

BOUNDARY LAYERS

WITH SWIRL

by

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ABSTRACT

The Patankar-Spalding finite-difference computation procedure has been extended to include within its capabilities the presence of swirl in boundary-layers. The use of the procedure, through its associated computer program, is demonstrated with reference to two types of boundary-layers: free and confined swirling jets; and laminar and turbulent swirling flow in smooth, straight circular pipes. The turbulence is modelled by a simple extension of Prandtl's mixing-length hypothesis.

Computations on the free swirling jet and on swirling pipe flows are compared with published experimental data, while those on the confined swirling jet are compared with data obtained as part of this investigation. The comparisons show that the mixing-length model enables predictions to be made which are in good qualitative agreement with most of the data. The model is, however, insufficiently universal to enable acceptable quantitative predictions to be made for all the flow situations considered.

PREFACE

This thesis is the outcome of my work lasting about four years and performed under the supervision of Professor D. B. Spalding in the department of Mechanical Engineering of Imperial College, London.

The thesis describes and demonstrates the use of a general computational procedure for the calculation of steady, two-dimensional, laminar or turbulent, internal or external, free or confined, boundary-layers with swirl. The procedure is a direct development of the Patankar-Spalding (P-S) finite-difference technique and its associated computer program. It is, to the best of my knowledge, unique in its ability to handle a wide variety of swirling flows that may be classed as boundary-layers.

Because the computation procedure and some of its applications have been described in detail in the book by Patankar and Spalding^{*}, I have had some difficulty in deciding how much detail to include here. To have included all that is relevant would have made the thesis very bulky and probably tedious as well. To have excluded all detail would, on the other hand, have rendered it incomprehensible. The particular balance I have tried to strike derives its justification from the hope that the reader will allow this thesis to be considered as an extension of, and adjunct to, the book.

I must now acknowledge the sources of the large amount of assistance I have received during the course of my work.

There are many fledgling Ph.D.'s working in the field of boundary-layer theory who would be envious of my having had Professor D.B. Spalding for a supervisor - and with much reason. Besides teaching me most of what

^{*} Patankar, S.V. and Spalding, D.B., Heat and Mass Transfer in Boundary Layers. 2nd Edition, Intertext Books Ltd., London, 1970.

I know about boundary-layers, he has endured with patience my numerous outbursts and tantrums and has been very generous with his time. He has been generous too with money for some of my travels within and outside the British Isles. If my semi-colons and hyphens are used to better effect now than when I joined him, the debt I owe Professor Spalding is considerable; for he spared no pains to correct or improve the prose in almost all of the written material I gave him.

I have benefited also from the advice and expertise of Doctors B.E. Launder and F.C. Lockwood who, with Professor Spalding, made up the membership of my thesis committee. They read and commented upon parts of thesis draft and thus helped me to re-write it. The other members of staff have, of course, been of no mean assistance.

Suhas Patankar endured my company in the early days of my stay at Imperial and patiently explained the computer program which bears his name.

Bill Froude and Alan Robinson helped me a great deal with the construction and commissioning of the experimental rig.

I was invariably late in returning books and periodicals to the libraries and both Edna Archer and Stephanie Fardouly have turned a blind eye towards these lapses. I must also thank Marjorie Steele for her help with a host of secretarial and administrative problems.

Many of the figures in this thesis have been drawn by Dev Raj Sharma, a true friend whose empathetic understanding meant much to me during the last year of my stay at Imperial.

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CHAPTER 0
INTRODUCTION

0. Introduction

0.1 The class of flow considered

Swirling flows may be described simply as those in which the fluid particles move in spiral paths. Tornadoes and whirlpools are examples of such flows as they occur in nature. The system of vortices emanating from the tips of aerofoils in a fluid stream is an example of a swirling flow occurring as an unintentioned by-product from the operation of a piece of engineering equipment.

Swirl may be introduced deliberately into fluid flows in order to influence the performance of engineering devices. Examples in this category include: the throttling of the mass flow rate through a rocket nozzle; the enhancement of mixing in a furnace burner by means of the imparting of swirl to the combustible gases; and the introduction of tape swirlers into pipes to increase the heat transfer to or from fluids flowing in them.

Restriction to boundary-layers

The above were some examples of general swirling flows. The exclusive concern of this thesis is with a special type of swirling flow: namely, the swirling boundary-layer. A swirling boundary-layer may be roughly defined as that region of the swirling flow where there is a single predominant direction of flow; and in which shear stresses, heat fluxes and diffusion fluxes are significant only in directions that are square to this predominant direction. The definition may be clarified by isolating the boundary-layer regions in some of general swirling flows quoted above. Thus, the stem of a tornado or whirlpool is a boundary-layer: the "eye" and the region of flow near the earth's crust

are not, for, in these regions, the surrounding fluid is drawn into or ejected from the vortical flow in directions at right-angles to the axis of swirl. There are thus not one but two major directions of flow; normal to the earth's crust and parallel to it. Near the "eye" and base of these vortical flows neither direction predominates.

Likewise, the region in the immediate vicinity of the tip of an aerofoil is not a boundary-layer, although the downstream region is a good example of one.

In the case of the furnace burner, strong swirl is often used to cause a region of the flow immediately downstream of the orifice of the burner to move along the axis in an upstream direction, thus providing a region of "recirculation" where the flame is stabilised. Only the flow downstream of such a region of recirculation is a boundary-layer.

Further restrictions

The swirling boundary-layers dealt with in this thesis are further restricted to being steady, though possibly turbulent, and two-dimensional. The last restriction means that all components of velocity and all scalar variables are functions of only two independent co-ordinates; specifically, the distances along and across the axis of symmetry. Furthermore, variations of fluid properties in the azimuthal direction are not allowed.

It is under these restrictions that reference to 'swirling flows' will be made in the rest of this thesis.

0.2 The thesis objective and outline of thesis content

There are two central purposes of the research reported in this thesis. The first is to develop and test a procedure for the computation of boundary-layers with swirl; to demonstrate the versatility of the

procedure by using it to calculate different types of swirling flows; and to develop a suitable effective-viscosity^{*} hypothesis for turbulent boundary-layers with swirl. The second is to determine, experimentally, the profiles, rate-of-spread and axial decay of velocity in a confined swirling jet so as to provide the empirical data for checking the effective-viscosity hypothesis.

The results of the research designed to achieve these two purposes are reported in this thesis in two parts, I and II.

Part I opens with a review of previous work on recirculation-free, two-dimensional swirling flows. The review reveals the advantages of the Patankar-Spalding (P-S) finite-difference procedure that justify its choice as the procedure used in the research reported here. The basis of the P-S procedure is outlined in Chapter 1. The effective-viscosity model for turbulent flows is developed in Chapter 2. The use of this model in the calculation procedure is demonstrated with reference to two flow geometries: two flows in the category of free swirling boundary-layers, namely free and confined swirling jets (Chapters 3 and 4); followed by two flows in the category of internal swirling flows in contact with walls, namely, laminar and turbulent swirling flows in pipes (Chapters 5 and 6).

Part II describes the apparatus and procedure used to obtain data on a confined swirling jet. These data are used in comparisons with computations displayed in Chapter 4.

To ease reference, figures appear at the end of each of the chapters with which they are associated while appendices and references appear at the end of the thesis.

* The concept of an 'effective-viscosity' is explained in Section 0.3-5.

PART I

O.3 Review of previous work on recirculation-free, two-dimensional, swirling flows

The reference list (pp.121 - 124) contains a selection of papers covering a range of theoretical work on two-dimensional swirling flows without recirculation. This section reviews the salient features of the work so far done on the subject.

The main details of the published work are displayed in tabular summary (next page). They are elaborated in a subsequent review (p. 17) which is divided into three sub-sections: namely, formulations of the equations of flow; solution procedures; and effective-viscosity hypotheses.

Table 0.3-1 Tabular Summary of Theoretical Work on Swirling Flows

Author(s)	Year	Flow situation considered	Formulation of equations of flow	Solution procedure adopted	Effective-viscosity hypothesis used
Binnie, A.M. and Harris, D.P.	1950	Laminar swirling liquid flow through a nozzle	Boundary-layer	Parametric-integral (Pohlhausen)	None
Taylor, G.I.	1950	Laminar swirling flow in a nozzle	Boundary-layer	Parametric-integral (Pohlhausen)	None
Loitsyanski, L.G.	1953	Laminar and turbulent swirling free-jets	Boundary-layer	Similarity	Constant effective-viscosity (unspecified value) replaces laminar viscosity
Görtler, H.	1954	Far-downstream region of swirling laminar free-jet	Boundary-layer	Similarity	None
Talbot, L.	1954	Laminar swirling pipe flow	Boundary-layer (linearised)	Analytical	None
Garbsch, von K.	1956	Laminar swirling flow through a nozzle	Boundary-layer	Cross-stream numerical integration	None

Author(s)	Year	Flow situation considered	Formulation of equations of flow	Solution procedure adopted	Effective-viscosity hypothesis used
Long, R.R.	1961	Laminar, 'tornado' vortex induced by a weak 'sink' on the axis of a tall rotating cylinder of water	Boundary-layer	Similarity	None
Steiger, M.H. and Bloom, M.H.	1961	Laminar, swirling free-jet in still surroundings	Boundary-layer	Parametric-integral (Karman)	None
Sibulkin, M.	1962	Temporal decay of a vortex	Boundary-layer (linearised)	Explicit finite-difference	None
Kreith, F. and Sonju, O.K.	1965	Decay of turbulent swirl flow in a pipe	Boundary-layer (linearised)	Perturbation technique	$\epsilon_\theta = 4.15 \times 10^{-3} \cdot N_R^{0.86}$ ϵ_θ is eddy diffusivity, N_R is axial-flow Reynolds number
Lee, S.L.	1965	Turbulent, swirling free-jet	Boundary-layer	Similarity plus empirically derived information on entrainment into jet	None required

Author(s)	Year	Flow situation considered	Formulation of equation of flow	Solution procedure adopted	Effective-viscosity hypothesis used
Chigier, N.A. and Chervinsky, A.	1966	Turbulent, swirling free-jets	Boundary-layer	Similarity	None required. Empirical relations for decay constants as functions of swirl
Hall, M.G.	1967	Wing-tip vortex	Boundary-layer	Implicit finite-difference	None
Kinney, R.B.	1967	Pure-rotational turbulent flows	Boundary-layer	Similarity	$\mu_{t,\theta} = \rho k^2 r^2 \left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right)$ ρ = density v_θ = tangential velocity r = radius k = empirical constant
Ghildayal, C.D.	1968	Turbulent swirling inlet flow in a circular pipe	Boundary-layer	Inviscid-core solution matched with outer 'law-of-the-wall' solution	As implied by laws-of-the-wall for axial and tangential velocity
Wilks, G.	1968	Swirling, laminar flow through a convergent funnel	Boundary-layer	Similarity and parametric-integral	None

Author(s)	Year	Flow situation considered	Formulation of equations of flow	Solution procedure adopted	Effective-viscosity hypothesis used
Back, L.H.	1969	Laminar, swirling boundary-layers over cooled surfaces	Boundary-layer	Similarity	None
Chervinsky, A.	1969	Turbulent, swirling jet diffusion flames	Boundary-layer (with additional approximations)	Non-similar solutions in von-Mises plane	Extension of Prandtl's formula: $\mu_{t,\theta} \propto (\text{width of jet}) \times (\text{maximum velocity difference})$ at any axial station
Rochino, A. and Lavan, Z.	1969	Turbulent, swirling flow in pipes	Navier-Stokes equations with approximations that retain elliptic nature of equations	Explicit finite-difference	As for Kinney (1967)
Cham, T.S. and Head, M.R.	1970	Turbulent, boundary-layer on a rotating cylinder in an axial stream	Boundary-layer	Explicit-integral	None required
Soo, S.L.	1970	Core velocity distribution in swirling pipe flow	Boundary-layer (neglecting viscosity in axial motion equation and axial convection in swirl equation)	Analytical	Uniform eddy diffusivity (in dimensional units for particular problem considered)

0.3-2. Formulations of the equations of flow

A glance at the fourth column of Table 0.3-1 reveals that the conservation equations for the axial and swirl momenta have most often been cast into their boundary-layer form. Among authors to do so are: Loitsyanski (1953); Steiger and Bloom (1961); Kreith and Sonju (1965); and Cham and Head (1970). Although Sibulkin (1962) tackles a time-dependent problem, his equations are still those of the two-dimensional boundary-layer: time replaces the axial co-ordinate in his study of the temporal decay of a vortex.

Some authors do not invoke all the boundary-layer assumptions. Thus Rochino and Lavan (1969) consider swirling flows in a pipe. They assume the non-dimensional axial velocity to be a function only of radial distance. They also neglect both the radial velocity and the axial convection of tangential velocity; but, in contrast with boundary-layer practice, they take into account the axial diffusion of tangential velocity. Chigier and Lorenz (1967) employ the full elliptic equations.

Distinguishing feature of the equations of the swirling boundary-layer

When the equations for swirling flows are formulated in their boundary-layer form,^{*} there is one feature which distinguishes them from the equations of the non-swirling boundary-layer. In the latter, cross-stream gradients in static pressure are justifiably neglected. In swirling flows, on the other hand, these gradients can be of considerable importance. The reason is that, as the swirl decays or

* The equations are formulated and written out in Chapter 1, Section 1.2.

build up,* the radial gradients of pressure tend either to diminish or to increase, thus causing axial gradients of static pressure. These gradients must be properly taken into account in the equation for the conservation of axial momentum. That their influence is not small is demonstrated in Chapter 5. An extended discussion on the influences of these gradients may be found in Morton (1969).

0.3-3 Solution procedures for the equations

The fifth column of Table 0.3-1 lists the solution procedures employed to solve the flow-equations whose formulations were discussed in the previous section. The solution procedures fall into two main categories: integral methods and finite-difference methods.

Integral procedures

Solution procedures of the integral type sub-divide into 'explicit' and 'parametric' or 'profile' -integral methods (as summarised by Spalding, 1967). In the explicit integral procedures, the differential equations use integral quantities (like momentum thickness) as dependent variables. Through these variables are expressed the auxiliary quantities appearing in the equations (like drag coefficient). Cham and Head (1970) employ this technique in their study of the turbulent boundary-layer on a rotating body in an axial stream. The integral-momentum equation is used in conjunction with an empirical relation connecting the 'shape factor', H , with the rate of entrainment of fluid from the main-stream into the boundary-layer. Further, the skin

* Examples for the two cases are:- A swirling free-jet and swirling flow through a conical nozzle with a substantial half-angle.

friction coefficient is empirically related to H and R , the Reynolds number based on the momentum thickness.

In the parametric-integral method, solutions to the equations of motion are obtained by processing the equations through the following stages:- At any longitudinal station in the boundary layer, the profiles of the dependent variables (e.g. axial velocity) are described by simple algebraic expressions (e.g. polynomials in distance from the axis of symmetry) containing free parameters (the coefficients of the polynomial, for example). Next, the partial differential equations are multiplied by each of a set of weighting functions of the dependent and/or independent variables. Then, by using the assumed profile specifications these equations are integrated across the boundary-layer; the result is a set of ordinary differential equations, in which the longitudinal distance appears as the independent variable while the several free parameters appear in linear combination as the dependent ones. Each parameter is then isolated in a first-order equation of its own by matrix inversion. Lastly, these equations are integrated by standard analytical or numerical procedures (e.g. Runge-Kutta); from the result, the desired properties of the solution may be obtained by algebraic means.

Practitioners of the parametric-integral method as applied to swirling boundary-layers include Steiger and Bloom (1961) and Wilks (1968).

Inadequacies of integral procedures

Briefly stated, the objections to integral procedures are these: procedures of the explicit-integral variety (e.g. Cham and Head, 1970) lack width of applicability because of the extent of empirical information

needed every time an increase is sought in the range of applicability. In the example quoted, the non-dimensional entrainment rate into the boundary-layer on a rotating body (already an empirically determined quantity in the non-swirling case) has to be further related empirically to a swirl parameter by performing experiments on the precise flow situation considered.

Procedures of the parametric-integral type are more general than explicit-integral procedures but they are liable to be inaccurate when the number of free parameters is small. Attempts to improve accuracy by increasing the number of free parameters becomes increasingly expensive of computer time and occasionally prone to failure on account of matrix singularity.*

Finite-difference procedures

A way out of the difficulties presented by integral procedures is offered by finite-difference (f.d.) methods. As one of these is used in this thesis, attention is now paid to previous applications of f.d. methods to solve the equations of the swirling boundary-layer.

Table O.3-2 below lists some details of f.d. procedures used to date.

* See Patankar and Spalding (1970) for a fuller discussion of integral procedures and their drawbacks.

Table 0.3-2

Author (Year)	Flow problem considered	Co-ordinates in which solved	Finite-difference procedure used
Garbsch (1956)	Laminar swirling flow in a conical nozzle.	Physical.	Cross-stream or 'shooting' integration.
Sibulkin (1962)	Unsteady vortex.	Several trans- formations and approximations. Linear p.d.e.'s.	Explicit. Forward differences for time interval and central differences for space intervals.
Hall (1967)	Wing-tip vortex.	Physical.	Implicit. First-order central differences. Step-by-step marching integration with matrix inversion.
Rochino and Lavan (1969)	Steady turbulent swirling flow in a pipe.	Physical, non- dimensional.	Explicit. Forward differences for first- order derivatives; central-differences for second-order derivatives.

Choice of finite-difference procedure

A good f.d. solution procedure is one that will:-

(i) Be usable for all swirling boundary-layer phenomena. For example, the procedure should be capable of handling free-swirling flows such as jets and wakes; swirling flows in ducts as also boundary-layers over rotating bodies.

(ii) Be easy and cheap to use: the computer program should be quickly adaptable to the specific problem at hand and should not be expensive of computer time.

(iii) Give accurate solutions and be free from numerical instabilities.

The f.d. procedures displayed in Table O.3-2 will satisfy one or another of these criteria: none will satisfy all of them. Thus, Hall's procedure may satisfy (iii) but the computer program is not easily adaptable to different flow geometries and the matrix inversion procedure is expensive of computer time. Rochino and Lavan restrict themselves to flow in pipes and their explicit scheme has a stability-imposed limit on step-size. Garbsch's procedure, being of the cross-stream integration variety, is very uneconomical of computer time while Sibulkin's is liable to be inaccurate because the boundary-layer equations are linearised before being solved.

A procedure that will satisfy all the criteria, and is therefore used in this thesis, is the one due to Patankar and Spalding. The 'implicit' formulation of the finite-difference equations releases the procedure from any stability-imposed limit on step-size. It is economical of computer time because, firstly, it is a marching- (as opposed to cross-stream) integration procedure; secondly, computing effort is expended in only those regions of the flow where there are significant gradients in the dependent variables. The computer program embodying the procedure is versatile: it is quickly adapted to handle a variety of free and confined boundary-layers.

O.3-4 Effective-viscosity hypotheses

The concept of an effective-viscosity

A swirling fluid has two principal components of mean motion: one axial; the other, tangential. Each of these components is constrained by the component of shear stress with which it is associated. The

diffusional momentum transport causing the above components of shear stress may be treated as obeying a viscosity law of the Newtonian type through the concept of an effective-viscosity, μ_{eff} .

It may be anticipated that such an effective-viscosity will not, in general, be isotropic. Consequently, two effective-viscosities, $\mu_{\text{eff},x}$ and $\mu_{\text{eff},\theta}$ * may be introduced through the stress-defining equations:

$$\tau_x = \mu_{\text{eff},x} \left(\frac{\partial v_x}{\partial r} \right) ; \quad (0.3-1)$$

$$\tau_\theta = \mu_{\text{eff},\theta} \cdot \left\{ r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) \right\} \quad (0.3-2)$$

where v_x and v_θ are the axial and tangential velocities respectively and r is the normal distance. Equation (0.3-2) is chosen instead of the dimensionally equivalent $\tau_\theta = \mu_{\text{eff},\theta} (\partial v_\theta / \partial r)$, in order to satisfy the condition that the shear stress vanishes when the fluid is in solid-body rotation (i.e. when $v_\theta / r = \text{constant}$).

The turbulent or 'eddy' viscosities are related to the laminar and effective viscosities by the defining equations:

$$\left. \begin{aligned} \mu_{\text{eff},x} &= \mu_{t,x} + \mu_l \\ \mu_{\text{eff},\theta} &= \mu_{t,\theta} + \mu_l \end{aligned} \right\} \quad (0.3-3)$$

where μ_l is the laminar viscosity.

For turbulent flows, before the equations of motion may be integrated, the μ_t 's must be known or be knowable through relations between calculable quantities.† The relations are invariably

* Subscripts x and θ denote the axial and tangential components, respectively of non-isotropic quantities.

† Solution procedures of the 'explicit-integral' type avoid making specific assumptions about $\mu_{\text{eff},x}$. They employ empirical information derived by generalising experimental findings which connect various integral properties of the boundary-layer. The drawbacks of these procedures have been stated in Section 0.3-3 above.

hypotheses and require confirmation through comparison of theoretical predictions with experimental data.

This section reviews some of the proposals that have been put forth for determining the μ_t 's. It concludes by examining the Prandtl (1925) mixing-length hypothesis and states the reasons for the choice of this hypothesis as the basis of the effective-viscosity model used in this thesis.

Proposals for the effective viscosities in swirling flows

The main proposals are displayed in Table 0.3-3 (next page). The first four entries in the table are past proposals; the last entry is the formula used in this thesis. The basis of this last formula and the reasons for its choice will be stated shortly. First, a brief description of past proposals is given and an assessment made of their relative success in predicting swirling flows.

The hypothesis of Kreith and Sonju is for swirling flow in a pipe. It is a simple empirical correlation between the tangential eddy diffusivity and the axial flow Reynolds number. By comparing their experimental measurements of swirl decay in a pipe with their analytical solutions, Kreith and Sonju arrive at the following relation:

$$\epsilon_{\theta} = 4.15 \times 10^{-3} \cdot N_R^{0.86} \quad (0.3-4)$$

in which $\epsilon_{\theta} = \mu_{t,\theta} / \mu_l$. N_R is the axial flow Reynolds number based on the pipe radius and the maximum of axial velocity at the longitudinal station where the analysis begins. The Reynolds number exponent in the above correlation is, according to these authors, in close agreement with the Martinelli (1947) analogy for non-swirling turbulent pipe flow which predicts an exponent of 0.9 for the variation of ϵ_x with N_R .

Table 0.3-3

Classification of Turbulent-viscosity Hypotheses
for Turbulent Flows with Swirl

(i) $\mu_{t,x}$	(ii) $\mu_{t,\theta}$	Relationship between (i) and (ii)	Reference
None specified*	Empirical correlation with axial flow Reynolds number. See text.	None specified, but see text.	Kreith and Sonju (1965)
None specified*	$= \rho k_{\theta}^2 r^2 \left \left(\frac{\partial v_{\theta}}{\partial r} - \frac{v_{\theta}}{r} \right) \right $ (Recommended $k_{\theta} = 0.028$)	None specified	Kinney (1967)
As implied by law-of- the-wall $u^+ = u^+ (y^+)$	As implied by law- of-the-wall $\frac{v_{\theta}^+}{r^+} = \frac{v_{\theta}^*}{u^* a^+} (y^+)$	Complex relat- ionship implied through solut- ions of flow equations	Ghildayal and Persen (1968)
None specified*	$= \rho k^2 r^2 \left \left(\frac{\partial v_{\theta}}{\partial r} - \frac{v_{\theta}}{r} \right) \right $ (Recommended $k = 0.034$)	None specified	Rochino and Lavan (1969)
$= \rho \ell_x^2 \left \frac{\partial u}{\partial r} \right $ $+ \rho \ell_{\theta}^2 r^2 \left \frac{\partial (v_{\theta}/r)}{\partial r} \right $	$= \frac{\mu_{t,x}}{Pr_{\theta}}$ where Pr_{θ} is a 'swirl Prandtl number'	$Pr_{\theta} = 1$ for free swirling flows; 5 for confined swirl- ing jets and 0.5 for swirl- ing pipe flows	This work

* The axial momentum equation is not solved.

Nomenclature:

a^+ = pipe radius
 k = empirical constant
 ρ = density
 r = cross stream radial distance
 u = axial velocity

ℓ_x = 'axial' mixing length
 ℓ_{θ}^x = 'tangential' mixing length
 v_{θ} = tangential velocity
 $u^* = \sqrt{\tau_x/\rho}$;
 $v_{\theta}^* = \sqrt{\tau_{\theta}^x/\rho}$;
 $u^+ = u/u^*$
 $y^+ = y u^*/\nu$

Thus, an implication of the hypothesis of Kreith and Sonju is that the 'axial' and 'tangential' eddy viscosities, $\mu_{t,x}$ and $\mu_{t,\theta}$ are in nearly constant proportion to each other.

Kinney (1967) extends von Karman's turbulence similarity hypothesis to a cylindrical geometry in order to establish the conditions under which universal velocity similarity will exist for two-dimensional rotational flows with no axial motion. Kinney finds a 'mixing-length' which is proportional to the radial co-ordinate. He determines the empirical constant of proportionality, k_θ , by comparing his computed universal velocity profiles and wall-shear-stress with the measurements of Taylor (1936).

By arguments similar to those of Kinney, Rochino and Lavan (1969) formulate an expression for the eddy diffusivity (ϵ_θ) in a swirling flow. Their formula, which is essentially the same as Kinney's, makes no allowance for any possible contribution to ϵ_θ by the mean axial motion.

In contrast to the above proposals is the work of Ghildayal and Persen (1968). These authors, while considering a swirling flow in the entrance region of a circular pipe, do not make any assumptions about the relationship between $\mu_{t,x}$ and $\mu_{t,\theta}$. They assume, instead, that Spalding's formulation of the law-of-the-wall (Spalding 1961) remains valid for flows with moderate swirl. They provide a similar formulation for a law-of-the-wall for tangential velocity using the same empirical constants as those in the law for axial velocity. It is not possible to extract from the analysis an explicit relationship between $\mu_{t,x}$ and $\mu_{t,\theta}$; for, this relationship (perhaps a non-unique one) is intimately bound-up with the solution of the equations of motion.

Successes and failures of the above proposals

An assessment of the success or failure of the first four proposals

displayed in Table 0.3-3 cannot be isolated from an assessment of the total analysis presented by their authors. This is so because the authors cited have not solved the equations of motion exactly. In consequence, mathematical approximations in the procedures used to solve the equations of motion introduce errors which cannot be distinguished from the errors caused by an inadequate effective-viscosity hypothesis. Thus, Ghildayal and Persen make no comparisons with experimental data and so no assessment can be made of the validity of their proposal. Kreith and Sonju make a prediction for the decay of average swirl in a pipe; the match with experimental data is excellent. Rochino and Lavan use empirical constants derived from the work of Kinney. For different degrees of swirl in a pipe, comparisons are made between calculated and experimental radial distributions of angular momentum at various axial stations. The agreement is good.

A judgement on the success or failure of an analysis, and an eddy viscosity hypothesis in particular, depends on what is expected of the analysis and the hypothesis. If one demands width of applicability with a minimum number of empirical factors that can be relied upon in different flow situations, then, as is evident from the foregoing discussion, none of the analyses presented satisfies the demand.

Choice of effective-viscosity hypothesis

The choice of a solution procedure for the equations of the swirling boundary-layer presented few difficulties (Section 0.3-3). The choice of an effective-viscosity hypothesis is less easy to make; for, there are several hypotheses which will go some way to satisfying the demand for width of applicability with a minimum number of empirical factors. Among them is the Prandtl (1925) mixing-length hypothesis used in this thesis.

The basic hypothesis and the reasons for its choice are set out below.

L. Prandtl (1925) proposed the following formula for the eddy-viscosity, μ_t , in non-swirling flows:

$$\mu_t = \rho \cdot \ell^2 \left| \frac{\partial v_x}{\partial r} \right| \quad (0.3-5)$$

Prandtl arrived at the above relation from equation (0.3-1) by arguing that, akin to the process by which molecular motions cause Newtonian viscous stress, the shear stress due to turbulence is generated by the transport of small globules of fluid, of density ρ , over a 'mixing-length', ℓ . The transported momentum per unit mass is the velocity of fluctuating motion (in the direction of mean flow) which Prandtl took to be the product of ℓ and $\left| \frac{\partial v_x}{\partial r} \right|$. Further, Prandtl related the mixing-length, ℓ , to some characteristic dimension of the flow by an equation of the type:

$$\ell = \text{constant} \times \text{distance from the wall}$$

The constant has, of course, to be determined empirically by comparing experimental data with predictions using the hypothesis.

The primary reason for the choice of Prandtl's mixing-length model is the fact that it is simple, is yet versatile and is the most thoroughly tested hypothesis of all. It has been used with success in predicting a wide variety of non-swirling boundary-layers.* This research has enabled the following conclusions to be drawn about the efficacy of the Prandtl hypothesis:

- (i) When the initial velocity and mixing-length profiles (the latter perhaps guessed) are not irregular (i.e. are not of

* A representative sample of such predictions may be found in Patankar and Spalding (1970).

the type one might find in wall jets, for example) a version of the Prandtl mixing-length model of turbulence will usually predict the data very well.

- (ii) By replacing the universal constants by universal functions of local flow properties, the 'universality' of the Prandtl hypothesis may be considerably extended. (See Pai (1969) and Siddhartha (1969) for examples.)
- (iii) For a fixed geometry, a single set of constants does not predict with adequate accuracy boundary-layers whose starting conditions are irregular (e.g. wall-jets).
- (iv) The mixing-length model does not allow flows of radically different geometries to be correctly predicted on the basis of a single set of empirical constants (or functions). Thus, different constants are required for plane and axisymmetric flows.

So we see that the mixing-length model has the advantage of being conceptually simple and requires for its use merely an algebraic specification of mixing-length or, not infrequently, a pair of empirical constants. But the model has the disadvantage of being insufficiently universal. Past experience with the model suggests that different types of swirling flows will, almost certainly, require different empirical constants to predict them correctly. There are, it is true, other models which hold greater promise in respect of universality - but they are more complex. A brief description of some of these models and the reasons why they have not been used in this thesis are given below.

The absence of universality in the Prandtl hypothesis has spurred the search for it in multi-equation models of turbulence. Some of these models assume μ_{eff} to be characterised by a length scale, l , and the

kinetic energy of turbulence, k , through the relation:

$$\mu_{\text{eff}} = \rho \cdot l \cdot k^{\frac{1}{2}} ;$$

where

$$k = \frac{1}{2}(u'^2 + v'^2 + w'^2)$$

u' , v' and w' are the fluctuating components of instantaneous velocity and ρ is the density.

Unlike the mixing-length model, the multi-equation models take into account the convection and diffusion of turbulence. Partial differential equations are formulated for the convective and diffusional transport of the quantities k and l (or for quantities algebraically related to these). These differential equations contain the empirical constants whose universality is under test.*

The merits of these multi-equation models in non-swirling flows were becoming apparent only towards the end of this research. They are still under test and there is no guarantee that their merits will be revealed in swirling flows as well. It would thus be premature to attempt to use them before extending the simpler and well-tried mixing-length hypothesis to swirling flows. They have, therefore, not been used in this research.

* A succinct summary of such models may be found in Spalding (1969).

CHAPTER 1THE EQUATIONS OF THE SWIRLING BOUNDARY-LAYER

Chapter 1

The equations of the swirling boundary-layer

The introduction has defined the class of flow considered in this thesis. Attention is restricted to steady, two-dimensional boundary-layer flow with swirl.

In this chapter are written down the governing momentum conservation equations of the two-dimensional swirling boundary-layer. Two equivalent forms of the equation for the conservation of angular momentum - called the swirl equation - are presented. The reasons for abandoning one form are stated; the alternative is cast into a form that renders it easily soluble by the P-S procedure.

1.1 The equations of the two-dimensional swirling boundary-layer

1.1-1 The co-ordinate system

Fig. 1.1-1 below shows the co-ordinate system. The flow region of interest lies between two imaginary surfaces designated by the subscripts I (for internal) and E (external).

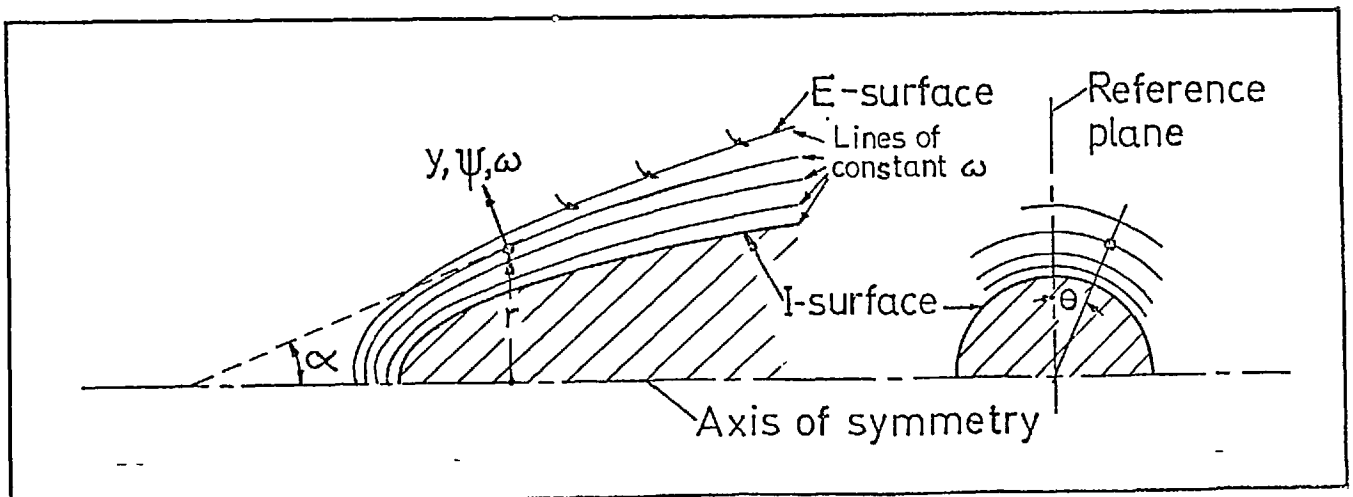


Fig. 1.1. The co-ordinate system

The stream-wise co-ordinate, x , is approximately parallel to the stream-lines. The x -direction makes an angle α with the axis of symmetry: α may vary with x but not too rapidly. The distance from the axis of symmetry, r , of any point in the flow field is given by:

$$r = r_I + y \cos \alpha \quad (1.1-1)$$

The ω co-ordinate

The equations of the boundary-layer, shortly to be written down, employ the dimensionless stream function, ω , as the cross-stream variable. By definition:

$$\omega = \frac{\psi - \psi_I}{\psi_E - \psi_I} \quad ; \quad (1.1-2)$$

where ψ_I and ψ_E are the values of the stream function, ψ , at the I and E boundaries of the layer; thus the entire cross-stream extent of the boundary-layer is restricted to an ω range between zero and unity.

ψ_E and ψ_I are made functions of x through the relations:

$$\frac{d\psi_I}{dx} = - r_I m''_I \quad ; \quad (1.1-3)$$

and

$$\frac{d\psi_E}{dx} = - r_E m''_E \quad (1.1-4)$$

where m''_I and m''_E are the rates of mass transfer through the I and E surfaces respectively. When either or both of these surfaces are free, these rates of mass transfer (or 'entrainment') into the boundary-layer are so controlled (in the calculation procedure) that all the important cross-stream variations in the dependent variables take place at ψ values between ψ_I and ψ_E .

1.1-2 The conservation equations

The conservation equations of the boundary-layer in the $x \sim w$ co-ordinate system may all be cast into the common form:

$$\frac{\partial \phi}{\partial x} + (a + bw) \frac{\partial \phi}{\partial w} = \frac{\partial}{\partial w} \left(c \frac{\partial \phi}{\partial w} \right) + d \quad (1.1-5)$$

Where

$$a = r_I m''_I / (\psi_E - \psi_I) \quad ; \quad (1.1-6)$$

$$b = (r_E m''_E - r_I m''_I) / (\psi_E - \psi_I) \quad ; \quad (1.1-7)$$

and

$$c = r^2 \rho V_x \mu_{\text{eff}} / (\psi_E - \psi_I)^2 \sigma_{\text{eff}} \quad (1.1-8)$$

In equation (1.1-5), ϕ stands for a dependent variable (axial velocity, enthalpy, mass fraction, etc.) and d is a 'source term' which does not contain $\partial \phi / \partial w$. σ_{eff} in relation (1.1-8) is the 'effective' Prandtl number appropriate to the variable ϕ .

When $\phi \equiv V_x$, the axial velocity, the expression for the 'source term' d , reads:

$$d = - \frac{1}{\rho V_x} \frac{\partial p}{\partial x} + \frac{V_\theta^2 \sin \alpha}{r V_x} \quad (1.1-9)$$

where 'p' is the static pressure which, in general, is a function of both the axial and cross-stream co-ordinates. V_θ is the swirl velocity.

The swirl equation

The conservation equation for the swirl velocity, V_θ , may be written in several ways. Of these, the one to choose is clearly that which falls most easily into the general form, equation (1.1-5); for then, a single set

* The derivation is available in Patankar and Spalding (1970).

of algorithms suffices for the solution of both the axial and swirl momenta. Accordingly, the momentum equation for the swirl direction may be expressed in the form:*

$$\frac{\partial}{\partial x} (rV_{\theta}) + (a + b_w) \frac{\partial}{\partial w} (rV_{\theta}) = \frac{\partial}{\partial w} \left[c \frac{\partial (rV_{\theta})}{\partial w} \right] - 2 \frac{\partial}{\partial w} \left(\frac{rV_{\theta} \mu_{\text{eff},\theta}}{\psi_E - \psi_I} \cos \alpha \right). \quad (1.1-10)$$

The last term represents the source term, d. $\mu_{\text{eff},\theta}$ is the effective-viscosity appropriate to the θ motion.

Equation 1.1-10 is not used directly in this thesis. To explain why not, it will be necessary to refer to the main features of the P-S procedure and the finite-difference formulation of the conservation equations. Such reference is made in Appendix I. Suffice it to say here that extreme care is necessary in expressing in finite-difference form the source term, d, of equation (1.1-10), so as to ensure the satisfaction of the conservation principle of angular momentum over the whole of the boundary-layer. It has been found preferable to suppress the source term entirely by re-writing the swirl equation in the alternative form:

$$\frac{\partial}{\partial x} (rV_{\theta}) + (a + b_w) \frac{\partial}{\partial w} (rV_{\theta}) = \frac{\partial}{\partial w} \left[c r^2 \frac{\partial}{\partial w} \left(\frac{V_{\theta}}{r} \right) \right] \quad (1.1-11)$$

Of course now, unlike the general equation (1.1-5), the above formulation of the swirl equation does not have identical operands in the differentiations on its left and right. This difference requires the finite-difference transform of (1.1-11) to vary from that valid for the general equation (1.1-5). The detailed derivation of this f.d formulation is given in Appendix I.

* A derivation is available in Koo Sin Lin and Lockwood (1969).

The radial momentum equation

It was pointed out in the Introduction that the feature distinguishing the analysis of swirling boundary-layers from non-swirling ones was the need to take account of the existence of cross-stream pressure gradients. These gradients must be determined from the cross-stream momentum equation. The actual relationship must, perforce, be an approximate one; for if it took account of all the terms in the equation, the problem would become elliptic in character and boundary-layer analysis would have to be abandoned.

The 'parabolic' form of the radial momentum equation is:

$$\frac{\partial p}{\partial y} = \frac{\rho v_{\theta}^2}{r} \cos \alpha \quad . \quad (1.1-12)$$

The finite-difference form of this equation and its use in the solution procedure is described in Appendix I.

CHAPTER 2

EFFECTIVE VISCOSITY RELATIONS FOR TURBULENT SWIRLING FLOWS

Chapter 2

Effective viscosity relations for turbulent swirling flows

2.1 Introduction

In the last chapter, the equations of the swirling boundary-layer have been written down. The swirl equation has been moulded into a form which renders it soluble by the Patankar-Spalding procedure. Before it can be employed to solve a real flow problem, the procedure requires information on the physics of the flow. For turbulent flow, this takes the form of effective-viscosities for the axial and tangential motions.

This chapter is devoted to the development of the equations which relate the effective viscosities appropriate to the axial and tangential motions to the characteristics of the mean motion in the two directions. In Sections 2.2 and 2.3 are derived the formulae for the shear stress in, respectively, non-rotating and rotational flows with no axial motion. The relations are based on Prandtl's mixing-length hypothesis. Next, in Section 2.4, is developed the formula for effective-viscosity in general swirling flows. In Section 2.5 is discussed the van-Driest modification to the formula when it is applied to the region of the flow in the vicinity of a wall. The chapter closes with some proposals relating the mixing-lengths to the characteristic dimensions of the flow.

2.2 Prandtl's formula for turbulent shear-stress in non-rotating flows

According to Prandtl, lumps of turbulent fluid retain their translational momentum over a short transverse distance ℓ_x called the mixing length. In moving from one lamina to another these lumps cause fluctuations in the translational momentum of the lamina to which they move. These fluctuations may be related to the gradient of the mean velocity.

Thus, if at any point in the flow, V_x , the translational velocity (momentum per unit mass) is composed of the mean velocity $\bar{V}_x$ and the fluctuating velocity V'_x such that:

$$V_x = \bar{V}_x + V'_x \quad ; \quad (2.1)$$

then

$$\overline{|V'_x|} = \ell_x \cdot \left| \frac{\partial \bar{V}_x}{\partial r} \right| \quad , \quad (2.2)$$

where r is the ordinal distance to the point in question.

The fluctuating component of the transverse velocity is presumed to be of the same order as V'_x , and thus:

$$\overline{|V'_r|} \simeq \overline{|V'_x|} = \ell_x \cdot \left| \frac{\partial \bar{V}_x}{\partial r} \right| \quad . \quad (2.3)$$

The correlation of V'_x and V'_r is the shear stress:

$$\tau_x = -\rho \overline{V'_x V'_r} \quad , \quad (2.4)$$

where ρ is the density.

Bearing in mind that the sign of the shear stress takes the sign of the mean velocity gradient, we get Prandtl's formula for the turbulent shear stress:

$$\tau_x = \rho \ell_x^2 \left| \frac{\partial \bar{V}_x}{\partial r} \right| \frac{\partial \bar{V}_x}{\partial r} \quad . \quad (2.5)$$

2.3 Formula for shear-stress in rotating flows with no axial motion

(pure rotational flows)

The formula for shear-stress in pure rotational flows may be derived by analogy with Prandtl's reasoning. If ℓ_θ be the mixing-length

appropriate to the tangential motion and $\bar{v}_\theta$ the mean tangential velocity; then, by analogy with Prandtl's reasoning, one may argue that the fluctuation in the tangential velocity, v'_θ , is caused by the radial transport of angular momentum per unit moment-of-inertia, i.e. by angular velocity.

Mathematically:

$$\left| v'_\theta \right| = l_\theta \left| r \frac{\partial(\bar{v}_\theta/r)}{\partial r} \right|, \quad (2.6)$$

and the tangential shear stress, τ_θ , is given by:

$$\tau_\theta = \rho l_\theta^2 \left| r \frac{\partial(\bar{v}_\theta/r)}{\partial r} \right| \left| r \frac{\partial(v_\theta/r)}{\partial r} \right| \quad (2.7)$$

Equation (2.7) satisfies the no-shear-stress requirement of a flow in solid-body rotation (i.e. when $\bar{v}_\theta/r = \text{constant}$).

2.4 Formulae for the Effective-Viscosities in Swirling Flows

In the last two sections formulae for the shear stress in pure axial and pure rotational flows have been derived. It now remains to combine them, in some plausible fashion, to obtain the relations for the shear stresses and effective viscosities in general swirling flows.

Experiments show that the rate of entrainment into a free swirling jet and the rate-of-decay of the radial maximum of axial velocity increase with increasing levels of swirl. This behaviour implies that the magnitude of the shear stress, and therefore also the effective viscosity, is enhanced by swirl. A plausible formula for $\mu_{\text{eff},x}$ that accommodates this fact is that the effective viscosity for pure axial flows is simply added to that for pure rotational flows.

By definition:

$$\tau_x = \mu_{\text{eff},x} \left(\frac{\partial v_x}{\partial r} \right); \quad (2.8)$$

* The bars denoting time averaging will be dropped for convenience.

which, when incorporated in (2.5), yields:

$$\mu_{\text{eff},x} \Big|_{\theta=0} = \rho \ell_x^2 \left| \frac{\partial V_x}{\partial r} \right|, \quad (2.9)$$

where $\theta = 0$ indicates the absence of swirl.

Similarly,

$$\mu_{\text{eff},\theta} \Big|_{x=0} = \rho \ell_\theta^2 \left| r \frac{\partial (V_\theta/r)}{\partial r} \right|; \quad (2.10)$$

where $x=0$ indicates the absence of axial motion.

The 'axial' effective viscosity in swirling flows is now simply written as:

$$\boxed{\mu_{\text{eff},x} = \rho \ell_x^2 \left| \frac{\partial V_x}{\partial r} \right| + \rho \ell_\theta^2 \left| r \frac{\partial (V_\theta/r)}{\partial r} \right|}. \quad (2.11)$$

This relation satisfies the requirement that $\mu_{\text{eff},x}$ tend to $\mu_{\text{eff},x} \Big|_{\theta=0}$ as either $V_\theta \rightarrow 0$ or as V_θ/r tends to a constant (solid-body rotation).

The effective viscosity appropriate to the tangential motion, $\mu_{\text{eff},\theta}$, is related to $\mu_{\text{eff},x}$ through the defining equation:

$$\mu_{\text{eff},\theta} = \frac{\mu_{\text{eff},x}}{\text{Pr}_\theta} \quad (2.12)$$

Pr_θ is the 'Prandtl number for swirl'; a concept which is an obvious read-across from convective heat-transfer theory. It is empirically determined.

Relation (2.11) satisfies the requirement that, as $V_x \rightarrow 0$, $\mu_{\text{eff},\theta}$ tend to $\mu_{\text{eff},\theta} \Big|_{x=0}$; a tendency that is also obtained in plug flow.

2.5 The region near a wall

2.5-1 Need for special treatment

The relations derived above for the 'axial' and 'tangential' effective viscosities take into account only the turbulent contribution as this, in the main, is the dominant part. In the immediate vicinity of a wall, however, the magnitude of the turbulent viscosity diminishes and becomes comparable with its laminar value. Because the axial and tangential components of the shear stress at a wall are of essential practical interest and because there exist steep gradients of velocities near a wall, a more detailed hypothesis is needed for μ_{eff} near a wall: one that describes the respective roles of the turbulent and laminar viscosities.

2.5-2 The van-Driest hypothesis

For flows without swirl, a generalisation of the van-Driest hypothesis may be made in the following form^{*}:

$$\mu_{eff,x} \Big|_{\theta=0} = \mu_l + \rho \cdot \ell_x^2 [1 - \exp\{-y\sqrt{\tau_x \rho} / (\mu_l A_{+,x})\}]^2 \cdot \left| \frac{\partial V_x}{\partial r} \right| \quad (2.13)$$

Here, y is the distance from the wall; and $A_{+,x}$ is a constant. The hypothesis implies that μ_{eff} is composed of the laminar viscosity μ_l and the turbulent viscosity $\rho \ell_x^2 \left| \frac{\partial V_x}{\partial r} \right|$; the latter is damped near the wall in the manner of an exponential.

For flows with swirl, a form of the corresponding van-Driest modification for $\mu_{eff,x}$ near a wall follows by analogy. Thus:

$$\mu_{eff,x} = \mu_l + \rho \ell_x^2 \left| \frac{\partial V_x}{\partial r} \right| D_x + \rho \ell_\theta^2 \left| r \frac{\partial (V_\theta/r)}{\partial r} \right| D_\theta \quad (2.14)$$

in which the two 'damping' terms D_x and D_θ are given by the relations:

$$D_x = [1 - \exp\{-y\sqrt{\tau_x \rho} / (\mu_l A_{+,x})\}]^2 \quad (2.15)$$

^{*} See Patankar and Spalding (1970), Section 1.3-5, p.21.

and,

$$D_{\theta} = [1 - \exp\{-Y \sqrt{\tau_{\theta} \rho} / (\mu_{\ell} A_{+, \theta})\}]^2 \quad (2.16)$$

$A_{+, x}$ and $A_{+, \theta}$ are constants. The fixing of their values is discussed in Chapter 6 where these relations are employed in the calculation of swirling turbulent flow in a pipe.

2.6 The mixing-lengths ℓ_x and ℓ_{θ}

Before the formulae (2.11) and (2.14) may be used, it will be necessary to estimate the magnitudes of the mixing-lengths ℓ_x and ℓ_{θ} .

For non-swirling flows, there have been many proposals for the mixing-length, ℓ_x . For flows remote from walls, ℓ_x is usually taken as uniform across, and proportional to, the thickness of the layer. An example of the use of this proposal may be found in Siddhartha (1969). For flows near walls, ℓ_x is taken as proportional to the distance from the wall. The applications of these proposals for ℓ_x to swirling flows are discussed in Chapters 4 and 6.

As for ℓ_{θ} , one of the simplest hypotheses is to presume it to be proportional to the local radius. The plausibility of this proposal may be tested by using it to predict the shape of the velocity profile in the turbulent region of incompressible flow between rotating concentric cylinders.

An angular momentum balance in such a flow shows the shear stress to be proportional to the inverse of the square of the radius (see von Karman 1951). Thus:

$$\tau_{\theta} \rho r^2 = \text{const} = A, \text{ say} \quad (2.17)$$

The proposal for the mixing-length ℓ_θ is:

$$\ell_\theta = \sqrt{k_\theta} \cdot r \quad (2.18)$$

With this relation, the formula for tangential shear-stress, equation (2.7), becomes:

$$\tau_\theta / \rho = k_\theta r^2 \left| r \frac{\partial (v_\theta / r)}{\partial r} \right| r \frac{\partial (v_\theta / r)}{\partial r}$$

Substituting for τ_θ from (2.17) we have:

$$\frac{A}{k_\theta r^6} = \left| \frac{\partial (v_\theta / r)}{\partial r} \right| \frac{\partial (v_\theta / r)}{\partial r}$$

$$\text{If } A > 0, \quad \frac{\partial (v_\theta / r)}{\partial r} = \left(\frac{A}{k_\theta} \right)^{\frac{1}{2}} \frac{1}{r^3}$$

$$\therefore \quad v_\theta / r = -\frac{1}{2} \left(\frac{A}{k_\theta} \right)^{\frac{1}{2}} \cdot \frac{1}{r^2} + B$$

$$\text{i.e.} \quad \boxed{v_\theta r = Br^2 - \frac{1}{2} \left(\frac{A}{k_\theta} \right)^{\frac{1}{2}}} \quad (2.19)$$

$$\text{If } A < 0, \quad \frac{\partial (v_\theta / r)}{\partial r} = -\frac{A^{\frac{1}{2}}}{k_\theta} \cdot \frac{1}{r^3}$$

$$\text{i.e.} \quad \boxed{v_\theta r = Br^2 + \frac{1}{2} \left(\frac{-A}{k_\theta} \right)^{\frac{1}{2}}} \quad (2.20)$$

When the outer cylinder rotates, B must be positive since v_θ approaches the velocity of the cylinder as r increases. The predicted profile from either of (2.19) and (2.20) corresponds to near solid-body rotation - in accord with expectation. When the outer cylinder does not rotate, on the other hand, v_θ must be small at large r; so B must be small. $v_\theta r$ is then virtually independent of the radius: a prediction in accord with the experimental data of Wattendorf (1935)*.

* Displayed in Kinney (1967).

CHAPTER 3

THE SWIRLING, FREE-TURBULENT JET IN STAGNANT SURROUNDINGS

Chapter 3

The swirling, free-turbulent jet in stagnant surroundings

3.1 Introduction

Chapter 2 has developed Prandtl's mixing length model of turbulence as extended to include the effects of swirl. The present chapter demonstrates an application of this model to the prediction of some properties of swirling, free-turbulent jets expanding into stagnant surroundings. The shapes of the profiles of axial and tangential velocity and the rate-of-spread of these jets are computed and compared with experimental data. This comparison enables one to fix the empirical constant, k_θ , appearing in the turbulence model through the mixing-length relation, equation (2.18).

The value of the 'swirl Prandtl number', Pr_θ , is established by making a comparison between the computed and experimentally determined decay of the radial maximum of tangential velocity, $V_{\theta, \max}$.

3.2 The swirling turbulent jet in stagnant surroundings

The computations were carried out with a version of the computer program described in Appendix II. The mathematical and physical inputs to the program were as follows.

3.2-1 Mathematical inputs

a. Grid: The number of cross-stream intervals was twenty - equally spaced in θ . The computations start from beyond the edge of the potential core at an axial station four orifice diameters away from the jet exit plane ($x/R = 4$).

b. Initial profiles: The initial axial velocity-profiles were the smooth exponential curves that Chigier and Chervinsky (1967) have drawn through their experimental data; the tangential velocity profiles were three part approximations to the experimental ones. The broken lines in Figure 3.3 and 3.4 show these initial profiles. Computations with input profiles of tangential velocity differing somewhat from the ones plotted showed no significant differences in the shapes of the profiles downstream or in the rates-of-spread of the jet.

3.2-2 Physical inputs

The Prandtl mixing-length model as given by equations (2.11) and (2.12) was used. The relations are repeated here for convenience.

$$\mu_{\text{eff},x} = \rho \ell_x^2 \left| \frac{\partial v_x}{\partial r} \right| + \rho \ell_\theta^2 \left| r \frac{\partial (v_\theta/r)}{\partial r} \right| \quad ; (2.11)$$

$$\mu_{\text{eff},\theta} = \mu_{\text{eff},x} / \text{Pr}_\theta \quad . (2.12)$$

Additionally, the following relations for the mixing-lengths, ℓ_x and ℓ_θ , were used:

$$\ell_x = k_x \cdot r_\ell \quad (3.1)$$

$$\ell_\theta = k_\theta \cdot r \quad (3.2)$$

In relations (3.1) and (3.2), r is the radial distance to the point at which $\mu_{\text{eff},x}$ is determined, while r_ℓ is the radial distance to that point on the axial velocity profile where the axial velocity is 1% of its maximum value at the axial station of interest. ' r_ℓ ' is thus an arbitrarily defined jet radius.

Preliminary computations on non-swirling free jets enabled k_x (equation (3.1)) to be fixed at 0.08.* The other empirical constant, k_θ , appearing in the effective-viscosity model was determined as follows. It was, in the first instance, made a simple constant and computations made for various values of k_θ . As will be shown below, comparison with experimental data revealed that such a constant value of k_θ did not quite yield predictions which compared well with the experimentally established trends. In the second set of computations, k_θ was made a linear function of the local 'swirl number'. The last mentioned quantity characterises the strength of swirl in the jet at any axial station and is defined as:

$$\text{Swirl number } (S_w) = \frac{\text{Axial flux of angular momentum}}{\text{Axial flux of axial momentum} \times r_\ell} \quad (3.3)$$

At the nozzle exit plane, the nozzle radius replaces r_ℓ and $S_w = S_{w,0}$ the initial swirl number.

3.2-3 The predictions

a. Predictions using a constant value of k_θ . Figure 3.1 shows some experimental data on the normalised 'half-jet radius', $r_{\frac{1}{2}}$, at a non-dimensional axial distance of $x/R = 23.4$ plotted against the initial swirl number, $S_{w,0}$. R is the radius of the exit nozzle and $r_{\frac{1}{2}}$ is the radial distance to that point on the axial velocity profile where the velocity is half its maximum value at the axial station of interest. $S_{w,0}$ is the initial swirl number of the jet.

* All swirling jets tend, asymptotically, towards the non-swirling free turbulent jet either as the input swirl decreases or as the distance from the nozzle increases. A study of non-swirling jets thus enables the fixing of k_x . The details of this earlier study are available in Siddhartha (1969).

The solid curves result from joining the computational points. Each curve represents results from a different but constant value of k_θ . The large scatter in the data (points) allows a liberal interpretation to be given to the satisfaction of the requirement of an 'adequate' prediction but, clearly, the predicted trends are not in accord with the experimental ones. If k_θ were made to vary with the local swirl number in a suitable way then, perhaps, the predictions would match the trend of the data rather better.

b. Predictions using a value of k_θ that is a function of the swirl number.

Figure 3.2 shows the same data as in Fig. 3.1 together with a prediction using the following relationship for k_θ :

$$k_\theta = 0.26 + 1.2 S_w$$

This second prediction is, arguably, rather more adequate than those displayed in Fig. 3.1.

c. Profile shapes. Figures 3.3 to 3.5 show the initial (broken lines) and computed (full lines) profiles of velocity and static pressure compared with experimental data (points).

The abscissa, $\xi = r/x + a$, of the plots is the radial distance, r , divided by the axial distance measured from the virtual origin of the jet as determined from the data*.

Profiles of axial velocity. In Fig. 3.3, the ordinate is the axial velocity non-dimensionalised with the radial maximum of this velocity. (For the swirl numbers considered, this occurred at the axis of the jet).

* Chigier and Chervinsky (1967) report this distance to be 2.3 nozzle diameters away from the nozzle exit plane independent of the input swirl number.

The computed profiles develop from the supplied initial profiles and approach similarity: the profiles hardly changing beyond twenty nozzle diameters from the exit plane. The discrepancy between the scattered data and the computations is, of course, a consequence of the particular turbulence model and the particular value(s) of k_θ .

Profiles of tangential velocity. Fig. 3.4 shows the profiles of tangential velocity normalised with its radial maximum.

The computed profiles predict correctly the trend of the data for the outer half of the profile; namely, a movement inwards with increasing distance downstream. For the highest swirl number ($S_{w,0} = .416$) the predicted behaviour is more complex, a movement outwards preceding the shift towards the centre-line.

In the inner region, the predictions are not in good agreement with the trend of the experimental data; whereas the latter show an outward movement with distance downstream, the computed profiles show a continuous inward movement whose extent diminishes with increasing initial swirl. The predicted behaviour is a consequence of the mixing-length hypothesis: the flow in the inner region being in near solid-body rotation, the swirl contributes little to the total effective viscosity. Consequently, the outwards diffusion of the swirl velocity is determined almost solely by the gradients in axial velocity, and these are not large in the central region of the jet. The calculated outwards diffusion of the angular momentum is thus insufficient.

Profiles of static pressure

Figure 3.5 shows the predicted profiles of static pressure, p , compared with data for two swirl numbers. p_c is the static pressure on the jet axis and p_∞ the ambient pressure.

The agreement with data of these predictions must be considered satisfactory in view of the discrepancies between the computations of swirl velocity and experimental data and the errors involved in the measurement of static pressure.

c. Axial decays and rates-of-spread

Figure 3.6 shows the decay of the maximum of tangential velocity, $V_{\theta, \max}$, against axial distance in nozzle diameters, D . $V_{\theta, \max, 0}$ is the value of $V_{\theta, \max}$ at the nozzle exit plane. The origin of the predicted curve has been shifted leftward to coincide with the origin of the plot of experimental data. Predictions were made for four swirl numbers, $S_{w, 0}$ from 0.068 through to 0.416. The differences between the predictions are of the order of the thickness of the solid line drawn in the figure. The agreement with the data is satisfactory.

An important conclusion to be drawn from this figure is that the rate of axial decay of the maximum of tangential velocity is largely independent of the input swirl number.

Figure 3.7 shows the rate-of-spread of the swirling jets. The trend of the data is predicted satisfactorily, all the curves approaching a linear spreading rate with increasing axial distance from the orifice. In this, as in other plots, the discrepancies between the data and the predictions are a consequence of the particular choice of K_{θ} in the turbulence model.

3.3 Conclusions

The computations presented in this chapter have demonstrated the use of the P-S procedure and the mixing-length hypothesis to predict some properties of the swirling, free turbulent jet in stagnant surroundings. Comparison with experimental data show that:-

- (i) The simple mixing-length model for effective-viscosity developed in chapter 2, as embodied in equations (2.11), (2.12), (3.1) and (3.2), is capable of predicting the gross features of the properties of the swirling turbulent jet with the three empirical constants set to the values: $K_x = 0.08$, $K_\theta = 0.26$ and $Pr_\theta = 1$
- (ii) Somewhat better agreement with data may be obtained by making the constant K_θ in the mixing-length specification (equation 2.18) a function of the local swirl number, S_w . A recommended formula is:

$$K_\theta = 0.26 + 1.2 S_w.$$

- (iii) The ratio of the effective-viscosities appropriate to the axial and tangential motions (Pr_θ) may be taken to be unity and uniform across the jet. Better agreement between the predicted profile shapes and the data could perhaps be obtained with non-uniform distributions of Pr_θ . The over-simple mixing-length model does not, it would seem, warrant such fine adjustment.

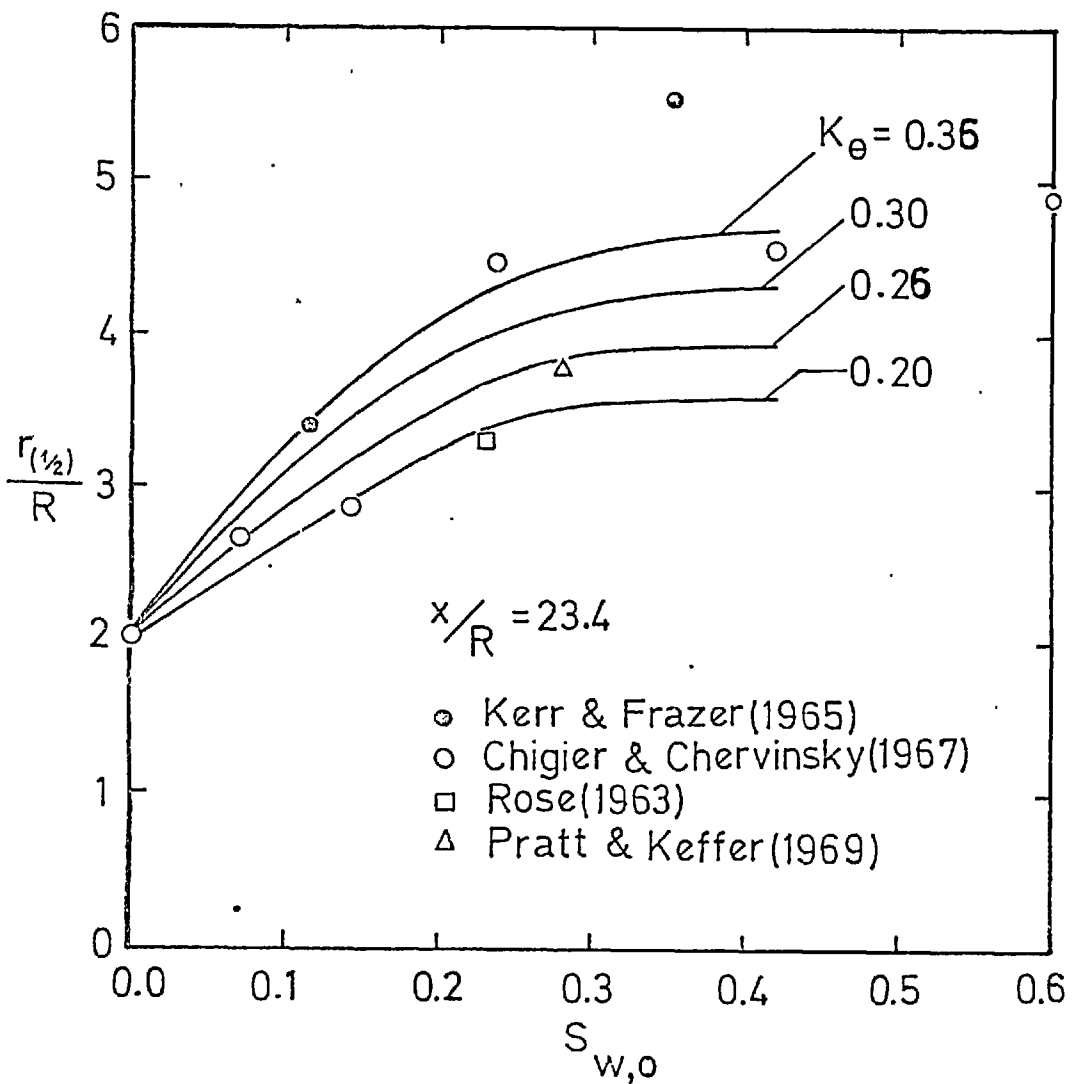


Fig. 3-1

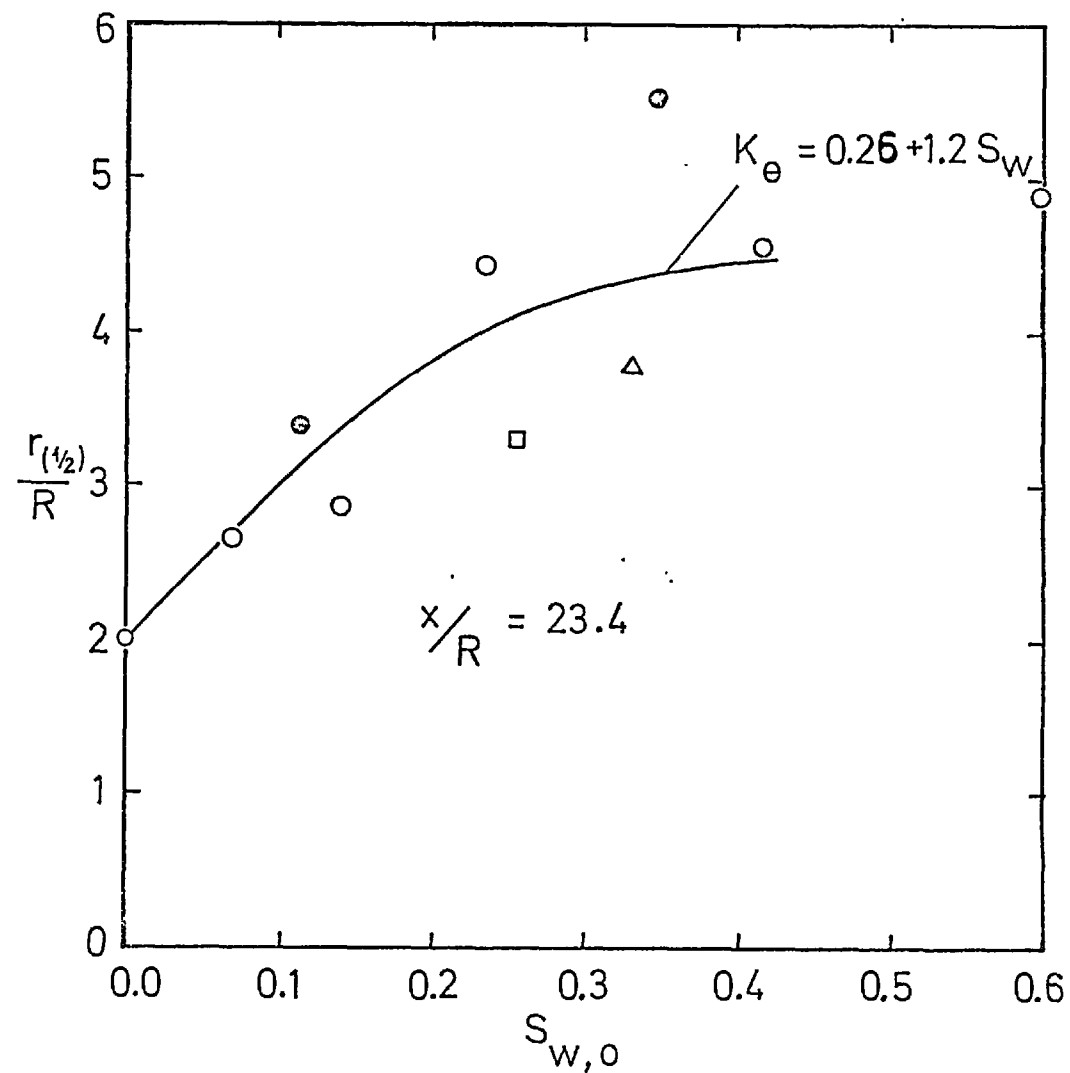


Fig. 3-2

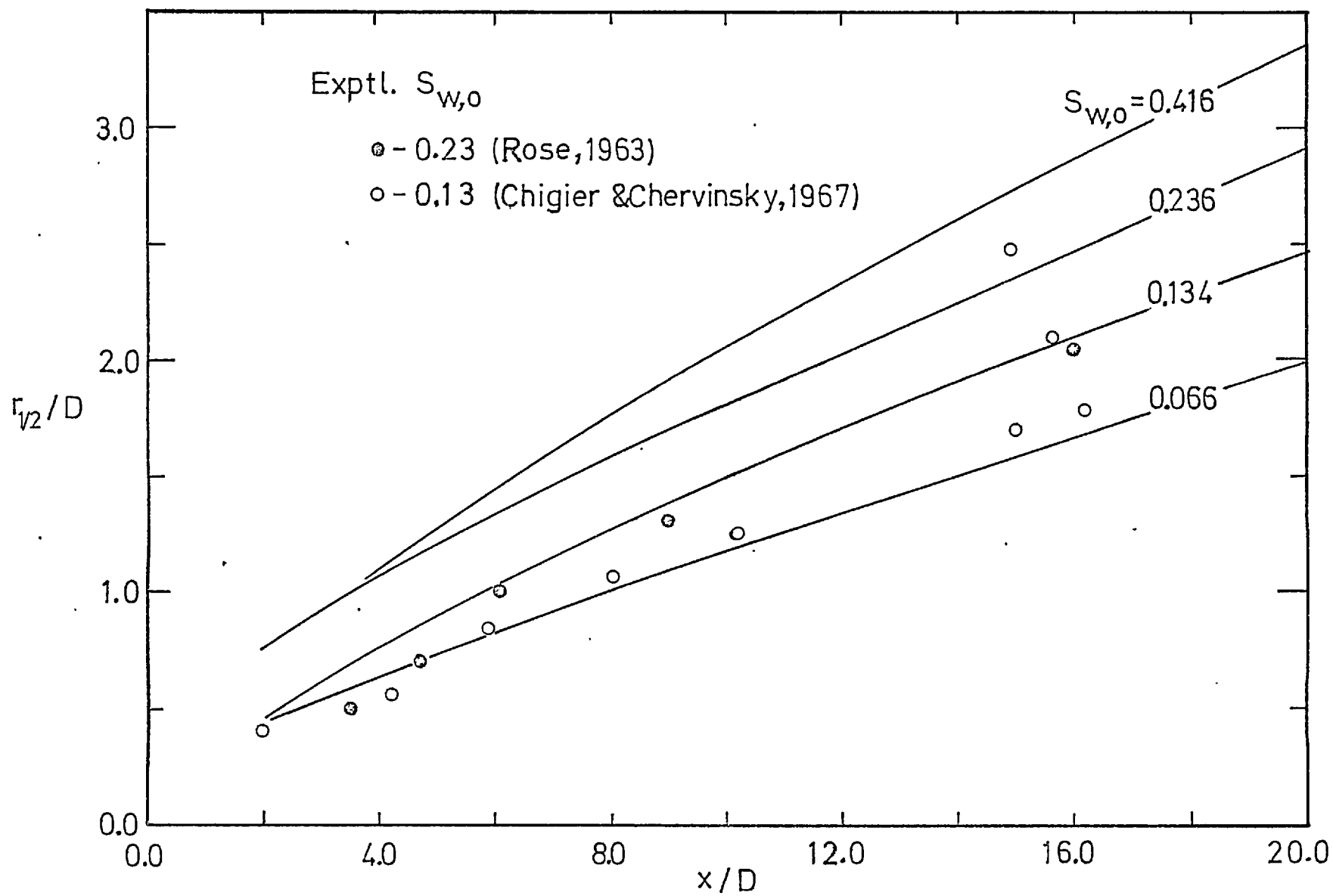


Fig. 3.7

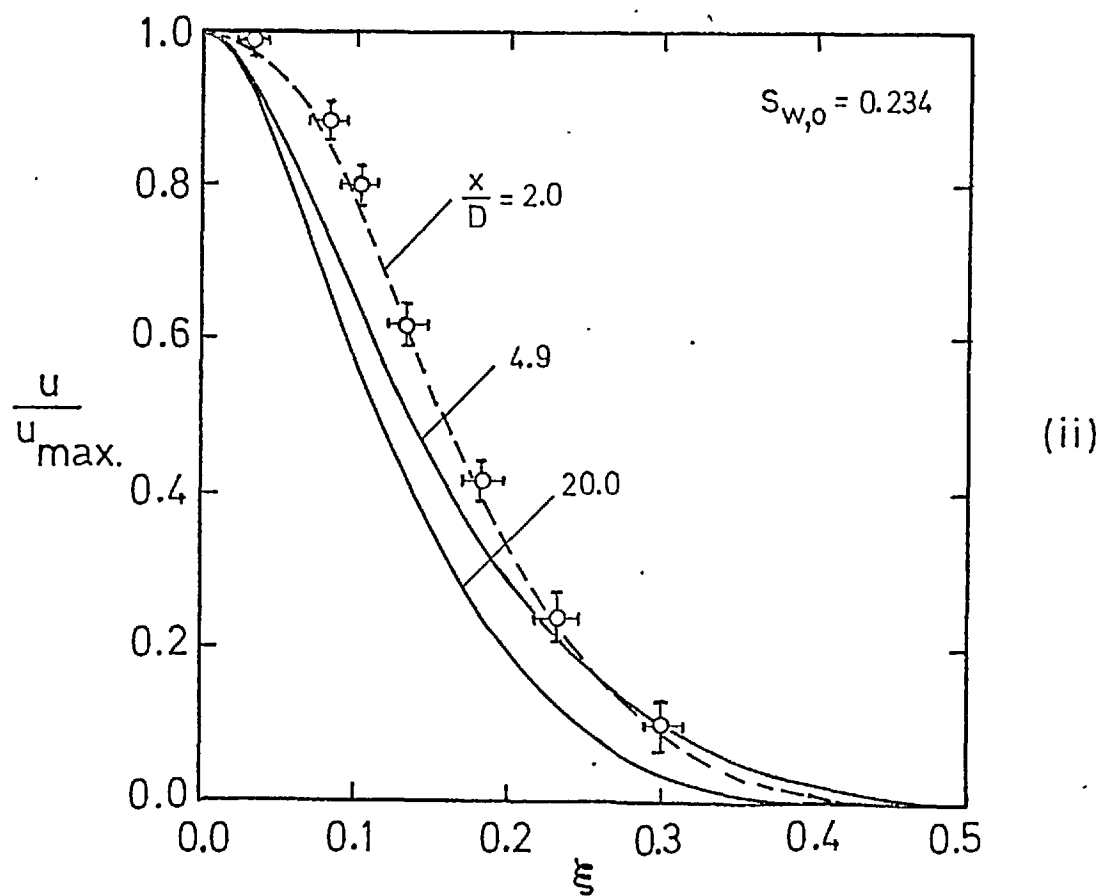
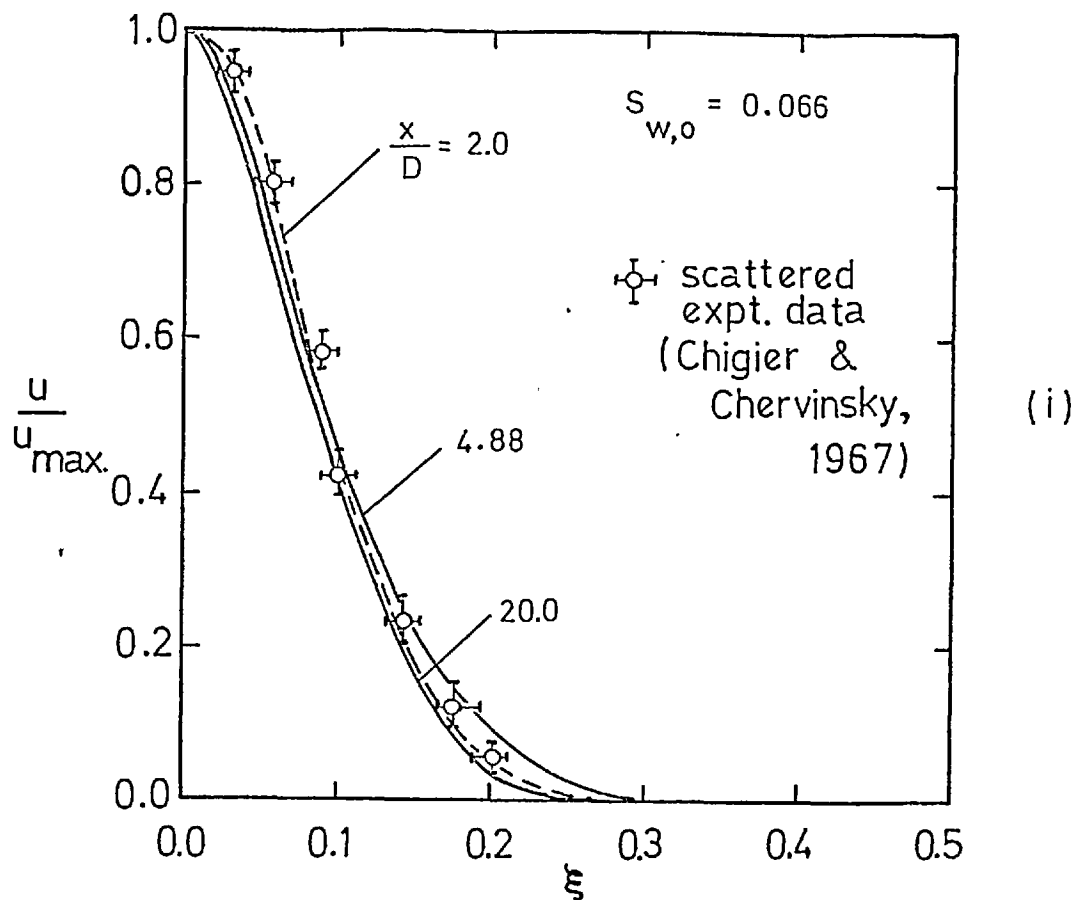


Fig. 3.3

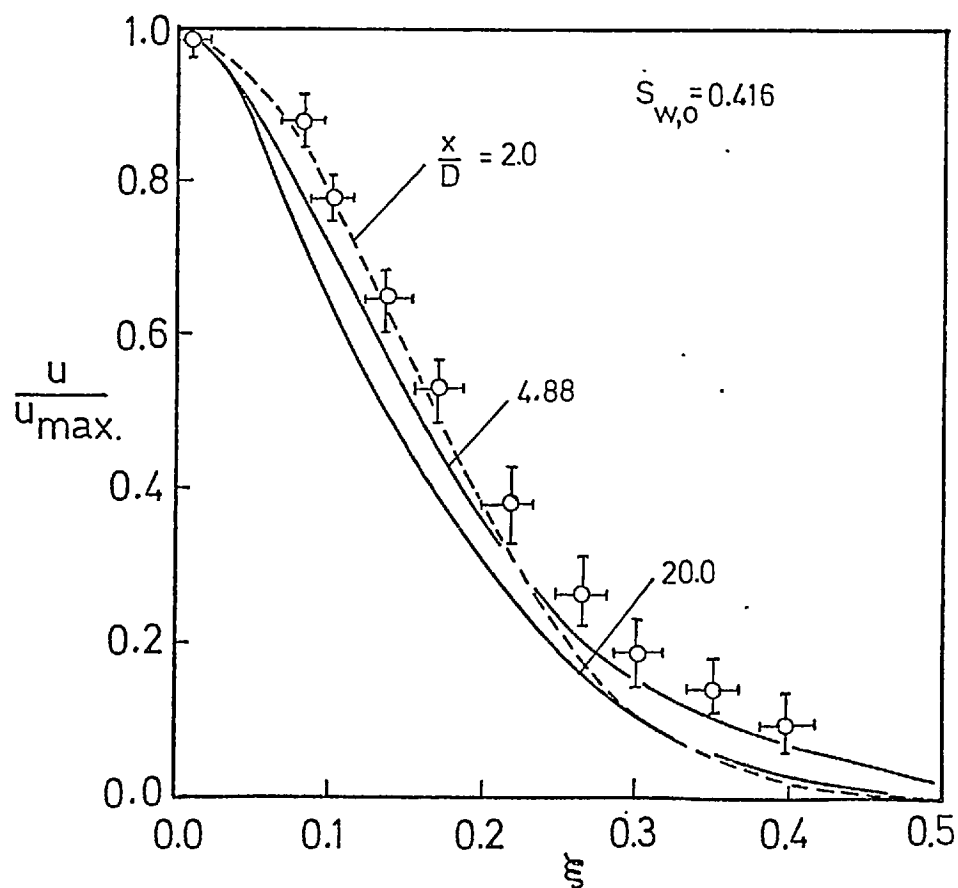
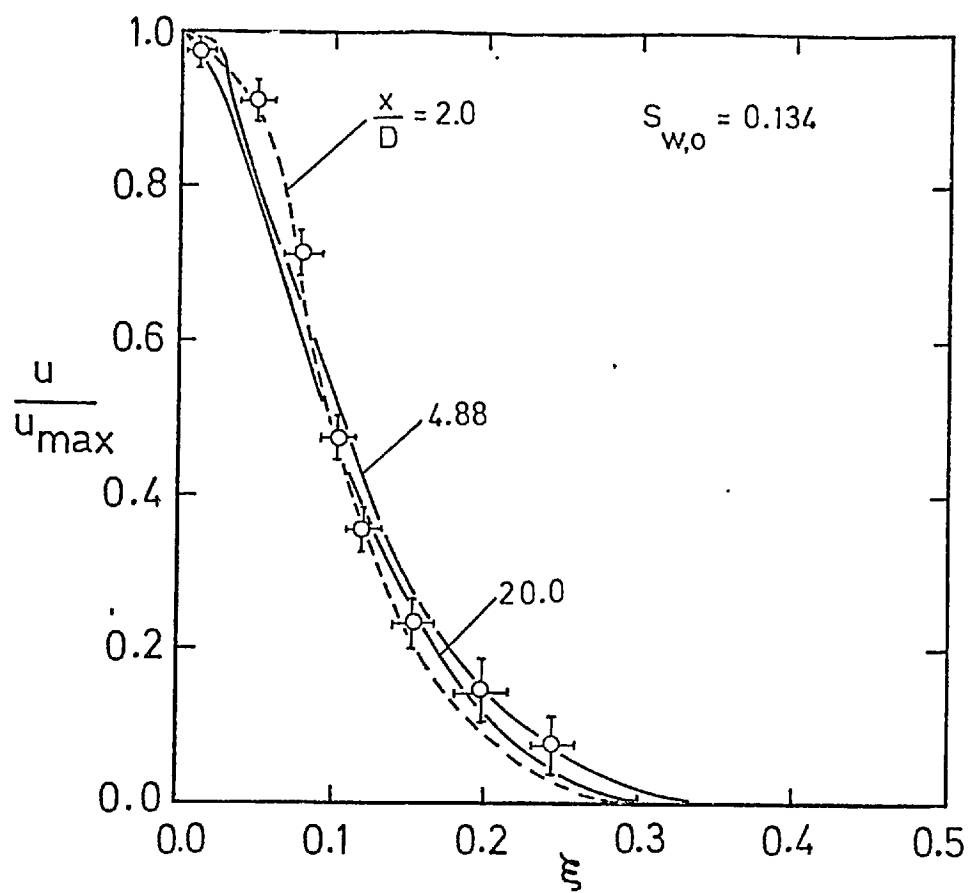


Fig. 3.3

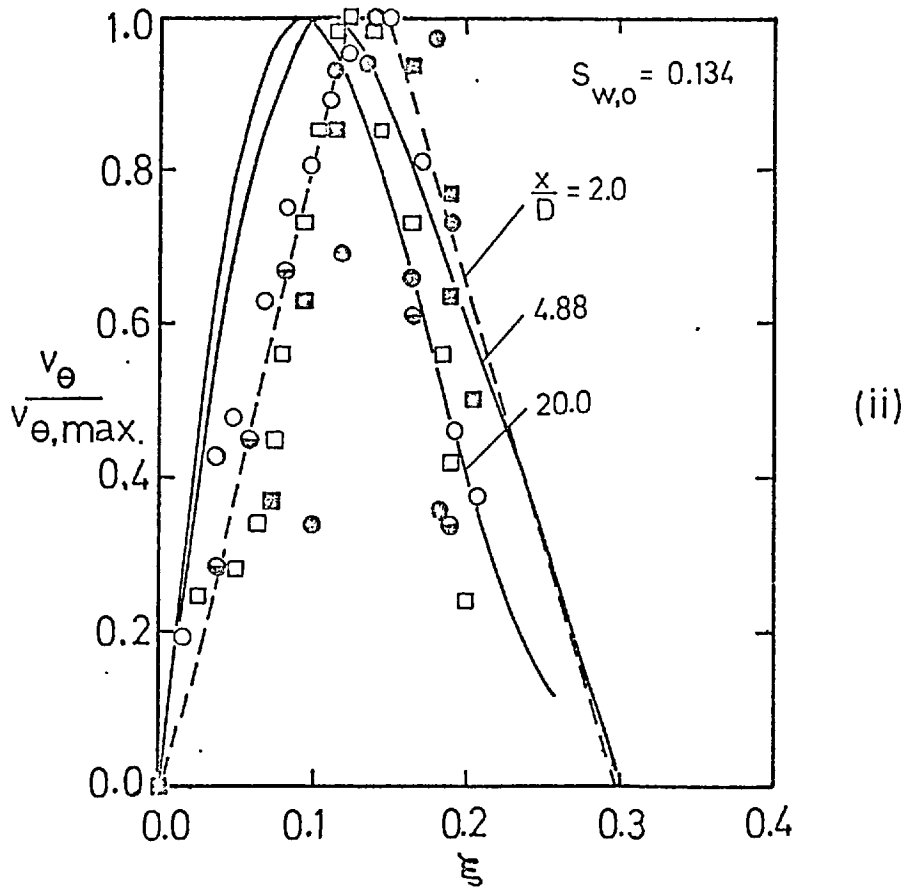
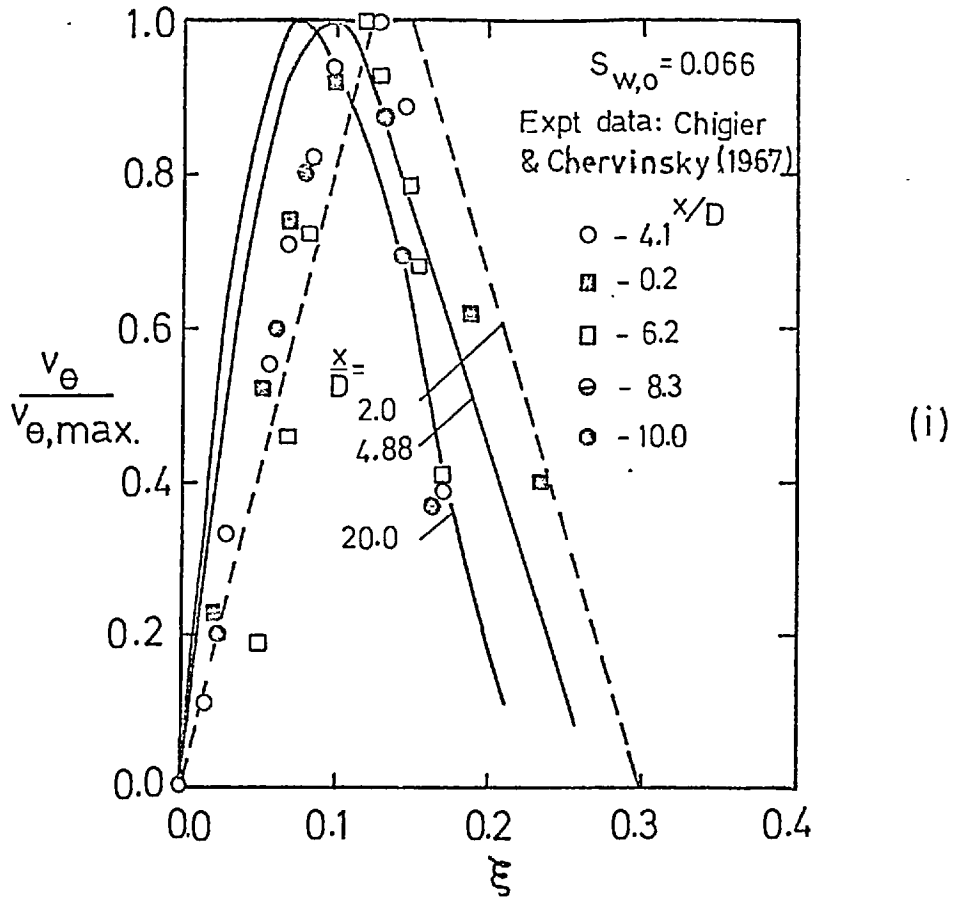


Fig. 3.4

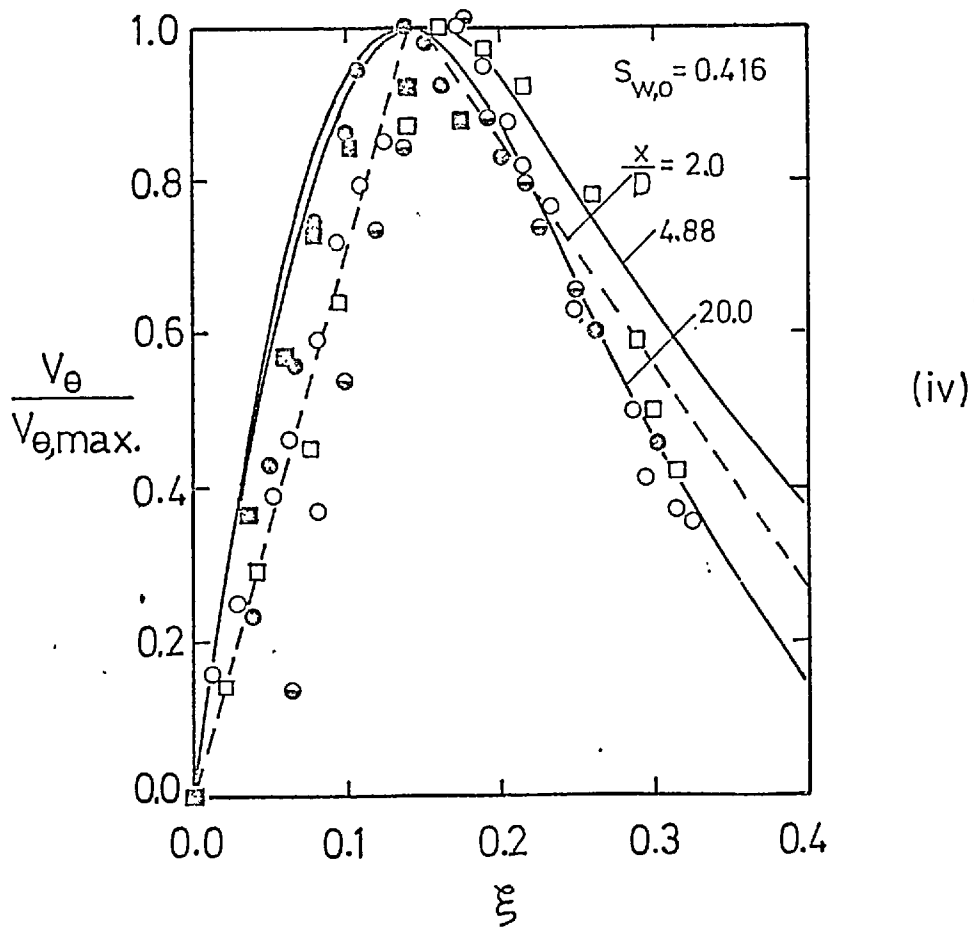
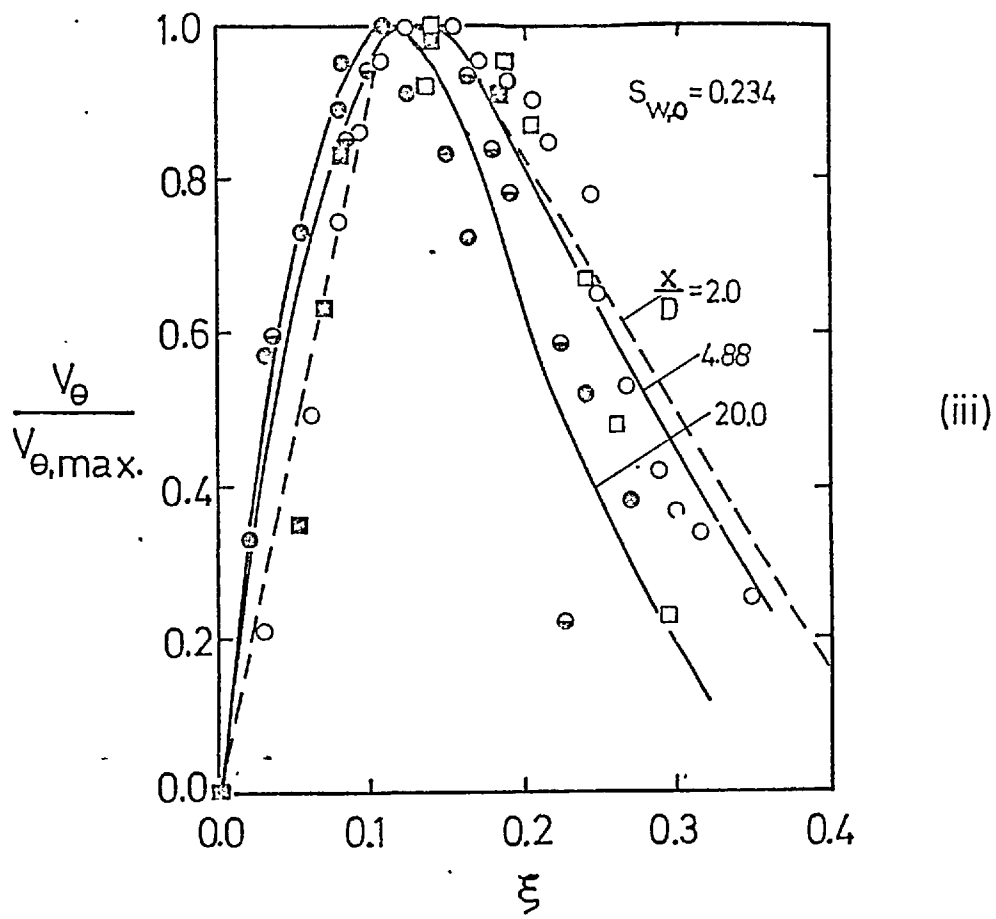


Fig. 34

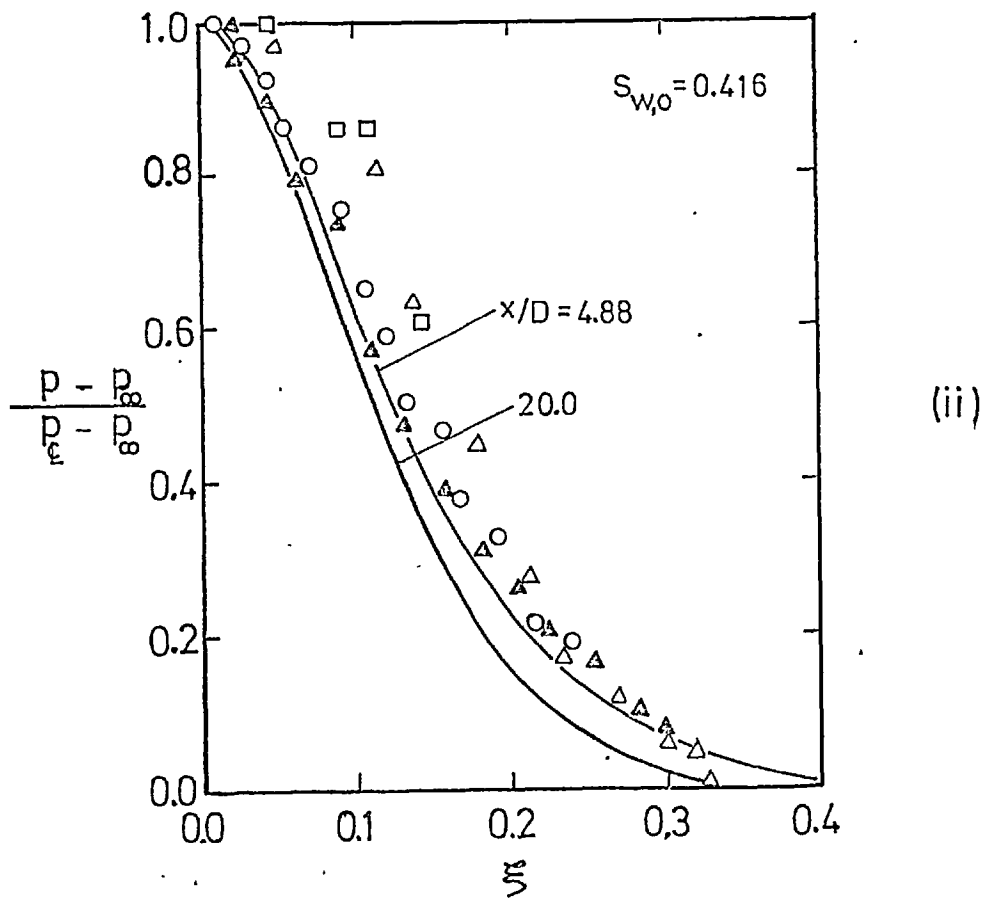
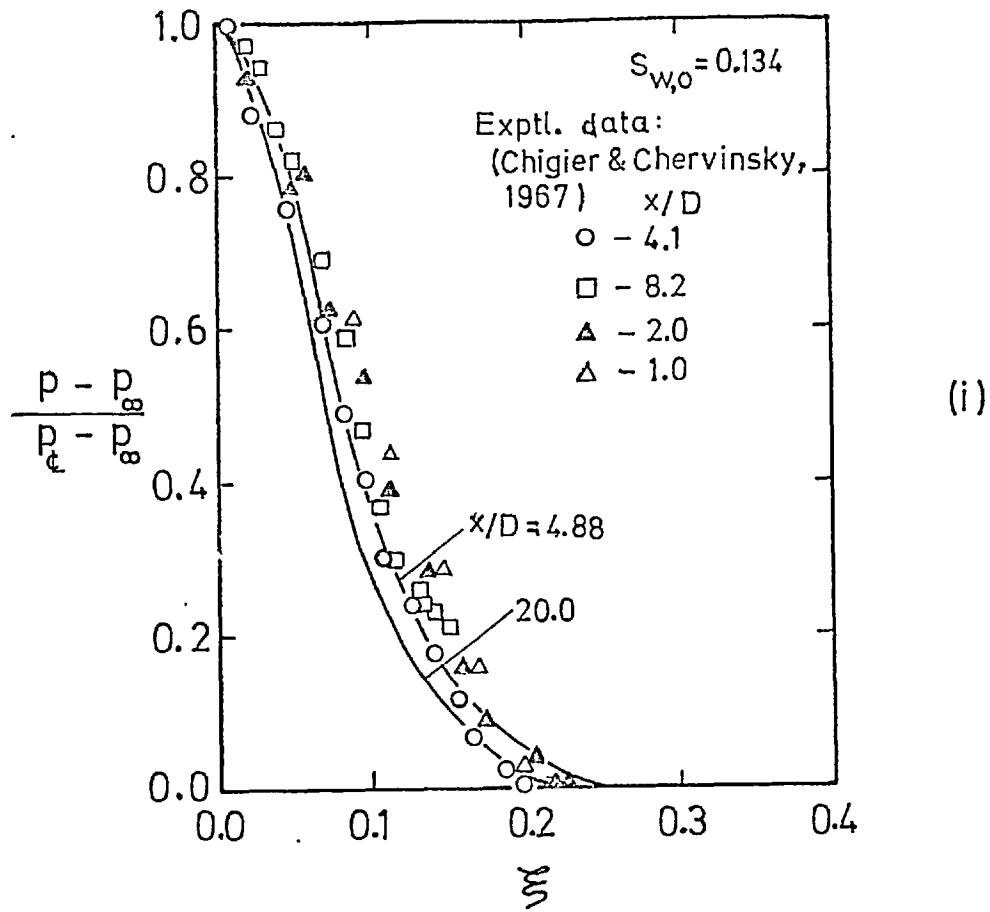


Fig. 3.5

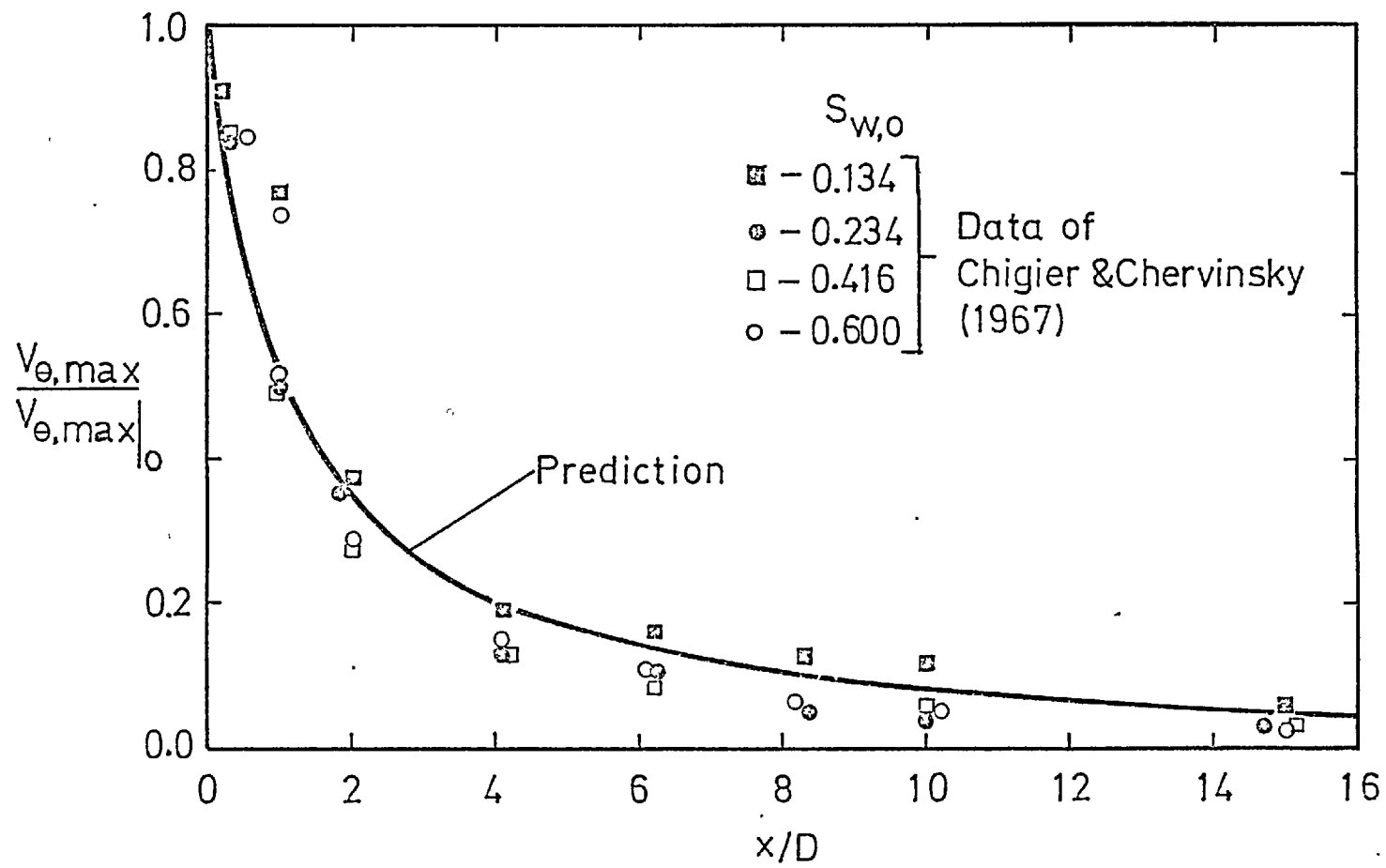


Fig. 3.6

CHAPTER 4

THE SWIRLING, CONFINED TURBULENT JET

Chapter 4

The swirling, confined turbulent jet

4.1 Introduction

Chapter 3 has demonstrated the use of Prandtl's mixing-length hypothesis to predict some properties of the swirling, free-turbulent jet in stagnant surroundings. In this chapter, computations are displayed of a swirling, turbulent jet in a co-flowing stream confined in a duct of varying area. The computations are compared with data obtained as part of this research (see Part II of the thesis).

The values of two of the empirical constants used in the present predictions are the same as those established in the last chapter (Section 3.2) as being appropriate to free swirling jets; namely, $K_x = 0.08$ and $K_\theta = 0.26$. The justification of using the same value of K_x as in free-jets is established by making comparisons between predictions of a confined, non-swirling jet and experimental data. However, satisfactory agreement between predictions and data for a confined swirling jet could be obtained only by setting $Pr_\theta = .5$, as against $Pr_\theta = 1$ established for free swirling jets.

4.2 The non-swirling, confined turbulent jet

4.2-1 The flow geometry and inputs

Figure 4.1 shows the geometry of the problem and defines the nomenclature used in the two figures that follow it.

The initial profiles and physical inputs were the same as those used for the free-turbulent jet, with one addition - account has been taken, in the following way, of the presence of free-stream turbulence.

In Prandtl's hypothesis, the quantity [mixing-length $\times$ velocity gradient]

is approximately the root-mean-square fluctuation velocity, $\sqrt{u'^2}$ (Chapter 2, Section 2.2). The latter, expressed as a fraction of the free-stream velocity, U_d , is the specified free-stream turbulence level. In the computational procedure, whenever the quantity [mixing length $\times$ velocity gradient] fell below the specified turbulence level, it was raised to equal this level.

This method of specifying a minimum turbulence level gave rise to another adjustable constant, but one that was fixed by experimental conditions. This constant was not altered to procure better agreement between the data and the predictions.

4.2-2 The predictions

The object of these predictions was to establish the value of K_x for the jet. So, as the experimental radius ratio R_d/R_j was quite large, the boundary-layer on the wall of the duct were neglected and computations performed for the jet region alone.

Figures 4.2 and 4.3 show, respectively, the computed profile shape and axial decay of velocity in a confined, swirl-free jet compared with the data (for this type of jet) of Curtet and Ricou (1961). The computations employed the same values of K_x as were used for the free-turbulent jet; namely, 0.065 and 0.08 before and after the end of potential core respectively. (See Siddhartha, 1969).

The satisfactory agreement between computations and data in spite of the neglect of the influence of the wall boundary-layer establishes the adequacy of these values of K_x for predicting the velocity field of a non-swirling, confined turbulent jet.

4.3 The confined, swirling, turbulent jet

4.3.1 The flow geometry and inputs

Figure 4.4 illustrates the geometry of the problem under consideration.

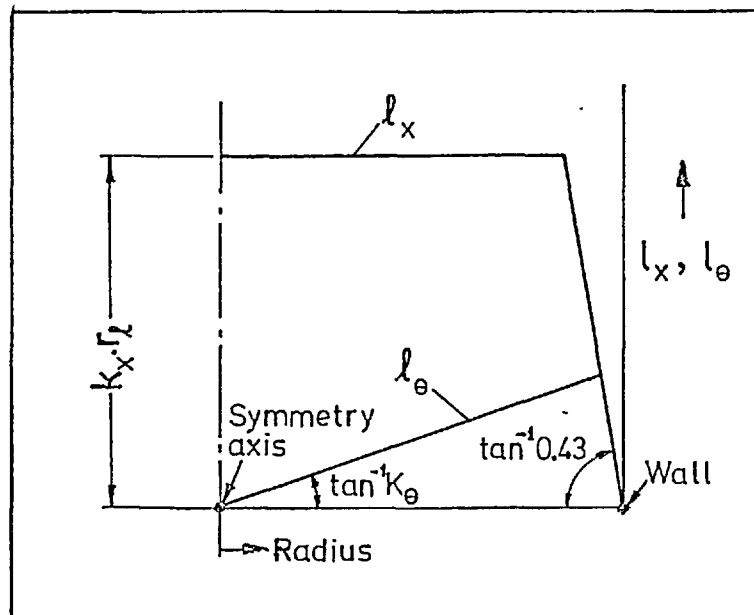
a. Mathematical inputs

The computations begin at an axial station 4" away from the exit plane of the jet pipe. The initial profiles of axial and tangential velocity are derived from smooth curves drawn through experimental data points. The development of the profiles downstream of this station is computed and compared with experimental data. Account is taken of the axi-symmetric boundary-layer developing along the walls of the diffuser by extending the computational grid all the way from the centre-line to the wall. However, the flow near the wall may be presumed to be ignorant of the presence of swirl; for, the swirling jet occupies only the central region of the duct in the early stages of its development and has dissipated its swirl by the time it diffuses out to the wall.*

*The experimental flow situation allows this presumption to be made (see Part II).

b. Physical inputs

At any one axial station, the mixing-length distributions were as in the sketch below.



These mixing-length distributions will be recognized as the same as those prevailing in the free-turbulent swirling jet (Chapter 3) except for the region near the wall. Here both l_x and l_θ are made proportional to the distance from the wall; the constant of proportionality having been established in earlier studies on flows near walls.*

* Patankar and Spalding (1970).

4.3-2 The predictions

Figure 4.5 shows the predicted axial velocity profiles compared with experimental data. The smooth line drawn through the data points for Station 1 represents the initial profile fed into the Program. The figure shows that the development of the axial velocity profiles is predicted fairly using the empirical constants $K_x = 0.08$ and $K_\theta = 0.26$. Varying the value of K_θ in the range 0.2 to 0.3 did not alter the profile development appreciably.

Figure 4.6 compares the predicted tangential velocity profiles with the data. The dashed line 2' is the predicted profile for Station 2 using a value of $Pr_\theta = 1$. (The value established in the last chapter as being appropriate to free swirling jets.) The curve compares very unfavourably with the data points for station 2 (marked X). Raising Pr_θ to 5 greatly improves the match between data and predictions. In particular, the radial position of the maximum of tangential velocity is well predicted although the profile in the centre of the duct (i.e. in the region $0.01 \leq r/r_d \leq 0.05$) is not predicted satisfactorily.

4.4 Conclusions

The computations displayed in the present chapter demonstrate that the problem of swirling jets in co-flowing streams confined in ducts can be handled by the P-S procedure. The empirical constants K_x and K_θ are the same as those established for free swirling jets; namely, 0.08 and 0.26 respectively. However, Pr_θ has to be raised to 5 (from unity for the free jet) before adequate comparisons between predictions and data can be made.

As in the case of free swirling jets, better agreement between the profiles of tangential velocity and data could perhaps be procured by employing non-uniform distributions of Pr_0 . The accuracy of the data (see Part II) and the over-simple mixing-length model make such fine adjustments hardly worthwhile.

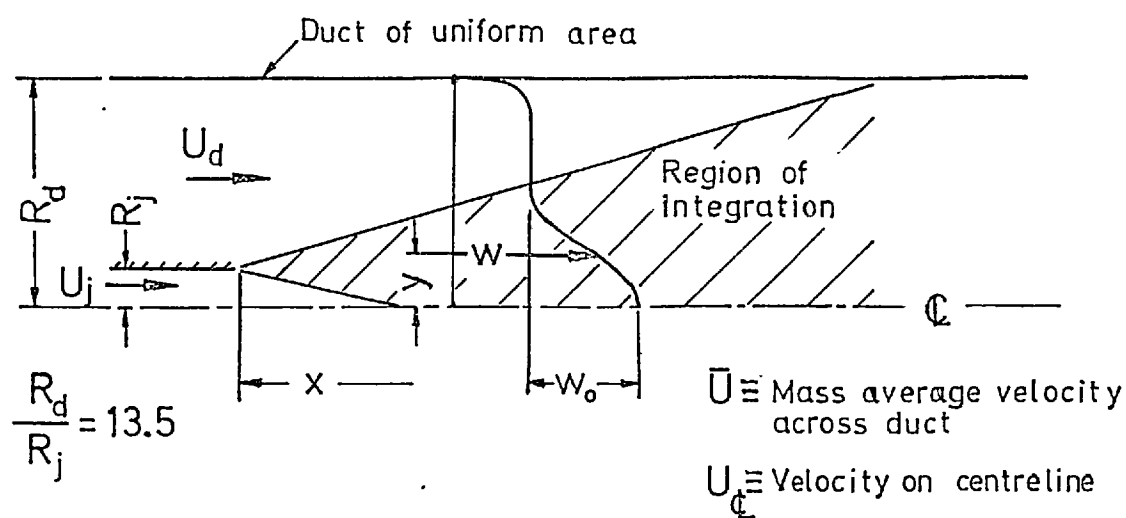


Fig. 4.1

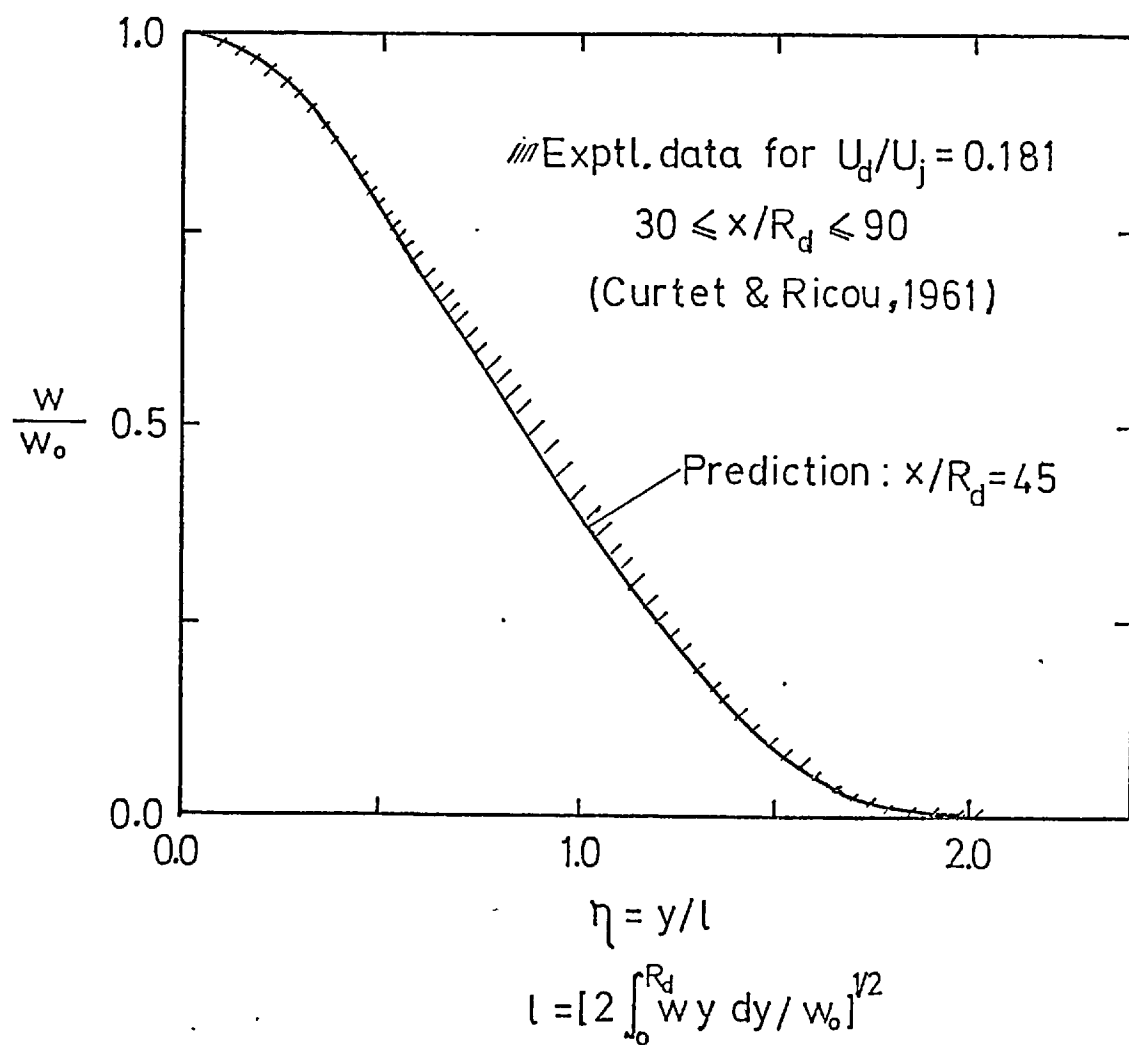


Fig. 4.2

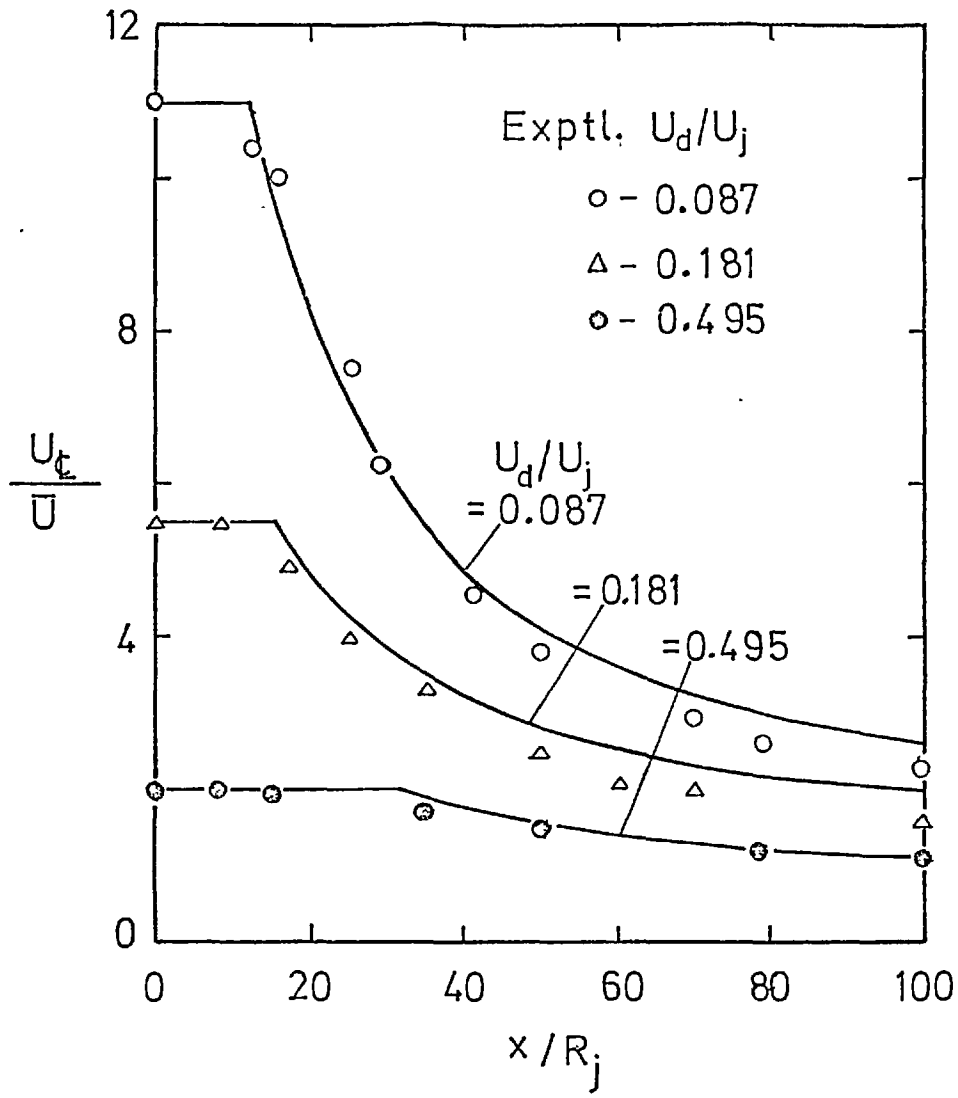


Fig. 4.3

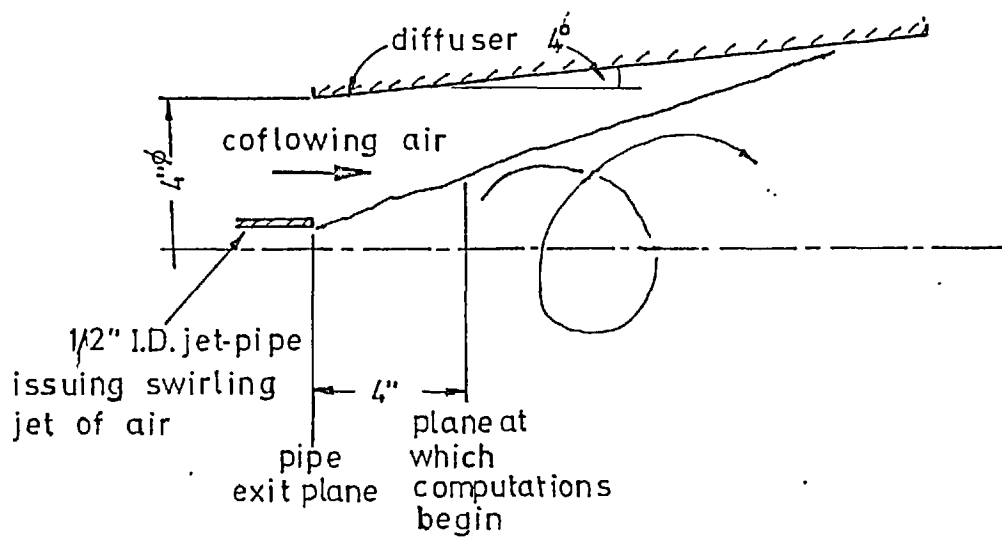


Fig. 4.4

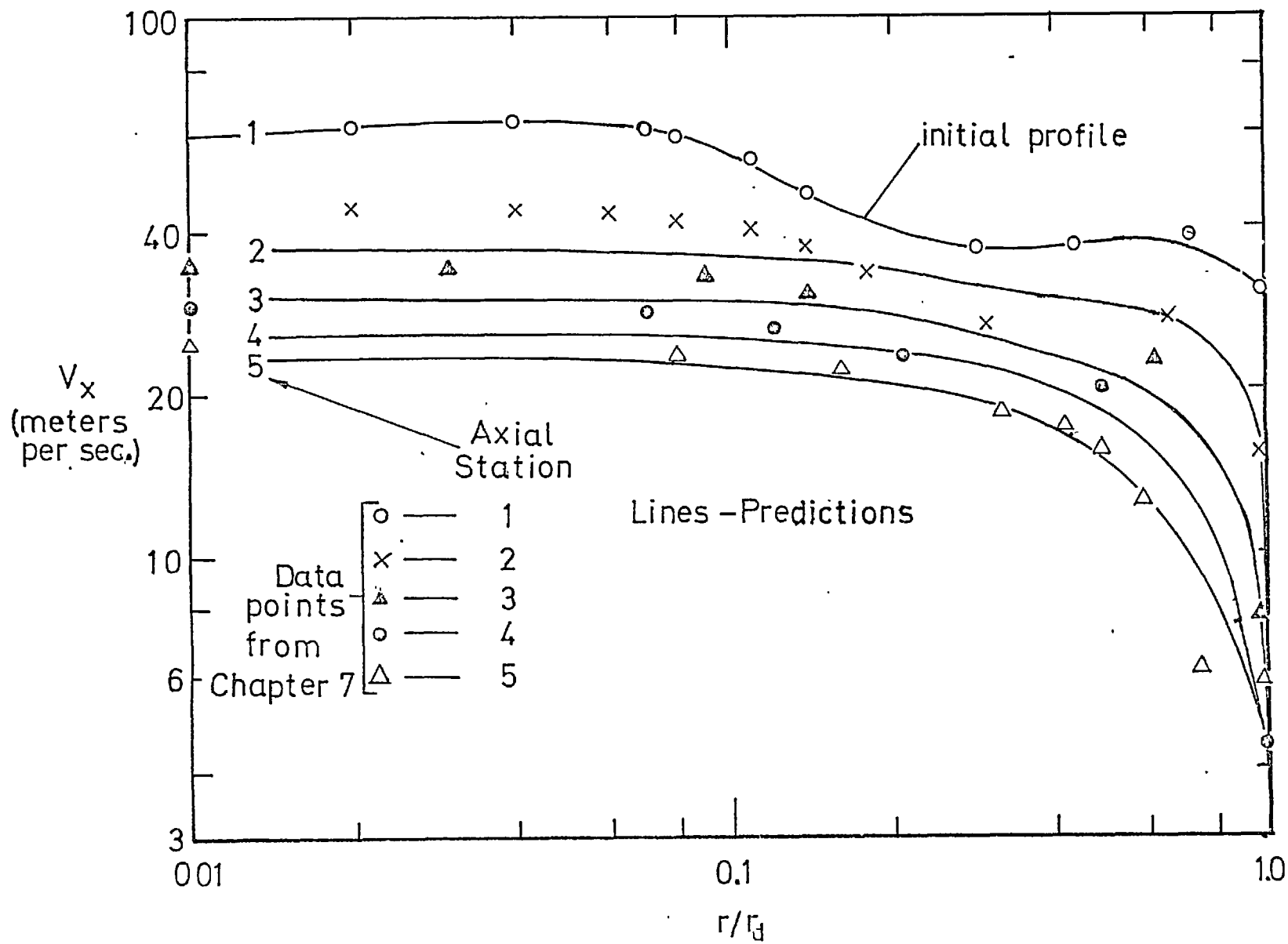


Fig. 4.5

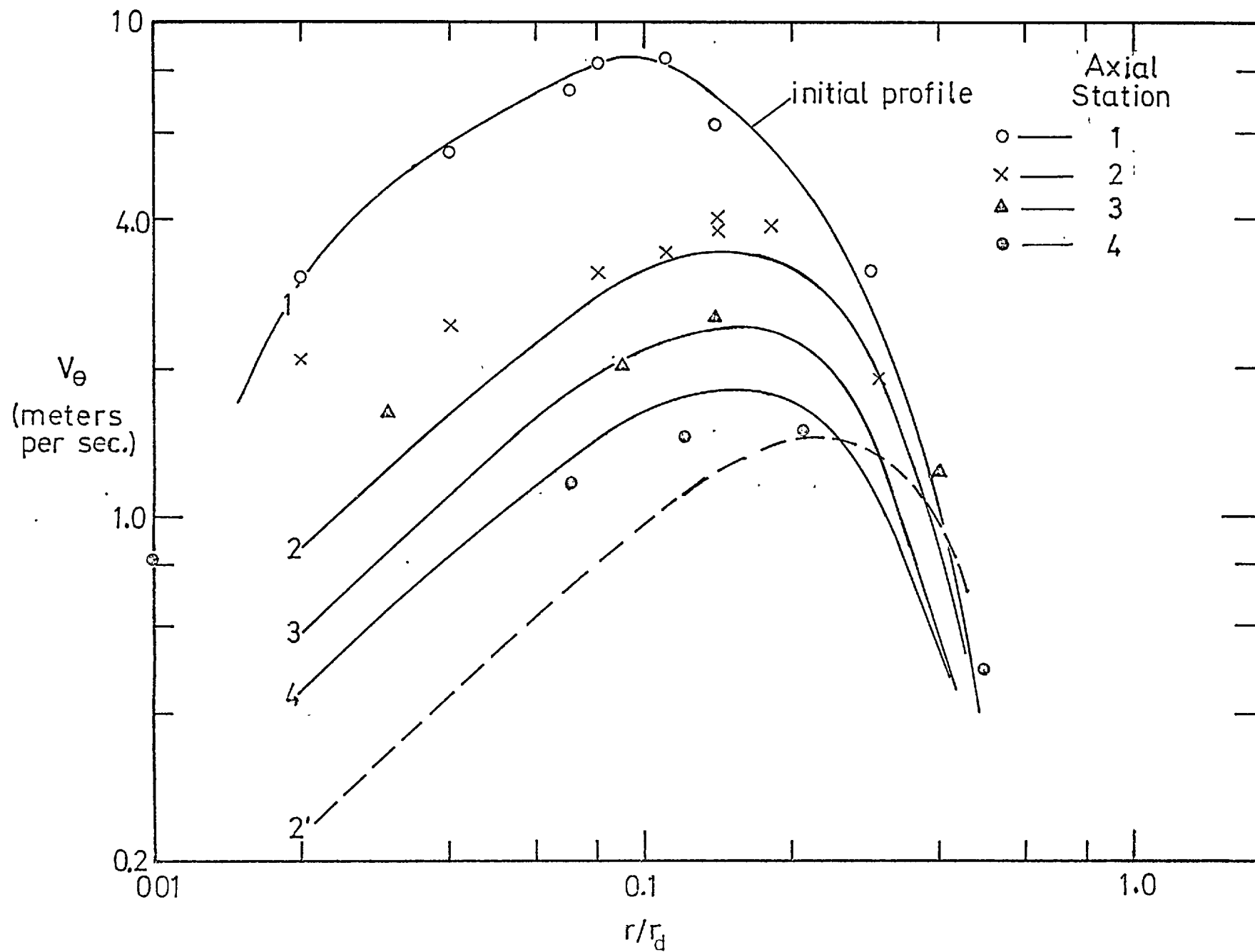


Fig. 4.6

CHAPTER 5

SWIRLING FLOW IN PIPES: LAMINAR FLOW

Chapter 5

Swirling, laminar flow in pipes

5.1 Introduction

An objective of this thesis is to demonstrate the versatility of the P-S procedure (and its associated computer program) by using it to predict different types of swirling flows. The last two chapters have demonstrated the use of the procedure in predicting free and confined turbulent jets. This chapter and the next demonstrate the use of the procedure in predicting, respectively, laminar and turbulent flow through pipes.

Whilst laminar swirling flows are perhaps not of great engineering importance, a study of such flows helps to elucidate the crucial role played by the cross - stream pressure gradients in modifying the axial-flow field and the rate of momentum transfer to the wall. When turbulent flows come to be studied, this information helps one to perceive with greater clarity the separate effects of turbulence hypotheses and cross - stream pressure gradients.

5.2 The problem considered

A fully - developed laminar flow in a pipe is considered. Initially, a linear profile for tangential velocity, i.e. a profile resulting from solid-body rotation, is superposed on a parabolic, axial-velocity profile. The influence is investigated of the intensity of swirl on the downstream development of the shapes of the profiles of axial and tangential velocity; of the axial- and tangential-shear-stress co-efficients and of the axial rate-of-decay of the radial maximum of swirl velocity. Also, the decay of swirl is characterised by a 'decay parameter' whose variation with axial flow Reynolds number is compared with the experimental data of Talbot (1954).

5.3 Nomenclature

The symbols used in the text and in the figures have the following meanings.

<u>Symbol</u>	<u>Meaning</u>	<u>Defining equation</u>
D	Diameter of pipe	-
r	Radial distance from pipe axis	-
R	Radius of pipe	-
Re	Axial flow Reynolds number	5.4
S_N	Swirl number	5.1
S_x	Axial-shear-stress coefficient	5.6
S_θ	Tangential-shear-stress coefficient	5.7
T_x	Axial shear-stress at pipe wall	-
T_θ	Tangential shear-stress at pipe wall	-
V_x	Axial velocity	-
V_θ	Tangential (swirl) velocity	-
x	Axial distance from section at which swirl is imposed	-
x^+	Transformed x	5.5
β	Decay parameter	5.8
ρ	Density	
ν	Kinematic viscosity	

Subscripts

0	At axial station where swirl is imposed
$f.d.$	Value when flow is swirl-free and fully-developed
l	local value
max	Maximum value at any axial station

5.4. Characterising the intensity of swirl input - the swirl number

A measure is needed for the intensity of swirl input into swirling pipe flows. The angular velocity of the initial solid-body rotation imparted to the flow is, by itself, an insufficient measure; for, the same angular velocity has not the same influence on flows with low mass-flow rates as on flows with high mass-flow rates. A satisfactory measure of the swirl input may be obtained, as in the case of swirling jets, by defining a swirl number,

$$S_N \equiv \frac{\text{Axial flux of angular momentum}}{\text{Axial flux of axial momentum}} \times \frac{1}{\text{pipe radius}}$$

$$= \frac{\int_0^R \rho \cdot v_x \cdot r \cdot (r v_\theta) dr}{\int_0^R \rho \cdot v_x^2 \cdot r \cdot dr} \times \frac{1}{R}, \quad (5.1)$$

at the axial station where the swirl is imposed. Equation (5.1) is used to determine the swirl number in the present computations.

5.5. Elimination of the axial-flow Reynolds number as a parameter of the problem

The general equations of the two-dimensional swirling boundary-layer have been given in chapter 2. For swirling flow in a straight circular pipes ($\alpha = 0$) with impermeable walls ($a = b = 0$), the three hydrodynamic equations reduce to:

$$\left. \begin{aligned} \frac{\partial v_x}{\partial x} &= \frac{\partial}{\partial \omega} \left(c \frac{\partial v_x}{\partial \omega} \right) - \frac{1}{\rho v_x} \cdot \frac{\partial p}{\partial x} \\ \frac{\partial (v_\theta \cdot r)}{\partial x} &= \frac{\partial}{\partial \omega} \left(c r^2 \frac{\partial (v_\theta / r)}{\partial \omega} \right) \\ \text{and} \quad \rho \frac{v_\theta^2}{r} &= \frac{\partial p}{\partial r} \end{aligned} \right\} \quad (5.2)$$

where the symbols have the meanings ascribed to them in Chapter 2.

If, in equations 6.5, the velocities, distances and pressures are normalised respectively by the fully-developed axial velocity, $V_{x,0}$; the radius of the pipe, R and $\rho V_{x,0}^2$; the equations of uniform property flow reduce to:

$$\left. \begin{aligned} \frac{\partial V'_x}{\partial x'} &= \frac{1}{Re} \frac{\partial}{\partial \omega'} \left(r'^2 V'_x \frac{\partial V'_x}{\partial \omega'} \right) - \frac{1}{V'_x} \frac{\partial p}{\partial x'} ; \\ \frac{\partial (V'_\theta r')}{\partial x'} &= \frac{1}{Re} \frac{\partial}{\partial \omega'} \left(r'^4 V'_x \frac{\partial (V'_\theta r')}{\partial \omega'} \right) ; \\ \text{and } \frac{V'_\theta}{r'} &= \frac{\partial p'}{\partial r'} . \end{aligned} \right\} \quad (5.3)$$

$$\text{Where } Re = \frac{V_{x,0} \cdot R}{\nu} \quad (5.4)$$

is the axial flow Reynolds number and primes indicate normalisation.

If, further, a new axial co-ordinate x^+ is defined such that

$$x^+ = x'/Re = \frac{x}{Re \cdot R} ; \quad (5.5)$$

the axial-flow Reynolds number may be eliminated from equations (5.3) by substituting x^+ for x' . The equations then become identical in form to equations (5.2).

Thus, provided the transformed co-ordinate, x^+ , is used; the solutions to equations (5.3) may be expected to depend only on the swirl number, defined in 5.4 above.

The results of the computations are presented in the next section as a family of curves with the swirl number, S_N , as the only parameter.

5.6 The influence of swirl on fully-developed, laminar pipe flows

5.6-1 The profiles of axial and tangential velocity

Figure 5.1 (a) and (b) display, respectively, the shapes of the axial and tangential velocity profiles for one swirl number.

The shapes of the profiles of axial velocity show the effect of the cross-stream pressure gradients. The centrifugal forces steepen the profile near the wall as a result of their tendency to increase the velocity there and to diminish it near the centre of the pipe. At large values of x^+ , where the flow is virtually swirl free, the profile settles on to the familiar parabolic profile of fully-developed, laminar flow. The actual value of x^+ at which the profile becomes indistinguishable (to the scale of the figures) from the parabolic profile depends on the swirl number and, of course, on one's visual acuity. The development of the axial velocity profile at other swirl numbers is similar to that shown in the figure.

A typical example of the development of the swirl velocity profile is displayed in figure 5.1 (b). Starting with a linear, solid-body rotation profile (with a sharp drop to zero velocity at the wall), the swirl velocity develops towards a 'similarity' profile whose shape and radial position of maximum swirl velocity (at $r/R = 0.425$) remain invariant with both x^+ and the swirl number, S_N . Of course, the value of x^+ at which this invariant profile is obtained increases with increasing S_N .

5.6-2 The axial velocity on the pipe axis

During discussion of the shape of the axial velocity profile above, mention was made of the tendency of the velocity near the pipe axis to diminish. This influence of swirl is demonstrated in figure 5.2.

Plotted against x^+ is the ratio on the axis of axial velocity with swirl to the axial velocity when no swirl is present. The numbers against the curves refer to the swirl number, S_N . As expected, for large values of x^+ , all the curves tend asymptotically to unity. An interesting facet of the behaviour of the velocity on the pipe axis is the tendency of the minimum to shift towards lower values of x^+ with increasing swirl number.

5.6-3 The axial-shear-stress co-efficient

As in non-swirling flows, the shear-stress co-efficient appropriate to the axial-shear-stress at the pipe wall is defined as:

$$S_x \equiv \frac{\tau_x}{\rho \bar{V}_x^2} \quad , \quad (5.6)$$

where τ_x is the axial-shear-stress at the wall and $\bar{V}_x$ is the mass-average axial velocity. The curves of figure 5.3 show plotted against x^+ , the ratio of the local shear-stress co-efficient, $S_{x, \ell}$, to $S_{x, f.d.}$, the shear-stress co-efficient obtained when the flow is fully-developed and devoid of swirl. The swirl-induced steepening of the gradient of axial velocity near the pipe wall was shown in 5.6-1. In figure 5.4 the consequence of this influence of swirl is clearly evident; $S_{x, \ell}$ increases because τ_x is enhanced by the increased gradient of axial velocity near the pipe wall. As is to be expected, at large values of x^+ , where the swirl has virtually disappeared, $S_{x, \ell}$ approaches $S_{x, f.d.}$

5.6-4 The tangential-shear-stress co-efficient

A convenient definition of the tangential-shear-stress co-efficient is:

$$S_\theta \equiv \frac{\tau_\theta \cdot R^2}{\int_0^R \rho (r V_\theta^2) dr} \cdot \frac{\bar{V}_\theta R}{v} \quad , \quad (5.7)$$

where $\bar{V}_\theta$ is an average swirl velocity at the axial station in question.

Figure 5.4 shows the decay of S_θ with x^+ . The influence of swirl on the decay rate is much smaller than its influence on the rate-of-decay of $S_{x, \ell}$. Irrespective of the swirl number, S_θ settles down to a value of 9.94 at $x^+ = 0.18$. Thus 9.94 appears to be the far-downstream tangential-shear-stress co-efficient for all laminar, swirling flows.

5.6-5 The decay of the radial maximum of the swirl velocity

Figure 5.5 shows the decay of $V_{\theta, \max}$ towards $V_{\theta, \max, 0}$ plotted against x^+ . The influence of the swirl number is quite small; so it was also in the case of free swirling jets (Chapter 3. Fig. 3.6).

5.6-6 The decay of swirl: comparison between computations and experimental data

Talbot (1954) provides some experimental data on the decay of laminar swirl in a pipe. His experimental set-up consisted of a long straight pipe in which a laminar flow was fully developed. Following this, a section of rotating pipe imparted solid-body rotation to the flow which was allowed to decay in a further, stationary, test-section. A constant angular velocity of 1.95 radians/second was imparted to the flow and the Reynolds number was changed by changing the throughput of water flowing in the pipes: thus, Talbot changed his swirl number as well.

Talbot defines a decay parameter

$$\beta \equiv - \frac{1}{V_{\theta}(r, x)} \cdot \frac{dV_{\theta}(r, x)}{dx} \quad , \quad (5.8)$$

and measures this parameter from motion-picture studies of oil droplets

introduced into the flow.[†] Figure 5.6 shows the computed (line) and measured (points) value of β plotted against the axial flow Reynolds number.

It is interesting to note that the predicted relationship between β and Re (solid line in Fig. 5.6) is independent of both x/D and swirl number. (We recall that in Talbot's data, every change in axial flow Reynolds's number was accompanied by a change in the swirl number).

The discrepancy between the experimental data and the computations remains to be explained. Although Talbot claims a $\pm 10\%$ accuracy for his data, he mentions that the chosen value of angular velocity (1.95 radians/sec.) for his rotating pipe section was close to one that caused 'instability' (i.e. incipient transition to turbulence) in the flow. It is not unlikely, therefore that Talbot's data refer really to transitional rather than strictly laminar flow.

5.7 A remark on the accuracy of the plotted results

The absolute accuracy of the computations is difficult to assess and there are no published exact analytical solutions to the problem of swirling laminar flow in a pipe. Nevertheless, some idea of the accuracy may be obtained by noting the discrepancies between the results of the computations for different Reynold's numbers. Ideally (as shown in 5.5 above), if the x^+ co-ordinate is used, there should be no difference whatever between these different computations. In fact, there are small differences. Their magnitudes vary with the swirl number but in no case do they exceed the magnitudes of the thicknesses of the lines drawn in the plots. As explained below, these discrepancies could have been reduced to vanishing point by taking much shorter forward steps. But then

[†] No details of the technique are given. It is presumed here that β was calculated from the measured decay of $V_{\theta, \max}$.

the penalty would have had to be paid of a consequent increase in computer time.

As detailed in section 6.5 of Patanker and Spalding (1970), the axial gradient of the average of the cross-stream pressures is calculated by a non-iterative procedure. At each forward step, the unknown pressure gradient is estimated from a one-dimensional flow formula. This estimate is modified by a correction procedure which strives, in arrears, to reduce the discrepancy between the radius corresponding to the calculated flow area and the specified pipe radius. Thus, confined flow calculations are made for a fictitious duct whose radius is very close to, but different from, the specified duct radius. Figure 5.9 shows a typical plot from the present calculations of the percentage radius discrepancy versus x/R . Beyond $x/R = 4$, the discrepancy drops below 1% - an acceptable level - and rapidly diminishes thereafter. For a given forward step this behaviour of the radius discrepancy differs somewhat for different Reynolds and swirl numbers. Hence the slight differences in the plots referred to in the previous paragraph. The differences decreased when the step-size was reduced, but only at the expense of an increase in computer time.

The non-iterative technique of calculating the axial pressure gradient also imposes an upper limit on the range of swirl numbers investigated. Beyond a swirl number of about 0.9, the cross-stream pressure gradients cause large changes to occur in the shape of the profile of axial velocity. The one-dimensional flow formula for axial pressure-gradient makes no allowances for these changes in profile shape. In consequence, the error between the pressure gradient given by the formula and the one actually required by the flow becomes too large to be handled in a stable manner by the simple correction procedure employed. As a result, the flow-area suffers large oscillations and the

calculations break down, i.e., negative velocities appear in the flow field. Attempts to devise more stable formulae and correction procedures were not successful. It would seem, therefore, that for the higher swirl numbers, iteration will be unavoidable.

5.8 Conclusions

This chapter has demonstrated the use of the P-S procedure in the calculation of swirling pipe flows. The influence of swirl on fully-developed, laminar pipe flows may be summarised as under.

- (i) The radial pressure gradients induced by swirl change the shape of profile of axial velocity; in particular, the velocity gradient near the pipe wall is increased. (Figure 5.1 (a)).
- (ii) The local value of the axial-shear-stress co-efficient is enhanced by swirl. Its rate-of-decay increases with the strength of the swirl. (Figure 5.3).
- (iii) The rates-of-decay of the maximum of swirl velocity and of the tangential-shear-stress co-efficient are not greatly influenced by the strength of the swirl. (Figures 5.4 and 5.5).
- (iv) The tangential velocity profile develops towards a 'similarity' profile whose shape is invariant with both the swirl number and axial distance. (Figure 5.1 (b)).

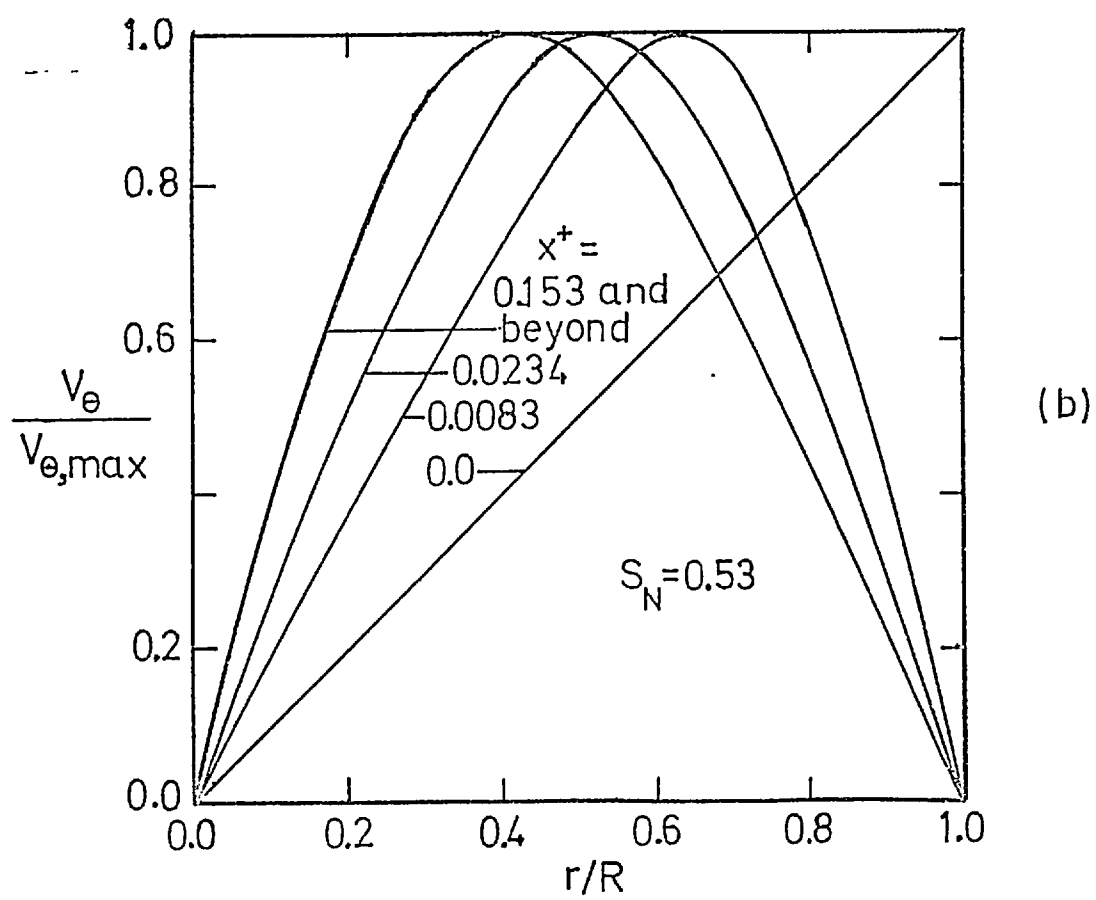
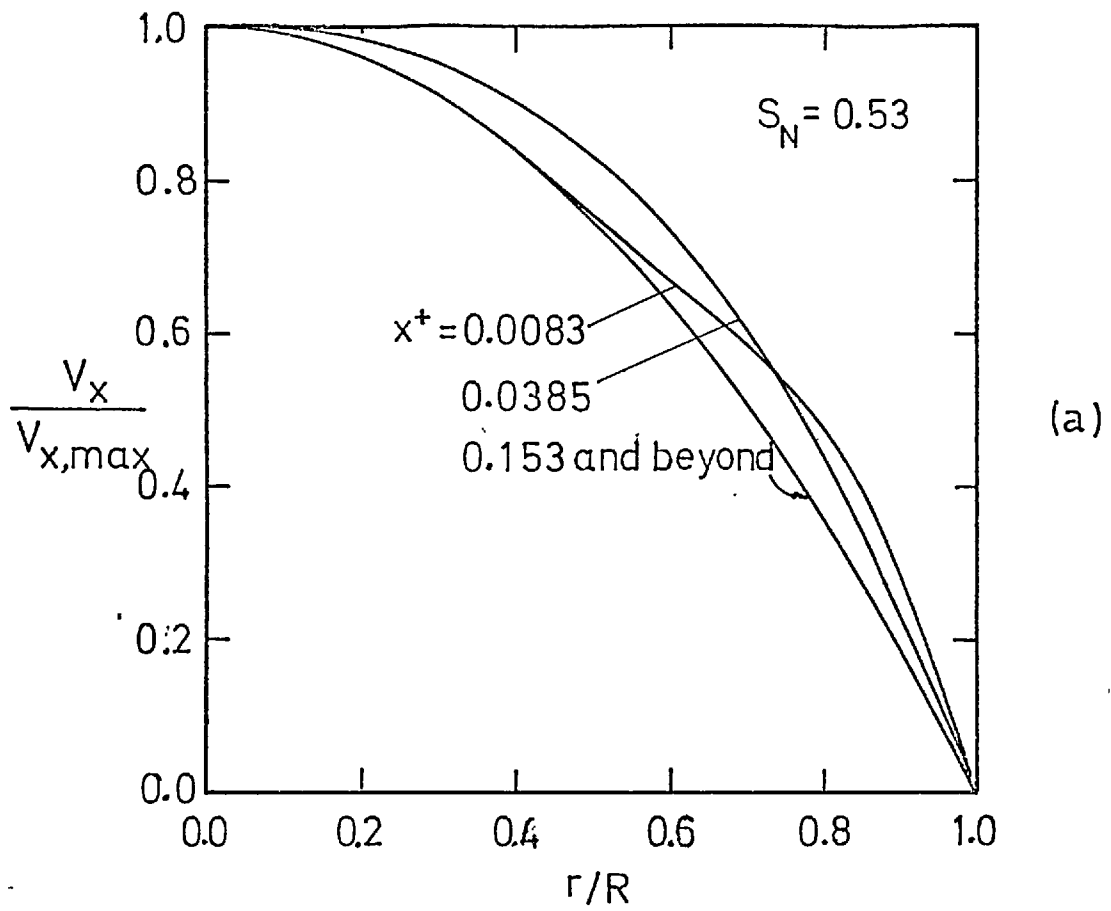


Fig. 5.1

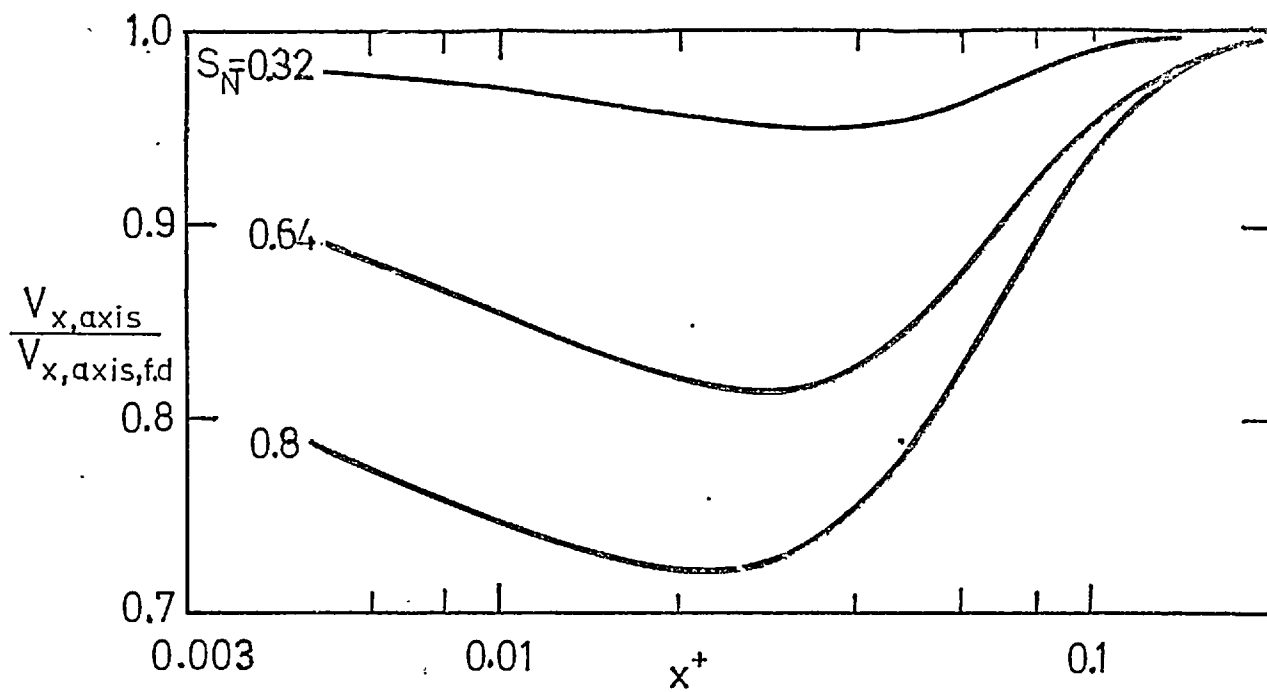


Fig. 5.2

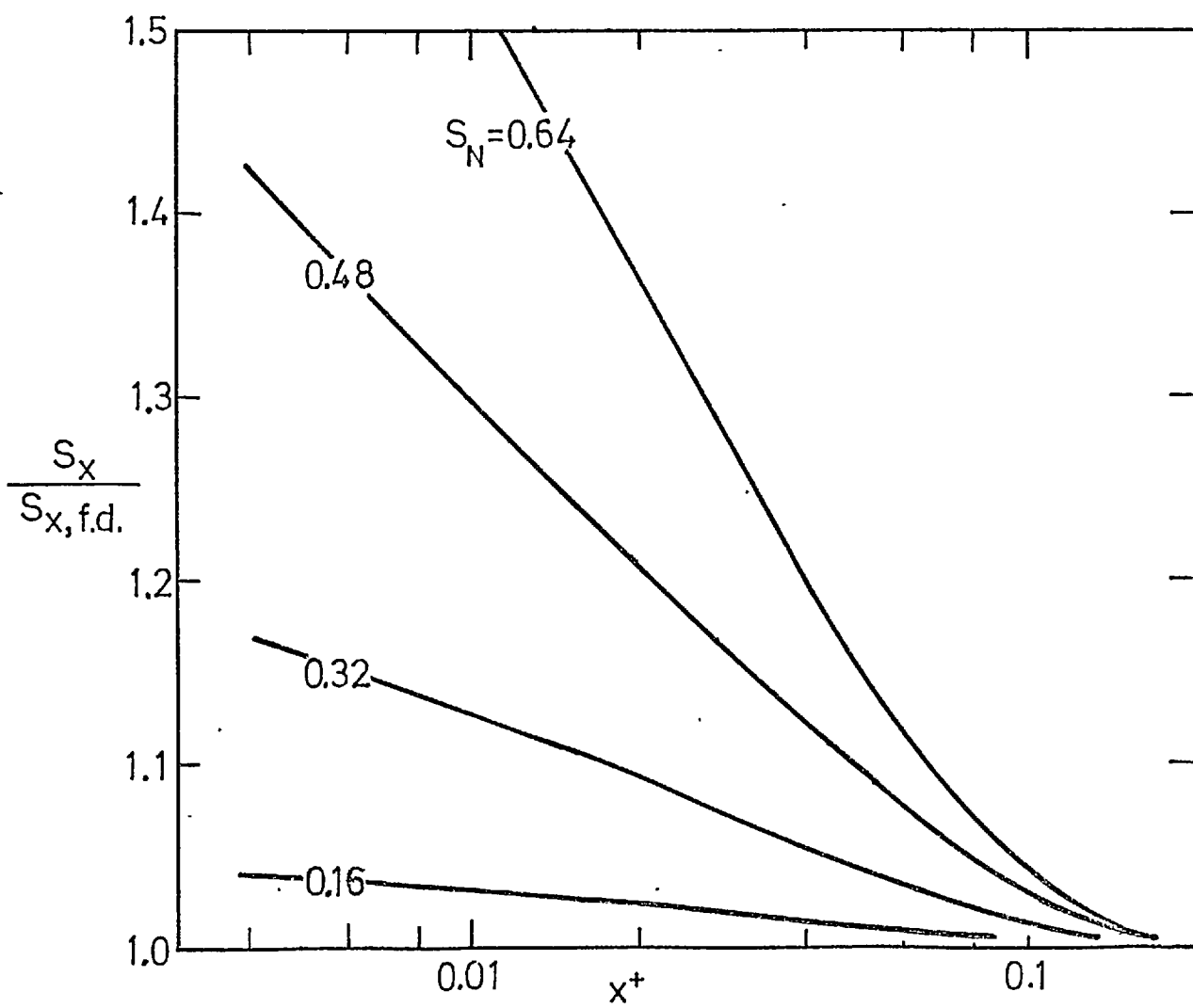


Fig. 5.3

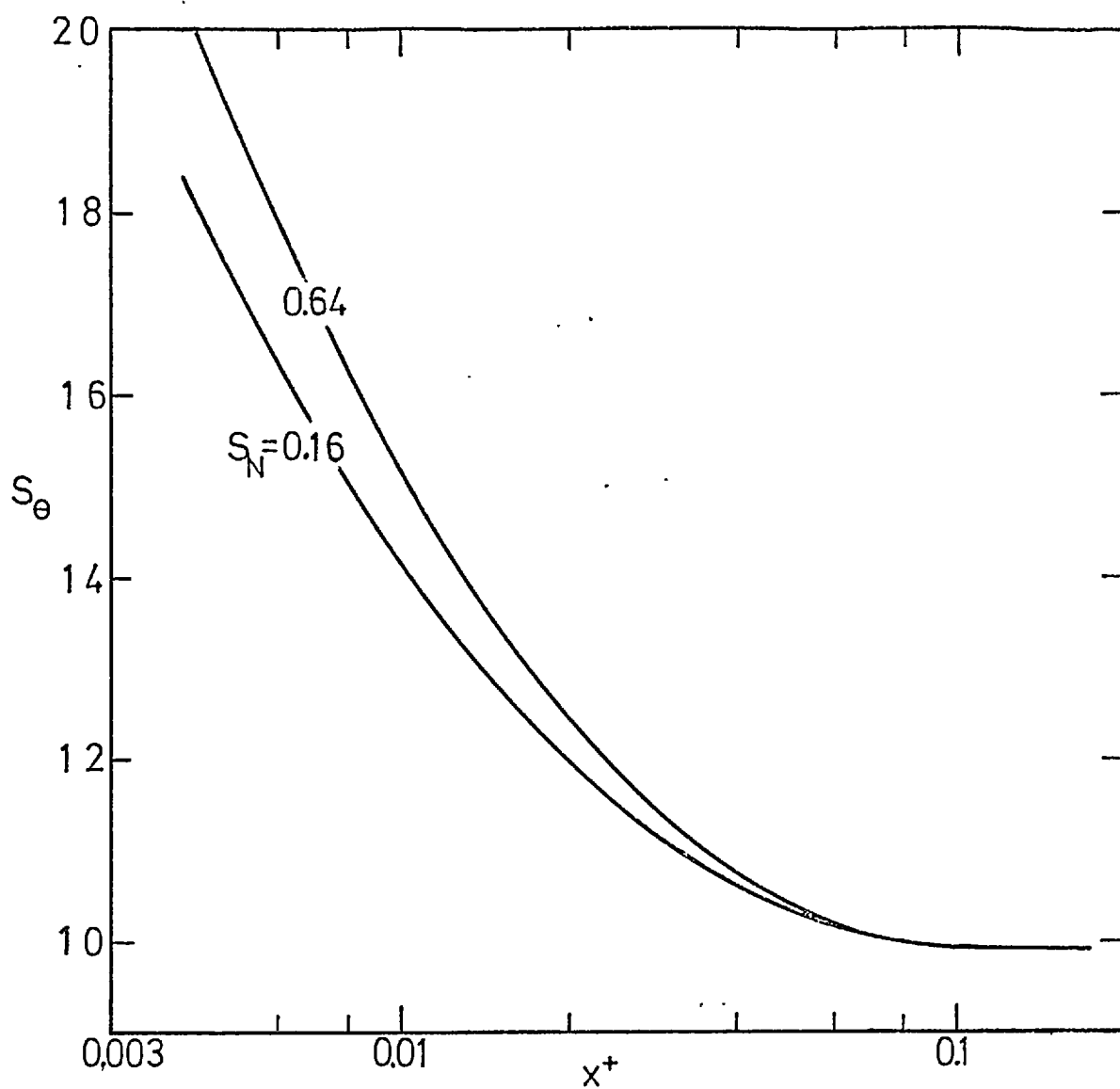


Fig. 5.4

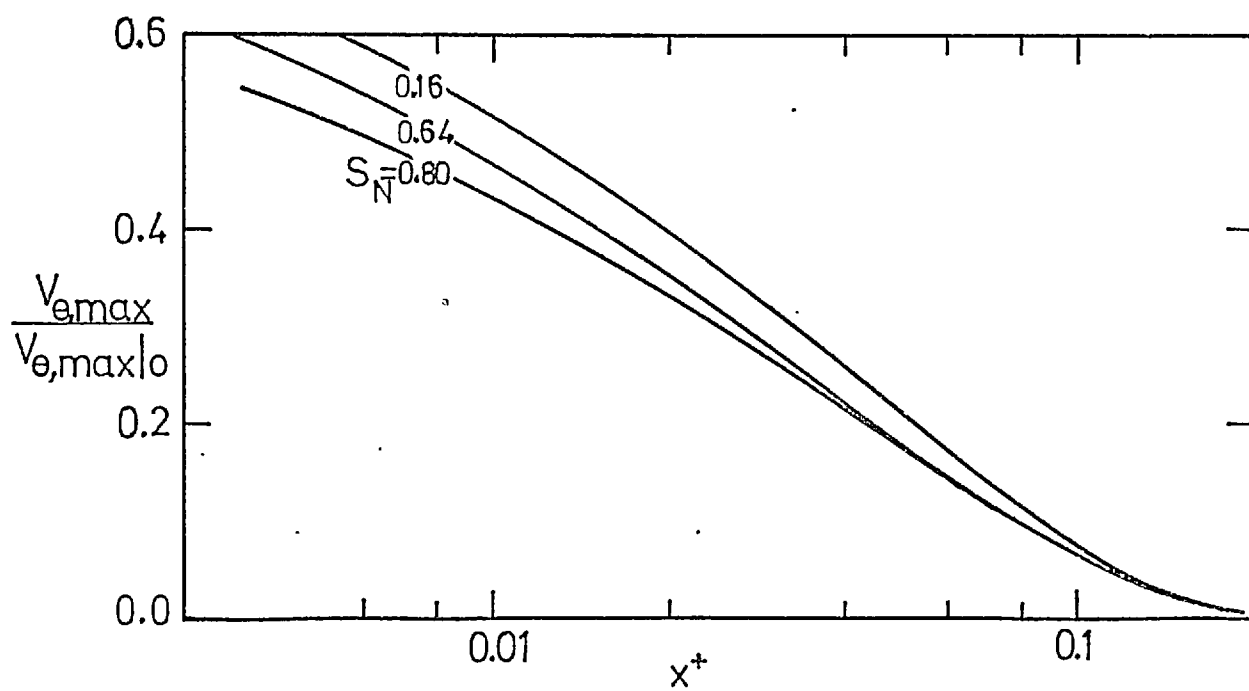


Fig. 5.5

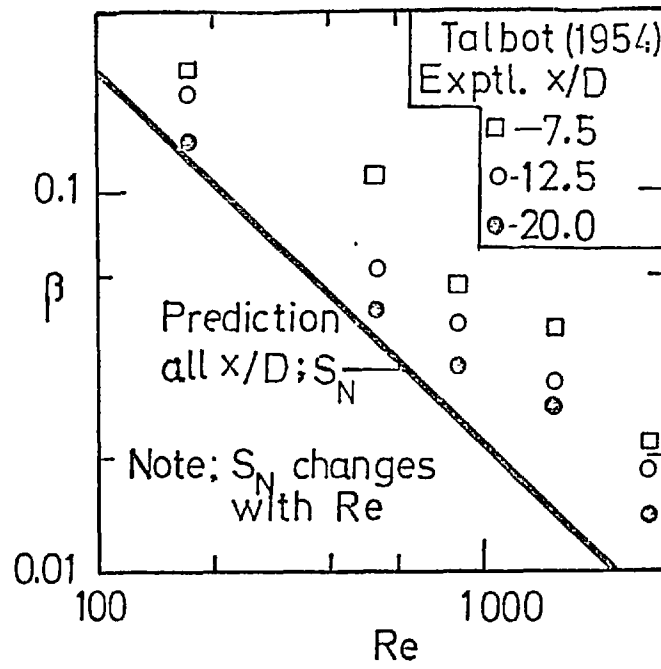


Fig. 5.6

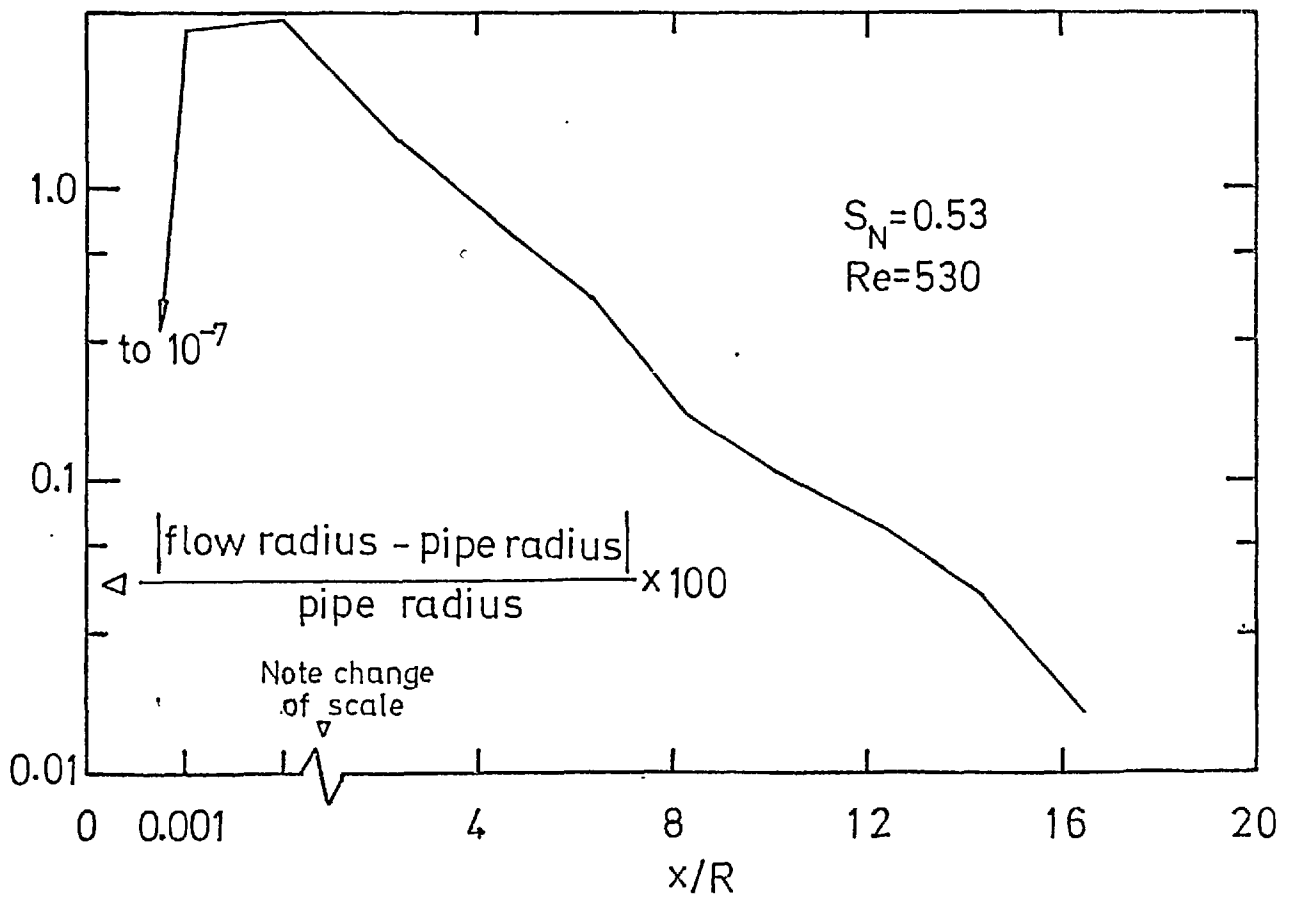


Fig. 5.7

CHAPTER 6

SWIRLING FLOW IN PIPES: TURBULENT FLOW

Chapter 6

Swirling, turbulent flow in pipes

6.1 Introduction

The last chapter has demonstrated the influence of swirl on fully-developed laminar flow in pipes. The cross-stream pressure gradients were shown to influence the shape of the axial-velocity profile and the decay of the axial-shear-stress co-efficient.

In this chapter, the implications are explored of the mixing-length model of turbulence (described in Chapter 2) for swirling turbulent flow in smooth pipes. Two mixing-length distributions and a 'swirl Prandtl Number' provide the empirical input to the problem. Nikuradse's mixing-length distribution for fully-developed pipe flows is superposed on a linear $l_0 \sim r$ mixing-length relation for the tangential motion. The swirl Prandtl Number, Pr_θ , is determined by comparing the predicted decay of swirl with experimental data.

6.2 The flow problem considered

A swirl velocity distribution, induced by a twisted tape, is superposed on an existing fully-developed axial velocity distribution in a pipe. The downstream development of the shape of the swirl velocity profile and the decay of the average swirl are compared with experimental data. Also compared with data is the ratio of the tangential and axial components of the wall-shear-stress (at one axial station) as a function of the swirl input.

6.3 Nomenclature

The symbols used in the text and in the figures have the following meanings.

<u>Symbol</u>	<u>Meaning</u>	<u>Defining equation</u>
A_+	van Driest constant	(2.18)
D	Diameter of pipe	-
H	Pitch of twisted tape	Axial distance for a 360 - degree twist of the tape
k	Mixing-length constant	Fig. 7.4-1
K	Angular momentum flux	$= 2\pi\rho \int_0^{r_b} v_x v_\theta r^2 dr$
K_o	K at $x/D = 10$	
K_o/MD	Baker's Swirl parameter	$= \frac{K_o}{\pi\rho \bar{v}_x^2 \frac{D}{4} \cdot D}$
l	Mixing-length	Fig. 7.6-1
Pr_θ	Swirl Prandtl number	(2.15)
r	Radial distance from pipe axis	-
r_b	Radius of blade of swirl meter	-
R	Radius of pipe	-
Re	Axial-flow Reynolds number	$= \frac{V_{x,o \max} \cdot R}{\nu}$
Re	Baker's axial-flow Reynolds number	$= \frac{\bar{V}_{x,o} \cdot D}{\nu}$
<u>Symbol</u>	<u>Meaning</u>	<u>Defining equation</u>
τ	Shear-stress at pipe wall	-
V_x	Axial velocity	-
V_θ	Tangential (swirl) velocity	-
x	Axial distance from end of tape swirler	-
Δx	Axial distance from $x/D = 10$	-

<u>Symbol</u>	<u>Meaning</u>	<u>Defining equation</u>
ρ	Density	-
ν	Kinematic viscosity	-

Subscripts

x	pertaining to axial component of flow
θ	pertaining to tangential component of flow
0	condition at downstream end of tape swirler
_{max}	radial maximum

A bar over the symbol indicates a spatial average value.

6.4 The mixing-length distributions

The principal input to the Program takes the form of two mixing-length distributions, with their associated constants, appropriate to the axial and tangential motions. Earlier work (Spalding, 1968) has established that the use of Nikuradse's mixing-length distribution enables excellent predictions to be made of the shape of the axial velocity profile and the wall-shear-stress co-efficient in fully-developed turbulent pipe flows without swirl.

For flows with swirl, a linear mixing-length distribution for the tangential motion is superposed upon the Nikuradse distribution as illustrated in Fig. 6.1 below.

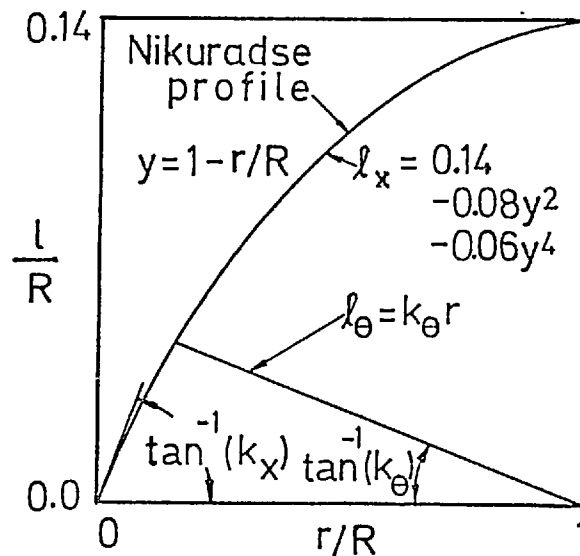


Fig. 6.1

Mixing-length
distributions

ℓ_x and ℓ_θ are the mixing-lengths appropriate to the axial and tangential motions respectively. R is the radius of the pipe and r the distance from the pipe axis to any radial point. Near the wall, beyond the point where the ℓ_θ distribution intersects the ℓ_x distribution, both ℓ_x and ℓ_θ are deemed to be represented by the Nikuradse profile.

The constants $k_x = 0.41$, for the slope near the wall and 0.14 for the maximum value, respectively, of the ℓ_x distribution have, of course, been fixed by reference to fully-developed swirl-free turbulent flow

(Spalding, 1968). It remains to determine k_θ as also the swirl Prandtl number, Pr_θ . How these are fixed is discussed in sections 6.6 and 6.7 below.

6.5 Initial profiles

Axial velocity. The axial velocity distribution was obtained by simply allowing the Program to compute a swirl-free fully-developed turbulent flow in a pipe.

Swirl velocity. The initial swirl velocity distribution was specified by using the following expression which approximates closely the experimental swirl velocity distribution immediately downstream of a tape-swirler ($H = 10$) in a pipe (Krieth and Sonju, 1965).

$$\frac{v_\theta}{v_{x,0}} = \frac{1}{10} \left[6.3 \frac{r}{R} - 0.013 \left(1.1 - \left(\frac{r}{R} \right)^{-2.68} \right) \right]$$

6.6 The predictions

6.6-1 The profiles of swirl velocity

Figure 6.3 displays the profile of tangential velocity, V_θ , non-dimensionalised with respect to the axial velocity, V_x , on the axis of the pipe compared with the data of Musolf (1963).

The influence of k_θ . The major part of the $\ell_\theta \sim r$ profile being remote from the wall (Fig. 6.1), the first choice for the constant k_θ was 0.26, i.e. the same as that used for the confined swirling jet (Chapter 4). The profiles displayed in figure 6.3 are those resulting from this value of k_θ . In the range $0.2 \leq k_\theta \leq 0.3$, the computations for different values of k_θ could not (to the scale of figure 6.3) be distinguished from each other. This is not surprising since a major part of the flow, where the ℓ_θ distribution applies, is in near solid-body rotation and contributes little to the effective

viscosity.

The influence of $A_{+, \theta}$. For the region of the flow in the immediate vicinity of a wall, chapter 2 described the van Driest modification to the mixing-length hypothesis. The two 'damping' constants $A_{+, x}$ (for the axial motion) and $A_{+, \theta}$ (for the swirl motion) appearing in the formula (equation 2.18) were left to be determined empirically.

Research into swirl-free flows has established that use of a value of 26 ± 1 for $A_{+, x}$ results in predictions that agree well with the majority of data on flows over smooth walls.[†] In particular, $A_{+, x} = 26$ serves to predict the wall-shear-stress in fully-developed flows in smooth pipes.

Figure 6.3 shows predictions using this value of $A_{+, x}$ and $A_{+, \theta}$. Over a wide range of $A_{+, \theta}$, from 10 through to 50, the predictions cannot be distinguished from each other to the scale of the figure.

6.6-2 The decay of the average swirl

Kreith and Sonju (1965) present data on the average swirl measured at various distances downstream of the end of their tape swirler. The swirl was measured by means of a swirl meter consisting of a flat blade free to rotate about an axial shaft in the centre of a short tube. The entire tube was inserted into the transparent test-section and the rotational speed of the blade was measured stroboscopically. The angular velocity of the blade was thus a measure of the average swirl at any axial station.

Figure 6.4 shows the swirl as a percentage of its initial value versus axial distance. The points (with maximum deviations) are the data

[†] See Patankar and Spalding (1970) for examples of and references to such research.

of Kreith and Sonju for an axial Reynolds number of 18,000. The lines represent the computed decay obtained for various values of Pr_θ .

It would appear from Fig. 6.4 that the most appropriate value of Pr_θ is 0.5; i.e. the effective-viscosity for the tangential motion is twice as large as that for the axial motion.

It is now necessary to see if $Pr_\theta = 0.5$ will result in adequate predictions for other Reynolds numbers. Figure 6.5 shows the predicted decay for a Reynolds number of 61000. The prediction does not match the data very well. We learn from this that the effective-viscosity model is not universal enough. Of course, a lower value of Pr_θ could be chosen to provide better agreement for this Reynolds number; a functional relationship could then be proposed for the variation of Pr_θ with Reynolds number and which could be tested out for intermediate Reynolds numbers. Kreith and Sonju do not present data for intermediate Reynolds numbers. Consequently, this last step in improving the match between the data and the predictions has not been carried out.

In keeping with the experimental findings of Kreith and Sonju, the computed rates of swirl decay showed no significant influence of the level of initial swirl.

6.6-3 The tangential shear-stress

Baker (1967) provides some measurements of the tangential component of the wall-shear-stress obtained indirectly by derivation from the integral momentum equations and the experimentally measured decay of swirl.

Figure 6.6 shows the ratio of the tangential shear-stress to the axial shear-stress (at one axial station) as a function of the swirl

parameter K_0/MD .[†] Clearly, the trend of the data is predicted well enough. The dotted lines are extrapolations into the higher levels of swirl, calculations for which were not possible for the reason already explained in section 5.7 of Chapter 5.

6.6-4 The axial-shear-stress co-efficient

The last chapter showed how the cross-stream pressure gradients in laminar swirling pipe flows enhanced the local value of the axial shear-stress co-efficient (section 5.7-3). In turbulent flows, this enhancement is caused by two separate factors: the cross-stream pressure gradients, and the contribution to the effective-viscosity by the swirling motion. Figure 6.7 shows the separate influences of these two factors on the ratio of the local value of the axial shear-stress co-efficient to its fully-developed value.

Curve A results from computations in which the influence of the swirl on the effective-viscosity has been suppressed; i.e. when the effective-viscosity formula consists of only the first two terms on the right of equation (2.14). Curve B results from computations which include the contribution of swirl to the effective-viscosity as embodied in equation (2.14). We see from these curves that the contribution of swirl to the effective-viscosity has not a great influence on the axial component of the wall-shear-stress.

6.7 Conclusions

The computations displayed in this chapter have revealed some of the implications of the mixing-length hypothesis for swirling turbulent flow in smooth pipes. The effective-viscosity formulae developed in Chapter 2 (equations (2.14) to (2.16)) with the following empirical constants:

[†] Defined in the nomenclature, section 6.3 above.

$k_x = 0.41$, $k_\theta = 0.26$, $A_{+,x} = 26$, $Pr_\theta = 0.5$, have been shown to enable computations to be made which predict correctly the trend of the data on the shape of the profile of tangential velocity; the axial rate-of-decay of the average swirl; and the increase with swirl of the tangential component of the wall-shear-stress. However, Pr_θ may require to be made a decreasing function of axial flow Reynolds number. More data are needed to establish the precise functional relationship between Pr_θ and the axial flow Reynolds number.

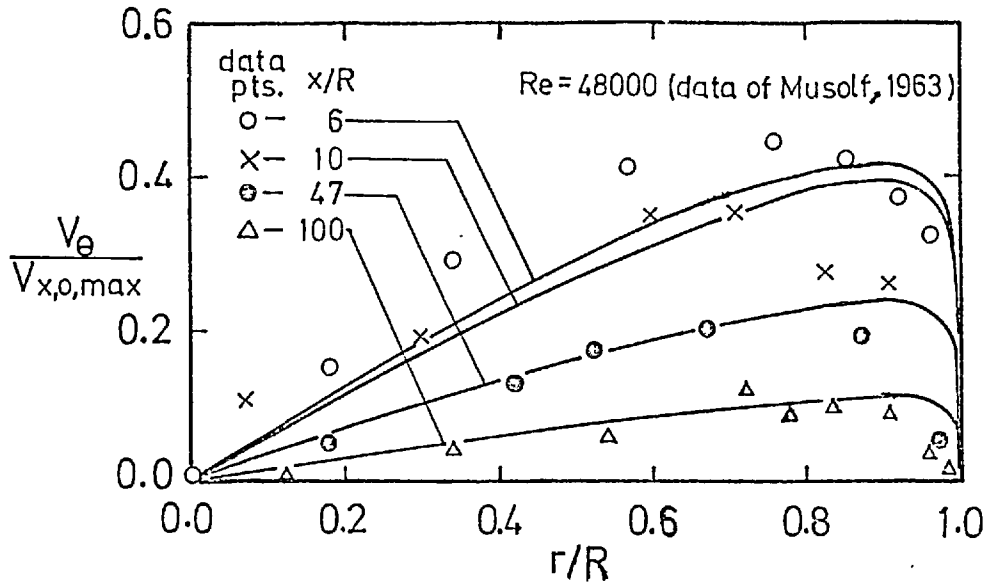


Fig. 6.3

Fig. 6.4
▽

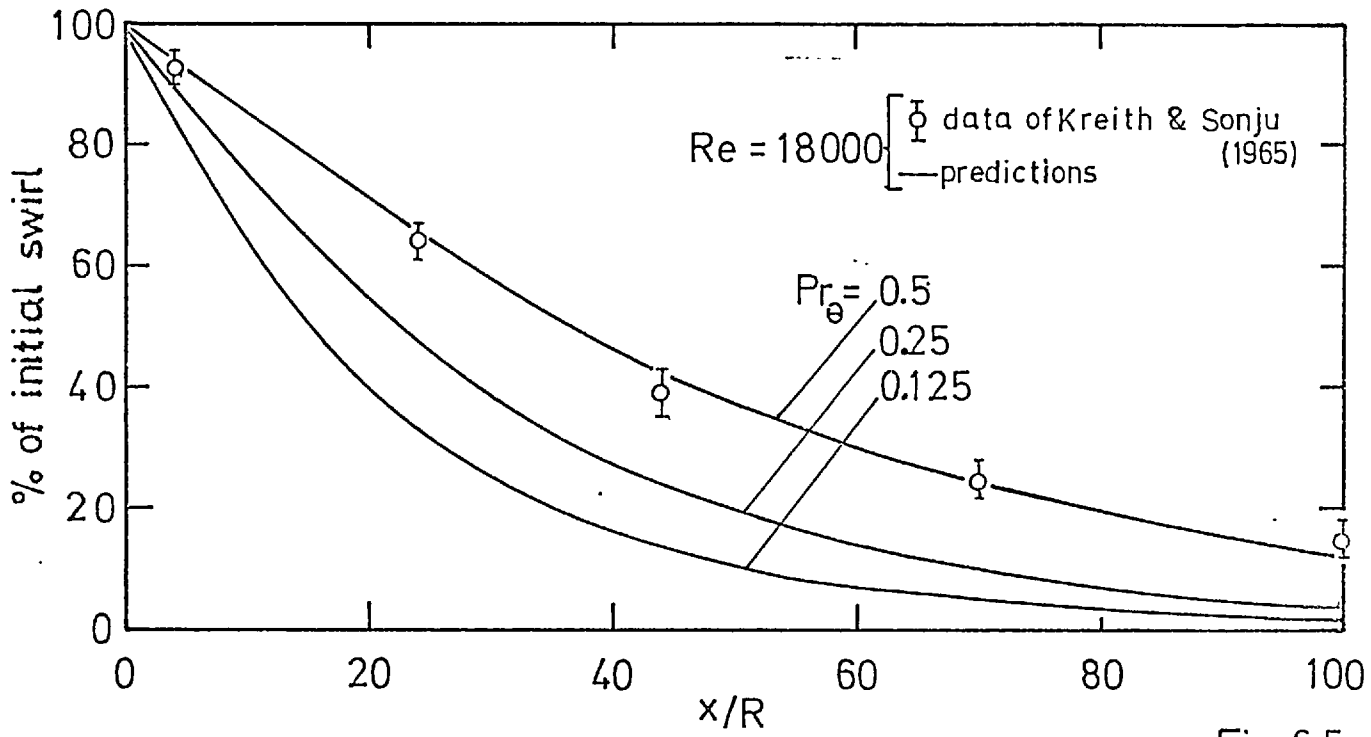
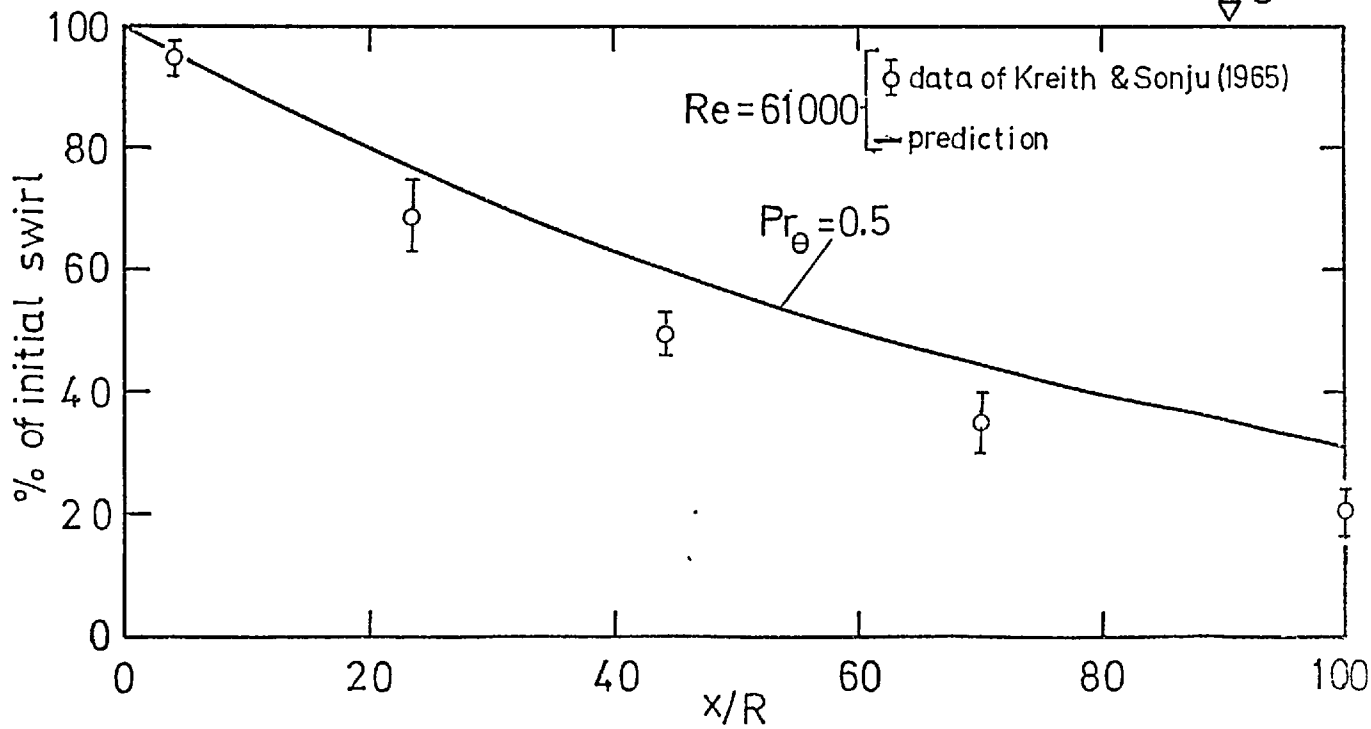


Fig. 6.5
▽



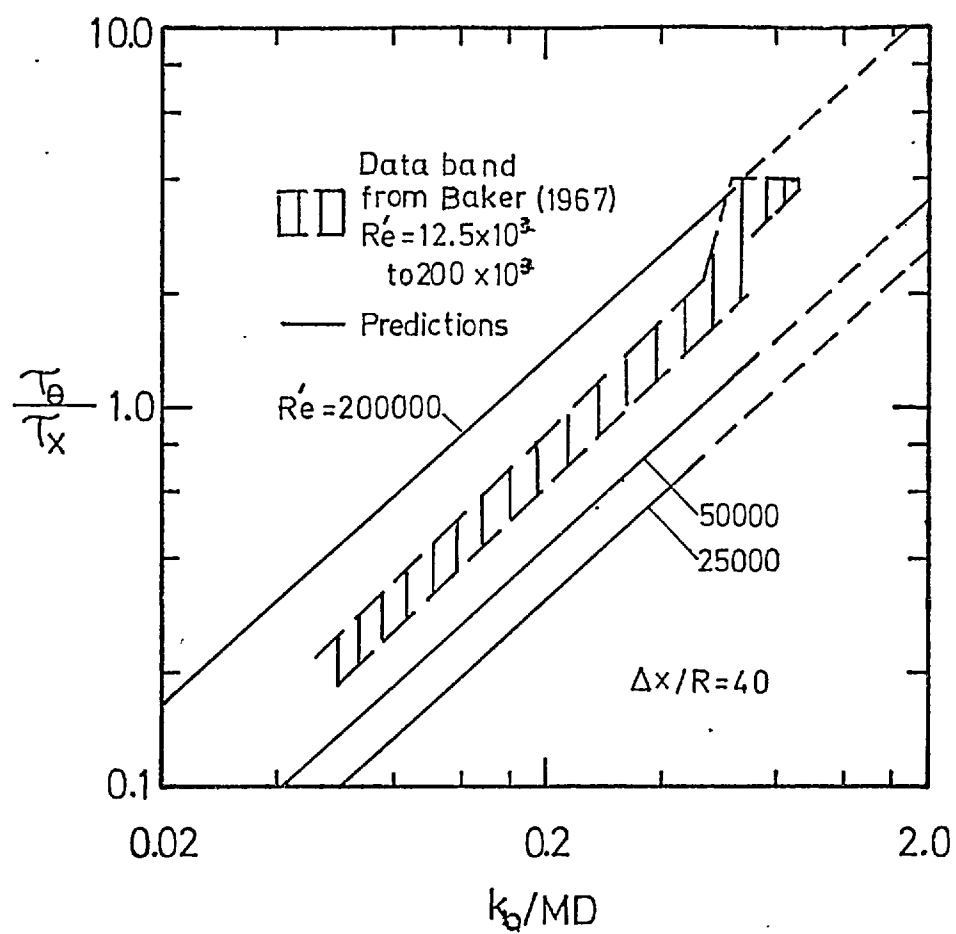


Fig. 6.6

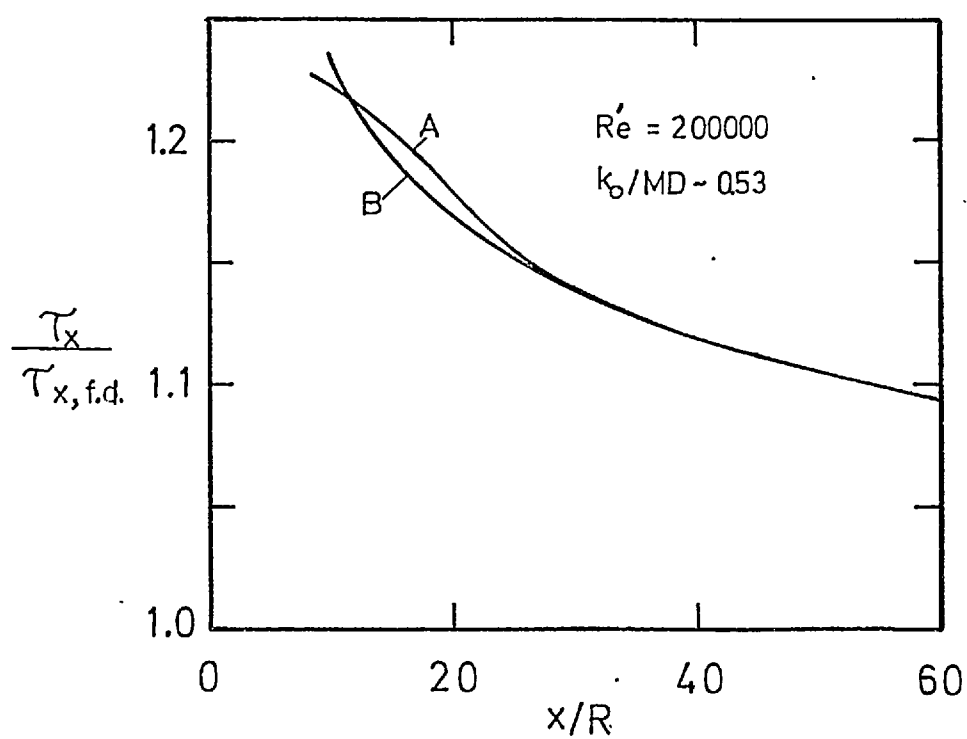


Fig. 6.7

PART II

EXPERIMENTAL INVESTIGATION INTO THE HYDRODYNAMICS

OF A CONFINED, SWIRLING JET

Chapter 7

Experimental investigation into the hydrodynamics of a confined swirling jet

7.1 Introduction

Part I of this thesis has described the development and use of a finite-difference computation procedure for swirling boundary-layers. Use has been made of Prandtl's mixing-length hypothesis to predict some properties of free and confined turbulent flows.

In Chapter 4, the predictions for the confined swirling jet were compared with experimental data obtained as part of this research programme. Part II now describes the details of the experiments.

Section 2 describes the experimental rig and data-logging equipment. Section 3 discusses the calibration and use of the five-hole pitot probe used to survey the velocity and static pressure fields. Section 4 summarises the experimental results and discusses them.

7.2 Description of experimental investigation

The purpose of the experiments was to survey the velocity (and static pressure) fields in a swirling jet in a co-flowing stream confined in a duct so as to provide a check for the computation procedure developed in Part I.

The type of experimental data obtained is summarised in the next section. The flow and the data-monitoring and aquisition systems are described in sections 2.3 and 2.4 respectively.

7.2-1 The type of data obtained

A swirling jet of air from a half-inch pipe expands into a stream of co-flowing air confined in a four-degree half-angle diffuser with an entrance diameter of four inches. At five axial stations six inches apart,

profiles have been obtained of the axial and tangential velocities and static pressure across the duct, using a 4mm. diameter five-holed pitometer.

The results are presented in computer output form in section 7.3-3. They have been used in graphical form (Figs. 4.5 and 4.6) in Chapter 4.

7.2-2 The flow system

Figure 7.1 is a self-explanatory sketch of the flow system. The swirl nozzle consisted of a tube with four tangential slots milled into the wall. This tube was sheathed in a blind outer tube, which carried the inlet for 'swirl air' and behind it, beyond the slots, the inlet for 'axial air'. These inlet tubes served also as the struts supporting the swirl nozzle in the outer flow. The struts were covered by a brass-shim aerofoil. A total-head traverse at the entrance to the diffuser checked that the wake from this aerofoil had been killed by the combination of honeycomb, wire mesh and contraction, thus ensuring a symmetrical total head profile at the exit and across the diameter of the contraction in the plane of the struts. The rotameter in the 'swirl air' line indicated the level of swirl for any one setting of the 'axial air' valve and was used for monitoring the flows.

7.2-3 The data-monitoring-and-aquisition system

This consisted of a differential pressure transducer in parallel with the Y - channel of a Hewlett-Packard X-Y plotter and three out of five channels of a 20 - channel Solatron data - logger.

Probe traversing gear

The radial location of the five-hole probe was transduced to an electrical signal by a helical potentiometer geared to a motorised probe-

traversing mechanism. This signal was fed to the X - channel of the X-Y plotter and two of five channels of the data logger.

Proving the system

The magnitude of the error in the probe position indication caused by backlash in the traversing gear amounted to one-tenth of a millimeter, i.e. one-fortieth of the probe diameter.

The X-Y plotter provided a visual trace of the total pressure profile across the duct, thus allowing a greater number of pressure data points to be logged in those regions of the flow that had significant gradients of static pressure.

7.3 The five-hole pitot probe

The velocity and static pressure field inside the conical diffuser were surveyed by a commercial, custom-built, five holed hemispherical pitotmeter. It had a uniform head and stem diameter of 4mm. (the smallest available). The pressure holes measured 0.38 mm. across.

The calibration and use of this probe followed closely the recommendations of M.A. Wright (1968). His report should be consulted for a complete description of the theoretical basis for and justification of the calibration procedure described briefly below.

7.3-1 Calibration of five-hole pitometer

The pitometer was calibrated at the bell-mouthed exit of a calibration tunnel capable of issuing a steady stream of low-turbulence air at approximately 20 metres/sec. The calibration stand was furnished with two protractors that measured the conical and dihedral angles, $\emptyset$ and δ , in Fig. 7.2 .

Calibration of the probe entailed measuring the differential

pressure between the four peripheral holes and the central hole for each combination of ϕ and δ . Also required was a measurement of the dynamic pressure of the jet. A preliminary measurement with a large pitot probe indicated that the pressure recorded at the central hole (P_o) of the five-hole probe, when $\phi = 0^\circ$, gave the stagnation pressure to the same accuracy as the 'standard'.

For two values of δ (0° and 30°) and various ϕ , the pressure differentials ($P_o - P_n$) between the central hole and each of four ($n = 1$ to 4) peripheral holes were measured with a sensitive transducer manometer as also ($P_o - P_{at}$), the differential pressure between the central hole and the ambient atmosphere. With the static pressure (P_s) at the exit of the tunnel assumed to be atmospheric, the following probe factors were computed, plotted against the conical angle (ϕ) and, following Wright (1968), polynomial 'lines of best fit' were calculated.

Probe factor	Line of best fit (after Wright)
Angle factor: $K_\phi = \left[1 - \sum_{n=1}^{n=4} (P_o - P_n) / 2 \left\{ \sum_{n=1}^{n=4} (P_o - P_n)^2 \right\}^{\frac{1}{2}} \right]^{\frac{1}{2}} \quad (7.1)$	ϕ = Fifth-order polynomial in K_ϕ
Velocity factor: $K_v = \left[\rho V^2 \left\{ \sum_{n=1}^{n=4} (P_o - P_n)^2 \right\}^{-\frac{1}{2}} \right]^{\frac{1}{2}} \quad (7.2)$	K_v = Fourth-order polynomial in ϕ
Pressure factor: $K_p = 2(P_o - P_s) / \rho V^2 \quad (7.3)$	K_p = Fourth-order polynomial in ϕ

In relations (7.2) and (7.3), ρ is the density and V is the total velocity of the air stream.

Figures 7.3 to 7.5 show the calibration probe factors against the conical angle ϕ . The equations for the 'lines of best fit' are:[†]

$$\phi = -0.0171 + 2.1721 K_\phi - 3.9021 K_\phi^2 + 4.7404 K_\phi^3 - 3.0802 K_\phi^4 + 0.8833 K_\phi^5; (7.4)$$

$$K_v = 0.1985 - 0.0600 \phi + 0.7117 \phi^2 - 0.8888 \phi^3 + 0.2769 \phi^4; (7.5)$$

$$K_p = 0.9863 + 0.0269 \phi - 1.8976 \phi^2 - 0.5077 \phi^3 + 0.7879 \phi^4. (7.6)$$

7.3-2 Using the five-hole pitot probe

The probe is inserted into the general swirling stream and the probe and stem axes aligned with whichever two axes in the flow serve as the reference axes. The pressure differential $(P_o - P_{at})$ and $(P_o - P_n)$, $n = 1$ to 4, are measured. Then the angle factor, K_ϕ is computed according to equation 7.1. This is injected into the calibration $\phi \sim K_\phi$ polynomial (equation 7.4) to give the flow conical angle, ϕ . This value of ϕ is inserted into the $K_v \sim \phi$ polynomial (equation 7.5) to yield the velocity factor K_v . The total velocity, V , is then obtained from the measured pressures and the just computed K_v using equation 7.2. Equation 7.3 and the $K_p \sim \phi$ polynomial then provides $(P_o - P_s)$ from the value of V determined above. Since $(P_o - P_{at})$ is a measured pressure differential, the differential between the static and atmospheric pressure $(P_s - P_{at})$ can be calculated. The dihedral angle δ is calculated in the manner indicated by Wright.

[†] The chosen highest power in the polynomials are the ones used by Wright (1968) except that, for K_p , a fourth order polynomial has replaced his exponential.

7.3-3 Estimate of the accuracy of measurements using the five-hole probe and its calibration

To get an estimate of the accuracy of a measurement using the five-hole probe, a further set of pressure measurements were made in the calibration tunnel at an arbitrary δ (15°) for various values of ϕ . The calibration curves were used to compute ϕ and the total velocity V , and these were then compared with the known values of ϕ set on the protractor and the known value of the tunnel velocity. Table 7.1 below summarises the results.

Table 7.1

Set and computed values of conical
angel ϕ and tunnel velocity, V

Tunnel velocity = 20.00 m/s. Differential static pressure ($P_s - P_{at}$) assumed = 0 mm. of water.

Conical angle ϕ ($^\circ$)		V (m/s) computed	$(P_s - P_{at})$ computed (mm. of water)
Set	Computed		
0	0.4	20.24	- 0.24
10	9.9	20.02	+ 0.24
20	19.4	20.18	- 0.14
30	29.6	20.25	- 0.25
40	39.4	19.80	- 0.14
50	50.7	20.29	+ 0.59
60	59.0	20.28	- 0.88

From the Table 7.1 one may conclude that, upto a flow conical angle of 60° ,

- (i) the flow conical angle, ϕ , may be measured to within $\pm 1^\circ$;
- (ii) the total flow velocity, V , may be measured to within $\pm 2\%$;
- and (iii) the static pressure differential with ambient may be measured to within ± 1 mm. of water.

Owing to inaccuracies in the construction of the probe, the computed dihedral angle, δ , differed from the set value by several hundred percent. An assumption about δ had, therefore, to be made for swirling jet measurements. As explained below, δ may be taken to be 0° without loss of accuracy.

7.3-4 The assumption about the value of δ

In a swirling jet without recirculation, the streamlines are curved to a significant extent only near the edge of a jet. Elsewhere they are nearly parallel. To assume that δ is zero is therefore very reasonable except near the edge of the jet. But here both the swirl axial velocities are small if the jet is in stagnant surroundings. If the jet is submerged in a co-flowing stream, the axial velocity is not small but the streamline curvature is. In these circumstances too, δ may be taken as zero.

7.4 Results and discussion

7.4-1 The processed results

The processed output from the measurements is displayed on pages 113-114. The wall static pressures in millimeters water guage and printed out in sequence from left to right starting with the pressure

near the entry to the diffuser and ending with the pressure near the exit. The physical location of the pressure tapings appear in Fig. 7.1. The third, ninth, fifteenth, twenty-first and twenty-seventh values in the matrix refer to the wall static pressures at the axial station numbers 1 through 5 at which profile traverses were made.

The print out of the axial station number is followed by a table of values of the radial co-ordinate non-dimensionalised with respect to the duct radius. AXVEL and TANVEL are the axial and tangential velocities in meters per second. P - ST is the local static pressure in millimeters of water below (-) or above (+) the wall static pressure at that axial location.

R/R-DUCT is positive from the duct centre-line towards the 'bottom' wall of the duct. Thus, negative values of R/RDUCT in the print out indicate that the probe has overshoot the jet centre-line in its traverse across the duct. The axial velocity on the duct centre-line is determined by plotting R/R-DUCT against velocity and extrapolating the curve to $R/R-DUCT = 0$.

7.4-2 Assessment of results

Accuracy. There are two aspects to an assessment of the accuracy of the measurements. The first question one asks is, "whatever has been measured, it is what was intended to be measured?". All measurements with probes disturb the flow and, in reality, measure the characteristics of the disturbed flow. Provided the probe is small in relation to the rest of the flow system, this disturbance is small. In the experiments performed, the probe diameter was 4mm. This must be compared with a duct diameter varying between 10 cms. and 30 cms. One way to establish the extent of disturbance to the flow by the probe would have been to

repeat the experiments with a smaller probe and/or a bigger duct. A smaller probe could not be made and a bigger duct was not readily available. The answer to the question must, therefore, remain a matter of personal judgement.

The second question to be answered is, "whatever has been measured, has it been measured accurately?". Several probe-diameters away from the wall, the accuracy of the measurements cannot be far from those indicated in section 7.3-3. Less than about five probe-diameters away from the wall, the calibration of the probe appears to change; for, otherwise, the value of the static pressure computed from the probe pressures could not be as different as they are from the wall static pressure recorded at the same axial station ($P - ST$ at the first radial position should be ≈ 0). Thus, except close to the wall, the measurements are accurate to within about $\pm 2\%$.

Usefulness. The measurements presented here are, so far as I know, the only ones of their kind. They establish the main trends of the flow considered. They thus provide a useful qualitative check on the computations performed and discussed in Chapter 4: a comparison with these measurements shows the turbulence model of Chapter 2 to be adequate to the task of predicting the rate of decay of the maximum of tangential velocity in this flow.

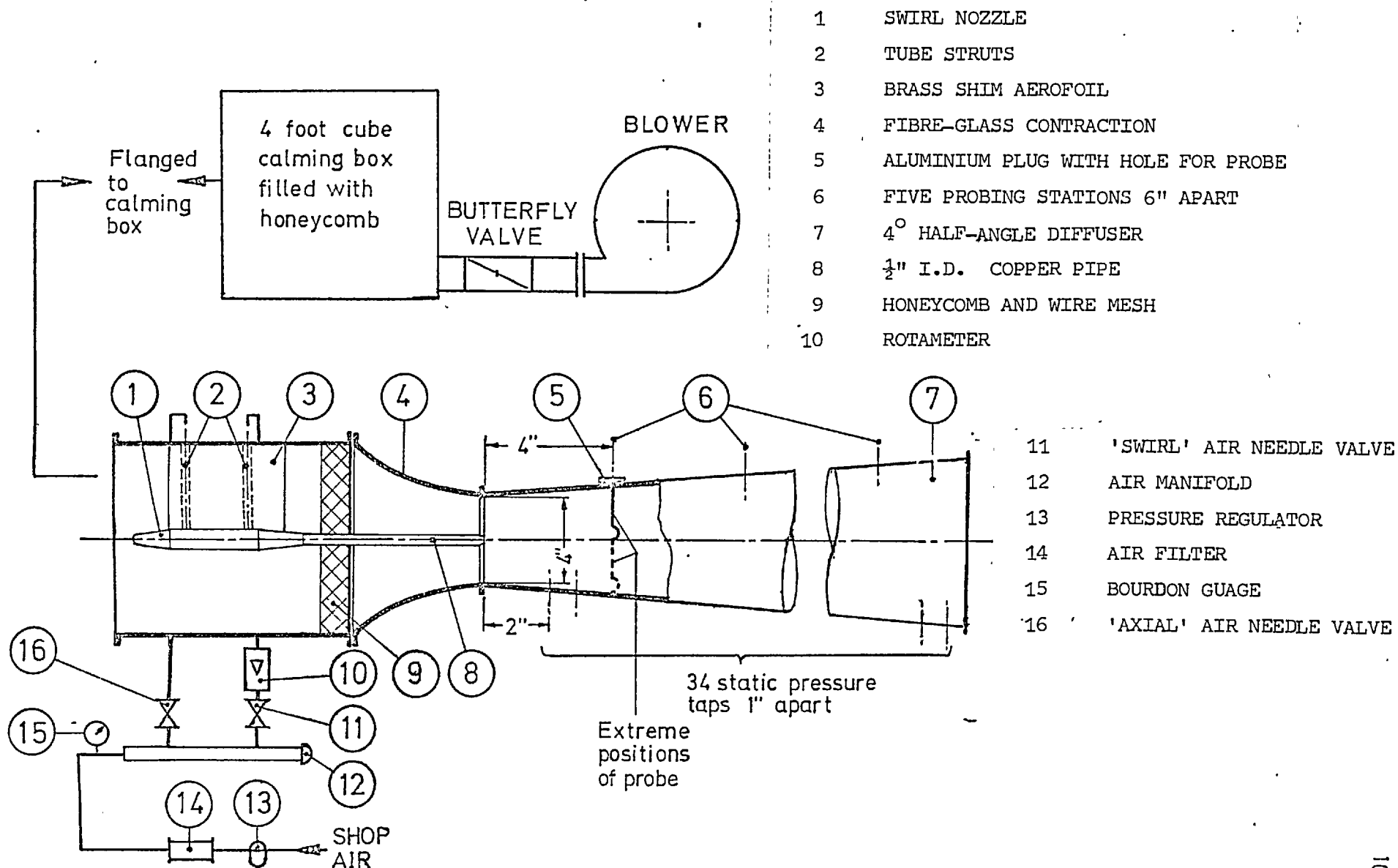


Fig 7.1 Flow System (Not to scale)

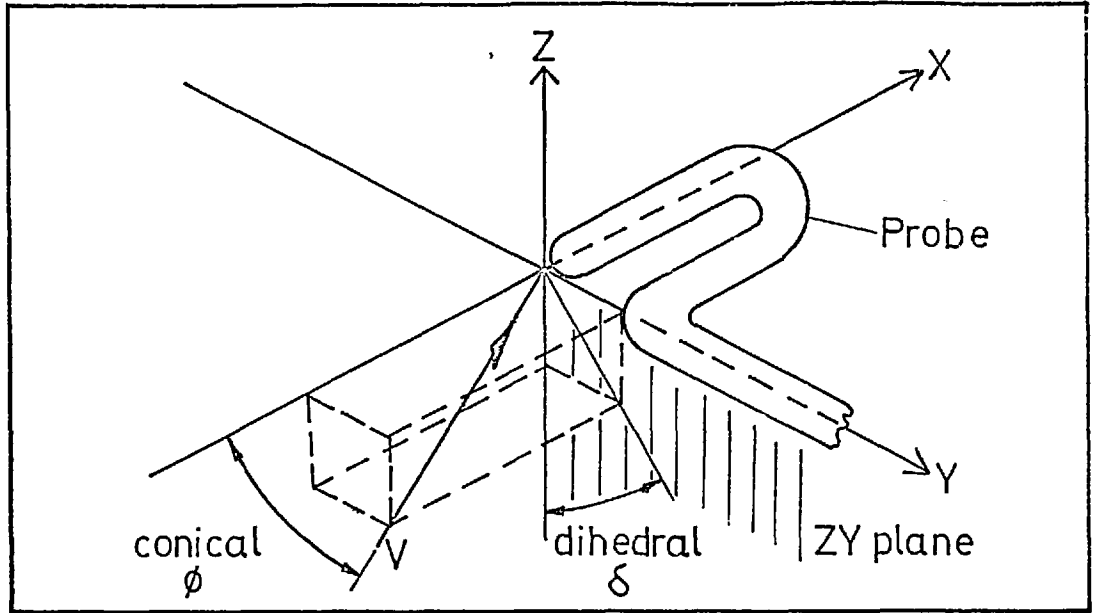


Fig. 7.2

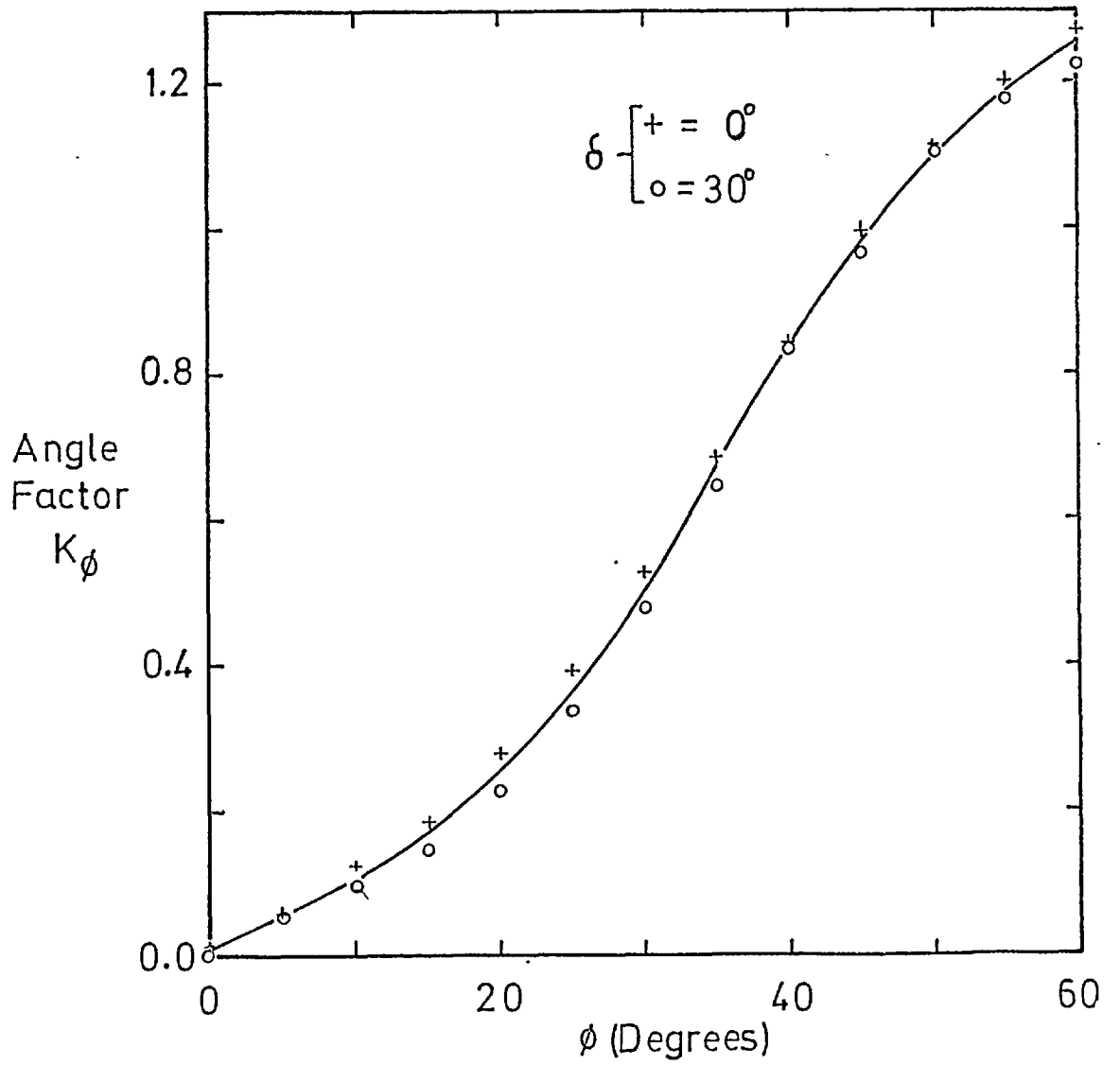


Fig 7.3

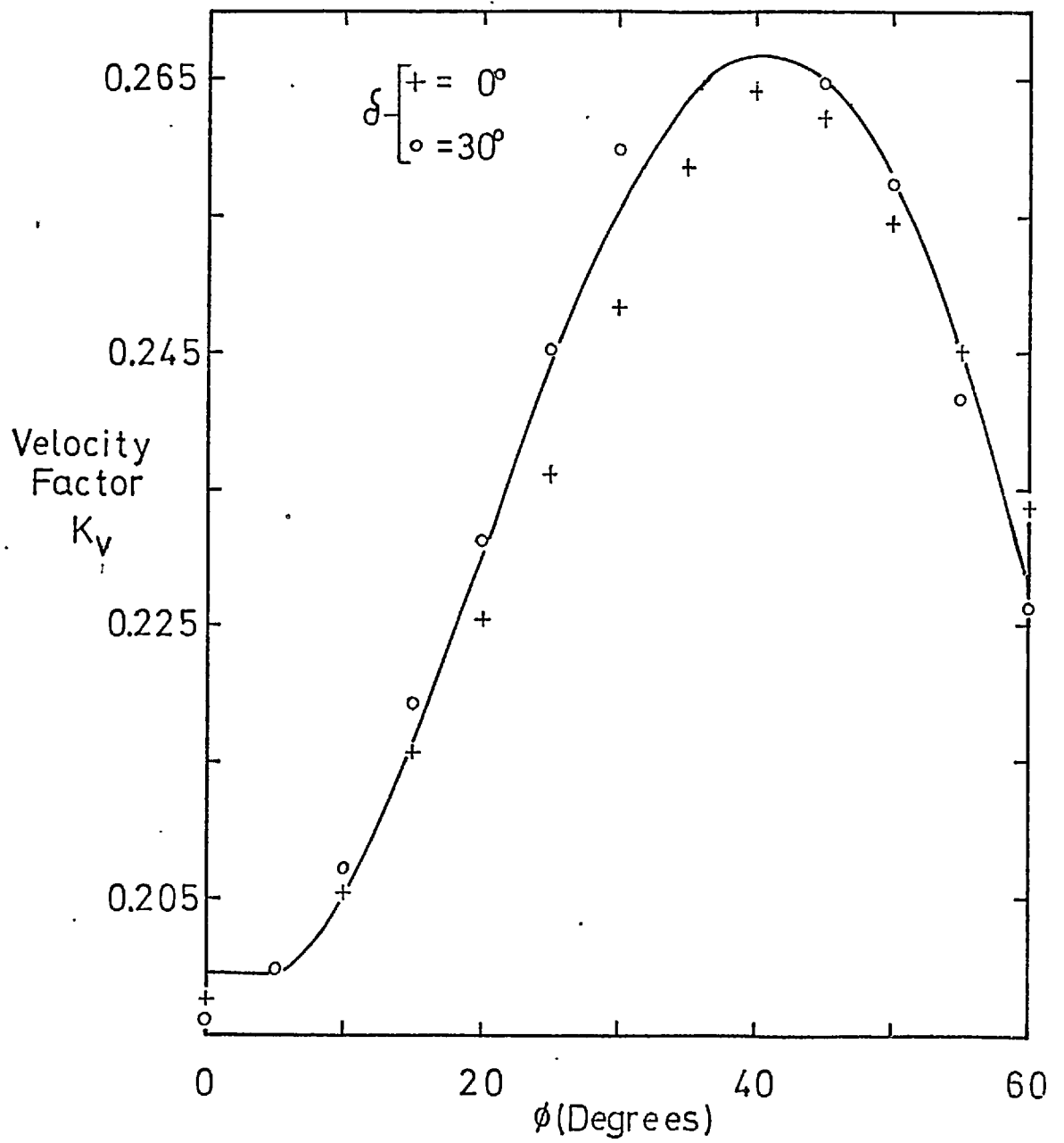


Fig. 7.4

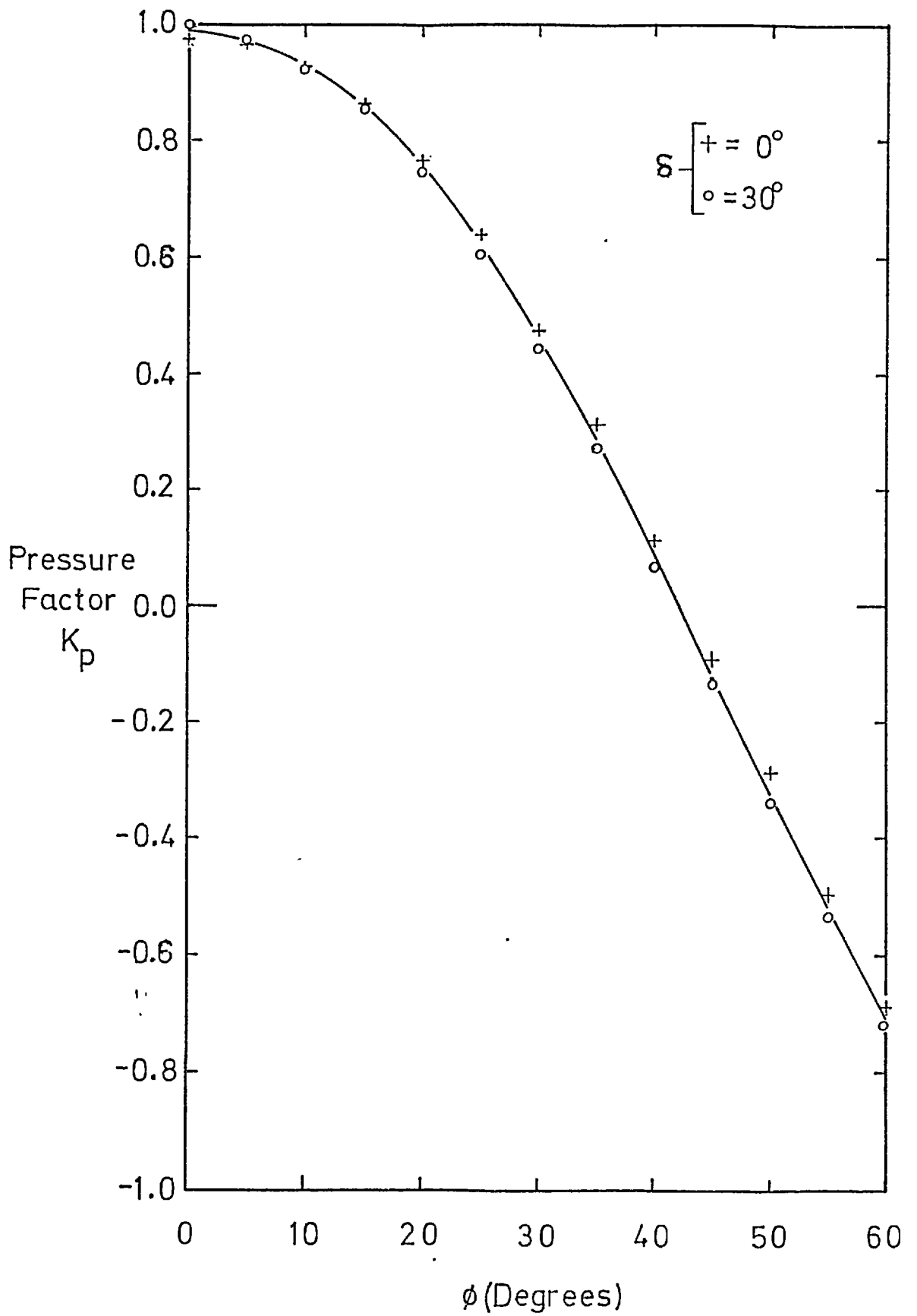


Fig. 7.5

AMBIENT PRESSURE=750.5 MM. HG.

MEAN TEMPERATURE OF AIR FLOW=295 DEG. KELVIN

WALL STATIC PRESSURES IN MM. WATER GAUGE

```

-7.737E+01 -6.679E+01 -5.800E+01 -5.079E+01 -4.421E+01
-3.908E+01 -3.392E+01 -3.303E+01 -2.625E+01 -2.292E+01
-2.029E+01 -1.800E+01 -1.583E+01 -1.396E+01 -1.262E+01
-1.102E+01 -9.833E+00 -8.542E+00 -7.792E+00 -7.125E+00
-6.292E+00 -5.750E+00 -5.042E+00 -4.792E+00 -3.875E+00
-3.657E+00 -3.167E+00 -2.583E+00 -2.375E+00 -2.042E+00
-1.458E+00 -1.250E+00 -7.917E-01

```

AXIAL STATION NUMBER 1

DUCT RADIUS= 5.75 CMS.

R/R-DUCT	AXVEL	TANVEL	P-ST
0.97	31.01	9.35	-2.9
0.72	38.58	4.72	-12.0
0.44	37.52	3.36	-11.2
0.29	37.14	3.15	-11.3
0.14	47.01	7.30	-16.5
0.11	54.84	8.53	-21.2
0.08	60.39	8.31	-24.5
0.07	62.98	7.30	-25.7
0.04	64.09	5.57	-25.4
0.02	63.41	3.06	-23.9
-0.01	61.77	0.87	-23.6
-0.04	60.32	1.81	-21.3
-0.05	59.98	2.73	-20.9

AXIAL STATION NUMBER 2

DUCT RADIUS= 6.85 CMS.

R/R-DUCT	AXVEL	TANVEL	P-ST
0.97	15.73	5.54	237.7
0.55	27.80	2.39	-4.5
0.30	27.00	1.93	-4.1
0.18	33.81	3.94	-6.3
0.14	37.56	4.06	-7.5
0.14	37.48	3.88	-7.1
0.11	40.16	3.47	-8.6
0.08	42.05	3.16	-9.1
0.06	43.39	2.93	-9.9
0.04	44.05	2.43	-9.7
0.02	44.36	2.13	-10.6
-0.01	44.24	1.57	-10.3
-0.04	43.76	1.25	-10.2
-0.10	41.58	0.84	-9.7
-0.14	38.77	1.29	-7.2

AXIAL STATION NUMBER 3

DUCT RADIUS= 7.90 CMS.

R/R-DUCT	AXVEL	TANVEL	P-ST
0.97	7.69	2.68	-0.1
0.62	23.06	1.43	-1.9
0.41	22.02	1.23	-1.9
0.14	30.98	2.56	-4.4
0.79	33.07	2.05	-4.6
0.23	34.51	1.63	-5.8
0.20	34.90	1.33	-5.8
-0.04	34.87	0.83	-5.9
-0.07	34.38	0.45	-6.4
-0.12	33.31	0.12	-5.6
-0.16	31.69	0.16	-4.8

AXIAL STATION NUMBER 4

DUCT RADIUS= 9.00 CMS.

R/R-DUCT	AXVEL	TANVEL	P-ST
0.98	4.43	1.45	-0.2
0.49	20.19	0.49	-1.4
0.21	23.58	1.50	-2.1
0.12	26.80	1.48	-2.5
0.07	28.28	1.16	-2.8
0.01	29.45	0.82	-3.6
-0.07	29.67	0.21	-4.1
-0.11	29.15	0.06	-4.2
-0.16	27.98	0.13	-4.3
-0.21	26.65	0.29	-3.0

AXIAL STATION NUMBER 5

DUCT RADIUS=10.10 CMS.

R/R-DUCT	AXVEL	TANVEL	P-ST
0.98	0.58	1.58	0.5
0.74	6.10	2.07	-0.2
0.58	12.67	2.62	-0.9
0.49	15.83	1.81	-0.9
0.42	17.28	1.38	-0.5
0.32	18.31	1.34	-0.8
0.16	22.07	1.72	-1.3
0.08	23.87	2.19	-1.0
-0.00	25.49	1.19	-1.8
-0.09	25.73	0.91	-2.3
-0.13	25.24	0.35	-2.3

CHAPTER 8

SUMMARY OF CONCLUSIONS

Chapter 8

Summary of conclusions

The Patankar - Spalding computation procedure has been extended to include within its capabilities the computation of boundary-layers with swirl (Chapter 1 and Appendices I and II). The procedure has been demonstrated by using it to calculate three different types of turbulent swirling flow; namely, the free-swirling jet (Chapter 3); the confined swirling jet (Chapter 4) and swirling flow in a pipe (Chapter 5).

The turbulence has been modelled by an extension of Prandtl's (1925) mixing-length hypothesis for effective-viscosity. Formulae for the two effective-viscosities, $\mu_{\text{eff},x}$ and $\mu_{\text{eff},\theta}$, appropriate to the axial and tangential motions respectively have been proposed. The formulae (developed in Chapter 2) are:

$$\mu_{\text{eff},x} = \mu_l + \rho l_x^2 \left| \frac{\partial v_x}{\partial r} \right| \cdot D_x + \rho l_\theta^2 \left| r \cdot \frac{\partial (v_\theta/r)}{\partial r} \right| \cdot D_\theta \quad ; \quad (2.14)$$

$$\text{and} \quad \mu_{\text{eff},\theta} = \frac{\mu_{\text{eff},x}}{\text{Pr}_\theta} \quad . \quad (2.12)$$

In equation (2.14),

$$D_x = [1 - \exp\{-\gamma \sqrt{\tau_x \rho} / (\mu_l A_{+,x})\}]^2 \quad . \quad (2.15)$$

$$D_\theta = [1 - \exp\{-\gamma \sqrt{\tau_\theta \rho} / (\mu_l A_{+,\theta})\}]^2 \quad . \quad (2.16)$$

The symbols have the following meanings:

$A_{+,x}$ and $A_{+,\theta}$ are empirical constants;

l_x and l_θ are mixing-lengths;

Pr_θ is the (empirically determined) 'swirl Prandtl number' ;

r is the radial distance from the axis;

V_x and V_θ are respectively the axial and tangential velocities;

y is the distance from the wall;

ρ is the point density;

τ_x and τ_θ are the axial and tangential components of the wall-shear-stress;

and μ_l is the laminar viscosity.

Table 8.1 (overpage) gives the relations for the mixing-lengths ℓ_x and ℓ_θ and the best values of the empirical constants for the three different types of flows studied.

Table 8.1 Relations for mixing-lengths and best values of empirical constants

Type of swirling flow	Relation for ℓ_x	Relation for ℓ_θ	k_x	k_θ	Pr_θ	$A_{+,x} = A_{+,\theta} =$	Chapter reference and remarks
Free jet	$\ell_x = k_x \cdot r_l$ (eqn. 3.1) r_l = characteristic width of jet defined in section 3.2-2	$\ell_\theta = k_\theta \cdot r$	0.08	0.26	1.	Not applicable ($D_x = D_\theta = 1$)	Chapter 3. Marginally better fit with data may be obtained by using $k_\theta = 0.26 + 1.2 S_W$. S_W is local swirl number
Confined jet	As above	As above	0.08	0.26	5.	26	Chapter 4
Pipe flow	$\frac{\ell_x}{R} = 0.14 - 0.08y^2 - 0.06y^4$ R = pipe radius y = distance from centre-line/ R (Section 6.4)	As above except near wall where $\ell_\theta = \ell_x$	0.41 (See remarks)	0.26	0.5	26	Chapter 6. The relation for ℓ_x implies $\frac{\ell_x}{R} = k_x(1-y)$ near a wall.

Closing remarks

The central purposes of this thesis have been to extend the Patanker - Spalding procedure to boundary-layers with swirl; to demonstrate the versatility of the procedure by using it to calculate different types of swirling flows; and to develop a suitable effective-viscosity hypothesis for turbulent, swirling boundary-layers.

Whether this thesis has served these purposes is, of course, a matter for the reader to decide. For the author, there always appears room for improvement in both the effective-viscosity hypothesis and in the computation procedure - more particularly in the former.

For reasons stated in the Introduction, the computations on turbulent flow have employed, as their physical input, an extension of Prandtl's mixing-length hypothesis. When this choice of hypothesis was made, it was expected that the values of the empirical constants in the effective-viscosity formulae would have to be changed from one flow situation to another. The table above shows the greatest change (by a factor of ten) to be in the ratio of the effective viscosities in the axial and tangential directions. Why this ratio should require to be changed by so large a factor remains to be explained.

Although other variants of the Prandtl hypothesis may be derived and used in an attempt to reduce the required change in the value of Pr_θ , and hence to obtain greater universality, the effort is perhaps better expended in the development of multi-equation models of turbulence. The Introduction has outlined the lines along which the development of these, more universal, models of turbulence is proceeding. The book by Patankar and Spalding lists several reports and publications which detail the use of these models to predict non-swirling, turbulent

boundary-layers. Recently, Roberts (1971) has employed a two-equation model for $\mu_{\text{eff},x}$ coupled with a mixing-length model for $\mu_{\text{eff},\theta}$ to predict swirling flows in pipes. The agreement of his predictions[†] with the experimental data is rather better than that obtained in Chapter 6. It is thus apparent from these studies that the next steps in research into turbulent swirling flows are to extend these models to include a swirl component of velocity and to test them out by incorporation into the P-S procedure.

As for the computation procedure, much time and effort have been spent in devising non-iterative schemes for the calculation of the axial pressure gradients (Appendix I). The reasons for eschewing iteration have been partly economic and partly aesthetic. Iteration is expensive of computer time and a blemish in the otherwise elegant marching-integration procedure.

Time and circumstance may erode both these justifications for the No-Iteration rule. After all, computers are becoming bigger, faster and cheaper to use and it is a moot point whether straightforward iteration is not less of a blemish than the several ad hoc schemes designed to avoid it (see Appendix I). In any case, iteration may become unavoidable if accuracy is to be had in strongly swirling regions of the flow. (see Section 5.7 of Chapter 5).

The devices described in this thesis to avoid iteration have served to demonstrate the efficacy of the computation procedure even when heavy demands are made upon it: the prospective user of the procedure may now rest assured that it will not fail him should he demand less of it.

[†] They were made with the complete elliptic equations rather than the boundary-layer ones presented here.

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Extension of the Pantankar-Spalding finite-difference
procedure to boundary-layers with swirl

I.1 Introduction

Chapter 1 set out the momentum - conservation equations of the swirling boundary-layer. The swirl equation had to be written in a form that required its finite-difference transform to vary from that valid for the equation of axial-momentum conservation. The main purpose of this appendix is to detail the steps necessary to cast the swirl equation into a finite-difference mould that renders it soluble by the P-S procedure. This done in Sections I.2 and I.3. The method of computation of the radial pressure gradients is set out in I.4. In section I.5 is described the procedure for taking account of the axial pressure gradients that ensue from the decay or build-up of these radial pressure gradients.

I.2 The finite-difference form of the swirl equation

I.2-1 Derivation of the swirl equation

As stated in Chapter 1, one form of the swirl equation is:

$$\frac{\partial(rV_{\theta})}{\partial x} + (a + bw) \frac{\partial(rV_{\theta})}{\partial w} = \frac{\partial}{\partial w} \left(c \frac{\partial(rV_{\theta})}{\partial w} \right) + d \quad , \quad (1.1-10)$$

where

$$d = -2 \frac{\partial}{\partial w} \left(\frac{rV_{\theta} \mu_{eff,\theta}}{\psi_E - \psi_I} \cos \alpha \right)$$

Equation (1.1-10) was abandoned in favour of an alternative form that suppressed the last term, d, on the right hand side of the above equation. To explain why this was done, it will be necessary to refer to the main features of the P-S procedure and the finite-difference formulation of the conservation equations.

The P-S procedure is a step-by-step marching integration method. Initial conditions are specified at some upstream station and boundary conditions on some bounding surface of revolution (in axisymmetric flows). At each forward step the conservation equations (in their f.d. form) are solved successively for each of the dependent variables.

The f.d. equations are derived from the partial differential ones by integration over a micro-region defined by the grid; the profile within each micro-region is composed of straight-line segments. This formulation guarantees strict satisfaction of the conservation principle over the whole of the boundary-layer only when the source term, $\underline{d}$, is known in advance of each forward step. Whenever this term is related to the as-yet-unknown downstream value of the dependent variable - as in equation (1.1-10) - $\underline{d}$, must be estimated, perhaps by use of its upstream value. This approximation entails an error which one may attempt to reduce to acceptable levels by various correction procedures.

Such attempts were made in some preliminary numerical experiments with equation (1.1-10). Different formulations of the source term, $\underline{d}$, were tried as also different correction procedures. No simple, universal procedure could be found which guaranteed the satisfaction of the overall conservation of the axial flux of angular momentum; particularly where there were substantial variations of effective viscosity across and along the boundary-layer. It thus became profitable to look for an alternative form of the swirl equation that would suppress the source term altogether. This form is derived below.

Alternative form of the swirl equation

The stream function ψ is defined by:

$$\psi = \int_0^y \rho V_x r \, dy \quad . \quad (I.2-1)$$

Using relations (1.1-1), (1.1-2), (1.1-6) and (I.2-1), the right hand side of equation (1.1-8) may be re-written as:

$$\begin{aligned}
 \text{R.H.S. of (1.1-8)} &= \frac{\partial}{\partial \omega} \left[c \left\{ \frac{\partial(rv_{\theta})}{\partial \omega} - 2v_{\theta} \frac{\partial r}{\partial \omega} \right\} \right] \\
 &= \frac{\partial}{\partial \omega} \left[c \left\{ r \frac{\partial v_{\theta}}{\partial \omega} - v_{\theta} \frac{\partial r}{\partial \omega} \right\} \right] \\
 &= \frac{\partial}{\partial \omega} \left[cr^2 \left\{ \frac{1}{r} \frac{\partial v_{\theta}}{\partial \omega} - \frac{v_{\theta}}{r} \frac{\partial r}{\partial \omega} \right\} \right] \\
 &= \frac{\partial}{\partial \omega} \left[cr^2 \frac{\partial}{\partial \omega} \left(\frac{v_{\theta}}{r} \right) \right]
 \end{aligned}$$

The swirl equation may now re-written in the alternative form:

$$\boxed{\frac{\partial}{\partial x} (rv_{\theta}) + (a+b\omega) \frac{\partial}{\partial \omega} (rv_{\theta}) = \frac{\partial}{\partial \omega} \left[cr^2 \frac{\partial}{\partial \omega} \left(\frac{v_{\theta}}{r} \right) \right]} \quad , \quad (1.1-9)$$

The finite-difference transform of this equation is derived below.

I.2-2 The central regions of the grid

The notation and procedure are identical to those adopted by Patankar and Spalding (1970). Figure I.1(p. 128) is a replica of Fig. 6.2-1 of their book. It shows a part of the central region of the finite-difference grid. The general dependent variable ϕ is distributed linearly between the grid lines. The shaded region of the $x \sim \omega$ grid marks the control volume over which integration takes place.

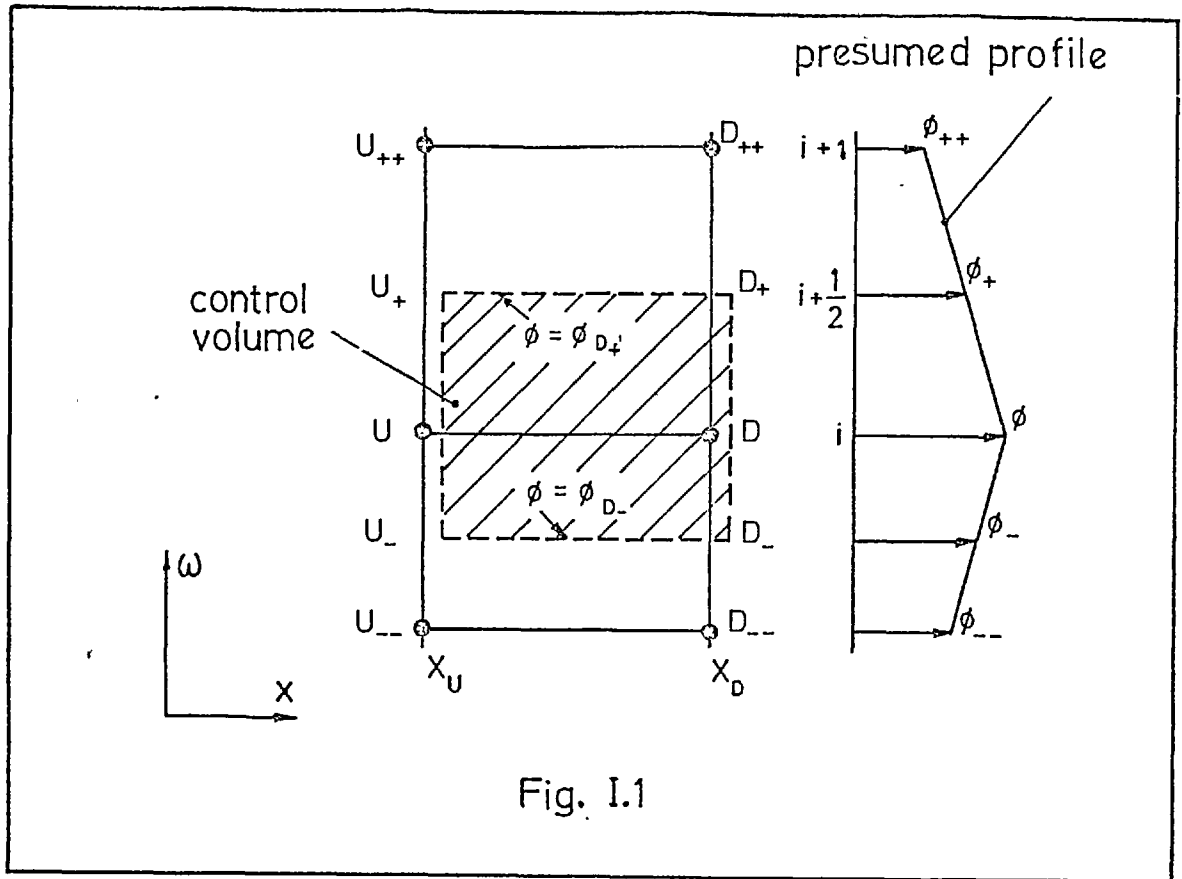


Fig. I.1

The micro-integral equation

Integration of equation (1.1-9) over the control volume of Fig. I-1 yields:

$$\begin{aligned}
 & \left[\int_{-}^{+} (rV_{\theta}) d\omega - \int_{-}^{+} (rV_{\theta})_U d\omega \right] / \delta x \\
 & + \left[\left\{ (a + b\omega)(rV_{\theta}) \right\}_{+} - \left\{ (a + b\omega)(rV_{\theta}) \right\}_{-} \right] - b \int_{-}^{+} (rV_{\theta}) d\omega \\
 & = \left[\left\{ cr^a \frac{\partial(V_{\theta} r)}{\partial \omega} - cr^a \frac{\partial(V_{\theta} r)}{\partial \omega} \right\}_{-} \right] \quad ; \quad (I.2-2)
 \end{aligned}$$

where $\delta x = (x_D - x_U)$.

After multiplying both sides of equation (I.1-1) by $2(\psi_E - \psi_I)$ and introducing the following symbols:

$$P = (\psi_E - \psi_I) \delta x \quad ;$$

$$G = -(\psi_E - \psi_I) b \quad ;$$

$$L_+ = r_I m_I - \omega_+ G \quad ;$$

$$L_- = r_I m_I - \omega_- G \quad ;$$

$$\Omega = \omega_{++} - \omega_{--} \quad ;$$

$$\Omega_+ = \omega_{++} - \omega_{--} \quad ;$$

$$\Omega_- = \omega - \omega_{--} \quad ;$$

$$T_+ = \frac{(r\mu_{\text{eff},\theta})_+}{(Y_{++} - Y)} \quad ;$$

$$T_- = \frac{(r\mu_{\text{eff},\theta})_-}{(Y - Y_{--})} \quad ;$$

equation (I.1-2) may be re-written in the form:

$$\begin{aligned} & \phi \left[\left\{ \frac{3}{4}(P + G)\Omega_+ + L_+ - L_- \right\} + 2T_+ \frac{r_+^2}{r} + 2T_- \left(\frac{r_-}{r} \right)^2 \right] \\ & + \phi_{++} \left[\left\{ \frac{1}{4}(P + G)\Omega_+ + L_+ \right\} - 2T_+ \left(\frac{r_+}{r_{++}} \right)^2 \right] \\ & + \phi_{--} \left[\left\{ \frac{1}{4}(P + G)\Omega_- - L_- \right\} - 2T_- \left(\frac{r_-}{r_{--}} \right)^2 \right] \\ & - \frac{1}{4}P[3\Omega\phi_U + \phi_{U++}\Omega_+ \phi_{U--}\Omega_-] = 0 \end{aligned} \quad (I.2-3)$$

Here, $\phi = r \cdot V_\theta$.

The co-efficients of the finite-difference equation

The value of the dependent variable, ϕ , at any grid node may be related to its value at the boundaries of the control volume by identifying equation (I.2-3) with

$$\phi = A_s \phi_{++} + B_s \phi_{--} + C_s \quad . \quad (I.2-4)$$

The identification results in the co-efficients of the finite-difference equation, A_s , B_s and C_s , being defined by:

$$\begin{aligned} A_s &= A'_s / D_s \\ B_s &= B'_s / D_s \\ C_s &= C'_s / D_s \\ A'_s &= 2T_+^* \left(\frac{r_+}{r_{++}} \right)^2 - L_+ - \frac{1}{4}(P + G)\Omega_+ \\ B'_s &= 2T_-^* \left(\frac{r_-}{r_{--}} \right)^2 + L_- - \frac{1}{4}(P + G)\Omega_- \\ C'_s &= \frac{1}{4}P(3\phi_U\Omega + \phi_{U++}\Omega_+ + \phi_{U--}\Omega_-) \\ D_s &= 2T_+^* \left(\frac{r_+}{r} \right)^2 + 2T_-^* \left(\frac{r_-}{r} \right)^2 + \frac{3}{4}(P + G)\Omega + L_+ + L_- \end{aligned} \quad (I.2-5)$$

In relations I.1-5, each T^* is related to T by the equation

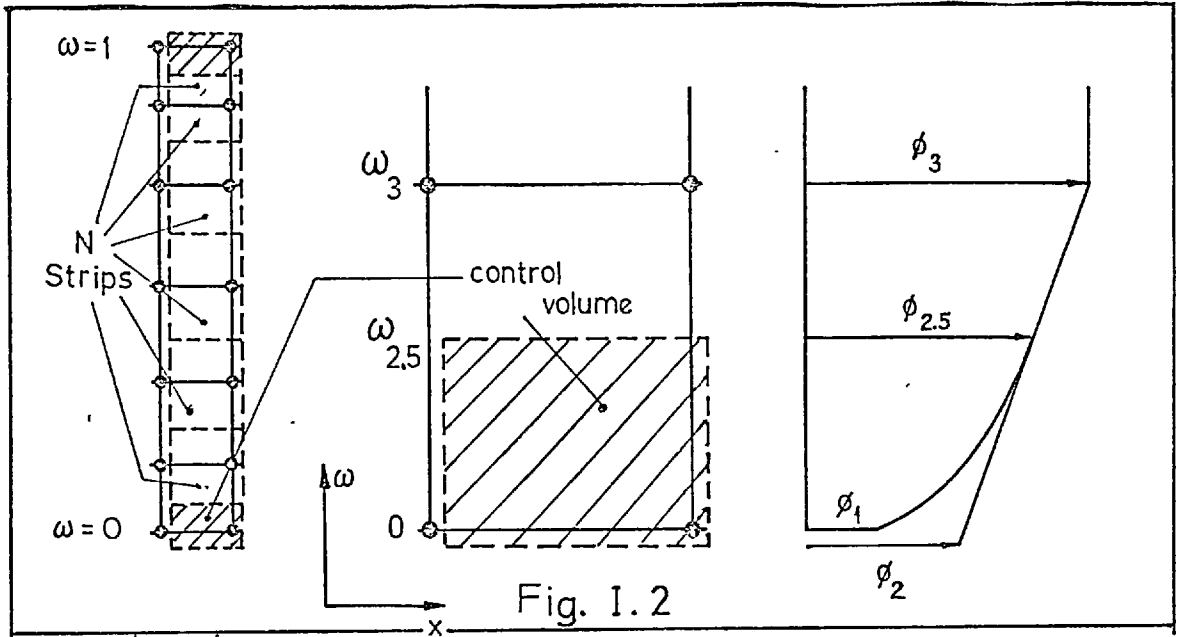
$$T^* = \frac{1}{2}[T + |\frac{1}{2}L| + |T - |\frac{1}{2}L||] \quad . \quad (I.2-6)$$

The motive and justification of this 'high-lateral-flux' modification are explained in the book by Patankar and Spalding (Section 6.2, p.95).

I.2-3. The half-intervals near the I and E boundaries

Fig. I-2 illustrates the region of the grid of interest. The task is to relate the fluxes across all the boundaries of the control volume

so as to connect ϕ_2 with ϕ_1 and ϕ_3 and ϕ_{N+2} to ϕ_{N+1} and ϕ_{N+3} ; N being the number of 'strips' in the grid.



At the I-boundary. The micro-integral equation here is:

$$\begin{aligned}
 & \left(\frac{3}{4} \phi_2 + \frac{1}{4} \phi_3 \right) (P + G) \Omega_+ - \left(\frac{3}{4} \phi_{2U} + \frac{1}{4} \phi_{3U} \right) P \Omega_+ \\
 & + L_+ (\phi_2 + \phi_3) - 2L_1 \phi_1 = 2T_+ (r^2 \Delta \phi')_{2.5} - 2T_1 (r^2 \Delta \phi')_I \quad . \quad (I.2-7)
 \end{aligned}$$

Here $\phi' = V_\theta / r$ and $\phi = V_\theta \cdot r$.

A tactical difficulty is encountered when the I-boundary is a line of symmetry. In this instance, the RHS of (I.2-7) cannot be evaluated satisfactorily; for, when the first term is put into the required form, namely:

$$2T_+ \left[\left(\frac{r_{2.5}}{r_3} \right)^2 \cdot \phi_3 - \left(\frac{r_{2.5}}{r_2} \right)^2 \cdot \phi_2 \right] ,$$

r_2 appears in the denominator and, as this is close to zero, attempts

to evaluate the term can result in frequent 'overflow' in the computer core.

To obviate this difficulty, the definition of $T_{+,2.5}$ may be changed so that it is evaluated by reference to values of quantities only at the points 2.5 and 3. Thus, if,

$$\begin{aligned} T'_+ &= \frac{(r\mu_{\text{eff},\theta})_{2.5}}{(y_3 - y_{2.5})} \\ &= 2T_+ \end{aligned} \quad , (I.2-8)$$

the RHS of (I.2-7) may be written as:

$$\text{RHS} = 2T'_+ \left[\left(\frac{r_{2.5}}{r_3} \right)^2 - \frac{1}{2} \phi_3 - \frac{1}{2} \phi_2 \right] - 2T_1 \left[\frac{1}{2}(\phi_2 + \phi_3) \left(\frac{r_I}{r_{2.5}} \right)^2 - \phi_1 \right] \quad (I.2-9)$$

Equating the LHS of (I.2-7) with (I.2-9) and identifying with

$$\phi_2 = A_{s,2} \phi_3 + B_{s,2} \phi_1 + C_{s,2} \quad , (I.2-10)$$

one obtains the following co-efficients:

$$\begin{aligned} A'_{s,2} &= A'_{s,2} / D_{s,2} & ; \\ B'_{s,2} &= B'_{s,2} / D_{s,2} & ; \\ C'_{s,2} &= C'_{s,2} / D_{s,2} & ; \\ A'_{s,2} &= 2T'_+ \left[\left(\frac{r_{2.5}}{r_3} \right)^2 - \frac{1}{2} \right] - T_1 \left(\frac{r_I}{r_{2.5}} \right)^2 - \frac{1}{4}(P + G)\Omega_+ - L_+ & ; \\ B'_{s,2} &= 2(L_1 + T_1) & ; \\ C'_{s,2} &= \frac{1}{4} P\Omega_+ (3\phi_{2v} + \phi_{3v}) & ; \\ D'_{s,2} &= T'_+ + T_1 \left(\frac{r_I}{r_{2.5}} \right)^2 + \frac{3}{4}(P + G)\Omega_+ + L_+ & . (I.2-11) \end{aligned}$$

Starred quantities indicate incorporation of the 'high-lateral-flux' modification.

Consequences for the co-efficients for point 3.

Because the '+' surface for point 2 is the '-' surface for point 3, any change in definition of quantities applicable to the '+' surface of point 2 will also have to appear in the co-efficients for point 3. It will be sufficient to focus on the diffusion transport terms; in particular, on $T'_{+,2} = T'_{-,3}$

The diffusion terms for point 3 read:

$$\begin{aligned} & 2T'_+ r_+^2 (\phi'_4 - \phi'_3) - 2T'_- r_-^2 \left\{ \phi'_3 - \phi'_{2.5} \right\} \\ &= 2T'_+ r_+^2 (\phi'_4 - \phi'_3) - 2T'_- \left[\left\{ \left(\frac{r_{2.5}}{r_3} \right)^2 - \frac{1}{2} \right\} \phi'_3 - \frac{1}{2} \phi'_2 \right] \end{aligned}$$

Recalling that equation with which to identify this expression is:

$$D'_{s,3} \phi'_3 = A'_{s,3} \phi'_4 + B'_{s,3} \phi'_2 + C'_{s,3}$$

it is evident by inspection that, in place of $\frac{r_-}{r_2} \cdot 2T'_-$ and $\left(\frac{r_-}{r_3} \right) \cdot 2T'_-$, $B'_{s,3}$ and $D'_{s,3}$ have respectively T'_- and $2T'_- \times \left[\left(\frac{r_{2.5}}{r_3} \right)^2 - \frac{1}{2} \right]$ have been incorporated in the Program to be described in Appendix II.

At the E boundary

The co-efficients for the half-interval at the E boundary may be derived analogously. Here, unlike at the I boundary, there is no need to contort the definition of T_- since r_{N+2} is never zero.

The co-efficients with the incorporation of the high-lateral-flux

modification are:

$$\begin{aligned}
 A_{S,N+2} &= A'_{S,N+2} / D_{S,N+2} & ; \\
 B_{S,N+2} &= B'_{S,N+2} / D_{S,N+2} & ; \\
 C_{S,N+2} &= C'_{S,N+2} / D_{S,N+2} & ; \\
 A'_{S,N+2} &= 2(T_{N+3} - L_{N+3}) & ; \\
 B'_{S,N+2} &= 2T_{-}^{*} \left(\frac{r_{N+1.5}}{r_{N+1}} \right)^2 - T_{N+3} \left(\frac{r_E}{r_{N+1.5}} \right)^2 + L_{-} - \frac{1}{4}(P + G)\Omega_{-} & ; \\
 C'_{S,N+2} &= \frac{1}{4} P \Omega_{-} (3\phi_{U,N+2} + \phi_{U,N+1}) & ; \\
 D_{S,N+2} &= 2T_{-}^{*} \left(\frac{r_{N+1.5}}{r_{N+2}} \right)^2 + T_{N+3} \left(\frac{r_E}{r_{N+1.5}} \right)^2 + \frac{3}{4}(P + G)\Omega_{-} - L_{-} & .
 \end{aligned}$$

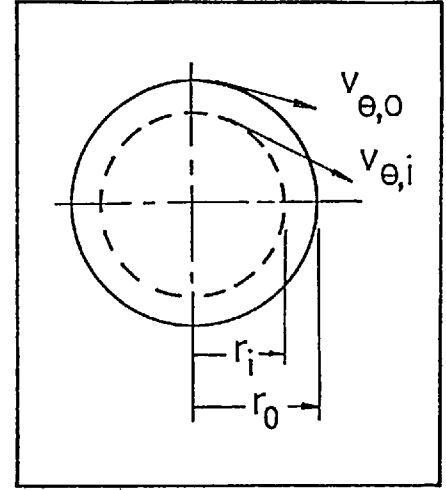
(I.2-12)

I.3 The wall function for swirl velocity

I.3-1 Purpose of the wall function. To evaluate the finite-difference co-efficients for the grid points near wall boundaries (relations I.2-5 and I.2-6), it is necessary to assign values to the quantities T_1 and T_{N+3} . These are calculated by way of a wall function. The relevant formulae are of two kinds: those that follow from the consequences of definitions; and those which express information concerning the physics of the flow near a wall. The formulae are derived by effecting a Couette-flow analysis for the flow near a wall.

I.3-2 Couette-flow analysis for the swirl velocity equation

Consider a thin ring of swirling fluid adjacent to a wall (full line in Fig. I.3) rotating with a tangential velocity $V_{\theta,0}$. The swirl velocity at the inner edge of the ring (broken line) is $V_{\theta,i}$. The axial motion is out of the plane of the paper.



Near a wall, the axial velocity is small and therefore, the x - wise convection is locally negligible. The flow within the ring in Fig. I.3 is thus a one-dimensional Couette flow.

Fig. I.3

Transforming the equation of swirl motion (equation 1.1-11) to cylindrical polar co-ordinates and neglecting the x - wise convection terms (the defining supposition for Couette flow), the one-dimensional equation of swirl motion within the ring reduces to:[†]

$$\frac{\partial}{\partial r} \left\{ \mu_{\text{eff},\theta} r^3 \frac{d}{dr} \left(\frac{v_{\theta}}{r} \right) \right\} = 0 \quad .(I.3-1)$$

The tangential stress, τ_{θ} , of the wall on the fluid is defined by the equation:

$$\tau_{\theta,0} = - \mu_{\text{eff},\theta} \cdot r \cdot \left. \frac{d}{dr} \left(\frac{v_{\theta}}{r} \right) \right|_{r=r_0} \quad .(I.3-2)$$

Integrating (I.3-1) once and using (I.3-2) we get:

$$\mu_{\text{eff},\theta} \cdot r^3 \frac{d}{dr} \left(\frac{v_{\theta}}{r} \right) = - \tau_{\theta,0} \cdot r_0^2 \quad .(I.3-3)$$

An assumption about the physics of the flow has been made here.

It is that $\mu_{\text{eff},\theta}$ does not vary with r . This is true in a turbulent flow only when the Couette-flow region lies wholly within the laminar sub-layer

[†] In order not to obscure the essentials, the analysis to be presented here does not take account of possible blowing or suction through the wall.

Should the Couette-flow region be partly turbulent, some hypothesis about the variation of $\mu_{\text{eff},\theta}$ with r would have to be used to effect the integration.[†] Assuming then that the flow is laminar, the subscript 'eff' may be dropped. Integrating (I.3-3) again, we get:

$$\frac{v_{\theta}}{r} = \frac{\tau_{\theta,0} \cdot r_0^2}{2\mu_{\theta} \cdot r^2} + \frac{v_{\theta,0}}{r_0} - \frac{\tau_{\theta,0}}{2\mu_{\theta}} \quad (\text{I.3-4})$$

The constant of integration having been determined from the boundary condition:

$$v_{\theta} = v_{\theta,0} \quad \text{at } r = r_0.$$

Using the second boundary condition,

$$v_{\theta} = v_{\theta,i} \quad \text{at } r = r_i,$$

we construct from (I.3-4) the expression for the tangential shear stress at the wall:

$$\tau_{\theta,0} = 2\mu_{\theta} \left[v_{\theta,i} - v_{\theta,0} \frac{r_i}{r_0} \right] \frac{r_i}{r_0^2 - r_i^2} \quad (\text{I.3-5})$$

If the broken line in Fig. I.3 were a wall and the full line represented the outer edge of the Couette-flow region, a torque balance on the ring would reveal the wall shear-stress to be:

$$\tau_{\theta,i} = \tau_{\theta,0} \left(\frac{r_0}{r_i} \right)^2 \quad (\text{I.3-6})$$

I.3-3 Wall function for the swirl velocity

Analogous to the definitions of T_{N+3} and T_1 for the axial velocity

[†] This need may be obviated by using a grid spacing near a wall that is sufficiently fine to ensure that the Couette-flow region is always within the laminar sub-layer.

these quantities for the swirl velocity are defined through the relations:

$$\tau_{\theta,0} r_{N+3}^2 = T_{N+3} \left[\phi_{\text{ref}} \cdot \left(\frac{r_{N+3}}{r_{\text{ref}}} \right)^2 \right] \quad ; \quad (\text{I.3-7})$$

and
$$\tau_{\theta,i} r_1^2 = T_1 \left[\phi_{\text{ref}} \cdot \left(\frac{r_1}{r_{\text{ref}}} \right)^2 \right] \quad . \quad (\text{I.3-8})$$

ϕ_{ref} and r_{ref} are respectively the values of $\phi \equiv rV_{\theta}$ and the radius, r , at the edge of the Couette-flow region.

Although relations (I.3-5) to (I.3-8) are sufficient to give the required values of T_{N+3} and T_1 , some interesting results can be derived if we press on to consider the case when the walls are stationary.

Let, for example, the E boundary be a stationary wall; i.e. $V_{\theta,0} = 0$. We now define a tangential-shear-stress co-efficient thus:

$$S_{\theta,0} = \frac{\tau_{\theta,0}}{\rho V_{\theta,i}^2} \quad . \quad (\text{I.3-9})$$

From (I.3-8) and (I.3-9) we get

$$S_{\theta,0} = \frac{2}{\text{Re},i} \cdot \frac{1}{(1 + r_0/r_i)} \quad , \quad (\text{I.3-10})$$

in which the Couette-flow Reynolds number, Re,i , is defined by:

$$\text{Re},i = \frac{\rho(r_0 - r_i)V_{\theta,i}}{\mu_{\theta}} \quad . \quad (\text{I.3-11})$$

Note that, as r_i tends to r_0 i.e. the Couette-flow region becomes thin, equation (I.3-10) tends to the form appropriate to plane-flow; $S = 1/\text{Re},i$. Combining (I.3-7) with (I.3.9), the expression for T_{N+3} simplifies to:

$$\boxed{T_{N+3} = S_{\theta,0} \cdot \rho \cdot \phi_{\text{ref}}} \quad , \quad (\text{I.3-12})$$

since $V_{\theta,\text{ref}} = V_{\theta,i}$ and $r_{\text{ref}} = r_i$.

It is convenient from the programming point of view to compute

T_{N+3} through the relations (I.3-10), (I.3-11) and (I.3-12). Appendix II lists and explains the Program embodiment of these relations.

I.4 The finite-difference form of the radial momentum equation

From Chapter 1, the radial momentum equation is:

$$\frac{\partial p}{\partial y} = \frac{\rho V_{\theta}^2}{r} \cos \alpha \quad . \quad (1.1-12)$$

This may be transformed to a finite-difference form as follows. In the notation of Fig. I-1,

$$p_i^* \approx p_{i+1}^* + \rho_{i+\frac{1}{2}} \frac{(rV_{\theta})_+^2}{r_+^3} \cdot (r_i - r_{i+1})$$

$$\text{and} \quad p_i = p_i^* \cos \alpha \quad . \quad (I.4-1)$$

In those swirling boundary-layers where the axial variation of pressure at a boundary is known explicitly, successive substitution in relations (I.4-1) yields the radial pressure distribution. The calculations on a free-swirling jet displayed in Chapter 3 employ this procedure.

In swirling flows where only the average pressure at any axial station is known, such as in the case of swirling pipe-flows, the procedure is as follows:- A guessed value of the pressure at the boundary is used in relations (I.4-1) to determine a nominal radial distribution of pressure. The average of this distribution is computed and compared with the known average pressure. A proportion of the discrepancy between the two is used to alter the guessed value of pressure at the boundary and the cycle repeated until the discrepancy between the known and calculated average pressures is acceptably small. The use of this procedure is demonstrated in Chapters 4,5 and 6.

I.5 The treatment of the source terms in the equation for axial momentum

I.5-1 Motive for special attention

From Chapter 1, the expression for the source, d , in the equation for the conservation of axial momentum is:

$$d = - \frac{1}{\rho V_x} \frac{\partial p}{\partial x} + \frac{v_\theta^2 \sin \alpha}{r V_x} \quad . \quad (1.1-9)$$

The first term on the right takes account of the axial pressure gradient; the second represents the axial component of the centrifugal force generated by swirl. The axial pressure gradient may be thought of as being composed of two parts: the first is independent of the presence of swirl; it may, for example, be an explicitly prescribed gradient in the case of an external flow or it may be the implicit gradient caused by confinement in a duct. The second part is a direct consequence of the possible decay or build-up of swirl. In either event, the radial gradients of pressure either to diminish or to increase with axial distance thus causing axial gradients of static pressure. It is to take account of these that special attention has to be paid to the source term, d .

I.5-2 Taking account of the swirl-induced gradients in axial pressure

If the radial distributions of pressure were known at two axial stations, subtraction of the pressures at corresponding grid nodes and division by the increment in axial distance would give the required radial distribution of axial pressure gradient which would then appear in (1.1-9) above. The snag is that, because the Patankar-Spalding procedure is of the marching-integration type, the downstream distribution of pressure is unknown and unknowable before the complete execution of each forward step; for, the required downstream radii are computed from the yet-to-be-determined downstream axial velocities.

The way out of this dilemma is to adopt the following procedure. Solve the swirl equation first,[†] thus obtaining the downstream values of rV_θ . Use these and the upstream values of r 's in relations (I.5-2) to obtain a first guess of the downstream pressure distribution. Use the resulting guess of axial pressure gradient in the solution of the axial momentum equation. Repeat this procedure iteratively until the values of the pressure distribution at successive iterations differ by no more than a small, specified amount.

The above procedure will undoubtedly work but only at the expense of computer storage and time. Patankar and Spalding present a non-iterative scheme for accounting for the axial pressure gradient in confined flows. Taking our cue from that scheme which preserves both the economy and elegance of the solution procedure, it is possible to devise non-iterative schemes that take account of the swirl - induced part of the axial pressure gradient.

For confined swirling flows, the average pressure at the downstream axial station is determined and corrected in arrears as in the P-S scheme. The radial distribution of pressure then is determined from relations (I.4-1) in the manner outlined in Section I.4 above. With this information and the known distribution of pressure at the upstream station, the distribution of axial pressure gradient is easily determined. The computations presented in Chapters 4 to 6 make use of this technique.

For free-swirling flows, an application of the conservation principle over the whole of the boundary-layer furnishes an integral constraint that makes possible the calculation of the distribution of axial pressure

[†] The co-efficients of the finite-difference equations employ the upstream values of the dependent variables and co-efficients of diffusive transport. The order in which the equations are solved can, therefore, be chosen to suit convenience.

gradient. The net momentum flux at any axial station, namely

$$\int_0^{\infty} (p + \rho \dot{V}_x^2) r dr, \text{ is an invariant of a free-swirling boundary-layer.}$$

As before, we compute an estimate of downstream pressure distribution using the calculated downstream values of rV_θ and the upstream values of the r 's. The integral momentum flux, $\int_0^{\infty} (p + \rho V_x^2) r dr$, is computed at the downstream station using this pressure distribution. It will not, in general, equal its upstream value. The discrepancy between the two may be employed to correct the distribution of axial pressure gradient for the next forward step in a manner which tends to diminish this discrepancy at each successive step. The computations presented in Chapter 3 employ this technique.

I.5-3 The axial component of the centrifugal force

It so happens that in both the flow situations dealt with in this thesis, it has been possible to neglect the second term on the right of relation (1.1-9). But in many flow situations capable of being handled by the computational procedure presented in this thesis, it would have to be taken into account. One such situation is that of swirling flow through a conical nozzle; here the average swirl velocity could increase with axial distance downstream. In this event, the second term would influence substantially the behaviour of the axial flow.

Although not a great deal of attention has been paid to the matter, there appears no obvious reason why the use of the downstream values of rV_θ and the upstream r 's should not provide an adequate treatment of this term. The Program to be described in Appendix II embodies such use.

The GENSWIRL Program for the solution of the equations
of the swirling boundary-layer

II.1 Introduction

Appendix I has set out the special relations and procedures required for the solution by the Patankar-Spalding procedure of the equations of the swirling boundary-layer. The execution of the procedure is effected by a FORTRAN IV computer program called GENSWIRL. The Program is capable of handling internal, external and free swirling flows.

The GEN series of computer programs has been described in some detail by Patankar and Spalding in their book. Insufficient time before publication had prevented the incorporation in these programs of two desirable developments: the explicit imbedding of the co-efficients of the swirl equation into the heart of the programs and the abandonment of according separate treatment to the variable u , the axial velocity.

GENSWIRL makes good these deficiencies. Furthermore, the computer store requirements have been greatly reduced. Whereas, previously, the storage required rose in proportion to the number of equations being solved, the storage demanded by GENSWIRL is almost independent of this number.

This appendix is a guided tour of the anatomy of GENSWIRL. The details of its various organs are too close to their GENMIX - 4[†] counterparts to warrant elaborate exposition. The sections to follow dwell on only those details of GENSWIRL which differ from the others in the GEN series.

II.2 The anatomy of GENSWIRL

Figure II.1 (overpage) is a block diagram showing the interconnection

[†] This Program is described in detail in Patankar and Spalding (1970), Chapters 7 and 8.

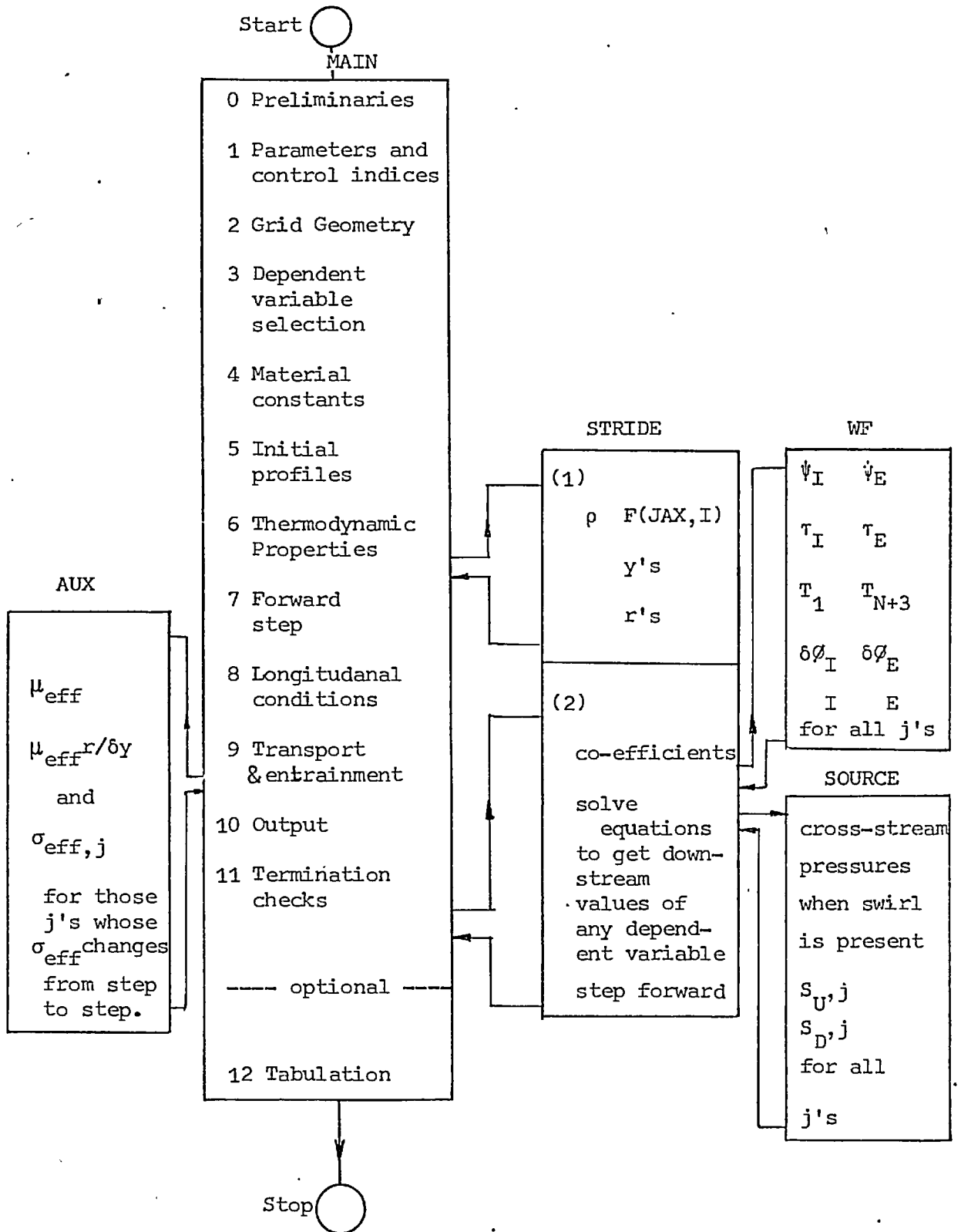


Figure II.1

between the various organs of GENSWIRL. The statements and symbols appearing inside the boxes of Figure II.1 summarise the quantities which each particular subroutine, or section thereof, calculates.

II.2-1 The MAIN program

The purpose of MAIN is to specify completely the flow problem considered. Figure II.1 shows MAIN divided into twelve sections; each specifies something of the problem. MAIN is specific to the particular problem being solved. The attached listing shows a version of MAIN used to compute confined swirling jet problem of Chapter 4.

II.2-2 The heart of GENSWIRL - subroutine STRIDE.[†]

a. The main structural change

The single most important change in the structure of STRIDE is the one that has enabled storage requirements to be slashed. Whereas in GENMIX-4 the co-efficients of all the equations were first computed and stored before being used to obtain the downstream values of the variables; in GENSWIRL, the downstream values of any dependent variable are obtained as soon as the co-efficients for that variable have been computed. This practice has enabled several two-dimensional arrays to be shrunk to one-dimensional ones and allowed some one-dimensional arrays to be reduced to single variables.

[†] See the appended listing.

The details of the changes follow.

b. New arrays and variables

Arrays: The arrays (AU(I), BU(I) and CU(I) have disappeared. U(I) has left its work to be done by one of the F(J,I)'s. Each of A(J,I), B(J,I), C(J,I), SU(J,I) and SD(S,I) has been reduced to a dimensional array of the form [variable] (I). FDIFE, FDIFI, GE, GI and TTPF replace their subscripted forms.

Two new arrays PY(I) and RP(I) are allocated to, respectively, the cross-stream pressures and the arithmetic mean of the radii at i and $i + 1$, i.e. to $r_{+,i}$ (r-plus, hence the mnemonic. RP(I) and RP(NP1) thus displace R25 and RN15.

Variables: JAX and JSW are the numbers respectively of the AXial and SWirl equations. Thus, F(JAX,I) represents the axial velocity and F(JSW,I) the variable ($V_\theta \times r$). TAUES and TAUIS (S for Swirl) represent the tangential component of the shear stress at the I - and E - boundaries respectively. SNALFA carries the sine of the angle α in Fig. 1.1. If α changes substantially from one grid node to the next, SNALFA may have to be subscripted.

c. Details of the structural changes

STRIDE is called only twice from MAIN, rather than thrice as in GENMIX-4. Control is deflected to the relevant section of STRIDE through the first executable statement by the argument ISW. The last DO loop in STRIDE (1) fills the RP array with r_+ 's.

STRIDE (2) computes the co-efficients for the entire grid and obtains all the downstream values of each of the dependent variables, in turn,

in one all-embracing DO loop; DO 3340 J = 1, NEQ.

En route, calls are made to SOURCE (J) for all the dependent variables J, and to WF if one or both boundaries happen to be walls. The special co-efficients for the swirl equation (derived in Appendix I) have been incorporated; their location is marked by comment cards.

The calculation of the wall-fluxes takes place after the computation of the downstream values of the dependent variables. This practice is now consistent with the practice for the interior part of the grid where the diffusional fluxes across the control volume surrounding each grid node are calculated from diffusional co-efficients multiplying the downstream value of the variable.

The flow of control through STRIDE (2) has been smoothed by the removal of some logical IF's. The execution of several statements repeatedly setting variables to zero, when already set once, may appear to be 'cruelty' to the computer but the practice is more economical of its time than the execution of fewer logical IF's and GO TO's.

II.2-3 Subroutine AUX

AUX has suffered some surgery. The entire section dealing with the filling of the 'source' arrays, SU and SD, has been removed; leaving only the filling of the EM U(I) array to be done by algorithms based on a particular model of turbulence.

The appended listing shows Program embodiment of the mixing-length model discussed in Chapter 2 and used in Chapter 6 to calculate turbulent swirling flow in a pipe.

II.3-4 Subroutine SOURCE

a. Purpose of SOURCE. This limb of GENSIWRL is additional to the ones

described by Patankar and Spalding. Whereas in GENMIX - 4, AUX filled the arrays SU and SD, this is now done in SOURCE for all values of J. SOURCE carries the Program embodiment of the schemes discussed in sections I.4 and I.5: the cross-stream pressures are calculated here and used to obtain an estimate of the pressure gradient applicable to the axial momentum equation. For free swirling flows, SOURCE is where a correction is applied to this estimate of axial pressure-gradient such that the discrepancy between the local and starting values of the integral momentum flux is kept within specified limits.

The appended listing is a version of SOURCE appropriate to the calculation of confined swirling flows.

b. The calculation of cross-stream pressures. The array PY(I) harbours the cross-stream pressures. These are calculated relative to PY(NP2) using relations (I.4-1) and the downstream values of $(V_{\theta} \cdot r)$. On the way, the previous values of the PY's are deposited in another array, DP(I). At the end of the DO operation, the PY array contains an estimate of the pressures at the downstream station whilst the DP array contains the pressures upstream. Their difference divided by the step length DX is added to any existing pressure gradient - externally applied or caused by confinement. The total pressure gradient and the axial component of centrifugal force appear in the formulae for the SU(I)'s: these are the source terms for the axial momentum equation.

II.2-5 Subroutine WF

This is the Wall Function subroutine. Its main purpose is to supply STRIDE(2) with T_1 and T_{N+3} (see Appendix I) for all the dependent variables.

The appended listing of WF is only one example of many possible

variants. It is appropriate to the calculation of the hydrodynamics of swirling boundary-layers in pipes. It is assumed that the first grid point near the wall (at the E - boundary in the quoted example) is maintained close enough to it to ensure that the Couette-flow region is wholly laminar.

For the axial momentum, the subroutine lists the FORTRAN translations of the laminar wall-function relations derived by Patankar and Spalding.

For the swirl momentum, the architecture of STRIDE (2) requires OUT1 to be set to zero. For ISTEP greater than JSTEP, the formula for OUT2 is a translation of equations (I.3-10) and (I.3-12) of Appendix I. Setting OUT2 to zero for all ISTEP less than JSTEP is a simple programming device that allows the analyst to develop an axial velocity profile and a solid-body-rotation tangential velocity profile, independently of each other, until the value of ISTEP reaches the index JSTEP which is set in the MAIN program.

[illegible]

```

      F(2,I)=U(I)
C   F(1,I)
      F(1,I)=VT(I)*R(I)
    40 CONTINUE
      DO 50 I=3,NP2
        OM(I)=OM(I-1)+.5*(RHO(I)*F(JAX,I)*R(I)+RHO(I-1)*F(JAX,I-1)*R(I-1))
          1                                     *(R(I)-R(I-1))
    50 CONTINUE
      PEI=OM(NP2)
C   OMEGA GRID
      DO 51 I=3,NP1
        OM(I)=OM(I)/PEI
    51 CONTINUE
C-----FLOW RATES
      PSII=0.
      PSIE=PSII+PEI
C-----TOTAL FLOW RATE/(2.*PI)
      FLOTOT=PSIE
C-----
CHAPTER66666666666666666666666666666666 THERMODYNAMIC PROPERTIES 666666
C INITIAL WALL STATIC PRESSURE IN NEWTONS PER METER SQUARED
      PRESS=1.E5
      DPDDX=0.
C ***** START OF MAIN LOOP *****
    60 CONTINUE
C-----NEW PRESSURES
      PRESS=PRESS+DPDDX*DX
C-----
      CALL STRIDE(1)
C-----
CHAPTER 77777777777777777777777777777777 TRANSPORT AND ENTRAINMENT PROPERTIES 77777
    90       DO 92 I=1,NP3
    92 EMU(I)=YL
C-----
      CALL AUX
C-----ENTRAINMENT CONTROL -----
C NO ENTRAINMENT AT EITHER BOUNDARY
C RMI AND RME HAVE BEEN SET TO ZERO IN CHAPTER FIVE
C-----
CHAPTER 88888888888888888888888888888888 FORWARD STEP 888888888888888888
    96 CONTINUE
      DX=FRA*RDUCT
      IF(ISTEP.LT.10) DX=.1*DX
    73 XD=XU+DX
C FURTHER ADJUSTMENTS TO DX ARE MADE IN CHAPTERS 8 AND 9
C-----
CHAPTER 99999999999999999999999999999999 ADJUST LONGITUDINAL CONDITIONS 9999999999
C***** SPECIFICATION OF VARIATION OF OUTER RADIUS *****
      ROUT=RDUCT+XD*TAN
      IF(ROUT.LT.RD(IAX)) GO TO 72
      XD=(RD(IAX)-RDUCT)/TAN
      DX=XD-XU
      IAX=IAX+1
      IF(IAX.GT.5) IFIN=1
      IPRINT=1
    72 CONTINUE
C-----PRELIMINARIES FOR PRESSURE GRADIENT
    86 UBAR=0.
      DO 83 I=2,NP1

```

```

83      UBAR=UBAR+(F(JAX,I)+F(JAX,I+1))*(OM(I+1)-OM(I))
        UBAR=.5*UBAR
        IF(KIN.EQ.2) UBAR=(UBAR-UA)*PEI/PSIE+UA
        AD2PI=ROUT**2*.5
        IF(KIN.EQ.1) AD2PI=AD2PI-R(1)**2*.5
        DYNHED=UBAR*FLOTOT/AD2PI
        REX=R(NP3)-ROUT
        REXD=REX/ROUT
C***** PRESSURE GRADIENT
        DA=-FAC*ROUT*(R(NP3)-ROUT)
        DPDDX=(DYNHED*DA/DX-TAU1*R(1)-TAUE*R(NP3))/AD2PI
87 DO 85 I=1,NP3
85 DPDX(I)=DPDDX
-----
CHAPTER 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 10 OUTPUT 10 10 10
        IF(ISTEP.GT.0) GO TO 901
        IPRINT=1
        L=4
        WRITE(6,991) (OM(I),I=1,NP3)
901     CONTINUE
        UBAR2=UBAR*UBAR
        TAU=2.*TAUE/UBAR2
        TAU=TAU/DEN
        PRES=PRES
C  WRITE STATION VARIABLES
        WRITE(6,990) XU,TAU,DPDDX,PRES,REXD,ISTEP
        IF(IPRINT.EQ.0) GO TO 1000
        IPRINT=0
C  WRITE PROFILES
        DO 905 IPRO=1,L
        GO TO (910,920,930,940), IPRO
910    CONTINUE
        OP(NP3)=ROUT
        DO 911 I=2,NP2
911    OP(I)=Y(I)/OP(NP3)
        OP(1)=R(1)
        GO TO 950
920    DO 921 I=1,NP3
921    OP(I)=F(2,I)
        GO TO 950
930    OP(1)=F(1,1)
        OP(2)=F(1,2)
        OP(NP3)=F(1,NP3)
        DO 931 I=3,NP3
931    OP(I)=F(1,I)/R(I)
        GO TO 950
940    DO 941 I=1,NP3
941    OP(I)=PY(I)
950    WRITE(6,993) OP(1),(OP(I),I=2,NP2),OP(NP3)
905    CONTINUE
990    FORMAT(1H0,
           13HXU=,1PE10.3,2X,
           24HTAU=,1PE10.3,2X,
           35HDPDX=,1PE10.3,2X,
           45HPRES=,1PE10.3,2X,
           45HRDIF=,1PE10.3,2X,
           46HISTEP=16
           9)
991    FORMAT(19HO0MEGA DISTRIBUTION/1H ,1P10E11.3/(1H ,10E11.3))

```


SUBROUTINE STRIDE(ISW)

```

C
C   MODIFIED FORM OF GENMIX4 VERSION OF STRIDE.          JANUARY 1970
C
      DIMENSION A(43),B(43),C(43)
      COMMON/GENRAL/ AJE(5),AJI(5),CSALFA,DPDX(43),DX,EMU(43),F(5,43),
1     FS(5,43),H,IFIN,INDE(5),INDI(5),ISTEP,ITEST,IUTRAP,JAX,JSW,
2     KEX,KIN,KRAD,N,NEQ,NPH,NP1,NP2,NP3,OM(43),PEI,PR(5),PREF(5,43),
3     PSIE,PSII,PY(43),R(43),RHO(43),RME,RMI,RP(43),RU(43),SNALFA,
4     SD(43),SU(43),TAUE,TAUES,TAUI,TAUIS,XD,XU,Y(43),YE,YI

      GO TO (1000,2000), ISW
C***** S T R I D E 1 *****
1000 IF(ISTEP.GT.0) GO TO 1100
      OM(1)=0.
      OM(2)=0.
      OM(NP2)=1.
      OM(NP3)=1.
      OMI=.5*OM(3)
      OME=.5*(1.-OM(NP1))
      BPE=1.
      BPI=1.
      Y(1)=0.
      IF(KRAD.EQ.1) GO TO 1100
      DO 1001 I=1,NP3
1001  R(I)=1.
      RP(2)=1.
      RP(NP1)=1.
      IF(ITEST.EQ.0) GO TO 9018
      WRITE(6,9010) (R(I),I=1,NP3),R25,RN15
9010  FORMAT(1H ,1P11E11.3)
9018  CONTINUE
C----- CALCULATION OF RHO*U 'S -----
1100 DO 1101 I=1,NP3
1101  RU(I)=RHO(I)*F(JAX,I)
      RU3=RU(3)
      RUN1=RU(NP1)
      DO 1102 I=2,NP1
      RU(I)=.5*(RU(I)+RU(I+1))
      IF(RU(I).GT.0.)GO TO 1102
      ITEST=1
      WRITE(6,1200)
1200  FORMAT(25HAN RU IS ZERO OR NEGATIVE)
      IFIN=1
1102  CONTINUE
      IF(ITEST.EQ.0) GO TO 9019
      WRITE(6,9010) (RU(I),I=1,NP3),RUN1,RUN2,RU2,RU3,PEI
9019  CONTINUE
C----- CALCULATION OF Y 'S AND R 'S -----
C ----- Y'S FOR PLANE GEOMETRY -----
      YI=PEI*OMI/(BPI*RU(2))
      Y(3)=YI+PEI*OM(3)/(RU(2)+RU3)
      Y(2)=2.*YI-Y(3)
      DO 1103 I=4,NP1
1103  Y(I)=Y(I-1)+PEI*(OM(I)-OM(I-1))/RU(I-1)
      YN15=Y(NP1)+PEI*(1.-OM(NP1))/(RU(NP1)+RUN1)
      YE=PEI*OME/(BPE*RU(NP1))
      Y(NP3)=YN15+YE
      Y(NP2)=2.*YN15-Y(NP1)

```

```

      IF(KRAD.EQ.0) RETURN
C ----- Y'S AND R'S FOR AXISYMMETRICAL GEOMETRY
      IF(CSALFA.EQ.0.) GO TO 1110
C ----- CSALFA NE ZERO
      COSD2=.5*CSALFA
      IF(R(1).NE.0.) GO TO 1105
C ----- R(1)=0.
      DO 1106 I=2,NP3
      Y(I)=SQRT(ABS(Y(I) /COSD2))
1106 R(I)=Y(I)*CSALFA
      YI=SQRT(ABS(YI/COSD2))
      YN15=SQRT(ABS(YN15/COSD2))
      GO TO 1107
C ----- R(1) NE 0.
1105 R1D2=.5*R(1)
      R1D2SQ=R1D2*R1D2
      DO 1104 I=2,NP3
      Y(I)=Y(I)/(R1D2+SQRT(ABS(R1D2SQ+COSD2*Y(I))))
1104 R(I)=R(1)+Y(I)*CSALFA
      YI=YI/(R1D2+SQRT(ABS(R1D2SQ+COSD2*YI)))
      YN15=YN15/(R1D2+SQRT(ABS(R1D2SQ+COSD2*YN15)))
1107 YE=Y(NP3)-YN15
      GO TO 1120
C ----- CSALFA EQ ZERO
1110 DO 1111 I=2,NP3
      Y(I)=Y(I)/R(1)
1111 R(I)=R(1)
      YI=YI/R(1)
      YN15=YN15/R(1)
      YE=Y(NP3)-YN15
C ----- FILL RP ARRAY-----
1120 DO 1121 I=2,NP1
1121 RP(I)=.5*(R(I)+R(I+1))
      RP(2)=R(1)+YI*CSALFA
      RP(NP1)=R(1)+YN15*CSALFA
      RETURN
C***** S T R I D E 2 *****
C ----- PRELIMINARIES FOR COEFFICIENTS
C ----- BPI AND BPE -----
2000 IF(KIN.EQ.1) GO TO 3350
      IF(KRAD.EQ.0) BPI=.33333+.66667*RU(1)/RU(2)
      IF(KRAD.EQ.1) BPI=(R(1)*(5.*RU(1)+RU(2))+3.*RP(2)*
1 (RU(1)+RU(2)))/6./(R(1)+RP(2))/RU(2)
3350 IF(KEX.EQ.1) GO TO 2001
      IF(KRAD.EQ.0) BPE=.33333+.66667*RU(NP3)/RU(NP1)
      IF(KRAD.EQ.1) BPE=(R(NP3)*(5.*RU(NP3)+RU(NP1))+3.*RP(NP1)*
1 (RU(NP3)+RU(NP1)))/6./(R(NP3)+RP(NP1))/RU(NP1)
2001 PX=PE1/DX
      G=RM1-RME
      PD8=.125*PX
      PD4=.25*PX
      PG=PX+G
      PGD8=.125*PG
      PGD4=PGD8+PGD8
      RMID2=.5*RM1
      GD4=.25*G
C -----START OF J LOOP
      DO 3340 J=1,NEQ
      BOMP=OM(3)-OM(2)

```

```

PGOMP=PGD4*BOMP
P4OMP=PD4*BOMP
C ----- GRID POINT 2
  T=0.
  FDIFI=0.
  GI=0.
  T1F=0.
  T1S=0.
2002      HLP=RMID2-GD4*(OM(2)+OM(3))
          AHLP=ABS(HLP)
          THLP=HLP+HLP
          TP=EM U(2)
C ----- BOUNDARY COEFFICIENTS FOR F'S
  CALL SOURCE(J)
C ----- GRID POINT 2
  TPF2=TP/PREF(J,2)
  TTPF=TPF2+AHLP+ABS(TPF2-AHLP)
  IF(KIN.NE.1) GO TO 2304
  IF(J.EQ.JAX) CALL WF(JAX,1,2,3,BPI,T1F)
  IF(J.EQ.JSW) CALL WF(JSW,1,2,3,T,T1S)
  IF(J.NE.JAX.AND.J.NE.JSW) CALL WF(J,1,2,3,FDIFI,T1F)
2304 IF(J.EQ.JSW) GO TO 2305
      IF(INDI(J).EQ.2) GO TO 2303
2302 ADF=TTPF-THLP-T1F-PGOMP+.5*SD(2)
      BDF=2.*(T1F+RMI)
      DF=ADF+BDF+PX*BOMP-2.*SD(2)
      T=-T1F*FDIFI
      GO TO 2305
2303 ADF=TTPF-THLP-PGOMP+.5*SD(2)
      BDF=0.
      DF=ADF+PX*BOMP-2.*SD(2)+RMI*2.
      T=RMI*F(J,1)+AJI*R(1)
2305 TT=3.*F(J,2)+F(J,3)
      CDF=P4OMP*TT+2.*(T+SU(2))
      IF(J.NE.JSW) GO TO 2306
C ----- COEFFICIENTS FOR SWIRL EQUATION
  T1SR=R(2)/RP(2)
  T1SR=T1S*T1SR*T1SR
  ADF=RP(2)/R(3)
  ADF=ADF*ADF
  ADF=TTPF*(2.*ADF-1.)-T1SR-PGOMP-THLP
  BDF=2.*(T1S+RMI)
  DF=TTPF+T1SR+3.*PGOMP+THLP
2306 A(2)=ADF/DF
      B(2)=BDF/DF
      C(2)=CDF/DF
C ----- GRID POINT NP2
  FDIFE=0.
  GE=0.
  TNP3F=0.
  TNP3S=0.
2310 BOMM=OM(NP2)-OM(NP1)
      HLM=RMID2-GD4*(OM(NP1)+OM(NP2))
      AHLM=ABS(HLM)
      THLM=HLM+HLM
      TM=EM U(NP1)
      PGOMM=PGD4*BOMM
      P4OMM=PD4*BOMM
      TMF=TM/PREF(J,NP1)

```

```

      TTMF=TMF+AHLM+ABS(TMFAHLM)
      IF(KEX.NE.1) GO TO 2314
      IF(J.EQ.JAX) CALL WF(JAX,NP3,NP2,NP1,BPE,TNP3F)
      IF(J.EQ.JSW) CALL WF(JSW,NP3,NP2,NP1,T,TNP3S)
      IF(J.NE.JAX.AND.J.NE.JSW) CALL WF(J,NP3,NP2,NP1,FDIFE,TNP3F)
2314  IF(J.EQ.JSW) GO TO 2315
      IF(INDE(J).EQ.2) GO TO 2313
2312  ADF=2.*(TNP3F-RME)
      BDF=TTMF+THLM-TNP3F-PGOMM+.5*SD(NP2)
      DF=ADF+BDF+PX*BOMM-2.*SD(NP2)
      T=-TNP3F*FDIFE
      GO TO 2315
2313  ADF=0.
      BDF=TTMF+THLM-PGOMM+.5*SD(NP2)
      DF=BDF+PX*BOMM-2.*SD(NP2)-RME*2.
      T=-RME*F(J,NP3)-AJE*R(NP3)
2315  TT=3.*F(J,NP2)+F(J,NP1)
      CDF=P40MM*TT+2.*(T+SU(NP2))
      IF(J.NE.JSW) GO TO 2316
C ----- COEFFICIENTS FOR SWIRL EQUATION
      TNP3SR=R(NP3)/R(NP1)
      TNP3SR=TNP3S*TNP3SR**2
      ADF=2.*(TNP3S-RME)
      BDF=RP(NP1)/R(NP1)
      BDF=BDF*BDF
      BDF=TTMF*BDF-TNP3SR+THLM-PGOMM
      DF=RP(NP1)/R(NP2)
      DF=DF*DF
      DF=TTMF*DF+TNP3SR+3.*PGOMM-THLM
2316  A(NP2)=ADF/DF
      B(NP2)=BDF/DF
      C(NP2)=CDF/DF
C ----- COEFFICIENTS
3000  DO 3005 I=3,NP1
      BOMM=BOMP
      BOMP=OM(I+1)-OM(I)
      BOM=BOMM+BOMP
      BOMT3=BOM*3.
      PGOMM=PGOMP
      PGOMP=PGD4*BOMP
      PBOM=PX*BOM
      RM=RP(I-1)
      THLM=THLP
      HLP=RMID2-GD4*(OM(I+1)+OM(I))
      THLP=HLP+HLP
      AHLP=ABS(HLP)
      TTMF=TPPF
      TPF=EM U(I)/PREF(J,I)
      TTPF=TPF+AHLP+ABS(TPF-AHLP)
      IF(J.EQ.JSW) GO TO 3002
      AD=TTPF-THLP-PGOMP
      BD=TTMF+THLM-PGOMM
      DF=AD+BD+PBOM-2.*SD(I)
3002  CONTINUE
      CD=PD4*(BOMT3*F(J,I)+BOMP*F(J,I+1)+BOMM*F(J,I-1))
      CD=CD+2.*SU(I)
      IF(J.NE.JSW) GO TO 3003
C ----- COEFFICIENTS FOR SWIRL EQUATION
      AD=RP(I)/R(I+1)

```

```

AD=AD*AD
AD=TTPF*AD-THLP-PGOMP
DF=RP(1)/R(1)
DF=DF*DF
DS=TTPF*DF+THLP-THLM+PGD4*BOMT3
DF=RM/R(1)
DF=DF*DF
IF(1.EQ.3) GO TO 3004
BD=RM/R(1-1)
BD=BD*BD
BD=TTMF*BD+THLM-PGOMM
DF=DS+TTMF*DF
GO TO 3003
3004 BD=TTMF+THLM-PGOMM
DF=DS+TTMF*(2.*DF-1.)
3003 A(1)=AD/DF
B(1)=BD/DF
C(1)=CD/DF
3005 CONTINUE
C----- SOLVE FOR DOWNSTREAM F'S -----
3006 B(2)=B(2)*F(J,1)+C(2)
DO 3110 I=3,NP2
T=1./((1.-B(1))*A(I-1))
A(I)=A(I)*T
3110 B(I)=(B(I)*B(I-1)+C(I))*T
DO 3150 IDASH=2,NP2
I=N+4-IDASH
3150 F(J,I)=A(I)*F(J,I+1)+B(I)
C----- ADJUST F(J,1) AND F(J,NP3) AND CALCULATE WALL FLUXES
GO TO (3210,3220,3230),KIN
3210 FAV=.5*(F(J,2)+F(J,3))
IF(J.EQ.JAX) TAU1=T1F*FAV/R(1)
IF(J.EQ.JSW) TAU1S=T1F*FAV/RP(2)**2
IF(J.NE.JAX.AND.J.NE.JSW) GO TO 3211
GO TO 3240
3211 IF(INDI(J).EQ.1) AJ1=T1F/R(1)*(F(J,1)-FAV-FDIF1)
IF(INDI(J).EQ.2) F(J,1)=FDIF1+FAV+AJ1*R(1)/T1F
GO TO 3240
3220 IF(J.EQ.JAX.AND.RU(1).NE.0.) F(JAX,1)=F(JAX,1)-DPDX(1)*DX/RU(1)
GO TO 3240
3230 F(J,1)=.5*(F(J,2)+F(J,3))
3240 GO TO (3310,3320,3330),KEX
3310 FAV=.5*(F(J,NP2)+F(J,NP1))
IF(J.EQ.JAX) TAU3=TNP3F*FAV/R(NP3)
IF(J.EQ.JSW) TAU3S=TNP3S*FAV/RP(NP1)**2
IF(J.NE.JAX.AND.J.NE.JSW) GO TO 3311
GO TO 3340
3311 IF(INDE(J).EQ.1) AJE=TNP3F/R(NP3)*(FAV-F(J,NP3)+FDIFE)
IF(INDE(J).EQ.2) F(J,NP3)=FDIFE+FAV-AJE*R(NP3)/TNP3F
GO TO 3340
3320 IF(J.EQ.JAX.AND.RU(NP3).NE.0.) F(JAX,NP3)=F(JAX,NP3)-DPDX(NP3)*DX/
1 RU(NP3)
GO TO 3340
3330 F(J,NP3)=.5*(F(J,NP1)+F(J,NP2))
3340 XU=XD
PSII=PSII-RMI*DX
PSIE=PSIE-RME*DX
PEI=PSIE-PSII
ISTEP=ISTEP+1
RETURN
END

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SUBROUTINE SOURCE(J)
  DIMENSION DP(43)
  COMMON/GENRAL/ AJE(5),AJI(5),CSALFA,DPDX(43),DX,EMU(43),F(5,43),
1  FS(5,43),H,IFIN,INDE(5),INDI(5),ISTEP,ITEST,IUTRAP,JAX,JSW,
2  KEX,KIN,KRAD,N,NEQ,NPH,NP1,NP2,NP3,OM(43),PEI,PR(5),PREF(5,43),
3  PSIE,PSII,PY(43),R(43),RHO(43),RME,RMI,RP(43),RU(43),SNALFA,
4  SD(43),SU(43),TAUE,TAUES,TAUI,TAUIS,XD,XU,Y(43),YE,YI
  COMMON/GS1/ AK,EWALL,FR,JSTEP,KX,KTHETA,PRESS,UFAC
1  /CKEA/ ANGKIN,PYP
2  /OMDIF/ DOM(43)
  IF(J.EQ.JSW) GO TO 60
  IF(ISTEP.LT.JSTEP) GO TO 23
C  CROSS STREAM PRESSURES
  DP(1)=PY(1)
  DO 21 K=2,NP1
  DP(K)=PY(K)
  FAV=.5*(F(1,K)+F(1,K+1))
  RHM=.5*(RHO(K)+RHO(K+1))
C  SU(K) IS USED FOR (R(K)-R(K+1))
  SU(K)=R(K)-R(K+1)
  SD(K)=RHM*(FAV**2)/RP(K)
C  SD(K) IS USED FOR DP(K)/DR
  SD(K)=SD(K)/(RP(K)**2)
21  CONTINUE
  PYP=PRESS
  PY(NP2)=PRESS
19  CONTINUE
  PY(NP2)=PY(NP2)+.5*(PRESS-PYP)
  DO 20 I=2,NP1
  K=NP1-I+2
  PY(K)=PY(K+1)+SD(K)*SU(K)
  PY(K)=PY(K)*CSALFA
20  CONTINUE
  PY(1)=PY(2)
  PYP=0.
  DO 22 I=3,NP2
  PYP=PYP+(PY(I)+PY(I-1))*DOM(I)
22  CONTINUE
  PYP=.5*PYP
  IF(ABS(1.-PYP/PRESS).GT..0001) GO TO 19
23  CONTINUE
  DO 50 I=3,NP1
  IF(ISTEP.GT.JSTEP) DPDX(I)=(PY(I)-DP(I))/DX -SD(I)*SNALFA
  RDR=RP(I)**2-RP(I-1)**2
  SU(I)=-.5*DPDX(I)*RDR
50  SD(I)=0.
  IF(ISTEP.GT.JSTEP) DPDX(2)=(PY(2)-DP(2))/DX -SD(2)*SNALFA
  RDR=RP(2)**2-RP(1)**2
  SU(2)=-.5*DPDX(2)*RDR
  SD(2)=0.
  RDR=R(NP3)**2-RP(NP1)**2
  SU(NP2)=-.5*DPDX(NP2)*RDR
  SD(NP2)=0.
  RETURN
60  DO 61 I=1,NP3
  SU(I)=0.
61  SD(I)=0.
  RETURN
END

```



```

81 T1=SQRT(EMUP(JSW,I))
   ARDUDY=ABS(RP(I)*DUDYL)*RHM
   A2=YM*SQRT(ARDUDY)/(APSW*EMU(I))
   T2=EXP(-A2*T1)
   T3=1.-T2
   IF(ALT.GT.AK*YM*T3.AND.I.GT.N/4) ALT=AK*YM*T3
   EMUP(JSW,I)=ALT**2*ARDUDY
21 CONTINUE
   EMUP(JSW,I)=EMUP(JSW,I)+EMU(I)
   EMU(I)=EMUP(JSW,I)+EMUT
20 CONTINUE

```

C----- MODIFICATION OF EMU ARRAY -----

```

      DO 24 I=2,NP1
24  EM U(I)=EMU(I)/(Y(I+1)-Y(I))
      IF(KRAD.EQ.0) GO TO 25
      DO 26 I=2,NP1
26  EMU(I)=EMU(I)*RP(I)
25 CONTINUE
      RETURN
      END

```

SUBROUTINE WF (J,I1,I2,I3,OUT1,OUT2)

COMMON/GENRAL/ AJE(5),AJI(5),CSALFA,DPDX(43),DX,EMU(43),F(5,43),
 1 FS(5,43),H,IFIN,INDE(5),INDI(5),ISTEP,ITEST,IUTRAP,JAX,JSW,
 2 KEX,KIN,KRAD,N,NEQ,NPH,NP1,NP2,NP3,OM(43),PEI,PR(5),PREF(5,43),
 3 PSIE,PSII,PY(43),R(43),RHO(43),RME,RMI,RP(43),RU(43),SNALFA,
 4 SD(43),SU(43),TAUE,TAUES,TAUI,TAUIS,XD,XU,Y(43),YE,YI
 COMMON/GS1/ AK,EWALL,FR,JSTEP,KX,KTHETA,PRESS,UFAC

C IF(J.GT.1) GO TO 100
 I25=I3-1/I1

C ----- VELOCITY

RHOREF=.5*RHO(I1)+.25*(RHO(I2)+RHO(I3))
 VREF=EMU(I1)
 YREF=YI+(YE-YI)*OM(I1)
 RREF=RP(I25)
 RRUREF=RREF*RU(I25)
 EM U(I25)=VREF*RREF/ABS(Y(I3)-Y(I2))
 100 UREF=.5*(F(J,I2)+F(J,I3))
 RE=UREF*RHOREF*YREF/VREF
 IF(J.EQ.JSW) GO TO 105

C ----- AXIAL MOMENTUM

F1=YREF*DPDDX/RHOREF/UREF/UREF
 S=1./RE -F1/2.
 OUT1=.5-F1*RE/12.
 103 OUT2=S*RRUREF
 RETURN

C ----- SWIRL MOMENTUM

105 CONTINUE
 OUT1=0.
 OUT2=0.
 IF(ISTEP.LE.JSTEP) RETURN
 RE=RE/RREF
 RRAT=R(NP3)/RREF
 S=2./RE*1./((1.+RRAT)
 RETURN
 END