

Dissipative dynamics for infinite lattice systems

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Abstract

We study dissipative dynamics constructed by means of non-commutative Dirichlet forms for various lattice systems with multiparticle interactions associated to CCR algebras. We give a number of explicit examples of such models. Using an idea of quasi-invariance of a state, we show how one can construct unitary representations of various groups. Moreover in models with locally conserved quantities associated to an infinite lattice we show that there is no spectral gap and the corresponding dissipative dynamics decay to equilibrium polynomially in time.

Contents

1	Introduction	2
2	The Infinite Quantum System	3
2.1	Domain Issues	6
2.2	Adjoint Operators	7
3	Quasi Invariance and Unitary Group Representations	8
4	Modular Dynamics and Finite Speed of Propagation of Information	9
5	Convergence of $\mathbb{L}_p$ Norms	11
6	Noncommutative Dirichlet forms and Markov Generators	13
7	Examples of models	14
7.1	Mean Field Models	15
7.2	Non-diagonal Dirichlet Forms	17
8	No Spectral Gap Property	24
9	Algebra of Invariant Derivations	25
10	Polynomial Decay to Equilibrium	26
11	Appendix: Γ_1	29

1 Introduction

The theory of quantum dissipative systems has grown rapidly in the last few decades (see e.g. [CZ] and references therein). Such systems can be described by an equation of the following form

$$\frac{\partial}{\partial t} P_t f = \mathfrak{L} P_t f, \quad P_0 = id \quad (1)$$

where $\mathfrak{L}$ is a Markov generator and $P_t \equiv e^{t\mathfrak{L}}$ is the associated Markov semigroup. The semigroup P_t is positivity and unit preserving. An easier way to prove closability and dissipativeness of $\mathfrak{L}$ is to consider a symmetric semibounded operator in a Hilbert space. A delicate problem in quantum case is how to simultaneously have symmetry in a Hilbert space associated to a state and positivity preservation of the generated semigroup. One of the solutions to this problem come via proving the closability and Markov property of the pre-Dirichlet form. The theory of Dirichlet forms originated in [AHK] for the trace state and fully developed later in [Cip1]; for further development see e.g. references in [Cip3], [P1],[Z],[CZ]. In this work, we assume that the generator $\mathfrak{L}$ is constructed using Quantum Dirichlet form given by

$$\mathcal{E}_\Lambda(f) \equiv \sum_{j \in \Lambda} \int_{\mathbb{R}} \left(\nu_j \langle \delta_{\alpha_t(X_j)}(f), \delta_{\alpha_t(X_j)}(f) \rangle_\omega + \mu_j \langle \delta_{\alpha_t(X_j^*)}(f), \delta_{\alpha_t(X_j^*)}(f) \rangle_\omega \right) \eta(t) dt \quad (2)$$

for ν_j, μ_j and $\Lambda \subseteq \mathbb{Z}^d$, with a scalar product

$$\langle f, g \rangle_\omega \equiv \lim_{\Lambda \rightarrow \mathbb{Z}^d} \langle f, g \rangle_{\omega_\Lambda} = \lim_{\Lambda \rightarrow \mathbb{Z}^d} \text{Tr}(\rho_\Lambda^{1/2} f^* \rho_\Lambda^{1/2} g)$$

where ω_Λ is a finite volume state corresponding to a density matrix $\rho_\Lambda \equiv e^{-\beta U_\Lambda} / \text{Tr}_\Lambda(e^{-\beta U_\Lambda})$, and the modular dynamics corresponding to ω is defined by

$$\alpha_t(B) = \lim_{\Lambda \rightarrow \mathbb{Z}^d} e^{-it\beta U_\Lambda} B e^{it\beta U_\Lambda}. \quad (3)$$

The function $\eta(t)$ is an admissible function in the sense of Park, [P1], (see Definition (6.1) below), and the operators X_j 's are chosen appropriately later for specific cases.

By the Beurling Deny theory([BD]), (with its noncommutative generalisation [Cip1, Cip3]), providing a one to one correspondence between Markov semigroups and Dirichlet forms, the problem of analysing the generator of the Markov semigroups is translated to studying their corresponding Dirichlet forms. We say that the semigroup P_t converges to equilibrium if $\lim_{t \rightarrow \infty} P_t(f) = \omega(f)$ where the limit is with respect to the noncommutative L_p spaces or the norm of the algebra. Frequently one can study decay to equilibrium in L_2 and L_p spaces using coercive inequalities of Poincare and Log Sobolev type that were studied e.g. in [OZ, CB] and references therein.

We discuss construction of Dirichlet forms for models of large interacting systems. As a starting point we use the quantum harmonic oscillator (refer [CFL]). Given a Hilbert space, say $\mathfrak{h}$ with $\{e_n\}$ as the orthonormal basis, one defines the creation and annihilation operators by

$$A^* e_n = \sqrt{n+1} e_{n+1} \quad (4)$$

$$A e_n = \sqrt{n} e_{n-1} \quad (5)$$

and the associated particle number operator

$$N e_n \equiv A^* A e_n = n e_n$$

with dense domains $D(A) = D(A^*) = D(\sqrt{N})$. For $U = N$ we define a density matrix $\rho = \frac{1}{Z} e^{-\beta U}$, with a normalization constant $Z \in (0, \infty)$. In commutative analysis, one considers an Ornstein-Uhlenbeck (O-U)

semigroups defined by the Dirichlet generator associated to a Gaussian measure. The noncommutative generalisation of O-U semigroup is introduced in [CFL],[KP] with the generator ([CM]) given by the extension of the Alicki's theorem

$$\mathfrak{L}f = \sum_j \left(-e^{-\frac{\beta}{2}} [V_j, f] V_j^* + e^{\frac{\beta}{2}} V_j^* [V_j, f] \right) \quad (6)$$

where V_j are the eigenvectors of the modular operator (9). The quantum Markov semigroups generated by the generators of the form (6) were studied in [CM] in the finite dimensional setting where the authors obtained some entropic dissipation inequalities for the necessarily infinite-dimensional Bose O-U semigroup.

In case of infinite systems with interaction, in order to be able to define Dirichlet forms and generators on dense domains, the finite speed of propagation of information for the hamiltonian dynamics needs to be satisfied. The case with bounded multi-particle interactions for quantum spin systems was established in [LR]. We generalise it to the systems with multiparticle interactions involving unbounded operators.

Some dissipative dynamics for the quantum spin systems on the lattice were earlier discussed in [MZ], [GM], [Z], where they show the existence of dynamics that have an exponential decay to equilibrium in the high-temperature region. General systems with quadratic interaction were studied in [OZa], [BKP] and [FQ]. Because of the presence of unbounded operators in the Markov generators, these issues are more complicated than otherwise.

In this work, we construct the models of interacting dissipative systems on a finite dimensional lattice $\mathbb{Z}^d$ with finite range interaction. We consider the multiparticle interaction and generalise the setup of quantum spin systems used e.g. in [MZ, LR, BR]. For some of our models with locally conserved quantities, we provide the detailed analysis proving that Poincaré inequality cannot be satisfied and the system converges to equilibrium with polynomial rate of convergence. In particular this extends commutative case considered in [INZ].

A quantum Brownian motion model provided in [CFL] has no spectral gap and equilibrium state. Some of our models are more general with no spectral gap at the bottom of spectrum of Markov generator along with the existence of an equilibrium state.

The organisation of this paper is as follows. In Section 2, we describe the infinite dimensional quantum system considered in this work and give some basic notations and definitions. We also address the corresponding domain issues of derivations, their adjoints and Dirichlet forms. Section 3 we explore the idea of quasi-invariance of the states to provide the unitary group representations related to certain transformations of the commutation relations. This includes in particular new representations of Minkowski group.

In Section 4, the modular dynamics is constructed and the finite speed of propagation of information for unbounded potentials is proven. In Section 5, the convergence of states and norms in the corresponding $\mathbb{L}_p$ space is established. We continue in Section 6 with definitions of non-commutative Dirichlet forms and corresponding Markov generators. In Section 7, we provide number of explicit models and their Dirichlet forms and Markov generators. Section 8 and Section 9 establishes that there is no spectral gap for some examples, including the case which is invariant with respect to modular operator derivations.

In Section 10 generalises the result of [INZ] by showing the algebraic decay to equilibrium for the corresponding quantum Markov semigroups. Finally, in the Appendix, we provide computations for the corresponding Γ_1 form and discuss the operator carré du champ type bounds and some of their decay to equilibrium implications.

2 The Infinite Quantum System

The Lattice

For $d \in \mathbb{N}$, let $\mathbb{Z}^d$ be the d -dimensional square lattice with the l_1 lattice metric $dist(\cdot, \cdot)$ defined by

$$dist(\mathbf{i}, \mathbf{j}) := |\mathbf{i} - \mathbf{j}|_1 \equiv \sum_{l=1}^d |i_l - j_l|$$

for $\mathbf{i} = (i_1, \dots, i_d), \mathbf{j} = (j_1, \dots, j_d) \in \mathbb{Z}^d$. Given $R \in (0, \infty)$, for $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$ we write $\mathbf{i} \sim \mathbf{j}$ whenever $0 \leq \text{dist}(\mathbf{i}, \mathbf{j}) \leq R$ and say that $\mathbf{i}$ and $\mathbf{j}$ are neighbours in the lattice. If O is a finite subset of $\mathbb{Z}^d$ we will write $O \subset\subset \mathbb{Z}^d$.

The configuration space

For $\Lambda \subseteq \mathbb{Z}^d$, we associate a separable Hilbert space $\mathcal{H}_\Lambda$ with a property that for bounded sets $\Lambda_1, \Lambda_2 \subset\subset \mathbb{Z}^d$, $\Lambda_1 \subset \Lambda_2$, we have $\mathcal{H}_{\Lambda_1} \subset \mathcal{H}_{\Lambda_2}$ and $\mathcal{H}_{\mathbb{Z}^d} = \overline{\bigcup_{\Lambda \subset\subset \mathbb{Z}^d} \mathcal{H}_\Lambda}$. For a finite set $O \subset\subset \Lambda$, let $\mathcal{A}_O \subset \mathcal{L}(\mathcal{H}_O)$ denote the algebra of bounded operators. Let $\mathcal{A} = \bigcup_{O \subset\subset \Lambda} \mathcal{A}_O$ be an inductive limit algebra [BR] generated by all local algebras.

We assume, we have a family of mutually commuting copies of creation and annihilation operators satisfying the CCR given by

$$[A_j, A_j^*] = id, \quad [A_j, A_k^\#] = 0, \quad j \neq k. \quad (7)$$

where $A_k^\# \in \{A_k, A_k^*\}$. For $O \subset\subset \mathbb{Z}^d$, let $\mathcal{D}_O$ denote the family of finite polynomials in the creation and annihilation operators $A_k^\#, k \in O$ and let $\mathcal{D} \equiv \bigcup_{O \subset\subset \mathbb{Z}^d} \mathcal{D}_O$. A particle number operator at $j \in \mathbb{Z}^d$ is defined by

$$N_j \equiv A_j^* A_j$$

Remark

The following relations will be useful in proving many claims involving creators/annihilators and particle number operators. For analytic function h , (using a basis of eigenvectors for N), one can see that

$$\begin{aligned} A^* h(N) &= h(N-1) A^* \quad \text{and} \quad h(N) A^* = A^* h(N+1) \\ A h(N-1) &= h(N) A \quad \text{and} \quad h(N+1) A = A h(N). \end{aligned} \quad (8)$$

For $\beta \in (0, \infty)$, let

$$Z_o \equiv \text{Tr}_j \left(e^{-\beta N_j} \right)$$

which is finite, positive and independent of $j \in \mathbb{Z}^d$, and for $\Lambda \subset\subset \mathbb{Z}^d$ let

$$\rho_{o, \Lambda} \equiv \prod_{j \in \Lambda} \frac{1}{Z_o} e^{-\beta N_j}$$

For $\Lambda \subset \mathbb{Z}^d$ define on $\mathcal{D} \cup \mathcal{A}$ the following linear maps

$$\mathbb{E}_{o, \Lambda, s, p}(f) \equiv \text{Tr}_\Lambda \left(\rho_{o, \Lambda}^{(1-s)/p} f \rho_{o, \Lambda}^{s/p} \right)$$

with $0 \leq s \leq 1$ and $p \in [1, \infty)$. On $\mathcal{D} \cup \mathcal{A}$ we consider an infinite product state

$$\omega_o(f) \equiv \lim_{\Lambda \rightarrow \mathbb{Z}^d} \mathbb{E}_{o, \Lambda, s}(f)$$

which is well defined on $\mathcal{D} \cup \mathcal{A}$. The corresponding $\mathbb{L}_p(\omega_o)$, $p \in [1, \infty)$, functional is well defined on $\mathcal{D}$ (and $\mathcal{A}$) by

$$\|f\|_{\omega_o, p, s}^p = \lim_{\Lambda \rightarrow \mathbb{Z}^d} \text{Tr}_\Lambda \left| \rho_{o, \Lambda}^{(1-s)/p} f \rho_{o, \Lambda}^{s/p} \right|^p$$

where $|g| \equiv (g^* g)^{\frac{1}{2}}$. Replacing the limits by $\Lambda \rightarrow O$ for any unbounded $O \subset \mathbb{Z}^d$, we can introduce the corresponding state $\omega_{o, O}$ and the corresponding $\mathbb{L}_p(\omega_{o, O})$, $p \in [1, \infty)$, norms.

Interaction

A potential $\Phi = \{(\Phi_O)_{O \subset\subset \mathbb{Z}^d}\}$ of finite range $R \in (0, \infty)$ is a family of selfadjoint densely defined operators $\Phi_O \in \mathcal{A}_O$ (or $\mathcal{D}_O$), such that

$$\Phi_O = 0 \text{ if } \text{diam}(O) \geq R.$$

The potential energy for a finite set $\Lambda \subset \mathbb{Z}^d$ is defined by

$$U_\Lambda = \sum_{O \subset \Lambda} \Phi_O.$$

We assume that the potential has the following thermodynamic stability property

$$\begin{aligned} 0 < Z_\Lambda &\equiv \text{Tr}_\Lambda e^{-U_\Lambda} < \infty \\ 0 < \left| \limsup_{\Lambda \rightarrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log Z_\Lambda \right| &< \infty \end{aligned}$$

In the setup of this paper it is natural to consider the bounded multiparticle interaction in the form of polynomials in bounded operators $a_k^\# \equiv (1 + \varepsilon N_k^{\frac{1}{2}})^{-1} A_k^\#$, $k \in \mathbb{Z}^d$, $\varepsilon \in (0, \infty)$,

$$\Phi_O \equiv \Phi_O(a_k^\#, k \in O).$$

A state and noncommutative $\mathbb{L}_p$ spaces

On $\mathcal{D} \cup \mathcal{A}$, we define a state with respect to the interaction U_Λ

$$\omega^{(\Lambda)}(f) = \omega_o \text{Tr}_\Lambda(\rho_\Lambda f)$$

where the density matrix is given by $\rho_\Lambda \equiv \frac{1}{Z_\Lambda} e^{-\beta U_\Lambda}$ and the associated modular dynamics is defined by

$$\alpha_{t,\Lambda}(B) = \lim_{\tilde{\Lambda} \rightarrow \mathbb{Z}^d} \rho_{o,\tilde{\Lambda} \cap \Lambda^c}^{it} \rho_\Lambda^{it} B \rho_\Lambda^{-it} \rho_{o,\tilde{\Lambda} \cap \Lambda^c}^{-it}. \quad (9)$$

The corresponding scalar product is given as follows

$$\langle f, g \rangle_{\omega^{(\Lambda)}} = \omega^{(\Lambda)}((\alpha_{-i/2,\Lambda}(f))^* g) = \omega_\Lambda(f^*(\alpha_{-i/2,\Lambda}(g))).$$

Under suitable conditions (discussed later, which hold e.g. for finite range uniformly bounded weak multiparticle potential or quadratic ones) the following limits exist, (see [BR] for bounded quantum spin systems),

$$\begin{aligned} \omega(f) &\equiv \lim_{\Lambda \rightarrow \mathbb{Z}^d} \omega_\Lambda(f) \\ \alpha_t(f) &\equiv \lim_{\Lambda \rightarrow \mathbb{Z}^d} \alpha_{t,\Lambda}(f) \end{aligned}$$

and consequently the following scalar product is well defined

$$\langle f, g \rangle_\omega = \lim_{\Lambda \rightarrow \mathbb{Z}^d} \langle f, g \rangle_{\omega_\Lambda} = \omega(f^*(\alpha_{-i/2}(g))) = \omega((\alpha_{-i/4}(f))^*(\alpha_{-i/4}(g))).$$

The corresponding $\mathbb{L}_p(\omega)$, for any $p \in [1, \infty)$, functional is well defined on $\mathcal{D} \cup \mathcal{A}$ by

$$\|f\|_{\omega,p,s}^p = \lim_{\Lambda \rightarrow \mathbb{Z}^d} \text{Tr}_\Lambda \left| \rho_\Lambda^{(1-s)/p} f \rho_\Lambda^{s/p} \right|^p$$

where $|g| \equiv (g^* g)^{\frac{1}{2}}$. Note that in particular case $p \in \mathbb{N}$ we have the following expression, see e.g. [MZ],

$$\|f\|_{\omega,p,1/2}^p = \omega(\alpha(i/2p)(f) \alpha(3i/2p)(f) \dots \alpha((2p-1)i/2p)(f))$$

where $\alpha(qi/2p)(f) \equiv \alpha_{qi/2p}(f)$, for a suitable class of operators f .

Derivations

A derivation δ_X in direction $X \in \mathcal{A}$ (resp. $\mathcal{D}$) on $D(\delta_X) = \mathcal{A}$ (resp. $\mathcal{D}$) is defined by

$$\delta_X(B) = i[X, B].$$

Proposition 2.1. *If $X^*X, XX^* \in \mathbb{L}_2(\omega)$, then the derivation is well defined on a domain*

$$D(\delta_X) \supset \{B^*B, BB^* \in \mathbb{L}_2(\omega)\}.$$

In particular for any $X \in \mathcal{D} \cup \mathcal{A}$

$$\mathcal{D} \cup \mathcal{A} \subset D(\delta_X).$$

The above condition holds if $X, \alpha_{\pm i/4}(X) \in \mathbb{L}_4(\omega)$ and then we have $D(\delta_X) \supset \{B, \alpha_{\pm i/4}(B) \in \mathbb{L}_4(\omega)\}$.

Proof. : We note that

$$\langle \delta_X(B), \delta_X(B) \rangle_\omega \leq 2\|XB\|_\omega^2 + 2\|BX\|_\omega^2$$

If $\tilde{\omega}(\cdot) = \text{Tr}(\tilde{\rho} \cdot)$, with a density matrix $\tilde{\rho}$, then using the definition of the norm and Cauchy-Schwartz inequality for trace we get

$$\begin{aligned} \|XB\|_\omega^2 &= \text{Tr}(\tilde{\rho}^{\frac{1}{2}} B^* X^* \tilde{\rho}^{\frac{1}{2}} XB) = \text{Tr}(B \tilde{\rho}^{\frac{1}{2}} B^* \cdot X^* \tilde{\rho}^{\frac{1}{2}} X) \\ &\leq \|B^*B\|_\omega^2 \|XX^*\|_\omega^2 \end{aligned}$$

Similarly we have

$$\begin{aligned} \|BX\|_\omega^2 &= \text{Tr}(\tilde{\rho}^{\frac{1}{2}} X^* B^* \tilde{\rho}^{\frac{1}{2}} BX) = \text{Tr}(B^* \tilde{\rho}^{\frac{1}{2}} B \cdot X \tilde{\rho}^{\frac{1}{2}} X^*) \\ &\leq \|BB^*\|_\omega^2 \|X^*X\|_\omega^2 \end{aligned}$$

Next we remark that, with $\tilde{\alpha}_{\frac{i}{4}}(X) \equiv \tilde{\rho}^{-\frac{1}{4}} X \tilde{\rho}^{\frac{1}{4}}$, we have

$$\begin{aligned} \|X^*X\|_\omega^2 &= \text{Tr}((\tilde{\rho}^{\frac{1}{8}} X^* \tilde{\rho}^{\frac{1}{8}})(\tilde{\rho}^{\frac{1}{8}} \tilde{\rho}^{-\frac{1}{4}} X \tilde{\rho}^{\frac{1}{4}} \tilde{\rho}^{\frac{1}{8}})(\tilde{\rho}^{\frac{1}{8}} X^* \tilde{\rho}^{\frac{1}{8}})(\tilde{\rho}^{\frac{1}{8}} \tilde{\rho}^{-\frac{1}{4}} X \tilde{\rho}^{\frac{1}{4}} \tilde{\rho}^{\frac{1}{8}})) \\ &= \text{Tr}((\tilde{\rho}^{\frac{1}{8}} X^* \tilde{\rho}^{\frac{1}{8}})(\tilde{\rho}^{\frac{1}{8}} \tilde{\alpha}_{\frac{i}{4}}(X) \tilde{\rho}^{\frac{1}{8}})(\tilde{\rho}^{\frac{1}{8}} X^* \tilde{\rho}^{\frac{1}{8}})(\tilde{\rho}^{\frac{1}{8}} \tilde{\alpha}_{\frac{i}{4}}(X) \tilde{\rho}^{\frac{1}{8}})) \\ &\leq \|X^*\|_{\omega,4}^2 \|\tilde{\alpha}_{\frac{i}{4}}(X)\|_{\omega,4}^2 \end{aligned}$$

and similarly

$$\|XX^*\|_\omega^2 \leq \|X\|_{\omega,4}^2 \|\tilde{\alpha}_{\frac{i}{4}}(X^*)\|_{\omega,4}^2$$

The rest follows by approximation of a state by normal states. $\square$

2.1 Domain Issues

We discuss briefly the domain questions when single point potential is given by $\Phi_j \equiv V(N_j)$ with unbounded real function V (and the rest of the potential is bounded of finite range). Consider the one point case with the state ω_V given by density matrix $\exp\{-\beta V(N)\}/Z$. Then we notice that, using (8), we have

$$\begin{aligned} \alpha_{is}(A^*) &\equiv e^{sV(N)} A^* e^{-sV(N)} = e^{s(V(N+1)-V(N))} A^* \\ \alpha_{-is}(A) &\equiv e^{-sV(N)} A e^{sV(N)} = e^{s(V(N+1)-V(N))} A \end{aligned}$$

If V is linear or sublinear the exponential multiplier is bounded. However for V growing faster than linearly, the exponential multiplier is an unbounded operator and $\alpha_{\pm is}(A^\sharp)$ may not be in $L_p(\omega_V)$ for some $s \neq 0$ and all sufficiently large $p \in [0, \infty)$. In this case it is necessary to consider a suitable replacement for the space $\mathcal{D}$. We propose to consider the following set

$$\mathcal{D}_{\varepsilon, \Lambda} \equiv \{F e^{-\sum_{j \in \Lambda} \varepsilon V(N_j) N_j^\delta} : F \in \mathcal{D}_\Lambda, \varepsilon, \delta \in (0, \infty)\}$$

and for infinite system

$$\mathcal{D}_\varepsilon \equiv \bigcup_{\Lambda \subset \mathbb{Z}^d} \mathcal{D}_{\varepsilon, \Lambda}.$$

Such set is dense in the closure of $\mathcal{D}$ with respect to L_p norms.

2.2 Adjoint Operators

We define left L_X , respectively right R_X , multiplication operator by an operator X as follows

$$L_X f \equiv Xf, \quad R_X f \equiv fX$$

provided $\text{Ran}(f) \subset D(X)$ and $\text{Ran}(X) \subset D(f)$, respectively.

We have the following expressions for corresponding adjoints with respect to the scalar product.

$$\begin{aligned} \langle L_X(g), f \rangle &= \text{Tr} \left(\rho^{\frac{1}{2}} (Xg)^* \rho^{\frac{1}{2}} f \right) = \text{Tr} \left(\rho^{\frac{1}{2}} g^* \rho^{\frac{1}{2}} \left(\rho^{-\frac{1}{2}} X^* \rho^{\frac{1}{2}} \right) f \right) \\ &= \langle g, L_{\alpha_{i/2}(X^*)} f \rangle \end{aligned}$$

and similarly we get

$$\langle R_X(g), f \rangle = \langle g, R_{\alpha_{-i/2}(X^*)} f \rangle.$$

Denoting the adjoint operation by $\star$, we summarise this as follows

$$L_X^\star = L_{\alpha_{i/2}(X^*)}, \quad R_X^\star = R_{\alpha_{-i/2}(X^*)}.$$

Next we discuss adjoint of derivation with respect to a scalar product associated to a normal state $\omega(\cdot) \equiv \text{Tr}(\rho \cdot)$, with density operator ρ with respect to a trace Tr . We have

$$\begin{aligned} \langle \delta_X(f), g \rangle_\omega &\equiv \text{Tr}(\rho^{1/2} (\delta_X(f))^* \rho^{1/2} g) = \text{Tr}(\rho^{1/2} (i[X^*, f^*]) \rho^{1/2} g) \\ &= \text{Tr}(\rho^{1/2} f^* \rho^{1/2} (g \rho^{1/2} iX^* \rho^{-1/2} - \rho^{-1/2} iX^* \rho^{1/2} g)) \end{aligned}$$

Alternatively using the fact that

$$\delta_X(f) = i(L_X(f) - R_X(f)),$$

we have

$$\delta_X^\star = -i(L_X^\star - R_X^\star) = i(R_{\alpha_{-i/2}(X^*)} - L_{\alpha_{i/2}(X^*)}).$$

Summarising, we have the following property.

Proposition 2.2. For $X \in \mathcal{D}_\varepsilon$

$$\begin{aligned} \delta_X^\star(g) &= i(R_{\alpha_{-i/2}(X^*)}(g) - L_{\alpha_{i/2}(X^*)}(g)) \\ &= i(g\alpha_{-i/2}(X^*) - \alpha_{i/2}(X^*)g) \\ &= -\delta_{\alpha_{-i/2}(X^*)}(g) - i(\alpha_{i/2}(X^*) - \alpha_{-i/2}(X^*))g \\ &= -\delta_{\alpha_{i/2}(X^*)}(g) - gi(\alpha_{i/2}(X^*) - \alpha_{-i/2}(X^*)) \\ &= -\delta_{\frac{1}{2}(\alpha_{i/2}(X^*) + \alpha_{-i/2}(X^*))}(g) + \left\{ \frac{1}{2}i(\alpha_{-i/2}(X^*) - \alpha_{i/2}(X^*)), g \right\} \end{aligned}$$

on a dense domain $\mathcal{D}(\delta_X^\star) \supset \mathcal{D}$. Moreover, for $f, g \in \mathcal{D}_\varepsilon$, we have modified Leibnitz rule

$$\begin{aligned} \delta_X^\star(fg) &= \delta_X^\star(f)g - f\delta_{\alpha_{-i/2}(X^*)}(g) \\ &= f\delta_X^\star(g) - \delta_{\alpha_{i/2}(X^*)}(f)g \\ &= \delta_X^\star(f)g + f\delta_X^\star(g) - if(\alpha_{-i/2}(X^*) - \alpha_{i/2}(X^*))g. \end{aligned}$$

Remark: We remark that in case when

$$\alpha_{\pm i/2}(X) = e^{\pm \xi} X$$

and so

$$\alpha_{\mp i/2}(X^*) = e^{\pm \xi} X^*$$

for some $\xi \in \mathbb{R}$, the above formulas simplify as follows

$$\begin{aligned}
\delta_X^*(g) &= i \left(g(e^\xi X^*) - (e^{-\xi} X^*)g \right) \\
&= -e^\xi \delta_{X^*}(g) + 2i \sinh(\xi) X^* g \\
&= -e^{-\xi} \delta_{X^*}(g) + 2i \sinh(\xi) g X^* \\
&= -\cosh(\xi) \delta_{X^*}(g) + i \sinh(\xi) \{X^*, g\}
\end{aligned}$$

and for the modified Leibnitz rule as follows

$$\begin{aligned}
\delta_X^*(fg) &= \delta_X^*(f)g - e^\xi f \delta_{X^*}(g) \\
&= f \delta_X^*(g) - e^{-\xi} \delta_{X^*}(f)g.
\end{aligned}$$

On the algebra associated to $\xi = 0$, as e.g. algebra of functions of N in case of quantum harmonic oscillator, we recover perfect classical formulas.

3 Quasi Invariance and Unitary Group Representations

In this section we discuss briefly unitary representations of some groups. One of the motivations for that is the fact that the generators of such representations can be used in constructions of dissipative dynamics discussed later on.

Suppose, $\theta, \tau \in \mathbb{C}$ satisfy $|\tau|^2 - |\theta|^2 = 1$. Then the operators

$$a \equiv a(\tau, \theta) \equiv \tau A + \theta A^*, \quad a^* \equiv a^*(\tau, \theta) \equiv \bar{\tau} A^* + \bar{\theta} A$$

satisfy CCR

$$[a, a^*] = id.$$

The transformation of this type is known in the mathematical/theoretical physics literature under the name of Bogolubov transformations.

We can consider here a class of transformations which includes the Lorentz group acting on two dimensional CCR vectors via matrices,

$$\mathbb{R} \ni t \mapsto \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}$$

(which can be extend to complex parameter). For any polynomial function f and interaction energy U for which $Tr(f(A, A^*)e^{-U(A, A^*)})$ is well defined, we have

$$Tr(f(a, a^*)e^{-U(a, a^*)}) = Tr(f(a, a^*)e^{-U(A, A^*)}).$$

For a differentiable function $\mathbb{R} \ni s \rightarrow \{(\tau(s), \theta(s)) : |\tau(s)|^2 - |\theta(s)|^2 = 1\}$, we define

$$a_s \equiv a(\tau(s), \theta(s)), \quad a_s^* \equiv a(\tau(s), \theta(s))^*,$$

with the initial condition $a_0 = A, a_0^* = A^*$. Given initial density $\rho \equiv \frac{1}{Z} e^{-U(A, A^*)}$, we define transformed density as follows

$$\rho_s \equiv \frac{1}{Z_s} e^{-U(a_s, a_s^*)}$$

with normalisation factor $Z_s = Tre^{-U(a_s, a_s^*)} = Tre^{-U(A, A^*)}$. Then we have the following result.

Theorem 3.1. *The following formula defines a unitary group representation in $\mathbb{L}_2(\rho)$.*

$$V_s(f(A, A^*)) \equiv \rho^{-\frac{1}{4}} \rho_s^{\frac{1}{4}} f(a_s, a_s^*) \rho_s^{\frac{1}{4}} \rho^{-\frac{1}{4}}$$

The generator of the group on polynomials is given by

$$\partial_s V_s(f)(A, A^*)|_{s=0} = \partial_s f(a_s, a_s^*)|_{s=0} - \left\{ \frac{1}{4} \int_0^1 d\lambda \rho^{\frac{\lambda}{4}} \partial_s U(a_s, a_s^*)|_{s=0} \rho^{\frac{1-\lambda}{4}}, f(A, A^*) - \frac{1}{2} \frac{Z'}{Z} f(A, A^*) \right\}.$$

where curly bracket denote anticommutator and $Z' \equiv (\partial_s Z_s)|_{s=0}$.

To get to the representation of Lorentz group in higher space dimension one needs to consider higher order quantisation of space-time in a form of product space of many independent harmonic oscillators $(A_i, A_i^*)_{i=0,1,\dots,n}$ in which we can consider the following CCR representation

$$(A_i, A_i^*)_{i=1,\dots,n} \mapsto S(\tau, \mathbf{x}) \equiv \tau \frac{1}{\sqrt{n}} \sum_{i=1,\dots,n} A_i + \sum_{i=1,\dots,n} x_i A_i^*.$$

[So in infinite dimensional limit time component is given by a "Gaussian random variable".] Then we have the following representation of the Minkowski scalar product

$$[S, S^*] = |\tau|^2 - |\mathbf{x}|^2.$$

Remark Given $n \in \mathbb{N}$, $n > 1$, CCR pairs $(A_i, A_i^*)_{i=0,1,\dots,n}$, we can define

$$\begin{aligned} A_i(\gamma, \kappa) &\equiv \sum_{j=1,\dots,n} (\gamma_{ij} A_j + \kappa_{ij} A_j^*) \\ A_i^*(\gamma, \kappa) &\equiv \sum_{j=1,\dots,n} (\tilde{\gamma}_{ij} A_j^* + \tilde{\kappa}_{ij} A_j) \end{aligned}$$

with the CCR condition $[A_i(\gamma, \kappa), A_i^*(\gamma, \kappa)] = id$ given by

$$\sum_{j=1,\dots,n} |\gamma_{ij}|^2 - \sum_{j=1,\dots,n} |\kappa_{ij}|^2 = 1$$

Hence we can introduce a group of linear transformations

$$\begin{aligned} \mathbf{T} &\equiv (T, \tilde{T}) : (\gamma, \kappa) \mapsto \mathbf{T}(\gamma, \kappa) \equiv (T(\gamma), \tilde{T}(\kappa)) \\ (\mathbf{T}(\gamma))_i &\equiv \sum_{j=1,\dots,n} (T_{ij} \gamma_{ij}), \quad (\mathbf{T}(\kappa))_i \equiv \sum_{j=1,\dots,n} (\tilde{T}_{ij} \kappa_{ij}) \end{aligned}$$

preserving the CCR condition, i.e. satisfying

$$\sum_{j=1,\dots,n} |T(\gamma)_{ij}|^2 - \sum_{j=1,\dots,n} |\tilde{T}(\gamma)_{ij}|^2 = 1.$$

We can use the idea of the quasi-invariance of a state to obtain unitary representations of this extended group in a similar fashion.

More representations of groups in multicomponent systems are considered later on, (see also [MZ]).

4 Modular Dynamics and Finite Speed of Propagation of Information

It is sufficient to define modular dynamics for operators for creation and anihilation operators. We construct the modular dynamics first on bounded operators defined by polynomials of the following modified creation and anihilation operators

$$a_j^\# \equiv \frac{1}{1 + \epsilon N_j^{\frac{1}{2}}} A_j^\#.$$

For $\Lambda \subset \mathbb{Z}^d$, $\alpha_{t,\Lambda}$ is well defined and for a given $j \in \Lambda$ and $k \notin \Lambda$, we have

$$\begin{aligned}\alpha_{t,\Lambda \cup \{k\}}(a_j^\#) - \alpha_{t,\Lambda}(a_j^\#) &= - \int_0^t ds \frac{d}{ds} \alpha_{s,\Lambda}(\alpha_{t-s,\Lambda \cup \{k\}}(a_j^\#)) \\ &= \int_0^t \sum_{\substack{O \ni k \\ O \neq \{k\}}} \alpha_{s,\Lambda} \delta_{\Phi_O}(\alpha_{t-s,\Lambda \cup \{k\}}(a_j^\#)) ds\end{aligned}$$

Assuming Φ_O are bounded if O is not one point set, this implies the following bound involving operator norm

$$\|\alpha_{t,\Lambda \cup \{k\}}(a_j^\#) - \alpha_{t,\Lambda}(a_j^\#)\| \leq \int_0^t \sum_{\substack{O \ni k \\ O \neq \{k\}}} \|\delta_{\Phi_O}(\alpha_{s,\Lambda \cup \{k\}}(a_j^\#))\| ds$$

To estimate the right hand side we use the following result where

$$c_\Phi \equiv 2 \sup_{|diam(O)| \leq 2R} \sum_{\tilde{O}' \subset O_R} \|\Phi_{O'}\|,$$

where the summation with $\tilde{\cdot}$ runs over sets different than one point sets and O_R denotes a set of points with distance from O bounded by R .

Theorem 4.1. (*Finite speed of propagation of information estimate*)

Assume the potential is of finite range $R \in (0, \infty)$ and $c_\Phi < \infty$. There exist constants $D, C, m \in \mathbb{R}^+$ such that for any $j \in \Lambda \subset \mathbb{Z}^d$ and any $t \in \mathbb{R}^+$ we have

$$\|\delta_{\Phi_O}(\alpha_{t,\Lambda}(a_j^\#))\| \leq D e^{Ct - md(O,j)}$$

where Φ_O is a bounded part of potential localised in a set of size $2R$. The estimate remains valid for $t \in \mathbb{C}$, $|\Im(t)| \leq 1$, with $\Re(t)$ on the right hand side, provided one point interaction $V(N_k)$ is at most linear or $a_j \in \mathcal{D}_\varepsilon$.

Proof. If the interaction is of range $R \in (0, \infty)$, we note that when $\text{dist}(O, j) > 2R$ and $j \in \Lambda \setminus O_R$, we have

$$\delta_{\Phi_O}(\alpha_{s,\Lambda \setminus O_R}(a_j^\#)) = 0,$$

where $O_R \equiv \{l : \text{dist}(l, O) \leq 2R\}$. Hence

$$\|\delta_{\Phi_O}(\alpha_{t,\Lambda}(a_j^\#))\| = \|\delta_{\Phi_O}(\alpha_{t,\Lambda}(a_j^\#) - \alpha_{t,\Lambda \setminus O_R}(a_j^\#))\|$$

We have

$$\begin{aligned}\alpha_{t,\Lambda}(a_j^\#) - \alpha_{t,\Lambda \setminus O_R}(a_j^\#) &= - \int_0^t ds \frac{d}{ds} \alpha_{t-s,\Lambda \setminus O_R}(\alpha_{s,\Lambda}(a_j^\#)) \\ &\quad - \int_0^t ds \alpha_{t-s,\Lambda \setminus O_R} \left(\sum_{\tilde{O}' \subset O_R} \delta_{\Phi_{O'}}(\alpha_{s,\Lambda}(a_j^\#)) \right)\end{aligned}$$

where $\tilde{\cdot}$ over the sum indicates summation over the bounded part of the potential, since $\alpha_{\tau,\Lambda'}$ is defined outside the set Λ' by using the one point potential only and this is cancelled out in the derivation with respect to s . (Here we may need to consider first bounded approximation of the one point potential.) Hence we get

$$\begin{aligned}\|\delta_{\Phi_O}(\alpha_{t,\Lambda}(a_j^\#))\| &\leq 2 \|\delta_{\Phi_O} \left(\int_0^t ds \alpha_{t-s,\Lambda \setminus O_R} \sum_{\tilde{O}' \subset O_R} \delta_{\Phi_{O'}}(\alpha_{s,\Lambda}(a_j^\#)) \right)\| \\ &\leq 2 \|\Phi_O\| \sum_{\tilde{O}' \subset O_R} \int_0^t ds \|\delta_{\Phi_{O'}}(\alpha_{s,\Lambda}(a_j^\#))\|\end{aligned}$$

Since by our assumption

$$c_\Phi \equiv 2 \sup_{|diam(O)| \leq 2R} \sum_{O' \subset O_R} \|\Phi_{O'}\| < \infty,$$

we get the following bound

$$\|\delta_{\Phi_O}(\alpha_{t,\Lambda}(a_j^\#))\| \leq c_\Phi \sum_{O' \subset O_R} \int_0^t ds \|\delta_{\Phi_{O'}}(\alpha_{s,\Lambda}(a_j^\#))\|$$

We iterate this bound, each time getting finite constant multiplier c_Φ , a finite sum of at most $2^{(2R)^d}$ terms (which is a maximal number of $O' \subseteq O_R$) and iterated integral. The iteration is terminated when one of the sets O'_R contains j . The minimal number of steps to reach j is at least $n \equiv [dist(O, j)/2R]$ (where the square bracket indicates the integer part). Thus, with $C \equiv 2^{(2R)^d} c_\Phi$, we obtain the following bound

$$\|\delta_{\Phi_O}(\alpha_{t,\Lambda}(a_j^\#))\| \leq C^n \frac{t^n}{n!} e^{Ct} \|a_j^\#\|$$

At this point one needs to use additional assumptions on $a_j^\#$. Using Stirling-de Moivre bound

$$n! \geq n^{n+1/2} e^{-n}$$

we get

$$\|\delta_{\Phi_O}(\alpha_{t,\Lambda}(a_j^\#))\| \leq \exp\{n(\log(Cet) - \log n)\} e^{Ct} \|a_j^\#\|$$

which for any fixed $t \in \mathbb{R}^+$ and $n \geq Cet \exp(m(2R)^d)$, yields

$$\|\delta_{\Phi_O}(\alpha_{t,\Lambda}(a_j^\#))\| \leq e^{Ct-mn} \|a_j^\#\|$$

□

Our result generalises that of [LR] ([Mat2]) who consider similar result for a system of bounded spin on a lattice.

5 Convergence of $\mathbb{L}_p$ Norms

In this section we provide a brief discussion of convergence of the sequence of states $\omega^{(\Lambda)}$, $\Lambda \subset \subset \mathbb{Z}^d$ and corresponding $\mathbb{L}_p$ norms. Let $\Lambda_0 \subset \Lambda$ and $\partial_{2R}\Lambda_0 \equiv \{k \notin \Lambda_0 : dist(k, \Lambda_0) \leq 2R\}$. We will use the following interpolation of the the potential

$$\Phi_s \equiv \{\Phi_O, O \subset \Lambda \setminus \partial_{2R}\Lambda_0, s\Phi_{O'}, O' \cap \partial_{2R}\Lambda_0 \neq \emptyset\}_\sim,$$

where $\sim$ signifies that we exclude one point potential, and denote by $\rho_{\Lambda,s} \equiv \rho_{\Lambda,\Lambda_0,s}$ the corresponding density matrix localised in Λ . Note that for $s = 0$, we have

$$\rho_{\Lambda,s=0} = \rho_{\Lambda \setminus \partial_{2R}\Lambda_0} \rho_{\Lambda_0} \rho_{O, \partial_{2R}\Lambda_0}$$

where the density matrices on the right hand side commute and the corresponding state is a product state. For $p \in \mathbb{N}$ and f localised in Λ_0 , we have

$$Tr_{\Lambda} \rho_{\Lambda,s=0} f = Tr_{\Lambda_0} \rho_{\Lambda_0} f$$

and

$$\|f\|_{p,\Lambda,s=0} = \|f\|_{p,\Lambda_0}.$$

Next we have

$$\begin{aligned} Tr_\Lambda \left((\rho_\Lambda)^{\frac{1}{4p}} f^* (\rho_\Lambda)^{\frac{2}{4p}} f (\rho_\Lambda)^{\frac{1}{4p}} \right)^{2p} - Tr_\Lambda \left((\rho_{\Lambda,s=0})^{\frac{1}{4p}} f^* (\rho_{\Lambda,s=0})^{\frac{2}{4p}} f (\rho_{\Lambda,s=0})^{\frac{1}{4p}} \right)^{2p} \\ = \int_0^1 ds \frac{d}{ds} Tr_\Lambda \left((\rho_{\Lambda,s})^{\frac{1}{4p}} f^* (\rho_{\Lambda,s})^{\frac{2}{4p}} f (\rho_{\Lambda,s})^{\frac{1}{4p}} \right)^{2p} \end{aligned}$$

Next we note that

$$\frac{d}{ds} (\rho_{\Lambda,s})^{\frac{1}{4p}} = \left(\frac{d}{ds} \exp\{-\frac{1}{4p} U_\Lambda(\Phi_s)\} \right) \frac{1}{Z_{\Lambda,s}^{\frac{1}{4p}}} - \frac{1}{4p} (\rho_{\Lambda,s})^{\frac{1}{4p}} \cdot \frac{1}{Z_{\Lambda,s}} \frac{d}{ds} Z_{\Lambda,s}$$

For the derivative of the first factor on the right hand side we have

$$\begin{aligned} \frac{d}{ds} \exp\{-\frac{1}{4p} U_\Lambda(\Phi_s)\} &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\exp\{-\frac{1}{4p} U_\Lambda(\Phi_{s+h})\} - \exp\{-\frac{1}{4p} U_\Lambda(\Phi_s)\} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \int_0^1 d\tau \frac{d}{d\tau} \exp\{-\frac{\tau}{4p} U_\Lambda(\Phi_{s+h})\} \exp\{-\frac{1-\tau}{4p} U_\Lambda(\Phi_s)\} \\ &= -\frac{1}{4p} \int_0^1 d\tau \exp\{-\frac{\tau}{4p} U_\Lambda(\Phi_{s+h})\} \lim_{h \rightarrow 0} \frac{1}{h} (U_\Lambda(\Phi_{s+h}) - U_\Lambda(\Phi_s)) \exp\{-\frac{1-\tau}{4p} U_\Lambda(\Phi_s)\} \\ &= -\frac{1}{4p} \sum_{O \subset \partial_{2R}\Lambda} \int_0^1 d\tau (\rho_{\Lambda,s})^{\frac{\tau}{4p}} (\Phi_O) (\rho_{\Lambda,s})^{-\frac{\tau}{4p}} \rho_{\Lambda,s}^{\frac{1}{4p}} \\ &\equiv -\frac{1}{4p} \sum_{O \subset \partial_{2R}\Lambda} \int_0^1 d\tau \left(\alpha_{\Lambda,s}(-i\frac{\tau}{4p})(\Phi_O) \right) \rho_{\Lambda,s}^{\frac{1}{4p}} \end{aligned}$$

where $\alpha_{\Lambda,s}(-i\frac{\tau}{4p})$ denotes the automorphism corresponding to Φ_s at time $-i\frac{\tau}{4p}$, and similarly for the second power of the density. Hence, we also get

$$\begin{aligned} \left| \frac{1}{Z_{\Lambda,s}} \frac{d}{ds} Z_{\Lambda,s} \right| &= \left| - \sum_{O \subset \partial_{2R}\Lambda_0} \int_0^1 d\tau \frac{1}{Z_{\Lambda,s}} Tr_\Lambda (\rho_{\Lambda,s})^\tau (\Phi_O) (\rho_{\Lambda,s})^{-\tau} \rho_{\Lambda,s} \right| \\ &= \left| - \sum_{O \subset \partial_{2R}\Lambda_0} \int_0^1 d\tau \omega_{\Lambda,s}(\Phi_O) \right| \leq \sum_{O \subset \partial_{2R}\Lambda_0} \|\Phi_O\| \end{aligned}$$

Using the above, we obtain

$$\begin{aligned} \frac{d}{ds} Tr_\Lambda \left((\rho_{\Lambda,s})^{\frac{1}{4p}} f^* (\rho_{\Lambda,s})^{\frac{2}{4p}} f (\rho_{\Lambda,s})^{\frac{1}{4p}} \right)^{2p} &\leq \\ &\leq Tr_\Lambda \left((\rho_{\Lambda,s})^{\frac{1}{4p}} f^* (\rho_{\Lambda,s})^{\frac{2}{4p}} f (\rho_{\Lambda,s})^{\frac{1}{4p}} \right)^{2p} \left(\sum_{O \subset \partial_{2R}\Lambda_0} \left(\|\Phi_O\| + \sup_{\Lambda,s,\tau \in [0,1]} \|\alpha_{\Lambda,s}(-i\frac{\tau}{4p})(\Phi_O)\| \right) \right) \end{aligned}$$

and hence we arrive at the following bound

$$\|f\|_{\Lambda,p} \leq \|f\|_{\Lambda_0,p} e^{C|\partial_{2R}\Lambda_0|} \quad (10)$$

with a constant

$$C \leq \sum_{O \subset \partial_{2R}\Lambda_0} \left(\|\Phi_O\| + \sup_{\Lambda,s,\tau \in [0,1]} \|\alpha_{\Lambda,s}(-i\frac{\tau}{4p})(\Phi_O)\| \right)$$

which according to the considerations of the previous subsection is finite under suitable assumptions on bounded part of the potential.

The above bounds (10) provide compactness of the set of states ω_Λ , $\Lambda \subset \mathbb{Z}^d$, and we have the following possibility of defining $\mathbb{L}_p(\omega)$ norms associated to a state $\omega \equiv \lim_{\Lambda_k \rightarrow \mathbb{Z}^d} \omega_{\Lambda_k}$ for some subsequence $\Lambda_k \subset \Lambda_{k+1}$,

$$\|f\|_{\omega,p} \equiv \limsup_{\Lambda \rightarrow \mathbb{Z}^d} \|f\|_{\omega_\Lambda,p}$$

Under additional assumptions on the interactions it is possible to use the ideas utilised above to prove convergence of the $\|f\|_{\omega_\Lambda,p}$ as $\Lambda \rightarrow \infty$.

Note that for positive f the symmetric $\mathbb{L}_{\omega_\Lambda,1}$ coincides with $\omega_\Lambda(f)$, so the problem of convergence is the same for both. On the other hand for the symmetric $\mathbb{L}_{\omega_\Lambda,2}$, given the convergence of the sequence of state and the modular operator, we get convergence for corresponding norms. Given $\mathbb{L}_{\omega,1}$ and $\mathbb{L}_{\omega,2}$ one can use interpolation theory to get all the intermediate norms and spaces and then by duality one can define the norms and $\mathbb{L}_{\omega,p}$ spaces for $p \in (2, \infty)$.

6 Noncommutative Dirichlet forms and Markov Generators

To give the definition of Dirichlet forms ([Cip3, CZ, P1, MZ, SQV]), we first describe the admissible function:

Definition 6.1. An analytic function $\eta : D \rightarrow \mathbb{C}$ on a domain D containing the strip $\text{Im } z \in [-1/4, 1/4]$ is said to be admissible function if the following holds:

1. $\eta(t) \geq 0$ for $t \in \mathbb{R}$,
2. $\eta(t + i/4) + \eta(t - i/4) \geq 0$ for $t \in \mathbb{R}$,
3. there exists $M > 0$ and $p > 1$ such that the bound

$$|\eta(t + is)| \leq M(1 + |t|)^{-p}$$

holds uniformly in $s \in [-1/4, 1/4]$.

A nice example of such function is as a gaussian smoothing of the following

$$\eta(t) \equiv \frac{e^{ikt}}{\cosh(2n\pi t)}. \quad (11)$$

For a collection of other examples see [CZ], [P1].

Given a family of local elements $X_j \in \mathcal{D}$, $j \in \mathbb{Z}^d$, we have discussed above, we can define derivations in directions $\alpha_t(X_j)$ and, on a dense domain containing $\mathcal{D}_\epsilon$, define the following Dirichlet form in $\mathbb{L}_{\omega,2}$

$$\mathcal{E}_j(f) \equiv \int_{\mathbb{R}} \left(\nu_j \langle \delta_{\alpha_t(X_j)}(f), \delta_{\alpha_t(X_j)}(f) \rangle_\omega + \mu_j \langle \delta_{\alpha_t(X_j^*)}(f), \delta_{\alpha_t(X_j^*)}(f) \rangle_\omega \right) \eta(t) dt$$

for ν_j, μ_j and a finite $\Lambda \subset \mathbb{Z}^d$

$$\mathcal{E}_\Lambda(f) \equiv \sum_{j \in \Lambda} \mathcal{E}_j(f)$$

Some other Dirichlet form can be defined, see e.g.[CM], as follows

$$\tilde{\mathcal{E}}_\Lambda(f) \equiv \sum_{j \in \Lambda} \left(\nu_j \langle \delta_{E_j}(f), \delta_{E_j}(f) \rangle_\omega + \mu_j \langle \delta_{E_j^*}(f), \delta_{E_j^*}(f) \rangle_\omega \right) \quad (\tilde{\mathcal{E}})$$

where E_j are the eigenvectors of the modular operator $\alpha_\omega(\pm \frac{i}{2})$, associated to the state ω at time $\pm \frac{i}{2}$, such that

$$\alpha_\omega(\pm \frac{i}{2})(E_j) = e^{\pm \xi_j} E_j,$$

for some $\xi_j \in \mathbb{R}$, and $\nu_j, \mu_j \in (0, \infty)$.

Now, the Markov generator corresponding to the first, respectively the second, Dirichlet form is formally give by

$$\langle f, -\tilde{\mathcal{Q}}_\Lambda f \rangle = \tilde{\mathcal{E}}_\Lambda(f), \quad \text{resp.} \quad \langle f, -\mathcal{Q}_\Lambda f \rangle = \mathcal{E}_\Lambda(f).$$

Using operation $\star$ of taking the adjoint (with respect to the scalar product), we have

$$-\mathcal{Q}_j(f) = \int \left(\nu_j \delta_{\alpha_t(X_j)}^\star \delta_{\alpha_t(X_j)}(f) + \mu_j \delta_{\alpha_t(X_j^*)}^\star \delta_{\alpha_t(X_j^*)}(f) \right) \eta(t) dt.$$

For the case of infinite number of eigenvectors

$$X_j = \left(\int \alpha_t(B_{j,l}) \eta(t) dt \right)_{l=1, \dots, K}, \quad j \in \mathbb{Z}^d,$$

the tilded generator is as follows

$$-\tilde{\mathcal{Q}}(f^* f) = \sum_{j \in \mathbb{Z}^d} \left(\kappa \delta_{X_j}^\star \delta_{X_j}(f^* f) + \mu \delta_{X_j^*}^\star \delta_{X_j^*}(f^* f) \right).$$

The corresponding Markovian form is defined by

$$\Gamma_{1,\Lambda}(f) = \frac{1}{2} (\mathcal{Q}_\Lambda(f^* f) - f^* \mathcal{Q}_\Lambda(f) - \mathcal{Q}_\Lambda(f^*) f)$$

and in our setup is given by

$$\begin{aligned} 2\Gamma_{1,\Lambda}(f) = \sum_{j \in \Lambda} \left\{ \int \left(\nu_j \delta_{\alpha_t(X_j)}^\star \delta_{\alpha_t(X_j)}(f^* f) + \mu_j \delta_{\alpha_t(X_j^*)}^\star \delta_{\alpha_t(X_j^*)}(f^* f) \right) \eta(t) dt \right. \\ \left. - f^* \left(\int \left(\nu_j \delta_{\alpha_t(X_j)}^\star \delta_{\alpha_t(X_j)}(f) + \mu_j \delta_{\alpha_t(X_j^*)}^\star \delta_{\alpha_t(X_j^*)}(f) \right) \eta(t) dt \right) \right. \\ \left. - \int \left(\nu_j \delta_{\alpha_t(X_j)}^\star \delta_{\alpha_t(X_j)}(f^*) + \mu_j \delta_{\alpha_t(X_j^*)}^\star \delta_{\alpha_t(X_j^*)}(f^*) \right) \eta(t) dt f \right\} \end{aligned}$$

and similarly in the tilded case.

Using Proposition 2.2 after lengthy computations (provided for the reader convenience in an Appendix), one gets (for every j) a result of the following form.

Proposition 6.1.

$$\Gamma_1(f) = -\frac{1}{2} \int \left(\left| \delta_{\alpha_{t-i/4}(X^*)}(f) \right|^2 + \left| \delta_{\alpha_{t-i/4}(X)}(f) \right|^2 \right) (\eta(t + i/4) + \eta(t - i/4)) dt$$

and respectively

$$\tilde{\Gamma}_1(f) = \sum_{j \in \mathbb{Z}^d} \left(\kappa e^{-\xi} |\delta_{X_j}(f)|^2 + \mu e^{\xi} |\delta_{X_j^*}(f)|^2 \right)$$

In the infinite dimensional case to secure the dense domain it is necessary to have the finite speed of propagation of information (as even for localised $B_{j,l}$, in general the eigenvectors are not localised).

7 Examples of models

In this section we provide few explicit examples of models with interesting phenomena.

7.1 Mean Field Models

Example 7.1. (Mean Field Model) With $[X_j, X_k^*] = \delta_{j,k}$, define

$$\mathbf{X}_\Lambda \equiv \frac{1}{\sqrt{|\Lambda|}} \sum_{k \in \Lambda} X_k$$

Then the following CCR holds

$$[\mathbf{X}_\Lambda, \mathbf{X}_\Lambda^*] = id.$$

Consider the following quadratic Hamiltonian

$$U_\Lambda = \mathbf{X}_\Lambda^* \mathbf{X}_\Lambda = \frac{1}{|\Lambda|} \sum_{j,k \in \Lambda} X_j^* X_k$$

Then the corresponding modular dynamics of $\mathbf{X}_\Lambda$ and $\mathbf{X}_\Lambda^*$ associated to U_Λ is given by

$$\alpha_{t,\Lambda}(\mathbf{X}_\Lambda) = e^{i\beta t} \mathbf{X}_\Lambda \quad \text{and} \quad \alpha_{t,\Lambda}(\mathbf{X}_\Lambda^*) = e^{-i\beta t} \mathbf{X}_\Lambda^*$$

and we have the following result.

Proposition 7.1. The Dirichlet form in the directions of $\mathbf{X}_\Lambda$ and $\mathbf{X}_\Lambda^*$ with respect to U_Λ is given by

$$\mathcal{E}_\Lambda(f) = \hat{\eta}(0) \left(\langle \delta_{\mathbf{X}_\Lambda}(f), \delta_{\mathbf{X}_\Lambda}(f) \rangle_\omega + \langle \delta_{\mathbf{X}_\Lambda^*}(f), \delta_{\mathbf{X}_\Lambda^*}(f) \rangle_\omega \right). \quad (12)$$

where $\hat{\eta}(0) = \int_{\mathbb{R}} \eta(t) e^{ist} dt$. This form has the dense domain $D(\mathcal{E}_\Lambda) \supset \mathcal{D}_\Lambda$.

Proof: We compute the derivations

$$\begin{aligned} \delta_{\alpha_t(\mathbf{X}_\Lambda)}(f) &= i[\alpha_t(\mathbf{X}_\Lambda), f] = ie^{i\beta t}[(\mathbf{X}_\Lambda), f] \\ (\delta_{\alpha_t(\mathbf{X}_\Lambda)}(f))^* &= -i[f^*, \alpha_t(\mathbf{X}_\Lambda)^*] = -e^{-i\beta t}i[f^*, (\mathbf{X}_\Lambda)^*] \end{aligned}$$

Inserting this into the formula for the Dirichlet form we arrive at (14) ■

The associated Markovian generator is given by

$$-\mathcal{L}_\Lambda(f) = \hat{\eta}(0) \left(-e^{-\frac{\beta}{2}} [\mathbf{X}_\Lambda, f] \mathbf{X}_\Lambda^* + e^{\frac{\beta}{2}} \mathbf{X}_\Lambda^* [\mathbf{X}_\Lambda, f] - e^{\frac{\beta}{2}} [\mathbf{X}_\Lambda^*, f] \mathbf{X}_\Lambda + e^{-\frac{\beta}{2}} \mathbf{X}_\Lambda [\mathbf{X}_\Lambda^*, f] \right) \quad (13)$$

i.e. the corresponding Dirichlet operator is the generator of quantum O - U semigroup which maps symmetric polynomials in creation/annihilation operators into itself. By [CFL] the Poincaré inequality holds and by [CaS] the Log Sobolev inequality is satisfied. The limiting theory can be described in the framework of [RW].

Example 7.2. (Mean Field Model 2) With $[X_j, X_k^*] = \delta_{j,k}$, an integer $n > 1$ and $\varepsilon \in [0, 1]$, define

$$\mathbf{X}_{n,\Lambda} \equiv \frac{1}{|\Lambda|^\varepsilon} \sum_{k \in \Lambda} X_k^n$$

Then we have

$$[\mathbf{X}_{n,\Lambda}, \mathbf{X}_{n,\Lambda}^*] = \frac{1}{|\Lambda|^{2\varepsilon}} \sum_{k \in \Lambda} [X_k^n, \sum_{j \in \Lambda} X_j^{*n}] = \frac{1}{|\Lambda|^{2\varepsilon}} \sum_{k \in \Lambda} [X_k^n, X_k^{*n}] = \frac{1}{|\Lambda|^{2\varepsilon}} \sum_{k \in \Lambda} P_n(N_k);$$

with some polynomial P_n .

Consider the following quadratic Hamiltonian

$$U_\Lambda = \mathbf{X}_{n,\Lambda}^* \mathbf{X}_{n,\Lambda} = \frac{1}{|\Lambda|} \sum_{j,k \in \Lambda} X_j^{*n} X_k^n$$

(which commutes with $\sum_{k \in \Lambda} U_k$). Now we obtain

$$\begin{aligned} [U_\Lambda, \mathbf{X}_{n,\Lambda}] &\equiv -i\delta_{U_\Lambda}(\mathbf{X}_{n,\Lambda}) = \frac{1}{|\Lambda|^{\frac{1}{2}+\varepsilon}} \sum_{j \in \Lambda} [X_j^*, X_j^n] \frac{1}{\sqrt{|\Lambda|}} \sum_{k \in \Lambda} X_k \\ &= -n \frac{1}{|\Lambda|^{\frac{1}{2}}} \mathbf{X}_{n-1,\Lambda} \mathbf{X}_\Lambda \end{aligned}$$

and

$$\begin{aligned} (-i\delta_{U_\Lambda})^2(\mathbf{X}_{n,\Lambda}) &= -i\delta_{U_\Lambda} \left(-n \frac{1}{|\Lambda|^{\frac{1}{2}}} \mathbf{X}_{n-1,\Lambda} \mathbf{X}_\Lambda \right) \\ &= n(n-1) \frac{1}{|\Lambda|} \mathbf{X}_{n-2,\Lambda} \mathbf{X}_\Lambda^2 + n \frac{1}{|\Lambda|^{\frac{1}{2}}} \mathbf{X}_{n-1,\Lambda} \mathbf{X}_\Lambda \end{aligned}$$

Using the formula

$$\delta_{U_\Lambda}(\mathbf{X}_\Lambda^m) = -m \mathbf{X}_\Lambda^m$$

we have

$$\begin{aligned} (-i\delta_{U_\Lambda})^3(\mathbf{X}_{n,\Lambda}) &= (-i\delta_{U_\Lambda}) \left(n(n-1) \frac{1}{|\Lambda|} \mathbf{X}_{n-2,\Lambda} \mathbf{X}_\Lambda^2 + n \frac{1}{|\Lambda|^{\frac{1}{2}}} \mathbf{X}_{n-1,\Lambda} \mathbf{X}_\Lambda \right) \\ &= \left(-n(n-1)(n-2) \frac{1}{|\Lambda|^{\frac{3}{2}}} \mathbf{X}_{n-3,\Lambda} \mathbf{X}_\Lambda^3 - 3n(n-1) \frac{1}{|\Lambda|} \mathbf{X}_{n-2,\Lambda} \mathbf{X}_\Lambda^2 - n \frac{1}{|\Lambda|^{\frac{1}{2}}} \mathbf{X}_{n-1,\Lambda} \mathbf{X}_\Lambda \right) \end{aligned}$$

By induction, for $k = n-1$, we get

$$\delta_{U_\Lambda}^{n-1}(\mathbf{X}_{n,\Lambda}) = \sum_{l=1}^{n-1} (-i)^l a_{k,l} \frac{1}{|\Lambda|^{\frac{l}{2}}} \mathbf{X}_{n-l,\Lambda} \mathbf{X}_\Lambda^l$$

with some coefficients $a_{k,l}$. For $k > n-1$ we will not produce different operators, just the coefficient will be changing according to the following formula

$$a_{k+1,l} = -(l-1)a_{k,l-1} - (n-l)a_{k,l},$$

and thus asymptotically they will be growing at most as $2^k n^k$. Additionally each term $\mathbf{X}_{n-l,\Lambda} \mathbf{X}_\Lambda^l$ comes accompanied by a factor $\frac{1}{|\Lambda|^{\frac{l}{2}}}$

Lemma 7.1. The modular dynamics of $\mathbf{X}_{n,\Lambda}$ and $\mathbf{X}_{n,\Lambda}^*$ associated to U_Λ given by

$$\alpha_{t,\Lambda}(\mathbf{X}_{n,\Lambda}^\#) = \sum_{k=0}^{\infty} \frac{1}{k!} (-it)^k \delta_{U_\Lambda}^k(\mathbf{X}_{n,\Lambda}^\#)$$

is well defined.

Hence we conclude with the following result.

Theorem 7.1. The Dirichlet form in the directions of $\mathbf{X}_\Lambda$ and $\mathbf{X}_\Lambda^*$ with respect to U_Λ is given by

$$\mathcal{E}_\Lambda(f) = \int_{\mathbb{R}} \left(\langle \delta_{\alpha_t(\mathbf{X}_{n,\Lambda})}(f), \delta_{\alpha_t(\mathbf{X}_{n,\Lambda})}(f) \rangle + \langle \delta_{\alpha_t(\mathbf{X}_{n,\Lambda}^*)}(f), \delta_{\alpha_t(\mathbf{X}_{n,\Lambda}^*)}(f) \rangle \right) \eta(t) dt. \quad (14)$$

is well defined on the dense domain $D(\mathcal{E}_\Lambda) \supset \mathcal{D}_\Lambda$ and closable, and hence defines a Markov generator.

Remark 7.1. In the limit $\Lambda \rightarrow \mathbb{Z}^d$ this form for $n > 1$ is becoming trivial. It is an interesting open question whether or not the limits $t \rightarrow \infty$ and $\Lambda \rightarrow \mathbb{Z}^d$ are interchangeable.

7.2 Non-diagonal Dirichlet Forms

In this part we consider the dissipative dynamics defined by a Dirichlet form with nonlocal derivations. In particular for an infinite product space we provide an example in which the generator has no spectral gap.

Example 7.3 (Z-type fields). For $\kappa \equiv \{\kappa_j \in \mathbb{C} : \sum_j |\kappa_j| < \infty\}$, define

$$Z_\kappa = \sum_j \kappa_j A_j \quad (15)$$

the series being convergent in any $\mathbb{L}_{p,\omega_0}$ for $p \in [1, \infty)$. Later on we assume that κ is not equal to a zero vector. Given two absolutely convergent sequences κ and ξ , we have the following CCR relation

$$[Z_\kappa, Z_\xi^*] = \sum_j \kappa_j \bar{\xi}_j. \quad (16)$$

The modular dynamics (associated to ω_0) of Z_κ is given

$$\alpha_t(Z_\kappa) = e^{i\beta t} Z_\kappa \text{ and } \alpha_t(Z_\kappa^*) = e^{-i\beta t} Z_\kappa^*$$

Given a translation $(T_j \kappa) \equiv (\kappa_{l-j})_{l \in \mathbb{Z}^d}$, we define translation of Z_κ by $Z_{T_j \kappa}$.

Theorem 7.2. Suppose $\kappa, \xi \neq \theta$. The Dirichlet form associated to the directions of $Z_{T_j \kappa}$ and $Z_{T_j \xi}^*$, $j \in \mathbb{Z}^d$, with respect to the state ω_0 is given by

$$\mathcal{E}(f) = \hat{\eta}(0) \sum_{j \in \mathbb{Z}^d} \left(\langle \delta_{Z_{T_j \kappa}}(f), \delta_{Z_{T_j \kappa}}(f) \rangle_\omega + \langle \delta_{Z_{T_j \xi}^*}(f), \delta_{Z_{T_j \xi}^*}(f) \rangle_\omega \right).$$

with a dense domain $D(\mathcal{E}) \supset \mathcal{D}_\varepsilon$ is closable and hence defines a Markov generator.

Remark: Similar conclusion holds for the Dirichlet forms associated to the Gibbs states ω .

Proof. Thanks to summability of κ and ξ our Dirichlet form can be dominated by a multiple of the one defined with derivations in directions A_j and A_j^* . For the latter Dirichlet form we already showed that the domain contains a dense set $\mathcal{D}_\varepsilon$ on which the adjoint operators are well defined and so one can define a pre-Markov generator on the same dense domain. The Friedrichs extension completes the construction of the Markov generator. In case of a Gibbs state different than a product state the Dirichlet form generally the integral with η does not factorises, however, as we discussed earlier in section 2, all the arguments go through thanks to finite speed of propagation of information and the fact that $\mathcal{D}_\varepsilon$ is a subset of $\mathbb{L}_{p,\omega}$ for $p \in [1, \infty)$. $\square$

Example 7.4. Here we consider an example involving infinite set of CCRs which are not independent in general. Given $j \sim k$, $j, k \in \mathbb{Z}^d$ and $\kappa_j, \varepsilon_k \in \mathbb{C}$ define

$$Z_{j,k} := \kappa_j A_j + \varepsilon_k A_k$$

and consider the following Hamiltonian

$$H_\Lambda = \sum_{\substack{j \sim k \\ j,k \in \Lambda}} Z_{j,k}^* Z_{j,k} = \sum_{\substack{j \sim k \\ j,k \in \Lambda}} (\bar{\kappa}_j A_j^* + \bar{\varepsilon}_k A_k^*) (\kappa_j A_j + \varepsilon_k A_k).$$

We note that the following derivation

$$\delta_{\mathbf{H}}(f) \equiv \lim_{\Lambda \rightarrow \mathbb{Z}^d} \delta_{H_\Lambda}(f).$$

is evidently well defined on all local polynomials in creators and annihilators $A_j^\#$, $j \in \mathbb{Z}^d$. In particular, for a fixed $l, m \in \Lambda$, we have

$$\begin{aligned} i\mathbf{H}(Z_{l,m}) &= [Z_{l,m}, \sum_{j \sim k} Z_{j,k}^* Z_{j,k}] = \sum_{j \sim k} [Z_{l,m}, Z_{j,k}^* Z_{j,k}] = \sum_{j \sim k} [Z_{l,m}, Z_{j,k}^*] Z_{j,k} \\ &= \sum_{j \sim k} (\delta_{l,j} |\kappa_j|^2 + \delta_{l,k} \kappa_k \bar{\varepsilon}_k + \delta_{m,j} \varepsilon_j \bar{\kappa}_j + \delta_{m,k} |\varepsilon_k|^2) Z_{j,k} \end{aligned}$$

Thus, for later application we note that the infinite dimensional hamiltonian evolution $\alpha_t(Z_{l,m}) \equiv \alpha_{t,\mathbf{H}}(Z_{l,m}) \equiv e^{-it\beta\delta_{\mathbf{H}}} Z_{l,m}$ is well defined and satisfies the following relation

$$\frac{d}{dt} \alpha_t(Z_{l,m}) = i\beta \left(\sum_{j \sim k} (\delta_{l,j} |\kappa_j|^2 + \delta_{l,k} \kappa_k \bar{\varepsilon}_k + \delta_{m,j} \varepsilon_j \bar{\kappa}_j + \delta_{m,k} |\varepsilon_k|^2) \right) \alpha_t(Z_{j,k})$$

Remark: To check that corresponding hamiltonian dynamics

$$\alpha_t(f) = e^{-it\delta_{\mathbf{H}}} f = \lim_{\Lambda \rightarrow \mathbb{Z}^d} e^{-it\delta_{\mathbf{H}_\Lambda}} f = \lim_{\Lambda \rightarrow \mathbb{Z}^d} e^{-itH_\Lambda} f e^{itH_\Lambda}$$

is well defined on local polynomials in creators and annihilators, note that we have

$$\begin{aligned} \frac{d}{dt} \alpha_{t,\Lambda}(A_l) &= -i\alpha_{t,\Lambda}([H_\Lambda, A_l]) \\ &= -i \sum_{\substack{m \in \Lambda \\ m \sim l}} \left((|\kappa_l|^2 + |\varepsilon_l|^2) \alpha_{t,\Lambda}(A_l) + (\bar{\kappa}_l \varepsilon_m + \bar{\varepsilon}_l \kappa_m) \alpha_{t,\Lambda}(A_m) \right). \end{aligned}$$

From that should be clear (via arguments given in the previous sections) that the infinite dimensional limit can be performed and that we have finite speed of propagation of interaction in the system.

To elucidate this a bit, note that the above equation can be solved in the algebra if mollified by dividing by a power of $(1 + \varepsilon N_\Lambda)^{-1}$ with $N_\Lambda \equiv \sum_{j \in \Lambda} N_j$. This is because

$$\delta_{H_\Lambda}(N_\Lambda) = 0.$$

Consequently for mollified problem one can use iteration scheme in operator norm, otherwise one needs to study convergence in $L_p(\omega_0)$ spaces, $p \in (1, \infty)$.

Remark Note that one has a generalisation of the last relation directly to the infinite dimensions in the sense of commutation of derivations as follows

$$[\delta_{\mathbf{H}}, \delta_{\mathbf{N}}] = 0$$

where

$$\delta_{\mathbf{N}}(f) \equiv \lim_{\Lambda \rightarrow \mathbb{Z}^d} \delta_{N_\Lambda}(f)$$

with the limit on the right hand side on local polynomials in creators and annihilators (weak or strong in a Hilbert space) or in operator norm on the algebra generated by the mollified local polynomials (with local mollification provided by N_O with $O \subset \subset \mathbb{Z}^d$). $\circ$

With the well defined hamiltonian dynamics for which finite speed of propagation of information property holds we have the following result.

Theorem 7.3. *The following Dirichlet form is well defined on the set of local polynomials*

$$\mathcal{E}(f) \equiv \sum_{\substack{j=k \\ j,k \in \mathbb{Z}^d}} \int \left(\langle \delta_{\alpha_t, \mathbf{H}(Z_{jk})}(f), \delta_{\alpha_t, \mathbf{H}(Z_{jk})}(f) \rangle_{\omega_0} + \langle \delta_{\alpha_t, \mathbf{H}(Z_{jk}^*)}(f), \delta_{\alpha_t, \mathbf{H}(Z_{jk}^*)}(f) \rangle_{\omega_0} \right) \eta(t) dt$$

and its closure defines a Markov generator in $\mathbb{L}_2(\omega_0)$.

A variation in the similar direction is provided in the following example where we use the mixed representation of CCRs.

Example 7.5 (Y-type fields). *For some $\kappa, \xi \neq \theta$,*

$$Y_{\kappa, \xi} = Z_{\kappa} - Z_{\xi}^*.$$

Then

$$[Y_{\kappa, \xi}, Y_{\kappa, \xi}^*] = (|\kappa|_2^2 - |\xi|_2^2) \mathbf{id}$$

where $|\kappa|_2^2 \equiv \sum_j |\kappa_j|^2$ and similarly for $|\xi|_2^2$. We also note that

$$[Y_{\kappa, \xi}, N_l] = \kappa_l A_l - \bar{\xi}_l A_l^*$$

Using the above, we have the following fact.

Lemma 7.2. *The modular dynamics of $Y_{\kappa, \xi}$ and $Y_{\kappa, \xi}^*$ with respect to the infinite product state is given*

$$\alpha_t(Y_{\kappa, \xi}) = e^{i\beta t} Z_{\kappa} - e^{-i\beta t} Z_{\xi}^*$$

Consider the following corresponding Dirichlet form

$$\mathcal{E}(f) = \sum_{j \in \mathbb{Z}^d} \int \langle \delta_{\alpha_t Y_{T_{j\kappa}, T_{j\xi}}}(f), \delta_{\alpha_t Y_{T_{j\kappa}, T_{j\xi}}}(f) \rangle \eta(t) dt$$

We have the following representation of this Dirichlet form.

Theorem 7.4. *The Dirichlet form in the directions of $Y_{T_{j\kappa}, T_{j\xi}}$ and $Y_{T_{j\kappa}, T_{j\xi}}^*$, $j \in \mathbb{Z}^d$, associated to the product state ω_0 is well defined on a dense set including $\mathcal{D}_{\varepsilon}$ and is given by*

$$\begin{aligned} \mathcal{E}(f) &= \sum_{j \in \mathbb{Z}^d} \hat{\eta}(0) \langle \delta_{Z_{T_{j\kappa}}}(f), \delta_{Z_{T_{j\kappa}}}(f) \rangle_{\omega_0} + \sum_{j \in \mathbb{Z}^d} \hat{\eta}(0) \langle \delta_{Z_{T_{j\xi}}}(f), \delta_{Z_{T_{j\xi}}}(f) \rangle_{\omega_0} \\ &+ \sum_{j \in \mathbb{Z}^d} \sqrt{2\pi} \hat{\eta}(-2\beta) \langle \delta_{Z_{T_{j\kappa}}}(f), \delta_{Z_{T_{j\xi}}^*}(f) \rangle_{\omega_0} + \sum_{j \in \mathbb{Z}^d} \sqrt{2\pi} \hat{\eta}(2\beta) \langle \delta_{Z_{T_{j\xi}}^*}(f), \delta_{Z_{T_{j\kappa}}}(f) \rangle_{\omega_0} \end{aligned}$$

On the domain $\mathcal{D}_{\varepsilon}$ it defines a pre-Markov generator.

Example 7.6. *Let*

$$W_{j,k} \equiv W_{j,k}^{(n,m)} \equiv A_j^{*n} A_k^m.$$

Then, using modular dynamics corresponding to $\mathbf{N}_{\Lambda} \equiv \sum_{l \in \Lambda} N_l$ with $j, k \in \Lambda$, we have

$$\alpha_t(W_{j,k}) = \lim_{\Lambda \rightarrow \mathbb{Z}^d} e^{-i\beta t \mathbf{N}_{\Lambda}} W_{j,k} e^{i\beta t \mathbf{N}_{\Lambda}} = \alpha_t(A_j^{*n}) \alpha_t(A_k^m) = e^{i(m-n)\beta t} W_{j,k}$$

Theorem 7.5. *The Dirichlet form with derivations generated by directions $W_{j,k}^{(n,m)}$ given as follows*

$$\mathcal{E}(f) = \hat{\eta}(0) \sum_{\substack{j,k \in \mathbb{Z}^d \\ j=k}} \left(\|\delta_{W_{j,k}}(f)\|_{2,\omega}^2 + \|\delta_{W_{j,k}^*}(f)\|_{2,\omega}^2 \right)$$

is well defined on the algebra of local polynomials in creators and annihilators. Its closure defines a Markov generator.

Note that for selfadjoint operators

$$W_{j,k} \equiv A_j^{*n} A_k^m + A_k^{*m} A_j^n = W_{j,k}^*.$$

and the same modular dynamics as before we have

$$\alpha_t(W_{j,k}) = e^{i(m-n)\beta t} A_j^{*n} A_k^m + e^{-i(m-n)\beta t} A_k^{*m} A_j^n$$

and hence a more complicated Dirichlet form given as follows

$$\begin{aligned} \mathcal{E}(f) = & \sum_{j,k} 2(\hat{\eta}(0) \langle \delta_{A_j^{*n} A_k^m}(f), \delta_{A_j^{*n} A_k^m}(f) \rangle_\omega + \hat{\eta}(2(n-m)\beta) \langle \delta_{A_j^{*n} A_k^m}(f), \delta_{A_k^{*m} A_j^n}(f) \rangle_\omega \\ & + \hat{\eta}(2(m-n)\beta) \langle \delta_{A_k^{*m} A_j^n}(f), \delta_{A_j^{*n} A_k^m}(f) \rangle_\omega + \hat{\eta}(0) \langle \delta_{A_k^{*m} A_j^n}(f), \delta_{A_k^{*m} A_j^n}(f) \rangle_\omega) \end{aligned}$$

In the special case when $m = n$, the time dependence factorises and for each term we get the same multiplier $\hat{\eta}(0)$. Moreover in the selfadjoint case $W_{j,k} = W_{j,k}^*$ the Dirichlet form is not ergodic, i.e. it vanishes on nonzero elements.

When $m = 1 = n$, we have

$$W_{j,k} \equiv A_j^* A_k + A_k^* A_j = W_{j,k}^*$$

and they are invariant with respect to modular dynamics. In general such operators with different indices do not commute, but in this special case we have the following nice commutation relations

$$\begin{aligned} [W_{j,k}, W_{n,m}^*] &= [A_j^* A_k + A_k^* A_j, A_n^* A_m + A_m^* A_n] \\ &= \delta_{jm} A_n^* A_k + \delta_{kn} A_j^* A_m + \delta_{jn} A_m^* A_k + \delta_{km} A_j^* A_n \\ &\quad + \delta_{km} A_n^* A_j + \delta_{jn} A_k^* A_m + \delta_{kn} A_m^* A_j + \delta_{jm} A_k^* A_n \\ &= \delta_{jm} W_{k,n} + \delta_{kn} W_{j,m} + \delta_{jn} W_{k,m} + \delta_{km} W_{j,n} \end{aligned} \tag{17}$$

The Dirichlet form is not ergodic and to change that one need to add terms with derivations related to

$$W_{j,k} \equiv i(A_j^* A_k - A_k^* A_j)$$

Example 7.7. Consider

$$Z_{j,k} \equiv A_j^n - A_k^m$$

With notation $A_j^{*n} \equiv (A_j^*)^n$, we have

$$[Z_{j,k}, Z_{j,k}^*] = [A_j^n - A_k^m, A_j^{*n} - A_k^{*m}] = [A_j^n, A_j^{*n}] + [A_k^m, A_k^{*m}]$$

Using formula (8) (or lemma 5.1 from [CZ]),

$$[A_j^n, A_j^{*n}] = (N+n)(N+n-1) \cdots (N+2)(N+1) - N(N-1)(N-2) \cdots (N-(n-1))$$

Hence

$$\begin{aligned} [Z_{j,k}, Z_{j,k}^*] &= (N+n)(N+n-1) \cdots (N+2)(N+1) - N(N-1)(N-2) \cdots (N-(n-1)) \\ &\quad + (N+m)(N+m-1) \cdots (N+2)(N+1) - N(N-1)(N-2) \cdots (N-(m-1)) \end{aligned}$$

Lemma 7.3. The modular dynamics in the directions of $Z_{j,k}$ and $Z_{j,k}^*$ with respect to the product state of a system of quantum harmonic oscillators is given by

$$\alpha_t(Z_{j,k}) = e^{in\beta t} A_j^n - e^{im\beta t} A_k^m \quad \text{and} \quad \alpha_t(Z_{j,k}^*) = e^{-in\beta t} A_j^{*n} - e^{-im\beta t} A_k^{*m}$$

Proof: We know that for $j, k \in \Lambda$ we have

$$\begin{aligned} [Z_{j,k}, \sum_{j \in \Lambda} N_j] &= [A_j^n - A_k^m, \sum_{j \in \Lambda} N_j] = [A_j^n, \sum_{j \in \Lambda} N_j] - [A_k^m, \sum_{j \in \Lambda} N_j] \\ &= nA_j^n - mA_k^m \end{aligned}$$

And

$$[Z_{j,k}^*, \sum_{j \in \Lambda} N_j] = -nA_j^{*n} + mA_k^{*m}$$

Then $\alpha_t(Z_{j,k}) = \alpha_t(A_j^n) - \alpha_t(A_k^m) = e^{in\beta t} A_j^n - e^{im\beta t} A_k^m$ and similarly $\alpha_t(Z_{j,k}^*) = \alpha_t(A_j^{*n}) - \alpha_t(A_k^{*m}) = e^{-in\beta t} A_j^{*n} - e^{-im\beta t} A_k^{*m}$.

Theorem 7.6. *The Dirichlet form in the directions of $Z_{j,k}$ and $Z_{j,k}^*$ with respect to the product state of a system of quantum harmonic oscillators given by*

$$\mathcal{E}_\Lambda(f) = \sum_{j \sim k} \hat{\eta}(0) \langle \delta_{A_j^n}(f), \delta_{A_j^n}(f) \rangle_\omega + \hat{\eta}(0) \langle \delta_{A_k^m}(f), \delta_{A_k^m}(f) \rangle_\omega + \hat{\eta}((m-n)\beta) \langle \delta_{A_j^n}(f), \delta_{A_k^m}(f) \rangle_\omega + \hat{\eta}((n-m)\beta) \langle \delta_{A_k^m}(f), \delta_{A_j^n}(f) \rangle_\omega +$$

$$\hat{\eta}(0) \langle \delta_{A_j^{*n}}(f), \delta_{A_j^{*n}}(f) \rangle_\omega + \hat{\eta}(0) \langle \delta_{A_k^{*m}}(f), \delta_{A_k^{*m}}(f) \rangle_\omega - \hat{\eta}((n-m)\beta) \langle \delta_{A_j^{*n}}(f), \delta_{A_k^{*m}}(f) \rangle_\omega - \hat{\eta}((m-n)\beta) \langle \delta_{A_k^{*m}}(f), \delta_{A_j^{*n}}(f) \rangle_\omega$$

with a dense domain $D(\mathcal{E}_\Lambda) \supset \mathcal{D}$ is closable and hence defines a Markov generator.

Proof: We compute the derivations in the direction of $\alpha_t(Z_{j,k})$

$$\begin{aligned} \delta_{\alpha_t(Z_{j,k})}(f) &= i[\alpha_t(Z_{j,k}), f] = ie^{in\beta t}[A_j^n, f] - ie^{im\beta t}[A_k^m, f] \\ (\delta_{\alpha_t(Z_{j,k})}(f))^* &= -i[f^*, \alpha_t(Z_{j,k})^*] = -ie^{-in\beta t}[f^*, A_j^{*n}] + ie^{-im\beta t}[f^*, A_k^{*m}] \end{aligned}$$

And hence

$$\begin{aligned} \int \langle \delta_{\alpha_t(Z_{j,k})}(f), \delta_{\alpha_t(Z_{j,k})}(f) \rangle_\omega \eta(t) dt &= \frac{1}{Z} \text{Tr}(e^{-\beta \sum_{j \in \Lambda} N_j/2} (\delta_{\alpha_t(Z_{j,k})}(f))^* e^{-\beta \sum_{j \in \Lambda} N_j/2} \delta_{\alpha_t(Z_{j,k})}(f)) \\ &= \frac{1}{Z} \text{Tr}(-e^{-\beta \sum_{j \in \Lambda} N_j/2} (-ie^{-in\beta t}[f^*, A_j^{*n}] + ie^{-im\beta t}[f^*, A_k^{*m}]) e^{-\beta \sum_{j \in \Lambda} N_j/2} (ie^{in\beta t}[A_j^n, f] - ie^{im\beta t}[A_k^m, f])) \\ &= \hat{\eta}(0) \langle \delta_{A_j^n}(f), \delta_{A_j^n}(f) \rangle_\omega + \hat{\eta}(0) \langle \delta_{A_k^m}(f), \delta_{A_k^m}(f) \rangle_\omega + \hat{\eta}((m-n)\beta) \langle \delta_{A_j^n}(f), \delta_{A_k^m}(f) \rangle_\omega + \hat{\eta}((n-m)\beta) \langle \delta_{A_k^m}(f), \delta_{A_j^n}(f) \rangle_\omega \end{aligned}$$

Similarly

$$\begin{aligned} \delta_{\alpha_t(Z_{j,k}^*)}(f) &= i[\alpha_t(Z_{j,k}^*), f] = ie^{-in\beta t}[A_j^{*n}, f] - ie^{-im\beta t}[A_k^{*m}, f] \\ (\delta_{\alpha_t(Z_{j,k}^*)}(f))^* &= -i[f^*, \alpha_t(Z_{j,k}^*)^*] = -ie^{in\beta t}[f^*, A_j^n] + ie^{im\beta t}[f^*, A_k^m] \end{aligned}$$

And hence

$$\begin{aligned} \int \langle \delta_{\alpha_t(Z_{j,k}^*)}(f), \delta_{\alpha_t(Z_{j,k}^*)}(f) \rangle_\omega \eta(t) dt \\ = \hat{\eta}(0) \langle \delta_{A_j^{*n}}(f), \delta_{A_j^{*n}}(f) \rangle_\omega + \hat{\eta}(0) \langle \delta_{A_k^{*m}}(f), \delta_{A_k^{*m}}(f) \rangle_\omega - \hat{\eta}((n-m)\beta) \langle \delta_{A_j^{*n}}(f), \delta_{A_k^{*m}}(f) \rangle_\omega - \hat{\eta}((m-n)\beta) \langle \delta_{A_k^{*m}}(f), \delta_{A_j^{*n}}(f) \rangle_\omega \end{aligned}$$

Then

$$\begin{aligned} \mathcal{E}_\Lambda(f) &= \sum_{j \sim k} \int_{-\infty}^{\infty} (\langle \delta_{\alpha_t(Z_{j,k})}(f), \delta_{\alpha_t(Z_{j,k})}(f) \rangle_\omega + \langle \delta_{\alpha_t(Z_{j,k}^*)}(f), \delta_{\alpha_t(Z_{j,k}^*)}(f) \rangle_\omega) \eta(t) dt \\ &= \hat{\eta}(0) \langle \delta_{A_j^n}(f), \delta_{A_j^n}(f) \rangle_\omega + \hat{\eta}(0) \langle \delta_{A_k^m}(f), \delta_{A_k^m}(f) \rangle_\omega + \hat{\eta}((m-n)\beta) \langle \delta_{A_j^n}(f), \delta_{A_k^m}(f) \rangle_\omega + \hat{\eta}((n-m)\beta) \langle \delta_{A_k^m}(f), \delta_{A_j^n}(f) \rangle_\omega + \\ &\quad \hat{\eta}(0) \langle \delta_{A_j^{*n}}(f), \delta_{A_j^{*n}}(f) \rangle_\omega + \hat{\eta}(0) \langle \delta_{A_k^{*m}}(f), \delta_{A_k^{*m}}(f) \rangle_\omega - \hat{\eta}((n-m)\beta) \langle \delta_{A_j^{*n}}(f), \delta_{A_k^{*m}}(f) \rangle_\omega - \hat{\eta}((m-n)\beta) \langle \delta_{A_k^{*m}}(f), \delta_{A_j^{*n}}(f) \rangle_\omega \end{aligned}$$

Example 7.8. Consider

$$Y_{j,k} = A_j^n - A_k^{*m}$$

then

$$\begin{aligned} [Y_{j,k}, Y_{j,k}^*] &= [A_j^n - A_k^{*m}, A_j^{*n} - A_k^m] = [A_j^n, A_j^{*n}] + [A_k^{*m}, A_k^m] \\ &= nA_j^{n-1}A_j^{*n-1} - mA_k^{m-1}A_k^{*m-1} \end{aligned}$$

and

$$[Y_{j,k}, \sum_j N_j] = [A_j^n - A_k^{*m}, \sum_j N_j] = nA_j^n + mA_k^{*m}$$

Lemma 7.4. The modular dynamics of $Y_{j,k}$ and $Y_{j,k}^*$ with respect to the infinite product state ω_0 of a system of quantum harmonic oscillators is given

$$\alpha_t(Y_{j,k}) = e^{in\beta t}A_j^n - e^{-im\beta t}A_k^{*m} \text{ and } \alpha_t(Y_{j,k}^*) = e^{-in\beta t}A_j^{*n} - e^{im\beta t}A_k^m$$

Proof: So

$$\alpha_t(Y_{j,k}) = \alpha_t(A_j^n) - \alpha_t(A_k^{*m}) = e^{in\beta t}A_j^n - e^{-im\beta t}A_k^{*m}$$

Similarly

$$\alpha_t(Y_{j,k}^*) = e^{-in\beta t}A_j^{*n} - e^{im\beta t}A_k^m$$

Theorem 7.7. The Dirichlet form in the directions of $Y_{j,k}$ and $Y_{j,k}^*$ with respect to the product state ω_0 is given

$$\begin{aligned} \mathcal{E}_\Lambda(f) &= \sum_{j \sim k} \hat{\eta}(0) \langle \delta_{A_j^n}(f), \delta_{A_j^n}(f) \rangle_\omega + \hat{\eta}(0) \langle \delta_{A_k^{*m}}(f), \delta_{A_k^{*m}}(f) \rangle_\omega - \hat{\eta}((-m-n)\beta) \langle \delta_{A_j^n}(f), \delta_{A_k^{*m}}(f) \rangle_\omega - \hat{\eta}((n+m)\beta) \langle \delta_{A_k^{*m}}(f), \delta_{A_j^n}(f) \rangle_\omega \\ &+ \hat{\eta}(0) \langle \delta_{A_j^{*n}}(f), \delta_{A_j^{*n}}(f) \rangle_\omega + \hat{\eta}(0) \langle \delta_{A_k^m}(f), \delta_{A_k^m}(f) \rangle_\omega - \hat{\eta}((n+m)\beta) \langle \delta_{A_j^{*n}}(f), \delta_{A_k^m}(f) \rangle_\omega - \hat{\eta}((-m-n)\beta) \langle \delta_{A_k^m}(f), \delta_{A_j^{*n}}(f) \rangle_\omega \end{aligned}$$

Proof: We compute the derivations in the direction of $\alpha_t(Y_{j,k})$

$$\begin{aligned} \delta_{\alpha_t(Y_{j,k})}(f) &= i[\alpha_t(Y_{j,k}), f] = ie^{in\beta t}[A_j^n, f] - ie^{-im\beta t}[A_k^{*m}, f] \\ (\delta_{\alpha_t(Y_{j,k})}(f))^* &= -i[f^*, \alpha_t(Y_{j,k})^*] = -ie^{-in\beta t}[f^*, A_j^{*n}] + ie^{im\beta t}[f^*, A_k^m] \end{aligned}$$

And hence

$$\begin{aligned} \int \langle \delta_{\alpha_t(Y_{j,k})}(f), \delta_{\alpha_t(Y_{j,k})}(f) \rangle_\omega \eta(t) dt &= \frac{1}{Z} Tr(e^{-\beta \sum_{j \in \Lambda} N_j/2} (\delta_{\alpha_t(Y_{j,k})}(f))^* e^{-\beta \sum_{j \in \Lambda} N_j/2} \delta_{\alpha_t(Y_{j,k})}(f)) \\ &= \frac{1}{Z} Tr(e^{-\beta \sum_{j \in \Lambda} N_j/2} (-ie^{-in\beta t}[f^*, A_j^{*n}] + ie^{im\beta t}[f^*, A_k^m]) e^{-\beta \sum_{j \in \Lambda} N_j/2} (ie^{in\beta t}[A_j^n, f] - ie^{-im\beta t}[A_k^{*m}, f])) \\ &= \hat{\eta}(0) \langle \delta_{A_j^n}(f), \delta_{A_j^n}(f) \rangle_\omega + \hat{\eta}(0) \langle \delta_{A_k^{*m}}(f), \delta_{A_k^{*m}}(f) \rangle_\omega - \hat{\eta}((-m-n)\beta) \langle \delta_{A_j^n}(f), \delta_{A_k^{*m}}(f) \rangle_\omega - \hat{\eta}((n+m)\beta) \langle \delta_{A_k^{*m}}(f), \delta_{A_j^n}(f) \rangle_\omega \end{aligned}$$

Similarly

$$\begin{aligned} \delta_{\alpha_t(Y_{j,k}^*)}(f) &= i[\alpha_t(Y_{j,k}^*), f] = ie^{-in\beta t}[A_j^{*n}, f] - ie^{im\beta t}[A_k^m, f] \\ (\delta_{\alpha_t(Y_{j,k}^*)}(f))^* &= -i[f^*, \alpha_t(Y_{j,k}^*)^*] = -ie^{in\beta t}[f^*, A_j^n] + ie^{-im\beta t}[f^*, A_k^{*m}] \end{aligned}$$

And hence

$$\begin{aligned} \int \langle \delta_{\alpha_t(Y_{j,k}^*)}(f), \delta_{\alpha_t(Y_{j,k}^*)}(f) \rangle_\omega \eta(t) dt \\ = \hat{\eta}(0) \langle \delta_{A_j^{*n}}(f), \delta_{A_j^{*n}}(f) \rangle_\omega + \hat{\eta}(0) \langle \delta_{A_k^m}(f), \delta_{A_k^m}(f) \rangle_\omega - \hat{\eta}((n+m)\beta) \langle \delta_{A_j^{*n}}(f), \delta_{A_k^m}(f) \rangle_\omega - \hat{\eta}((-m-n)\beta) \langle \delta_{A_k^m}(f), \delta_{A_j^{*n}}(f) \rangle_\omega \end{aligned}$$

Then

$$\begin{aligned} \mathcal{E}_\Lambda(f) &= \sum_{j \sim k} \int_{-\infty}^{\infty} \left(\langle \delta_{\alpha_i(Y_{jk})}(f), \delta_{\alpha_i(Y_{jk})}(f) \rangle_\omega + \langle \delta_{\alpha_i(Y_{jk}^*)}(f), \delta_{\alpha_i(Y_{jk}^*)}(f) \rangle_\omega \right) \eta(t) dt \\ &= \sum_{j \sim k} \hat{\eta}(0) \langle \delta_{A_j^n}(f), \delta_{A_j^n}(f) \rangle_\omega + \hat{\eta}(0) \langle \delta_{A_k^m}(f), \delta_{A_k^m}(f) \rangle_\omega - \hat{\eta}((-m-n)\beta) \langle \delta_{A_j^n}(f), \delta_{A_k^m}(f) \rangle_\omega - \hat{\eta}((n+m)\beta) \langle \delta_{A_k^m}(f), \delta_{A_j^n}(f) \rangle_\omega \\ &\quad + \hat{\eta}(0) \langle \delta_{A_j^{*n}}(f), \delta_{A_j^{*n}}(f) \rangle_\omega + \hat{\eta}(0) \langle \delta_{A_k^{*m}}(f), \delta_{A_k^{*m}}(f) \rangle_\omega - \hat{\eta}((n+m)\beta) \langle \delta_{A_j^{*n}}(f), \delta_{A_k^{*m}}(f) \rangle_\omega - \hat{\eta}((-m-n)\beta) \langle \delta_{A_k^{*m}}(f), \delta_{A_j^{*n}}(f) \rangle_\omega \end{aligned}$$

Remark 7.2. In Examples 7.5 and 7.8, the operators Y_{jk} 's satisfy systems of CCRs but they are not necessarily commuting for different pairs of $j, k \in \Lambda$.

Remark 7.3. The Markov generators corresponding to these Dirichlet forms are defined on a dense set and have Friedrich's extension to a self adjoint operator denoted by $-\mathfrak{Q}$, which generates a strongly continuous semigroup $P_t = e^{t\mathfrak{Q}}$.

Example 7.9 (G-Models).

For some $\kappa, \xi \neq 0$, let $Y_{\kappa, \xi} = \kappa A + \xi A^*$ and

$$\mathcal{N} \equiv Y_{\kappa, \xi}^* Y_{\kappa, \xi}.$$

Consider

$$G_{\kappa, \xi} \equiv \frac{1}{2} Y_{\kappa, \xi}^2 = \frac{1}{2} (\kappa A - \xi A^*)^2.$$

Then, with $\mathcal{R} \equiv |\kappa|^2 - |\xi|^2$, we have

$$[G_{\kappa, \xi}, G_{\kappa, \xi}^*] = \frac{1}{2} \mathcal{R}^2 + \mathcal{R} \mathcal{N}$$

and

$$[G_{\kappa, \xi}, \mathcal{N}] = 2\mathcal{R} G_{\kappa, \xi}, \quad [G_{\kappa, \xi}^*, \mathcal{N}] = -2\mathcal{R} G_{\kappa, \xi}^*.$$

Please note that in this model $\mathcal{R} \equiv |\kappa|^2 - |\xi|^2$ can take on positive as well negative values.

It provides another example of type [ShS]. More on the corresponding algebra is discussed in [MsZ1], [MsZ2].

To introduce infinite dimensional models we consider a product state ω_θ associated to one particle interaction of the form $\mathcal{N}_j \equiv \mathcal{N}_j(\kappa_j, \xi_j)$, $j \in \mathbb{Z}^d$, as follows.

$$\omega_\theta \equiv \lim_{\Lambda \rightarrow \mathbb{Z}^d} \frac{1}{Z_\Lambda} \text{Tr}_\Lambda e^{-\sum_{j \in \Lambda} \beta_j \mathcal{N}_j(\kappa_j, \xi_j)}$$

where $\beta_j \in (0, \infty)$ and $0 < Z_\Lambda < \infty$ is the normalisation constant. Thus the constants $\mathcal{R}_j$ may change with $j \in \mathbb{Z}^d$ (and the integer lattice can be replaced by a triangulation of a manifold). Then the modular and Hamiltonian automorphisms' is defined by the following (non inner) densely defined derivation

$$ad_H(f) \equiv \lim_{\Lambda \rightarrow \mathbb{Z}^d} \sum_{j \in \Lambda} [\mathcal{N}_j, f]$$

This setup includes a large family of eigenvectors of modular operator given as follows: For a finite set $J \subset \mathbb{Z}^d$

$$W_{J,O}(\kappa, \xi, \mathbf{n}) \equiv \prod_{j \in O} G_{\kappa_j, \xi_j}^{n(j)} \prod_{k \in J \setminus O} G_{\kappa_k, \xi_k}^{*n(k)}$$

where indices j, k indicate independent (commuting) copies of $G_{\kappa, \xi}$, and $\kappa \equiv (\kappa_j)_{j \in \mathbb{Z}^d}$, $\xi \equiv (\xi_j)_{j \in \mathbb{Z}^d}$, $\mathbf{n} \equiv \{n(l) \in \mathbb{N}, l \in \mathbb{Z}^d\}$.

Each of $W_{J,O}$ allows to introduce a densely defined Dirichlet form as follows

$$\mathcal{E}_{J,O}(f) \equiv \sum_{j \in \mathbb{Z}^d} \left(\|\delta_{W_{J+j,O+j}} f\|_2 + \|\delta_{W_{J+j,O+j}^*} f\|_2 \right)$$

where $J + j, O + j$ denote a shift of J, O by a vector $j \in \mathbb{Z}^d$. Note that in case when $|n(O)| = |n(J \setminus O)|$, the corresponding $W_{J,O}$ is invariant with respect to modular operator.

One can also consider Dirichlet forms with the following non-inner derivations

$$\delta_{W(JO)} \equiv \sum_{j \in \mathbb{Z}^d} \delta_{W_{J+j, O+j}}.$$

8 No Spectral Gap Property

We can prove, generalising the results of commutative case obtained in [INZ], that the quantum dissipative semigroups involving Dirichlet forms $\mathcal{E}$ defined with $Z_{j,k} = A_j - A_k$ and $Y_{j,k} = A_j - A_k^*$, for nearest neighbours' pairs (j, k) , decay to equilibrium only polynomially in time. We have the following result.

Theorem 8.1. *The Poincaré inequality does not hold for Dirichlet forms $\mathcal{E}$ defined with $Z_{j,k} = A_j - A_k$ as well as with $Y_{j,k} = A_j - A_k^*$.*

We remark that so far the only quantum example where Poincaré inequality failed was provided in [CFL] for O-U generator with equal coefficients for both directions of the derivations. For that example however an equilibrium state does not exist, while in our example it does. The idea of the proof of the theorem is to show that no constant $m \in (0, \infty)$ exists such that

$$m \|f - \omega(f)\|_2^2 \leq \mathcal{E}(f).$$

This is done by constructing a sequence of operators f_n localised in boxes Λ_n of size $2n + 1$ with variance growing as the volume of the box and the Dirichlet form only as a surface of the box.

Proof. For $Z_{j,k} = \frac{1}{2}(A_j - A_k)$, consider an increasing sequence of a finite set $\Lambda_n \equiv [-n, n]^d$ and corresponding sequence of the following operators

$$F_n \equiv \sum_{k \in \Lambda_n} A_k^*.$$

Then we have

$$\delta_{Z_{j,k}}^*(F_n) = 0$$

and

$$\delta_{Z_{j,k}}(F_n) = i \left[\frac{1}{2}(A_j - A_k), \sum_{l \in [-n, n]^d} A_l^* \right] = \frac{i}{2} \sum_{l \in [-n, n]^d} (\delta_{j,l} - \delta_{k,l})$$

with

$$\sum_{l \in [-n, n]^d} (\delta_{j,l} - \delta_{k,l}) = 0 \text{ if both } j, k \text{ are outside or both inside } \Lambda_n$$

and otherwise $\sum_{l \in [-n, n]^d} |\delta_{j,l} - \delta_{k,l}| \leq 2d$. Hence we get

$$\mathcal{E}(F_n) = \hat{\eta}(0) \sum_{j,k \in \mathbb{Z}} \left(\|\delta_{Z_{j,k}}(F_n)\|_{2,\omega}^2 + \|\delta_{Z_{j,k}}^*(F_n)\|_{2,\omega}^2 \right) \leq \text{Const} |\partial \Lambda_n|$$

On the other hand, for the product state,

$$\|F_n - \langle F_n \rangle_\omega\|_{2,\omega}^2 \sim \sum_{j \in \Lambda_n} \|A_j^* - \langle A_j^* \rangle_\omega\|_{2,\omega}^2 \sim \text{Vol}(\Lambda_n) \|A_0^* - \langle A_0^* \rangle_\omega\|_{2,\omega}^2$$

Since the boundary $|\partial \Lambda_n| / \text{Vol}(\Lambda_n) \rightarrow_{n \rightarrow \infty} 0$, the Poincaré inequality cannot hold.

□

Further examples

Similar conclusion holds for Dirichlet form defined with some other derivations.

Example 8.1. For $Z_{j,k} = A_j^m - A_k^m$, one can consider a sequence of the following operators

$$F_n \equiv \sum_{k \in \Lambda_n} A_k^*.$$

for a finite set Λ_n .

Then we have

$$\begin{aligned} \delta_{Z_{j,k}}(F_n) &= i[A_j^m - A_k^m, \sum_{l \in [-n,n]^d} A_l^*] = im \sum_{l \in [-n,n]^d} (\delta_{j,l} A_j^{m-1} - \delta_{k,l} A_k^{m-1}) \\ &\sim \begin{cases} mA_j^{m-1} \text{ or } -mA_k^{m-1} & \text{if either } j \text{ or } k \text{ is outside } \Lambda_n \\ 0 & \text{if both are outside } \Lambda_n \end{cases} \end{aligned}$$

and

$$\delta_{Z_{j,k}}^*(F_n) = 0$$

Hence we conclude that

$$\mathcal{E}(F_n) \sim |\partial \Lambda_n|, \quad \text{and} \quad \|F_n - \langle F_n \rangle\|^2 \sim |\Lambda_n|.$$

which precludes Poincare inequality. Another important possibility of a sequence of test operators is provided by

$$F_n \equiv \sum_{k \in \Lambda_n} N_k.$$

In this case we have

$$\delta_{Z_{j,k}}^*(F_n) = \text{sign}(\sharp) i[A_j^{\sharp m} - A_k^{\sharp m}, \sum_{l \in [-n,n]} N_l] = \text{sign}(\sharp) im (\delta_{j,l} A_j^{\sharp m-1} - \delta_{k,l} A_k^{\sharp m-1})$$

where $\text{sign}(\sharp)$ equals ± 1 , (1 for A and -1 for A^*), respectively. Thus we reach the same conclusion.

Example 8.2. For $Y_{j,k} = A_j + A_k^*$, consider a sequence

$$F_n = \sum_{\substack{l, l' \in \Lambda_n \\ l \sim l'}} (A_l + A_{l'}^*)$$

Remark One could generalise this example to include $Y_{\kappa, \xi}$, with κ and ξ being absolutely summable sequences satisfying the following mean zero condition

$$\sum_{j \in \mathbb{Z}^d} \kappa_j = 0, \quad \sum_{j \in \mathbb{Z}^d} \xi_j = 0$$

9 Algebra of Invariant Derivations

In the setup associated to the state ω_0 , consider the operators of the following form

$$A(I, J) \equiv \prod_{i \in I} A_i \prod_{j \in J} A_j^*.$$

One can show using e.g. (8) that in case $|I| = |J|$ such operators are invariant with respect to the modular dynamics. (Note that for $I = J$, the operator $A(I, J)$ is a polynomial in N_j , $j \in J$.) Choosing $I \cap J = \emptyset$, we can define a Dirichlet form as follows

$$\mathcal{E}_{I,J}(f) \equiv \sum_{k \in \mathbb{Z}^d} \|\delta_{A(I+k, J+k)} f\|_{2, \omega_0}^2.$$

We have the following general result.

Theorem 9.1. *The Dirichlet form $\mathcal{E}_{I,J}$ does not satisfy Poincaré inequality.*

Proof. One can prove this general result choosing a sequence of operators

$$F_n \equiv \sum_{j \in \Lambda_n} N_j$$

and noticing that if $I + k, J + k \subset \Lambda_n$, then

$$\delta_{A(I+k, J+k)}(F_n) = 0.$$

Hence

$$\mathcal{E}_{I,J}(F_n) \sim |\partial \Lambda_n|$$

while

$$\|F_n - \omega_0(F_n)\|_{2, \omega_0}^2 \sim |\Lambda_n|.$$

and hence the Poincaré inequality with any fixed positive constant would break down for large Λ_n . $\square$

In fact in the present setup we have an infinite dimensional algebra which is invariant with respect to the modular operator. It includes operators $A(I, J)$, but also more general ones. For example denoting by

$$A_I^{n(I)} \equiv \prod_{i \in I} A_i^{n(i)}$$

with multi-index $n(I) \equiv (n(i) \in \mathbb{N})_{i \in I}$, and setting $|n(I)| \equiv \sum_{i \in I} n(i)$, we can consider operators of the form

$$P_{I,J}(N_I, N_J) \left(A_I^{n(I)} \left(A_J^{n(J)} \right)^* \right)$$

with $|n(I)| = |n(J)|$ and a polynomial $P_{I,J}(N_I, N_J)$.

This provides a possibility to define an infinite number of Dirichlet forms which cannot satisfy Poincaré inequality.

Besides the inner derivations indicated above, one can consider also exterior derivations and an infinite dimensional Lie algebra of them. For example the following limit provides non-inner derivations well defined on a dense set of local elements in $\mathbb{L}_{2, \omega_0}$

$$\delta_{I,J}(f) \equiv \lim_{\Lambda \rightarrow \mathbb{Z}^d} \sum_{k \in \Lambda} \delta_{A(I+k, J+k)}(f).$$

An interesting open problem is to provide a constructive examples of such algebra for Gibbs states obtained with a nontrivial interaction.

The Dirichlet forms defined with derivations which are non-inner provide a new challenging area of noncommutative analysis.

10 Polynomial Decay to Equilibrium

In this section we consider a model with derivations in direction of $Z_{j,k} = A_j - A_k$, for $j \sim k$, $j, k \in \mathbb{Z}^d$ in the space associated to the product state ω_0 describing a system of infinite number of quantum harmonic oscillators. We show the following result generalising the commutative one of [INZ].

Theorem 10.1. *A quantum system described by the family of generators of the form in the above decays to equilibrium algebraically in time in the sense that*

$$\sum_j |\delta_{A_j^\#}(e^{-tL} f)|^2 \rightarrow_{t \rightarrow \infty} 0$$

with algebraic rate.

Proof:
Z-Case

For the model with derivations in direction of $Z_{jk} = A_j - A_k$, for $j \sim k$, $j, k \in \mathbb{Z}^d$, set $f_t = P_t f$. Same as writing the derivative of the scalar product

$$\begin{aligned} \frac{d}{dt} \langle \delta_{A_j}(P_t f), \delta_{A_j}(P_t f) \rangle_\omega &= \langle \frac{d}{dt} \delta_{A_j}(P_t f), \delta_{A_j}(P_t f) \rangle_\omega + \langle \delta_{A_j}(P_t f), \frac{d}{dt} \delta_{A_j}(P_t f) \rangle_\omega \\ &= \langle \delta_{A_j} L(P_t f), \delta_{A_j}(P_t f) \rangle_\omega + \langle \delta_{A_j}(P_t f), \delta_{A_j} \mathfrak{L}(P_t f) \rangle_\omega \\ &= \langle L \delta_{A_j}(P_t f) + [\delta_{A_j}, L](P_t f), \delta_{A_j}(P_t f) \rangle_\omega + \langle \delta_{A_j}(P_t f), L \delta_{A_j}(P_t f) + [\delta_{A_j}, L](P_t f) \rangle_\omega \\ &\leq \langle [\delta_{A_j}, L](P_t f), \delta_{A_j}(P_t f) \rangle_\omega + \langle \delta_{A_j}(P_t f), [\delta_{A_j}, L](P_t f) \rangle_\omega \end{aligned} \quad (18)$$

Next we use the following fact.

Lemma 10.1.

$$[\delta_{A_j}, L] = 4\hat{\eta}(0) \sinh(\beta/2) \sum_{k \sim j} (\delta_{A_k} f - \delta_{A_j} f).$$

Proof. (of Lemma) Since

$$L = -\hat{\eta}(0) \sum_{j \sim k} (\delta_{Z_{jk}}^* \delta_{Z_{jk}} + \delta_{Z_{jk}^*}^* \delta_{Z_{jk}^*})$$

and A_l commute with Z_{jk} , for fixed l , we have

$$[\delta_{A_l}, L] = -\hat{\eta}(0) \sum_{j \sim k} ([\delta_{A_l}, \delta_{Z_{jk}}^*] \delta_{Z_{jk}} + [\delta_{A_l}, \delta_{Z_{jk}^*}^*] \delta_{Z_{jk}^*}).$$

Since, by Proposition 2.2, we have

$$\begin{aligned} \delta_{Z_{jk}}^*(g) &= -\delta_{\alpha_{-i/2}(Z_{jk}^*)}(g) + i(\alpha_{-i/2}(Z_{jk}^*) - \alpha_{i/2}(Z_{jk}^*))g \\ &= -e^{-\frac{\beta}{2}} \delta_{Z_{jk}^*}(g) - 2i \sinh(\frac{\beta}{2}) Z_{j,k}^* g \end{aligned}$$

In our setup we have

$$[\delta_{A_l}, \delta_{Z_{jk}^*}^*] = i\delta_{[A_l, Z_{jk}^*]} = 0.$$

On the other hand for the left multiplication operator $l_{Z_{jk}^*}^*$ by $Z_{j,k}^*$, thanks to Leibnitz rule for the derivation

$$[\delta_{A_l}, l_{Z_{jk}^*}^*] = \delta_{A_l}(Z_{j,k}^*) = \delta_{l,j} - \delta_{l,k}$$

Hence we obtain

$$\begin{aligned} [\delta_{A_l}, \delta_{Z_{jk}^*}^*](g) &= [\delta_{A_l}, -e^{-\frac{\beta}{2}} \delta_{Z_{jk}^*} - 2i \sinh(\frac{\beta}{2}) l_{Z_{jk}^*}^*](g) = [\delta_{A_l}, -2i \sinh(\frac{\beta}{2}) l_{Z_{jk}^*}^*](g) \\ &= 2 \sinh(\frac{\beta}{2}) (\delta_{l,j} - \delta_{l,k}) g \end{aligned}$$

as $[A_l, Z_{j,k}^*] = \delta_{l,j} - \delta_{l,k}$. Thus we have

$$[\delta_{A_l}, \delta_{Z_{jk}^*}^*] \delta_{Z_{jk}}(f) = 2 \sinh(\beta/2) (\delta_{l,j} - \delta_{l,k}) \delta_{Z_{jk}}(f)$$

and hence for a fixed l ,

$$\begin{aligned} [\delta_{A_l}, L] &= -2 \sinh(\beta/2) \hat{\eta}(0) \sum_{j \sim k} (\delta_{l,j} - \delta_{l,k}) \delta_{Z_{jk}}(f) \\ &= -4 \sinh(\beta/2) \hat{\eta}(0) \sum_{j \sim l} \delta_{Z_{l,j}}(f) \\ &= -4 \sinh(\beta/2) \hat{\eta}(0) \sum_{j \sim l} (\delta_{A_l} f - \delta_{A_j} f) \end{aligned}$$

□

Using (18) and lemma 10.1, for any $j \in \mathbb{Z}^d$ we have

$$\frac{d}{dt} \langle \delta_{A_j}(P_t f), \delta_{A_j}(P_t f) \rangle_\omega \leq 8 \sinh(\beta/2) \hat{\eta}(0) \sum_{k \sim j} \Re \langle \delta_{A_j}(P_t f), (\delta_{A_k} - \delta_{A_j})(P_t f) \rangle.$$

Using Cauchy-Schwartz inequality on the right hand side, this implies the following relation

$$\frac{d}{dt} \|\delta_{A_j}(P_t f)\| \leq 4\hat{\eta}(0) \sinh(\beta/2) \sum_{k \sim j} \left(\|\delta_{A_k}(P_t f)\| - \|\delta_{A_j}(P_t f)\| \right).$$

with the norm given by the scalar product.

Denoting by Δ the lattice Laplacian on $\mathbb{Z}^d$ and setting $F(t, j) = \|\delta_{A_j}(P_t f)\|$, with $C \equiv 4\hat{\eta}(0) \sinh(\beta/2)$, we get the following differential inequality

$$\frac{d}{dt} F(t, j) \leq C (\Delta F)(t, j)$$

Integrating both sides

$$\begin{aligned} \int_0^t dF(s) &\leq \int_0^t C \Delta F(s) ds \\ F(t) &\leq F(0) + \int_0^t C \Delta F(s) ds, \end{aligned}$$

by iteration we arrive at the following bound

$$F(t) \leq e^{Ct\Delta} F(0)$$

from which the algebraic decay follows, see e.g. [INZ]. Similar arguments work and so we get the same algebraic bound for derivation with respect to A_j^* .

Let us note that similarly as in the case of functions, [INZ], the space of linear combinations of A_j, A_j^* , $j \in \mathbb{Z}^d$, is mapped into itself by the generator L . Explicitly we have

$$\begin{aligned} LA_l &= -\hat{\eta}(0) \sum_{j \sim k} \left(\delta_{Z_{jk}}^* \delta_{Z_{jk}} \right) A_l \\ &= i\hat{\eta}(0) \sum_{j \sim k} \left(\delta_{Z_{jk}}^* (\delta_{jl} - \delta_{kl}) \right) = 4\hat{\eta}(0) \sinh\left(\frac{\beta}{2}\right) \sum_{j \sim l} (A_j^* - A_l^*) \\ LA_l^* &= -i\hat{\eta}(0) \sum_{j \sim k} \left(\delta_{Z_{jk}}^* (\delta_{jl} - \delta_{kl}) \right) = 4\hat{\eta}(0) \sinh\left(\frac{\beta}{2}\right) \sum_{j \sim l} (A_j - A_l) \end{aligned}$$

where we used

$$\begin{aligned} \delta_{Z_{jk}}^* (g) &= -e^{-\frac{\beta}{2}} \delta_{Z_{jk}}^* (g) - 2i \sinh\left(\frac{\beta}{2}\right) Z_{jk}^* g \\ \delta_{Z_{jk}}^* (g) &= -e^{\frac{\beta}{2}} \delta_{Z_{jk}} (g) + 2i \sinh\left(\frac{\beta}{2}\right) Z_{jk} g \end{aligned}$$

Hence, with $C \equiv 4\hat{\eta}(0) \sinh(\frac{\beta}{2})$, we get

$$\begin{aligned} L(A_j \pm A_j^*) &= C \sum_{j \sim l} ((A_j \pm A_j^*) - (A_l \pm A_l^*)) \\ &\equiv C(\Delta(A. \pm A^*))_j \end{aligned}$$

Thus for an operator linear in creators and annihilators

$$f \equiv \sum_j \kappa_j(t)(A_j \pm A_j^*)$$

we have

$$\begin{aligned} \partial_t f &= C \sum_j \dot{\kappa}_j(t)(A_j \pm A_j^*) \\ &= C \sum_j \kappa_j(t)(\Delta(A \cdot \pm A^*))_j = C \sum_j (\Delta \kappa)_j(t)(A_j \pm A_j^*) \end{aligned}$$

which holds iff

$$\dot{\kappa}_j(t) = C \sum_j (\Delta \kappa)_j(t)$$

i.e.

$$\kappa(t) = e^{tC\Delta} \kappa_0.$$

This provides explicit algebraic decay to equilibrium for operators linear in creators and annihilators.

Y -case is similar. □

11 Appendix: Γ_1

For the generator

$$-\mathfrak{L}(f) = \int \left(\delta_{\alpha_t(X)}^* \delta_{\alpha_t(X)}(f) + \delta_{\alpha_t(X^*)}^* \delta_{\alpha_t(X^*)}(f) \right) \eta(t) dt$$

associated to a state ω and the corresponding Dirichlet form in this appendix we demonstrate that

$$\begin{aligned} 2\Gamma_1(f) &= \mathfrak{L}(f^* f) - f^* \mathfrak{L}(f) - \mathfrak{L}(f^*) f \\ &= \int \left(\left| \delta_{\alpha_{t-i/4}(X^*)}(f) \right|^2 + \left| \delta_{\alpha_{t-i/4}(X)}(f) \right|^2 \right) (\eta(t + i/4) + \eta(t - i/4)) dt \end{aligned}$$

for all operators $f \in \mathcal{D}(\mathfrak{L})$ such that $f^* f \in \mathcal{D}(\mathfrak{L})$ and $f^* \mathfrak{L}(f) \in \mathbb{L}_2$.

Proof. Using the Leibnitz rule for the derivation for the first part of the integrant, (the second will be analogous), we have

$$\delta_{\alpha_t(X)}^* \delta_{\alpha_t(X)}(f^* f) = \delta_{\alpha_t(X)}^* (\delta_{\alpha_t(X)}(f^*) f + f^* \delta_{\alpha_t(X)}(f))$$

Since the 1st, respectively 2nd, formula of Proposition 2 for modified Leibnitz property of the adjoint, for $f, g \in \mathcal{D}_\varepsilon$, we have

$$\begin{aligned} \delta_Y^*(fg) &= \delta_Y^*(f)g - f\delta_{\alpha_{-i/2}(Y^*)}(g) \\ &= f\delta_Y^*(g) - \delta_{\alpha_{i/2}(Y^*)}(f)g, \end{aligned}$$

hence

$$\begin{aligned} \delta_{\alpha_t(X)}^* \delta_{\alpha_t(X)}(f^* f) &= \left(\delta_{\alpha_t(X)}^* \delta_{\alpha_t(X)}(f^*) \right) f - \delta_{\alpha_t(X)}(f^*) \cdot \delta_{\alpha_{t-i/2}(X^*)}(f) \\ &\quad + f^* \left(\delta_{\alpha_t(X)}^* \delta_{\alpha_t(X)}(f) \right) - \delta_{\alpha_{t+i/2}(X^*)}(f^*) \cdot \delta_{\alpha_t(X)}(f) \end{aligned}$$

Similarly for the term with X^* replacing X , we get

$$\begin{aligned}\delta_{\alpha_t(X^*)}^* \delta_{\alpha_t(X^*)}(f^* f) &= \left(\delta_{\alpha_t(X^*)}^* \delta_{\alpha_t(X^*)}(f^*) \right) f - \delta_{\alpha_t(X^*)}(f^*) \cdot \delta_{\alpha_{t-i/2}(X)}(f) \\ &\quad + f^* \left(\delta_{\alpha_t(X^*)}^* \delta_{\alpha_t(X^*)}(f) \right) - \delta_{\alpha_{t+i/2}(X)}(f^*) \cdot \delta_{\alpha_t(X^*)}(f)\end{aligned}$$

Adding together both formulas we have

$$\begin{aligned}\mathfrak{Q}(f^* f) &= \int \left(\delta_{\alpha_t(X)}^* \delta_{\alpha_t(X)}(f^* f) + \delta_{\alpha_t(X^*)}^* \delta_{\alpha_t(X^*)}(f^* f) \right) \eta(t) dt \\ &= f^* \mathfrak{Q}(f) + \mathfrak{Q}(f^*) f \\ &\quad + \int \left(\delta_{\alpha_t(X)}(f^*) \cdot \delta_{\alpha_{t-i/2}(X^*)}(f) \right) \eta(t) dt \\ &\quad + \int \left(\delta_{\alpha_{t+i/2}(X^*)}(f^*) \cdot \delta_{\alpha_t(X)}(f) \right) \eta(t) dt \\ &\quad + \int \left(\delta_{\alpha_t(X^*)}(f^*) \cdot \delta_{\alpha_{t-i/2}(X)}(f) \right) \eta(t) dt \\ &\quad + \int \left(\delta_{\alpha_{t+i/2}(X)}(f^*) \cdot \delta_{\alpha_t(X^*)}(f) \right) \eta(t) dt\end{aligned}\tag{A.1}$$

Shifting the integration variable in first and third term by $-i/4$ and in the second and fourth by $-i/4$, we get

$$\begin{aligned}\mathfrak{Q}(f^* f) &= \int \left(\delta_{\alpha_t(X)}^* \delta_{\alpha_t(X)}(f^* f) + \delta_{\alpha_t(X^*)}^* \delta_{\alpha_t(X^*)}(f^* f) \right) \eta(t) dt \\ &= f^* \mathfrak{Q}(f) + \mathfrak{Q}(f^*) f \\ &\quad + \int \left(\delta_{\alpha_{t+i/4}(X)}(f^*) \cdot \delta_{\alpha_{t-i/4}(X^*)}(f) \right) \eta(t + i/4) dt \\ &\quad + \int \left(\delta_{\alpha_{t+i/4}(X^*)}(f^*) \cdot \delta_{\alpha_{t-i/4}(X)}(f) \right) \eta(t - i/4) dt \\ &\quad + \int \left(\delta_{\alpha_{t+i/4}(X^*)}(f^*) \cdot \delta_{\alpha_{t-i/4}(X)}(f) \right) \eta(t + i/4) dt \\ &\quad + \int \left(\delta_{\alpha_{t+i/4}(X)}(f^*) \cdot \delta_{\alpha_{t-i/4}(X^*)}(f) \right) \eta(t - i/4) dt\end{aligned}$$

Now the terms in the brackets can be written as squares of operators. After adding corresponding terms with the same weight, we obtain

$$\begin{aligned}\mathfrak{Q}(f^* f) &= \int \left(\delta_{\alpha_t(X)}^* \delta_{\alpha_t(X)}(f^* f) + \delta_{\alpha_t(X^*)}^* \delta_{\alpha_t(X^*)}(f^* f) \right) \eta(t) dt \\ &= f^* \mathfrak{Q}(f) + \mathfrak{Q}(f^*) f \\ &\quad + \int \left(\left| \delta_{\alpha_{t-i/4}(X^*)}(f) \right|^2 + \left| \delta_{\alpha_{t-i/4}(X)}(f) \right|^2 \right) \eta(t + i/4) dt \\ &\quad + \int \left(\left| \delta_{\alpha_{t-i/4}(X)}(f) \right|^2 + \left| \delta_{\alpha_{t-i/4}(X^*)}(f) \right|^2 \right) \eta(t - i/4) dt\end{aligned}$$

Since the brackets are the same in both integral expressions, finally this can be rearranged as follows

$$\begin{aligned}\mathfrak{Q}(f^* f) &= \int \left(\delta_{\alpha_t(X)}^* \delta_{\alpha_t(X)}(f^* f) + \delta_{\alpha_t(X^*)}^* \delta_{\alpha_t(X^*)}(f^* f) \right) \eta(t) dt \\ &= f^* \mathfrak{Q}(f) + \mathfrak{Q}(f^*) f \\ &\quad + \int \left(\left| \delta_{\alpha_{t-i/4}(X^*)}(f) \right|^2 + \left| \delta_{\alpha_{t-i/4}(X)}(f) \right|^2 \right) (\eta(t + i/4) + \eta(t - i/4)) dt\end{aligned}$$

Hence we conclude

$$(A.2) \quad \begin{aligned} 2\Gamma_1(f) &= \mathfrak{Q}(f^*f) - f^*\mathfrak{Q}(f) - \mathfrak{Q}(f^*)f \\ &= \int \left(|\delta_{\alpha_{t-i/4}(X^*)}(f)|^2 + |\delta_{\alpha_{t-i/4}(X)}(f)|^2 \right) (\eta(t+i/4) + \eta(t-i/4)) dt \end{aligned}$$

□

If $\alpha_{\pm i/4}(X) = e^{\pm \frac{1}{2}\xi}X$ then the formula (A.2) yields

$$(A.3) \quad \frac{1}{C}\Gamma_1(f) = e^\xi |\delta_{X^*}(f)|^2 + e^{-\xi} |\delta_X(f)|^2$$

with a constant $C \equiv \int (\eta(t+i/4) + \eta(t-i/4)) dt$.

For example this will be the case of modular dynamics associated to ω_0 and derivations associated with $A_{I,J}$. Generalisation to the case of infinite number of eigenvectors is as follows $X_j, j \in \mathbb{Z}^d$, is as follows

$$-\tilde{\mathfrak{Q}}(f^*f) = \sum_{j \in \mathbb{Z}^d} \left(\kappa_j \delta_{X_j}^* \delta_{X_j}(f^*f) + \mu_j \delta_{X_j}^* \delta_{X_j^*}(f^*f) \right)$$

with some constants $\kappa_j, \mu_j \in \mathbb{R}^+$. Then similar computations as above yield

$$\tilde{\Gamma}_1(f) = \sum_{j \in \mathbb{Z}^d} \left(\kappa_j e^{-\xi_j} |\delta_{X_j}(f)|^2 + \mu_j e^{\xi_j} |\delta_{X_j^*}(f)|^2 \right)$$

In the infinite dimensional case on a lattice to secure the dense domain it is necessary to have the finite speed of propagation of information.

Remark A.1:

Recall that using Leibnitz rule for the derivations and modified Leibnitz rule for its adjoint in $\mathbb{L}_2$

$$\delta_Y^*(fg) = \delta_Y^*(f)g - f\delta_{\alpha_{-i/2}(Y^*)}(g),$$

we have the following calculations

$$\begin{aligned} \delta_{X_t}^* \delta_{X_t}(f^*f) &= \delta_{X_t}^* (\delta_{X_t}(f^*)f + f^* \delta_{X_t}(f)) \\ &= (\delta_{X_t}^* \delta_{X_t}(f^*))f - f^* \delta_{\alpha_{-i/2}(X_t^*)}(f) + f^* (\delta_{X_t}^* \delta_{X_t}(f)) - \delta_{\alpha_{i/2}(X_t^*)}(f^*)f \delta_{X_t}^*(f^*) \delta_{X_t}(f) \end{aligned}$$

which implies

$$\mathfrak{Q}(f^*f) = \mathfrak{Q}(f^*)f - f^* \int (\delta_{\alpha_{-i/2}(X_t^*)}(f) + \delta_{\alpha_{-i/2}(X_t)}(f)) \eta_t dt + f^* \mathfrak{Q}(f) - \int (\delta_{\alpha_{i/2}(X_t^*)}(f^*) + \delta_{\alpha_{i/2}(X_t)}(f^*)) \eta_t dt f.$$

provided operator f satisfies suitable conditions so that the operations on the right hand side make sense. In particular $f^*f \in \mathcal{D}(\mathfrak{Q})$, for bounded operators $f \in \mathcal{D}(\mathfrak{Q})$, if $\exists C \in (0, \infty)$ such that we have

$$\| \int (\delta_{\alpha_{i/2}(X_t^*)}(f^*) + \delta_{\alpha_{i/2}(X_t)}(f^*)) \eta_t dt \|^2 \leq 2 \int (\| \delta_{\alpha_{\pm i/2}(X_t^*)}(f^*) \|_2^2 + \| \delta_{\alpha_{\pm i/2}(X_t)}(f^*) \|_2^2) \eta_t dt \leq C \mathcal{E}(f)$$

The last condition can be satisfied if η is analytic in a strip $[-i/2, +i/2]$ and $|\eta(t \pm i/2)| \leq C\eta(t)$. In the models considered before a domain of validity of the above relation could contain local polynomials of elements in $\mathcal{D}_\varepsilon$. For a positivity preserving contraction semigroup in $\mathbb{L}_2$, for any $t > 0$ and $f \in \mathbb{L}_\infty$, we have $P_t f \in \mathcal{D}(\mathfrak{Q}) \cap \mathbb{L}_\infty$. Thus for bounded operators, we can identify Γ_1 as follows: For $0 < s < t$

$$\frac{d}{ds} (P_{t-s} |P_s f|^2 - |P_s f|^2) = 2P_{t-s} \Gamma_1(P_s f) \geq 0.$$

Using this, we have

$$P_t f^2 - |P_t f|^2 = \int_0^t \frac{d}{ds} P_{t-s} (P_s f)^2 ds = \int_0^t ds \frac{d}{ds} P_{t-s} (P_s f)^2 = \int_0^t ds 2P_{t-s} \Gamma_1(P_s f) \geq 0.$$

Hence, under the above conditions, for Markov semigroup in $\mathbb{L}_2$ the following strong positivity conditions holds.

Proposition 11.1. *For the diffusion semigroup $P_t \equiv e^{t\mathbb{L}}$ in $\mathbb{L}_2(\omega)$, for any $t > 0$ and $f \in \mathbb{L}_2 \cap \mathbb{L}_\infty$, the following Schwartz inequality is true.*

$$|P_t f|^2 \leq P_t |f|^2.$$

Remark A.2:

Next we note that using the formula for Γ_1 , with suitable function $\gamma(t) \geq 0$, in general with some positive constants $\nu_\sharp$, we have

$$\begin{aligned} \Gamma_1(\alpha_{-i/4}(f)) &= \int \sum_{\sharp} \nu_{\sharp} (|\alpha_{-i/4}(\delta_{\alpha_t(X^\sharp)}(f))|^2) \gamma(t) dt \\ &= \int \sum_{\sharp} \nu_{\sharp} (\rho^{-1/4} (\delta_{\alpha_t(X^\sharp)}(f))^* \rho^{1/2} (\delta_{\alpha_t(X^\sharp)}(f)) \rho^{-1/4}) \gamma(t) dt \end{aligned}$$

Hence we have the following property.

Proposition 11.2.

$$\omega \Gamma_1(\alpha_{-i/4}(f)) = \int \sum_{\sharp} \nu_{\sharp} \langle \delta_{\alpha_t(X^\sharp)}(f), \delta_{\alpha_t(X^\sharp)}(f) \rangle \gamma(t) dt \simeq \mathcal{E}_\gamma(f).$$

In case when $X^\sharp$ are eigenvectors of modular operator to the power $\pm 1/4$, with some positive constants ν, μ , we have

$$\Gamma_1(\alpha_{-i/4}(f)) = \nu e^{-\xi/2} |\delta_X(\alpha_{-i/4}(f))|^2 + \mu e^{\xi/2} |\delta_{X^*}(\alpha_{-i/4}(f))|^2 = \nu |\alpha_{-i/4}(\delta_X(f))|^2 + \mu |\alpha_{-i/4}(\delta_{X^*}(f))|^2$$

and hence

$$\omega \tilde{\Gamma}_1(\alpha_{-i/4}(f)) = \nu \langle \delta_X(f), \delta_X(f) \rangle + \mu \langle \delta_{X^*}(f), \delta_{X^*}(f) \rangle \sim \mathcal{E}(f)$$

Hence, we have the following property.

Proposition 11.3. *For $f_t \equiv P_t f \equiv e^{-t\mathbb{L}} f$, if*

$$\Gamma_1(f_t) \leq e^{-2mt} \text{Const} \Gamma_1(f),$$

then we have

$$\mathcal{E}(f_t) \leq e^{-2mt} \text{Const} \mathcal{E}(f)$$

and if Poincaré inequality holds, we get

$$\|f_t - \omega(f)\|_2^2 \leq e^{-2mt} \|f - \omega(f)\|_2^2.$$

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•Back to Contents