

Two-dimensional SCFTs from D3-branes

Francesco Benini⁽¹⁾, Nikolay Bobev⁽²⁾ and P. Marcos Cricigno⁽³⁾

⁽¹⁾ Blackett Laboratory, Imperial College London
South Kensington Campus, London SW7 2AZ, United Kingdom

⁽²⁾ Instituut voor Theoretische Fysica, KU Leuven
Celestijnenlaan 200D, B-3001 Leuven, Belgium

⁽³⁾ Institute for Theoretical Physics and Spinoza Institute, Utrecht University
Leuvenlaan 4, 3854 CE Utrecht, The Netherlands

f.benini@imperial.ac.uk, nikolay@itf.fys.kuleuven.be, p.m.crichigno@uu.nl

Abstract

We find a large class of two-dimensional $\mathcal{N} = (0, 2)$ SCFTs obtained by compactifying four-dimensional $\mathcal{N} = 1$ quiver gauge theories on a Riemann surface. We study these theories using anomalies and c -extremization. The gravitational duals to these fixed points are new AdS_3 solutions of IIB supergravity which we exhibit explicitly. Along the way we uncover a universal relation between the conformal anomaly coefficients of four-dimensional and two-dimensional SCFTs connected by an RG flow across dimensions. We also observe an interesting novel phenomenon in which the superconformal R-symmetry mixes with baryonic symmetries along the RG flow.

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1 Introduction

Understanding the space of consistent conformal field theories (CFTs) is of great importance since this would provide insight into a classification of the possible phases of quantum field theories. One can hope that this hard problem becomes more manageable if one introduces additional symmetries, such as supersymmetry, to restrict the class of possible theories. In two spacetime dimensions there is a further simplification since the conformal group is infinite-dimensional. Despite this favorable circumstance the classification of two-dimensional superconformal field theories (SCFTs) is far from complete. In view of this it is important to understand the space of consistent two-dimensional SCFTs and to sharpen our tools to study such theories. The goal of this work is to provide evidence for the existence of a novel class of 2d SCFTs with $\mathcal{N} = (0, 2)$ supersymmetry which arise from the twisted compactification of 4d SCFTs on a Riemann surface and to employ a variety of techniques to understand their physics.

Two-dimensional CFTs are also very interesting for a different reason. Gravity in three-dimensional asymptotically AdS space is supposed to be one of the simplest toy models for understanding quantum gravity; see for example [1]. Thus constructing and classifying possible AdS_3 solutions of string theory, and understanding their holographic duals, is of great importance for uncovering the structure of quantum gravity in three dimensions. In addition to that gravitational theories in AdS_3 spaces also provide good laboratories to test and explore the AdS/CFT correspondence in detail – in fact this system was the precursor of holography [2]. These two alternative vantage points provide a different perspective and further motivation for the work presented here.

Our goal is to study four-dimensional superconformal field theories (SCFTs) with $\mathcal{N} = 1$ supersymmetry compactified on a Riemann surface with a partial topological twist. The main tools we use are anomalies, c -extremization, and holography. The basic idea is simple and dates back to the work of Witten [3]. On a general curved manifold supersymmetry will be broken because there are no covariantly constant spinors. If however the supersymmetric QFT at hand has a continuous R-symmetry one can turn on a background field for it which cancels the spin connection on the curved manifold. This procedure of preserving supersymmetry on curved spaces is called the “topological twist.” We will be interested in studying four-dimensional $\mathcal{N} = 1$ theories on $\mathbb{R}^2 \times \Sigma_{\mathfrak{g}}$ where $\Sigma_{\mathfrak{g}}$ is a smooth Riemann surface of genus $\mathfrak{g}$. Since the four-dimensional theory has a $U(1)_R$ R-symmetry and the structure group of $\Sigma_{\mathfrak{g}}$ is $SO(2)$ we generically preserve $(0, 2)$ supersymmetry on $\mathbb{R}^2$ and thus at energies below the scale set by the size of the Riemann surface we have a two-dimensional supersymmetric field theory. These two-dimensional theories are the main subject of our work. In particular, we will argue that generically these theories will be superconformal and by using the anomalies of the four-dimensional theory we will be able to calculate the anomalies of its two-dimensional “offspring.” An interesting generalization is possible if the four-dimensional theory has continuous flavor symmetries. Then supersymmetry is preserved even when one turns on background magnetic gauge fields on the Riemann surface for these symmetries. In this way from a single four-dimensional SCFT one can obtain a multi-parameter family of candidate two-dimensional theories labeled by the genus of the Riemann surface and the choice of background magnetic flavor fluxes. Since the magnetic flux on a compact Riemann surface must be appropriately quantized this leads to a discrete family of theories. While anomalies provide a powerful calculational tool they are not always well-suited to answering dynamical questions so in general it is hard to rigorously argue that the two-dimensional SCFTs in question actually exist. One possible approach to remedy this situation is to employ holography and construct explicit AdS_3 vacua which are holographic duals to the SCFTs of interest. This is often possible if the parent four-dimensional theory has itself a holographic dual description as we demonstrate explicitly.

These general ideas were made very concrete in [4–8] where they were applied to the case of four-dimensional $\mathcal{N} = 4$ SYM theory. Here we argue that the setup is much more general and provide evidence for this claim by analyzing in detail the $Y^{p,q}$ family of superconformal quiver gauge theories [9]. Using the knowledge of the ’t Hooft anomalies for these theories we calculate the central charges of the two-dimensional theories obtained from them upon twisted compactification on $\mathbb{R}^2 \times \Sigma_g$. An important role in this analysis is played by c -extremization [7, 8] which is a tool that allows us to unambiguously determine the superconformal R-symmetry in two dimensions and thus the correct conformal anomalies. The reason we choose this class of theories is that they have explicit AdS_5 holographic duals, constructed in [10]. This provides us with the reasonable expectation that the two-dimensional SCFTs will also have weakly-coupled duals in type IIB supergravity. This expectation indeed bears fruit and we are able to construct new explicit warped $\text{AdS}_3 \times_w \mathcal{M}_7$ solutions of IIB supergravity which are dual to the 2d SCFTs of interest.

A novel phenomenon that arises from the study of this class of field theories is that the R-symmetry generically mixes along the RG flow not only with usual mesonic flavor symmetries, but also with the baryonic flavor symmetry available in all $Y^{p,q}$ quivers. This is rather surprising from the supergravity perspective because unlike mesonic symmetries, the baryonic symmetry does not arise from isometries of the metric, but from the RR five-form flux on a topological three-cycle.

Finally, we should point out that the AdS_3 solutions we construct can be thought of as the near-horizon limit of BPS black strings in five dimensions. The entropy density of these black strings is related to the central charge of the dual 2d CFT and thus our successful match of the supergravity and field theory central charges can also be viewed as a microscopic counting of the degrees of freedom of the black string.

The ideas and techniques discussed in this paper are similar to the ones employed by Maldacena-Núñez in [6] as well as in the more recent literature [7, 8, 11–13], see also [14, 15] for relevant recent work. The supersymmetric AdS_3 solutions of type IIB supergravity we find have only five-form flux turned on. These backgrounds fall under the classification of [16] and indeed some of our solutions have been studied previously in [17–23]. More recently, AdS_3 solutions arising from string and M-theory have also been analyzed in [24–29]. On the field theory side there have been interesting constructions of 2d $\mathcal{N} = (0, 2)$ SCFTs and dualities between them by employing compactifications of a higher-dimensional SCFT in [30–35].

We begin our exploration in the next section with a brief review of the $Y^{p,q}$ quiver gauge theories and we then proceed to compactify these theories on a Riemann surface and study the system at low energies. We also discuss a universal feature of RG flows connecting four-dimensional $\mathcal{N} = 1$ and two-dimensional $\mathcal{N} = (0, 2)$ SCFTs. As an illustration of this general

result in Section 2.4 we consider the four-dimensional $\mathcal{N} = 1$ SCFTs arising from D3-branes at del Pezzo singularities. In Section 3 we switch gears and discuss the construction of explicit AdS_3 solutions of type IIB supergravity which are holographic duals to the two-dimensional SCFTs of interest. We conclude in Section 4 with a short summary and a number of directions for future work. In the various appendices we present technical details which pertain to the construction and analysis of the supergravity solutions discussed in Section 3.

2 Field theory

2.1 $Y^{p,q}$ quivers

Let us first summarize some of the salient features of the $Y^{p,q}$ family of four-dimensional $\mathcal{N} = 1$ superconformal field theories. We will follow the notation and conventions of [9] and take the coprime integers p, q to satisfy, $p > 0$ and $0 \leq q \leq p$. The theories are quiver gauge theories, with $2p$ nodes each representing an $SU(N)$ gauge group. The matter fields are in chiral multiplets and transform in bifundamental representations of pairs of gauge groups, as dictated by the quiver diagram. The theories have an $SU(2)_1 \times U(1)_2 \times U(1)_B \times U(1)_R$ continuous global symmetry, where $SU(2)_1 \times U(1)_2$ is a mesonic flavor symmetry (and we denote the Cartan of $SU(2)_1$ with $U(1)_1$), $U(1)_B$ is a baryonic symmetry and $U(1)_R$ is the superconformal R-symmetry. The matter fields can be organized into four groups, dubbed $\{Y, Z, U^\alpha, V^\alpha\}$ with $\alpha = 1, 2$, according to their charges under the global symmetry as we summarize in the following table:

Fields	multiplicity	$U(1)_1$	$U(1)_2$	$U(1)_R$	$U(1)_B$
Y	$p + q$	0	-1	R_Y	$p - q$
Z	$p - q$	0	1	R_Z	$p + q$
U^1	p	1	0	R_U	$-p$
U^2	p	-1	0	R_U	$-p$
V^1	q	1	1	R_V	q
V^2	q	-1	1	R_V	q
λ	$2p$	0	0	1	0

(2.1)

By λ we denoted the gaugini in vector multiplets, transforming in the adjoint representation of the gauge groups. The R-charges of the matter chiral multiplets are

$$\begin{aligned}
R_Y &= \frac{(2p - q)w + 2pq - w^2}{3q^2}, & R_U &= \frac{4p^2 - 2pw}{3q^2}, \\
R_Z &= \frac{(2p + q)w - 2pq - w^2}{3q^2}, & R_V &= \frac{3q - 2p + w}{3q},
\end{aligned}
\tag{2.2}$$

where we have defined

$$w \equiv \sqrt{4p^2 - 3q^2} . \quad (2.3)$$

One should keep in mind that the fermions in chiral multiplets have R-charge 1 less than that of the multiplet. When $w \in \mathbb{Z}$, the central charges of the 4d theory are rational.

The conformal anomaly coefficients, or central charges, a and c of the $Y^{p,q}$ theories, can be computed using the well-known relation [36] between conformal and R-symmetry 't Hooft anomalies in $\mathcal{N} = 1$ SCFTs:

$$a = \frac{9}{32} \text{Tr}(R^3) - \frac{3}{32} \text{Tr}(R) , \quad c = \frac{9}{32} \text{Tr}(R^3) - \frac{5}{32} \text{Tr}(R) . \quad (2.4)$$

Using the charges in (2.1) and (2.2), one finds

$$a(Y^{p,q}) + \frac{3p}{8} = c(Y^{p,q}) + \frac{p}{4} = \frac{3p^2(3q^2 - 2p^2 + pw)}{4q^2(2p + w)} N^2 . \quad (2.5)$$

This is obtained by noticing that the bifundamentals have implicit multiplicity N^2 , while the gaugini have multiplicity $N^2 - 1$. In particular if some chiral multiplet is in the adjoint rather than in the bifundamental, the implicit multiplicity is $N^2 - 1$ and the $\mathcal{O}(1)$ terms are different. At leading order in N , the two central charges are equal because for this class of quiver gauge theories and at that order, the linear R-symmetry 't Hooft anomaly vanishes: $\text{Tr } R = \mathcal{O}(1)$.

There are some cases of special interest. The theory $Y^{p,0}$ is a $\mathbb{Z}_p$ orbifold of the Klebanov-Witten (KW) theory [37] and has central charges

$$a(Y^{p,0}) \simeq c(Y^{p,0}) \simeq \frac{27p}{64} N^2 , \quad (2.6)$$

at leading order in N . The theory $Y^{p,p}$ is a $\mathbb{Z}_p$ orbifold of the $\mathcal{N} = 2$ quiver theory which itself is obtained by a $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM. The central charges for this theory are

$$a(Y^{p,p}) \simeq c(Y^{p,p}) \simeq \frac{p}{2} N^2 \simeq 2p a_{\mathcal{N}=4} , \quad (2.7)$$

where in the last equality we have emphasized the relation to the central charge of $\mathcal{N} = 4$ SYM at leading order.

It is worth collecting here the explicit expressions for the linear and cubic 't Hooft anomalies for the quiver gauge theories of interest. After a straightforward algebraic calculation one finds that the 20 independent cubic 't Hooft anomalies are

$$\begin{aligned} k_{111} &= k_{122} = k_{12B} = k_{12R} = k_{1BB} = k_{1BR} = k_{1RR} = 0 , \\ k_{112} &= 2qN^2 , \quad k_{11B} = 2(q^2 - p^2)N^2 , \quad k_{11R} = \frac{2}{3q^2}(pw^2 + (q^2 - 2p^2)w - 2pq^2)N^2 \\ k_{222} &= k_{2RR} = 0 , \quad k_{22B} = 2p^2N^2 , \quad k_{2BB} = 2p^2qN^2 , \\ k_{22R} &= \frac{2}{3q^2}(2p^2 + pq + q^2)(w - 2p)N^2 , \quad k_{2BR} = \frac{2p^2}{3q}(w - 2p)N^2 \\ k_{BBB} &= k_{BRR} = 0 , \quad k_{BBR} = -\frac{2}{3}p^2(p + w)N^2 , \quad k_{RRR} = \frac{8p^2}{9q^4}(w^3 + 9pq^2 - 8p^3)N^2 - 2p . \end{aligned} \quad (2.8)$$

The linear 't Hooft anomalies are

$$k_1 = k_2 = k_B = 0, \quad k_R = -2p. \quad (2.9)$$

The identity $9k_{JRR} = k_J$ valid for any global non-R symmetry in a 4d $\mathcal{N} = 1$ SCFT is clearly obeyed [38]. As pointed out in [9] for a baryonic symmetry we must have that $k_{BBB} = k_B = 0$. For general flavor symmetries this is not necessary, but the $Y^{p,q}$ theories seem to be a special case since $k_{111} = k_{222} = k_1 = k_2 = 0$.

2.2 2d central charges

In this section we consider compactifications of generic four-dimensional $\mathcal{N} = 1$ field theories on compact (*i.e.* with no punctures) Riemann surfaces $\Sigma_{\mathfrak{g}}$ of genus $\mathfrak{g}$, performing a partial topological twist so as to preserve $\mathcal{N} = (0, 2)$ supersymmetry in two dimensions. Under the assumption that the theories flow to interacting SCFTs (which could be tested holographically, for instance), we would like to compute their central charges. To do this, we exploit the fact that in two-dimensional $\mathcal{N} = (0, 2)$ SCFTs the R-symmetry can be identified by a c -extremization principle [7, 8], and then the central charges are related to its 't Hooft anomalies. We begin by providing explicit examples in the case of $Y^{p,q}$ quivers and then discuss an approach for generic four-dimensional $\mathcal{N} = 1$ field theories.

The calculation proceeds as in [7]. To perform the partial topological twist, we turn on a background gauge field along the generator

$$T = b_1 T_1 + b_2 T_2 + B T_B + \frac{\kappa}{2} T_R, \quad (2.10)$$

where $T_{1,2}$, T_B are the generators of $U(1)_{1,2}$, $U(1)_B$, respectively and T_R is the generator of the $U(1)_R$ *superconformal* R-symmetry. We have defined κ as the normalized curvature of the Riemann surface: $\kappa = 1$ for $\mathfrak{g} = 0$; $\kappa = 0$ for $\mathfrak{g} = 1$; and $\kappa = -1$ for $\mathfrak{g} > 1$. When the flavor flux b_1 is nonzero, the $SU(2)_1$ flavor symmetry of the system is broken to $U(1)_1$. For $b_1 = 0$ the $SU(2)_1$ symmetry is intact.

An important point is that the background flux (2.10) must be properly and carefully quantized. We turn on an external flux

$$F = T d\text{vol}_{\Sigma_{\mathfrak{g}}}, \quad (2.11)$$

where the volume form is normalized $\int d\text{vol}_{\Sigma_{\mathfrak{g}}} = 2\pi\eta_{\Sigma}$ and $\eta_{\Sigma} = 2|\mathfrak{g} - 1|$ for $\mathfrak{g} \neq 1$, $\eta_{\Sigma} = 1$ for $\mathfrak{g} = 1$. Then for every gauge invariant operator $\mathcal{O}$, the effective number n of flux units felt by the associated particles and defined by

$$\frac{1}{2\pi} \int_{\Sigma_{\mathfrak{g}}} F \cdot \mathcal{O} = \eta_{\Sigma} T \cdot \mathcal{O} \equiv n \mathcal{O}, \quad (2.12)$$

should be an integer: $n \in \mathbb{Z}$. This is the standard Dirac quantization condition. Since we have fixed the origin of the flavor flux around the 4d superconformal R-symmetry, which in the case of $Y^{p,q}$ quivers typically assigns irrational charges, one generically needs an irrational flavor flux to balance it. In particular, zero flavor flux is generically not allowed unless the superconformal R-charges are rational. When a twist by the pure superconformal R-symmetry is in fact possible, we refer to it as the “universal twist,” for reasons that will become clear below.

Next, we define the trial 2d R-symmetry to be a general linear combination of the 4d R-symmetry and the Abelian flavor symmetries, *i.e.*,

$$T_{\text{tr}} = \epsilon_1 T_1 + \epsilon_2 T_2 + \epsilon_B T_B + T_R , \quad (2.13)$$

where the real parameters ϵ_i ’s are unfixed at the moment and we construct the trial central charge

$$c_r^{\text{tr}} = -3\eta_\Sigma \sum_{\sigma} m_{\sigma} t_{\sigma} (q_R^{(\sigma)})^2 . \quad (2.14)$$

The sum above is over the 4d fermionic fields labelled by σ , m_{σ} is their multiplicity, $q_R^{(\sigma)}$ is the charge under the trial R-symmetry in (2.13), and t_{σ} is the charge under the background gauge field in (2.10). Here we have used the relation $c_r^{\text{tr}} = 3k_{RR}$ (see [7, 8] for details) and that the net number of right-moving minus left-moving 2d chiral massless fermions is computed by the index theorem:

$$n_r^{(\sigma)} - n_{\ell}^{(\sigma)} = -t_{\sigma} \eta_{\Sigma} , . \quad (2.15)$$

For the case of $Y^{p,q}$ quivers, the various parameters are summarized in the following table:

Fields	m_{σ}	t_{σ}	$q_R^{(\sigma)}$
Y	$(p+q)N^2$	$\frac{\kappa}{2}(R_Y - 1) - b_2 + B(p-q)$	$R_Y - 1 - \epsilon_2 + \epsilon_B(p-q)$
Z	$(p-q)N^2$	$\frac{\kappa}{2}(R_Z - 1) + b_2 + B(p-q)$	$R_Z - 1 + \epsilon_2 + \epsilon_B(p-q)$
U^1	pN^2	$\frac{\kappa}{2}(R_U - 1) + b_1 - Bp$	$R_U - 1 + \epsilon_1 - \epsilon_B p$
U^2	pN^2	$\frac{\kappa}{2}(R_U - 1) - b_1 - Bp$	$R_U - 1 - \epsilon_1 - \epsilon_B p$
V^1	qN^2	$\frac{\kappa}{2}(R_V - 1) + b_2 + b_1 + Bq$	$R_V - 1 + \epsilon_2 + \epsilon_1 + \epsilon_B q$
V^2	qN^2	$\frac{\kappa}{2}(R_V - 1) + b_2 - b_1 + Bq$	$R_V - 1 + \epsilon_2 - \epsilon_1 + \epsilon_B q$
λ	$2p(N^2 - 1)$	$\frac{\kappa}{2}$	1

(2.16)

We recall that if some chiral multiplet is in the adjoint, rather than bifundamental (as in the $Y^{1,1}$ theory) then the total multiplicity is proportional to $N^2 - 1$, rather than N^2 .

At this point we invoke the principle of c -extremization, stating that the 2d superconformal R-symmetry is the one extremizing the trial central charge (2.14), whose value at the extremum is the actual right-moving central charge c_r of the 2d SCFT. With the ingredients given above, these can be calculated for any $Y^{p,q}$ quiver, Riemann surface, and background fluxes. In full

generality the result is lengthy, so in the following subsections we discuss some cases of particular interest. When carrying out the extremization procedure, one must often treat the cases $\kappa = 0$ ($\mathfrak{g} = 1$) and $\kappa \neq 0$ ($\mathfrak{g} \neq 1$) separately, as we do below.

2.2.1 $Y^{p,0}$ on $\Sigma_{\mathfrak{g} \neq 1}$

We begin with the special case $q = 0$. For $p = 1$ this corresponds to the KW theory, while for general values of p we have a $\mathbb{Z}_p$ orbifold of it that preserves $\mathcal{N} = 1$ supersymmetry. Assuming $\kappa \neq 0$ (and thus $\kappa^2 = 1$) the trial central charge is extremized at

$$\begin{aligned} \epsilon_1 &= \frac{2b_1\kappa(16b_2^2 - (4Bp - \kappa)^2)}{1 - 8(b_1^2 + b_2^2 + 2B^2p^2) + 32Bp\kappa(b_1^2 - b_2^2)} , \\ \epsilon_2 &= \frac{2b_2\kappa(16b_1^2 - (4Bp + \kappa)^2)}{1 - 8(b_1^2 + b_2^2 + 2B^2p^2) + 32Bp\kappa(b_1^2 - b_2^2)} , \\ \epsilon_B &= \frac{2[2(b_1^2 - b_2^2) - Bp\kappa(1 + 8b_1^2 + 8b_2^2 - 16B^2p^2)]}{p(1 - 8(b_1^2 + b_2^2 + 2B^2p^2) + 32Bp\kappa(b_1^2 - b_2^2))} . \end{aligned} \quad (2.17)$$

We note, rather surprisingly, that even when the background baryonic flux B vanishes, we have $\epsilon_B \neq 0$ and thus the two-dimensional superconformal R-symmetry is mixed with the baryonic symmetry. Only when the flavor fluxes $b_{1,2}$ are also set to zero there is no mixing and the two-dimensional and four-dimensional R-symmetry coincide. This is a generic feature of all the examples we will discuss below.

Evaluating the trial central charge at the extremum we find

$$c_r = -3p\kappa\eta_\Sigma \left[\frac{3 - 16(b_1^2 + b_2^2 + 2B^2p^2) - 256(b_1^2b_2^2 + B^4p^4 - B^2p^2(b_1^2 + b_2^2))}{4(1 - 8(b_1^2 + b_2^2 + 2B^2p^2) + 32Bp\kappa(b_1^2 - b_2^2))} N^2 - 1 \right] . \quad (2.18)$$

An interesting case is obtained by setting the mesonic flavor fluxes to zero, *i.e.*, $b_1 = b_2 = 0$, for which

$$c_r = -3p\kappa\eta_\Sigma \left[\frac{1}{4}(3 + 16B^2p^2)N^2 - 1 \right] , \quad (2.19)$$

this can be positive only for $\kappa = -1$.

Another useful specialization is obtained by setting $B = 0$, for which

$$c_r = -3p\kappa\eta_\Sigma \left[\frac{3 - 16(b_1^2 + b_2^2 + 16b_1^2b_2^2)}{4(1 - 8(b_1^2 + b_2^2))} N^2 - 1 \right] . \quad (2.20)$$

Interestingly, there is a region in the $\{b_1, b_2\}$ plane where c_r is positive for both $\kappa = 1$ and $\kappa = -1$.

Finally, we note that setting $B = b_1 = b_2 = 0$ (*i.e.* when the twist is purely along the superconformal R-symmetry in the UV), which requires $\kappa = -1$ one has

$$c_r = \frac{32}{3}(\mathfrak{g} - 1)a(Y^{p,0}) - 2p(\mathfrak{g} - 1) , \quad (2.21)$$

where $a(Y^{p,0})$ is the 4d central charge of the $Y^{p,0}$ theory, given in (2.6). We will see that this simple relation between the 2d central charge and the 4d anomaly coefficient a is a *universal* feature that holds for a large class of theories justifying the name “universal twist”.

Before moving to other examples, let us analyze the $Y^{p,0}$ theory, with $\kappa = -1$ in more detail since this is one of the examples that we will revisit holographically in Section 3. Specifically, we set $b_1 = b_2 = 0$, but admit a nonzero baryonic flux B . The R-charges of the fields (Y, Z, U^1, U^2) are $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and the baryonic charges are $(p, p, -p, -p)$, respectively. The baryonic charges are such that all gauge-invariant operators have integer charges. It is easy to see that requiring that the background R-flux (2.10) is quantized leads to the quantization condition

$$pNB = \frac{h}{4(\mathfrak{g} - 1)} N_B, \quad (2.22)$$

where N_B is an integer and $h = \text{hcf}[2, N(\mathfrak{g} - 1)]$. To leading order in N equation (2.19) can be written as

$$c_r = \frac{32}{3}(\mathfrak{g} - 1)a(Y^{p,0}) + \frac{3ph^2N_B^2}{2(\mathfrak{g} - 1)} - 2p(\mathfrak{g} - 1), \quad (2.23)$$

with $a(Y^{p,0})$ given in (2.6). We will reproduce this result holographically in Section 3.

2.2.2 $Y^{p,0}$ on T^2

Setting $\kappa = 0$ and when all background fluxes are non-vanishing one finds

$$\begin{aligned} \epsilon_1 &= \frac{b_1(b_2^2 - B^2p^2)}{Bp(b_1^2 - b_2^2)}, \\ \epsilon_2 &= \frac{b_2(b_1^2 - B^2p^2)}{Bp(b_1^2 - b_2^2)}, \\ \epsilon_B &= \frac{2B^2p^2 - (b_1^2 + b_2^2)}{2p(b_1^2 - b_2^2)}, \end{aligned} \quad (2.24)$$

which leads to the central charge

$$c_r = 6\eta_\Sigma \frac{(b_1^2 - B^2p^2)(b_2^2 - B^2p^2)}{B(b_1^2 - b_2^2)} N^2. \quad (2.25)$$

When $B = 0$ the trial central charge is linear in the $\epsilon_{1,2}$ parameters and one cannot apply c -extremization. When $B \neq 0$ but one of the fluxes $b_{1,2}$ vanishes we find

$$\begin{aligned} \epsilon_1 = 0, \quad \epsilon_2 = \frac{Bp}{b_2}, \quad \epsilon_B = \frac{1 - 2B^2p^2}{2pb_2^2}, \quad c_r = 6\eta_\Sigma \frac{Bp^2(b_2^2 - B^2p^2)}{b_2^2} N^2 \quad & b_1 = 0, b_2 \neq 0, \\ \epsilon_1 = -\frac{Bp}{b_1}, \quad \epsilon_2 = 0, \quad \epsilon_B = \frac{-1 + 2B^2p^2}{2pb_1^2}, \quad c_r = 6\eta_\Sigma \frac{Bp^2(-b_1^2 + B^2p^2)}{b_1^2} N^2 \quad & b_2 = 0, b_1 \neq 0. \end{aligned} \quad (2.26)$$

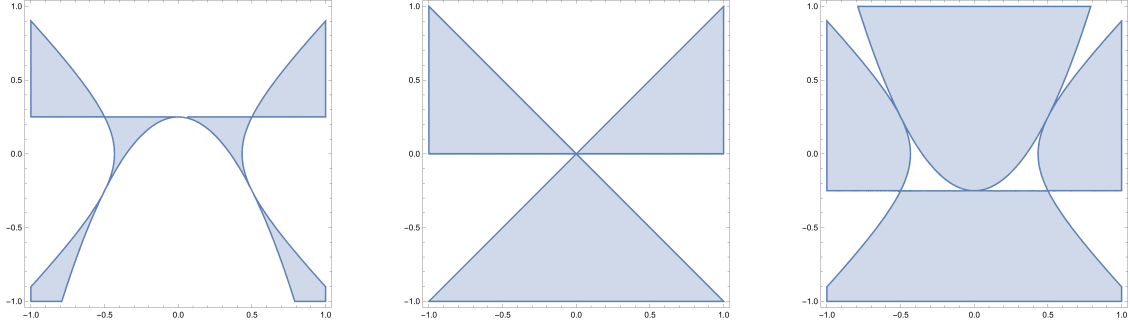


Figure 1: The region in the (b_2, pB) plane (in blue) for which the central charge for $Y^{p,0}$ with $b_1 = 0$ is positive. The cases $\kappa = \{1, 0, -1\}$ are presented from left to right and the horizontal and vertical axes represent b_2 and B respectively.

The case $b_1 = 0$ is special because the $SU(2)_1$ factor in the flavor symmetry is restored, and the analysis of Section 3 will focus on this case. As one can see from the expressions above, there are always values (in fact whole regions) for b_2, B such that the central charge is positive, for $\kappa = 0, \pm 1$. This is illustrated in Figure 1. A few comments about the boundaries of the colored regions in Figure 1 are in order. The special points $(b_2, pB) = (\pm 1/2, \pm 1/4)$ in the plots for $\kappa = \pm 1$ are excluded, as is the origin in the case $\kappa = 0$. For $\kappa = 1$ the point $(0, 1/4)$ is excluded. For $\kappa = -1$ the central charge is not analytic in the (b_2, pB) plane at the point $(0, -1/4)$. Setting first $b_2 = 0$ and then $pB = -1/4$ gives central charge $c_r = 1$, while taking the limit in the opposite order gives $c_r = 0$. All other points on the boundaries of the colored regions in Figure 1 are excluded.

2.2.3 $Y^{p,p}$ on $\Sigma_{g \neq 1}$

Another special case of interest is $q = p$. For $p = 1$ one has a quiver with two nodes with $\mathcal{N} = 2$ supersymmetry which is a $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM [39]. In this case the chiral field Y is in the adjoint. For general values of p we have a $\mathbb{Z}_p$ orbifold of this $\mathcal{N} = 2$ theory which now preserves only $\mathcal{N} = 1$ supersymmetry.

Assuming $\kappa \neq 0$, the trial central charge (2.14) is extremized for

$$\begin{aligned} \epsilon_1 &= -\frac{2b_1\kappa(6b_2 + \kappa)}{1 - 12(b_1^2 + b_2^2 + B^2p^2 + b_2Bp)} , \\ \epsilon_2 &= \frac{12b_2^2 - 2b_2\kappa - 8(b_1^2 + b_2^2 + B^2p^2 + b_2Bp)}{1 - 12(b_1^2 + b_2^2 + B^2p^2 + b_2Bp)} , \\ \epsilon_B &= \frac{4(b_1^2 + b_2^2 + B^2p^2 + b_2Bp) - (12b_2^2 + 12b_2Bp + 2Bp\kappa)}{p(1 - 12(b_1^2 + b_2^2 + B^2p^2 + b_2Bp))} , \end{aligned} \tag{2.27}$$

and the right-moving central charge reads

$$c_r = 3p\kappa\eta_\Sigma \left[\frac{72(1 - 3\kappa b_2)(b_1^2 + B^2 p^2 + b_2 B p) - 8(1 - 9b_2^2)}{9(1 - 12(b_1^2 + b_2^2 + B^2 p^2 + b_2 B p))} N^2 + 1 \right] . \quad (2.28)$$

For $b_1 = b_2 = 0$ this simplifies to

$$c_r = -3p\kappa\eta_\Sigma \left[\frac{8(1 - 9B^2 p^2)}{9(1 - 12B^2 p^2)} N^2 - 1 \right] . \quad (2.29)$$

On the other hand for $B = 0$ one finds

$$c_r = -3p\kappa\eta_\Sigma \left[\frac{8(1 - 9(b_1^2 + b_2^2) + 27\kappa b_1^2 b_2)}{9(1 - 12(b_1^2 + b_2^2))} N^2 - 1 \right] . \quad (2.30)$$

2.2.4 $Y^{p,p}$ on T^2

Setting $\kappa = 0$ and when all background fluxes are non-vanishing one finds

$$\begin{aligned} \epsilon_1 &= \frac{b_1 b_2}{b_1^2 + b_2^2 + B^2 p^2 + b_2 B p} , \\ \epsilon_2 &= \frac{2}{3} - \frac{b_2^2}{b_1^2 + b_2^2 + B^2 p^2 + b_2 B p} , \\ \epsilon_B &= \frac{-b_1^2 + 2b_2^2 - B^2 p^2 + 2b_2 B p}{3p(b_1^2 + b_2^2 + B^2 p^2 + b_2 B p)} , \end{aligned} \quad (2.31)$$

which leads to the central charge

$$c_r = 6\eta_\Sigma \frac{b_2 p(b_1^2 + B^2 p^2 + b_2 B p)}{b_1^2 + b_2^2 + B^2 p^2 + b_2 B p} N^2 . \quad (2.32)$$

If $B = b_1 = 0$ the c -extremization procedure seems to be applicable but one finds that $c_r = 0$ thus we do not get a good candidate unitary CFT. When $B = b_2 = 0$ or $b_1 = b_2 = 0$ the trial central charge is linear in the parameters ϵ_i so one cannot apply c -extremization. Thus we can conclude that for $\kappa = 0$ we have good candidate CFTs only when at least two of the background fluxes are non-trivial. We summarize some of the results for the $Y^{p,p}$ quivers with $b_1 = 0$ in Figure 2. The boundaries of the colored regions in Figure 2 deserve some further comments. We note that for $\kappa = \pm 1$ there are a number of interesting points at the intersection of the various regions. For $\kappa = 1$ all these points are excluded, but for $\kappa = -1$ the situation is more subtle. For $\kappa = -1$ the intersection points of the straight lines, located at $(-1/3, 2/3)$, $(-1/3, -1/3)$ and $(2/3, -1/3)$, are excluded. The intersection points of the ellipse with the straight lines are located at $A = (-1/3, 1/6)$, $B = (1/6, -1/3)$ and $C = (1/6, 1/6)$. At points A, B , the central charge is not analytic, giving either $c_r = 0$ or $c_r = 1$, depending on the order in which the limit is taken. At point C the central charge is 1, independent of the order of the limits. All other points on the boundaries of the colored regions in Figure 2 are excluded.

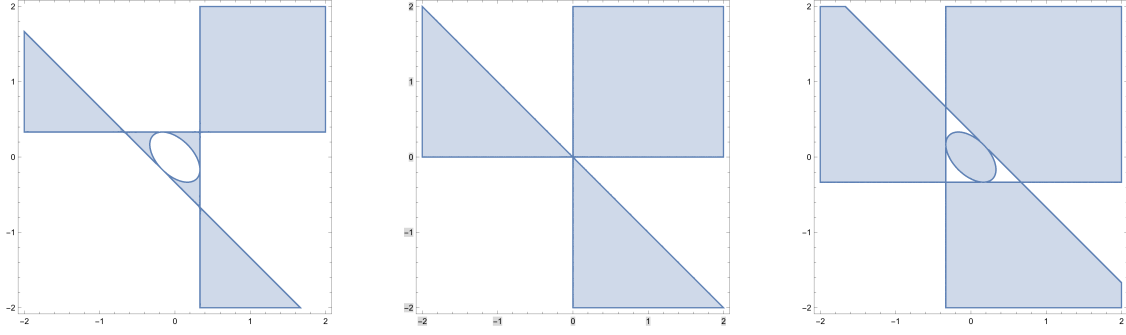


Figure 2: The region in the (b_2, pB) plane (in blue) for which the central charge for $Y^{p,p}$ with $b_1 = 0$ is positive. The cases $\kappa = \{1, 0, -1\}$ are presented from left to right and again b_2 is on the horizontal axis.

One might have naively expected that the central charges in (2.28) with $B = 0$ can be compared to the ones derived in [7], since the theories considered here arise as the IR fixed points of $\mathbb{Z}_2 \times \mathbb{Z}_p$ orbifolds of $\mathcal{N} = 4$ SYM further placed on a Riemann surface, while the theories in [7] came from pure $\mathcal{N} = 4$ SYM on a Riemann surface. However this is not the case and the central charges in (2.28) differ from the ones in [7]. This suggests that the RG flow from four to two dimensions does not commute with the orbifold action. An important reason for this is the role played by the $U(1)_B$ symmetry which is absent for $\mathcal{N} = 4$ SYM (and therefore in the setup of [7]), but clearly plays a crucial role in the present construction since it mixes along the RG flow with the $U(1)_R$ symmetry.

2.2.5 $Y^{p,q}$ on T^2

Let us now take $\kappa = 0$ and keep p and q general. For general values of the flavor and baryonic fluxes the central charges are lengthy and we will refrain from presenting them here. When we set $b_1 = 0$ we get an enhanced $SU(2)$ flavor symmetry and this will be the case of interest in the supergravity analysis. Below we will focus on this choice of background flux.

The trial central charge (2.14) is extremized for

$$\begin{aligned} \epsilon_1 &= 0, \\ \epsilon_2 &= \frac{p+w}{3q} - \frac{3pb_2^2}{3q(b_2^2 + b_2Bq + B^2q^2)}, \\ \epsilon_B &= \frac{4p-2w}{3q^2} - \frac{pB^2}{b_2^2 + b_2Bq + B^2q^2}. \end{aligned} \quad (2.33)$$

The right-moving central charge is particularly simple (recall that for $\kappa = 0$ we have $\eta_\Sigma = 1$)

$$c_r = 6p^2B \left[1 - \frac{B^2p^2}{b_2^2 + b_2Bq + B^2q^2} \right] N^2. \quad (2.34)$$

If in addition we set $b_2 = 0$ the result is

$$c_r = -\frac{6Bp^2(p^2 - q^2)}{q^2}N^2, \quad (2.35)$$

which is positive only for $B < 0$. This result looks very similar to the central charges found in supergravity in Section 4.1 of [20]. Indeed after the redefinition $p = Q$, $q = P + Q$ the central charge in (2.35) becomes

$$c_r = 6\frac{PQ^2(P + 2Q)}{(P + Q)^2}BN^2, \quad (2.36)$$

which is identical to equation (4.18) in [20] upon the identification $p_{\text{DGK}} = P$, $q_{\text{DGK}} = Q$, $BN^2 = N_{\text{DGK}}M_{\text{DGK}}$.¹

2.2.6 $Y^{p,q}$ on $\Sigma_{g \neq 1}$

Finally, we discuss the generic case of $Y^{p,q}$ on a Riemann surface with $\kappa \neq 0$. For general values of p and q and general background fluxes it is straightforward to apply the general c -extremization procedure as outlined above, but the results are too unwieldy to present explicitly. Therefore we will restrict ourselves to a few special values of the background fluxes while keeping p and q general.

For $b_1 = b_2 = 0$ the expression for the central charge is relatively complicated and takes the form

$$\begin{aligned} c_r = & -3\eta_\Sigma \kappa \left[\frac{16}{9}a(Y^{p,q}) \right. \\ & + \frac{4p^2q^2B^2[w^2(2p^3 - p^2w + 3q^2w) - 36\kappa Bq^2(p^2 - q^2)(pw + w^2 + 6q^2B^2(q^2 - pw - 2p^2))]}{3(2p^2 + pw - q^2)(2pw - w^2 + 6\kappa Bq^2(p - w) - 12B^2q^4)^2}N^2 \\ & \left. + \frac{48p^2q^4B^4(p + w)[2p^4 - p(p^2 + q^2)w + (p^2 - 4q^2)w^2 + pw^3]}{3(2p^2 + pw - q^2)(2pw - w^2 + 6\kappa Bq^2(p - w) - 12B^2q^4)^2}N^2 - \frac{p}{3} \right]. \end{aligned} \quad (2.37)$$

If instead we set $b_1 = B = 0$ we find

$$c_r = -3\eta_\Sigma \kappa \left[\frac{16}{9}a(Y^{p,q}) + \frac{8b_2^2p^2w}{w(2p + w) - 4b_2^2(2p + w)^2}N^2 - \frac{p}{3} \right], \quad (2.38)$$

where $a(Y^{p,q})$ is the central charge of the four-dimensional $Y^{p,q}$ SCFT (2.5).

Finally, we note that by setting the remaining flux $b_2 = 0$ (which requires $\kappa = -1$) one finds to leading order in N again the relation

$$c_r = \frac{32}{3}(\mathfrak{g} - 1)a(Y^{p,q}) + \mathcal{O}(1). \quad (2.39)$$

¹It is clear from the analysis of [20] that there is an allowed range for the parameters in which $p_{\text{DGK}} \leq 0$, $q_{\text{DGK}} > 0$ and $q_{\text{DGK}} \geq |p_{\text{DGK}}|$. This is the range compatible with the values of the parameters p and q in $Y^{p,q}$, i.e. with $p > 0$ and $p \geq q \geq 0$.

As we explain in the next section, this relation holds not only for $Y^{p,q}$ quivers at large N , but quite generally for a large class of 4d $\mathcal{N} = 1$ SCFTs on Riemann surfaces, twisted by the four-dimensional superconformal R-symmetry (when this is possible).

2.3 A universal RG flow across dimensions

Here we would like to show that when a four-dimensional $\mathcal{N} = 1$ SCFT is placed on a Riemann surface with a partial topological twist, there is a universal relation between the conformal anomalies in two and four-dimensions. Our result is valid under the assumption that the 2d theory in the IR is indeed a SCFT with normalizable vacuum, and that there are no accidental IR symmetries. Whether this is true or not is a dynamical question which we will not be able to address in general. However if the four-dimensional theory has a gravitational dual we will establish the existence of the two-dimensional SCFT holographically.

Suppose that we have a 4d $\mathcal{N} = 1$ supersymmetric theory (not necessarily conformal) with global symmetry $U(1)_R \times U(1)_F \times G_F$ where $U(1)_R$ is an R-symmetry, $U(1)_F$ is a flavor symmetry, and G_F is some additional non-Abelian global symmetry.² The 't Hooft anomalies of this theory are encoded in the following six-form anomaly polynomial:

$$I_6 = \frac{k_{RRR}}{6} c_1(\mathcal{F}_R)^3 + \frac{k_{FFF}}{6} c_1(\mathcal{F}_F)^3 + \frac{k_{RRF}}{2} c_1(\mathcal{F}_R)^2 c_1(\mathcal{F}_F) + \frac{k_{RFF}}{2} c_1(\mathcal{F}_R) c_1(\mathcal{F}_F)^2 - \frac{k_R}{24} c_1(\mathcal{F}_R) p_1(\mathcal{T}_4) - \frac{k_F}{24} c_1(\mathcal{F}_F) p_1(\mathcal{T}_4) . \quad (2.40)$$

Here k_{ABC} and k_A are the cubic and linear 't Hooft anomalies, $c_1(\mathcal{F})$ is the Chern class of the bundle with curvature $\mathcal{F}$, $p_1(\mathcal{T}_4)$ is the Pontryagin class of the four-manifold on which the theory is placed, and the powers of all characteristic classes are with respect to the wedge product. When the theory has a Lagrangian description, one can easily compute the anomalies as $k_{ABC} = \text{Tr}(ABC)$ and $k_A = \text{Tr}(A)$ where the trace is over all chiral fermions in the theory.³

In a similar fashion one can encode the anomalies of a 2d theory with $\mathcal{N} = (0, 2)$ supersymmetry in the four-form anomaly polynomial⁴

$$I_4 = \frac{k_{RR}}{2} c_1(\mathcal{F}_R)^2 + \frac{k_{FF}}{2} c_1(\mathcal{F}_F)^2 + k_{RF} c_1(\mathcal{F}_R) c_1(\mathcal{F}_F) - \frac{k}{24} p_1(\mathcal{T}_2) , \quad (2.41)$$

where all the Chern and Pontryagin classes are the ones in 2d. The coefficients k_{AB} are the quadratic 't Hooft anomalies, while k is the gravitational anomaly. In a theory with Lagrangian

²The results below generalize easily to the case where there is more than one Abelian factor in the flavor group. We refrain from discussing the general case to avoid clutter in the formulae.

³One should represent all fermions with right-moving chiral fields. Otherwise, the correct formulae should be $k_{ABC} = \text{Tr} \gamma_5 ABC$ and $k_A = \text{Tr} \gamma_5 A$, where γ_5 is the 4d chirality matrix.

⁴For simplicity we again assume that the 2d theory has only a single Abelian factor in the flavor group.

description they are given by the formula $k_{AB} = \text{Tr } \gamma_3 AB$ and $k = \text{Tr } \gamma_3$, where the trace is over all complex chiral fermions in the theory and γ_3 is the 2d chirality matrix (positive on right-movers).

If the theories are actually superconformal and R is the superconformal R-symmetry, the relations between conformal and 't Hooft anomalies in 4d and 2d take the following form:

$$a = \frac{9}{32}k_{RRR} - \frac{3}{32}k_R, \quad c = \frac{9}{32}k_{RRR} - \frac{5}{32}k_R, \quad c_r = 3k_{RR}, \quad c_r - c_l = k. \quad (2.42)$$

Here $c_{l,r}$ are the 2d left- and right-moving central charges. Superconformal symmetry also enforces $9k_{RRF} = k_F$ in 4d [38] and $k_{RF} = 0$ in 2d [8].

We place the 4d theory on a compact Riemann surface and implement a partial topological twist which preserves $\mathcal{N} = (0, 2)$ supersymmetry in the remaining two dimensions. At the level of R-symmetry and flavor symmetry line bundles, this topological twist amounts to the following replacement:

$$\mathcal{F}_R^{(4D)} \rightarrow \mathcal{F}_R^{(2D)} - \frac{\kappa}{2} t_{\mathfrak{g}}, \quad \mathcal{F}_F^{(4D)} \rightarrow \mathcal{F}_F^{(2D)} + \epsilon \mathcal{F}_R^{(2D)} + \mathfrak{b} t_{\mathfrak{g}}. \quad (2.43)$$

Here $t_{\mathfrak{g}}$ is the Chern class of the tangent bundle to the Riemann surface normalized in such a way that $\int_{\Sigma_{\mathfrak{g}}} t_{\mathfrak{g}} = \eta_{\Sigma}$. The R-symmetry background is fixed by supersymmetry. The parameter $\mathfrak{b}$, instead, represents the freedom to turn on a background magnetic flux through the Riemann surface for the $U(1)_F$ symmetry (such a parameter should be properly quantized). We are interested in flows that lead to 2d fixed points. We have introduced the parameter ϵ because by $\mathcal{F}_R^{(2D)}$ we now mean the 2d *superconformal* R-symmetry, which in general is a mix between some R-symmetry derived from four dimensions and the Abelian flavor symmetries. As in Section 2.2, the value of ϵ at the 2d fixed point is fixed by c -extremization.

To calculate the anomalies of the IR 2d SCFT, we plug the background (2.43) into the six-form (2.40), integrate the result over $\Sigma_{\mathfrak{g}}$ (notice that $t_{\mathfrak{g}}^2 = 0$) and then read off the I_4 anomaly polynomial of the 2d theory. Extremizing the trial value of $k_{RR}(\epsilon)$ with respect to ϵ we find

$$\epsilon = -\frac{\kappa k_{RRF} - 2\mathfrak{b} k_{RFF}}{\kappa k_{RFF} - 2\mathfrak{b} k_{FFF}}, \quad (2.44)$$

and the right-moving central charge is

$$c_r = \frac{3\eta_{\Sigma}}{2} \left[-\kappa k_{RRR} + \frac{(\kappa k_{RRF} - 2\mathfrak{b} k_{RFF})^2 + 2\mathfrak{b} k_{RRF}(\kappa k_{RFF} - 2\mathfrak{b} k_{FFF})}{\kappa k_{RFF} - 2\mathfrak{b} k_{FFF}} \right]. \quad (2.45)$$

The values of the other 2d anomalies are

$$k_{FF} = \frac{\eta_{\Sigma}}{2} (2\mathfrak{b} k_{FFF} - \kappa k_{RFF}), \quad c_r - c_l = \frac{\eta_{\Sigma}}{2} (2\mathfrak{b} k_F - \kappa k_R), \quad k_{RF} = 0. \quad (2.46)$$

The relation $k_{RF} = 0$ precisely corresponds to the fact that we have extremized c_r .

Consider now the case of a 4d SCFT with $k_F = 0$, and perform the partial topological twist using the exact 4d superconformal R-symmetry, *i.e.* R is the 4d superconformal R-symmetry and take $\mathfrak{b} = 0$ (in cases where the R-symmetry flux on $\Sigma_{\mathfrak{g}}$ is properly quantized). Since $9k_{RRF} = k_F = 0$, from (2.44) it follows that $\epsilon = 0$. This means that the IR 2d superconformal R-symmetry coincides with the UV 4d one, and no mixing with $U(1)_F$ occurs along the RG flow. For such an RG flow across dimensions, which is unitary only for $\kappa = -1$, we obtain a universal relation

$$\begin{pmatrix} c_r \\ c_l \end{pmatrix} = \frac{16}{3}(\mathfrak{g} - 1) \begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix}. \quad (2.47)$$

This result is reminiscent of the universal RG flow between four-dimensional $\mathcal{N} = 2$ and $\mathcal{N} = 1$ SCFTs discussed in [40]. In our case, the RG flow is between four-dimensional $\mathcal{N} = 1$ SCFTs and two-dimensional $\mathcal{N} = (0, 2)$ SCFTs.

We note that for c_r to be positive, the four-dimensional theory should satisfy

$$\frac{a}{c} > \frac{3}{5}, \quad (2.48)$$

or $k_{RRR} > 0$. This lower bound is compatible with the Hofman-Maldacena (HM) [41] window $\frac{3}{2} \geq \frac{a}{c} \geq \frac{1}{2}$ for $\mathcal{N} = 1$ SCFTs, but it places a restriction on the class of theories for which this RG flow leads to unitary 2d SCFTs with a normalizable vacuum in the IR. On the other hand, the upper bound of the HM window implies that the 2d SCFTs at hand obey the bound $c_r/c_l \leq 9/4$. We emphasize that these inequalities hold only for the universal twist of four-dimensional theories with $k_F = 0$.

Finally, we note that for theories with $a = c$ (*i.e.* with $k_R = 0$), one has $c_r = c_l$ and

$$c_r = \frac{32}{3}(\mathfrak{g} - 1) a. \quad (2.49)$$

Notice that if $k_F \simeq 0$ or $k_R \simeq 0$ only at leading order in N , the statements above are still true at leading order. Since the $Y^{p,q}$ quivers have $k_F \simeq k_R \simeq 0$, the result in (2.49) is an explanation of the universal result observed at large N in many of the examples seen above, as in (2.19). For CFTs with weakly coupled supergravity duals we have $a \simeq c$ and thus the universal relation in (2.49) will hold. In Section 3.2 and Appendix B we indeed show how this comes about on the supergravity side.

Since the $Y^{p,q}$ quiver gauge theories usually have *two* Abelian flavor symmetries for generic p, q (one mesonic and one baryonic), the formula (2.45) is not applicable. Of course, the approach taken above is valid and one can repeat the analysis in the case of several Abelian flavor symmetries, but we do not provide the results here. The case of $Y^{p,0}$ with purely baryonic flux $\mathfrak{b} = B$ is an exception; in this case the full flavor symmetry is enhanced to $SU(2)_1 \times SU(2)_2 \times U(1)_B$ and there can be no mixing of R-symmetry with any flavor symmetry other than the baryonic one.

In this case using the values for the anomaly coefficients k_{RBB} , etc. one can verify that (2.45) reproduces (2.19).

Some interesting examples of $\mathcal{N} = 1$ SCFTs with holographic duals for which we have $k_R \neq 0$ are discussed in [11, 12, 42]. For these theories one should use the general formula in (2.47).

2.4 D3-branes at del Pezzo singularities

To illustrate the utility of our general result in (2.47) let us consider the $\mathcal{N} = 1$ SCFT describing the low-energy dynamics of D3-branes at the tip of Calabi-Yau three-folds which are complex cones over del Pezzo singularities, dP_k , $k = 0, 1, \dots, 8$. These theories were originally introduced in [43] and admit a dual holographic description. For $k = 3, \dots, 8$ the theories do not have any flavor symmetry, have rational R-charges and thus should provide an ideal testing ground for our universal formula. In the case of $dP_0 \cong \mathbb{P}^2$, the theory has $SU(3)$ flavor symmetry which however cannot mix with the R-symmetry. We can also add $\mathbb{P}^1 \times \mathbb{P}^1$ to the list, which has $SU(2)^2$ flavor symmetry not mixable with the R-symmetry. Finally, we can also consider D3-branes in flat spacetime—of which the $\mathbb{P}^2$ case is a $\mathbb{Z}_3$ orbifold—giving $\mathcal{N} = 4$ SYM at low energies. The cases of $dP_1 \cong Y^{2,1}$ and dP_2 , instead, are different because they have an Abelian flavor symmetry that can mix with the R-symmetry, and in fact the 4d R-charges are irrational: these theories cannot be placed on a Riemann surface in the “universal way” (although they can if we allow for flavor fluxes).

The conformal anomalies for the quiver gauge theories arising from the $dP_{k=3,\dots,8}$ singularities were computed for example in [44]. At leading order in N —or formally for gauge group $U(N)$ —they are given by

$$a_{dP_k} = c_{dP_k} = \frac{27}{4(9-k)} N^2. \quad (2.50)$$

The conformal anomalies of $\mathcal{N} = 4$ SYM are

$$a_{\mathcal{N}=4} = c_{\mathcal{N}=4} = \frac{1}{4} N^2. \quad (2.51)$$

The case of $dP_0 \cong \mathbb{P}^2$ gives a $\mathbb{Z}_3$ orbifold of $\mathcal{N} = 4$ SYM with conformal anomalies

$$a_{\mathbb{P}^2} = c_{\mathbb{P}^2} = \frac{3}{4} N^2. \quad (2.52)$$

Finally, the line bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ gives the Klebanov-Witten theory:

$$a_{\text{KW}} = c_{\text{KW}} = \frac{27}{64} N^2. \quad (2.53)$$

From the universal formula (2.49) we find the central charges of the two-dimensional SCFTs that arise from the compactification of the 4d SCFTs on a Riemann surface with $U(1)_R$ twist:

$$c_{\mathcal{N}=4}^{(2d)} = \frac{8(\mathfrak{g}-1)}{3} N^2, \quad c_{\text{KW}}^{(2d)} = \frac{9(\mathfrak{g}-1)}{2} N^2, \quad c_{dP_k}^{(2d)} = \frac{72(\mathfrak{g}-1)}{(9-k)} N^2. \quad (2.54)$$

These field theory results nicely reproduce a dual supergravity calculation presented in [18] as we now show.

In Section 6.1 of [18] the authors found a class of AdS_3 solutions of type IIB supergravity based on the six del Pezzo surfaces $dP_{k=3,\dots,8}$, on $\mathbb{P}^2$ and on $\mathbb{P}^1 \times \mathbb{P}^1$. The internal seven-dimensional manifold is topologically a Sasaki-Einstein 5d manifold fibered over a closed Riemann surface of genus $\mathfrak{g} > 1$. The Sasaki-Einstein manifold is in turn a $U(1)$ bundle over the four-dimensional Kähler-Einstein base. One can think of these solutions as arising from the backreaction of D3-branes transverse to the 5-manifold which wrap the Riemann surface. The central charges of the $\mathcal{N} = (0, 2)$ SCFTs dual to these solutions were computed in [18]:

$$c_{\text{sugra}} = \frac{36M|\chi|}{m^2h^2l}n^2. \quad (2.55)$$

Here $\chi = 2 - 2\mathfrak{g}$ is the Euler number of the Riemann surface, $l = \text{hcf}\{m, |\chi|\}$ and $h = \text{hcf}\{\frac{M}{m}, |\chi|\}$. The numbers (M, m) are as follows: for $\mathbb{CP}^2$ we have $(M, m) = (9, 3)$, for $\mathbb{P}^1 \times \mathbb{P}^1$ we have $(M, m) = (8, 2)$, and for dP_k with $k = 3, \dots, 8$ we have $(M, m) = (9 - k, 1)$. Finally, the number n is expressed in terms of N through

$$N = -\frac{M}{mh}n. \quad (2.56)$$

The integer N is the quantized five-form flux through the five-cycle transverse to the Riemann surface wrapped by the D3-branes, and should then be identified with the rank of the gauge group in the dual field theory. We can rewrite the supergravity central charge as

$$c_{\text{sugra}} = \frac{72|\mathfrak{g} - 1|}{Ml}N^2. \quad (2.57)$$

Using the values of M and l given above, we find perfect agreement with the field theory result in (2.54). For $dP_{3\dots 8}$ and $\mathbb{P}^1 \times \mathbb{P}^1$ we have immediate matching. For the circle bundle over $\mathbb{P}^2$ that gives S^5 , *i.e.* for $\mathcal{N} = 4$ SYM, one notices that the adjoints have R-charge $\frac{2}{3}$ and so there are gauge-invariant mesonic operators of fractional R-charge: the twist is only possible on surfaces whose $\mathfrak{g} - 1$ is a multiple of 3, then $l = 3$ and the central charges match. Alternatively, for the line bundle over $\mathbb{P}^2$ which leads to $\mathbb{C}^3/\mathbb{Z}_3$, the field theory has bi-fundamentals of R-charge $\frac{2}{3}$ but the gauge-invariant mesons have integer R-charge 2, and the twist is possible for any genus; then $l = 1$ and the central charges match.

This agreement between field theory and gravitational calculations provides strong evidence that the supergravity solutions found in Section 6.2 of [18] are dual to the 2d $\mathcal{N} = (0, 2)$ SCFTs which arise from a twisted compactification on $\Sigma_{\mathfrak{g}}$ of the 4d $\mathcal{N} = 1$ dP_k SCFTs.⁵ In fact, we will

⁵We were informed by Jerome Gauntlett that he has arrived at the same conclusion by an independent field theory calculation of the two-dimensional central charges.

show in the next section that this matching holds for twisted compactifications on Σ_g of a general class of 4d $\mathcal{N} = 1$ SCFTs with gravity duals, whenever twisting by the pure 4d superconformal R-symmetry is possible. We will also provide new examples of gravity duals to field theories twisted by baryonic flux and match their central charges. A generalization to include flavor flux is also possible, and although we provide the local backgrounds explicitly, we leave a global analysis of these solutions and a matching of their central charge for future work.

3 Supergravity solutions

We are interested in constructing type IIB supergravity solutions of the form $\text{AdS}_3 \times_w \mathcal{M}_7$ preserving $\mathcal{N} = (0, 2)$ supersymmetry. The concrete four-dimensional $\mathcal{N} = 1$ SCFTs discussed above arise from D-branes at the tip of conical Calabi-Yau manifolds. This suggests that the only non-vanishing flux in the supergravity solutions of interest is the self-dual five-form. Thus, we search for solutions of the form

$$\begin{aligned} ds_{10}^2 &= L^2 [e^{2\lambda} ds^2(\text{AdS}_3) + ds^2(\mathcal{M}_7)] , \\ g_s F_{(5)} &= L^4 (1 + *_10) d\text{volAdS}_3 \wedge F_{(2)} , \end{aligned} \quad (3.1)$$

where $F_{(2)}$ is a two-form on $\mathcal{M}_7$. The most general solution with these properties was analyzed in [16, 22], where it was shown that the internal manifold $\mathcal{M}_7$ must locally be a $U(1)$ bundle over a six-dimensional Kähler manifold, whose Kähler potential satisfies a fourth-order nonlinear partial differential equation. Explicit solutions were further studied in [19, 20]. Here, rather than searching for explicit solutions for the six-dimensional base, motivated by the field theory analysis we assume that $\mathcal{M}_7$ is a five-dimensional fibration over a Riemann surface with $SU(2) \times U(1) \times U(1)$ isometry and derive a set of BPS and Bianchi equations for this Ansatz. Of course, the final solution can be written in the form derived in [16, 22], as we have checked.

When only the metric and five-form flux are turned on (*i.e.* without any non-trivial axio-dilaton or three-form flux), the supersymmetry variations of the spin- $\frac{1}{2}$ fermions in type IIB supergravity vanish identically and the gravitino variations are given by⁶

$$\delta\psi_\mu = \partial_\mu \epsilon + \frac{1}{4} \omega_{\mu ab} \Gamma^{ab} \epsilon + \frac{i}{192} F_{\mu\nu_1\nu_2\nu_3\nu_4} \Gamma^{\nu_1\nu_2\nu_3\nu_4} \epsilon = 0 , \quad (3.2)$$

where ϵ is a complex ten-dimensional spinor satisfying the chirality condition $\Gamma^{12345678910} \epsilon = \epsilon$. The self-dual five-form $F_{(5)} = (1 + *_10) G_{(5)}$ must satisfy the Bianchi identity $dF_{(5)} = 0$, and we will make a choice for $G_{(5)}$ such that

$$dG_{(5)} = d *_10 G_{(5)} = 0 . \quad (3.3)$$

⁶We follow the conventions of [45].

In principle, solutions to (3.2) and (3.3) are not necessarily solutions to the equations of motion. In our setup, however, we have checked that solving (3.2) and (3.3) for the Ansatz in (3.1) leads to a solution of the equations of motion.

Once we have constructed a globally well-defined supergravity solution of the form (3.1) the central charge of the dual CFT is given by the Brown-Henneaux formula [2]

$$c_{\text{sugra}} = \frac{3L}{2G_N^{(3)}} , \quad (3.4)$$

where $G_N^{(3)}$ is the 3d Newton constant (see Appendix B for conventions and explicit formulas).

The Ansatz

The field theory setup suggests that we should be looking for solutions in which $\mathcal{M}_7$ is a five-dimensional fibration over a Riemann surface Σ_g :

$$\begin{array}{ccc} \mathcal{M}_5 & \longrightarrow & \mathcal{M}_7 \\ & & \downarrow \\ & & \Sigma_g \end{array} .$$

In addition, we require $\mathcal{M}_7$ to have $SU(2) \times U(1) \times U(1)$ isometry, corresponding to the flavor and R-symmetry of the dual field theory. We denote the coordinates of AdS_3 by $\{t, z, r\}$, the coordinates of Σ_g by $\{x_1, x_2\}$, and the remaining coordinates by $\{y, \theta, \phi, \beta, \psi\}$. The most general Ansatz compatible with these requirements is⁷

$$\begin{aligned} ds_{10}^2 &= f_1(y)^2 ds_{\text{AdS}_3}^2 + f_2(y)^2 ds_{\Sigma_g}^2 + f_3(y)^2 ds_{S^2}^2 + f_4(y)^2 dy^2 + f_5(y)^2 (D\beta)^2 \\ &\quad + f_6(y)^2 (D\psi + f_7(y) D\beta)^2 , \\ G_{(5)} &= d\text{vol}_{\text{AdS}_3} \wedge \left[G_1(y) d\text{vol}_{\Sigma_g} + G_2(y) d\text{vol}_{S^2} + G_3(y) dy \wedge D\beta \right. \\ &\quad \left. + G_4(y) dy \wedge (D\psi + f_7(y) D\beta) \right] , \end{aligned} \quad (3.5)$$

where

$$\begin{aligned} ds_{\text{AdS}_3}^2 &= \frac{-dt^2 + dz^2 + dr^2}{r^2} , \quad ds_{\Sigma_g}^2 = e^{2h(x_1, x_2)} (dx_1^2 + dx_2^2) , \quad ds_{S^2}^2 = d\theta^2 + \sin^2 \theta d\phi^2 , \\ D\beta &= d\beta + c \cos \theta d\phi + a_2 A_g(x_1, x_2) , \quad D\psi = d\psi + b \cos \theta d\phi + a_3 A(x_1, x_2) , \quad dA_g = d\text{vol}_{\Sigma_g} , \end{aligned}$$

and $d\text{vol}_X$ is the volume form on X (see Appendix A for details). The real parameters $\{b, c, a_2, a_3\}$ are for the moment free but will be constrained by the BPS equations. We choose $0 \leq \theta \leq \pi$

⁷Here we omit the overall scale factors of L and g_s from (3.1). These must be reinstated when computing the central charge.

and $0 \leq \phi \leq 2\pi$ so that $ds_{S^2}^2$ is the metric on the round S^2 . The range of the other coordinates will be determined by requiring that the metric is globally compact and smooth and it depends on the details of the particular solution. We will discuss this in more detail for some concrete examples below. The parameters a_2 and a_3 specify the fibration of the five-manifold over $\Sigma_{\mathfrak{g}}$ and thus we expect them to be related to the flavor flux b_2 and the R-symmetry flux fixed to $\kappa/2$ by supersymmetry (2.10). Since we impose an $SU(2)$ isometry on $\mathcal{M}_7$ our solutions should capture supergravity duals of the field theory setup in Section 2 with vanishing flavor flux b_1 in (2.10).

In principle, a term of the form $G_5(y)D\beta \wedge D\psi$ in the flux is allowed, but it is easy to show that $G_5 = 0$ follows from $\delta\psi_5 = 0$ (see Appendix A). The function $h(x_1, x_2)$ encodes the constant curvature metric on the genus $\mathfrak{g}$ Riemann surface and is given by

$$h(x_1, x_2) = \begin{cases} -\log \frac{1+x_1^2+x_2^2}{2} & \text{for } \mathfrak{g} = 0 \\ \frac{1}{2} \log 2\pi & \text{for } \mathfrak{g} = 1 \\ -\log x_2 & \text{for } \mathfrak{g} > 1. \end{cases} \quad (3.6)$$

We define the normalized curvature $\kappa = 1$ for $\mathfrak{g} = 0$, $\kappa = 0$ for $\mathfrak{g} = 1$, and $\kappa = -1$ for $\mathfrak{g} > 1$. The symmetries of the problem suggest that we impose the following projectors on ϵ :

$$\Gamma^{12}\epsilon = -\epsilon, \quad \Gamma^{45}\epsilon = i\epsilon, \quad \Gamma^{67}\epsilon = i\epsilon, \quad \Gamma^{89}\epsilon = i\epsilon. \quad (3.7)$$

As shown in Appendix A, the BPS equations impose $f_6(y) = \alpha_2 f_1(y)$, with α_2 a non-vanishing constant. The function $f_4(y)$ in (3.5) can be freely adjusted by choosing an appropriate coordinate y . It is convenient to make a choice such that

$$f_5(y) = \frac{1}{f_1^2(y)f_4(y)}. \quad (3.8)$$

To simplify the BPS equations for the remaining functions f_1, f_2, f_3, f_4, f_7 it is instructive to rewrite them in terms of the functions $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{Q}, \mathcal{P}_7$, defined by:

$$\begin{aligned} f_1^2(y) &= \sqrt{\frac{\mathcal{P}_2(y)\mathcal{P}_3(y)}{\mathcal{P}_1(y)}}, & f_2^2(y) &= \sqrt{\frac{\mathcal{P}_2(y)\mathcal{P}_1(y)}{\mathcal{P}_3(y)}}, & f_3^2(y) &= \sqrt{\frac{\mathcal{P}_3(y)\mathcal{P}_1(y)}{\mathcal{P}_2(y)}}, \\ f_4^2(y) &= \frac{\sqrt{\mathcal{P}_1(y)\mathcal{P}_2(y)\mathcal{P}_3(y)}}{\mathcal{Q}(y)}, & f_7(y) &= \frac{\mathcal{P}_7(y)}{\mathcal{P}_2(y)\mathcal{P}_3(y)}. \end{aligned} \quad (3.9)$$

This form of the Ansatz combined with reality and positivity of the metric requires that

$$\text{sign}(\mathcal{P}_1(y)\mathcal{P}_2(y)\mathcal{P}_3(y)) = +, \quad \text{sign } \mathcal{Q}(y) = +. \quad (3.10)$$

The range of y will be restricted by the zeros of the function $\mathcal{Q}(y)$, between which $\mathcal{Q}(y) > 0$. We shall assume that y takes values in the finite range $[y_1, y_2]$ between two such zeros and that in

this range⁸

$$\mathcal{P}_1(y) > 0, \quad \mathcal{P}_2(y) > 0, \quad \mathcal{P}_3(y) > 0, \quad y \in [y_1, y_2]. \quad (3.11)$$

In what follows we will often omit the argument y in all the functions.

3.1 General solution

As shown in Appendix A, the BPS equations imply that the functions $\mathcal{P}_{2,3}$ are linear in y :

$$\mathcal{P}_2 = a_2 y + C_2, \quad \mathcal{P}_3 = -c y + C_3, \quad (3.12)$$

where $C_{2,3}$ are two integration constants. The functions $\mathcal{P}_{1,7}$ are fixed in terms of $\mathcal{P}_{2,3}$ and $\mathcal{Q}$ by

$$\mathcal{P}_1 = \frac{\alpha_2 (a_3 \mathcal{P}_3 - b \mathcal{P}_2 + \mathcal{P}_7')}{2}, \quad \mathcal{P}_7 = \frac{4\alpha_1 \mathcal{P}_2 \mathcal{P}_3 - \mathcal{Q}'}{4\alpha_2}, \quad (3.13)$$

where α_1 is another integration constant⁹ and prime denotes derivative with respect to y . For consistency of the BPS equations one must impose the constraints

$$c \alpha_1 + b \alpha_2 = -\frac{1}{2}, \quad a_2 \alpha_1 + a_3 \alpha_2 = \frac{\kappa}{2}. \quad (3.14)$$

Similarly, the five-form flux in the Ansatz (3.5) is determined by:

$$\begin{aligned} G_1 &= \frac{8 \mathcal{P}_1 \mathcal{P}_2 - 2\kappa \mathcal{P}_2 \mathcal{P}_3 + a_2 \mathcal{Q}'}{4 \mathcal{P}_1}, & G_2 &= \frac{8 \mathcal{P}_1 \mathcal{P}_3 - 2 \mathcal{P}_2 \mathcal{P}_3 - c \mathcal{Q}'}{4 \mathcal{P}_1}, \\ G_3 &= \frac{2 \mathcal{P}_2 \mathcal{P}_3 (\mathcal{P}_2 + \kappa \mathcal{P}_3) + (c \mathcal{P}_2 - a_2 \mathcal{P}_3) \mathcal{Q}'}{4 \mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_3}, & G_4 &= -\alpha_2 \left(\frac{\mathcal{P}_2 \mathcal{P}_3}{\mathcal{P}_1} \right)'. \end{aligned} \quad (3.15)$$

Thus, the metric and the five-form are completely determined in terms of the integration constants and the single (yet unknown) function $\mathcal{Q}$. The final constraint is the Bianchi identity (3.3), which implies a fourth-order ODE for the function $\mathcal{Q}$. Remarkably this ODE can be integrated twice into the following second-order ODE:

$$\mathcal{Q}'' - 2\mathcal{Q}(\mathcal{Q}'' - 2(\mathcal{P}_2 + \kappa \mathcal{P}_3)) + \mathcal{P}_2 \mathcal{P}_3 (-4y^2 \kappa + \delta_1 + \delta_2 y) = 0, \quad (3.16)$$

where $\delta_{1,2}$ are new integration constants. Thus, the supergravity backgrounds we are after are completely characterized by solutions to (3.16). Although we have not found the most general solution to this equation, it is easy to see that the most general *polynomial* solution is at most a cubic:

$$\mathcal{Q} = q_3 y^3 + q_2 y^2 + q_1 y + q_0. \quad (3.17)$$

⁸Another option is that two of the three functions $\mathcal{P}_{1,2,3}$ are negative and the remaining one is positive. However, by simple redefinitions one can choose them all to be positive. For instance, if $\mathcal{P}_{1,3} < 0$ and $\mathcal{P}_2 > 0$, one may redefine $(\mathcal{P}_1, \mathcal{P}_3, \mathcal{P}_7) \rightarrow (-\mathcal{P}_1, -\mathcal{P}_3, -\mathcal{P}_7)$, which leaves the Ansatz invariant.

⁹As discussed in Appendix A.2 it is always possible to set $\alpha_1 = 0$ by a coordinate transformation.

In this case the functions $\mathcal{P}_1, \mathcal{P}_7$ in (3.13) become linear and quadratic in y , respectively. Specifically, we have

$$\mathcal{P}_1 = C_1 y + C_0, \quad \mathcal{P}_2 = a_2 y + C_2, \quad \mathcal{P}_3 = -c y + C_3, \quad \mathcal{P}_7 = \frac{4\alpha_1 \mathcal{P}_2 \mathcal{P}_3 - \mathcal{Q}'}{4\alpha_2}, \quad (3.18)$$

where $C_0 \equiv \frac{1}{4}(C_2 + C_3 \kappa - q_2)$ and $C_1 \equiv \frac{1}{4}(a_2 - c\kappa - 3q_3)$.

The solution seems to depend on the parameters $\{\alpha_1, \alpha_2, a_2, a_3, b, c, \kappa, q_i, C_2, C_3, \delta_1, \delta_2\}$. However, these are not all independent. The parameters $\alpha_{1,2}$ can be set to a convenient value by a choice of coordinates (see Appendix A.2), and we consider a_3, b fixed in terms of other parameters by (3.14). Finally, plugging in the expressions for $\mathcal{Q}, \mathcal{P}_2, \mathcal{P}_3$ into the Bianchi identity (3.16) leads to a number of nonlinear constraints among the remaining parameters $\{a_2, c, \kappa, q_i, C_2, C_3, \delta_1, \delta_2\}$, with many different branches of solutions, depending on the values of a_2, c, κ . All the explicit supergravity solutions that we discuss below and in Appendix A.1 arise from different solutions to these constraints.

For the purpose of comparison with the field theory analysis of Section 2, we are interested in solutions describing $Y^{p,q}$ manifolds fibered over the Riemann surface. Of course, our general Ansatz captures not only those solutions, but all solutions with (at least) the same isometry, including for instance $\text{AdS}_3 \times S^3 \times T^4$. In this section we will focus on the solutions relevant to the field theory analysis. Before we do this however it is important to understand whether $\mathcal{M}_7$ can have any conical singularities prior to specifying any particular solution.

3.1.1 Absence of conical singularities

One may worry that at zeros of the function $\mathcal{Q}$ the metric might be singular. It is easy to see, however, that such potential singularities are at most conical singularities, and—due to a certain identity—can always be removed by an appropriate choice of coordinate periodicity. This follows from the form of the Ansatz and the BPS equations and thus holds for any solution in this class. We first make a linear change of variables $\beta = w_1 \tilde{\beta} + w_2 \tilde{\psi}$ and $\psi = w_3 \tilde{\beta} + w_4 \tilde{\psi}$ in (3.5), where $w_{1,2,3,4}$ are real constants, and study the metric in the $(y, \tilde{\beta})$ subspace. As shown in Appendix A.2:

$$ds_2^2 = \frac{\sqrt{\mathcal{P}}}{|\mathcal{Q}|} \left[dy^2 + w^2 \frac{\alpha_2^2 \mathcal{Q}^2}{w_2^2 \mathcal{P}_1 \mathcal{Q} + \alpha_2^2 (w_4 \mathcal{P}_2 \mathcal{P}_3 + w_2 \mathcal{P}_7)^2} (d\tilde{\beta} + \dots)^2 \right], \quad (3.19)$$

where we defined $\mathcal{P} \equiv \mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_3$ and $w^2 \equiv (w_1 w_4 - w_2 w_3)^2$. Near a zero of $\mathcal{Q}$ at $y = y_i$ we expand $\mathcal{Q}(y) \approx \mathcal{Q}'(y_i)(y - y_i)$ and, defining the new radial variable

$$r^2 = 2(y - y_i), \quad (3.20)$$

we have

$$ds_2^2 \approx \frac{2\sqrt{\mathcal{P}(y_i)}}{|\mathcal{Q}'(y_i)|} \left[dr^2 + \frac{w^2}{4} \frac{\mathcal{Q}'(y_i)^2}{(w_4 \mathcal{P}_2(y_i) \mathcal{P}_3(y_i) + w_2 \mathcal{P}_7(y_i))^2} r^2 (d\tilde{\beta} + \dots)^2 \right]. \quad (3.21)$$

To avoid conical singularities one has to ensure that the coefficient of $r^2(d\tilde{\beta} + \dots)^2$ is the same constant at *all* the zeros y_i of $\mathcal{Q}$ (and choose the periodicity of $\tilde{\beta}$ accordingly). The functional identity relating $\mathcal{P}_7$ to $\mathcal{Q}'$ in (3.18) ensures that this is indeed the case. Note that this identity follows from the BPS equations in general, not just in the case of polynomial $\mathcal{Q}$. Hence, choosing

$$w_4 = -\alpha_1, \quad w_2 = \alpha_2, \quad (3.22)$$

and using (3.18), the 2D metric near a zero of $\mathcal{Q}$ becomes

$$ds_2^2 \approx \frac{2\sqrt{\mathcal{P}(y_i)}}{|\mathcal{Q}'(y_i)|} \left[dr^2 + 4(\alpha_1 w_1 + \alpha_2 w_3)^2 r^2 (d\tilde{\beta} + \dots)^2 \right]. \quad (3.23)$$

All conical singularities are avoided by fixing the periodicity of $\tilde{\beta}$ to be 2π and choosing, say, w_3 such that

$$\alpha_1 w_1 + \alpha_2 w_3 = \pm \frac{1}{2}. \quad (3.24)$$

3.2 $Y^{p,q}$ on $\Sigma_{g>1}$ with universal twist

As we have seen on the field theory side, when the flavor flux b_2 vanishes, *i.e.* for a twist performed using the UV superconformal R-symmetry, the RG flow is special and universal. It is natural to expect that $b_2 = 0$ is mapped to $a_2 = 0$ in our supergravity Ansatz. Indeed in this case the supergravity solutions considerably simplify as we now show. After setting $a_2 = 0$ there are still various branches of solutions depending on the values of c and κ . Assuming $\kappa \neq 0$ and $c \neq 0$, one such solution is (see Appendix A):

$$\begin{aligned} ds_{10}^2 = & ds_{AdS_3}^2 + \frac{3}{4} ds_{\Sigma_{g>1}}^2 + \frac{3|1-cy|}{8} (d\theta^2 + \sin^2 \theta d\phi^2) \\ & + \frac{9}{8} \frac{|1-cy|}{(2cy^3 - 3y^2 + a)} dy^2 + \frac{1}{8} \frac{(2cy^3 - 3y^2 + a)}{|1-cy|} (d\beta + c \cos \theta d\phi)^2 \\ & + \frac{1}{4} \left(d\psi - \cos \theta d\phi + y(d\beta + c \cos \theta d\phi) - \frac{dx_1}{x_2} \right)^2, \quad (3.25) \\ G_{(5)} = & d\text{vol}_{AdS_3} \wedge \left(2d\text{vol}_{\Sigma_{g>1}} + \frac{|1-cy|}{4} d\text{vol}_{S^2} + \frac{1}{4} dy \wedge (d\beta + c \cos \theta d\phi) \right), \end{aligned}$$

where a is the only remaining integration constant. This solution exists only for $\kappa = -1$ (*i.e.* $g > 1$). The internal metric is precisely the metric on $Y^{p,q}$ written in canonical form as in [10], fibered over $\Sigma_{g>1}$ in such a way that the fibration is non-trivial only along the Reeb vector ∂_ψ . This is a consequence of setting $a_2 = 0$. As in the case of the standard $Y^{p,q}$, since we have assumed¹⁰ $c \neq 0$ it can be rescaled to 1 and $0 < a < 1$. In fact, this metric looks like those found

¹⁰It is in fact possible to set $c = 0$ in this solution, corresponding to $Y^{p,0}$ fibered over $\Sigma_{g>1}$. However, in this case there is a more general solution which we discuss in Section 3.3.

in Section 6.1 of [18], namely

$$ds_{10}^2 = ds_{AdS_3}^2 + \frac{3}{4} ds_{\Sigma_{\mathfrak{g}>1}}^2 + \frac{9}{4} ds^2(SE_5) , \quad (3.26)$$

with $ds^2(SE_5)$ a five-dimensional Sasaki-Einstein manifold fibered over $\Sigma_{\mathfrak{g}>1}$. In the case at hand, $SE_5 = Y^{p,q}$. Using the general formulas for the supergravity central charge presented in Appendix B, the volume of the $Y^{p,q}$ manifolds computed in [10], and the AdS_5/CFT_4 relation $a_{4D}^{\text{sugra}} = \frac{\pi^3 N^2}{4 \text{Vol}(SE_5)}$, the central charge of the two-dimensional CFT dual to the AdS_3 solution in (3.26) can be written as

$$c_{\text{sugra}} = \frac{32}{3} (\mathfrak{g} - 1) a_{4D}^{\text{sugra}} . \quad (3.27)$$

This is in perfect agreement with the universal field theory result (2.49) obtained by c -extremization. This is strong evidence that the background in (3.25) describes the IR fixed point of 4d $Y^{p,q}$ SCFTs with AdS_5 gravity duals, placed on $\mathbb{R}^2 \times \Sigma_{\mathfrak{g}>1}$ with a partial-topological twist along the UV superconformal R-symmetry. In fact, in this case there exist a consistent truncation of type IIB supergravity to five-dimensional minimal supergravity [46]. Within the five-dimensional theory it is possible to construct the entire RG flow connecting the AdS_5 and AdS_3 backgrounds at hand analytically [6, 7, 47]. We believe that these supergravity and field theory results amount to very strong evidence for the proposed duality.

3.3 $Y^{p,0}$ on $\Sigma_{\mathfrak{g}>1}$ with baryonic flux

As seen on the field theory side, in the case of $Y^{p,q}$ quivers an interesting generalization of the universal twist arises by turning on a background baryonic flux B (we will discuss the possibility of turning on flavor flux below). We first discuss the special case $Y^{p,0}$ which includes the well-studied KW theory corresponding to $Y^{1,0} \cong T^{1,1}$. Setting $a_2 = c = 0$ in (3.16) and solving for the remaining parameters one finds that only $\kappa = -1$ is allowed. After some coordinate redefinitions and an overall rescaling, the metric and five-form read

$$ds_{10}^2 = ds_{AdS_3}^2 + \frac{v^2 + v + 1}{4v} ds_{\Sigma_{\mathfrak{g}>1}}^2 + \frac{v^2 + v + 1}{4(v+1)} \left(d\theta^2 + \sin^2 \theta d\phi^2 + \frac{1}{v} (dw^2 + \sin^2 w d\nu^2) \right) + \frac{1}{4} \left(d\psi - \cos \theta d\phi - \cos w d\nu - \frac{dx_1}{x_2} \right)^2 , \quad (3.28)$$

$$G_{(5)} = d\text{vol} AdS_3 \wedge \left(\frac{(v+1)^2}{2v} d\text{vol} \Sigma_{\mathfrak{g}>1} + \frac{1}{2(v+1)} \left(v^2 d\text{vol} S_{\theta\phi}^2 + \frac{1}{v} d\text{vol} S_{w\nu}^2 \right) \right) ,$$

where $v > 0$ is a parameter controlling the relative size of $\Sigma_{\mathfrak{g}}$ and the two S^2 's in the metric. Let us understand the geometry of $\mathcal{M}_7$. It is a $U(1)$ bundle over $\Sigma_{\mathfrak{g}} \times S^2 \times S^2$. Let $\psi \cong \psi + 2\pi\ell_\psi$,

then redefine $\psi' = \psi/\ell_\psi$, and compute the first Chern classes to be $-2(\mathfrak{g}-1)/\ell_\psi$, $2/\ell_\psi$, $2/\ell_\psi$ respectively. The maximal length is

$$\psi \cong \psi + 4\pi \quad (3.29)$$

and it corresponds to a fibration of $T^{1,1}$ over $\Sigma_{\mathfrak{g}}$. To arrive at $Y^{p,0}$ we need to take $\psi \cong \psi + 4\pi/p$. The central charge for the background (3.28) is given by (for conventions see Appendix B)

$$c_{\text{sugra}} = 6p(\mathfrak{g}-1) \frac{v^2 + v + 1}{(1+v)^2} N^2. \quad (3.30)$$

The 2-forms $d\text{vol}_\Sigma$, $d\text{vol}_{\theta\phi}$ and $d\text{vol}_{w\nu}$ are closed and potentially in cohomology, however because of de_ψ one linear combination is zero in cohomology. Correspondingly, there two 5-cycles we can easily construct in the geometry: 1) Since $\mathcal{M}_7$ is a fibration of $T^{1,1}$ over $\Sigma_{\mathfrak{g}}$, the fiber $T^{1,1}$ at a fixed point on $\Sigma_{\mathfrak{g}}$ is a good 5-cycle; 2) Since $T^{1,1} \cong S^3 \times S^2$, a 5-cycle is given by the fibration of S^3 over $\Sigma_{\mathfrak{g}}$. Two representatives of S^3 are at fixed (θ, ϕ) or fixed (w, ν) . By integrating $*G_5$ on those 5-cycles, we get:

$$N = 4\pi^3 \Lambda \frac{v^2 + v + 1}{v}, \quad N_1 = \frac{\mathfrak{g}-1}{v+1} N, \quad N_2 = (\mathfrak{g}-1) \frac{v}{v+1} N, \quad (3.31)$$

where $\Lambda \equiv \frac{1}{(2\pi l_s)^4} \frac{L^4}{pg_s}$. Notice that

$$N_1 + N_2 = (\mathfrak{g}-1)N, \quad \tilde{N}_B \equiv N_2 - N_1 = (\mathfrak{g}-1) \frac{v-1}{v+1} N. \quad (3.32)$$

The first relation is precisely the relation in homology: if N and N_1 are quantized, then N_2 is quantized as well. We interpret N (in some units) as the number of D3-branes, and $\tilde{N}_B$ as the baryonic flux. Notice that N is even and $\tilde{N}_B$ is odd under $v \rightarrow 1/v$. The solution with $v = 1$, which is the universal twist of the conifold over $\Sigma_{\mathfrak{g}}$, corresponds to setting $\tilde{N}_B = 0$. Solutions with non-trivial baryonic flux have other values of v , fixed by

$$v = \frac{(\mathfrak{g}-1)N + \tilde{N}_B}{(\mathfrak{g}-1)N - \tilde{N}_B}. \quad (3.33)$$

Using (3.33) in (3.30) gives

$$c_{\text{sugra}} = \frac{32}{3}(\mathfrak{g}-1)a(Y^{p,0}) + \frac{3p\tilde{N}_B^2}{2(\mathfrak{g}-1)}. \quad (3.34)$$

We recall that the KW theory with baryonic flux was considered in Section 2.2.1. Indeed, to make contact with the field theory result (2.39), we note that if $N(\mathfrak{g}-1)$ is odd, so is $\tilde{N}_B$ and we identify $\tilde{N}_B = N_B$. On the other hand, if $N(\mathfrak{g}-1)$ is even, so is $\tilde{N}_B$ and we identify $\tilde{N}_B = 2N_B$. Thus, we can write

$$c_{\text{sugra}} = \frac{32}{3}(\mathfrak{g}-1)a(Y^{p,0}) + \frac{3ph^2N_B^2}{2(\mathfrak{g}-1)}, \quad (3.35)$$

where $h = \text{hcf}[2, N(\mathfrak{g} - 1)]$. This matches precisely the field theory result (2.23) at large N . The solution with $N_B = 0$ corresponds to setting $v = 1$, and corresponds to the universal twist of the $Y^{p,0}$ theory over $\Sigma_{\mathfrak{g}}$.

3.4 $Y^{p,q}$ on T^2 with baryonic flux

An interesting solution is found by setting $a_2 = 0$ but keeping $c \neq 0$ (which we rescale to 1). For simplicity we discuss here only the case $\kappa = 0$. Solving the constraints among the remaining parameters imposed by (3.16), and some redefinitions the metric reads:

$$ds_{10}^2 = \frac{b}{\sqrt{\tilde{y}}} ds_{AdS_3}^2 + \frac{\sqrt{\tilde{y}}}{b} ds_{T^2}^2 + \frac{1}{4b\sqrt{\tilde{y}}} ds_{S^2}^2 + \frac{b}{4\tilde{y}^{5/2}(b^2 - (1 - \tilde{y})^2)} d\tilde{y}^2 \\ + \frac{(b^2 - (1 - \tilde{y})^2)}{4b\sqrt{\tilde{y}}(b^2 - 1 + 2\tilde{y})} (d\tilde{\beta} + \cos\theta d\phi)^2 + \frac{b^2 - 1 + 2\tilde{y}}{4b\sqrt{\tilde{y}}} \left(d\tilde{\psi} - \frac{\tilde{y}}{b^2 - 1 + 2\tilde{y}} (d\tilde{\beta} + \cos\theta d\phi) \right)^2. \quad (3.36)$$

The flux is given in the Appendix (see (A.93) and discussion below). This solution was found previously in [20] (see Section 4.1 of [20]), where its central charge was computed and shown to depend on four integers $p_{\text{DGK}}, q_{\text{DGK}}, M_{\text{DGK}}, N_{\text{DGK}}$ and is given by

$$c_{\text{sugra}} = \frac{6p_{\text{DGK}}q_{\text{DGK}}^2(p_{\text{DGK}} + 2q_{\text{DGK}})}{(p_{\text{DGK}} + q_{\text{DGK}})^2} M_{\text{DGK}} N_{\text{DGK}}. \quad (3.37)$$

By making the identifications $p = q_{\text{DGK}}, q = p_{\text{DGK}} + q_{\text{DGK}}$ and $BN^2 = M_{\text{DGK}}N_{\text{DGK}}$, the central charge matches exactly the field theory result (2.35). This is strong evidence that (3.36) is the gravity dual in the IR to $Y^{p,q}$ quiver gauge theories on T^2 , twisted by the superconformal R-symmetry and baryonic flux.

3.5 Solutions with flavor flux

Finally, we comment on turning the parameter a_2 on. There are several branches of solutions to the Bianchi identity, which we give in detail in Appendix A.4. Here we give only the local form of the solution, leaving a careful analysis of global properties, quantization of the five-form flux, and computation of the central charge for future work. Assuming $\mathfrak{g} > 1$ and $c \neq 0$ the metric is

given by

$$\begin{aligned}
ds_{10}^2 = & \sqrt{\frac{y(3+4a_2y)}{3(1-a)a_2^2-4C_1y}} ds_{AdS_3}^2 + \frac{1}{4} \sqrt{\frac{(3+4a_2y)(3(1-a)a_2^2-4C_1y)}{y}} ds_{\Sigma_{\mathfrak{g}>1}}^2 \\
& + \sqrt{\frac{y(3(1-a)a_2^2-4C_1y)}{3+4a_2y}} ds_{S^2}^2 + \frac{1}{4} \frac{\sqrt{y(3+4a_2y)(3(1-a)a_2^2-4C_1y)}}{\mathcal{Q}} dy^2 \\
& + \frac{4\mathcal{Q}\sqrt{y(3(1-a)a_2^2-4C_1y)}}{y^2(3+4a_2y)^{3/2}} D\beta^2 + \frac{1}{4} \sqrt{\frac{y(3+4a_2y)}{3(1-a)a_2^2-4C_1y}} \left(D\psi + \frac{2\mathcal{Q}'}{y(3+4a_2y)} D\beta \right)^2,
\end{aligned} \tag{3.38}$$

where $D\beta = d\beta + \cos\theta + a_2 A_{\mathfrak{g}}$, $D\psi = d\psi - \cos\theta - A_{\mathfrak{g}}$, and

$$\begin{aligned}
\mathcal{Q} = & \frac{-3(a-1)(7+a_2^2+8C_1-4a_2(1+C_1))}{16(a_2-1)^2} - \frac{3a_2(a-1)(a_2-4C_1-5)}{4(a_2-1)} y \\
& + \frac{3}{4} (1-4(a-1)a_2^2) y^2 + \frac{1}{3} (1+a_2-4C_1) y^3,
\end{aligned} \tag{3.39}$$

and $C_1 = -\frac{1}{4}(1+a_2+2\sqrt{1-a_2+a_2^2})$ and a is the only remaining integration constant and $a_2 \neq 1$. The five-form flux, which we do not write here, is determined by the formulas in Appendix A.4.4. In the special case $a_2 = 0$ the solution coincides with (3.25) with $c = 1$, which is seen by a simple change of coordinates. The case $a_2 = 1$ is a special branch (see Appendix A.4). For $\mathfrak{g} = 1$ there are two other branches of solutions, given in Appendix A.4.3.

4 Discussion

In this paper we have argued for the existence of a vast landscape of two-dimensional conformal field theories with $\mathcal{N} = (0, 2)$ supersymmetry. These theories arise through twisted compactifications of four-dimensional $\mathcal{N} = 1$ SCFTs on a smooth Riemann surface. If the four-dimensional theory has a weakly-coupled supergravity dual we have argued that the two-dimensional SCFT will also have one. We have illustrated in detail how these general ideas work for the case of $Y^{p,q}$ quiver gauge theories which arise from D3-branes probing toric Calabi-Yau singularities. We have also argued that there is universal RG flow across dimensions connecting two-dimensional $\mathcal{N} = (0, 2)$ and four-dimensional $\mathcal{N} = 1$ SCFTs. This flow bears a resemblance to the universal flow between four-dimensional $\mathcal{N} = 2$ and $\mathcal{N} = 1$ SCFTs discussed in [40].

Our supergravity solutions for the general $Y^{p,q}$ theories suggest that the two-dimensional SCFTs have, in general, large conformal manifolds. Some of the exactly marginal deformations are easy to identify. For $\mathfrak{g} > 1$ there are the $3\mathfrak{g}-3$ complex structure deformations of the Riemann surface¹¹ (in the case $\mathfrak{g} = 1$ there is one complex structure deformation). Besides, one can turn

¹¹We conjecture that similarly to the analysis in [48] the Kähler moduli of the Riemann surface correspond to irrelevant deformations.

on a flat connection for the $SU(2)$ flavor symmetry group that does not receive magnetic flux. As discussed in [42], the $SU(2)$ flavor group leads to $3\mathfrak{g} - 3$ independent complex moduli (one for $\mathfrak{g} = 1$). There might be other marginal deformations, for instance coming from flat connections for the remaining $U(1)$ mesonic and $U(1)_B$ baryonic flavor symmetry depending on the flux turned on, as well as other less manifest moduli. For $\mathfrak{g} = 0$ there are no complex structure deformations nor flat connections, and it is plausible that the corresponding two-dimensional SCFTs are isolated. This is surely an issue that deserves further study. Let us also remark that, although we have not studied supergravity solutions in which the $SU(2)$ flavor symmetry is broken, the field theory analysis in Section 2 suggests that there are two-dimensional SCFTs with only $U(1)_1 \times U(1)_2$ flavor and $U(1)_B$ baryonic symmetry for any $\mathfrak{g}$ in some range of the parameters $\{p, q, b_1, b_2, B\}$.

It is certainly desirable to have a more direct understanding of the 2d SCFTs uncovered by our construction. One way of thinking about these two-dimensional systems is to start from the four-dimensional theory on $\mathbb{R}^2 \times \Sigma_{\mathfrak{g}}$ with a partial topological twist on $\Sigma_{\mathfrak{g}}$ and write down the BPS equations following from the Lagrangian of the theory. Then the two-dimensional theory at low energies will be a nonlinear sigma model on the moduli space of solutions to these BPS equations. A similar analysis has been performed for four-dimensional $\mathcal{N} = 2$ SCFTs in [49]. The difficulty in this approach stems from the fact that the BPS equations for these four-dimensional theories are some appropriate generalizations of the Hitchin equations on $\Sigma_{\mathfrak{g}}$ and the moduli space of solutions is not known. An alternative approach would be to find a suitable two-dimensional gauged linear sigma model which in the IR describes the SCFTs of interest. It would be interesting to explore also whether there is a connection with the recent work in [31, 50, 51].

The current work as well as the construction in [7] leads to the natural question of whether one can establish a useful correspondence between 2d CFTs and some 2d TQFT on the compactification Riemann surface. This correspondence should be in the spirit of similar proposals that relate four- and three-dimensional SCFTs with two and three-dimensional TQFTs, respectively [52–54]. In the same spirit it is natural to extend our construction to Riemann surfaces with punctures.

It would also be interesting to compute the $T^2 \times S^2$ partition function for our theories and see whether we can match the result with the supergravity calculation. The field theory analysis should be accessible through the techniques developed recently in [30, 55].

The manifolds we constructed provide infinite-dimensional families of explicit 7d metrics of the type studied in [16, 22]. These manifolds seem to provide a natural generalization to Sasaki-Einstein geometry and it would be very interesting to understand their geometry further. The principle of c -extremization in the dual SCFT suggests that these type of manifolds enjoy some type of volume-minimization principle analogous to [56–58]. The analysis however will probably

be more complicated due to the important role played by the non-geometric baryonic $U(1)$ symmetries.

We have restricted our supergravity analysis to AdS_3 solutions with $SU(2) \times U(1) \times U(1)$ isometry. It should be possible to relax this assumption and look for solutions with lower amount of symmetry. While this will be technically complicated since the BPS equations will reduce to PDEs, rather than ODEs, our field theory analysis suggests that these PDEs should have interesting solutions. Among them should be the solutions dual to $Y^{p,q}$ theories on Σ_g with non-zero b_1 and b_2 flavor flux. In addition it is natural to expect that there is a generalization of our analysis to the $L^{p,q,r}$ quiver gauge theories which possess only $U(1) \times U(1) \times U(1)$ global symmetry [59–61].

The field theory calculation of the central charges of the two-dimensional SCFTs performed in Section 2 is exact while the supergravity results are valid only to leading order in the rank of the gauge group, N . It will certainly be very interesting to understand how the $1/N^2$ corrections to the central charge arise on the supergravity side. This should amount to understanding higher-curvature corrections to our type IIB supergravity backgrounds along the lines of [62].

It would be nice to extend our analysis and find similar AdS_2 solutions of eleven-dimensional supergravity. These should fall in the classification of [63] and be dual to M2-branes at the tip of a conical singularity, wrapping a compact Riemann surface. These solutions can be viewed as M2-brane black holes and the microscopic understanding of their entropy will be facilitated by the techniques recently developed in [64, 65].

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A BPS equations and Bianchi identities

Assuming that only the metric and 5-form flux are turned on, the type IIB gravitino variations read¹²

$$\delta\psi_\mu = \partial_\mu\epsilon + \frac{1}{4}\omega_{\mu ab}\Gamma^{ab}\epsilon + \frac{i}{192}F_{\mu\nu_1\nu_2\nu_3\nu_4}\Gamma^{\nu_1\nu_2\nu_3\nu_4}\epsilon = 0, \quad (\text{A.1})$$

where ϵ is a complex ten-dimensional spinor satisfying the chirality condition

$$\Gamma^{12345678910}\epsilon = \epsilon. \quad (\text{A.2})$$

The supersymmetry variation of the spin-1/2 fermion in type IIB supergravity vanishes identically when there are no non-trivial dilaton-axion and 3-form fluxes turned on. In addition there is the Bianchi identity for the self-dual five-form flux $F_{(5)} = (1 + *_{10})G_{(5)}$:

$$dF_{(5)} = 0 \quad \Rightarrow \quad dG_{(5)} = d *_ {10} G_{(5)} = 0. \quad (\text{A.3})$$

The symmetries of the problem suggest that we impose the following projectors on the spinor ϵ

$$\Gamma^{12}\epsilon = -\epsilon, \quad \Gamma^{45}\epsilon = i\epsilon, \quad \Gamma^{67}\epsilon = i\epsilon, \quad \Gamma^{89}\epsilon = i\epsilon, \quad (\text{A.4})$$

which together with the ten-dimensional chirality condition implies

$$\Gamma^{310}\epsilon = -i\epsilon. \quad (\text{A.5})$$

The most general metric Ansatz compatible with our expectations is

$$ds_{10}^2 = f_1(y)^2 ds_{AdS_3}^2 + f_2(y)^2 ds_{\Sigma_g}^2 + f_3(y)^2 ds_{S^2}^2 + f_4(y)^2 dy^2 + f_5(y)^2 (D\beta)^2 + f_6(y)^2 (D\psi + f_7(y)D\beta)^2, \\ G_{(5)} = e^1 \wedge e^2 \wedge e^3 \wedge [g_1(y)e^4 \wedge e^5 + g_2(y)e^6 \wedge e^7 + g_3(y)e^8 \wedge e^9 + g_4(y)e^8 \wedge e^{10} + g_5(y)e^9 \wedge e^{10}], \quad (\text{A.6})$$

where

$$ds_{AdS_3}^2 = \frac{-dt^2 + dz^2 + dr^2}{r^2}, \quad ds_{\Sigma_g}^2 = e^{2h(x_1, x_2)} (dx_1^2 + dx_2^2), \quad ds_{S^2}^2 = d\theta^2 + \sin^2\theta d\phi^2, \\ D\beta = d\beta + c \cos\theta d\phi + a_2 A_g(x_1, x_2), \quad D\psi = d\psi + b \cos\theta d\phi + a_3 A_g(x_1, x_2), \quad dA_g = d\text{vol}\Sigma_g,$$

and we defined the vielbein

$$e^1 = \frac{f_1}{r} dt, \quad e^2 = \frac{f_1}{r} dz, \quad e^3 = \frac{f_1}{r} dr, \quad e^4 = f_2 e^h dx_1, \quad e^5 = f_2 e^h dx_2, \\ e^6 = f_3 d\theta, \quad e^7 = f_3 \sin\theta d\phi, \quad e^8 = f_4 dy, \quad e^9 = f_5 D\beta, \quad e^{10} = f_6 (D\psi + f_7 D\beta). \quad (\text{A.7})$$

¹²We follow the conventions of [45].

The function $h(x_1, x_2)$ is defined in (3.6). We denote the volume forms by

$$d\text{vol}AdS_3 = \frac{1}{r^3} dt \wedge dz \wedge dr, \quad d\text{vol}\Sigma_{\mathfrak{g}} = e^{2h(x_1, x_2)} dx_1 \wedge dx_2, \quad d\text{vol}S^2 = \sin \theta d\theta \wedge d\phi. \quad (\text{A.8})$$

In our normalization

$$\text{vol}\Sigma_{\mathfrak{g}} = \int_{\Sigma_{\mathfrak{g}}} d\text{vol}\Sigma_{\mathfrak{g}} = \begin{cases} 4\pi|\mathfrak{g} - 1|, & \mathfrak{g} \neq 1 \\ 2\pi, & \mathfrak{g} = 1 \end{cases} \quad (\text{A.9})$$

Now we can use the gravitino variation in (A.1) along with the projectors in (A.2), (A.4) to derive a set of differential equations for the unknown functions f_i, g_i appearing in the Ansatz (A.6). Before writing out all the equations we show that $g_5 = 0$. This follows from the $\delta\psi_5$ component of the gravitino variation¹³

$$\delta\psi_5 = \frac{x_2}{f_2} \partial_{x_2} \epsilon + \frac{1}{8} \left[4 \frac{f'_2}{f_2 f_4} - 2a_2 \frac{f_5}{f_2^2} - g_4 - ig_5 \right] \Gamma^{49} \epsilon - \frac{i}{8} \left[2 \frac{f_6(a_3 + a_2 f_7)}{f_2^2} + g_1 - g_2 - g_3 \right] \Gamma^{34} \epsilon = 0. \quad (\text{A.10})$$

For the spinor of interest we should have $\partial_{x_2} \epsilon = 0$.¹⁴ Then the equation $\delta\psi_5 = 0$ is of the form

$$iA\Gamma^{34}\epsilon + (B + iC)\Gamma^{49}\epsilon = [-iA + (B + iC)\Gamma^{39}] \epsilon = 0, \quad (\text{A.11})$$

with $\{A, B, C\}$ real and $C = -g_5/8$. This equation implies $A^2 - B^2 + C^2 = BC = 0$, which in turn implies $C = 0$ and thus $g_5 = 0$ in order to have nontrivial solutions. From now on we set $g_5 = 0$ in the remaining equations. The explicit form of all gravitino variations is:

$$\delta\psi_1 = \frac{r}{f_1} \partial_t \epsilon + \frac{1}{8} \left[\frac{4}{f_1} - (g_1 + g_2 + g_3) \right] \Gamma^{13} \epsilon - \frac{1}{8} \left[\frac{4f'_1}{f_1 f_4} + g_4 \right] \Gamma^{18} \epsilon, \quad (\text{A.12})$$

$$\delta\psi_2 = \frac{r}{f_1} \partial_z \epsilon - \frac{1}{8} \left[\frac{4}{f_1} - (g_1 + g_2 + g_3) \right] \Gamma^{13} \epsilon + \frac{1}{8} \left[\frac{4f'_1}{f_1 f_4} + g_4 \right] \Gamma^{18} \epsilon, \quad (\text{A.13})$$

$$\delta\psi_3 = \frac{r}{f_1} \partial_r \epsilon + \frac{1}{8} (g_1 + g_2 + g_3) \epsilon + \frac{1}{8} \left[\frac{4f'_1}{f_1 f_4} + g_4 \right] \Gamma^{38} \epsilon, \quad (\text{A.14})$$

¹³One can also argue that $g_5 = 0$ using the equations of motion.

¹⁴Here we assume that the metric on the Riemann surface is the constant curvature one and thus the spinors do not depend on the coordinates on the Riemann surface. This assumptions could in principle be relaxed but the general analysis is more involved. The results of [48] however suggest that the constant curvature metric is capturing all interesting physics.

$$\begin{aligned}\delta\psi_4 = & \frac{1}{2f_2e^h} [2\partial_{x_1} - 2A_1a_2\partial_\beta - 2A_1a_3\partial_\psi + i\partial_{x_2}h] \epsilon + \frac{1}{8} \left[-4\frac{f'_2}{f_2f_4} + 2a_2\frac{f_5}{f_2^2} + g_4 \right] \Gamma^{59} \epsilon \\ & + \frac{1}{8} \left[\frac{2f_6(a_3 + a_2f_7)}{f_2^2} + g_1 - g_2 - g_3 \right] \Gamma^{510} \epsilon ,\end{aligned}\quad (\text{A.15})$$

$$\begin{aligned}\delta\psi_5 = & \frac{1}{2f_2e^h} [2\partial_{x_2} - 2A_2a_2\partial_\beta - 2A_2a_3\partial_\psi - i\partial_{x_1}h] \epsilon + \frac{1}{8} \left[4\frac{f'_2}{f_2f_4} - 2a_2\frac{f_5}{f_2^2} - g_4 \right] \Gamma^{49} \epsilon \\ & - \frac{i}{8} \left[2\frac{f_6(a_3 + a_2f_7)}{f_2^2} + g_1 - g_2 - g_3 \right] \Gamma^{34} \epsilon ,\end{aligned}\quad (\text{A.16})$$

$$\delta\psi_6 = \frac{1}{f_3} \partial_\theta \epsilon + \frac{1}{8} \left[-4\frac{f'_3}{f_3f_4} - 2c\frac{f_5}{f_3^2} + g_4 \right] \Gamma^{79} \epsilon + \frac{1}{8} \left[-2\frac{f_6(cf_7 + b)}{f_3^2} - g_1 + g_2 - g_3 \right] \Gamma^{710} \epsilon , \quad (\text{A.17})$$

$$\begin{aligned}\delta\psi_7 = & \frac{1}{f_3 \sin \theta} [\partial_\phi + \cos \theta (-b\partial_\psi - c\partial_\beta - \frac{i}{2})] \epsilon + \frac{1}{8} \left[4\frac{f'_3}{f_3f_4} + 2c\frac{f_5}{f_3^2} - g_4 \right] \Gamma^{69} \epsilon \\ & + \frac{1}{8} \left[2\frac{f_6(cf_7 + b)}{f_3^2} + g_1 - g_2 + g_3 \right] \Gamma^{610} \epsilon ,\end{aligned}\quad (\text{A.18})$$

$$\delta\psi_8 = \left[\frac{1}{f_4} \partial_y \epsilon + \frac{1}{8} g_4 \epsilon \right] + \frac{1}{8} \left[2\frac{f_6f'_7}{f_4f_5} - g_1 - g_2 + g_3 \right] \Gamma^{910} \epsilon , \quad (\text{A.19})$$

$$\begin{aligned}\delta\psi_9 = & \frac{1}{f_5} \left[\partial_\beta - f_7\partial_\psi - \frac{i}{2}\frac{f'_5}{f_4} - \frac{i}{4}f_5^2 \left(\frac{a_2}{f_2^2} - \frac{c}{f_3^2} \right) + \frac{i}{8}f_5g_4 \right] \epsilon \\ & - \frac{1}{8} \left[2\frac{f_6f'_7}{f_4f_5} - g_1 - g_2 + g_3 \right] \Gamma^{810} \epsilon ,\end{aligned}\quad (\text{A.20})$$

$$\begin{aligned}\delta\psi_{10} = & \left[\frac{1}{f_6} \partial_\psi - \frac{i}{4}f_6 \left(\frac{(a_3 + a_2f_7)}{f_2^2} - \frac{(cf_7 + b)}{f_3^2} \right) - \frac{i}{8} \left(2\frac{f_6f'_7}{f_4f_5} + g_1 + g_2 + g_3 \right) \right] \epsilon \\ & - \frac{1}{8} \left[4\frac{f'_6}{f_4f_6} + g_4 \right] \Gamma^{810} \epsilon .\end{aligned}\quad (\text{A.21})$$

The gravitino variations are of the form $(A + B \Gamma^{c_1 c_2}) \epsilon = 0$ for some real A and B , $c_1 \neq c_2$. If ϵ and $\Gamma^{c_1 c_2} \epsilon$ are independent spinors (*i.e.*, $\Gamma^{c_1 c_2}$ is none of the projectors appearing in A.4), then $A = B = 0$. Equipped with this fact we are ready to analyze the gravitino variations in detail.

We first focus on the terms proportional to gamma matrices in $\delta\psi_\mu$; this leads to a total of

eight independent equations:

$$4 - f_1(g_1 + g_2 + g_3) = 0, \quad (\text{A.22})$$

$$-g_1 + g_2 + g_3 - \frac{2f_6(a_2f_7 + a_3)}{f_2^2} = 0, \quad (\text{A.23})$$

$$g_1 - g_2 + g_3 + \frac{2f_6(cf_7 + b)}{f_3^2} = 0, \quad (\text{A.24})$$

$$g_4 + \frac{4f'_1}{f_1f_4} = 0, \quad (\text{A.25})$$

$$g_4 + \frac{4f'_6}{f_4f_6} = 0, \quad (\text{A.26})$$

$$-\frac{2f_6f'_7}{f_4f_5} + g_1 + g_2 - g_3 = 0, \quad (\text{A.27})$$

$$-\frac{2a_2f_5}{f_2^2} + \frac{4f'_2}{f_2f_4} - g_4 = 0, \quad (\text{A.28})$$

$$\frac{2cf_5}{f_3^2} + \frac{4f'_3}{f_3f_4} - g_4 = 0. \quad (\text{A.29})$$

From (A.22-A.26) we can algebraically solve for the functions g_i :

$$\begin{aligned} g_1 &= \frac{2}{f_1} - \frac{f_6(a_3 + a_2f_7)}{f_2^2}, \\ g_2 &= \frac{2}{f_1} + \frac{f_6(b + cf_7)}{f_3^2}, \\ g_3 &= f_6 \left(\frac{a_2f_7 + a_3}{f_2^2} - \frac{b + cf_7}{f_3^2} \right), \\ g_4 &= -\frac{4f'_6}{f_4f_6} = -\frac{4f'_1}{f_4f_1}, \end{aligned} \quad (\text{A.30})$$

and we also find

$$f_6 = \alpha_2 f_1, \quad (\text{A.31})$$

with α_2 a constant. From now on we consider the functions g_i determined in terms of the f_i by (A.30). The remaining equations (A.27-A.29) read

$$\frac{2}{f_1} + \alpha_2 f_1 \left(\frac{-a_3 - a_2f_7}{f_2^2} + \frac{b + cf_7}{f_3^2} - \frac{f'_7}{f_4f_5} \right) = 0, \quad (\text{A.32})$$

$$\frac{2f'_1}{f_1f_4} + \frac{2f'_2}{f_2f_4} - \frac{a_2f_5}{f_2^2} = 0, \quad (\text{A.33})$$

$$\frac{2f'_1}{f_1f_4} + \frac{2f'_3}{f_3f_4} + \frac{cf_5}{f_3^2} = 0. \quad (\text{A.34})$$

Now we turn to the equations arising from terms proportional to the identity in $\delta\psi_\mu$. There are ten such equations in total. From (A.12), (A.13), (A.14), (A.16), part of (A.18), and (A.19) one immediately concludes that

$$\partial_t\epsilon = \partial_z\epsilon = \partial_\theta\epsilon = \partial_\phi\epsilon = 0. \quad (\text{A.35})$$

In addition, we assume that $\partial_{x_1}\epsilon = \partial_{x_2}\epsilon = 0$. Then

$$\epsilon = \left(\frac{f_1(y)}{r}\right)^{1/2} \tilde{\epsilon}(\beta, \psi), \quad (\text{A.36})$$

where $\tilde{\epsilon}(\beta, \psi)$ is a spinor that will be fixed shortly. In addition, we have the equations

$$(i\partial_{x_2}h - 2a_2A_1\partial_\beta - 2a_3A_1\partial_\psi)\epsilon = 0, \quad (\text{A.37})$$

$$(-i\partial_{x_1}h - 2a_2A_2\partial_\beta - 2a_3A_2\partial_\psi)\epsilon = 0, \quad (\text{A.38})$$

$$[-i - 2c\partial_\beta - 2b\partial_\psi]\epsilon = 0, \quad (\text{A.39})$$

$$\left[2\left(\frac{a_2}{f_2^2} - \frac{c}{f_3^2}\right)f_5 - g_4 + \frac{4f_5'}{f_4f_5} + \frac{8i(\partial_\beta - f_7\partial_\psi)}{f_5}\right]\epsilon = 0, \quad (\text{A.40})$$

$$\left[\frac{4}{f_1} + \frac{8i\partial_\psi}{f_6} - 2f_6\left(\frac{-a_3 - a_2f_7}{f_2^2} + \frac{b + cf_7}{f_3^2} - \frac{f_7'}{f_4f_5}\right)\right]\epsilon = 0. \quad (\text{A.41})$$

Combining (A.32) and (A.41) implies $\partial_\psi\epsilon = i\alpha_2\epsilon$ and from (A.39) we have

$$\epsilon = e^{i(\alpha_1\beta + \alpha_2\psi)} \left(\frac{f_1(y)}{r}\right)^{1/2} \epsilon_0, \quad (\text{A.42})$$

where ϵ_0 is a constant spinor obeying the projection conditions (A.2), (A.4) and

$$c\alpha_1 + b\alpha_2 = -\frac{1}{2}. \quad (\text{A.43})$$

Thus, we are left with the set of BPS equations

$$\frac{2f_1'}{f_1f_4} + \frac{2f_2'}{f_2f_4} - \frac{a_2f_5}{f_2^2} = 0, \quad (\text{A.44})$$

$$\frac{2f_1'}{f_1f_4} + \frac{2f_3'}{f_3f_4} + \frac{cf_5}{f_3^2} = 0, \quad (\text{A.45})$$

$$\frac{2}{f_1} + \alpha_2f_1\left(\frac{-a_3 - a_2f_7}{f_2^2} + \frac{b + cf_7}{f_3^2} - \frac{f_7'}{f_4f_5}\right) = 0, \quad (\text{A.46})$$

$$\left[2\left(\frac{a_2}{f_2^2} - \frac{c}{f_3^2}\right)f_5 - g_4 + \frac{4f_5'}{f_4f_5} - \frac{8(\alpha_1 - f_7\alpha_2)}{f_5}\right]\epsilon = 0, \quad (\text{A.47})$$

$$(i\partial_{x_2}h - 2iA_1(\alpha_1a_2 + a_3\alpha_2))\epsilon = 0, \quad (\text{A.48})$$

$$(-i\partial_{x_1}h - 2iA_2(\alpha_1a_2 + a_3\alpha_2))\epsilon = 0. \quad (\text{A.49})$$

It is convenient to choose the coordinate y such that

$$f_1^2 f_4 f_5 = D, \quad (\text{A.50})$$

where D is a constant. With this coordinate choice (A.44) and (A.45) simplify to

$$(f_1^2 f_2^2)' = a_2 D, \quad (f_1^2 f_3^2)' = -c D. \quad (\text{A.51})$$

Let us analyze equations (A.48), (A.49). Taking the derivative of these equations one has

$$2(a_2 \alpha_1 + a_3 \alpha_2) dA_{\mathfrak{g}} = -(\partial_{x_1}^2 + \partial_{x_2}^2) h dx_1 \wedge dx_2. \quad (\text{A.52})$$

For $\mathfrak{g} \neq 1$ (*i.e.* $\kappa \neq 0$) the background R-symmetry flux is set to $dA = d\text{vol}\Sigma_{\mathfrak{g}} = e^{2h} dx_1 \wedge dx_2$ in order to preserve supersymmetry. From (3.6), (A.52) implies the consistency condition

$$a_2 \alpha_1 + a_3 \alpha_2 = \frac{\kappa}{2}. \quad (\text{A.53})$$

Thus equations (A.48), (A.49) imply

$$A_{\mathfrak{g} \neq 1} = \kappa (\partial_{x_2} h dx^1 - \partial_{x_1} h dx^2), \quad (\text{A.54})$$

which is compatible with $dA_{\mathfrak{g}} = d\text{vol}\Sigma_{\mathfrak{g}}$. Of course, the connection is defined up to gauge transformations $A_{\mathfrak{g}} \rightarrow A_{\mathfrak{g}} + d\lambda$. In the case $\mathfrak{g} = 1$, we can choose a gauge in which:

$$A_{\mathfrak{g}=1} = \frac{1}{2\pi} x_1 dx_2. \quad (\text{A.55})$$

Up to this point the only assumptions we have made are the Ansatz (A.6), the projectors (A.4) and that the spinor ϵ is independent of $\{x_1, x_2\}$.

Bianchi

In addition to the BPS equations there are a total of four Bianchi identities from (A.3) which read:

$$\partial_y(g_1 f_1^3 f_2^2) - a_2 g_3 f_1^3 f_4 f_5 - g_4 f_1^3 f_4 f_6 (a_2 f_7 + a_3) = 0, \quad (\text{A.56})$$

$$\partial_y(g_2 f_1^3 f_3^2) + c g_3 f_1^3 f_4 f_5 + g_4 f_1^3 f_4 f_6 (c f_7 + b) = 0, \quad (\text{A.57})$$

$$\partial_y(f_2^2 f_3^2 g_3 f_6) + f_4 f_5 f_6 (c g_2 f_2^2 - a_2 g_1 f_3^2) = 0, \quad (\text{A.58})$$

$$\partial_y(g_3 f_2^2 f_3^2 f_6 f_7) - \partial_y(g_4 f_2^2 f_3^2 f_5) + f_4 f_5 f_6 (a_3 g_1 f_3^2 - b g_2 f_2^2) = 0. \quad (\text{A.59})$$

Here we have used $dA_{\mathfrak{g}} = d\text{vol}\Sigma_{\mathfrak{g}}$ to simplify the equations.

A.1 General solution

To solve the BPS equations and Bianchi it is convenient to trade the functions f_1, f_2, f_3, f_4, f_7 by the functions $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{Q}, \mathcal{P}_7$, defined by:

$$\begin{aligned} f_1^2(y) &= \sqrt{\frac{\mathcal{P}_2(y)\mathcal{P}_3(y)}{\mathcal{P}_1(y)}}, & f_2^2(y) &= \sqrt{\frac{\mathcal{P}_2(y)\mathcal{P}_1(y)}{\mathcal{P}_3(y)}}, & f_3^2(y) &= \sqrt{\frac{\mathcal{P}_3(y)\mathcal{P}_1(y)}{\mathcal{P}_2(y)}}, \\ f_4^2(y) &= D^2 \frac{\sqrt{\mathcal{P}_1(y)\mathcal{P}_2(y)\mathcal{P}_3(y)}}{\mathcal{Q}(y)}, & f_7(y) &= \frac{\mathcal{P}_7(y)}{\mathcal{P}_2(y)\mathcal{P}_3(y)}. \end{aligned} \quad (\text{A.60})$$

Recall that f_5, f_6 are given by (A.50), (A.31), respectively. This form of the Ansatz requires that

$$\text{sign}(\mathcal{P}_1(y)\mathcal{P}_2(y)\mathcal{P}_3(y)) = +, \quad \mathcal{Q}(y) > 0. \quad (\text{A.61})$$

We assume that the range of interest for $y \in [y_1, y_2]$ is finite and that in this range

$$\mathcal{P}_2(y), \mathcal{P}_3(y) > 0, \quad y \in [y_1, y_2], \quad (\text{A.62})$$

which also implies that $\mathcal{P}_1(y) > 0$ in this range. From now on we omit the argument in the functions $\mathcal{P}, \mathcal{Q}$. From (A.51) it follows that

$$\mathcal{P}_2 = a_2 D y + C_2, \quad \mathcal{P}_3 = -c D y + C_3. \quad (\text{A.63})$$

The remaining BPS equations are (A.46) and (A.47), which using (A.60) can be used to write $\mathcal{P}_{1,7}$ in terms of the known functions $\mathcal{P}_{2,3}$ and the (yet undermined) function $\mathcal{Q}$ as:

$$\mathcal{P}_1 = \frac{\alpha_2 (a_3 D \mathcal{P}_3 - b D \mathcal{P}_2 + \mathcal{P}_7')}{2D}, \quad \mathcal{P}_7 = \frac{4D\alpha_1 \mathcal{P}_2 \mathcal{P}_3 - \mathcal{Q}'}{4D\alpha_2}. \quad (\text{A.64})$$

Thus, given a function $\mathcal{Q}$ all local solutions to the BPS equations are given by (A.60,A.63,A.64) and are characterized by the parameters $\{a_2, a_3, b, c, C_2, C_3, D, \alpha_1, \alpha_2, \kappa\}$, subject to the constraints (A.43, A.53). Not all these parameters are physical, *e.g.*, by a rescaling of y one can set $D = 1$, and other parameters may also be absorbed by coordinate transformations. We will analyze this in more detail below.

The only equations that remain to be solved are the Bianchi identities (A.56-A.59). The first three of these equations are automatically satisfied, assuming the BPS equations. Thus, the only remaining equation is (A.59), which using the BPS equations can be written as a fourth order differential equation for the function $\mathcal{Q}$ and has the form

$$\text{Bianchi} = \mathcal{B}_2 \times \mathcal{B}_4 = 0, \quad (\text{A.65})$$

with $\mathcal{B}_2 = \mathcal{Q}'' - 2D^2(\mathcal{P}_2 + \kappa\mathcal{P}_3)$ and $\mathcal{B}_4$ depends on up to fourth order derivatives of $\mathcal{Q}$. There are two branches: $\mathcal{B}_2 = 0$ and $\mathcal{B}_4 = 0$. The former leads to

$$\mathcal{Q} = \frac{D^3}{3} y^3 (a_2 - c\kappa) + D^2 y^2 (C_2 + C_3 \kappa) + \gamma_2 y + \gamma_1, \quad (\text{A.66})$$

where $\gamma_{1,2}$ are constants. However, using this solution in (A.64) implies $\mathcal{P}_1 = 0$ which is clearly singular. Thus, the solutions of interest arise from the branch $\mathcal{B}_4 = 0$. This is a fourth order differential equation for $\mathcal{Q}$, which is rather complicated. However, it has the remarkable property that it can be integrated twice into the second order equation:¹⁵

$$\frac{1}{\mathcal{P}_2 \mathcal{P}_3} (\mathcal{Q}'^2 - 2\mathcal{Q}(\mathcal{Q}'' - 2D^2(\mathcal{P}_2 + \kappa \mathcal{P}_3))) - 4D^4 y^2 \kappa + \delta_1 + \delta_2 y = 0, \quad (\text{A.67})$$

where $\delta_{1,2}$ are integration constants.¹⁶ Thus, general solutions to the BPS equations and Bianchi are characterized completely by solutions to this differential equation. Although we have not found the most general solution to (A.67), it is straightforward to show that the most general *polynomial* solution can be at most cubic, *i.e.*,

$$\mathcal{Q} = \sum_{i=0}^3 q_i y^i. \quad (\text{A.68})$$

Plugging (A.68) into (A.67) leads to a system of five algebraic equations, which determine the four coefficients q_i plus one constraint among the remaining parameters. The generic solution can be written in the iterative form¹⁷

$$\begin{aligned} q_3 &= -\frac{2}{3} \left(-a_2 + c\kappa \pm \sqrt{a_2^2 + a_2 \kappa c + c^2 \kappa^2} \right), \\ q_2 &= C_2 + \kappa C_3 + \frac{-4a_2 C_2 + a_2 c \delta_2 + 4C_3 c \kappa^2}{4(a_2 - c\kappa - q_3)}, \\ q_1 &= \frac{a_2 c \delta_1 + \delta_2 (c C_2 - a_2 C_3) + 4C_2 C_3 \kappa - 4(C_2 + \kappa C_3) q_2}{4a_2 - 4c\kappa - 6q_3}, \\ q_0 &= -\frac{C_2 C_3 \delta_1 + q_1^2}{4(C_2 + C_3 \kappa - q_2)}, \end{aligned} \quad (\text{A.69})$$

together with the constraint

$$\frac{a_2 c (a_2 + c\kappa) \delta_1 + (a_2^2 C_3 - C_2 c^2 \kappa) \delta_2 + 4\kappa (2C_2 C_3 (a_2 + \kappa c) + c C_2^2 + a_2 \kappa C_3^2)}{16(a_2 + c\kappa)^2 (a_2^2 + a_2 c \kappa + c^2 \kappa^2)} = 0. \quad (\text{A.70})$$

Since the denominator in (A.70) vanishes for the special values $a_2 = -\kappa c$ or $a_2 = 0 = c$ or $a_2 = 0 = \kappa$, these cases must be analyzed separately.

We note that when $\mathcal{Q}$ is a cubic polynomial, the function $\mathcal{P}_1$ given in (A.64) becomes linear:

$$\mathcal{P}_1 = \frac{1}{4} (C_2 + C_3 \kappa - q_2) + \frac{1}{4} (a_2 - c\kappa - 3q_3) y, \quad (\text{A.71})$$

¹⁵We note that the differential equation below has the form $\mathcal{Q}(y)^{\prime 2} - 2\mathcal{Q}(y)\mathcal{Q}(y)'' + 2\mathcal{Q}(y)F(y) + G(y) = 0$. Defining $q(y) = \mathcal{Q}(y)^{1/2}$, it becomes $q(y)'' - \frac{G(y)}{4q(y)^3} - \frac{F(y)}{2q(y)} = 0$.

¹⁶It is worth pointing out that a similar differential equation controls a class of supersymmetric AdS_3 solutions of eleven-dimensional supergravity studied in [18].

¹⁷We have used a rescaling of the y coordinate to set $D = 1$ for convenience.

and $\mathcal{P}_7$ becomes a quadratic function of y .

Finally, we give the expression for the five-form $G_{(5)}$. Using (A.30) and the expression for $\mathcal{P}_7$ in (A.64) and the relations (A.43,A.53), we find:

$$G_{(5)} = d\text{vol}AdS_3 \wedge [G_1(y) d\text{vol}\Sigma_{\mathfrak{g}} + G_2(y) d\text{vol}S^2 + G_3(y)dy \wedge D\beta + G_4(y)dy \wedge (D\psi + f_7(y) D\beta)] \quad (\text{A.72})$$

where

$$\begin{aligned} G_1(y) &= \frac{1}{4D\mathcal{P}_1}(8D\mathcal{P}_1\mathcal{P}_2 - 2D\kappa\mathcal{P}_2\mathcal{P}_3 + a_2\mathcal{Q}'), & G_2(y) &= \frac{1}{4D\mathcal{P}_1}(8D\mathcal{P}_1\mathcal{P}_3 - 2D\mathcal{P}_2\mathcal{P}_3 - c\mathcal{Q}'), \\ G_3(y) &= \frac{2D\mathcal{P}_2\mathcal{P}_3(\mathcal{P}_2 + \kappa\mathcal{P}_3) + (c\mathcal{P}_2 - a_2\mathcal{P}_3)\mathcal{Q}'}{4\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3}, & G_4(y) &= -\alpha_2 \left(\frac{\mathcal{P}_2\mathcal{P}_3}{\mathcal{P}_1} \right)'. \end{aligned} \quad (\text{A.73})$$

This completes our analysis of the local solutions to the BPS equations and Bianchi identities. It is worth emphasizing that despite the fact that our backgrounds preserve only 4 out of the 32 supercharges of eleven-dimensional supergravity we have managed to solve the BPS equations in full detail analytically.

A.2 Linear transformations on (β, ψ)

Let us focus on the two-dimensional part of the metric (A.6) corresponding to the (β, ψ) plain:

$$\begin{aligned} ds_2^2 &= f_5(y)^2 (d\beta + c \cos \theta d\phi + a_2 A_{\mathfrak{g}})^2 \\ &\quad + f_6(y)^2 \left(d\psi + b \cos \theta d\phi + f_7(y) \left(d\beta + c \cos \theta d\phi + \frac{a_2}{x_2} dx_1 \right) + a_3 A_{\mathfrak{g}} \right)^2, \end{aligned} \quad (\text{A.74})$$

and perform the linear transformation

$$(d\beta, d\psi)^\top = \mathcal{W} (\tilde{d}\beta, \tilde{d}\psi)^\top, \quad \mathcal{W} \equiv \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}. \quad (\text{A.75})$$

The Killing vectors transform as $(\partial_\beta, \partial_\psi) = (\tilde{\partial}_\beta, \tilde{\partial}_\psi) \mathcal{W}^{-1}$. It is easy to see that the metric takes the same form as in (A.74), *i.e.*,

$$\begin{aligned} \tilde{ds}_2^2 &= \tilde{f}_5(y)^2 \left(d\tilde{\beta} + \tilde{c} \cos \theta d\phi + \tilde{a}_2 A_{\mathfrak{g}} \right)^2 \\ &\quad + \tilde{f}_6(y)^2 \left(d\tilde{\psi} + \tilde{b} \cos \theta d\phi + \tilde{f}_7(y) \left(d\tilde{\beta} + \tilde{c} \cos \theta d\phi + \frac{\tilde{a}_2}{x_2} dx_1 \right) + \tilde{a}_3 A_{\mathfrak{g}} \right)^2, \end{aligned} \quad (\text{A.76})$$

where the new parameters are given by

$$(\tilde{a}_2, \tilde{a}_3)^\top = \mathcal{W}^{-1} (a_2, a_3)^\top, \quad (\tilde{c}, \tilde{b})^\top = \mathcal{W}^{-1} (c, b)^\top, \quad (\text{A.77})$$

and the functions by

$$\begin{aligned}\tilde{f}_5^2 &= (\det \mathcal{W})^2 \frac{f_5^2 f_6^2}{w_2^2 f_5^2 + f_6^2 (w_4 + w_2 f_7)^2} , \\ \tilde{f}_6^2 &= w_2^2 f_5^2 + f_6^2 (w_4 + w_2 f_7)^2 , \\ \tilde{f}_7 &= \frac{w_1 w_2 f_5^2 + f_6^2 (w_3 + w_1 f_7)(w_4 + w_2 f_7)}{w_2^2 f_5^2 + f_6^2 (w_4 + w_2 f_7)^2} .\end{aligned}\tag{A.78}$$

These expressions are useful in proving the absence of conical singularities, as shown in Section 3.1.1.

Setting $\alpha_1 = 0$

Let us perform a linear coordinate transformation as above with

$$w_1 = w_4 = 1 , \quad w_2 = 0 , \tag{A.79}$$

while keeping w_3 arbitrary for now. One can see that

$$\tilde{a}_2 = a_2 , \quad \tilde{c} = c , \quad \tilde{f}_5^2 = f_5^2 , \quad \tilde{f}_6^2 = f_6^2 . \tag{A.80}$$

Moreover one has

$$\tilde{a}_3 = a_3 - w_3 a_2 , \quad \tilde{b} = b - w_3 c , \quad \tilde{f}_7 = f_7 + w_3 . \tag{A.81}$$

Note that a_3 and b appear in the differential BPS equations and Bianchi identity only through the combinations

$$a_3 + a_2 f_7 = \tilde{a}_3 + \tilde{a}_2 \tilde{f}_7 , \quad b + c f_7 = \tilde{b} + \tilde{c} \tilde{f}_7 . \tag{A.82}$$

Thus the only effect of the arbitrary constant w_3 on the system of differential equations is in the expression

$$\alpha_1 - \alpha_2 f_7 = (\alpha_1 + \alpha_2 w_3) - \alpha_2 \tilde{f}_7 . \tag{A.83}$$

The phase of the spinor is also modified:

$$\alpha_1 \beta + \alpha_2 \psi = (\alpha_1 + \alpha_2 w_3) \tilde{\beta} + \alpha_2 \tilde{\psi} . \tag{A.84}$$

Finally, the algebraic constraints from the BPS equations read

$$a_3 \alpha_2 + a_2 \alpha_1 = \tilde{a}_3 \alpha_2 + \tilde{a}_2 (\alpha_1 + \alpha_2 w_3) = \frac{\kappa}{2} , \quad b \alpha_2 + c \alpha_1 = \tilde{b} \alpha_2 + \tilde{c} (\alpha_1 + \alpha_2 w_3) = -\frac{1}{2} . \tag{A.85}$$

Since we assume $\alpha_2 \neq 0$ (as we must, since $f_6 = \alpha_2 f_1$) we can always choose the arbitrary constant $w_3 = -\alpha_1/\alpha_2$, thus eliminating the constant α_1 . We conclude from this analysis that we can safely set $\alpha_1 = 0$ from the beginning, which is often convenient. Finally, by a rescaling of the coordinate ψ it is possible to set α_2 to any nonzero value. Note that in this argument we have not made any assumptions on the solutions to the BPS equations or the values of the parameters $\{a_{2,3}, b, c\}$.

A.3 Solutions with no flavor flux: $a_2 = 0$

Setting $a_2 = 0$ leads to many simplifications. Nonetheless, the system is still quite rich and there are four cases that must be analyzed separately: (i) $\kappa \neq 0, c = 0$; (ii) $\kappa \neq 0, c \neq 0$; (iii) $\kappa = 0, c = 0$ and; (iv) $\kappa = 0, c \neq 0$.

A.3.1 Case (i): $\kappa \neq 0, c = 0$

One finds that only $\kappa = -1$ is allowed. After some coordinate redefinitions and an overall rescaling of the metric, the solution reads

$$ds_{10}^2 = ds_{AdS_3}^2 + \frac{v^2 + v + 1}{4v} ds_{\Sigma_{g>1}}^2 + \frac{v^2 + v + 1}{4(v+1)} \left(d\theta^2 + \sin^2 \theta d\phi^2 + \frac{1}{v} (dw^2 + \sin^2 w d\nu^2) \right) + \frac{1}{4} \left(d\psi - \cos \theta d\phi - \cos w d\nu - \frac{dx_1}{x_2} \right)^2, \quad (A.86)$$

$$G_{(5)} = d\text{vol} AdS_3 \wedge \left(\frac{(v+1)^2}{2v} d\text{vol} \Sigma_{g>1} + \frac{1}{2(v+1)} \left(v^2 d\text{vol} S_{\theta\phi}^2 + \frac{1}{v} d\text{vol} S_{w\nu}^2 \right) \right), \quad (A.86)$$

where $v > 0$ is a real parameter. This is the solution discussed in some detail in Section 3.3. For the special value $v = 1$ the metric of the solution can be written as

$$ds_{10}^2 = ds_{AdS_3}^2 + \frac{3}{4} ds_{\Sigma_{g>1}}^2 + \frac{9}{4} ds_{T^{1,1}}^2, \quad (A.87)$$

where $ds_{T^{1,1}}^2$ is the metric for the conifold with a Reeb vector fibered over Σ_g . This solution is an example of the solutions discussed in Section 6.1 of [18] for $\mathbb{P}^1 \times \mathbb{P}^1$ as the Kähler-Einstein base.

A.3.2 Case (ii): $\kappa \neq 0, c \neq 0$

There are two branches, corresponding to $q_3 \neq 0$ and $q_3 = 0$.

Branch $q_3 \neq 0$: Fibered $Y^{p,q}$

In this branch, only $\kappa = -1$ is allowed. We set $\alpha_1 = 0$ and $\alpha_2 = 1/2$. After an appropriate coordinate redefinition and overall rescaling the background takes the form:

$$ds_{10}^2 = ds_{AdS_3}^2 + \frac{3}{4} ds_{\Sigma_{g>1}}^2 + \frac{3|1-cy|}{8} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{9}{8} \frac{|1-cy|}{(2cy^3 - 3y^2 + a)} dy^2 + \frac{1}{8} \frac{(2cy^3 - 3y^2 + a)}{|1-cy|} (d\beta + c \cos \theta d\phi)^2 + \frac{1}{4} \left(d\psi - \cos \theta d\phi + y (d\beta + c \cos \theta d\phi) - \frac{dx_1}{x_2} \right)^2, \quad (A.88)$$

$$G_{(5)} = d\text{vol}AdS_3 \wedge \left(2d\text{vol}\Sigma_{\mathfrak{g}>1} + \frac{|1-cy|}{4}d\text{vol}S^2 + \frac{1}{4}dy \wedge (d\beta + c \cos \theta d\phi) \right). \quad (\text{A.89})$$

This solution is again of the form presented in Section 6.1 of [18], *i.e.*,

$$ds_{10}^2 = ds_{AdS_3}^2 + \frac{3}{4}ds_{\Sigma_{\mathfrak{g}>1}}^2 + \frac{9}{4}ds_{Y^{p,q}}^2, \quad (\text{A.90})$$

where $ds_{Y^{p,q}}^2$ is the local form of the metric on $Y^{p,q}$ (fibered over $\Sigma_{\mathfrak{g}>1}$), written in canonical form [10]. We discuss this solution in Section 3.2.

Branch $q_3 = 0$: $AdS_3 \times S^3 \times T^4$

In this case, we find that only $\kappa = 1$ is allowed and we denote this sphere by $\tilde{S}^2$, with coordinates $(\tilde{\theta}, \tilde{\phi})$. After an appropriate change of coordinates we find

$$ds_{10}^2 = ds_{AdS_3}^2 + \frac{1}{4}ds_{\tilde{S}^2}^2 + \frac{y}{4}ds_{S^2}^2 + \frac{y}{4(a+y^2)}dy^2 + \frac{(a+y^2)}{4y}(d\beta + \cos \theta d\phi)^2 + \frac{1}{4}(d\tilde{\psi} - \cos \tilde{\theta} d\tilde{\phi})^2, \\ G_{(5)} = d\text{vol}AdS_3 \wedge \left(\frac{y}{2}d\text{vol}S^2 - \frac{1}{2}dy \wedge (d\beta + \cos \theta d\phi) \right), \quad (\text{A.91})$$

where we defined $\tilde{\psi} = \psi + \beta$. This solution corresponds to $AdS_3 \times S^3 \times T^4$ (with only 5-form flux turned on), for any value of a . This is easy to see in the case $a = 0$; defining $y = \rho^2$ the terms $ds_{S^2}^2$, $d\rho^2$ and the $(D\beta)^2$ combine into the metric on T^4 while the Riemann surface and the $(D\psi)^2$ term combine into an S^3 , and the solution is $AdS_3 \times S^3 \times T^4$. One can check that for generic a the four-dimensional metric has vanishing Riemann tensor and is therefore always T^4 . Usually the $AdS_3 \times S^3 \times T^4$ background of type IIB supergravity is associated with the D1-D5 system and thus it has only 3-form flux. Here we see the same solution sourced only by 5-form flux. The two backgrounds should be related by two T-duality transformations.

A.3.3 Case (iii): $c = \kappa = 0$

After appropriate coordinate redefinitions, we find

$$ds_{10}^2 = ds_{AdS_3}^2 + \frac{1}{4}ds_{\Sigma_{\mathfrak{g}=1}}^2 + \frac{1}{4}ds_{S^2}^2 + \frac{dy^2}{4y} + \frac{y}{4}d\beta^2 + \frac{1}{4}(d\psi - \cos \theta d\phi - \frac{1}{2}d\beta)^2, \\ G_{(5)} = d\text{vol}AdS_3 \wedge \left(\frac{1}{2}d\text{vol}\Sigma_{\mathfrak{g}=1} + \frac{1}{2}dy \wedge d\beta \right). \quad (\text{A.92})$$

Again, by the simple change of coordinates $y = \rho^2$ one sees that the genus-one Riemann surface combines with $d\rho^2$ and $d\beta^2$ into a T^4 and the S^2 combines with the $(D\psi)^2$ part to give an S^3 . Thus, the solution is again $AdS_3 \times S^3 \times T^4$ with five-form flux.

A.3.4 Case (iv): $\kappa = 0, c \neq 0$

After appropriate redefinitions and simple coordinate transformations¹⁸ the metric and 5-form flux read:

$$\begin{aligned}
ds_{10}^2 &= \frac{b}{\sqrt{\tilde{y}}} ds_{AdS_3}^2 + \frac{\sqrt{\tilde{y}}}{b} ds_{\Sigma_{\mathfrak{g}=1}}^2 + \frac{1}{4b\sqrt{\tilde{y}}} ds_{S^2}^2 + \frac{b}{4\tilde{y}^{5/2}(b^2 - (1 - \tilde{y})^2)} d\tilde{y}^2 \\
&\quad + \frac{(b^2 - (1 - \tilde{y})^2)}{4\sqrt{\tilde{y}}b^3} (d\beta + \cos\theta d\phi)^2 + \frac{b}{4\sqrt{\tilde{y}}} \left(d\psi - \cos\theta d\phi + \frac{b^2 + \tilde{y} - 1}{b^2} (d\beta + \cos\theta d\phi) \right)^2, \\
G_{(5)} &= d\text{vol}AdS_3 \wedge \left(2d\text{vol}\Sigma_{\mathfrak{g}=1} + \frac{1}{2} d\text{vol}S^2 - \frac{\tilde{y} - 1}{2\tilde{y}^2} d\tilde{y} \wedge (d\beta + \cos\theta d\phi) \right. \\
&\quad \left. + \frac{b^2}{2\tilde{y}^2} d\tilde{y} \wedge \left(d\psi - \cos\theta d\phi + \frac{b^2 + \tilde{y} - 1}{b^2} (d\beta + \cos\theta d\phi) \right) \right). \tag{A.93}
\end{aligned}$$

This coincides with the solution in section 4.1 of [20]. To see this one must make the linear change of coordinates:

$$\beta = -\tilde{\beta} + \tilde{\psi}, \quad \psi = \tilde{\beta}. \tag{A.94}$$

Using the formulae in Appendix A.2 it is easy to see that the metric reads:

$$\begin{aligned}
ds_{10}^2 &= \frac{b}{\sqrt{\tilde{y}}} ds_{AdS_3}^2 + \frac{\sqrt{\tilde{y}}}{b} ds_{\Sigma_{\mathfrak{g}=1}}^2 + \frac{1}{4b\sqrt{\tilde{y}}} ds_{S^2}^2 + \frac{b}{4\tilde{y}^{5/2}(b^2 - (1 - \tilde{y})^2)} d\tilde{y}^2 \\
&\quad + \frac{(b^2 - (1 - \tilde{y})^2)}{4b\sqrt{\tilde{y}}(b^2 - 1 + 2\tilde{y})} (d\tilde{\beta} + \cos\theta d\phi)^2 + \frac{b^2 - 1 + 2\tilde{y}}{4b\sqrt{\tilde{y}}} \left(d\tilde{\psi} - \frac{\tilde{y}}{b^2 - 1 + 2\tilde{y}} (d\tilde{\beta} + \cos\theta d\phi) \right)^2, \tag{A.95}
\end{aligned}$$

which coincides exactly with the solution found in [20], where its central charge was also computed. We discuss this solution further in Section 3.4.

A.4 Solutions with flavor flux: $a_2 \neq 0$

Here there are also a few special cases: 1) $a_2 = -\kappa c \neq 0$ and; 2) $c = 0$.

A.4.1 Case $c = 0$

When solving the Bianchi identity there are two branches: $q_3 = 0$ and $q_3 \neq 0$. The latter, however is inconsistent as it leads to $\mathcal{P}_1/\mathcal{P}_2 < 0$, in contradiction with the assumption that all $\mathcal{P}_i > 0$. The former branch leads to

$$\begin{aligned}
ds_{10}^2 &= ds_{AdS_3}^2 + y ds_{\Sigma_{\mathfrak{g}}}^2 + \frac{1}{4} ds_{S^2}^2 + \frac{y}{\kappa y^2 + \lambda} dy^2 + \frac{\kappa y^2 + \lambda}{y} (d\beta + A_{\mathfrak{g}})^2 + \frac{1}{4} (d\psi - \cos\theta d\phi)^2, \\
G_{(5)} &= d\text{vol}AdS_3 \wedge (2y d\text{vol}\Sigma_{\mathfrak{g}} + 2dy \wedge (d\beta + A_{\mathfrak{g}})). \tag{A.96}
\end{aligned}$$

¹⁸For the y coordinate the change of variables is of the form $\tilde{y} = \gamma/y + \delta$, with δ, γ some specific constants.

Here $\kappa = \{0, \pm 1\}$ and λ is a real parameter and $y > 0$. For $\kappa = \{1, 0, -1\}$, positivity of the internal manifold requires for $y^2 > -\lambda$, $\lambda > 0$, and $y^2 < \lambda$, respectively. The connection $A_{\mathfrak{g}}$ is given in (A.54) for $\kappa = \pm 1$ and for $\kappa = 0$ in (A.55). The case $\kappa = -1$ is not topologically allowed since the Riemann surface $\Sigma_{\mathfrak{g}}$ shrinks to zero size at $y = 0$. One can see that the cases $\kappa = \{1, 0\}$ lead again to $AdS_3 \times S^3 \times T^4$ for any value of λ .

A.4.2 Case $a_2 = -\kappa c \neq 0$

There are two branches: branch A with $q_3 = -\frac{2}{3}c\kappa$; and branch B with $q_3 = -2c\kappa$.

Branch A

For this branch, we find that only $\kappa = 1$ is allowed and after appropriate coordinate redefinitions

$$\begin{aligned}
ds_{10}^2 &= y ds_{AdS_3}^2 + \frac{3}{8} \left(ds_{\Sigma_{\mathfrak{g}=0}}^2 + ds_{S^2}^2 \right) + \frac{9y}{4(4y^3 - 9y^2 + 6ay - a^2)} dy^2 \\
&\quad + \frac{4y^3 - 9y^2 + 6ay - a^2}{16y^3} (d\beta + \cos \theta d\phi - A_{\mathfrak{g}})^2 + y \left(d\psi + \frac{a - 3y}{4y^2} (d\beta + \cos \theta d\phi - A_{\mathfrak{g}}) \right)^2, \\
G_{(5)} &= d\text{vol} AdS_3 \wedge \left(\frac{a}{4} d\text{vol} \Sigma_{\mathfrak{g}=0} + \frac{a}{4} d\text{vol} S^2 + \frac{(a - 3y)}{2y} dy \wedge (d\beta + \cos \theta d\phi - A_{\mathfrak{g}}) \right. \\
&\quad \left. - 2y dy \wedge \left(d\psi + \frac{a - 3y}{4y^2} (d\beta + \cos \theta d\phi - A_{\mathfrak{g}}) \right) \right). \tag{A.97}
\end{aligned}$$

This matches the solution presented in [17], with $KE_4 = S^2 \times S^2$. We were not able to identify a candidate dual field theory to this solution.

Branch B

After appropriate coordinate redefinitions we find that in this branch the metric reads:

$$\begin{aligned}
ds_{10}^2 &= \sqrt{\frac{y(y-1)\kappa}{a-y}} ds_{AdS_3}^2 + \sqrt{\frac{y(a-y)\kappa}{y-1}} ds_{\Sigma_{\mathfrak{g} \neq 1}}^2 + \sqrt{\frac{(y-1)(a-y)\kappa}{y}} ds_{S^2}^2 \\
&\quad + \frac{\sqrt{y(y-1)(a-y)\kappa}}{(2y-1)(a-2ay+y^2)} dy^2 \\
&\quad + \frac{(2y-1)(a-2ay+y^2) \sqrt{y(y-1)(a-y)\kappa}}{y^2(y-1)^2} (d\beta + \cos \theta d\phi - \kappa A_{\mathfrak{g}})^2 \\
&\quad + \frac{1}{4} \sqrt{\frac{y(y-1)\kappa}{a-y}} \left(d\psi' + \frac{2(a-2ay+y^2)}{y(y-1)} (d\beta + \cos \theta d\phi - \kappa A_{\mathfrak{g}}) \right)^2, \tag{A.98}
\end{aligned}$$

where a is a real parameter and the flux reads:

$$\begin{aligned}
G_{(5)} = d\text{vol}AdS_3 \wedge & \left(\frac{a-y^2}{a-y} d\text{vol}\Sigma_{\mathfrak{g} \neq 1} - \frac{(a-2y+y^2)\kappa}{a-y} d\text{vol}S^2 \right. \\
& + \frac{(2y-1)(a-2ay+y^2)\kappa}{y(y-1)(a-y)} dy \wedge (d\beta + \cos\theta d\phi - \kappa A_{\mathfrak{g}}) \\
& \left. + \frac{(a-2ay+y^2)\kappa}{2(a-y)^2} dy \wedge \left(d\psi' + \frac{2(a-2ay+y^2)}{y(y-1)} (d\beta + \cos\theta d\phi - \kappa A_{\mathfrak{g}}) \right) \right). \tag{A.99}
\end{aligned}$$

The connection $A_{\mathfrak{g}}$ is given in (A.54). For the metric to be positive definite one needs¹⁹

$$\begin{aligned}
\kappa = 1 : & \left(0 < y < a + \sqrt{a(a-1)}, a \leq 0 \right) \quad \text{or} \quad \left(a < y < 1, \frac{1}{2} < a < 1 \right) \quad \text{or} \quad \left(\frac{1}{2} < y < 1, a \leq \frac{1}{2} \right) \\
\kappa = -1 : & \left(\frac{1}{2} < y < a, \frac{1}{2} < a \leq 1 \right) \quad \text{or} \quad \left(\frac{1}{2} < y < a - \sqrt{a(a-1)}, a > 1 \right).
\end{aligned}$$

We note that for $\kappa = 1$ there are no real zeroes for the metric functions; thus the solutions for $\kappa = 1$ are not compact and we must take $\kappa = -1$. In the case $\kappa = -1$ and $a = 1$ the space is topologically $AdS_3 \times \Sigma_{\mathfrak{g} > 1} \times S^5$. This is easy to see; setting $a = 1$ the metric can be written as

$$ds_{10}^2 = \sqrt{y} \left[ds_{AdS_3}^2 + ds_{\Sigma_{\mathfrak{g} > 1}}^2 + ds_5^2 \right], \tag{A.100}$$

and with the change of variables $\sin^2 \mu = \frac{1-y}{y}$ the five-dimensional metric reads

$$\begin{aligned}
\frac{1}{4} ds_5^2 = & \frac{d\mu^2}{(1 + \sin^2 \mu)^2} + \frac{1}{4} \sin^2 \mu \left(ds_{S^2}^2 + \cos^2 \mu (d\beta + \cos\theta d\phi + A_{\mathfrak{g}})^2 \right) \\
& + \frac{1}{16} \left(d\psi' - 2 \sin^2 \mu (d\beta + \cos\theta d\phi + A_{\mathfrak{g}}) \right)^2. \tag{A.101}
\end{aligned}$$

This metric looks like the metric on S^5 written as a $U(1)$ bundle over $\mathbb{CP}^2$. Due to the denominator in the $d\mu^2$ term this is not the Einstein metric on S^5 . Since we have $a_2 \neq 0$ we believe that this supergravity solution is dual to the $Y^{p,p}$ theory compactified on $\Sigma_{\mathfrak{g}}$ with some particular value of the background flavor flux b_2 and possibly non-zero baryonic flux. It will be of course interesting to understand better this supergravity background, compute the supergravity value of the central charge and compare to the expressions in Section 2.2.3.

A.4.3 Case $\kappa = 0$, $c \neq 0$

There are two branches:

¹⁹For $\kappa = -1$ one also finds the possibilities $(a + \sqrt{a(a-1)} < y, a > 1)$ or $(y > 1, a \leq 1)$. However, we exclude these since the warp factor is not bounded.

Branch A: $q_3 = 0$

In this branch the metric reads

$$\begin{aligned}
ds_{10}^2 = & \sqrt{\frac{y(b-a_2y)}{b-1-a_2y}} ds_{AdS_3}^2 + \frac{1}{4} \sqrt{\frac{(b-a_2y)(b-1-a_2y)}{y}} ds_{T^2}^2 + \frac{1}{4} \sqrt{\frac{y(b-1-a_2y)}{b-a_2y}} ds_{S^2}^2 \\
& + \frac{dy^2 \sqrt{(b-a_2y)(b-1-a_2y)}}{4y^{3/2}} + \frac{\sqrt{y(b-1-a_2y)}}{4(b-a_2y)^{3/2}} D\beta^2 \\
& + \frac{1}{4} \sqrt{\frac{y(b-a_2y)}{b-1-a_2y}} \left(D\psi + \frac{1}{b-a_2y} D\beta \right)^2,
\end{aligned} \tag{A.102}$$

where $D\psi = d\psi - \cos\theta d\phi$, $D\beta = d\beta + \cos\theta d\phi + a_2 A$ and the five-form flux:

$$\begin{aligned}
G_{(5)} = & \frac{1}{2} d\text{vol}AdS_3 \wedge \left(\left(1 + b - a_2y - \frac{b-1}{b-1-a_2y} \right) d\text{vol}T^2 \right. \\
& - \left(1 + \frac{a_2y}{(b-a_2y)(b-1-a_2y)} \right) dy \wedge D\beta \\
& \left. - \frac{1}{2} \left(1 - \frac{b-1}{b-1-a_2y} \right) dy \wedge (D\psi + \frac{1}{b-a_2y} D\beta) \right).
\end{aligned} \tag{A.103}$$

Branch B: $q_3 \neq 0$

We find:

$$\begin{aligned}
ds^2 = & \sqrt{\frac{y(y-1)}{q_0-a_2y}} ds_{AdS_3}^2 + \frac{3}{4} \sqrt{\frac{(q_0-a_2y)(y-1)}{y}} ds_{T^2}^2 + \frac{3}{4} \sqrt{\frac{y(q_0-a_2y)}{a_2^2(y-1)}} ds_{S^2}^2 \\
& + \frac{9dy^2 \sqrt{y(y-1)(q_0-a_2y)}}{4w(y)} + \frac{w(y) \sqrt{y(y-1)(q_0-a_2y)}}{4a_2^2 y^2 (y-1)^2} D\beta^2 \\
& + \frac{1}{4} \sqrt{\frac{y(y-1)}{q_0-a_2y}} \left(D\psi - \frac{w'(y)}{6a_2y(y-1)} D\beta \right)^2,
\end{aligned} \tag{A.104}$$

where q_0 is a parameter and we assumed $a_2 > 0$, $D\psi = d\psi - \cos\theta d\phi$, $D\beta = d\beta + \cos\theta d\phi + a_2 A$ and defined

$$w(y) = q_0(2-3y)^2 + a_2y^2(3-4y), \tag{A.105}$$

and the flux is given by

$$\begin{aligned}
G_{(5)} = & d\text{vol}AdS_3 \wedge \left(\frac{q_0 + a_2y(y-2)}{2(q_0-a_2y)} d\text{vol}T^2 + \frac{q_0-a_2y^2}{a_2(q_0-a_2y)} d\text{vol}S^2 \right. \\
& + \frac{(q_0-2q_0y+a_2y^2)}{2(q_0-a_2y)^2} \left(-1 + \frac{w'(y)}{6a_2y(y-1)} \right) dy \wedge D\beta \\
& \left. + \frac{(q_0-2q_0y+a_2y^2)}{2(q_0-a_2y)^2} dy \wedge \left(D\psi - \frac{w'(y)}{6a_2y(y-1)} D\beta \right) \right),
\end{aligned} \tag{A.106}$$

which can also be written as

$$G_{(5)} = d\text{vol}AdS_3 \wedge \left(\frac{q_0 + a_2 y(y-2)}{2(q_0 - a_2 y)} d\text{vol}T^2 + \frac{q_0 - a_2 y^2}{a_2(q_0 - a_2 y)} d\text{vol}S^2 \right. \\ \left. + \frac{(q_0 - 2q_0 y + a_2 y^2)}{2(q_0 - a_2 y)^2} dy \wedge (D\psi - D\beta) \right). \quad (\text{A.107})$$

We leave the analysis of the global properties of the solutions in branches A and B above for future work.

A.4.4 Case $\kappa \neq 0$, $c \neq 0$

Here we assume $\kappa = -1$ and $c \neq 0$, which we rescale to 1. As discussed in Appendix A.2 it is always possible to set $\alpha_1 = 0$ and $\alpha_2 = \frac{1}{2}$ by a coordinate transformation, which we do. Since $c \neq 0$ we may shift y to set $C_3 = 0$. The parameter C_2 can be similarly rescaled away, thus we can set C_2 to any nonzero numerical value (we set $C_2 = \frac{3}{4}$). The Bianchi identity leads to a number of constraints among the remaining parameters. Finally, the solution is given by

$$\mathcal{P}_1 = \frac{3}{4}(a-1)a_2^2 + C_1 y, \quad \mathcal{P}_2 = \frac{3}{4} + a_2 y, \quad \mathcal{P}_3 = -y, \quad \mathcal{P}_7 = -\frac{1}{2}Q', \quad (\text{A.108})$$

$$\mathcal{Q} = \frac{-3(a-1)(7+a_2^2+8C_1-4a_2(1+C_1))}{16(a_2-1)^2} - \frac{3a_2(a-1)(a_2-4C_1-5)}{4(a_2-1)}y \\ + \frac{3}{4}(1-4(a-1)a_2^2)y^2 + \frac{1}{3}(1+a_2-4C_1)y^3, \quad (\text{A.109})$$

with $C_1 = -\frac{1}{4}(1+a_2+2\sqrt{1-a_2+a_2^2})$. Explicitly, the metric reads

$$ds_{10}^2 = \sqrt{\frac{y(3+4a_2 y)}{3(1-a)a_2^2-4C_1 y}} ds_{AdS_3}^2 + \frac{1}{4} \sqrt{\frac{(3+4a_2 y)(3(1-a)a_2^2-4C_1 y)}{y}} ds_{\Sigma_{\mathfrak{g}>1}}^2 \\ + \sqrt{\frac{y(3(1-a)a_2^2-4C_1 y)}{3+4a_2 y}} ds_{S^2}^2 + \frac{1}{4} \frac{\sqrt{y(3+4a_2 y)(3(1-a)a_2^2-4C_1 y)}}{\mathcal{Q}} dy^2 \\ + \frac{4\mathcal{Q}\sqrt{y(3(1-a)a_2^2-4C_1 y)}}{y^2(3+4a_2 y)^{3/2}} D\beta^2 + \frac{1}{4} \sqrt{\frac{y(3+4a_2 y)}{3(1-a)a_2^2-4C_1 y}} \left(D\psi + \frac{2\mathcal{Q}'}{y(3+4a_2 y)} D\beta \right)^2, \quad (\text{A.110})$$

where $D\beta = d\beta + \cos\theta + a_2 A_{\mathfrak{g}}$, $D\psi = d\psi - \cos\theta - A_{\mathfrak{g}}$. The five-form, which we do not write explicitly here, is given by (A.72) and (A.73). Setting $a_2 = 0$ and sending $y \rightarrow -\frac{3}{8}(y+1)$ and redefining the parameter a , the metric coincides with the solution with no flavor flux (A.88), with c scaled to 1. It is also possible to take the limit $a_2 \rightarrow 1$. Taking this limit and performing the change of variables $y \rightarrow 3/4(y-1)$ and changing $\psi \rightarrow \psi - \beta$ the metric coincides with (A.98) with $\kappa = -1$. Thus, this solution contains all previous cases.

Case $a = 1$. Let us consider the local form of the metric more carefully for $a = 1$. In this case the cubic $\mathcal{Q}$ becomes

$$\mathcal{Q} = \frac{3}{4y_1}y^2(y_1 - y), \quad y_1 \equiv -\frac{9}{8(1 + a_2 + \sqrt{1 + a_2^2 - a_2})}. \quad (\text{A.111})$$

There is a negative root at $y = y_1$ and a doubly-degenerate root at $y = 0$. It is convenient to go to a basis $(\tilde{\beta}, \tilde{\psi})$ in which it is manifest that there are no conical singularities, provided $\tilde{\beta}$ period 2π . As discussed in Section 3.1.1 this is accomplished by a coordinate transformation mixing (ψ, β) . Setting $\alpha_1 = 0$, $\alpha_2 = 1/2$ and $c = -b = 1$ we may choose $w_1 = -1, w_2 = \frac{1}{2}, w_3 = 1, w_4 = 0$, *i.e.*,

$$\beta = \frac{1}{2}\tilde{\psi} - \tilde{\beta}, \quad \psi = \tilde{\beta}. \quad (\text{A.112})$$

It is also convenient to make the coordinate transformation

$$y = \frac{2y_1 \sin^2 \mu}{1 + \sin^2 \mu}, \quad \mu \in [0, \pi/2], \quad (\text{A.113})$$

in terms of which $\mathcal{Q} = \frac{3y_1^2 \cot^2 \mu}{(1 + \csc^2 \mu)^3}$, with zeros at $\mu = \{0, \pi/2\}$. In these coordinates the metric reads

$$ds_{10}^2 = f_1^2 \left[ds_{AdS_3}^2 + |C_1| ds_{\Sigma_{\mathfrak{g}>1}}^2 + \frac{32|C_1 y_1|}{3} ds_5^2 \right], \quad (\text{A.114})$$

with

$$\begin{aligned} ds_5^2 = & \frac{d\mu^2}{(1 + \sin^2 \mu)^2} + \frac{1}{4g(\mu)} \sin^2 \mu \left(ds_{S^2}^2 + \frac{1}{f(\mu)} \cos^2 \mu (d\tilde{\beta} + \cos \theta d\phi + A_{\mathfrak{g}})^2 \right) \\ & + \frac{3}{8|C_1|} \frac{f(\mu)}{16g(\mu)} \left(\frac{d\tilde{\psi}}{2} + (a_2 - 1)A_{\mathfrak{g}} - \left(\frac{9 + 8y_1}{3} \right) \frac{\sin^2 \mu}{f(\mu)} (d\tilde{\beta} + \cos \theta d\phi + A_{\mathfrak{g}}) \right)^2, \end{aligned} \quad (\text{A.115})$$

where we defined the functions

$$f(\mu) \equiv 1 + \left(1 + \frac{8y_1}{3} \right) \sin^2 \mu, \quad g(\mu) \equiv 1 + \left(1 + \frac{8a_2 y_1}{3} \right) \sin^2 \mu, \quad (\text{A.116})$$

and the overall warp factor is given by $f_1^2 = \left(\frac{3g(\mu)}{4|C_1|(1 + \sin^2 \mu)} \right)^{1/2}$. We note that for the special value $a_2 = 1$ then $y_1 = -\frac{3}{8}, C_1 = -1$ and $g(\mu) = f(\mu) = 1$ and the metric coincides with (A.101) with $\tilde{\psi} = 2\psi'$. Similarly to the metric in (A.101) the metric in (A.115) looks like a squashed metric on S^5 written as a $U(1)$ bundle over $\mathbb{CP}^2$. We believe that this background is dual to the $Y^{p,p}$ theory on $\Sigma_{\mathfrak{g}}$ with general value of the flavor flux b_2 which is related to the value of the supergravity parameter a_2 . However we leave a global analysis of this background and a supergravity calculation of the central charge for future work.

B General formulas for the central charge

Here we provide some useful normalizations and formulas for computing the gravitational central charge for the class of solutions considered in the paper.

Consider a metric and flux of the form

$$ds^2 = L^2 e^{2\lambda} ds^2(AdS_3) + L^2 ds^2(\mathcal{M}_7) , \quad g_s F_{(5)} = L^4 (1 + *_{10}) G_{(5)} . \quad (\text{B.1})$$

The quantization condition for $F_{(5)}$ is (here we follow the conventions of [17])

$$N(D) = \frac{1}{(2\pi\ell_s)^4} \int_D F_{(5)} , \quad (\text{B.2})$$

where D is any five-cycle in $\mathcal{M}_7$ and $N(D)$ is an integer. The type IIB supergravity action is

$$S_{\text{IIB}} = \frac{1}{16\pi G_N^{(10)}} \int d^{10}x \sqrt{-g^{(10)}} R^{(10)} + \dots , \quad (\text{B.3})$$

where the dots stand for other terms in the action that will not be important for our discussion. The normalization we use (see for example Appendix D in [66]) is such that

$$16\pi G_N^{(10)} = (2\pi)^7 g_s^2 \ell_s^8 , \quad (\text{B.4})$$

where g_s is the string coupling constant and ℓ_s is the string length. The central charge of the dual CFT is given by the Brown-Henneaux formula [2]

$$c_{\text{sugra}} = \frac{3L}{2G_N^{(3)}} , \quad (\text{B.5})$$

where L is the same as in (B.1) and $G_N^{(3)}$ is the 3d Newton constant which can be read off from the 3d effective gravitational action

$$S_{3d} = \frac{1}{16\pi G_N^{(3)}} \int d^3x \sqrt{-g^{(3)}} R^{(3)} + \dots . \quad (\text{B.6})$$

The goal now is to find $G_N^{(3)}$ by reducing the type IIB action on the manifold $\mathcal{M}_7$. To do this one has to plug in the metric (B.1) in the type IIB action (B.3) which leads to²⁰

$$\frac{1}{16\pi G_N^{(3)}} = \frac{L^7}{16\pi G_N^{(10)}} \int d^7x \sqrt{g(\mathcal{M}_7)} e^\lambda , \quad (\text{B.7})$$

and therefore

$$c_{\text{sugra}} = \frac{3L^8}{2G_N^{(10)}} \int d^7x \sqrt{g(\mathcal{M}_7)} e^\lambda . \quad (\text{B.8})$$

²⁰Under a conformal transformation $\tilde{g} = e^{2\lambda}g$, $\sqrt{\tilde{g}}\tilde{R} = e^{(D-2)\lambda}R\sqrt{g} + \dots$, where D is the dimension; this leads to the factor e^λ in the integrand.

Now we specialize these general expressions to the Ansatz in equation (3.5). Using (3.9) we have

$$\begin{aligned} \int d^7x \sqrt{g(\mathcal{M}_7)} e^\lambda &= \int d^7x \sqrt{g_{\Sigma_g}} \sqrt{g_{S^2}} f_1 f_2^2 f_3^2 f_4 f_5 f_6 \\ &= \alpha_2 (4\pi)^2 (\mathfrak{g} - 1) \Delta\beta \Delta\psi \int_{y_1}^{y_2} dy \mathcal{P}_1(y), \end{aligned} \quad (\text{B.9})$$

where $\Delta\beta = 2\pi l_\beta$ and $\Delta\psi = 2\pi l_\psi$ denote the periods of the corresponding coordinate and the integral over y is between two roots y_1, y_2 of $\mathcal{Q}$, between which the function $\mathcal{Q}$ is positive.

Now we look at the quantization condition for the five-form $F_{(5)}$. In general there can be several five-cycles in $\mathcal{M}_7$, one of them being the manifold $\mathcal{M}_5$, spanned by $\{\theta, \phi, y, \beta, \psi\}$, itself. The five-form flux through $\mathcal{M}_5$ corresponds to the number N of D3-branes. The only term that contributes to this integral is

$$\int_{\mathcal{M}_5} (1 + *_10 G_{(5)}) = \int_{\mathcal{M}_5} g_1 f_3^2 f_4 f_5 f_6 d\text{vol} S^2 \wedge dy \wedge d\beta \wedge d\psi. \quad (\text{B.10})$$

Using (A.30), the expression for $\mathcal{P}_7$ in (A.64), and the relations (A.43), (A.53) we can write

$$g_1 f_3^2 f_4 f_5 f_6 = \frac{\alpha_2}{4} \partial_y \left(2y - \frac{\mathcal{Q}'}{\mathcal{P}_2} \right). \quad (\text{B.11})$$

Thus,

$$\int_{\mathcal{M}_5} (1 + *_10 G_{(5)}) = \pi \alpha_2 \Delta\beta \Delta\psi \left(2y - \frac{\mathcal{Q}'}{\mathcal{P}_2} \right) \Big|_{y_1}^{y_2}. \quad (\text{B.12})$$

Thus, the quantization condition reads

$$N = \frac{1}{(2\pi\ell_s)^4} \frac{L^4}{g_s} \pi \alpha_2 \Delta\beta \Delta\psi S(y_1, y_2), \quad S(y_1, y_2) \equiv \left(2y - \frac{\mathcal{Q}'}{\mathcal{P}_2} \right) \Big|_{y_1}^{y_2}. \quad (\text{B.13})$$

Putting everything together, the central charge is given by:

$$c_{\text{sugra}} = \frac{192(\mathfrak{g} - 1)N^2}{\alpha_2 l_\beta l_\psi} \frac{1}{S(y_1, y_2)^2} \int_{y_1}^{y_2} dy \mathcal{P}_1(y). \quad (\text{B.14})$$

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