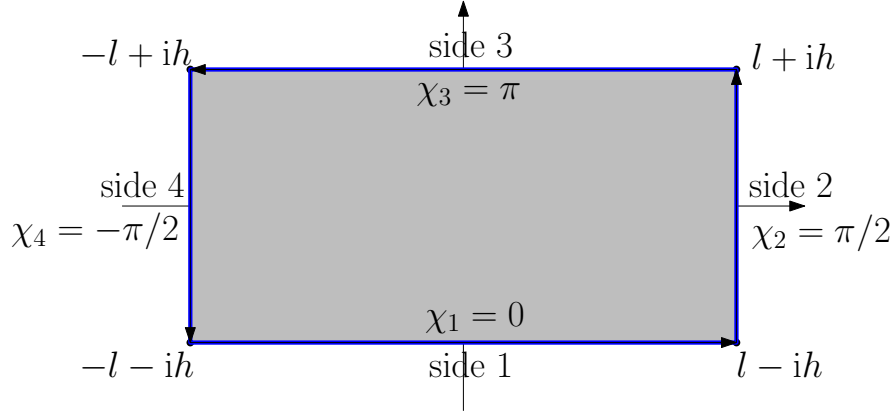


Figure 7: The fundamental polygon for the Weierstrass $\wp$ function with periods $2l$ and $2hi$. The angle χ_j made by side j to the positive real axis is indicated.

Remark: We pause to point out that we could, alternatively, make the choice

$$f_s(z) = \frac{1}{z^2} \quad (61)$$

at this point and more will be said on this in §6.

We know that since $\hat{f}(z)$ is analytic in P then it has the integral representation

$$\hat{f}(z) = \sum_{j=1}^4 \int_{L_j} \rho_j(z) e^{ikz} dk, \quad (62)$$

where the integration rays are defined in Figure 8 and the spectral functions are defined by

$$\rho_j(k) \equiv \int_{\text{side } j} \hat{f}(z) e^{-ikz} dz, \quad j = 1, 2, 3, 4, \quad (63)$$

and where sides 1–4 are labelled in Figure 7. The global relation in this case is

$$\rho_1(k) + \rho_2(k) + \rho_3(k) + \rho_4(k) = 0, \quad k \in \mathbb{C}. \quad (64)$$

To find the spectral functions note that since $f_s(z)$ is even, and the required $f(z)$ is even, then so is $\hat{f}(z)$. On multiplication of (58) by e^{-ikz} and integration over side 1,

$$\int_{-l-ih}^{l-ih} \hat{f}(-z) e^{-ikz} dz = \int_{-l-ih}^{l-ih} \hat{f}(z) e^{-ikz} dz. \quad (65)$$

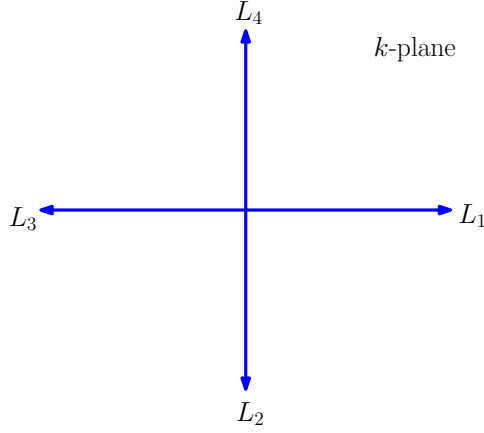


Figure 8: Integration rays L_j for the plane-wave representation (62). The ray L_j makes angle $-\chi_j$ to the real k -axis where the values of χ_j for each side are shown in Figure 7.

The change of variable $z \mapsto -z$ in the integral on the left hand side leads to the deduction that

$$-\rho_3(-k) = \rho_1(k). \quad (66)$$

Similarly, multiplication of (58) by e^{-ikz} and subsequent integration of the resulting relation over side 2,

$$\int_{l-ih}^{l+ih} \hat{f}(-z)e^{-ikz} dz = \int_{l+ih}^{l+ih} \hat{f}(z)e^{-ikz} dz \quad (67)$$

together with the change of variable $z \mapsto -z$ in the left-side integral leads to

$$-\rho_4(-k) = \rho_2(k). \quad (68)$$

Next we multiply (60) by e^{-ikz} and integrate the resulting relation over side 1:

$$\int_{-l-ih}^{l-ih} f(z+2ih)e^{-ikz} dz = \int_{-l-ih}^{l-ih} f(z)e^{-ikz} dz. \quad (69)$$

On substitution of (56) into (69),

$$\int_{-l-ih}^{l-ih} \hat{f}(z)e^{-ikz} dz - \int_{-l-ih}^{l-ih} \hat{f}(z+2ih)e^{-ikz} dz = R(k) \quad (70)$$

where

$$R(k) = c \left[\int_{-l-ih}^{l-ih} \operatorname{cosec}^2 \left(\frac{\pi(z+2ih)}{2l} \right) e^{-ikz} dz - \int_{-l-ih}^{l-ih} \operatorname{cosec}^2 \left(\frac{\pi z}{2l} \right) e^{-ikz} dz \right]. \quad (71)$$

Now, on making the change of variable $z \mapsto z + 2ih$ in (69), we find

$$\rho_1(k) + e^{-2kh} \rho_3(k) = R(k). \quad (72)$$

The periodicity condition (59) can equivalently be written

$$f(z-2l) = f(z). \quad (73)$$

On multiplication of this relation by e^{-ikz} and integration over side 2, we find

$$\int_{l-ih}^{l-ih} f(z-2l) e^{-ikz} dz = \int_{l+ih}^{l-ih} f(z) e^{-ikz} dz. \quad (74)$$

Substitution of (56) and use of the $2l$ -periodicity of $f_s(z)$ leads to

$$\int_{l-ih}^{l-ih} \hat{f}(z-2l) e^{-ikz} dz = \int_{l+ih}^{l-ih} \hat{f}(z) e^{-ikz} dz. \quad (75)$$

The change of variables $z \mapsto z - 2l$ in the left hand side integral leads to

$$\rho_2(k) + e^{-2ikl} \rho_4(k) = 0. \quad (76)$$

The global relation for this problem is given in (64). On substituting for $\rho_3(k)$ and $\rho_4(k)$ from (72) and (76) into the global relation we find

$$\rho_1(k)[1 - e^{2kh}] + \rho_2(k)[1 - e^{2ikl}] = -e^{2kh} R(k). \quad (77)$$

On use of the change of variable $k \mapsto -k$ in the last two integrals in the integral representation (62) for $\hat{f}(z)$ we can write

$$\hat{f}(z) = \int_{L_1} [\rho_1(k) e^{ikz} + \rho_3(-k) e^{-ikz}] dk + \int_{L_2} [\rho_2(k) e^{ikz} + \rho_4(-k) e^{-ikz}] dk. \quad (78)$$

From (66) and (68) we find

$$\hat{f}(z) = \int_{L_1} \rho_1(k) [e^{ikz} + e^{-ikz}] dk + \int_{L_2} \rho_2(k) [e^{ikz} + e^{-ikz}] dk. \quad (79)$$

It is therefore enough to find $\rho_1(k)$ and $\rho_2(k)$.

We now show how to determine these spectral functions. On side 1 we let

$$\hat{f}(z) = \sum_n a_n e^{in\pi x/l} \quad (80)$$

for some set of Fourier coefficients $\{a_n | n \in \mathbb{Z}\}$ to be determined. Multiplication of (80) by $e^{-im\pi x/l}$ for some integer m and integration over the range $x \in [-l, l]$ leads to

$$\int_{-l}^l \hat{f}(z) e^{-im\pi x/l} = 2la_m. \quad (81)$$

Now let $z = x - ih$ in the left hand side. We deduce

$$\int_{-l-ih}^{l-ih} \hat{f}(z) e^{-im\pi(z+ih)/l} = 2la_m, \quad (82)$$

or

$$a_m = \frac{e^{m\pi h/l} \rho_1(k_m)}{2l}, \quad k_m = \frac{m\pi}{l}. \quad (83)$$

Hence to determine $\rho_1(k)$ we need to find the set of values $\{\rho_1(k_m)\}$. But these are available from the global relation (77) since

$$1 - e^{2ik_m l} = 0, \quad (84)$$

so that

$$\rho_1(k_m) = \frac{e^{k_m h} R(k_m)}{e^{k_m h} - e^{-k_m h}}, \quad m \in \mathbb{Z}. \quad (85)$$

But, as shown in appendix Appendix A, $R(k)$ is given by the explicit (and rapidly convergent) sum

$$R(k) = \frac{4\pi}{l} \sin(kl) e^{-kh} \sum_{m \geq 1}^{\infty} \frac{(-1)^m m^2 \rho^m}{m^2 - k^2 l^2 / \pi^2}, \quad (86)$$

where

$$\rho = e^{-\pi h/l} < 1. \quad (87)$$

Note that appropriate limiting values of (86) must be found to evaluate the entire function (86) at $k = k_m$. Hence, for any integer $m \geq 0$, it is found from (83), (85) and (86), that

$$a_m = -\left(\frac{\pi}{l}\right)^2 \frac{m\rho^m}{1 - \rho^{2m}}, \quad a_{-m} = -\left(\frac{\pi}{l}\right)^2 \frac{m\rho^{3m}}{1 - \rho^{2m}}. \quad (88)$$

Hence, since

$$\rho_1(k) = \int_{-l-ih}^{l-ih} \hat{f}(z) e^{-ikz} dz = \int_{-l-ih}^{l-ih} \left\{ \sum_n a_n e^{in\pi x/l} \right\} e^{-ikz} dz \quad (89)$$

it follows, on evaluating the integrals, that

$$\rho_1(k) = -2\pi^2 e^{-kh} \sin(kl) \sum_{m \geq 1} \frac{(-1)^m m \rho^m}{k^2 l^2 - m^2 \pi^2} \left\{ \frac{1 + \rho^{2m}}{1 - \rho^{2m}} k + \frac{m\pi}{l} \right\}. \quad (90)$$

Finally, from (77) we also know that

$$\rho_2(k) = -\frac{e^{2kh} R(k)}{1 - e^{2ikl}} - \rho_1(k) \left[\frac{1 - e^{2kh}}{1 - e^{2ikl}} \right] \quad (91)$$

or, after some algebra,

$$\rho_2(k) = i\pi^2 e^{-kh} e^{-ikl} \sum_{m \geq 1} \frac{(-1)^m m \rho^m}{k^2 l^2 - m^2 \pi^2} \left[\frac{1 + \rho^{2m}}{1 - \rho^{2m}} k (1 - e^{2kh}) + \frac{m\pi}{l} (1 + e^{2kh}) \right]. \quad (92)$$

In summary we have found the following plane-wave integral representation of the Weierstrass $\wp$ function:

$$\begin{aligned} \wp(z) = & \left(\frac{\pi}{2l} \right)^2 \operatorname{cosec}^2 \left(\frac{\pi z}{2l} \right) + \frac{1}{2\pi} \int_{L_1} \rho_1(k) [e^{ikz} + e^{-ikz} - 2] dk \\ & + \frac{1}{2\pi} \int_{L_2} \rho_2(k) [e^{ikz} + e^{-ikz} - 2] dk - \frac{\pi^2}{12l^2}, \end{aligned} \quad (93)$$

where we have added a z -independent term to ensure that (55) is satisfied.

But on taking a derivative with respect to z of the earlier result (39) from our first example in §3 we find the following plane-wave integral representation of the squared cosecant function:

$$\left(\frac{\pi}{2l} \right)^2 \operatorname{cosec}^2 \left(\frac{\pi z}{2l} \right) = \frac{1}{z^2} + \int_{L_2} \frac{k(e^{ikz} + e^{-ikz})}{1 - e^{2ikl}} dk. \quad (94)$$

This representation is valid everywhere in the strip shown in Figure 6 and, consequently, certainly everywhere in the fundamental rectangle of Figure 7.

115 (94) can now be substituted for the first term on the right hand side of (93) to give the modified plane-wave representation (5) where we have added the required z -independent term to ensure that (55) is satisfied.

6. Discussion

It is natural to ask how (1) relates to the new representation (5) derived here. After some algebraic manipulations the result (1) of Dienstfrey & Huang [7] is found to be equivalent to

$$\wp(z) = \frac{1}{z^2} + \int_{L_1} \tilde{\rho}_1(k)[e^{ikz} + e^{-ikz}]dk + \int_{L_2} \tilde{\rho}_2(k)[e^{ikz} + e^{-ikz}]dk + \text{constant}, \quad (95)$$

where

$$\begin{aligned} \tilde{\rho}_1(k) &= -2ke^{i\tau k} f_2(k, \tau) = \frac{k}{2} \left[\frac{(e^{ik} + 1)^2}{(e^{ik} - e^{-ik\tau})(e^{ik} - e^{ik\tau})} \right], \\ \tilde{\rho}_2(k) &= -2ke^{ik} f_1(-ik, \tau) = \frac{k}{2} \left[\frac{(e^{ik\tau} + 1)^2}{(e^{ik\tau} - e^{ik})(e^{ik\tau} - e^{-ik})} \right]. \end{aligned} \quad (96)$$

It is clear that the functional form of (95) is identical with that of the newly-
120 derived (5): the physical singularity in the z -plane is the same in both, and
the complementary plane-wave integrals in the spectral plane are taken over
precisely the same integration rays L_1 and L_2 . But the spectral functions are not
the same. The easiest way to see this is to note that the spectral function $\rho_1(k)$
in (5), as given explicitly in (90), is an *entire* function while $\tilde{\rho}_1(k)$ appearing in
125 (95), and given explicitly in (96), is a meromorphic function. This difference in
the spectral functions is consistent with the fact that (95) is only valid in $D(\tau)$
while our newly-derived (5) is valid everywhere in the fundamental rectangle.

Dienstfrey & Huang [7] noted that the Weierstrass $\wp$ function can be evaluated by means of rapidly convergent Fourier series representation recorded in equation (37) of [7] which, they concede, presents “stiff competition from a numerical perspective”. On this matter, note that for $z = x - ih$ (that is, on side 1 of our fundamental polygon) on combining (56), (57), (80) and (88) we deduce that

$$\begin{aligned} f(z) &= \left(\frac{\pi}{2l}\right)^2 \operatorname{cosec}^2\left(\frac{\pi z}{2l}\right) \\ &\quad - \left(\frac{\pi}{l}\right)^2 \left\{ \sum_{m \geq 1} \frac{m\rho^m}{1 - \rho^{2m}} e^{im\pi x/l} + \sum_{m \geq 1} \frac{m\rho^{3m}}{1 - \rho^{2m}} e^{-im\pi x/l} \right\} \end{aligned} \quad (97)$$

which, on analytic continuation off side 1, can be shown to coincide with the rapidly convergent Fourier series referred to by Dienstfrey & Huang [7]. (Alter-

130 natively, (97) can be shown to be the Fourier series given in (37) of [7] evaluated on side 1 of the fundamental polygon). In this way we see that the integral representations (93) we have derived here are the *natural plane-wave analogues* of this rapidly convergent Fourier series discussed in equation (37) of [7].

It should be clear from our results that *many different* plane-wave integral representations of a given function are possible depending on the demands one makes on the analyticity properties of the spectral functions and on its required domain of validity. Note that, owing to the nature of our construction of (5), the spectral function over the contour L_2 is *not* entire (even though $\rho_2(k)$ is); the extra term, $k/(1 - e^{2ikl})$, in the integral over L_2 is clearly meromorphic. As remarked on earlier, we reiterate that *yet another* plane-wave integral representation of the Weierstrass $\wp$ function can be derived, using the methodology of this paper, if we make the alternative choice (61) for $f_s(z)$ in (56) rather than our chosen (57). The analogous construction leads to *both* spectral functions over the rays L_1 and L_2 being entire functions and the function that would replace our function $R(k)$ is

$$\int_{-l-ih}^{l-ih} \left[\frac{1}{(z + 2ih)^2} - \frac{1}{z^2} \right] e^{-ikz} dz \quad (98)$$

leading to spectral functions involving incomplete gamma functions.

135 Dienstfrey, Hang & Huang [8] have extended the ideas of the work in [7] to finding a more efficient numerical scheme for the evaluation of the periodic Green's function for the two-dimensional Helmholtz equation. It is important to remark that the Fokas transform method [10] has also been applied to the linear Helmholtz equation – as well as to many other PDEs including the modified
140 Helmholtz equation [3] and the biharmonic equation [6] – and, therefore, it is clearly of interest to adapt the ideas expounded in this paper to find plane-wave representations in that case too.

In summary we believe all our observations here, which complements other recent related work by Fokas & Glasser [12], show that the ideas implicit in
145 the Fokas transform approach offer a new theoretical window on many special functions and, indeed, the practical matter of their numerical evaluation.

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By definition,

$$R(k) = c \left[\int_{-l-ih}^{l-ih} \operatorname{cosec}^2 \left(\frac{\pi(z+2ih)}{2l} \right) e^{-ikz} dz - \int_{-l-ih}^{l-ih} \operatorname{cosec}^2 \left(\frac{\pi z}{2l} \right) e^{-ikz} dz \right]. \quad (\text{A.1})$$

Writing $z = x - ih$ in the both integrals leads to

$$R(k) = ce^{-kh} [I_1(k) - I_2(k)], \quad (\text{A.2})$$

where

$$\begin{aligned} I_1(k) &= \int_{-l}^l e^{-ikx} \operatorname{cosec}^2 \left(\frac{\pi(x+ih)}{2l} \right) dx, \\ I_2(k) &= \int_{-l}^l e^{-ikx} \operatorname{cosec}^2 \left(\frac{\pi(x-ih)}{2l} \right) dx. \end{aligned} \quad (\text{A.3})$$

$I_1(k)$ can be rewritten as

$$I_1(k) = \int_{-l}^l \frac{-4\rho e^{-ikx}}{[e^{-i\pi x/l} - \rho e^{i\pi x/l}]^2} dx, \quad (\text{A.4})$$

where

$$\rho = e^{-\pi h/l} < 1. \quad (\text{A.5})$$

Now introduce the change of variable

$$\zeta = e^{i\pi x/2l}, \quad dx = \frac{2ld\zeta}{i\pi\zeta} \quad (\text{A.6})$$

then

$$\begin{aligned} I_1(k) &= -\frac{8\rho l}{\pi i} \oint_{|\zeta|=1} \frac{1}{\zeta^{2kl/\pi}} \frac{\zeta}{(1-\rho\zeta^2)^2} d\zeta = -\frac{4l}{\pi i} \oint_{|\zeta|=1} \frac{1}{\zeta^{2kl/\pi}} \frac{d}{d\zeta} \left[\frac{1}{1-\rho\zeta^2} \right] d\zeta, \\ &= -\frac{4l}{\pi i} \oint_{|\zeta|=1} \frac{1}{\zeta^{2kl/\pi}} \frac{d}{d\zeta} \sum_{j \geq 0} \rho^j \zeta^{2j} d\zeta, \\ &= -\frac{8l}{\pi i} \sum_{j \geq 0} \oint_{|\zeta|=1} j \rho^j \zeta^{2j-2kl/\pi-1} d\zeta. \end{aligned} \quad (\text{A.7})$$

Provided that $kl/\pi \neq 2j$ for some integer $j \geq 0$ then each integral in the sum can be performed explicitly leading, after some algebra, to

$$I_1(k) = \frac{8l}{\pi} \sin(kl) \sum_{j \geq 0} \frac{(-1)^j j \rho^j}{j - kl/\pi} \quad (\text{A.8})$$

Similarly, provided $kl/\pi \neq -2j$ for some integer $j \geq 0$

$$I_2(k) = -\frac{8l}{\pi} \sin(kl) \sum_{j \geq 0} \frac{(-1)^j j \rho^j}{j + kl/\pi} \quad (\text{A.9})$$

Hence, from (A.2), we arrive at

$$R(k) = \frac{4\pi}{l} \sin(kl) e^{-kh} \sum_{m \geq 1}^{\infty} \frac{(-1)^m m^2 \rho^m}{m^2 - k^2 l^2 / \pi^2}. \quad (\text{A.10})$$