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Gopakumar–Vafa invariants in genus 0 and 1

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Declaration of Originality

I hereby declare that all material in this dissertation which is not my own work has been properly acknowledged.

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Abstract

In this thesis we explore two possible approaches to the study of Gopakumar-Vafa invariants in genus 0 and 1, mostly concentrating on their relation with certain moduli spaces of morphisms.

The work is divided in two parts. In the first part we study via a new technique inspired by Hitchin's spectral curve construction, a version of Donaldson-Thomas invariants which Katz has proposed as a mathematical definition of genus 0 GV invariants. The spectral curve point of view leads us, in the reduced class case, to express the local invariants of a fixed curve in terms of maps from curves homeomorphic to genus 0 ones. At the end of the first part we look at some examples in the non reduced class case and explore the spectral construction techniques in this set up.

In the second part of the thesis I discuss a joint work with Luca Battistella and Cristina Manolache. We first recall Li-Zinger definition of reduced genus 1 invariants, which coincide with genus 1 GV for the quintic 3-folds for the reduced class case, and prove that such invariants coincide with those arising from the moduli space of 1-stable maps. In such moduli space elliptic tails are destabilised and replaced by non degenerate maps from cuspidal curves. We think of this as a first step in a longer project, with the aim is that to define alternate compactifications of the moduli space of maps in higher genus whose associated invariants do not take into account degenerate contributions.

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Introduction

Gromov-Witten theory is the first example of the modern approach to enumerative geometry and, as the first deformation invariant theory constructed, it has been intensively studied for more than 20 years. The theory presents several serious difficulties: it is extremely hard to compute, the invariants are *rational* due to the presence of finite automorphisms of the maps in the case of *multiple cover*, and in genus $g > 0$ the different boundary components of the moduli space give *degenerate contributions*.

When X is a smooth projective Calabi-Yau 3-fold, Gopakumar-Vafa conjectured the existence of an integer valued theory underlying Gromov-Witten counting; via M-theoretic methods they defined the BPS invariants $n_{g,\beta}$ which should count the embedded curves in geometric genus g underlying the GW and should satisfy:

$$\sum_{\beta,g} N_{\beta}^g q^{\beta} \lambda^{2g-2} = \sum_{\beta,g,m} \frac{n_{g,\beta}}{m} \left(2 \sin \left(\frac{m\lambda}{2} \right) \right)^{2g-2} q^{m\beta}, \quad (0.1)$$

where we denoted by N_{β}^g the Gromov-Witten invariants of X . A direct mathematical definition for these invariants in all genus has been recently proposed in [MT16] via the perverse cohomology of the DT sheaf on the moduli space $\mathcal{M}_{1,\beta}(X)$ of pure 1-dimensional stable sheaves $\mathcal{F}$ with support in class $\beta \in H_2(X, \mathbb{Z})$ and holomorphic Euler characteristic $\chi(\mathcal{F}) = 1$.

In this thesis we explore, via different techniques, Gopakumar-Vafa invariants in genus zero and one and their relations with moduli spaces of maps.

Katz invariants and spectral curve construction

The numerical invariants arising from $\mathcal{M}_{1,\beta}(X)$ were already proposed as mathematical definition of genus 0 BPS invariants by Katz [Kat08]:

$$n_{0,\beta} = \int_{[\mathcal{M}_{1,\beta}(X)]^{vir}} 1 = \chi(\mathcal{M}_{1,\beta}(X), \nu)$$

where ν denotes the Behrend function of the moduli space. It is a priori not clear from the definition why these invariants should capture only the genus 0 curves and our main motivation was to understand whether there exists a technique to reduce Katz invariants to genus 0 curves and thus reveal the relation with maps from genus 0 curves.

We will actually work with local contribution : for a fixed curve $i: C \rightarrow X$ and γ a cycle on C satisfying $i_*\gamma = \beta$, we can define the contributions of C to Katz's invariants as

$$n_{0,\gamma} = \chi(\mathcal{M}_{1,\gamma}(C), \nu|_{\mathcal{M}_{1,\gamma}(C)}).$$

where $\mathcal{M}_{1,\gamma}(C)$ is the subspace of $\mathcal{M}_{1,\beta}(X)$ of sheaves set theoretically supported on C whose associated fundamental cycle $[\mathcal{F}]$ (see (1.7)) coincide with γ .

In this thesis we develop a new technique to study Katz invariants and their localisation to rational curves; the technique is inspired by Hitchin's spectral curve construction. Before giving an outline of the main idea, we collect the main results.

Reduced class case

In the reduced case, i.e. $\gamma = [C]$ and C reduced, we give an interpretation of the invariants in terms of curves homeomorphic to genus 0 curve. We first define (see Definition 2.13) the moduli functor $\underline{PN}^m$ of partial normalisations of C of fixed genus (which is indeed m in the connected case) and show that:

Theorem. $\underline{PN}^m$ is represented by a closed subscheme PN^m of the Quot scheme $\text{Quot}_C(\mathcal{S}, m)$ parametrising length m quotient of the zero dimensional sheaf on $\mathcal{S} \cong v_*\mathcal{O}_{\bar{C}}/\mathcal{O}_C$. We denoted by $\bar{C} \xrightarrow{v} C$ the normalisation.

Let $T^m \subseteq PN_{\text{conn}}^m \subseteq PN^m$ denote the locally closed sub-variety parametrising partial normalisations homeomorphic to trees of $\mathbb{P}^1$.

Theorem. Let C be a connected, reduced projective curve of arithmetic genus g and normalisation $\sqcup_1^N \mathbb{P}^1$. Let $C \xhookrightarrow{i} X$ denote its embedding in a smooth Calabi-Yau 3-fold X . The contribution $n_{0,[C]}$ to the invariant $n_{0,i_*[C]}$ is given by:

$$n_{0,[C]} = \sum_{j=0}^{g(C)} \chi(T^{g(C)-j}, (s_j)_* \nu|_{\mathcal{M}_{1,[C]}(C)}) \quad (0.2)$$

where s_j is, roughly speaking, defined by mapping a point $\mathcal{F} \in \mathcal{M}_{1,[C]}(C)$ to the curve $\tilde{C}$ which is the maximal partial normalisation from which $\mathcal{F}$ is pushed-down.

Assuming that the normalisation of C is $\sqcup_1^N \mathbb{P}^1$ is not a restrictive hypothesis. Indeed, it is not hard to see that for any curve class β , curves with positive geometric genus do not contribute to Katz invariants [JS12, Proposition 6.19], [Tod11, Lemma 2.11]. To understand the contribution of singular curves with normalisation a disjoint union of $\mathbb{P}^1$, the idea is to exploit the action of $\text{Pic}^0(\widehat{C})$ (where $\widehat{C}$ is a formal neighbourhood of C in X) on $\mathcal{M}_{1,\gamma}(C)$, given by $\widehat{\mathcal{L}} \cdot \mathcal{F} = \widehat{\mathcal{L}} \otimes \mathcal{F}$ and understand the orbits of this action via Hitchin's spectral construction. For example, $\mathcal{F}$ supported on C is a fixed point if and only if for any $\mathcal{L} \in \text{Pic}^0(C)$ there exists an isomorphism

$$\mathcal{F} \xrightarrow{\phi} \mathcal{F} \otimes \mathcal{L} \quad (0.3)$$

In Hitchin's language (0.3) is a so called Higgs field. Given such a triple $(\mathcal{F}, \phi, \mathcal{L})$, we get a unique compactly supported sheaf $\mathcal{F}_\phi$ on $\text{Tot}(\mathcal{L}) \xrightarrow{\pi} C$ called eigen-sheaf of ϕ , whose fiber over a point $x \in \text{Tot}(\mathcal{L})$ corresponds to an eigen-space of $\mathcal{F}_{\pi(x)} \xrightarrow{\phi_{\pi(x)}} \mathcal{F}_{\pi(x)}$. We denote by $C_{\mathcal{F}}$ the scheme theoretic support of $\mathcal{F}_\phi$. From $\mathcal{F}_\phi$ we recover $\mathcal{F}$ as $\nu_* \mathcal{F}_\phi$ where $\nu: C_{\mathcal{F}} \rightarrow C$ shall be thought of as a finite cover where the fiber over $y \in C$ corresponds to the eigenvalues x_i of ϕ_y . We will make this description more precise in § 1.3.

We want to exploit the fact that, being $\mathcal{L}$ of degree 0, a Higgs-field with values in $\mathcal{L}$, which is moreover an isomorphism, is rare and its existence constrains the geometry of the spectral curve. In fact, when C is smooth, unless $\mathcal{L}$ is a torsion point, a Higgs-field isomorphism with values in a degree 0 line bundle does not even exist.

Nodal curve Example

Let us look at the simplest example of singular curve: an irreducible nodal cubic C . It is well known that the pure rank 1 sheaves on C are either line bundle or push-down $\nu_* \mathcal{O}_{\mathbb{P}^1}(d)$ from the normalization $\mathbb{P}^1$. In a local (analytic) neighbourhood U around the singular point, $\nu_* \mathcal{O}_{\mathbb{P}^1}(d)|_U \cong \mathcal{O}_{B_1} \oplus \mathcal{O}_{B_2}$ where B_1, B_2 are the two branches coming together at 0. Then locally we have an isomorphism of the form:

$$\mathcal{O}_{B_1} \oplus \mathcal{O}_{B_2} \xrightarrow{(\alpha, \beta)} \mathcal{O}_{B_1} \oplus \mathcal{O}_{B_2}, \quad \alpha \neq \beta \in \mathbb{C}^*.$$

which is not given by a regular function on U since we have chosen $\alpha \neq \beta$, but it is given by a regular function on the normalisation. The above local isomorphism glues to a global Higgs-field $\nu_* \mathcal{O}_{\mathbb{P}^1}(d) \xrightarrow{\phi} \nu_* \mathcal{O}_{\mathbb{P}^1}(d) \otimes \mathcal{L}$ for any $\mathcal{L}$ and in the associated spectral curve the two branches are told apart, i.e. $\mathcal{F}$ is a $\mathbb{P}^1$ embedded in $\text{Tot}(\mathcal{L})$. The Higgs sheaf ϕ is uniquely determined up to scalar and once such a scalar is fixed, say λ , $\mathbb{P}^1$ intersects the fiber over the node in λ and $\alpha\lambda$ where α is the scalar corresponding to $\mathcal{L}$ in the identification $\text{Pic}^0(C) \cong \mathbb{C}^*$.

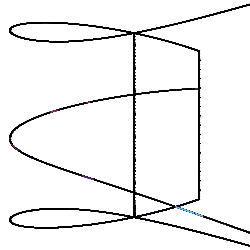


Figure 1: Spectral curve nodal, rank 1 case

Orbits and fixed points

Pursuing this idea, in the reduced class case we are able to describe all the orbits and in particular the fixed points:

Proposition. *Let C be a connected, reduced, projective curve and $\mathcal{F}$ a sheaf of rank 1 on each irreducible component of C . $\mathcal{F}$ has stabiliser $K \subseteq \text{Pic}^0(C)$ if and only if $\mathcal{F} \cong v_{\tilde{C}*}\mathcal{E}$ where $\tilde{C} \xrightarrow{v} C$ is a partial normalisation of C with $\text{Pic}^0(\tilde{C}) \cong \text{Pic}^0(C)/K$ and $\text{Stab}(\mathcal{E}) = \{e\}$.*

In particular, the fixed points are push-down from curves of arithmetic genus 0:

Corollary. *In the previous hypothesis, $\mathcal{F}$ is fixed by $\text{Pic}^0(C)$ if and only if $\mathcal{F} \cong v_{D*}\mathcal{E}_D$, where $D \xrightarrow{v} C$ is a partial normalization of C whose connected components are trees of $\mathbb{P}^1$ and $\mathcal{E}_D$ a pure rank 1 sheaf on it.*

Moreover, under stability hypothesis:

Corollary. *If $\mathcal{F}$ is stable and $\chi(\mathcal{F}) = 1$ then D is a tree of $\mathbb{P}^1$ and $\mathcal{E}_D = \mathcal{O}_D$. Furthermore, for any reduced, projective curve C , there are finitely many partial normalisations with source a tree of $\mathbb{P}^1$. In particular the fixed point of the $\text{Pic}^0(C)$ action on $\mathcal{M}_{1,[C]}(C)$ are finite.*

Some examples in the non reduced case

In the case $\gamma = r[C]$ we can try to exploit the same idea, but in this case the geometry of the spectral curve, although still constrained, is harder to describe. So far we only have partial results and just for higher rank sheaves supported on the reduced curve C underlying γ .

Proposition. *Let $\mathcal{F}$ be a rank r stable sheaf supported on the reduced curve C underlying γ and let fix $\chi(\mathcal{F}) = 1$.*

1. *Let C be an irreducible nodal curve with a single node and $\mathcal{F}$ as before. $\mathcal{F}$ is fixed by $\text{Pic}^0(C)$ if and only if $\mathcal{F} \cong v_{r*}\mathcal{O}_{C^r}$, where $v_r: C^r \rightarrow C$ is a $r : 1$ morphism from a chain of $\mathbb{P}^1$ which does not factor through any sub-chain.*
2. *Let C be an irreducible curve with a single ordinary cusp and $\mathcal{F}$ as before. $\mathcal{F}$ is fixed by $\text{Pic}^0(C)$ if and only if $\mathcal{F} \cong v_{r*}\mathcal{O}_{C_r}$, where C_r is a length r , genus 0 thickening of $\mathbb{P}^1$ with v_r not factoring through any sub-scheme C_k .*

As far as we know, the spectral construction technique is a new approach to this problem and we hope therefore that it will lead to a different point of view to take to understand Katz invariants and their relation to GW. However, at the moment it is not yet clear the generality in which this approach can be pursued and if it is going to lead us any closer to an understanding in elementary terms of the GV/GW relation.

Reduced genus 1 invariants from 1-stable maps

While genus 0 Gromov–Witten theory is understood fairly well, the higher genus theory is still unknown in general and its enumerative meaning is affected by more degenerate contributions. *Reduced Gromov–Witten invariants* [Zin09a], [VZ08], [VZ07], [LZ07], [LZ09], [Zin08], [Zin09b] solve two problems at the same time: they give computations

of genus 1 Gromov–Witten invariants and they have a better enumerative meaning. The only unpleasant aspect of reduced invariants is that their modular interpretation is complicated. In this paper we propose a way to fix this issue by working with a different moduli space.

Reduced invariants

The locus of maps from smooth elliptic curves to $\mathbb{P}^r$ is irreducible; we call its closure the *main component* of $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$. In [VZ08] R. Vakil and A. Zinger construct a desingularisation $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)^{\text{main}}$ of the main component of $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ via an iterated blow-up construction. Let

$$\begin{array}{ccc} \widetilde{\mathcal{C}} & \xrightarrow{\tilde{f}} & \mathbb{P}^r \\ \tilde{\pi} \downarrow & & \\ \widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)^{\text{main}} & & \end{array}$$

be the universal curve over $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)^{\text{main}}$ and let $l \in \mathbb{N}$. Then the sheaf $\tilde{\pi}_* \tilde{f}^* \mathcal{O}_{\mathbb{P}^r}(l)$ is a vector bundle. Given X_l a hypersurface of degree l in $\mathbb{P}^r$, *reduced genus 1 invariants* of X_l (see [VZ08], [HL10a]) are defined as

$$c_{\text{top}}(\tilde{\pi}_* \tilde{f}^*(\mathcal{O}_{\mathbb{P}^r}(l))) \cap [\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)^{\text{main}}].$$

Standard and reduced invariants of the quintic threefold $X_5 \subseteq \mathbb{P}^4$ are related by the Li–Zinger formula [LZ07]:

$$N_1(X_5, d) = N_1^{\text{red}}(X_5, d) + \frac{1}{12} N_0(X_5, d). \quad (0.4)$$

Cuspidal Gromov–Witten invariants

Based on D.I. Smyth’s work on the birational geometry of $\overline{\mathcal{M}}_{1,n}$ [Smy11a, Smy11b], M. Viscardi [Vis12] has introduced a series of *alternate compactifications* of the moduli space of maps from smooth elliptic curves. The first instance of these alternate compactifications is $\overline{\mathcal{M}}_{1,n}^{(1)}(X, \beta)$, where elliptic tails are made unstable and cuspidal singularities are allowed in the source curve. These spaces carry perfect obstruction theories and lead to invariants; we call them cuspidal Gromov–Witten invariants.

The main result we prove in Chapters § 5,6 is the following:

Theorem. *Let $X_5 \subseteq \mathbb{P}^4$ be a generic smooth quintic threefold. Then:*

$$N_1^{\text{red}}(X_5, d) = N_1^{\text{cusp}}(X_5, d),$$

where:

$$N_1^{\text{red}}(X_5, d) := \deg \left(c_{\text{top}}(\tilde{\pi}_* \tilde{f}^*(\mathcal{O}_{\mathbb{P}^4}(5))) \cap [\widetilde{\mathcal{M}}_1(\mathbb{P}^4, d)^{\text{main}}] \right)$$

and

$$N_1^{\text{cusp}}(X_5, d) := \deg[\overline{\mathcal{M}}_1^{(1)}(X_5, d)]^{\text{vir}}.$$

The proof of the main Theorem

The moduli space $\overline{\mathcal{M}}_1(\mathbb{P}^r, d)$ has many boundary components. From the proof of the Li–Zinger formula we see that the genus 0 contribution to the right hand side of (0.4) comes from the boundary component with a single rational tail. This suggests that discarding this component would provide a more direct approach to reduced invariants for the quintic threefold. This is exactly what the Smyth–Viscardi space $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^r, d)$ does.

Here is a rough sketch of the proof of our result. We first show:

Theorem. *There exists a well-defined 1-stabilisation morphism at the level of weighted-stable curves:*

$$\mathfrak{M}_{1,n}^{\text{wt,st}} \rightarrow \mathfrak{M}_{1,n}^{\text{wt,st}}(1)$$

replacing elliptic tails of weight 0 with cusps.

We then consider the following cartesian diagram:

$$\begin{array}{ccc} \mathcal{Z}_{X_5} & \longrightarrow & \overline{\mathcal{M}}_1^{(1)}(X_5, d) \\ \downarrow & \square & \downarrow \\ \mathfrak{M}_1^{\text{wt}} & \longrightarrow & \mathfrak{M}_1^{\text{wt}}(1). \end{array}$$

and prove that $\mathcal{Z}_{X_5}$ is a substack of $\overline{\mathcal{M}}_1(X_5, d)$ that has no component with contracted elliptic tails. The diagram above endows $\mathcal{Z}_{X_5}$ with a virtual class which, by Costello’s virtual push-forward formula [Cos06], has the same enumerative content as $[\overline{\mathcal{M}}_1^{(1)}(X_5, d)]^{\text{vir}}$.

In order to compare the degree of $[\mathcal{Z}_{X_5}]^{\text{vir}}$ with the reduced Gromov–Witten invariants we follow in Chang and Li’s footsteps. We first introduce the moduli space of 1-stable maps with p -fields; this has the advantage of having a simpler geometry and admits a cosection-localised virtual class [KL13] with the same degree as $[\overline{\mathcal{M}}_1^{(1)}(X_5, d)]^{\text{vir}}$ [CL12]. We then consider

$$\mathfrak{M}_1^{\text{wt,st}} \times_{\mathfrak{M}_1^{\text{wt,st}}(1)} \overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p,$$

which admits, thanks to a variation of Costello’s construction 6.17, a cosection localised class with the same enumerative information as $[\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p]_{\text{loc}}^{\text{vir}}$. We finally perform a desingularisation of this auxiliary p -fields space via the study of its local equations [HL10a]. In the end we analyse a splitting of the intrinsic normal cone of the desingularised space as in [CL15]. All these steps deliver the theorem.

Smoothability

On our way there, we review Vakil’s characterisation of maps lying in the boundary of $\overline{\mathcal{M}}_1(\mathbb{P}^r, d)^{\text{main}}$ [VZ07] and we give a new proof using Hu–Li’s equations [HL10a]. Using

Smyth's work on genus 1 singularities [Smy13, Appendix A] we also rephrase smoothability as follows:

Theorem. *A map $[f] \in \overline{\mathcal{M}}_1(\mathbb{P}^r, d)$ contracting the minimal genus 1 subcurve is smoothable if and only if it factors through a map from a genus 1 singularity that does not contract any component.*

Chapter Contents

Chapter 1

Here we introduce the notations, definitions and basic results we need in the study of local Katz invariants: we collect basic facts about Picard groups of singular groups, introduce the notion of Higgs field, spectral curve and give some examples, and finally recall the definition of local contributions.

Chapter 2

Using the spectral curve construction we characterise the orbits and in particular the fixed points of the Pic action on the moduli space of rank 1 sheaves. We then give an interpretation of the invariants in terms of moduli spaces of unibranch partial normalisations; the spaces are defined in Section 2 of this chapter and we prove that they admit a natural algebraic structure as locally closed sub-schemes of a certain quot scheme.

Chapter 3

We study the geometry of spectral curve associated to Picard fixed sheaves of higher rank on a reduced curve with a single node and on a reduced curve with a single cusp. We make some conjecture about the geometry structure of the spectral curve associated to Pic invariant higher rank sheaves on general reduced curve which suggest there should be also in the non reduced case an interpretation of the invariants in terms of curves homeomorphic to trees of $\mathbb{P}^1$.

Chapter 4

We review some classical results about the irreducible components of $\overline{\mathcal{M}}_1(\mathbb{P}^r, d)$ and local equations for this moduli stack in a smooth ambient space, and recall the definition of reduced invariants. We recall Smyth's m -elliptic singularities and discuss two different proofs of Vakil's characterisation of smoothable maps to projective space: one is based on Hu-Li's local equations; for the other one we give a classification of genus 1 singularities.

Chapter 5

We introduce Viscardi's compactifications, study their components, and define the invariants arising from them. We show the existence of a comparison morphism (the so

called 1-stabilisation) defined at the level of weighted pre-stable curves, which contracts elliptic tails to cusps.

Chapter 6

We adapt all the theory developed by [HL10a],[CL12],[CL15] in order to give an algebraic proof of the Li-Zinger formula to the case of cuspidal stable maps. Finally we use the comparison morphism and standard virtual pull-back techniques or their localised version to compare cuspidal invariant to standard GW and to reduced GW.

1

Picard groups, Higgs sheaves and Katz invariants

In this first chapter we introduce the dictionary, notation and basic results we keep using through the first three chapters.

1.1. Partial normalisations and simultaneous normalisations

Definition 1.1. Let C be a connected, reduced, projective curve. We say that $C' \xrightarrow{\nu} C$ is a *partial normalisation* if ν is birational on any irreducible component of C .

Notice that, due to its universal property, the normalisation $\overline{C} \xrightarrow{\bar{\nu}} C$ admits a factorisation through $C' \xrightarrow{\nu} C$ for any partial normalisation.

Definition 1.2. Let $C' \xrightarrow{\nu} C$ be as in Definition 1.1. For any $x \in S \subseteq \text{Sing}(C)$, where $S \subset C$ is the finite set of points where ν is not an isomorphism, we define the *relative delta invariant* of x to be the quantity:

$$\delta_x^{C'} := \dim_{\mathbb{C}} (\nu_* \mathcal{O}_{C',x} / \mathcal{O}_{C,x}).$$

If C' is the full *normalisation* of C the above is the usual definition of delta invariant of a curve singularity; in the latter case we suppress the super script C' .

Definition 1.3. We define the *arithmetic genus* of a projective curve C to be the quantity $g_a(C) := \dim_{\mathbb{C}} H^1(C, \mathcal{O}_C)$. If C is reduced this is related to the genus of the normalisation in the following way:

$$g_a(C) = g(\overline{C}) + 1 - N + \sum_{x \in \text{Sing}(C)} \delta_x$$

where N is the number of irreducible components of C . For C reduced, projective with normalisation $\overline{C}$, the *geometric genus* is the dimension of the $H^1(\overline{C}, \mathcal{O}_{\overline{C}})$.

Remark 1.4. Some authors define the arithmetic genus of a curve C to be the quantity $1 - \chi(C, \mathcal{O}_C)$. This agrees with our definition when C is connected and reduced, and it is sometimes preferred as it is invariant in flat families. Our choice allows us to identify the arithmetic genus of a curve with the dimension of its Pic^0 .

1.1.1. Semi-normalisations and Simultaneous normalisations

Given a flat family of reduced curves $\mathcal{C} \rightarrow S$ we want to recall what are sufficient conditions for it to admit a simultaneous normalisation, i.e. a morphism $\overline{\mathcal{C}} \rightarrow \mathcal{C} \rightarrow S$ such that for any closed point $s \in S$ the morphism $\overline{\mathcal{C}}_s \rightarrow \mathcal{C}_s$ is the normalisation. In order to state the result we are interested in, let us first recall the definition of semi-normalisation.

Definition 1.5. [Kol96, Definition 7.2.1] Let X be a reduced scheme. The *semi-normalisation* of X is a scheme X' together with a birational map $X' \xrightarrow{m} X$ such that for each $x \in X$ there exist a unique $x' \in X'$ with $m(x') = x$, i.e. m is a homeomorphism. Furthermore, if $Y \rightarrow X$ is any other birational map which is bijective on closed points, then there exists a unique factorisation $X' \rightarrow Y \rightarrow X$. In other words, the semi-normalisation is the maximal partial normalisation among the ones which are bijections. We say that X is semi-normal if it is isomorphic to its semi-normalisation.

Example 1.6. Let C be a reduced, projective curve whose only singularities are ordinary nodes. Then C is semi-normal: any partial normalisation $C' \rightarrow C$ is the normalisation at one of the nodes and thus is not bijective. Hence the maximal bijective partial normalisation is C itself.

Let instead C be an integral curve with a cusp singularity; then C is not semi-normal. Indeed, the normalisation $\mathbb{P}^1 \xrightarrow{\nu} C$ is also the semi-normalisation as ν is an homeomorphism.

Remark 1.7. Semi-normal curves play an important role in the theory of generalised Jacobians; indeed, semi-normal rational curves (i.e. whose normalisation is a disjoint union of $\mathbb{P}^1$) are the only curves whose Pic^0 is an algebraic torus $(\mathbb{C}^*)^n$ for some n . We will go back to semi-normal curves and describe their analytic model in § 1.2.

The following result, which is [Kol11, Theorem 12], gives a characterisation of families of curves over semi-normal bases which admit a simultaneous normalisation. We state Kollár's Theorem only for families of curves, see [Kol11] for the higher dimensional case and for the proof.

Theorem 1.8. *Let $\mathcal{C} \rightarrow S$ be a flat family of reduced, projective curves and S a semi-normal scheme. Then the following are equivalent:*

1. $\mathcal{C} \rightarrow S$ admits a simultaneous normalisation $\overline{\mathcal{C}} \rightarrow \mathcal{C} \rightarrow S$;

2. The Hilbert polynomial of the normalisation of the fibers $\chi(\overline{\mathcal{C}}_s, \mathcal{O}(nH))$ is locally constant on S , namely all the fibers have the same number of irreducible components and the same geometric genus.

1.2. Jacobian of singular curves

Most of what appears in this section is taken from [Liu02, § 7.5], and full details for the proofs we do not give can be found in there .

Definition 1.9. [Liu02, Definition 5.9] Let C be a projective curve over $\mathbb{C}$ with irreducible components $C_1, \dots, C_N$. We define $\text{Pic}^0(C)$ to be the set of isomorphism classes of line bundles $\mathcal{L}$ such that $\deg(\mathcal{L}|_{C_i^{\text{red}}}) = 0$ for every $1 \leq i \leq N$. $\text{Pic}^0(C)$ is a group with respect to the tensor product $\otimes$.

Abusing terminology we will sometimes refer to $\text{Pic}^0(C)$ as the Jacobian of C .

Lemma 1.10. [Liu02, Lemma 5.11] Let C be a projective connected curve over $\mathbb{C}$ and let $C' \xrightarrow{i} C$ be a 1-dimensional closed sub-scheme defined by a nilpotent ideal. Then the pull-back map $i^*: \text{Pic}^0(C) \rightarrow \text{Pic}^0(C')$ is surjective with unipotent kernel of dimension $\dim_{\mathbb{C}} H^1(C, \mathcal{O}_C) - \dim_{\mathbb{C}} H^1(C', \mathcal{O}_{C'})$.

Remark 1.11. Since the Pic^0 is commutative and we are working over $\mathbb{C}$, for us a unipotent group is just a group isomorphic to $\mathbb{G}_a^k$ for some k .

The group of line bundles $\text{Pic}^0(\widehat{C})$

Let X be a smooth projective variety and let $C \hookrightarrow X$ be a projective, connected curve with defining ideal $\mathcal{I}$. We denote by C_n the thickening of C defined by $\mathcal{I}^n$ and by $\widehat{C}$ its completion, i.e. $\widehat{C}$ is endowed with the sheaf of rings $\varprojlim_n \mathcal{O}_X/\mathcal{I}^n$.

Definition 1.12. We define $\text{Pic}^0(\widehat{C})$ to be the group under $\otimes$ of line bundles on $\widehat{C}$ whose restriction to C has degree 0 on any irreducible component C_i of C .

By Lemma 1.10, the pull-back morphisms along restriction maps

$$\text{Pic}^0(C_n) \xrightarrow{j_{n-1}^*} \text{Pic}^0(C_{n-1}) \xrightarrow{j_0^*} \text{Pic}^0(C)$$

are surjective for each n . In particular, each line bundle on C extends to a line bundle $\widehat{\mathcal{L}}$ on $\widehat{C}$, [Har77, Proposition 9.6], i.e. $\text{Pic}^0(\widehat{C}) \twoheadrightarrow \text{Pic}^0(C)$.

Remark 1.13. $\widehat{C}$ is a formal neighbourhood of C in X and in particular any finite length thickening $\mathbf{C}$ of C in X is contained in such neighbourhood $\mathbf{C} \subseteq \widehat{C}$.

Pic^0 for reduced curves

In the notation of § 1.1, given a partial normalisation $C' \xrightarrow{v} C$, we denote by $\mathcal{K}$ the 0-dimensional sheaf $\prod_{x \in S} v_* \mathcal{O}_{C',x}^* / \mathcal{O}_{C,x}^*$ fitting in the short exact sequence:

$$0 \longrightarrow \mathcal{O}_C^* \longrightarrow v_* \mathcal{O}_{C'}^* \longrightarrow \mathcal{K} \longrightarrow 0.$$

Identifying $H^1(C, \mathcal{O}_C^*)$, $H^1(C', v_* \mathcal{O}_{C'}^*)$ with $\text{Pic}(C)$ and $\text{Pic}(C)'$ respectively, we get an induced sequence of groups:

$$H^0(C', v_* \mathcal{O}_{C'}^*) \rightarrow H^0(\mathcal{K}) \rightarrow \text{Pic}(C) \xrightarrow{v^*} \text{Pic}(C') \rightarrow 0. \quad (1.1)$$

Since pull-back along birational morphisms preserves the degree of line bundles, for each partial normalisation there is a surjective morphism of groups:

$$\text{Pic}^0(C) \xrightarrow{v^*} \text{Pic}^0(C') \quad (1.2)$$

The kernel of v^* and thus the structure of $\text{Pic}^0(C)$ can be explicitly described. We will be only interested in curves with normalisation a disjoint union of $\mathbb{P}^1$, thus we do not recall the structure of $\text{Pic}^0(C)$ for C of positive geometric genus.

Proposition 1.14. [Liu02, Lemma 5.12, Theorem 5.19, Remark 5.20] *Let C be a reduced, connected projective curve, $C' \xrightarrow{v} C$ a partial normalisation with k connected components, and S the set of points where v is not an isomorphism. Then*

$$\text{Ker}(v^*) \cong \bigtimes_{x \in S} \left((\mathbb{C}^*)^{|\{v^{-1}(x)\}| - 1} \times \mathbb{C}^{\delta_x^{C'} - |\{v^{-1}(x)\}| + 1} \right) / (\mathbb{C}^*)^{k-1}$$

In particular, if $\bigsqcup_1^N \mathbb{P}^1$ is the normalisation of C then $\text{Pic}^0(C)$ is isomorphic to:

$$\bigtimes_{p \in \text{Sing}(C)} \left((\mathbb{C}^*)^{b(x)-1} \times \mathbb{C}^{\delta_x - b(x) + 1} \right) / (\mathbb{C}^*)^{N-1}.$$

Remark 1.15. If $\mathcal{C} \rightarrow T$ is a morphism of schemes, it is well defined the relative Picard group $\text{Pic}_T^0(\mathcal{C}) \rightarrow T$, [BLR12, § 8, § 9]. This is the sub-functor of $\text{Pic}_{\mathcal{C}/T}$ ([BLR12, Definition 2, § 8] consisting of all elements whose restriction $\mathcal{L}_t$ belongs to $\text{Pic}^0(C_t)$ for all fibers of $\mathcal{C} \rightarrow T$. If $\mathcal{C} \rightarrow T$ is a flat family of reduced, projective curves, then $\text{Pic}_T^0(\mathcal{C}) \rightarrow T$ is a smooth family of commutative connected groups over T [BLR12, § 8.4]. If we moreover assume that all the fibers of $\mathcal{C} \rightarrow T$ have geometric genus 0, then the above groups are affine.

Curves whose $\text{Pic}^0(C)$ is an algebraic torus

Definition 1.16. Let C be a reduce curve; a point $p \in C$ is a **genus 0 singularity** if $\widehat{\mathcal{O}}_{C,p}$ is isomorphic to $\mathbb{C}[[x_1, \dots, x_k]] / (x_i x_j \mid i < j)$ for some k (these are also called **ordinary k-fold**

singularity [Liu02, Definition 5.13]), see Figure 1.1. A reduced curve with only genus 0 singularities is semi-normal in the sense of definition 1.5.

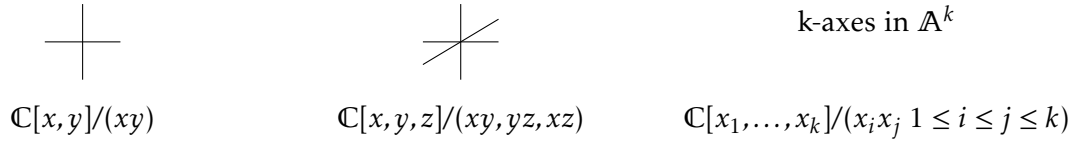


Figure 1.1: Genus 0 singularities

Remark 1.17. If p is a genus 0 singularity, its delta invariant is $\delta(p) = b(p) - 1$ where $b(p)$ is the number of branches. For a fixed number of branches, there exists only one genus 0 singularity up to isomorphism ([Smy11a, Proposition A.2]).

Lemma 1.18. [Liu02, Lemma 5.18] *Let C be a reduced, projective curve. $\text{Pic}^0(C)$ is an algebraic torus, i.e. $\text{Pic}^0(C) \cong \mathbb{G}_m^k$ for some k , if and only if all the singularities of C are genus 0 singularities and the normalisation of C is a disjoint union of $\mathbb{P}^1$.*

A special case is that of curves of arithmetic genus 0 and thus with trivial $\text{Pic}^0(C)$:

Definition 1.19. We call a curve C a *generalised tree* if it is **connected** and it has trivial $\text{Pic}^0(C)$. If all the singularity of C are ordinary nodes we say that C is a *tree*. Furthermore, if a tree of $\mathbb{P}^1$ satisfy the following condition: any irreducible component of the tree meets at most two other irreducible component, we say that it is a *chain* of $\mathbb{P}^1$.

Remark 1.20. A generalised tree has only genus 0 singularities, the normalisation at any singular point disconnects the curve, i.e. the irreducible components are smooth $\mathbb{P}^1$, and there are no cycles, i.e. $\pi_1(C) = 0$.

Remark 1.21. Notice that for a curve C which is homeomorphic to a generalised tree D the $\text{Pic}^0(C)$ is unipotent of dimension the arithmetic genus of C . Any curve with normalisation $\mathbb{P}^1$ and whose singularities are all unibranch is an example of the above.

We will need the following fact about generalised trees:

Lemma 1.22. *Let C be a reduced, projective curve. There exists a finite number of partial normalisations $D \rightarrow C$, with D a generalised tree.*

Lemma 1.23. *Let C be a reduced projective curve. There exists a unique partial normalization $C' \xrightarrow{\nu_2} C$ such that: ν_2 is an homeomorphism, and C' has only genus 0 singularities. C' is the semi-normalization of C (Definition 1.5).*

Proof. Let $\bar{C} \xrightarrow{\nu} C$ be the full normalisation. For any $x \in \text{Sing}(C)$ consider its pre-image under ν , $\{q_1, \dots, q_m\} = \nu^{-1}(x)$, and choose an affine neighbourhood V_x of x which does not contain any other singular point. Denote by W_q the affine curve defined as Spec of the $\mathcal{O}_C(V_x)$ algebra:

$$\{f \in \mathcal{O}_{\bar{C}}(\nu^{-1}(V_x)) \mid f(q_1) = \dots = f(q_m)\}. \quad (1.3)$$

Then we have that ν decomposes as $\nu^{-1}(V_x) \xrightarrow{\nu_1} W_q \xrightarrow{\nu_2} V_x$ and both ν_1, ν_2 are isomorphisms outside x , which ensures we can glue together the local charts W_q to get C' . By construction ν_2 is bijective and since it is also proper it is a homeomorphism, this proves (1).

The second statement then follows directly from the local definition of C' (1.3) and [Liu02, Proposition 5.15].

Finally the uniqueness follows from the universal property of semi-normalisations. $\square$

proof of Lemma 1.22. If $D \xrightarrow{\nu_D} C$ is a generalised tree which is a partial normalisation of C , then ν_D factors through the semi-normalisation, which is unique for its universal property. So it is enough to argue that a curve C' with only genus 0 singularities admits finitely many partial normalisations, and so, in particular, finitely many connected ones.

Let $p_1, \dots, p_N$ be the singular points of C' ; to give a partial normalisation $D \rightarrow C'$ we need to specify a partial normalisation at any singular point p_i . It suffices to show that a curve A_m with an ordinary m -fold singularity p admits a finite number K_m of partial normalisations $A' \rightarrow A$, and then for a semi-normal curve C' the number of partial normalisation will be given by $\prod_{\{p_1, \dots, p_N\}} K_{m_i}$.

Let $\mathbb{P}_1^1 \sqcup \dots \sqcup \mathbb{P}_m^1 \rightarrow A_m$ be the normalisation of the m -fold singularity, and $(k_1, \dots, k_l)$ a partition of m . Any choice in

$$\binom{m}{k_1} \binom{m-k_1}{k_2} \dots \binom{k_l+k_{l-1}}{k_{l-1}} \quad (1.4)$$

defines a curve B (unique by Remark 1.17) with l connected components $B_1, \dots, B_l$ having respectively a $k_1, \dots, k_l$ -ordinary fold, and admitting a birational map $B \rightarrow A_m$. In fact such a map is unique up to isomorphism and thus (1.4) is indeed the number of partial normalisations of A_m . The uniqueness follows from the fact that B is constructed from the full normalisation contracting a divisor [Ser00, § 4.4], i.e. as a push-out in the category of schemes [Sch05, § 3] and thus the universal property of push-out [Sch05, Theorem 2.3] ensures the uniqueness of the map to A_m . $\square$

Remark 1.24. Not every singular curve is defined by contracting a closed sub-scheme of the normalisation, e.g. the ramphoidal cusp, locally defined by $y^2 - x^5 = 0$. is not. In particular, in the general case, there could be a unique (up to isomorphism) curve C' admitting an infinity of non isomorphic birational maps to a fixed curve. We will see an explicit example 2.19 in § 2.2.

1.3. Spectral curve construction

The theory of Higgs bundles on smooth curves of genus $g \geq 2$ has been broadly studied since Simpson and Hitchin seminal works on non abelian Hodge theory [Hit87],[Sim92], and later it found applications in several other field. We are not going into this beautiful

piece of geometry, but we just borrow the idea of the spectral curve construction to then exploit it in the singular setting we are interested in.

Definition 1.25. Let C be a projective, connected curve. Given a line bundle $\mathcal{L}$ on C we define an $\mathcal{L}$ valued Higgs sheaf on C to be a pair $(\mathcal{E}, \phi)$ where $\mathcal{E}$ is a sheaf of $\mathcal{O}_C$ modules and ϕ is an $\mathcal{L}$ valued endomorphism of $\mathcal{E}$, i.e. $\phi \in \text{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{L})$.

Given a Higgs sheaf $(\mathcal{E}, \phi)$ there is a natural action of $\text{Sym}(\mathcal{L}^\vee)$ on $\mathcal{E}$ defined through $\phi: \mathcal{E} \otimes \mathcal{L}^{-i} \xrightarrow{\phi^i} \mathcal{E}$. Hence $\mathcal{E}$ has a structure of $\text{Sym}(\mathcal{L}^\vee) = \pi_* \mathcal{O}_S$ module, where we denoted by π the natural projection from $S := \text{Tot}(\mathcal{L})$. Since π is affine, the category of quasi-coherent $\mathcal{O}_S$ modules is equivalent to the category of quasi-coherent $\pi_* \mathcal{O}_S = \text{Sym}(\mathcal{L}^\vee)$ modules [Har77, Exercise II.5.17]. We denote by $\mathcal{E}_\phi$ the sheaf of $\mathcal{O}_S$ -modules which uniquely correspond to the Higgs sheaf $(\mathcal{E}, \phi)$. We call $\mathcal{E}_\phi$ the *eigensheaf* of $(\mathcal{E}, \phi)$. Viceversa, given a sheaf $\mathcal{F}$ on S , $\pi_* \mathcal{F}$ has naturally a $\text{Sym}(\mathcal{L}^\vee)$ module structure defined by: $\mathcal{F} \otimes \mathcal{L}^{-i} \xrightarrow{\lambda^i} \mathcal{F}$, where λ is the tautological section of $\pi^* \mathcal{L}$; this defined a Higgs sheaf $(\pi_* \mathcal{F}, \pi_* \lambda)$.

A morphism of Higgs sheaves $(\mathcal{E}, \phi), (\mathcal{F}, \psi)$ is a morphism of sheaves $\mathcal{E} \xrightarrow{f} \mathcal{F}$ inducing a commutative diagram:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{f} & \mathcal{F} \\ \downarrow \phi & & \downarrow \psi \\ \mathcal{E} \otimes \mathcal{L} & \xrightarrow{f \otimes \text{id}_\mathcal{L}} & \mathcal{F} \otimes \mathcal{L} \end{array} \quad (1.5)$$

This correspond to a λ linear morphism $\mathcal{E}_\phi \rightarrow \mathcal{F}_\psi$ on $\text{Tot}(\mathcal{L})$.

Notice that $\mathcal{E}$ on C is coherent if and only if $\mathcal{E}_\phi$ is a compactly supported coherent sheaf on S . In conclusion, we have an equivalence of categories: $\text{Higgs}_\mathcal{L}(C) \cong \text{Coh}_{\text{cpt}}(\text{Tot}(\mathcal{L}))$.

We denote by $C_\mathcal{E}$ the scheme theoretic support of the eigensheaf and we call it the *spectral curve* associated to $(\mathcal{E}, \phi)$.

Note 1.26. The scheme structure of the spectral curve depends on the Higgs-field, not just on the sheaf $\mathcal{E}$ so it would be more correct to use the notation $C_{\mathcal{E}, \phi}$, but we find it heavy, thus we will almost always suppress the ϕ .

Examples in the smooth setting

Example 1.27. Let C be a smooth curve and $(\mathcal{E}, \phi)$ a locally free rank 1 Higgs sheaf with values in $\mathcal{L}$. Then ϕ is a section s of $\mathcal{L}$, possibly the zero section, and thus $C \cong C_\mathcal{E} = \{\lambda - s = 0\} \subseteq \text{Tot } \mathcal{L}$, and $\mathcal{E}_\phi$ is just $\mathcal{E}$ itself.

Example 1.28. Let us consider $C = \mathbb{P}^1$, $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ with $a \geq b+1$, and $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}(1)$. An $\mathcal{L}$ valued endomorphism of $\mathcal{E}$ is represented by a matrix:

$$\phi = \begin{pmatrix} s_1 & s_{ab} \\ 0 & s'_1 \end{pmatrix} \text{ with } s_1, s'_1 \in H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \text{ and } s_{ab} \in H^0(\mathcal{O}_{\mathbb{P}^1}(a-b+1)).$$

Let us fix s_1, s'_1 to be x_0, x_1 respectively; we can explicitly write down the equation of the spectral curve $C_{\mathcal{E}}$ in the two charts of the line bundle:

$$C_{\mathcal{E}} \cap U_{\lambda, t=\frac{x_1}{x_0}} = \{(\lambda - 1)(\lambda - t) = 0\}, \quad C_{\mathcal{E}} \cap U_{\lambda', s=\frac{1}{t}} = \{(\lambda' - 1)(\lambda' - s) = 0\}.$$

So $C_{\mathcal{E}}$ is a union of two $\mathbb{P}^1$ meeting in a simple node in the point $P = ([1 : 1], 1)$. In order to describe $\mathcal{E}_{\phi}$, which we already know is a line bundle outside the node, we need to understand its stalk at P and its degree on the components $C_2 := C_{(\lambda-1)=0}$ and $C_1 := C_{(\lambda-t)=0}$.

The stalk of the eigen-sheaf at P depends on s_{ab} : if the latter section vanishes in $\pi(P)$, where π is the natural projection to $\mathbb{P}^1$ then $\phi_{\pi(P)}$ is diagonalisable and $\mathcal{E}_{\phi} \cong \mathcal{O}_{C_1}(a'') \oplus \mathcal{O}_{C_2}(b'')$; otherwise, the Higgs sheaf is not diagonalisable in $\pi(P)$ and $\mathcal{E}_{\phi} \cong \mathcal{O}(a', b')$ is a line bundle on $C_{\mathcal{E}}$.

Finally, to understand the degree of $\mathcal{E}_{\phi}$ on the two components, we look at how λ acts on the eigen-sheaf and we see that: if the first case above occur, then $a'' = a, b'' = b$, and if the second case occur, $a' = a + P, b' = b$.

Example 1.29. Let $\mathcal{E} = \oplus_i \mathcal{O}_{\mathbb{P}^1}$ be a rank r trivial vector bundle on $\mathbb{P}^1$ and $\phi \in \text{Hom}(\mathcal{E}, \mathcal{E})$. We can assume that ϕ is represented by an upper triangular matrix whose entries a_{ij} are constant sections; in particular the eigenvalues of ϕ are all constant. So the spectral curve is a disjoint union of multiple $\mathbb{P}^1$:

$$C_{\mathcal{E}} = \{(\lambda - \alpha_1)^{k_1} \cdots (\lambda - \alpha_m)^{k_m}\} \subseteq \mathbb{P}^1 \times \mathbb{C},$$

where k_i is the multiplicity of α_i in the minimal polynomial of ϕ and the eigen-sheaf $\mathcal{E}_{\phi}$ encode the normal form of the Higgs field at the generic point of C . Denote $C_{\alpha_i}^{k_i} = \{(\lambda - \alpha_1)^{k_1} = 0\}$

$$\mathcal{E}_{\phi}|_{C_{\alpha_i}^{k_i}} = \mathcal{O}_{C_{\alpha_i}^{k_i}} \oplus \mathcal{O}_{C_{\alpha_i}^{k_i}^1} \cdots \oplus \mathcal{O}_{C_{\alpha_i}^{k_i}^m},$$

where the k_i^j are the dimensions of Jordan blocks for the eigenvalue α_i in the normal form decomposition of ϕ .

The above examples can be summed up in the following description: when $(\mathcal{E}, \phi)$ is a $\mathcal{L}$ Higgs bundle on C smooth, the spectral curve $C_{\mathcal{E}}$ is a divisor in $\text{Tot}(\mathcal{L})$ cut out by the minimal polynomial of $\phi : p(\lambda) \in \mathcal{O}_{\text{Tot}(\mathcal{L})}$, whose coefficients are sections of powers of $\mathcal{L}$. The points $x_1, \dots, x_k$ in the fiber of $C_{\mathcal{E}} \rightarrow C$ over a point $x \in C$ are the eigenvalues of $\phi_x : \mathcal{E}_x \rightarrow \mathcal{E}_x \otimes \mathcal{L}_x$. If ϕ has generically distinct eigenvalues, then the map $C_{\mathcal{E}} \xrightarrow{\eta} C$ is a $\text{rk}(\mathcal{E}) : 1$ cover, branched over the points where different eigenvalues come together, and the eigensheaf $\mathcal{E}_{\phi}$ is generically a line bundle on $C_{\mathcal{E}}$, whose rank may jump up on the ramification point if ϕ_x for x a branched point is diagonalisable. Otherwise to interpret η as a $\text{rk}(\mathcal{E}) : 1$ we have to take into account both the rank of the eigen-sheaf and the multiplicity of the irreducible components of the spectral curve.

Examples in the singular setting

In this section, C is a projective and integral curve, possibly very singular. Let $(\mathcal{F}, \phi)$ be an $\mathcal{L}$ valued Higgs field, with $\mathcal{F}$ torsion free of rank r . We have already seen an example of the geometry of the spectral curve in the singular setting in the introduction, recall Figure 1. A general way to study the geometry of the spectral curve associated to a singular Higgs sheaf is to relate it to a smooth Higgs bundle in the following way.

Let $\bar{C} \xrightarrow{\nu} C$ be the normalisation. Then we have a $\nu^*\mathcal{L}$ valued Higgs bundle on $\bar{C}$ given by the torsion free part of the pull-back of $\mathcal{F}$, i.e. $(\mathcal{E} := \nu^*\mathcal{F}/\text{tor}, \phi_0 = \nu^*\phi/\text{tor})$. This gives a short exact sequence of Higgs sheaves (1.5):

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F} & \xrightarrow{\nu^*} & \nu_*\mathcal{E} & \longrightarrow & \mathcal{T} \longrightarrow 0 \\ & & \downarrow \phi & & \downarrow \nu_*\phi_0 & & \downarrow \\ 0 & \longrightarrow & \mathcal{F} \otimes \mathcal{L} & \xrightarrow{\nu^*\text{id}_{\mathcal{L}}} & \nu_*\mathcal{E} \otimes \mathcal{L} & \longrightarrow & \mathcal{T} \otimes \mathcal{L} \longrightarrow 0 \end{array} \quad (1.6)$$

where $\mathcal{T}$ is supported on the zero dimensional singular locus of C . Let us denote by $\text{Tot}(\nu^*\mathcal{L}) \xrightarrow{\tilde{\nu}} \text{Tot}(\mathcal{L})$ the natural map induced by the normalisation. The morphisms in (1.6) define, under the equivalence $\text{Higgs}_{\mathcal{L}}(C) \cong \text{Coh}_{\text{cpt}}(\text{Tot}(\mathcal{L}))$, a short exact sequence of compactly supported sheaves on $\text{Tot}(\mathcal{L})$:

$$0 \rightarrow \mathcal{F}_{\phi} \rightarrow \tilde{\nu}_*\mathcal{E}_{\phi_0} \rightarrow \mathcal{T}' \rightarrow 0$$

where $\mathcal{T}'$ is supported on a zero dimensional sub-scheme contained in the fibers over $\text{Sing}(C)$. This implies that the spectral curve $C_{\mathcal{F}} \cong \text{Supp}(\mathcal{F}_{\phi})$ is the scheme theoretic image under $\tilde{\nu}$ of $C_{\mathcal{E}} \cong \text{Supp}(\mathcal{E}_{\phi_0})$, i.e. the ideal sheaf of $C_{\mathcal{F}} \subseteq \text{Tot}(\mathcal{L})$ is given by

$$\mathcal{I} = \text{Ker} \left(\mathcal{O}_{\text{Tot}(\mathcal{L})} \hookrightarrow \tilde{\nu}_*\mathcal{O}_{\text{Tot}(\nu^*\mathcal{L})} \twoheadrightarrow \tilde{\nu}_*\mathcal{O}_{C_{\mathcal{E}}} \right)$$

It is in general quite difficult to compute $\mathcal{I}$ and, whenever possible, we will find alternative ways to study the geometry of $C_{\mathcal{F}}$.

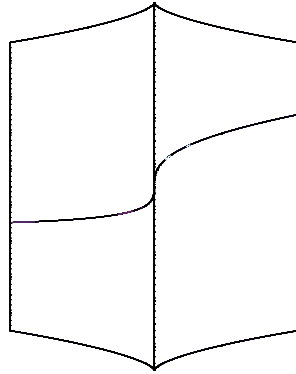


Figure 1.2: Spectral curve cusp

Example 1.30. Let C be a reduced, projective curve with a cusp singularity, and normalisation $\mathbb{P}^1 \xrightarrow{\nu} C$. Consider on C the pure rank 1 sheaf $\mathcal{F} = \nu_* \mathcal{O}_{\mathbb{P}^1}(d)$ and let $\mathcal{L}_a \in \text{Pic}^1(C) \cong (\mathbb{C}, +)$ a line bundle of degree 1 on C . Notice that $\dim_{\mathbb{C}} \text{Hom}(\mathcal{F}, \mathcal{F} \otimes \mathcal{L}_a) = 2$ while $\dim_{\mathbb{C}} H^0(\mathcal{L}_a) = 1$. Thus given a Higgs sheaf $(\mathcal{F}, \phi)$ with values in $\mathcal{L}_a$, ϕ is not necessarily a section of $\mathcal{L}_a$.

In the above notation, the Higgs bundle $(\mathcal{E}, \phi_0)$ on $\mathbb{P}^1$ associated to $(\mathcal{F}, \phi)$ is $(\mathcal{O}_{\mathbb{P}^1}(d), s)$ where $s \in H^0(\mathcal{O}_{\mathbb{P}^1}(1))$, and the corresponding spectral curve $C_{\mathcal{E}}$ is cut out in $\text{Tot}(\mathcal{O}_{\mathbb{P}^1})(1)$ by the equation $\lambda' - s = 0$, where λ' is the tautological section (i.e. the fiber coordinate) of $\pi^* \mathcal{O}_{\mathbb{P}^1}(1)$ on $\text{Tot}(\mathcal{O}_{\mathbb{P}^1})(1)$. Since $C_{\mathcal{F}}$ and $C_{\mathcal{E}}$ are isomorphic away from the singularity, in order to understand $\mathcal{F}$, or equivalently $C_{\mathcal{F}}$ we can work locally around the singular point.

Let $P_0 = \nu^{-1}(0)$ be the pre-image of the cuspidal point and let $0 \in U_0$ be a trivialising chart for $\mathcal{L}_a$, e.g. $U_0 = \text{Spec}(\mathbb{C}[t^2, t^3, (1 - a^2 t^2)^{-1}])$, and $U_{P_0} = \nu^{-1}(U_0)$ the corresponding trivialising chart for $\mathcal{O}_{\mathbb{P}^1}(1) = \nu^* \mathcal{L}_a$. In these charts, the morphism $\tilde{\nu}$ is given by:

$$\begin{aligned} \mathbb{C}[t^2, t^3, (1 - a^2 t^2)^{-1}, \lambda] &\xrightarrow{\tilde{\nu}} \mathbb{C}[t, (1 - a^2 t^2)^{-1}, \lambda'] \\ \lambda &\rightarrow \lambda' + at\lambda' \end{aligned}$$

and the section s is given by $a_0 + b_0 t$ for some $a_0, b_0 \in \mathbb{C}$. Then we see that s is in the image of $\tilde{\nu}$, i.e. it descends to a section of $\mathcal{L}_a$ if and only if $b_0/a_0 = a$. In this case $\tilde{\nu}(C_{\mathcal{E}}) \cong C$ and $\mathcal{E}_{\phi} \cong \mathcal{F}$. If instead s does not satisfy the previous condition, then it does not define a section on the cuspidal curve and $C_{\mathcal{F}}$, given by the kernel of

$$\mathbb{C}[t^2, t^3, (1 - a^2 t^2)^{-1}, \lambda] \xrightarrow{\tilde{\nu}} \mathbb{C}[t, (1 - a^2 t^2)^{-1}, \lambda'] \xrightarrow{\pi} \mathbb{C}[t, (1 - a^2 t^2)^{-1}, \lambda'] / (\lambda' - (a_0 + b_0 t)),$$

has parametrisation $t \rightarrow (t^2, t^3, a_0 + (b_0 + aa_0)t + ab_0 t^2) \in \text{Tot}(\mathcal{L}_a)$. In this case $C_{\mathcal{F}} \cong \mathbb{P}^1$, see Figure (1.2), and the natural projection to C is the normalisation map, and $\mathcal{E}_{\phi} \cong \mathcal{O}_{\mathbb{P}^1}(d)$.

1.4. Moduli of stable sheaves and DT invariants

Let X be a smooth projective Calabi-Yau 3-fold over $\mathbb{C}$, i.e. $K_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. In what follows we denote by ω a polarisation $\mathcal{O}_X(1)$ on X .

Definition 1.31. A coherent sheaf $\mathcal{F}$ on X is *pure* of dimension d if the support of $\mathcal{F}$ has dimension d and for any sub-sheaf $\mathcal{G} \subseteq \mathcal{F}$, the dimension of the support of $\mathcal{G}$ is still d .

For $\mathcal{F}$ a pure 1-dimensional coherent sheaf on X we denote by $[\mathcal{F}]$ the fundamental 1-cycle associated to $\mathcal{F}$ defined by:

$$[\mathcal{F}] = \sum_{\eta} \text{length}_{\mathcal{O}_{X,\eta}} \mathcal{F}_{\eta} [\bar{\eta}] \quad (1.7)$$

where η runs over the codimension 2 points in X . The rational equivalence class of the latter is the Poincaré dual of $ch_2(\mathcal{F})$.

Definition 1.32. Let ω be a fixed polarisation on X . The *slope* of a pure sheaf $\mathcal{F}$ with respect to ω is defined as:

$$\mu_\omega(\mathcal{F}) = \frac{\chi(\mathcal{F})}{\text{ch}_2(\mathcal{F}) \cdot \omega}$$

where $\chi(\mathcal{F})$ denotes the holomorphic Euler characteristic. We say that $\mathcal{F}$ is μ_ω *semistable* if for any proper subsheaf $\mathcal{G} \subseteq \mathcal{F}$, the inequality $\mu_\omega(\mathcal{G}) \leq \mu_\omega(\mathcal{F})$ holds, and *stable* if the above inequality is strict.

For fixed $n \in \mathbb{Z}$, $\beta \in H_2(X, \mathbb{Z})$, we denote by $\mathfrak{M}_{n,\beta}(X)$ the algebraic stack of pure 1-dimensional, ω semistable sheaves $\mathcal{F}$ with $\chi(\mathcal{F}) = n$ and fundamental 1 cycle $[\mathcal{F}]$ in class β .

Theorem 1.33. [HL10b, Theorem 4.3.4] *If β and n be such that there are not strictly semistable sheaves, then $\mathfrak{M}_{n,\beta}(X)$ is a projective scheme; in this case we will use the notation $\mathcal{M}_{n,\beta}(X)$.*

Theorem 1.34. [Tho00] *The moduli space $\mathcal{M}_{n,\beta}(X)$ admits a symmetric perfect obstruction theory given by $R\pi_* R\mathcal{H}om_{X \times \mathcal{M}_{n,\beta}(X)}(\mathbf{F}, \mathbf{F})[2]$ where $\mathbf{F}$ is the universal sheaf on $X \times \mathcal{M}_{n,\beta}(X)$ and π denotes the second projection.*

The above obstruction theory defines a 0-dimensional virtual class on $\mathcal{M}_{n,\beta}(X)$. We denote the associated DT invariant with:

$$DT_{n,\beta} = \int_{[\mathcal{M}_{n,\beta}(X)]^{vir}} 1.$$

In [Kat08] the genus 0 GV invariants are defined as the DT invariants arising from the moduli space of pure 1-dimensional stable sheaves with $\chi(\mathcal{F}) = 1$. The latter is a technical assumption to ensure there are not strictly semistable and the stability condition is independent from the choice of polarisation:

Lemma 1.35. *For $\chi(\mathcal{F}) = 1$, there are no strictly semi-stable sheaves and $\mathcal{F}$ is stable if and only if for any proper subsheaf $\mathcal{G} \subseteq \mathcal{F}$ the inequality $\chi(\mathcal{G}) \leq 0$ holds.*

Proof. If for any proper subsheaf $\mathcal{G} \subseteq \mathcal{F}$ the inequality $\chi(\mathcal{G}) \leq 0$ holds, in particular $\mu_\omega(\mathcal{G}) < \mu_\omega(\mathcal{F})$ for any ω and $\mathcal{F}$ is strictly stable independently of the choice of polarisation by definition.

Vivecersa, let $\mathcal{F} \in \mathcal{M}_{1,\beta}(X)$ be semi-stable object, i.e. any sub-sheaf $\mathcal{G} \subseteq \mathcal{F}$,

$$\mu_\omega(\mathcal{G}) = \frac{\chi(\mathcal{G})}{\text{ch}_2(\mathcal{G}) \cdot \omega} \leq \mu_\omega(\mathcal{F}) = \frac{1}{\text{ch}_2(\mathcal{F}) \cdot \omega}.$$

As $\mathcal{G}$ is a sub-sheaf, for any choice of polarisation:

$$\text{length}_{\mathcal{O}_{X,\eta}} \mathcal{G}_\eta \leq \text{length}_{\mathcal{O}_{X,\eta}} \mathcal{F}_\eta \text{ for any } \eta$$

which implies $ch_2(\mathcal{G}) \cdot \omega \leq ch_2(\mathcal{F}) \cdot \omega$ for any ω . Then $\mathcal{F}$ semistable implies that any proper subsheaf $\mathcal{G}$ has $\chi(\mathcal{G}) \leq 1$ and in fact we will now show that the inequality has to be strict. Suppose $\chi(\mathcal{G}) = 1$, then since we assumed $\mathcal{F}$ semi-stable,

$$\frac{1}{ch_2(\mathcal{G}) \cdot \omega} \leq \frac{1}{ch_2(\mathcal{F}) \cdot \omega} \quad \text{and} \quad ch_2(\mathcal{G}) \cdot \omega \leq ch_2(\mathcal{F}) \cdot \omega$$

which implies $ch_2(\mathcal{G}) \cdot \omega = ch_2(\mathcal{F}) \cdot \omega$ for any choice of ω and thus $ch_2(\mathcal{G}) = ch_2(\mathcal{F})$. This, in turns, implies that the quotient $\mathcal{F}/\mathcal{G}$ is supported in dimension 0 and has 0 holomorphic Euler characteristic, i.e. $\mathcal{G} \cong \mathcal{F}$ and thus $\mathcal{G}$ is not a proper subsheaf. □

We will refer to the invariants:

$$n_{0,\beta} = \int_{[\mathcal{M}_{1,\beta}(X)]^{vir}} 1$$

as the *Katz invariants* of X .

1.4.1. Behrend function and Local invariants

Let Y be a Deligne-Mumford stack, denote by $Z_*(Y)$ the free abelian group generated by cycles on Y and by $\text{Con}(Y)$ the group of integer valued constructible functions on Y . Given a morphism $X \xrightarrow{g} Y$ and constructible functions φ on X and ψ on Y we will denote by $g^*\psi$ and $g_*\varphi$, respectively, the pull-back constructible function, defined by $x \rightarrow \psi(g(x))$, and the push-forward constructible function, defined by $y \rightarrow \chi(X_y, \varphi|_{X_y})$. Recall that $\chi(X, \varphi) = \chi(Y, g_*\varphi)$ and if g is a homeomorphism, then $\chi(Y, \psi) = \chi(X, g^*\psi)$.

Definition 1.36. We call Behrend function of the DM stack Y and denote by ν_Y the constructible function

$$\nu_Y = \text{Eu}(\mathfrak{c}_Y)$$

where Eu is the MacPherson's local Euler obstruction and $\mathfrak{c}_Y$ is the signed support of the intrinsic normal cone of Y , often called the *distinguished cycle* on Y .

We refer to [Ful98, Example 4.2.9] for the definition of Eu and to [Beh09, § 1.1] for the explicit local expression of $\mathfrak{c}_Y$.

Theorem 1.37. [Beh09, Theorem 4.18] *Let Y be a proper DM stack endowed with a symmetric perfect obstruction theory, then*

$$\int_{[Y]^{vir}} 1 = \chi(Y, \nu_Y) = \sum_{n \in \mathbb{Z}} n \chi(\{\nu_Y = n\}).$$

The obstruction theory for $\mathcal{M}_{n,\beta}(X)$ is symmetric, thus the above Theorem implies we can express the invariants as a weighted Euler characteristic.

We recall some properties of ν_Y and their consequences; full details can be found in [Beh09, Proposition 1.8]:

- The constructible function ν_Y only depends on the scheme structure of Y . It follows in particular that the DT type invariants do not depend from the choice of a symmetric perfect obstruction theory.
- If $f: Z \rightarrow Y$ is étale then $f^* \nu_Y \cong \nu_Z$. Thus $\nu_Y(P)$ is an invariant of the singularity of Y at P .
- if $Z = Z_1 \cup Z_2$ is the disjoint union of two locally closed substack, then $\chi(Z, \nu_Y|_Z) = \chi(Z_1, \nu_Y|_{Z_1}) + \chi(Z_2, \nu_Y|_{Z_2})$.
- If Y is a disjoint union of a closed substack Z and a open substack U then $\chi(Y, \nu_Y) = \chi(U, \nu_U) + \chi(Z, \nu_Y|_Z)$. In particular we can compute DT invariants over stratifications of the moduli space.

Definition 1.38. For a fixed reduced curve $i: C \rightarrow X$ and $\gamma \in \oplus_i \mathbb{Z}[C_i]$ a cycle on C satisfying $i_* \gamma = \beta$, we define the *local contributions* of C to the invariants to be:

$$n_{0,\gamma} = \chi(\mathcal{M}_{1,\gamma}(C), \nu|_{\mathcal{M}_{1,\gamma}(C)})$$

where $\mathcal{M}_{1,\gamma}(C)$ is closed locus in $\mathcal{M}_{1,\beta}(X)$ whose sheaves satisfy $[\mathcal{F}] = \gamma$, (this is an equality as cycles) i.e $\mathcal{F}$ are scheme theoretically supported either on C or on thickenings of C .

Remark 1.39. As explained in [PT10, § 3.2] we can recover the global invariants from the local contributions: the subspaces $\mathcal{M}_{1,\gamma}(C)$ are the fibers of the Hilbert-Chow morphism

$$\mathcal{M}_{1,\beta}(X) \rightarrow \text{Chow}(X, \beta)$$

sending $\mathcal{F}$ to its fundamental 1-cycle, where we denoted by $\text{Chow}(X, \beta)$ the parameter space of 1 cycles in X of class β . Then we have a constructible function

$$\begin{aligned} \tilde{n}_{0,\beta}: \text{Chow}(X, \beta) &\rightarrow \mathbb{Z} \\ \gamma &\rightarrow n_{0,\gamma} \end{aligned}$$

such that $n_{0,\beta} = \chi(\text{Chow}(X, \beta), \tilde{n}_{0,\beta})$.

1.4.2. Reduction to curves of geometric genus 0

We prefer working with local contributions since the spaces $\mathcal{M}_{1,\gamma}(C)$ admit an action of the group $\text{Pic}^0(\widehat{C})$, given by:

$$\mathcal{L} \cdot \mathcal{F} := \mathcal{F} \otimes \mathcal{L}. \tag{1.8}$$

The action is in fact defined on the formal neighbourhood $\mathcal{M}_{1,\beta}(\widehat{C})$ of $\mathcal{M}_{1,\gamma}(C)$ in $\mathcal{M}_{1,\beta}(X)$; and it preserves the closed subspace $\mathcal{M}_{1,\gamma}(C)$. Any thickening $\mathbf{C}$ of C with $[\mathbf{C}] = \gamma$ is

contained in the formal neighbourhood $\widehat{C}$ of C and the action (1.8) restricts to the action of $\text{Pic}^0(\mathbf{C})$ for sheaves supported on $\mathbf{C}$.

Notice that the Behrend function $v_{\mathcal{M}}$ is invariant under the $\text{Pic}^0(\widehat{C})$ action since tensoring with a line bundle is locally an isomorphism on $\mathcal{M}_{1,\beta}(X)$; in particular $v_{\mathcal{M}}$ is constant on the orbits and, if we act by a sub-torus of $\text{Pic}^0(\widehat{C})$, we can restrict the invariants to the fixed points.

Using this action, it has already been shown the invariants $DT_{n,\gamma}$ vanishes unless the normalisation of the reduced curve underlying γ is a disjoint union of $\mathbb{P}^1$:

Lemma 1.40. *Let C be a reduced projective curve and $C_1, \dots, C_N$ its irreducible components. Suppose that for some $1 \leq i \leq N$ C_i has geometric genus bigger or equal then one, then for every cycle γ supported on C the invariant $DT_{n,\gamma}$ vanishes.*

We refer to [JS12, Proposition 6.19], [Tod11, Lemma 2.11] for the proof. We just want to remark that the point is that, as soon as we have a component of positive geometric genus, $\chi(\text{Pic}^0(C)) = 0$, and the same will be true for all the orbits of the action; in particular this means that the action does not admits fixed points (see also Corollary 2.7).

If instead C has all irreducible component with high arithmetic genus and normalisation $\mathbb{P}^1$, then C contributes to the invariants and we want to understand how. In particular, we would like to understand if it is possible to localise the invariants to curves of arithmetic genus 0, namely to generalised tree 1.19.

2

Contribution of singular curves: reduced cycle case

In this chapter we give an interpretation in terms of curves homeomorphic to generalised trees of $\mathbb{P}^1$ of the invariant $n_{0,\gamma}$ in the case $\gamma = [C]$ and C reduced. In such case, $\mathcal{F} \in \mathcal{M}_{1,[C]}(C)$ is scheme theoretically supported on C and it has rank 1 on each irreducible component of the curve. The group $\text{Pic}^0(\widehat{C})$ acts on $\mathcal{F} \in \mathcal{M}_{1,[C]}(C) \subset \mathcal{M}_{1,\beta}(\widehat{C})$ through $\text{Pic}^0(C)$.

2.1. Characterisation of fixed point of the action and description of the orbits

Let C be a reduced, projective curve and let $\mathcal{F} \in \text{Coh}(C)$ be a pure sheaf of rank 1 on each irreducible component of C . Let us assume furthermore that the stabiliser $\text{Stab}(\mathcal{F})$ of $\mathcal{F}$ with respect to the $\text{Pic}^0(C)$ action is non trivial. Then, for each $\mathcal{L} \in \text{Stab}(\mathcal{F})$ there exists an isomorphism: $\mathcal{F} \xrightarrow{\phi} \mathcal{F} \otimes \mathcal{L}$. In the language of § 1.3, ϕ is an $\mathcal{L}$ -valued Higgs field.

Lemma 2.1. *Let C be a reduced, projective curve, and let $\mathcal{F}$ be a pure sheaf of rank 1 on each irreducible component of C . If for a fixed $\mathcal{O}_C \neq \mathcal{L} \in \text{Pic}^0(C)$ there exists an isomorphism $\mathcal{F} \xrightarrow{\phi} \mathcal{F} \otimes \mathcal{L}$, then the spectral curve $C_{\mathcal{F}}$ is a non trivial partial normalisation of C .*

Proof. Denote by $\nu: C_{\mathcal{F}} \rightarrow C$ the natural map coming from the spectral construction. First we show that $C_{\mathcal{F}} = \text{Supp}(\mathcal{F}_{\phi})$ does not have embedded points, equivalently that $\mathcal{O}_{C_{\mathcal{F}}}$ is pure. As $\mathcal{F}$ is pure, the eigensheaf $\mathcal{F}_{\phi}$ is pure as well, otherwise, given a 0 dimensional sub-sheaf $\mathcal{G} \subseteq \mathcal{F}_{\phi}$ pushing down we would get a zero dimensional subsheaf $\nu_*\mathcal{G}$ of $\mathcal{F}$, contradicting the purity of $\mathcal{F}$. Then $\mathcal{O}_{C_{\mathcal{F}}}$ is pure as well. In fact, suppose there

was a dimension 0 sub-sheaf $\mathcal{I}_Z$, then

$$\mathcal{I}_Z \subseteq \mathcal{O}_{C_{\mathcal{F}}} \implies \mathcal{I}_Z \cdot \mathcal{F}_{\phi} \subseteq \mathcal{F}_{\phi} \implies \mathcal{I}_Z \cdot \mathcal{F}_{\phi} = 0$$

where the last implication follows by purity of $\mathcal{F}_{\phi}$. This means that $\mathcal{F}_{\phi}$ is supported on Z , i.e., $Z = C_{\mathcal{F}}$ and $\mathcal{I}_Z = 0$. Moreover, the fact that $\mathcal{F}$ has rank 1 on every irreducible component of C implies that ν is birational, so $C_{\mathcal{F}}$ is reduced with N irreducible components.

Finally, to prove that the spectral curve is a non trivial partial normalisation, let's prove that the tautological section $\lambda|_{C_{\mathcal{F}}}$ of $\nu^*\mathcal{L}$ is nowhere vanishing and thus trivialises $\nu^*\mathcal{L}$. It follows from the definition of $\mathcal{F}_{\phi}$ that the push-down of $\lambda \otimes \text{id}_{\mathcal{F}_{\phi}}$ along ν is the isomorphism ϕ . As ν is affine the push-forward ν_* is exact, thus $\lambda \otimes \text{id}_{\mathcal{F}_{\phi}}$ is an isomorphism, hence nowhere 0. But then so is λ which therefore gives a trivialization for $\nu^*\mathcal{L}$. $\square$

Lemma 2.2. *Let C be reduced and projective, $\mathcal{F}$ pure of rank 1, and let $\mathcal{F} \xrightarrow{\phi} \mathcal{F} \otimes \mathcal{L}$ be an isomorphism, with $\mathcal{O}_C \neq \mathcal{L} \in \text{Stab}(\mathcal{F})$. Denote by $C_{\mathcal{F}} \xrightarrow{\nu} C$ the spectral curve, and by $\mathcal{F}_{\phi}$ the eigensheaf of $(\mathcal{F}, \phi)$. Then we have a surjection $\text{Stab}(\mathcal{F}) \xrightarrow{\nu^*} \text{Stab}(\mathcal{F}_{\phi})$. In particular if $\mathcal{F}$ is $\text{Pic}^0(C)$ invariant, $\mathcal{F}_{\phi}$ is $\text{Pic}^0(C_{\mathcal{F}})$ invariant.*

Proof. Let us first prove that the map is well defined, namely that if $\mathcal{M} \in \text{Stab}(\mathcal{F})$ then $\nu^*\mathcal{M} \in \text{Stab}(\mathcal{F}_{\phi})$, i.e. we want to show that for any $\mathcal{M} \in \text{Stab}(\mathcal{F})$ there exists an isomorphism

$$\mathcal{F}_{\phi} \xrightarrow{\cong} \mathcal{F}_{\phi} \otimes \nu^*\mathcal{M}. \quad (2.1)$$

Notice that we have an isomorphism of $\mathcal{O}_{\text{Tot } \mathcal{L}}$ modules $\mathcal{F}_{\phi} \otimes \nu^*\mathcal{M} \cong (\mathcal{F} \otimes \mathcal{M})_{\phi \otimes \text{id}_{\mathcal{M}}}$, where $(\mathcal{F} \otimes \mathcal{M})_{\phi \otimes \text{id}_{\mathcal{M}}}$ denotes the eigensheaf defined by $\mathcal{F} \xrightarrow{\phi \otimes \text{id}_{\mathcal{M}}} \mathcal{F} \otimes \mathcal{L} \otimes \mathcal{M}$. So the existence of the isomorphism (2.1) is equivalent to the existence of the commutative diagram:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\phi} & \mathcal{F} \otimes \mathcal{L} \\ \downarrow \psi & & \downarrow \psi \otimes \text{id}_{\mathcal{L}} \\ \mathcal{F} \otimes \mathcal{M} & \xrightarrow{\phi \otimes \text{id}_{\mathcal{M}}} & \mathcal{F} \otimes \mathcal{M} \otimes \mathcal{L}. \end{array} \quad (2.2)$$

The isomorphism ψ surely exists because we chose $\mathcal{M} \in \text{Stab}(\mathcal{F})$; we only need to show that the diagram commutes. On the dense open set $U = C \setminus \text{Sing}(C)$ where $\mathcal{F}|_U$ is a line bundle, the morphisms in (2.2) are multiplication by a scalar and thus the diagram restricted to U is commutative. In other words the homomorphism

$$F := \phi \circ \psi \otimes \text{id}_{\mathcal{L}} - \psi \circ \phi \otimes \text{id}_{\mathcal{M}}: \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{L} \otimes \mathcal{M}$$

is zero when restricted to U . This implies that $\text{Supp}(\mathcal{I}m(F)) \subseteq (C \setminus U)$, i.e. it has dim 0. But $\mathcal{F} \otimes \mathcal{L} \otimes \mathcal{M}$ is pure, so $\mathcal{I}m(F) = 0$, i.e. $F = 0$ and the diagram (2.2) commutes.

We conclude arguing that $\text{Stab}(\mathcal{F}) \xrightarrow{\nu^*} \text{Stab}(\mathcal{F}_\phi)$ is surjective. By Lemma 2.1 the spectral curve is a partial normalisation, which implies the pull-back $\text{Pic}^0(C) \rightarrow \text{Pic}^0(C_\mathcal{F})$ is surjective. In particular any $\mathcal{L}' \in \text{Stab}(\mathcal{F}_\phi)$ is a pull-back $\mathcal{L}' \cong \nu^* \mathcal{M}$ for some $\mathcal{M} \in \text{Pic}^0(C)$. For any $\mathcal{L}' \cong \nu^* \mathcal{M} \in \text{Stab}(\mathcal{F}_\phi)$ we have an isomorphism $\mathcal{F}_\phi \xrightarrow{\cong} \mathcal{F}_\phi \otimes \mathcal{L}'$, whose push-down give an isomorphism $\mathcal{F} \cong \mathcal{F} \otimes \mathcal{M}$, thus $\mathcal{M} \in \text{Stab}(\mathcal{F})$. $\square$

Proposition 2.3. *Let C be a reduced, projective curve and $\mathcal{F}$ a pure sheaf of rank 1 on each irreducible component of C . $\mathcal{F}$ has stabiliser $K \subseteq \text{Pic}^0(C)$ if and only if:*

1. $\mathcal{F} \cong \nu_* \mathcal{E}$ where $\widetilde{C} \xrightarrow{\nu} C$ is a partial normalisation of C with $\text{Pic}^0(\widetilde{C}) \cong \text{Pic}^0(C)/K$,
2. $\mathcal{E}$ a pure rank 1 sheaf on $\widetilde{C}$ with trivial stabiliser with respect to the $\text{Pic}^0(\widetilde{C})$ action.

Proof. Suppose first $\mathcal{F} \cong \nu_* \mathcal{E}$ satisfy conditions (1) and (2) above. Then for any $\mathcal{L} \in K$ we have by projection formula that $\nu_* \mathcal{E} \otimes \mathcal{L} \cong \nu_*(\mathcal{E} \otimes \nu^* \mathcal{L})$. Since we assumed $K \cong \text{Ker}(\nu^*)$, we have $\nu_*(\mathcal{E} \otimes \nu^* \mathcal{L}) \cong \nu_* \mathcal{E}$ and thus $K \subseteq \text{Stab}(\mathcal{F})$. On the other hand, since we assumed that $\mathcal{E}$ has not stabiliser with respect to the $\text{Pic}^0(\widetilde{C})$ action, $K \cong \text{Stab}(\mathcal{F})$.

To prove the other implication we proceed inductively on the dimension of the stabiliser. If the action of $\text{Pic}^0(C)$ on $\mathcal{F}$ is free, i.e. K is trivial, conditions (1) and (2) are satisfied.

If not, choose $\mathcal{O}_C \neq \mathcal{L}_1 \in \text{Pic}^0(C)$ and consider an isomorphism $\mathcal{F} \xrightarrow{\phi^1} \mathcal{F} \otimes \mathcal{L}_1$. By Lemma 2.1 the spectral curve $C_\mathcal{F} \xrightarrow{\nu^1} C$ associated to $(\mathcal{F}, \phi^1)$ is a non trivial partial normalisation of C . Furthermore, by Lemma 2.2 the eigen-sheaf $\mathcal{F}_{\phi^1}$ is a pure rank 1 sheaf on $C_\mathcal{F}$ with stabiliser $K_1 = \nu_1^* K / \text{Ker}(\nu_1^*)$ with respect to the $\text{Pic}^0(C_\mathcal{F})$ action. If K_1 is zero, then the action of $\text{Pic}^0(C_\mathcal{F})$ on $\mathcal{F}_{\phi^1}$ is free and we are done. Otherwise there exists a $\mathcal{O}_{C_\mathcal{F}} \neq \mathcal{L}_2 \in \text{Stab}(\mathcal{F}_{\phi^1})$ and we can repeat construction.

Since $C_{\mathcal{F}_{\phi^k}}$ is always a non trivial partial normalisation of $C_{\mathcal{F}_{\phi^{k-1}}}$, after finitely many steps the pull-back of K along $C_{\mathcal{F}_{\phi^k}} \xrightarrow{\nu_k} C$ is zero. Indeed, if $\widetilde{C} \xrightarrow{\bar{\nu}} C$ is the normalisation, then pure sheaves on $\widetilde{C}$ are line bundles and the action of $\text{Pic}^0(\widetilde{C})$ on them is free. So the stabiliser K of a pure rank 1 sheaf on C is a subgroup of the $\text{Ker}\left(\text{Pic}^0(C) \xrightarrow{\bar{\nu}^*} \text{Pic}^0(\widetilde{C})\right)$ and the spectral curve construction process terminates after at most $g(C) - g(\widetilde{C})$ steps. $\square$

Remark 2.4. The fact that any pure rank 1 sheaf on a reduced curve is push-forward of a pure rank 1 sheaf from some partial normalisation is well known by people studying compactifications of $\text{Pic}^0(C)$.

The spectral curve construction gives a way to produce the maximal partial normalisation from which my rank 1 pure sheaf is push-forward, but another way to construct $\widetilde{C}$ given $\mathcal{F}$, which does not involve the spectral curve, is the following: if $\mathcal{F}$ is pure of rank 1 on any irreducible component of a reduced curve C , then it is generically a line bundle. This ensures that the sheaf of $\mathcal{O}_C$ algebras $\mathcal{E}nd_C(\mathcal{F}, \mathcal{F})$ is commutative. Clearly $\mathcal{O}_C$ is a sub-ring of $\mathcal{E}nd_C(\mathcal{F}, \mathcal{F})$; on the other hand, any $\phi \in \mathcal{E}nd_C(\mathcal{F}, \mathcal{F})$ is annihilated by

a polynomial with coefficient in $\mathcal{O}_C$, thus $\mathcal{E}nd_C(\mathcal{F}, \mathcal{F})$ is an integral extension of $\mathcal{O}_C$, i.e. $\widetilde{C} = \text{Spec}_C(\mathcal{E}nd_C(\mathcal{F}, \mathcal{F})) \rightarrow C$ is a partial normalisation. Obviously $\mathcal{F}$ is an $\mathcal{O}_{\widetilde{C}}$ module.

From this point of view we also see that $\widetilde{C}$ only depends on $\mathcal{F}$ and not on the choices we do in the spectral construction process.

Corollary 2.5. *Each orbit of the action of $\text{Pic}^0(C)$ on $\mathcal{M}_{1,[C]}(C)$ is isomorphic to $\text{Pic}^0(\widetilde{C})$ for $\widetilde{C} \xrightarrow{\eta} C$ some partial normalisation.*

Proof. Given $\mathcal{F} \in \mathcal{M}_{1,[C]}(C)$, its orbit $O_{\mathcal{F}}$ is isomorphic to $\text{Pic}^0(C)/\text{Stab}(\mathcal{F}) \cong \text{Pic}^0(\widetilde{C})$ where $\widetilde{C}$ is the partial normalisation constructed in Proposition 2.3. $\square$

Remark 2.6. Since we are interested in computing $n_{0,[C]}$, or also its naive counterpart $\chi(\mathcal{M}_{1,[C]}(C))$, we need to know which orbits, or equivalently which partial normalisations, contribute to the invariants. By Proposition 1.14 $\text{Pic}^0(\widetilde{C})$ is a product of an algebraic torus $(\mathbb{C}^*)^m$ with a commutative unipotent group $(\mathbb{C}, +)^n$ for some n, m , thus we want to distinguish partial normalisations whose Pic^0 has a $\mathbb{C}^*$ factor from those which don't. In the next section we will deal with this problem; we construct parameter spaces of partial normalisations with unipotent Pic^0 and show that the invariant can be localised to these curves, which are homeomorphic to genus 0 curves.

Corollary 2.7. *Under the hypotheses of Proposition 2.3, $\mathcal{F}$ is fixed by $\text{Pic}^0(C)$ if and only if $\mathcal{F} \cong v_{D*}\mathcal{E}_D$, where $D \xrightarrow{v} C$ is a partial normalization of C whose connected components are generalised trees and $\mathcal{E}_D$ is a pure rank 1 sheaf on it. In particular, the normalization of C is a disjoint union of $\mathbb{P}^1$.*

Remark 2.8. In Proposition 2.3 and Corollary 2.7 we did not impose any stability condition on $\mathcal{F}$. In particular, the above characterisation holds if we want to work with moduli spaces $\mathfrak{M}_{n,\beta}(X)$ parametrising strictly semistable objects.

Corollary 2.9. *Let C be a connected, reduced projective curve; $\mathcal{F} \in \mathcal{M}_{1,[C]}(C)$ is a fixed point of the $\text{Pic}^0(C)$ action if and only if $\mathcal{F} \cong v_*\mathcal{O}_D$ where $D \xrightarrow{v} C$ is partial normalisation with D a generalised tree. In particular, there are finitely many fixed points.*

Proof. By Corollary 2.7 it is enough to show that for any generalised tree D , $M_{1,[D]}(D)$ is a reduced point represented by $\mathcal{O}_D$ and then the claim follows from the fact that there are finitely many generalised trees D which partially normalised a fixed curve C , see Lemma 1.22. Notice that we are just considering connected partial normalisations as $\mathcal{M}_{1,[C]}(C)$ does not have strictly semi-stables. Let D be a generalised tree and $\mathcal{F}$ a pure rank 1 sheaf on D with $\chi(\mathcal{F}) = h^0(\mathcal{F}) - h^1(\mathcal{F}) = 1$. Then there exists at least one non zero section $\mathcal{O}_D \xrightarrow{s} \mathcal{F}$ and thus an inclusion $\mathcal{O}_D/\text{Ker}(s) \hookrightarrow \mathcal{F}$. Notice that $\mathcal{O}_Z \cong \mathcal{O}_D/\text{Ker}(s)$ is the structure sheaf of a sub-variety of D and at the same time a sub-sheaf of $\mathcal{F}$ which is pure. Then Z is a union (possibly disconnected) of generalised trees.

In particular $\chi(\mathcal{O}_Z) \geq 1$ and so $\mathcal{O}_Z$ destabilizes $\mathcal{F}$ (see Lemma 1.35). It follows that s is injective, i.e. $\mathcal{O}_D \subseteq \mathcal{F}$. But $\chi(\mathcal{O}_D) = 1$, which implies by strict stability $\mathcal{O}_D \cong \mathcal{F}$. Finally, the fact that $M_{1,[D]}(D)$ is reduced simply follows from the fact that $\text{Ext}_D^1(\mathcal{O}_D, \mathcal{O}_D) = 0$. $\square$

For C a projective reduced curve with only genus 0 singularities, $\text{Pic}^0(C) \cong (\mathbb{C}^*)^{g_a(C)}$ where $g_a(C)$ is the arithmetic genus of C . Recall the following result for torus actions:

Proposition 2.10. [BB73] *Let X be an algebraic scheme endowed with an action of an algebraic torus $T = (\mathbb{C}^*)^r$, then the topological Euler characteristic of X coincides with the topological Euler characteristic of the fixed locus X^T*

$$\chi(X) = \chi(X^T). \quad (2.3)$$

For each $\widehat{\mathcal{L}} \in (\mathbb{C}^*)^{g_a(C)}$ the morphism $-\otimes \widehat{\mathcal{L}}$ is in fact an automorphism of $\mathcal{M}_{1,\beta}(\widehat{C})$; so the Behrend function, which only depend from the scheme structure of the neighbourhood $\mathcal{M}_{1,\beta}(\widehat{C})$, is invariant under such action. It follows that a weighted version of (2.3) holds. Moreover, such action preserve the closed sub-scheme $\mathcal{M}_{1,[C]}(C)$ and we have:

Corollary 2.11. *Let C be a reduced projective curve with only genus 0 singularities, then:*

$$\chi(\mathcal{M}_{1,[C]}(C)) = \sum_{D \xrightarrow{\eta_D} C} 1$$

where the sum run over the arithmetic genus 0 connected partial normalisations of C . If C is embedded on a smooth Calabi-Yau 3-fold X , then the local invariant is given by $n_{0,[C]} = \sum_{D \xrightarrow{\eta_D} C} \nu_{\mathcal{M}}(\eta_{D*}\mathcal{O}_D)$ where we wrote $\nu_{\mathcal{M}}(\eta_{D*}\mathcal{O}_D)$ for the value of the Behrend function of $\mathcal{M}_{1,\beta}(X)$ at the point $i_*\eta_{D*}\mathcal{O}_D$.

Proof. The proof follows immediately from the results of this section and the fact that by Corollary 2.9 the sum is finite. $\square$

Remark 2.12. For curves with nodal singularities a result more general than this was already known via Stable-Pairs. Indeed in [PT10, § 3.6] the authors prove that all the BPS numbers $n_{h,C}$ of a nodal curve C can be expressed in terms of partial normalisation of fixed arithmetic genus h , or equivalently, h dimensional sub-tori of $\mathcal{M}_{1,[C]}(C)$. An analogous result for higher genus BPS number of curves with ordinary 3-folds points (Definition 1.16).

For curves with singularities other than genus 0, two complications arise: $\text{Pic}^0(C)$ has unipotent factors and thus in general positive dimensional orbits of the action contribute to the invariants, and in general there are infinitely many of such orbits. In the next section we discuss a way to describe the contributions coming from positive dimensional orbits.

2.2. Moduli spaces of partial normalisations

In this section C is a connected, reduced, projective and rational curve, i.e. with normalisation $\overline{C} = \sqcup_{i=1}^N \mathbb{P}^1 \xrightarrow{\eta} C$. We denote by $\mathcal{S} = \eta_*\mathcal{O}_{\sqcup_{i=1}^N \mathbb{P}^1}/\mathcal{O}_C = \bigoplus_{x \in \text{Sing}(C)} \mathcal{O}_x^{\delta_x}$ the zero dimensional sheaf supported on the singular locus whose length is $g_a(C) + N - 1$.

Definition 2.13. Let C be as above. For $0 \leq m \leq \text{length}(\mathcal{S})$ we define a functor

$$\underline{PN}^m: (Sch) \rightarrow (Sets)$$

as follows: for any $T \rightarrow (Sch)$ we define $\underline{PN}^m(T)$ to be the set of isomorphism classes of diagrams

$$\begin{array}{ccc} \overline{C} \times T & \xrightarrow{\quad} & \mathcal{C}_T \\ & \searrow \eta \times \text{id}_T & \downarrow \pi_T \\ & & C \times T \end{array}$$

where $\mathcal{C}_T \rightarrow T$ is a flat family of reduced projective curves with $1 - \chi(\mathcal{O}_{\mathcal{C}_t}) = 1 - N + m$, and for every geometric point the morphism π is a partial normalisation $\overline{C} \rightarrow \mathcal{C}_t \xrightarrow{\pi} C$ factoring η . For a morphism $T' \xrightarrow{p} T$ we get an induced map $\underline{PN}^m(T') \rightarrow \underline{PN}^m(T)$ given by pull-back.

We want to show that $\underline{PN}^m$ is represented by a closed subscheme of the quot scheme $\text{Quot}_C(\mathcal{S}, k)$, parametrising length k quotients of $\mathcal{S}$. See [HL10b, § 2] for a proof of the fact that $\text{Quot}_C(\mathcal{S}, k)$ is a projective scheme over $\mathbb{C}$. Let us denote by $\mathcal{O}_{\mathcal{Q}} \otimes \mathcal{S} \xrightarrow{q} \mathcal{W}_{\mathcal{Q}}$ the universal quotient module over $\text{Quot}_C(\mathcal{S}, k) \times C$.

Theorem 2.14. *In the above notation, $\underline{PN}^m$ is represented by a closed subscheme PN^m of $\text{Quot}_C(\mathcal{S}, m)$.*

Proof. For any $\overline{C} \times T \rightarrow \mathcal{C}_T \xrightarrow{\pi} C \times T$ we have the following commutative diagram of T -flat sheaves:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{C \times T} & \longrightarrow & (\eta \times \text{id}_T)_* \mathcal{O}_{\overline{C} \times T} & \longrightarrow & \mathcal{S} \otimes \mathcal{O}_T \longrightarrow 0 \\ & & \downarrow & & \downarrow = & & \downarrow q \\ 0 & \longrightarrow & \pi_* \mathcal{O}_{\mathcal{C}_T} & \longrightarrow & (\eta \times \text{id}_T)_* \mathcal{O}_{\overline{C} \times T} & \longrightarrow & \mathcal{W}_T \longrightarrow 0 \end{array}$$

and thus $\mathcal{S} \otimes \mathcal{O}_T \xrightarrow{q} \mathcal{W}_T \in \text{Quot}_C(\mathcal{S}, m)(T)$. Moreover two partial normalisations $\mathcal{C}_{T,1} \xrightarrow{\pi_1} C \times T$, $\mathcal{C}_{T,2} \xrightarrow{\pi_2} C \times T$, are isomorphic as $\mathcal{O}_{C \times T}$ algebras if and only if $\mathcal{O}_{\mathcal{C}_{T,1}}$ and $\mathcal{O}_{\mathcal{C}_{T,2}}$ are isomorphic as $\mathcal{O}_{C \times T}$ modules. Indeed, given an isomorphism as $\mathcal{O}_{C \times T}$ modules $\varphi: \pi_{1,*} \mathcal{O}_{\mathcal{C}_{T,1}} \xrightarrow{\cong} \pi_{2,*} \mathcal{O}_{\mathcal{C}_{T,2}}$ there is an induced isomorphism of $\mathcal{O}_{C \times T}$ algebras

$$\mathcal{E}nd_{\mathcal{O}_{C \times T}}(\pi_{2,*} \mathcal{O}_{\mathcal{C}_{T,2}}, \pi_{2,*} \mathcal{O}_{\mathcal{C}_{T,2}}) \xrightarrow{\varphi^{-1} \circ (-) \circ \varphi} \mathcal{E}nd_{\mathcal{O}_{C \times T}}(\pi_{1,*} \mathcal{O}_{\mathcal{C}_{T,1}}, \pi_{1,*} \mathcal{O}_{\mathcal{C}_{T,1}})$$

and we also have the isomorphisms $\pi_{i,*} \mathcal{O}_{\mathcal{C}_T} \cong \mathcal{E}nd_{\mathcal{O}_{C \times T}}(\pi_{i,*} \mathcal{O}_{\mathcal{C}_{T,i}}, \pi_{i,*} \mathcal{O}_{\mathcal{C}_{T,i}})$ explicitly given associating to the ring element x the endomorphism m_x defined by multiplication by x .

This implies that two families of partial normalisations are isomorphic if and only if there exists an isomorphism $\mathcal{W}_1 \cong \mathcal{W}_2$ such that $\mathcal{S} \otimes \mathcal{O}_T \xrightarrow{q_1} \mathcal{W}_1$, $\mathcal{S} \otimes \mathcal{O}_T \xrightarrow{q_2} \mathcal{W}_2$ commutes with the identity on $\mathcal{S}$, i.e. they are the same point in the Quot scheme. This define an injective morphism $\overline{PN}^m \hookrightarrow \text{Quot}_C(\mathcal{S}, m)$.

Fixed a quotient $\mathcal{S} \otimes \mathcal{O}_T \xrightarrow{q} \mathcal{W}_T$, we get a finite $\mathcal{O}_{C \times T}$ module

$$\mathcal{M}_{\mathcal{W}_T} := \text{Ker} \left((\eta \times \text{id}_T)_* \mathcal{O}_{\overline{C} \times T} \rightarrow \mathcal{S} \otimes \mathcal{O}_T \xrightarrow{q} \mathcal{W}_T \right)$$

such that $\mathcal{O}_{C \times T} \subseteq \mathcal{M}_{\mathcal{W}_T} \subseteq (\eta \times \text{id}_T)_* \mathcal{O}_{\overline{C} \times T}$.

Then $PN^m(T) \subseteq \text{Quot}_C(\mathcal{S}, m)(T)$ is identified with the locus in T where $\mathcal{M}_{\mathcal{W}_T}$ is an $\mathcal{O}_{C \times T}$ algebra with respect to the product $\mu \circ \iota^{\otimes 2}$ defined on $\mathcal{M}_{\mathcal{W}_T}^{\otimes 2}$ via the inclusion $\iota: \mathcal{M}_{\mathcal{W}_T} \hookrightarrow (\eta \times \text{id}_T)_* \mathcal{O}_{\overline{C} \times T}$.

Thus we want to identify the locus of $t \in T$ such that the product $\mu \circ \iota^{\otimes 2}$, which lies naturally in $(\eta \times \text{id}_T)_* \mathcal{O}_{\overline{C} \times T}$, actually lies in $\mathcal{M}_{\mathcal{W}_T}$. Equivalently, $PN^m(T)$ is the locus of $t \in T$ where the projection

$$\mathcal{M}_{\mathcal{W}_T}^{\otimes 2} \xrightarrow{q \circ \mu \circ \iota^{\otimes 2}} \mathcal{W}_T$$

is identically zero. In conclusion, $PN^m(T)$ is identified with the locally closed subscheme $T_P \subseteq T$ in the flattening stratification of $\text{coker}(q \circ \mu \circ \iota^{\otimes 2})$ corresponding to Hilbert polynomial P equal to the (constant) Hilbert polynomial of the fibers of the flat sheaf $\mathcal{W}_T$ [HL10b, Theorem 2.1.5]. In fact T_P is closed since it is the stratum corresponding to the maximal Hilbert polynomial. Alternatively, we can see that T_P is closed showing that it is stable under specialization. □

Proposition 2.15. *In the hypothesis of Theorem 2.14:*

1. *The locus $PN_{\text{conn}}^m \subseteq PN^m$ parametrising connected partial normalisations of arithmetic genus $1 - N + m$ is open in PN^m ;*
2. *The locus $T^m \subseteq PN_{\text{conn}}^m$ parametrising partial normalisation homeomorphic¹ to a generalised tree (Definition 1.19) is closed in PN_{conn}^m*

Proof. 1. Let $\mathcal{M}_{\mathcal{W}}|_{PN^m}$ be the universal $\mathcal{O}_{C \times PN^m}$ algebra on PN^m . Then $PN_{\text{conn}}^m \subseteq PN^m$ is identified with the locus where $\dim H^0(C, (\mathcal{M}_{\mathcal{W}}|_{PN^m})_p) < 2$, which is open in PN^m by upper semicontinuity of cohomology.

2. A partial normalisation $[C_p]$ is in $T^m \subseteq PN_{\text{conn}}^m$ if and only if $\text{Pic}^0(C_p)$ is unipotent. Let $\text{Pic}_S^0(\mathcal{C}) \rightarrow S$ be the relative Pic^0 of the flat family $\mathcal{C} \rightarrow S$ defined by some $S \rightarrow PN_{\text{conn}}^m$. Then (Remark 1.15) $\text{Pic}_S^0(\mathcal{C}) \rightarrow S$ is a smooth family of affine commutative groups and $T^m(S)$ is the locus of $s \in S$ such that $\text{Pic}^0(C_s)$ does not contain a multiplicative subgroup. The latter is a closed condition in the base by [Con14, Proposition 3.1.6], [Ber, XI, 5.8(a)]. □

We sketch a couple of examples.

¹These curves are really just bijective to generalised tree and not isomorphic to them unless $m = 0$.

Example 2.16. Let A_{n-1} be an integral, projective curve with normalisation $\mathbb{P}^1$ and a unique singular point p_0 such that $\mathcal{O}_{A_{n-1}, p_0} \cong \mathbb{C}[t^2, t^n]$, $n \geq 3$ odd, namely p_0 is a A_{n-1} singularity and $\delta(p_0) = \frac{n-1}{2}$. Notice that $\mathcal{O}_{A_{n-1}}$ can be described as the Kernel of:

$$\begin{aligned} \mathcal{O}_{\mathbb{P}^1} &\rightarrow \mathbb{C}^\delta \\ p(t) &\rightarrow (p^{(1)}(0), p^{(3)}(0), \dots, p^{(n-2)}(0)) \end{aligned}$$

where $p^{(j)}(0)$ denotes the j -th derivative evaluated at 0. We claim that T^m is a reduced point for each $0 \leq m \leq \frac{n-1}{2}$. For $n = 3$, A_2 has a cuspidal singularity, $\mathcal{S}$ has length 1 and thus it is just the structure sheaf $\mathcal{O}_0$ of the singular point, and the only non empty T^m are T^1 and T^0 , both isomorphic to a reduced point.

Next we show that for a fixed A_{n+1} , the variety underlying the scheme $T^{\delta-1}$, it's a point represented by the partial normalisation $[\mathbb{P}^1 \rightarrow A_{n-1} \rightarrow A_{n+1}]$. We are actually convinced that $T^{\delta-1}$ is scheme theoretically a point, and this should be proved studying infinitesimal deformations of $\mathbb{P}^1 \rightarrow A_{n-1} \rightarrow A_{n+1}$ with fixed sources and targets.

By Theorem 2.14, $T^{\delta-1} \subseteq \text{Quot}(\mathcal{O}_{p_0}^\delta, \delta-1) \cong \mathbb{P}^{\delta-1}$, where the last equality follows from the fact that the module structure on $\mathcal{O}_{p_0}^\delta$ is the trivial one and thus the Quot scheme can be identified with the Grassmanian $\text{Gr}(\mathbb{C}^\delta, \delta-1)$. All we have to understand is for which $[W] \in \text{Gr}(\mathbb{C}^\delta, \delta-1)$ the module

$$\text{Ker}(\varphi_W) = \{p \in \mathbb{C}[t] \mid (p^{(1)}(0), p^{(3)}(0), \dots, p^{(n-2)}(0)) \in W^{\perp} = L\}$$

is closed under multiplication. We can write down this condition for a generic W : given $p(t) = \sum_{i \geq 0} a_i t^i$, $q(t) = \sum_{i \geq 0} b_i t^i \in \text{Ker}(\varphi_W)$, namely:

$$\begin{pmatrix} (p)^{(1)}(0) \\ (p)^{(3)}(0) \\ \vdots \\ (p)^{(n-2)}(0) \end{pmatrix} = \lambda \cdot \begin{pmatrix} v_1 \\ v_3 \\ \vdots \\ v_{n-2} \end{pmatrix}, \quad \begin{pmatrix} (q)^{(1)}(0) \\ (q)^{(3)}(0) \\ \vdots \\ (q)^{(n-2)}(0) \end{pmatrix} = \nu \cdot \begin{pmatrix} v_1 \\ v_3 \\ \vdots \\ v_{n-2} \end{pmatrix}, \quad \lambda, \nu \in \mathbb{C} \quad (2.4)$$

where $(\underline{v}) = (v_j)$ is the spanning vector of the line L , we want that the derivatives of the product to lie on the same line L as p and q , i.e. if there exist $\mu \in \mathbb{C}$ such that for all $a_i, b_i \in \mathbb{C}$

$$\begin{pmatrix} (p \cdot q)^{(1)}(0) \\ (p \cdot q)^{(3)}(0) \\ \vdots \\ (p \cdot q)^{(n-2)}(0) \end{pmatrix} = \begin{pmatrix} a_0 \nu v_1 + b_0 \lambda v_1 \\ a_0 \nu v_3 + b_0 \lambda v_3 + a_2 \nu v_1 + b_2 \lambda v_1 \\ \vdots \\ a_0 \nu v_{n-2} + b_0 \lambda v_{n-2} + \sum_{i+j=n-2} a_i \nu v_j + b_i \lambda v_j \end{pmatrix} = \mu \cdot \begin{pmatrix} v_1 \\ v_3 \\ \vdots \\ v_{n-2} \end{pmatrix}. \quad (2.5)$$

where the expression on the right hand side is deduced from (2.4). The only possibility to satisfy (2.5) for any choice of coefficients is $v_1 = \dots = v_{n-4} = 0$, thus $T^{\delta-1}$ is the point in $[0 : \dots : 0 : 1] \in \mathbb{P}^{\delta-1}$ and the explicit form of (2.5) shows that it corresponds, as expected, to the natural inclusions $\mathbb{C}[t^2, t^{n+2}] \hookrightarrow \mathbb{C}[t^2, t^n] \hookrightarrow \mathbb{C}[t]$, alias $\mathbb{P}^1 \rightarrow A_{n-1} \rightarrow A_{n+1}$. Notice that in this case $T^{\delta-1} \cong PN_{\text{conn}}^{\delta-1} \cong PN^{\delta-1}$.

Notice that any partial normalisation $C \xrightarrow{\nu} A_{n+1}$ of cogenus 2 factors through a partial normalisation of cogenus 1. Indeed, the pull-back along ν of the maximal ideal $\mathfrak{m}_0$ of the singular point defines a colength 3 ideal J in $\mathcal{O}_C$. Choosing a colength 2 ideal I containing

J and contained in the maximal ideal of the singular point in C , i.e. $J \subseteq I \subseteq \mathfrak{m}'_0$, a cogenus 1 partial normalisation which factors ν is given by $\mathbb{C}[I]$. In our case, since there is just one partial normalisation of cogenus 1, this imply $T^{\delta-2}(A_{n+1}) \cong T^{\delta-1}(A_{n-1})$ and thus inductively we get that T^m is a point for all m .

The Example 2.16 can be generalised to all ADE singularities (see § 2.3 for the definition and the analytic local model of ADE and their partial normalisations). In the example below we write $PN^m \cong [C']$ (or $T^m \cong [C']$) implying that there exists, up to isomorphism, a unique partial normalisation (or partial normalisation homeomorphic to a generalised tree) in the sense of Definition 2.13.

Example 2.17. Let C be an integral, projective curve with normalisation $\mathbb{P}^1$ and a unique singular point p_0 which is of type ADE:

1. Let p_0 be an A_n singularity.
 - (a) if n is even, for each $0 \leq m \leq n/2$ we have $PN^m \cong T^m \cong [A_{2m}]$;
 - (b) if n is odd, for each $1 \leq m \leq n+1/2$, we have $PN^m \cong [A_{2m+1}]$, $T^m = \emptyset$, and $PN^0 \cong T^0 \cong [\mathbb{P}^1]$. The latter is true for any integral curve of geometric genus one, thus we don't repeat it in what follows.
2. Let p_0 be a D_n singularity.
 - (a) If n is odd, for each $1 < m < \frac{n+1}{2} - 1$, $PN^m = [A_{2m-2} \vee L] \sqcup [A_{2m}]$, thus $T^m = [A_{2m}]$; for $m = \frac{n+1}{2} - 1$, $PN^m = [A_{2m} \vee L]$ and thus $T^m = \emptyset$; for $m = 1$, $PN^1 = [A_2] \sqcup [A_1]$ and thus $T^1 = [A_2]$.
 - (b) If n is even, $1 < m < \frac{n}{2} - 1$, $PN^m = [A_{2m-1} \vee L] \sqcup [A_{2m+1}]$, and thus $T^m = \emptyset$; for $m = \frac{n}{2}$, $PN^m = [A_{2m+1} \vee L]$ and thus $T^m = \emptyset$; for $m = 1$, $PN^1 = [A_1]$ and thus $T^1 = \emptyset$.
3. Let p_0 be a singularity of type E_6, E_7, E_8 .
 - (a) in the first case we have: $PN^2 \cong T^2 \cong [E_6(1)]$, $PN^1 \cong T^1 \cong [A_1]$.
 - (b) In the second case we have $PN^3 = [E_7(1)]$ and thus $[T^3] = \emptyset$, $PN^2 = [A_2 \vee L]$ and thus $[T^2] = \emptyset$, $PN^1 = [A_2] \sqcup [A_1]$ and thus $[T^1] = [A_2]$.
 - (c) Finally for E_8 we have $PN^3 \cong T^3 = [E_8(1)]$, $PN^2 \cong T^2 = [E_6(1)]$ and $PN^1 \cong T^1 = [A_2]$.

To compute this example we would follow the steps of Example 2.16 and we thus avoid giving further details for the sake of conciseness.

Remark 2.18. The statement of Example 2.17, at least at the level of varieties, can be deduced by [GK85], [Coo98], where a complete classification of the (finitely many!) pure rank 1 modules on ADE local rings is given.

We now consider an easy example of uni-branch curve singularity with infinitely many partial normalisations of fixed cogenus.

Example 2.19. Let C be the integral projective curve with normalisation $\mathbb{P}^1$ and a unique singular point p_0 such that $\mathcal{O}_{C,p_0} \cong \mathbb{C}[t^4, t^5, t^6, t^7]$. Then $\mathcal{O}_C$ can be described as the Kernel of:

$$\begin{aligned} \mathcal{O}_{\mathbb{P}^1} &\rightarrow \mathbb{C}^3 \\ p(t) &\rightarrow (p^{(1)}(0), p^{(2)}(0), p^{(3)}(0)). \end{aligned}$$

Let us start by finding the equation of $T^2 \subseteq \text{Gr}(\mathbb{C}^3, 2) \cong \mathbb{P}^2$. As before we need to understand for which $[W] \in \mathbb{P}^2$ the $\mathcal{O}_C$ module $\mathcal{M}_W$ has a ring structure. As before, $p(t) = \sum_{i \geq 0} a_i t^i$, $q(t) = \sum_{i \geq 0} b_i t^i \in \text{Ker}(\varphi_W)$, if and only if:

$$\begin{pmatrix} (p)^{(1)}(0) \\ (p)^{(2)}(0) \\ (p)^{(3)}(0) \end{pmatrix} = \lambda \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \begin{pmatrix} (q)^{(1)}(0) \\ (q)^{(2)}(0) \\ (q)^{(3)}(0) \end{pmatrix} = \nu \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \lambda, \nu \in \mathbb{C}$$

where $\underline{v} = (v_j)$ is the spanning vector of the line $L = W^{\perp}$. In order for $\mathcal{M}_W$ to be a ring, the derivatives of the product should lie on the same line, i.e. there should exist $\mu \in \mathbb{C}$ such that for all $a_i, b_i \in \mathbb{C}$:

$$\begin{pmatrix} (p \cdot q)^{(1)}(0) \\ (p \cdot q)^{(2)}(0) \\ (p \cdot q)^{(3)}(0) \end{pmatrix} = \begin{pmatrix} a_0 \nu v_1 + b_0 \lambda v_1 \\ a_0 \nu v_2 + b_0 \lambda v_2 + 2 \nu \lambda v_1^2 \\ a_0 \nu v_3 + b_0 \lambda v_3 + 2 \nu \lambda v_1 v_2 \end{pmatrix} = \mu \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}. \quad (2.6)$$

We see that (2.6) is satisfied for any choice of coefficients as soon as $v_1 = 0$ and so $\mathbb{P}^1 \cong T^2 = \{x = 0\} \subseteq \mathbb{P}_{[x:y:z]}^2$ and we can write explicitly the $\mathbb{P}^1$ family of genus 2 partial normalisations: $C_{[a:b]} = \mathbb{C}[at^2 + bt^3, t^4, t^5]$. Notice that for $a \neq 0$ the $C_{[1:b']}$ are all isomorphic to A_4 as $\mathbb{C}$ -algebras, but they are not isomorphic as $\mathcal{O}_C$ algebras. A computation similar to the ones we just did proves that T^1 is again only one point and the corresponding partial normalisation is a curve with a cuspidal singularity.

This example gives the motivation for the next short subsection, where we revisit the spectral curve construction in a way which will be useful in the next section:

2.2.1. One Step Spectral Construction

Let $\mathcal{F}$ be a pure rank 1 sheaf on a reduced curve C with non trivial stabiliser with respect to the $\text{Pic}^0(C)$ action. We have seen in Proposition 2.3 how to construct, iterating the spectral curve construction a finite number of times, the partial normalisation $\text{Spec}_C(\mathcal{E}nd_C(\mathcal{F}, \mathcal{F})) := \widetilde{C} \xrightarrow{\widetilde{\nu}} C$ such that $\mathcal{F} = \widetilde{\nu}_* \widetilde{\mathcal{F}}$, and $\widetilde{\mathcal{F}}$ has not stabiliser with respect to the $\text{Pic}^0(\widetilde{C})$ action. By construction $\widetilde{C}$ has cogenus $\dim_{\mathbb{C}} \text{Stab}(\mathcal{F})$.

Sometimes, choosing $\mathcal{L} \in \text{Stab}(\mathcal{F})$ generically, we can get the curve $\widetilde{C}$ performing the spectral construction only once. Here generic means not contained in $\text{Ker}\left(\text{Pic}^0(C) \xrightarrow{\nu^*} \text{Pic}(C')\right)$ for any $C' \xrightarrow{\nu} C$ such that $\widetilde{\nu}$ factors through ν . However, this is not always possible. For

example in the case of 2.19, if we consider $\mathcal{F} = \nu_* \mathcal{O}_{C_{\text{cusp}}}$, then $\text{Stab}(\mathcal{F}) \cong \mathbb{C}^2$ and any point in this plane belongs to the one dimensional kernel $L_{a,b}$ of the pull-back along $C_{a,b} \xrightarrow{\nu_{a,b}} C$ for one of the $\mathbb{P}^1_{[a:b]}$ cogenus 1 partial normalisations.

Construction 2.20. Let C be projective, reduced, with normalisation a disjoint union of $\mathbb{P}^1$ and $\mathcal{F}$ a pure rank 1 sheaf on C and let $\text{Stab}(\mathcal{F}) \subseteq \text{Pic}^0(C)$ a j -dimensional subgroup of the $g(C)$ -dimensional group $\text{Pic}^0(C)$.

If we choose $\mathcal{L}_1, \dots, \mathcal{L}_j \in \text{Stab}(\mathcal{F})$ generically, then the smallest algebraic subgroup of $\text{Stab}(\mathcal{F})$ spanned² by the $\mathcal{L}_i$, is $\text{Stab}(\mathcal{F})$ itself. For each $\mathcal{L}_i$ we have an isomorphism $\mathcal{F} \xrightarrow{\phi_i} \mathcal{F} \otimes \mathcal{L}_i$ and any two such isomorphism commutes (see Lemma 2.2). Thus we have a morphism

$$\underline{\phi} = (\phi_1, \dots, \phi_j): \mathcal{F} \rightarrow \mathcal{F} \otimes (\mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_j)$$

which endows $\mathcal{F}$ with a structure of $\text{Sym}((\mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_j)^\vee)$ module.

Then there exist a unique compactly supported sheaf $\tilde{\mathcal{E}}$ on $X := \text{Tot}(\mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_j) \xrightarrow{\pi} C$ such that $\pi_* \tilde{\mathcal{E}} = \mathcal{F}$ and the push-down of the tautological section $\underline{\lambda} = (\lambda_1, \dots, \lambda_j): \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}} \otimes \pi^*(\mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_j)$ recovers $\underline{\phi}$ (Compare with § 1.3).

Let $\tilde{C}$ be the scheme theoretic support of $\tilde{\mathcal{F}}$; the same argument of Lemma 2.1 shows that $\tilde{C} \xrightarrow{\tilde{\nu}} C$ is a partial normalisation of C . Furthermore, the tautological sections $\mathcal{O}_X \xrightarrow{\lambda_i} \pi^* \mathcal{L}_i$ restricted to $\tilde{C}$ trivialise $\tilde{\nu}^* \mathcal{L}_i$ so $\tilde{C}$ is a non trivial partial normalisation. Finally, the stabiliser $\text{Stab}(\tilde{\mathcal{E}})$ of $\tilde{\mathcal{E}}$ with respect to the $\text{Pic}^0(\tilde{C})$ action is $\tilde{\nu}^* \text{Stab}(\mathcal{F}) / \text{Ker}(\tilde{\nu}^*)$ (Compare with Lemma 2.2), but the latter is trivial if we are in the assumptions of 2.20.

In conclusion, $\tilde{C} \xrightarrow{\tilde{\nu}} C$ is the unique curve of genus $g(C) - j$ such that $\mathcal{F} \cong \tilde{\nu}_* \tilde{\mathcal{E}}$ with $\tilde{\mathcal{E}}$ stabiliser-free with respect to the $\text{Pic}^0(\tilde{C})$ action.

Remark 2.21. The uniqueness follows from the fact that, as we already remarked, we recover $\tilde{C}$ from $\mathcal{F}$ as $\text{Spec}_C(\mathcal{E}nd_C(\mathcal{F}, \mathcal{F}))$. In fact, clearly $\mathcal{O}_{\tilde{C}} \subseteq \tilde{\nu}_* \mathcal{E}nd_{\tilde{C}}(\tilde{\mathcal{E}}, \tilde{\mathcal{E}}) \cong \mathcal{E}nd_C(\mathcal{F}, \mathcal{F})$. On the other hand, if the inclusion is proper then $\tilde{\mathcal{E}}$ is push-down from a further partial normalisation, but this contradicts the fact that $\tilde{\mathcal{E}}$ stabiliser-free with respect to the $\text{Pic}^0(\tilde{C})$ action.

2.3. Interpretation of the invariants in terms of generalised tree

Let C be a connected, reduced projective curve of arithmetic genus g and with normalisation a disjoint union of $\mathbb{P}^1$.

To simplify notation from now on we denote $\mathcal{M}_{1,[C]}(C)$ simply by $\mathcal{M}$. Consider the Cartesian diagram:

$$\begin{array}{ccc} \text{Stab} & \longrightarrow & \text{Pic}^0(C) \times \mathcal{M} \\ \downarrow p & & \downarrow a \\ \mathcal{M} & \xrightarrow{\Delta} & \mathcal{M} \times \mathcal{M} \end{array}$$

²We mean the smallest algebraic subgroup containing all the powers of the $\mathcal{L}_i$ and all their tensor products.

where a is the action, i.e. $a(\mathcal{L}, \mathcal{F}) = (\mathcal{L} \otimes \mathcal{F}, \mathcal{F})$ and Δ the embedding of the diagonal. Then $\text{Stab} \xrightarrow{p} \mathcal{M}$ is a group scheme whose fiber over a point $m \in \mathcal{M}$ is the stabiliser $\text{Stab}(m) \subseteq \text{Pic}^0(C)$ of that point. Let us denote by $\mathcal{M}_j \subseteq \mathcal{M}$ the locus where the dimension of the fibers of p is equal j , i.e. the locus of points whose stabiliser has dimension j .

The fiber dimension of a morphism of schemes (locally of finite type) is always an upper semicontinuous function on the source scheme [GD65, Theorem 13.1.3], but since Stab is a group scheme and therefore it has a section, namely the identity, the fiber dimension is also upper semicontinuous on the target $\mathcal{M}$ [Ber, Proposition 4.11].

It follows that each $\mathcal{M}_j$ is locally closed in $\mathcal{M}$; we endowed the $\mathcal{M}_j$ with their reduced scheme structure. In particular, the locus $\mathcal{M}_{1,[C]}(C)^\circ \subseteq \mathcal{M}_{1,[C]}(C)$ where the $\text{Pic}^0(C)$ action is free, which in this case coincide with the locus of points with finite stabiliser, is open.

Notice that $\mathcal{M}_j$ can be also thought as the union of all orbits of dimension $g(C) - j$, and thus it is $\text{Pic}^0(C)$ invariant.

Remark 2.22. The locus $\mathcal{M}_{1,[C]}(C)^\circ$ coincides with $\text{Pic}^0(C)$ if C has only locally planar singularities, but not in general. For example, if C has a non Gorenstein singularity, then the dualising sheaf ω_C is pure of rank 1, and $\mathcal{E}nd_C(\omega_C, \omega_C) \cong \mathcal{O}_C$ by local duality. In particular this implies that $\text{Stab}(\omega_C) = \{e\}$. So there is at least a second irreducible component $U_{\omega_C} \cong \text{Pic}^0(C) \otimes \omega_C \subseteq \mathcal{M}_{1,[C]}(C)^\circ$ and in general this is not the only one [Kas12].

Definition 2.23. Let D be a connected, reduced, projective curve; we define the *weight* of D to be the positive integer $\chi(\mathcal{M}_{1,[D]}(D)^\circ)$.

The next result shows that the local contributions to the invariants can be localised to curves homeomorphic to rational connected curves, i.e to generalised tree of $\mathbb{P}^1$:

Theorem 2.24. Let C be a connected, reduced projective curve of arithmetic genus g and normalisation $\sqcup_1^N \mathbb{P}^1$. Then the Euler characteristic of its Simpson compactified Jacobian $\mathcal{M}_{1,[C]}(C)$ is given by:

$$\chi(\mathcal{M}_{1,[C]}(C)) = \sum_{j=0}^{g(C)} \chi(T^{g(C)-j}, (s_j)_* 1)$$

where $(s_j)_* 1$ is the constructible function whose value at a point $[D] \in T^{g(C)-j}$, is the weight of D .

If $C \xhookrightarrow{i} X$ is embedded in a smooth Calabi-Yau 3-fold X , then the contribution $n_{0,[C]}$ to the invariant $n_{0,i,[C]}$ is given by:

$$n_{0,[C]} = \sum_{j=0}^{g(C)} \chi(T^{g(C)-j}, (s_j)_* v_{\mathcal{M}(X)}) \quad (2.7)$$

where $(s_j)_* v_{\mathcal{M}}$ takes the value $\chi(\mathcal{M}_{1,[D]}(D)^\circ, v_{\mathcal{M}(X)})$ at a point $[D]$.

Proof. We begin by constructing the morphisms s_j which induces the constructible functions $(s_j)_* 1$ and $(s_j)_* v_{\mathcal{M}}$.

Let consider the restriction $\mathcal{F}_{\mathcal{M}_j}$ of the universal sheaf of $\mathcal{M}$ to $C \times \mathcal{M}_j$ and the pull-back $\text{Stab}_j \xrightarrow{p} \mathcal{M}_j$ of the group scheme $\text{Stab} \rightarrow \mathcal{M}$. Given a closed point $m \in \mathcal{M}_j$ we can pick generic elements $\mathcal{L}_1, \dots, \mathcal{L}_j$ in the fiber $p^{-1}(m) = \text{Stab}(m)$ in such a way that the smallest algebraic subgroup spanned by the $\mathcal{L}_i$ is $\text{Stab}(m)$. Since the latter is an open condition, on a neighbourhood U_m of m we can choose local sections $\underline{\mathcal{L}}_1, \dots, \underline{\mathcal{L}}_j$ of the smooth map $\text{Stab}_j \xrightarrow{p} \mathcal{M}_j$ such that for any $m' \in U_m$ the smallest algebraic subgroup spanned by $\underline{\mathcal{L}}_1|_{m'}, \dots, \underline{\mathcal{L}}_j|_{m'}$ is $\text{Stab}(m')$. Let $\mathcal{U} = \{U_m\}$ be a cover of $\mathcal{M}_j$ whose opens satisfy the above property. Then for any U_m we have a morphism of $\mathcal{O}_{C \times U_m}$ modules

$$\Phi_{U_m} : \mathcal{F}_{\mathcal{M}_j}|_{U_m} \xrightarrow{\phi_1, \dots, \phi_j} \mathcal{F}_{\mathcal{M}_j}|_{U_m} \otimes (\underline{\mathcal{L}}_1 \oplus \dots \oplus \underline{\mathcal{L}}_j) \quad (2.8)$$

where each ϕ_i is an isomorphism. Construction 2.20 for the family of Higgs fields (2.8) produces an eigen-sheaf sheaf $\widetilde{\mathcal{E}}_{U_m}$ on $\text{Tot}_{C \times U_m}(\underline{\mathcal{L}}_1 \oplus \dots \oplus \underline{\mathcal{L}}_j)$, flat over U_m , with compact support $\widetilde{\mathcal{E}}_{U_m}$. Notice that furthermore, all the $\underline{\mathcal{L}}_i$ pull-back to trivial line bundles on the spectral curve $\widetilde{\mathcal{C}}_{U_m}$. The construction commutes with taking pull-backs; in particular, the pull-back to any closed $m' \in U_m$ gives a partial normalisation $\widetilde{C} \xrightarrow{\widetilde{v}} C$ of cogenus j , and a sheaf $\widetilde{\mathcal{E}}$ on $\widetilde{C}$ such that: $\widetilde{v}_* \widetilde{\mathcal{E}} \cong \mathcal{F}_{m'}$ and $\widetilde{\mathcal{E}}$ is stabiliser free with respect to the $\text{Pic}^0(\widetilde{C})$ action.

We have in this way constructed on any open U_m a flat family $\widetilde{\mathcal{C}}_{U_m} \rightarrow U_m$ of cogenus j connected partial normalisations. The flatness follows from the constance of the arithmetic genus, which in the case of curves it's equivalent to the constance of the Hilbert polynomial of the structure sheaf of the fibers; the connectedness follows from the fact that $\mathcal{F}_{m'} = \widetilde{v}_* \widetilde{\mathcal{E}}$ is stable and thus it cannot be the push-forward of a sheaf on a disconnected curve.

The families $\widetilde{\mathcal{C}}_{U_m} \rightarrow U_m$ constructed above are unique up to isomorphism, as they can be recovered from $\mathcal{F}$ as $\mathcal{E}nd_{C \times U_m}(\mathcal{F}_{\mathcal{M}_j}|_{U_m}, \mathcal{F}_{\mathcal{M}_j}|_{U_m})$ (see Remark 2.25). It follows that we can glue the $\widetilde{\mathcal{C}}_{U_m} \rightarrow U_m$ on the intersections $U_m \cap U_{m'}$ to get a flat family of cogenus j connected partial normalisations $\widetilde{\mathcal{C}} \xrightarrow{\widetilde{v}} \mathcal{M}_j$.

Notice furthermore that the normalisation of each fiber of $\widetilde{\mathcal{C}} \xrightarrow{\widetilde{v}} \mathcal{M}_j$ is isomorphic to $\sqcup_1^N \mathbb{P}^1$. Then, up to taking the semi-normalisation of the base $\mathcal{M}_j^{sn} \xrightarrow{sn} \mathcal{M}_j$, Theorem 1.8[Kol11] ensures that $sn^* \widetilde{\mathcal{C}} \rightarrow \mathcal{M}_j^{sn}$ admits a simultaneous normalisation $\overline{\mathcal{C}} \rightarrow \mathcal{M}_j^{sn}$. Moreover, since $\overline{\mathcal{C}} \rightarrow C \times \mathcal{M}_j^{sn}$ and $\sqcup_1^N \mathbb{P}^1 \times \mathcal{M}_j^{sn} \rightarrow C \times \mathcal{M}_j^{sn}$ are two simultaneous normalisations of $C \times \mathcal{M}_j^{sn}$ over a reduced base they are isomorphic, [Kol11]. In conclusion the triple

$$\sqcup_1^N \mathbb{P}^1 \times \mathcal{M}_j^{sn} \rightarrow sn^* \widetilde{\mathcal{C}} \rightarrow C \times \mathcal{M}_j^{sn}$$

defines a morphism $\mathcal{M}_j^{sn} \xrightarrow{s_j} PN_{\text{con}}^{g(C)-j}$ and thus a map of topological spaces $\mathcal{M}_j \rightarrow PN_{\text{con}}^{g(C)-j}$.

Recall that, by definition of semi-normalisation, (Definition 1.5) the morphism $\mathcal{M}_j^{sn} \xrightarrow{sn}$

$\mathcal{M}_j$ is a homeomorphism, and thus

$$\chi(\mathcal{M}_j^{sn}) = \chi(\mathcal{M}_j), \quad \chi(\mathcal{M}_j^{sn}, sn^* v_{\mathcal{M}}(X)) = \chi(\mathcal{M}_j, v_{\mathcal{M}}(X)) \quad (2.9)$$

where we denoted by $v_{\mathcal{M}}(X)$ the restriction to $\mathcal{M}_j$ of the Behrend function of the ambient space $\mathcal{M}_{1,[C]}(X)$. Let $T^{g(C)-j} \subseteq PN_{\text{con}}^{g(C)-j}$ be the closed subscheme, endowed with the reduced structure, parametrising partial normalisation of genus $g(C) - j$ homeomorphic to a generalised tree and U its open complement. Then we have:

$$\chi(\mathcal{M}_j^{sn}) = \chi(PN_{\text{con}}^{g(C)-j}, s_{j*}1) = \chi(T^{g(C)-j}, s_{j*}1|_{T^{g(C)-j}}) + \chi(U, s_{j*}1|_U) \quad (2.10)$$

$$\begin{aligned} \chi(\mathcal{M}_j^{sn}, sn^* v_j) &= \chi(PN_{\text{con}}^{g(C)-j}, s_{j*}(sn^* v_j)) \\ &= \chi(T^{g(C)-j}, s_{j*}(sn^* v_j)|_{T^{g(C)-j}}) + \chi(U, s_{j*}(sn^* v_j)|_U) \end{aligned}$$

and thus to conclude it suffices to argue that $s_{j*}1|_U, s_{j*}(sn^* v_{\mathcal{M}}(X))|_U$ are identically zero on U .

By definition, the value at a point $p \in U$ of the constructible functions above is given, respectively, by $\chi(s_j^{-1}(p))$ and $\chi(s_j^{-1}(p), sn^* v_{\mathcal{M}}(X)|_{s_j^{-1}(p)})$. On the other hand, the fiber over $p = [C_{\mathcal{F}}]$ is the open locus $\mathcal{M}_{1,[C_{\mathcal{F}}]}^{\circ}(C_{\mathcal{F}}) \subseteq \mathcal{M}_{1,[C_{\mathcal{F}}]}(C_{\mathcal{F}})$ where the $\text{Pic}^0(C_{\mathcal{F}})$ is free, and since we have chosen $p \in U$, $\text{Pic}^0(C_{\mathcal{F}})$ contains a non trivial torus $(\mathbb{C}^*)^k$ for some $1 \leq k \leq g(c) - j$ so $\chi(\mathcal{M}_{1,[C_{\mathcal{F}}]}^{\circ}(C_{\mathcal{F}})) = 0$ by Proposition 2.10. The same holds true for the weighted Euler characteristic since $v_{\mathcal{M}}(X)$ is invariant under the action of tensoring with a line bundle. $\square$

Remark 2.25. We did not construct $\widetilde{\mathcal{C}}_{U_m}$ directly as $\text{Spec}(\mathcal{E}nd_{C \times U_m}(\mathcal{F}_{\mathcal{M}_j}|_{U_m}, \mathcal{F}_{\mathcal{M}_j}|_{U_m}))$ because we could not argue directly that the sheaf of endomorphism of $\mathcal{F}$ is flat over U_m . In order to prove flatness it would be enough to prove that $(\dagger): i_m^* \mathcal{E}nd_{C \times U_m}(\mathcal{F}_{\mathcal{M}_j}|_{U_m}, \mathcal{F}_{\mathcal{M}_j}|_{U_m}) = \mathcal{E}nd_C(i_m^* \mathcal{F}_{\mathcal{M}_j}, i_m^* \mathcal{F}_{\mathcal{M}_j})$ and then use the constancy of the Hilbert polynomial for the right hand side as m' varies in $\mathcal{M}_j$. In fact, if one manages to prove $(\dagger)$ one can then define $\widetilde{C} \rightarrow \mathcal{M}_j$ directly and simplify the above argument.

In the case of curves with ADE singularities this theorem has an easier expression, directly in terms of the finitely many orbits of the action. Unfortunately at the moment we do not have a technique to compute the weight of curves other than planar ones, so it is hard to compute $\chi(\mathcal{M}_{1,[C]}(C))$ explicitly.

We have computed in Example 2.17 that curves with ADE singularity admit finitely many partial normalisation in any fixed genus. In fact, more is known; the main result of [GK85] can be restated in the following way: for any curve C with a unique singular point p_0 which is a finite representation type singularity (See Appendix 2.3 below for the definition), the only pure rank 1 sheaves on C with trivial $\text{Pic}^0(C)$ stabiliser are line bundles when C is Gorenstein, line bundles and sheaves of the form $\omega_C \otimes \mathcal{L}$, if C is not

Gorenstein. In particular, $\mathcal{M}_{1,[C]}^\circ(C)$ is isomorphic to $\text{Pic}^0(C)$ in the Gorenstein case and it is a disjoint union of two connected components, both isomorphic to $\text{Pic}^0(C)$, in the non Gorenstein case.

Corollary 2.26. *Let C be a connected, reduced projective curve with only ADE singularities, then*

$$n_{0,[C]} = \sum_{D \rightarrow C} \left(\begin{cases} \nu(\mathcal{O}_D(g(D))), & \text{if } \omega_D \cong \mathcal{L} \\ \nu(\mathcal{O}_D((g(D))) + \nu(\omega_D(2 - g(D))) \end{cases} \right)$$

where the sum runs over the finitely many partial normalisations homeomorphic to a generalised tree (Definition 1.19) of $\mathbb{P}^1$.

Proof. The only claim we are left to prove is that if $O_{\mathcal{F}}$ denotes the orbit under the $\text{Pic}^0(C)$ action of $\mathcal{F} \cong \mathcal{O}_D(g(D))$ or $\mathcal{F} \cong \omega_D(2 - g(D))$ then $\chi(O_{\mathcal{F}}, \nu|_{O_{\mathcal{F}}}) = \nu(\mathcal{F})$. First notice that ν is constant on the orbits of the action because we can extend the action to a local isomorphism around $\mathcal{M}_{1,[C]}(C) \hookrightarrow \mathcal{M}_{1,\beta}(X)$ [PT10, Lemma 3.11]. Second, recall that D homeomorphic to a generalised tree means $\text{Pic}^0(D) \cong (\mathbb{C}, +)^{g(D)}$ and clearly $\mathcal{O}_{\mathcal{F}} \cong \text{Pic}^0(D)$. In conclusion $\chi(O_{\mathcal{F}}, \nu|_{O_{\mathcal{F}}}) = \nu(\mathcal{F}) \cdot \chi((\mathbb{C}, +)^{g(D)}) = \nu(\mathcal{F})$. $\square$

Remarks about the relation with genus 0 Gromov-Witten invariants

The above theorem gives an interpretation of Katz invariants in terms of birational morphisms from curves homeomorphic to genus 0 curves but it does not seem to give any new insight on the expected relation between genus 0 GW invariant and genus 0 BPS invariants.

Another possible point of view to consider, which we didn't explore yet but we think might be worth pursuing, is proposed in what follows (without any claim of it being rigorous).

Let us consider $C \hookrightarrow X$ an integral, rational rigid curve in a Calabi-Yau 3-fold X . Under the rigidity hypothesis, $\mathcal{M}_{1,[C]}(C)$ is a connected component of $\mathcal{M}_{1,\beta}(X)$ and it also make sense to define the contribution of C to genus 0 GW invariants as the length of the zero dimensional scheme $M_{0,0}(C, [C])$, whose unique reduce point represent the normalisation of C .

If $\mathcal{C} \cong \mathbb{P}^1 \times M_{0,0}(C, [C])$ and F denote respectively the universal curve and the universal map over $M_{0,0}(C, [C])$, then $F_*\mathcal{O}_{\mathcal{C}}$ is an object in $\mathcal{M}_{1,[C]}(C)(M_{0,0}(C, [C]))$. This means that we have a morphism $M_{0,0}(C, [C]) \rightarrow \mathcal{M}_{1,[C]}(C)$ and the image of $M_{0,0}(C, [C])$ is a certain thickening Z of the point $[f_*\mathcal{O}_{\mathbb{P}^1}]$, where f is the normalisation.

On the other hand, given the action of $\text{Pic}^0(C)$ on the subspace $\mathcal{M}_{1,[C]}(C)$, for a generic choice of j in $\text{Lie}(\text{Pic}^0(C))$, we get a vector field V on the moduli space:

$$\mathcal{O}_{\mathcal{M}_{1,\gamma}(C)} \xrightarrow{V} T\mathcal{M}_{1,\gamma}(C) \quad (2.11)$$

which vanishes on the fixed points of the action. Suppose this extends to a vector field $\overline{V}$ on the whole moduli space $\mathcal{M}_{1,\beta}(X) := \mathcal{M}$, e.g. if C is rigid as we assumed. Dualising $\overline{V}$

and remembering that, since $\mathcal{M}$ admits a symmetric obstruction theory with obstruction sheaf $\Omega_{\mathcal{M}}$, we get a cosection

$$\text{Ob}_{\mathcal{M}} \xrightarrow{\overline{V}^\vee} \mathcal{O}_{\mathcal{M}}$$

whose degeneracy locus, around $\mathcal{M}_{1,\gamma}(C)$, coincides with the fixed locus of the $\text{Pic}^0(C)$. Then by [KL13], we get a cosection localized virtual $[D(\overline{V}^\vee)]_{\overline{V}^\vee}^{\text{vir}}$ cycle supported only on the fixed points of the $\text{Pic}^0(C)$ action.

But for our integral rigid curve with normalisation $\mathbb{P}^1$ the fixed locus is $[f_*\mathcal{O}_{\mathbb{P}^1}]$, so the cosection localised cycle $[D(\overline{V}^\vee)]_{\overline{V}^\vee}^{\text{vir}}$, should probably be (if we assume the local version of the GW/BPS correspondence in genus 0 and in the base case $d = 1$) the cycle $[Z]$.

Appendix: curves of finite representation type and in particular curves with ADE singularities

Definition 2.27. [Kas12] Let $p \in C$ be a singular point of C ; the singularity at p is said to be of **finite representation type** if there exist finitely many isomorphism classes of rank 1 pure modules on $\hat{\mathcal{O}}_{C,p}$.

A curve singularity is of finite presentation type if and only if its analytic local ring is one of those listed in Table 2.1 or in Table 2.2 [GK85]. We say that a curve C has a singularity of type ADE in p if locally analytically around p the ring of regular function is one of those in Table 2.1.

The remaining finite representation type singularities are all non Gorenstein and they are partial normalisations of curves of type ADE, we recall their parametrisation in Table 2.2:

Remark 2.28. Example 2.19 and more in general all curves admitting infinitely many non isomorphic partial normalisations are not of finite presentation type. However, it is possible to have finitely many isomorphism classes of partial normalisations, but infinitely many isomorphism classes of pure rank one modules. This is the case for any curve with a genus zero singularity 1.16 with 4 or more branches.

Table 2.1: Singularities of type ADE

Singularity	Parameterization
$A_n, n \geq 2$ and even	$k[[t^2, t^{n+1}]]$
$A_n, n \geq 1$ and odd	$k[(t, t), (t^{\frac{n+1}{2}}, 0)]$
$D_n, n \geq 3$ and odd	$k[(t^{n-2}, t), (t^2, 0)]$
$D_n, n \geq 3$ and even	$k[(t, t, 0), (t^{\frac{n-2}{2}}, 0, t)]$
E_6	$k[[t^3, t^4]]$
E_7	$k[(t^2, t), (t^3, 0)]$
E_8	$k[[t^3, t^5]]$

Table 2.2: NG Singularities of finite representation type

Singularity	Parameterization
$A_n \vee L, n \geq 2$ and even	$k[(t^{n+1}, 0), (t^2, 0), (0, t)]$
$A_n \vee L, n \geq 1$ and odd	$k[(t^{(n+1)/2}, -t^{(n+1)/2}, 0), (t, t, 0), (0, 0, t)]$
$E_6(1)$	$k[[t^3, t^4, t^5]]$
$E_7(1)$	$k[(t^2, t), (t^3, 0), (t^4, 0)]$
$E_8(1)$	$k[[t^3, t^5, t^7]]$

3

Towards non reduced class contribution

We now want to study the contributions $n_{0,\gamma}$ for γ non reduced. In this case the moduli space $\mathcal{M}_{1,\gamma}(C)$ is more complicated: $\mathcal{F} \in \mathcal{M}_{1,\gamma}(C)$ could be a higher rank sheaf on the reduced curve C as well as a sheaf supported on some thickening of C .

We keep looking for an interpretation of the fixed points of the $\text{Pic}^0(\widehat{C})$ action on $\mathcal{M}_{1,\gamma}(C)$ as sheaves pushed-forward from genus 0 curves, and the idea is once again to understand the spectral curve associated to a Higgs field isomorphism $\mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{L}$.

We did not explore the spectral construction on non reduced curves and already in the case of higher rank sheaves supported on reduced curves we do not have yet an efficient procedure to study the geometry of $C_{\mathcal{F}}$ in general.

In this chapter we just report the very few examples we worked out so far; most likely to treat the general case we need a different idea than the one we pursued here.

We will see that $\text{Pic}^0(C)$ invariant higher rank sheaves supported on a (reduced) curve with a node or a with a cusp are indeed push down from a genus 0 curve. However we have to include in the picture non reduced curves, thus before going into the example we recall the definition of generalised rank, which we will denote by Rk , for a sheaf supported on a non reduced curve.

Filtrations of sheaves in $\mathcal{M}_{1,\gamma}(C)$ and associated graded

Let C be an integral projective curve with normalisation $\mathbb{P}^1$, and $\gamma = l[C]$ a cycle supported on C . Fix $\mathcal{F} \in \mathcal{M}_{1,\gamma}(C)$ whose support is a thickening C_r of C of length r , i.e. such

that $[C_r] = r[C]$ with $r \leq l$. We define a filtration:

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots \subseteq \mathcal{F}_r = \mathcal{F}$$

constructed iteratively taking the Kernel of the natural restriction map $\mathcal{F}_k \twoheadrightarrow \mathcal{F}_k|_C$, and a graded object supported on the reduced curve associated to $\mathcal{F}$, namely $G(\mathcal{F}) = \bigoplus_i^r \mathcal{F}_i|_C$ of rank l and Euler characteristic $\chi(G(\mathcal{F})) = \chi(\mathcal{F})$.

Remark 3.1. Notice that if $\mathcal{F}$ is fixed by the $\text{Pic}^0(\widehat{C})$ action so is each sheaf $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots \subseteq \mathcal{F}_r$ on its filtration and its associated graded $G(\mathcal{F}) = \bigoplus_i^r \mathcal{F}_i|_C$.

So we start by investigating $\text{Pic}^0(\widehat{C})$ invariant sheaves of higher rank on reduced curves. In what follows we will abuse terminology and say that a sheaf $\mathcal{F}$ on a non reduced curve C has rank r if the associated graded has rank r on the underlying reduced curve; we will use the notation $\text{Rk}(\mathcal{F})$ for the rank of the associated graded.

Reminder of Section 1.3

Let C be an integral, rational, projective curve. Recall that, given a pure rank r sheaf $\mathcal{F}$ and a Higgs-field $\mathcal{F} \xrightarrow{\phi} \mathcal{F} \otimes \mathcal{L}$ on C with $\mathcal{L} \in \text{Pic}^0(C)$, we have an induced $\mathcal{O}_{\mathbb{P}^1}$ valued Higgs-field:

$$\mathcal{E} := (v^* \mathcal{F}) / \text{tors} \xrightarrow{\phi_0} \mathcal{E} \quad (3.1)$$

on the normalisation $\mathbb{P}^1 \xrightarrow{v} C$; the eigenvalues of ϕ_0 are constant. Thus the associated spectral curve is a disjoint union of multiple $\mathbb{P}^1$:

$$C_{\mathcal{E}} = \{(\lambda' - \alpha_1)^{k_1} \dots (\lambda' - \alpha_m)^{k_m} = 0\} \subseteq \mathbb{P}^1 \times \mathbb{C}. \quad (3.2)$$

Then we argued in Section 1.3, the spectral curve $C_{\mathcal{F}}$ for $(\mathcal{F}, \phi, \mathcal{L})$, namely the scheme theoretic support of the eigen-sheaf $\mathcal{E}_{\phi}$ is the scheme theoretic image of $C_{\mathcal{E}}$ along the morphism

$$\tilde{v}: \text{Tot}(v^* \mathcal{L}) \cong \mathbb{P}^1 \times \mathbb{C} \rightarrow \text{Tot}(\mathcal{L}), \quad (3.3)$$

i.e. $C_{\mathcal{F}}$ is the closed subscheme of $\text{Tot}(\mathcal{L})$ whose ideal sheaf is the Kernel of:

$$\mathcal{O}_{\text{Tot} \mathcal{L}} \rightarrow \tilde{v}_* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{C}} \twoheadrightarrow \tilde{v}_* \mathcal{O}_{\mathcal{E}} \quad (3.4)$$

Also recall that we call *pure* a projective curve whose structure sheaf is pure, i.e. it does not have torsion sub-sheaves. This is equivalent to say that the curve does not have embedded points, i.e. is Cohen-Macaulay. We have seen in Lemma 2.1 that the spectral curve of a pure Higgs sheaf $\mathcal{F} \xrightarrow{\phi} \mathcal{F} \otimes \mathcal{L}$ is pure; here another way to argue for $\text{Pic}^0(C)$ fixed ones:

Remark 3.2. The scheme theoretic image $C_{\mathcal{F}}$ of $C_{\mathcal{E}}$ along $\tilde{v}$ is pure. Indeed, since $\tilde{v}$ is birational, we have an injective morphism of sheaves $\mathcal{O}_{C_{\mathcal{F}}} \hookrightarrow \tilde{v}_* \mathcal{O}_{C_{\mathcal{E}}}$. Suppose that $\mathcal{O}_{C_{\mathcal{F}}}$

has a torsion sub-sheaf $\mathcal{T}$. Then

$$0 \neq \mathrm{Hom}_{C_{\mathcal{E}}}(\mathcal{T}, \tilde{\nu}_* \mathcal{O}_{C_{\mathcal{E}}}) \cong \mathrm{Hom}(\nu^* \mathcal{T}, \mathcal{O}_{C_{\mathcal{E}}}) = 0$$

where the last equality follows from the fact that $C_{\mathcal{E}}$ is pure.

3.1. Invariant higher rank sheaves on rational curves with a single node

In this section, C is an integral curve with a unique singular point p_0 which is an ordinary node and normalisation $\mathbb{P}^1 \xrightarrow{\nu} C$. We will use the following notation: we denote by $\{x, y\} = \nu^{-1}(p_0)$ the pre-image of the singular point, by $\mathcal{L}_{\alpha}$, with $\alpha \in \mathbb{C}^*$, the line bundle obtained from $\mathbb{P}^1 \times \mathbb{C}$ choosing $\{x\} \times \mathbb{C} \xrightarrow{\alpha} \{y\} \times \mathbb{C}$ as gluing automorphism.

Lemma 3.3. *For any pure rank r sheaf $\mathcal{F}$ on C and any Higgs field $\mathcal{F} \xrightarrow{\phi} \mathcal{F} \otimes \mathcal{L}_{\alpha}$, which is an isomorphism, the spectral curve $C_{\mathcal{F}}$ is a pure curve of arithmetic genus 0. The underlying reduced curve has chains of $\mathbb{P}^1$ as connected components (Definition 1.19).*

Proof. In order to study the geometry of $C_{\mathcal{F}}$, we look at the Higgs-field $\mathcal{E} \xrightarrow{\phi_0} \mathcal{E}$ (3.1) on $\mathbb{P}^1$ associated with ϕ , and we describe the scheme theoretic image $C_{\mathcal{F}}$ of $C_{\mathcal{E}}$ (3.4) along $\tilde{\nu}$ (3.3). Since outside of the singularity p_0 both ν and $\tilde{\nu}$ are isomorphisms, we can work analytically locally around the singularity as elsewhere $C_{\mathcal{F}}$ is isomorphic to $C_{\mathcal{E}}$. In the following we abuse notation and keep denoting by $C_{\mathcal{F}}$ and $C_{\mathcal{E}}$ their analytic neighbourhoods around $\pi^{-1}(p_0) \subset \mathrm{Tot}(\mathcal{L}_{\alpha})$ and $\tilde{\nu}^{-1}\pi^{-1}(p_0) \subseteq \mathbb{P}^1 \times \mathbb{C}$ respectively.

As in Example 1.30, we look at an explicit local description of the morphism $\tilde{\nu}$ in the relevant charts. Denoting by λ the fiber coordinate of $\mathrm{Tot}(\mathcal{L}_{\alpha})$ and by λ' the fiber coordinate on $\mathbb{P}^1 \times \mathbb{C}$, we have that $\tilde{\nu}$ is analytically locally given by:

$$\mathbb{C}[t_1, t_2, \lambda]/(t_1 \cdot t_2) \xrightarrow{\tilde{\nu}} \mathbb{C}[t_1, \lambda'] \oplus \mathbb{C}[t_2, \lambda'] \quad (3.5)$$

$$\lambda \mapsto (\lambda', \alpha \cdot \lambda'). \quad (3.6)$$

We want to understand the image under $\tilde{\nu}$ of the general $C_{\mathcal{E}}$ (3.2), for a fixed $\mathcal{L}_{\alpha}$.

First let us look at the case $C_{\mathcal{E}} = \{(\lambda' - \alpha_1) = 0\}$.¹ Second we look at the case of a multiple section, i.e., $C_{\mathcal{E}} = \{(\lambda' - \alpha_1)^k = 0\}$. Finally we will look at the case of a disjoint union of two (possibly multiple) sections, i.e. $C_{\mathcal{E}} = \{(\lambda' - \alpha_1)^{k_1}(\lambda' - \alpha_2)^{k_2} = 0\}$.

1. In the first case $C_{\mathcal{F}}$ is the union of the two disjoint lines in $\mathrm{Tot}(\mathcal{L}_{\alpha})$ defined by the ideal

$$\mathcal{I}_{C_{\mathcal{F}}} = (t_1, \lambda - \alpha_1) \cap (t_2, \lambda - \frac{1}{\alpha} \cdot \alpha_1).$$

¹This is for example the case that occurs when we look at rank one sheaves, so we already know from the previous chapter that $C_{\mathcal{F}}$ is isomorphic to a $\mathbb{P}^1$, so analytically locally around the singularity we have two disjoint smooth branches. The computation we perform now explains the way such a $\mathbb{P}^1$ is embedded in the fixed $\mathcal{L}_{\alpha}$.

Looking at the global picture, this tells us that the image of a constant section is still isomorphic to a $\mathbb{P}^1$ and such $\mathbb{P}^1$ intersects the fiber over the node at (p_0, α_1) and $(p_0, \frac{1}{\alpha} \alpha_1)$ (Look back at Figure 1). In particular the spectral curve is not a section of $\text{Tot}(\mathcal{L}_\alpha) \rightarrow C$, unless $\alpha = 1$ and the line bundle is the trivial one.

2. In the second case, the image $C_{\mathcal{F}}$ of the k -th thickening $C_{\mathcal{E}} := \{(\lambda' - \alpha_1)^k = 0\}$ is the unique pure thickening of length k of $\tilde{\nu}(C_{\mathcal{E}}^{\text{red}})$, where the image is again in the sense of (3.4). Then $C_{\mathcal{F}}$ is defined by the ideal

$$(t_1, (\lambda - \alpha_1)^k) \cap (t_2, (\lambda - \frac{1}{\alpha} \cdot \alpha_1)^k) \cong \left(\mathcal{I}_{C_{\mathcal{L}_\alpha}}^k \right)^{\text{pure}}.$$

Looking at the global picture, this tells us that the image of the k -th thickening $k\mathbb{P}^1 \subseteq \mathbb{P}^1 \times \mathbb{C}$ of the constant section is once again isomorphic to the source. In particular this means that $\chi(\mathcal{O}_{C_{\mathcal{F}}}) = k$.

3. In the third case, the ideal of $C_{\mathcal{F}}$ is given by:

$$\mathcal{I}_{C_{\mathcal{F}}} = (t_1, (\lambda - \alpha_1)^{k_1}) \cap (t_2, (\lambda - \frac{1}{\alpha} \alpha_1)^{k_2}) \cap (t_1, (\lambda - \alpha_2)^{k_1}) \cap (t_2, (\lambda - \alpha_2 \alpha)^{k_2}).$$

Thus, unless $\alpha_2 = \frac{1}{\alpha} \cdot \alpha_1$ or $\alpha_2 = \alpha \alpha_1$, $C_{\mathcal{F}}$ is again isomorphic to the disjoint union of the two multiple sections. Let us instead assume that $\alpha_2 = \frac{1}{\alpha} \cdot \alpha_1$ (the other case is completely symmetric); then two of the four separate branches of the analytic local picture meet in the point $S = (p_0, \alpha_1)$ in the fiber over the node; to complete the description of $C_{\mathcal{F}}$, we need to compute the multiplicity of the intersection in S . To do that, we compute the zero dimensional cokernel of the inclusion $\mathcal{O}_{C_{\mathcal{F}}} \rightarrow \tilde{\nu}_* \mathcal{O}_{C_{\mathcal{E}}}$ and see that it is generated over $\mathbb{C}$ by the functions $\{(1, 0), (\lambda', 0), \dots, (\lambda'^{(k_1-1)}, 0)\}$, where k_1 is the min between k_1 and k_2 . Furthermore, from the "normalisation" exact sequence

$$0 \rightarrow \mathcal{O}_{C_{\mathcal{F}}} \rightarrow \tilde{\nu}_* \mathcal{O}_{C_{\mathcal{E}}} \rightarrow \mathcal{O}_S^{k_1} \rightarrow 0$$

we see that the arithmetic genus of $C_{\mathcal{F}}$ is zero also in this last case and $\chi(\mathcal{O}_{C_{\mathcal{F}}}) = k_2$.

Finally notice that each irreducible (multiple) component $C_i = \tilde{\nu}(\{(\lambda' - \alpha_i)^{k_i} = 0\})$ of $C_{\mathcal{F}}$ meets at most two other irreducible components and that there are no cycles. Indeed, C_i and C_j meets if and only if $\alpha_j = \alpha \alpha_i$ or $\alpha_j = \alpha_i / \alpha$, so there are at most two possible j and obviously no cycles.

□

Remark 3.4. Notice that the curves $C_{\mathcal{F}}$ obtained in the above Lemma as scheme theoretic images under $\tilde{\nu}$ of some $C_{\mathcal{E}}$ have holomorphic Euler characteristic $\chi(\mathcal{O}_{C_{\mathcal{F}}}) \geq 1$, where the equality holds if and only if $C_{\mathcal{F}}$ is reduced and connected, i.e. a chain of $\mathbb{P}^1$.

Corollary 3.5. *Let $\mathcal{F}$ be a rank r sheaf on a reduced integral curve C with a single node. Then $\mathcal{F}$ is $\text{Pic}^0(C)$ -fixed if and only if it is push-down of a sheaf $\mathcal{E}_\phi$ from a curve $\mathbf{C} = \mathbf{C}_1 \cup \dots \cup$*

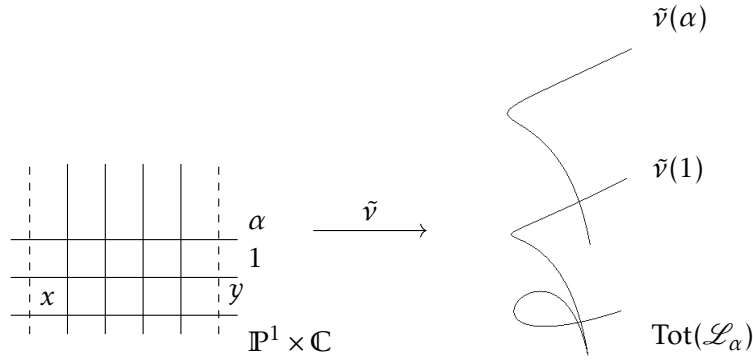


Figure 3.1: Spectral curve case (3) Lemma 3.3

$\mathbf{C}_N \xrightarrow{\pi} \mathbf{C}$ such that: two components $\mathbf{C}_i$ and $\mathbf{C}_j$ meet at most in a point (possibly non reduced), the arithmetic genus $g(\mathbf{C}) = h^1(\mathcal{O}_{\mathbf{C}}) = 0$, the connected components of $\mathbf{C}^{\text{red}}$ are chains of $\mathbb{P}^1$. Moreover $\mathcal{E}_\phi$ is such such that: $r = \text{Rk}(\mathcal{E}_\phi|_{\mathbf{C}_1}) + \dots + \text{Rk}(\mathcal{E}_\phi|_{\mathbf{C}_N})$.²

Proof. If $\mathcal{F} = \pi_* \mathcal{E}$ is push-down from such a curve $\mathbf{C}$ satisfying the conditions above, then $\mathcal{F}$ is fixed by $\text{Pic}^0(\mathbf{C})$ since by projection formula $\pi_* \mathcal{E} \otimes \mathcal{L} \cong \pi_*(\mathcal{E} \otimes \pi^* \mathcal{L})$ and by assumption $\text{Pic}^0(\mathbf{C}) = 0$.

Viceversa, if $\mathcal{F}$ is fixed by $\text{Pic}^0(\mathbf{C})$, then for each $\mathcal{L} \in \text{Pic}^0(\mathbf{C})$ we have an isomorphism $\mathcal{F} \xrightarrow{\phi} \mathcal{F} \otimes \mathcal{L}$ and thus we can think of $\mathcal{F}$ as being push-down from of the eigensheaf $\mathcal{E}_\phi$ from the spectral curve $C_{\mathcal{F}}$ of $(\mathcal{F}, \phi, \mathcal{L})$. As we reminded in § 3, the spectral curve $C_{\mathcal{F}}$ is the scheme theoretic image (in the sense of (3.4)) along $\tilde{v}: \mathbb{P}^1 \times \mathbb{C} \rightarrow \text{Tot}(\mathcal{L})$ of the spectral curve $C_{\mathcal{E}}$ for Higgs field $\phi_0: \mathcal{E} = v^* \mathcal{F} / \text{tors} \rightarrow \mathcal{E}$ on the normalisation $\mathbb{P}^1$. Lemma 3.3 describes the geometry of such a curve $C_{\mathcal{F}}$ and in particular proves that the latter satisfy the hypothesis assumed for $\mathbf{C}$. $\square$

If we don't impose any stability condition we can't be any more precise about the geometry of $C_{\mathcal{F}}$ or about the eigen-sheaf $\mathcal{E}_\phi$. However, for $\mathcal{F} \in \mathcal{M}_{1,\gamma}(\mathbf{C})$ of rank r supported on $\mathbf{C}$ we can say more.

Lemma 3.6. *Let $\mathbf{C}$ be as in Corollary 3.5. A pure sheaf $\mathcal{F}$ supported on $\mathbf{C}$ is stable with $\chi(\mathcal{F}) = 1$ if and only if $\mathcal{F} \cong \mathcal{O}_{C^k}$ with $C^k \subseteq \mathbf{C}$ a chain of $\mathbb{P}^1$.*

Proof. If $\mathcal{F} \cong \mathcal{O}_{C^k}$ then it is obviously stable of holomorphic Euler characteristic one.

Let's now suppose that $\mathcal{F}$ on $\mathbf{C}$ is stable with $\chi(\mathcal{F}) = 1$. First, $\mathbf{C}$ is connected, as there are not strictly stable sheaves supported on a disconnected curve. Moreover, since $\chi(\mathcal{F}) = h^0(\mathbf{C}, \mathcal{F}) - h^1(\mathbf{C}, \mathcal{F}) = 1$, then $\mathcal{F}$ admits at least one non zero section $\mathcal{O}_{\mathbf{C}} \xrightarrow{s} \mathcal{F}$ and thus $\mathcal{O}_Z = \mathcal{O}_{\mathbf{C}} / \text{Ker}(s) \subseteq \mathcal{F}$ with Z a pure sub-curve of $\mathbf{C}$. In particular, Z itself has $h^1(\mathcal{O}_Z) = 0$ and thus $\chi(\mathcal{O}_Z) \geq 1$, and the equality holds only when Z is a chain of $\mathbb{P}^1$. It thus follows by stability of $\mathcal{F}$ and by Lemma 1.35 that Z must be a chain C_k for some k and in that case $\mathcal{O}_k \cong \mathcal{F}$ again by Lemma 1.35. $\square$

²See §3 for the definition of Rk

Proposition 3.7. *Let $\mathcal{F}$ be a pure rank r stable sheaf on C with $\chi(\mathcal{F}) = 1$; $\mathcal{F}$ is fixed by the $\text{Pic}^0(C)$ action if and only if $\mathcal{F} \cong \nu_{r*}\mathcal{O}_{C^r}$, where $\nu_r: C^r \rightarrow C$ is a morphism from a chain of $\mathbb{P}^1$ which does not factor through any sub-chain, in particular ν_r is $r:1$.*

Proof. Let first suppose that $\mathcal{F} \cong \nu_{r*}\mathcal{O}_{C^r}$; then, as usual, we easily see by projection formula that it is fixed by $\text{Pic}^0(C)$ and since ν_r is affine $\chi(\mathcal{F}) = \chi(\mathcal{O}_{C^r}) = 1$. We are left to prove that under the hypothesis that ν_r does not factor through any sub-chain the push-forward $\nu_{r*}\mathcal{O}_{C^r}$ is stable. We do this by induction on r ; for $r = 1$ it is obvious. Let us suppose that the claim is true for the push-down of the structure sheaf from a $r-1$ chain. Let now consider $C_r = C_{r-1} \cup C_1$; notice that $\mathcal{O}_{C^r}$ is the unique up to scalar non trivial extension e in $\text{Ext}_{C^r}^1(i_*\mathcal{O}_{C^{r-1}}, j_*\mathcal{O}_{C^1}(-1))$. The hypothesis that ν_r does not factor trough any sub-chain ensure that the push-down:

$$\nu_{r*}(e): 0 \rightarrow \nu_{r*}j_*\mathcal{O}_{C^1}(-1) \xrightarrow{j} \nu_{r*}\mathcal{O}_{C^r} \xrightarrow{i} \nu_{(r-1)*}\mathcal{O}_{C^{r-1}} \rightarrow 0$$

is still a non trivial extension. Moreover by the inductive hypothesis, both $\nu_{r*}j_*\mathcal{O}_{C^1}(-1)$ and $\nu_{(r-1)*}\mathcal{O}_{C^{r-1}}$ are stable of holomorphic Euler characterstic 0 and 1 respectively.

Let consider $\mathcal{G} \xrightarrow{k} \mathcal{F}$; we want to show (see Lemma 1.35) that $\chi(\mathcal{G}) \leq 0$. If $i \circ k = 0$ then the inclusion k factors trough j , i.e. $\mathcal{G}$ is a sub-sheaf of $\nu_{r*}j_*\mathcal{O}_{C^1}(-1)$ and we are done since the latter is stable of Euler characteristic 0. Otherwise, first notice that from the Kernel-Cokernel sequence we get that

$$\mathcal{K} = \text{Ker}(\mathcal{G} \xrightarrow{i \circ k} \nu_{(r-1)*}\mathcal{O}_{C^{r-1}}) \subseteq \nu_{r*}j_*\mathcal{O}_{C^1}(-1),$$

which implies $\chi(\mathcal{K}) \leq 0$. Moreover, $\mathcal{G}/\mathcal{K} \subseteq \nu_{(r-1)*}\mathcal{O}_{C^{r-1}}$ and since the latter is stable of Euler characteristic 1 by inductive hypothesis, $\chi(\mathcal{G}/\mathcal{K}) \leq 0$ and thus we are done as the homolorphic Euler characteristic is additive.

Viceversa, let $\mathcal{F}$ of rank r and $\chi(\mathcal{F}) = 1$ fixed by $\text{Pic}^0(C)$. Then for any $\mathcal{L} \neq \mathcal{O}_C$, if we consider the (unique up to scalar) Higgs field $\mathcal{F} \xrightarrow{\phi} \mathcal{F} \otimes \mathcal{L}$ we can think of $\mathcal{F}$ as the push-down of its eigen-sheaf $\mathcal{E}_\phi$. By Lemma 3.6 $C_\mathcal{F}$ is a chain of $\mathbb{P}^1$ of length $k \leq r$ obtained as shown in Lemma 3.3 from k copies of the normalisation C_i, x_i, y_i gluing y_i to x_{i+1} and since $\mathcal{E}_\phi$ is stable with $\chi(\mathcal{E}_\phi) = 1$ on C^k we in fact have $\mathcal{E}_\phi \cong \mathcal{O}_{C^k}$, which in particular forces $k = r$ as every component of the chain maps birationally to C . □

I think that using a technique similar to the one adopted in this section it would be possible to show that:

Conjecture 3.8. *If $\mathcal{F}$ is a stable rank r sheaf with $\chi(\mathcal{F}) = 1$ on an integral curve C whose only singularities are genus 0 singularities, then $\mathcal{F} \cong p_*\mathcal{O}_{C_r}$ where C_r is generalised tree with r irreducible components, and the morphism $C_r \xrightarrow{p} C$ does not factor trough any sub-tree and restrict to the normalisation on each irreducible component.*

However, since I hope in the future to find a more elegant way to describe the structure of the spectral curve also in the higher rank case, I do not want to torture the reader any further with computations of the above type (for this section).

Remark 3.9. Another point of view worth exploring in order to understand higher rank sheaves on singular curves could be the following: inspired by the rank 1 case, we might want to think of a sheaf $\mathcal{F}$ on C as a module over $\mathcal{A} = \mathcal{E}nd_C(\mathcal{F}, \mathcal{F})$. In the higher rank case the latter is non commutative, but $\mathcal{F}$ is also a module over the center of $\mathcal{A}$. I think it might be interesting to understand whether the structure sheaf of the spectral curve is indeed such center.

Remark 3.10. Let $C \hookrightarrow X$ be the embedding of an irreducible nodal curve with a single node in X a CY 3-fold, and let us assume that the normal bundle of C in X is $N_{C/X} \cong \mathcal{L} \oplus \mathcal{L}^\vee$ with $\mathcal{L}$ a non torsion line bundles of degree 0; in particular C is rigid. The contribution to genus 0 GW invariants in class $r[C]$ for such a curve has been computed in [BKL01, Theorem 1.2] to be $\sum_k 1/k^3$. Thus accordingly to the genus 0 GW/GV correspondence the contributions $n_{0,r[C]}$ are equal 1 for all r .

Furthermore, looking at [BKL01] proof we see that the contribution is computed in the following way: there is an isomorphism of moduli spaces

$$\bigsqcup_{(d_1, \dots, d_n), \sum d_i = r} M_{0,0}(C_n, (d_1, \dots, d_n)) \rightarrow M_{0,0}(C, r)$$

where $M_{0,0}(C_n, (d_1, \dots, d_n))$ are genus 0 stable maps with target a length n chain of $\mathbb{P}^1$ and fixed degree on each irreducible component. The contribution of $M_{0,0}(C_n, (d_1, \dots, d_n))$ is then computed to be $1/k^3$ when all the d_i are equal k and zero otherwise.

We have seen that on the reduced curve C the only stable sheaf which is $\mathbb{C}^*$ invariant is $\nu_{r*}\mathcal{O}_{C_r}$. In fact, under the above rigidity hypothesis, we expect that if $\mathcal{F} \in \mathcal{M}_{1,r[C]}(C)$ is $\mathbb{C}^*$ fixed, then $\mathcal{F}$ is push-forward from the reduced curve C and thus is $\nu_{r*}\mathcal{O}_{C_r}$. If this is true, then $n_{0,r[C]}$ is essentially identified with the contribution of $M_{0,0}(C_r, (1, \dots, 1))$.

For $r = 2$ we can actually prove that all stable sheaves of Euler characteristic 1 are in fact pushed forward from C , but it is not yet clear how to proceed in the general case.

3.2. Invariant higher rank sheaves on the cusp

In this section C denotes an integral, rational projective curve with a unique singular point p_0 which is an ordinary cusp. As in the previous section, $\nu: \mathbb{P}^1 \rightarrow C$ denotes the normalisation. Moreover we denote by $\mathcal{L}_m$ for $m \in \mathbb{G}_a$ the degree 0 line bundle on C whose sheaf of section is the kernel of $\nu_*\mathcal{O}_{\mathbb{P}^1} \xrightarrow{p'(p_0) - mp(p_0)} \mathcal{O}_{p_0}$ and by $\tilde{\nu}$ the morphism $\mathbb{P}^1 \times \mathbb{C} \rightarrow \text{Tot}(\mathcal{L}_m)$.

Remark 3.11. In general, if C is an integral curve with only unibranch singularities, then both the normalisation morphism $\nu: \mathbb{P}^1 \rightarrow C$ and the morphism $\tilde{\nu}: \mathbb{P}^1 \times \mathbb{C} \rightarrow \text{Tot}(\mathcal{L})$ for

$\mathcal{L}$ a degree zero line bundle on C , are homeomorphisms. In particular, if $Z \subseteq \mathbb{P}^1 \times \mathbb{C}$ is a one dimensional sub-scheme and $\tilde{v}(Z)$ is its scheme theoretic image (always in the sense of 3.4), then the number of connected components of Z and $\tilde{v}(Z)$ is the same.

Lemma 3.12. *Let C be a curve with a cusp singularity, $\mathbb{P}^1 \xrightarrow{\nu} C$ its normalisation and $\tilde{v}: \mathbb{P}^1 \times \mathbb{C} \rightarrow \text{Tot}(\mathcal{L})$ the induced morphism. The scheme theoretic image $\mathbf{C}_k = \tilde{v}(k\mathbb{P}^1)$ of the k -th thickening of $\mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{C}$ is a pure curve of arithmetic genus $g(\mathbf{C}_k) = \dim H^1(\mathbf{C}_k, \mathcal{O}_{\mathbf{C}_k}) = 0$, and $\mathbf{C}_k^{\text{red}} \cong \mathbb{P}^1$.*

Proof. We first prove that $\mathbf{C}_k$ has "delta invariant" $k-1$ with respect to its "normalisation" $k\mathbb{P}^1$, i.e. there is a short exact sequence:

$$0 \rightarrow \mathcal{O}_{\mathbf{C}_k} \rightarrow \tilde{v}_* \mathcal{O}_{k\mathbb{P}^1} \rightarrow \mathcal{O}_Q^{k-1} \rightarrow 0 \quad (3.7)$$

where Q is a point in the fiber over p_0 . As in the previous section, outside the singular point $\tilde{v}$ is an isomorphism and thus we only have to understand the geometry of $\mathbf{C}_k$ locally around $\tilde{v}^{-1}(p_0)$. Unfortunately we do not have a geometric insightful way to do that and we proceed by brute force algebra. Recall from Example 1.30 that the morphism $\tilde{v}$ is given, in a trivialising chart U_0 for $\mathcal{L}_m$ around the singular point, by the inclusion:

$$\begin{aligned} \mathbb{C}[t^2, t^3, (1 - m^2 t^2)^{-1}, \lambda] &\xrightarrow{\tilde{v}} \mathbb{C}[t, (1 - m^2 t^2)^{-1}, \lambda'] \\ \lambda &\rightarrow \lambda' + mt\lambda'. \end{aligned}$$

We want to understand the kernel of the morphism F given by the composition

$$\mathbb{C}[t^2, t^3, (1 - m^2 t^2)^{-1}, \lambda] \xrightarrow{\tilde{v}} \mathbb{C}[t, (1 - m^2 t^2)^{-1}, \lambda'] \rightarrow \mathbb{C}[t, (1 - m^2 t^2)^{-1}, \lambda'] / (\lambda' - \alpha)^k.$$

However, to prove (3.7), it is enough to argue that the cokernel $\mathcal{K} = \tilde{v}_* \mathcal{O}_{k\mathbb{P}^1} / \text{Im}(F)$ has length $k-1$. If $k = 1$ then $\text{Im}(F) \cong \mathbb{C}[t^2, t^3, (1 - m^2 t^2)^{-1}, \alpha + mt\alpha] \cong \mathbb{C}[t, (1 - m^2 t^2)^{-1}]$ and thus we are done. If $k \geq 1$, we will show that the $\mathcal{K}$ is generated over $\mathbb{C}$ by $\{\lambda', \dots, (\lambda')^{k-1}\}$. The algebra $\mathbb{C}[t, (1 - m^2 t^2)^{-1}, \lambda'] / (\lambda' - \alpha)^k$ it is infinitely generated over $\mathbb{C}$ by the monomials

$$\frac{(\lambda')^a t^b}{(1 - m^2 t^2)^c} \quad b, c \in \mathbb{N}, 0 \leq a \leq k-1.$$

We want to understand which among those generate the cokernel. Since $\text{coker } \tilde{v} \cong \mathbb{C}[\lambda']$, we get that $\{\lambda', \dots, (\lambda')^{k-1}\} \subseteq \text{coker}(F)$. Now notice that for each a, c , and for $b \geq 2$

$$\frac{(\lambda')^a t^b}{(1 - m^2 t^2)^c} = F \left(\frac{\lambda^a t^b (1 - mt)^a}{(1 - m^2 t^2)^{a+c}} \right). \quad (3.8)$$

Consequently we have that for any $a > 1, c$ the class of $(\lambda')^a t / (1 - m^2 t^2)^c$ in $\mathcal{K}$ is the same as the class of $(\lambda')^a$, indeed, first notice that from (3.8):

$$(\lambda')^a t / (1 - m^2 t^2)^c - (\lambda')^a t \in \text{Im}(F)$$

and thus the class of $(\lambda')^a t / (1 - m^2 t^2)^c$ in the quotient $\mathcal{K}$ is the same as for $(\lambda')^a t$ independently of c . Finally observe that:

$$(\lambda')^a t = \frac{1}{am} (\lambda^a - \sum_{i \geq 2} \binom{a}{i} m^i t^i (\lambda')^a - (\lambda')^a). \quad (3.9)$$

Last, we have to check that the class of t in the $\mathcal{K}$ can be written as linear combination of λ'^a , but this will follow from the previous points and from the following explicit expression of $F((\lambda - \alpha)^k)$:

$$\begin{aligned} F((\lambda - \alpha)^k) &= (\lambda' + mt\lambda' - \alpha)^k = \\ &= \underbrace{mt(\lambda')^k + mt\lambda' \sum_{i=1}^{k-2} \binom{k-1}{i} (\lambda')^i \alpha^{k-1-i}}_{\in \langle \lambda', \dots, (\lambda')^{k-1} \rangle \text{ by (3.9)}} + \underbrace{\sum_{i=0}^{k-2} \binom{k}{i} (mt\lambda')^{k-i} \sum_{j=0}^i \binom{i}{j} (\lambda')^j \alpha^{j-i}}_{\in \text{Im}(F) \text{ by (3.8)}} \end{aligned}$$

Finally $mt(\lambda')^k = mt\alpha_k + mt \sum_{i=0}^{k-1} \binom{k-1}{i} (\lambda')^i (-\alpha)^{k-1-i}$ in $\text{Im}(F)$ and this allow us to conclude again by (3.9). We have finally proved (3.7), and now the fact that the arithmetic genus is 0 follows computing inductively $H^1(C_k, \mathcal{O}_{C_k})$ using the left most colon of the commutative diagram:

$$\begin{array}{ccccc} \mathcal{O}_{C_1}(-1) & \xhookrightarrow{\tilde{v}} & \tilde{v}_* \mathcal{O}_{\mathbb{P}^1} & \twoheadrightarrow & \mathcal{O}_Q \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_{C_k} & \xhookrightarrow{\tilde{v}} & \tilde{v}_* \mathcal{O}_{k\mathbb{P}^1} & \twoheadrightarrow & \mathcal{O}_Q^{k-1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_{C_{k-1}} & \xhookrightarrow{\tilde{v}} & \tilde{v}_* \mathcal{O}_{(k-1)\mathbb{P}^1} & \twoheadrightarrow & \mathcal{O}_Q^{k-2}. \end{array}$$

□

Proposition 3.13. *Let $\mathcal{F}$ be a pure rank r stable sheaf on C with $\chi(\mathcal{F}) = 1$; then $\mathcal{F}$ is fixed by the $\text{Pic}^0(C)$ action if and only if $\mathcal{F} \cong v_{r*} \mathcal{O}_{C_r}$ where C_r is a length r , genus 0 thickening of $\mathbb{P}^1$ with v_r not factoring through any sub-scheme C_k .*

The proof of Proposition 3.7 can be adapted to prove the above statement; we thus refrain from repeating the arguments.

Higher rank $\text{Pic}^0(C)$ invariant sheaves on general reduce curve

Given a $\text{Pic}^0(C)$ invariant rank r sheaf $\mathcal{F}$ on a curve C with a unibranch singularity worst than cuspidal, the spectral curve $C_{\mathcal{F}}$ associated to an Higgs field isomorphism $\mathcal{F} \xrightarrow{\phi} \mathcal{F} \otimes \mathcal{L}$ can be a thickened $\mathbb{P}^1$ with strictly positive arithmetic genus.

For example, let C be a integral curve with normalisation $\mathbb{P}^1$ and a unique singular point p_0 with analytic local ring given by $\mathbb{C}[[t^3, t^4, t^5]]$. In this case $\text{Pic}^0(C) \cong \mathbb{C}^2$ and for

a line bundle $\mathcal{L}_{(a,b)}$ which is generic, i.e. $a \neq b \neq 0$, a computation similar to the one of Lemma 3.12 shows that the scheme theoretic image $\tilde{v}(2\mathbb{P}^1) \subseteq \text{Tot}(\mathcal{L}_{a,b})$ is a thickened $\mathbb{P}^1$ of arithmetic genus 1, with structure sheaf $\mathcal{O}_{\tilde{v}(2\mathbb{P}^1)}$ determined by an extension $e \in \text{Ext}_{\text{Tot}(\mathcal{L}_{a,b})}^1(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-2))$.

However, $C_{\mathcal{F}}$ is still a curve homeomorphic to a genus 0 curve and this is true in general. Indeed, the results of the previous chapter imply that for any integral reduced curve with normalisation $\mathbb{P}^1$ the scheme theoretic image along $\tilde{v}: \mathbb{P}^1 \times \mathbb{C} \rightarrow \text{Tot}(\mathcal{L})$ of a reduced section $\mathbb{P}^1 \cong \{\lambda' - \alpha = 0\} \subseteq \mathbb{P}^1 \times \mathbb{C}$ is isomorphic to $\mathbb{P}^1$. Combining the last observation with the claim of 3.8, we expect a statement like the following to hold true:

Conjecture 3.14. *Let C be a reduced projective curve with normalisation a disjoint union of $\mathbb{P}^1$. A pure sheaf $\mathcal{F}$ on C is $\text{Pic}^0(C)$ fixed if and only if it is push-down from a curve $\mathbf{C}$ whose connected components $\mathbf{C}_i$, endowed with their reduced scheme structure, are generalised tree.*

4

Moduli space of stable maps in genus 1 and Li-Zinger reduced invariants

This second part of the thesis is essentially the content of [BCM18], joint work with L. Battistella and C. Manolache.

In this first chapter we will: recall the description of the irreducible components of $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$, review Hu-Li equations and desingularisation of the moduli space of maps to $\mathbb{P}^r$ (which allows to define reduced invariants), and finally give two different proofs of the smoothability criterion for maps in genus 1. Let us start by fixing some notation we are going to use in the next 3 chapters:

Notation and Conventions

We denote by $\overline{\mathcal{M}}_{1,n}(X, \beta)$ denote Kontsevich's moduli stack of n -pointed genus 1 stable maps to X in the homology class β . We may use the shorthand M for a moduli space of stable maps, and always denote by $(\pi, f): \mathcal{C}_M \rightarrow M \times X$ the universal curve and the universal stable map.

We denote $\mathfrak{M}_{1,n}$ the algebraic stack of prestable n -marked curves of arithmetic genus, and denote by $\mathcal{C} \xrightarrow{\pi} \mathfrak{M}_{1,n}$ the universal curve. We will later define the algebraic stacks $\mathfrak{M}_{1,n}^{\text{wt=d, st}}$ (Definition 4.4) and $\mathfrak{M}_{1,n}^{\text{div}}$ (Subsection 4.1.1) parametrizing respectively pre-stable pointed curves decorated with weights, and pre-stable pointed curves together with a Cartier divisor, and see that there are natural forgetful morphisms $\mathfrak{M}_{1,n}^{\text{div}} \rightarrow \mathfrak{M}_{1,n}^{\text{wt=d, st}} \rightarrow \mathfrak{M}_{1,n}$. We will often abuse notation and keep denoting by $\mathcal{C}$ and π the pullbacks of the universal curve.

We shall also consider the universal Picard stack $\mathfrak{P} = \text{Pic}_{1,n}^{\text{totdeg=d, st}}$ of $\pi: \mathcal{C} \rightarrow \mathfrak{M}_{1,n}$ with universal line bundle $\mathcal{L}$, subject to the stability condition that $\omega_{\pi}^{\log} \otimes \mathcal{L}^{\otimes 2}$ is π -

ample, and the requirement that the total degree of $\mathcal{L}$ on the fibers of $\mathcal{C}$ is d . Above we denoted by $\omega_\pi^{\log} = \omega_{\mathcal{C}/\mathfrak{P}}(\sigma_1 + \cdots + \sigma_n)$ where σ_i are the universal sections of $\mathcal{C} \xrightarrow{\pi} \mathfrak{P}$. Notice that we denoted again by $(\mathcal{C}, \pi)$ the pullback of the universal curve to the Picard stack along the forgetful map $\mathfrak{P} \rightarrow \mathfrak{M}_{1,n}$. Through the chapter we will keep denoting by $(\mathcal{C}, \pi)$ also its pull-backs to all relevant algebraic stacks mapping on $\mathfrak{M}_{1,n}$.

Let λ denote the projection $\mathfrak{Pic}_{1,n}^{\text{totdeg}=d, \text{st}} \rightarrow \mathfrak{M}_{1,n}^{\text{wt}=d, \text{st}}$, where the weight comes from the degree of the line bundle. The forgetful map $M = \overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d) \rightarrow \mathfrak{M}_{1,n}^{\text{wt}=d, \text{st}}$ factors through λ by declaring $\mathcal{L}_M := f^* \mathcal{O}_{\mathbb{P}^r}(1)$. We shall often use shorthand notation $\mathfrak{M}$ for $\mathfrak{M}_{1,n}^{\text{wt}=d, \text{st}}$ and $\mathfrak{P}$ for $\mathfrak{Pic}_{1,n}^{\text{totdeg}=d, \text{st}}$.

There is a "cuspidal version" of all the spaces introduced in the above section; we will denote by $\mathfrak{M}_{1,n}(1)$ the stack pre-stable, at worst cuspidal curves, by $\widehat{\mathfrak{M}} := \mathfrak{M}_{1,n}^{\text{wt}=d, \text{st}}(1)$ the weight-decorated version, by $\mathfrak{M}_1^{\text{div}}(1)$ the stack of pre-stable curves together with a relative Cartier divisor, and by $\widehat{\mathfrak{P}} := \mathfrak{Pic}^{\text{totdeg}=d, \text{st}}(1)$ the connected component of line bundles of total degree d the universal Picard stack over the space of pre-stable (or weighted pre-stable) cuspidal curve. (See Chapter 5 for the precise definitions).

4.1. Components and Hu-Li equations

The global geometry of the moduli space $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ is well understood. In this section we recall how to describe the irreducible components of $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^r, d)$ and Hu-Li's procedure to find local equations for the moduli space in a smooth ambient space [HL10a]. Now that we have introduced all the necessary notation, we can indeed sum up Hu-Li method for finding equations of the moduli space of maps in a smooth ambient. We will then be more precise in Section § 4.1.1.

Since giving a map in $\mathbb{P}^r$ is the same as giving a base point free line bundle together with $r+1$ sections, the moduli space of maps to $\mathbb{P}^r$ can be described as an open inside the cone $\pi_* \mathcal{L}^{\oplus r+1}$, which lives over $\mathfrak{P}$. In section § 4.1.1 we will see how to find equations for $\pi_* \mathcal{L}^{\oplus r+1}$ in a smooth ambient space. In fact, following Hu-Li original approach we will work over $\mathfrak{M}_{1,n}^{\text{div}}$ rather than $\mathfrak{P}$, which we can do once we fix coordinates on $\mathbb{P}^r$. It is also possible to work on $\mathfrak{P}$; we will see how in Chapter 6 when we will study equations for a space similar to the space of maps.

Components

The *main component* of the moduli space, denoted by $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)^{\text{main}}$, is defined to be the closure of the irreducible locus of stable maps with source a smooth elliptic curve to $\mathbb{P}^r$.

Moreover, for every positive integer k and partition $\lambda \vdash d$ into k positive parts, there is an irreducible *boundary component* $D_\lambda(\mathbb{P}^r, d)$ defined to be the closure of the locus where:

- (i) the source curve is obtained by gluing a smooth elliptic curve E with k rational tails $R_i \cong \mathbb{P}^1$,
- (ii) the map contracts the elliptic curve E to a point,

(iii) the map has degree λ_i on the rational tail R_i .

The irreducibility follows from the fact that $D_\lambda(\mathbb{P}^r, d)$ is the image of the gluing morphism from the fiber product:

$$\left(\overline{\mathcal{M}}_{1,k} \times \prod_{i=1}^k \overline{\mathcal{M}}_{0,1}(\mathbb{P}^r, \lambda_i) \right) \times_{(\mathbb{P}^r)^k} \mathbb{P}^r.$$

We will denote by D^k the union of all the $D_\lambda(\mathbb{P}^r, d)$ where λ has k parts. We will use the following terminology:

Definition 4.1. We call *core* of a connected genus 1 curve the unique smallest (by inclusion) sub-curve of arithmetic genus 1.

Proposition 4.2. (1) *These are all the irreducible components of $\overline{\mathcal{M}}_1(\mathbb{P}^r, d)$:*

$$\overline{\mathcal{M}}_1(\mathbb{P}^r, d) = \overline{\mathcal{M}}_1(\mathbb{P}^r, d)^{\text{main}} \cup \bigcup_{\lambda} D_\lambda(\mathbb{P}^r, d).$$

(2) *A map $[f]$ lies in the boundary of the main component if and only if:*

- *either f is non-constant on at least one irreducible component of the core,*
- *or if f contracts the core, write $C = E \sqcup_{\mathbf{p} \sqcup \mathbf{q}} \bigsqcup_{i=1}^k R_i$, where E is the maximal contracted genus 1 subcurve, then $\{df(T_{q_i} R_i)\}_{i=1}^k$ are linearly dependent in $T_{f(E)} \mathbb{P}^r$.*

In this case we say that $[f]$ is smoothable.

This is due to R. Vakil and A. Zinger, see [Vak00, Lemma 5.9] or [VZ07]. We shall later discuss a proof of this fact based on local equations for the moduli space.

4.1.1. Hu-Li equations

Recall that a map $C \rightarrow \mathbb{P}^r$ is given by a line bundle L on C together with $r+1$ sections in $H^0(C, L)$ that generate the line bundle, i.e. they don't have a base locus.

It is therefore natural to look for an embedding of $\overline{\mathcal{M}}_1(\mathbb{P}^r, d)$ as an open inside $\pi_* \mathcal{L}^{\oplus r+1}$, where we denoted: by $\mathcal{L}$ universal line bundle of total degree d on the universal Picard stack $\mathfrak{P} := \mathfrak{Pic}_{1,n}^{\text{totdeg}=d, \text{st}}$ of $\pi: \mathcal{C} \rightarrow \mathfrak{M}$. Let $\omega_\pi^{\log}$ be the relative canonical bundle twisted by the n universal sections of π ; then $\mathcal{L}$ is subject to the following stability condition:

$$\omega_\pi^{\log} \otimes \mathcal{L}^{\otimes 2} \text{ is } \pi\text{-ample.}$$

The stack $\mathfrak{M}_{1,n}^{\text{div}}$

Hu and Li actually work over a local chart of $\mathfrak{M}_1^{\text{div}}$, which is the stack parametrising families of nodal genus 1 curves together with a relative Cartier divisor; these have to

satisfy a stability condition, namely that $\omega_\pi \otimes \mathcal{O}_{\mathcal{C}}(2D)$ is a π -ample line bundle, where D is the universal Cartier divisor.

We can construct $\mathfrak{M}_{1,n}^{\text{div}}$ as the open locus inside

$$C(\pi_* \mathcal{L}) = \text{Spec}_{\mathfrak{p}} \text{Sym}^\bullet(R^1 \pi_* \mathcal{L}^\vee \otimes \omega_\pi)$$

(see [CL12] and Section 6.2.1), where the section of $\mathcal{L}$ is not 0 on any irreducible component of the curve. The morphism to the base $\mathfrak{M}_{1,n}^{\text{div}} \rightarrow \mathfrak{p}$ is obtained by declaring $\mathcal{L}_{\mathfrak{M}_{1,n}^{\text{div}}} := \mathcal{O}_{\mathcal{C}}(D)$.

An étale chart over $\mathfrak{M}_{1,n}^{\text{div}}$ for the moduli space of maps

Let $[f: C \rightarrow \mathbb{P}^r] = ([u_0 : \dots : u_r], u_i \in H^0(C, f^* \mathcal{O}_{\mathbb{P}^r}(1)))$ be a point of $\overline{\mathcal{M}}_1(\mathbb{P}^r, d)$: we may fix homogeneous coordinates on $\mathbb{P}^r$ in such a way that $D := f^{-1}\{x_0 = 0\} \subseteq C$ is a simple divisor supported on the smooth locus of C , i.e. a d -uple of distinct smooth points. The latter is an open condition and it will thus hold in a neighbourhood of $[f]$.

This gives a morphism from an étale chart $\mathcal{U}$ of $\overline{\mathcal{M}}_1(\mathbb{P}^r, d)$ to an étale chart $\mathcal{V}$ of $\mathfrak{M}_1^{\text{div}}$. We may assume that $\mathcal{V}$ is contained in the locus where the divisor consists of d distinct smooth points; notice that such locus is smooth over $\mathfrak{M}_1$.

A map to $\mathbb{P}^r$ shall now be thought of as a pair (C, D) curve-divisor, together with r sections of $\mathcal{O}_C(D)$; the map can be written as $[1 : u_1 : \dots : u_r]$, where 1 denotes the constant section of $H^0(\mathcal{O}_C) \subseteq H^0(\mathcal{O}_C(D))$.

Furthermore, étale locally on $\overline{\mathcal{M}}_1(\mathbb{P}^r, d)$, we may pick two extra sections $\mathcal{A}$ and $\mathcal{B}$ of the universal curve $\mathcal{C} \rightarrow \overline{\mathcal{M}}_1(\mathbb{P}^r, d)$ such that:

1. they pass through the core;
2. they are distinct smooth points disjoint from the support of D .

Explicit form of the equations

Now $R^1 \pi_* \mathcal{O}_{\mathcal{C}}(D + \mathcal{A}) = 0$ and thus $\pi_* \mathcal{O}_{\mathcal{C}}(D + \mathcal{A})$ is a vector bundle on $\mathcal{V}$ and $\pi_* \mathcal{O}_{\mathcal{C}}(D)$ is defined inside it by the vanishing of the restriction map $\pi_* \mathcal{O}_{\mathcal{C}}(D + \mathcal{A}) \rightarrow \pi_* \mathcal{O}_{\mathcal{C}}(D + \mathcal{A})|_{\mathcal{A}}$.

Proposition 4.3. (1) *There is a locally closed embedding of an étale chart of $\overline{\mathcal{M}}_1(\mathbb{P}^r, d)$ inside the vector bundle $V := \text{Spec}_{\mathcal{V}}(\pi_* \mathcal{O}_{\mathcal{C}}(D + \mathcal{A})^{\oplus r})$:*

$$\begin{array}{ccccc} \overline{\mathcal{M}}_1(\mathbb{P}^r, d) & \xleftarrow{\text{ét}} & \mathcal{U} & \hookrightarrow & V \\ & & \downarrow & \nearrow & \\ \mathfrak{M}_1^{\text{div}} & \xleftarrow{\text{ét}} & \mathcal{V} & & \end{array}$$

Notice that V is smooth since $\mathcal{V}$ is.

(2) Assume furthermore that $\mathcal{V}$ is affine; on $\mathcal{V}$ we then have:

$$\pi_* \mathcal{O}_{\mathcal{C}}(\mathcal{D} + \mathcal{A}) \cong \pi_* \mathcal{O}_{\mathcal{C}}(\mathcal{D} + \mathcal{A} - \mathcal{B}) \oplus \pi_* \mathcal{O}_{\mathcal{C}}(\mathcal{D} + \mathcal{A})|_{\mathcal{B}}$$

and restricting to $\mathcal{A}$ is zero on the second factor.

(3) Call $\varphi: \pi_* \mathcal{O}_{\mathcal{C}}(\mathcal{D} + \mathcal{A} - \mathcal{B}) \rightarrow \pi_* \mathcal{O}_{\mathcal{C}}(\mathcal{D} + \mathcal{A})|_{\mathcal{A}}$ the map induced on the first factor by restricting to $\mathcal{A}$. Locally on $\mathcal{V}$, $\pi_* \mathcal{O}_{\mathcal{C}}(\mathcal{D} + \mathcal{A} - \mathcal{B})$ can be written as a sum of line bundles and thus the map φ can be rewritten as follows: let $\mathcal{D} = \sum_{i=1}^d \delta_i$, then

$$\varphi = \oplus \varphi_i: \oplus_{i=1}^d \pi_* \mathcal{O}_{\mathcal{C}}(\delta_i + \mathcal{A} - \mathcal{B}) \rightarrow \pi_* \mathcal{O}_{\mathcal{C}}(\mathcal{D} + \mathcal{A})|_{\mathcal{A}}.$$

(4) After choosing a trivialisation for each of the line bundles above, $\varphi_i: \mathcal{O}_{\mathcal{V}} \rightarrow \mathcal{O}_{\mathcal{V}}$ is given by multiplication by

$$\prod_{q \in [\mathcal{A}, \delta_i]} \zeta_q,$$

where ζ_q is the smoothing coordinate on $\mathfrak{M}_1$ corresponding to the node q , and $[\mathcal{A}, \delta_i]$ denotes the set of nodes separating $\mathcal{A}$ (the core) from the point δ_i .

We refer the reader to [HL10a, Lemma 4.10, Proposition 4.13] for the details; we are going to review the key ideas in §6.4, where we describe, following these ideas, local equations for the moduli space of 1-stable maps $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^r, d)$.

We can now describe the equations as follows: first choose coordinates $(w_i^j)_{i=1, \dots, d}^{j=1, \dots, r}$ on the fiber of $V' = \text{Spec}_{\mathcal{V}}(\pi_* \mathcal{O}_{\mathcal{C}}(\mathcal{D} + \mathcal{A} - \mathcal{B})^{\oplus r}) \subseteq V \rightarrow \mathcal{V}$ compatible with the basis given in Proposition 4.3.(3). Then $\mathcal{U}$ is an open inside $\{F^1 = \dots = F^r = 0\} \subseteq V$, where:

$$F^j = \sum_{i=1}^d \left(\prod_{q \in [\mathcal{A}, \delta_i]} \zeta_q \right) w_i^j.$$

Desingularisation of the main component and reduced invariants

The main motivation for finding local equations of the moduli space of maps was to study a desingularisation of the main component of the latter. We briefly recall the blow-up construction giving the aforementioned desingularisation and conclude the section defining reduced invariants.

Definition 4.4. $\mathfrak{M}_{1,n}^{\text{wt}=d, \text{st}}$ denotes the stack of *weighted-stable* curves of genus 1 with n markings and total weight d : geometric points of $\mathfrak{M}_{1,n}^{\text{wt}=d, \text{st}}$ represent connected, reduced, nodal, projective curves of arithmetic genus 1 with n distinct smooth markings and an integer-valued function wt on the dual graph.

Such a function assumes only nonnegative values, the sum of which on all the vertices of the dual graph is d ; wt is compatible with the specialisation maps, and we further impose the following stability condition:

- every $p_a = 0$ component of weight 0 has at least three special points;

- every $p_a = 1$ component of weight 0 has at least one special point.

There is an étale, non-separated morphism $\mathfrak{M}_{1,n}^{\text{wt}=\text{d},\text{st}} \rightarrow \mathfrak{M}_{1,n}$. The stability condition is such that the forgetful map $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d) \rightarrow \mathfrak{M}_{1,n}$ factors through $\mathfrak{M}_{1,n}^{\text{wt}=\text{d},\text{st}}$, and the weight is coming from the degree of the map to $\mathbb{P}^r$.

Hu-Li consider an iterated blow up of $\Theta_k \subseteq \mathfrak{M}_{1,n}^{\text{wt}=\text{d},\text{st}}$ $k \geq 2$, defined as the closure of the loci where the minimal elliptic subcurve E has weight 0 and $|\overline{C} \setminus E \cap E| = k$.

Notice that after the k -th blow-up, the strict transform of Θ_{k+1} is smooth, so the final result $\widetilde{\mathfrak{M}}_1^{\text{wt},\text{st}}$ is smooth as well. The stability condition on the weighted curves ensures that the process terminates.

The Vakil-Zinger space is then defined by taking the following Cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d) & \longrightarrow & \overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d) \\ \downarrow & & \downarrow \\ \widetilde{\mathfrak{M}}_{1,n}^{\text{wt}} & \longrightarrow & \mathfrak{M}_{1,n}^{\text{wt}} \end{array} \quad \begin{array}{c} f: (C, x) \rightarrow \mathbb{P}^r \\ \downarrow \\ (C, x, \deg(f^* \mathcal{O}(1))). \end{array}$$

Remark 4.5. Notice that $\mathfrak{M}_{1,n}^{\text{div}} \rightarrow \mathfrak{M}_{1,n}^{\text{wt}}$ is smooth, so we can think that the equation described in the previous section are equation in a smooth ambient over the space of pre-stable weighted curves. The reason why it is preferable to blow-up $\mathfrak{M}_{1,n}^{\text{wt}}$ rather than $\mathfrak{M}_{1,n}^{\text{div}}$ is that the combinatorics of the former is easier.

Theorem 4.6. [HL10a, Theorem 5.24, 2.11] (i) *The blow-up of the main component $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)^{\text{main}}$ is smooth.*

(ii) *Let $\pi^\circ: \mathcal{C}^{\text{main}} \rightarrow \widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)^{\text{main}}$ be the universal curve over the main component of $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ and let $f^\circ: \mathcal{C}^{\text{main}} \rightarrow \mathbb{P}^r$ be the universal morphism. Then $\mathcal{E}_{ad} := \pi_*^\circ (f^\circ)^* \mathcal{O}_{\mathbb{P}^r}(a)$ is a vector bundle of rank ad .*

For a hypersurface $X \subseteq \mathbb{P}^r$ of degree a , the main component is defined by the Cartesian diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{1,n}(X, d)^{\text{main}} & \longrightarrow & \overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)^{\text{main}} \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{M}}_{1,n}(X, d) & \longrightarrow & \overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d) \end{array}$$

We then define $\widetilde{\mathcal{M}}_{1,n}(X, d)^{\text{main}}$ to be the cartesian product $\widetilde{\mathcal{M}}_{1,n}(X, d) \times_{\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)} \overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)^{\text{main}}$, where $\widetilde{\mathcal{M}}_{1,n}(X, d)$ it's the proper transform of the blow up.

Then we have that $\mathcal{E}_{ad}$ is a relative perfect obstruction theory for $\widetilde{\mathcal{M}}_{1,n}(X, d)^{\text{main}} \hookrightarrow \overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)^{\text{main}}$, hence we have a virtual class

$$[\widetilde{\mathcal{M}}_{1,n}(X, d)^{\text{main}}]^{\text{vir}} = c_{ad}(\mathcal{E}_{ad}) \cap [\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)^{\text{main}}] \quad (4.1)$$

of homological degree $d(r+1-a)+n$.

Definition 4.7. For a hypersurface $X \subseteq \mathbb{P}^r$, the genus 1 *reduced invariants* are defined by integrating against $[\widetilde{\mathcal{M}}_{1,n}(X, d)^{\text{main}}]^{\text{vir}}$.

For a Calabi-Yau complete intersection X_5 , the relation between reduced invariants and standard invariant is explained by the Li-Zinger formula [LZ07]:

$$N_1(X_5, d) = N_1^{\text{red}}(X_5, d) + \frac{1}{12}N_0(X_5, d). \quad (4.2)$$

In fact there exists various generalisations of [LZ07], but we shall not mention them here as we are not going to explore invariants other than the ones of the quintic 3-fold in $\mathbb{P}^4$.

Remark 4.8. Formula (4.1) should be thought of as a *hyperplane property for reduced invariants*.

4.2. Genus 1 singularities and smoothability

In this section we provide two different proof of the smoothability criterion, Proposition 4.2.(2).

Smoothability from the equations

We provide a proof of Proposition 4.2.(2) based on Hu-Li's equations.

Proof. Let us start with the easiest degenerate situation: $[C = E \cup R \xrightarrow{f} \mathbb{P}^r]$ is a stable map from a curve with a single separating node which contracts the elliptic curve E and has degree d on the rational tail. Let denote the single node of C by q . We saw that equations for the moduli space of maps around such a point look like:

$$\zeta_q \sum_{i=1}^d w_i^j = 0, \quad \text{for } j = 1, \dots, r.$$

Since ζ_q is the smoothing parameter of the node q , the point $[f]$ defines a smoothable map if and only if the equations admit a solution with $\zeta_q \neq 0$, that is $\sum_{i=1}^d w_i^j = 0$ for every j . Taking a coordinate z on $\mathbb{P}^1$ centred at the node q , the i -th basis vector of the basis given in Proposition 4.3.(3) corresponds to a polynomial vanishing at q and at δ_l , $\forall l \neq i$. This can be written as:

$$e_i(z) = z \prod_{l \neq i} \frac{(z - \delta_l)}{-\delta_l},$$

where we have chosen a convenient normalisation. So the restriction to the rational tail of the map corresponding to the point of coordinates $(w_i^j)_{i=1, \dots, d}^{j=1, \dots, r}$ can be written as:

$$[1 : \sum_{i=1}^d w_i^1 e_i(z) : \dots : \sum_{i=1}^d w_i^r e_i(z)].$$

Differentiating with respect to z we see that the image of the tangent vector at q is given in affine coordinates around $f(E)$ by:

$$\left(\sum_{i=1}^d w_i^1, \dots, \sum_{i=1}^d w_i^r\right).$$

Hence smoothability is equivalent to the image of the tangent vector being zero.

More generally we may assume that the dual graph is terminally weighted [HL10a, §3.1]: essentially this means that the weight is positive only on the leaves of the dual graph and it satisfy the stability condition. We can make this assumption because the moduli space of map of genus 0 with target $\mathbb{P}^r$ is smooth and this says that every dual graph is in the closure of a terminally weighted one.

Assume there are k rational tails of positive weight R_h , $h = 1, \dots, k$. Denote by $D(h)$ the set of indices i such that δ_i belongs to the R_h , and by $E(h)$ the set of nodes separating the core from R_h . The equations will then take the following form:

$$\sum_{h=1}^k \left(\prod_{q \in E(h)} \zeta_q \right) \left(\sum_{i \in D(h)} w_i^j \right) = 0, \quad j = 1, \dots, r,$$

which can be assembled in matrix form as follows:

$$W \cdot \underline{\zeta} := \left(\sum_{i \in D(h)} w_i^j \right)_{j,h} \cdot \left(\prod_{q \in E(h)} \zeta_q \right)_h = 0.$$

A map is smoothable if and only if the above equation as a solution for the smoothing parameters $\underline{\zeta} \neq 0$, if and only if W is not invertible, so if and only if we have linear dependence of the rows of the above matrix W . On the other hand we can choose a suitable coordinate z_h around the node q_h on R_h and write the map as:

$$[1 : p_h^1(z_h) : \dots : p_h^r(z_h)],$$

where:

$$p_h^j(z_h) = \sum_{i \in D(h)} w_i^j e_i^h(z_h) \quad \text{and} \quad e_i^h(z_h) = z_h \prod_{l \in D(h) \setminus \{i\}} \frac{(z_h - \delta_l)}{-\delta_l}.$$

The elliptic curve is contracted to the point $[1 : 0 : \dots : 0]$ and the tangent vector to R_h at q_h is mapped to the h -th row of W (in affine coordinates around $f(E)$). Again we see that the map is smoothable if and only if the image of the tangent vectors to the rational tails at the nodes are linearly dependent in $T_{f(E)}\mathbb{P}^r$. $\square$

Smoothability as factorization property

Before introducing Smyth-Viscardi alternate compactification, we recall the classification of genus 1 singularities and propose a different characterisation of smoothability, which

is close in spirit to [RSW17] modular interpretation of $\widetilde{\mathcal{M}}_{1,m}(\mathbb{P}^r, d)^{\text{main}}$.

Definition 4.9. Let C be a reduced curve over $\mathbf{k}$ and $p \in C$ a singular point. We define the *genus of the singularity* in p to be the quantity:

$$g(p) = \delta(p) - b(p) + 1,$$

where $\delta(p) = \dim_{\mathbf{k}}(\nu_* \mathcal{O}_{\overline{C}} / \mathcal{O}_C) \otimes k(p)$, $\nu: \overline{C} \rightarrow C$ is the normalisation of C at p , and $b(p)$ is the number of branches of C at p .

Smyth proved [Smy11a, Appendix A] that once the number of branches $m = m(p)$ is fixed, there is a unique Gorenstein genus 1 singularity up to analytic isomorphism. A $\mathbf{k}$ -rational $p \in C$ with a Gorenstein genus 1 singularity is called an *elliptic m -fold point*. The analytic local model for these singularities are:

$$\hat{\mathcal{O}}_{C,p} \cong \begin{cases} \mathbf{k}[[x, y]] / (y^2 - x^3) & m = 1 \\ \mathbf{k}[[x, y]] / (x(x - y^2)) & m = 2 \\ \mathbf{k}[[x, y]] / I_m & m \geq 3 \end{cases}$$

where $I_m = (x_h x_i - x_h x_j : i, j, h \in \{1, \dots, m-1\})$ and i, j, h are distinct.

The proof of the classification is a commutative algebra computation; we outline the way it works completing the classification of the non Gorenstein ones.

Lemma 4.10. (*Classification of genus 1 singularities*) *The genus 1 singularities with m branches are obtained by gluing a genus 0 singularity with k branches together with a Smyth's $(m - k)$ -elliptic fold. In the following, we will call these $(k, m - k)$ singularities. Notice that k may be 0 (i.e. a point) or 1 (i.e. a smooth rational curve).*

Proof. Adopting Smyth's notation: R denotes the completion of the local ring of C at the singularity; $\widetilde{R} = \mathbf{k}[[t_1]] \oplus \dots \oplus \mathbf{k}[[t_m]]$ its integral closure; $\mathfrak{m}_R$ the maximal ideal of R and $\mathfrak{m}_{\widetilde{R}}$ that of $\widetilde{R}$.

Let us recall that to describe R as a quotient polynomial ring, it is enough to find a $\mathbf{k}$ -basis for $\mathfrak{m}_R / \mathfrak{m}_R^2 = \langle e_1, \dots, e_s \rangle_{\mathbf{k}}$ where the e_i are some elements in $\widetilde{R}$. Indeed, once given such a basis, it is a consequence of completeness that R can be recognised as $\mathbf{k}[[x_1, \dots, x_s]] / I$, where I is the kernel of the ring homomorphism

$$\begin{aligned} \mathbf{k}[[x_1, \dots, x_s]] &\rightarrow R \subset \mathbf{k}[[t_1]] \oplus \dots \oplus \mathbf{k}[[t_m]] \\ x_i &\mapsto e_i \end{aligned}$$

Smyth observes that $\widetilde{R}/R$ is graded by:

$$(\widetilde{R}/R)_i := \mathfrak{m}_{\widetilde{R}}^i / (\mathfrak{m}_{\widetilde{R}}^i \cap R) + \mathfrak{m}_{\widetilde{R}}^{i+1}$$

Furthermore:

1. $m = \delta(p) = \sum_{i \geq 0} \dim_{\mathbf{k}}(\widetilde{R}/R)_i$;
2. $1 = g(p) = \sum_{i \geq 1} \dim_{\mathbf{k}}(\widetilde{R}/R)_i$;
3. if $(\widetilde{R}/R)_i = (\widetilde{R}/R)_j = 0$ then $(\widetilde{R}/R)_{i+j} = 0$.

From (2) and (3) it follows that $\dim_{\mathbf{k}}(\widetilde{R}/R)_1 = 1$ and $\dim_{\mathbf{k}}(\widetilde{R}/R)_i = 0$ for $i \geq 2$. Then the exact sequence:

$$0 \rightarrow \frac{\mathfrak{m}_{\widetilde{R}}^2}{\mathfrak{m}_{\widetilde{R}}^2 \cap R} \rightarrow \frac{\mathfrak{m}_{\widetilde{R}}}{\mathfrak{m}_{\widetilde{R}} \cap R} \rightarrow (\widetilde{R}/R)_1 \rightarrow 0$$

entails that:

$$\mathfrak{m}_{\widetilde{R}}^2 \subseteq \mathfrak{m}_R, \quad \mathfrak{m}_R/\mathfrak{m}_{\widetilde{R}}^2 \subseteq \mathfrak{m}_{\widetilde{R}}/\mathfrak{m}_{\widetilde{R}}^2 \text{ is a codimension 1 } \mathbf{k}\text{-subspace.}$$

Obviously $\mathfrak{m}_{\widetilde{R}}^2 \subseteq \mathfrak{m}_R^2$. Hence s is at least $m-1$. After relabelling we may assume $e_1, \dots, e_{m-1}$ generate $\mathfrak{m}_R/\mathfrak{m}_{\widetilde{R}}^2$, and after Gaussian elimination they take the following form:

$$\begin{aligned} e_1 &= (t_1, 0, \dots, 0, a_1 t_m) \\ e_2 &= (0, t_2, \dots, 0, a_2 t_m) \\ &\dots \\ e_{m-1} &= (0, 0, \dots, t_{m-1}, a_{m-1} t_m) \end{aligned}$$

with $a_1, \dots, a_{m-1} \in \mathbf{k}$. At this point Smyth restricts his attention to Gorenstein singularities and shows that under this assumption he can choose all the a_i to be 1. Furthermore $\mathfrak{m}_{\widetilde{R}}^2 = \mathfrak{m}_R^2$ holds if $m \geq 3$, thus the above are generators for $\mathfrak{m}_R/\mathfrak{m}_{\widetilde{R}}^2$ and the goal is reached. For $m = 1$ (resp. 2) he finds extra generators and shows they satisfy the equation of a cusp (resp. a tacnode).

Removing the Gorenstein restriction we have three possibilities for $m \geq 3$:

1. At least two of the a_i are non-zero, say $i = 1, 2$: then $a_1 a_2 t_m^2 = e_1 \cdot e_2$ so $\mathfrak{m}_{\widetilde{R}}^2 = \mathfrak{m}_R^2$ and we are done as above. It is easily seen that $e_1, \dots, e_{m-1}$ satisfy the equations of a $(k, m-k)$ singularity where k is the number of a_i that are zero.
2. Only one of the a_i is non-zero, say $a_1 = 1$: then $\mathfrak{m}_{\widetilde{R}}^2 = \mathfrak{m}_R^2 + (t_m^2)$ and by adjoining $e_m = t_m^2$ to the e_i we see that they generate $\mathfrak{m}_R/\mathfrak{m}_{\widetilde{R}}^2$ and they satisfy the equation of a tacnode $(e_1^2 - e_m)e_m = 0$, and $e_i e_j = 0$ for $i \neq j$ and $(i, j) \neq (1, m)$, so this is an $(m-2, 2)$ singularity.
3. Finally all the a_i are zero: then we have to add t_m^2, t_m^3 to generate $\mathfrak{m}_R/\mathfrak{m}_{\widetilde{R}}^2$ and we find an $(m-1, 1)$ singularity.

Similarly for $m = 2$ there are two possibilities: the tacnode (corresponding to $a_1 \neq 0$) and the union of a cusp with a non-coplanar line (for $a_1 = 0$), with equations:

$$\mathbf{k}[[x, y, z]]/(xz, yz, y^2 - x^3).$$

For $m = 1$ the only possibility is the cusp; the maximal ideal is given by $\mathfrak{m}_R^2 = \mathfrak{m}_R$. $\square$

Remark 4.11. The genus and number of branches are no more sufficient to tell these singularities apart, neither is the embedding dimension. A numerical invariant that distinguishes them is $\dim_{\mathbf{k}} R/I_p$ where

$$I_p = \text{Ann}_R(\widetilde{R}/R) = \{f \in R \mid f \cdot \widetilde{R} \subseteq R\}$$

is the conductor ideal at the singular point. Using the explicit description of R given in the lemma above, we can easily find generators of I_p and check that for a $(k, m - k)$ singularity we have:

$$\dim_{\mathbf{k}} R/I_p = m - k.$$

Definition 4.12. Let C be a genus 1 curve and let E a genus 1 sub-curve. We say that a rational component R is at distance l from E if there are l rational components between R and E . A genus 1 sub-curve $E \subseteq C$ is called balanced if all the marked rational components of C have same distance from E .

We can then rephrase smoothability as the following factorisation property:

Proposition 4.13. Let $[f : C \rightarrow \mathbb{P}^r]$ be a stable map from a genus 1 curve $C = E_{\mathbf{p}} \sqcup_{\mathbf{q}} \bigsqcup_{i=1}^m R_i$, where E is the maximal genus 1 subcurve contracted by f , and the R_i are rational tails on which f has positive degree.

Then f is smoothable if and only if it factors through a curve with a genus 1 singularity $f : C \xrightarrow{\phi} \widehat{C} \xrightarrow{\hat{f}} \mathbb{P}^r$, such that ϕ contracts $\text{Exc}(f) = \{D \subset C \mid f(D) = pt\}$ and is an isomorphism outside it.

Lemma 4.14. Every genus 1 singularity is smoothable.

Proof. We explicitly construct a (semi)stable model with a contraction to the given singularity. Assume we start with a $(k, m - k)$ -singularity $\widehat{C}$. Pick any m -pointed elliptic curve $(E, q_1, \dots, q_m)$ and glue E along the markings with m copies of $\mathbb{P}^1$ at their respective points 0; call the rational tails $R_1, \dots, R_m$. It is useful to consider the resulting curve C as a point of $\overline{\mathcal{M}}_{1,2m}$ with markings on the rational tails given by 1 and ∞ . We now choose a smoothing $\mathcal{C}_{\Delta}$ of C over a DVR scheme Δ , such that the total space $\mathcal{C}_{\Delta}$ has an A_1 -singularity at the nodes $q_1, \dots, q_k$ and is regular everywhere else; furthermore extend the markings to get a horizontal Cartier divisor Σ . Resolving the singularities we obtain a fibered surface $\pi^{\text{ss}} : \mathcal{C}_{\Delta}^{\text{ss}} \rightarrow \Delta$ such that in the central fiber the strict transforms $\widetilde{R}_1, \dots, \widetilde{R}_k$ are at distance 1 from the core E , while the other rational tails are adjacent to it. Call $F_1, \dots, F_k$ the exceptional divisors of the resolution. As in [Smy11a, Lemma 2.12] we may contract E , which is obviously balanced, by means of the line bundle:

$$\mathcal{L}_1 := \omega_{\pi^{\text{ss}}}(E + \Sigma),$$

which results in $\overline{\mathcal{C}}_\Delta$ with an m -elliptic singularity in the central fiber, k of whose branches are the images of $F_1, \dots, F_k$ and are thus unmarked. We may finally perform a second contraction by using the line bundle $\mathcal{L}_2 := \mathcal{O}_{\overline{\mathcal{C}}_\Delta}(\overline{\Sigma})$; let $\widehat{\mathcal{C}}_\Delta$ denote the resulting family of curves, with singularity $\hat{q}$.

Notice that for each $t \in \Delta$ we have $H^1(C_t, \mathcal{L}_2|_{C_t}) = 0$. One only needs to check on $t = 0$; for C_0 let us then look at the normalisation sequence

$$0 \rightarrow \mathcal{L}_2|_{C_0} \nu^* \mathcal{L}_2|_{C_0} \rightarrow \nu^* \mathcal{L}_2|_{C_0} / \mathcal{L}_2|_{C_0} \rightarrow 0$$

and notice that the second map is surjective at the level of global sections as soon as $\mathcal{L}_2|_{C_0}$ has positive degree on at least one branch, which is the case for us. It follows that $\mathcal{L}_2$ satisfies cohomology and base-change, hence we may simply check on the central fiber both its relative semi-ampleness and the behaviour of its sections. The sections will be constant along $\overline{F}_1, \dots, \overline{F}_k$, so the linear relation among their derivatives at the Smyth's singularity will only imply the linear dependence of the tangent vectors of $\widehat{R}_{k+1}, \dots, \widehat{R}_m$ at $\hat{q}$; on the other hand sections of $\mathcal{L}_2$ along $\overline{R}_1, \dots, \overline{R}_k$ are completely independent, and thus they embed these $\mathbb{P}^1$. We deduce that the central fiber of $\widehat{\mathcal{C}}_\Delta$ has a $(k, m - k)$ -singularity. $\square$

Proof. 4.13 The argument that if f is smoothable then it has to factor through a genus 1 singularity was essentially given by Vakil in [Vak00, Lemma 5.9]. We review it here in some detail.

Pick a smoothing $F: \mathcal{C}_\Delta \rightarrow \mathbb{P}^r$ of $f = F_0$. After base-change we may assume that Δ is the spectrum of a complete DVR. The line bundle $\mathcal{L} := F^* \mathcal{O}_{\mathbb{P}^r}(1)$ is trivial on every irreducible component of the central fiber contracted by f . Since every connected component of $\text{Exc}(\mathcal{L}) = \{D \subset C : \mathcal{L}|_D \simeq \mathcal{O}_D\}$ has arithmetic genus 0 or 1, we may use the argument in [Smy11a, Lemma 2.12] to show that $\mathcal{L}$ is π_Δ -semi-ample. We thus get a contraction $\phi: \mathcal{C}_\Delta \rightarrow \overline{\mathcal{C}}_\Delta$ and notice that F factors through ϕ by the construction of $\overline{\mathcal{C}}_\Delta = \text{Proj}_\Delta \left(\bigoplus_{n \geq 0} \pi_{\Delta,*} \mathcal{L}^{\otimes n} \right)$. Being a smoothing of a reduced curve, $\mathcal{C}_\Delta$ is normal, thus ϕ factors through the normalisation $\nu: \widehat{\mathcal{C}}_\Delta \rightarrow \overline{\mathcal{C}}_\Delta$, which is a finite map by [Liu02, §8.2]. It is now clear that $\widehat{\mathcal{C}}_\Delta \rightarrow \Delta$ is a flat family of genus 1 curves with reduced fibers, and the map $\overline{F} \circ \nu$ has positive degree on all the components of $\widehat{\mathcal{C}}_0$, which for flatness reasons needs to have a genus 1 singularity. In conclusion we have constructed a factorisation $\bar{f}$ of f through a curve with a genus 1 singularity.

Vice versa let us suppose that we have a factorisation:

$$\begin{array}{ccc} C & \xrightarrow{f} & \mathbb{P}^r \\ & \searrow \phi & \nearrow \hat{f} \\ & \widehat{C} & \end{array}$$

such that ϕ contracts $\text{Exc}(f)$ and is an isomorphism everywhere else. We shall below make the point that f is smoothable as soon as it is not constant on the core (compare

also with [RSW17, Theorem 4.5.1]), so in this case we have no use for ϕ . Otherwise ϕ contracts at least the core, and $\widehat{C}$ has a genus 1 singularity at a certain point q . We first show that:

Claim 4.15. *Under these assumptions, we can produce smoothings $\mathcal{C}_\Delta$ of C and $\widehat{\mathcal{C}}_\Delta$ of $\widehat{C}$ with a contraction map ϕ_Δ extending the given ϕ .*

The proof of this claim follows the lines of the two-step contraction described in the proof of Lemma 4.14. It will be useful to treat C as a stable curve marked with $f^{-1}(Q)$ for a generic quadric $Q \subseteq \mathbb{P}^r$.

Proof. Assume that $\widehat{C}$ has a $(k, m-k)$ -singularity at q ; denote by $E := \phi^{-1}(q) \subset C$, which is the maximal genus 1 unmarked (i.e. contracted by f) connected subcurve of C . By assumption the components of C adjacent to E are marked (i.e. not contracted by f), and they can be partitioned into $R_1, \dots, R_k$ mapping to the genus 0 part of the $(k, m-k)$ -singularity, and $R_{k+1}, \dots, R_m$ parametrising the branches of the $(m-k)$ -elliptic fold. Let l_i denote the distance of R_i from the core Z , which can be determined from the marked dual graph. Let $l = \max\{l_{k+1}, \dots, l_m\}$ and $L = \max\{l_1, \dots, l_m\}$. Consider a smoothing $\mathcal{C}_\Delta$ of C with an A_{l-l_i} singularity at $R_i \cap E$ for $i = k+1, \dots, m$, and an A_{L-l_i+1} -singularity at $R_i \cap E$ for $i = 1, \dots, k$. Extend the markings $f^{-1}(Q)$ to a horizontal Cartier divisor Σ . Let $\mathcal{C}_\Delta^{\text{ss}}$ be the semistable model with regular total space; notice that in the central fiber the strict transforms of $R_{k+1}, \dots, R_m$ are at distance l from the core, while all other marked components are further apart. Let E^{bal} be the maximal unmarked balanced subcurve inside $\mathcal{C}_0^{\text{ss}}$, including all the components at distance less than l from Z . We may now use Smyth's line bundle:

$$\mathcal{L}_1 = \omega_{\mathcal{C}_\Delta^{\text{ss}}/\Delta} \left(\sum_{F \subseteq E^{\text{bal}}} (l+1-l(F, Z))F \right) \otimes \mathcal{O}_{\mathcal{C}_\Delta^{\text{ss}}}(\Sigma),$$

to perform the contraction of E^{bal} , obtaining a family $\overline{\mathcal{C}}_\Delta \rightarrow \Delta$ whose central fiber has an m -elliptic fold point. Notice that the contraction is otherwise an isomorphism, due to the stability condition. As in Lemma 4.14 we may then perform a second contraction by using the line bundle $\mathcal{L}_2 := \mathcal{O}_{\overline{\mathcal{C}}_\Delta}(\overline{\Sigma})$; we denote by $\widehat{\mathcal{C}}_\Delta$ the resulting family of curves. It may be seen that the central fiber $\widehat{\mathcal{C}}_0$ is isomorphic to $\widehat{C}$, and so is the resulting contraction up to an automorphism of C . $\square$

Let $\widehat{\mathcal{L}}$ be an extension of $\widehat{L} := \hat{f}^* \mathcal{O}_{\mathbb{P}^r}(1)$ on $\widehat{\mathcal{C}}$, which exists because deforming line bundles on curves is unobstructed. In order to extend $\hat{f}$ to $\widehat{F}: \widehat{\mathcal{C}}_\Delta \rightarrow \mathbb{P}^r$ all we have to show is that the $r+1$ sections $\hat{s}_0, \dots, \hat{s}_{r+1}$ representing the map $\hat{f}$ extend to sections of $\widehat{\mathcal{L}}$. Thus it is enough to verify that $H^1(\widehat{C}, \widehat{L}) = 0$ [Wan12]. Once this is done we are going to obtain the smoothing of the original map by precomposing with the contraction $F: \mathcal{C}_\Delta \xrightarrow{\phi} \widehat{\mathcal{C}}_\Delta \xrightarrow{\widehat{F}} \mathbb{P}^r$.

Since all the curves we are considering are Cohen-Maculay, they have a dualising sheaf $\omega_{\widehat{C}}$ such that for any line bundle $\widehat{L}$:

$$H^1(\widehat{C}, \widehat{L}) = (H^0(\widehat{C}, \widehat{L}^{-1} \otimes \omega_{\widehat{C}}))^{\vee}.$$

It is known [Ser00, IV, § 3] that $\omega_{\widehat{C}}$ can be described as the subsheaf of $\nu_* \omega_{\bar{C}} \otimes K(\bar{C})$ (where $\nu: \bar{C} \rightarrow \widehat{C}$ is the normalisation) satisfying: for every open $U \subseteq \widehat{C}$ a rational 1-form $\omega \in \nu_* (\omega_{\bar{C}} \otimes K(\bar{C}))(U)$ is a section of $\omega_{\widehat{C}}(U)$ if and only if:

$$\sum_{q \in \nu^{-1}(0)} \text{Res}(\nu^*(f)\omega) = 0 \quad \forall f \in \mathcal{O}_{\widehat{C}}(U)$$

From this description it is patent that the pullback of $\omega_{\widehat{C}}$ to the normalisation restricts to a line bundle of degree -1 or 0 on every branch, according to it being one of the k or $m-k$ components. Hence it can be seen from the normalisation exact sequence and Serre duality that $H^1(\widehat{C}, \widehat{L}) = 0$ as soon as $\widehat{L}$ has positive degree on one of the branches. $\square$

5

Viscardi Alternate compactification and 1-stabilisation map

5.1. Definition of the moduli space and components

The moduli spaces of m -stable maps give alternate compactifications of $\mathcal{M}_{1,n}(X, \beta)$.

Definition 5.1. Let C be a connected, reduced, projective curve of arithmetic genus 1 over $\mathbf{k}$, and let $p_1, \dots, p_n \in C$ be smooth, distinct points. A map $f: C \rightarrow X$ is said to be m -stable if the following conditions hold:

- (1) C has only nodes and elliptic l -fold points, $l \leq m$, as singularities.

Stability:

- (2) For any subcurve $E \subset C$ of arithmetic genus 1 on which f is constant,

$$\left| \left\{ E \cap \overline{C \setminus E} \right\} \cup \{i : p_i \in E\} \right| > m.$$

- (3) f has no non-trivial infinitesimal automorphisms.

Remark 5.2. While in the case of Kontsevich's moduli space of stable maps the stability condition is equivalent to the map having finite automorphisms, in this case, it is strictly stronger. So already intuitively we do expect to have fewer stable objects.

Viscardi's main result [Vis12, Theorem 3.6] is the following:

Theorem 5.3. *The moduli functor of m -stable maps $\overline{\mathcal{M}}_{1,n}^{(m)}(X, \beta)$ is represented by a proper Deligne-Mumford stack of finite type over $\mathbf{k}$.*

Remark 5.4. We think that the algorithm proposed by Viscardi to prove the properness of his moduli spaces oversees a case. The issue is that, given a map $f: \mathcal{C}_\Delta \rightarrow \mathbb{P}_\Delta^r$ over a

DVR scheme Δ such that $\mathcal{C}_\eta$ is smooth and f_0 is constant on a genus 1 connected subcurve $E \subseteq \mathcal{C}_0$, it is not always true that f descends to a map $\hat{f}: \widehat{\mathcal{C}}_\Delta \rightarrow \mathbb{P}_\Delta^r$, where $\widehat{\mathcal{C}}_0$ has a genus 1 Gorenstein singularity.

Consider the stable map $[f]$ in $\overline{\mathcal{M}}_1(\mathbb{P}^3, 4)$ from an elliptic bridge

$$C := R_1 \sqcup_{q_1} E \sqcup_{q_2} R_2$$

to $\mathbb{P}^3$ that maps R_1 to the z -axis, contracts E to the origin, and makes (R_2, q_2) into the normalisation of a cusp in the (x, y) -plane, i.e. its image is the non-Gorenstein singularity:

$$D := \text{Spec}(\mathbf{k}[x, y, z]/(x, y)) \cup \text{Spec}(\mathbf{k}[x, y, z]/(z, y^2 - x^3)).$$

Notice that $df(T_{q_2}R_2) = 0$, so there is a non-trivial linear relation:

$$0 \cdot df(T_{q_1}R_1) + 1 \cdot df(T_{q_2}R_2) = 0,$$

and hence the map is smoothable. Viscardi claims that the map factors through the family $\widehat{\mathcal{C}}_\Delta$, obtained by contracting E to a tacnode. Notice that the image of $\hat{f}$ would still be D , since there is at most one indeterminacy point of $\widehat{\mathcal{C}}_\Delta \dashrightarrow \mathbb{P}_\Delta^r$, located at the singularity. However in our example f cannot factor through the tacnode. Indeed in that case we would have a birational morphism between two singular curves with the same δ invariant and the same normalisation, so $\hat{f}: \widehat{C} \rightarrow D$ would be an isomorphism, which is a contradiction. We suggest that in this case the correct procedure would be to sprout (R_1, q_1) .

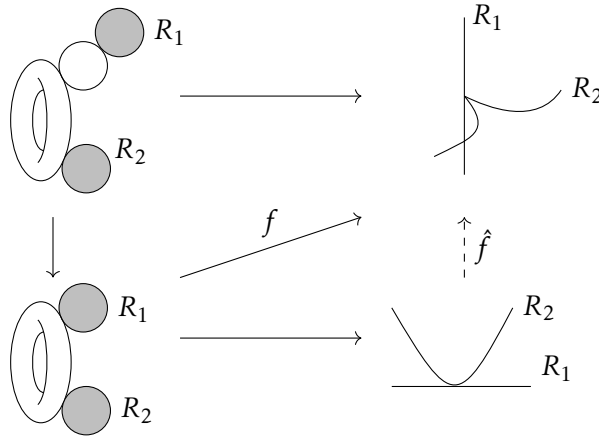


Figure 5.1: An example of non-factorisation.

However Viscardi's argument can be fixed. Let $(\mathcal{C}_\eta, F_\eta)$ be a stable map to $\mathbb{P}^r$, defined on the generic point of a DVR scheme Δ ; we may assume that $\mathcal{C}_\eta$ is smooth [Vis12, Section 3.2.1]. As described in [Vis12, Theorem 3.6, Step 1], after applying semi-stable reduction we get a map $F: \mathcal{C}_\Delta \rightarrow \mathbb{P}_\Delta^r$, for which we may suppose that $C := \mathcal{C}_0$ is nodal and reduced, and $f := F_0$ is stable (the point is that we already know that the Kontsevich

moduli space of stable maps is proper).

If f is not constant on the minimal genus 1 subcurve, then it is already m -stable and there is nothing to say. Otherwise let $E \subset C$ be the maximal genus 1 subcurve where f is constant and let $R_1 \sqcup \dots \sqcup R_m = \overline{C/E}$.

By Proposition 4.2 (2) we know there is a non-trivial linear relation among the $\mathrm{df}(T_{q_i} R_i)$'s. Consider a maximal one with all non-zero coefficients. Possibly after relabelling, such a relation looks like:

$$\alpha_1 \mathrm{df}(T_{q_1} R_1) + \dots + \alpha_j \mathrm{df}(T_{q_j} R_j) = 0.$$

In this case we blow-up $\mathcal{C}_\Delta$ in $q_{j+1}, \dots, q_m$. The induced map $\tilde{F}_0$ is constant on the exceptional divisors $G_{j+1}, \dots, G_m$ and we can complete the above linear relation to

$$\alpha_1 \mathrm{d}\tilde{f}(T_{q_1} \tilde{R}_1) + \dots + \alpha_j \mathrm{d}\tilde{f}(T_{q_j} \tilde{R}_j) + \beta_{j+1} \mathrm{d}\tilde{f}(T_{q_{j+1}} G_{j+1}) + \dots + \beta_m \mathrm{d}\tilde{f}(T_{q_m} G_m) = 0$$

with $\beta_i = 1$ for all i . Now this *sprouting* [Smy11b, Section 2.3] ensures that the map descends to the corresponding elliptic m -fold singularity. Notice that, for every $m_1 \leq m_2$, there is a morphism from an m_2 -elliptic singularity to an m_1 -elliptic point that is birational on the target and contracts $m_2 - m_1$ branches of the source to the singular point. Proceed now with Step 2 of Viscardi's algorithm.

The irreducible components of Viscardi's moduli space $\overline{\mathcal{M}}_{1,n}^{(m)}(\mathbb{P}^r, d)$ are also well understood [Vis12, Thm. 5.9]; indeed they have a similar description of the ones of Kontsevich's space. The main advantage of the m -stable compactification is that the number of components drops as m increases:

Proposition 5.5. *As in Proposition 4.2, define $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)^{\mathrm{main}}$, to be the closure in $\overline{\mathcal{M}}_{1,n}^{(m)}(\mathbb{P}^r, d)$ of the irreducible locus of stable maps with source a smooth elliptic curve to $\mathbb{P}^r$, and, for every positive integer k and partition $\lambda \vdash d$ into k positive parts, let $D_\lambda(\mathbb{P}^r, d)$ be the closure of the locus with source $E \cup R_1 \cup \dots \cup R_k$ and $f(E) = \mathrm{pt}$. Then*

$$\overline{\mathcal{M}}_1^{(m)}(\mathbb{P}^r, d) = \overline{\mathcal{M}}_1(\mathbb{P}^r, d)^{\mathrm{main}} \cup \bigcup_{\lambda} D_\lambda(\mathbb{P}^r, d)$$

for $0 \leq m+1 \leq k \leq \min\{r, d\}$.

In particular, in the moduli space $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^r, d)$ we will consider in the next sections the boundary component D^1 has been destabilised, and for m big enough the space of m -stable maps it is actually irreducible.

Obstruction theory and m -reduced invariants

The Behrend-Fantechi obstruction theory $R^\bullet \hat{\pi}_* \hat{f}^* T_X$ for spaces of morphisms endows $\overline{\mathcal{M}}_{1,n}^{(m)}(X, \beta)$ with a perfect obstruction theory relative to $\mathfrak{M}_{1,n}(m)$.

Indeed, $\overline{\mathcal{M}}_{1,n}^{(m)}(X, \beta)$ is open in $\mathrm{Mor}_{\mathfrak{M}_{1,n}(m)}(\widehat{\mathcal{C}}, X \times \mathfrak{M}_{1,n}(m))$ and thus the natural relative

obstruction theory $R^\bullet \hat{\pi}_* \hat{f}^* T_X$ on the latter [BF97, Proposition 6.2] restricts to an obstruction theory on the moduli space of m -stable maps.

Moreover, Behrend shows that $R^\bullet \hat{\pi}_*(\hat{f}^* T_X \otimes \mathcal{L}^N)$ for $\mathcal{L} \cong \omega_{\hat{\pi}}(\Sigma)(\hat{f}^* \mathcal{O}_X(1))$ is a resolution in locally free sheaves for the obstruction theory (see [Beh97, Proposition 5]), thus the relative obstruction theory is perfect.

Finally, notice that the base is an irreducible Artin stack for every m (since every point is limit of a family of smooth curves and the locus of smooth curves is itself smooth) and it is a smooth stack for $m \leq 5$. Irreducibility ensures that $\mathfrak{M}_{1,n}(m)$ has a fundamental class and thus have a virtual class ([BF97], [LT98]) on $\overline{\mathcal{M}}_{1,n}^{(m)}(X, \beta)$ can be defined in the usual way, by virtual-pull-back [Man12].

Then m -stable invariants are defined in the usual way capping $[\overline{\mathcal{M}}_{1,n}^{(m)}(X, \beta)]^{\text{vir}}$ with pull-backs under the evaluation maps of cohomology classes on X .

Remark 5.6. For $m > d + n$, the space $\overline{\mathcal{M}}_{1,n}^{(m)}(\mathbb{P}^r, d)$ is smooth relatively to $\mathfrak{M}_{1,n}(m)$ since the stability condition ensures that not genus one curves are contracted. Moreover for a complete intersection $X \hookrightarrow \mathbb{P}^r$ cut out by a regular section of $\mathcal{E} = \bigoplus \mathcal{O}(d_i)$, we have that $\hat{\pi}_* \hat{f}^* \mathcal{E}$ is a vector bundle, and thus $\overline{\mathcal{M}}_{1,n}^{(m)}(X, \beta) \subseteq \overline{\mathcal{M}}_{1,n}^{(m)}(\mathbb{P}^r, d)$ should satisfy a hyperplane property as in the genus 0 case [KKP03] or in the case of reduced invariants.

However since the base $\mathfrak{M}_{1,n}(m)$ does not seem to have a natural perfect obstruction theory for $m > 5$, we in fact do not understand the absolute obstruction theory of the moduli space of m -stable maps, and thus we actually do not know how to compute the invariants by localisation techniques.

Main Theorem

We can now state the main theorem of the second part:

Theorem 5.7. *Let X be a generic smooth quintic three-fold. Then:*

$$N_1^{\text{red}}(X, d) = N_1^{\text{cusp}}(X, d)$$

where:

$$N_1^{\text{red}}(X, d) := \deg\left([\widetilde{\mathcal{M}}_1(\mathbb{P}^4, d)^{\text{main}}] \cap c_{\text{top}}(\tilde{\pi}_* \tilde{f}^*(\mathcal{O}_{\mathbb{P}^4}(5)))\right)$$

and

$$N_1^{\text{cusp}}(X, d) := \deg[\overline{\mathcal{M}}_1^{(1)}(X, d)]^{\text{vir}}.$$

In the remaining of this section and for all the next chapter we built up the necessary machinery to prove the above Theorem.

5.2. 1-stabilisation map

In this section we construct the 1-stabilisation map from the Artin stack of weighted pre-stable curves to the Artin stack of weighted 1-pre-stable curves (defined below). This will

allow us to compare the 1-stable invariants defined in the previous section with the usual GW invariants, and with the reduced ones via the Li-Zinger formula (0.4).

Before we begin, we recall for the benefit of the reader a result of Smyth, [Smy11a, Lemma 2.13] about contracting genus 1 curves in a one parameter smoothing, which will be quoted often in this section:

Lemma 5.8. *Let $\pi: \mathcal{C} \rightarrow \Delta$ be a projective and flat family of curves over a dvr scheme Δ , with smooth general fiber and connected reduced, nodal special fiber. Let $\mathcal{L}$ be a line-bundle on $\mathcal{C}$ with positive degree on the generic fiber and non-negative degree on each irreducible component of the special fiber. Consider*

$$E = \{\text{Irreducible components } F \subset C \mid \deg \mathcal{L}|_F = 0\},$$

and assume that: E is connected of arithmetic genus one, $\mathcal{L}|_E \simeq \mathcal{O}_E$, Then $\mathcal{L}$ is π -semiample and there exists a contraction $\mathcal{C} \xrightarrow{\phi} \widehat{\mathcal{C}}$, where ϕ is proper, birational, and $\text{Exc}(\phi) = E$. Furthermore, $\widehat{\mathcal{C}} \rightarrow \Delta$ is flat with reduced fibers and the number of branches and the δ -invariant of the isolated curve singularity $p \in \widehat{\mathcal{C}}$ are given by

$$\begin{aligned} m &= |\overline{C \setminus E} \cap E| \\ \delta &= p_a(E) + m - 1. \end{aligned}$$

In analogy with Definition 4.4 we define:

Definition 5.9. Let $\mathfrak{M}_{1,n}^{\text{wt}=d, \text{st}}(1)$ be the stack of *at worst cuspidal* connected, reduced, n -marked, projective curves of arithmetic genus 1 that are weighted-stable with total weight d , i.e. the weight is nonnegative and such that:

- every $p_a = 0$ component of weight 0 has at least three special points;
- every $p_a = 1$ component of weight 0 has at least *two* special points.

In this definition, by *special point* we mean the preimage of a node or a marking in the normalisation.

In the following we will sometimes denote $\mathfrak{M}_{1,n}^{\text{wt}=d, \text{st}}(1)$ with $\widehat{\mathfrak{M}}$ and with $\widehat{\mathcal{C}}$ its universal curve.

Theorem 5.10. *There exists a morphism $\mathfrak{M}_{1,n}^{\text{wt}=d, \text{st}} \rightarrow \mathfrak{M}_{1,n}^{\text{wt}=d, \text{st}}(1)$ which extends the identity on the smooth locus.*

The idea behind this construction is essentially due to Hassett [HH09, §2] and it has recently been reviewed and simplified in [RSW17, §3.7]. For simplicity we will work with unmarked curves, which is indeed the case of interest when looking at the Gromov-Witten theory of a Calabi-Yau threefold.

We shall construct the contraction over $\mathfrak{M}_1^{\text{div}}$ first, and then show that it descends to $\mathfrak{M}_1^{\text{wt}, \text{st}}$. The weighted stability condition is implicit in the notation.

Let $\mathcal{E}$ be the locus inside the universal curve spanned by elliptic tails of weight 0; this is a Cartier divisor in the universal curve over $\mathfrak{M}_1^{\text{wt, st}}$; we will abuse notation and denote by $\mathcal{E}$ all its pullbacks. Moreover we denote by $\mathcal{D}^1$ its image in $\mathfrak{M}_1^{\text{wt, st}}$, which is a Cartier divisor as well.

Consider the following line bundle on the universal curve over $\mathfrak{M}_1^{\text{div}}$:

$$\mathcal{N} := \omega_\pi(\mathcal{E}) \otimes \mathcal{O}_{\mathcal{C}}(2\mathcal{D}), \quad (5.1)$$

where $\mathcal{D}$ is the universal Cartier divisor over $\mathfrak{M}_1^{\text{div}}$. Notice that $\mathcal{N}$ is trivial on the locus of elliptic tails, so the Proj construction applied to $\mathcal{N}$ will contract this locus.

Proposition 5.11. *Let $\widehat{\mathcal{C}} = \underline{\text{Proj}}_{\mathfrak{M}_1^{\text{div}}}(\bigoplus_{n \geq 0} \pi_* \mathcal{N}^{\otimes n})$. Then $\widehat{\mathcal{C}}$ is a family of weighted 1-stable curves and ϕ is a regular morphism:*

$$\begin{array}{ccc} (\mathcal{C}, \mathcal{D}) & \xrightarrow{\phi} & (\widehat{\mathcal{C}}, \phi(\mathcal{D})) \\ \pi \searrow & & \nwarrow \hat{\pi} \\ & \mathfrak{M}_1^{\text{div}} & \end{array}$$

This defines the 1-stabilisation morphism $\mathfrak{M}_1^{\text{div}} \rightarrow \mathfrak{M}_1^{\text{div}}(1)$.

We need to prove that $\mathcal{N}$ is π -semi-ample, namely that $\pi^* \pi_* \mathcal{N}^{\otimes n} \rightarrow \mathcal{N}^{\otimes n}$ is surjective for n big enough, as this ensures regularity of ϕ , and that $\pi_* \mathcal{N}$ is locally free, in order to have flatness of $\hat{\pi}$. Both these facts are clearly true generically, but less so on points of $\mathcal{E}$ and $\mathcal{D}^1$. We shall check this by exploiting the next lemma, which is a nice technical gadget taken from [RSW17].

Lemma 5.12 (pullback with a boundary). [RSW17, Lemmma 3.7.2.2] *Let $\pi: \mathcal{C} \rightarrow S$ be a proper family of curves over a smooth base, and let $\mathcal{N}$ be a line bundle on $\mathcal{C}$ such that $R^1 \pi_* \mathcal{N}$ is a line bundle supported on a Cartier divisor $\mathcal{D} \subseteq S$. Then for every DVR scheme Δ with closed point 0 and generic point η , and for every morphism $f: \Delta \rightarrow S$ such that $f(0) \in \mathcal{D}$ and $f(\eta) \in S \setminus \mathcal{D}$ we have*

$$f^* \pi_* \mathcal{N} \cong \pi_{\Delta, *} f_{\mathcal{C}}^* \mathcal{N}.$$

Now recall that in our case $\mathcal{N}$ is trivial on elliptic tails and of positive degree everywhere else, and that (by cohomology and base change) the rank of $R^1 \pi_* \mathcal{N}$ can be checked on the fibers, [Har77, Theorem III.12.11], so we see that it is 0 outside $\mathcal{D}^1$ and 1 on it.

Lemma 5.13. *The line bundle $\mathcal{N}$ is π -semi-ample, i.e. the natural map*

$$\pi^* \pi_* \mathcal{N}^{\otimes n} \rightarrow \mathcal{N}^{\otimes n}$$

is surjective for $n \gg 0$.

Proof. Outside the locus of elliptic tails $\mathcal{N}$ is π -ample. We are left to check on points belonging to an elliptic tail; thanks to the above Lemma we can reduce to the case that C is the central fiber of a one-parameter smoothing of the elliptic tail. This has been dealt with by Smyth [Smy11a, Lemma 2.13] (see Lemma 5.8). $\square$

Lemma 5.14. $\pi_*\mathcal{N}$ is locally free on $\mathfrak{M}_1^{\text{div}}$.

Proof. [RSW17, Proposition 3.7.2.1] We have to check that $\pi_*\mathcal{N}$ has constant rank. On $\mathfrak{M}_1^{\text{div}} \setminus \mathcal{D}^1$ we see that $R^1\pi_*\mathcal{N} = 0$, so $\pi_*\mathcal{N}$ satisfies Cohomology and Base Change and its rank is determined by Riemann-Roch. Given a point x on the boundary $\mathcal{D}^1$, we can pick a DVR scheme Δ whose closed point maps to x and whose generic point maps to $\mathfrak{M}_1^{\text{div}} \setminus \mathcal{D}^1$. Then we are in the hypotheses of Lemma 5.12 and we can check the rank at x by looking at $\pi_*f^*\mathcal{N}$ over Δ . Now $f^*\mathcal{N}$ is flat over Δ , so $\pi_*f^*\mathcal{N}$ is as well, which implies torsion-free and thus constant rank. $\square$

Proof. 5.11 Let $S \rightarrow \mathfrak{M}_1^{\text{div}}$ be a smooth atlas, then we have:

$$\begin{array}{ccc} (\mathcal{E}_S, \mathcal{D}_S) & \xrightarrow{\phi_S} & (\widehat{\mathcal{E}}_S, \phi(\mathcal{D}_S)) \\ & \searrow \pi_S \quad \swarrow \hat{\pi}_S & \\ & S & \end{array}$$

where $\widehat{\mathcal{E}}_S = \underline{\text{Proj}}_S(\bigoplus_{n \geq 0} \pi_{S,*}\mathcal{N}^{\otimes n})$, ϕ_S is a proper and birational morphism since $\mathcal{N}$ is π_S -semi-ample and $\hat{\pi}_S$ is flat since $\pi_{S,*}\mathcal{N}$ is locally free. To verify that this defines a morphism $S \rightarrow \mathfrak{M}_1^{\text{div}}(1)$ we have to argue that $\widehat{\mathcal{E}}_S$ has reduced fibers and only nodes and cusps as singularities.

Since these properties only concern the fibers of $\hat{\pi}$ we can verify them after base change to a DVR scheme Δ chosen as in Lemma 5.12, so that the construction commutes with base-change. Furthermore we can pick $f: \Delta \rightarrow S$ such that the total space $\mathcal{E}_\Delta$ is regular, so we may just apply Smyth's Contraction Lemma [Smy11a, Lemma 2.13] (see Lemma 5.8).

Finally to conclude that this defines a morphism $\mathfrak{M}_1^{\text{div}} \rightarrow \mathfrak{M}_1^{\text{div}}(1)$ it is enough to verify that there is an isomorphism $\text{pr}_1^*\widehat{\mathcal{E}}_S \cong \text{pr}_2^*\widehat{\mathcal{E}}_S$ satisfying the cocycle condition, where $\text{pr}_i: S' = S \times_{\mathfrak{M}_1^{\text{div}}} S \rightrightarrows S$.

This follows from the fact that $\text{pr}_i^*\widehat{\mathcal{E}}_S$ are obtained from applying the Proj construction to $\text{pr}_i^*\pi_{S,*}\mathcal{N} \cong \pi_{S',*}\text{pr}_i^*\mathcal{N}$, by flatness of $S' \rightarrow S$. Thus it is enough to show that $\text{pr}_1^*\mathcal{N} \cong \text{pr}_2^*\mathcal{N}$. But $\mathcal{N}$ is the pullback of a line bundle on $\mathfrak{M}_1^{\text{div}}$, thus the desired isomorphism follows from the commutativity of the diagram

$$\begin{array}{ccc} S \times_{\mathfrak{M}_1^{\text{div}}} S & \xrightarrow{\text{pr}_1} & S \\ \downarrow \text{pr}_2 & & \downarrow \\ S & \longrightarrow & \mathfrak{M}_1^{\text{div}} \end{array}$$

The cocycle condition is derived similarly. $\square$

Proposition 5.15. *The 1-stabilisation for curves with a divisor induces an analogue morphism on weighted curves:*

$$\begin{array}{ccc} \mathfrak{M}_1^{\text{div}} & \longrightarrow & \mathfrak{M}_1^{\text{div}}(1) \\ \downarrow & \cup & \downarrow \\ \mathfrak{M}_1^{\text{wt,st}} & \xrightarrow{\exists} & \mathfrak{M}_1^{\text{wt,st}}(1) \end{array}$$

Proof. Étale locally on $\mathfrak{M}_1^{\text{wt,st}}$ we can choose smooth sections s_i of the universal curve so that the Cartier divisor $\mathcal{D} = \sum s_i$ has degree compatible with the weight function, so in particular it makes $\mathcal{N} = \omega_\pi(\mathcal{E}) \otimes \mathcal{O}_{\mathcal{C}}(2\mathcal{D})$ trivial on the elliptic tails and π -ample elsewhere. For a smooth atlas $S \rightarrow \mathfrak{M}_1^{\text{wt,st}}$, this observation allows us to define a lifting $S \rightarrow \mathfrak{M}_1^{\text{div}}$, and thus a morphism $\xi: S \rightarrow \mathfrak{M}_1^{\text{wt,st}}(1)$ through the construction of Proposition 5.11.

In order to show that this descends to a morphism $\mathfrak{M}_1^{\text{wt,st}} \rightarrow \mathfrak{M}_1^{\text{wt,st}}(1)$ we need to verify that there exists $\text{pr}_1^*(\xi) \cong \text{pr}_2^*(\xi)$ satisfying the cocycle condition, where $\text{pr}_i: S' = S \times_{\mathfrak{M}_1^{\text{wt,st}}} S \rightrightarrows S$.

This boils down to checking that for two different choices of a lifting $\mathcal{D}_1, \mathcal{D}_2: S \rightarrow \mathfrak{M}_1^{\text{div}}$ there exists a unique isomorphism

$$\widehat{\mathcal{C}}_1 = \underline{\text{Proj}}_S \left(\bigoplus_{n \geq 0} \pi_*(\mathcal{N}_1)^{\otimes n} \right) \cong \underline{\text{Proj}}_S \left(\bigoplus_{n \geq 0} \pi_*(\mathcal{N}_2)^{\otimes n} \right) = \widehat{\mathcal{C}}_2.$$

By construction there is a birational map ψ :

$$\begin{array}{ccc} & \mathcal{C}_S & \\ \phi_1 \swarrow & & \searrow \phi_2 \\ \widehat{\mathcal{C}}_1 & \xrightarrow{\psi} & \widehat{\mathcal{C}}_2. \end{array}$$

We want to show that ψ extends to a regular morphism. Notice that $\widehat{\mathcal{C}}_i$ is normal, $i = 1, 2$. Indeed since S is smooth and the singularities of the fibers are in codimension 1, $\widehat{\mathcal{C}}_i$ is regular in codimension 1. Moreover since both S (smooth) and the fibers (Cohen-Macaulay) satisfy Serre's condition S_2 , so does the total space of $\widehat{\mathcal{C}}_i$ by [Mat89, Theorem 23.9]. Then, since the geometric fibers of ϕ_i are either points or $p_a = 1$ subcurves, and in either case connected, Zariski's connectedness theorem implies that

$$\phi_{i,*} \mathcal{O}_{\mathcal{C}_S} \cong \mathcal{O}_{\widehat{\mathcal{C}}_i}.$$

Moreover by construction $\text{Exc}(\phi_1) = \text{Exc}(\phi_2)$ is the locus of elliptic tails of weight 0, so in particular ϕ_2 contracts all fibers of ϕ_1 . Then [Deb13, Lemma 1.15] implies that ϕ_2 factors through ϕ_1 , and viceversa. This proves the regularity of ψ and its inverse. Notice that ψ is unique as it is the only extension of $\phi_2 \circ \phi_1^{-1}$. $\square$

Moreover the 1-stabilisation map is proper:

Lemma 5.16. *The 1-stabilisation $\mathfrak{M}_1^{\text{wt, st}} \rightarrow \mathfrak{M}_1^{\text{wt, st}}(1)$ is a proper morphism.*

Proof. Use the valuative criterion: let Δ be a DVR scheme with generic point η ; we have to fill in the upper right part of the following diagram:

$$\begin{array}{ccc} \mathcal{C}_\eta & \dashrightarrow & \mathcal{C} \\ \downarrow \phi_\eta & & \downarrow \phi \\ \widehat{\mathcal{C}}_\eta & \longrightarrow & \widehat{\mathcal{C}} \end{array}$$

We may in fact assume that ϕ_η is an isomorphism, and then it is enough to take the weighted stable model of $\widehat{\mathcal{C}}$. □

Future direction: comparing m-reduced invariants

The m -stabilisation for $m \geq 2$ does not extend to a regular morphism; the following is an example with stable curves:

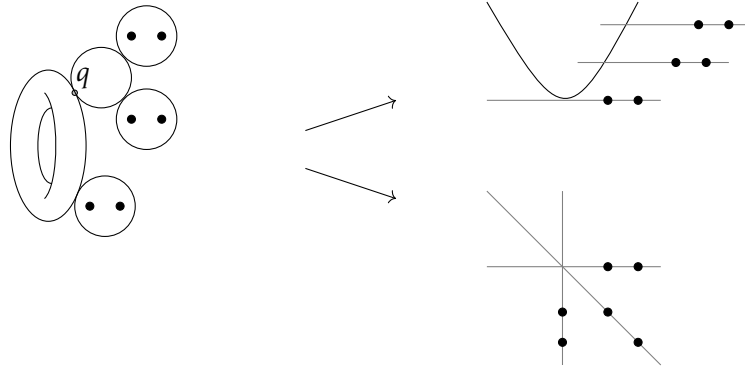


Figure 5.2: An example of different plausible 3-stabilisations.

Both curves on the right are 3-stable, so it is unclear how to define the 3-stabilisation already on points.

Here is some heuristics about why the natural choices in families do not work. Indeed suppose we try contracting the minimal genus 1 unmarked subcurve. Consider a 1-parameter smoothing of the node q : the generic fiber is a smooth elliptic curve with three marked $\mathbb{P}^1$ attached. If a contraction of the core existed, we would get a family with a Smyth's 3-fold elliptic point degenerating to the tacnode, but such a family cannot exist; the miniversal deformation of the tacnode contains only cusps and nodes as singularities.

On the other hand, try to define the 3-stabilisation by contracting the maximal unmarked genus 1 subcurve. Choose a generic smoothing of our stable curve in such a way that the total space $\mathcal{C}_\Delta$ is smooth. Notice that the maximal unmarked subcurve of the central fiber is not balanced [Smy11a, Definition 2.11], so contracting it we would get a non-Goreinstein singularity.

However, the indeterminacy locus has been resolved by [RSW17]. They consider the moduli space $\mathfrak{M}_{1,n}^{\text{rad}}$ parametrising *radially aligned logarithmic curves*, which, in very rough terms, means that the logarithmic structure is such that it is possible to define a total order on the vertices of the dual graph of the curve.

They then show that the latter resolve the indeterminacy, i.e. we have a diagram:

$$\begin{array}{ccc} & \mathfrak{M}_{1,n}^{\text{rad,wt}} & \\ s_0 \swarrow & & \searrow s_m \\ \mathfrak{M}_{1,n}^{\text{wt}} & \dashrightarrow & \mathfrak{M}_{1,n}^{\text{wt}}(m). \end{array}$$

We have seen in the above example that, given a genus 1 semi-stable curve C (in fact stable in the example), the m -(semi)-stable curve $\widehat{C}$ we should contract C to seems to depend on: a choice of smoothing, and a choice of balanced genus 1 sub-curve to contract. Giving a radially aligned log structure to the curve fixes both these choices and thus determine the contraction s_m .

We expect that using the above morphism and a generalisation of the arguments we are going to see in the next chapter it will be possible to prove:

Conjecture 5.17. *If X is a smooth projective Calabi-Yau 3-fold. Then for any $m \geq 1$:*

$$GW_{1,d}^{\text{red}}(X) = \deg\left([\overline{\mathcal{M}}_{1,n}^{(m)}(X, \beta)]^{\text{vir}}\right).$$

If X is a smooth projective Fano 3-fold, then for any choice of insertions $\gamma_1, \dots, \gamma_n$ the reduced invariants coincide with the m -invariants for any $m \geq n$.

6

Reduced vs Cuspidal invariants

6.1. The main idea and Chang-Li solution

We exploit the 1-stabilisation morphism constructed before to introduce some auxiliary spaces, endowed with natural virtual classes, that allow us the comparison between cuspidal and reduced invariants of the quintic threefold. Let $\mathcal{Z}$ be defined by the pullback diagram:

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{b'} & \overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d) \\ \downarrow & \square & \downarrow \rho \\ \mathfrak{M}_1^{\text{wt, st}} & \xrightarrow{b} & \mathfrak{M}_1^{\text{wt, st}}(1) \end{array} \quad (6.1)$$

Objects of $\mathcal{Z}$ over a scheme S consist of diagrams:

$$\begin{array}{ccccc} C & \xrightarrow{\phi} & \widehat{C} & \xrightarrow{f} & \mathbb{P}^4 \\ & \searrow \pi & \swarrow \hat{\pi} & & \\ & & S & & \end{array}$$

where f is a 1-stable map and ϕ is the weighted 1-stabilisation; arrows over id_S are commutative diagrams:

$$\begin{array}{ccccc} C & \xrightarrow{\phi} & \widehat{C} & \xrightarrow{f} & \mathbb{P}^4 \\ \downarrow \psi & & \downarrow \hat{\psi} & & \downarrow \text{id}_{\mathbb{P}} \\ C' & \xrightarrow{\phi'} & \widehat{C}' & \xrightarrow{f'} & \mathbb{P}^4 \end{array}$$

where ψ and $\hat{\psi}$ are isomorphisms. Recall that $\hat{\psi}$ is determined by ψ due to the required

commutativity with the dominant “birational” morphism ϕ .

Forgetting $\widehat{C}$ and keeping $f \circ \phi: C \rightarrow \mathbb{P}^4$, we obtain a morphism

$$i: \mathcal{Z} \rightarrow \overline{\mathcal{M}}_1(\mathbb{P}^4, d).$$

Lemma 6.1. *The morphism $i: \mathcal{Z} \hookrightarrow \overline{\mathcal{M}}_1(\mathbb{P}^4, d)$ is a closed immersion. In particular $\mathcal{Z}$ is a proper DM stack.*

Proof. From the above description of arrows in $\mathcal{Z}$, i is representable (i.e. faithful) and a monomorphism (i.e. full).

We can check properness using the valuative criterion. We argue as in [RSW17, Theorem 4.3]. Let Δ be a DVR scheme with generic point η ; consider a diagram:

$$\begin{array}{ccccc} \mathcal{C}_\eta & \longrightarrow & \mathcal{C}_\Delta & \xrightarrow{f} & \mathbb{P}^4 \\ \phi_\eta \downarrow & & \phi_\Delta \downarrow & \nearrow & \uparrow \\ \widehat{\mathcal{C}}_\eta & \xrightarrow{j} & \widehat{\mathcal{C}}_\Delta & \xrightarrow{g} & \mathbb{P}^4 \end{array}$$

Notice that there is an open dense substack of $\mathcal{Z}$ where ϕ is an isomorphism. Indeed the generic point of either the main component or any boundary component is already 1-stable. Thus we can assume that ϕ_η in the above diagram is an isomorphism.

Observe that f is constant on the fibers of ϕ_Δ , so it factors topologically through $\widehat{\mathcal{C}}_\Delta$. We can conclude proving that $\phi_* \mathcal{O}_{\mathcal{C}_\Delta} \cong \mathcal{O}_{\widehat{\mathcal{C}}_\Delta}$, which ensures we are in the hypothesis of Debarre factorisation Lemma [Deb13, Lemma 1.15]. To prove the above isomorphism of sheaves, consider the exact sequence:

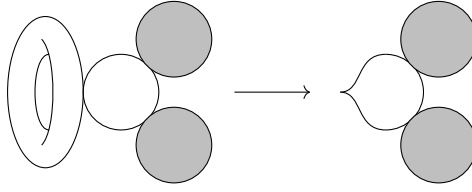
$$0 \rightarrow \mathcal{O}_{\widehat{\mathcal{C}}_\Delta} \rightarrow \phi_* \mathcal{O}_{\mathcal{C}_\Delta} \rightarrow \phi_* \mathcal{O}_{\mathcal{C}_\Delta} / \mathcal{O}_{\widehat{\mathcal{C}}_\Delta} \rightarrow 0$$

Since ϕ is an isomorphism away from the cuspidal point, the cokernel is supported in dimension 0. However $\chi(\mathcal{O}_{\widehat{\mathcal{C}}_\eta}) = \chi(\phi_* \mathcal{O}_{\mathcal{C}_\eta})$ implies the same equality holds on the whole of Δ , since the Euler characteristic is constant in flat families. So $\chi(\phi_* \mathcal{O}_{\mathcal{C}_\Delta} / \mathcal{O}_{\widehat{\mathcal{C}}_\Delta}) = \text{length}(\phi_* \mathcal{O}_{\mathcal{C}_\Delta} / \mathcal{O}_{\widehat{\mathcal{C}}_\Delta}) = 0$. $\square$

So we may add the commutative diagram to the left:

$$\begin{array}{ccccc} \overline{\mathcal{M}}_1(\mathbb{P}^4, d) & \xleftarrow{i} & \mathcal{Z} & \longrightarrow & \overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d) \\ \downarrow & & \downarrow & \square & \downarrow \\ \mathfrak{M}_1^{\text{wt, st}} & \xleftarrow{\sim} & \mathfrak{M} & \longrightarrow & \mathfrak{M}_1^{\text{wt, st}}(1) \end{array}$$

A description of the irreducible components of $\mathcal{Z}$ can be easily obtained from the inclusion i : there is a main component $\mathcal{Z}^{\text{main}}$ which is the closure of the locus of maps from a smooth elliptic curve, and for every $k \geq 2$ a boundary component $D^k \mathcal{Z}$, whose

Figure 6.1: A typical element of $D^1 \cap D^2$.

general point represents a contracted elliptic curve with k many rational tails of positive degree.

Remark 6.2. Each component of $\mathcal{Z}$ is isomorphic to the corresponding one in $\overline{\mathcal{M}}_1(\mathbb{P}^4, d)$. Indeed given any stable map there is at most one factorisation through the weighted 1-stabilisation of the curve and more in details: objects of $\overline{\mathcal{M}}_1(\mathbb{P}^4, d) \setminus D^1$ are 1-stable already; objects of $D^1 \cap \overline{\mathcal{M}}_1(\mathbb{P}^4, d)^{\text{main}}$ factor through the cusp thanks to the smoothability criterion, which implies the node is a ramification point of the map; and objects of $D^1 \cap D^k$ ($k \geq 2$) factor through a map which is constant on the cusp.

On the other hand, objects of $D^{1,\circ} = D^1 \setminus (\overline{\mathcal{M}}_1(\mathbb{P}^4, d)^{\text{main}} \cup \bigcup_{k \geq 2} D^k)$ do not admit any factorisation, so $\mathcal{Z}$ has no corresponding component.

If we replace $\mathbb{P}^4$ with X the quintic 3-fold in (6.1), then by Lemma 6.1 we get a closed sub-stack $\mathcal{Z}_X \hookrightarrow \overline{\mathcal{M}}_1(X, \beta)$ (in the proof we never use the fact that the target is $\mathbb{P}^4$).

Moreover, $\mathcal{Z}_X$ is naturally endowed with a zero-dimensional virtual class $[\mathcal{Z}_X]^{\text{vir}}$ whose degree coincide with the one of $[\overline{\mathcal{M}}_1^{(1)}(X, \beta)]^{\text{vir}}$. Indeed, given the inclusions $\mathcal{C}_{\mathcal{Z}_X/\mathfrak{M}} \hookrightarrow \mathcal{C}_{\widehat{M}(X)/\widehat{\mathfrak{M}}} \hookrightarrow b^* \mathbb{E}_{\rho_X}$, where $\mathbb{E}_{\rho_X}$ is the relative obstruction theory for 1-stable maps to X , we can define $[\mathcal{Z}_X]^{\text{vir}}$ by virtual pull-back of $[\mathfrak{M}]$. Moreover, since the 1-stabilisation map is a proper birational morphism and virtual pull-back commutes with proper push-forward [Man12, Theorem 4.1], $b_*[\mathcal{Z}_X]^{\text{vir}} = [\overline{\mathcal{M}}_1^{(1)}(X, \beta)]^{\text{vir}}$.

So if we were able to understand how the virtual class $[\overline{\mathcal{M}}_1(X, \beta)]^{\text{vir}}$ was supported on the different components of the moduli space, the comparison between GW and cuspidal invariants will be quickly done.

Unfortunately, the geometry of $\overline{\mathcal{M}}_1(X, \beta)$ is hard to understand and neither we can try to study it through the easier space $\overline{\mathcal{M}}_1(\mathbb{P}^4, d)$ as there is not a perfect obstruction theory for $\overline{\mathcal{M}}_1(X, \beta) \hookrightarrow \overline{\mathcal{M}}_1(\mathbb{P}^4, d)$.

Chang-Li theory of maps with p -fields is a beautiful way around these difficulties.

6.2. 1-stable maps with p fields

We adapt Chang-Li's theory of p -fields to cuspidal maps. This is a word-by-word repetition of the arguments in [CL12] once noticed that they carry over to families of at worst cuspidal curves. We provide the non-expert reader with a résumé of some of the key ideas contained in [CL12]; this section can otherwise be skipped.

6.2.1. Moduli of sections

Definition 6.3. Let B be an algebraic stack and let $\pi: \mathcal{C} \rightarrow B$ be a flat proper morphism of finite presentation, which is representable by algebraic spaces, and whose geometric fibers are reduced l.c.i. curves. Let $\mathcal{Z}$ be an algebraic stack, representable, quasi-projective, and smooth over $\mathcal{C}$. The *cone of sections* of $\mathcal{Z}$ over $\mathcal{C}$ is a B -stack $\mathfrak{S}$ defined by:

$$\mathfrak{S}(S \rightarrow B) = \{\text{sections of } \mathcal{Z}_S \rightarrow \mathcal{C}_S\};$$

The groupoid $\mathfrak{S}$ is an algebraic stack representable and quasi-projective over B [CL15, Proposition 2.3].

To fix the notation let us draw the following diagram:

$$\begin{array}{ccc} & & \mathcal{Z} \\ & \nearrow \epsilon & \downarrow \\ \mathcal{C}_{\mathfrak{S}} & \longrightarrow & \mathcal{C} \\ \downarrow \pi_{\mathfrak{S}} & & \downarrow \pi \\ \mathfrak{S} & \longrightarrow & B \end{array} \quad (6.2)$$

where ϵ denotes the universal section. Let us see some examples of the above construction we will be interested in.

Direct image cones

When $\mathcal{Z} = \text{Vb}(\mathcal{L})$ for a line bundle $\mathcal{L}$ on $\mathcal{C}$, the algebraic stack $\mathfrak{S} \rightarrow B$ representing sections of $\mathcal{L}$ can be constructed as:

$$C(\pi_* \mathcal{L}) := \text{Spec}_B \text{Sym}^\bullet (R^1 \pi_* \mathcal{L}^\vee \otimes \omega_\pi).$$

This is essentially because $R^1 \pi_* \mathcal{L}^\vee \otimes \omega_\pi$ commutes with pullbacks, and it has the desired modular interpretation by Serre duality.

Moduli of stable maps and 1-stable maps

Recall that $\mathfrak{p}$ denotes the universal Picard stack. Let $X \subseteq \mathbb{P}^r$ be a projective variety and let $\mathcal{Z}_X$ be defined by the following cartesian diagram over $\mathcal{C}_{\mathfrak{p}}$:

$$\begin{array}{ccc} \mathcal{Z}_X & \xrightarrow{/\mathbb{G}_m} & X \times \mathcal{C}_{\mathfrak{p}} \\ \downarrow & \square & \downarrow \\ \mathcal{L}^{\oplus r+1} \setminus 0_{\mathcal{C}} & \xrightarrow{/\mathbb{G}_m} & \mathbb{P}^r \times \mathcal{C}_{\mathfrak{p}} \end{array}$$

Then $\overline{\mathcal{M}}_1(X, \beta)$ is the open substack of the moduli space $\mathfrak{S}_X$ of sections of $\mathcal{Z}_X \rightarrow \mathcal{C}_{\mathfrak{p}}$ defined by the stability condition. This is the point of view we have already taken for

$X = \mathbb{P}^n$ in Proposition 4.3 to describe the local equations of the moduli space of maps.

Analogously the moduli space of 1-stable maps $\overline{\mathcal{M}}_1^{(1)}(X, \beta)$ can be thought of as an open inside the moduli space of sections of

$$\widehat{\mathcal{X}}_X = (\mathrm{Vb}(\widehat{\mathcal{L}}_{\widehat{\mathbb{P}}}^{\oplus r+1}) \setminus 0_{\widehat{\mathcal{C}}}) \times_{\mathbb{P}^r} X$$

over $\widehat{\mathcal{C}}_{\widehat{\mathbb{P}}}$.

Moduli of 1-stable maps with p -fields

In this section we denote by $\widehat{\mathcal{M}}$ the moduli space of 1-stable maps $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)$ and by $\widehat{\mathcal{P}}_{\widehat{\mathcal{M}}} = \widehat{\mathcal{L}}_{\widehat{\mathcal{M}}}^{\otimes -5} \otimes \omega_{\widehat{\pi}_{\widehat{\mathcal{M}}}}$.

The moduli space of p -fields is defined as the cone of sections of the line bundle $\widehat{\mathcal{P}}_{\widehat{\mathcal{M}}}$ over $\widehat{\mathcal{C}}_{\widehat{\mathcal{M}}}$:

Definition 6.4. The moduli space of 1-stable maps with p -fields:

$$\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p := C(\widehat{\pi}_{\widehat{\mathcal{M}},*}(\widehat{\mathcal{P}}_{\widehat{\mathcal{M}}}))$$

parametrises 1-stable maps:

$$\begin{array}{ccc} \widehat{\mathcal{C}}_S & \xrightarrow{\hat{f}_S} & \mathbb{P}^4 \\ \downarrow \hat{\pi}_S & & \\ S & & \end{array}$$

with a p -field $\hat{\psi} \in H^0(\widehat{\mathcal{C}}_S, \hat{f}_S^* \mathcal{O}_{\mathbb{P}^4}(-5) \otimes \omega_{\hat{\pi}_S})$.

We use the abbreviation $\widehat{\mathcal{P}} := \overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p$. Employing the above description of the moduli space of 1-stable maps, $\widehat{\mathcal{P}}$ can also be thought of as an open in the moduli space of sections of the vector bundle $\mathrm{Vb}(\widehat{\mathcal{L}}_{\widehat{\mathbb{P}}}^{\oplus 5} \oplus \widehat{\mathcal{P}}_{\widehat{\mathbb{P}}})$.

6.2.2. Obstruction theories

With the notation introduced in section 6.2.1 (in particular see diagram (6.2)), the morphism $\mathfrak{S} \rightarrow B$ admits a relative dual perfect obstruction theory:

$$\phi_{\mathfrak{S}/B}: \mathbb{T}_{\mathfrak{S}/B} \rightarrow \mathbb{E}_{\mathfrak{S}/B} := R^\bullet \pi_{\mathfrak{S},*} \mathfrak{e}^* T_{\mathcal{X}/\mathcal{C}}$$

For the proof see [CL12, Proposition 2.5] and notice that it relies on general properties of obstruction theories and the cotangent complex, and the fact that $\mathcal{X} \rightarrow \mathcal{C}$ is smooth, but never on the specification that $\mathcal{C} \rightarrow B$ is a family of *nodal* curves.

Let us review the examples above:

- for the direct image cone of a line bundle $\mathcal{L}$ the dual obstruction theory is $\mathbb{E}_{C(\pi_* \mathcal{L})/B} = R^\bullet \pi_* \mathcal{L}$;

- the moduli space $\widehat{M} = \overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)$ has a dual obstruction theory relative to $\widehat{\mathfrak{p}}$ given by $\mathbb{E}_{\widehat{M}/\widehat{\mathfrak{p}}} = R^\bullet \hat{\pi}_{\widehat{M},*}(\bigoplus_0^4 \widehat{\mathcal{L}}_{\widehat{M}})$;
- in the case of 1-stable maps with p -fields $\widehat{\mathcal{P}} = \overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p$, we get $\mathbb{E}_{\widehat{\mathcal{P}}/\widehat{\mathfrak{p}}} = R^\bullet \hat{\pi}_{\widehat{\mathcal{P}},*}(\widehat{\mathcal{L}}_{\widehat{\mathcal{P}}}^{\oplus 5} \oplus \widehat{\mathcal{P}}_{\widehat{\mathcal{P}}})$.

We review the compatibility of various obstruction theories for the moduli spaces mentioned above.

Lemma 6.5. *$\widehat{\mathfrak{p}}$ is a smooth Artin stack of dimension 0. Furthermore there is a compatible triple of dual perfect obstruction theories:*

$$\begin{array}{ccccccc}
 \hat{\lambda}^* \mathbb{T}_{\widehat{\mathfrak{p}}/\widehat{\mathfrak{m}}}[-1] & \longrightarrow & \mathbb{T}_{\widehat{M}/\widehat{\mathfrak{p}}} & \longrightarrow & \mathbb{T}_{\widehat{M}/\widehat{\mathfrak{m}}} & \xrightarrow{[1]} & \longrightarrow \\
 \downarrow \wr & & \downarrow & & \downarrow & & \\
 R^\bullet \hat{\pi}_{\widehat{M},*}(\mathcal{O}_{\widehat{\mathcal{C}}}) & \longrightarrow & R^\bullet \hat{\pi}_{\widehat{M},*}(\bigoplus_0^4 \widehat{\mathcal{L}}_{\widehat{M}}) & \longrightarrow & R^\bullet \hat{\pi}_{\widehat{M},*}(f_{\widehat{M}}^* T_{\mathbb{P}^4}) & \xrightarrow{[1]} & \longrightarrow
 \end{array}$$

implying that $\mathbb{E}_{\widehat{M}/\widehat{\mathfrak{p}}} = R^\bullet \hat{\pi}_{\widehat{M},*}(\bigoplus_0^4 \widehat{\mathcal{L}}_{\widehat{M}})$ gives the standard Behrend-Fantechi virtual class on $\widehat{M}$. We denoted by $\hat{\lambda}$ the natural forgetful map from $\widehat{M} \rightarrow \widehat{\mathfrak{p}}$.

Proof. The first statement follows from deformation theory: the projection $\hat{\rho}: \widehat{\mathfrak{p}} \rightarrow \widehat{\mathfrak{m}}$ is unobstructed of relative dimension 0 and $\widehat{\mathfrak{m}}$ is a smooth Artin stack of dimension 0, since both nodal and cuspidal singularities are l.c.i., so obstructions to their deformations are contained in $\text{Ext}_{\mathcal{O}_{\widehat{\mathcal{C}}}}^2(\Omega_{\widehat{\mathcal{C}}}, \mathcal{O}_{\widehat{\mathcal{C}}}) = 0$.

The fact that $\mathbb{T}_{\widehat{M}/\widehat{\mathfrak{m}}} \rightarrow \mathbb{E}_{\widehat{M}/\widehat{\mathfrak{m}}} := R^\bullet \hat{\pi}_{\widehat{M},*}(f_{\widehat{M}}^* T_{\mathbb{P}^4})$ is a perfect obstruction theory when $\widehat{\mathcal{C}} \rightarrow \widehat{M}$ is a family of Gorenstein curves is proved in [BF97, Proposition 6.3].

The lower row in the above diagram is induced by the Euler sequence of $\mathbb{P}^4$. The middle column comes from identifying the space of stable maps as an open substack of the cone of sections (see above) of $\text{Vb}(\bigoplus_0^4 \widehat{\mathcal{L}})$ over $\widehat{\mathfrak{p}}$. The existence of such a commutative diagram is [CL12, Lemma 2.8]. The final claim follows from functoriality of virtual pullbacks [Man12]. $\square$

Lemma 6.6. *There is a compatible triple of dual perfect obstruction theories*

$$(R^\bullet \hat{\pi}_{\widehat{\mathcal{P}},*}(\widehat{\mathcal{P}}_{\widehat{\mathcal{P}}}), R^\bullet \hat{\pi}_{\widehat{\mathcal{P}},*}(\widehat{\mathcal{L}}_{\widehat{\mathcal{P}}}^{\oplus 5} \oplus \widehat{\mathcal{P}}_{\widehat{\mathcal{P}}}), R^\bullet \hat{\pi}_{\widehat{\mathcal{P}},*}(\widehat{\mathcal{L}}_{\widehat{\mathcal{P}}}^{\oplus 5}))$$

for the triangle:

$$\begin{array}{ccc}
 \widehat{\mathcal{P}} & \xrightarrow{\hat{\rho}} & \widehat{M} \\
 & \searrow & \swarrow \\
 & \widehat{\mathfrak{p}} &
 \end{array}$$

Notice that the virtual rank of $\mathbb{E}_{\widehat{\mathcal{P}}/\widehat{\mathfrak{p}}} := R^\bullet \hat{\pi}_{\widehat{\mathcal{P}},*}(\widehat{\mathcal{L}}_{\widehat{\mathcal{P}}}^{\oplus 5} \oplus \widehat{\mathcal{P}}_{\widehat{\mathcal{P}}})$ is 0, hence it endows the moduli space of 1-stable maps with p -fields with a cycle class of dimension 0. However

$\widehat{\mathcal{P}}$ is not proper. In the next section we describe Chang-Li's cosection (depending on the choice of $\mathbf{w} \in \mathbf{k}[x_0, \dots, x_4]_5$) of the obstruction bundle, and show that the corresponding localised cycle is supported on the proper substack $\overline{\mathcal{M}}_1^{(1)}(X, d)$, where $X = V(\mathbf{w})$ is the quintic threefold, that is the degeneracy locus of the cosection.

6.2.3. Cosection localisation and virtual pullback

Recall Kiem-Li's machinery of cosection localised virtual classes [KL13, Theorem 1.1]:

Theorem 6.7 (Localisation by cosection). *Let $Y \rightarrow S$ be a morphism of DM type between algebraic stacks, with Y Deligne-Mumford and S smooth, endowed with a perfect obstruction theory. Suppose the obstruction sheaf Ob_Y admits a surjective homomorphism $\sigma : \text{Ob}_Y|_U \rightarrow \mathcal{O}_U$ over an open $U \subseteq Y$.*

Let $\iota : Y(\sigma) := Y \setminus U \hookrightarrow Y$, then (Y, σ) has a localised virtual cycle:

$$[Y]_{\text{loc}}^{\text{vir}} \in A_* Y(\sigma).$$

This cycle enjoys the usual properties of the virtual cycles; it relates to the usual virtual cycle $[Y]^{\text{vir}}$ via $[Y]^{\text{vir}} = \iota_* [Y]_{\text{loc}}^{\text{vir}} \in A_* Y$.

We now review a slight generalisation of this, that combines it with Manolache's virtual pullback construction [Man12]; this can also be found in [CL12, § 5].

Let $Y \rightarrow S$ be as in the hypotheses of the theorem, or more generally such that there is a triple of compatible obstruction theories for a triangle $Y \rightarrow S \rightarrow T$ with T smooth. Assume for simplicity that the obstruction theory $\mathbb{E}_{Y/S}$ is a vector bundle E admitting a cosection:

$$E \xrightarrow{\sigma} \mathcal{O}_Y$$

surjective on an open $U \subseteq Y$; denote by $Y(\sigma)$ the complement $Y \setminus U$. Recall the following notation from [KL13]:

$$G = \text{Ker}(E|_U \rightarrow \mathcal{O}_Y|_U), \quad E(\sigma) = E|_{Y(\sigma)} \cup G.$$

Kiem and Li define a localised Gysin map:

$$s_{\sigma, \text{loc}}^! : A_*(E(\sigma)) \rightarrow A_*(Y(\sigma)),$$

which we explain in what follows. Let $[B] \in Z_*(E(\sigma))$ be a cycle represented by a closed integral substack. If $B \subset E|_{Y(\sigma)}$ then they use the standard Gysin morphism:

$$s_{\sigma, \text{loc}}^![B] := s_{E|_{Y(\sigma)}}^![B] \in A_*(Y(\sigma)).$$

Suppose instead that B is not contained in $E|_{Y(\sigma)}$. Then we may choose a variety with a regularising morphism $\widetilde{Y} \xrightarrow{\nu} Y$ such that:

- ν is proper and $\nu(\widetilde{Y}) \cap U \neq \emptyset$;
- the pullback along ν of the cosection $\nu^*\sigma =: \tilde{\sigma}$ extends to a surjective morphism

$$\nu^*E \xrightarrow{\tilde{\sigma}} \mathcal{O}_{\widetilde{Y}}(D) \rightarrow 0$$

for a Cartier divisor $D \subseteq \widetilde{Y}$.

- there is a closed integral $\widetilde{B} \subset \widetilde{G} = \text{Ker}(\tilde{\sigma})$ such that $\tilde{\nu}_*[\widetilde{B}] = k[B]$ for some $k \in \mathbb{N}$.

We denote by $\nu(\sigma): D \rightarrow Y(\sigma)$ the restriction of ν to the divisor, and by $\tilde{\nu}: \widetilde{G} \rightarrow E(\sigma)$ the restriction of the natural map $\nu^*E \rightarrow E$. The localised Gysin pullback is then defined as:

$$s_{\sigma, \text{loc}}^![B] := \frac{1}{k} \nu(\sigma)_*([D] \cdot s_{\widetilde{G}}^![\widetilde{B}]) \in A_*(Y(\sigma)).$$

In [KL13, § 2] they prove that: such $(\widetilde{Y}, \nu, \widetilde{B})$ always exist; the cycle $s_{\sigma, \text{loc}}^![B]$ is independent of the above choices; the construction preserves rational equivalences and thus defines the desired morphism on Chow groups.

Remark 6.8. If we consider $[Y]$ as a cycle in $E(\sigma)$, thought of as the zero section of E , notice that a natural choice for $\widetilde{Y}$ is $\text{Bl}_{Y(\sigma)} Y$.

Kiem and Li then proceed to extend the cosection-localised Gysin pullback to vector bundle stacks $h^1/h^0(\mathbb{E})$ (this is relatively easy in the presence of a global resolution, which is the case in the situation at hand); they show that the intrinsic normal cone $\mathfrak{C}_{Y/S}$ is contained in the closed substack $h^1/h^0(\mathbb{E})(\sigma)$, at least when the cosection of the relative obstruction bundle lifts to a cosection of the absolute obstruction bundle [KL13, Corollary 4.5].

Definition 6.9. Consider a cartesian diagram of stacks:

$$\begin{array}{ccc} Y' & \xrightarrow{\varphi'} & Y \\ \downarrow \rho' & \square & \downarrow \rho \\ S' & \xrightarrow{\varphi} & S \end{array}$$

with ρ as above. Assume furthermore that we have a cosection

$$\text{Ob}_{Y/S} \rightarrow \mathcal{O}_Y$$

that lifts to a cosection of the absolute obstruction sheaf Ob_Y . Then we can define a localised virtual pullback operation:

$$(\rho')_{\mathbb{E}_{Y'/S'}, \sigma'}^!: A_*(S') \rightarrow A_{*- \text{rk}(\mathbb{E})}(Y'(\sigma'))$$

where $\mathbb{E}_{Y'/S'} = (\varphi')^* \mathbb{E}_{Y/S}$, $\sigma' = (\varphi')^*(\sigma)$ and $Y'(\sigma') = Y(\sigma) \times_Y Y'$.

Replace S' by a primitive cycle in it, i.e. assume that S' is irreducible and reduced. Recall that pulling back $\mathbb{E}_{Y/S}$ along φ' endows ρ' with a relative perfect obstruction theory. Furthermore we can pullback the cosection:

$$\mathrm{Ob}_{Y'/S'} \cong \varphi'^* \mathrm{Ob}_{Y/S} \xrightarrow{\sigma'} \mathcal{O}_{Y'}.$$

The degeneracy locus of σ' is $Y'(\sigma')$, which in turn implies that:

$$h^1/h^0(\mathbb{E}_{Y'/S'})(\sigma') = (\varphi')^* h^1/h^0(\mathbb{E}_{Y/S})(\sigma).$$

By [KL13, Corollary 4.5] and from the cartesian diagram above,

$$\mathbb{C}_{Y'/S'} \subseteq (\varphi')^* \mathbb{C}_{Y/S} \subseteq \mathbb{E}_{Y'/S'}(\sigma')$$

thus we can define the localised virtual pullback of $[S']$ to be:

$$(\rho')^!_{\mathbb{E}_{Y'/S'}, \sigma'} [S'] = s^!_{\sigma', \mathrm{loc}} [\mathbb{C}_{Y'/S'}].$$

6.2.4. A cosection for p -fields

We are going to construct the cosection paralleling [CL12, §§3.2-3.4]. There is a morphism of vector bundles on $\widehat{\mathfrak{p}}$ induced by iterated tensoring of line bundles:

$$h_1: \mathrm{Vb}(\widehat{\mathcal{L}}_{\widehat{\mathfrak{p}}}^{\oplus 5} \oplus \widehat{\mathcal{P}}_{\widehat{\mathfrak{p}}}) \rightarrow \mathrm{Vb}(\omega_{\widehat{\pi}_{\widehat{\mathfrak{p}}}}), \quad h_1(x, p) = p\mathbf{w}(x_0, \dots, x_4)$$

where x_i and p denotes the tautological sections of the pull-back of $\widehat{\mathcal{L}}_{\widehat{\mathfrak{p}}}^{\oplus 5}$, and $\widehat{\mathcal{P}}_{\widehat{\mathfrak{p}}}$ to the vector bundle, and $\mathbf{w}$ is the fixed degree 5 polynomial defining the quintic 3-fold X .

Differentiating h_1 gives a morphism of tangent sheaves, and thus pulling it back along the universal evaluation

$$\begin{array}{ccc} & & \mathrm{Vb}(\widehat{\mathcal{L}}_{\widehat{\mathfrak{p}}}^{\oplus 5}) \setminus \{0\} \oplus \mathrm{Vb}(\widehat{\mathcal{P}}_{\widehat{\mathfrak{p}}}) \\ & \nearrow \epsilon & \downarrow \\ \widehat{\mathcal{E}}_{\widehat{\mathcal{P}}} & \longrightarrow & \widehat{\mathcal{E}}_{\widehat{\mathfrak{p}}} \\ \downarrow \widehat{\pi}_{\widehat{\mathcal{P}}} & & \downarrow \widehat{\pi}_{\widehat{\mathfrak{p}}} \\ \widehat{\mathcal{P}} & \longrightarrow & \widehat{\mathfrak{p}} \end{array}$$

we obtain a cosection of the relative obstruction sheaf (see § 6.2.2):

$$\begin{aligned} \sigma_1 : \text{Ob}_{\widehat{\mathcal{P}}/\widehat{\mathcal{P}}} &= R^1 \hat{\pi}_{\widehat{\mathcal{P}},*}(\widehat{\mathcal{L}}_{\widehat{\mathcal{P}}}^{\oplus 5} \oplus \widehat{\mathcal{P}}_{\widehat{\mathcal{P}}}) \rightarrow R^1 \hat{\pi}_{\widehat{\mathcal{P}},*}(\omega_{\widehat{\mathcal{P}}}) \simeq \mathcal{O}_{\widehat{\mathcal{P}}} \\ \sigma_1|_{(u,\psi)}(\dot{x}, \dot{p}) &= \dot{p}\mathbf{w}(u) + \psi \sum_{i=0}^4 \partial_i \mathbf{w}(u) \dot{x}_i, \quad \text{where} \\ [(u, \psi)] &\in \widehat{\mathcal{P}}, \quad (\dot{x}, \dot{p}) \in H^1(u^* \widehat{\mathcal{L}}_{\widehat{\mathcal{P}}}^{\oplus 5}) \oplus H^1(u^* \widehat{\mathcal{P}}_{\widehat{\mathcal{P}}}) \end{aligned} \quad (6.3)$$

The degeneracy locus of this cosection is $\overline{\mathcal{M}}_1^{(1)}(X, d)$: indeed, by Serre duality if $\mathbf{w}(u) \neq 0$ then we can find a p such that the cosection does not vanish; similarly we can do if $\psi \sum_{i=0}^4 \partial_i \mathbf{w}(u) \neq 0$. But $\mathbf{w}(u) = 0$ and $\partial_i \mathbf{w}(u) = 0$ never happen simultaneously by smoothness of X , so it has to be $\mathbf{w}(u) = \psi = 0$.

Moreover Chang and Li prove ([CL12, Proposition 3.5]) that σ_1 lifts to a cosection of the absolute obstruction bundle $\text{Ob}_{\widehat{\mathcal{P}}} \xrightarrow{\sigma} \mathcal{O}_{\widehat{\mathcal{P}}}$, i.e. σ_1 factors as $\text{Ob}_{\widehat{\mathcal{P}}/\widehat{\mathcal{P}}} \rightarrow \text{Ob}_{\widehat{\mathcal{P}}} \xrightarrow{\sigma} \mathcal{O}_{\widehat{\mathcal{P}}}$. Notice that σ and σ_1 have the same degeneracy locus because $\text{Ob}_{\widehat{\mathcal{P}}/\widehat{\mathcal{P}}} \rightarrow \text{Ob}_{\widehat{\mathcal{P}}}$ is surjective.

Being able to lift the cosection to the absolute obstruction sheaf is a sufficient condition for the relative intrinsic normal cone $\mathcal{C}_{\widehat{\mathcal{P}}/\widehat{\mathcal{P}}}$ to be contained in the closed substack $h^1/h^0(\mathbb{E})(\sigma_1)$ of the obstruction bundle $h^1/h^0(\mathbb{E})$ (determined by the cosection, see [KL13]).

We may thus endow $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p$ with a localised virtual cycle:

$$[\widehat{\mathcal{P}}]_{\text{loc}}^{\text{vir}} = 0_{\sigma_1, \text{loc}}^! [\mathcal{C}_{\widehat{\mathcal{P}}/\widehat{\mathcal{P}}}] \in A_0\left(\overline{\mathcal{M}}_1^{(1)}(X, d)\right).$$

We want to show that it gives the same numerical invariants as the cuspidal Gromov-Witten theory of X , up to a sign:

Theorem 6.10.

$$\deg[\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p]_{\text{loc}}^{\text{vir}} = (-1)^{5d} \deg[\overline{\mathcal{M}}_1^{(1)}(X, d)]^{\text{vir}}$$

6.2.5. From p -fields to the quintic threefold

This is achieved in two steps: we first compare the invariants of $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p$ with the ones coming from $\overline{\mathcal{M}}_1^{(1)}(N_{X/\mathbb{P}^4}, d)^p$, where $N_{X/\mathbb{P}^4}$ is the normal bundle of the quintic in $\mathbb{P}^4$.

Second we compare the latter invariants with $\deg[\overline{\mathcal{M}}_1^{(1)}(X, d)]^{\text{vir}}$.

Deformation to the normal cone

This is attained in [CL12, §§4-5] by a family version of the p -fields construction applied to the deformation to the normal cone of $X \subseteq \mathbb{P}^4$; let us denote the latter by $V \rightarrow \mathbb{A}_t^1$, so that $V_{t \neq 0} = \mathbb{P}^4$ and $V_0 = N_{X/\mathbb{P}^4}$. The idea is to proceed as follows: construct a perfect obstruction theory for $\overline{\mathcal{M}}_1^{(1)}(V, d)^p$ relative to $\widehat{\mathcal{P}}$ which is compatible with the obstruction theories of the fibers $\overline{\mathcal{M}}_1^{(1)}(V_t, d)^p$; then construct a family version of the cosection and

use the fact that for compatible obstruction theories taking cosection localized cycles commutes with Gysin pull-back to conclude the deformation to the normal cone.

In order to construct the obstruction theory for $\overline{\mathcal{M}}_1^{(1)}(V, d)^p$, we want to express it as an open inside a cone of sections (then we can use the general result in § 6.2.2), to such goal we will find useful the following elementary Lemma:

Lemma 6.11. *The deformation to the normal cone V is cut inside $\text{Vb}(\mathcal{O}_{\mathbb{P}^4}(5)) \times \mathbb{A}_t^1$ by the equation $\mathbf{w}(x) - ty = 0$, where $[x_0 : \dots : x_4]$ are the basis coordinates and y is fiber coordinate. If $C(V)$ denotes the affine cone over V , then $C(V)$ is cut out inside $\mathbb{C}_x^5 \times \mathbb{C}_y \times \mathbb{A}_t^1$ by the same equation, and its tangent bundle is determined by the following exact sequences:*

$$0 \rightarrow T_{C(V)/\mathbb{A}_t^1} \rightarrow \mathcal{O}_{C(V)}^{\oplus 5} \oplus \mathcal{O}_{C(V)} \xrightarrow{\sum_i \partial_i \mathbf{w}(x) x_i - t y} \mathcal{O}_{C(V)} \rightarrow 0 \quad (6.4)$$

$$0 \rightarrow T_{C(V)} \rightarrow \mathcal{O}_{C(V)}^{\oplus 5} \oplus \mathcal{O}_{C(V)} \oplus \mathcal{O}_{C(V)} \xrightarrow{\sum_i \partial_i \mathbf{w}(x) x_i - t y - y t} \mathcal{O}_{C(V)} \rightarrow 0 \quad (6.5)$$

The proof is just a computation in local coordinates.

This allows a description of the moduli space of maps to V as the cone of sections of a certain smooth object $\mathcal{Z}'$ over $\widehat{\mathcal{C}}_{\widehat{\mathbb{P}} \times \mathbb{A}_t^1}$:

$$\begin{array}{ccc}
 & \mathcal{Z}' & \xrightarrow{\quad} V \\
 & \downarrow & \square \\
 & \text{Vb}(\mathcal{L}_{\widehat{\mathbb{P}}}^{\oplus 5}) \setminus \{0\} \oplus \text{Vb}(\mathcal{L}_{\widehat{\mathbb{P}}}^{\otimes 5}) & \longrightarrow \text{Vb}(\mathcal{O}_{\mathbb{P}^4}(5)) \times \mathbb{A}_t^1 \\
 & \downarrow & \\
 \widehat{\mathcal{C}}_{\overline{\mathcal{M}}_1^{(1)}(V)} & \xrightarrow{\quad} & \widehat{\mathcal{C}}_{\widehat{\mathbb{P}} \times \mathbb{A}_t^1} \\
 \downarrow \hat{\pi}_{\overline{\mathcal{M}}_1^{(1)}(V)} & & \downarrow \hat{\pi}_{\widehat{\mathbb{P}} \times \mathbb{A}_t^1} \\
 \overline{\mathcal{M}}_1^{(1)}(V, (d, 0)) & \xrightarrow{\quad} & \widehat{\mathbb{P}} \times \mathbb{A}_t^1
 \end{array} \quad (6.6)$$

Similarly $\widehat{\mathcal{V}} := \overline{\mathcal{M}}_1^{(1)}(V, (d, 0))^p$ can be defined as the cone of sections of $\mathcal{Z} := \mathcal{Z}' \oplus \text{Vb}(\widehat{\mathcal{P}}_{\widehat{\mathbb{P}}})$. The general theory explained above tells us that $R^\bullet \hat{\pi}_* e^* T_{\mathcal{Z}/\widehat{\mathbb{P}} \times \mathbb{A}_t^1}$ provides an obstruction theory for $\widehat{\mathcal{V}} \rightarrow \widehat{\mathbb{P}} \times \mathbb{A}_t^1$. Moreover the fact that $\mathcal{Z}'$ is defined via the cartesian square (6.6) together with Lemma 6.11 allows us to give a more explicit description of the above perfect obstruction theory ([CL12, Proposition 4.2]):

Lemma 6.12. *A dual perfect obstruction theory is given by:*

$$\phi_{\widehat{\mathcal{V}}/\widehat{\mathbb{P}} \times \mathbb{A}_t^1} : \mathbb{T}_{\widehat{\mathcal{V}}/\widehat{\mathbb{P}} \times \mathbb{A}_t^1} \rightarrow \mathbb{E}_{\widehat{\mathcal{V}}/\widehat{\mathbb{P}} \times \mathbb{A}_t^1} := R^\bullet \hat{\pi}_{\widehat{\mathcal{V}}}^* (f_{\widehat{\mathcal{V}}}^* \mathcal{H} \oplus \widehat{\mathcal{P}}_{\widehat{\mathcal{V}}})$$

where $f_{\widehat{\mathcal{V}}} : \widehat{\mathcal{C}}_{\widehat{\mathcal{V}}} \rightarrow V$ is the universal map and $\mathcal{H}$ is the vector bundle on V defined by the exact

sequence, see (6.4):

$$0 \rightarrow \mathcal{H} \rightarrow \mathrm{pr}_{\mathbb{P}^4}^* \left(\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 5} \oplus \mathcal{O}_{\mathbb{P}^4}(5) \right) \xrightarrow{\sum_i \partial_i \mathbf{w}(x) \hat{x}_i - t \hat{y}} \mathrm{pr}_{\mathbb{P}^4}^* \mathcal{O}_{\mathbb{P}^4}(5) \rightarrow 0.$$

The restriction of $\phi_{\widehat{\mathcal{V}}/\widehat{\mathfrak{p}} \times \mathbb{A}_t^1}$ to a fiber

$$\widehat{\mathcal{V}}_t = \begin{cases} \widehat{\mathcal{P}} & t \neq 0 \\ \overline{\mathcal{M}}_1^{(1)}(N_{X/\mathbb{P}^4}, d)^p & t = 0 \end{cases}$$

gives the standard obstruction theory of $\widehat{\mathcal{V}}_t \rightarrow \widehat{\mathfrak{p}}$.

We would like to conclude that the restriction of the virtual cycle to the fibers is the standard virtual cycle on the fiber. The techniques of functoriality in intersection theory teach us that we should look for a triple of compatible obstruction theories for the triangle:

$$\begin{array}{ccc} \widehat{\mathcal{V}}_t & \xrightarrow{\iota_t} & \widehat{\mathcal{V}} \\ & \searrow & \swarrow \\ & \widehat{\mathfrak{p}} & \end{array}$$

The cone of sections interpretation provides us with an obstruction theory relative to $\widehat{\mathcal{V}} \rightarrow \widehat{\mathfrak{p}}$ given by:

$$\mathbb{E}'_{\widehat{\mathcal{V}}/\widehat{\mathfrak{p}}} := R^\bullet \hat{\pi}_{\widehat{\mathcal{V}}}^* (f_{\widehat{\mathcal{V}}}^* \mathcal{H} \oplus \widehat{\mathcal{P}}_{\widehat{\mathcal{V}}})$$

where $\mathcal{H}$ is determined by the following exact sequence on V , see (6.5):

$$0 \rightarrow \mathcal{H} \rightarrow \mathrm{pr}_{\mathbb{P}^4}^* \left(\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 5} \oplus \mathcal{O}_{\mathbb{P}^4}(5) \oplus \mathcal{O}_{\mathbb{P}^4} \right) \xrightarrow{\sum_i \partial_i \mathbf{w}(x) \hat{x}_i - t \hat{y} - y \hat{t}} \mathrm{pr}_{\mathbb{P}^4}^* \mathcal{O}_{\mathbb{P}^4}(5) \rightarrow 0$$

Of course $h^0(\mathbb{E}'_{\widehat{\mathcal{V}}/\widehat{\mathfrak{p}}}) \simeq \mathbb{T}_{\widehat{\mathcal{V}}/\widehat{\mathfrak{p}}}$, but the previous lemma and the difference between $\mathcal{H}$ and $\mathcal{H}$ hint at the fact that the obstruction sheaf $h^1(\mathbb{E}'_{\widehat{\mathcal{V}}/\widehat{\mathfrak{p}}})$ contains one factor $R^1 \hat{\pi}_{\widehat{\mathcal{V},*}} \mathcal{O}_{\widehat{\mathcal{E}}_{\widehat{\mathcal{V}}}} \simeq \mathcal{O}_{\widehat{\mathcal{V}}}$ too many, so that restricting $\mathbb{E}'_{\widehat{\mathcal{V}}/\widehat{\mathfrak{p}}}$ to the fibers we would not find their standard obstruction theory. A confirmation of this fact is given by observing that $\mathbb{E}'_{\widehat{\mathcal{V}}/\widehat{\mathfrak{p}}}$ equips $\widehat{\mathcal{V}}$ with a 0-dimensional cycle, while we are looking for a 1-dimensional cycle such that restricting to any fiber, i.e. applying $\iota_t^!$, we get $[\widehat{\mathcal{V}}_t]^{\mathrm{vir}} \in A_0(\widehat{\mathcal{V}}_t)$.

This issue is solved in [CL12, §§4.5-6] by lifting the standard obstruction theory $\phi'_{\widehat{\mathcal{V}}/\widehat{\mathfrak{p}}}$ to a different, specifically tailored one:

$$\phi_{\widehat{\mathcal{V}}/\widehat{\mathfrak{p}}}: \mathbb{T}_{\widehat{\mathcal{V}}/\widehat{\mathfrak{p}}} \rightarrow \mathbb{E}_{\widehat{\mathcal{V}}/\widehat{\mathfrak{p}}},$$

where the latter two-term complex fits into an exact triangle:

$$R^1 \hat{\pi}_{\widehat{\mathcal{V},*}} \mathcal{O}_{\widehat{\mathcal{E}}_{\widehat{\mathcal{V}}}}[-2] \rightarrow \mathbb{E}_{\widehat{\mathcal{V}}/\widehat{\mathfrak{p}}} \xrightarrow{\nu} \mathbb{E}'_{\widehat{\mathcal{V}}/\widehat{\mathfrak{p}}} \xrightarrow{[1]}$$

Furthermore $\phi_{\widehat{\mathcal{V}}/\widehat{\mathfrak{p}}}$ is compatible with the standard obstruction theory for the fibers:

Lemma 6.13. *For every $t \in \mathbb{A}_k^1$ we have a commutative diagram:*

$$\begin{array}{ccccc} \hat{\pi}_{\widehat{\mathcal{V}}_c} \mathcal{O}_{\widehat{\mathcal{E}}_{\widehat{\mathcal{V}}_c}}[-1] & \longrightarrow & \mathbb{E}_{\widehat{\mathcal{V}}_c/\widehat{\mathcal{P}}} & \longrightarrow & \mathbb{E}_{\widehat{\mathcal{V}}/\widehat{\mathcal{P}}}|_{\widehat{\mathcal{V}}_c} \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{T}_{\widehat{\mathcal{V}}_c/\widehat{\mathcal{V}}}^{\leq 1} & \longrightarrow & \mathbb{T}_{\widehat{\mathcal{V}}_c/\widehat{\mathcal{P}}}^{\leq 1} & \longrightarrow & \mathbb{T}_{\widehat{\mathcal{V}}/\widehat{\mathcal{P}}}|_{\widehat{\mathcal{V}}_c}^{\leq 1} \end{array}$$

For the proof see [CL15, § 4.6]. Notice that the lower row is not a distinguished triangle, yet Chang and Li prove in [CL12, A.4] that this weaker (truncated) notion of compatibility is enough to prove functoriality.

Since we are working with cosection localised cycles, we need a family version of the cosection. This is induced by differentiating the following vector bundle morphism on $\widehat{\mathcal{P}} \times \mathbb{A}_t^1$:

$$\mathrm{Vb}(\mathcal{L}_{\widehat{\mathcal{P}}}^{\oplus 5} \oplus \mathcal{L}_{\widehat{\mathcal{P}}}^{\otimes 5} \oplus \widehat{\mathcal{P}}_{\widehat{\mathcal{P}}}) \xrightarrow{(\mathrm{pr}_2, \mathrm{pr}_3)} \mathrm{Vb}(\mathcal{L}_{\widehat{\mathcal{P}}}^{\otimes 5} \oplus \widehat{\mathcal{P}}_{\widehat{\mathcal{P}}}) \rightarrow \mathrm{Vb}(\omega_{\widehat{\pi}_{\widehat{\mathcal{P}}}})$$

The cosection takes then the following form:

$$\begin{aligned} \mathrm{Ob}_{\widehat{\mathcal{V}}/\widehat{\mathcal{P}} \times \mathbb{A}^1} &\subseteq \mathrm{R}^1 \hat{\pi}_*(\mathcal{L}_{\widehat{\mathcal{V}}}^{\oplus 5} \oplus \mathcal{L}_{\widehat{\mathcal{V}}}^{\otimes 5} \oplus \widehat{\mathcal{P}}_{\widehat{\mathcal{V}}}) \rightarrow \mathrm{R}^1 \hat{\pi}_* \omega_{\widehat{\pi}} \\ \bar{\sigma}_1|_{(u,v,\psi)}(\hat{x}, \hat{y}, \hat{p}) &= \psi \hat{y} + v \hat{p} \end{aligned}$$

It is showed in [CL12, §4.7] that $\bar{\sigma}_1$ lifts to a cosection $\bar{\sigma}: \mathrm{Ob}_{\widehat{\mathcal{V}}} \rightarrow \mathcal{O}_{\widehat{\mathcal{V}}}$ and that the degeneracy locus of $\bar{\sigma}$ is

$$\overline{\mathcal{M}}_1^{(1)}(X, d) \times \mathbb{A}_t^1.$$

Recall that the sections (u, v) are required to satisfy $\mathbf{w}(u) - tv = 0$. So $\bar{\sigma}_1$ coincides up to a non-zero scalar with the above defined cosection σ_1 for $\widehat{\mathcal{P}}$ when $t \neq 0$. It is proved in [KL13, Theorem 5.2] that, given compatible perfect obstruction theories, the construction of a cosection localised virtual cycle is compatible with Gysin pullbacks, so that by [CL12, Proposition 4.9]:

$$\iota_{t \neq 0}^! [\widehat{\mathcal{V}}]_{\bar{\sigma}}^{\mathrm{vir}} = [\widehat{\mathcal{P}}]_{\sigma}^{\mathrm{vir}} \in A_0(\widehat{\mathcal{Q}}), \quad \iota_0^! [\widehat{\mathcal{V}}]_{\bar{\sigma}}^{\mathrm{vir}} = [\overline{\mathcal{M}}_1^{(1)}(N_{X/\mathbb{P}^4}, d)^p]_{\bar{\sigma}_0}^{\mathrm{vir}} \in A_0(\widehat{\mathcal{Q}})$$

where we have denoted by $\widehat{\mathcal{Q}} := \overline{\mathcal{M}}_1^{(1)}(X, d)$.

Functoriality of cosection-localised pullback

We are left with proving that $\deg[\overline{\mathcal{M}}_1^{(1)}(N_{X/\mathbb{P}^4}, d)^p]_{\bar{\sigma}_0}^{\mathrm{vir}}$ coincides up to a sign with $\deg[\overline{\mathcal{M}}_1^{(1)}(X, d)]^{\mathrm{vir}}$.

In the rest of the section we use the following notation: $\widehat{\mathcal{N}} := \overline{\mathcal{M}}_1^{(1)}(N_{X/\mathbb{P}^4}, d)^p$, and $\hat{v}: \widehat{\mathcal{N}} \rightarrow \widehat{\mathcal{Q}}$.

First, Chang and Li prove that there is a perfect obstruction theory

$$\mathbb{E}_{\widehat{\mathcal{N}}/\widehat{\mathcal{Q}}} := \mathrm{R}^\bullet \hat{\pi}_{\widehat{\mathcal{N}},*}(\mathcal{L}_{\widehat{\mathcal{N}}}^{\otimes 5} \oplus \widehat{\mathcal{P}}_{\widehat{\mathcal{N}}})$$

compatible with $\mathbb{E}_{\widehat{\mathcal{N}}/\widehat{\mathcal{P}}}$ and $\hat{v}^*\mathbb{E}_{\widehat{\mathcal{Q}}/\widehat{\mathcal{P}}}$, so that $\mathbb{E}_{\widehat{\mathcal{N}}/\widehat{\mathcal{Q}}}$ inherits a cosection σ'_0 with degeneracy locus $D(\sigma'_0) = \widehat{\mathcal{Q}}$. So they have a localised virtual pullback:

$$\hat{v}_{\mathbb{E}_{\widehat{\mathcal{N}}/\widehat{\mathcal{Q}}}, \text{loc}}^! : A_*(\widehat{\mathcal{Q}}) \rightarrow A_*(D(\sigma'_0) = \widehat{\mathcal{Q}})$$

which they then prove to satisfy functoriality using the techniques of [KKP03], so:

$$\hat{v}_{\mathbb{E}_{\widehat{\mathcal{N}}/\widehat{\mathcal{Q}}}, \text{loc}}^![\widehat{\mathcal{Q}}]^{\text{vir}} = [\overline{\mathcal{M}}_1^{(1)}(N_{X/\mathbb{P}^4}, d)^p]_{\sigma'_0}^{\text{vir}}.$$

To conclude it is enough to compute the degree of $\hat{v}_{\mathbb{E}_{\widehat{\mathcal{N}}/\widehat{\mathcal{Q}}}, \text{loc}}^!$ on $A_0(\widehat{\mathcal{Q}})$; we just need to compute $\deg(\hat{v}_{\mathbb{E}_{\widehat{\mathcal{N}}/\widehat{\mathcal{Q}}}, \text{loc}}^![\zeta])$ for a closed point ζ . This is done in [CL12, Theorem 5.7] and the same considerations work in our case.

Hopefully we have managed to convince the reader that the subtle intersection theory perpetuated in [CL12] does not rely at all on the hypothesis that the families of curves we are working with are *nodal*, but *l.c.i.* curves are well-behaved enough so that all the proofs carry over to the situation of our interest.

6.3. Auxiliary space of p-fields

Now that we have 1-stable maps with fields, we would like to compare them with ordinary stable maps with fields, in order to finally achieve the wanted comparison between cuspidal and reduced invariants. We thus define an analogue of the space $\mathcal{Z}$ in § 6.1.

Let $\mathcal{XP}$ and $\mathcal{Z}^p$ be the algebraic stacks defined by the following cartesian diagram:

$$\begin{array}{ccc} \mathcal{Z}^p & \xrightarrow{\alpha} & \overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p \\ \downarrow \gamma & \square & \downarrow \\ \mathcal{XP} & \longrightarrow & \text{Pic}_1(1) \\ \downarrow & \square & \downarrow \\ \mathfrak{M}_1^{\text{wt}, \text{st}} & \longrightarrow & \mathfrak{M}_1^{\text{wt}, \text{st}}(1) \end{array}$$

Remark 6.14. The stack $\mathcal{XP}$ parametrises

$$\begin{array}{ccc} \mathcal{C}_S & \xrightarrow{\phi_S} & \widehat{\mathcal{C}}_S \\ & \searrow & \swarrow \\ & S & \end{array}$$

with a line bundle $\widehat{\mathcal{L}}_S$ on $\widehat{\mathcal{C}}_S$. Notice that by pulling back $\widehat{\mathcal{L}}_S$ via ϕ_S we obtain a line bundle on $\mathcal{C}_S$, hence a morphism $\mathcal{XP} \rightarrow \text{Pic}_1$. This is generically an isomorphism, but has 1-dimensional fibers over the locus of elliptic tails, due to the fact that $\text{Pic}(\widehat{C}) \rightarrow \text{Pic}(C)$ has kernel $\mathbb{G}_a$ when $\widehat{C}$ has a cusp.

Remark 6.15. Similarly the stack $\mathcal{Z}^p$ parametrises

$$\begin{array}{ccccc} \mathcal{C}_S & \xrightarrow{\phi_S} & \widehat{\mathcal{C}}_S & \xrightarrow{f_S} & \mathbb{P}_S^4 \\ & \searrow \pi_S & \swarrow \hat{\pi}_S & & \\ & & S & & \end{array}$$

with a p -field $\psi \in H^0(\widehat{\mathcal{C}}_S, f_S^* \mathcal{O}_{\mathbb{P}^4}(-5) \otimes \omega_{\hat{\pi}_S})$.

Caution 6.16. Unfortunately we are not able to compare $\mathcal{Z}^p$ with $\overline{\mathcal{M}}_1(\mathbb{P}^4, d)^p$ directly, since, contrary to the space of maps case, we don't have a morphism between the two spaces of p -fields.

Indeed, denote by $\widehat{\mathcal{L}} = f^* \mathcal{O}_{\mathbb{P}^4}(1)$ and by $\mathcal{L} = \phi^* \widehat{\mathcal{L}}$, a morphism $\mathcal{Z}^p \rightarrow \overline{\mathcal{M}}_1(\mathbb{P}^4, d)^p$ is equivalent to a morphism $R^1 \pi_* \mathcal{L}^{\otimes 5} \rightarrow R^1 \hat{\pi}_* \widehat{\mathcal{L}}^{\otimes 5}$.

Instead, we only have a map the wrong way around: $R^1 \hat{\pi}_* \widehat{\mathcal{L}} \rightarrow R^1 \pi_* \mathcal{L}$ on $\mathcal{Z}^p$ which is not an isomorphism. To see this, consider the dual picture: since $\phi^* \omega_{\hat{\pi}} = \omega_{\pi}(\mathcal{E})$, the p -fields on $\mathcal{C}$ that we get by pulling back from $\widehat{\mathcal{C}}$ vanish on elliptic tails.

This meant that we could not directly look at [CL15] decomposition of the intrinsic cone, but we had to use the machinery they built to study instead the cone of $\mathcal{Z}^p$. The comparison with reduced invariants is possible because $\mathcal{Z}^p$ and $\mathcal{Z}$ (and all of their resolutions) have the same main component, and the latter is comparable with standard GW.

Before starting the study of the geometry of $\mathcal{Z}^p$ which will lead us to the final comparison, let us show that this auxiliary space of p -fields can be endowed with a virtual class whose degree coincide with cuspidal invariants of the quintic:

The perfect obstruction theory $R^\bullet \hat{\pi}_*(\widehat{\mathcal{L}}^{\oplus 5} \oplus \widehat{\mathcal{P}})$ for the morphism $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p \rightarrow \text{Pic}_1(1)$ together with its cosection induce a localised virtual class on $\mathcal{Z}^p$ by localised virtual pullback, see § 6.2.3:

$$[\mathcal{Z}^p]_{\alpha^* \sigma, \text{loc}}^{\text{vir}} := \gamma_{\mathbb{E}_{\mathcal{Z}^p/\mathcal{X}^p, \alpha^* \sigma}}^! [\mathcal{X}^p].$$

In order to show that this gives the same invariants as the space of 1-stable maps with p -fields, we need to prove that localised virtual pullback commutes with proper pushforward.

Lemma 6.17. (*Localised virtual pullback commutes with proper pushforward.*) *With the same setup and notation as in § 6.2.3, suppose furthermore that φ is proper and $\varphi'(Y') \cap U \neq \emptyset$. For every cycle $[B'] \in A_*(S')$,*

$$\varphi'_* (\rho')_{\mathbb{E}_{Y'/S'}}^! [B'] = (\rho)_{\mathbb{E}_{Y/S}}^! \circ \varphi_* [B] \in A_*(Y(\sigma)).$$

Proof. Consider the following diagram:

$$\begin{array}{ccccccc}
 A_*(S') & \longrightarrow & A_*(\mathfrak{C}_{Y'/S'}) & \longrightarrow & A_*(h^1/h^0(\mathbb{E}_{Y'/S'})(\sigma')) & \xrightarrow{s_{\sigma',\text{loc}}^!} & A_*(Y'(\sigma')) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 A_*(S) & \longrightarrow & A_*(\mathfrak{C}_{Y/S}) & \longrightarrow & A_*(h^1/h^0(\mathbb{E}_{Y/S})(\sigma)) & \xrightarrow{s_{\sigma,\text{loc}}^!} & A_*(Y(\sigma))
 \end{array}$$

Only the right-most square is not classical. Hence we are left with showing that for every cycle $[B'] \in A_*(h^1/h^0(\mathbb{E}_{Y'/S'})(\sigma'))$ we have:

$$\varphi'_*(s_{\sigma',\text{loc}}^![B']) = s_{\sigma,\text{loc}}^!(\varphi_*^{\mathbb{E}}[B']) \in A_*(Y(\sigma)), \quad (6.7)$$

where $\varphi^{\mathbb{E}}: h^1/h^0(\mathbb{E}_{Y'/S'})(\sigma') \rightarrow h^1/h^0(\mathbb{E}_{Y/S})(\sigma)$ is induced by the morphism of vector bundle stacks.

If $B' \subset h^1/h^0(\mathbb{E}_{Y'/S'})|_{Y'(\sigma')}$ then $\varphi^{\mathbb{E}}(B') \subset h^1/h^0(\mathbb{E}_{Y/S})|_{Y(\sigma)}$, thus the localised Gysin morphisms coincide with $s_{h^1/h^0(\mathbb{E}_{Y'/S'})|_{Y'(\sigma')}}^!$ and $s_{h^1/h^0(\mathbb{E}_{Y/S})|_{Y(\sigma)}}^!$, and their commutativity with proper pushforward is well-known, see [Kre99] if we assume the existence of global resolutions).

Assume then that B' is a prime cycle in $h^1/h^0(\mathbb{E}_{Y'/S'})(\sigma')$ not contained in $h^1/h^0(\mathbb{E}_{Y'/S'})|_{Y'(\sigma')}$. Then recall that:

$$s_{\sigma',\text{loc}}^![B'] := \frac{1}{k} \nu'(\sigma')_*([D] \cdot s_{\widetilde{G}}^![\widetilde{B}]) \in A_*(Y'(\sigma')),$$

where $\nu': \widetilde{Y} \rightarrow Y'$, $(\nu')^*(\sigma')$ and $\widetilde{B}$ satisfy the conditions of a regularising morphism, see § 6.2.3. Notice that if we consider the composition $\nu: \widetilde{Y} \xrightarrow{\nu'} Y' \xrightarrow{\varphi'} Y$ and the pull-back $\nu^*(\sigma)$, the first two of those conditions are always satisfied. How about $\widetilde{B} \rightarrow B' \rightarrow \varphi^{\mathbb{E}}(B') =: B$? Suppose first that $B' \rightarrow B$ is generically finite of degree h ; then $\widetilde{B} \rightarrow B$ is generically finite of degree hk , so:

$$\begin{aligned}
 \varphi'_*(s_{\sigma',\text{loc}}^![B']) &= \frac{1}{k} \varphi'_* \circ \nu'(\sigma')_*([D] \cdot s_{\widetilde{G}}^![\widetilde{B}]) \\
 &= \frac{1}{k} \nu(\sigma)_*([D] \cdot s_{\widetilde{G}}^![\widetilde{B}]) \\
 &= h s_{\sigma,\text{loc}}^![B] \\
 &= s_{\sigma,\text{loc}}^!(\varphi_*^{\mathbb{E}}[B']).
 \end{aligned}$$

If instead $B' \rightarrow B$ is not generically finite, then the right-hand side in formula (6.7) is zero. In order to show that the same is true for the left-hand side it is enough to appeal to [KL13, Lemma 2.7], since $\nu_*^{\mathbb{E}}[\widetilde{B}] = (\nu')_*^{\mathbb{E}} \circ \varphi_*^{\mathbb{E}}[\widetilde{B}] = 0$. $\square$

Corollary 6.18. $\deg[\mathcal{Z}^p]_{\text{loc}}^{\text{vir}} = \deg[\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p]_{\text{loc}}^{\text{vir}}$

This follows immediately from the fact that the 1-stabilisation is proper and birational, Lemma 5.16 and thus $\mathcal{XP}$ and $\text{Pic}_1(1)$ are birational.

6.4. Local equations and desingularisation

6.4.1. Equations for $\mathcal{Z}^p$ relative to $\mathcal{XP}$

We are going to need a description of the normal cone $\mathbb{C}_{\mathcal{Z}^p/\mathcal{XP}}$ in order to perform a splitting. Since $\mathcal{Z}^p$ is a line bundle over the boundary of $\mathcal{Z}$, we find instead equations for the latter.

Recall that $\mathcal{Z}$ can be embedded as an open inside $C(\hat{\pi}_*\widehat{\mathcal{L}}^{\oplus 5})$ over $\mathcal{XP}$. We are going to find an embedding of $C(\hat{\pi}_*\widehat{\mathcal{L}}^{\oplus 5})$ in a smooth ambient space over $\mathcal{XP}$, that will be a vector bundle obtained by suitably twisting $\widehat{\mathcal{L}}$.

This is a revisited version of § 4.1.1, so the ideas all come from [HL10a]. We work locally on $\mathcal{Z}$: start with a point $\xi = [(C \xrightarrow{\phi} \widehat{C} \xrightarrow{f} \mathbb{P}^4)] \in \mathcal{Z}$ and choose coordinates on $\mathbb{P}^4$ such that $f^{-1}\{x_0 = 0\}$ is a simple smooth divisor $D = \sum_{i=1}^d \delta_i$ on $\widehat{C}$. This continues to be true on a neighbourhood $\mathcal{U}$ of ξ .

Locally the morphism $\mathcal{Z} \rightarrow \mathcal{XP}$ can be written as $\xi \mapsto [C \rightarrow \widehat{C}, \mathcal{O}_{\widehat{C}}(D)]$, which admits a local lifting $\mathcal{U} \rightarrow \mathcal{X}^{\text{div}} := \mathfrak{M}_1^{\text{div}}(1) \times_{\widehat{\mathfrak{M}}} \mathfrak{M}$, and in fact hits the smooth locus of the latter.

Remark 6.19. The space $\mathfrak{M}_1^{\text{div}}$ maps (actually includes) in $\mathfrak{M}_1^{\text{div}}(1) \times_{\widehat{\mathfrak{M}}} \mathfrak{M}$ but it is not isomorphic to the latter; pulling back $\widehat{D}$ from $\widehat{C}$ to C might give a divisor vanishing identically on the elliptic tail.

Remark 6.20. The projection $\mathcal{X}^{\text{div}} \rightarrow \mathcal{XP}$ is *not* smooth. In fact, when the line bundle is trivial on the minimal elliptic subcurve E , it may be deformed to a degree 0, non-effective line bundle on such a subcurve, so that sections of $\mathcal{O}_{\widehat{C}}(D)$ which are constant and non-zero on E are obstructed. The locus where such a morphism is smooth, in $\mathcal{XP}$ is the locus where the line bundle is effective. However the latter is not locally closed, so even working in local charts we can't avoid the issue.

This tells us the equations in a smooth ambient relatively to $\mathcal{X}^{\text{div}}$ will differ from the equations in a smooth ambient relatively to $\mathcal{XP}$.

However we will now see that the difference is minimal and, most important, does not affect the results of [HL10a].

In a neighbourhood $\mathcal{V} \subseteq \mathcal{XP}$ of $[C \rightarrow \widehat{C}, \mathcal{O}_{\widehat{C}}(D)]$ we can write the universal line bundle $\widehat{\mathcal{L}}_{\mathcal{V}}$ as $\mathcal{O}_{\widehat{\mathcal{C}}_{\mathcal{V}}}(\mathcal{D} + p - p_0)$, where p, p_0 are sections through the core, with p free to move on the core, i.e. p as a local coordinate on $\mathcal{XP}$ relative to $\mathfrak{M}$. Then in the chosen charts, the deformation theory of $C \rightarrow \widehat{C} \rightarrow \mathbb{P}^4$ is the same as the deformation of $C \rightarrow \widehat{C}, \widehat{\mathcal{L}}_{\mathcal{V}}, \underline{s} \in H^0(\widehat{\mathcal{L}}_{\mathcal{V}}^{\otimes 5})$. Notice that another difference with respect to §4.1.1 is that, working without having fixed a divisor, we have to consider deformation of all the $4 + 1$ sections defining the map to $\mathbb{P}^4$.

Locally on $\mathcal{V}$ we can pick another smooth section $\mathcal{A}$ of the core not intersecting p_0 , neither the support of $\mathcal{D} + p$.

Lemma 6.21. $C(\hat{\pi}_* \widehat{\mathcal{L}}_{\mathcal{V}})$ is the kernel of the vector bundle map:

$$\hat{\pi}_* \mathcal{O}_{\widehat{\mathcal{C}}}(\mathcal{A} + \mathcal{D} + p - p_0) \xrightarrow{\varphi} \hat{\pi}_* \mathcal{O}_{\mathcal{A}}(\mathcal{A})$$

where, up to shrinking $\mathcal{V}$, we may write:

$$\hat{\pi}_* \mathcal{O}_{\widehat{\mathcal{C}}}(\mathcal{A} + \mathcal{D} + p - p_0) \cong \bigoplus_{i=1}^d \hat{\pi}_* \mathcal{O}_{\widehat{\mathcal{C}}}(\mathcal{A} + \delta_i - p_0) \oplus \hat{\pi}_* \mathcal{O}_{\widehat{\mathcal{C}}}(\mathcal{A} + p - p_0) \quad (6.8)$$

where $\mathcal{D} = \sum_{i=1}^d \delta_i$, and thus

$$\varphi = \bigoplus_{i=1}^d \varphi_i \oplus \varphi_p$$

where we denoted by φ_* the composition

$$\pi_* \mathcal{O}_{\widehat{\mathcal{C}}}(\mathcal{A} + * - p_0) \hookrightarrow \pi_* \mathcal{O}_{\widehat{\mathcal{C}}}(\mathcal{D} + \mathcal{A} + p - p_0) \rightarrow \hat{\pi}_* \mathcal{O}_{\mathcal{A}}(\mathcal{A})$$

The proof goes as in [HL10a, Lemma 4.10], to which we refer for the details. The idea is the following: both sides in 6.8 are vector bundles and we do have a map between them, so we just have to check the isomorphism fiberwise and for that we can base change to a point and check that the global sections of right hand side form a basis for the global sections of the left hand side.

Let us introduce some more notation: around a point $[\widehat{C}] \in \mathfrak{M}_1^{\text{wt, st}}(1)$, for every node q of $\widehat{C}$ there is a coordinate ζ_q whose vanishing locus is the divisor where such a node is *not* smoothed. These functions can be pulled back to $\mathcal{V}$. Denote by

$$\zeta_{[\delta_i, \mathcal{A}]} = \prod \zeta_q$$

where the product runs over all the nodes separating δ_i from the core.

Lemma 6.22. *There are trivialisations for the line bundles $\pi_* \mathcal{O}_{\widehat{\mathcal{C}}}(\mathcal{A} + \delta_i - p_0)$, $\hat{\pi}_* \mathcal{O}_{\widehat{\mathcal{C}}}(\mathcal{A} + p - p_0)$, $\pi_* \mathcal{O}_{\mathcal{A}}(\mathcal{A})$ such that locally we have the following explicit expression for φ_i and φ_p :*

$$\varphi_i = \zeta_{[\delta_i, \mathcal{A}]}, \quad \varphi_p = (p - p_0)$$

This is essentially [HL10a, Proposition 4.13]. We only remark the idea and then refer to [HL10a, Proposition 4.13] for details. Proving that two sections coincide (up to multiplication by a scalar) is equivalent to say that their zero loci coincide. Let R be a point in $\mathcal{XP}$, let us for example show that $\varphi_{p,R}$ vanishes if and only if $R \in \{p - p_0 = 0\}$. The Kernel of $\varphi_{p,R}$ is given by $\hat{\pi}_{R,*} \mathcal{O}_{\widehat{\mathcal{C}}_R}(p - p_0)$. Thus we have that, if $p = p_0$ the latter is isomorphic to $\mathcal{O}_R \cong \hat{\pi}_R * \mathcal{O}_{\widehat{\mathcal{C}}_R}(\mathcal{A} + p - p_0)$ and thus $\varphi_{p,R} = 0$, otherwise $\hat{\pi}_{R,*} \mathcal{O}_{\widehat{\mathcal{C}}_R}(p - p_0) = 0$ and $\varphi_{p,R}$ is an isomorphism.

Remark 6.23. The vanishing locus of $(p - p_0)$ on the boundary means that the line bundle

restricts to the trivial one on the core.

Lemma 6.24. *A local chart $\mathcal{U}$ for $\mathcal{Z}$ can be embedded as an open inside:*

$$(F_0 = \dots = F_4 = 0) \subseteq \mathrm{Vb}(\hat{\pi}_* \widehat{\mathcal{L}}_{\mathcal{V}}(\mathcal{A})^{\oplus 5})$$

where

$$F_j = \sum_{i=1}^d \zeta_{[\delta_i, \mathcal{A}]} w_i^j + (p - p_0) w_{d+1}^j$$

and w_i^j are coordinates on the fiber of the j -th copy of $\mathrm{Vb}(\hat{\pi}_* \widehat{\mathcal{L}}_{\mathcal{V}}(\mathcal{A}))$ over $\mathcal{V}$.

Compare with [HL10a, Theorems 2.17-19].

6.4.2. Hu-Li blow-up and desingularisation

We perform a modular blow-up of $\mathfrak{M}_1^{\mathrm{wt}, \mathrm{st}}(1)$: we successively blow up $\widehat{\Theta}_k$, $k \geq 2$, defined as the closure of the loci where the minimal elliptic subcurve E has weight 0 and $|\overline{C \setminus E} \cap E| = k$.

Notice that after the k -th blow-up, the strict transform of $\widehat{\Theta}_{k+1}$ is smooth, so the final result $\widetilde{\mathfrak{M}}_1^{\mathrm{wt}, \mathrm{st}}(1)$ is smooth as well.

Remark 6.25. The fiber product

$$\widetilde{\mathfrak{M}}_1^{\mathrm{wt}, \mathrm{st}}(1) \times_{\mathfrak{M}_1^{\mathrm{wt}, \mathrm{st}}(1)} \mathfrak{M}_1^{\mathrm{wt}, \mathrm{st}}$$

recovers the Hu-Li blow-up $\widetilde{\mathfrak{M}}_1^{\mathrm{wt}, \mathrm{st}}$. The key observation is that Θ_1 is already a Cartier divisor and the inverse image of $\widehat{\Theta}_k$ is precisely Θ_k ; using the universal property of the blow-up, it can be shown that there are maps in both directions, and they are inverse to one another.

Remark 6.26. After blowing up, the equations in 6.24 are simplified and assume the following form:

$$\tilde{\zeta} \tilde{w} + (p - p_0) w_{d+1} = 0$$

where $\tilde{\zeta}$ is one of the newly created boundary divisors $\widetilde{\Theta}_k$ from $\widetilde{\mathfrak{M}}_1^{\mathrm{wt}, \mathrm{st}}$ (i.e. one of the exceptional divisors produced by the blow-up process), and $\tilde{w}$ is a suitably defined coordinate on the fiber of $\mathrm{Vb}(\hat{\pi}_* \widehat{\mathcal{L}}(\mathcal{A})) \times_{\mathcal{X}^{\mathcal{P}}} \widetilde{\mathcal{X}^{\mathcal{P}}}$.

Notice that the components of $\widetilde{\mathcal{Z}}^p$ are in bijection with those of $\mathcal{Z}^p$, however all the boundary ones have the same dimension $5d + 4$, and their intersection with main, which is now smooth, is a divisor in the latter.

Summing up, we get:

$$\begin{array}{ccccc}
 & & \widetilde{\mathcal{Z}}^p & \longrightarrow & \widetilde{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p \\
 & & \downarrow & \square & \downarrow \\
 \widetilde{\mathcal{M}}_1(\mathbb{P}^4, d) & \xleftarrow{i} & \widetilde{\mathcal{Z}} & \longrightarrow & \widetilde{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d) \\
 \downarrow & & \downarrow & \square & \downarrow \\
 \widetilde{\text{Pic}}_1 & \xleftarrow{\quad} & \widetilde{\mathcal{XP}} & \longrightarrow & \widetilde{\text{Pic}}_1(1) \\
 \downarrow & & \downarrow & \square & \downarrow \\
 \widetilde{\mathfrak{m}}_1^{\text{wt, st}} & \xleftarrow{\sim} & \widetilde{\mathfrak{m}}_1^{\text{wt, st}} & \longrightarrow & \widetilde{\mathfrak{m}}_1^{\text{wt, st}}(1)
 \end{array}$$

We conclude this brief section remarking that the blow-up procedure does not affect the invariants:

Lemma 6.27. *We have the identity:*

$$\deg[\widetilde{\mathcal{Z}}^p]_{\text{loc}}^{\text{vir}} = \deg[\mathcal{Z}^p]_{\text{loc}}^{\text{vir}}.$$

Follows again from 6.17; compare also with [CL15, Proposition 2.5].

6.5. Splitting the cone and proof of the Main Theorem

We are finally able to study the cone $\mathbb{C}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{XP}}}$. This is going to be a word-by-word repetition of the arguments in [CL12].

Lemma 6.28. *The map $\tilde{\rho}: \widetilde{\mathcal{Z}}^p \rightarrow \widetilde{\mathcal{XP}}$ has a relative perfect obstruction theory $\mathbb{E}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{XP}}} = \mathbf{R}^* \hat{\pi}_*(\widehat{\mathcal{L}}^{\oplus 5} \oplus \widehat{\mathcal{P}})$. The intrinsic normal cone $\mathbb{C}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{XP}}}$ has the following properties:*

1. *Its restriction to the open $\widetilde{\mathcal{Z}}^{p, \circ} = \widetilde{\mathcal{Z}}^{p, \text{main}} \setminus \bigcup_{k \geq 2} D^k \widetilde{\mathcal{Z}}^p$ can be described as the zero section of $h^1/h^0(\mathbb{E}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{XP}}})|_{\widetilde{\mathcal{Z}}^{p, \circ}}$*
2. *Its restriction to the open locus of the boundary $\widetilde{\mathcal{Z}}^{p, \text{gst}, \circ} = \widetilde{\mathcal{Z}}^p - \widetilde{\mathcal{Z}}^{p, \text{main}}$ is a rank 2 subbundle stack of $h^1/h^0(\mathbb{E}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{XP}}})|_{\widetilde{\mathcal{Z}}^{p, \text{gst}, \circ}}$*

Proof. Compare with [CL12, Lemma 4.3]

1. Observe that $\widetilde{\mathcal{Z}}^{p, \circ} \cong \widetilde{\mathcal{Z}}^{\circ}$ because here $H^0(\widehat{C}, \widehat{L}^{\otimes -5} \otimes \omega_{\widehat{C}}) = 0$. Moreover $\widetilde{\mathcal{Z}}^{\circ}$ is unobstructed on its image, which is an open of $\widetilde{\mathcal{XP}}$, because $\mathbf{R}^1 \hat{\pi}_* \widehat{\mathcal{L}} = 0$. So the normal cone is $[\widetilde{\mathcal{Z}}^{p, \circ}/\hat{\pi}_* \widehat{\mathcal{L}}^{\oplus 5}]$, which is the zero section of $h^1/h^0(\mathbb{E}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{XP}}})|_{\widetilde{\mathcal{Z}}^{p, \circ}} = [0 \oplus \mathbf{R}^1 \hat{\pi}_* \widehat{\mathcal{P}}/\hat{\pi}_* \widehat{\mathcal{L}}^{\oplus 5} \oplus 0]$.
2. We know that $\widetilde{\mathcal{Z}}^{p, \text{gst}, \circ}$ is a line bundle over $\widetilde{\mathcal{Z}}^{\text{gst}, \circ}$. From the equations 6.26 we see that the latter is smooth over its image $\mathcal{W}$ in $\widetilde{\mathcal{XP}}$, which is the codimension 2 locus where the core has weight 0 and the line bundle is trivial on it. Recall that every

smooth morphism $A \rightarrow B$ of relative dimension n factors as $A \xrightarrow{\acute{e}t} B \times \mathbb{A}^n \xrightarrow{\text{pr}_1} B$. So we have

$$\begin{array}{ccc} \widetilde{\mathcal{Z}}^{p,\text{gst},\circ} & \xrightarrow{\acute{e}t} & \mathcal{W} \times \mathbb{A}^{5d+6} \hookrightarrow \widetilde{\mathcal{X}}\mathcal{P} \times \mathbb{A}^{5d+6} \\ & \downarrow q & \downarrow \\ & \mathcal{W} & \hookrightarrow \widetilde{\mathcal{X}}\mathcal{P} \end{array}$$

where the bottom horizontal arrow is a codimension 2 regular embedding. Thus

$$\mathfrak{C}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{X}}\mathcal{P}}|_{\widetilde{\mathcal{Z}}^{p,\text{gst},\circ}} \cong \left[q^* C_{\mathcal{W}/\widetilde{\mathcal{X}}\mathcal{P}}/\hat{\pi}_* \widehat{\mathcal{L}}^{\oplus 5} \oplus \hat{\pi}_* \widehat{\mathcal{P}} \right]$$

is a rank 2 subbundle stack of $h^1/h^0(\mathbb{E}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{X}}\mathcal{P}})|_{\widetilde{\mathcal{Z}}^{p,\text{gst},\circ}}$.

□

Notice that the image of $\widetilde{\mathcal{Z}}^\circ$ in $\widetilde{\mathcal{M}}_1(\mathbb{P}^4)$ contains $\widetilde{\mathcal{M}}_1(\mathbb{P}^4)^{\text{main}} \cap \widetilde{D}^1$.

6.5.1. Contribution of the main component

Recall the definition of the *closure of the zero section of a vector bundle stack*: let B be an integral algebraic stack and let $\mathbb{F} = [F_0 \xrightarrow{d} F_1]$ be a complex of locally free sheaves on B . The zero section is $0_{\mathbb{F}}: [F_0/F_0] \rightarrow h^1/h^0(\mathbb{F}) = [F_1/F_0]$, which is in general not a closed embedding; its closure is defined as:

$$\overline{0}_{\mathbb{F}} = [\text{cl}(dF_0)/F_0]$$

$\overline{0}_{\mathbb{F}}$ is an integral stack.

Example 6.29. When $h^0(\mathbb{F}) = 0$, the closure of the zero section looks like B with some further stacky structure on the vanishing locus of d . Consider for example $B = \mathbb{P}^1$ and $\mathbb{F} = [\mathcal{O}_{\mathbb{P}^1} \xrightarrow{x} \mathcal{O}_{\mathbb{P}^1}(1)]$. Then the action of $e \in F_0$ on F_1 is given by $f \mapsto f + xe$. Clearly $\text{cl}(dF_0)$ is the whole line bundle F_1 ; the F_0 -action is transitive on the fibers over $\{x \neq 0\}$ and trivial on the $\{x = 0\}$ -fiber. Hence $\overline{0}_{\mathbb{F}}$ is isomorphic to $\mathbb{P}^1 \setminus \{x = 0\}$ with a gerbe $[\mathbb{A}^1/\mathbb{G}_a]$ replacing the point $\{x = 0\}$.

We may now split the cone $\mathfrak{C}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{X}}\mathcal{P}}$ in the following manner: we denote by $\mathfrak{C}^{\text{main}}$ the closure of the zero section of $h^1/h^0(\mathbb{E}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{X}}\mathcal{P}})|_{\widetilde{\mathcal{Z}}^{p,\text{main}}}$, which is an irreducible cone supported on the main component. All the rest is supported on the boundary components, possibly on their intersection with the main one, and we are going to pack all the components supported on $D^k \widetilde{\mathcal{Z}}^p$ together and label them $\mathfrak{C}^k$ accordingly, so in the end we obtain a splitting:

$$\mathfrak{C}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{X}}\mathcal{P}} = \mathfrak{C}^{\text{main}} + \sum_{k \geq 2} \mathfrak{C}^k$$

We are going to show that:

1. the contribution of $\mathfrak{C}^{\text{main}}$ is exactly the reduced invariants of X ;

2. the other cones $\mathfrak{C}^k$, $k \geq 2$, are enumeratively meaningless.

In order to prove the first claim we proceed as in [CL12, §5]; let us start by noticing that the obstruction theory $\mathbb{E} := \mathbb{E}_{\widetilde{\mathcal{Z}}^p/\widetilde{\mathcal{X}}^p}$ splits as $\mathbb{E}_1 \oplus \mathbb{E}_2$ where $\mathbb{E}_1 = R^\bullet \hat{\pi}_*(\widehat{\mathcal{L}}^{\otimes 5})$ and $\mathbb{E}_2 = R^\bullet \hat{\pi}_*(\widehat{\mathcal{P}})$. When we restrict to $\widetilde{\mathcal{Z}}^{p,\text{main}}$ we see that $h^1/h^0(\mathbb{E}_1)$ is the closure of its own zero section; it follows that:

$$\mathfrak{C}^{\text{main}} = \overline{0}_{\mathbb{E}|_{\widetilde{\mathcal{Z}}^{p,\text{main}}}} = h^1/h^0(\mathbb{E}_1)|_{\widetilde{\mathcal{Z}}^{p,\text{main}}} \oplus \overline{0}_{\mathbb{E}_2|_{\widetilde{\mathcal{Z}}^{p,\text{main}}}}$$

Then by standard intersection theory (pullback the right-hand side by $\hat{\pi}_{\mathbb{E}}^* = \hat{\pi}_{\mathbb{E}_1}^* \circ \hat{\pi}_{\mathbb{E}_2}^*$):

$$0_{\mathbb{E}}^![\mathfrak{C}^{\text{main}}] = 0_{\mathbb{E}_2}^![\overline{0}_{\mathbb{E}_2|_{\widetilde{\mathcal{Z}}^{p,\text{main}}}}]$$

At this point we recall the following [CL12, Lemma 5.3]:

Lemma 6.30. *Let $\mathbb{E} = [E_0 \rightarrow E_1]$ be a complex of locally free sheaves on an integral Deligne-Mumford stack B such that $h^1(\mathbb{E})$ is a torsion sheaf on B and the image sheaf of $E_0 \rightarrow E_1$ is locally free. Let $U \subseteq B$ be the complement of the support of $h^1(\mathbb{E})$, and let $\mathbf{B} \subseteq h^1/h^0(\mathbb{E}^\vee[-1])$ be the closure of the zero section of the vector bundle $h^1/h^0(\mathbb{E}^\vee[-1]|_U) = h^0(\mathbb{E}|_U)^\vee$. Then*

$$0_{\mathbf{B}}^![\mathbf{B}] = e(h^0(\mathbb{E})^\vee) \in A_*(B).$$

We apply this lemma to $R^\bullet \hat{\pi}_* \widehat{\mathcal{L}}^{\otimes 5} = \mathbb{E}_2^\vee$ on $\widetilde{\mathcal{Z}}^{p,\text{main}}$. Notice that it satisfies the hypotheses by virtue of the equations in Remark 6.26: indeed the question may be addressed locally; looking at the resolution of $\mathbb{E}_2^\vee$:

$$\hat{\pi}_* \widehat{\mathcal{L}}^{\otimes 5}(\mathcal{A}) \rightarrow \hat{\pi}_* \widehat{\mathcal{L}}^{\otimes 5}(\mathcal{A})|_{\mathcal{A}}$$

we deduce from the equation that the image of this map is $\pi_* \widehat{\mathcal{L}}^{\otimes 5}(\mathcal{A})|_{\mathcal{A}} \otimes \mathcal{O}_{\widetilde{\mathcal{Z}}^{p,\text{main}}}(-\Xi)$, where Ξ denotes the Cartier divisor $\Xi = \widetilde{\mathcal{Z}}^{p,\text{main}} \cap (\bigcup_{k \geq 2} D^k \widetilde{\mathcal{Z}}^p)$. Then $\hat{\pi}_* \widehat{\mathcal{L}}^{\otimes 5}$ is a vector bundle, being the kernel of a vector bundle map.

Lemma 6.31. *If we let i be the inclusion of $\widetilde{\mathcal{Z}}$ in $\widetilde{\mathcal{M}}_1(\mathbb{P}^4, d)$, then:*

$$i_*(c_{\text{top}}(\hat{\pi}_* \widehat{\mathcal{L}}^{\otimes 5}) \cap [\widetilde{\mathcal{Z}}^{p,\text{main}}]) = c_{\text{top}}(\pi_* \mathcal{L}^{\otimes 5}) \cap [\widetilde{\mathcal{M}}_1(\mathbb{P}^4, d)^{p,\text{main}}]$$

Proof. Recall that the projection $(-)^p \rightarrow (-)$ is an isomorphism on main, so it makes sense to write $i_*[\widetilde{\mathcal{Z}}^{p,\text{main}}] = [\widetilde{\mathcal{M}}_1(\mathbb{P}^4, d)^{p,\text{main}}]$, which follows from Lemma 6.1. On the other hand notice that on $\widetilde{\mathcal{Z}}^{p,\text{main}}$ we have:

$$\pi_* \mathcal{L} = \hat{\pi}_* \phi_* \phi^* \widehat{\mathcal{L}} = \hat{\pi}_* \widehat{\mathcal{L}}$$

by projection formula and since $\phi_* \mathcal{O}_{\widetilde{\mathcal{Z}}^{p,\text{main}}} = \mathcal{O}_{\widehat{\mathcal{Z}}^{p,\text{main}}}$ by Zariski connectedness theorem. The result follows from the projection formula for Chern classes. $\square$

We are left with showing that the rest of the $\mathfrak{C}^k$ do not contribute to the invariants. This is essentially a dimensional computation. The arguments of [CL12, §§6-8] may be adapted; we shall outline them for the benefit of the reader. We introduce the notation $\widetilde{\mathcal{Z}}^{p,\text{gst}} := \bigcup_{k \geq 2} D^k \widetilde{\mathcal{Z}}^p$ for the union of the boundary components, and $\mathfrak{C}^{\text{gst}} = \bigcup_{k \geq 2} \mathfrak{C}^k$.

Step I: reduction to the case of a cone inside a vector bundle

This section deals with removing the technicalities of working with a cone stack inside a vector bundle stack and reducing to the case of a cone inside a vector bundle.

The key point is that $\mathbb{E} := \mathbb{E}|_{\widetilde{\mathcal{Z}}^{p,\text{gst}}}$ has locally free h^0 and h^1 : the equations in Remark 6.26 are automatically satisfied on the boundary, without imposing any condition on the fiber coordinates.

When we fix a resolution by locally free sheaves $\mathbb{E} = [F_0 \xrightarrow{d} F_1]$, the image sheaf $d(F_0)$, which is the kernel of $F_1 \rightarrow h^1(\mathbb{E})$, is a subbundle of F_1 . Consider the projections:

$$\beta: F_1 \rightarrow h^1/h^0(\mathbb{E}) \quad \text{and} \quad \beta': F_1 \rightarrow \tilde{V} := R^1 \hat{\pi}_*(\widehat{\mathcal{L}}^{\oplus 5} \oplus \widehat{\mathcal{P}});$$

the second is also flat since $d(F_0)$ is a vector bundle. The cone stack $\mathfrak{C}^{\text{gst}}$ can be descended to a cone $C^{\text{gst}} \subseteq \tilde{V}$ by taking the quotient of $\beta^{-1}\mathfrak{C}^{\text{gst}}$ by the free action of $d(F_0)$; C^{gst} should then be thought of as the coarse moduli of $\mathfrak{C}^{\text{gst}}$. Recall that we started with a cosection σ_1 of V (see Eq. (6.3)) and pulled it back to all relevant spaces. It follows from the commutativity of localised Gysin pullback with flat pullback that:

$$s_{h^1/h^0(\mathbb{E}), \tilde{\sigma}_1}^! [\mathfrak{C}^{\text{gst}}] = s_{\tilde{V}, \tilde{\sigma}_1}^! [C^{\text{gst}}]$$

See [CL12, Proposition 6.3].

Step II: from cosection localised to standard Gysin map

Recall that the construction of a localised virtual class refines the standard one, namely:

$$\iota_* [Y]_{\text{loc}}^{\text{vir}} = [Y]^{\text{vir}},$$

where $\iota: Y(\sigma) \hookrightarrow Y$ is the degeneray locus of the cosection. In particular, when Y itself is a proper Deligne-Mumford stack, the degree of the localised virtual class can be computed after pushing it forward to Y .

Then to compute $s_{\tilde{V}, \tilde{\sigma}_1}^! [C^{\text{gst}}]$ we are going to compactify $\widetilde{\mathcal{Z}}^{p,\text{gst}}$, extend the cone, the vector bundle and the cosection to it, and then make use of the fact we just recalled.

Since $\widetilde{\mathcal{Z}}^{p,\text{gst}} \cong \text{Vb}_{\widetilde{\mathcal{Z}}^{\text{gst}}}(\hat{\pi}_* \widehat{\mathcal{P}})$, we take its standard copactification:

$$\bar{\gamma}: \overline{\mathcal{Z}}^{p,\text{gst}} := \mathbb{P}(\hat{\pi}_* \widehat{\mathcal{P}} \oplus \mathcal{O}_{\widetilde{\mathcal{Z}}^{\text{gst}}}) \rightarrow \widetilde{\mathcal{Z}}^{\text{gst}}.$$

We want to pullback the cosection:

$$\sigma_{1|(u,\psi)}(\mathring{x}, \mathring{p}) = \mathring{p}\mathbf{w}(u) + \psi \sum_{i=0}^4 \partial_i \mathbf{w}(u) \mathring{x}_i;$$

noticing that in the compactification the p -field ψ can go to infinity, we define the vector bundles on $\overline{\mathcal{Z}}^{p,\text{gst}}$:

$$\tilde{V}_1^{\text{cpt}} = \tilde{\gamma}^* V_1(-D_\infty), \quad \tilde{V}_2^{\text{cpt}} = \tilde{\gamma}^* V_2.$$

We are now able to extend the cosection and get:

$$\bar{\sigma}: \tilde{V}^{\text{cpt}} = \tilde{V}_1^{\text{cpt}} \oplus \tilde{V}_2^{\text{cpt}} \rightarrow \mathcal{O}_{\overline{\mathcal{Z}}^{p,\text{gst}}}.$$

The compatibility of $\bar{\sigma}$ with $\bar{\sigma}$, and the fact that $\overline{\mathcal{Z}}^{p,\text{gst}}$ is proper, together with the observation at the beginning of this section, explain the following:

Proposition 6.32. *Let $\iota_!: Z_*(\tilde{V}(\bar{\sigma})) \rightarrow Z_*(\tilde{V}^{\text{cpt}})$ be defined by $\iota_![C] = [\bar{C}]$. And $i: D(\bar{\sigma}) \rightarrow \tilde{\mathcal{Z}}^{\text{gst}}$ the inclusion. Then*

$$\tilde{\gamma}_* \circ s_{\tilde{V}^{\text{cpt}}}^! \circ \iota_! = i_* \circ s_{\bar{\sigma}, \text{loc}}^!: Z_*(\tilde{V}(\bar{\sigma})) \rightarrow A_*(\tilde{\mathcal{Z}}^{\text{gst}}).$$

See [CL15, Proposition 6.4] for full details.

Furthermore, from functoriality of Gysin pullbacks and the deformation to the normal cone it follows that:

$$s_{\tilde{V}^{\text{cpt}}}^![\bar{C}] = s_{\tilde{V}_2^{\text{cpt}}}^! \circ s_{\tilde{V}_1^{\text{cpt}}}^![N_{\bar{C} \cap 0 \oplus \tilde{V}_2^{\text{cpt}}} \bar{C}].$$

Step III: homogeneous cones

Chang and Li introduce the notion of *homogeneity* for substacks of $\tilde{V}$ on $\overline{\mathcal{Z}}^{p,\text{gst}}$: write

$$\tilde{V} = \tilde{V}_1 \oplus \tilde{V}_2 \quad \text{with} \quad \tilde{V}_1 = \mathbb{R}^1 \hat{\pi}_*(\mathcal{L}^{\oplus 5}) \quad \text{and} \quad \tilde{V}_2 = \mathbb{R}^1 \hat{\pi}_*(\mathcal{P}),$$

and $\gamma: \overline{\mathcal{Z}}^{p,\text{gst}} \rightarrow \tilde{\mathcal{Z}}^{\text{gst}}$ the projection. Since $\overline{\mathcal{Z}}^{p,\text{gst}}$ is the total space of a line bundle over $\tilde{\mathcal{Z}}^{\text{gst}}$, it comes with a natural $\mathbb{G}_m$ -action on the fibers of γ . Moreover, since the highest pushforwards satisfy cohomology and base-change:

$$\tilde{V}_i = \gamma^* V_i$$

for the corresponding vector bundles V_i on $\tilde{\mathcal{Z}}^{\text{gst}}$, so the total space of $\tilde{V}_i$ can be endowed with a $\mathbb{G}_m$ -action that makes the projection to $\tilde{\mathcal{Z}}^{p,\text{gst}}$ equivariant. We say that a closed substack of $\tilde{V}$ is 0-homogeneous if it is the pullback of a closed substack of V along γ . In fact there are different $\mathbb{G}_m$ -actions on $\tilde{V}$ that make the projection to $\overline{\mathcal{Z}}^{p,\text{gst}}$ equivariant: namely, we can twist the trivial action on the fibers by two characters of $\mathbb{G}_m$, one for each

$\tilde{V}_i$. Then we say that a substack of $\tilde{V}$ is (l_1, l_2) -homogeneous if it is invariant with respect to such an action. Here is how we are going to use the homogeneity:

Lemma 6.33. *Let $C \subseteq \tilde{V}$ be an (l_1, l_2) -homogeneous subcone of $\tilde{V}$; then the cone $\overline{C} \cap (0 \oplus \tilde{V}_2^{\text{cpt}})$ is pulled back from a cone in V_2 on $\tilde{\mathcal{Z}}^{\text{gst}}$.*

Proof. Locally we may pick coordinates t on the fibers of γ , $x_1, \dots, x_5$ on the fibers of $\tilde{V}_1^{\text{cpt}}$, and $y_1, \dots, y_{5d+5}$ on the fibers of $\tilde{V}_2^{\text{cpt}}$, such that the ideal of $\overline{C}$ is generated by separately homogeneous polynomials p_j in $t^{-l_1}x_i$ and $t^{-l_2}y_i$. The ideal of $\overline{C} \cap (0 \oplus \tilde{V}_2^{\text{cpt}})$ is then given by $\langle x_1, \dots, x_5, p_j(0, t^{-l_2}y) \rangle_j$, where $p_j(0, t^{-l_2}y)$ results from setting $x_i = 0$ in p_j . Notice now that $\overline{C}$ being a cone, it is invariant by scalar multiplication on the fibers of $\tilde{V}^{\text{cpt}}$, so we may as well say that $\overline{C} \cap (0 \oplus \tilde{V}_2^{\text{cpt}})$ is cut by the ideal $\langle x_1, \dots, x_5, p_j(0, y) \rangle_j$. This makes it clear that $\overline{C} \cap (0 \oplus \tilde{V}_2^{\text{cpt}})$ is pulled back from $(0 \oplus V_2)$ on $\tilde{\mathcal{Z}}^{\text{gst}}$. $\square$

Finally Chang and Li point out that the coarse moduli cone C^{gst} is $(0, 1)$ -homogeneous [CL12, Proposition 6.7].

Step IV: reduction of the support of the cone

The final goal is to show that $s_{\tilde{V}^{\text{cpt}}}^![\overline{C^{\text{gst}}}]$ is zero, and the way we will do that, it's by a dimension count.

In order to do that, we will in this section consider the following cycle: $\overline{C^{\text{gst}}} \cap 0 \oplus \tilde{V}_2^{\text{cpt}}$. The reason for this is that, using deformation to the normal cone and functoriality of the Gysin pull-back, we get an identity of cycles:

$$s_{\tilde{V}^{\text{cpt}}}^![\overline{C^{\text{gst}}}] = s_{\tilde{V}_2^{\text{cpt}}}^! \circ s_{\tilde{V}_1^{\text{cpt}}}^![N_{\overline{C^{\text{gst}}} \cap (0 \oplus \tilde{V}_2^{\text{cpt}})} \overline{C^{\text{gst}}}].$$

On the other hand, the support of the cycle we obtain from the first Gysin pull-back $s_{\tilde{V}_1^{\text{cpt}}}^!$ is contained in the support of the cone $\overline{C^{\text{gst}}} \cap (0 \oplus \tilde{V}_2^{\text{cpt}})$, so it will be enough to study the dimension of the latter.

The reason why the latter should be easier to understand is that, as we will see below, the inclusion $\overline{C^{\text{gst}}} \cap (0 \oplus \tilde{V}_2^{\text{cpt}}) \subseteq \tilde{V}_2^{\text{cpt}}$ factors through the (coarse moduli of the) intrinsic normal cone $\mathcal{C}_{\tilde{\mathcal{Z}}^p/\tilde{\mathcal{Z}}}$ for $\tilde{\mathcal{Z}}^p \rightarrow \tilde{\mathcal{Z}}$, and this in turn can be described quite explicitly since $\tilde{\mathcal{Z}}^p$ is a line bundle on $\tilde{\mathcal{Z}}$, and we have explicit equations for the latter.

Using the above observation, one can see that $\overline{C^{\text{gst}}} \cap (0 \oplus \tilde{V}_2^{\text{cpt}})$ is contained inside a substack of $\tilde{V}_2^{\text{cpt}}$ which is the union of the zero-section $0_{\tilde{V}_2^{\text{cpt}}}$ (i.e. $\tilde{\mathcal{Z}}^{p, \text{gst}}$) and a line subbundle $\tilde{F}$ of $\tilde{V}_2^{\text{cpt}}$ supported on the intersection of the boundary with main $\tilde{\Xi} = \tilde{\mathcal{Z}}^{p, \text{main}} \cap \tilde{\mathcal{Z}}^{p, \text{gst}}$.

Even better, using the notion of homogeneity we reviewed in § 6.5.1 we can then show that such a line bundle actually comes from $\Xi = \tilde{\mathcal{Z}}^{\text{main}} \cap \tilde{\mathcal{Z}}^{\text{gst}}$, i.e. $\tilde{F} = \gamma^*F$ (this last fact together with push-pull techniques for intersections of cycles will simplify even more the computations).

We give the precise statement and a sketch of the proof, see [CL12, Proposition 7.1] for more details.

Lemma 6.34. *There is a line subbundle F of $V_2|_{\Xi}$ such that:*

$$\overline{C^{\text{gst}}} \cap 0 \oplus \tilde{V}_2^{\text{cpt}} \subseteq 0_{\tilde{V}_2^{\text{cpt}}} \cup \tilde{F} := \overline{\mathcal{Z}^{p,\text{gst}}} \cup \tilde{\gamma}^* F$$

It is enough to show this before taking the closure (since C^{gst} is integral). First they use the fact that there is a triple of compatible obstruction theories for the triangle:

$$\begin{array}{ccc} \tilde{\mathcal{Z}}^p & \xrightarrow{\gamma} & \tilde{\mathcal{Z}} \\ & \searrow & \swarrow \\ & \tilde{\mathcal{X}}\tilde{\mathcal{P}} & \end{array}$$

such that their restrictions to $\tilde{\mathcal{Z}}^{p,\text{gst}}$ have locally free h^0 and h^1 . By taking h^1 of the dual obstruction theories we obtain a commutative diagram:

$$\begin{array}{ccccc} h^1(\mathbb{L}_{\tilde{\mathcal{Z}}^p/\tilde{\mathcal{Z}}}^\vee|_{\tilde{\mathcal{Z}}^{p,\text{gst}}}) & \longrightarrow & h^1(\mathbb{L}_{\tilde{\mathcal{Z}}^p/\tilde{\mathcal{X}}\tilde{\mathcal{P}}}^\vee|_{\tilde{\mathcal{Z}}^{p,\text{gst}}}) & \longrightarrow & h^1(\gamma^*\mathbb{L}_{\tilde{\mathcal{Z}}/\tilde{\mathcal{X}}\tilde{\mathcal{P}}}^\vee|_{\tilde{\mathcal{Z}}^{p,\text{gst}}}) \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{V}_2 & \xrightarrow{i_2} & \tilde{V}_1 \oplus \tilde{V}_2 & \xrightarrow{\text{pr}_1} & \tilde{V}_1 \end{array}$$

We know that the vertical arrows are injective, essentially by the definition of an obstruction theory, and the bottom triangle is exact.

Notice that $0 \oplus \tilde{V}_2$ is precisely the kernel of pr_1 . It follows that, in order to understand the support of $\overline{C^{\text{gst}}} \cap 0 \oplus \tilde{V}_2$, it is enough to study that of N , where N is the coarse moduli cone of $\mathcal{C}_{\tilde{\mathcal{Z}}^p/\tilde{\mathcal{Z}}}$, living in the upper left corner of the above diagram.

This is an easier task, since we know that $\tilde{\mathcal{Z}}^p/\tilde{\mathcal{Z}}$ is a line bundle on $\tilde{\mathcal{Z}}^{\text{gst}}$ and an isomorphism on $\tilde{\mathcal{Z}}^{\text{main},\circ}$.

Hence we can always find a local chart $S \rightarrow \tilde{\mathcal{Z}}$ and a diagram as follows:

$$\begin{array}{ccccc} \tilde{\mathcal{Z}}^p & \longleftarrow & T & \xrightarrow{V(\tilde{\zeta}t)} & S \times \mathbb{A}_t^1 \\ \downarrow & & \downarrow & \swarrow & \\ \tilde{\mathcal{Z}} & \xleftarrow{\text{ét}} & S & & \end{array} \quad \square$$

where $\tilde{\zeta}$ is a local equation for the boundary. Then

$$\tau^{\geq -1} \mathbb{L}_{\tilde{\mathcal{Z}}^p/\tilde{\mathcal{Z}}}|_T = [I/I^2 \xrightarrow{\delta} \Omega_{\mathbb{A}_S^1/S}]$$

where I is generated by the equation $\tilde{\zeta}t$, whose image under δ is $\tilde{\zeta}dt$, which restricts to 0 on $\tilde{\mathcal{Z}}^{p,\text{gst}} \times_{\tilde{\mathcal{Z}}^p} T = \{\tilde{\zeta} = 0\}$. So the action is trivial, and the coarse moduli cone is precisely $\text{Spec}_{T^{\text{gst}}} \text{Sym}^\bullet I/I^2$, which is a line bundle supported on $\tilde{\Xi} \times_{\tilde{\mathcal{Z}}^p} T$ and trivial otherwise. By gluing different charts we get the line bundle $\tilde{F}$ on $\tilde{\Xi}$.

The last part of the statement, namely that $\tilde{F}$ is the pull-back of a line bundle F on Ξ is proved using Lemma 6.33: the normal cone of $\tilde{\mathcal{Z}}^p/\tilde{\mathcal{Z}}$ is homogeneous with respect

to the $\mathbb{G}_m$ -action with character 1 on the fibers (because so is the obstruction theory) of $\tilde{V}_2 \rightarrow \tilde{\mathcal{Z}}^{p,\text{gst}}$; furthermore, being a cone, it is 0-homogeneous as well (see the above discussion of homogeneity), so it is $\tilde{\gamma}^*F$ for some line bundle F on $\Xi \subseteq \tilde{\mathcal{Z}}$.

Step V: the boundary pushes forward to zero

Recall that we need to show that the degree of the following class is 0:

$$s_{\tilde{V}_2^{\text{cpt}}}^![\overline{C^{\text{gst}}}] = s_{\tilde{V}_2^{\text{cpt}}}^! \circ s_{\tilde{V}_1^{\text{cpt}}}^![N_{\overline{C^{\text{gst}}} \cap (0 \oplus \tilde{V}_2^{\text{cpt}})} \overline{C^{\text{gst}}}]$$

It follows from the previous section that $s_{\tilde{V}_1^{\text{cpt}}}^![N_{\overline{C^{\text{gst}}} \cap (0 \oplus \tilde{V}_2^{\text{cpt}})} \overline{C^{\text{gst}}}]$ can be represented by the sum of two cycles, one (call it N_1) supported on a line subbundle of $\tilde{V}_2^{\text{cpt}}$ on Ξ , the other one (call it N_2) supported on the zero section of $\tilde{V}_2^{\text{cpt}}$.

Lemma 6.35. *Both N_1 and N_2 are $5d + 1$ -dimensional cycles, and for $i = 1, 2$:*

$$\deg(s_{\tilde{V}_2^{\text{cpt}}}^![N_i]) = 0.$$

Proof. Compare with [CL12, Lemma 8.1]. The dimension of $\tilde{\mathcal{Z}}^{\text{gst}}$ is $5d + 3$, being locally a $5(d + 1)$ vector bundle over a dimension -2 stack; so $\tilde{\mathcal{Z}}^{p,\text{gst}}$, which is a line bundle on the former, has dimension $5d + 4$. The coarse moduli cone has then dimension $5d + 6$, as can be argued from Lemma 6.28. $\tilde{V}_1^{\text{cpt}}|_{\tilde{\mathcal{Z}}^{p,\text{gst}}}$ has rank 5, so $s_{\tilde{V}_1^{\text{cpt}}}^![\overline{C^{\text{gst}}}]$ is represented by a cycle of dimension $5d + 1$. We shall exploit the commutativity of Gysin pullback with proper pushforward.

For N_1 we conclude easily from the above facts, since $\deg(s_{\tilde{V}_2^{\text{cpt}}}^![N_1]) = \deg(s_{\tilde{V}_2}^! \tilde{\gamma}_*[N_1])$, but $\tilde{\gamma}_*[N_1] \in A_{5d+1}(F)$ must be trivial, since F has dimension $5d$, being a line bundle on Ξ which is a divisor in $\mathcal{Z}^{\text{main}}$.

On the other hand $N_2 \subseteq \tilde{V}_2^{\text{cpt}}$ admits a further splitting into $N_{2,\mu} \subseteq \tilde{V}_{2,\mu}^{\text{cpt}}$ according to the component $D^\mu \tilde{\mathcal{Z}}^{p,\text{gst}}$ on which they are supported, with $\mu \vdash d$ in k parts. There exists a comparison morphism:

$$\beta_\mu: D^\mu \tilde{\mathcal{Z}}^{p,\text{gst}} \rightarrow D^\mu \mathcal{Z} = D^\mu \overline{\mathcal{M}}_1(\mathbb{P}^4, d) \rightarrow \mathcal{W}_\mu$$

where $\mathcal{W}_\mu := \prod_{i=1}^k \overline{\mathcal{M}}_{0,1}(\mathbb{P}^4, d_i) \times_{(\mathbb{P}^4)^k} \mathbb{P}^4$. The map β_μ is given by forgetting the p -field, the Vakil-Zinger blow-up and the k -pointed elliptic curve contracted by the map to $\mathbb{P}^4$. This has the nice property that $\tilde{V}_2^{\text{cpt}}$ is the pullback along β_μ of a vector bundle on $\mathcal{W}_\mu$. First construct a connected curve $\overline{\mathcal{C}}_\mu$ by gluing the universal curve over each factor along the given sections, producing a genus 0 non-Gorenstein (unless $k = 2$) singularity to which

the universal maps to $\mathbb{P}^4$ descend by the property of pushouts:

$$\begin{array}{ccc} \overline{\mathcal{C}}_\mu & \xrightarrow{\tilde{f}} & \mathbb{P}^4 \\ \downarrow \tilde{\pi} & & \\ \mathcal{W}_\mu & & \end{array}$$

Notice now that the sheaf $V_\mu := \tilde{\pi}_* \tilde{f}^* \mathcal{O}_{\mathbb{P}^4}(5)$ is a vector bundle of rank $5d + 1$ on $\mathcal{W}_\mu$, as can be checked by Riemann-Roch and the normalisation sequence, and $\tilde{V}_{2,\mu}^{\text{cpt}} = \beta_\mu^*(V_\mu^\vee)$. Finally the actual dimension of $\mathcal{W}_\mu$ is $5d + 4 - 2k < 5d + 1$ for $k \geq 2$ by Kleiman-Bertini theorem, so $\beta_{\mu,*}[N_{2,\mu}] = 0$. $\square$

Proof. (Main Theorem) For the benefit of the reader we summarise here the key results which allow us to conclude:

- p -fields give the same invariants as the quintic up to a sign, see Theorem 6.10:

$$\deg[\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p]_{\text{loc}}^{\text{vir}} = (-1)^{5d} \deg[\overline{\mathcal{M}}_1^{(1)}(X, d)]^{\text{vir}}.$$

- $\mathcal{Z}^p$ and $\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p$ are virtually birational, see Corollary 6.18:

$$\deg[\mathcal{Z}^p]_{\text{loc}}^{\text{vir}} = \deg[\overline{\mathcal{M}}_1^{(1)}(\mathbb{P}^4, d)^p]_{\text{loc}}^{\text{vir}}.$$

- The desingularisation $\tilde{\mathcal{Z}}^p \rightarrow \mathcal{Z}^p$ does not alter the invariants, see Lemma 6.27

$$\deg[\tilde{\mathcal{Z}}^p]_{\text{loc}}^{\text{vir}} = \deg[\mathcal{Z}^p]_{\text{loc}}^{\text{vir}}.$$

- The main component of $\tilde{\mathcal{Z}}^p$ contributes with the reduced invariants up to a sign, while the boundary is numerically irrelevant, see Lemmas 6.31 and 6.35:

$$\deg[\tilde{\mathcal{Z}}^p]_{\text{loc}}^{\text{vir}} = (-1)^{5d} \deg\left(c_{\text{top}}(\tilde{\pi}_* \tilde{f}^* \mathcal{O}_{\mathbb{P}^4}(5) \cap [\widetilde{\mathcal{M}}_1(\mathbb{P}^4, d)^{\text{main}}])\right).$$

$\square$

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