NUMERICAL ANALYSIS OF THIN SHELLS

by

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Volume I

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ABSTRACT

An extended Levy method for the solution of shallow doubly curved shells with general boundary conditions along their edges is presented. The analysis is based on a linear theory for thin, elastic, isostropic, shallow shells formulated by Vlasov. The shells are assumed to be of constant thickness and to have a rectangular planform.

A Navier type solution using double trigonometric expansions is used as the particular solution. The complementary solution is made up of two groups of linearly independent Levy type solutions using trigonometric expansions in the two orthogonal curvilinear coordinate directions respectively. The particular solution and each Levy type solution forming the complementary solution satisfies the governing equations but not all the boundary conditions. The method of solution consists of combining suitable sets of Levy type solutions with the particular solution to satisfy all the boundary conditions simultaneously.

The theory for the Levy type solution of a spherical cap (equal curvatures) is presented.

A digital computer programme for the solution of translational shells (plates, cylinders, elliptic paraboloids and hyperbolic paraboloids) with a variety of

boundary conditions(normal gables, oblique gables, hinged, clamped, free, flexible edge beams) along their translational generators is presented. Uniform rectangular patch loadings in the three coordinate directions and thermal loading on the shell, and uniform line loads on the edge beams are considered.

A digital computer programme for the solution of hyperbolic paraboloid shells with a variety of boundary conditions (free-guided, normal gables, hinged, clamped and free) along their straight edges is presented. Uniform rectangular patch loadings in the three coordinate directions are considered.

Several checks made on the computer programmes are included. The method of solution and the convergence of the solutions are discussed. Solution of certain problems whose range extends beyond those normally included in the standard literature is presented.

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NOTATION

Shell

α, β	Curvilinear coordinates of the middle surface of the shell
γ	Coordinate normal to the middle surface of the shell
t	Thickness of the shell
l_α, l_β	Arc lengths of the shell in the α and β directions
L_α, L_β	Plan lengths of the shell in the α and β directions
a, b	'Rise' of the shell in the α and β directions
$k_\alpha, k_{\alpha\beta}, k_\beta$	Curvatures of the middle surface of the shell
E	Young's modulus of the shell material
ν	Poisson's ratio of the shell material
λ	Coefficient of linear thermal expansion of the shell material
$D = \dfrac{Et^3}{12(1-\nu^2)}$	Flexural rigidity of the shell material
$u_\alpha, u_\beta, u_\gamma$	Displacements of the middle surface of the shell
$n_\alpha, n_{\alpha\beta}, n_{\beta\alpha}, n_\beta$	Extensional actions
$m_\alpha, m_{\alpha\beta}, m_{\beta\alpha}, m_\beta$	Flexural actions
q_α, q_β	

$q'_\alpha = q_\alpha + \frac{\partial m_{\alpha\beta}}{\partial \beta}$ Modified transverse shear and extensional shear actions along a free boundary (Kirchhoff boundary conditions)

$q'_\beta = q_\beta + \frac{\partial m_{\alpha\beta}}{\partial \alpha}$

$n'_{\alpha\beta} = n_{\alpha\beta} - k_\alpha\ m_{\alpha\beta}$

$n'_{\beta\alpha} = n_{\beta\alpha} - k_\beta\ m_{\alpha\beta}$

ϕ	Stress function
$G_\alpha, G_\beta, G_\gamma$	Components of external loading on the shell
T_1, T_2	Temperatures on shell surfaces $\gamma = -t/2$ and $\gamma = +t/2$
(x_i, y_i)	Coordinates of the centre of patch loading on the shell
(u_i, v_i)	Dimensions of the loaded area (patch loading)
$\bar{P}$	Intensity of loading in the $(\beta-\gamma)$ plane for a cylindrical shell
ψ	Anticlockwise angle measured from the positive direction of loading $\bar{P}$ to the positive direction of shell axis γ at the origin of coordinates (α, β, γ).
ϕ_1, ϕ_4	Anticlockwise angles measured from oblique gable surfaces to positive direction of shell axis γ along edges 1 and 4
ϕ_2, ϕ_3	Clockwise angles measured from oblique gable surfaces to positive direction of shell axis γ along edges 2 and 3

Edge beams

I, II, III	Beam axes
E_b	Young's modulus of edge beam material
U_1, U_2, U_3, U_4	Edge beam displacements
N_1, S_1, S_2	Edge beam actions
M_1, M_2, M_3	
Z_1, Z_2, Z_3	Components of external loading on edge beam
I_1^k	Second moment of area of edge beam k about beam axis II
I_2^k	Second moment of area of edge beam k about beam axis III
I_3^k	'Twisting' moment of area of edge beam k
A^k	Cross-sectional area of edge beam k
a^k, b^k	Distances measured from the centroid of edge beam k to the intersection of shell and edge beam k in the direction of beam axes II and III
ψ_1, ψ_4	Anticlockwise angles measured from the positive direction of beam axis III to the positive direction of shell axis γ along edges 1 and 4

ψ_2, ψ_3	Clockwise angles measured from the positive direction of beam axis III to the positive direction of shell axis γ along edges 2 and 3

General

p,q	Positive integers representing the harmonics in the α and β directions
$i = j(k)l$	The notation represents i taking the sequence of values j, j+k, j+2k,...,l-k,l
x, y	The notation represents x x 10^y (floating point representation)
$P = a_1(b_1)c_1$	Complementary solution harmonics considered in the α direction
$Q = a_2(b_2)c_2$	Complementary solution harmonics considered in the β direction
$P' = a_3(b_3)c_3$	Particular solution harmonics considered in the α direction
$Q = a_4(b_4)c_4$	Particular solution harmonics considered in the β direction

$m = \frac{p\pi}{l_\alpha}$

$n = \frac{q\pi}{l_\beta}$

$\bar{\alpha} = \frac{\pi\alpha}{l_\alpha}$

$\bar{\beta} = \frac{\pi\beta}{l_\beta}$

$$\alpha_1 = \frac{\alpha}{l_\alpha}$$

$$\beta_1 = \frac{\beta}{l_\beta}$$

$$x_i = \bar{x}_i \; l_\alpha$$

$$y_i = \bar{y}_i \; l_\beta$$

$$u_i = \bar{u}_i \; l_\alpha$$

$$v_i = \bar{v}_i \; l_\beta$$

$$F(\;) = \sin(\;) \text{ or } \cos(\;)$$

$$\nabla_R^{\,2} = k_\beta \frac{\partial^2}{\partial\alpha^2} - 2k_{\alpha\beta} \frac{\partial^2}{\partial\alpha\partial\beta} + k_\alpha \frac{\partial^2}{\partial\beta^2}$$

$$\nabla_R^{\,4} = \nabla_R^{\,2} \, \nabla_R^{\,2}$$

$$\nabla^2 = \frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}$$

$$\nabla^4 = \nabla^2 \nabla^2$$

$$\nabla^8 = \nabla^4 \nabla^4$$

CHAPTER 1

INTRODUCTION AND SCOPE OF RESEARCH

1.1 Introduction

The object of this section is to describe briefly some of the developments in the solution of the shallow curved plate equations.

Plates

The first solution to the problem of the bending of plates is due to Navier, in 1820, using double Fourier series. In 1899, Levy suggested the use of trigonometric functions in one coordinate direction and a trigonometric-hyperbolic solution in the other coordinate direction. Since then, the Navier, Levy and extended Levy methods of solution have been extensively used in the solution of plate problems.

Cylindrical shells

In the 1930's Dischinger, Schorer(46) and others applied the Levy method to the solution of cylindrical shells with deep thin inextensible gables along their curved boundaries. In 1947, Jenkins(6) introduced a matrix formulation of the Levy method for laterally continuous cylindrical shells with flexible edge beams along their straight generators. Digital computer programmes have been written by Gibson(42), Booth and

Morice[41] and others for the Levy type solution of cylindrical shells. Morice[19] and Munro[2] have solved the Schorer and 'intermediate equations' respectively for longitudinally continuous cylindrical shells using Rayleigh functions. More recently, cylindrical shells have been solved, by Newman[18] and Lu[17] using an extended Levy method, by Chuang and Veletsos[50] using a finite difference technique and by Hrennikoff and Tezcan[49] using a finite element method.

Translational shells

The Navier method has been used by Ambartsumyan[43] and Flugge and Conrad[45]. The Levy method has been used by Bouma[22] and Apeland[23]. Design tables have been published by Apeland and Popov[34]. Digital computer programmes have been written by Apeland and Tocher[24] for anisotropic shells and by Gibson[32] for laterally continuous shells. Noor and Veletsos[20] have studied the application of variational and finite difference techniques. Aass[11] has used a variational method for the solution of elliptic paraboloid shells. Padilla and Schnobrich[16] have used a finite difference technique for the solution of translational shells with flexible edge beams along the four edges of the shell. A discrete element method has been used by Mohraz and Schnobrich[15] for the analysis of shallow shells.

Hyperbolic paraboloid shells supported along their straight edges

The Navier method has been used by Munro[2] and the Levy method by Apeland and Popov[21] and Apeland[25]. However, these methods lead to unrealistic boundary conditions. Approximate methods of solution have been used by Loof, Gerard[47] and Bleich and Salvadori[44]. Variational methods have been used by Tottenham[28],[29],[40] and Chetty[36]. Finite difference methods have been used by Das Gupta[48] and others. Duddeck[33] has used an extended Levy method to solve a hyperbolic paraboloid shell with tangential shear forces applied along the edges of the shell.

Although many methods have been used to solve the shallow curved plate equations, the availability of design tables or digital computer programmes have been, in most cases, limited to shells with boundary conditions corresponding to standard Levy type solutions. In most cases only one case of loading, a uniform transverse loading acting over the total surface area of the shell, is considered.

I.2 Scope of research

1. To write a digital computer programme for the analysis of translational shells, with a variety of boundary conditions (including flexible edge members) along their

translational generators, under the action of a general system of loading (including thermal loading).

2. To write a digital computer programme for the analysis of hyperbolic paraboloid shells, with a variety of boundary conditions along their straight edges, under the action of a general system of loading.

CHAPTER II

STATEMENT OF THE PROBLEM AND OUTLINE OF THE METHOD OF SOLUTION

II.1. Geometry of the middle surface of the shell

Let the middle surface of a shell referred to a fixed orthogonal right handed Cartesian reference frame (x,y,z) have the form:-

$$z = ax^2 + by^2 + cxy + dx + ey + f \qquad (2.1)$$

where a,b,c,d,e and f are constants.

Let the (α,β) curvilinear coordinate set be defined by the intersection of the y=constant and the x=constant planes with the middle surface of the shell. Let γ be the direction mutually orthogonal to the (α,β) set. The positive direction of γ is chosen such that (α,β,γ) form a right handed triad (figure 2.1).

$$\text{If } \left(\frac{\partial z}{\partial x}\right)^2 \ll 1, \ \left(\frac{\partial z}{\partial y}\right)^2 \ll 1 \text{ and } \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} \ll 1 \qquad (2.2)$$

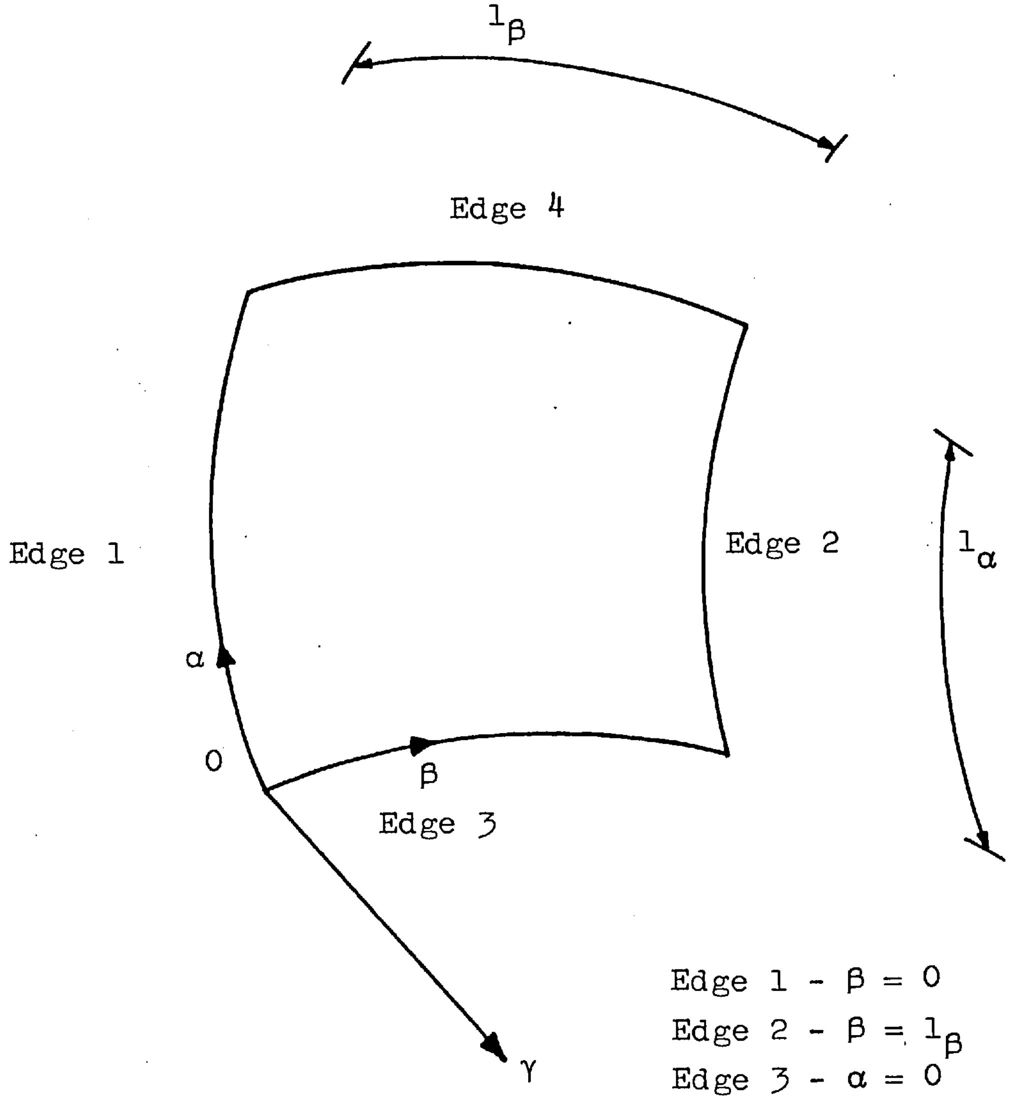
the (α,β) set will then be sensibly orthogonal and the first fundamental form of the middle surface of the shell can be approximated to(3):-

$$(ds)^2 = A^2(d\alpha)^2 + B^2\ (d\beta)^2 \qquad (2.3)$$

where A and B are constants.

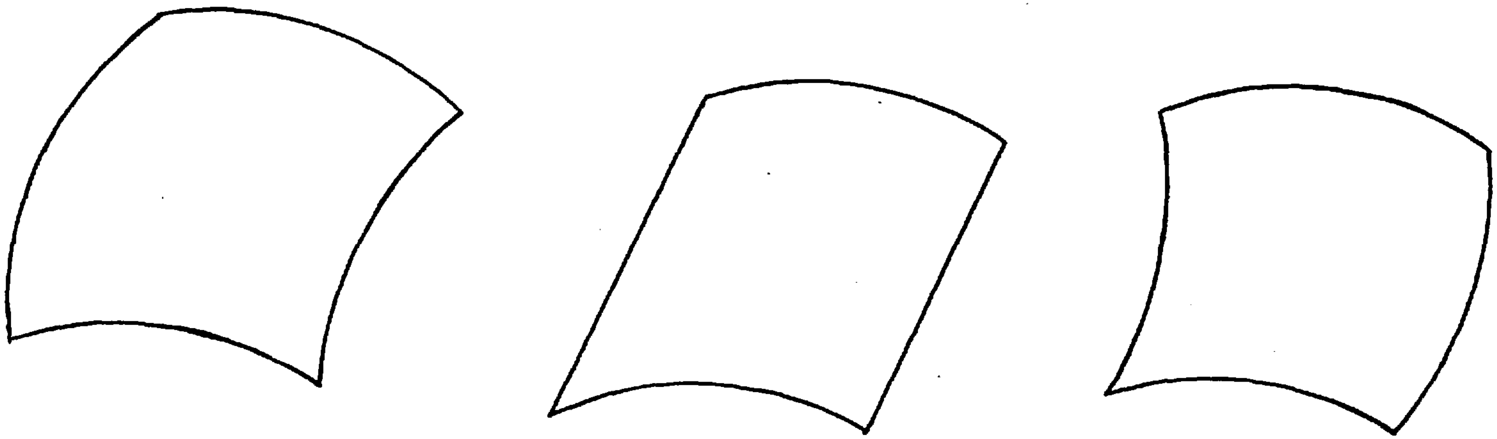
The (α,β) set is chosen such that A and B are unity.

Further the curvatures of the surface will be



Edge 1 - $\beta = 0$
Edge 2 - $\beta = l_\beta$
Edge 3 - $\alpha = 0$
Edge 4 - $\alpha = l_\alpha$

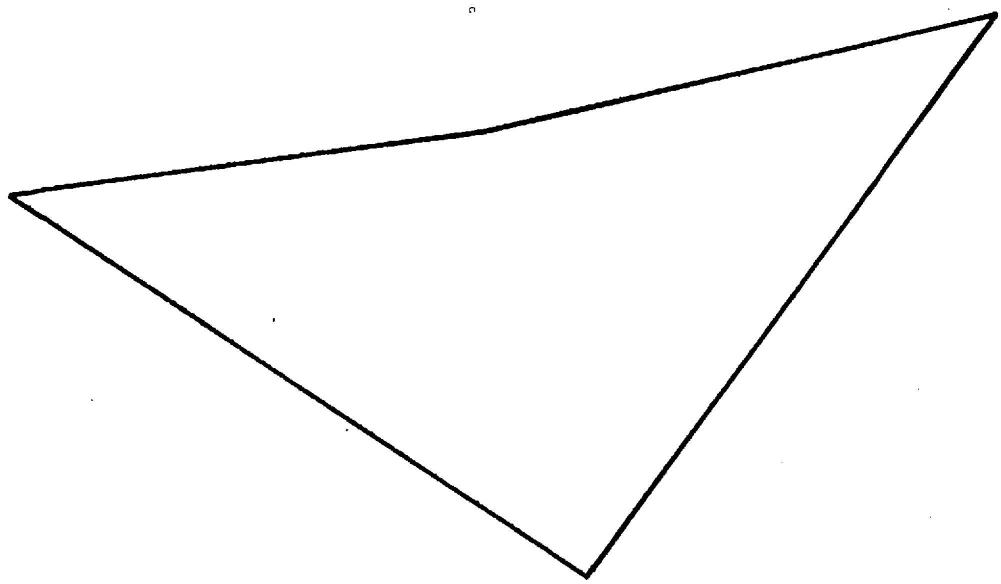
Figure 2.1



Elliptic paraboloid　　　　Cylinder　　　　Hyperbolic paraboloid

Translational shells ($k_{\alpha\beta} = 0$)

Figure 2.2



Hyperbolic paraboloid

Ruled surface ($k_\alpha = 0 = k_\beta$)

Figure 2.3

approximately constant and are given by:-

$$k_\alpha = 2a$$

$$k_{\alpha\beta} = c \tag{2.4}$$

$$k_\beta = 2b$$

Two classes of shells are considered.

Class A - $k_{\alpha\beta} = 0$ (figure (2.2)

(i) $k_\alpha \, k_\beta > 0$

Represents an elliptic paraboloid where the (α,β) set coincides with the parabolic generators of the surface.

(ii) $k_\alpha \, k_\beta = 0$

Represents a parabolic cylinder where the (α,β) set coincides with the parabolic and straight line generators of the surface. When $k_\alpha = 0$, $k_\beta = 0$ the surface is a flat plate. For a circulate cylinder* and a flat plate equations (2.3) and (2.4) are exact.

(iii) $k_\alpha \, k_\beta < 0$

Represents a hyperbolic paraboloid where the (α,β) set coincides with the parabolic generators of the surface.

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*The geometrical hypothesis cannot distinguish between a parabolic cylinder and a circular cylinder. In practice the curvatures k_α and k_β are determined on the basis that the normal sections are circular.

In the context of this thesis, a shell with $k_{\alpha\beta} = 0$ will be referred to as a translational shell.

Class B - $k_\alpha = 0$, $k_\beta = 0$ (figure 2.3)

Represents a hyperbolic paraboloid (or a flat plate when $k_{\alpha\beta} = 0$) where the (α, β) set coincides with the straight line generators of the surface.

In the context of this thesis, a shell with $k_\alpha = 0$, $k_\beta = 0$ will be referred to as a ruled surface.

In this thesis shells of constant thickness, whose middle surface is represented by equation (2.1) and satisfying (2.3) and (2.4), classified under A and B are considered. The shells are assumed to have a rectangular planform.

II.2. Shallow curved plate theory

The analysis is based on a linear elastic theory of thin shallow shells formulated by Vlasov[1] together with the geometrical assumptions made in section II.1. A theory based on these assumptions is referred to as a shallow curved plate theory by Munro[2]. It is generally accepted that the theory is sufficiently accurate for most engineering purposes when the maximum (rise to span) ratio of the shell is limited to 1/5[1].

Equations of equilibrium

The equations of equilibrium for a shell element referred to the orthogonal curvilinear coordinate system

(α,β,γ) defined in section I I.1 are given by[2]:-

$$\frac{\partial n_\alpha}{\partial \alpha} + \frac{\partial n_{\alpha\beta}}{\partial \beta} + G_\alpha = 0 \qquad (2.5)$$

$$\frac{\partial n_{\alpha\beta}}{\partial \alpha} + \frac{\partial n_\beta}{\partial \beta} + G_\beta = 0 \qquad (2.6)$$

$$\frac{\partial m_\alpha}{\partial \alpha} + \frac{\partial m_{\alpha\beta}}{\partial \beta} - q_\alpha = 0 \qquad (2.7)$$

$$\frac{\partial m_{\alpha\beta}}{\partial \alpha} + \frac{\partial m_\beta}{\partial \beta} - q_\beta = 0 \qquad (2.8)$$

$$\frac{\partial q_\alpha}{\partial \alpha} + \frac{\partial q_\beta}{\partial \beta} + k_\alpha\, n_\alpha + 2k_{\alpha\beta}\, n_{\alpha\beta} + k_\beta\, n_\beta + G_\gamma = 0 \qquad (2.9)$$

Terms involving the product of a flexural action and a curvature are neglected in the equations of equilibrium. The sixth equation of equilibrium is then identically equal to zero. The positive directions of loading, extensional actions and flexural actions are defined in figures 2.4, 2.5 and 2.6 respectively.

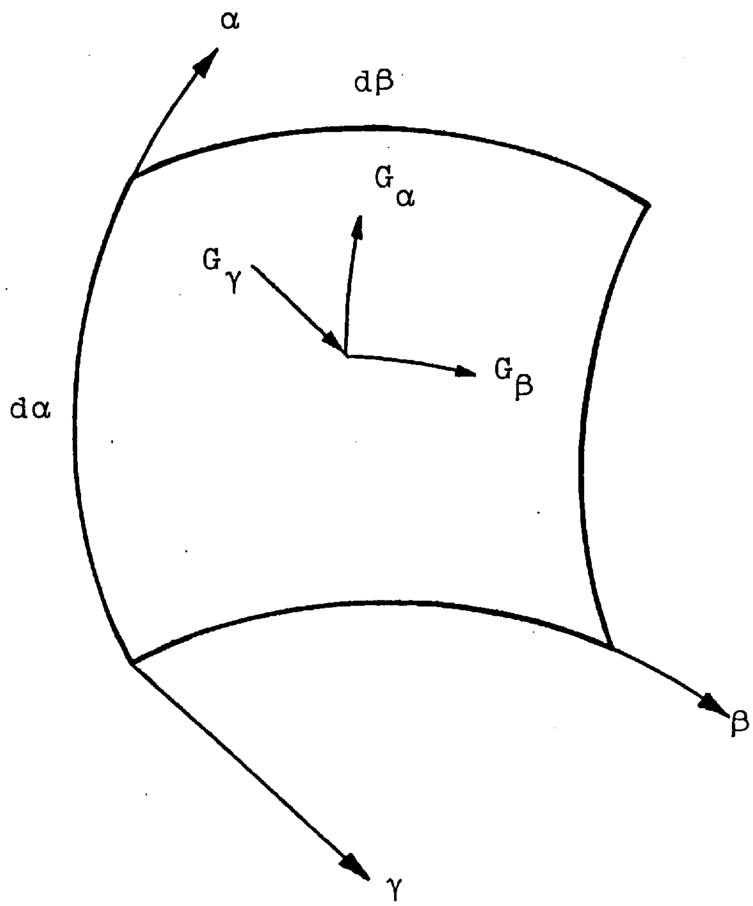
Action-middle surface displacement relations

Under isothermal conditions the action-middle surface displacement relations are given by[2],[4],[5],[12],[35]:-

$$n_\alpha = \frac{Et}{1-\nu^2}\left[\left(\frac{\partial u_\alpha}{\partial \alpha} - k_\alpha\, u_\gamma\right) + \nu\left(\frac{\partial u_\beta}{\partial \beta} - k_\beta\, u_\gamma\right)\right] + n_T \qquad (2.10)$$

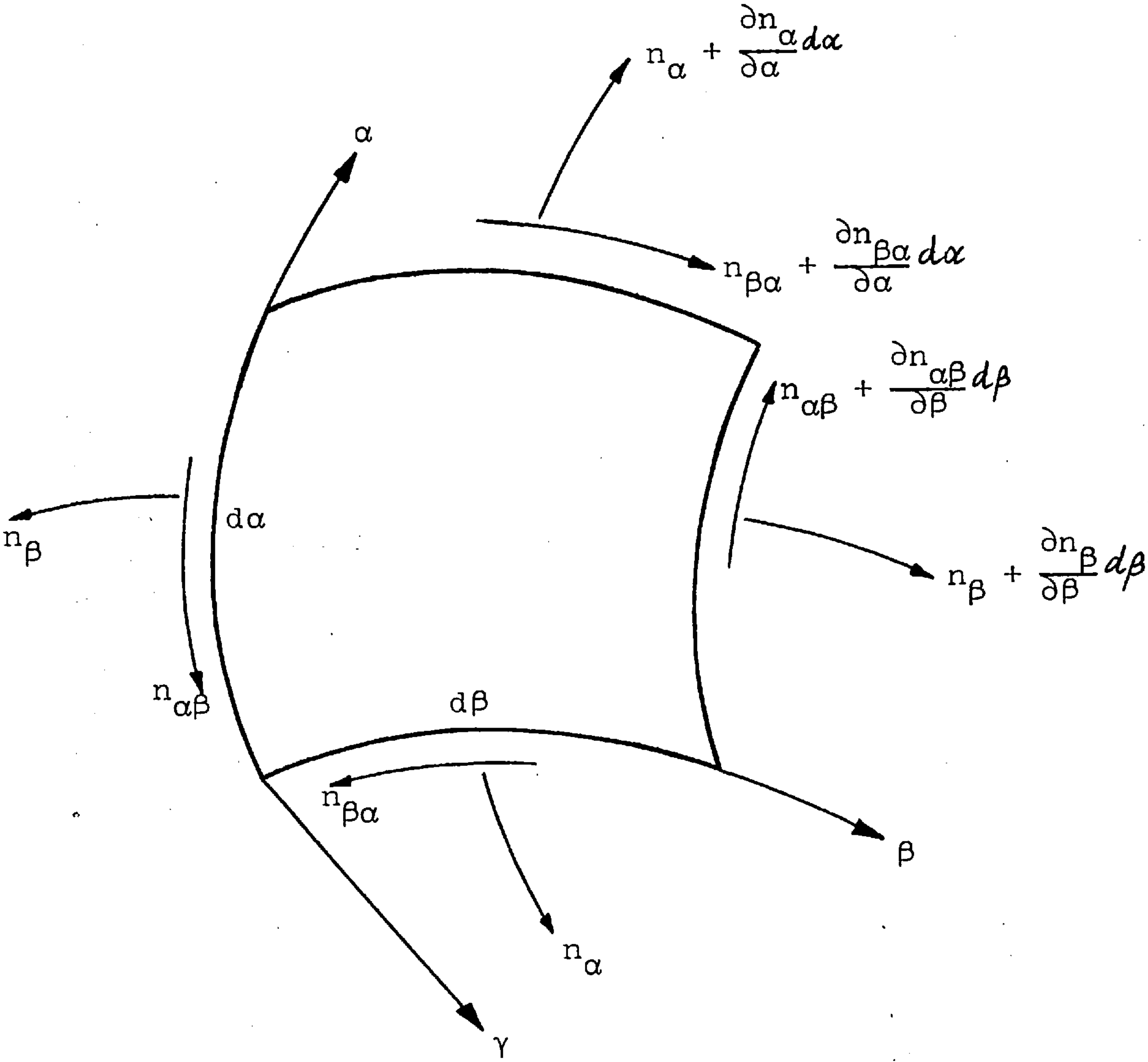
$$n_{\alpha\beta} = n_{\beta\alpha} = \frac{Et}{2(1+\nu)}\left[\left(\frac{\partial u_\beta}{\partial \alpha} + \frac{\partial u_\alpha}{\partial \beta}\right) - 2k_{\alpha\beta}\, u_\gamma\right] \qquad (2.11)$$

$$n_\beta = \frac{Et}{1-\nu^2}\left[\left(\frac{\partial u_\beta}{\partial \beta} - k_\beta\, u_\gamma\right) + \nu\left(\frac{\partial u_\alpha}{\partial \alpha} - k_\alpha\, u_\gamma\right)\right] + n_T \qquad (2.12)$$



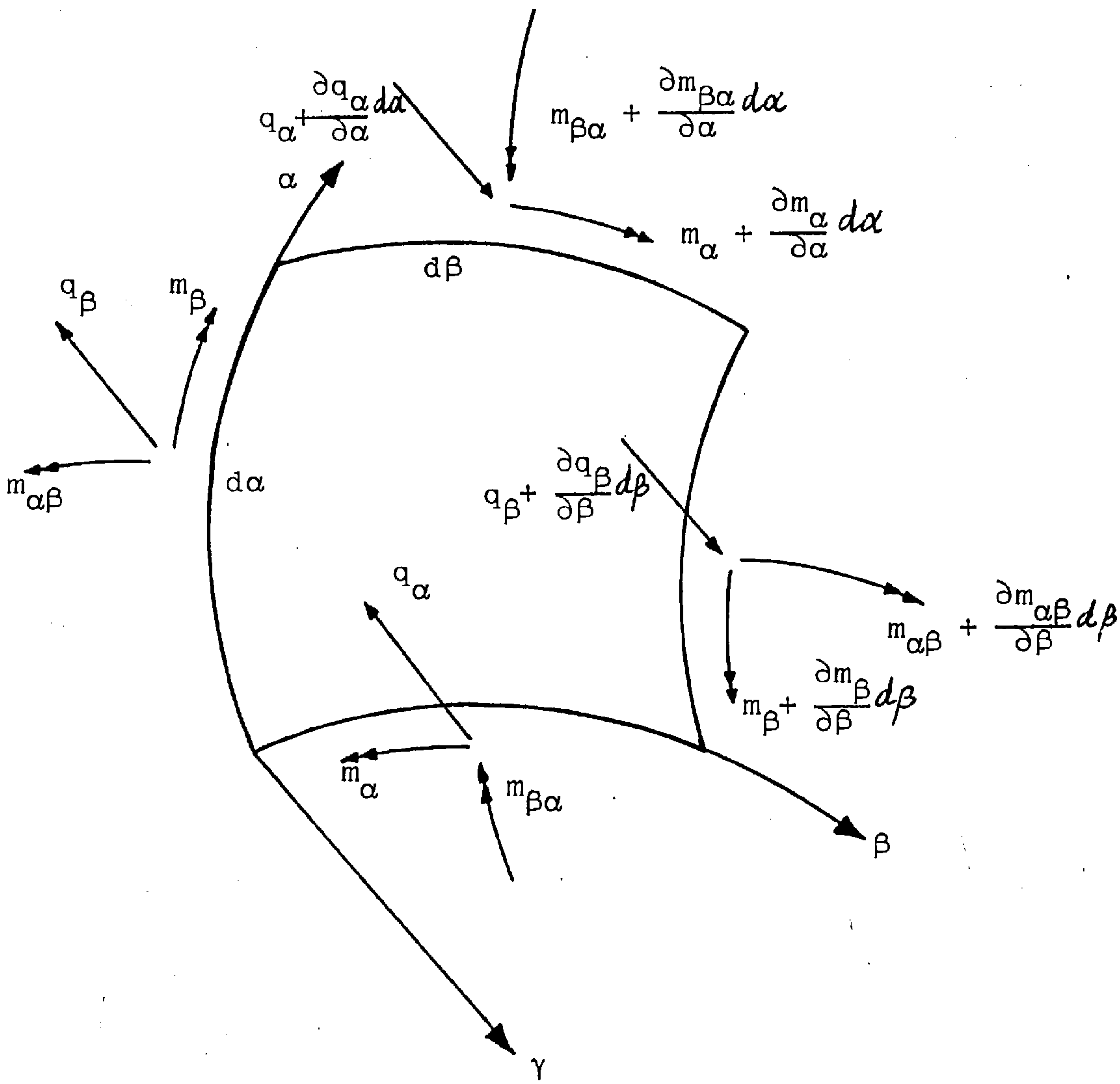
External loading on shell element

Figure 2.4



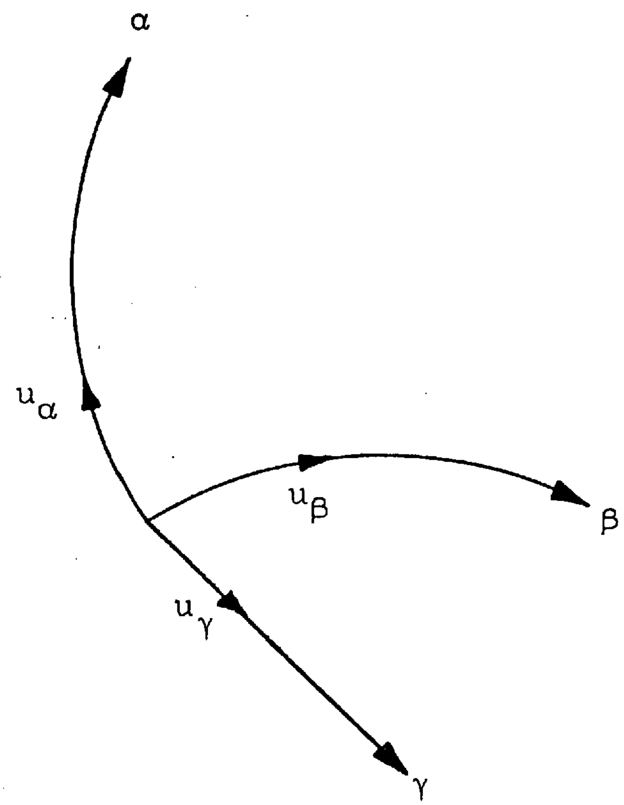
Extensional actions

Figure 2.5



Flexural actions

Figure 2.6



Middle surface displacements

Figure 2.7

$$m_\alpha = -D\left[\frac{\partial^2 u_\gamma}{\partial\alpha^2} + \nu\frac{\partial^2 u_\gamma}{\partial\beta^2}\right] + m_T \tag{2.13}$$

$$m_{\alpha\beta} = m_{\beta\alpha} = -D(1-\nu)\frac{\partial^2 u_\gamma}{\partial\alpha\partial\beta} \tag{2.14}$$

$$m_\beta = -D\left[\frac{\partial^2 u_\gamma}{\partial\beta^2} + \nu\frac{\partial^2 u_\gamma}{\partial\alpha^2}\right] + m_T \tag{2.15}$$

where t = thickness of the shell

E = Young's modulus of the shell material

ν = Poisson's ratio of the shell material

and $D = \dfrac{Et^3}{12(1-\nu^2)}$

$$n_T = -\frac{E\lambda}{1-\nu}\int_{-\frac{t}{2}}^{+\frac{t}{2}} T(\alpha,\beta,\gamma)d\gamma \tag{2.16}$$

$$m_T = -\frac{E\lambda}{1-\nu}\int_{-\frac{t}{2}}^{+\frac{t}{2}} T(\alpha,\beta,\gamma)\gamma d\gamma \tag{2.17}$$

$T(\alpha,\beta,\gamma)$ = temperature distribution in the shell space*.

λ = coefficient of linear thermal expansion of the shell material

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*The temperature $T(\alpha,\beta,\gamma)$ represents the increment of the temperature from the initial stressless state when $T(\alpha,\beta,\gamma) = 0$.

The positive directions of the middle surface displacements are shown in figure 2.7.

Let T_1, T_2 be the temperatures on the shell surfaces $\gamma = -t/2$ and $\gamma = +t/2$ respectively. If the temperature varies linearly in the γ direction (across the thickness of the shell),

$$\text{then } T(\alpha,\beta,\gamma) = \frac{T_1 + T_2}{2} + \frac{\gamma}{t}(T_2 - T_1) \tag{2.18}$$

$$\text{and } n_T = -\frac{E\lambda t}{2(1-\nu)}(T_1 + T_2)$$

$$m_T = -\frac{E\lambda t^2}{12(1-\nu)}(T_2 - T_1) \tag{2.19}$$

Introducing a stress function (ϕ) defined by:-

$$n_\alpha = \frac{\partial^2\phi}{\partial\beta^2} - \int G_\alpha \, d\alpha \tag{2.20}$$

$$n_{\alpha\beta} = -\frac{\partial^2\phi}{\partial\alpha\partial\beta} \tag{2.21}$$

$$n_\beta = \frac{\partial^2\phi}{\partial\alpha^2} - \int G_\beta \, d\beta \tag{2.22}$$

equations (2.5) to (2.22) can be reduced to two simultaneous fourth order partial differential equations in u_γ and ϕ:-

$$D\nabla^4 u_\gamma - \nabla_R^2\phi = G_\gamma - k_\alpha \int G_\alpha \, d_\alpha - k_\beta \int G_\beta \, d\beta + \nabla^2 m_T \tag{2.23}$$

$$\nabla^4\phi + Et\nabla_R^2 u_\gamma = \frac{\partial^2}{\partial\beta^2}\int G_\alpha \, d\alpha + \frac{\partial^2}{\partial\alpha^2}\int G_\beta \, d\beta - \nu\left(\frac{\partial G_\beta}{\partial\beta} + \frac{\partial G_\alpha}{\partial\alpha}\right) + (1-\nu)\nabla^2 n_T \tag{2.24}$$

where $$\nabla_R^{\,2} = k_\beta \frac{\partial^2}{\partial\alpha^2} - 2k_{\alpha\beta}\frac{\partial^2}{\partial\alpha\partial\beta} + k_\alpha \frac{\partial^2}{\partial\beta^2}$$

$$\nabla^{\,2} = \frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}$$

Equations (2.23) and (2.24) can be reduced to a single eighth order partial differential equation in u_γ:-

$$D\nabla^8 u_\gamma + Et\nabla_R^{\,4} u_\gamma = \nabla^4 \left[G_\gamma - k_\alpha \int G_\alpha d\alpha - k_\beta \int G_\beta d\beta\right] + \nabla_R^{\,2}\left[\frac{\partial^2}{\partial\beta^2}\int G_\alpha d\alpha + \frac{\partial^2}{\partial\alpha^2}\int G_\beta d\beta - \nu\left(\frac{\partial G_\beta}{\partial\beta} + \frac{\partial G_\alpha}{\partial\alpha}\right)\right] + \nabla^6 m_T + (1-\nu)\nabla_R^{\,2}\nabla^2 n_T \qquad (2.25)$$

II.3 Loading on shell

The following cases of loading are considered:-

(i) Uniformly distributed patch loads in the α, β and γ directions acting over rectangular areas of the shell surface. The intensity of the loads and the rectangular areas over which these loads act may be specified independently and simultaneously in the three directions.

(ii) For cylindrical shells, uniform rectangular patch loading in a fixed direction in the $(\beta-\gamma)$ plane.

(iii) Loads as in (i) but the loads being symmetric or antimetric about the axis $\alpha = \frac{l_\alpha}{2}$ and symmetric or antimetric about the axis $\beta = \frac{l_\beta}{2}$.

(iv) For translational shells, a uniform temperature field T_1 on the shell surface $\gamma = -t/2$, and a uniform temperature field T_2 on the shell surface $\gamma = -t/2$ (the temperature varying linearly across the thickness of the shell), in conjunction with loading case (i) or (ii).

II.4. Loading on edge beams

Uniformly distributed loads acting over the total length of the beam, in the directions of the principal axes of a transverse section of the beam are considered.

II.5. Boundary conditions

Shell edges 1,2,3 and 4 are defined in figure 2.1

Translational shells

The boundary conditions considered are listed in table 2.1.

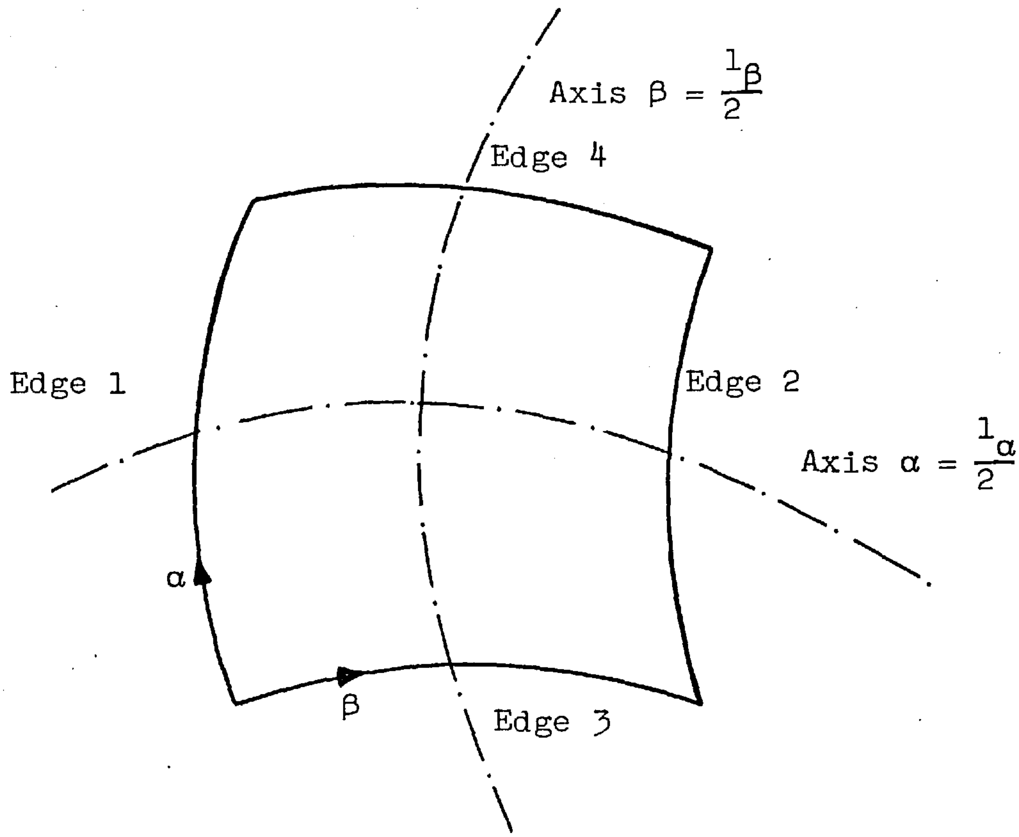
ϕ_1 = Anticlockwise angle measured from gable surface to positive direction of γ at intersection of shell and gable.

ϕ_2 = Clockwise angle measured from gable surface to positive direction of γ at intersection of shell and gable.

Any combination of boundary conditions may be specified. However, when there is a free edge or an edge with a flexible beam, the boundary conditions and loading on the shell and edge beams must be symmetric about an axis as shown in figure 2.8.

Ruled surfaces

The boundary conditions considered are listed in table 2.2.



Edge 1 or Edge 2 - flexible beam support or free, then boundary conditions and loading must be symmetric about the axis $\alpha = \frac{l_\alpha}{2}$.

Edge 3 or Edge 4 - flexible beam support or free, then boundary conditions and loading must be symmetric about the axis $\beta = \frac{l_\beta}{2}$.

These restrictions do not apply to a flat plate.

Figure 2.8

Table 2.1

Notation	Code Number	Conditions to be satisfied along edges 1 and 2*
Deep thin inextensible normal gable (simply supported)	1	$u_\alpha = 0$ $u_\gamma = 0$ $n_\beta = 0$ $m_\beta = 0$
Clamped	2	$u_\alpha = 0$ $u_\gamma = 0$ $u_\beta = 0$ $\frac{\partial u_\gamma}{\partial \beta} = 0$
Free	3	$n'_{\alpha\beta} = 0$ $q'_\beta = 0$ $n_\beta = 0$ $m_\beta = 0$
Hinged	4	$u_\alpha = 0$ $u_\gamma = 0$ $u_\beta = 0$ $m_\beta = 0$
Deep thin inextensible oblique gable	5	$u_\gamma \cos(\phi_i)+(-1)^i u_\beta \sin(\phi_i) = 0$ $q'_\beta \sin(\phi_i)-(-1)^i n_\beta \cos(\phi_i) = 0$ $u_\alpha = 0$ $m_\beta = 0$ i = 1 for edge 1 i = 2 for edge 2
Flexible edge beam	0	$(u_\alpha)_s = (u_\alpha)_b$ $(u_\gamma)_s = (u_\gamma)_b$ $(u_\beta)_s = (u_\beta)_b$ $(\frac{\partial u_\gamma}{\partial \beta})_s = (\frac{\partial u_\gamma}{\partial \beta})_b$ at intersection of shell and edge beam**

*For conditions to be satisfied along edges 3 and 4, interchange α and β and replace i by j (j = 3,4).

**Suffixes s and b refer to the shell and edge beam respectively.

Table 2.2

Notation	Code Number	Conditions to be satisfied along edges 1 and 2*
Deep thin inextensible normal gable (simply supported)	1	$n_\beta = 0$ $\quad u_\gamma = 0$ $u_\alpha = 0$ $\quad m_\beta = 0$
Clamped	2	$u_\beta = 0$ $\quad u_\gamma = 0$ $u_\alpha = 0$ $\quad \frac{\partial u_\gamma}{\partial \beta} = 0$
Free	3	$n_\beta = 0$ $\quad q'_\beta = 0$ $n_{\alpha\beta} = 0$ $\quad m_\beta = 0$
Hinged	4	$u_\beta = 0$ $\quad u_\gamma = 0$ $u_\alpha = 0$ $\quad m_\beta = 0$
Free-guided	10	$u_\beta = 0$ $\quad u_\gamma = 0$ $n_{\alpha\beta} = 0$ $\quad m_\beta = 0$
-	6	$n_\beta = 0$ $\quad u_\gamma = 0$ $n_{\alpha\beta} = \text{constant}$ $\quad m_\beta = 0$

*For conditions to be satisfied along edges 3 and 4 interchange α and β.

Any combination of boundary conditions may be specified. However, when there is a free edge, the boundary conditions and loading on the shell must be symmetric about an axis as shown in figure 2.8.

II.6. Method of solution

II.6.1. Outline of the method

A standard Navier type particular solution using double trigonometric expansions is used. The complementary solution is made up of two groups of linearly independent Levy type solutions. The first group consists of a set of linearly independent Levy type solutions using a trigonometric expansion in the α direction, and the second group consists of a set of linearly independent Levy type solutions using a trigonometric expansion in the β direction. The particular solution and each Levy type solution forming the complementary solution satisfies the governing equations but not all the boundary conditions. The procedure is to combine suitable sets of Levy type solutions with the particular solution to satisfy all the boundary conditions simultaneously. The method is referred to as an extended Levy method of solution.

II.6.2. Particular solution

The external loads and temperature fields acting on the shell are expressed in the form of half range double trigonometric series in the α and β directions.

A solution for u_γ is sought in the form:-

$$u_\gamma = \sum_{p=o}^{\infty} \sum_{q=o}^{\infty} a^{pq} \sin(p\bar{\alpha}) \sin(q\bar{\beta}) \tag{2.26}$$

where a^{pq} is a constant.

All shell quantities* can be expressed in the form:-

$$Y^o = \sum_{p=o}^{\infty} \sum_{q=o}^{\infty} (y^o)^{pq} F(p\bar{\alpha})F(q\bar{\beta}) \tag{2.27}$$

where $(y^o)^{pq}$ is a constant

and F( ) = sin( ) or cos( )

Along edges 1 and 2 ($\beta = 0, 1_\beta$) Y^o takes the form of a half range trigonometric series in the α direction given by:-

$$(Y^o)^i = \sum_{p=o}^{\infty} (y^o)^i_p F(p\bar{\alpha}) \tag{2.28}$$

where $(y^o)^i_p$ is a constant

and superscript i refers to the edge (i = 1,2).

Along edges 3 and 4 ($\alpha = 0, 1_\alpha$) Y^o takes the form of a half range trigonometric series in the β direction given by:-

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*All actions, displacements and rotations.

$$(Y^{o})^{j} = \sum_{q=o}^{\infty} (y^{o})_{q}^{j} \, F(q\overline{\beta}) \qquad (2.29)$$

where $(y^{o})_{q}^{j}$ is a constant

and superscript j refers to the edge ($j = 3,4$).

The particular solution satisfies certain homogeneous boundary conditions, some of which are common with the edge conditions considered in section II.5. These common conditions are listed in table 2.3.

II.6.3. <u>Complementary solution</u>

The complementary solution consists of two groups of linearly independent Levy type solutions referred to as type 1 and type 2.

<u>Levy solutions type 1</u>

Consists of solutions where a solution for u_{γ} is sought in the form:-

$$u_{\gamma} = \sum_{p=1}^{\infty} \psi_{p}(\beta) \sin(p\overline{\alpha}) \qquad (2.30)$$

where $\psi_{p}(\beta)$ is a function of β.

All shell quantities Y^{1} can be expressed in the form:-

$$Y^{1} = \sum_{p=1}^{\infty} \psi_{p}(\beta) \, F(p\overline{\alpha}) \qquad (2.31)$$

where $F(p\overline{\alpha}) = \sin(p\overline{\alpha})$ or $\cos(p\overline{\alpha})$.

<u>Using this type of solution any independent set of boundary conditions can be satisfied along edges 1 and 2,</u>

Table 2.3

Edge condition	Boundary conditions satisfied by the particular solution along edges 1 and 2*	
	Translational shells	Ruled surfaces
Deep thin inextensible normal gable (simply supported)	$u_\alpha = 0$ $u_\gamma = 0$ $n_\beta = 0$ $m_\beta = 0$	$u_\gamma = 0$ $m_\beta = 0$
Clamped	$u_\alpha = 0$ $u_\gamma = 0$	$u_\beta = 0$ $u_\gamma = 0$
Free	$n_\beta = 0$ $m_\beta = 0$	$n_{\alpha\beta} = 0$ $m_\beta = 0$
Deep thin inextensible oblique gable	$u_\alpha = 0$ $m_\beta = 0$	not considered
Free-guided	not considered	$u_\beta = 0$ $u_\gamma = 0$ $n_{\alpha\beta} = 0$ $m_\beta = 0$
Flexible edge beam	none	not considered
Code number 6 (refer table 2.2)	not considered	$u_\gamma = 0$ $m_\beta = 0$

*For edges 3 and 4 interchange α and β.

<u>while along edges 3 and 4 the same set of homogeneous boundary conditions as in the particular solution is satisfied</u>.

Consider $(Y^1)_p = \psi_p(\beta)\ F(p\bar{\alpha})$ (2.32)

Subscript p refers to the harmonic considered.

Along edges 1 and 2 ($\beta = 0,\ l_\beta$), $(Y^1)_p$ takes the form:-

$$(Y^1)_p^i = c\ F(p\bar{\alpha}) \tag{2.33}$$

where c is a constant

and superscript i refers to the edge (i = 1,2).

Equation (2.33) is in the same form as equation (2.28) corresponding to the particular solution along edges 1 and 2.

Along edges 3 and 4 ($\alpha = 0,\ l_\alpha$), $(Y^1)_p$ takes the form:-

$$(Y^1)_p^j = c\ \psi_p(\beta) \tag{2.34}$$

where c is a constant

and superscript j refers to the edge (j = 3,4).

$(Y^1)_p^j$ can be written in a form similar to the particular solution along edges 3 and 4 (equation (2.29)) by expanding $(Y^1)_p^j$ into a half range trigonometric series in the β direction of the form:-

$$(Y^1)_p^j = \sum_{q=o}^{\infty} (y^1)_q^j\ F(q\bar{\beta}) \tag{2.35}$$

where $(y^1)^j_q$ is a constant*.

Levy solutions type 2

Consists of solutions where a solution for u_γ is sought in the form:-

$$u_\gamma = \sum_{q=1}^{\infty} \psi_q(\alpha) \sin (q\overline{\beta}) \tag{2.36}$$

where $\psi_q(\alpha)$ is a function of α.

All shell quantities Y^2 can be expressed in the form:-

$$Y^2 = \sum_{q=1}^{\infty} \psi_q(\alpha)\, F(q\overline{\beta}) \tag{2.37}$$

where $F(q\overline{\beta}) = \sin(q\overline{\beta})$ or $\cos(q\overline{\beta})$.

Using this type of solution any independent set of boundary conditions can be satisfied along edges 3 and 4, while along edges 1 and 2 the same set of homogeneous boundary conditions as in the particular solution is satisfied.

Consider $(Y^2)_q = \psi_q(\alpha)\, F(q\overline{\beta})$ (2.38)

Subscript q refers to the harmonic considered.

---

*Note that any 'edge disturbance' of the form represented by equation (2.33) applied along edge 1 or 2 generates a whole series of 'disturbances' along edges 3 and 4 of the form represented by equation (2.35).

Along edges 3 and 4 ($\alpha = 0,\ 1_\alpha$), $(Y^2)_q$ takes the form:-

$$(Y^2)_q^j = c\ F(q\bar{\beta}) \tag{2.39}$$

where c is a constant
and superscript j refers to the edge ($j = 3,4$).

Equation (2.39) is in the same form as equation (2.29) corresponding to the particular solution along edges 3 and 4.

Along edges 1 and 2 ($\beta = 0,\ 1_\beta$), $(Y^2)_q$ takes the form:-

$$(Y^2)_q^i = c\ \psi_q(\alpha) \tag{2.40}$$

where c is a constant
and superscript i refers to the edge ($i = 1,2$).

$(Y^2)_q^i$ can be written in a form similar to the particular solution along edges 1 and 2 (equation (2.28)) by expanding $(Y^2)_q^i$ into a half range trigonometric series in the α direction of the form:-

$$(Y^2)_q^i = \sum_{p=o}^{\infty} (y^2)_p^i\ F(p\bar{\alpha}) \tag{2.41}$$

where $(y^2)_p^i$ is a constant*.

---

*Note that any 'edge disturbance' of the form represented by equation (2.39) applied along edge 3 or 4 generates a whole series of 'disturbances' along edges 1 and 2 of the form represented by equation (2.41).

II.6.4. <u>An example</u>

Consider a translational shell with the following boundary conditions:-

Edge 1 - Clamped

Edge 2 - Clamped

Edge 3 - Free

Edge 4 - Hinged

acted on by a uniform load in the γ direction acting over the whole surface area of the shell.

Suppose the loading on the shell is satisfactorily represented by a truncated series of the form given in appendix 1, for

$$p = 1\ (1)\ 10$$
$$q = 1\ (1)\ 10^{*}$$

The particular solution does not satisfy all the boundary conditions (table 2.3). Hence it is necessary to add a complementary solution to the particular solution to obtain the total solution.

The boundary conditions satisfied by the particular solution, which can also be satisfied by each Levy type solution forming the complementary solution, are listed in table 2.4. The complementary solution 'edge disturbances'

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*The notation $i = j(k)l$ represents i taking the sequence of values $j,\ j+k,\ j+2k, \ldots, l-k,\ l$.

to be applied, to satisfy the remaining boundary conditions, are also listed in table 2.4.

Table 2.4

Edge	Boundary conditions satisfied by the particular solution	'Edge disturbances' to be applied
1	$u_\alpha = 0 \qquad u_\gamma = 0$	u_β , $\frac{\partial u_\gamma}{\partial \beta}$
2	$u_\alpha = 0 \qquad u_\gamma = 0$	u_β , $\frac{\partial u_\gamma}{\partial \beta}$
3	$n_\alpha = 0 \qquad m_\alpha = 0$	$n'_{\beta\alpha}$, q'_α
4	$u_\beta = 0 \quad u_\gamma = 0 \quad m_\alpha = 0$	u_α

In the Levy solutions type 1, the boundary values along edges 3 and 4 are expanded into half range trigonometric series in the β direction (equation (2.35)). Suppose, for a suitable representation of these boundary values, it is necessary to consider values of q corresponding to $q=1(1)15$. Similarly, in the Levy solutions of type 2, let the values of p considered be $p = 1(1)15$ (equation (2.41)).

Since 10 harmonics are used in each direction to represent the external loading on the shell, the boundary conditions not satisfied by the particular solution will be in the form of 10 harmonics ($p = 1(1)10$) along edges 1 and 2 (equation (2.28)), and 10 harmonics ($q = 1(1)10$) along edges 3 and 4 (equation (2.29)). However, when the complementary solution 'edge disturbances' are applied along edges 1 and 2 to satisfy these boundary conditions, each of these 'disturbances' will generate a series of 'disturbances' corresponding to $q = 1(1)15$ along edges 3 and 4 (equation (2.35)). Similarly complementary solution 'edge disturbances' applied along edges 3 and 4 will generate 'disturbances' corresponding to $p = 1(1)15$ along edges 1 and 2 (equation (2.41)).

Hence, to satisfy all the boundary conditions simultaneously, complementary solution 'edge disturbances' corresponding to $p = 1(1)15$ must be applied along edges 1 and 2, and complementary solution 'edge disturbances', corresponding to $q = 1(1)15$ must be applied along edges 3 and 4.

Let the boundary conditions not satisfied by the particular solution along edges 1,2,3 and 4 be represented by vectors V^1, V^2, V^3 and V^4 respectively.

$$V^1 = \begin{bmatrix} V_1{}^1 \\ V_2{}^1 \\ \vdots \\ V_p{}^1 \\ \vdots \\ V_{15}{}^1 \end{bmatrix} \qquad V^2 = \begin{bmatrix} V_1{}^2 \\ V_2{}^2 \\ \vdots \\ V_p{}^2 \\ \vdots \\ V_{15}{}^2 \end{bmatrix} \tag{2.42}$$

$$V^3 = \begin{bmatrix} V_1^{\,3} \\ V_2^{\,3} \\ \vdots \\ V_q^{\,3} \\ \vdots \\ V_{15}^{\,3} \end{bmatrix} \qquad V^4 = \begin{bmatrix} V_1^{\,4} \\ V_2^{\,4} \\ \vdots \\ V_q^{\,4} \\ \vdots \\ V_{15}^{\,4} \end{bmatrix}$$

where $$V_p^{\,1} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad V_p^{\,2} = \begin{bmatrix} a_3 \\ a_4 \end{bmatrix} \qquad (2.43)$$

$$V_q^{\,3} = \begin{bmatrix} a_5 \\ a_6 \end{bmatrix} \qquad V_q^{\,4} = \begin{bmatrix} a_7 \end{bmatrix}$$

and a_1, a_2 are the coefficients of the p^{th} harmonic corresponding to the particular solution values of u_β and $\frac{\partial u_\gamma}{\partial \beta}$ along edge 1,

a_3, a_4 are the coefficients of the p^{th} harmonic corresponding to the particular solution values of u_β and $\frac{\partial u_\gamma}{\partial \beta}$ along edge 2,

a_5, a_6 are the coefficients of the q^{th} harmonic corresponding to the particular solution values of $n'_{\beta\alpha}$ and q'_α along edge 3,

a_7 is the coefficient of the q^{th} harmonic corresponding to the particular solution value of u_α along edge 4.

Since the particular solution is in the form of only 10 harmonics in each direction,

$$V_p^{\ i} = 0 \text{ for } p > 10 \qquad (i = 1,2)$$

and $$V_q^{\ j} = 0 \text{ for } q > 10 \qquad (j = 3,4)$$

Let the complementary solution 'edge disturbances' to be applied along edges 1,2,3 and 4 be represented by the vectors G^1, G^2, G^3 and G^4 respectively.

$$G^1 = \begin{bmatrix} G_1^{\ 1} \\ G_2^{\ 1} \\ \vdots \\ G_p^{\ 1} \\ \vdots \\ G_{15}^{\ 1} \end{bmatrix} \qquad G^2 = \begin{bmatrix} G_1^{\ 2} \\ G_2^{\ 2} \\ \vdots \\ G_p^{\ 2} \\ \vdots \\ G_{15}^{\ 2} \end{bmatrix}$$

$$G^3 = \begin{bmatrix} G_1^{\ 3} \\ G_2^{\ 3} \\ \vdots \\ G_q^{\ 3} \\ \vdots \\ G_{15}^{\ 3} \end{bmatrix} \qquad G^4 = \begin{bmatrix} G_1^{\ 4} \\ G_2^{\ 4} \\ \vdots \\ G_q^{\ 4} \\ \vdots \\ G_{15}^{\ 4} \end{bmatrix} \qquad (2.44)$$

$$\text{where}\quad G_p^{\,1} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad G_p^{\,2} = \begin{bmatrix} b_3 \\ b_4 \end{bmatrix}$$

$$G_q^{\,3} = \begin{bmatrix} b_5 \\ b_6 \end{bmatrix} \qquad G_q^{\,4} = \begin{bmatrix} b_7 \end{bmatrix} \tag{2.45}$$

and b_1, b_2 are the coefficients of the p^{th} harmonic corresponding to u_β and $\frac{\partial u_\gamma}{\partial \beta}$ applied along edge 1,

b_3, b_4 are the coefficients of the p^{th} harmonic corresponding to u_β and $\frac{\partial u_\gamma}{\partial \beta}$ applied along edge 2,

b_5, b_6 are the coefficients of the q^{th} harmonic corresponding to $n'_{\beta\alpha}$ and q'_α applied along edge 3,

b_7 is the coefficient of the q^{th} harmonic corresponding to u_α applied along edge 4.

<u>Influence coefficient matrices</u>

<u>Edge disturbances applied along edge 1</u>

Let the matrix E_{11} be defined by:-

$$E_{11} = \begin{bmatrix} A_1^{\,1,1} & & & & & & \\ & A_2^{\,1,2} & & & & \bigcirc & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & \bigcirc & & & A_p^{\,1,p} & & \\ & & & & & \cdot & \\ & & & & & & A_{15}^{\,1,15} \end{bmatrix} \tag{2.46}$$

where $A_p^{1,p} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ (2.47)

and a_{11}, a_{21} (a_{12}, a_{22}) are the coefficients of the p^{th} harmonic corresponding to u_β and $\frac{\partial u_\gamma}{\partial \beta}$ along edge 1, due to the application of $u_\beta (\frac{\partial u_\gamma}{\partial \beta})$ of unit amplitude corresponding to the p^{th} harmonic along edge 1. A Levy solution of type 1 satisfying the following boundary conditions:-

Edge 1 - $u_\alpha = 0$, $u_\gamma = 0$, $\frac{\partial u_\gamma}{\partial \beta} = 0$ $(u_\alpha=0,\ u_\gamma=0,\ u_\beta=0)$

Edge 2 - clamped

is used.

Hence $a_{11} = 1 = a_{22}$, $a_{21} = 0 = a_{12}$

and $E_{11} = I$ (2.48)

where I is a unit matrix of order (30 x 30).

Since all the boundary conditions along edge 2 are satisfied, the matrix E_{21}, defining the influence coefficients along edge 2 due to edge disturbances applied along edge 1, is a null matrix.

$$E_{21} = \underline{0} \tag{2.49}$$

where $\underline{0}$ is a null matrix of order (30 x 30).

The application of these disturbances along edge 1 generates a series of disturbances along edges 3 and 4 defined by matrices E_{31} and E_{41}.

$$E_{31} = \begin{bmatrix} C_1^{1,1} & C_1^{1,2} \; . \; . \; . & C_1^{1,p} \; . \; . \; . & C_1^{1,15} \\ C_2^{1,1} & C_2^{1,2} \; . \; . \; . & C_2^{1,p} \; . \; . \; . & C_2^{1,15} \\ \vdots & & & \\ C_q^{1,1} & C_q^{1,2} \; . \; . \; . & C_q^{1,p} \; . \; . \; . & C_q^{1,15} \\ \vdots & & & \\ C_{15}^{1,1} & C_{15}^{1,2} \; . \; . \; . & C_{15}^{1,p} \; . \; . \; . & C_{15}^{1,15} \end{bmatrix} \quad (2.50)$$

$$E_{41} = \begin{bmatrix} D_1^{1,1} & D_1^{1,2} \; . \; . \; . & D_1^{1,p} \; . \; . \; . & D_1^{1,15} \\ D_2^{1,1} & D_2^{1,2} \; . \; . \; . & D_2^{1,p} \; . \; . \; . & D_2^{1,15} \\ \vdots & & & \\ D_q^{1,1} & D_q^{1,2} \; . \; . \; . & D_q^{1,p} \; . \; . \; . & D_q^{1,15} \\ \vdots & & & \\ D_{15}^{1,1} & D_{15}^{1,2} \; . \; . \; . & D_{15}^{1,p} \; . \; . \; . & D_{15}^{1,15} \end{bmatrix} \quad (2.51)$$

where $C_q^{1,p} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \qquad D_q^{1,p} = \begin{bmatrix} d_{11} & d_{12} \end{bmatrix}$ (2.52)

and c_{11}, c_{21} (c_{12}, c_{22}) are the coefficients of the q^{th} harmonic corresponding to $n'_{\beta\alpha}$ and q'_α along edge 3, due to the application of u_β ($\frac{\partial u_\gamma}{\partial \beta}$) of unit amplitude corresponding to the p^{th} harmonic along edge 1,

d_{11} (d_{12}) is the coefficient of the q^{th} harmonic corresponding to u_α along edge 4, due to the application of

u_β $(\frac{\partial u_\gamma}{\partial \beta})$ of unit amplitude corresponding to the p^{th} harmonic along edge 1.

Edge disturbances applied along edge 2

Let the matrix E_{22} be defined by:-

$$E_{22} = \begin{bmatrix} B_1^{2,1} & & & & & \bigcirc \\ & B_2^{2,2} & & & & \\ & & \cdot & & & \\ & \bigcirc & & \cdot\ B_p^{2,p} & & \\ & & & & \cdot & \\ & & & & & \cdot\ B_{15}^{2,15} \end{bmatrix} \tag{2.53}$$

$$\text{where} \quad B_p^{2,p} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \tag{2.54}$$

and b_{11}, b_{21}, $(b_{12}$, $b_{22})$ are the coefficients of the p^{th} harmonic corresponding to u_β and $\frac{\partial u_\gamma}{\partial \beta}$ along edge 2, due to the application of u_β $(\frac{\partial u_\gamma}{\partial \beta})$ of unit amplitude corresponding to the p^{th} harmonic along edge 2. A Levy solution of type 1 satisfying the following boundary conditions:-

Edge 1 - clamped

Edge 2 - $u_\alpha=0$, $u_\gamma=0$, $\frac{\partial u_\gamma}{\partial \beta}=0$ $(u_\alpha=0,\ u_\gamma=0,\ u_\beta=0)$

is used.

Hence $\quad E_{22} = I \qquad (2.55)$

where I is a unit matrix of order (30 x 30), and $E_{12} = \underline{0}$ (2.56)

where $\underline{0}$ is a null matrix of order (30 x 30).

The application of these disturbances along edge 2 generates a series of disturbances along edges 3 and 4 defined by the matrices E_{32} and E_{42}.

$$E_{32} = \begin{bmatrix} C_1^{2,1} & C_1^{2,2} & \dots & C_1^{2,p} & \dots & C_1^{2,15} \\ C_2^{2,1} & C_2^{2,2} & \dots & C_2^{2,p} & \dots & C_2^{2,15} \\ \vdots & & & & & \\ C_q^{2,1} & C_q^{2,2} & \dots & C_q^{2,p} & \dots & C_q^{2,15} \\ \vdots & & & & & \\ C_{15}^{2,1} & C_{15}^{2,2} & \dots & C_{15}^{2,p} & \dots & C_{15}^{2,15} \end{bmatrix} \qquad (2.57)$$

$$E_{42} = \begin{bmatrix} D_1^{2,1} & D_1^{2,2} & \dots & D_1^{2,p} & \dots & D_1^{2,15} \\ D_2^{2,1} & D_2^{2,2} & \dots & D_2^{2,p} & \dots & D_2^{2,15} \\ \vdots & & & & & \\ D_q^{2,1} & D_q^{2,2} & \dots & D_q^{2,p} & \dots & D_q^{2,15} \\ \vdots & & & & & \\ D_{15}^{2,1} & D_{15}^{2,2} & \dots & D_{15}^{2,p} & \dots & D_{15}^{2,15} \end{bmatrix} \qquad (2.58)$$

where $C_q^{2,p} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \qquad D_q^{2,p} = \begin{bmatrix} d_{11} & d_{12} \end{bmatrix}$ (2.59) and c_{11}, c_{21} (c_{12}, c_{22}) are the coefficients of the q^{th} harmonic corresponding to $n'_{\beta\alpha}$ and q'_{α} along edge 3, due to the application of u_{β} $(\frac{\partial u_{\gamma}}{\partial \beta})$ of unit amplitude corresponding to the p^{th} harmonic along edge 2,

d_{11} (d_{12})is the coefficient of the q^{th} harmonic corresponding to u_{α} along edge 4, due to the application of u_{β} $(\frac{\partial u_{\gamma}}{\partial \beta})$ of unit amplitude corresponding to the p^{th} harmonic along edge 2.

Edge disturbances applied along edge 3

Let the matrix E_{33} be defined by:-

$$E_{33} = \begin{bmatrix} C_1^{3,1} & & & & & \\ & C_2^{3,2} & & & \bigcirc & \\ & & \ddots & & & \\ & \bigcirc & & C_q^{3,q} & & \\ & & & & \ddots & \\ & & & & & C_{15}^{3,15} \end{bmatrix} \tag{2.60}$$

$$\text{where } C_q^{3,q} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \tag{2.61}$$

and c_{11}, c_{21} (c_{12}, c_{22}) are the coefficients of the q^{th} harmonic corresponding to $n'_{\beta\alpha}$ and q'_{α} along edge 3, due to the application of $n'_{\beta\alpha}$ (q'_{α}) of unit amplitude corresponding to the q^{th} harmonic along edge 3. A Levy solution of type 2 satisfying the following boundary conditions:-

Edge 3 - $q'_\alpha = 0$, $n_\alpha = 0$, $m_\alpha = 0$ ($n'_{\beta\alpha} = 0$, $n_\alpha = 0$, $m_\alpha = 0$)

Edge 4 - hinged

is used.

Hence $E_{33} = I$ (2.62)

where I is a unit matrix of order (30 x 30),

and $E_{43} = \underline{0}$ (2.63)

where $\underline{0}$ is a null matrix of order (15 x 30).

The application of these disturbances along edge 3 generates a series of disturbances along edges 1 and 2 defined by the matrices E_{13} and E_{23}.

$$E_{13} = \begin{bmatrix} A_1^{3,1} & A_1^{3,2} & \dots & A_1^{3,q} & \dots & A_1^{3,15} \\ A_2^{3,1} & A_2^{3,2} & \dots & A_2^{3,q} & \dots & A_2^{3,15} \\ \vdots & & & & & \\ A_p^{3,1} & A_p^{3,2} & \dots & A_p^{3,q} & \dots & A_p^{3,15} \\ \vdots & & & & & \\ A_{15}^{3,1} & A_{15}^{3,2} & \dots & A_{15}^{3,q} & \dots & A_{15}^{3,15} \end{bmatrix} \tag{2.64}$$

$$E_{23} = \begin{bmatrix} B_1^{3,1} & B_1^{3,2} & \dots & B_1^{3,q} & \dots & B_1^{3,15} \\ B_2^{3,1} & B_2^{3,2} & \dots & B_2^{3,q} & \dots & B_2^{3,15} \\ \vdots & & & & & \\ B_p^{3,1} & B_p^{3,2} & \dots & B_p^{3,q} & \dots & B_p^{3,15} \\ \vdots & & & & & \\ B_{15}^{3,1} & B_{15}^{3,2} & \dots & B_{15}^{3,q} & \dots & B_{15}^{3,15} \end{bmatrix} \tag{2.65}$$

where $A_p^{3,q} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B_p^{3,q} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ (2.66)

and a_{11}, a_{21} (a_{12}, a_{22}) are the coefficients of the p^{th} harmonic corresponding to u_β and $\frac{\partial u_\gamma}{\partial \beta}$ along edge 1, due to the application of $n'_{\beta\alpha}$ (q'_α) of unit amplitude corresponding to the q^{th} harmonic along edge 3. b_{11}, b_{21} (b_{12}, b_{22}) are the coefficients of the p^{th} harmonic corresponding to u_β and $\frac{\partial u_\gamma}{\partial \beta}$ along edge 2, due to the application of $n'_{\beta\alpha}$ (q'_α) of unit amplitude corresponding to the q^{th} harmonic along edge 3.

<u>Edge disturbances applied along edge 4.</u>

Let the matrix E_{44} be defined by:-

$$E_{44} = \begin{bmatrix} D_1^{4,1} & & & & & \\ & D_2^{4,2} & & & & \\ & & \ddots & & & \\ & & & D_q^{4,q} & & \\ & & & & \ddots & \\ & & & & & D_{15}^{4,15} \end{bmatrix} \tag{2.67}$$

$$\text{where } D_q^{4,q} = \begin{bmatrix} d_{11} \end{bmatrix} \tag{2.68}$$

and d_{11} is the coefficient of the q^{th} harmonic corresponding to u_α along edge 4, due to the application of u_α of unit amplitude corresponding to the q^{th} harmonic along edge 4. A Levy solution of type 2 satisfying the following boundary conditions:-

Edge 3 - Free

Edge 4 - $u_\beta = 0$, $u_\gamma = 0$, $m_\alpha = 0$

is used.

$$\text{Hence } E_{44} = I \tag{2.69}$$

where I is a unit matrix of order (15 x 15),

$$\text{and } E_{34} = \underline{0} \tag{2.70}$$

where $\underline{0}$ is a null matrix of order (30 x 15).

The application of these disturbances along edge 4 generates a series of disturbances along edges 1 and 2 defined by the matrices E_{14} and E_{24}.

$$E_{14} = \begin{bmatrix} A_1^{4,1} & A_1^{4,2} & \dots & A_1^{4,q} & \dots & E_1^{4,15} \\ A_2^{4,1} & A_2^{4,2} & \dots & A_2^{4,q} & \dots & A_2^{4,15} \\ \vdots & & & & & \\ A_p^{4,1} & A_p^{4,2} & \dots & A_p^{4,q} & \dots & A_p^{4,15} \\ \vdots & & & & & \\ A_{15}^{4,1} & A_{15}^{4,2} & \dots & A_{15}^{4,q} & \dots & A_{15}^{4,15} \end{bmatrix} \tag{2.71}$$

$$E_{24} = \begin{bmatrix} B_1^{4,1} & B_1^{4,2} & \dots & B_1^{4,q} & \dots & B_1^{4,15} \\ B_2^{4,1} & B_2^{4,2} & \dots & B_2^{4,q} & \dots & B_2^{4,15} \\ \vdots & & & & & \\ B_p^{4,1} & B_p^{4,2} & \dots & B_p^{4,q} & \dots & B_p^{4,15} \\ \vdots & & & & & \\ B_{15}^{4,1} & B_{15}^{4,2} & \dots & B_{15}^{4,q} & \dots & B_{15}^{4,15} \end{bmatrix} \quad (2.72)$$

where $A_p^{4,q} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \qquad B_p^{4,q} = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} \quad (2.73)$

and a_{11}, a_{21} are the coefficients of the p^{th} harmonic corresponding to u_β and $\frac{\partial u_\gamma}{\partial \beta}$ along edge 1, due to the application of u_α of unit amplitude corresponding to the q^{th} harmonic along edge 4,

b_{11}, b_{21} are the coefficients of the p^{th} harmonic corresponding to u_β and $\frac{\partial u_\gamma}{\partial \beta}$ along edge 2, due to the application of u_α of unit amplitude corresponding to the q^{th} harmonic along edge 4.

The complementary solution edge disturbances G^1, G^2, G^3 and G^4 are obtained as the solution of the matrix equation:-

$$\begin{bmatrix} I & 0 & E_{13} & E_{14} \\ 0 & I & E_{23} & E_{24} \\ E_{31} & E_{32} & I & 0 \\ E_{41} & E_{42} & 0 & I \end{bmatrix} \begin{bmatrix} G^1 \\ G^2 \\ G^3 \\ G^4 \end{bmatrix} = - \begin{bmatrix} V^1 \\ V^2 \\ V^3 \\ V^4 \end{bmatrix} \qquad (2.74)$$

The total solution is obtained as the algebraic sum of the following:-

(i) a Navier type particular solution corresponding to harmonics $p = 1(1)10$ and $q = 1(1)10$,

(ii) a set of Levy solutions of type 1 corresponding to edge disturbances G^1 and G^2 applied along edges 1 and 2 respectively,

(iii) a set of Levy solutions of type 2 corresponding to edge disturbances G^3 and G^4 applied along edges 3 and 4 respectively.

In the example presented the condition of symmetry of loading and boundary conditions about the axis $\beta = \frac{{}^1\beta}{2}$ was not taken into account. However, the computer programmes are written to take into account conditions of symmetry or antimetry about the axis $\beta = \frac{{}^1\beta}{2}$ and/or the axis $\alpha = \frac{{}^1\alpha}{2}$.

CHAPTER III

TRANSLATIONAL SHELLS - AN EXTENDED LEVY METHOD OF SOLUTION

III.1. Shell particular solution

A Navier type particular solution is used. The external loads and temperature fields acting on the shell are expanded into double Fourier series of the form (refer Appendix 1):-

$$
\begin{aligned}
G_\alpha &= \sum_{p=o}^{\infty} \sum_{q=o}^{\infty} g_\alpha^{pq} \cos(p\bar{\alpha}) \sin(q\bar{\beta}) \\
G_\beta &= \sum_{p=o}^{\infty} \sum_{q=o}^{\infty} g_\beta^{pq} \sin(p\bar{\alpha}) \cos(q\bar{\beta}) \\
G_\gamma &= \sum_{p=o}^{\infty} \sum_{q=o}^{\infty} g_\gamma^{pq} \sin(p\bar{\alpha}) \sin(q\bar{\beta}) \qquad (3.1) \\
T_1 &= \sum_{p=o}^{\infty} \sum_{q=o}^{\infty} t_1^{pq} \sin(p\bar{\alpha}) \sin(q\bar{\beta}) \\
T_2 &= \sum_{p=o}^{\infty} \sum_{q=o}^{\infty} t_2^{pq} \sin(p\bar{\alpha}) \sin(q\bar{\beta})
\end{aligned}
$$

where p,q are positive integers

g_α^{pq}, g_γ^{pq}, g_β^{pq}, t_1^{pq}, t_2^{pq} are constants

$$
\begin{aligned}
\text{and } \bar{\alpha} &= \frac{\pi}{l_\alpha} \alpha \\
\bar{\beta} &= \frac{\pi}{l_\beta} \beta
\end{aligned} \qquad (3.2)
$$

In practice the loads are expressed in the form of truncated series. Let the maximum values of p and q considered be r and s respectively.

<u>Case 1 - $p \neq 0$, $q \neq 0$</u>

Consider a typical set of loading terms

$$
\begin{aligned}
G_\alpha^{pq} &= g_\alpha^{pq} \cos(p\bar{\alpha}) \sin(q\bar{\beta}) \\
G_\beta^{pq} &= g_\beta^{pq} \sin(p\bar{\alpha}) \cos(q\bar{\beta}) \\
G_\gamma^{pq} &= g_\gamma^{pq} \sin(p\bar{\alpha}) \sin(q\bar{\beta}) \\
T_1^{pq} &= t_1^{pq} \sin(p\bar{\alpha}) \sin(q\bar{\beta}) \\
T_2^{pq} &= t_2^{pq} \sin(p\bar{\alpha}) \sin(q\bar{\beta})
\end{aligned}
\tag{3.3}
$$

A solution for u_γ is sought in the form

$$u_\gamma = a^{pq} \sin(p\bar{\alpha}) \sin(q\bar{\beta}) \tag{3.4}$$

where a^{pq} is a constant.

Substituting (3.3) and (3.4) in equation (2.25)*:-

---

*All integration functions are assumed to be zero.

$$a^{pq} = \frac{1}{D(m^2+n^2)^4 + Et(k_\beta m^2+k_\alpha n^2)^2} \Bigg[(m^2+n^2)^2 \left[g_\gamma{}^{pq} - \frac{k_\alpha}{m} g_\alpha{}^{pq} - \frac{k_\beta}{n} g_\beta{}^{pq} \right] + (k_\beta m^2+k_\alpha n^2) \left[\frac{n^2}{m} g_\alpha{}^{pq} + \frac{m^2}{n} g_\beta{}^{pq} - \nu\,(ng_\beta{}^{pq}+mg_\alpha{}^{pq}) \right]$$

$$+ \frac{E\lambda t^2}{12(1-\nu)} (t_2{}^{pq} - t_1{}^{pq})(m^2+n^2)^3$$

$$- \frac{E}{2} \lambda t(t_1{}^{pq}+t_2{}^{pq})(m^2+n^2)(k_\beta m^2+k_\alpha n^2) \Bigg] \qquad (3.5)$$

$$\text{where } m = \frac{p\pi}{l_\alpha}, \qquad n = \frac{q\pi}{l_\beta} \qquad (3.6)$$

A solution for the stress function ϕ can then be obtained in the form:-

$$\phi = b^{pq} \sin(p\bar{\alpha}) \sin(q\bar{\beta}) \qquad (3.7)$$

where b^{pq} is a constant.

Substituting (3,3), (3.4) and (3.7) in equation (2.24):-

$$b^{pq} = \frac{1}{(m^2+n^2)^2} \Bigg[Et\, a^{pq} \left(k_\beta m^2 + k_\alpha n^2\right) - \left[\frac{n^2}{m} g_\alpha{}^{pq} + \frac{m^2}{n} g_\beta{}^{pq} - \nu(n\, g_\beta{}^{pq} + mg_\alpha{}^{pq}) \right]$$

$$+ \frac{E\lambda t}{2} (t_1{}^{pq} + t_2{}^{pq})(m^2+n^2) \Bigg] \qquad (3.8)$$

Using equations (2.5) to (2.22) all particular solution shell quantities Y^o can then be calculated.

<u>Case 2 - $p = 0,\ q \neq 0$</u>

Consider a typical set of loading terms

$$
\begin{aligned}
G_\alpha^{oq} &= g_\alpha^{oq}\sin(q\bar{\beta}) \\
G_\beta^{oq} &= 0 \\
G_\gamma^{oq} &= 0 \\
T_1^{oq} &= 0 \\
T_2^{oq} &= 0
\end{aligned}
\tag{3.9}
$$

A particular solution can be obtained where the only non zero shell quantities $n_{\alpha\beta}$ and u_α are given by:-

$$
\begin{aligned}
n_{\alpha\beta} &= \frac{g_\alpha^{oq}}{n}\cos(q\bar{\beta}) \\
u_\alpha &= \frac{2(1+\nu)}{n^2Et}\ g_\alpha^{oq}\sin(q\bar{\beta})
\end{aligned}
\tag{3.10}
$$

<u>Case 3 - $p \neq o,\ q = 0$</u>

Consider a typical set of loading terms

$$
\begin{aligned}
G_\alpha^{po} &= 0 \\
G_\beta^{po} &= g_\beta^{po}\sin(p\bar{\alpha}) \\
G_\gamma^{po} &= 0 \\
T_1^{po} &= 0 \\
T_2^{po} &= 0
\end{aligned}
\tag{3.11}
$$

A particular solution can be obtained where the only non zero shell quantities $n_{\alpha\beta}$ and u_β are given by:-

$$\begin{aligned} n_{\alpha\beta} &= \frac{g_\beta^{po}}{m} \cos(p\bar{\alpha}) \\ u_\beta &= \frac{2(1+\nu)}{m^2 Et} g_\beta^{po} \sin(p\bar{\alpha}) \end{aligned} \qquad (3.12)$$

Case 4 - p = 0, q = 0

All loading terms are zero. Hence a null particular solution could be used.

All particular solution shell quantities Y^o can be written in the form:-

$$Y^o = \sum_{p=o}^{r} \sum_{q=o}^{s} c_k^{pq} F(p\bar{\alpha}) F(q\bar{\beta}) \qquad (3.13)$$

where c_k^{pq} is a constant

and F( ) = sin( ) or cos( ).

The results are listed in tables 3.1 and 3.2.

In table 3.1 a^{pq}, b^{pq} are defined by equations (3.5) and (3.8) respectively, and

$$\begin{aligned} q'_\alpha &= q_\alpha + \frac{\partial m_{\alpha\beta}}{\partial\beta} \\ q'_\beta &= q_\beta + \frac{\partial m_{\alpha\beta}}{\partial\alpha} \\ n'_{\alpha\beta} &= n_{\alpha\beta} - k_\alpha m_{\alpha\beta} \\ n'_{\beta\alpha} &= n_{\beta\alpha} - k_\beta m_{\alpha\beta}^{\mathbf{x}} \end{aligned} \qquad (3.14)$$

Table 3.1

Y^o	C_k	$F(p\bar{\alpha})$	$F(q\bar{\beta})$	C_k^{pq} $(p \neq 0,\ q \neq 0)$
u_γ	C_1	$\mathrm{Sin}(p\bar{\alpha})$	$\mathrm{Sin}(q\bar{\beta})$	$+\ a^{pq}$
m_α	C_2	$\mathrm{Sin}(p\bar{\alpha})$	$\mathrm{Sin}(q\bar{\beta})$	$+Da^{pq}\left(m^2+\nu n^2\right) - \frac{E\lambda}{12(1-\nu)}\ t^2\left(t_2^{pq} - t_1^{pq}\right)$
$m_{\alpha\beta}$	C_3	$\mathrm{Cos}(p\bar{\alpha})$	$\mathrm{Cos}(q\bar{\beta})$	$-Dmna^{pq}(1 - \nu)$
m_β	C_4	$\mathrm{Sin}(p\bar{\alpha})$	$\mathrm{Sin}(q\bar{\beta})$	$+Da^{pq}\left(n^2+\nu m^2\right) - \frac{E\lambda}{12(1-\nu)}\ t^2\left(t_2^{pq} - t_1^{pq}\right)$
n_α	C_5	$\mathrm{Sin}(p\bar{\alpha})$	$\mathrm{Sin}(q\bar{\beta})$	$-n^2b^{pq} - \frac{g_\alpha^{pq}}{m}$
$n_{\alpha\beta}$	C_6	$\mathrm{Cos}(p\bar{\alpha})$	$\mathrm{Cos}(q\bar{\beta})$	$-mnb^{pq}$
n_β	C_7	$\mathrm{Sin}(p\bar{\alpha})$	$\mathrm{Sin}(q\bar{\beta})$	$-m^2b^{pq} - \frac{g_\beta^{pq}}{n}$
u_α	C_8	$\mathrm{Cos}(p\bar{\alpha})$	$\mathrm{Sin}(q\bar{\beta})$	$-\ \frac{1}{mEt}\left(c_5^{pq} - \nu c_7^{pq}\right) - \frac{1}{m}\ k_\alpha\ a^{pq} - \frac{\lambda}{2m}\ \left(t_1^{pq}+t_2^{pq}\right)$
u_β	C_9	$\mathrm{Sin}(p\bar{\alpha})$	$\mathrm{Cos}(q\bar{\beta})$	$-\ \frac{1}{nEt}\left(c_7^{pq} - \nu c_5^{pq}\right) - \frac{1}{n}\ k_\beta\ a^{pq} - \frac{\lambda}{2n}\ \left(t_1^{pq}+t_2^{pq}\right)$

Table 3.1 (continued)

Y^o	C_k	$F(p\bar{\alpha})$	$F(q\bar{\beta})$	c_k^{pq} $(p \neq 0,\ q \neq 0)$
q_α	C_{10}	$\mathrm{Cos}(p\bar{\alpha})$	$\mathrm{Sin}(q\bar{\beta})$	$+ mc_2^{pq} - nc_3^{pq}$
q_β	C_{11}	$\mathrm{Sin}(p\bar{\alpha})$	$\mathrm{Cos}(q\bar{\beta})$	$+ nc_4^{pq} - mc_3^{pq}$
q'_α	C_{12}	$\mathrm{Cos}(p\bar{\alpha})$	$\mathrm{Sin}(q\bar{\beta})$	$+ c_{10}^{pq} - nc_3^{pq}$
q'_β	C_{13}	$\mathrm{Sin}(p\bar{\alpha})$	$\mathrm{Cos}(q\bar{\beta})$	$+ c_{11}^{pq} - mc_3^{pq}$
$n'_{\alpha\beta}$	C_{14}	$\mathrm{Cos}(p\bar{\alpha})$	$\mathrm{Cos}(q\bar{\beta})$	$+ c_6^{pq} - k_\alpha c_3^{pq}$
$n'_{\beta\alpha}$	C_{15}	$\mathrm{Cos}(p\bar{\alpha})$	$\mathrm{Cos}(q\bar{\beta})$	$+ c_6^{pq} - k_\beta\ c_3^{pq}$
$\frac{\partial u_\gamma}{\partial \alpha}$	C_{16}	$\mathrm{Cos}(p\bar{\alpha})$	$\mathrm{Sin}(q\bar{\beta})$	$+ ma^{pq}$
$\frac{\partial u_\gamma}{\partial \beta}$	C_{17}	$\mathrm{Sin}(p\bar{\alpha})$	$\mathrm{Cos}(q\bar{\beta})$	$+ na^{pq}$

Table 3.2

Y^o	c_k	c_k^{pq}	
		$p = 0, \quad q \neq 0$	$p \neq 0, \quad q = 0$
$n_{\alpha\beta}$	c_6	$+ \dfrac{g_\alpha{}^{oq}}{n}$	$+ \dfrac{g_\beta{}^{po}}{m}$
u_α	c_8	$+ \dfrac{2(1+\nu)}{n^2Et} g_\alpha{}^{oq}$	0
u_β	c_9	0	$+ \dfrac{2(1+\nu)}{m^2Et} g_\beta{}^{po}$
$n'_{\alpha\beta}$	c_{14}	$+ c_6{}^{oq}$	$+ c_6{}^{po}$
$n'_{\beta\alpha}$	c_{15}	$+ c_6{}^{oq}$	$+ c_6{}^{po}$

All coefficients c_k not listed in table 3.2 are zero when $p = 0$ or $q = 0$.

---

*Although terms of the order of $k_\alpha\, m_{\alpha\beta}$, $k_\beta\, m_{\alpha\beta}$ are neglected in comparison with $n_{\alpha\beta}$ in the equilibrium equations, these quantities are retained in the formulation of the boundary conditions.

Equation (3.13) can be written in matrix form

$$Y^o = P_1 \; K \; P_2 \tag{3.15}$$

where P_1, K and P_2 are matrices of order $\left[1 \times (r+1)\right]$, $\left[(r+1) \times (s+1)\right]$ and $\left[(s+1) \times 1\right]$ respectively defined by:-

$$P_1 = \begin{bmatrix} F(0\bar{\alpha}) & F(1\bar{\alpha}) & F(2\bar{\alpha}) & \dots & F(r\bar{\alpha}) \end{bmatrix}$$

$$K = \begin{bmatrix} C^{oo} & C^{o1} & \dots & C^{os} \\ C^{10} & C^{11} & \dots & C^{1s} \\ \vdots & \vdots & & \vdots \\ C^{ro} & C^{r1} & \dots & C^{rs} \end{bmatrix} \tag{3.16}$$

$$P_2 = \begin{bmatrix} F(o\bar{\beta}) \\ F(1\bar{\beta}) \\ F(2\bar{\beta}) \\ \vdots \\ F(s\bar{\beta}) \end{bmatrix}$$

<u>Matrices defining shell actions and displacements along the edges</u>

All shell quantities defined in this section are the values along an edge.

<u>Notation</u>

The first superscript associated with matrices x and u refers to the type of solution. A superscript (0) will be used to denote the particular solution.

The second superscript refers to the edge. The superscript (i) is used to denote edge 1 or 2, and (j) to denote edge 3 or 4.

The subscript p or q refers to the harmonic considered.

Let $(x^o)^i_p$, $(u^o)^i_p$, $(x^o)^j_q$ and $(u^o)^j_q$ be vectors of order (4 x 1) defined by:-

$$(x^o)^i_p = \begin{bmatrix} n'_{\alpha\beta} \\ q'_{\beta} \\ n_{\beta} \\ m_{\beta} \end{bmatrix} = \begin{bmatrix} \mathrm{Cos}(p\bar{\alpha}) & & \bigcirc & \\ & \mathrm{Sin}(p\bar{\alpha}) & & \\ \bigcirc & & \mathrm{Sin}(p\bar{\alpha}) & \\ & & & \mathrm{Sin}(p\bar{\alpha}) \end{bmatrix} (\bar{x}^o)^i_p$$

$$(u^o)^i_p = \begin{bmatrix} u_{\alpha} \\ u_{\gamma} \\ u_{\beta} \\ \dfrac{\partial u_{\gamma}}{\partial \beta} \end{bmatrix} = \begin{bmatrix} \mathrm{Cos}(p\bar{\alpha}) & & \bigcirc & \\ & \mathrm{Sin}(p\bar{\alpha}) & & \\ \bigcirc & & \mathrm{Sin}(p\bar{\alpha}) & \\ & & & \mathrm{Sin}(p\bar{\alpha}) \end{bmatrix} (\bar{u}^o)^i_p$$

and

$$(x^{o})_q^j = \begin{bmatrix} n'_{\beta\alpha} \\ q'_{\alpha} \\ n_{\alpha} \\ m_{\alpha} \end{bmatrix} = \begin{bmatrix} \mathrm{Cos}(q\bar{\beta}) & & & \bigcirc \\ & \mathrm{Sin}(q\bar{\beta}) & & \\ & & \mathrm{Sin}(q\bar{\beta}) & \\ \bigcirc & & & \mathrm{Sin}(q\bar{\beta}) \end{bmatrix} (\bar{x}^{o})_q^j \tag{3.18}$$

$$(u^{o})_q^j = \begin{bmatrix} u_{\beta} \\ u_{\gamma} \\ u_{\alpha} \\ \dfrac{\partial u_{\gamma}}{\partial \alpha} \end{bmatrix} = \begin{bmatrix} \mathrm{Cos}(q\bar{\beta}) & & & \bigcirc \\ & \mathrm{Sin}(q\bar{\beta}) & & \\ & & \mathrm{Sin}(q\bar{\beta}) & \\ \bigcirc & & & \mathrm{Sin}(q\bar{\beta}) \end{bmatrix} (\bar{u}^{o})_q^j$$

<u>Edge 1</u>

$$(\bar{x}^{o})_p^1 = \begin{bmatrix} d_p^1 \\ d_p^2 \\ 0 \\ 0 \end{bmatrix} \quad ; \quad (\bar{u}^{o})_p^1 = \begin{bmatrix} 0 \\ 0 \\ d_p^3 \\ d_p^4 \end{bmatrix} \tag{3.19}$$

where $d_p^1 = \sum_{q=o}^{s} C_{14}^{pq}$

$$d_p^2 = \sum_{q=o}^{s} C_{13}^{pq}$$

$$d_p^3 = \sum_{q=o}^{s} C_9^{pq} \qquad (3.20)$$

$$d_p^4 = \sum_{q=o}^{s} C_{17}^{pq}$$

C_{14}^{pq}, C_{13}^{pq}, C_9^{pq}, C_{17}^{pq} are listed in tables 3.1 and 3.2

Edge 2

$$(\bar{x}^o)_p^2 = \begin{bmatrix} d_p^5 \\ d_p^6 \\ 0 \\ 0 \end{bmatrix} \quad ; \quad (\bar{u}^o)_p^2 = \begin{bmatrix} 0 \\ 0 \\ d_p^7 \\ d_p^8 \end{bmatrix} \qquad (3.21)$$

where $d_p^5 = \sum_{q=o}^{s} C_{14}^{pq} \operatorname{Cos}(q\pi)$

$$d_p^6 = \sum_{q=o}^{s} C_{13}^{pq} \operatorname{Cos}(q\pi) \qquad (3.22)$$

$$d_p^7 = \sum_{q=o}^{s} C_9^{pq} \operatorname{Cos}(q\pi)$$

$$d_p^8 = \sum_{q=o}^{s} C_{17}^{pq} \operatorname{Cos}(q\pi)$$

Edge 3

$$(\bar{x}^o)_q^3 = \begin{bmatrix} d_q^{\,9} \\ d_q^{\,10} \\ 0 \\ 0 \end{bmatrix} \qquad (\bar{u}^o)_q^3 = \begin{bmatrix} 0 \\ 0 \\ d_q^{\,11} \\ d_q^{\,12} \end{bmatrix} \tag{3.23}$$

$$\text{where } d_q^{\,9} = \sum_{p=o}^{r} C_{15}^{\,pq}$$

$$d_q^{\,10} = \sum_{p=o}^{r} C_{12}^{\,pq}$$

$$d_q^{\,11} = \sum_{p=o}^{r} C_{8}^{\,pq} \tag{3.24}$$

$$d_q^{\,12} = \sum_{p=o}^{r} C_{16}^{\,pq}$$

$C_{15}^{\,pq}$, $C_{12}^{\,pq}$, $C_{8}^{\,pq}$, $C_{16}^{\,pq}$ are listed in tables 3.1 and 3.2.

Edge 4

$$(\bar{x}^o)_q^4 = \begin{bmatrix} d_q^{\,13} \\ d_q^{\,14} \\ 0 \\ 0 \end{bmatrix} \; ; \quad (\bar{u}^o)_q^4 = \begin{bmatrix} 0 \\ 0 \\ d_q^{\,15} \\ d_q^{\,16} \end{bmatrix} \tag{3.25}$$

where
$$d_q^{13} = \sum_{p=o}^{r} c_{15}^{pq} \cos(p\pi)$$
$$d_q^{14} = \sum_{p=o}^{r} c_{12}^{pq} \cos(p\pi)$$
$$d_q^{15} = \sum_{p=o}^{r} c_{8}^{pq} \cos(p\pi) \qquad (3.26)$$
$$d_q^{16} = \sum_{p=o}^{r} c_{16}^{pq} \cos(p\pi)$$

All particular solution 'edge disturbances' can now be determined (in the standard form defined in Section II.6.2) as a linear combination of matrices, $(x^o)_p^i$, $(u^o)_p^i$ along edges 1 and 2, and $(x^o)_q^j$, $(u^o)_q^j$ along edges 3 and 4.

III.2. Shell complementary solution

III.2.1. Levy solutions type 1*

A solution for u_γ is sought in the form**:-

*The formulation is based on a method used by Jenkins(6) for cylindrical shells. A similar method has been used by Powell(14).

**A solution of this form is possible because equation (3.28) contains only even derivatives of α, and $\sin(p\bar{\alpha})$ repeats its form after an even number of differentiations with respect to α.

$$u_\gamma = \bar{u}_\gamma(\beta)\ \sin(p\bar{\alpha}) \tag{3.27}$$

where $\bar{u}_\gamma(\beta)$ is a function of β.

When $G_\alpha = 0$, $G_\beta = 0$, $G_\gamma = 0$, $T_1 = 0$, $T_2 = 0$ equation (2.25) reduces to:-

$$D\nabla^8 u_\gamma + Et\,\nabla_R^{\ 4} u_\gamma = 0 \tag{3.28}$$

Substituting (3.27) in (3.28)

$$\left[\left[-m^2 + \frac{d^2}{d\beta^2}\right]^4 + \frac{12(1-\nu^2)}{t^2}\left[-k_\beta m^2 + k_\alpha \frac{d^2}{d\beta^2}\right]^2\right]\bar{u}_\gamma(\beta) = 0 \tag{3.29}$$

$$\text{Let } \bar{u}_\gamma(\beta) = C\ e^{\rho\beta} \tag{3.30}$$

Substuting (3.30) in (3.29) yields the auxiliary equation for u_γ:-

$$(\rho^2 - m^2)^4 + (k_2\rho^2 - k_3)^2 = 0 \tag{3.31}$$

$$\begin{aligned} \text{where } k_1 &= \frac{2\left[3 - 3\nu^2\right]^{1/2}}{t} \\ k_2 &= k_1\ k_\alpha \\ k_3 &= k_1\ k_\beta\ m^2 \end{aligned} \tag{3.32}$$

The eight roots of the auxiliary equation are (refer appendix 2):-

(i) when $k_\alpha \neq k_\beta$

$$\rho = \pm\ (\sigma_1 \pm i\ \lambda_1)\ ;\ \ \pm(\sigma_2 \pm i\lambda_2) \tag{3.33}$$

(ii) when $k_\alpha = k_\beta$

$$\rho = \pm (\sigma_1 \pm i \lambda_1) \;\; ; \;\; \pm m \; ; \;\; \pm m \tag{3.34}$$

where σ_1, σ_2, λ_1, λ_2, i are defined in appendix 2.

Case 1 - $k_\alpha \neq k_\beta$

The solutions for u_γ and ϕ are (refer appendix 3):-

$$\begin{aligned} u_\gamma = \sin(p\bar{\alpha}) \Big[& e^{-\sigma_1\beta}(\mu_1 \cos(\lambda_1\beta) + \mu_2 \sin(\lambda_1\beta)) \\ & + e^{-\sigma_2\beta}(\mu_3 \cos(\lambda_2\beta) + \mu_4 \sin(\lambda_2\beta)) \\ & + e^{-\sigma_1 z}(\mu_5 \cos(\lambda_1 z) + \mu_6 \sin(\lambda_1 z)) \\ & + e^{-\sigma_2 z}(\mu_7 \cos(\lambda_2 z) + \mu_8 \sin(\lambda_2 z)) \Big] \end{aligned} \tag{3.35}$$

and

$$\begin{aligned} \phi = \frac{Et}{k_1} \sin(p\bar{\alpha}) \Big[& e^{-\sigma_1\beta}(\mu_2 \cos(\lambda_1\beta) - \mu_1 \sin(\lambda_1\beta)) \\ & + e^{-\sigma_2\beta}(\mu_4 \cos(\lambda_2\beta) - \mu_3 \sin(\lambda_2\beta)) \\ & + e^{-\sigma_1 z}(\mu_6 \cos(\lambda_1 z) - \mu_5 \sin(\lambda_1 z)) \\ & + e^{-\sigma_2 z}(\mu_8 \cos(\lambda_2 z) - \mu_7 \sin(\lambda_2 z)) \Big] \end{aligned} \tag{3.36}$$

where μ_1, μ_2,....,μ_8 are arbitrary constants

and $\quad z = l_\beta - \beta \qquad (3.37)$

Using the relations

(i) If $f = f_1(\beta) + f_1(z)$

then $$\frac{df}{d\beta} = \frac{df_1(\beta)}{d\beta} - \frac{df_1(z)}{dz} \tag{3.38}$$

(ii) If $f = e^{-\sigma\beta}\left[(a_1\ c_1 + a_2\ c_2)\cos(\lambda\beta) + (a_1\ c_2 - a_2\ c_1)\sin(\lambda\beta)\right]$

where a_1, a_2, c_1, c_2 are constants.

then $\frac{df}{d\beta} = e^{-\sigma\beta}\left[(a_3\ c_1 + a_4\ c_2)\cos(\lambda\beta) + (a_3\ c_2 - a_4\ c_1)\sin(\lambda\beta)\right]$

$$\begin{aligned} \text{where } a_3 &= -\sigma a_1 - \lambda a_2 \\ a_4 &= +\lambda a_1 - \sigma a_2 \end{aligned} \qquad (3.39)$$

all shell quantities Y^1 can be written in the form:-

$$\begin{aligned} Y^1 = F(p\bar{\alpha}) \Big[& e^{-\sigma_1\beta}\left[(a_1\mu_1+a_2\mu_2)\cos(\lambda_1\beta)+(a_1\mu_2-a_2\mu_1)\sin(\lambda_1\beta)\right] \\ & + e^{-\sigma_2\beta}\left[(a_3\mu_3+a_4\mu_4)\cos(\lambda_2\beta)+(a_3\mu_4-a_4\mu_3)\sin(\lambda_2\beta)\right] \\ & + e^{-\sigma_1 z}\left[(a_5\mu_5+a_6\mu_6)\cos(\lambda_1 z)+(a_5\mu_6-a_6\mu_5)\sin(\lambda_1 z)\right] \\ & + e^{-\sigma_2 z}\left[(a_7\mu_7+a_8\mu_8)\cos(\lambda_2 z)+(a_7\mu_8-a_8\mu_7)\sin(\lambda_2 z)\right] \Big] \end{aligned} \qquad (3.40)$$

where a_1, a_2,....,a_8 are constants

and $F(p\bar{\alpha}) = \sin(p\bar{\alpha})$ or $\cos(p\bar{\alpha})$.

Equation (3.40) can be written in matrix form

$$Y^1 = \left[A\ L_1{}^{\beta}f + B\ L_1{}^{z}g\right] F(p\bar{\alpha}) \qquad (3.41)$$

$$\text{where } f = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} \qquad g = \begin{bmatrix} \mu_5 \\ \mu_6 \\ \mu_7 \\ \mu_8 \end{bmatrix} \tag{3.42}$$

$$A = \begin{bmatrix} a_1 & a_2 & \bar{a}_1 & \bar{a}_2 \end{bmatrix} \tag{3.43}$$

$$B = c \begin{bmatrix} a_1 & a_2 & \bar{a}_1 & \bar{a}_2 \end{bmatrix} \tag{3.44}$$

$$L_1^{\beta} = \begin{bmatrix} +e^{-\sigma_1\beta}\cos(\lambda_1\beta) & +e^{-\sigma_1\beta}\sin(\lambda_1\beta) & 0 & 0 \\ -e^{-\sigma_1\beta}\sin(\lambda_1\beta) & +e^{-\sigma_1\beta}\cos(\lambda_1\beta) & 0 & 0 \\ 0 & 0 & +e^{-\sigma_2\beta}\cos(\lambda_2\beta) & +e^{-\sigma_2\beta}\sin(\lambda_2\beta) \\ 0 & 0 & -e^{-\sigma_2\beta}\sin(\lambda_2\beta) & +e^{-\sigma_2\beta}\cos(\lambda_2\beta) \end{bmatrix} \tag{3.45}$$

$$L_1^{z} = \begin{bmatrix} +e^{-\sigma_1 z}\cos(\lambda_1 z) & +e^{-\sigma_1 z}\sin(\lambda_1 z) & 0 & 0 \\ -e^{-\sigma_1 z}\sin(\lambda_1 z) & e^{-\sigma_1 z}\cos(\lambda_1 z) & 0 & 0 \\ 0 & 0 & +e^{-\sigma_2 z}\cos(\lambda_2 z) & +e^{-\sigma_2 z}\sin(\lambda_2 z) \\ 0 & 0 & -e^{-\sigma_2 z}\sin(\lambda_2 z) & +e^{-\sigma_2 z}\cos(\lambda_2 z) \end{bmatrix} \tag{3.46}$$

The results are listed in table 3.3. Coefficients $\bar{a}_1$, $\bar{a}_2$ are obtained from a_1, a_2 by replacing σ_1, λ_1 by σ_2, λ_2 respectively.

Table 3.3

Y^1	$F(p\bar{\alpha})$	c	a_1	a_2
u_γ	$\sin(p\bar{\alpha})$	+1	$+1$	0
m_α	$\sin(p\bar{\alpha})$	+1	$+D\left[m^2-\nu(\sigma_1{}^2-\lambda_1{}^2)\right]$	$+2D\nu\sigma_1\lambda_1$
$m_{\alpha\beta}$	$\cos(p\bar{\alpha})$	-1	$+Dm(1-\nu)\sigma_1$	$-Dm(1-\nu)\lambda_1$
m_β	$\sin(p\bar{\alpha})$	+1	$+D\left[\nu m^2-(\sigma_1{}^2-\lambda_1{}^2)\right]$	$+2D\sigma_1\lambda_1$
n_α	$\sin(p\bar{\alpha})$	+1	$+2\dfrac{Et}{k_1}\sigma_1\lambda_1$	$+\dfrac{Et}{k_1}(\sigma_1{}^2-\lambda_1{}^2)$
$n_{\alpha\beta}$	$\cos(p\bar{\alpha})$	-1	$+\dfrac{Et}{k_1}m\lambda_1$	$+\dfrac{Et}{k_1}m\sigma_1$
n_β	$\sin(p\bar{\alpha})$	+1	0	$-\dfrac{m^2}{k_1}Et$
u_α	$\cos(p\bar{\alpha})$	+1	$-\dfrac{1}{mk_1}\left[2\sigma_1\lambda_1+k_1k_\alpha\right]$	$-\dfrac{1}{mk_1}\left[\sigma_1{}^2-\lambda_1{}^2+\nu m^2\right]$
u_β	$\sin(p\bar{\alpha})$	-1	$+\dfrac{1}{k_1(\sigma_1{}^2+\lambda_1{}^2)}\left[-k_1k_\beta\sigma_1-m^2\lambda_1+\nu\lambda_1(\sigma_1{}^2+\lambda_1{}^2)\right]$	$+\dfrac{1}{k_1(\sigma_1{}^2+\lambda_1{}^2)}\left[-k_1k_\beta\lambda_1+m^2\sigma_1+\nu\sigma_1(\sigma_1{}^2+\lambda_1{}^2)\right]$

Table 3.3 (continued)

Y^o	$F(p\bar{\alpha})$	c	a_1	a_2
q_α	$\cos(p\bar{\alpha})$	+1	$+Dm\left[m^2 - (\sigma_1{}^2 - \lambda_1{}^2)\right]$	$+ 2Dm\ \sigma_1\lambda_1$
q_β	$\sin(p\bar{\alpha})$	-1	$+D\sigma_1\left[\sigma_1{}^2 - 3\lambda_1{}^2 - m^2\right]$	$+ D\lambda_1\left[\lambda_1{}^2 - 3\sigma_1{}^2 + m^2\right]$
q'_α	$\cos(p\bar{\alpha})$	+1	$+Dm\left[m^2 - (2-\nu)(\sigma_1{}^2-\lambda_1{}^2)\right]$	$+ 2Dm\sigma_1\lambda_1\ (2-\nu)$
q'_β	$\sin(p\bar{\alpha})$	-1	$+ D\sigma_1\left[\sigma_1{}^2- 3\lambda_1{}^2- m^2(2-\nu)\right]$	$+ D\lambda_1\left[\lambda_1{}^2-3\sigma_1{}^2+ m^2(2-\nu)\right]$
$n'_{\alpha\beta}$	$\cos(p\bar{\alpha})$	-1	$+ \frac{Et}{k_1}\ m\ \lambda_1 - k_\alpha\ Dm(1-\nu)\sigma_1$	$+ \frac{Et}{k_1}\ m\sigma_1 + k_\alpha Dm(1-\nu)\lambda_1$
$n'_{\beta\alpha}$	$\cos(p\bar{\alpha})$	-1	$+ \frac{Et}{k_1}\ m\ \lambda_1 - k_\beta Dm(1-\nu)\sigma_1$	$+ \frac{Et}{k_1}\ m\sigma_1 + k_\beta Dm(1-\nu)\lambda_1$
$\frac{\partial u_\gamma}{\partial\alpha}$	$\cos(p\bar{\alpha})$	+1	$+m$	0
$\frac{\partial u_\gamma}{\partial\beta}$	$\sin(p\bar{\alpha})$	-1	$-\sigma_1$	$+\lambda_1$

Case 2 - $k_\alpha = k_\beta$

The solutions for u_γ and ϕ are (refer appendix 3):-

$$u_\gamma = \sin(p\bar{\alpha}) \left[e^{-\sigma_1\beta} \left[\mu_1 \cos(\lambda_1\beta) + \mu_2 \sin(\lambda_1\beta) + \mu_4 e^{-m\bar{\beta}} \right] + e^{-\sigma_1 z} \left[\mu_5 \cos(\lambda_1 z) + \mu_6 \sin(\lambda_1 z) + \mu_8 e^{-m\bar{z}} \right] \right] \tag{3.47}$$

and

$$\phi = \frac{Et}{k_1} \sin(p\bar{\alpha}) \left[e^{-\sigma_1\beta} \left[\mu_2 \cos(\lambda_1\beta) - \mu_1 \sin(\lambda_1\beta) + \mu_3 e^{-m\beta} \right] + e^{-\sigma_1 z} \left[\mu_6 \cos(\lambda_1 z) - \mu_5 \sin(\lambda_1 z) + \mu_7 e^{-mz} \right] \right] \tag{3.48}$$

where $\mu_1, \mu_2, \ldots, \mu_8$ are arbitrary constants and

$$z = l_\beta - \beta \tag{3.49}$$

Using the relations given by (3.38) and (3.39) all shell quantities Y^l can be written in the form:-

$$\begin{aligned} Y^l = F(p\bar{\alpha}) \Big[& e^{-\sigma_1\beta} \Big[(a_1\mu_1 + a_2\mu_2)\cos(\lambda_1\beta) + (a_1\mu_2 - a_2\mu_1)\sin(\lambda_1\beta) \\ & + a_3\mu_3 e^{-m\beta} + a_4\mu_4 e^{-m\beta} \Big] \\ & + e^{-\sigma_1 z} \Big[(a_5\mu_5 + a_6\mu_6)\cos(\lambda_1 z) + (a_5\mu_6 - a_6\mu_5)\sin(\lambda_1 z) \\ & + a_7\mu_7 e^{-mz} + a_8\mu_8 e^{-mz} \Big] \Big] \end{aligned} \tag{3.50}$$

where $a_1, a_2, \ldots, a_8$ are constants

and $F(p\bar{\alpha}) = \sin(p\bar{\alpha})$ or $\cos(p\bar{\alpha})$.

Equation (3.50) can be written in matrix form

$$Y^1 = \left[AL_2^{\beta} f + BL_2^{z} g \right] F(p\bar{\alpha}) \tag{3.51}$$

where

$$f = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} \qquad g = \begin{bmatrix} \mu_5 \\ \mu_6 \\ \mu_7 \\ \mu_8 \end{bmatrix} \tag{3.52}$$

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \tag{3.53}$$

$$B = c\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \tag{3.54}$$

$$L_2^{\beta} = \begin{bmatrix} +e^{-\sigma_1\beta}\cos(\lambda_1\beta) & +e^{-\sigma_1\beta}\sin(\lambda_1\beta) & 0 & 0 \\ -e^{-\sigma_1\beta}\sin(\lambda_1\beta) & +e^{-\sigma_1\beta}\cos(\lambda_1\beta) & 0 & 0 \\ 0 & 0 & +e^{-m\beta} & 0 \\ 0 & 0 & 0 & +e^{-m\beta} \end{bmatrix} \tag{3.55}$$

$$L_2^{z} = \begin{bmatrix} +e^{-\sigma_1 z}\cos(\lambda_1 z) & +e^{-\sigma_1 z}\sin(\lambda_1 z) & 0 & 0 \\ -e^{-\sigma_1 z}\sin(\lambda_1 z) & +e^{-\sigma_1 z}\cos(\lambda_1 z) & 0 & 0 \\ 0 & 0 & +e^{-mz} & 0 \\ 0 & 0 & 0 & +e^{-mz} \end{bmatrix} \tag{3.56}$$

$F(p\bar{\alpha})$, a_1, a_2, c are listed in table 3.3 and a_3, a_4 in table 3.4.

TABLE 3.4

Y^o	a_3	a_4
u_γ	0	+1
m_α	0	$+ Dm^2(1-\nu)$
$m_{\alpha\beta}$	0	$+ Dm^2(1-\nu)$
m_β	0	$- Dm^2(1-\nu)$
n_α	$+ \frac{Et}{k_1} m^2$	0
$n_{\alpha\beta}$	$+ \frac{Et}{k_1} m^2$	0
u_α	$- \frac{m}{k_1} (1+\nu)$	$- \frac{k_\alpha}{m}$
u_β	$+ \frac{m}{k_1} (1+\nu)$	$- \frac{k_\beta}{m}$
q_α	0	0
q_β	0	0
q'_α	0	$- Dm^3(1-\nu)$
q'_β	0	$- Dm^3(1-\nu)$
$n'_{\alpha\beta}$	$+ \frac{Et}{k_1} m^2$	$- Dm^2(1-\nu)k_\alpha$
$n'_{\beta\alpha}$	$+ \frac{Et}{k_1} m^2$	$- Dm^2(1-\nu)k_\beta$
$\frac{\partial u_\gamma}{\partial \alpha}$	0	$+m$
$\frac{\partial u_\gamma}{\partial \beta}$	0	$-m$
n_β	$\frac{- Etm^2}{k_1}$	0

Let $(x^1)_p$ and $(u^1)_p$ be vectors of order (4 x 1) defined by:-

$$(x^1)_p = \begin{bmatrix} n'_{\alpha\beta} \\ q'_{\beta} \\ n_{\beta} \\ m_{\beta} \end{bmatrix} = \begin{bmatrix} \cos(p\bar{\alpha}) & & & \bigcirc \\ & \sin(p\bar{\alpha}) & & \\ & \bigcirc & \sin(p\bar{\alpha}) & \\ & & & \sin(p\bar{\alpha}) \end{bmatrix} (\bar{x}^1)_p \qquad (3.57)$$

and

$$(u^1)_p = \begin{bmatrix} u_{\alpha} \\ u_{\gamma} \\ u_{\beta} \\ \dfrac{\partial u_{\gamma}}{\partial \beta} \end{bmatrix} = \begin{bmatrix} \cos(p\bar{\alpha}) & & & \bigcirc \\ & \sin(p\bar{\alpha}) & & \\ & \bigcirc & \sin(p\bar{\alpha}) & \\ & & & \sin(p\bar{\alpha}) \end{bmatrix} (\bar{u}^1)_p \qquad (3.58)$$

$$\text{where } (\bar{x}^1)_p = A_1 L^{\beta} f + A_2 L^{z} g$$
$$(\bar{u}^1)_p = A_3 L^{\beta} f + A_4 L^{z} g \qquad (3.59)$$

A_1, A_2, A_3, A_4, L^{β}, L^{z} are matrices of order (4 x 4) defined in appendix 4.

The boundary conditions along edges 1 and 2 can, in general, be specified as a linear combination of matrices $(x^1)_p$ and $(u^1)_p$[*]. Let these boundary conditions be defined

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[*]The case where there are flexible beams along the edges is considered in Section III.3.1.

by matrices $(\bar{y})_p^1$ and $(\bar{y})_p^2$ where:-

$$\begin{aligned}(\bar{y})_p^1 &= B_1 \; I \; f + B_2 \; L \; g \\ (\bar{y})_p^2 &= B_3 \; L \; f + B_4 \; I \; g\end{aligned} \qquad (3.60)$$

where B_1, B_2, B_3, B_4 are matrices of order (4 x 4)

I is a unit matrix of order (4 x 4)

and L represents the matrix L^β and L^z when $\beta = 1_\beta$ and $z = 1_\beta$ respectively.

Equations (3.60) can be written

$$\begin{bmatrix} (\bar{y})_p^1 \\ (\bar{y})_p^2 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 L \\ B_3 L & B_4 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} \qquad (3.61)$$

Hence

$$\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} B_1 & B_2 L \\ B_3 L & B_4 \end{bmatrix}^{-1} \begin{bmatrix} (\bar{y})_p^1 \\ (\bar{y})_p^2 \end{bmatrix} \qquad (3.62)$$

Once the boundary conditions along edges 1 and 2 are specified the eight arbitrary constants represented by matrices f and g can be calculated from equation (3.62). All shell quantities Y^1 can then be calculated using equation (3.41) when $k_\alpha \neq k_\beta$, and equation (3.51) when $k_\alpha = k_\beta$.

It can be shown that:-

(i) When the boundary conditions and loading on the shell are symmetric about the axis $\beta = \frac{1_\beta}{2}$, $f = g$,

(ii) when the boundary conditions are symmetric and the loading on the shell antimetric about the axis $\beta = \frac{l_\beta}{2}$, $f = -g$.

Matrices defining shell actions and displacements along the edges

Notation

The first superscript associated with matrices x and u refers to the type of solution. A superscript (1) will be used to denote Levy solutions of type 1.

The second superscript refers to the edge.

The subscript p or q refers to the harmonic considered.

Edge 1

$$(x^1)^1_p = \begin{bmatrix} n'_{\alpha\beta} \\ q'_\beta \\ n_\beta \\ m_\beta \end{bmatrix} = \begin{bmatrix} \cos(p\bar{\alpha}) & & & \\ & \sin(p\bar{\alpha}) & & \\ & & \sin(p\bar{\alpha}) & \\ & & & \sin(p\bar{\alpha}) \end{bmatrix} (\bar{x}^1)^1_p \tag{3.63}$$

$$(u^1)^1_p = \begin{bmatrix} u_\alpha \\ u_\gamma \\ u_\beta \\ \frac{\partial u_\gamma}{\partial \beta} \end{bmatrix} = \begin{bmatrix} \cos(p\bar{\alpha}) & & & \\ & \sin(p\bar{\alpha}) & & \\ & & \sin(p\bar{\alpha}) & \\ & & & \sin(p\bar{\alpha}) \end{bmatrix} (\bar{u}^1)^1_p$$

$$\text{where } (\bar{x}^1)_p^1 = A_1 f + A_2 L\, g$$
$$(\bar{u}^1)_p^1 = A_3 f + A_4 L\, g \qquad (3.64)$$

A_1, A_2, A_3, A_4 are matrices of order (4 x 4) defined in appendix 4

$$\text{and } L = L^{\beta = l_\beta} = L^{z = l_\beta} \qquad (3.65)$$

<u>Edge 2</u>

$$(x^1)_p^2 = \begin{bmatrix} n'_{\alpha\beta} \\ q'_\beta \\ n_\beta \\ m_\beta \end{bmatrix} = \begin{bmatrix} \cos(p\bar{\alpha}) & & & \bigcirc \\ & \sin(p\bar{\alpha}) & & \\ & & \sin(p\bar{\alpha}) & \\ \bigcirc & & & \sin(p\bar{\alpha}) \end{bmatrix} (\bar{x}^1)_p^2 \qquad (3.66)$$

$$(u^1)_p^2 = \begin{bmatrix} u_\alpha \\ u_\gamma \\ u_\beta \\ \dfrac{\partial u_\gamma}{\partial\beta} \end{bmatrix} = \begin{bmatrix} \cos(p\bar{\alpha}) & & & \bigcirc \\ & \sin(p\bar{\alpha}) & & \\ & & \sin(p\bar{\alpha}) & \\ \bigcirc & & & \sin(p\bar{\alpha}) \end{bmatrix} (\bar{u}^1)_p^2$$

$$\text{where } (\bar{x}^1)_p^2 = A_1 L\, f + A_2\, g$$
$$(\bar{u}^1)_p^2 = A_3 L\, f + A_4\, g \qquad (3.67)$$

Edge 3

Let $(x^1)_p^j$ and $(u^1)_p^j$ be vectors of order (4 x 1) defined by:-

$$(x^1)_p^j = \begin{bmatrix} n'_{\beta\alpha} \\ q'_\alpha \\ n_\alpha \\ m_\alpha \end{bmatrix} \; ; \quad (u^1)_p^j = \begin{bmatrix} u_\beta \\ u_\gamma \\ u_\alpha \\ \dfrac{\partial u_\gamma}{\partial \alpha} \end{bmatrix} \tag{3.68}$$

where superscript j refers to edge 3 or 4.

Along edge 3,

$$(x^1)_p^3 = \begin{bmatrix} \cos(po) & & & \\ & \cos(po) & & \bigcirc \\ \bigcirc & & \sin(po) & \\ & & & \sin(po) \end{bmatrix} (\bar{x}^1)_p^3 \tag{3.69}$$

$$(u^1)_p^3 = \begin{bmatrix} \sin(po) & & & \\ & \sin(po) & & \bigcirc \\ \bigcirc & & \cos(po) & \\ & & & \cos(po) \end{bmatrix} (\bar{u}^1)_p^3$$

where $(\bar{x}^1)_p^3 = A_5 L^\beta f + A_6 L^z g$

$$(\bar{u}^1)_p^3 = A_7 L^\beta f + A_8 L^z g \tag{3.70}$$

A_5, A_6, A_7, A_8, L^β, L^z are matrices of order (4 x 4) defined in appendix 4.

The elements of vectors $(x^1)^3_p$, $(u^1)^3_p$ are general functions of the coordinate β. It is necessary to express these functions as half range trigonometric series of the form (refer section II.6.3):-

$$(x^1)^3_p = \sum_{q=o}^{\infty} d_q \ (\bar{x}^1)^3_q$$

$$(u^1)^3_p = \sum_{q=o}^{\infty} d_q \ (\bar{u}^1)^3_q \qquad (3.71)$$

where $d_q = \begin{bmatrix} \cos(q\bar{\beta}) & & & \bigcirc \\ & \sin(q\bar{\beta}) & & \\ & & \sin(q\bar{\beta}) & \\ \bigcirc & & & \sin(q\bar{\beta}) \end{bmatrix} \qquad (3.72)$

$(\bar{x}^1)^3_q$, $(\bar{u}^1)^3_q$ are vectors of order (4 x 1) defined in appendix 4.

<u>Edge 4</u>

$$(x^1)^4_p = \begin{bmatrix} \cos(p\pi) & & & \bigcirc \\ & \cos(p\pi) & & \\ & & \sin(p\pi) & \\ \bigcirc & & & \sin(p\pi) \end{bmatrix} (\bar{x}^1)^4_p \qquad (3.73)$$

$$(u^1)_p^4 = \begin{bmatrix} \sin(p\pi) & 0 & 0 & 0 \\ 0 & \sin(p\pi) & 0 & 0 \\ 0 & 0 & \cos(p\pi) & 0 \\ 0 & 0 & 0 & \cos(p\pi) \end{bmatrix} (\bar{u}^1)_p^4$$

$$\text{where } (\bar{x}^1)_p^4 = A_5 L^{\beta} f + A_6 L^{z} g$$

$$(\bar{u}^1)_p^4 = A_7 L^{\beta} f + A_8 L^{z} g \qquad (3.74)$$

Matrices $(x^1)_p^4$, $(u^1)_p^4$ are expressed as half range trigonometric series of the form (refer section II.6.3):-

$$(x^1)_p^4 = \sum_{q=o}^{\infty} d_q \; (\bar{x}^1)_q^4$$

$$(u^1)_p^4 = \sum_{q=o}^{\infty} d_q \; (\bar{u}^1)_q^4 \qquad (3.75)$$

where $(\bar{x}^1)_q^4$, $(\bar{u}^1)_q^4$ are matrices of order (4 x 4) defined in appendix 4, and d_q is defined by equation (3.72).

All 'edge disturbances' can now be obtained (in the standard form defined in section II.6) as a linear combination of the matrices, $(x^1)_p^i$, $(u^1)_p^i$ along edges 1 and 2, and $(x^1)_p^j$, $(u^1)_p^j$ along edges 3 and 4 respectively.

III.2.2. Levy solutions type 2

A solution for u_γ is sought in the form:-

$$u_\gamma = \bar{u}_\gamma(\alpha)\,\sin(q\bar{\beta}) \tag{3.76}$$

where $\bar{u}_\gamma(\alpha)$ is a function of α.

This solution can be directly obtained from the results for Levy solutions type 1 (section III.2.1) by making the following changes:-

(Change α into β) $\alpha \to \beta$

$\beta \to \alpha$

$p \to q$

$q \to p$

superscript $i \to j$

superscript $j \to i$

$m \to n$

$n \to m$

Edge 1 → Edge 3

Edge 2 → Edge 4

Edge 3 → Edge 1

Edge 4 → Edge 2

and the first superscript associated with matrices x and u from 1 to 2.

III.3.1. Flexible beams along shell edges 1 and 2*

Beams of constant rectangular cross section are considered.

Beam axes

The origin is at the centroid of a transverse section of the beam. Direction I is taken along the longitudinal axis of the beam. Directions II and III are the directions of the principal axes of a transverse section of the beam. The positive direction of axis III is chosen such that the included angle between axis III and the shell axis γ (at a point of intersection of the shell and a transverse section of the edge beam) is less than ninety degrees. Axes I, II and III form a right handed orthogonal triad (figure 3.1).

A clockwise rotation about axis I is considered a positive rotation.

a^i, b^i are the distances measured from the centroid of the edge beam to a point of intersection of the shell and a transverse section of the edge beam in the directions II and III respectively.

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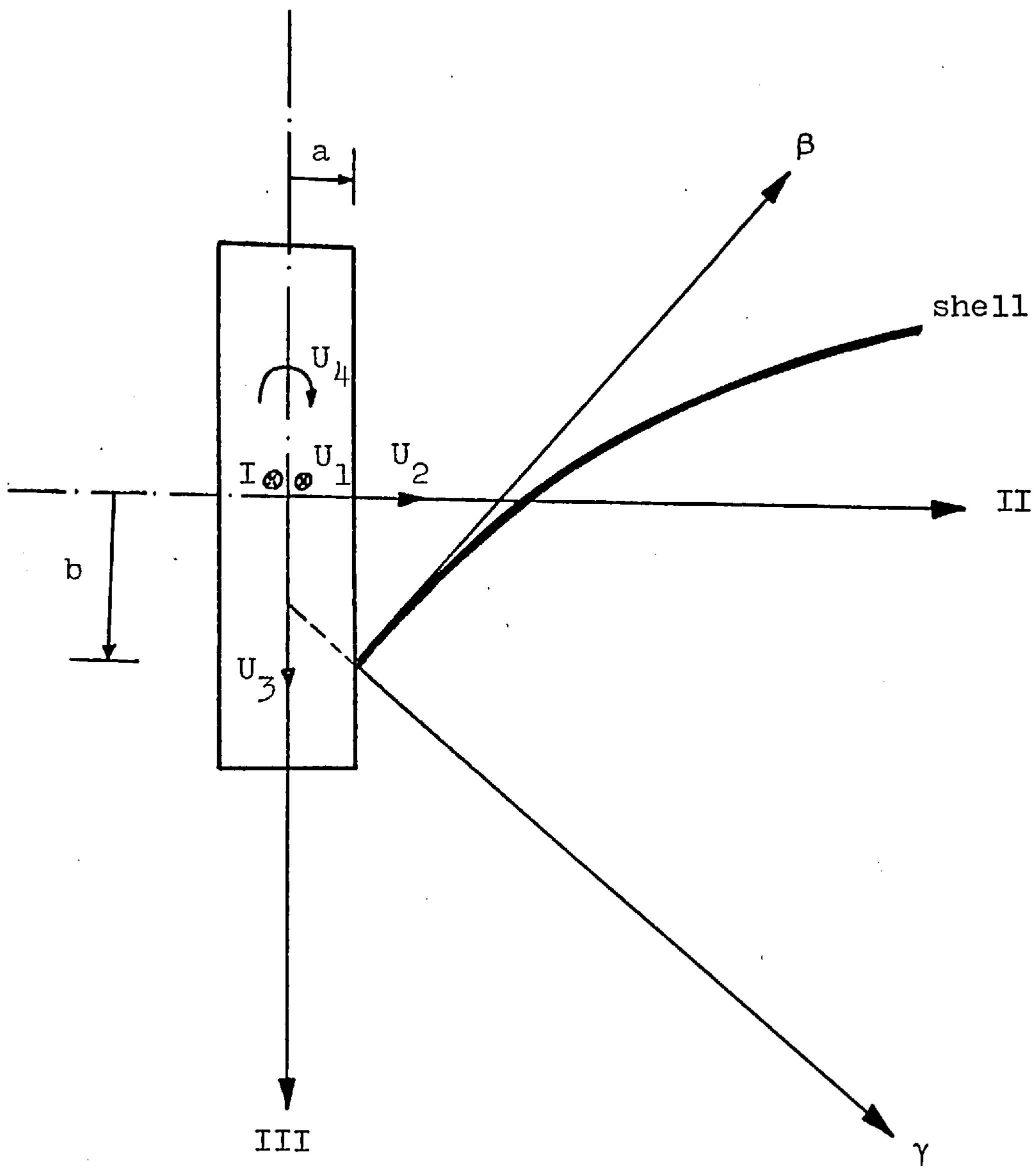
*The formulation is based on a method used by Jenkins for the solution of cylindrical shells with flexible beams along the straight edges of the shell[6].



Beam axes and displacements (edges 1 and 2)

Figure 3.1

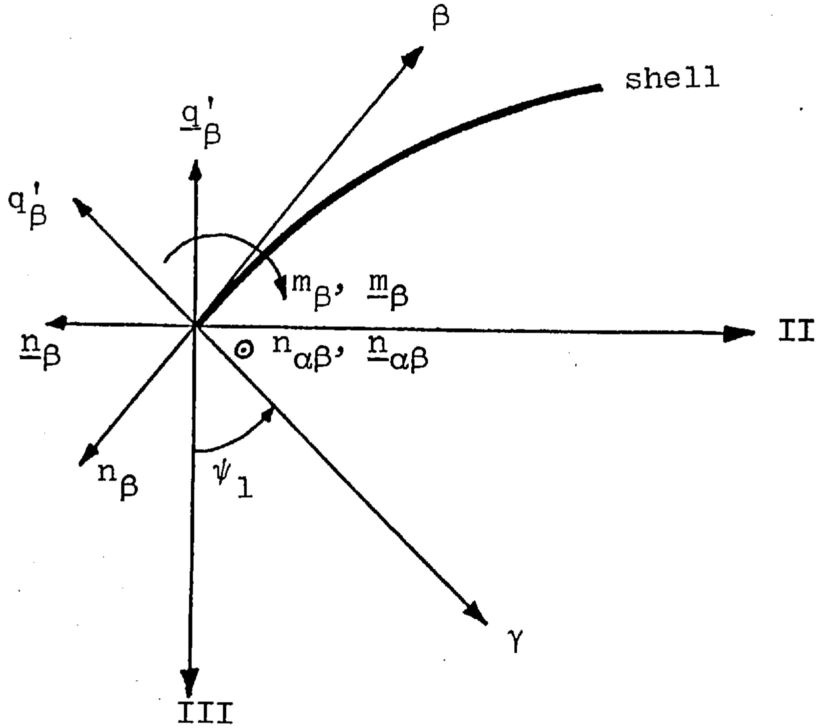


Figure 3.2

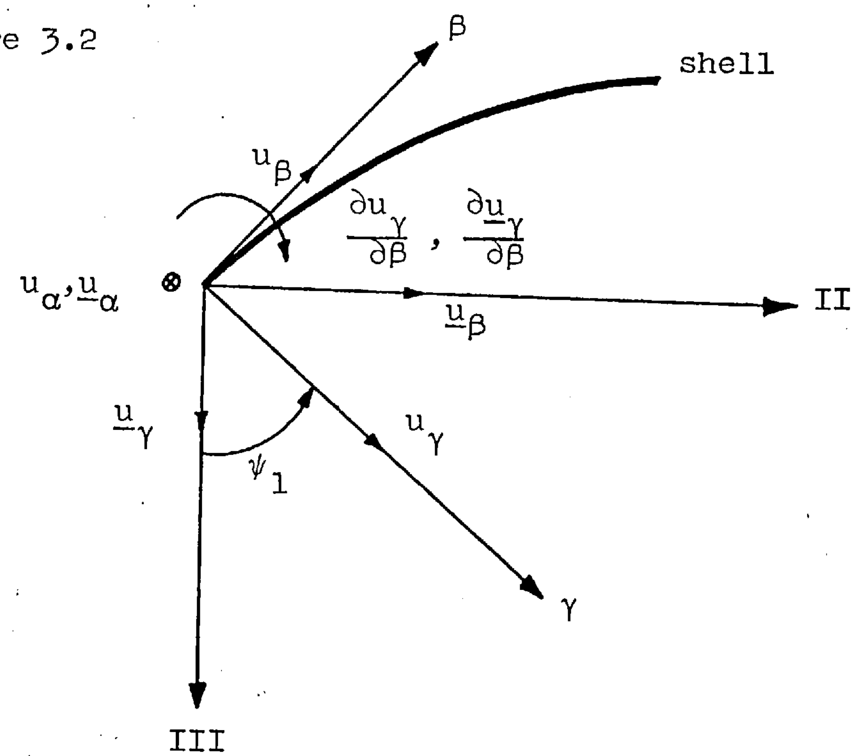


Figure 3.3

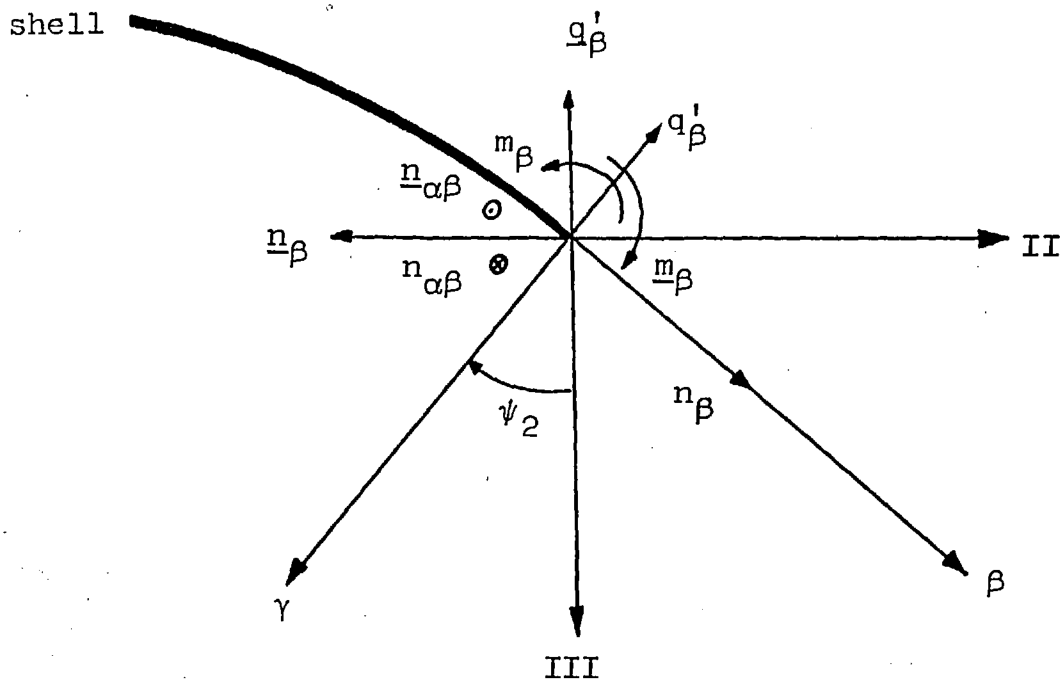


Figure 3.4

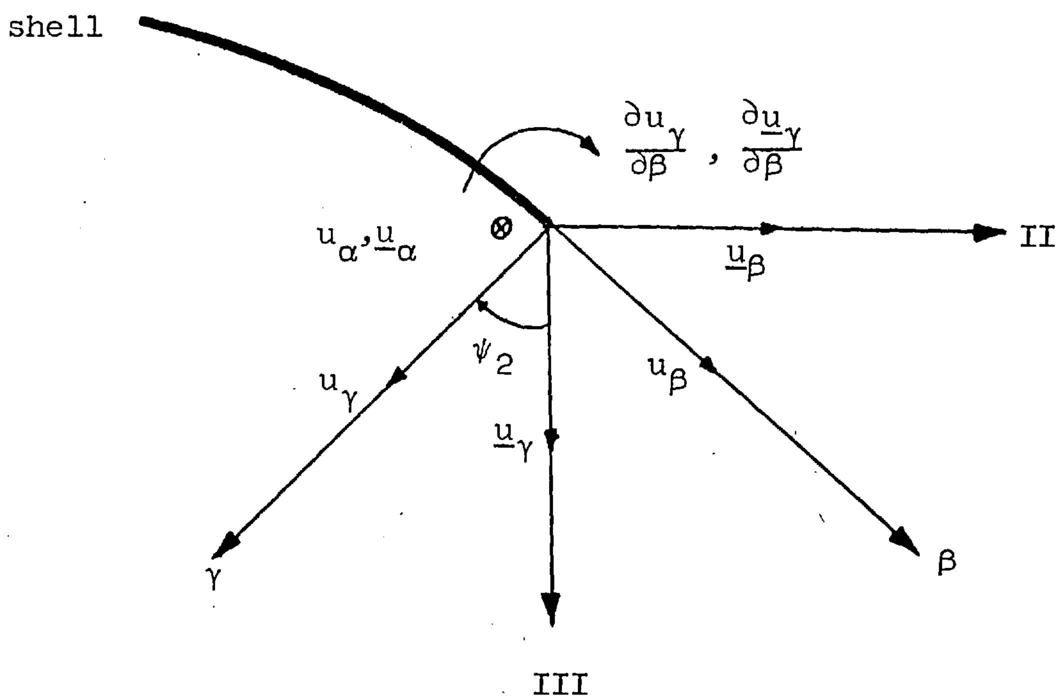


Figure 3.5

Junction axes

The origin is at a point of intersection of the shell and a transverse section of the edge beam. The axes are taken parallel to the beam axes.

Rotational transformation of shell actions and displacements from shell axes to junction axes

Edge 1

Let the shell actions be $n'_{\alpha\beta}$, q'_{β}, n_{β} and m_{β} in shell axes and $\underline{n}'_{\alpha\beta}$, $\underline{q}'_{\beta}$, $\underline{n}_{\beta}$ and $\underline{m}_{\beta}$ in junction axes (figure 3.2).

Then $\underline{x}^1 = +H^1 \; x^1$ (3.77)

$$\text{where } \underline{x}^1 = \begin{bmatrix} \underline{n}'_{\alpha\beta} \\ \underline{q}'_{\beta} \\ \underline{n}_{\beta} \\ \underline{m}_{\beta} \end{bmatrix} \; ; \qquad x^1 = \begin{bmatrix} n'_{\alpha\beta} \\ q'_{\beta} \\ n_{\beta} \\ m_{\beta} \end{bmatrix} \tag{3.78}$$

$$\text{and } H^1 = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & +\cos(\psi_1) & -\sin(\psi_1) & 0 \\ 0 & +\sin(\psi_1) & +\cos(\psi_1) & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \tag{3.79}$$

where ψ_1 = the anticlockwise angle measured from the positive direction of beam axis III to the positive direction of shell axis γ at a point of intersection of the shell and a transverse section of the beam 1 (figure 3.2).

Let the shell displacements be u_α, u_γ, u_β, $\frac{\partial u_\gamma}{\partial \beta}$ in shell axes and $\underline{u}_\alpha$, $\underline{u}_\gamma$, $\underline{u}_\beta$, $\frac{\partial \underline{u}_\gamma}{\partial \beta}$ in junction axes (figure 3.3).

Then $\underline{u}^1 = +H^1\, u^1$ (3.80)

$$\text{where } \underline{u}^1 = \begin{bmatrix} \underline{u}_\alpha \\ \underline{u}_\gamma \\ \underline{u}_\beta \\ \frac{\partial \underline{u}_\gamma}{\partial \beta} \end{bmatrix} \; ; \qquad u^1 = \begin{bmatrix} u_\alpha \\ u_\gamma \\ u_\beta \\ \frac{\partial \underline{u}_\gamma}{\partial \beta} \end{bmatrix} \tag{3.81}$$

and matrix H^1 is defined by (3.79).

Edge 2

Along edge 2 (figures 3.4 and 3.5):-

$$\underline{x}^2 = -H^2\, x^2 \tag{3.82}$$

$$\underline{u}^2 = +H^2\, u^2 \tag{3.83}$$

$$\text{where } H^2 = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & +\cos(\psi_2) & +\sin(\psi_2) & 0 \\ 0 & -\sin(\psi_2) & +\cos(\psi_2) & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \tag{3.84}$$

and ψ_2 = the clockwise angle measured from the positive direction of beam axis III to the positive direction of shell axis γ at a point of intersection of the shell and a transverse section of beam 2 (figure 3.4).

Equations of equilibrium for edge beam 1 (or 2)

Let the components of external loading on the edge beam be Z_1, Z_2 and Z_3 per unit length of beam in directions I, II and III respectively.

The reactions from the shell on the beam are shown in figure 3.6.

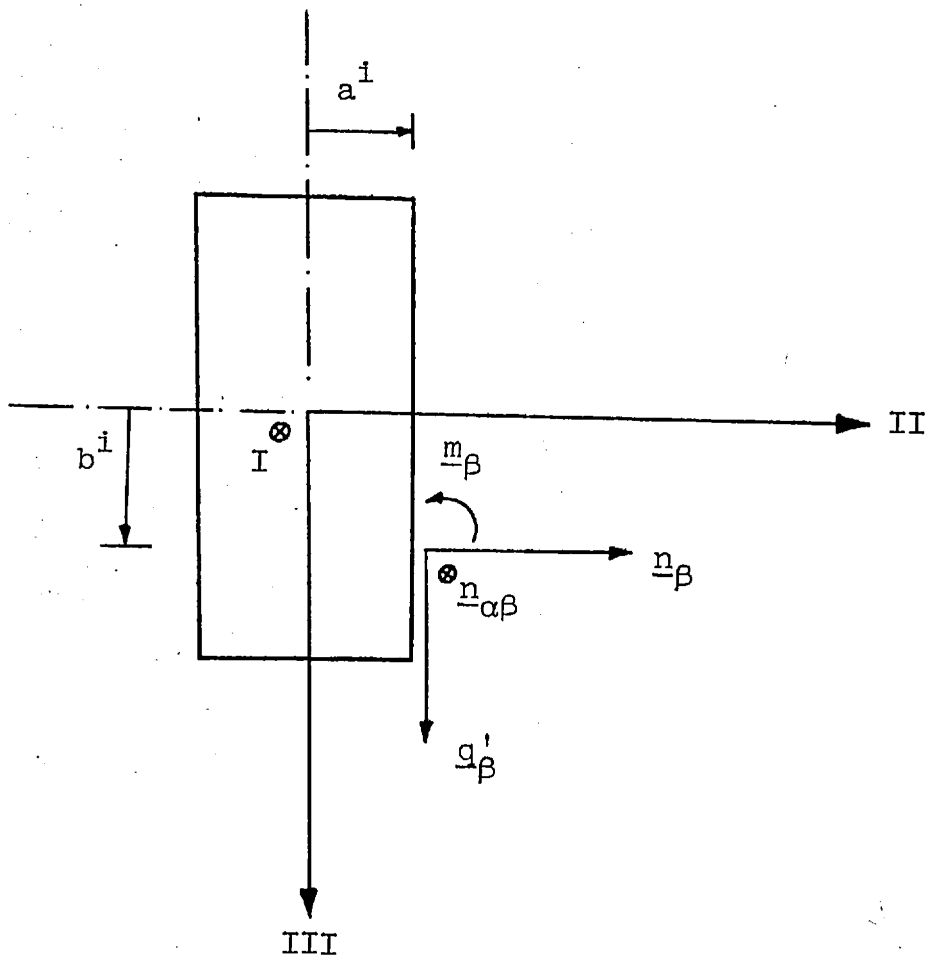
The equations of equilibrium are:-

$$
\begin{aligned}
&\frac{dN_1}{d\alpha} - k_\alpha S_1 + \underline{n}_{\alpha\beta} + Z_1 = 0 \\
&\frac{dS_1}{d\alpha} + k_\alpha N_1 + \underline{q}'_\beta + Z_3 = 0 \\
&\frac{dS_2}{d\alpha} + \underline{n}_\beta + Z_2 \qquad = 0 \\
&\frac{dM_1}{d\alpha} - S_1 + b^i\,\underline{n}_{\alpha\beta} \qquad = 0 \\
&k_\alpha M_3 + \frac{dM_2}{d\alpha} + S_2 - a^i\,\underline{n}_{\alpha\beta} = 0 \\
&- k_\alpha M_2 + \frac{dM_3}{d\alpha} - \underline{m}_\beta - b^i\underline{n}_\beta + a^i\underline{q}'_\beta = 0
\end{aligned}
\qquad (3.85)
$$

The positive directions of beam actions are shown in figure 3.7.

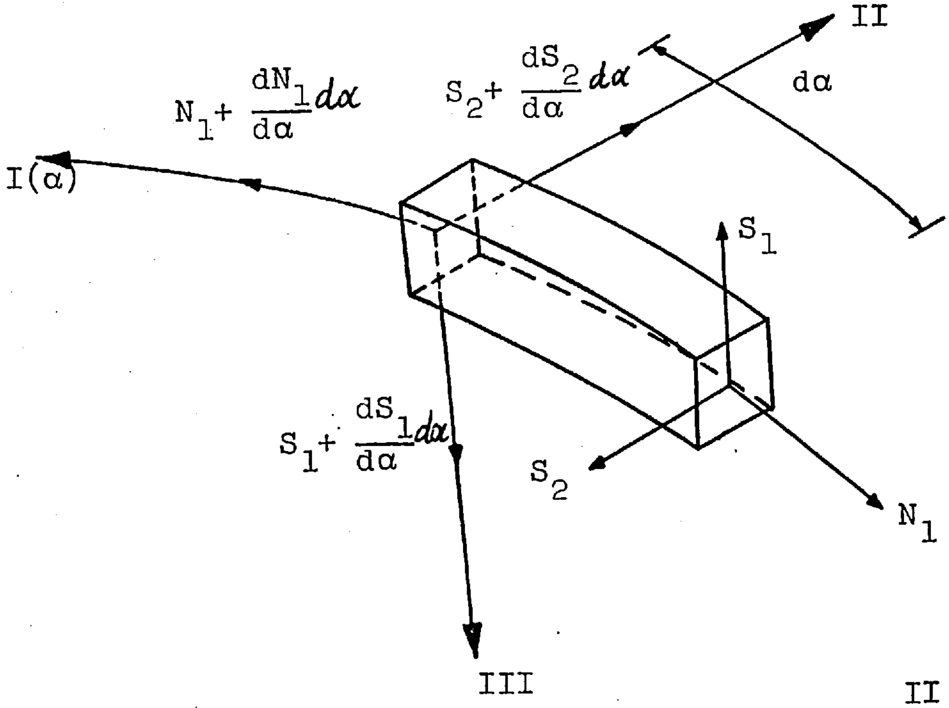
Superscript i takes the values 1 and 2 for edges beams 1 and 2 respectively.

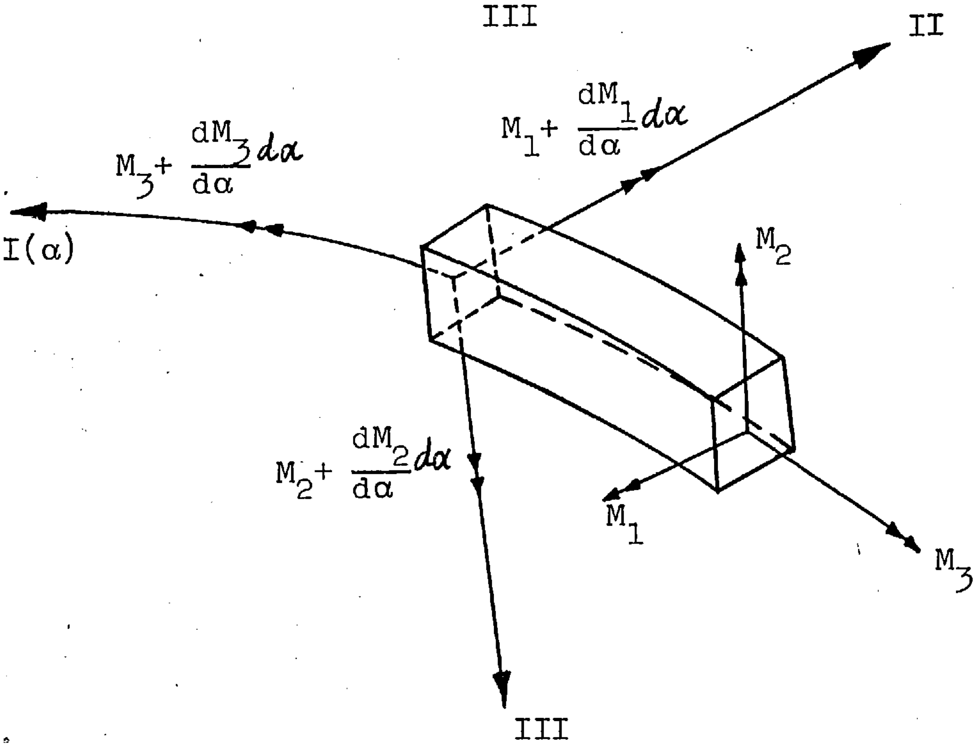
The reactions from the shell on the beam are in the form of half range trigonometric expansions. If the



a^i
b^i
I
II
III
$\underline{m}_\beta$
$\underline{n}_\beta$
$\underline{n}_{\alpha\beta}$
$\underline{q}'_\beta$

Figure 3.6





Beam actions (edges 1 and 2)

Figure 3.7

components of external loading on the beam are expanded into half range trigonometric series (refer appendix 1), a solution for the beam actions can be obtained, where:-

(i) N_1, M_1, M_2 are expressed as half range sine series,

(ii) S_1, S_2, M_3 are expressed as half range cosine series.

Edge beam complementary solution

The edge beam complementary solution is defined to be the solution corresponding to reactions on the beam due to complementary solution shell actions.

Putting $Z_1 = 0$, $Z_2 = 0$, $Z_3 = 0$ and eliminating S_1 and S_2 from equations (3.85):-

$$(\bar{X})_p^i = C_p^i \, (\underline{\bar{x}})_p^i \tag{3.86}$$

$$\text{where } (X)_p^i = \begin{bmatrix} N_1 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} \sin(p\bar{\alpha}) & & & \\ & \sin(p\bar{\alpha}) & & \\ & & \sin(p\bar{\alpha}) & \\ & & & \cos(p\bar{\alpha}) \end{bmatrix} (\bar{X})_p^i \tag{3.87}$$

$$(\underline{x})_p^i = \begin{bmatrix} \underline{n}'_{\alpha\beta} \\ \underline{q}'_{\beta} \\ \underline{n}_{\beta} \\ \underline{m}_{\beta} \end{bmatrix} = \begin{bmatrix} \cos(p\bar{\alpha}) & & & \\ & \sin(p\bar{\alpha}) & & \\ & & \sin(p\bar{\alpha}) & \\ & & & \sin(p\bar{\alpha}) \end{bmatrix} (\underline{\bar{x}})_p^i \tag{3.88}$$

$$\text{and } C_p^i = \frac{1}{(m^2 - k_\alpha{}^2)} \begin{bmatrix} c_{11} & c_{12} & 0 & 0 \\ c_{21} & c_{22} & 0 & 0 \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix} \tag{3.89}$$

where $c_{11} = -m$ $\qquad c_{12} = +k_\alpha$

$$c_{21} = +b^i \left[\frac{k_\alpha{}^2}{m} - m\right] - \frac{k_\alpha}{m} \qquad c_{22} = +1$$

$$c_{31} = +ma^i \qquad c_{32} = -k_\alpha a^i$$

$$c_{33} = +k_\alpha b^i - 1 \qquad c_{34} = +k_\alpha$$

$$c_{41} = -k_\alpha a^i \qquad c_{42} = +ma^i$$

$$c_{43} = + \frac{k_\alpha}{m} - mb^i \qquad c_{44} = -m$$

Superscript i refers to the edge and the subscript p to the harmonic considered.

Equation (3.86) gives a set of beam actions in beam axes, resulting from a set of shell actions in junction axes.

From equations (3.77) and (3.82):-

$$\begin{aligned} (\underline{\overline{x}})_p^1 &= +H^1(\overline{x})_p^1 \\ (\underline{\overline{x}})_p^2 &= -H^2(\overline{x})_p^2 \end{aligned} \tag{3.90}$$

where matrices $(\overline{x})_p^1$ and $(\overline{x})_p^2$ are defined by equations (3.64) and (3.67) respectively for Levy solutions of type 1.

Edge beam particular solution

The edge beam particular solution consists of the algebraic sum of solutions due to:-

(i) external loading on edge beam,

(ii) reactions on edge beam due to particular solution shell actions.

Let the external loading on the beam be expressed in the form (refer appendix 1):-

$$
\begin{aligned}
Z_1 &= 0^{*} \\
Z_2 &= z_2{}^{p} \sin(p\bar{\alpha}) \\
Z_3 &= z_3{}^{p} \sin(p\bar{\alpha})
\end{aligned}
\tag{3.91}
$$

The edge beam particular solution corresponding to the p^{th} harmonic is given by:-

$$(\bar{X}^{o})^{i}_{p} = (\bar{E}^{o})^{i}_{p} + C^{i}_{p}\ (\underline{\bar{x}}^{o})^{i}_{p} \tag{3.92}$$

where $$(\bar{E}^{o})^{i}_{p} = \frac{1}{(m^2-k_\alpha{}^2)} \begin{bmatrix} +k_\alpha z_3{}^{p} \\ +z_3{}^{p} \\ -z_2{}^{p} \\ +\frac{k_\alpha}{m}\ z_2{}^{p} \end{bmatrix} \tag{3.93}$$

and matrix C^{i}_{p} is defined by equation (3.89)

---

*Axial loadings on the edge beam are not considered.

From equations (3.77) and (3.82):-

$$(\underline{\bar{x}}^o)^1_p = + H^1(\bar{x}^o)^1_p$$
$$(\underline{\bar{x}}^o)^2_p = - H^2\ (\bar{x}^o)^2_p \tag{3.94}$$

where matrices $(\bar{x}^o)^1_p$ and $(\bar{x}^o)^2_p$ are defined by equations (3.19) and (3.21) respectively.

Edge beam action-displacement relations

The analysis follows on similar lines to that used by Aass(11) and Powell(10). The edge beam action-displacement relations are given by:-

$$N_1 = +E_b A^i \left[\frac{dU_1}{d\alpha} - k_\alpha\ U_3\right] - E_b I_1{}^i\ k_\alpha \left[k_\alpha{}^2 U_3 + \frac{d^2U_3}{d\alpha^2}\right]$$

$$M_1 = -E_b I_1{}^i \left[\frac{d^2U_3}{d\alpha^2} + k_\alpha{}^2\ U_3\right]$$

$$M_2 = +E_b I_2{}^i \left[\frac{d^2U_2}{d\alpha^2} - k_\alpha\ U_4\right] \tag{3.95}$$

$$M_3 = + \frac{E_b}{2}\ I_3{}^i \left[\frac{dU_4}{d\alpha} + k_\alpha \frac{dU_2}{d\alpha}\right]$$

where U_1, U_2, U_3, U_4 are the edge beam displacements (figure 3.1) and

E_b = Young's modulus of the edge beam material.

A^i = Area of a transverse section of beam i.

$I_1{}^i$ = Second moment of area of a transverse section of beam i about the beam axis II.

$I_2^{\ i}$ = Second moment of area of a transverse section of beam i about the beam axis III.

$I_3^{\ i}$ = 'Twisting moment of area' of a transverse section of beam i.

Timoshenko and Goodier[38] have tabulated values of I_3 for rectangular cross sections.

$$I_3 = 12k\ I_2$$

where k is a numerical factor depending on the (breadth to depth) ratio of the beam. Several values of this factor are listed in table 3.5.[38]

Table 3.5

c	k
1.0	0.1406
1.5	0.196
2.0	0.229
2.5	0.249
3.0	0.263
4.0	0.281
5.0	0.291
10.0	0.312
∞	0.333

$$\frac{1}{c} = \frac{\text{breadth of beam}}{\text{depth of beam}}$$

Equations (3.95) can be written in matrix form:-

$$(\bar{X})_p^i = (S)_p^i (\bar{U})_p^i \tag{3.96}$$

where
$$(U)_p^i = \begin{bmatrix} U_1 \\ U_3 \\ U_2 \\ U_4 \end{bmatrix} = \begin{bmatrix} \cos(p\bar{\alpha}) & & & \\ & \sin(p\bar{\alpha}) & & \\ & & \sin(p\bar{\alpha}) & \\ & & & \sin(p\bar{\alpha}) \end{bmatrix} (\bar{U})_p^i \tag{3.97}$$

$$(S)_p^i = E_b \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \tag{3.98}$$

where $a_{11} = -A^i m$ $\qquad$ $a_{12} = -I_1{}^i k_\alpha \left[k_\alpha{}^2 - m^2\right] - A^i k_\alpha$

$a_{22} = +I_1{}^i \left[m^2 - k_\alpha{}^2\right]$ $\qquad$ $a_{33} = -I_2{}^i\, m^2$

$a_{34} = -I_2{}^i k_\alpha$

$\qquad\qquad\qquad$ $a_{43} = +\dfrac{I_3{}^i}{2}\, m k_\alpha$

$a_{44} = +\dfrac{I_3}{2}\, m$

and matrix $(\bar{X})_p^i$ is defined by equation (3.87). From equation (3.97) it is seen that U_2, U_3 and U_4 are expressed as half range sine series, and U_1 is expressed as a half range cosine series.

<u>Translatory transformation of edge beam displacements from beam axes to junction axes</u>

Let the edge beam displacements be U_1, U_2, U_3 and U_4 in beam axes and $\underline{U}_1$, $\underline{U}_2$, $\underline{U}_3$ and $\underline{U}_4$ in junction axes.

Then $\quad \underline{U}^i = D^i\, U^i \qquad (3.99)$

$$\text{where } \underline{U}^i = \begin{bmatrix} \underline{U}_1 \\ \underline{U}_3 \\ \underline{U}_2 \\ \underline{U}_4 \end{bmatrix} ; \quad U^i = \begin{bmatrix} U_1 \\ U_3 \\ U_2 \\ U_4 \end{bmatrix} \qquad (3.100)$$

$$\text{where } D^i = \begin{bmatrix} +(1-k_\alpha b^i) & -mb^i & -ma^i & 0 \\ 0 & +1 & 0 & +a^i \\ 0 & 0 & +1 & -b^i \\ 0 & 0 & 0 & +1 \end{bmatrix} \qquad (3.101)$$

<u>Summary of results</u>

<u>Edge 1</u>

$$\begin{aligned} \underline{x}^1 &= + H^1\, x^1 \\ \underline{u}^1 &= + H^1\, u^1 \end{aligned} \qquad (3.102)$$

<u>Edge 2</u>

$$\begin{aligned} \underline{x}^2 &= - H^2\, x^2 \\ \underline{u}^2 &= + H^2\, u^2 \end{aligned} \qquad (3.103)$$

Edges 1 and 2

$$(\bar{X})^i_p = (C)^i_p \, (\underline{\bar{x}})^i_p \tag{3.104}$$

$$(\bar{X})^i_p = (S)^i_p \, (\bar{U})^i_p \tag{3.105}$$

$$\underline{U}^i = D^i \, \bar{U}^i \tag{3.106}$$

$$(\bar{X}^o)^i_p = (\bar{E}^o)^i_p + (C)^i_p \, (\underline{\bar{x}}^o)^i_p \tag{3.107}$$

From equation (3.105):-

$$(\bar{U})^i_p = \left[(S)^i_p\right]^{-1} (\bar{X})^i_p \tag{3.108}$$

Substituting (3.108) in (3.104):-

$$(\bar{U})^i_p = \left[(S)^i_p\right]^{-1} (C)^i_p \, (\underline{\bar{x}})^i_p \tag{3.109}$$

From (3.106):-

$$(\underline{\bar{U}})^i_p = D^i \left[(S)^i_p\right]^{-1} (C)^i_p \, (\underline{\bar{x}})^i_p \tag{3.110}$$

'Gap' between shell and edge beam

In the method of solution employed, the compatibility of displacements of the shell and edge beam is, in general, not satisfied by each individual solution forming the total solution. The algebraic difference between the shell displacements and the beam displacements (at the intersection of shell and beam), measured in junction axes, is referred to as the 'gap' between shell and beam.

Particular solution gaps between shell and beams

Edge 1

$$(\underline{\overline{V}}^{o})^{1}_{p} = (\underline{\overline{u}}^{o})^{1}_{p} - (\underline{\overline{U}}^{o})^{1}_{p}$$

$$= +H^{1}(\overline{u}^{o})^{1}_{p} - D^{1}\left[(S)^{1}_{p}\right]^{-1}\left[(\overline{E}^{o})^{1}_{p} + (C)^{1}_{p}H^{1}(\overline{x}^{o})^{1}_{p}\right] \qquad (3.111)$$

Edge 2

$$(\underline{\overline{V}}^{o})^{2}_{p} = (\underline{\overline{u}}^{o})^{2}_{p} - (\underline{\overline{U}}^{o})^{2}_{p}$$

$$= +H^{2}(\overline{u}^{o})^{2}_{p} - D^{2}\left[(S)^{2}_{p}\right]^{-1}\left[(\overline{E}^{o})^{2}_{p} - (C)^{2}_{p}H^{2}(\overline{x}^{o})^{2}_{p}\right] \qquad (3.112)$$

Complementary solution gaps between shell and beams

Edge 1

$$(\underline{\overline{V}}^{1})^{1}_{p} = +H^{1}(\overline{u}^{1})^{1}_{p} - D^{1}\left[(S)^{1}_{p}\right]^{-1}(C)^{1}_{p}H^{1}(\overline{x}^{1})^{1}_{p} \qquad (3.113)$$

Edge 2

$$(\underline{\overline{V}}^{1})^{2}_{p} = +H^{2}(\overline{u}^{1})^{2}_{p} + D^{2}\left[(S)^{2}_{p}\right]^{-1}(C)^{2}_{p}H^{2}(\overline{x}^{1})^{2}_{p} \qquad (3.114)$$

Levy solutions type 1 with flexible beams along shell edges 1 and 2

The boundary conditions to be satisfied along edges 1 and 2 are:-

$$\begin{aligned} (\underline{\overline{V}}^{1})^{1}_{p} + (\underline{\overline{V}}^{o})^{1}_{p} &= 0 \\ (\underline{\overline{V}}^{1})^{2}_{p} + (\underline{\overline{V}}^{o})^{2}_{p} &= 0 \end{aligned} \qquad (3.115)$$

$$\begin{aligned} \text{Let } F^{1}_{p} &= +D^{1}\left[(S)^{1}_{p}\right]^{-1}(C)^{1}_{p}H^{1} \\ F^{2}_{p} &= -D^{2}\left[(S)^{2}_{p}\right]^{-1}(C)^{2}_{p}H^{2} \end{aligned} \qquad (3.116)$$

Using equations (3.64), (3.67), (3.113), (3.114) and (3.116) equations (3.115) can be written[x]:-

$$\begin{bmatrix} (H^1A_3-F_p{}^1A_1) & (H^1A_4-F_p{}^1A_2)L \\ (H^2A_3-F_p{}^2A_1)L & (H^2A_4-F_p{}^2A_2) \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = - \begin{bmatrix} (\underline{\overline{V}}^o)_p^1 \\ (\underline{\overline{V}}^o)_p^2 \end{bmatrix} \qquad (3.117)$$

The eight arbitrary constants represented by matrices f and g can be determined from equation (3.117).

II.3.2. Flexible beams along shell edges 3 and 4

The results for this case can be directly obtained from section III.3.1 by making the following changes:-

(Change α into β) $\quad \alpha \rightarrow \beta$

$\beta \rightarrow \alpha$

$p \rightarrow q$

superscript $i \rightarrow j \quad (j = 3,4)$

Edge $1 \rightarrow$ Edge 3

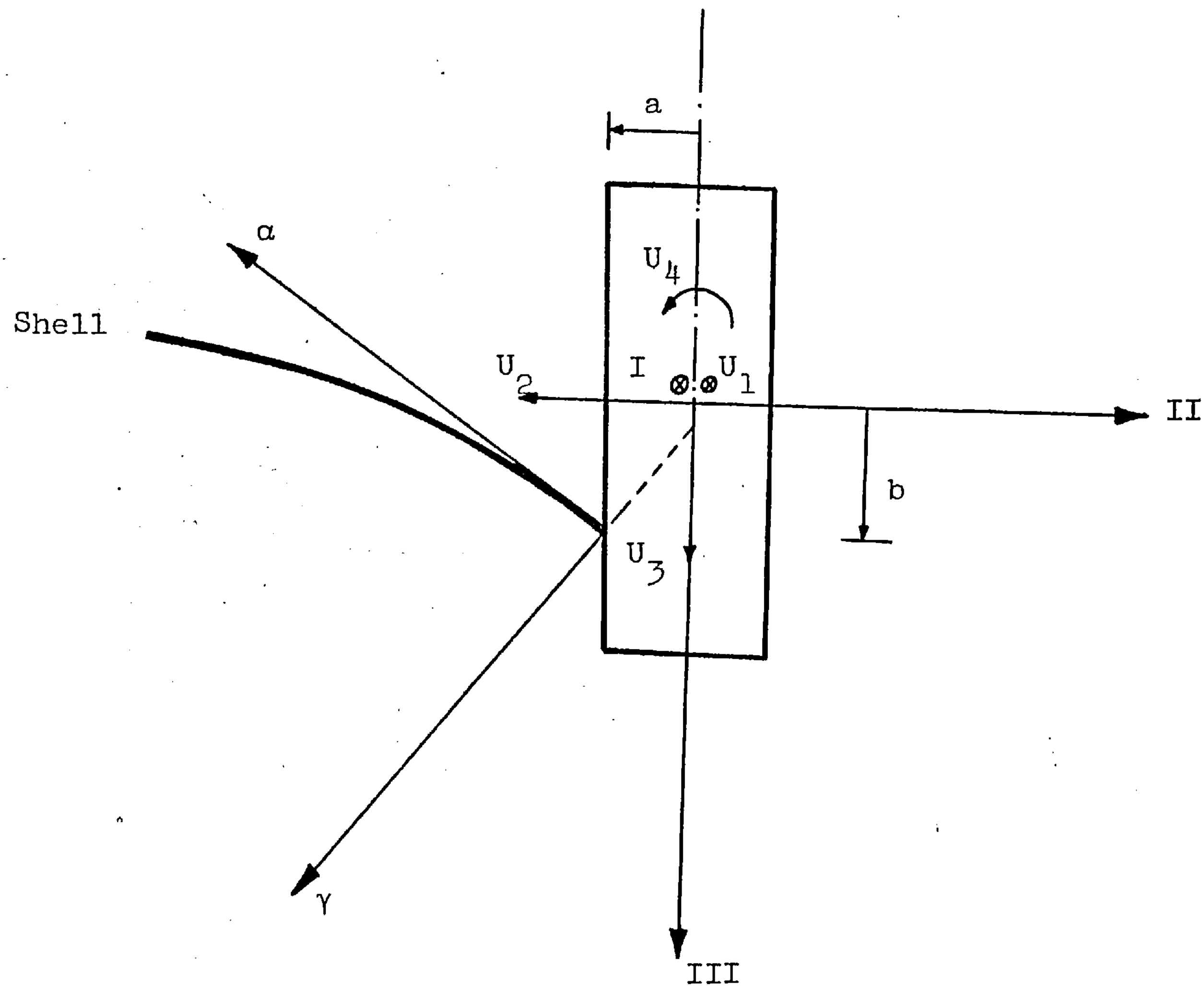
Edge $2 \rightarrow$ Edge 4

and where

ψ_3 = Clockwise angle measured from the positive direction of beam axis III to the positive direction of shell axis γ at a point of intersection of the shell and a transverse section of the beam 3.

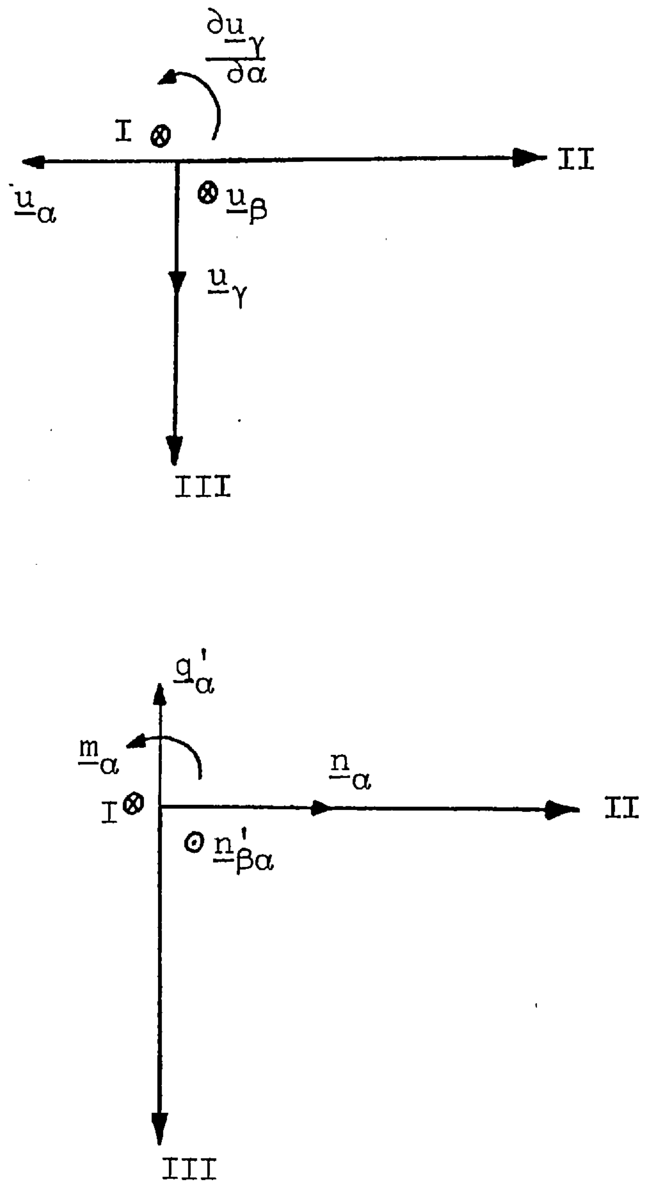
---

[x]Compare equation (3.117) with equation (3.61)



Beam axes and displacements (edges 3 and 4)

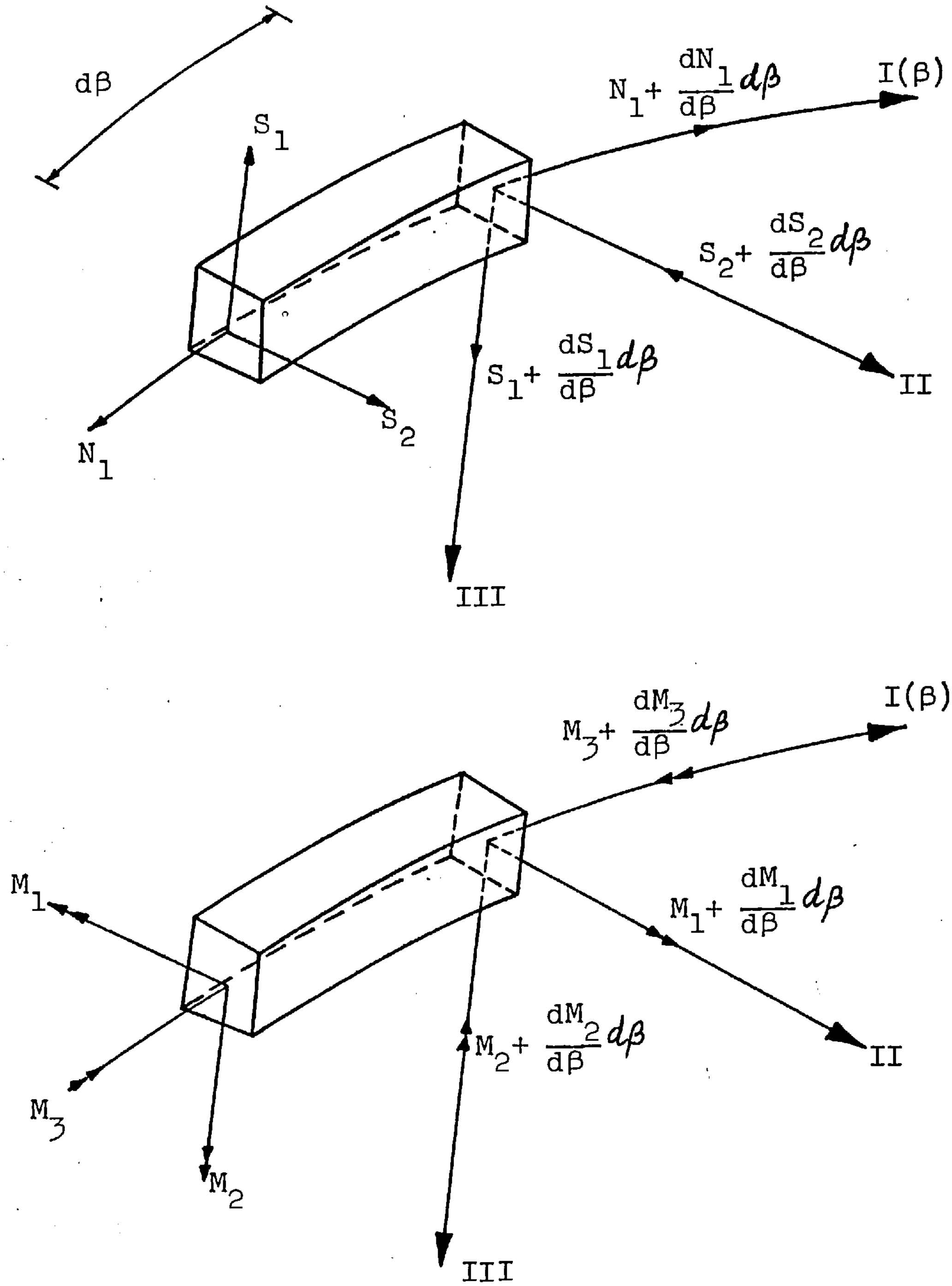
Figure 3.8



Shell actions and displacements in junction axes

(edges 3 and 4)

Figure 3.9



Beam actions (edges 3 and 4)

Figure 3.10

ψ_4 = anticlockwise angle measured from the positive direction of beam axis III to the positive direction of shell axis γ at a point of intersection of the shell and a transverse section of the beam 4.

The positive directions of the beam axes and beam displacements, shell actions and displacements in junction axes, beam actions, are shown in figures 3.8, 3.9 and 3.10 respectively.

CHAPTER IV

RULED SURFACES - AN EXTENDED LEVY METHOD OF SOLUTION

The solution follows on similar lines to that for translational shells (Chapter III).

IV.1. Shell particular solution

A Navier type particular solution is used. The external loads acting on the shell are expanded into double Fourier series of the form (refer appendix 5):-

$$
\begin{aligned}
G_\alpha &= \sum_{p=o}^{\infty} \sum_{q=o}^{\infty} g_\alpha^{pq} \sin(p\bar{\alpha}) \cos(q\bar{\beta}) \\
G_\beta &= \sum_{p=o}^{\infty} \sum_{q=o}^{\infty} g_\beta^{pq} \cos(p\bar{\alpha}) \sin(q\bar{\beta}) \qquad (4.1) \\
G_\gamma &= \sum_{p=o}^{\infty} \sum_{q=o}^{\infty} g_\gamma^{pq} \sin(p\bar{\alpha}) \sin(q\bar{\beta})
\end{aligned}
$$

where p,q are positive integers

g_α^{pq}, g_β^{pq}, g_γ^{pq} are constants

and $\bar{\alpha}$, $\bar{\beta}$ are defined by equations (3.2).

In practice the loads are expressed in the form of truncated series. Let the maximum values of p and q considered be r and s respectively.

All particular solution shell quantities Y^o can be written in the form:-

$$Y^o = \sum_{p=o}^{r} \sum_{q=o}^{s} c_k^{pq} F(p\bar{\alpha}) F(q\bar{\beta}) \qquad (4.2)$$

where c_k^{pq} is a constant

and F( ) = cos( ) or sin( ).

The results are listed in tables 4.1 and 4.2, where

$$a^{pq} = \frac{1}{D(m^2+n^2)^4+4tm^2n^2k_{\alpha\beta}{}^2E}\left[(m^2+n^2)^2 g_\gamma{}^{pq}\right.$$

$$\left. - 2mnk_{\alpha\beta}\left[\left(\frac{n^2}{m} - \nu m\right)g_\alpha{}^{pq} + \left(\frac{m^2}{n} - \nu n\right)g_\beta{}^{pq}\right]\right] \qquad (4.3)$$

$$b^{pq} = \frac{1}{(m^2+n^2)^2}\left[2Etmnk_{\alpha\beta}a^{pq} + \left[\frac{n^2}{m} - \nu m\right] g_\alpha{}^{pq} + \left[\frac{m^2}{n} - \nu n\right] g_\beta{}^{pq}\right] \qquad (4.4)$$

All coefficients c_k not listed in table 4.2 are zero when p = 0 or q = 0.

Also $q_\alpha' = q_\alpha + \dfrac{\partial m_{\alpha\beta}}{\partial\beta}$

$$q_\beta' = q_\beta + \frac{\partial m_{\alpha\beta}}{\partial\alpha} \qquad (4.5)$$

$$n_{\alpha\beta}' = n_{\alpha\beta}$$

$$n_{\beta\alpha}' = n_{\alpha\beta}$$

Table 4.1

Y^{o}	c_k	$F(p\bar{\alpha})$	$F(q\bar{\beta})$	c_k^{pq} $(p \neq 0,\ q \neq 0)$
u_γ	c_1	$\mathrm{Sin}(p\bar{\alpha})$	$\mathrm{Sin}(q\bar{\beta})$	$+a^{pq}$
m_α	c_2	$\mathrm{Sin}(p\bar{\alpha})$	$\mathrm{Sin}(q\bar{\beta})$	$+Da^{pq}\left[m^2 + \nu n^2\right]$
$m_{\alpha\beta}$	c_3	$\mathrm{Cos}(p\bar{\alpha})$	$\mathrm{Cos}(q\bar{\beta})$	$-Dmna^{pq}(1 - \nu)$
m_β	c_4	$\mathrm{Sin}(p\bar{\alpha})$	$\mathrm{Sin}(q\bar{\beta})$	$+Da^{pq}\left[n^2 + \nu m^2\right]$
n_α	c_{15}	$\mathrm{Cos}(p\bar{\alpha})$	$\mathrm{Cos}(q\bar{\beta})$	$- n^2 b^{pq} + \dfrac{g_\alpha^{\ pq}}{m}$
$n_{\alpha\beta}$	c_6	$\mathrm{Sin}(p\bar{\alpha})$	$\mathrm{Sin}(q\bar{\beta})$	$- mnb^{pq}$
n_β	c_{14}	$\mathrm{Cos}(p\bar{\alpha})$	$\mathrm{Cos}(q\bar{\beta})$	$- m^2 b^{pq} + \dfrac{g_\beta^{\ pq}}{n}$
u_α	c_9	$\mathrm{Sin}(p\bar{\alpha})$	$\mathrm{Cos}(q\bar{\beta})$	$+ \dfrac{1}{mEt}\left[c_{15}^{\ pq} - \nu c_{14}^{\ pq}\right]$
u_β	c_8	$\mathrm{Cos}(p\bar{\alpha})$	$\mathrm{Sin}(q\bar{\beta})$	$+ \dfrac{1}{nEt}\left[c_{14}^{\ pq} - \nu c_{15}^{\ pq}\right]$

Table 4.1 (continued)

Y^o	c_k	$F(p\bar{\alpha})$	$F(q\bar{\beta})$	c_k^{pq} $(p \neq 0,\ q \neq 0)$
q_α	c_{10}	$\mathrm{Cos}(p\bar{\alpha})$	$\mathrm{Sin}(q\bar{\beta})$	$+ m\, c_2^{pq} - n\, c_3^{pq}$
q_β	c_{11}	$\mathrm{Sin}(p\bar{\alpha})$	$\mathrm{Cos}(q\bar{\beta})$	$+ n\, c_4^{pq} - m\, c_3^{pq}$
q_α'	c_{12}	$\mathrm{Cos}(p\bar{\alpha})$	$\mathrm{Sin}(q\bar{\beta})$	$+ c_{10}^{pq} - n\, c_3^{pq}$
q_β'	c_{13}	$\mathrm{Sin}(p\alpha)$	$\mathrm{Cos}(q\bar{\beta})$	$+ c_{11}^{pq} - m\, c_3^{pq}$
$n_{\alpha\beta}'$	c_7	$\mathrm{Sin}(p\bar{\alpha})$	$\mathrm{Sin}(q\bar{\beta})$	$+ c_6^{pq}$
$n_{\beta\alpha}'$	c_5	$\mathrm{Sin}(p\bar{\alpha})$	$\mathrm{Sin}(q\bar{\beta})$	$+ c_6^{pq}$
$\frac{\partial u_\gamma}{\partial \alpha}$	c_{16}	$\mathrm{Cos}(p\bar{\alpha})$	$\mathrm{Sin}(q\bar{\beta})$	$+ m a^{pq}$
$\frac{\partial u_\gamma}{\partial \beta}$	c_{17}	$\mathrm{Sin}(p\bar{\alpha})$	$\mathrm{Cos}(q\bar{\beta})$	$+ n a^{pq}$

Table 4.2

Y^o	c_k	c_k^{pq}	
		$p = 0,\ q \neq 0$	$p \neq 0,\ q = 0$
n_α	c_{15}	$+\dfrac{\nu g_\beta^{oq}}{n}$	$+\dfrac{g_\alpha^{po}}{m}$
n_β	c_{14}	$+\dfrac{g_\beta^{oq}}{n}$	$+\dfrac{\nu g_\alpha^{po}}{m}$
u_α	c_9	0	$+\dfrac{(1-\nu^2)}{m^2Et}\, g_\alpha^{po}$
u_β	c_8	$+\dfrac{(1-\nu^2)}{n^2Et}\, g_\beta^{oq}$	0

Matrices defining shell actions and displacements along the edges

The details are the same as for translational shells (Chapter III) where $(x^o)^i_p$, $(u^o)^i_p$, $(x^o)^j_q$ and $(u^o)^j_q$ are redefined as :-

$$(x^o)^i_p = \begin{bmatrix} n_\beta \\ q'_\beta \\ n'_{\alpha\beta} \\ m_\beta \end{bmatrix} ; \quad (u^o)^i_p = \begin{bmatrix} u_\beta \\ u_\gamma \\ u_\alpha \\ \frac{\partial u_\gamma}{\partial \beta} \end{bmatrix} \tag{4.6}$$

$$(x^o)^j_q = \begin{bmatrix} n_\alpha \\ q'_\alpha \\ n'_{\beta\alpha} \\ m_\alpha \end{bmatrix} ; \quad (u^o)^j_q = \begin{bmatrix} u_\alpha \\ u_\gamma \\ u_\beta \\ \frac{\partial u_\gamma}{\partial \alpha} \end{bmatrix} \tag{4.7}$$

and the coefficient c_k are listed in tables 4.1 and 4.2.

IV.2. Shell complementary solution

IV.2.1. Levy solutions type 1

A solution for u_γ is sought in the form:-

$$u_\gamma = \bar{u}_\gamma(\beta)\ \sin(p\bar{\alpha}) \tag{4.8}$$

where $\bar{u}_\gamma(\beta)$ is a function of β.

When $G_\alpha = 0$, $G_\beta = 0$, $G_\gamma = 0$ equation (2.25) reduces to:-

$$D\nabla^8 u_\gamma + 4Etk_{\alpha\beta}^2 \frac{\partial^4 u_\gamma}{\partial\alpha^2\partial\beta^2} = 0 \qquad (4.9)$$

Substituting (4.8) in (4.9)

$$\left[\left[-m^2 + \frac{d^2}{d\beta^2}\right]^4 - \frac{4Et}{D} m^2 k_{\alpha\beta}^2 \frac{d^2}{d\beta^2}\right] \bar{u}_\gamma(\beta) = 0 \qquad (4.10)$$

$$\text{Let } \bar{u}_\gamma(\beta) = C\, e^{\rho\beta} \qquad (4.11)$$

Substituting (4.11) in (4.10) yields the auxiliary equation for u_γ:-

$$(\rho^2 - m^2)^4 - k_4 m^2 \rho^2 = 0 \qquad (4.12)$$

$$\text{where } k_4 = \frac{4Et}{D} k_{\alpha\beta}^2 \qquad (4.13)$$

The eight roots of the auxiliary equation are (refer appendix 6):-

$$\rho = \pm\, \sigma_1 \;;\; \pm\, \sigma_2 \;;\; \pm\, (\sigma_3 \pm i\lambda_3) \qquad (4.14)$$

where σ_1, σ_2, σ_3, λ_3 and i are defined in appendix 6.

The solutions for u_γ and ϕ (refer appendix 7):-

$$u_\gamma = \sin(p\bar{\alpha}) \Big[\mu_1 e^{-\sigma_1\beta} + \mu_2 e^{-\sigma_2\beta} + e^{-\sigma_3\beta}\left[\mu_3\cos(\lambda_3\beta)+\mu_4\sin(\lambda_3\beta)\right]$$
$$+ \mu_5 e^{-\sigma_1 z} + \mu_6 e^{-\sigma_2 z} + e^{-\sigma_3 z}\left[\mu_7\cos(\lambda_3 z)+\mu_8\sin(\lambda_3 z)\right]\Big] \qquad (4.15)$$

and

$$\phi = -\frac{Dk_4^{1/2}}{2k_{\alpha\beta}}\cos(p\bar{\alpha})\Big[\mu_1 e^{-\sigma_1\beta}+\mu_2 e^{-\sigma_2\beta}+e^{-\sigma_3\beta}\left[-\mu_3\cos(\lambda_3\beta)-\mu_4\sin(\lambda_3\beta)\right]$$
$$-\mu_5 e^{-\sigma_1 z}-\mu_6 e^{-\sigma_2 z}+e^{-\sigma_3 z}\left[\mu_7\cos(\lambda_3 z)+\mu_8\sin(\lambda_3 z)\right]\Big] \tag{4.16}$$

where $\mu_1, \mu_2, \ldots, \mu_8$ are arbitrary constants

and $z = l_\beta - \beta$ (4.17)

Using the relations given by (3.38) and (3.39) all shell quantities Y^l can be written in the form:-

$$Y^l = F(p\bar{\alpha})\Big[a_1\mu_1 e^{-\sigma_1\beta}+a_2\mu_2 e^{-\sigma_2\beta}$$
$$+e^{-\sigma_3\beta}\left[(a_3\mu_3+a_4\mu_4)\cos(\lambda_3\beta)+(a_3\mu_4-a_4\mu_3)\sin(\lambda_3\beta)\right]$$
$$+a_5\mu_5 e^{-\sigma_1 z}+a_6\mu_6 e^{-\sigma_2 z}$$
$$+e^{-\sigma_3 z}\left[(a_7\mu_7+a_8\mu_8)\cos(\lambda_3 z)+(a_7\mu_8-a_8\mu_7)\sin(\lambda_3 z)\right]\Big] \tag{4.18}$$

where $a_1, a_2, \ldots, a_8$ are constants

and $F(p\bar{\alpha}) = \sin(p\bar{\alpha})$ or $\cos(p\bar{\alpha})$.

Equation (4.18) can be written in matrix form:-

$$Y^1 = \left[AL^{\beta}f + BL^{z}g\right] F(p\bar{\alpha}) \tag{4.19}$$

$$\text{where } f = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} \qquad g = \begin{bmatrix} \mu_5 \\ \mu_6 \\ \mu_7 \\ \mu_8 \end{bmatrix} \tag{4.20}$$

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \tag{4.21}$$

$$B = c\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \tag{4.22}$$

$$L^{\beta} = \begin{bmatrix} +e^{-\sigma_1\beta} & 0 & 0 & 0 \\ 0 & +e^{-\sigma_2\beta} & 0 & 0 \\ 0 & 0 & +e^{-\sigma_3\beta}\cos(\lambda_3\beta) & +e^{-\sigma_3\beta}\sin(\lambda_3\beta) \\ 0 & 0 & -e^{-\sigma_3\beta}\sin(\lambda_3\beta) & +e^{-\sigma_3\beta}\cos(\lambda_3\beta) \end{bmatrix} \tag{4.23}$$

$$L^{z} = \begin{bmatrix} +e^{-\sigma_1 z} & 0 & 0 & 0 \\ 0 & +e^{-\sigma_2 z} & 0 & 0 \\ 0 & 0 & +e^{-\sigma_3 z}\cos(\lambda_3 z) & +e^{-\sigma_3 z}\sin(\lambda_3 z) \\ 0 & 0 & -e^{-\sigma_3 z}\sin(\lambda_3 z) & +e^{-\sigma_3 z}\cos(\lambda_3 z) \end{bmatrix} \tag{4.24}$$

The results are listed in tables 4.3 and 4.4.

In tables 4.3 and 4.4

$$k_5 = - \frac{Dk_4^{1/2}}{2\ k_{\alpha\beta}} \tag{4.25}$$

It can be shown that:-

(i) when the boundary conditions and loading on the shell are symmetric about the axis $\beta = \frac{l_\beta}{2}$, $f = g$,

(ii) when the boundary conditions are symmetric and the loading on the shell antimetric about the axis $\beta = \frac{l_\beta}{2}$, $f = -g$.

<u>Matrices defining shell actions and displacements along the edges</u>

The details are the same as for translational shells (Chapter III) where $(x^1)^i_p$, $(u^1)^i_p$, $(x^1)^j_p$ and $(u^1)^j_p$ are redefined as:-

$$(x^1)^i_p = \begin{bmatrix} n_\beta \\ q'_\beta \\ n_{\alpha\beta} \\ m_\beta \end{bmatrix} ; \qquad (u^1)^i_p = \begin{bmatrix} u_\beta \\ u_\gamma \\ u_\alpha \\ \frac{\partial u_\gamma}{\partial \beta} \end{bmatrix} \tag{4.26}$$

Table 4.3

Y^1	$F(p\bar{\alpha})$	c	a_1	a_2
u_γ	$\mathrm{Sin}(p\bar{\alpha})$	+1	$+1$	$+1$
m_α	$\mathrm{Sin}(p\bar{\alpha})$	+1	$+D\left[m^2-\nu\sigma_1^{\,2}\right]$	$+D\left[m^2-\nu\sigma_2^{\,2}\right]$
$m_{\alpha\beta}$	$\mathrm{Cos}(p\bar{\alpha})$	-1	$+D(1-\nu)m\sigma_1$	$+D(1-\nu)m\sigma_2$
m_β	$\mathrm{Sin}(p\bar{\alpha})$	+1	$+D\left[\nu m^2-\sigma_1^{\,2}\right]$	$+D\left[\nu m^2-\sigma_2^{\,2}\right]$
n_α	$\mathrm{Cos}(p\bar{\alpha})$	-1	$+k_5\ \sigma_1^{\,2}$	$+k_5\ \sigma_2^{\,2}$
$n_{\alpha\beta}$	$\mathrm{Sin}(p\bar{\alpha})$	+1	$-k_5\ m\ \sigma_1$	$-k_5\ m\ \sigma_2$
n_β	$\mathrm{Cos}(p\bar{\alpha})$	-1	$-k_5\ m^2$	$-k_5\ m^2$
u_α	$\mathrm{Sin}(p\bar{\alpha})$	-1	$+\dfrac{k_5}{mEt}\left[\sigma_1^{\,2}-\nu m^2\right]$	$+\dfrac{k_5}{mEt}\left[\sigma_2^{\,2}-\nu m^2\right]$
u_β	$\mathrm{Cos}(p\bar{\alpha})$	+1	$-\dfrac{k_5}{Et\sigma_1}\left[m^2+\nu\sigma_1^{\,2}\right]$	$-\dfrac{k_5}{Et\sigma_1}\left[m^2+\nu\sigma_2^{\,2}\right]$
q'_α	$\mathrm{Cos}(p\bar{\alpha})$	+1	$+Dm\left[m^2-(2-\nu)\sigma_1^{\,2}\right]$	$+Dm\left[m^2-(2-\nu)\sigma_2^{\,2}\right]$
q'_β	$\mathrm{Sin}(p\bar{\alpha})$	-1	$+D\sigma_1\left[\sigma_1^{\,2}-(2-\nu)m^2\right]$	$+D\sigma_2\left[\sigma_2^{\,2}-(2-\nu)m^2)\right]$
q_α	$\mathrm{Cos}(p\bar{\alpha})$	+1	$+Dm\left[m^2-\sigma_1^{\,2}\right]$	$+Dm\left[m^2-\sigma_2^{\,2}\right]$
q_β	$\mathrm{Sin}(p\bar{\alpha})$	-1	$+D\sigma_1\left[\sigma_1^{\,2}-m^2\right]$	$+D\sigma_2\left[\sigma_2^{\,2}-m^2\right]$
$\dfrac{\partial u_\gamma}{\partial\alpha}$	$\mathrm{Cos}(p\bar{\alpha})$	+1	$+m$	$+m$
$\dfrac{\partial u_\gamma}{\partial\beta}$	$\mathrm{Sin}(p\bar{\alpha})$	-1	$-\sigma_1$	$-\sigma_2$

Table 4.4

Y^1	a_3	a_4
u_γ	$+1$	0
m_α	$+ D\left[m^2-\nu(\sigma_3{}^2-\lambda_3{}^2)\right]$	$+ 2 D\nu\sigma_3\lambda_3$
$m_{\alpha\beta}$	$+ D(1-\nu)\ m\ \sigma_3$	$- D(1-\nu)m\ \lambda_3$
m_β	$+ D\left[\nu m^2-(\sigma_3{}^2-\lambda_3{}^2)\right]$	$+ 2\ D\ \sigma_3\ \lambda_3$
n_α	$- k_5\ (\sigma_3{}^2 - \lambda_3{}^2)$	$+ 2\ k_5\ \sigma_3\ \lambda_3$
$n_{\alpha\beta}$	$+ k_5\ m\ \sigma_3$	$- k_5\ m\ \lambda_3$
n_β	$+ k_5\ m^2$	0
u_α	$+ \frac{k_5}{mEt}\left[\nu m^2 - (\sigma_3{}^2-\lambda_3{}^2)\right]$	$+ \frac{2k_5}{mEt}\ \sigma_3\ \lambda_3$

Table 4.4 (continued)

Y^1	a_3	a_4
u_β	$+\dfrac{k_5\ \sigma_3}{Et(\sigma_3{}^2+\lambda_3{}^2)}\left[\nu(\sigma_3{}^2+\lambda_3{}^2)+m^2\right]$	$-\dfrac{k_5\ \lambda_3}{Et(\sigma_3{}^2+\lambda_3{}^2)}\left[\nu(\sigma_3{}^2+\lambda_3{}^2)-m^2\right]$
q'_α	$+\ Dm\left[m^2-\ (2-\nu)(\sigma_3{}^2-\lambda_3{}^2)\right]$	$+\ 2\ Dm(2-\nu)\ \sigma_3\ \lambda_3$
q'_β	$+\ D\sigma_3\left[\sigma_3{}^2-3\lambda_3{}^2-\ (2-\nu)m^2\right]$	$+\ D\lambda_3\left[\lambda_3{}^2-\ 3\sigma_3{}^2+\ (2-\nu)m^2\right]$
q_α	$+\ Dm\left[m^2\ -\ (\sigma_3{}^2\ -\ \lambda_3{}^2)\right]$	$+\ 2\ D\ m\ \sigma_3\ \lambda_3$
q_β	$+\ D\sigma_3\left[\sigma_3{}^2-\ 3\lambda_3{}^2\ -\ m^2\right]$	$+\ D\lambda_3\left[\lambda_3{}^2-\ 3\sigma_3{}^2\ +\ m^2\right]$
$\dfrac{\partial u_\gamma}{\partial \alpha}$	$+m$	0
$\dfrac{\partial u_\gamma}{\partial \beta}$	$-\sigma_3$	$+\lambda_3$

$$(x^1)^j_p = \begin{bmatrix} n_\alpha \\ q'_\alpha \\ n_{\alpha\beta} \\ m_\alpha \end{bmatrix} \quad ; \quad (u^1)^j_p = \begin{bmatrix} u_\alpha \\ u_\gamma \\ u_\beta \\ \dfrac{\partial u_\gamma}{\partial \alpha} \end{bmatrix} \tag{4.27}$$

and the matrices A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, $(\bar{x}^1)^3_q$, $(\bar{u}^1)^3_q$, $(\bar{x}^1)^4_q$ and $(\bar{u}^1)^4_q$ are defined in appendix 8.

IV.2.2. Levy solutions type 2

A solution for u_γ is sought in the form:-

$$u_\gamma = \bar{u}_\gamma(\alpha)\ \sin(q\bar{\beta}) \tag{4.28}$$

where $\bar{u}_\gamma(\alpha)$ is a function of α.

This solution can be directly obtained from the results for Levy solutions of type 1 (section IV.2.1) by making the following changes:-

(Change α into β) $\quad \alpha \rightarrow \beta$

$\beta \rightarrow \alpha$

$p \rightarrow q$

$q \rightarrow p$

$m \rightarrow n$

$n \rightarrow m$

superscript $i \rightarrow j$

superscript $j \rightarrow i$

Edge 1 —> Edge 3

Edge 2 —> Edge 4

Edge 3 —> Edge 1

Edge 4 —> Edge 2

and the first superscript associated with matrices x and u from 1 to 2.

CHAPTER V

TRANSLATIONAL SHELLS - CHECKS ON THE COMPUTER PROGRAMME AND SOLUTION OF CERTAIN PROBLEMS WHOSE RANGE EXTENDS BEYOND THOSE NORMALLY INCLUDED IN THE STANDARD LITERATURE

P,Q represent the complementary solution harmonics considered in the α and β directions respectively.

P', Q' represent the particular solution harmonics considered in the α and β directions respectively.

All computations are made using the arc lengths l_α and l_β unless otherwise stated.

α_1, β_1 are defined by:-

$$\alpha_1 = \frac{\alpha}{l_\alpha}$$

$$\beta_1 = \frac{\beta}{l_\beta}$$

V.1 Assumptions made in the solution of the shallow curved plate equations and their effect on the accuracy of the numerical results

The shallow curved plate equations have been solved making the following assumptions:-

(i) the representation of external loads and thermal effects on the shell and edge beams by truncated Fourier expansions,

(ii) the representation of certain complementary solution 'edge disturbances' by truncated Fourier expansions.

The first assumption affects only the particular solution and the second only the complementary solution. Hence, the influence of these assumptions on the accuracy of the numerical results can be studied separately.

Particular solution

Consider the following examples:-

Example 5.1. An elliptic paraboloid with the following data:-

$t = 0.208333$	$L_\alpha = 60$	$L_\beta = 80$
$l_\alpha = 61.59$	$l_\beta = 81.62$	
$k_\alpha = 1.2821, -2$	$k_\beta = 8.4900, -3$	
$E = 4.5, 8$	$\nu = 0.15$	$G_\gamma = 50$

The particular solution is presented for the following cases:-

(i) $P' = Q' = 1(2)9$

(ii) $P' = Q' = 1(2)19$

(iii) $P' = Q' = 1(2)29$

in table 5.1.

TABLE 5.1

	α_1	β_1	case(i)	case(ii)	case(iii)
u_γ	1/8	1/8	6.327, -3	5.819, -3	5.832, -3
u_γ	1/2	1/2	5.484, -3	5.234, -3	5.252, -3
u_α	1/4	1/8	-3.859, -4	-3.828, -4	-3.828, -4
m_β	1/8	1/8	3.157, 1	8.286, 0	8.968, 0
m_β	1/2	1/2	9.509, 0	-1.864, 0	1.548, 0
n_β	1/8	1/4	-4.764, 3	-4.624, 3	-4.627, 3
n_β	1/2	1/2	-3.333, 3	-3.252, 3	-3.255, 3
$n_{\alpha\beta}$	0	0	-7.685, 3	-8.003, 3	-8.027, 3
$n_{\alpha\beta}$	1/4	0	-2.429, 3	-2.400, 3	-2.400, 3
$n_{\alpha\beta}$	1/8	1/4	-1.980, 3	-1.960, 3	-1.960, 3
n_α	1/4	1/8	-3.132, 3	-2.763, 3	-2.775, 3
n_α	1/2	1/2	-1.909, 3	-1.732, 3	-1.747, 3
$m_{\alpha\beta}$	0	0	-7.575, 1	-1.093, 2	-1.173, 2
q_β	1/8	0	2.282, 1	4.392, 1	5.690, 1

<u>Example 5.2</u> A hyperbolic paraboloid with the following data:-

$t = 0.208333$ $\qquad L_\alpha = 60$ $\qquad L_\beta = 80$

$l_\alpha = 61.59$ $\qquad l_\beta = 81.62$

$k_\alpha = -1.2821,\ -2$ $\qquad k_\beta = 8.4900,\ -3$

$E = 4.5,\ 8$ $\qquad \nu = 0.15$

$G_\alpha = 10$ $\qquad G_\beta = 10$ $\qquad G_\gamma = 50$

The particular solution is presented for the following cases:-

(i) $P' = Q' = 0(1)9$

(ii) $P' = Q' = 0(1)14$

(iii) $P' = Q' = 0(1)19$

in table 5.2

TABLE 5.2

	α_1	β_1	case (i)	case(ii)	case(iii)
u_γ	1/4	1/4	6.591, -1	6.592, -1	6.592, -1
u_γ	1/2	1/2	1.308, 0	1.308, 0	1.308, 0
u_α	0	1/2	3.307, -1	3.307, -1	3.307, -1
u_β	1/2	0	-2.703, -1	-2.703, -1	-2.704, -1
m_α	1/4	1/4	6.921, 2	6.956, 2	6.973, 2
m_α	1/2	1/2	1.391, 3	1.385, 3	1.378, 3
m_β	1/2	1/2	9.224, 2	9.194, 2	9.111, 2
n_α	3/4	1/4	-1.684, 4	-1.677, 4	-1.677, 4
n_α	1/2	1/2	-3.537, 4	-3.547, 4	-3.554, 4
$n_{\alpha\beta}$	0	0	-4.349, 4	-4.337, 4	-4.331, 4
$n_{\alpha\beta}$	1/4	1/4	-2.221, 4	-2.222, 4	-2.222, 4
n_β	1/2	1/2	-5.800, 4	-5.794, 4	-5.792, 4
$m_{\alpha\beta}$	0	0	-9.534, 2	-9.755, 2	-9.895, 2
q_β	1/2	0	6.171, 1	7.219, 1	8.412, 1

Complementary solution

The influence of the second assumption on the accuracy of the numerical results can be studied by taking a constant number of harmonics for the particular and varying the number of harmonics considered for the complementary solution.

Consider the following examples:-

Example 5.3 An elliptic paraboloid with the following data:-

Edge 1 - clamped Edge 2 - clamped

Edge 3 - clamped Edge 4 - clamped

$t = 0.25$ $L_\alpha = 80$ $L_\beta = 80$

$l_\alpha = 82.12$ $l_\beta = 82.12$

$k_\alpha = 9.6154, -3$ $k_\beta = 9.6154, -3$

$E = 4.5, 8$ $\nu = 0.15$ $G_\gamma = 50$

$P' = 1(2)9$ $Q' = 1(2)9$

The total solution is presented for the following cases:-

(i) $P = Q = 1(2)9$

(ii) $P = Q = 1(2)19$

(iii) $P = Q = 1(2)29$

in table 5.3.

TABLE 5.3

	α_1	β_1	case (i)	case(ii)	case(iii)
u_γ	1/8	1/8	3.090, -3	3.092, -3	3.092, -3
u_γ	1/2	1/2	2.488, -3	2.488, -3	2.488, -3
u_α	1/8	1/2	-6.497, -5	-6.501, -5	-6.501. -5
m_β	3/8	0	-9.702, 1	-9.888, 1	-9.839, 1
m_β	1/2	1/2	1.419, 1	1.419, 1	1.419, 1
n_β	1/4	0	-2.445, 3	-2.475, 3	-2.458, 3
n_β	1/2	1/2	-2.743, 3	-2.743, 3	-2.743, 3
n_β	1/2	1/2	-2.743, 3	-2.743, 3	-2.743, 3
$n_{\alpha\beta}$	1/8	0	-3.827, 2	-3.517, 2	-3.704, 2
$n_{\alpha\beta}$	1/4	1/4	-7.328, 1	-7.343, 1	-7.344, 1
q_β	1/8	0	1.971, 1	1.745, 1	1.884, 1

Example 5.4 A hyperbolic paraboloid with the following data:-

Edge 1 - clamped

Edge 2 - oblique gable, $\phi_2 = 17.98$

Edge 3 - hinged

Edge 4 - clamped

$t = 0.25$ $L_\alpha = 80$ $L_\beta = 60$

$l_\alpha = 81.62$ $l_\beta = 61.11$

$k_\alpha = -8.4900, -3$ $k_\beta = 1.0811, -2$

$E = 4.5, 8$ $\nu = 0.15$ $G_\gamma = 50$

$P' = 1(2)7$ $Q' = 1(2)7$

The total solution is presented for the following cases:-

(i) $P = Q = 1(1)7$

(ii) $P = Q = 1(1)11$

(iii) $P = Q = 1(1)15$

in table 5.4.

TABLE 5.4

	α_1	β_1	case (i)	case(ii)	case(iii)
u_γ	1/2	1/2	6.632, -3	6.632, -3	6.632, -3
u_γ	1/2	1	-7.375, -4	-7.368, -4	-7.364, -4
u_β	1/2	1	2.258, -3	2.258, -3	2.258, -3
m_β	1/4	0	-1.569, 2	-1.818, 2	-1.716, 2
m_β	1/2	1/2	-6.252, 0	-6.252, 0	-6.253, 0
m_α	1	1/4	-1.061, 2	-1.046, 2	-1.119, 2
n_β	1/4	0	-2.643, 3	-2.650, 3	-2.685, 3
n_β	1/2	1/2	-1.933, 3	-1.934, 3	-1.934, 3
n_α	1	1/2	4.427, 3	4.449, 3	4.446, 3
n_α	1/2	1/2	2.904, 3	2.903, 3	2.903, 3
$n_{\alpha\beta}$	0	1/2	2.558, 3	2.639, 3	2.651, 3
$n_{\alpha\beta}$	1/4	1/4	3.738, 2	3.693, 2	3.695, 2
q_β	1/4	1	-3.854, 1	-3.855, 1	-3.867, 1

Example 5.5 A cylindrical shell with the following data:-

Edge 1 - hinged Edge 2 - hinged

Edge 3 - clamped Edge 4 - free

$t = 0.25$ $L_\alpha = 70$ $L_\beta = 40$

$l_\alpha = 70.00$ $l_\beta = 43.19$

$k_\alpha = 0$ $k_\beta = 3.1180, -2$

$E = 4.5, 8$ $\nu = 0.15$ $G_\gamma = 50$

$P' = 1(2)9$ $Q' = 1(2)9$

The total solution is presented for the following cases:-

(i) $P = 1(1)9$, $Q = 1(2)9$

(ii) $P = 1(1)19$, $Q = 1(2)19$

(iii) $P = 1(1)29$, $Q = 1(2)29$

in table 5.5.

Example 5.6 An elliptic paraboloid with the following data:-

Edge 1 - free Edge 2 - free

Edge 3 - clamped Edge 4 - clamped

$t = 0.25$ $L_\alpha = 60$ $L_\beta = 40$

$l_\alpha = 61.59$ $l_\beta = 41.06$

$k_\alpha = 1.2821, -2$ $k_\beta = 1.9231, -2$

$E = 4.5, 8$ $\nu = 0.15$ $G_\gamma = 50$

$P' = 1(2)19$ $Q' = 1(2)19$

TABLE 5.5

	α_1	β_1	case (i)	case(ii)	case(iii)
u_γ	1/2	1/2	5.837, -4	5.713, -4	5.706, -4
u_γ	1	1/2	8.871, -4	9.245, -4	9.297, -4
u_α	1	1/2	1.032, -4	1.024, -4	1.024, -4
u_β	1	1/4	-1.697, -4	-1.757, -4	-1.767, -4
m_α	0	3/8	-6.742, 1	-6.788, 1	-6.788, 1
m_β	1/2	1/8	1.819, 1	1.852, 1	1.854, 1
n_β	1/2	0	-1.681, 3	-1.560, 3	-1.605, 3
n_β	1/2	1/2	-1.690, 3	-1.690, 3	-1.691, 3
n_α	0	3/8	-1.132, 2	-1.315, 2	-1.344, 2
n_α	1/2	3/8	-9.394, 1	-9.982, 1	-1.006, 2
$n_{\alpha\beta}$	0	1/8	-1.397, 2	-1.196, 2	-1.302, 2
$n_{\alpha\beta}$	3/4	1/8	7.016, 1	8.071, 1	7.986, 1
q_α	0	1/2	3.656, 1	3.582, 1	3.589, 1
q_β	1/8	0	-4.179, 1	-4.063, 1	-4.009, 1

The total solution is presented for the following cases:-

(i) $P = Q = 1(2)19$

(ii) $P = Q = 1(2)39$

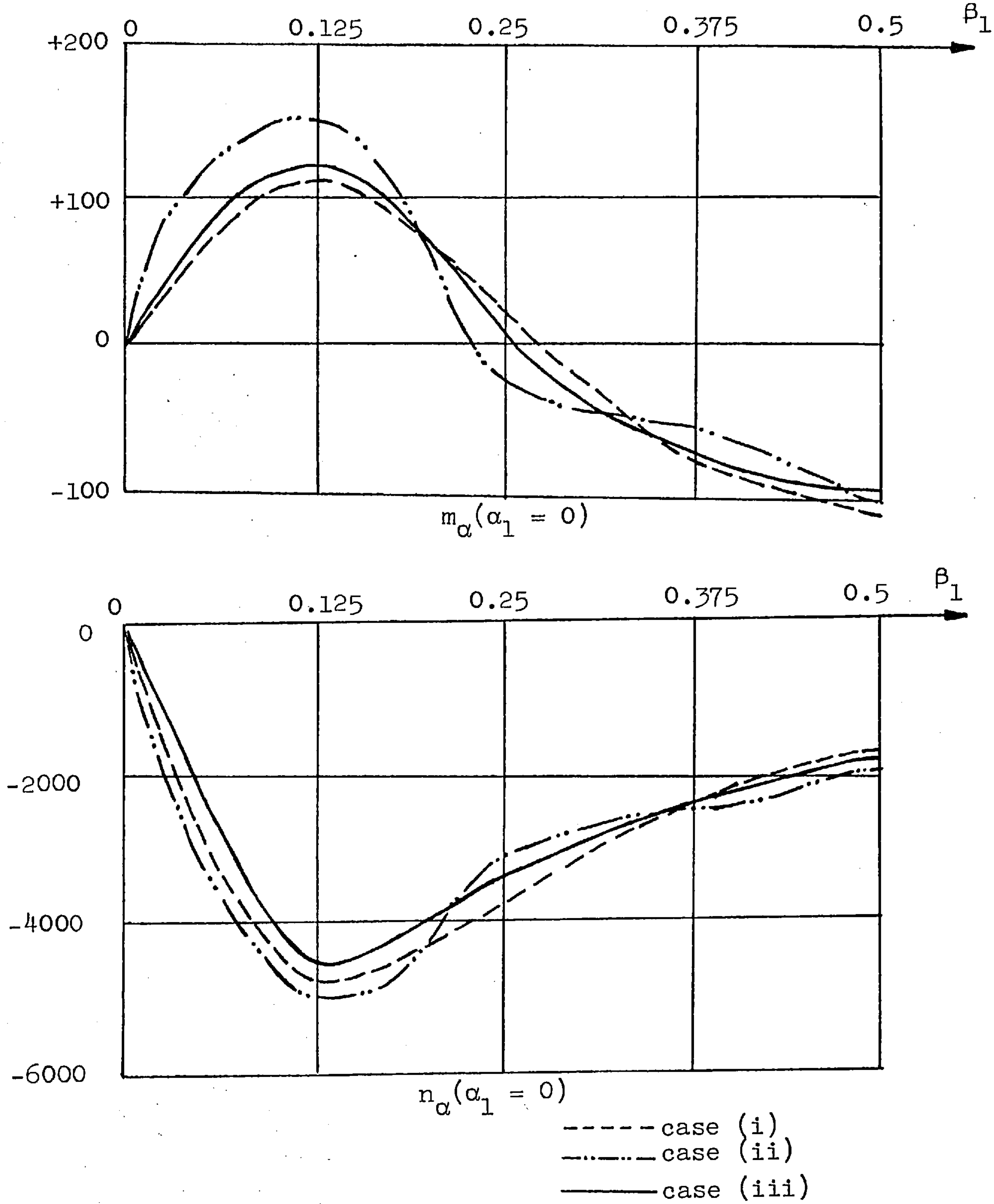
(iii) $P = Q = 1(2)59$

in table 5.6.

TABLE 5.6

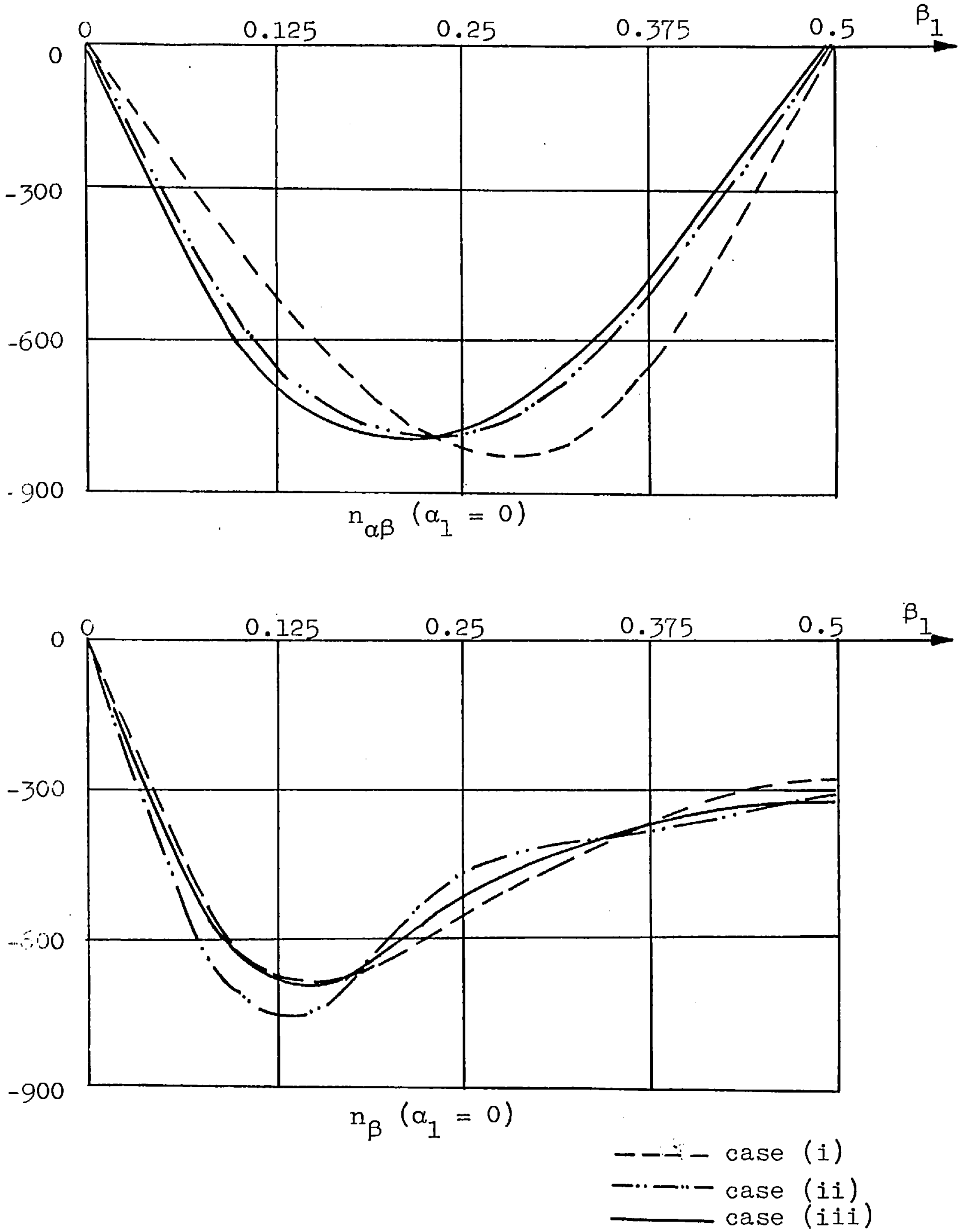
	α_1	β_1	case (i)	case(ii)	case(iii)
u_γ	1/2	0	1.130, -2	1.145, -2	1.148, -2
u_γ	1/2	1/2	3.751, -3	3.765, -3	3.768, -3
u_β	1/2	0	-2.490, -3	-2.509, -3	-2.513, -3
m_α	1/8	0	-1.044, 2	-9.379, 1	-1.065, 2
m_α	1/2	0	6.855, 1	7.216, 1	7.372, 1
m_β	1/2	1/4	-3.028, 1	-3.080, 1	-3.091, 1
n_α	3/8	0	-1.973, 3	-1.975, 3	-1.948, 3
n_α	1/2	1/2	-3.541, 3	-3.539, 3	-3.539, 3
$n_{\alpha\beta}$	1/8	1/4	-6.300, 2	-6.238, 2	-6.226, 2
$n_{\alpha\beta}$	3/8	1/8	-1.659, 2	-1.666, 2	-1.669, 2
n_β	1/8	1/8	1.270, 2	1.159, 2	1.145, 2
n_β	1/2	1/2	-2.248, 2	-2.258, 2	-2.259, 2
$m_{\alpha\beta}$	1/4	0	4.323, 1	4.390, 1	4.424, 1
u_α	1/4	0	-1.099, -3	-1.113, -3	-1.117, -3

The convergence of the solution along the clamped edges is slow (refer figures 5.1 and 5.2).



(Example 5.6)

Figure 5.1



(Example 5.6)

Figure 5.2

Example 5.7 A hyperbolic paraboloid with the following data:-

Edge 1 - flexible beam   Edge 2 - clamped

Edge 3 - oblique gable, $\phi_3 = 22.62$ Edge 4 - oblique gable, $\phi_4 = 22.62$

$t = 0.25$ $L_\alpha = 80$ $L_\beta = 70$

$l_\alpha = 82.12$ $l_\beta = 71.85$

$k_\alpha = 9.6154, -3$ $k_\beta = -1.0989, -2$

$E = 4.5, 8$ $\nu = 0.15$

$G_\alpha = 0$ $G_\beta = 10$ $G_\gamma = 50$

$P' = 1(2)9$ $Q' = 0(1)9$

Edge beam

$A = 5.0$ $I_1 = 10.417$ $I_2 = 0.4167$

$I_3 = 1.455$ $a^1 = 0.5$ $b^1 = 0$

$Z_3 = 750$ $Z_2 = 0$ $\psi_1 = -22.62$

$E_b = 4.5, 8$

The total solution for the shell and the edge beam is presented for the following cases:-

(i) $P = 1(2)9$, $Q = 1(1)10$

(ii) $P = 1(2)15$, $Q = 1(1)15$

(iii) $P = 1(2)19$, $Q = 1(1)20$

in tables 5.7(a) and 5.7(b) respectively.

TABLE 5.7(a)

	α_1	β_1	case (i)	case(ii)	case(iii)
u_γ	0	1/2	-2.103, -3	-2.000, -3	-2.008, -3
u_γ	1/2	0	2.182, -2	2.181, -2	2.178, -2
u_γ	1/2	1/2	2.127, -2	2.128, -2	2.128, -2
u_β	1/2	0	7.785, -3	7.783, -3	7.783, -3
u_α	0	1/2	-4.897, -3	-4.899, -3	-4.898, -3
m_β	1/8	0	-5.232, 2	-5.805, 2	-5.187, 2
m_β	1/2	1	4.940, 2	5.074, 2	5.065, 2
m_α	1/4	1/4	4.522, 1	4.532, 1	4.536, 1
n_α	3/8	0	-2.872, 3	-2.858, 3	-2.873, 3
n_α	1/2	1/2	1.140, 3	1.137, 3	1.132, 3
n_β	1/2	1	1.109, 4	1.100, 4	1.100, 4
n_β	1/2	1/2	5.621, 3	5.617, 3	5.612, 3
$n_{\alpha\beta}$	0	3/4	4.036, 3	4.041, 3	4.038, 3
$n_{\alpha\beta}$	1/4	1/2	5.191, 3	5.191, 3	5.191, 3
n_β	1/2	1	3.065, 2	3.109, 2	3.107, 2

TABLE 5.7(b)

	α_1	case (i)	case(ii)	case(iii)
S_1	0	1.139, 4	1.178, 4	1.178, 4
S_2	0	3.175, 3	3.035, 3	2.916, 3
M_3	0	1.299, 4	1.270, 4	1.236, 4
U_3	1/4	1.678, -2	1.677, -2	1.675, -2
U_3	1/2	2.286, -2	2.285, -2	2.282, -2
U_2	1/2	-1.211, -3	-1.205, -3	-1.196, -3
M_1	1/4	1.205, 5	1.205, 5	1.205, 5
M_1	1/2	1.263, 5	1.258, 5	1.257, 5
M_2	1/2	4.773, 2	4.284, 2	4.355, 2

Example 5.8 An elliptic paraboloid with the following data:-

Edge 1 - flexible beam Edge 2 - flexible beam

Edge 3 - flexible beam Edge 4 - flexible beam

$t = 0.25$ $L_\alpha = 50$ $L_\beta = 50$

$l_\alpha = 50.54$ $l_\beta = 50.54$

$k_\alpha = 1.0000, -2$ $k_\beta = 1.0000, -2$

$E = 7.68, 8$ $\nu = 0$ $G_\gamma = 50$

$P' = 1(2)9$ $Q' = 1(2)9$

<u>Edge beams</u>

All edge beams are identical and

$A = 1.25$ $\qquad$ $I_1 = 15.625$ $\qquad$ $I_2 = 1.0, -6$

$I_3 = 1.0, -6$ $\qquad$ $a = 0$ $\qquad$ $b = 0$

$Z_3 = 0$ $\qquad$ $Z_2 = 0$ $\qquad$ $\psi = 14.48$

$E_b = 7.68, 8$

The total solution is presented for the shell and edge beam 1 for the following cases:-

(i) $P = Q = 1(2)9$

(ii) $P = Q = 1(2)19$

(iii) $P = Q = 1(2)29$

in tables 5.8(a) and 5.8(b) respectively.

<u>Check on the overall equilibrium of the shell and edge beams</u>

Total normal load on the shell and edge

beams = 50 (50.54 x 50.54)

= 127715 lb

Total normal load transferred to the edge beam

supports = 4 (2 x 15880)

= 127040 lb

The solution obtained is in agreement with a solution published by Padilla and Schnobrich[15] using a modified finite difference technique.

TABLE 5.8(a)

	α_1	β_1	case (i)	case(ii)	case(iii)
u_γ	1/2	0	4.584, -3	4.622, -3	4.591, -3
u_γ	1/2	1/2	1.131, -2	1.130, -2	1.128, -2
u_β	1/2	0	-2.576, -3	-2.575, -3	-2.571, -3
m_β	1/2	0	-4.553, -3	5.312, -3	-7.865, -3
m_β	1/2	1/8	2.177, 2	2.171, 2	2.170, 2
m_β	1/2	1/4	-1.923, 1	-1.923, 1	-1.921, 1
m_β	1/2	3/8	-1.450, 1	-1.449, 1	-1.448, 1
m_β	1/2	1/2	8.145, 0	8.144, 0	8.143, 0
n_α	1/8	0	6.051, 3	5.992, 3	5.956, 3
n_α	1/4	0	9.448, 3	9.465, 3	9.422, 3
n_α	1/2	0	1.150, 4	1.145, 4	1.147, 4
n_α	1/2	1/8	-4.351, 3	-4.350, 3	-4.349, 3
n_α	1/2	1/4	-3.715, 3	-3.714, 3	-3.714, 3
n_α	1/2	1/2	-2.494, 3	-2.494, 3	-2.494, 3
n_β	1/2	0	-7.067, 1	-5.559, 1	-8.950, 1
n_β	1/2	1/8	-1.027, 3	-1.030, 3	-1.030, 3
n_β	1/2	1/2	-2.494, 3	-2.494, 3	-2.494, 3
$n_{\alpha\beta}$	0	0	-2.758, 3	-2.401, 3	-2.042, 3
$n_{\alpha\beta}$	1/8	0	-3.802, 3	-3.721, 3	-3.805, 3
$n_{\alpha\beta}$	1/4	0	-1.772, 3	-1.800, 3	-1.830, 3
$n_{\alpha\beta}$	3/8	0	-7.507, 2	-7.558, 2	-7.520, 2

TABLE 5.8(b)

	α_1	case (i)	case(ii)	case(iii)
N_1	1/2	5.474, 4	5.489, 4	5.476, 4
M_1	1/4	1.553, 4	1.652, 4	1.649, 5
M_1	1/2	2.284, 5	2.281, 5	2.279, 5
M_2	1/2	1.768, -2	-8.535, -2	8.741, -2
M_3	0	1.587, -1	1.911, -1	2.035, -1
S_1	0	1.527, 4	1.552, 4	1.588, 4
U_3	1/4	3.624, -3	3.250, -3	3.615, -3
U_3	1/2	5.110, -3	5.105, -3	5.097, -3
U_2	1/2	-1.456, -3	-1.283, -3	-1.378, -3

<u>Example 5.9</u> An elliptic paraboloid with the following data:-

Edge 1 - flexible beam Edge 2 - flexible beam

Edge 3 - clamped Edge 4 - clamped

$t = 0.25$ $L_\alpha = 70$ $L_\beta = 40$

$l_\alpha = 70.77$ $l_\beta = 41.06$

$k_\alpha = 7.2275, -3$ $k_\beta = 1.9231, -2$

$E = 4.5, 8$ $\nu = 0.15$ $G_\gamma = 50$

$P' = 1(2)19$ $Q' = 1(2)19$

The total solution for the shell and edge beam 1 is presented for the following cases:-

(i) $P = Q = 1(2)29$

(ii) $P = Q = 1(2)39$

(iii) $P = Q = 1(2)49$

in tables 5.9(a) and 5.9(b) respectively.

TABLE 5.9(a)

	α_1	β_1	case (i)	case(ii)	case(iii)
u_γ	1/4	0	8.942, -3	8.662, -3	8.479, -3
u_γ	1/2	0	1.298, -2	1.261, -2	1.237, -2
u_γ	1/2	1/2	7.713, -3	7.510, -3	7.377, -3
u_β	1/2	0	-4.238, -3	-4.129, -3	-4.050, -3
m_α	1/8	1/8	3.068, 1	2.884, 1	2.763, 1
m_α	1/2	1/8	2.095, 1	2.089, 1	2.045, 1
m_β	1/4	0	-6.382, 1	-7.893, 1	-6.400, 1
m_β	1/2	0	-1.034, 1	-2.235, 1	-1.597, 1
m_β	1/2	1/2	-3.322, 1	-3.111, 1	-3.209, 1
n_β	1/2	0	-3.014, 2	-3.458, 2	-3.253, 2
n_β	1/2	1/2	-5.758, 2	-6.862, 2	-6.338, 2
n_β	1/8	3/8	-1.258, 3	-1.300, 3	-1.328, 3
n_α	1/8	1/8	-7.221, 3	-6.971, 3	-6.809, 3
$n_{\alpha\beta}$	1/8	1/4	-1.865, 3	-1.851, 3	-1.841, 3
$n_{\alpha\beta}$	1/4	1/4	-7.581, 2	-7.782, 2	-7.913, 2
n_α	1/2	0	-2.573, 3	-2.423, 3	-2.551, 3
n_α	1/2	1/2	-5.372, 3	-5.279, 3	-5.218, 3

TABLE 5.9(b)

	α_1	case (i)	case(ii)	case(iii)
U_3	1/4	9.390, -3	9.097, -3	8.906, -3
U_3	1/2	1.356, -2	1.317, -2	1.292, -2
U_2	1/2	1.101, -3	1.067, -3	1.043, -3
M_1	1/4	5.970, 4	5.760, 4	5.616, 4
M_1	1/2	9.439, 4	9.257, 4	9.130, 4
M_2	1/2	-3.685, 2	-8.493, 2	-5.051, 2
M_3	0	-7.061, 2	-7.232, 2	-7.313, 2
S_1	0	4.076, 3	3.737, 3	3.461, 3
U_1	0	-9.342, -4	-8.496, -4	-7.938, -4

The convergence of the solution is slow especially along the clamped edges (refer figures 5.3 and 5.4).

Example 5.10 An elliptic paraboloid with the following data:-

Edge 1 - flexible beam Edge 2 - flexible beam

Edge 3 - flexible beam Edge 4 - flexible beam

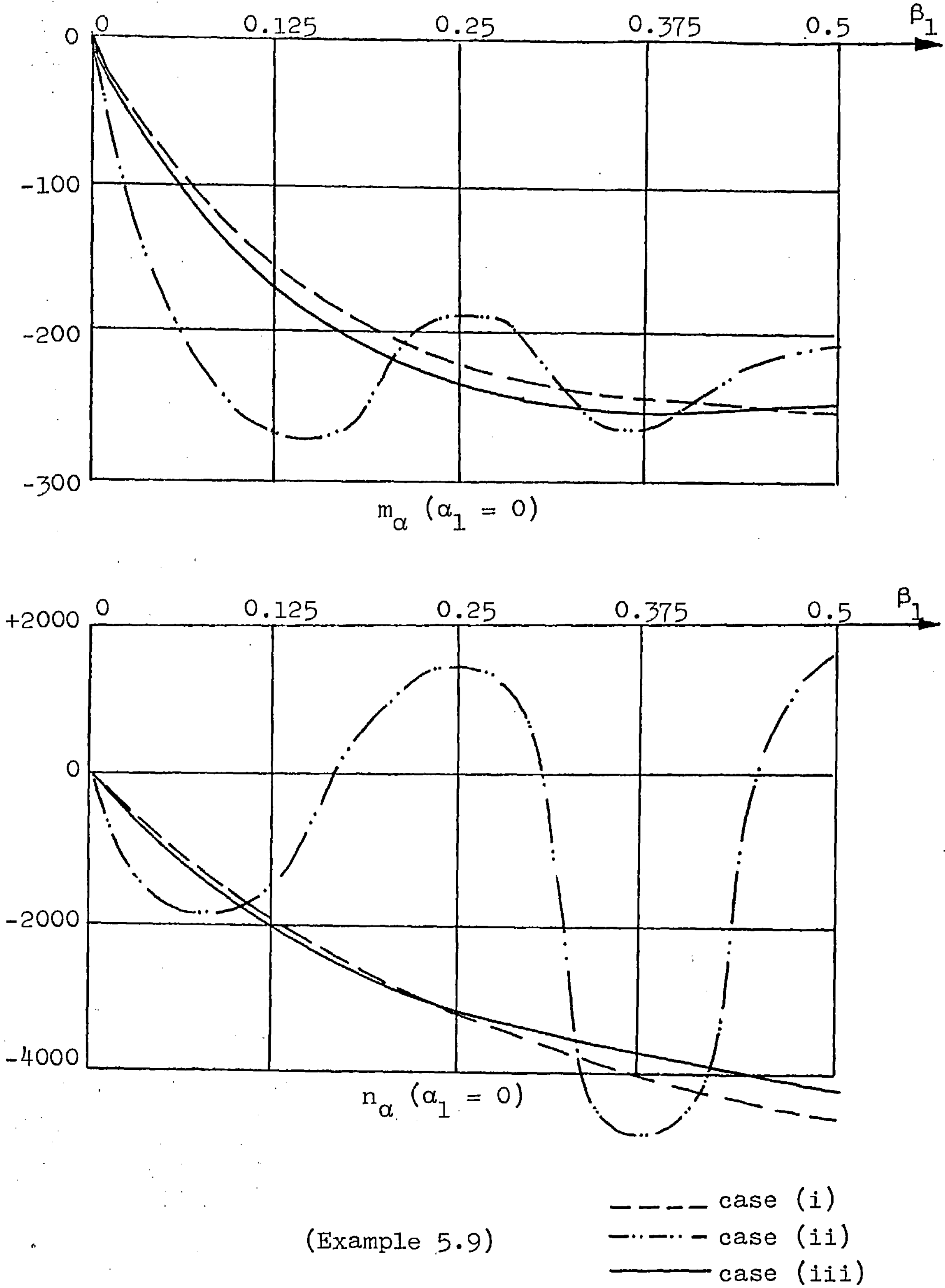
$t = 0.25$ $L_\alpha = 50$ $L_\beta = 50$

$l_\alpha = 50.54$ $l_\beta = 50.54$

$k_\alpha = 1.0000, -2$ $k_\beta = 1.0000, -2$

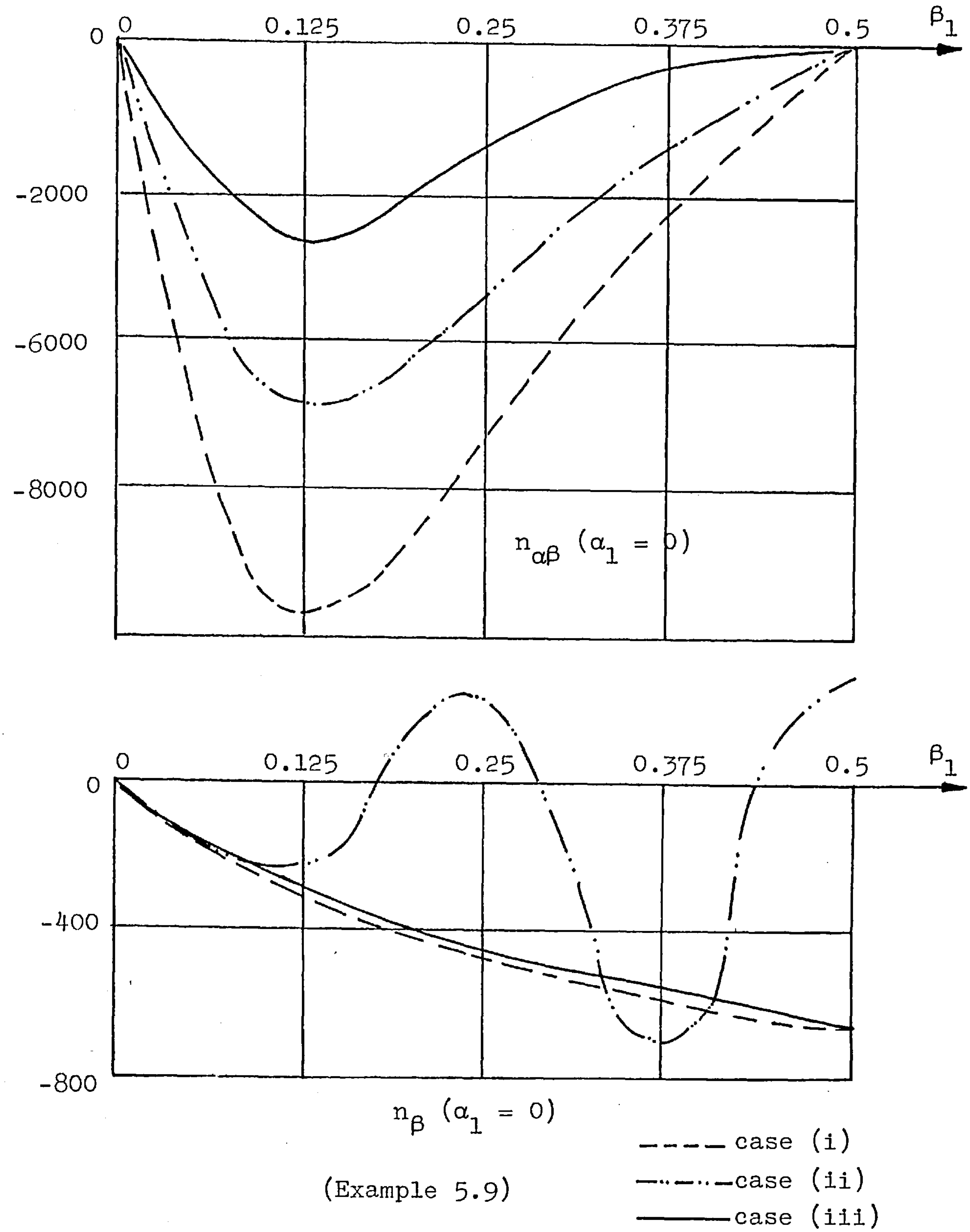
$E = 4.5, 8$ $\nu = 0.15$ $G_\gamma = 50$

$P' = 1(2)19$ $Q' = 1(2)19$



(Example 5.9)

Figure 5.3



(Example 5.9)

Figure 5.4

Edge beams

All edge beams are identical and

$A = 4.5$	$I_1 = 7.58$	$I_2 = 0.375$
$I_3 = 1.287$	$a^1 = a^3 = 0.5$	$a^2 = a^4 = -0.5$
$b = 0.25$	$Z_3 = 675$	$Z_2 = 0$
$\psi = 14.48$	$E_b = 4.5,\ 8$	

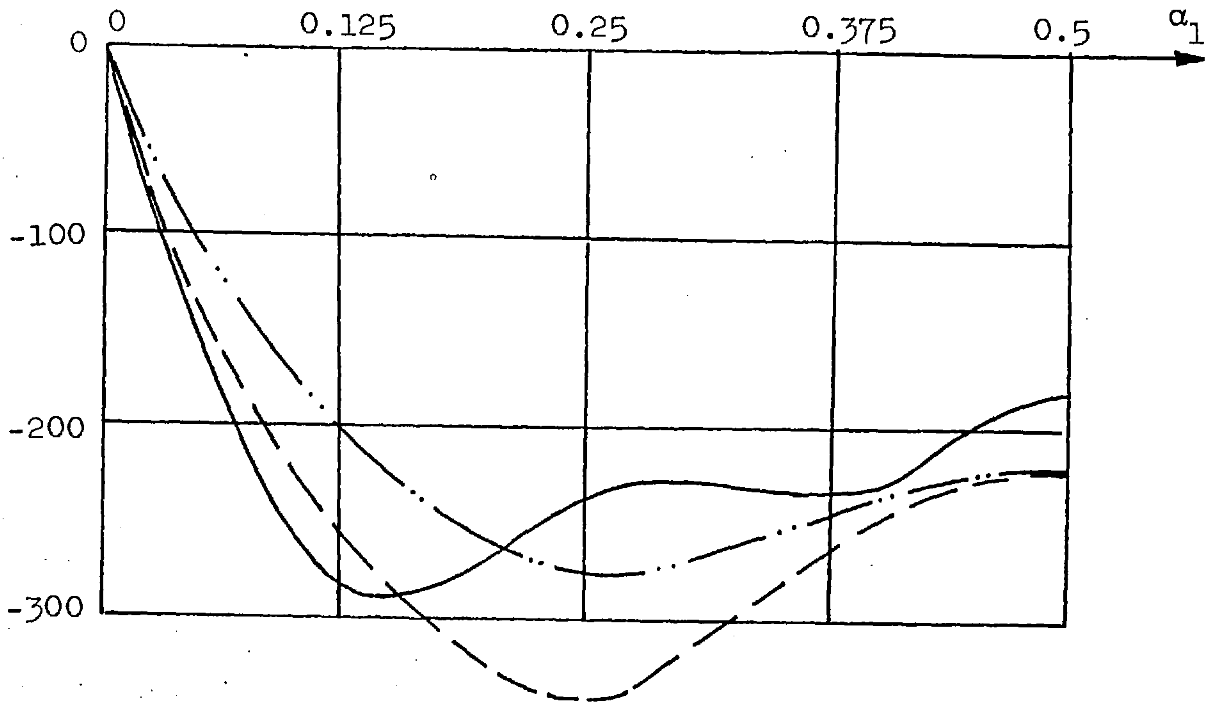
The solution for the shell and edge beam 1 is presented for the following cases:-

(i) $P = Q = 1(2)19$

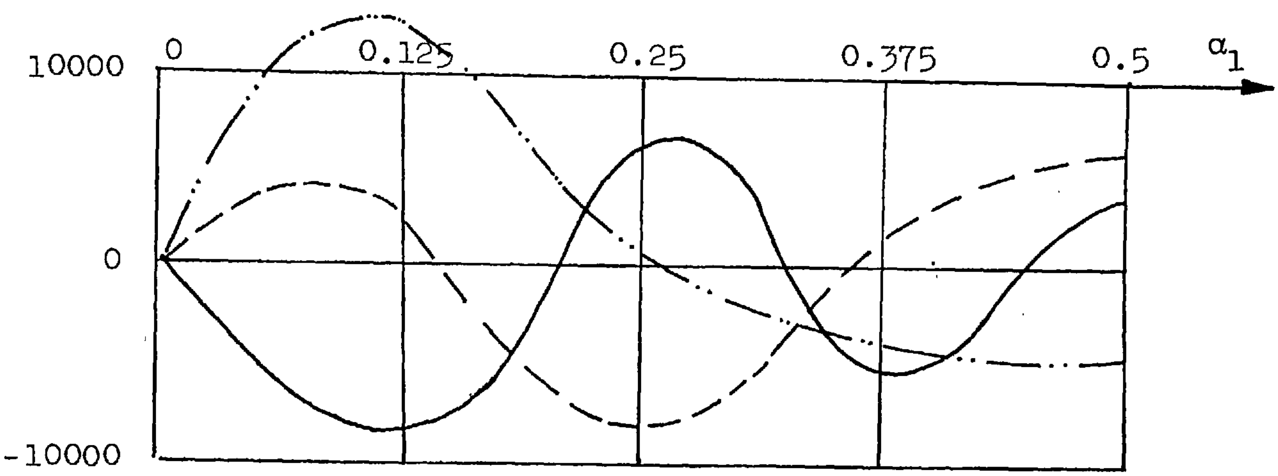
(ii) $P = Q = 1(2)29$

(iii) $P = Q = 1(2)39$

in tables 5.10(a) and 5.10(b) respectively. The convergence of the solution is extremely slow especially along the shell boundaries (refer figures 5.5 and 5.6).



m_β (β_1 = 0)

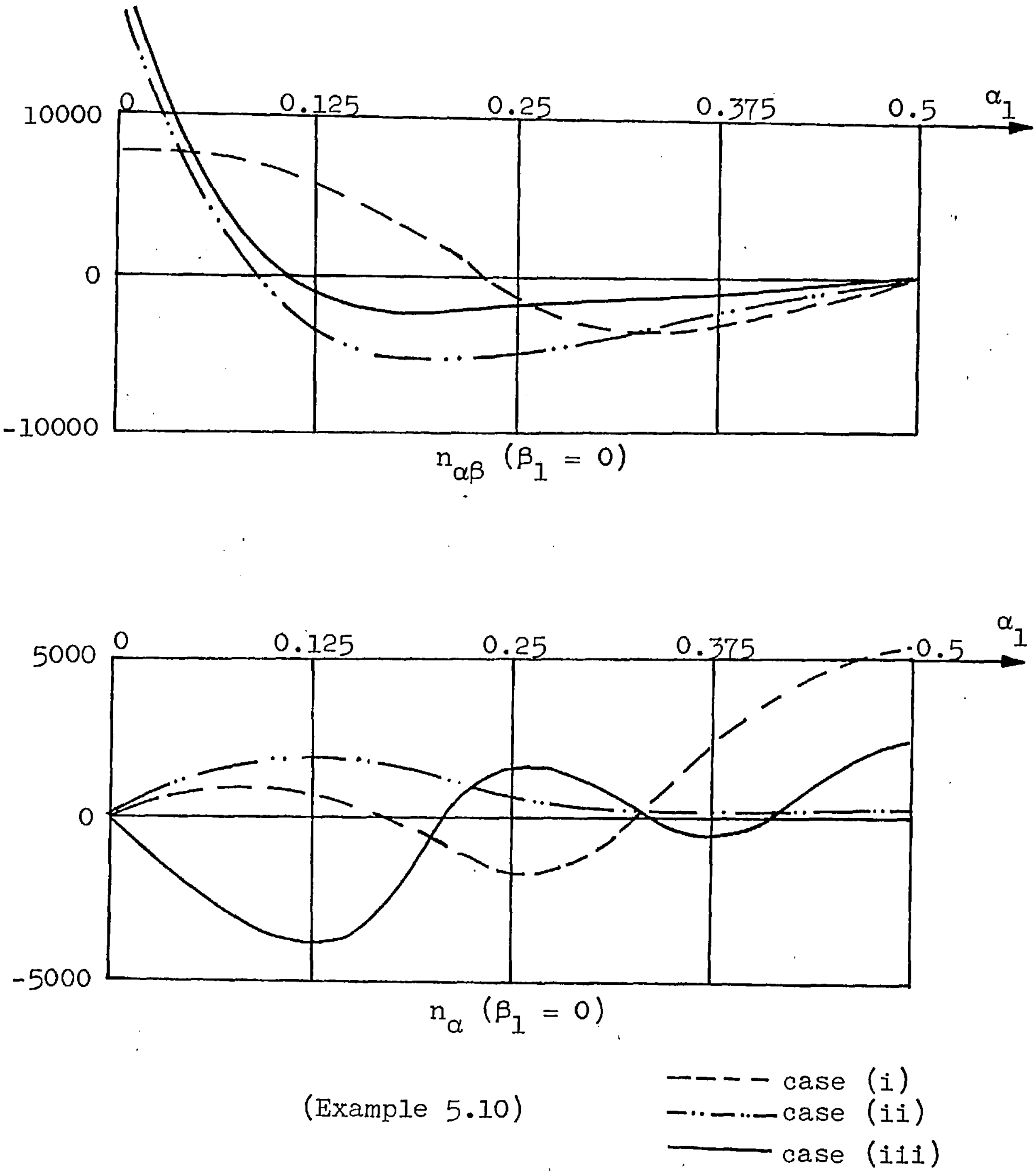


n_β (β_1 = 0)

- - - - case (i)
-··-··- case (ii)
—— case (iii)

(Example 5.10)

Figure 5.5



(Example 5.10)

Figure 5.6

TABLE 5.10(a)

	α_1	β_1	case (i)	case(ii)	case(iii)
u_γ	1/4	0	1.282, -2	1.032, -2	9.121, -3
u_γ	1/2	0	1.841, -2	1.486, -2	1.317, -2
u_γ	1/2	1/2	2.029, -2	1.665, -2	1.490, -2
u_β	1/2	0	-4.997, -3	-3.942, -3	-3.560, -3
m_α	1/4	1/8	4.639, 1	3.795, 1	3.381, 1
m_α	1/2	1/8	4.336, 1	3.596, 1	3.252, 1
m_α	1/8	1/2	8.287, 1	7.175, 1	6.640, 1
m_β	1/4	1/8	6.809, 1	5.935, 1	5.516, 1
m_β	3/8	1/8	8.083, 1	6.974, 1	6.442, 1
m_β	1/2	1/2	-1.425, 0	-1.404, 0	-1.394, 0
n_α	1/8	1/8	-1.981, 3	-2.016, 3	-2.030, 3
n_α	3/8	1/4	-3.297, 3	-3.252, 3	-3.230, 3
n_α	1/2	1/2	-2.476, 3	-2.479, 3	-2.481, 3
n_β	1/8	1/4	-3.058, 3	-3.084, 3	-3.089, 3
n_β	1/4	1/8	-1.264, 3	-1.337, 3	-1.383, 3
n_β	1/2	1/8	-1.069, 3	-1.076, 3	-1.072, 3
$n_{\alpha\beta}$	1/8	1/8	-5.565, 3	-5.092, 3	-4.831, 3
$n_{\alpha\beta}$	1/4	1/4	-1.444, 3	-1.420, 3	-1.408, 3
$n_{\alpha\beta}$	3/8	1/2	-6.413, 2	-6.433, 2	-6.441, 2

TABLE 5.10(b)

	α_1	case (i)	case(ii)	case(iii)
N_1	1/4	-1.453, 4	-2.268, 4	-2.642, 4
N_1	1/2	8.105, 2	-1.134, 4	-1.438, 4
M_1	1/4	1.720, 5	1.369, 5	1.205, 5
M_1	1/2	2.412, 5	1.976, 5	1.763, 5
M_2	1/2	-6.637, 3	4.831, 1	-2.677, 3
M_3	0	2.904, 4	3.306, 4	3.506, 4
U_3	1/4	1.334, -2	1.073, -2	9.474, -3
U_3	1/2	1.886, -2	1.522, -2	1.348, -2
U_2	1/2	-5.204, -5	-6.146, -5	-5.218, -5
U_1	0	-2.880, -3	-2.153, -3	-1.802, -3
S_1	0	3.199, 4	3.199, 4	3.198, 4

Discussion on the convergence of the solutions

The convergence of the solutions is mainly dependent on the boundary conditions. The convergence of the solutions is extremely rapid with any combination of simply supported, oblique gable, hinged and clamped boundaries (examples 5.3 and 5.4). Ten non zero harmonics in each direction for the particular solution and the complementary solution suffice. With the introduction of one or more free boundaries the convergence of the solutions is slower (examples 5.5 and 5.6). The rate of convergence of the solutions of a shell with flexible edge members is difficult to predict. In examples 5.7 and 5.8 the convergence of the solutions is extremely rapid, while in examples 5.9 and 5.10 the convergence of the solutions, especially along the shell boundaries, is extremely slow.

V.2. Check on the computer programme by comparing with solutions given in the literature

Example 5.11 A flat plate has been analysed with the following data:-

Edge 1 - clamped	Edge 2 - clamped	
Edge 3 - clamped	Edge 4 - clamped	
$t = 0.25$	$L_\alpha = 75$	$L_\beta = 50$
$k_\alpha = 1.65, -11$	$k_\beta = 1.65, -11$	
$E = 4.32, 8$	$\nu = 0.3$	$G_\gamma = 50$
$P = 1(2)13$	$Q = 1(2)13$	
$P' = 1(2)9$	$Q' = 1(2)9$	

The solution is compared with that given by Timoshenko(31) in table 5.11.

TABLE 5.11

	α_1	β_1	Programme	Timoshenko
m_α	0	0.5	-7.101, 3	-7.120, 3
m_β	0.5	0	-9.439, 3	-9.460, 3
u_γ	0.5	0.5	1.111, 0	1.110, 0
m_α	0.5	0.5	2.553, 3	2.540, 3
m_β	0.5	0.5	4.613, 3	4.600, 3

Example 5.12 A flat plate has been analysed with the following data:-

Edge 1 - clamped   Edge 2 - clamped

Edge 3 - clamped   Edge 4 - simply supported

$t = 0.20$ $L_\alpha = 40$ $L_\beta = 30$

$k_\alpha = 2.94, -11$ $k_\beta = 2.94, -11$

$E = 4.32, 8$ $\nu = 0.3$ $G_\gamma = 50$

$P = 1(1)9$ $Q = 1(2)13$

$P' = 1(2)9$ $Q' = 1(2)9$

The solution is compared with that given by Timoshenko(31) in table 5.12.

Examples 5.11 and 5.12 have been computed using the computer programme for ruled surfaces giving identical results (refer section VI.2).

TABLE 5.12

	α_1	β_1	Programme	Timoshenko
m_α	0	0.5	-2.564, 3	-2.570, 3
m_β	0.5	0	-3.368, 3	-3.378, 3
u_γ	0.5	0.5	2.769, -1	2.750, -1

Example 5.13 A flat plate has been analysed with the following data:-

Edge 1 - flexible beam   Edge 2 - flexible beam

Edge 3 - flexible beam   Edge 4 - flexible beam

$t = 0.25$ $L_\alpha = 100$ $L_\beta = 100$

$k_\alpha = 1.03, -11$ $k_\beta = 1.03, -11$

$E = 7.2, 8$ $\nu = 0.25$ $G_\gamma = 50$

$P = 1(2)7$ $Q = 1(2)7$

$P' = 1(2)7$ $Q' = 1(2)7$

Edge beams

All edge beams are identical and

$A = 100$ $I_1 = 1.0$ $I_2 = 100$

$I_3 = 1.0, -6$ $a = 0$ $b = 0$

$Z_3 = 0$ $Z_2 = 0$ [illegible] $= 0$

$E_b = 5.0, 8$

The solution is compared with that given by Timoshenko(31) in table 5.13.

TABLE 5.13

	α_1	β_1	Programme	Timoshenko
m_β	0	0.5	3.918, 3	3.250, 3
u_γ	0.5	0.5	2.558, 1	2.595, 1
m_α	0.5	0.5	2.453, 4	2.470, 4

Example 5.14 A cylinder has been analysed with the following data:-

Edge 1 - free   Edge 2 - free

Edge 3 - simply supported   Edge 4 - simply supported

$t = 7$ cm $L_\alpha = 1800$ cm $L_\beta = 1300$ cm

$l_\alpha = 1800$ cm $l_\beta = 1380.6$ cm

$k_\alpha = 0$ $k_\beta = 8.6486, -4\ cm^{-1}$

$E = 2.0, 5\ kg/cm^2$ $\nu = 0$ $G_\gamma = 1.9, -2\ kg/cm^2$

$P = 1(1)1$ $Q = 0(0)0$

$P' = 1(1)1$ $Q' = 0(0)0$

The solution is compared with that given by Scriven-er[30] and Bouma[22] in table 5.14.

TABLE 5.14

	α_1	β_1	Programme	Scrivener	Bouma
u_γ	0.5	0	15	15	17
n_α	0.5	0	915	916	875
u_γ	0.5	0.5	-2	-2	<5
m_β	0.5	0.5	-522	-522	-500
n_α	0.5	0.5	94	94	70

Example 5.15 A cylinder has been analysed with the following data:-

Edge 1 - flexible beam Edge 2 - flexible beam

Edge 3 - simply supported Edge 4 - simply supported

$t = 0.208333$ $L_\alpha = 60$ $L_\beta = 40$

$l_\alpha = 60$ $l_\beta = 41.89$

$k_\alpha = 0$ $k_\beta = 2.5000, -2$

$E = 7.2, 8$ $\nu = 0$

$\bar{P} = 100$ $\psi = 30$

$P = 1(1)1$ $Q = 0(0)0$

$P' = 1(1)1$ $Q' = 0(0)0$

Edge beams

Edge beams are identical and

$A = 1.75$ $I_1 = 1.78646$ $I_2 = 0.0364583$

$I_3 = 0.145833$ $a = 0$ $b = 0$

$Z_3 = 200$ $\qquad Z_2 = 0$ $\qquad \psi = 30$

$E_b = 7.2,\ 8$

The shell and edge beam solutions are compared with that given by Scrivener[30] in table 5.15(a) and 5.15(b) respectively.

TABLE 5.15(a)

	α_1	β_1	Programme	Scrivener
$n_{\alpha\beta}$	0	0	-8536	-8570
$n_{\alpha\beta}$	0	0.25	-6478	-6470
u_{γ}	0.5	0	0.0739	0.0755
m_{α}	0.5	0	110	112
m_{β}	0.5	0	-193	-181
n_{α}	0.5	0	19471	19500
n_{β}	0.5	0	-226	-224
q'_{β}	0.5	0	399	397
m_{β}	0.5	0.25	211	206
n_{α}	0.5	0.25	-14298	-14400
n_{β}	0.5	0.25	-4656	-4660
q'_{β}	0.5	0.25	-161	-162
m_{β}	0.5	0.5	-746	-751
n_{α}	0.5	0.5	-9426	-9350
n_{β}	0.5	0.5	-6108	-6110

TABLE 5.15(b)

	α_1	Programme	Scrivener
N_1	0.5	1.63, 5	1.64, 5
M_1	0.5	2.60, 5	2.66, 5
M_2	0.5	-1.43, 3	-1.48, 3
M_3	0	3.69, 3	3.45, 3

<u>Example 5.16</u> A cylinder has been analysed with the following data:-

Edge 1 - clamped Edge 2 - clamped

Edge 3 - clamped Edge 4 - clamped

$t = 4$ in $L_\alpha = 600$ in $L_\beta = 480$ in

$l_\alpha = 600$ in $l_\beta = 497.40$ in

$k_\alpha = 0$ $k_\beta = 1.8519, -3\ \text{in}^{-1}$

$E = 3.0, 6\ \text{lb/in}^2$ $\nu = 0$ $G_\gamma = 0.555\ \text{lb/in}^2$

$P = 1(2)19$ $Q = 1(2)19$

$P' = 1(2)19$ $Q' = 1(2)19$

The solution is compared with that given by Lu[17] in table 5.15. The method used in the thesis is the same as that used by Lu.

TABLE 5.16

	α_1	β_1	Programme	Lu
u_α	0.2	0.5	1.85, -4	1.85, -4
u_β	0.5	0.2	-1.40, -2	-1.40, -2
u_γ	0.5	0.5	2.037, -2	2.037, -2
m_α	0	0.2	-3.755, 2	-3.752, 2
m_α	0	0.5	-3.648, 2	-3.647, 2
m_α	0.1	0.2	7.29, 1	7.29, 1
m_α	0.1	0.5	6.62, 1	6.62, 1
m_β	0.2	0	-1.601, 2	-1.614, 2
m_β	0.2	0.2	3.61, 1	3.61, 1
m_β	0.5	0	-9.41, 1	-9.46, 1
n_α	0	0.5	3.57, 1	3.57, 1
n_α	0.5	0.5	-1.97, 1	-1.97, 1
n_β	0.5	0	-2.878, 2	-2.877, 2
n_β	0.5	0.2	-2.934, 2	-2.934, 2
n_β	0.5	0.5	-3.016, 2	-3.016, 2
q_α	0	0.5	1.67, 1	1.67, 1

Example 5.17 An elliptic paraboloid has been analysed with the following data:-

Edge 1 - free   Edge 2 - free

Edge 3 - simply supported   Edge 4 - simply supported

$t = 7$ cm $L_\alpha = 1739.3$ cm $L_\beta = 1300$ cm

$k_\alpha = 5.0232, -4\ \text{cm}^{-1}$ $k_\beta = 8.6486, -4\ \text{cm}^{-1}$

$E = 2.0, 5\ \text{kg/cm}^2$ $\nu = 0$ $G_\gamma = 1.9, -2\ \text{kg/cm}^2$

$P = 1(1)1$ $Q = 0(0)0$

$P' = 1(1)1$ $Q' = 0(0)0$

The computations are made using:-

$l_\alpha = 1739.3$ cm $l_\beta = 1380.6$ cm

The solution is compared with that given by Scrivener(30) and Bouma(22) in table 5.17.

TABLE 5.17

	α_1	β_1	Programme	Scrivener	Bouma
u_γ	0.5	0	126	126	132
u_γ	0.5	0.5	48	48	51
n_α	0.5	0	2021	2020	2094
n_α	0.5	0.5	-2	-1	<10
m_β	0.5	0.5	-1550	-1550	-1619

<u>Example 5.18</u> A hyperbolic paraboloid has been analysed with the following data:-

Edge 1 - free   Edge 2 - free

Edge 3 - simply supported   Edge 4 - simply supported

$t = 7$ cm $L_\alpha = 1796.3$ cm $L_\beta = 1300$ cm

$l_\alpha = 1800$ cm $l_\beta = 1380.6$ cm

$k_\alpha = -1.2358, -4\ cm^{-1}$ $k_\beta = 8.6486, -4\ cm^{-1}$

$E = 2.0, 5\ kg/cm^2$ $\nu = 0$

$G_\gamma = 1.90, -2\ kg/cm^2$

$P = 1(1)1$ $Q = 0(0)0$

$P' = 1(1)1$ $Q' = 0(0)0$

The solution is compared with that given by Scrivener(30) and Bouma(22) in table 5.18.

TABLE 5.18

	α_1	β_1	Programme	Scrivener	Bouma
n_α	0.5	0	284	285	271
u_γ	0.5	0.25	0.8	0.8	0.9
m_β	0.5	0.5	-42	-47	-58
n_α	0.5	0.5	-153	-152	-157

Example 5.19 A hyperbolic paraboloid has been analysed with the following data:-

Edge 1 - free Edge 2 - free

Edge 3 - simply supported Edge 4 - simply supported

$t = 7$ cm $L_\alpha = 1739.3$ cm $L_\beta = 1300$ cm

$k_\alpha = -5.0232, -4\ cm^{-1}$ $k_\beta = 8.6486, -4$ cm

$E = 2.0, 5\ kg/cm^2$ $\nu = 0$ $G_\gamma = 1.90, -2\ kg/cm^2$

$P = 1(1)1$ $Q = 0(0)0$

$P' = 1(1)1$ $Q' = 0(0)0$

The computations are made using:-

$$l_\alpha = 1739.3 \text{ cm} \qquad l_\beta = 1380.6 \text{ cm}$$

The solution is compared with that given by Scrivener[30] and Bouma[22] in table 5.19.

TABLE 5.19

	α_1	β_1	Programme	Scrivener	Bouma
u_γ	0.5	0	-3.5	-3.5	<5
n_α	0.5	0	473	472	498
u_γ	0.5	0.5	66	65	70
m_β	0.5	0.5	2120	2120	2235
n_α	0.5	0.5	-84	-84	-91

V.3. Check on the computer programme by using the principle of superposition

Example 5.20 A hyperbolic paraboloid has been analysed with the following data:-

Edge 1 - clamped   Edge 2 - hinged

Edge 3 - hinged   Edge 4 - clamped

$t = 0.208333$ $L_\alpha = 60$ $L_\beta = 40$

$l_\alpha = 61.59$ $l_\beta = 41.06$

$k_\alpha = 1.2821, -2$ $k_\beta = 1.9231, -2$

$E = 4.5, 8$ $\nu = 0.15$

$P = 1(1)11 \qquad Q = 1(1)11$

$P' = 1(1)11 \qquad Q' = 1(1)11$

for the following cases of loading:-

(i) $G_\gamma = 50$, acting over the total surface area of the shell.

(ii) $G_\gamma = 50$, patch loading defined by $(\bar{x}_3, \bar{y}_3) = (0.5, 0.15)$, $(\bar{u}_3, \bar{v}_3) = (1.0, 0.3)$

(iii) $G_\gamma = 50$, patch loading defined by $(\bar{x}_3, \bar{y}_3) = (0.5, 0.45)$, $(\bar{u}_3, \bar{v}_3) = (1.0, 0.3)$

(iv) $G_\gamma = 50$, patch loading defined by $(\bar{x}_3, \bar{y}_3) = (0.5, 0.8)$, $(\bar{u}_3, \bar{v}_3) = (1.0, 0.4)$

Loading case (i) = Loading cases (i)+(ii)+(iii)+(iv)

The solution for loading cases (i), (ii), (iii), (iv) and [(ii)+(iii)+(iv)] are presented in table 5.20.

V.4. Solution of further problems

Example 5.21 A north light cylindrical shell (figure 5.7) has been analysed with the following data:-

Edge 1 - flexible beam Edge 2 - oblique gable, $\phi_2 = 36.0$

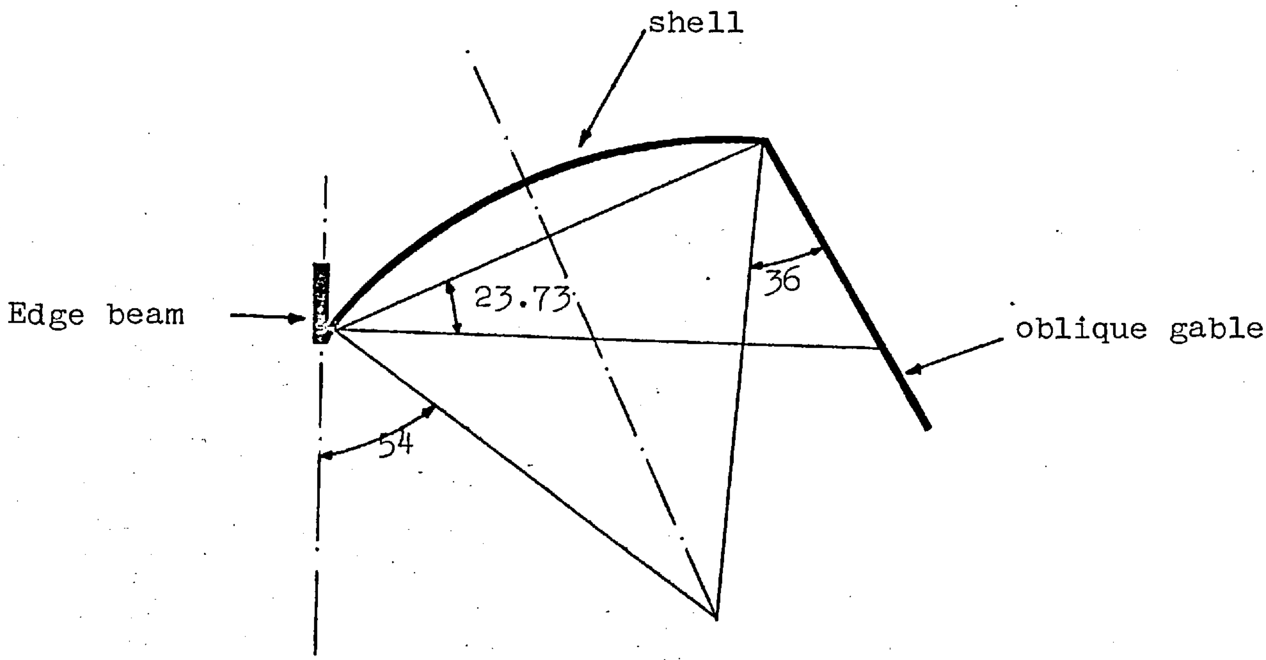
Edge 3 - clamped Edge 4 - clamped

$t = 0.208333 \qquad L_\alpha = 70 \qquad L_\beta = 30$

$l_\alpha = 70.00 \qquad l_\beta = 31.40$

TABLE 5.20

	α_1	β_1	case (i)	case(ii)	case(iii)	case(iv)	(ii)+(iii)+iv
u_γ	1/2	1/2	1.296, -3	-6.569, -4	2.718, -3	-7.658, -4	1.295, -3
u_γ	1/4	3/4	1.329, -3	-3.381, -4	-9.923, -4	2.659, -3	1.329, -3
m_β	1/2	0	-7.341, 1	-1.916, 2	7.251, 1	4.565, 1	-7.344, 1
m_β	1/2	1/4	4.482, 0	4.564, 1	-3.642, 1	-4.736, 0	4.484, 0
m_α	1	1/2	-9.402, 1	-8.301, 0	-9.519, 1	9.472, 0	-9.402, 1
n_α	1	1/4	1.209, 3	1.911, 3	1.858, 2	-8.883, 2	1.209, 3
n_α	1/4	3/4	1.204, 3	-4.114, 2	-9.697, 2	2.585, 3	1.204, 3
n_β	1/4	1/4	-1.883, 3	-6.709, 2	-7.666, 2	-4.456, 2	-1.883, 3
$n_{\alpha\beta}$	1/2	1	1.654, 1	-5.297, -1	8.078, 0	8.993, 0	1.654, 1
$n_{\alpha\beta}$	3/4	1/4	-1.963, 1	-1.696, 2	-2.144, 2	3.644, 2	-1.960, 1
q_β	1/4	0	2.595, 1	8.852, 1	-4.331, 1	-1.925, 1	2.596, 1
v_β	1/4	3/4	1.480, -5	7.327, -5	1.770, -4	-2.354, -4	1.487, -5



North light cylindrical shell (example 5.21)

Figure 5.7

$k_\alpha = 0$ $\qquad k_\beta = 3.3195, -2$

$\ddot{P} = 50$ $\qquad \psi = 54.0$

$E = 4.5, 8$ $\qquad \nu = 0.15$

$P = 1(2)19$ $\qquad Q = 1(1)19$

$P' = 1(2)19$ $\qquad Q' = 0(1)19$

Edge beam

$A = 4.5$ $\qquad I_1 = 7.58$ $\qquad I_2 = 0.375$

$I_3 = 1.287$ $\qquad a = 0.5$ $\qquad b = 2.0$

$Z_3 = 675$ $\qquad Z_2 = 0$ $\qquad \psi_1 = 54.0$

$E_b = 4.5, 8$

The solution for the shell and edge beam is presented in tables 5.21(a) and 5.21(b) respectively.

TABLES 5.21(a)

α_1	β_1	m_α	n_α	$n_{\alpha\beta}$
0	0	0	0	8.324, 4
0	1/4	-9.187, 2	-1.189, 4	-2.670, 4
0	1/2	-2.558, 2	1.892, 4	-2.570, 4
0	3/4	-3.524, 2	4.103, 3	-1.383, 4
0	1	0	0	-1.703, 4
1/8	1/2	-4.272, 1	4.810, 3	-1.712, 4
1/4	1/2	-2.571, 1	-1.583, 3	-1.189, 4
3/8	1/2	-1.406, 1	-5.555, 3	-5.983, 3
1/2	1/2	-1.148, 1	-6.818, 3	0

α_1	β_1	m_β	n_β	$n_{\alpha\beta}$
1/8	0	-7.158, 2	3.323, 2	-7.066, 3
1/4	0	-2.630, 2	-1.216, 3	-5.873, 3
3/8	0	-1.683, 2	-3.951, 2	-2.856, 3
1/2	0	-1.575, 2	1.623, 2	0
1/2	1/4	2.895, 2	-6.133, 2	0
1/2	1/2	-2.304, 2	-1.854, 3	0
1/2	3/4	-3.060, 1	-1.486, 3	0

α_1	β_1	u_γ	u_β
1/4	0	4.743, -2	-2.710, -2
1/2	0	7.228, -2	-3.947, -2
1/2	1/2	2.322, -2	-9.311, -3
1/2	1	2.816, -3	-3.877, -3

TABLE 5.21(b)

	α_1	Solution
N_1	1/4	-6.074, 4
N_1	1/2	-3.828, 3
M_1	1/4	3.122, 5
M_1	1/2	5.928, 5
M_2	1/2	-1.767, 4
M_3	0	-3.063, 3
S_1	0	2.735, 4
U_3	1/4	4.961, -2
U_3	1/2	7.402, -2
U_2	1/2	3.679, -2
U_1	0	1.073, -3

Example 5.22 An elliptic paraboloid has been analysed with the following data:-

Edge 1 - free Edge 2 - clamped

Edge 3 - simply supported Edge 4 - simply supported

$t = 0.208333$ $L_\alpha = 60$ $L_\beta = 60$

$l_\alpha = 61.59$ $l_\beta = 61.59$

$k_\alpha = 1.2821, -2$ $k_\beta = 1.2821, -2$

$E = 4.5, 8$ $\nu = 0.15$ $G_\gamma = 50$

$P = 1(1)1$ $Q = 0(0)0$

$P' = 1(1)1$ $Q' = 0(0)0$

This is a standard Levy type solution for a spherical cap ($k_\alpha = k_\beta$).

The solution is presented in tables 5.22.

TABLES 5.22

α_1	β_1	u_γ	m_α	m_β
1/2	0	1.013, 0	8.941, 2	0
1/2	1/8	6.952, -1	5.331, 2	-5.352, 2
1/2	1/4	4.700, -1	3.604, 2	-3.612, 2
1/2	3/8	3.189, -1	2.453, 2	-2.403, 2
1/2	1/2	2.169, -1	1.670, 2	-1.627, 2
1/2	5/8	1.481. -1	1.144, 2	-1.090, 2
1/2	3/4	1.022, -1	7.443, 1	-1.050, 2
1/2	7/8	7.136, -2	1.232, 2	4.017, 2
1/2	1	0	-6.325, 2	-4.217, 3

α_1	β_1	n_α	n_β
1/2	0	9.661, 3	0
1/2	1/8	-6.696, 3	1.658, 2
1/2	1/4	-6.152, 3	1.292, 3
1/2	3/8	-8.364, 3	3.396, 3
1/2	1/2	-1.177, 4	6.805, 3
1/2	5/8	-1.697, 4	1.205, 4
1/2	3/4	-2.526, 4	1.995, 4
1/2	7/8	-3.726, 4	3.185, 4
1/2	1	7.265, 3	4.843, 4

α_1	β_1	u_α	$n_{\alpha\beta}$	$m_{\alpha\beta}$
0	0	-2.567, -1	8.620, 0	6.724, 2
0	1/8	-1.733, -1	1.684, 3	5.358, 2
0	1/4	-1.168, -1	4.021, 3	3.570, 2
0	3/8	-7.829, -2	6.840, 3	2.410, 2
0	1/2	-5.184, -2	1.074, 4	1.625, 2
0	5/8	-3.330, -2	1.632, 4	1.097, 2
0	3/4	-1.978, -2	2.444, 4	7.010, 1
0	7/8	-9.144, -3	3.703, 4	7.721, 1
0	1	0	4.382, 4	0

Example 5.23 An elliptic paraboloid has been analysed with the following data:-

$t = 0.208333$ $\quad L_\alpha = 60$ $\quad L_\beta = 60$

$l_\alpha = 61.59$ $\quad l_\beta = 61.59$

$k_\alpha = 1.2821, -2$ $\quad k_\beta = 1.2821, -2$

$E = 4.5, 8$ $\quad \nu = 0.15$ $\quad G_\gamma = 50$

$P = 1(2)19$ $\quad Q = 1(2)19$

$P' = 1(2)19$ $\quad Q' = 1(2)19$

The following boundary conditions are considered:-

(i) All edges simply supported

(ii) All edges supported on oblique gables

$\phi_i = 22.62$ (i = 1,2,3,4)

(iii) All edges hinged

TABLE 5.23

	α_1	β_1	case (i)	case(ii)	case(iii)	case(iv)
u_γ	1/2	0	0	-3.755, -4	0	0
	1/2	1/8	3.451, -3	3.452, -3	1.527, -3	1.524, -3
	1/2	1/4	3.240, -3	3.218, -3	1.446, -3	1.525, -3
	1/2	3/8	3.241, -3	3.219, -3	1.446, -3	1.517, -3
	1/2	1/2	3.238, -3	3.216, -3	1.442, -3	1.514, -3
m_β	1/8	0	0	0	0	-1.224, 2
	1/4	0	0	0	0	-1.046, 2
	3/8	0	0	0	0	-1.029, 2
	1/2	0	0	0	0	-1.031, 2
	1/2	1/8	1.063, 1	1.168, 1	3.893, 0	1.085, 1
	1/2	1/4	4.990, -1	6.295, -2	1.116, 0	9.269, -1
	1/2	3/8	2.446, -1	6.342, -3	-6.735, -2	-7.655, -2
	1/2	1/2	-1.431, 0	-1.669, 0	-1.660, 0	-1.651, 0
n_β	1/8	0	0	-2.327, 1	-1.845, 3	-1.663, 3
	1/4	0	0	-2.265, 1	-1.908, 3	-1.837, 3
	3/8	0	0	-2.262, 1	-1.898, 3	-1.847, 3
	1/2	0	0	-2.220, 1	-1.887, 3	-1.843, 3
	1/2	1/8	-7.811, 2	-7.945, 2	-1.914, 3	-1.884, 3
	1/2	1/4	-1.415, 3	-1.422, 3	-1.931, 3	-1.917, 3
	1/2	3/8	-1.812, 3	-1.814, 3	-1.942, 3	-1.938, 3
	1/2	1/2	-1.946, 3	-1.946, 3	-1.945, 3	-1.946, 3
$n_{\alpha\beta}$	0	0	-6.460, 3	-6.248. 3	-4.143, 2	-2.967, 2
	0	1/8	-3.760, 3	-3.689, 3	-1.832, 2	-3.078, 2
	0	1/4	-1.893, 3	-1.854, 3	-4.612, 1	-1.067, 2
	0	3/8	-8.421, 3	-8.247, 2	-2.363, 1	-4.332, 1

(iv) All edges clamped

The solutions are presented in table 5.23.

Example 5.24 An elliptic paraboloid has been analysed with the following data:-

Edge 1 - clamped Edge 2 - clamped

Edge 3 - hinged Edge 4 - hinged

$t = 0.208333$ $L_\alpha = 60$ $L_\beta = 60$

$l_\alpha = 61.59$ $l_\beta = 61.59$

$k_\alpha = 1.2821, -2$ $k_\beta = 1.2821, -2$

$E = 4.5, 8$ $\nu = 0.15$

$T_1 = -25$ $T_2 = 15$ $\lambda = 1.0, -5$

$P = 1(2)19$ $Q = 1(2)19$

$P' = 1(2)19$ $Q' = 1(2)19$

The solution is presented in tables 5.24

TABLES 5.24

α_1	β_1	m_β	n_β	$n_{\alpha\beta}$
1/8	0	-7.382, 2	-1.661, 2	5.657, 2
1/4	0	-7.698, 2	1.483, 2	2.581, 2
3/8	0	-7.394, 2	8.078, 1	6.390, 1
1/2	0	-7.131, 2	3.575, 1	0

α_1	β_1	u_γ	m_β	n_β
1/2	1/8	4.046, -3	-7.335, 2	9.970, 1
1/2	1/4	3.831, -3	-7.458, 2	1.578, 2
1/2	3/8	3.711, -3	-7.453, 2	1.914, 2
1/2	1/2	3.627, -3	-7.467, 2	2.021, 2

α_1	β_1	n_α	$n_{\alpha\beta}$
0	0	0	3.752, 2
0	1/8	-6.490, 2	-4.730, 2
0	1/4	-3.138, 2	-7.562, 1
0	3/8	-3.002, 2	-4.968, 1
0	1/2	-3.388, 2	0

α_1	β_1	u_γ	m_α	n_α
1/8	1/2	4.301, -3	-7.833, 2	-3.697, 2
1/4	1/2	3.746, -3	-7.440, 2	-3.620, 2
3/8	1/2	3.691, -3	-7.450, 2	-3.410, 2

Example 5.25 A hyperbolic paraboloid has been analysed with the following data:-

$t = 0.208333$ $\qquad L_\alpha = 50$ $\qquad L_\beta = 60$

$l_\alpha = 51.32$ $\qquad l_\beta = 61.59$

$k_\alpha = -1.5385, -2$ $\qquad k_\beta = 1.2821, -2$

$E = 4.5, 8$ $\qquad \nu = 0.15$ $\qquad G_\gamma = 50$

$P = 1(2)19$ $\qquad Q = 1(2)19$

$P' = 1(2)19$ $\qquad Q' = 1(2)19$

The following boundary conditions are considered:-

(i) All edges simply supported

(ii) All edges supported on oblique gables

$\phi_i = 22.62$ $(i = 1,2,3,4)$

(iii) All edges hinged

(iv) All edges clamped.

The solutions are presented in table 5.25.

Example 5.26 A hyperbolic paraboloid has been analysed with the following data:-

Edge 1 - hinged $\qquad$ Edge 2 - hinged

Edge 3 - hinged $\qquad$ Edge 4 - hinged

$t = 4$ cm $\qquad L_\alpha = 4960$ cm $\qquad L_\beta = 1240$cm

$l_\alpha = 5297.2$ cm $\qquad l_\beta = 1261.4$ cm

$k_\alpha = 2.3563, -4$ cm^{-1} $\qquad k_\beta = -5.0710, -4$ cm^{-1}

$E = 3.4, 5$ kg/cm^2 $\qquad \nu = 0.1667$

$T_1 = 51$ C $\qquad T_2 = 51$ C $\qquad \lambda = 1.0, -5/C$

TABLE 5.25

	α_1	β_1	case (i)	case(ii)	case(iii)	case(iv)
u_γ	1/2	0	0	-3.753, -2	0	0
	1/2	1/8	1.856, -1	1.442, -1	1.590, -3	1.542, -3
	1/2	1/4	3.469, -1	2.665, -1	1.567, -3	1.613, -3
	1/2	3/8	4.594, -1	3.542, -1	1.680, -3	1.824, -3
	1/2	1/2	4.999, -1	3.860, -1	1.724, -3	1.915, -3
m_β	1/8	1/4	0	0	0	-1.476, 2
	1/4	0	0	0	0	-1.282, 2
	3/8	0	0	0	0	-1.251, 2
	1/2	0	0	0	0	-1.260, 2
	1/2	1/8	1.551, 2	1.987, 2	1.600, 0	8.581, 0
	1/2	1/4	3.489, 2	2.533, 2	9.882, -1	1.623, -1
	1/2	3/8	5.211, 2	4.060, 2	4.715, -1	9.580, -1
	1/2	1/2	5.865, 2	4.597, 2	-8.113, -1	-9.905, -2
n_β	1/8	0	0	-2.536, 2	-1.638, 3	-1.403, 3
	1/4	0	0	-3.908, 2	-1.610, 3	-1.656, 3
	3/8	0	0	-4.861, 2	-1.553, 3	-1.589, 3
	1/2	0	0	-5.178, 2	-1.553, 3	-1.572, 3
	1/2	1/8	-1.223, 4	-9.858, 3	-1.551, 3	-1.568, 3
	1/2	1/4	-2.295, 4	-1.801, 4	-1.549, 3	-1.563, 3
	1/2	3/8	-3.083, 4	-2.417, 4	-1.554, 3	-1.575, 3
	1/2	1/2	-3.385, 4	-2.649, 4	-1.563, 3	-1.594, 3
$n_{\alpha\beta}$	0	0	-2.886, 4	-2.303, 4	3.216, 1	1.815, 1
	0	1/8	-2.787, 4	-1.747, 4	-4.002, 1	2.882, 0
	0	1/4	-2.130, 4	-1.371, 4	-8.384, -1	-9.522, 0
	0	3/8	-1.127, 4	-7.178, 3	4.520, 0	1.635, 0

$$P = 1(2)29 \qquad Q = 1(2)29$$
$$P' = 1(2)19 \qquad Q' = 1(2)19$$

The solution is presented in tables 5.26.

This problem has been solved by Beles and Soare[13] assuming u_γ to be of the form

$$u_\gamma = c \sin(\bar{\alpha}) \sin(\bar{\beta})$$

where c is a constant.

However, the profile of u_γ obtained by the author is totally different from that assumed by Beles and Soare (tables 5.26).

TABLES 5.26

α_1	β_1	n_β	$n_{\alpha\beta}$
1/8	0	-4.134, 2	-5.834, 1
1/4	0	-4.381, 2	-1.992, 1
3/8	0	-4.531, 2	6.017, 0
1/2	0	-4.569, 2	0

α_1	β_1	u_γ	m_β	n_β
1/2	1/8	5.812, -1	4.435, 1	-4.423, 2
1/2	1/4	6.006, -1	-5.189, 0	-4.452, 2
1/2	3/8	5.833, -1	-2.769, 0	-4.458, 2
1/2	1/2	5.843, -1	1.181, 0	-4.455, 2

α_1	β_1	n_α	$n_{\alpha\beta}$
0	0	0	-2.152, 2
0	1/8	-8.205, 2	-6.392, 1
0	1/4	-9.118, 2	-2.470, 2
0	3/8	-9.099, 2	-8.132, 0
0	1/2	-9.077, 2	0

α_1	β_1	u_γ	m_β	n_α
1/8	1/2	6.771, -1	5.590, 0	-9.086, 2
1/4	1/2	6.816, -1	1.015, 1	-9.389, 2
3/8	1/2	6.069, -1	2.937, -2	-9.402, 2
1/2	1/2	5.843, -1	-6.766, 0	-9.409, 2

CHAPTER VI

RULED SURFACES - CHECKS ON THE COMPUTER PROGRAMME AND SOLUTION OF CERTAIN PROBLEMS WHOSE RANGE EXTENDS BEYOND THOSE NORMALLY INCLUDED IN THE STANDARD LITERATURE

P, Q represent the complementary solution harmonics considered in the α and β directions respectively.

P', Q' represent the particular solution harmonics considered in the α and β directions respectively.

All computations are made on the assumption that $l_\alpha = L_\alpha$ and $l_\beta = L_\beta$.

α_1, β_1 are defined by:-

$$\alpha_1 = \frac{\alpha}{l_\alpha}$$

$$\beta_1 = \frac{\beta}{l_\beta}$$

VI.1. Assumptions made in the solution of the shallow curved plate equations and their effect on the accuracy of the numerical results

The shallow curved plate equations have been solved making the following assumptions:-

(i) the representation of external loads on the shell by truncated Fourier expansions,

(ii) the representation of certain complementary solution 'edge disturbances' by truncated Fourier expansions.

The first assumption affects only the particular solution and the second only the complementary solution. Hence, the influence of these assumptions on the accuracy of the numerical results can be studied separately.

Particular solution

Consider the following examples:-

Example 6.1. A hyperbolic paraboloid with the following data:-

$t = 0.25^{x}$ $\quad L_\alpha = 80$ $\quad L_\beta = 60$

$k_{\alpha\beta} = 0.005$ $\quad E = 3, 8^{xx}$ $\quad \nu = 0.15$

$G_\gamma = 50$

The particular solution is presented for the following cases:-

(i) $P' = Q' = 1(2)9$

(ii) $P' = Q' = 1(2)19$

(iii) $P' = Q' = 1(2)29$

in table 6.1

---

[x]All units are pound, feet units unless otherwise stated.

[xx]The notation x, y represents $x \times 10^{y}$ (floating point notation).

TABLE 6.1

	α_1	β_1	case(i)	case(ii)	case(iii)
u_γ	1/8	1/4	4.847, -2	4.798, -2	4.799, -2
u_γ	1/2	1/2	3.501, -3	3.099, -3	3.114, -3
u_α	1/8	1/4	-5.701, -3	-5.704, -3	-5.704, -3
m_β	1/8	1/8	3.693, 2	3.365, 2	3.373, 2
m_β	1/2	1/2	-1.722, 2	-1.882, 2	-1.859, 2
n_β	0	0	-1.651, 4	-1.669, 4	-1.670, 4
$n_{\alpha\beta}$	1/8	1/8	-3.977, 3	-3.949, 3	-3.950, 3
$n_{\alpha\beta}$	1/2	1/2	-5.063, 3	-5.062, 3	-5.062, 3
$m_{\alpha\beta}$	0	1/8	-1.489, 2	-1.548, 2	-1.536, 2
$m_{\alpha\beta}$	1/4	1/4	5.118, 1	5.173, 1	5.172, 1
q_β	1/8	0	1.406, 2	1.606, 2	1.707, 2
q_β	1/4	1/4	-3.747, 1	-4.513, 1	-4.629, 1

Example 6.2. A hyperbolic paraboloid with the following data:-

$t = 0.25 \qquad L_\alpha = 80 \qquad L_\beta = 60$

$k_{\alpha\beta} = 0.005 \qquad E = 3, 8 \qquad \nu = 0.15$

$G_\alpha = 10 \qquad G_\beta = 12 \qquad G_\gamma = 50$

The particular solution is presented for the following cases:-

(i) $P' = Q' = 0(1)9$

(ii) $P' = Q' = 0(1)14$

(iii) $P' = Q' = 0(1)19$

in table 6.2.

TABLE 6.2

	α_1	β_1	case(i)	case(ii)	case(iii)
u_γ	1/4	1/4	4.231, -2	4.253, -2	4.251, -2
u_γ	1/2	1/2	3.501, -3	3.189, -3	3.099, -3
u_β	0	1/2	-3.190, -3	-3.203, -3	3.206, -3
u_β	3/4	1/4	-3.215, -4	-3.304, -4	3.291, -4
m_β	1/4	1/4	8.325, 1	8.945, 1	9.058, 1
m_β	1/2	1/2	-1.722, 2	-1.811, 2	-1.882, 2
n_α	1	1	2.109, 4	2.115, 4	2.117, 4
n_α	1/4	1/4	-1.611, 3	-1.616, 3	-1.614, 3
$n_{\alpha\beta}$	1/4	1/4	-5.770, 3	-5.772, 3	-5.772, 3
$n_{\alpha\beta}$	1/2	1/2	-5.063, 3	-5.062, 3	-5.062, 3
$m_{\alpha\beta}$	0	0	-3.927, 2	-4.196, 2	-4.341, 2
$m_{\alpha\beta}$	1/4	1/4	5.118, 1	5.164, 1	5.173, 1
q_β	1/4	0	1.268, 2	1.459, 2	1.600, 2
q_β	1/2	3/4	4.386, 1	5.656, 1	5.203, 1

Complementary solution

The influence of the second assumption on the accuracy of the numerical results can be studied by taking a constant number of harmonics for the particular solution and varying the number of harmonics considered for the complementary solution.

Consider the following examples:-

Example 6.3. A hyperbolic paraboloid with the following data:-

Edge 1 - Clamped Edge 2 - Clamped

Edge 3 - Clamped Edge 4 - Clamped

$t = 0.25$ $L_\alpha = 80$ $L_\beta = 60$

$k_{\alpha\beta} = 0.005$ $E = 4.5, 8$ $\nu = 0.15$

$G_\gamma = 50$

$P' = 1(2)9$ $Q' = 1(2)9$

The total solution is presented for the following cases:-

(i) $P = Q = 1(2)9$

(ii) $P = Q = 1(2)19$

(iii) $P = Q = 1(2)29$

in table 6.3.

TABLE 6.3

	α_1	β_1	case(i)	case(ii)	case(iii)
u_γ	1/4	1/4	1.568, -2	1.568, -2	1.568, -2
u_γ	1/2	1/2	1.069, -2	1.069, -2	1.069, -2
u_β	1/4	1/4	2.425, -4	2.425, -4	2.425, -4
m_β	1/2	0	-2.686, 2	-2.718, 2	-2.692, 2
m_β	1/2	1/2	-2.602, 1	-2.602, 1	-2.602, 1
m_β	1/4	1/8	9.599, 1	9.599, 1	9.600, 1
m_α	0	1/4	-5.043, 2	-4.983, 2	-5.031, 2
n_β	1/8	0	-4.356, 3	-4.360, 3	-4.359, 3
n_β	3/8	1/4	2.816, 2	2.817, 2	2.817, 2
n_α	0	1/8	-4.514, 3	-4.511, 3	-4.514, 3
n_α	1/4	1/8	-1.052, 3	-1.053, 3	-1.053, 3
$n_{\alpha\beta}$	3/8	0	-4.375, 3	-4.366, 3	-4.366, 3
$n_{\alpha\beta}$	1/2	1/2	-5.115, 3	-5.115, 3	-5.115, 3
$n_{\alpha\beta}$	0	1/2	-4.141, 3	-4.159, 3	-4.155, 3
q_α	0	1/8	1.354, 2	1.457, 2	1.489, 2
q_β	1/4	0	4.702, 1	4.576, 1	4.789, 1

Example 6.4. A hyperbolic paraboloid with the following data:-

Edge 1 - Clamped Edge 2 - hinged

Edge 3 - Clamped Edge 4 - free-guided

$t = 0.25$ $L_\alpha = 80$ $L_\beta = 60$

$k_{\alpha\beta} = 0.005$ $E = 4.5, 8$ $\nu = 0.15$

$G_\gamma = 50$

$P' = 1(2)9$ $Q' = 1(2)9$

The total solution is presented for the following cases:-

(i) $P = Q = 1(1)9$

(ii) $P = Q = 1(1)14$

(iii) $P = Q = 1(1)19$

in table 6.4.

Example 6.5. A hyperbolic paraboloid with the following data:-

Edge 1 - hinged Edge 2 - hinged

Edge 3 - Clamped Edge 4 - free

$t = 0.25$ $L_\alpha = 80$ $L_\beta = 60$

$k_{\alpha\beta} = 0.005$ $E = 4.5, 8$ $\nu = 0.15$

$G_\gamma = 50$

$P' = 1(2)9$ $Q' = 1(2)9$

The total solution is presented for the following cases:-

(i) $P = 1(1)9$, $Q = 1(2)9$

(ii) $P = 1(1)19$, $Q = 1(2)19$

(iii) $P = 1(1)29$, $Q = 1(2)29$

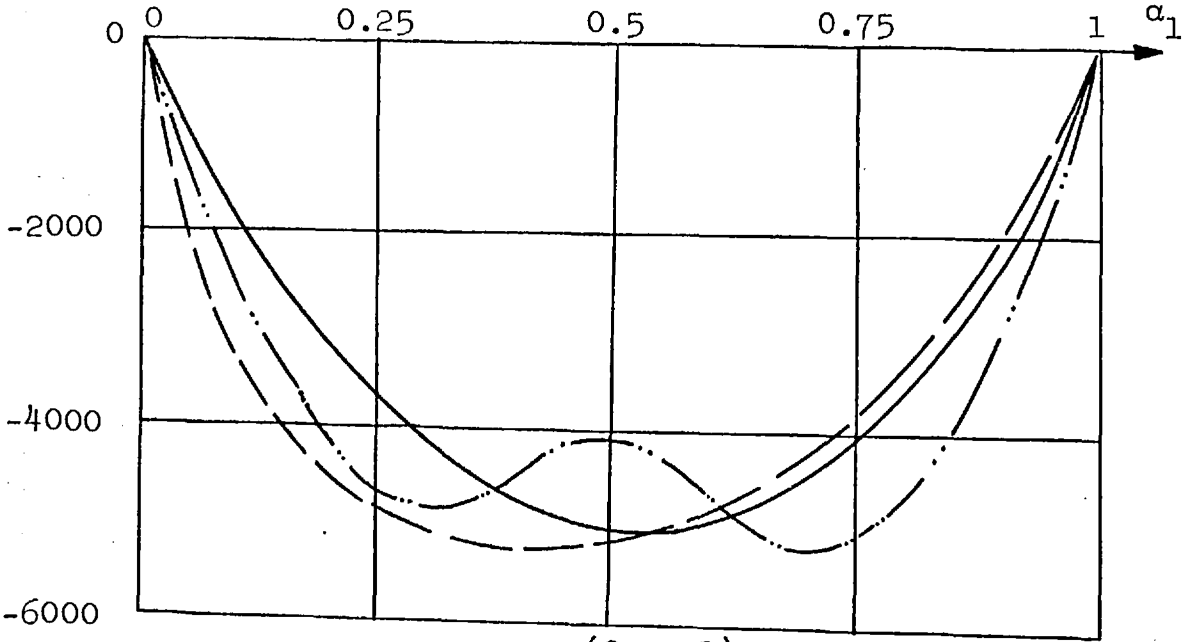
in table 6.5.

TABLE 6.4

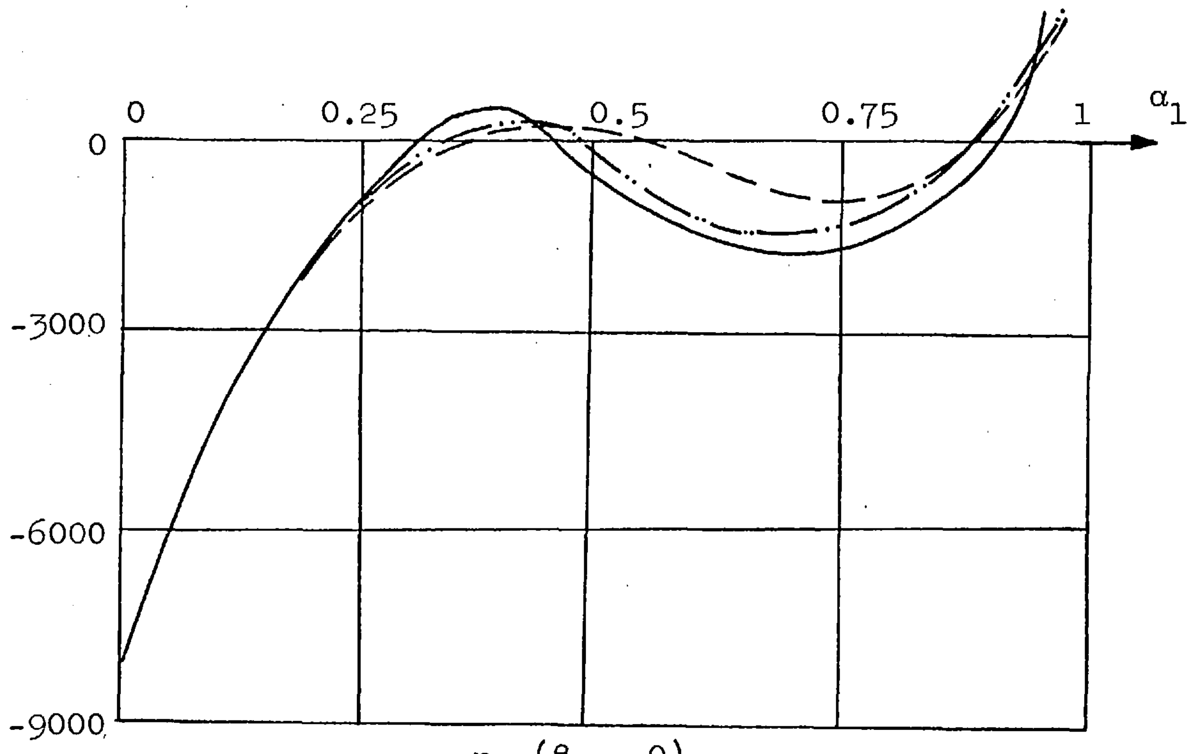
	α_1	β_1	case(i)	case(ii)	case(iii)
u_γ	3/4	1/4	1.640, -2	1.640, -2	1.640, -2
u_γ	1/2	1/2	9.525, -3	9.525, -3	9.525, -3
u_β	1	1/4	1.433, -3	1.433, -3	1.433, -3
u_α	3/4	1/4	-1.743, -4	-1.743, -4	-1.743, -4
m_β	1/4	0	-4.075, 2	-4.026, 2	-3.965, 2
m_β	1/4	1/4	7.693, 1	7.691, 1	7.692, 1
m_α	0	3/4	-5.042, 2	-5.116, 2	-5.147, 2
m_α	3/4	1/2	-8.699, 1	-8.700, 1	-8.700, 1
n_α	0	0	-2.966, 3	-3.035, 3	-3.044, 3
n_α	3/4	3/4	-6.903, 2	-6.903, 2	-6.903, 2
n_β	0	0	-5.565, 3	-5.617, 3	-5.667, 3
n_β	1	1/4	9.327, 3	9.331, 3	9.331, 3
n_β	1/4	1/4	-2.664, 2	-2.664, 2	-2.664, 2
$n_{\alpha\beta}$	1/2	1	-4.676, 3	-4.678, 3	4.674, 3
$n_{\alpha\beta}$	3/4	1/2	-5.592, 3	-5.592, 3	-5.592, 3
$n_{\alpha\beta}$	0	1/4	-2.217, 3	-2.217, 3	-2.218, 3
q_β	1/4	1	2.267, 1	2.263, 1	2.266, 1
q_α	0	1/4	1.023, 2	1.011, 2	9.387, 1

TABLE 6.5

	α_1	β_1	case(i)	case(ii)	case(iii)
u_γ	1/4	1/2	1.408, -2	1.418, -2	1.424, -2
u_γ	1/2	1/2	1.138, -2	1.157, -2	1.167 ,-2
u_γ	3/4	1/2	9.685, -3	9.812, -3	9.884, -3
u_γ	1	1/2	8.124, -2	8.145, -2	8.147, -2
u_γ	1	1/4	5.930, -2	6.131, -2	6.231, -2
u_α	1	1/4	2.778, -4	4.442, -4	5.293, -4
u_β	1	1/2	4.687, -3	4.695, -3	4.709, -3
m_α	0	1/8	-5.580, 2	-5.523, 2	-5.557, 2
m_α	0	1/4	-5.151, 2	-5.218, 2	-5.236, 2
m_α	0	3/8	-4.199, 2	-4.221, 2	-4.214, 2
m_α	0	1/2	-3.931, 2	-3.831, 2	-3.813, 2
$n_{\alpha\beta}$	0	1/8	7.822, 0	-1.189, 2	-1.340, 2
$n_{\alpha\beta}$	0	1/4	-1.957, 3	-1.964, 3	-1.996, 3
$n_{\alpha\beta}$	0	3/8	-3.530, 3	-3.526, 3	-3.535, 3
$n_{\alpha\beta}$	0	1/2	-4.005, 3	-4.085, 3	-4.080, 3
n_α	0	0	-2.497, 2	-1.088, 3	-1.422, 3
n_α	0	1/8	-2.740, 3	-3.352, 3	-3.718, 3
n_α	0	1/4	-1.498, 3	-1.846, 3	-2.025, 3
n_α	0	3/8	-5.751, 2	-6.544, 2	-6.993, 2
m_α	3/4	1/4	-1.281, 2	-1.242, 2	-1.224, 2
m_β	1/2	1/8	8.595, 1	1.038, 2	1.137, 2
n_α	1/2	1/8	5.425, 2	-5.973, 1	-3.756, 2
n_β	3/4	1/4	-1.192, 3	-1.191, 3	-1.196, 3
$n_{\alpha\beta}$	1/2	3/8	-4.980, 3	-4.997, 3	-5.006, 3



$n_{\alpha\beta}$ $(\beta_1 = 0)$

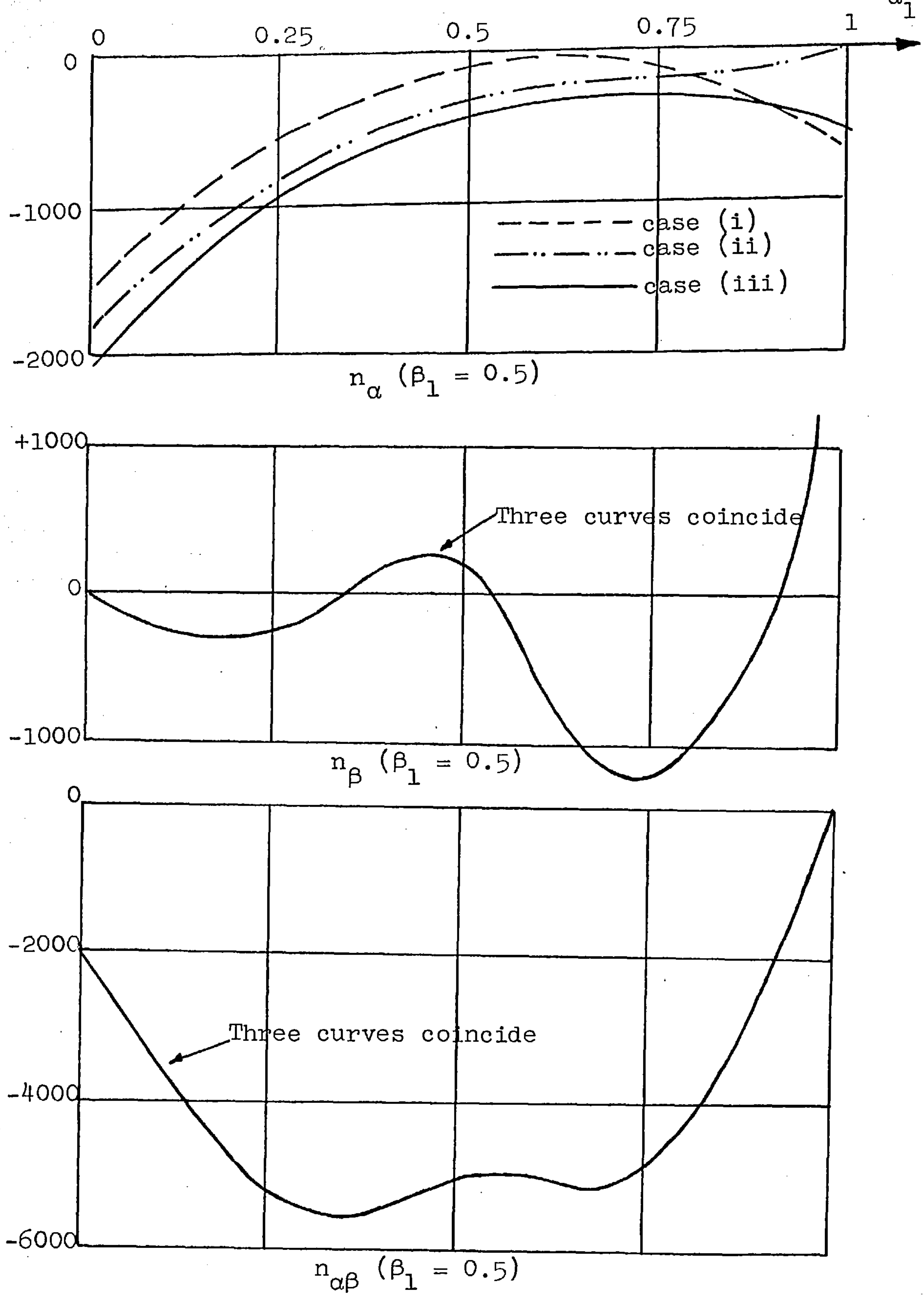


n_{β} $(\beta_1 = 0)$

case (i)
case (ii)
case (iii)

(Example 6.5)

Figure 6.1



(Example 6.5)

Figure 6.2

The congergence of n_α to zero is extremely slow along the free edge (edge 4). This affects the distribution of n_α throughout the shell and the distribution of $n_{\alpha\beta}$ and n_β along the edges adjacent to the free edge (edges 1 and 2) (see figures 6.1 and 6.2). However, the convergence of the moments, n_β and $n_{\alpha\beta}$ (other than along edges 1 and 2) is satisfactory. This difficulty is also encountered with a simply supported edge.

Discussion on the convergence of the solutions

For a shell with any combination of clamped, hinged and free-guided boundaries, under the action of patch loadings in the α, β and γ directions, the convergence of the solution is extremely rapid. In this case 10 harmonics in each direction are adequate. However, when the loading and boundary conditions are symmetric about the axes $\alpha = \frac{l_\alpha}{2}$ and $\beta = \frac{l_\beta}{2}$ 5 harmonics in each direction may suffice.

When there is a free edge or a simply supported edge the convergence of certain extensional actions is extremely slow (refer example 6.5).

VI.2. Check on the computer programme by comparing with solutions given in the literature

Example 6.6. A flat plate has been analysed with the following data:-

Edge 1 - clamped   Edge 2 - clamped

Edge 3 - clamped   Edge 4 - clamped

$t = 0.25$ $L_\alpha = 75$ $L_\beta = 50$

$k_{\alpha\beta} = 1, -10$ $E = 4.32, 8$ $\nu = 0.3$

$G_\gamma = 50$

$P = 1(2)13$ $Q = 1(2)13$

$P' = 1(2)9$ $Q' = 1(2)9$

The solution is compared with that given by Timoshenko(31) in table 6.6.

TABLE 6.6

	α_1	β_1	Computer programme	Timoshenko
m_α	0	0.5	-7.101, 3	-7.120, 3
m_β	0.5	0	-9.439, 3	-9.460, 3
u_γ	0.5	0.5	1.111, 0	1.110, 0
m_α	0.5	0.5	2.553, 3	2.540, 3
m_β	0.5	0.5	4.613, 3	4.600, 3

Example 6.7. A flat plate has been analysed with the following data:-

Edge 1 - clamped Edge 2 - clamped

Edge 3 - clamped Edge 4 - simply supported

$t = 0.20$ $L_\alpha = 40$ $L_\beta = 30$

$k_{\alpha\beta} = 1, -10$ $E = 4.32, 8$ $\nu = 0.3$

$G_\gamma = 50$

$P = 1(1)9$ $Q = 1(2)13$

$P' = 1(2)9$ $Q' = 1(2)9$

The solution is compared with that given by Timoshenko[31] in table 6.7.

TABLE 6.7

	α_1	β_1	computer programme	Timoshenko
m_α	0	0.5	-2.564, 3	-2.570, 3
m_β	0.5	0	-3.368, 3	-3.378, 3
u_γ	0.5	0.5	2.769, -1	2.750, -1

Examples 6.6 and 6.7 were also solved using the computer programme for translational shells and identical results were obtained (refer section V.2).

Example 6.8. A hyperbolic paraboloid has been analysed with the following data:-

Edge 1 - clamped Edge 2 - clamped

Edge 3 - clamped Edge 4 - clamped

$t = 0.8$ cm $L_\alpha = 100$ cm $L_\beta = 100$ cm

$k_{\alpha\beta} = 4, -3\ \text{cm}^{-1}$ $E = 3, 4\ \text{kg/cm}^2$ $\nu = 0.4$

$G_\gamma = 0.1\ \text{kg/cm}^2$

$P = 1(2)19$ $Q = 1(2)19$

$P' = 1(2)19$ $Q' = 1(2)19$

The solution is in good agreement with the experimental and theoretical results obtained by Brebbia[27]. Some of the results obtained are presented in table 6.8. The solutions obtained by Brebbia were presented in the form of graphs and are not reproduced here.

Example 6.9. A hyperbolic paraboloid has been analysed with the following data:-

Edge 1 - clamped Edge 2 - clamped

Edge 3 - clamped Edge 4 - clamped

$t = 0.25$ in $L_\alpha = 12.92$ in $L_\beta = 12.92$ in

$k_{\alpha\beta} = -3.1008, -2\ \text{in}^{-1}$ $E = 5, 5\ \text{lb/in}^2$ $\nu = 0.39$

$G_\gamma = 1.0\ \text{lb/in}^2$

$P = 1(2)19$ $Q = 1(2)19$

$P' = 1(2)19$ $Q' = 1(2)19$

The solution is compared with that given by Chetty[36] in table 6.9.

TABLE 6.8

α_1	β_1	u_γ	m_α	m_β	n_α	$n_{\alpha\beta}$	n_β
0	0	0	0	0	-16.1	0	-16.1
0	1/8	0	-4.3	-1.7	-13.9	1.6	-10.3
0	1/4	0	-5.6	-2.2	- 9.2	-0.8	- 1.9
0	3/8	0	-5.0	-2.0	- 3.8	-6.6	2.0
0	1/2	0	-4.6	-1.9	0	-8.5	0
1/4	0	0	-2.2	-5.6	-0.8	-1.9	- 9.2
1/4	1/8	.132	0.8	1.0	-7.0	-7.5	- 6.5
1/4	1/4	.234	1.4	1.4	-4.9	-10.8	- 4.9
1/4	3/8	.245	1.2	0.6	-1.5	-11.8	- 2.6
1/4	1/2	.237	1.0	0.2	0	-11.9	0
1/2	0	0	-1.9	-4.6	0	- 8.5	0
1/2	1/8	.126	0.07	0.5	0	- 9.0	0
1/2	1/4	.237	0.23	1.0	0	-11.9	0
1/2	3/8	.244	-0.24	0.03	0	-13.8	0
1/2	1/2	.231	-0.47	-0.5	0	-14.3	0

TABLE 6.9

α_1	β_1	u_γ		m_β		$n_{\alpha\beta}$	
		Programme	Chetty	Programme	Chetty	Programme	Chetty
1/2	0	0	0	-2.361	-2.451	6.093	6.173
1/2	1/16	8.013, -4	8.079, -4	-1.039	-0.998	6.676	7.035
1/2	1/8	2.526, -3	2.507, -3	-0.1743	-0.1118	8.236	8.639
1/2	3/16	4.450, -3	4.361, -3	0.3243	0.3628	10.32	10.63
1/2	1/4	6.164, -3	5.987, -3	0.5561	0.5632	12.51	12.64
1/2	5/16	7.493, -3	7.232, -3	0.6144	0.6036	14.47	14.41
1/2	3/8	8.402, -3	8.076, -3	0.5961	0.5718	15.97	15.76
1/2	7/16	8.921, -3	8.554, -3	0.5598	0.5290	16.91	16.60
1/2	1/2	9.088, -3	8.708, -3	0.5410	0.5108	17.22	16.88

Example 6.10. A hyperbolic paraboloid has been analysed with the following data:-

Edges 1 and 2 - $n_\beta = 0$, $u_\gamma = 0$, $m_\beta = 0$, $n_{\alpha\beta} = 10$ kg/m

Edges 3 and 4 - $n_\alpha = 0$, $u_\gamma = 0$, $m_\alpha = 0$, $n_{\alpha\beta} = 10$ kg/m

$t = 0.2$ m $\quad L_\alpha = 20$ m $\quad L_\beta = 20$ m

$k_{\alpha\beta} = -2.5, -2$ m^{-1} $\quad E = 1.0, 6$ kg/m^2 $\quad \nu = 0$

$G_\gamma = 0$

$P = 1(2)9$ $\quad Q = 1(2)9$

$P' = 1(2)9$ $\quad Q' = 1(2)9$

The solution is compared with that given by Duddeck[33] in table 6.10. The method of solution employed in this thesis is the same as that used by Duddeck.

TABLE 6.10

α_1	β_1	u_γ		m_α		n_α	
		Programme	Duddeck	Programme	Duddeck	Programme	Duddeck
0.1	0	0	0	0	0	136	135
0.1	0.1	-0.03	-0.03	-0.01	0	- 23	- 23
0.1	0.2	-0.06	-0.07	-1.6	-1.6	- 27	- 27
0.1	0.3	-0.09	-0.09	-3.7	-3.7	- 12	- 15
0.1	0.4	-0.11	-0.11	-5.2	-5.2	- 3	- 3
0.1	0.5	-0.11	-0.11	-5.7	-5.7	0	0
0.5	0	0	0	0	0	0	0
0.5	0.1	-0.11	-0.11	-1.7	-1.7	0	0
0.5	0.2	-0.19	-0.19	-2.5	-2.4	0	0
0.5	0.3	-0.24	-0.24	-2.5	-2.4	0	0
0.5	0.4	-0.26	-0.26	-2.2	-2.0	0	0
0.5	0.5	-0.27	-0.26	-2.0	-1.9	0	0

TABLE 6.10 (continued)

α_1	β_1	$n_{\alpha\beta}$		$m_{\alpha\beta}$		q_α	
		Programme	Duddeck	Programme	Duddeck	Programme	Duddeck
0	0	0	0	5.2	5.1	0	0
0	0.1	12	12	5.7	5.7	0.5	0.5
0	0.2	9	9	5.5	5.4	-1.7	-1.7
0	0.3	11	11	3.9	3.9	-3.7	-3.7
0	0.4	9	9	1.9	1.9	-4.8	-4.7
0	0.5	11	10	0	0	-5.0	-5.0
0.3	0	11	11	3.9	3.9	0	0
0.3	0.1	27	27	2.8	2.8	-1.1	-1.0
0.3	0.2	12	12	0.9	0.9	-0.7	-0.7
0.3	0.3	2	2	-0.1	-0.1	0.2	0.2
0.3	0.4	- 2	- 2	-0.2	-0.2	0.8	0.8
0.3	0.5	- 3	- 3	0	0	1.1	1.1

VI.3. Check on the computer programme by using the principle of superposition

Example 6.11. A hyperbolic paraboloid has been analysed with the following data:-

Edge 1 - clamped Edge 2 - clamped

Edge 3 - hinged Edge 4 - hinged

$t = 0.25$ $L_\alpha = 80$ $L_\beta = 60$

$k_{\alpha\beta} = 5.0, -3$ $E = 4.5, 8$ $\nu = 0.15$

for the following cases of loading:-

(i) $G_\gamma = 50$, acting over the total surface area of the shell.

$$P = Q = P' = Q' = 1(2)11$$

(ii) $G_\gamma = 50$, patch loading defined by $(\bar{x}_3, \bar{y}_3) = (0.3, 0.3)$, $(\bar{u}_3, \bar{v}_3) = (0.6, 0.6)$

$$P = Q = P' = Q' = 1(1)11$$

(iii) $G_\gamma = 50$, patch loading defined by $(\bar{x}_3, \bar{y}_3) = (0.3, 0.8)$, $(\bar{u}_3, \bar{v}_3) = (0.6, 0.4)$

$$P = Q = P' = Q' = 1(1)11$$

(iv) $G_\gamma = 50$, patch loading defined by $(\bar{x}_3, \bar{y}_3) = (0.8, 0.5)$, $(\bar{u}_3, \bar{v}_3) = (0.4, 1.0)$

$$P = Q = P' = Q' = 1(1)11$$

Loading (i) = Loadings $\left[(ii) + (iii) + (iv)\right]$

The solution for loading cases (i), (ii), (iii),

(iv) and $\left[(ii) + (iii) + (iv)\right]$ are presented in table 6.11.

VI.4. Solution of further problems

Example 6.12. A hyperbolic paraboloid shell has been analysed with the following data:-

$t = 0.25$ $\quad L_\alpha = 50$ $\quad L_\beta = 50$

$k_{\alpha\beta} = -8.0, -3$ $\quad E = 4.5, 8$ $\quad \nu = 0.15$

$G_\gamma = 50$

$P = 1(2)19$ $\quad Q = 1(2)19$

$P' = 1(2)19$ $\quad Q' = 1(2)19$

for the following boundary conditions:-

(i) all edges free-guided

(ii) all edges hinged

(iii) all edges clamped

The solutions are presented in table 6.12

TABLE 6.11

	α_1	β_1	case(i)	case(ii)	case(iii)	case(iv)	(ii)+(iii)+(iv)
u_γ	1/2	1/2	1.072, -2	2.273, -2	-4.062, -3	-7.947, -3	1.072, -2
u_γ	3/4	1/4	1.544, -2	-2.392, -3	-2.566, -3	2.039, -2	1.543, -2
m_β	1/2	1	-3.019, 2	6.114, 1	-5.962, 2	2.331, 2	-3.020, 2
m_β	1/2	1/4	9.642, 1	6.785, 1	4.859, 1	-2.001, 1	9.643, 1
m_α	3/4	3/4	2.338, 1	-3.725, 1	-3.339, 1	9.402, 1	2.338, 1
n_α	1	1/4	2.758, 3	6.906, 2	1.524, 2	1.915, 3	2.758, 3
n_α	1/4	3/4	1.191, 3	2.086, 3	-1.355, 3	4.602, 2	1.191, 3
n_β	1/4	1	4.653, 2	6.258, 2	1.886, 2	-3.491, 2	4.653, 2
n_β	1/4	1/4	1.054, 2	7.220, 1	-2.907, 2	3.239, 2	1.054, 2
$n_{\alpha\beta}$	1/2	1	-4.367, 3	4.261, 2	-3.225, 3	-1.568, 3	-4.367, 3
$n_{\alpha\beta}$	1/2	1/2	-5.138, 3	-3.796, 3	-8.127, 2	-5.287, 2	-5.137, 3
q_β	1/4	0	7.443, 1	4.565, 1	2.632, 1	2.463, 0	7.443, 1
u_α	1/4	1/4	-1.223, -4	-1.671, -4	7.824, -5	-3.342, -5	-1.223, -4

TABLE 6.12

	α_1	β_1	case(i)	case(ii)	case(iii)
u_γ	1/2	1/8	1.039, -2	4.653, -3	3.423, -3
	1/2	1/4	7.194, -3	5.029, -3	5.396, -3
	1/2	3/8	1.797, -3	3.661, -3	4.426, -3
	1/2	1/2	-4.877, -5	3.036, -3	3.677, -3
m_β	1/8	0	0	0	-3.107, 2
	1/4	0	0	0	-3.169, 2
	3/8	0	0	0	-2.450, 2
	1/2	0	0	0	-2.156, 2
	1/2	1/8	2.415, 2	7.507, 1	4.264, 1
	1/2	1/4	1.608, 1	2.427, 1	4.806, 1
	1/2	3/8	-6.881, 1	-1.696, 1	-1.039, 1
	1/2	1/2	-6.349, 1	-2.383, 1	-3.107, 1
$n_{\alpha\beta}$	1/8	0	0	1.009, 3	6.664, 2
	1/4	0	0	2.169, 3	1.755, 3
	3/8	0	0	2.772, 3	2.467, 3
	1/2	0	0	2.924, 3	2.677, 3
	1/2	1/8	2.434, 3	2.913, 3	2.717, 3
	1/2	1/4	3.395, 3	3.156, 3	3.082, 3
	1/2	3/8	3.248, 3	3.196, 3	3.256, 3
	1/2	1/2	3.055, 3	3.160, 3	3.261, 3

TABLE 6.12 (continued)

	α_1	β_1	case(i)	case(ii)	case(iii)
n_β	0	0	1.100, 4	3.521, 3	2.426, 3
	1/8	0	6.045, 3	2.555, 3	2.853, 3
	1/4	0	3.858, 2	5.798, 2	1.366, 3
	3/8	0	-9.646, 2	-1.215, 2	2.506, 2
q_β	1/8	0	1.240, 2	6.682, 1	1.467, 2
	1/4	0	1.192, 2	2.811, 1	9.969, 1
	3/8	0	1.200, 2	4.945, 0	5.295, 1
	1/2	0	1.225, 2	-1.469, 0	3.847, 1

Example 6.13. A hyperbolic paraboloid has been analysed with the following data:-

Edge 1 - hinged　　Edge 2 - clamped

Edge 3 - hinged　　Edge 4 - clamped

$t = 0.25$　　$L_\alpha = 50$　　$L_\beta = 60$

$k_{\alpha\beta} = -8.0, -3$　　$E = 4.5, 8$　　$\nu = 0.15$

$G_\alpha = -10$　　$G_\beta = -10$　　$G_\gamma = 50$

$P = 1(1)19$　　$Q = 1(1)19$

$P' = 0(1)19$　　$Q' = 0(1)19$

The solution is presented in tables 6.13.

TABLES 6.13

α_1	β_1	u_γ	m_β	n_β	$n_{\alpha\beta}$
1/2	0	0	0	-2.352, 2	2.872, 3
1/2	1/4	4.787, -3	1.704, -1	-1.352, 2	3.244, 3
1/2	1/2	3.410, -3	-1.151, 1	-1.980, 0	3.163, 3
1/2	3/4	5.445, -3	2.428, 1	1.326, 2	3.189, 3
1/2	1	0	-2.268, 2	2.506, 2	2.712, 3

α_1	β_1	n_α	n_β	$n_{\alpha\beta}$	q_α
0	0	3.959, 3	3.061, 3	0	0
0	1/4	-1.334, 2	-4.208, 2	2.447, 3	1.726, 1
0	1/2	-1.749, 2	-2.034, 2	2.858, 3	-2.580, 0
0	3/4	-8.433, 2	-2.352, 2	2.547, 3	1.485, 1
0	1	-4.414, 3	-1.582, 3	0	-1.222, 2

α_1	β_1	m_α	n_α	$n_{\alpha\beta}$	q_α
1	0	0	-2.571, 3	0	Q
1	1/4	-2.733, 2	-1.743, 2	2.129, 3	-7.289, 1
1	1/2	-1.828, 2	1.775, 2	2.817, 3	-2.395, 1
1	3/4	-2.727, 2	1.095, 3	2.224, 3	-6.982, 1
1	1	0	3.375, 3	0	3.379, 1

CHAPTER VII

CONCLUSIONS, DISCUSSION AND SUGGESTIONS FOR FURTHER RESEARCH

The rate of convergence of the solutions is mainly dependent on the boundary conditions and not on the type of loading.

Translational shells

For translational shells with any combination of simply supported, oblique gable, hinged and clamped boundaries the convergence of the solutions is extremely rapid irrespective of the shell dimensions. With the introduction of one or two free boundaries the convergence of the solution along the edges adjacent to the free boundaries is slower than elsewhere in the shell domain.

The convergence of the solution with any combination of the above boundary conditions and one flexible edge member is satisfactory. With the introduction of two or four flexible edge members, it is difficult to predict the rate of convergence of the solution. However, for flat plates and very shallow shells the convergence is usually good, while for steeper shells the convergence is unsatisfactory.

'Edge disturbances' q'_β, u_β, $\frac{\partial u_\gamma}{\partial \beta}$ along edges 1 and 2, and q'_α, u_α, $\frac{\partial u_\gamma}{\partial \alpha}$ along edges 3 and 4 are represented by half range sine series over the length of the shell in the α and β directions respectively. When there are free edges or edges with flexible beam supports some of these 'edge disturbances' have finite values at the ends of the half range of representation. This often leads to a very slow convergence of the half range sine series representing these edge disturbances, and hence to a slow convergence of the complementary solution. This difficulty may be overcome by using a full range Fourier expansion (of sine and cosine terms) over the length of the shell in the α and β directions.

The boundary conditions at the edges of the flexible beams are:-

$$N_1 = 0, \quad M_1 = 0, \quad U_3 = 0, \quad U_2 = 0$$

In practice, usually one of the boundary conditions is $U_1 = 0$ rather than $N_1 = 0$. This condition may be partially simulated by increasing the effective cross-sectional area of the edge member in the computations. Various other edge conditions may be simulated by suitably adjusting the stiffness of the edge members.

Ruled Surfaces

For hyperbolic paraboloid shells with any combination of free-guided, hinged and clamped boundary conditions along their straight edges, the rate of convergence of the solutions is extremely rapid irrespective of the shell dimensions. With the introduction of simply supported or free boundaries the rate of convergence of the solutions can be extremely slow. The reason for this slow convergence is the same as that for translational shells with free edges or edges with flexible beam supports.

Suggestions for further research

1. Extend the method to the analysis of orthotropic shells.
2. Improve the rate of convergence of the solutions for translational shells with flexible edge beam supports by using full range Fourier expansions.
3. Improve the rate of convergence of the solutions for ruled surfaces with simply supported and free edges by using full range Fourier expansions. This extension will enable flexible edge beam supports to be considered.

4. Extend suggestions 2 and 3 to shells with prestressed edge members.

APPENDIX 1

FOURIER EXPANSIONS OF EXTERNAL LOADING ON TRANSLATIONAL SHELLS

(i) Uniform rectangular patch loads

The loading on the shell is expressed in the form:-

$$G_\alpha = \sum_{p=o}^{\infty} \sum_{q=o}^{\infty} g_\alpha^{pq} \cos(m\alpha) \sin(n\beta)$$

$$G_\beta = \sum_{p=o}^{\infty} \sum_{q=o}^{\infty} g_\beta^{pq} \sin(m\alpha) \cos(n\beta) \qquad (1)$$

$$G_\gamma = \sum_{p=o}^{\infty} \sum_{q=o}^{\infty} g_\gamma^{pq} \sin(m\alpha) \sin(n\beta)$$

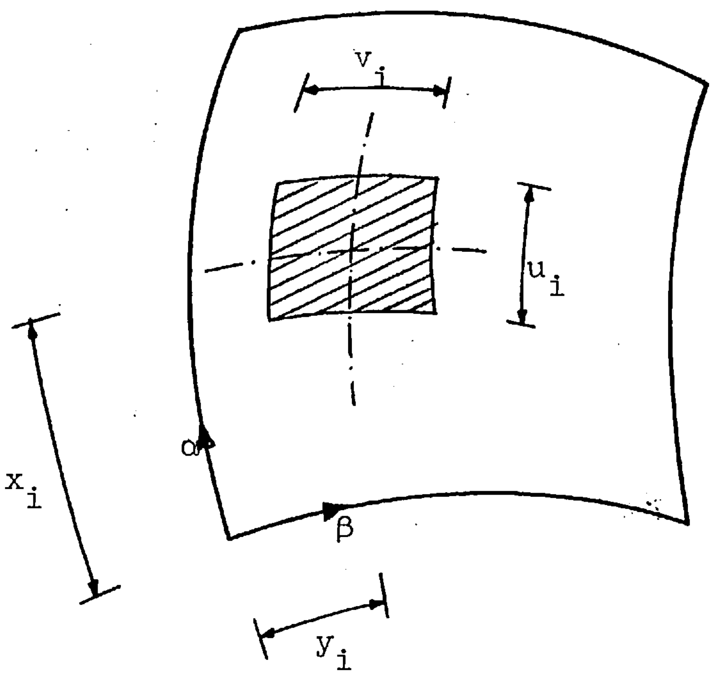
Let the coordinates of the centres and the dimensions of the loaded areas be (x_i, y_i) and (u_i, v_i) respectively (figure A1.1). Suffix i takes the values 1,2 and 3 and corresponds to the loading in the α,β and γ directions respectively.

Then $\quad g_j^{pq} = a_j \, b_j^{\,p} \, c_j^{\,q} \qquad (2)$

$$(j = \alpha, \beta, \gamma)$$

where $a_j = \dfrac{16P_j}{l_\alpha l_\beta} \qquad (3)$

and P_j is the intensity of loading in the j direction,



Patch loading - dimensions of loaded area

Figure A1.1

and the values of $b_j^{\,p}$, $c_j^{\,q}$ are listed in tables A1.1 and A1.2 respectively.

(ii) <u>Cylindrical shell - uniform patch loading in a fixed direction in the $(\beta - \gamma)$ plane</u>

Let $\overline{P}$ = intensity of loading

ψ = anticlockwise angle measured from the positive direction of loading to the positive direction of shell axis γ at the origin of coordinates $(\alpha,\ \beta,\ \gamma)$

$(x,\ y)$ = Coordinates of the centre of the loaded area (figure A1.1)

$(u,\ v)$ = Dimensions of loaded area (figure A1.1)

The loading is expressed in the form given by equations (1), where

$$\begin{aligned} g_\alpha^{\,pq} &= 0 \\ g_\beta^{\,pq} &= -\frac{8\overline{P}}{l_\alpha l_\beta}\sin(mx)\sin\left(\frac{mu}{2}\right) I_1 \\ g_\gamma^{\,pq} &= \frac{8\overline{P}}{l_\alpha l_\beta}\sin(mx)\sin\left(\frac{mu}{2}\right) I_2 \end{aligned} \tag{4}$$

and I_1, I_2 are given by:-

(i) when $n = 0$

$$\begin{aligned} I_1 &= \frac{l_\beta}{2k_\beta}\Big[\cos(\psi - k_\beta\,\beta)\Big]_{\beta=y-\frac{v}{2}}^{\beta=y+\frac{v}{2}} \\ I_2 &= 0 \end{aligned} \tag{5}$$

Table A1.1

j	p	b_j^p distribution of loading about axis $\alpha = \frac{l_\alpha}{2}$		
		general	symmetric	antimetric
α	$\neq 0$	$\frac{1}{m}\cos(mx_1)\sin(\frac{mu_1}{2})$	$\frac{1}{m}\left[\sin(mx_1)-\sin(mx_1-\frac{mu_1}{2})\right]$	$\frac{1}{m}\cos(mx_1)\sin(\frac{mu_1}{2})$
	0	$\frac{u_1}{4}$	0	$\frac{u_1}{4}$
β	$\neq 0$	$\frac{1}{m}\sin(mx_2)\sin(\frac{mu_2}{2})$	$\frac{1}{m}\sin(mx_2)\sin(\frac{mu_2}{2})$	$\frac{1}{m}\left[-\cos(mx_2)+\cos(mx_2-\frac{mu_2}{2})\right]$
	0	0	0	0
γ	$\neq 0$	$\frac{1}{m}\sin(mx_3)\sin(\frac{mu_3}{2})$	$\frac{1}{m}\sin(mx_3)\sin(\frac{mu_3}{2})$	$\frac{1}{m}\left[-\cos(mx_3)+\cos(mx_3-\frac{mu_3}{2})\right]$
	0	0	0	0

Table A1.2

j	q	$c_j^{\,q}$		
		Distribution of loading about axis $\beta = \frac{l_\beta}{2}$		
		general	symmetric	antimetric
α	$\neq 0$	$\frac{1}{n}\sin(ny_1)\sin(\frac{nv_1}{2})$	$\frac{1}{n}\sin(ny_1)\sin(\frac{nv_1}{2})$	$\frac{1}{n}\left[-\cos(ny_1)+\cos(ny_1-\frac{nv_1}{2})\right]$
	0	0	0	0
β	$\neq 0$	$\frac{1}{n}\cos(ny_2)\sin(\frac{nv_2}{2})$	$\frac{1}{n}\left[\sin(ny_2)-\sin(ny_2-\frac{nv_2}{2})\right]$	$\frac{1}{n}\cos(ny_2)\sin(\frac{nv_2}{2})$
	0	$\frac{v_2}{4}$	0	$\frac{v_2}{4}$
γ	$\neq 0$	$\frac{1}{n}\sin(ny_3)\sin(\frac{nv_3}{2})$	$\frac{1}{n}\sin(ny_3)\sin(\frac{nv_3}{2})$	$\frac{1}{n}\left[-\cos(ny_3)+\cos(ny_3-\frac{nv_3}{2})\right]$
	0	0	0	0

(ii) When $(n - k_\beta) = 0$

$$I_1 = \left[\frac{1}{2}\,\beta\,\sin(\psi) + \frac{\cos((n+k_\beta)\,\beta-\psi)}{2(n+k_\beta)} \right]_{\beta=y-\frac{v}{2}}^{\beta=y+\frac{v}{2}}$$

$$I_2 = \left[\frac{1}{2}\,\beta\,\sin(\psi) - \frac{\cos((n+k_\beta)\,\beta-\psi)}{2(n+k_\beta)} \right]_{\beta=y-\frac{v}{2}}^{\beta=y+\frac{v}{2}} \qquad (6)$$

(iii) When $(n + k_\beta) = 0$

$$I_1 = \left[\frac{1}{2}\,\beta\,\sin(\psi) - \frac{\cos((n-k_\beta)\,\beta+\psi)}{2(n-k_\beta)} \right]_{\beta=y-\frac{v}{2}}^{\beta=y+\frac{v}{2}}$$

$$(7)$$

$$I_2 = \left[-\,\frac{1}{2}\beta\,\sin(\psi) - \frac{\cos((n-k_\beta)\,\beta+\psi)}{2(n-k_\beta)} \right]_{\beta=y-\frac{v}{2}}^{\beta=y+\frac{v}{2}}$$

(iv) When $n \neq 0$, $(n \pm k_\beta) \neq 0$

$$I_1 = \left[-\,\frac{\cos((n-k_\beta)\,\beta+\psi)}{2(n-k_\beta)} + \frac{\cos((n+k_\beta)\beta-\psi)}{2(n+k_\beta)} \right]_{\beta=y-\frac{v}{2}}^{\beta=y+\frac{v}{2}}$$

$$I_2 = \left[-\,\frac{\cos((n-k_\beta)\,\beta+\psi)}{2(n-k_\beta)} - \frac{\cos((n+k_\beta)\beta-\psi)}{2(n+k_\beta)} \right]_{\beta=y-\frac{v}{2}}^{\beta=y+\frac{v}{2}}$$

$$(8)$$

(iii) Thermal loading

A uniform temperature field T_1 acting on the shell surface $\gamma = -t/2$ (outer) and a uniform temperature field T_2 acting on the shell surface $\gamma = +t/2$ (inner) are considered.

T_1 and T_2 are expressed in the form:-

$$T_1 = \sum_{p=o}^{\infty} \sum_{q=o}^{\infty} t_1^{pq} \sin(m\alpha)\sin(n\beta)$$
$$T_2 = \sum_{p=o}^{\infty} \sum_{q=o}^{\infty} t_2^{pq} \sin(m\alpha)\sin(n\beta) \tag{9}$$

where t_1^{pq}, t_2^{pq} are given by:-

(i) when $p = 0$ or $q = 0$

$$t_1^{pq} = 0 = t_2^{pq} \tag{10}$$

(ii) when $p \neq 0$ and $q \neq 0$

$$t_1^{pq} = \frac{4T_1}{\pi^2 pq}\left[1 - \cos(p\pi)\right]\left[1 - \cos(q\pi)\right]$$
$$t_2^{pq} = \frac{4T_2}{\pi^2 pq}\left[1 - \cos(p\pi)\right]\left[1 - \cos(q\pi)\right] \tag{11}$$

Fourier expansions of external loading on edge beams 1 and 2

Uniform loads acting over the total length of the beams in the direction of beam axes II and III are considered.

These loads are expressed in the form:-

$$
\left.
\begin{aligned}
Z_1 &= 0 \\
Z_2 &= \sum_{p=1}^{\infty} z_2{}^p \sin(m\alpha) \\
Z_3 &= \sum_{p=1}^{\infty} z_3{}^p \sin(m\alpha)
\end{aligned}
\right\} \qquad (12)
$$

$$
\left.
\begin{aligned}
\text{where } z_1{}^p &= \frac{2Z_2}{p\pi}\left[1 - \cos(p\pi)\right] \\
z_3{}^p &= \frac{2Z_3}{p\pi}\left[1 - \cos(p\pi)\right]
\end{aligned}
\right\} \qquad (13)
$$

APPENDIX 2

SOLUTION OF THE AUXILIARY EQUATION FOR u_γ FOR TRANSLATIONAL SHELLS

The auxiliary equation for u_γ is

$$(\rho^2-m^2)^4 + (k_2\rho^2-k_3)^2 = 0 \tag{1}$$

where $k_1 = \dfrac{2(3-3\nu^2)^{1/2}}{t}$

$$k_2 = k_1\ k_\alpha \tag{2}$$

$$k_3 = k_1\ m^2\ k_\beta$$

Case 1 - $k_\alpha \neq k_\beta$

Factorizing equation (1):-

$$\left[(\rho^2-m^2)^2+i(k_2\rho^2-k_3)\right]\ \left[(\rho^2-m^2)^2-i(k_2\rho^2-k_3)\right] = 0 \tag{3}$$

where $i = \left[-1\right]^{1/2}$

Consider the first factor of (3):-

$$(\rho^2-m^2)^2 + i(k_2\rho^2-k_3) = 0 \tag{4}$$

$$(\rho^2)^2 + (ik_2-2m^2)\rho^2 + (m^4-ik_3) = 0$$

$$\rho^2 = \frac{(2m^2-ik_2) \pm \left[-k_2{}^2+4i(k_3-m^2k_2)\right]^{1/2}}{2} \tag{5}$$

$$\text{Let } \sigma' + i\lambda' = \frac{\left[-k_2{}^2 + 4i(k_3 - m^2k_2)\right]^{1/2}}{2} \tag{6}$$

where σ', λ' are real and σ' is positive.

Then $(\sigma')^2 - (\lambda')^2 + 2i\sigma'\lambda' = -\frac{k_2^{\,2}}{4} + i(k_3 - m^2 k_2)$

Equating real and imaginary parts:-

$$(\sigma')^2 - (\lambda')^2 = -\frac{k_2^{\,2}}{4} \qquad (7)$$

$$2\sigma'\lambda' = k_3 - k_2 m^2$$

Eliminating λ' from (7)

$$(\sigma')^4 + \frac{k_2^{\,2}}{4}(\sigma')^2 - \frac{1}{4}(k_3 - k_2 m^2)^2 = 0$$

$$(\sigma')^2 = \frac{-\frac{k_2^{\,2}}{4} \pm \left[\frac{k_2^{\,4}}{16} + (k_3 - k_2 m^2)^2\right]^{1/2}}{2} \qquad (8)$$

Since σ' is defined to be real and positive

$$\sigma' = + \left[\frac{-\frac{k_2^{\,2}}{4} + \left[\frac{k_2^{\,4}}{16} + (k_3 - k_2 m^2)^2\right]^{1/2}}{2}\right]^{1/2} \qquad (9)$$

From equation (7)

$$\lambda' = \frac{k_3 - k_2 m^2}{2\sigma'} \qquad (10)$$

From equation (6)

$$\rho^2 = \frac{2m^2 - ik_2}{2} \pm (\sigma' + i\lambda')$$

$$\rho^2 = (m^2 \pm \sigma') + i(\pm \lambda' - \frac{k_2}{2}) \qquad (11)$$

Consider $\rho^2 = (m^2 + \sigma') + i(+\lambda' - \frac{k_2}{2}) \qquad (12)$

Let $\rho = \pm (\sigma_1 + i\lambda_1) \qquad (13)$

where σ_1, λ_1 are real and σ_1 is positive.

Then $\sigma_1{}^2 - \lambda_1{}^2 + 2i\sigma_1\lambda_1 = m^2 + \sigma' + i(\lambda' - \frac{k_2}{2})$

Equating real and imaginary parts:-

$$\left.\begin{aligned} \sigma_1{}^2 - \lambda_1{}^2 &= m^2 + \sigma' \\ 2\sigma_1\,\lambda_1 &= \lambda' - \frac{k_2}{2} \end{aligned}\right\} \tag{14}$$

Eliminating λ' from (14)

$$(\sigma_1{}^2)^2 - (m^2+\sigma')\sigma_1{}^2 - \frac{1}{4}(\lambda' - \frac{k_2}{2})^2 = 0 \tag{15}$$

$$\sigma_1{}^2 = \frac{(m^2+\sigma') \pm \left[(m^2+\sigma')^2 + (\lambda' - \frac{k_2}{2})^2\right]^{1/2}}{2} \tag{16}$$

Since σ_1 is defined to be real and positive

$$\sigma_1 = + \left[\frac{(m^2+\sigma') + \left[(m^2+\sigma')^2 + (\lambda' - \frac{k_2}{2})^2\right]^{1/2}}{2}\right]^{1/2} \tag{17}$$

From equation (14)

$$\lambda_1 = \frac{\lambda' - \frac{k_2}{2}}{2\sigma_1} \tag{18}$$

Consider $\rho^2 = (m^2-\sigma') + i(-\lambda' - \frac{k_2}{2})$ (19)

Let $\rho = \pm\, (\sigma_2 + i\lambda_2)$ (20)

where σ_2, λ_2 are real and σ_2 is positive.

Then $\sigma_2{}^2 - \lambda_2{}^2 + 2i\sigma_2\lambda_2 = m^2 - \sigma' + i(-\lambda' - \frac{k_2}{2})$

Equating real and imaginary parts:-

$$\begin{aligned} \sigma_2^{\,2} - \lambda_2^{\,2} &= m^2 - \sigma' \\ 2\sigma_2 \lambda_2 &= -\lambda' - \frac{k_2}{2} \end{aligned} \tag{21}$$

Eliminating λ_2 from (21)

$$(\sigma_2^{\,2})^2 - (m^2 - \sigma')\sigma_2^{\,2} - \tfrac{1}{4}(\lambda' + \frac{k_2}{2})^2 = 0 \tag{22}$$

$$\sigma_2^{\,2} = \frac{(m^2-\sigma') \pm \left[(m^2-\sigma')^2 + (\lambda' + \frac{k_2}{2})^2\right]^{1/2}}{2} \tag{23}$$

Since σ_2 is defined to be real and positive

$$\sigma_2 = + \left[\frac{(m^2-\sigma') + \left[(m^2-\sigma')^2 + (\lambda' + \frac{k_2}{2})^2\right]^{1/2}}{2}\right]^{1/2} \tag{24}$$

From equation (21)

$$\lambda_2 = - \frac{\lambda' - \frac{k_2}{2}}{2\sigma_2} \tag{25}$$

Hence four roots of the auxiliary equation (1) are:-

$$\rho = \pm (\sigma_1 + i\lambda_1) \; ; \; \pm (\sigma_2 + i\lambda_2) \tag{26}$$

However, since the coefficients of the auxiliary equation are real,

$$\rho = \pm (\sigma_1 - i\lambda_1) \; ; \; \pm (\sigma_2 - i\lambda_2) \tag{27}$$

are also roots of the auxiliary equation.

Therefore the eight roots of the auxiliary equation (1) are:-

$$\rho = \pm (\sigma_1 \pm i\lambda_1) \; ; \; \pm (\sigma_2 \pm i\lambda_2) \tag{28}$$

(when $k_\alpha \neq k_\beta$)

where $i = \left[-1\right]^{1/2}$

$$\sigma_1 = \left[\frac{(m^2+\sigma') + \left[(m^2+\sigma')^2 + (\lambda' - \frac{k_2}{2})^2\right]^{1/2}}{2}\right]^{1/2}$$

$$\lambda_1 = \frac{\lambda' - \frac{k_2}{2}}{2\sigma_1} \tag{29}$$

$$\sigma_2 = \left[\frac{(m^2-\sigma') + \left[(m^2-\sigma')^2 + (\lambda' + \frac{k_2}{2})^2\right]^{1/2}}{2}\right]^{1/2}$$

$$\lambda_2 = -\frac{\lambda' - \frac{k_2}{2}}{2\sigma_2}$$

and

$$\sigma' = \left[\frac{-\frac{k_2^2}{4} + \left[\frac{k_2^4}{16} + (k_3 - k_2 m^2)^2\right]^{1/2}}{2}\right]^{1/2} \tag{30}$$

$$\lambda' = \frac{k_3 - k_2 m^2}{2\sigma'}$$

k_1, k_2 and k_3 are defined by equations(2).

Hence $$u_\gamma = \sin(m\alpha) \sum_{j=1}^{8} c_j \, e^{\rho_j \beta} \tag{31}$$

where $c_1, c_2, \ldots, c_8$ are arbitrary constants and $\rho_1, \rho_2, \ldots, \rho_8$ are the eight roots of the auxiliary equation.

Case 2 - $k_\alpha = k_\beta$

Factorizing equation (1):-

$$\left[(\rho^2-m^2)^2 + ik_2\ (\rho^2-m^2)\right]\ \left[(\rho^2-m^2)^2 - ik_2(\rho^2-m^2)\right] = 0 \tag{32}$$

Consider the first factor of (32):-

$$(\rho^2-m^2)^2 + ik_2\ (\rho^2-m^2) = 0 \tag{33}$$

$$(\rho^2)^2 + (ik_2 - 2m^2)\rho^2 + (m^4 - ik_2m^2) = 0$$

$$\rho^2 = \frac{(2m^2-ik_2) \pm \left[(ik_2-2m^2)^2-4(m^4-ik_2m^2)\right]^{1/2}}{2}$$

$$\rho^2 = +m^2 - ik_2\ ;\quad +m^2 \tag{34}$$

$$\text{Consider } \rho^2 = m^2 - ik_2 \tag{35}$$

$$\text{Let } \rho = \pm\ (\sigma_1 + i\lambda_1) \tag{36}$$

where σ_1, λ_1 are real and σ_1 is positive.

Then $\sigma_1{}^2 - \lambda_1{}^2 + 2i\sigma_1\lambda_1 = m^2 - ik_2$

Equating real and imaginary parts:-

$$\left.\begin{aligned} \sigma_1{}^2 - \lambda_1{}^2 &= m^2 \\ 2\sigma_1\ \lambda_1 &= -k_2 \end{aligned}\right\} \tag{37}$$

Eliminating λ_1 from (37)

$$(\sigma_1{}^2)^2 - m^2\sigma_1{}^2 - \frac{k_2}{4} = 0 \tag{38}$$

$$\sigma_1{}^2 = \frac{m^2 \pm \left[m^4+k_2{}^2\right]^{1/2}}{2} \tag{39}$$

Since σ_1 is defined to be real and positive

$$\sigma_1 = + \left[\frac{m^2 + \left[m^4+k_2^2\right]^{1/2}}{2} \right]^{1/2} \qquad (40)$$

From equation (37)

$$\lambda_1 = - \frac{k_2}{2\sigma_1} \qquad (41)$$

Hence four roots of the auxiliary equation are:-

$$\rho = \pm (\sigma_1 + i\lambda_1) \quad ; \quad \pm m \qquad (42)$$

Consider the second factor of (32):-

$$(\rho^2-m^2)^2 - ik_2(\rho^2-m^2) = 0$$

$$(\rho^2)^2 - (ik_2+2m^2)\rho^2 + (m^4+ik_2m^2) = 0 \qquad (43)$$

$$\rho^2 = \frac{(2m^2+ik_2) \pm \left[(ik_2+2m^2)^2-4(m^4+ik_2m^2)\right]^{1/2}}{2}$$

$$\rho^2 = (m^2+ik_2) \quad ; \quad +m^2 \qquad (44)$$

It can be seen from equations (35) and (36) that

$$\rho = \pm (\sigma_1 - i\lambda_1) \quad ; \quad \pm m \qquad (45)$$

Therefore the eight roots of the auxiliary equation (1) are:-

$$\rho = \pm(\sigma_1 \pm i\lambda_1) \quad ; \quad \pm m \quad ; \quad \pm m \qquad (46)$$

$$(\text{when } k_\alpha = k_\beta)$$

where $i = \left[-1\right]^{1/2}$

$$\sigma_1 = + \left[\frac{m^2 + \left[m^4 + k_2{}^2 \right]^{1/2}}{2} \right]^{1/2}$$

$$\lambda_1 = - \frac{k_2}{2\sigma_1} \tag{47}$$

$$k_2 = \frac{2k_\alpha \left[3-3\nu^2 \right]^{1/2}}{t}$$

Hence

$$u_\gamma = \sin(m\alpha) \Big[c_1 e^{(\sigma_1 + i\lambda_1)\beta} + c_2 e^{(\sigma_1 - i\lambda_1)\beta}$$
$$+ c_3 e^{-(\sigma_1 + i\lambda_1)\beta} + c_4 e^{-(\sigma_1 - i\lambda_1)\beta} + (c_5\beta + c_6) e^{m\beta} + (c_7\beta + c_8) e^{-m\beta} \Big] \tag{48}$$

where $c_1, c_2, \ldots, c_8$ are arbitrary constants.

APPENDIX 3

COMPLEMENTARY SOLUTIONS FOR u_γ AND ϕ FOR TRANSLATIONAL SHELLS

Case 1 - $k_\alpha \neq k_\beta$

The solution for u_γ given by equation (31) of appendix 2 can be written

$$u_\gamma = \sin(m\alpha)\left[e^{-\sigma_1\beta}\left[\bar{\mu}_1\cos(\lambda_1\beta) + \bar{\mu}_2\sin(\lambda_1\beta)\right] + e^{-\sigma_2\beta}\left[\bar{\mu}_3\cos(\lambda_2\beta) + \bar{\mu}_4\sin(\lambda_2\beta)\right] + e^{+\sigma_1\beta}\left[\bar{\mu}_5\cos(\lambda_1\beta) + \bar{\mu}_6\sin(\lambda_1\beta)\right] + e^{+\sigma_2\beta}\left[\bar{\mu}_7\cos(\lambda_2\beta) + \bar{\mu}_8\sin(\lambda_2\beta)\right]\right] \tag{1}$$

where $\bar{\mu}_1, \bar{\mu}_2, \ldots, \bar{\mu}_8$ are arbitrary constants.

Introducing a new variable z defined by:-

$$z = l_\beta - \beta \tag{2}$$

equation (1) can be written

$$u_\gamma = \sin(m\alpha)\left[e^{-\sigma_1\beta}\left[\mu_1\cos(\lambda_1\beta) + \mu_2\sin(\lambda_1\beta)\right] + e^{-\sigma_2\beta}\left[\mu_3\cos(\lambda_2\beta) + \mu_4\sin(\lambda_2\beta)\right] + e^{-\sigma_1 z}\left[\mu_5\cos(\lambda_1 z) + \mu_6\sin(\lambda_1 z)\right] + e^{-\sigma_2 z}\left[\mu_7\cos(\lambda_2 z) + \mu_8\sin(\lambda_2 z)\right]\right] \tag{3}$$

where $\mu_1, \mu_2, \ldots, \mu_8$ are arbitrary constants.

The auxiliary equation for $\emptyset$ is identical in form to the auxiliary equation for u_γ. Hence the complementary solution for $\emptyset$ will be of the form:-

$$\begin{aligned}\emptyset = \sin(m\alpha)\Big[& e^{-\sigma_1\beta}\left[c_1\cos(\lambda_1\beta) + c_2\sin(\lambda_1\beta)\right] \\ & + e^{-\sigma_2\beta}\left[c_3\cos(\lambda_2\beta) + c_4\sin(\lambda_2\beta)\right] \\ & + e^{-\sigma_1 z}\left[c_5\cos(\lambda_1 z) + c_6\sin(\lambda_1 z)\right] \\ & + e^{-\sigma_2 z}\left[c_7\cos(\lambda_2 z) + c_8\sin(\lambda_2 z)\right]\Big]\end{aligned} \tag{4}$$

where $c_1, c_2, \ldots, c_8$ are constants to be determined from the equation coupling u_γ and $\emptyset$:-

$$\nabla^4\emptyset + Et\nabla_R^2\, u_\gamma = 0 \tag{5}$$

Consider

$$\begin{aligned}(u_\gamma)_1 &= e^{-\sigma_1\beta}\left[\mu_1\cos(\lambda_1\beta) + \mu_2\sin(\lambda_1\beta)\right]\sin(m\alpha) \\ \text{and } (\emptyset)_1 &= e^{-\sigma_1\beta}\left[c_1\cos(\lambda_1\beta) + c_2\sin(\lambda_1\beta)\right]\sin(m\alpha)\end{aligned} \tag{6}$$

$$\begin{aligned}\nabla^4(\emptyset)_1 &= \left[\frac{\partial^4}{\partial\alpha^4} + 2\,\frac{\partial^4}{\partial\alpha^2\partial\beta^2} + \frac{\partial^4}{\partial\beta^4}\right](\emptyset)_1 \\ &= \left[m^4 - 2m^2\,\frac{\partial^2}{\partial\beta^2} + \frac{\partial^4}{\partial\beta^4}\right](\emptyset)_1 \\ &= e^{-\sigma_1\beta}\Big[(a_1c_1 + a_2c_2)\cos(\lambda_1\beta) \\ &\qquad + (a_1c_2 - a_2c_1)\sin(\lambda_1\beta)\Big]\sin(m\alpha)\ ^{*}\end{aligned} \tag{7}$$

---

*Refer equation (3.39)

$$\text{where } a_1 = m^4 - 2m^2(\sigma_1{}^2-\lambda_1{}^2) + (\sigma_1{}^2-\lambda_1{}^2)^2 - 4\sigma_1{}^2\lambda_1{}^2 \tag{8}$$

$$a_2 = 4m^2\sigma_1\lambda_1 - 4\sigma_1\lambda_1\ (\sigma_1{}^2 - \lambda_1{}^2)$$

$$\begin{aligned}\nabla_R{}^2(u_\gamma)_1 &= \left[k_\beta \frac{\partial^2}{\partial\alpha^2} + k_\alpha \frac{\partial^2}{\partial\beta^2}\right] (u_\gamma)_1 \\ &= \left[-m^2k_\beta + k_\alpha \frac{\partial^2}{\partial\beta^2}\right] (u_\gamma)_1 \\ &= e^{-\sigma_1\beta} \Big[(b_1\mu_1+b_2\mu_2) \cos(\lambda_1\beta) \\ &\qquad + (b_1\mu_2 - b_2\mu_1) \sin(\lambda_1\beta)\Big] \sin(m\alpha)\end{aligned} \tag{9}$$

$$\text{where } b_1 = -k_\beta m^2 + k_\alpha(\sigma_1{}^2-\lambda_1{}^2) \tag{10}$$

$$b_2 = -\ 2k_\alpha\sigma_1\lambda_1$$

Substituting for $(\sigma_1{}^2 - \lambda_1{}^2)$, $\sigma_1\lambda_1$, $\left[(\sigma')^2-(\lambda')^2\right]$ and $\sigma'\lambda'$ from equations (14) and (7) of appendix 2:-

$$\begin{aligned} a_1 &= k_2(\lambda' - \frac{k_2}{2}) \\ a_2 &= k_2m^2 - k_3 + k_2\sigma' \\ b_1 &= \frac{a_2}{k_1} \\ b_2 &= -\frac{a_1}{k_1} \end{aligned} \tag{11}$$

Substituting these values in equation (5) and comparing coefficients of $\cos(\lambda_1\beta)$ and $\sin(\lambda_1\beta)$:-

$$a_1c_1 + a_2c_2 = -\frac{Et}{k_1}\left[a_2\mu_1 - a_1\mu_2\right]$$
$$a_1c_2 - a_2c_1 = -\frac{Et}{k_1}\left[a_2\mu_2 + a_1\mu_1\right] \qquad (12)$$

From equations (12):-

$$c_1 = +\frac{Et}{k_1}\mu_2$$
$$c_2 = -\frac{Et}{k_1}\mu_1 \qquad (13)$$

Consider

$$(u_\gamma)_2 = e^{-\sigma_2\beta}\left[\mu_3\cos(\lambda_2\beta) + \mu_4\sin(\lambda_2\beta)\right]\sin(m\alpha)$$

and
$$(\emptyset)_2 = e^{-\sigma_2\beta}\left[c_3\cos(\lambda_2\beta) + c_4\sin(\lambda_2\beta)\right]\sin(m\alpha) \qquad (14)$$

$$\nabla^4(\emptyset)_2 = e^{-\sigma_2\beta}\Big[(a_3c_3 + a_4c_4)\cos(\lambda_2\beta) + (a_3c_4 - a_4c_3)\sin(\lambda_2\beta)\Big]\sin(m\alpha) \qquad (15)$$

where
$$a_3 = m^4 - 2m(\sigma_2{}^2-\lambda_2{}^2) + (\sigma_2{}^2-\lambda_2{}^2) - 4\sigma_2{}^2\lambda_2{}^2$$
$$a_4 = 4m^2\sigma_2\lambda_2 - 4\sigma_2\lambda_2(\sigma_2{}^2-\lambda_2{}^2) \qquad (16)$$

$$\nabla_R{}^2(\ddot{u}_\gamma)_2 = e^{-\sigma_2\beta}\Big[(b_3\mu_3 + b_4\mu_4)\cos(\lambda_2\beta) + (b_3\mu_4 - b_4\mu_3)\sin(\lambda_2\beta)\Big]\sin(m\alpha) \qquad (17)$$

where
$$b_3 = -k_\beta m^2 + k_\alpha(\sigma_2{}^2 - \lambda_2{}^2)$$
$$b_4 = -2k_\alpha\,\sigma_2\,\lambda_2 \qquad (18)$$

Substituting for $(\sigma_2^2 - \lambda_2^2)$, $\sigma_2\lambda_2$, $\left[(\sigma')^2-(\lambda')^2\right]$ and $\sigma'\lambda'$ from equations (21) and (7) of appendix 2:-

$$\left.\begin{aligned} a_3 &= -k_2(\lambda' + \frac{k_2}{2}) \\ a_4 &= -k_3 + k_2(m^2-\sigma') \\ b_3 &= \frac{a_4}{k_1} \\ b_4 &= -\frac{a_3}{k_1} \end{aligned}\right\} \qquad (19)$$

Substituting these results in equation (5) and comparing coefficients of $\cos(\lambda_2\beta)$ and $\sin(\lambda_2\beta)$:-

$$\begin{aligned} a_3c_3 + a_4c_4 &= -\frac{Et}{k_1}\left[a_4\mu_3 - a_3\mu_4\right] \\ a_3c_4 - a_4c_3 &= -\frac{Et}{k_1}\left[a_4\mu_4 + a_3\mu_3\right] \end{aligned} \qquad (20)$$

From equations (20):-

$$\begin{aligned} c_3 &= +\frac{Et}{k_1}\mu_4 \\ c_4 &= -\frac{Et}{k_1}\mu_3 \end{aligned} \qquad (21)$$

Consider

$$(u_\gamma)_3 + (u_\gamma)_4 = \sin(m\alpha)\left[e^{-\sigma_1 z}(\mu_5\cos(\lambda_1 z)+\mu_6\sin(\lambda_1 z)) + e^{-\sigma_2 z}(\mu_7\cos(\lambda_2 z)+\mu_8\sin(\lambda_2 z))\right]$$

and $$(\emptyset)_3 + (\emptyset)_4 = \sin(m\alpha)\left[e^{-\sigma_1 z}(c_5\cos(\lambda_1 z)+c_6\sin(\lambda_1 z)) + e^{-\sigma_2 z}(c_7\cos(\lambda_2 z)+c_8\sin(\lambda_2 z))\right] \qquad (22)$$

Since only even derivatives of α occur in equation (5), the solution for $\left[(\phi)_3 + (\phi)_4\right]$ follows on exactly similar lines to the solution for $\left[(\phi)_1 + (\phi)_2\right]$ (refer equation (3.39)).

Therefore
$$\begin{aligned} c_5 &= + \frac{Et}{k_1}\mu_6 \\ c_6 &= - \frac{Et}{k_1}\mu_5 \\ c_7 &= + \frac{Et}{k_1}\mu_8 \\ c_8 &= - \frac{Et}{k_1}\mu_7 \end{aligned} \tag{23}$$

Hence the complementary solutions for u_γ and ϕ are (when $k_\alpha \neq k_\beta$):-

$$\begin{aligned} u_\gamma = \sin(m\alpha) \Big[& e^{-\sigma_1\beta}(\mu_1\cos(\lambda_1\beta) + \mu_2\sin(\lambda_1\beta)) \\ & + e^{-\sigma_2\beta}(\mu_3\cos(\lambda_2\beta) + \mu_4\sin(\lambda_2\beta)) \\ & + e^{-\sigma_1 z}(\mu_5\cos(\lambda_1 z) + \mu_6\sin(\lambda_1 z)) \\ & + e^{-\sigma_2 z}(\mu_7\cos(\lambda_2 z) + \mu_8\sin(\lambda_2 z)) \Big] \end{aligned} \tag{24}$$

$$\begin{aligned} \text{and } \phi = \frac{Et}{k_1}\sin(m\alpha) \Big[& e^{-\sigma_1\beta}(\mu_2\cos(\lambda_1\beta) - \mu_1\sin(\lambda_1\beta)) \\ & + e^{-\sigma_2\beta}(\mu_4\cos(\lambda_2\beta) - \mu_3\sin(\lambda_2\beta)) \\ & + e^{-\sigma_1 z}(\mu_6\cos(\lambda_1 z) - \mu_5\sin(\lambda_1 z)) \\ & + e^{-\sigma_2 z}(\mu_8\cos(\lambda_2 z) - \mu_7\sin(\lambda_2 z)) \Big] \end{aligned} \tag{25}$$

where $\mu_1, \mu_2, \ldots, \mu_8$ are arbitrary constants.

Case 2 - $k_\alpha = k_\beta$

The solution for u_γ given by equation (48) of appendix 2 can be written

$$u_\gamma = \sin(m\alpha)\Big[e^{-\sigma_1\beta}\left[\bar{\mu}_1\cos(\lambda_1\beta) + \bar{\mu}_2\sin(\lambda_1\beta)\right] + e^{-m\beta}\left[\bar{\mu}_3\beta + \bar{\mu}_4\right] + e^{+\sigma_1\beta}\left[\bar{\mu}_5\cos(\lambda_1\beta) + \bar{\mu}_6\sin(\lambda_1\beta)\right] + e^{+m\beta}\left[\bar{\mu}_7\beta + \bar{\mu}_8\right]\Big] \tag{26}$$

where $\bar{\mu}_1, \bar{\mu}_2, \ldots, \bar{\mu}_8$ are arbitrary constants.

Introducing a new variable z defined by:-

$$z = l_\beta - \beta \tag{27}$$

equation (26) can be written

$$u_\gamma=\sin(m\alpha)\Big[e^{-\sigma_1\beta}\left[\mu_1\cos(\lambda_1\beta) + \mu_2\sin(\lambda_1\beta)\right] + e^{-m\beta}\left[\mu_3\beta + \mu_4\right] + e^{-\sigma_1 z}\left[\mu_5\cos(\lambda_1 z) + \mu_6\sin(\lambda_1 z)\right] + e^{-mz}\left[\mu_7 z + \mu_8\right]\Big] \tag{28}$$

where $\mu_1, \mu_2, \ldots, \mu_8$ are arbitrary constants.

The auxiliary equation for ϕ is identical in form to the auxiliary equation for u_γ. Hence the complementary solution for ϕ will be of the form:-

$$\phi = \sin(m\alpha)\Big[e^{-\sigma_1\beta}\left[c_1\cos(\lambda_1\beta) + c_2\sin(\lambda_1\beta)\right] + e^{-m\beta}\left[c_3\beta + c_4\right] + e^{-\sigma_1 z}\left[c_5\cos(\lambda_1 z) + c_6\sin(\lambda_1 z)\right] + e^{-mz}\left[c_7 z + c_8\right]\Big] \tag{29}$$

where $c_1, c_2, \ldots, c_8$ are constants to be determined from the equation coupling u_γ and ϕ:-

$$\nabla^4\phi + Et\nabla_R^2\, u_\gamma = 0 \tag{30}$$

Consider

$$(u_\gamma)_1 = e^{-\sigma_1\beta}\left[\mu_1\cos(\lambda_1\beta)+\mu_2\sin(\lambda_1\beta)\right]\sin(m\alpha)$$

and $$(\phi)_1 = e^{-\sigma_1\beta}\left[c_1\cos(\lambda_1\beta)+c_2\sin(\lambda_1\beta)\right]\sin(m\alpha) \tag{31}$$

From equations (7) and (8):-

$$\nabla^4(\phi)_1 = e^{-\sigma_1\beta}\left[(a_1c_1+a_2c_2)\cos(\lambda_1\beta)+(a_1c_2-a_2c_1)\sin(\lambda_1\beta)\right]\sin(m\alpha) \tag{32}$$

where $$a_1 = m^4-2m^2(\sigma_1^2-\lambda_1^2)+(\sigma_1^2-\lambda_1^2)^2-4\sigma_1^2\lambda_1^2$$

$$a_2 = 4m^2\sigma_1\lambda_1 - 4\sigma_1\lambda_1(\sigma_1^2-\lambda_1^2) \tag{33}$$

From equations (9) and (10):-

$$\nabla_R^2(u_\gamma)_1 = e^{-\sigma_1\beta}\Big[(b_1\mu_1+b_2\mu_2)\cos(\lambda_1\beta)+(b_1\mu_2-b_2\mu_1)\sin(\lambda_1\beta)\Big]\sin(m\alpha) \tag{34}$$

$$\begin{aligned} \text{where } b_1 &= -k_\beta m^2 + k_\alpha(\sigma_1{}^2-\lambda_1{}^2) \\ b_2 &= -2k_\alpha\sigma_1\lambda_1 \end{aligned} \tag{35}$$

Substituting for $(\sigma_1{}^2-\lambda_1{}^2)$ and $\sigma_1\lambda_1$ from equations (37) of appendix 2:-

$$\begin{aligned} a_1 &= -k_2{}^2 \\ a_2 &= 0 \\ b_1 &= 0 \\ b_2 &= k_\alpha k_2 = -\frac{a_1}{k_1} \end{aligned} \tag{36}$$

Substituting these values in equation (30) and comparing coefficients of $\cos(\lambda_1\beta)$ and $\sin(\lambda_1\beta)$:-

$$\begin{aligned} a_1c_1 &= -\frac{Et}{k_1}\Big[-a_1\mu_2\Big] \\ a_1c_2 &= -\frac{Et}{k_1}\Big[+\ a_1\mu_1\Big] \end{aligned} \tag{37}$$

From equations (37):-

$$\begin{aligned} c_1 &= +\frac{Et}{k_1}\mu_2 \\ c_2 &= -\frac{Et}{k_1}\mu_1 \end{aligned} \tag{38}$$

$$\text{Consider } (u_\gamma)_2 = e^{-m\beta}\left[\mu_3\beta + \mu_4\right]\sin(m\alpha)$$
$$\text{and } \quad (\phi)_2 = e^{-m\beta}\left[c_3\beta + c_4\right]\sin(m\alpha) \tag{39}$$

$$\nabla^4(\phi)_2 = \left[\frac{\partial^4}{\partial\alpha^4} + \frac{\partial^4}{\partial\alpha^2\partial\beta^2} + \frac{\partial^4}{\partial\alpha^4}\right](\phi)_2$$

$$= \left[m^4 - 2m^2\frac{\partial^2}{\partial\beta^2} + \frac{\partial^4}{\partial\beta^4}\right](\phi)_2$$

$$\nabla^4(\phi)_2 = 0 \tag{40}$$

$$\nabla_R^2(u_\gamma)_2 = \left[k_\beta\frac{\partial^2}{\partial\alpha^2} + k_\alpha\frac{\partial^2}{\partial\beta^2}\right](u_\gamma)_2$$

$$= \left[-m^2k_\beta + k_\alpha\frac{\partial^2}{\partial\beta^2}\right](u_\gamma)_2$$

$$\nabla_R^2(u_\gamma)_2 = -2mk_\alpha\mu_3e^{-m\beta}\sin(m\alpha) \tag{41}$$

Substituting (40) and (41) in equation (30):-

$$\mu_3 = 0 \tag{42}$$

Since $(\phi)_2$ must be of the same form as $(u_\gamma)_2$,

$$c_3 = 0 \tag{43}$$

and c_4 can be specified independently of μ_4 and μ_3.

Consider

$$(u_\gamma)_3 + (u_\gamma)_4 = \sin(m\alpha)\left[e^{-\sigma_1 z}\left[\mu_5\cos(\lambda_1 z) + \mu_6\sin(\lambda_1 z)\right] + e^{-mz}\left[\mu_7 z + \mu_8\right]\right]$$

and

$$(\emptyset)_3 + (\emptyset)_4 = \sin(m\alpha)\left[e^{-\sigma_1 z}\left[c_5\cos(\lambda_1 z) + c_6\sin(\lambda_1 z)\right] + e^{-mz}\left[c_7 z + c_8\right]\right] \tag{44}$$

Since only even derivatives of α occur in equation (30), the solution for $\left[(\emptyset)_3 + (\emptyset)_4\right]$ follows on exactly similar lines to the solution for $\left[(\emptyset)_1 + (\emptyset)_2\right]$ (refer equation (3.39)).

Therefore
$$\begin{aligned} c_5 &= + \frac{Et}{k_1}\mu_6 \\ c_6 &= - \frac{Et}{k_1}\mu_5 \\ \mu_7 &= 0 \\ c_7 &= 0 \end{aligned} \tag{45}$$

and c_8 can be specified independently of μ_8 and μ_7.

Hence the complementary solutions for u_γ and $\emptyset$ are (when $k_\alpha = k_\beta$):-

$$\begin{aligned} u_\gamma = \sin(m\alpha)\Big[& e^{-\sigma_1\beta}\left[\mu_1\cos(\lambda_1\beta) + \mu_2\sin(\lambda_1\beta)\right] \\ & + \mu_4\, e^{-m\beta} \\ & + e^{-\sigma_1 z}\left[\mu_5\cos(\lambda_1 z) + \mu_6\sin(\lambda_1 z)\right] \\ & + \mu_8\, e^{-mz}\Big] \end{aligned} \tag{46}$$

$$\text{and } \phi = \frac{Et}{k_1}\sin(m\alpha)\Big[e^{-\sigma_1\beta}\left[\mu_2\cos(\lambda_1\beta) - \mu_1\sin(\lambda_1\beta)\right] + \mu_3 e^{-m\beta} + e^{-\sigma_1 z}\left[\mu_6\cos(\lambda_1 z) - \mu_5\sin(\lambda_1 z)\right] + \mu_7 e^{-mz}\Big] \tag{47}$$

where $\mu_1, \mu_2, \ldots, \mu_8$ are arbitrary constants.

APPENDIX 4

MATRICES DEFINED IN CHAPTER III FOR TRANSLATIONAL SHELLS

In this appendix:-

$c_1 = 1, \quad c_2 = 0 \quad \text{when } k_\alpha \neq k_\beta$

$c_1 = 0, \quad c_2 = 1 \quad \text{when } k_\alpha = k_\beta$

Matrices L^β, L^z

$$L^\beta = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \tag{1}$$

where $a_{11} = +e^{-\sigma_1\beta}\cos(\lambda_1\beta)$

$a_{12} = +e^{-\sigma_1\beta}\sin(\lambda_1\beta)$

$a_{21} = -a_{12}$

$a_{22} = +a_{11}$

$a_{33} = +c_1\, e^{-\sigma_2\beta}\cos(\lambda_2\beta) + c_2\, e^{-m\beta}$

$a_{34} = +c_1\, e^{-\sigma_2\beta}\sin(\lambda_2\beta)$

$a_{43} = -a_{34}$

$a_{44} = +a_{33}$

$$L^z = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \qquad (2)$$

where $a_{11} = +e^{-\sigma_1 z}\cos(\lambda_1 z)$

$a_{12} = +e^{-\sigma_1 z}\sin(\lambda_1 z)$

$a_{21} = -a_{12}$

$a_{22} = +a_{11}$

$a_{33} = +c_1\ e^{-\sigma_2 z}\cos(\lambda_2 z) + c_2 e^{-mz}$

$a_{34} = +c_1\ e^{-\sigma_2 z}\sin(\lambda_2 z)$

$a_{43} = -a_{34}$

$a_{44} = +\ a_{33}$

<u>Matrices A_1, A_2</u>

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \qquad (3)$$

where $a_{11} = \frac{Et}{k_1} m\lambda_1 - k_\alpha Dm(1-\nu)\sigma_1$

$$a_{12} = \frac{Et}{k_1} m\sigma_1 + k_\alpha Dm(1-\nu)\lambda_1$$

$$a_{13} = c_1\left[\frac{Et}{k_1} m\lambda_2 - k_\alpha Dm(1-\nu)\sigma_2\right] + c_2m^2 \frac{Et}{k_1}$$

$$a_{14} = c_1\left[\frac{Et}{k_1} m\sigma_2 + k_\alpha Dm(1-\nu)\lambda_2\right] - c_2m^2Dk_\alpha(1-\nu)$$

$$a_{21} = D\sigma_1 \left[\sigma_1{}^2 - 3\lambda_1{}^2 - m^2(2-\nu)\right]$$

$$a_{22} = D\lambda_1 \left[\lambda_1{}^2 - 3\sigma_1{}^2 + m^2(2-\nu)\right]$$

$$a_{23} = c_1D\sigma_2 \left[\sigma_2{}^2 - 3\lambda_2{}^2 - m^2(2-\nu)\right]$$

$$a_{24} = c_1D\lambda_2 \left[\lambda_2{}^2 - 3\sigma_2{}^2 + m^2(2-\nu)\right] - c_2Dm^3(1-\nu)$$

$$a_{31} = 0$$

$$a_{32} = -\frac{m^2Et}{k_1}$$

$$a_{33} = -c_2m^2 \frac{Et}{k_1}$$

$$a_{34} = -c_1m^2 \frac{Et}{k_1}$$

$$a_{41} = D\left[\nu m^2 - \sigma_1{}^2 + \lambda_1{}^2\right]$$

$$a_{42} = 2D\sigma_1\lambda_1$$

$$a_{43} = c_1D\left[\nu m^2 - \sigma_2{}^2 + \lambda_2{}^2\right]$$

$$a_{44} = 2c_1D\sigma_2\lambda_2 - c_2Dm^2(1-\nu)$$

$$A_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} A_1 \tag{4}$$

<u>Matrices A_3, A_4</u>

$$A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \tag{5}$$

where $a_{11} = -\frac{1}{mk_1}\left[2\sigma_1\lambda_1 + k_1k_\alpha\right]$

$$a_{12} = -\frac{1}{mk_1}\left[\sigma_1{}^2 - \lambda_1{}^2 + \nu m^2\right]$$

$$a_{13} = -\frac{c_1}{mk_1}\left[2\sigma_2\lambda_2 + k_1k_\alpha\right] - \frac{m}{k_1}(1+\nu)c_2$$

$$a_{14} = -\frac{c_1}{mk_1}\left[\sigma_2{}^2 - \lambda_2{}^2 + \nu m^2\right] - \frac{k_\alpha}{m}c_2$$

$$a_{21} = 1$$

$$a_{22} = 0$$

$$a_{23} = c_1$$

$$a_{24} = c_2$$

$$a_{31} = \frac{1}{k_1(\sigma_1{}^2+\lambda_1{}^2)}\left[-k_1k_\beta\sigma_1 - m^2\lambda_1 + \nu\lambda_1(\sigma_1{}^2+\lambda_1{}^2)\right]$$

$$a_{32} = \frac{1}{k_1(\sigma_1{}^2+\lambda_1{}^2)} \left[-k_1k_\beta\lambda_1 + m^2\sigma_1+\nu\sigma_1(\sigma_1{}^2+\lambda_1{}^2)\right]$$

$$a_{33} = \frac{1}{k_1(\sigma_2{}^2+\lambda_2{}^2)} \left[-k_1k_\beta\sigma_2-m^2\lambda_2+\nu\lambda_2(\sigma_2{}^2+\lambda_2{}^2)\right] + \frac{m}{k_1}(1+\nu)c_2$$

$$a_{34} = \frac{c_1}{k_1(\sigma_2{}^2+\lambda_2{}^2)} \left[-k_1k_\beta\lambda_2+m^2\sigma_2+\nu\sigma_2(\sigma_2{}^2+\lambda_2{}^2)\right] - \frac{k_\beta}{m} c_2$$

$$a_{41} = -\sigma_1$$

$$a_{42} = +\lambda_1$$

$$a_{43} = -c_1\sigma_2$$

$$a_{44} = c_1\lambda_1 - c_2m$$

$$A_4 = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} A_3 \qquad (6)$$

<u>Matrices A_5, A_6</u>

$$A_5 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \qquad (7)$$

where $a_{11} = \frac{Et}{k_1} m\lambda_1 - k_\beta Dm(1-\nu)\sigma_1$

$$a_{12} = \frac{Et}{k_1} m\sigma_1 + k_\beta Dm(1-\nu)\lambda_1$$

$$a_{13} = c_1 \left[\frac{Et}{k_1} m\lambda_2 - k_\beta Dm(1-\nu)\sigma_2 \right] + c_2 m^2 \frac{Et}{k_1}$$

$$a_{14} = c_1 \left[\frac{Et}{k_1} m\sigma_2 + k_\beta Dm(1-\nu)\lambda_2 \right] - c_2 m^2 Dk_\beta (1-\nu)$$

$$a_{21} = Dm \left[m^2 - (2-\nu)(\sigma_1{}^2 - \lambda_1{}^2) \right]$$

$$a_{22} = 2Dm\sigma_1\lambda_1(2-\nu)$$

$$a_{23} = c_1 Dm \left[m^2 - (2-\nu)(\sigma_2{}^2 - \lambda_2{}^2) \right]$$

$$a_{24} = 2c_1 Dm\sigma_2\lambda_2(2-\nu) - c_2 Dm^3(1-\nu)$$

$$a_{31} = 2 \frac{Et}{k_1} \sigma_1 \lambda_1$$

$$a_{32} = \frac{Et}{k_1} (\sigma_1{}^2 - \lambda_1{}^2)$$

$$a_{33} = 2c_1 \frac{Et}{k_1} \sigma_2 \lambda_2 + c_2 m^2 \frac{Et}{k_1}$$

$$a_{34} = c_1 \frac{Et}{k_1} (\sigma_2{}^2 - \lambda_2{}^2)$$

$$a_{41} = D \left[m^2 - \nu(\sigma_1{}^2 - \lambda_1{}^2) \right]$$

$$a_{42} = 2D\nu\sigma_1 \lambda_1$$

$$a_{43} = c_1 D \left[m^2 - \nu(\sigma_2{}^2 - \lambda_2{}^2) \right]$$

$$a_{44} = 2c_1 D\nu\sigma_2\lambda_2 + c_2 Dm^2(1-\nu)$$

$$A_6 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} A_5 \qquad (8)$$

Matrices A_7, A_8

$$A_7 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \tag{9}$$

where

$$a_{11} = \frac{1}{k_1(\sigma_1{}^2+\lambda_1{}^2)}\left[-k_1k_\beta\sigma_1 - m^2\lambda_1 + \nu\lambda_1(\sigma_1{}^2+\lambda_1{}^2)\right]$$

$$a_{12} = \frac{1}{k_1(\sigma_1{}^2+\lambda_1{}^2)}\left[-k_1k_\beta\lambda_1 + m^2\sigma_1 + \nu\sigma_1(\sigma_1{}^2+\lambda_1{}^2)\right]$$

$$a_{13} = \frac{c_1}{k_1(\sigma_2{}^2+\lambda_2{}^2)}\left[-k_1k_\beta\sigma_2 - m^2\lambda_2+\nu\lambda_2(\sigma_2{}^2+\lambda_2{}^2)\right] + \frac{m}{k_1}(1+\nu)c_2$$

$$a_{14} = \frac{c_1}{k_1(\sigma_2{}^2+\lambda_2{}^2)}\left[-k_1k_\beta\lambda_2 + m^2\sigma_2+\nu\sigma_2(\sigma_2{}^2+\lambda_2{}^2)\right] - \frac{k_\beta}{m}\, c_2$$

$$a_{21} = 1$$

$$a_{22} = 0$$

$$a_{23} = c_1$$

$$a_{24} = c_2$$

$$a_{31} = \frac{-1}{mk_1}\left[2\sigma_1\lambda_1 + k_1k_\alpha\right]$$

$$a_{32} = \frac{-1}{mk_1}\left[\sigma_1{}^2 - \lambda_1{}^2 + \nu m^2\right]$$

$$a_{33} = \frac{-c_1}{mk_1}\left[2\sigma_2\lambda_2 + k_1k_\alpha\right] - \frac{m}{k_1}\,(1+\nu)c_2$$

$$a_{34} = -\frac{c_1}{mk_1}\left[\sigma_2{}^2 - \lambda_2{}^2 + \nu m^2\right] - \frac{k_\alpha}{m}\, c_2$$

$$a_{41} = +m$$

$$a_{42} = 0$$

$$a_{43} = c_1 m$$

$$a_{44} = c_2 m$$

$$A_8 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} A_7 \qquad (10)$$

<u>Matrices $(\bar{x}^1)^3_q$, $(\bar{u}^1)^3_q$</u>

$$(\bar{x}^1)^3_q = \begin{bmatrix} b^1_q \\ b^2_q \\ 0 \\ 0 \end{bmatrix} \quad ; \quad (\bar{u}^1)^3_q = \begin{bmatrix} 0 \\ 0 \\ b^3_q \\ b^4_q \end{bmatrix} \qquad (11)$$

where $b^1_q = \dfrac{2}{l_\beta} \displaystyle\int_0^{l_\beta} \left[D_1 L^\beta\ f + D_2 L^z\ g\right] \cos(q\bar{\beta})\, d\beta$

$$b^2_q = \frac{2}{l_\beta} \int_0^{l_\beta} \left[D_3 L^\beta\ f + D_4 L^z\ g\right] \sin(q\bar{\beta})\, d\beta$$

$$b_q^3 = \frac{2}{l_\beta} \int_0^{l_\beta} \left[D_5 L^\beta \; f + D_6 L^z \; g \right] \sin(q\bar{\beta}) d\beta \qquad (12)$$

$$b_q^4 = \frac{2}{l_\beta} \int_0^{l_\beta} \left[D_7 L^\beta f + D_8 L^z \; g \right] \sin(q\bar{\beta}) d\beta$$

Matrices f and g are defined in Chapter III, and

D_1, D_2 are (1 x 4) matrices corresponding to the first row of matrices A_5, A_6 respectively,

D_3, D_4 are (1 x 4) matrices corresponding to the second row of matrices A_5, A_6 respectively,

D_5, D_6 are (1 x 4) matrices corresponding to the third row of matrices A_7, A_8 respectively,

D_7, D_8 are (1 x 4) matrices corresponding to the fourth row of matrices A_7, A_8 respectively.

From equations (12):-

$$\begin{aligned} b_q^1 &= D_1 M_1 f + D_2 M_2 \; g \\ b_q^2 &= D_3 N_1 f + D_4 N_2 g \\ b_q^3 &= D_5 N_1 f + D_6 N_2 g \\ b_q^4 &= D_7 N_1 f + D_8 N_2 g \end{aligned} \qquad (13)$$

where $M_1 = \frac{2}{l_\beta} \int_0^{l_\beta} \cos(q\bar{\beta}) L^\beta d\beta$

$$M_2 = \frac{2}{l_\beta} \int_0^{l_\beta} \cos(q\bar{\beta}) L^z d\beta$$

$$N_1 = \frac{2}{l_\beta} \int_0^{l_\beta} \sin(q\bar{\beta}) L^\beta d\beta \tag{14}$$

$$N_2 = \frac{2}{l_\beta} \int_0^{l_\beta} \sin(q\bar{\beta}) L^z d\beta$$

Using the following integration formulae[7]:-

$$\int e^{-ax}\cos(bx)dx = \frac{e^{-ax}}{a^2+b^2}\Big[-a\cos(bx)+b\sin(bx)\Big]$$

$$\int e^{-ax}\sin(bx)dx = \frac{e^{-ax}}{a^2+b^2}\Big[-a\sin(bx) - b\cos(bx)\Big]$$

$$\int e^{-ax}\cos(bx)\sin(cx)dx = \frac{e^{-ax}}{2\left[a^2+(b+c)^2\right]}\Big[-a\sin(bx+cx) - (b+c)\cos(bx+cx)\Big] - \frac{e^{-ax}}{2\left[a^2+(b-c)^2\right]}\Big[-a\sin(bx-cx) - (b-c)\cos(bx-cx)\Big]$$

$$\int e^{-ax}\cos(bx)\cos(cx)dx = \frac{e^{-ax}}{2\left[a^2+(b+c)^2\right]}\Big[-a\cos(bx+cx) + (b+c)\sin(bx+cx)\Big] + \frac{e^{-ax}}{2\left[a^2+(b-c)^2\right]}\Big[-a\cos(bx-cx) + (b-c)\sin(bx-cx)\Big]$$

$$\int e^{-ax}\sin(bx)\sin(cx)dx = \frac{e^{-ax}}{2\left[a^2+(b-c)^2\right]}\Big[-a\cos(bx-cx) + (b-c)\sin(bx-cx)\Big] - \frac{e^{-ax}}{2\left[a^2+(b+c)^2\right]}\Big[-a\cos(bx+cx) + (b+c)\sin(bx+cx)\Big] \tag{15}$$

$$M_1 = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \tag{16}$$

where
$$a_{11} = \frac{1}{l_\beta\left[\sigma_1{}^2+(\lambda_1+n)^2\right]}\left[(-1)^q e^{-\sigma_1 l_\beta}\left[-\sigma_1\cos(\lambda_1 l_\beta) + (\lambda_1+n)\sin(\lambda_1 l_\beta)\right] + \sigma_1\right]$$

$$+ \frac{1}{l_\beta\left[\sigma_1{}^2+(\lambda_1-n)^2\right]}\left[(-1)^q e^{-\sigma_1 l_\beta}\left[-\sigma_1\cos(\lambda_1 l_\beta) + (\lambda_1-n)\sin(\lambda_1 l_\beta)\right] + \sigma_1\right]$$

$$a_{12} = \frac{1}{l_\beta\left[\sigma_1{}^2+(\lambda_1+n)^2\right]}\left[(-1)^q e^{-\sigma_1 l_\beta}\left[-\sigma_1\sin(\lambda_1 l_\beta) - (\lambda_1+n)\cos(\lambda_1 l_\beta)\right] + (\lambda_1+n)\right]$$

$$- \frac{1}{l_\beta\left[\sigma_1{}^2+(\lambda_1-n)^2\right]}\left[(-1)^q e^{-\sigma_1 l_\beta}\left[+\sigma_1\sin(\lambda_1 l_\beta) + (\lambda_1-n)\cos(\lambda_1 l_\beta)\right] - (\lambda_1-n)\right]$$

$$a_{21} = -a_{12}$$

$$a_{22} = +\, a_{11}$$

$$a_{33} = \frac{c_1}{l_\beta\left[\sigma_2^{\,2}+(\lambda_2+n)^2\right]}\left[(-1)^q e^{-\sigma_2 l_\beta}\left[-\sigma_2\cos(\lambda_2 l_\beta) + (\lambda_2+n)\sin(\lambda_2 l_\beta)\right] + \sigma_2\right]$$

$$+ \frac{c_1}{l_\beta\left[\sigma_2^{\,2}+(\lambda_2-n)^2\right]}\left[(-1)^q e^{-\sigma_2 l_\beta}\left[-\sigma_2\cos(\lambda_2 l_\beta) + (\lambda_2-n)\sin(\lambda_2 l_\beta)\right] + \sigma_2\right]$$

$$+ \frac{2c_2 m}{l_\beta\left[m^2+n^2\right]}\left[1 - (-1)^q e^{-m l_\beta}\right]$$

$$a_{34} = \frac{c_1}{l_\beta\left[\sigma_2^{\,2}+(\lambda_2+n)^2\right]}\left[(-1)^q e^{-\sigma_2 l_\beta}\left[-\sigma_2\sin(\lambda_2 l_\beta) - (\lambda_2+n)\cos(\lambda_2 l_\beta)\right] + (\lambda_2+n)\right]$$

$$- \frac{c_1}{l_\beta\left[\sigma_2^{\,2}+(\lambda_2-n)^2\right]}\left[(-1)^q e^{-\sigma_2 l_\beta}\left[+\sigma_2\sin(\lambda_2 l_\beta) + (\lambda_2-n)\cos(\lambda_2 l_\beta)\right] - (\lambda_2-n)\right]$$

$$a_{43} = -\,a_{34}$$

$$a_{44} = +\,a_{33}$$

$$N_1 = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \qquad (17)$$

where $a_{11} = \dfrac{1}{l_\beta\left[\sigma_1{}^2+(\lambda_1+n)^2\right]} \left[(-1)^q e^{-\sigma_1 l_\beta}\left[-\sigma_1 \sin(\lambda_1 l_\beta) - (\lambda_1+n)\cos(\lambda_1 l_\beta)\right] + (\lambda_1+n)\right]$

$$- \frac{1}{l_\beta\left[\sigma_1{}^2+(\lambda_1-n)^2\right]} \left[(-1)^q e^{-\sigma_1 l_\beta}\left[-\sigma_1 \sin(\lambda_1 l_\beta) - (\lambda_1-n)\cos(\lambda_1 l_\beta)\right] + (\lambda_1-n)\right]$$

$$a_{12} = \frac{1}{l_\beta\left[\sigma_1{}^2+(\lambda_1-n)^2\right]} \left[(-1)^q e^{-\sigma_1 l_\beta}\left[-\sigma_1 \cos(\lambda_1 l_\beta) + (\lambda_1-n)\sin(\lambda_1 l_\beta)\right] + \sigma_1\right]$$

$$- \frac{1}{l_\beta\left[\sigma_1{}^2+(\lambda_1+n)^2\right]} \left[(-1)^q e^{-\sigma_1 l_\beta}\left[-\sigma_1 \cos(\lambda_1 l_\beta) + (\lambda_1+n)\sin(\lambda_1 l_\beta)\right] + \sigma_1\right]$$

$$a_{21} = - a_{12}$$

$$a_{22} = + a_{11}$$

$$a_{33} = \frac{c_1}{l_\beta\left[\sigma_2{}^2+(\lambda_2+n)^2\right]} \left[(-1)^q e^{-\sigma_2 l_\beta}\left[-\sigma_2 \sin(\lambda_2 l_\beta) - (\lambda_2+n)\cos(\lambda_2 l_\beta)\right] + (\lambda_2+n)\right]$$

$$- \frac{c_1}{l_\beta\left[\sigma_2{}^2+(\lambda_2-n)^2\right]} \left[(-1)^q e^{-\sigma_2 l_\beta}\left[-\sigma_2 \sin(\lambda_2 l_\beta) - (\lambda_2-n)\cos(\lambda_2 l_\beta)\right] + (\lambda_2-n)\right]$$

$$+ \frac{2c_2 n}{l_\beta\left[m^2+n^2\right]} \left[1 - (-1)^q e^{-m l_\beta}\right]$$

$$a_{34} = \frac{c_1}{l_\beta\left[\sigma_2{}^2+(\lambda_2-n)^2\right]}\left[(-1)^q e^{-\sigma_2 l_\beta}\left[-\sigma_2\cos(\lambda_2 l_\beta) + (\lambda_2-n)\sin(\lambda_2 l_\beta)\right] + \sigma_2\right]$$
$$+ \frac{c_1}{l_\beta\left[\sigma_2{}^2+(\lambda_2+n)^2\right]}\left[(-1)^q e^{-\sigma_2 l_\beta}\left[-\sigma_2\cos(\lambda_2 l_\beta) + (\lambda_2+n)\sin(\lambda_2 l_\beta)\right] + \sigma_2\right]$$

$$a_{43} = -\ a_{34}$$

$$a_{44} = +\ a_{33}$$

Substituting

$$z = l_\beta - \beta$$

in equations (14):-

$$\begin{aligned} M_2 &= +(-1)^q\ M_1 \\ N_2 &= -(-1)^q\ N_1 \end{aligned} \qquad (18)$$

<u>Matrices $(\overline{x}^1)_q^4$, $(\overline{u}^1)_q^4$</u>

$$\begin{aligned} (\overline{x}^1)_q^4 &= \cos(p\pi)\ (\overline{x}^1)_q^3 \\ (\overline{u}^1)_q^4 &= \cos(p\pi)\ (\overline{u}^1)_q^3 \end{aligned} \qquad (19)$$

where matrices $(\overline{x}^1)_q^3$ and $(\overline{u}^1)_q^3$ are defined by equations (11), (12), (13), (15), (16), (17) and (18).

APPENDIX 5

FOURIER EXPANSIONS OF EXTERNAL LOADING ON RULED SURFACES

Uniform rectangular patch loads

The loading on the shell is expressed in the form:-

$$G_\alpha = \sum_{p=o}^{\infty} \sum_{q=o}^{\infty} g_\alpha^{pq} \sin(m\alpha) \cos(n\beta)$$

$$G_\beta = \sum_{p=o}^{\infty} \sum_{q=o}^{\infty} g_\beta^{pq} \cos(m\alpha) \sin(n\beta)$$

$$G_\gamma = \sum_{p=o}^{\infty} \sum_{q=o}^{\infty} g_\gamma^{pq} \sin(m\alpha) \sin(n\beta)$$

Let the coordinates of the centres and the dimensions of the loaded areas be (x_i, y_i) and (u_i, v_i) respectively (figure A1.1). Suffix i takes the values 1,2 and 3 and corresponds to the loading in the α, β and γ directions respectively.

Then $g_j^{pq} = a_j \, b_j^{p} \, c_j^{q}$

$(j = \alpha, \beta, \gamma)$

where $a_j = \dfrac{16P_j}{l_\alpha l_\beta}$

and P_j is the intensity of loading in the j direction, and the values of b_j^p, c_j^q are listed in tables A5.1 and A5.2.

Table A5.1

j	p	b_j^p — distribution of loading about axis $\alpha = \frac{1_\alpha}{2}$: general	symmetric	antimetric
α	$\neq 0$	$\frac{1}{m}\sin(mx_1)\sin(\frac{mu_1}{2})$	$\frac{1}{m}\sin(mx_1)\sin(\frac{mu_1}{2})$	$\frac{1}{m}\left[-\cos(mx_1)+\cos(mx_1-\frac{mu_1}{2})\right]$
	0	0	0	0
β	$\neq 0$	$\frac{1}{m}\cos(mx_2)\sin(\frac{mu_2}{2})$	$\frac{1}{m}\left[\sin(mx_2)-\sin(mx_2-\frac{mu_2}{2})\right]$	$\frac{1}{m}\cos(mx_2)\sin(\frac{mu_2}{2})$
	0	$\frac{u_2}{4}$	0	$\frac{u_2}{4}$
γ	$\neq 0$	$\frac{1}{m}\sin(mx_3)\sin(\frac{mu_3}{2})$	$\frac{1}{m}\sin(mx_3)\sin(\frac{mu_3}{2})$	$\frac{1}{m}\left[-\cos(mx_3)+\cos(mx_3-\frac{mu_3}{2})\right]$
		0	0	0

Table A5.2

j	q	$c_j^{\,q}$ — distribution of loading about axis $\beta = \frac{{}^1\beta}{2}$: general	symmetric	antimetric
α	$\neq 0$	$\frac{1}{n}\cos(ny_1)\sin(\frac{nv_1}{2})$	$\frac{1}{n}\left[\sin(ny_1)-\sin(ny_1-\frac{nv_1}{2})\right]$	$\frac{1}{n}\cos(ny_1)\sin(\frac{nv_1}{2})$
	0	$\frac{v_1}{4}$	0	$\frac{v_1}{4}$
β	$\neq 0$	$\frac{1}{n}\sin(ny_2)\sin(\frac{nv_2}{2})$	$\frac{1}{n}\sin(ny_2)\sin(\frac{nv_2}{2})$	$\frac{1}{n}\left[-\cos(ny_2)+\cos(ny_2-\frac{nv_2}{2})\right]$
		0	0	0
γ	$\neq 0$	$\frac{1}{n}\sin(ny_3)\sin(\frac{nv_3}{2})$	$\frac{1}{n}\sin(ny_3)\sin(\frac{nv_3}{2})$	$\frac{1}{n}\left[-\cos(ny_3)+\cos(ny_3-\frac{nv_3}{2})\right]$
		0	0	0

APPENDIX 6

SOLUTION OF THE AUXILIARY EQUATION FOR u_γ FOR RULED SURFACES

The auxiliary equation of u_γ is

$$(\rho^2 - m^2)^4 - k_4 m^2 \rho^2 = 0 \tag{1}$$

$$\text{where } k_4 = \frac{4Et}{D} \; k_{\alpha\beta}{}^2 \tag{2}$$

Factorizing equation (1):-

$$\left[(\rho^2-m^2)^2 + k_4{}^{1/2} m\rho\right]\left[(\rho^2-m^2)^2 - k_4{}^{1/2} m\rho\right] = 0$$

$$\rho^4 - 2m^2\rho^2 \pm k_4{}^{1/2} m\rho + m^4 = 0 \tag{3}$$

The two biquadratic equations (3) are solved by Descartes' method.

Solution of a biquadratic equation by Descartes' method[9]

$$\text{Let } x^4 + ax^2 + bx + c = 0 \tag{4}$$

$$\text{Suppose } x^4+ax^2+bx+c = (x^2+ex+f)(x^2-ex+g) \tag{5}$$

Then

$$\begin{aligned} g + f &= a + e^2 \\ g - f &= \frac{b}{e} \\ gf &= c \end{aligned} \tag{6}$$

From (6):-

$$\begin{aligned} (a + e^2 + \tfrac{b}{e})(a + e^2 - \tfrac{b}{e}) &= 4c \\ e^6 + 2ae^4 + (a^2-4c)e^2 - b^2 &= 0 \end{aligned} \tag{7}$$

From (4) and (5):-

$$x = \frac{-e \pm \left[e^2-4f\right]^{1/2}}{2} \quad ; \quad \frac{+ e \pm \left[e^2-4g\right]^{1/2}}{2} \tag{8}$$

Substituting for f and g in terms of a, b and e

$$x = -\frac{e}{2} \pm \left[-\frac{e^2}{4} + \frac{b}{2e} - \frac{a}{2}\right]^{1/2} \quad ; \quad +\frac{e}{2} \pm \left[-\frac{e^2}{4} - \frac{b}{2e} - \frac{a}{2}\right]^{1/2} \tag{9}$$

where e^2 is a solution of equation (7). The positive (or negative) square root of e^2 may be taken for the value of e in (9).

$$\text{Let } e^2 = y \tag{10}$$

Equation (7) can then be written

$$y^3 + 3a_1y^2 + 3b_1y + c_1 = 0 \tag{11}$$

$$\text{where } a_1 = \frac{2a}{3}$$

$$b_1 = \frac{a^2 - 4c}{3} \tag{12}$$

$$c_1 = - b^2$$

$$\text{Let } y = z - a_1 \tag{13}$$

Equation (11) reduces to

$$z^3 + rz + s = 0 \tag{14}$$

$$\text{where } r = 3b_1 - 3a_1{}^2 = -\frac{a^2}{3} - 4c$$

$$s = c_1 - 3a_1b_1 + 2a_1{}^3 = -\frac{2a^3}{27} - b^2 + \frac{8ac}{3} \tag{15}$$

The cubic equation (14) is solved by Cardan's method.

$$\text{If } u^3 = -\frac{s}{2} + \left[\frac{s^2}{4} + \frac{r^3}{27}\right]^{1/2}$$
$$v^3 = -\frac{s}{2} - \left[\frac{s^2}{4} + \frac{r^3}{27}\right]^{1/2} \tag{16}$$

$$\text{Then } z = u + v \tag{17}$$

$$\text{where } uv = -\frac{r}{3} \tag{18}$$

$$\text{If} \quad u^3 > 0$$
$$v^3 > 0 \tag{19}$$

and $-r > 0$

$$\text{Then } z = \left[-\frac{s}{2} + \left(\frac{s^2}{4} + \frac{r^3}{27}\right)^{1/2}\right]^{1/3} + \left[-\frac{s}{2} - \left(\frac{s^2}{4} + \frac{r^3}{27}\right)^{1/2}\right]^{1/3} \tag{20}$$

The positive cube roots are taken in (20).

Hence the roots of the biquadratic equation (4) are:-

$$x = -\frac{e}{2} \pm \left[-\frac{e^2}{4} + \frac{b}{2e} - \frac{a}{2}\right]^{1/2} ; \quad +\frac{e}{2} \pm \left[-\frac{e^2}{4} - \frac{b}{2e} - \frac{a}{2}\right]^{1/2} \tag{21}$$

$$\text{where } e^2 = -\frac{2a}{3} + \left[-\frac{s}{2} + \left(\frac{s^2}{4} + \frac{r^3}{27}\right)^{1/2}\right]^{1/3} + \left[-\frac{s}{2} - \left(\frac{s^2}{4} + \frac{r^3}{27}\right)^{1/2}\right]^{1/3} \tag{22}$$

In (22) the positive cube roots are taken and e is defined to be the positive square root of e^2.

<u>A note on the roots of a biquadratic and its associated cubic equation</u>[9],[8]

(i) If the cubic has one positive root and two complex roots, the associated biquadratic equation has two real and two complex roots.

(ii) Every cubic equation has at least one real root of a sign opposite to that of its last term (c_1 in equation (11)).

(iii) If $G^2 + 4H^3 > 0$, the cubic has two complex roots.

$$G = c_1 - 3a_1b_1 + 2a_1^3$$

$$H = b_1 - a_1^2$$

in equation (11)

<u>Consider $\rho^4 - 2m^2\rho^2 + k_4^{1/2}m\rho + m^4 = 0$</u> (23)

$$a = -2m^2 \qquad b = k_4^{1/2}m \qquad c = m^4$$

$$c_1 = -\frac{4m^2}{3} \quad r = -\frac{16m^4}{3} \qquad s = k_4m^2 + \frac{128}{27}m^6$$

$$G^2 + 4H^3 = k_4^2m^4 + \frac{256}{27}k_4 m^8$$

The associated cubic equation is

$$y^3 - 4m^2y^2 - m^2k_4 = 0$$

Since $c_1 < 0$ and $G^2 + 4H^3 > 0$, the associated cubic has one real positive root and two complex roots. Hence the biquadratic has two real and two complex roots.

Also $u^3 > 0$

$v^3 > 0$

and $-r > 0$

$$\text{Therefore } \rho = -\frac{e}{2} \pm \left[-\frac{e^2}{4} + mk_4^{1/2} + m^2\right]^{1/2};$$

$$+\frac{e}{2} \pm \left[-\frac{e^2}{4} - mk_4^{1/2} + m^2\right]^{1/2} \qquad (24)$$

$$\text{where } e^2 = \frac{4m^2}{3} + \left[-\frac{s}{2} + \left(\frac{s^2}{4} + \frac{r^3}{27}\right)^{1/2}\right]^{1/3} + \left[-\frac{s}{2} - \left(\frac{s^2}{4} + \frac{r^3}{27}\right)^{1/2}\right]^{1/3} \qquad (25)$$

In (24) the first pair of roots corresponds to the real roots and the second pair to the complex roots of the biquadratic equation (23).

<u>Consider $\rho^4 - 2m^2\rho^2 - k_4^{1/2}m\rho + m^4 = 0$</u> (26)

$a = -2m^2 \qquad b = -k_4^{1/2}m \qquad c = m^4$

$c_1 = -\frac{4m^2}{3} \qquad r = -\frac{16m^4}{3} \qquad s = -k_4m^2 - \frac{128}{27}m^6$

The associated cubic equation is identical to that in the previous case.

$$\text{Therefore } \rho = -\frac{e}{2} \pm \left[-\frac{e^2}{4} - \frac{mk_4^{1/2}}{2e} + m^2 \right]^{1/2} ;$$

$$+\frac{e}{2} \pm \left[-\frac{e^2}{4} + \frac{mk_4^{1/2}}{2e} + m^2 \right]^{1/2} \qquad (27)$$

where e^2 is given by equation (25).

In (27) the first pair of roots corresponds to the complex roots and the second pair to the real roots of the biquadratic equation (26).

Hence the eight roots of the auxiliary equation (1) are:-

$$\rho = \pm\, \sigma_1 \; ; \quad \pm\, \sigma_2 \, , \quad \pm(\sigma_3 \pm i\lambda_3) \qquad (28)$$

where $i = \left[-1 \right]^{1/2}$

and σ_1, σ_2, σ_3, λ_3 are positive and defined by:-

$$\sigma_1 = \frac{e}{2} + \sigma'$$

$$\sigma_2 = \left| \frac{e}{2} - \sigma' \right|$$

$$\sigma_3 = \frac{e}{2}$$

$$\lambda_3 = + \left[-m^2 + \frac{mk_4^{1/2}}{2e} + \frac{e^2}{4} \right]^{1/2} \qquad (29)$$

$$\sigma' = + \left[+m^2 + \frac{mk_4^{1/2}}{2e} - \frac{e^2}{4} \right]^{1/2}$$

and

$$e^2 = \frac{4m^2}{3} + \left[\frac{s}{2} + \left(\frac{s^2}{4} + \frac{r^3}{27}\right)^{1/2} \right]^{1/3} + \left[\frac{s}{2} - \left(\frac{s^2}{4} + \frac{r^3}{27}\right)^{1/2} \right]^{1/3} \qquad (30)$$

$$\text{where } r = -\frac{16m^4}{3}$$

$$s = k_4 m^2 + \frac{128}{27} m^6 \tag{31}$$

$$k_4^{1/2} = +\left[\frac{4Et}{D} k_{\alpha\beta}^2\right]^{1/2}$$

In (30) the positive cube and square roots are taken, and e is defined to be the positive square root of e^2.

$$\text{Therefore } u_\gamma = \sin(m\alpha) \sum_{j=1}^{8} c_j e^{\rho_j \beta} \tag{32}$$

where c_1, $c_2, \ldots, c_8$ are arbitrary constants and $\rho_1, \rho_2, \ldots, \rho_8$ are the roots of the auxiliary equation (1) defined by (28).

APPRNDIX 7

COMPLEMENTARY SOLUTIONS FOR u_γ AND $\emptyset$ FOR RULED SURFACES

The solution for u_γ given by equation (32) of Appendix 6 can be written

$$u_\gamma = \sin(m\alpha)\left[\bar{\mu}_1 e^{-\sigma_1\beta}+\bar{\mu}_2 e^{-\sigma_2\beta}+e^{-\sigma_3\beta}\left[\bar{\mu}_3\cos(\lambda_3\beta)+\bar{\mu}_4\sin(\lambda_3\beta)\right]\right.$$
$$\left.+\bar{\mu}_5 e^{+\sigma_1\beta}+\bar{\mu}_6 e^{+\sigma_2\beta}+e^{+\sigma_3\beta}\left[\bar{\mu}_7\cos(\lambda_3\beta)+\bar{\mu}_8\sin(\lambda_3\beta)\right]\right] \tag{1}$$

where $\bar{\mu}_1$, $\bar{\mu}_2,\ldots,\bar{\mu}_8$ are arbitrary constants.

Introducing a new variable z defined by:-

$$z = l_\beta - \beta \tag{2}$$

equation (1) can be written

$$u_\gamma = \sin(m\alpha)\left[\mu_1 e^{-\sigma_1\beta}+\mu_2 e^{-\sigma_2\beta}+e^{-\sigma_3\beta}\left[\mu_3\cos(\lambda_3\beta)+\mu_4\sin(\lambda_3\beta)\right]\right.$$
$$\left.+\mu_5 e^{-\sigma_1 z}+\mu_6 e^{-\sigma_2 z}+e^{-\sigma_3 z}\left[\mu_7\cos(\lambda_3 z)+\mu_8\sin(\lambda_3 z)\right]\right] \tag{3}$$

where μ_1, $\mu_2,\ldots,\mu_8$ are arbitrary constants.

The auxiliary equation for $\emptyset$ is identical in form to the auxiliary equation for u_γ. Hence the complementary solution for $\emptyset$ will be of the form:-

$$\emptyset=\cos(m\alpha)\left[c_1e^{-\sigma_1\beta}+c_2e^{-\sigma_2\beta}+e^{-\sigma_3\beta}\left[c_3\cos(\lambda_3\beta)+c_4\sin(\lambda_3\beta)\right]\right.$$

$$\left.+\ c_5e^{-\sigma_1 z}+c_6e^{-\sigma_2 z}+e^{-\sigma_3 z}\left[c_7\cos(\lambda_3 z)+c_8\sin(\lambda_3 z)\right]\right] \qquad (4)$$

where c_1, $c_2,\ldots,c_8$ are constants to be determined from the equations coupling u_γ and $\emptyset$:-

$$D\nabla^4 u_\gamma - 2k_{\alpha\beta}\frac{\partial^2\emptyset}{\partial\alpha\partial\beta} = 0 \qquad (5)$$

$$\nabla^4\emptyset + 2Etk_{\alpha\beta}\frac{\partial^2 u_\gamma}{\partial\alpha\partial\beta} = 0 \qquad (6)$$

$$\text{Consider} \quad (u_\gamma)_1 = \sin(m\alpha)\ \mu_1 e^{-\sigma_1\beta}$$

$$\text{and} \quad (\emptyset)_1 = \cos(m\alpha)\ c_1 e^{-\sigma_1\beta} \qquad (7)$$

Substituting (7) in (5) and (6):-

$$c_1 = -\frac{D}{2k_{\alpha\beta}}\frac{(m^2-\sigma_1{}^2)^2}{m\sigma_1}\mu_1 \qquad (8)$$

$$c_1 = -\ 2Et\ k_{\alpha\beta}\frac{m\sigma_1}{(m^2-\sigma_1{}^2)^2}\mu_1$$

From (8):-

$$c_1 = -\frac{D}{2k_{\alpha\beta}}k_4{}^{1/2}\ \mu_1 \qquad (9)$$

$$\text{Similarly} \quad c_2 = -\frac{D}{2k_{\alpha\beta}}k_4{}^{1/2}\ \mu_2 \qquad (10)$$

Consider $(u_\gamma)_3 = \sin(m\alpha)\left[e^{-\sigma_3\beta}\left[\mu_3\cos(\lambda_3\beta)+\mu_4\sin(\lambda_3\beta)\right]\right]$

and $(\phi)_3 = \cos(m\alpha)\left[e^{-\sigma_3\beta}\left[c_3\cos(\lambda_3\beta)+c_4\sin(\lambda_3\beta)\right]\right]$ (11)

$$\nabla^4(u_\gamma)_3 = \sin(m\alpha)\, e^{-\sigma_3\beta}\left[(a_3\mu_3+a_4\mu_4)\cos(\lambda_3\beta) + (a_3\mu_4 - a_4\mu_3)\sin(\lambda_3\beta)\right]^* \tag{12}$$

where
$$a_3 = m^4 - 2m(\sigma_3^2-\lambda_3^2) + (\sigma_3^2-\lambda_3^2)^2 - 4\sigma_3^2\lambda_3^2$$
$$a_4 = 4m^2\sigma_3\lambda_3 - 4\sigma_3\lambda_3(\sigma_3^2-\lambda_3^2) \tag{13}$$

$$\frac{\partial^2(\phi)_3}{\partial\alpha\partial\beta} = m\sin(m\alpha)\, e^{-\sigma_3\beta}\left[(b_3c_3+b_4c_4)\cos(\lambda_3\beta) + (b_3c_4 - b_4c_3)\sin(\lambda_3\beta)\right] \tag{14}$$

where
$$b_3 = -\sigma_3$$
$$b_4 = \lambda_3 \tag{15}$$

From equations (29) of appendix 6:-

$$\sigma_3 = \frac{e}{2}$$
$$\lambda_3 = +\left[-m^2 + \frac{mk_4^{1/2}}{2e} + \frac{e^2}{4}\right]^{1/2} \tag{16}$$

---

*Refer equation (3.39)

Substituting for σ_3, λ_3 from (16) in (13):-

$$a_3 = -\frac{1}{4e^2}\left[(e^6-4m^2e^4-m^2k_4) + 2mk_4^{1/2}e^3\right]$$

$$a_3 = -\frac{mk_4^{1/2}e}{2} \quad * \tag{17}$$

$$a_4 = +mk_4^{1/2}\lambda_3$$

From equations (15) and (17):-

$$\begin{aligned} a_3 &= mk_4^{1/2}\, b_3 \\ a_4 &= mk_4^{1/2}\, b_4 \end{aligned} \tag{18}$$

Substituting these results in equation (5) and comparing coefficients of $\cos(\lambda_3\beta)$ and $\sin(\lambda_3\beta)$:-

$$\begin{aligned} Dmk_4^{1/2}\left[b_3\mu_3 + b_4\mu_4\right] &= 2k_{\alpha\beta}\, m\left[b_3c_3 + b_4c_4\right] \\ Dmk_4^{1/2}\left[b_3\mu_4 - b_4\mu_3\right] &= 2k_{\alpha\beta}\, m\left[b_3c_4 - b_4c_3\right] \end{aligned} \tag{19}$$

From equations (19):-

$$\begin{aligned} c_3 &= \frac{Dk_4^{1/2}}{2k_{\alpha\beta}}\,\mu_3 \\ c_4 &= \frac{Dk_4^{1/2}}{2k_{\alpha\beta}}\,\mu_4 \end{aligned} \tag{20}$$

---

* $e^6 - 2m^2e^4 - m^2k_4 = 0$, since e^2 is determined as a root of this equation (refer appendix 5).

The solution for c_5, c_6, c_7, c_8 follows on similar lines to the solution for c_1, c_2, c_3 and c_4 respectively. However, since equation (5) contains only even derivatives of u_γ and odd derivatives of ϕ with respect to β, an extra negative sign is introduced in the solutions (refer equation 3.39).

Therefore
$$
\begin{aligned}
c_5 &= \frac{D}{2k_{\alpha\beta}} k_4^{1/2} \mu_5 \\
c_6 &= \frac{D}{2k_{\alpha\beta}} k_4^{1/2} \mu_6 \\
c_7 &= -\frac{D}{2k_{\alpha\beta}} k_4^{1/2} \mu_7 \\
c_8 &= -\frac{D}{2k_{\alpha\beta}} k_4^{1/2} \mu_8
\end{aligned}
\tag{21}
$$

Hence the complementary solutions for u_γ and ϕ are:-

$$
\begin{aligned}
u_\gamma = \sin(m\alpha)\Big[&\mu_1 e^{-\sigma_1\beta} + \mu_2 e^{-\sigma_2\beta} + e^{-\sigma_3\beta}\left[\mu_3\cos(\lambda_3\beta) + \mu_4\sin(\lambda_3\beta)\right] \\
&+ \mu_5 e^{-\sigma_1 z} + \mu_6 e^{-\sigma_2 z} + e^{-\sigma_3 z}\left[\mu_7\cos(\lambda_3 z) + \mu_8\sin(\lambda_3 z)\right]\Big]
\end{aligned}
\tag{22}
$$

and

$$
\begin{aligned}
\phi = -\frac{Dk_4^{1/2}}{2k_{\alpha\beta}}\cos(m\alpha)\Big[&\mu_1 e^{-\sigma_1\beta} + \mu_2 e^{-\sigma_2\beta} + e^{-\sigma_3\beta}\left[-\mu_3\cos(\lambda_3\beta) - \right. \\
&\left. - \mu_4\sin(\lambda_3\beta)\right] \\
&- \mu_5 e^{-\sigma_1 z} - \mu_6 e^{-\sigma_2 z} + e^{-\sigma_3 z}\left[\mu_7\cos(\lambda_3 z) + \mu_8\sin(\lambda_3 z)\right]\Big]
\end{aligned}
\tag{23}
$$

where μ_1, μ_2, ..., μ_8 are arbitrary constants.

APPENDIX 8

MATRICES DEFINED IN CHAPTER IV FOR RULED SURFACES

$$\text{Let } k_5 = -\frac{Dk_4^{1/2}}{2k_{\alpha\beta}} = -(EtD)^{1/2}\text{sign}(k_{\alpha\beta}) \tag{1}$$

$$\text{where sign}(k_{\alpha\beta}) = +1 \text{ if } k_{\alpha\beta} \geq 0$$

$$= -1 \text{ if } k_{\alpha\beta} < 0$$

Matrices A_1, A_2

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \tag{2}$$

where $a_{11} = -m^2k_5$

$a_{12} = -m^2k_5$

$a_{13} = +m^2k_5$

$a_{14} = 0$

$a_{21} = D\sigma_1 \left[\sigma_1^{\,2} - (2-\nu)m^2\right]$

$a_{22} = D\sigma_2 \left[\sigma_2^{\,2} - (2-\nu)m^2\right]$

$a_{23} = D\sigma_3 \left[\sigma_3^{\,2} - 3\lambda_3^{\,2} - (2-\nu)m^2\right]$

$$a_{24} = D\lambda_3 \left[\lambda_3^2 - 3\sigma_3^2 + (2-\nu)m^2\right]$$

$$a_{31} = -k_5 m\sigma_1$$

$$a_{32} = -k_5 m\sigma_2$$

$$a_{33} = +k_5 m\sigma_3$$

$$a_{34} = -k_5 m\lambda_3$$

$$a_{41} = -D\left[\sigma_1^2 - \nu m^2\right]$$

$$a_{42} = -D\left[\sigma_2^2 - \nu m^2\right]$$

$$a_{43} = -D\left[\sigma_3^2 - \lambda_3^2 - \nu m^2\right]$$

$$a_{44} = +2D\sigma_3\lambda_3$$

$$A_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} A_1 \qquad (3)$$

<u>Matrices A_3, A_4</u>

$$A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \qquad (4)$$

where $a_{11} = \dfrac{k_5}{Et\sigma_1}\left[-m^2 - \nu\sigma_1^2\right]$

$$a_{12} = \frac{k_5}{Et\sigma_2}\left[-m^2 - \nu\sigma_2{}^2\right]$$

$$a_{13} = \frac{k_5\sigma_3}{Et(\sigma_3{}^2+\lambda_3{}^2)}\left[\nu\sigma_3{}^2+\nu\lambda_3{}^2+ m^2\right]$$

$$a_{14} = \frac{k_5\lambda_3}{Et(\sigma_3{}^2+\lambda_3{}^2)}\left[-\nu\lambda_3{}^2- \nu\sigma_3{}^2+ m^2\right]$$

$$a_{21} = 1$$

$$a_{22} = 1$$

$$a_{23} = 1$$

$$a_{24} = 0$$

$$a_{31} = \frac{k_5}{Etm}\left[\sigma_1{}^2 - \nu m^2\right]$$

$$a_{32} = \frac{k_5}{Etm}\left[\sigma_2{}^2 - \nu m^2\right]$$

$$a_{33} = \frac{k_5}{Etm}\left[\nu m^2 - \sigma_3{}^2 + \lambda_3{}^2\right]$$

$$a_{34} = \frac{2k_5}{Etm}\ \sigma_3\ \lambda_3$$

$$a_{41} = -\sigma_1$$

$$a_{42} = -\sigma_2$$

$$a_{43} = -\sigma_3$$

$$a_{44} = +\lambda_3$$

$$A_4 = \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} A_3 \qquad (5)$$

Matrices A_5, A_6

$$A_5 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \tag{6}$$

where $a_{11} = k_5 \; \sigma_1^2$

$$a_{12} = k_5 \; \sigma_2^2$$

$$a_{13} = k_5 \left[-\sigma_3^2 + \lambda_3^2\right] \; .$$

$$a_{14} = 2k_5 \; \sigma_3 \; \lambda_3$$

$$a_{21} = Dm \left[m^2 - (2-\nu)\sigma_1^2\right]$$

$$a_{22} = Dm \left[m^2 - (2-\nu)\sigma_2^2\right]$$

$$a_{23} = Dm \left[m^2 - (2-\nu)(\sigma_3^2 - \lambda_3^2)\right]$$

$$a_{24} = 2Dm(2-\nu)\sigma_3 \; \lambda_3$$

$$a_{31} = -k_5 m\sigma_1$$

$$a_{32} = -k_5 m\sigma_2$$

$$a_{33} = +k_5 m\sigma_3$$

$$a_{34} = -k_5 m\lambda_3$$

$$a_{41} = D \left[m^2 - \nu\sigma_1^2\right]$$

$$a_{42} = D \left[m^2 - \nu\sigma_2^2\right]$$

$$a_{43} = D \left[m^2 - \nu(\sigma_3^2 - \lambda_3^2)\right]$$

$$a_{44} = 2D\nu\sigma_3 \lambda_3$$

$$A_6 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} A_5 \qquad (7)$$

<u>Matrices A_7, A_8</u>

$$A_7 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \qquad (8)$$

where $a_{11} = \frac{k_5}{Etm} \left[\sigma_1{}^2 - \nu m^2\right]$

$$a_{12} = \frac{k_5}{Etm} \left[\sigma_2{}^2 - \nu m^2\right]$$

$$a_{13} = \frac{k_5}{Etm} \left[\nu m^2 - \sigma_3{}^2 + \lambda_3{}^2\right]$$

$$a_{14} = \frac{2k_5}{Etm} \sigma_3 \lambda_3$$

$$a_{21} = 1$$

$$a_{22} = 1$$

$$a_{23} = 1$$

$$a_{24} = 0$$

$$a_{31} = \frac{k_5}{Et\sigma_1} \left[-m^2 - \nu\sigma_1{}^2\right]$$

$$a_{32} = \frac{k_5}{Et\sigma_2}\left[-m^2 - \nu\sigma_2{}^2\right]$$

$$a_{33} = \frac{-k_5\sigma_3}{Et(\sigma_3{}^2+\lambda_3{}^2)}\left[-\nu\lambda_3{}^2 - \nu\sigma_3{}^2 - m^2\right]$$

$$a_{34} = \frac{k_5\lambda_3}{Et(\sigma_3{}^2+\lambda_3{}^2)}\left[-\nu\lambda_3{}^2 - \nu\sigma_3{}^2 + m^2\right]$$

$$a_{41} = m$$

$$a_{42} = m$$

$$a_{43} = m$$

$$a_{44} = 0$$

$$A_8 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} A_7 \tag{9}$$

<u>Matrices $(\overline{x}^1)^3_q$, $(\overline{u}^1)^3_q$</u>

$$(\overline{x}^1)^3_q = \begin{bmatrix} b^1_q \\ b^2_q \\ 0 \\ 0 \end{bmatrix} \quad ; \quad (\overline{u}^1)^3_q = \begin{bmatrix} 0 \\ 0 \\ b^3_q \\ b^4_q \end{bmatrix} \tag{10}$$

where $b^1_q = \frac{2}{l_\beta} \int_0^{l_\beta} \left[D_1 L^\beta f + D_2 L^z g\right] \cos(q\overline{\beta})d\beta$

$$b_q^2 = \frac{2}{l_\beta} \int_0^{l_\beta} \left[D_3 L^\beta f + D_4 L^z g \right] \sin(q\bar{\beta}) d\beta$$

$$b_q^3 = \frac{2}{l_\beta} \int_0^{l_\beta} \left[D_5 L^\beta f + D_6 L^z g \right] \sin(q\bar{\beta}) d\beta \qquad (11)$$

$$b_q^4 = \frac{2}{l_\beta} \int_0^{l_\beta} \left[D_7 L^\beta f + D_8 L^z g \right] \sin(q\bar{\beta}) d\beta$$

Matrices L^β, L^z, f and g are defined in Chapter IV, and

D_1, D_2 are (1 x 4) matrices corresponding to the first row of matrices A_5, A_6 respectively,

D_3, D_4 are (1 x 4) matrices corresponding to the second row of matrices A_5, A_6 respectively,

D_5, D_6 are (1 x 4) matrices corresponding to the third row of matrices A_7, A_8 respectively,

D_7, D_8 are (1 x 4) matrices corresponding to the fourth row of matrices A_7, A_8 respectively.

From equations (11):-

$$b_q^1 = D_1\, M_1\, f + D_2\, M_2\, g$$

$$b_q^2 = D_3\, N_1\, f + D_4\, N_2\, g \qquad (12)$$

$$b_q^3 = D_5\, N_1\, f + D_6\, N_2\, g$$

$$b_q^4 = D_7\, N_1\, f + D_8\, N_2\, g$$

where $$M_1 = \frac{2}{l_\beta} \int_0^{l_\beta} \cos(q\bar{\beta})\, L^\beta \, d\beta$$

$$M_2 = \frac{2}{l_\beta} \int_0^{l_\beta} \cos(q\bar{\beta})\, L^z \, d\beta$$

$$N_1 = \frac{2}{l_\beta} \int_0^{l_\beta} \sin(q\bar{\beta})\, L^\beta \, d\beta \qquad (13)$$

$$N_2 = \frac{2}{l_\beta} \int_0^{l_\beta} \sin(q\bar{\beta})\, L^z \, d\beta$$

Using the integration formulae given by equations (15) of appendix 4:-

$$M_1 = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \qquad (14)$$

where $$a_{11} = \frac{2\sigma_1}{l_\beta\left[\sigma_1{}^2+n^2\right]} \left[1 - (-1)^q \, e^{-\sigma_1 l_\beta}\right]$$

$$a_{22} = \frac{2\sigma_2}{l_\beta\left[\sigma_2{}^2+n^2\right]} \left[1 - (-1)^q \, e^{-\sigma_2 l_\beta}\right]$$

$$a_{33} = \frac{1}{l_\beta\left[\sigma_3{}^2+(\lambda_3+n)^2\right]}\left[(-1)^q e^{-\sigma_3 l_\beta}\left[-\sigma_3\cos(\lambda_3 l_\beta) + (\lambda_3+n)\sin(\lambda_3 l_\beta)\right] + \sigma_3\right]$$

$$+ \frac{1}{l_\beta\left[\sigma_3{}^2+(\lambda_3-n)^2\right]}\left[(-1)^q e^{-\sigma_3 l_\beta}\left[-\sigma_3\cos(\lambda_3 l_\beta) + (\lambda_3-n)\sin(\lambda_3 l_\beta)\right] + \sigma_3\right]$$

$$a_{34} = \frac{1}{l_\beta\left[\sigma_3{}^2+(\lambda_3+n)^2\right]}\left[(-1)^q e^{-\sigma_3 l_\beta}\left[-\sigma_3\sin(\lambda_3 l_\beta) - (\lambda_3+n)\cos(\lambda_3 l_\beta)\right] + (\lambda_3+n)\right]$$

$$- \frac{1}{l_\beta\left[\sigma_3{}^2+(\lambda_3-n)^2\right]}\left[(-1)^q e^{-\sigma_3 l_\beta}\left[\sigma_3\sin(\lambda_3 l_\beta) + (\lambda_3-n)\cos(\lambda_3 l_\beta)\right] - (\lambda_3-n)\right]$$

$$a_{43} = - a_{34}$$

$$a_{44} = + a_{33}$$

$$N_1 = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{bmatrix} \tag{15}$$

where $a_{11} = \frac{2n}{l_\beta\left[\sigma_1{}^2+n^2\right]}\left[1 - (-1)^q e^{-\sigma_1 l_\beta}\right]$

$$a_{22} = \frac{2n}{l_\beta\left[\sigma_2{}^2+n^2\right]}\left[1 - (-1)^q e^{-\sigma_2 l_\beta}\right]$$

$$a_{33} = \frac{1}{l_\beta\left[\sigma_3{}^2+(\lambda_3+n)^2\right]}\left[(-1)^q e^{-\sigma_3 l_\beta}\left[-\sigma_3 \sin(\lambda_3 l_\beta) - (\lambda_3+n)\cos(\lambda_3 l_\beta)\right] + (\lambda_3+n)\right]$$

$$- \frac{1}{l_\beta\left[\sigma_3{}^2+(\lambda_3-n)^2\right]}\left[(-1)^q e^{-\sigma_3 l_\beta}\left[-\sigma_3 \sin(\lambda_3 l_\beta) - (\lambda_3-n)\cos(\lambda_3 l_\beta)\right] + (\lambda_3-n)\right]$$

$$a_{34} = \frac{1}{l_\beta\left[\sigma_3{}^2+(\lambda_3-n)^2\right]}\left[(-1)^q e^{-\sigma_3 l_\beta}\left|-\sigma_3 \cos(\lambda_3 l_\beta) + (\lambda_3-n)\sin(\lambda_3 l_\beta)\right] + \sigma_3\right]$$

$$- \frac{1}{l_\beta\left[\sigma_3{}^2+(\lambda_3+n)^2\right]}\left[(-1)^q e^{-\sigma_3 l_\beta}\left[-\sigma_3 \cos(\lambda_3 l_\beta) + (\lambda_3+n)\sin(\lambda_3 l_\beta)\right] + \sigma_3\right]$$

$$a_{43} = - a_{34}$$

$$a_{44} = + a_{33}$$

Substituting

$$z = l_\beta - \beta$$

in equations (13):-

$$\left.\begin{aligned} M_2 &= +(-1)^q M_1 \\ N_2 &= -(-1)^q N_1 \end{aligned}\right\} \qquad (16)$$

Matrices $(\overline{x}^1)^4_q$, $(\overline{u}^1)^4_q$

$$
\begin{aligned}
(\overline{x}^1)^4_q &= \cos(p\pi)(\overline{x}^1)^3_q \\
(\overline{u}^1)^4_q &= \cos(p\pi)(\overline{u}^1)^3_q
\end{aligned}
\tag{17}
$$

where matrices $(\overline{x}^1)^3_q$, $(\overline{u}^1)^3_q$ are defined by equations (10), (11), (12), (14), (15) and (16).

APPENDIX 9

A STANDARD LEVY TYPE SOLUTION FOR TRANSLATIONAL SHELLS. EDGES 3 and 4 ARE SUPPORTED ON DEEP THIN INEXTENSIBLE NORMAL GABLES

Although this case can be solved by the extended Levy method outlined in Chapters II and III, it can be solved more efficiently by a standard Levy type solution*.

Particular solution

A standard Nevier type particular solution is used.

(i) Loading on shell expanded as double trigonometric series

Details are given Chapter III section 1 (III.1).

(ii) Uniform band load in γ direction expanded as a trigonometric series in the α direction

$$\text{Let } G_\gamma = \sum_{p=1}^{\infty} g_\gamma{}^p \sin(p\bar{\alpha}) \qquad (1)$$

---

*If p harmonics are considered in the α direction, the extended Levy method solves the problem as 8p coupled equations. The standard Levy method solves the same problem as p uncoupled sets of 8 equations.

If P_γ = intensity of loading

$(x, \cdot \frac{l_\beta}{2})$ = coordinates of the centre of the loaded area

(u, l_β) = dimensions of loaded area

then $$g_\gamma{}^p = \frac{4P_\gamma}{p\pi} \sin(mx) \sin(\frac{mu}{2}) \qquad (2)$$

All particular solution shell quantities Y^o can be written in the form:-

$$Y^o = \sum_{p=1}^{\infty} c_k{}^p F(p\bar{\alpha}) \qquad (3)$$

where $c_k{}^p$ is a constant

and $F(p\bar{\alpha}) = \sin(p\bar{\alpha})$ or $\cos(p\bar{\alpha})$.

The results are listed in table A9.1

$$a^p = \frac{g_\gamma{}^p}{\left[Dm^4 + Etk_\beta{}^2\right]} \qquad (4)$$

$$b^p = \frac{Etk_\beta g_\gamma{}^p}{m^2\left[Dm^4 + Etk_\beta{}^2\right]} \qquad (5)$$

(iii) <u>Cylindrical shell - uniform load acting over the total surface area of the shell in a fixed direction in the $(\beta-\gamma)$ plane</u>.

Let $\overline{P}$ = intensity of loading

ψ = anticlockwise angle measured from the positive

Table A9.1

Y^o	c_k	$F(p\bar{\alpha})$	$c_k^{\,p}$
u_γ	c_1	$\sin(p\bar{\alpha})$	$+ a^p$ (refer equation (4))
m_α	c_2	$\sin(p\bar{\alpha})$	$+ Dm^2a^p$
$m_{\alpha\beta}$	c_3	$\cos(p\bar{\alpha})$	0
m_β	c_4	$\sin(p\bar{\alpha})$	$+ D\nu m^2a^p$
n_α	c_5	$\sin(p\bar{\alpha})$	0
$n_{\alpha\beta}$	c_6	$\cos(p\bar{\alpha})$	0
n_β	c_7	$\sin(p\bar{\alpha})$	$- m^2b^p$ (refer equation (5))
u_α	c_8	$\cos(p\bar{\alpha})$	$+ \frac{1}{mEt}\left[\nu c_7^{\,p} - Etk_\alpha a^p\right]$
u_β	c_9	$\sin(p\bar{\alpha})$	0
q_α	c_{10}	$\cos(p\bar{\alpha})$	$+ mc_2^{\,p}$
q_β	c_{11}	$\sin(p\bar{\alpha})$	0
q'_α	c_{12}	$\cos(p\bar{\alpha})$	$+ mc_2^{\,p}$
q'_β	c_{13}	$\sin(p\bar{\alpha})$	0
$n'_{\alpha\beta}$	c_{14}	$\cos(p\bar{\alpha})$	0
$n'_{\beta\alpha}$	c_{15}	$\cos(p\bar{\alpha})$	0
$\frac{\partial u_\gamma}{\partial \alpha}$	c_{16}	$\cos(p\bar{\alpha})$	$+ ma^p$
$\frac{\partial u_\gamma}{\partial \beta}$	c_{17}	$\sin(p\bar{\alpha})$	0

direction of loading to the positive direction of shell coordinates (α, β, γ) (figure A9.1).

Then $G_\alpha = 0$

$$G_\beta = -\bar{P} \sin(\psi - k_\beta \beta) \qquad (6)$$

$$G_\gamma = +\bar{P} \cos(\psi - k_\beta \beta)$$

$$G_\beta = -\sin(\psi - k_\beta \beta) \sum_{p=1}^{\infty} g^p \sin(p\bar{\alpha})$$

$$(7)$$

$$G_\gamma = +\cos(\psi - k_\beta \beta) \sum_{p=1}^{\infty} g^p \sin(p\bar{\alpha})$$

where $g^p = \frac{2\bar{P}}{p\pi} \left[1 - \cos(p\pi) \right] \qquad (8)$

All shell quantities Y^o can be written in the form:-

$$Y^o = F(\psi - k_\beta \beta) \sum_{p=1}^{\infty} c_k{}^p F(p\bar{\alpha}) \qquad (9)$$

where $c_k{}^p$ is a constant

and F( ) = sin( ) or cos( ).

The results are listed in table A9.2.

$$a^p = \frac{g^p \left[2(m^2+k_\beta{}^2)^2 - m^4 + \nu m^2 k_\beta{}^2 \right]}{\left[D(m^2+k_\beta{}^2)^4 + Etk_\beta{}^2 \, m^4 \right]} \qquad (10)$$

$$b^p = \frac{\left[m^2 Etk_\beta a^p + g^p\left(\frac{m^2}{k_\beta} - \nu k_\beta\right) \right]}{(m^2 + k_\beta{}^2)^2} \qquad (11)$$

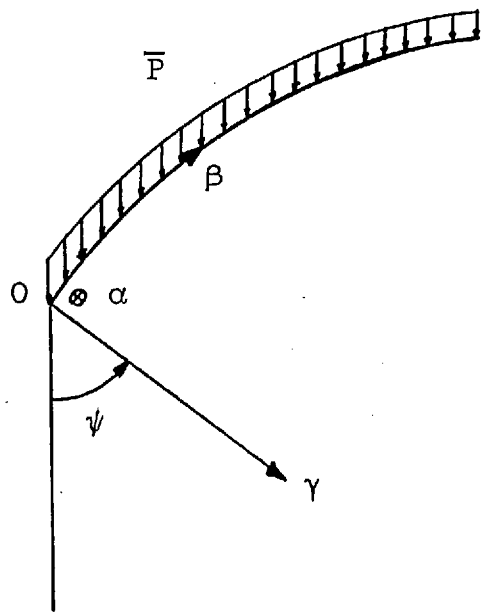


Figure A9.1

Table A9.2

Y^o	c_k	$F(p\bar{\alpha})$	$F(\psi - k_\beta \beta)$	$c_k{}^p$
u_γ	c_1	$\sin(p\bar{\alpha})$	$\cos(\psi - k_\beta \beta)$	$+a^p$ (refer equation (10)
m_α	c_2	$\sin(p\bar{\alpha})$	$\cos(\psi - k_\beta \beta)$	$+Da^p \left[m^2 + \nu k_\beta{}^2\right]$
$m_{\alpha\beta}$	c_3	$\cos(p\bar{\alpha})$	$\sin(\psi - k_\beta \beta)$	$-D(1-\nu)m\ k_\beta a^p$
m_β	c_4	$\sin(p\bar{\alpha})$	$\cos(\psi - k_\beta \beta)$	$+Da^p \left[k_\beta{}^2 + \nu m^2\right]$
n_α	c_5	$\sin(p\bar{\alpha})$	$\cos(\psi - k_\beta \beta)$	$-b^p k_\beta{}^2$ (refer equation (11))
$n_{\alpha\beta}$	c_6	$\cos(p\bar{\alpha})$	$\sin(\psi - k_\beta \beta)$	$-b^p\ m\ k_\beta$
n_β	c_7	$\sin(p\bar{\alpha})$	$\cos(\psi - k_\beta \beta)$	$-m^2 b^p + \frac{g^p}{k_\beta}$
u_α	c_8	$\cos(p\bar{\alpha})$	$\cos(\psi - k_\beta \beta)$	$-\frac{1}{mEt}\left[c_5{}^p - \nu c_7{}^p\right]$
u_β	c_9	$\sin(p\bar{\alpha})$	$\sin(\psi - k_\beta \beta)$	$-\frac{1}{k_\beta Et}\left[c_7{}^p - \nu c_5{}^p + Etk_\beta a^p\right]$
q_α	c_{10}	$\cos(p\bar{\alpha})$	$\cos(\psi - k_\beta \beta)$	$m\ c_2{}^p - k_\beta\ c_3{}^p$
q_β	c_{11}	$\sin(p\bar{\alpha})$	$\sin(\psi - k_\beta \beta)$	$k_\beta\ c_4{}^p - m\ c_3{}^p$
q'_α	c_{12}	$\cos(p\bar{\alpha})$	$\cos(\psi - k_\beta \beta)$	$m\ c_2{}^p - 2\ k_\beta\ c_3{}^p$
q'_β	c_{13}	$\sin(p\bar{\alpha})$	$\sin(\psi - k_\beta \beta)$	$k_\beta\ c_4{}^p - 2mc_3{}^p$
$n'_{\alpha\beta}$	c_{14}	$\cos(p\bar{\alpha})$	$\sin(\psi - k_\beta \beta)$	$+ c_6{}^p$
$n'_{\beta\alpha}$	c_{15}	$\cos(p\bar{\alpha})$	$\sin(\psi - k_\beta \beta)$	$+ c_6{}^p - k_\beta\ c_3{}^p$
$\frac{\partial u_\gamma}{\partial \alpha}$	c_{16}	$\cos(p\bar{\alpha})$	$\cos(\psi - k_\beta \beta)$	$+ ma^p$
$\frac{\partial u_\gamma}{\partial \beta}$	c_{17}	$\sin(p\bar{\alpha})$	$\sin(\psi - k_\beta \beta)$	$+ k_\beta a^p$

Complementary solution

A set of Levy solutions of type 1 defined in Chapter III section 2.1 is used.

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