

UNCONSTRAINED SUPERFIELD FORMULATION

OF N=2 (~~EXTENDED~~) SUPERGRAVITY THEORY

by

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اسرارِ ازل را نه تو دانی و نه من
 حل این معما نه تو خوانی و نه من
 همت از پس پرده گفتگوی من و تو
 چون پرده برافت نه تو دانی و نه من
 حکیم خیرام

There was a door to which I found no key,

There was the veil through which I might not see:

Some little talk awhile of me and thee,

There was - and then no more of thee and me.

Omar Khayam

To

Christopher John Isham

in

friendship

and

devotion

Abstract

In this thesis, we exhibit the construction of off-shell $N=2$ superfield supergravity using $N=1$ superspace geometry. We solve up to all orders, the off-shell $N=2$ supergravity constraints in terms of unconstrained $N=1$ superfields.

We start by a general introduction to supersymmetry and show how to combine it with standard field theory models. In particular, we consider scalar field theory whose supersymmetric extension is known as the Wess-Zumino model, which describes the supersymmetric coupling of scalars with spinors.

In the second chapter, the above coupling is extended to more complicated models where the $N=1$ superspace geometry is used to formulate both $N=1$ super Yang-Mills and $N=1$ supergravity in terms of unconstrained $N=1$ superfields.

The techniques used in the second chapter motivate those developed in Chapter 3 where we explicitly solve the $N=2$ off-shell supergravity constraints in terms of unconstrained $N=1$ superfields.

Finally, we discuss the formulation of extended supersymmetric field theories from a higher dimensional point of view. We prove that Sokatchev's $N=2$ supergravity constraints in four dimensions can be obtained from a single constraint in five dimensions, hence exploiting the geometrical structure of the $N=2$ supergravity constraints in four dimensions and emphasising the importance of central charge invariance of the constraints.

We also solve, to all orders in the Yang-Mills coupling constant, g , the $N=2$ super Yang-Mills constraints in six dimensions in terms of unconstrained $N=2$ superfields. To achieve this, we use the method of power expansions in the coupling constant, g .

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INTRODUCTION1.0 A brief discussion on Global Supersymmetric Models

The main aims of physics are to understand nature and relate the different forces that govern the geometry and the existence of our universe. So far there are only four known forces, namely the Gravitational, Electromagnetic, Weak and Strong forces. It has always been the objective of physicists to unify all these fundamental forces. In the modern history of physics this goes back to 1886, when Maxwell successfully unified the Electric and Magnetic forces, now known as the Electromagnetic forces. Subsequently a major breakthrough in the scheme of unification was achieved when Salam and Weinberg unified the Electromagnetic and Weak forces into Electro Weak interaction in 1967. Since then physicists have been trying hard to unify all the other interactions, in particular the general theory of relativity, which describes the gravitational field (Einstein's theory) with the strong and electroweak forces. The discovery of supersymmetry in 1971-74 [1] eventually made it possible to unify Einstein's gravity consistently with other interactions in nature. However, there are other schemes to unify strong and electroweak forces (i.e. without gravity). These are known as Grand Unified Theories (i.e. GUTs) [2].

Supersymmetry is the new type of symmetry which rotates bosons into fermions and fermions into bosons according to the schematic formulas:

$$\delta B = \bar{\epsilon} F \quad (a)$$

$$\delta F = \not{\partial} B \epsilon \quad (b) \quad (1.1)$$

where ϵ is an anti-commuting parameter which is an element of a Grassman algebra, B stands for bosons and F for fermions having dimensions $-\frac{1}{2}$, 1, $\frac{3}{2}$ respectively. We notice that (1.1) a) and b) can include higher order terms. For example, in (1.1b) we can have a term proportional to B^2 which has the correct dimension of 3/2. However, we discard such terms at the linearized level. Motivated by (1.1), it is then obvious that in a supersymmetric theory every bosonic field is associated with a fermionic companion field and vice versa. As an example consider a scalar field theory. The Klein Gordon action in four dimensions is given by:

$$I_{K.G} = -\frac{1}{2} \int [(\partial_\mu A)^2 + (\partial_\mu B)^2] d^4x. \quad (1.2)$$

where A and B are scalar and pseudo-scalar fields respectively.

If we want to introduce supersymmetry into the above action then, according to equation (1.1), the A and B fields have to transform into a fermionic field λ . Similarly, λ transforms into derivatives of the A and B fields

$$\delta A = \frac{i}{2} \bar{\epsilon} \lambda \quad (a)$$

$$\delta B = -\frac{1}{2} \bar{\epsilon} \gamma_5 \lambda \quad (b)$$

$$\delta \lambda = \not{\partial} (A + i \gamma_5 B) \epsilon \quad (c) \quad (1.3)$$

where γ_5 in (1.3b) indicates that B is a pseudo-scalar field.

In order for the action (1.2) to be invariant under the

transformations (1.3) one has to modify (1.2) by all possible terms which involve the field λ . Usually such terms contain in addition to the kinetic term of λ , all possible λ interactions.

The supersymmetric extension of (1.2) is [1]

$$I = -\frac{1}{2} \int [(\partial_\mu A)^2 + (\partial_\mu B)^2 - \frac{i}{2} \bar{\lambda} \not{\partial} \lambda] d^4x \quad (1.4)$$

This action is globally supersymmetric (where ξ^α is a constant).

Other examples of globally supersymmetric theories are Super Yang Mills Theories [3], [8].

In the case where $\xi = \xi(x)$, depending locally on the coordinates of the four dimensional space time, one is obliged to introduce gravity, which is the theory invariant under general coordinate transformations (G.C.T.). The new model which we obtain is then locally supersymmetric and it is called Supergravity (S.G.) Theory [4].

The simplest model of S.G. is the N=1 S.G., which contains the following N=1 massless supermultiplet (1 graviton spin 2 field e_μ^α , 1 gravitino spin $\frac{3}{2}$ field ψ_μ^α) where the spin $\frac{3}{2}$ field is the fermionic companion of the graviton, the massless spin 2 field.

The massless supermultiplet of N=2 S.G. contains (1 e_μ^α , 2 ψ_μ^α , 1 A_μ) where A_μ is the Abelian U(1) vector field which is the electromagnetic field. The maximal theory is the N=8 theory which contains the following massless supermultiplet:

(1 graviton, 8 gravitinos, 28 gauge vector bosons, 56 spin $\frac{1}{2}$ fields, 35 scalars, 35 pseudo-scalars).

In addition to supersymmetry invariance, S.G. theories have internal $SO(N)$ symmetries where N is identified as the number of gravitinos. The internal symmetry of any S.G. theory may be enlarged depending on whether or not there are hidden symmetries in the theory. For example, the $N=8$ S.G. [5] has a hidden global symmetry, the exceptional group $E(7)$. The maximal $N=8$ S.G. theory presents a model which was hoped to unify all the fundamental forces in nature. However, this theory still suffers from very serious problems such as ultra-violet divergencies, non-chirality of its fermionic spectrum and the fact that the internal symmetry group of the $N=8$ S.G. theory does not contain the $SU(3) \times SU(2) \times U(1)$ symmetry of strong and weak forces. For this reason it is now believed that superstring theories [6] (which only exist in ten dimensions and contain $N=8$ supergravity) stand a chance of being theories which unify all the fundamental interactions in nature. Moreover, superstrings solve the problem of chirality and their symmetries (in the case of the $E_8 \times E_8$ heterotic strings) break down in four dimensions to groups which contain the standard model's symmetry group $SU(3) \times SU(2) \times U(1)$.

Supersymmetric field theories can also be formulated in terms of superfields (S.F.) which provide an elegant linear and compact description of supersymmetry representation. In fact, all supersymmetric (global and local) theories take a very simple form when expressed in terms of superfields.

Superfields are defined over superspaces. A superspace is the space of points $Z^A = (x^a, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ where x^a are the four dimensional space time coordinates and θ^α ($\alpha = 1, 2$) is an anti-commuting Grassmannian coordinate[7] .

The coordinates z^A are usually considered to be coordinates in the coset space:

Super $ISO(3,1)/SO(3,1)$, where super $ISO(3,1)$ is the superpoincare group and $SO(3,1)$ is the Lorentz group. An element of this coset space is usually parametrized as:

$$h(z^A) = \exp z^A T_A \quad (1.5)$$

where T_A are supergenerators of the superpoincare group and they satisfy:

$$\begin{aligned} [T_A, T_B] &= T_A T_B + (-1)^{ab} T_B T_A \\ &= C_{AB}^C T_C \end{aligned} \quad (1.6)$$

where C_{AB}^C are structure constants of super $ISO(3,1)$. We leave further details to the next chapter.

A simple example of superfields is a scalar S.F. which is identified as $\phi = \phi(x, \theta, \bar{\theta})$. Similarly, a supervector, a supertensor, etc., can be defined in the same way. Let us take a scalar S.F. $\phi(x, \theta, \bar{\theta})$ and expand it as a polynomial in θ . We get:

$$\begin{aligned} \phi(x, \theta, \bar{\theta}) &= A(x) + \theta^\alpha \lambda_\alpha(x) + \bar{\theta}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}(x) \\ &\quad + \theta\theta F(x) + \bar{\theta}\bar{\theta} D(x) + \theta^\alpha \bar{\theta}_{\dot{\beta}} V_\alpha^{\dot{\beta}}(x) \\ &\quad + \theta\theta \bar{\theta}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}(x) + \bar{\theta}\bar{\theta} \theta^\alpha \lambda_\alpha(x) + \theta\theta \bar{\theta}\bar{\theta} D(x). \end{aligned}$$

Because of the anti-commuting property of θ 's the other terms vanish and we have a terminating polynomial expansion. Moreover, in addition to vector derivatives, one also defines spinorial covariant derivatives as follows:

$$D_\alpha = \frac{i}{2} (\partial_\alpha + i\bar{\theta}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}}) \quad (a)$$

$$\bar{D}_{\dot{\alpha}} = -\frac{i}{2} (\bar{\partial}_{\dot{\alpha}} + i\theta^\alpha \partial_{\alpha\dot{\alpha}}) \quad (b) \quad (1.8)$$

Therefore the components of the S.F. $\phi(x, \theta, \bar{\theta})$ in (1.7) can be defined as successive differentiation of ϕ evaluated at $\theta = \bar{\theta} = 0$ i.e.

$$\begin{aligned} A(x) &= \phi(x, \theta, \bar{\theta}) \Big|_{\theta=\bar{\theta}=0} \\ \lambda^\alpha(x) &= -2i D^\alpha \phi(x, \theta, \bar{\theta}) \Big|_{\theta=\bar{\theta}=0} \\ F(x) &= D^2 \phi(x, \theta, \bar{\theta}) \Big|_{\theta=\bar{\theta}=0} \\ &\vdots \end{aligned} \quad (1.9)$$

The supersymmetry transformation law for a scalar superfield is given by:

$$\delta_\epsilon \phi = (\epsilon^\alpha Q_\alpha + \bar{\epsilon}^{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}) \phi \quad (1.10)$$

where $\epsilon^\alpha (\bar{\epsilon}^{\dot{\alpha}})$ are the supersymmetry parameters and $Q^\alpha (\bar{Q}_{\dot{\alpha}})$ are the supersymmetry generators given by

$$Q_\alpha = \partial_\alpha - i\bar{\theta}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \quad (a)$$

$$\bar{Q}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} - i\theta^\alpha \partial_{\alpha\dot{\alpha}} \quad (b) \quad (1.11)$$

They satisfy the following properties:

$$\{0, Q\} = \{\bar{0}, \bar{Q}\} = \{\bar{0}, Q\} = \{0, \bar{Q}\} = 0 \quad (1.12)$$

using the expansion of ϕ in (1.7) and equations (1.11) one can prove that (1.10) yields the component supersymmetry transformation rules of ϕ .

The general scalar superfield ϕ is complex. Equation (1.10) shows that ϕ transforms under a representation of the superpoincare group. However, this representation is reducible. In order to have irreducible representations of the superpoincare group one has to impose additional conditions. A particularly useful irreducible representation is called the chiral representation. In this representation the chiral scalar superfield ϕ satisfies the following equations:

$$\bar{D}\phi(z) = 0 \quad (a)$$

(1.13)

$$D\bar{\phi}(z) = 0 \quad (b)$$

Note that a real superfield cannot be chiral because

$$D\phi = \bar{D}\bar{\phi} = \bar{D}\phi = 0 \Rightarrow \{Q_\alpha, \bar{D}_{\dot{\alpha}}\}\phi = \frac{i}{2}\partial_{\alpha\dot{\alpha}}\phi = 0$$

which forces ϕ to be a constant.

Equations (1.13) can be simply solved to give:

$$\begin{aligned} \phi(x, \theta, \bar{\theta}) &= \phi(y, \theta) \\ &= A(y) + \theta^\alpha \chi_\alpha(y) + \theta\theta B(y) \end{aligned} \quad (1.14)$$

where $y^\alpha = x^\alpha - i\bar{\theta}\gamma^\alpha\theta$

Consider the following action:

$$I = \int d^4x d^4\theta (\phi \bar{\phi}) \quad (1.15)$$

The measure $\int d^4\theta$ is defined such that $\int d\theta = 0$, $\int \theta d\theta = 1$ which is known as the Berezinian integral. In other words $\int d\theta \propto \frac{\partial}{\partial \theta}$ i.e. in superspace integration and differentiation are equivalent. Consequently one takes:

$$\int d^4x d^4\theta = \int d^4x d^2\theta d^2\bar{\theta} = \int D^2 \bar{D}^2 d^4x \quad (1.16)$$

Equation (1.16) reduces (1.15) to:

$$I = \int d^4x [\phi D^2 \bar{D}^2 \bar{\phi} + 2(D^\alpha \phi)(D_\alpha \bar{D}^2 \bar{\phi}) + (D^2 \phi)(\bar{D}^2 \bar{\phi})] \quad (1.17)$$

To evaluate (1.17) we use the projection method (see eqn. 1.9) and the following identities:

$$D^2 \bar{D}^2 \bar{\phi} = \square \bar{\phi}$$

$$D_\alpha \bar{D}^2 \bar{\phi} = -i \partial_{\alpha\dot{\alpha}} \bar{D}^{\dot{\alpha}} \bar{\phi}$$

We obtain

$$I \Big|_{\theta=\bar{\theta}=0} = \int d^4x (A \square \bar{A} - \frac{i}{2} \lambda^\alpha \partial_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} + F \bar{F}) \quad (1.18)$$

which is the free massless action for a complex spin 0 field A , a Weyl fermi field λ^α and a non-propagating auxiliary

field F .

Let $A = A_1 + iA_2$ and insert this in (1.18). Then we get exactly the off-shell Wess-Zumino action given by equation (1.4).

One draws two important conclusions from this example:

- i) The action written in terms of superfields in superspace can indeed take a very simple form, as seen in equation (1.15) (compared with (1.4)).
- ii) We get the off-shell formulation of supersymmetric field theories in a straightforward manner, as can be seen from (1.18) with the off-shell fields F and $\bar{F}$. The on-shell action (1.4), can be obtained by solving the equations of motion of the F and $\bar{F}$ fields and substituting them back into the action (we can do this since F and $\bar{F}$ are non-propagating).

Another example of global supersymmetric theories is Super Yang Mills (S.Y.M.) theories. The pure Yang Mills action is:

$$I = \int -\frac{1}{4} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) d^4x \quad (1.19)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$

Equation (1.19) is invariant under the following non-abelian gauge transformation:

$$A'_\mu = \Omega A_\mu \Omega^{-1} + \Omega \partial_\mu \Omega^{-1} \quad (1.20)$$

where Ω is an element of a non-abelian gauge group. We supersymmetrize (1.19) by introducing a fermi field χ_α such that

(using general principles given by equations (1.1))

$$\delta \chi_\alpha = \frac{1}{2} F_{\rho\sigma} (\gamma^{\rho\sigma})_{\alpha\beta} \epsilon_\beta$$

$$\delta A_\mu = \bar{\epsilon} \gamma_\mu \chi \quad (1.21)$$

By requiring (1.19) to be invariant under (1.21) we have to add to (1.19) the kinetic term for χ . Consequently we obtain the supersymmetrically invariant action:

$$I = \int \left[-\frac{1}{4} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) - \frac{i}{2} \bar{\chi} \not{D} \chi \right] d^4x \quad (1.22)$$

Since we have one fermi field associated with the gauged vector boson field A , the above model is called the Simple N=1 S.Y.M. Theory.

In order to express (1.22) in terms of N=1 superfields, we introduce a prepotential superfield V in the Wess-Zumino gauge [7] i.e.:

$$\begin{aligned} V_{W-Z} = & \theta^\alpha \bar{\theta}^{\dot{\alpha}} A_{\alpha\dot{\alpha}}(x) - \frac{1}{2} \theta^2 \bar{\theta}^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}}(x) \\ & - \frac{1}{2} \bar{\theta}^{\dot{\alpha}} \theta^\alpha \chi_\alpha(x) + \frac{1}{4} \theta^2 \bar{\theta}^2 D(x) \end{aligned} \quad (1.23)$$

using also the following identities:

$$D_\alpha \bar{D}^2 = -i \partial_{\alpha\dot{\alpha}} \bar{D}^{\dot{\alpha}} + \bar{D}^2 D_\alpha \quad (a)$$

$$D_\alpha D_\beta = -\frac{1}{2} \epsilon_{\alpha\beta} D^2 \quad (b) \quad (1.24)$$

we find that:

$$\left. D_\alpha \bar{D}^2 D_\beta V \right|_{\theta=\bar{\theta}=0} = -i \partial_{\beta\dot{\alpha}} A_\alpha^{\dot{\alpha}}(x) - \frac{1}{2} \epsilon_{\alpha\beta} D(x) \quad (1.25)$$

Similarly we find that

$$\left. \bar{D}^2 \bar{D}^2 D_\beta V \right|_{\theta=\bar{\theta}=0} = i \partial_{\beta\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}(x) \quad (1.26)$$

Now by calling:

$$W_\alpha = \bar{D}^2 D_\alpha V(x) \quad (1.27)$$

we find that the linearized version of (1.22) in terms of superfields is exactly given by:

$$\left. \mathcal{I} \right|_{\theta=\bar{\theta}=0} = \int d^4x \left[(D_\alpha W^\beta)(D^\alpha W_\beta) + W^\beta D^2 W_\beta \right] \quad (1.28)$$

which can be rewritten as:

$$\mathcal{I} = \frac{1}{2} \int d^4x d^2\theta W^\alpha W_\alpha \quad (1.29)$$

This is the N=1 superfield version of linearized (1.22). However, the above result is just at the linearized level and we would like to extend it to the non-linear level by finding the non-linear version of (1.27). This will be the subject of the next chapter.

The main theme of this thesis is to construct type I off-shell $N=2$ Poincare Supergravity [21] in terms of unconstrained $N=1$ superfields. The motivation for doing so is to provide a scheme for quantizing the theory in terms of $N=1$ superFeynmann rules which so far have been proved to be the most adequate way for calculating supergraphs.

Supergravity as it stands is a classical theory and needs to be quantized in order to get useful physics out of it. However, quantizing supergravity is not an easy task and one either has to develop new techniques or a more flexible approach to quantization. Superfield formulations of supersymmetric theories have provided such a flexibility as we have seen in the previous discussion. In fact, the proof of the finiteness of $N=4$ super-Yang-Mills theory was carried out using superfield perturbation theory [26]. With the development of superfield techniques we now have efficient methods for quantum calculations in supersymmetric theories. Unfortunately, such techniques are only in terms of $N=1$ superfields and therefore in supergravity, quantum calculations have been considered only for simple $N=1$ supergravity. To go to higher N extended theories one faces two options: either to develop an N extended superfield technique for quantum calculations or to express the higher extended supergravity theories in terms of $N=1$ superfields and then quantize.

In this thesis we will consider the second option, being encouraged by the existence of $N=1$ superFeynmann rules. Hence we will concentrate on the $N=2$ supergravity formulation in terms of $N=1$ superfields. Note that this formulation should be in terms of unconstrained or chiral prepotentials for practical quantum calculations.

The strategy of formulating extended supersymmetric superfield theories of $N=1$ or any N superfields relies on some basic facts. Supersymmetric theories can only go smoothly if the theory is offshell i.e. in which the algebra of infinitesimal transformations on the fields closes without imposing the field equations. In superfield language, this is equivalent to off-shell constraints (on shell constraints impose the field equations). The motivations behind

imposing these constraints are to restrict the superfields to the locally irreducible representations of supersymmetry and to give the correct degrees of freedom for a particular supersymmetric theory. However off-shell constraints are completely known only for $N=1$ and $N=2$ supergravity and super-Yang-Mills theories.

The plan of the rest of the thesis is as follows:
Chapter 2 will be mainly a general review of $N=1$ super symmetric field theories. Such a review is necessary because the mathematical techniques and methods involved serve as a basis for those in Chapter 3.

We discuss (as an introduction to Chapter 3) the algebraic and differential constraints imposed on the superfields. We sketch the solutions to these constraints and we show how to build the corresponding density actions. In particular we consider $N=1$ S.Y.M. and $N=1$ S.G. theories.

In Chapter 3 we formulate the $N=2$ S.G. theory in terms of unconstrained $N=1$ superfields. We do this by studying first the $N=2$ S.G. constraints projected onto $N=1$ superspace. By choosing a specific gauge, "the chiral gauge", the constraints take simpler forms which allow them to be solved in terms of $N=1$ superfields to all orders. Finally, we construct the corresponding action, and we close the chapter by some discussion on the importance of these results in the content of quantization of the $N=2$ model. Speculations on extending our results are given.

In Chapter 4 we construct the $N=2$ supergravity constraints from the five dimensional point of view. Using central charge invariance, we determine exactly the second fundamental $N=2$ S.G. constraint in $N=2$ chiral superspace. Moreover, we consider $N=2$ Super Yang Mills constraints in six dimensions and we show how to solve them to all orders in the S.Y.M. coupling constant using the power expansion method [26].

Finally we attach three appendices for relevant identities and details of Chapters 2, 3 and 4.

SUPERSPACE

2.1 Flat superspace

In the first chapter we gave a preliminary introduction to some supersymmetric field theories. By introducing the notion of "superspace", the Wess-Zumino and N=1 S.Y.M. actions took particularly simple forms which were manifestly supersymmetric.

In this chapter we will further elaborate on the construction of specific types of superspaces which turn out to be useful in the formulation of locally supersymmetric field theories.

The simplest type of superspace is "flat" superspace, which is defined as the coset space M.

$$M = \text{Super ISO}(3,1) / \text{SO}(3,1) \quad (2.1.1)$$

where superISO(3,1) is the superpoincare group and SO(3,1) is the Lorentz group. Using two component notation [9] a coordinate z^A of M is given by

$$z^A = (x^a, \theta^a, \bar{\theta}^{\dot{a}})$$

where x^a are the coordinates of the coset space M which is a 4 dimensional Minkowski space. θ^a and $\bar{\theta}^{\dot{a}}$ are anti-commuting coordinates taking values in the odd part of the Grassmann algebra.

An element of the superpoincare group can always be parameterized in the following way:

$$g = \exp(i(z^A T_A + \frac{1}{2} \lambda^{ab} M_{ab})) \quad (2.1.2)$$

where T_A are the super translations satisfying equation (1.6), M_{ab} are the Lorentz rotation generators and λ^{ab} are their associated parameters.

The coset manifold M in (2.1.1) is defined [3] as the set of cosets of g (or orbits) with respect to the Lorentz group, i.e.

$$M = \{gH ; g \in \text{Super ISO}(3,1) , H = \text{SO}(3,1)\} \quad (2.1.3)$$

An element of M can be parameterized as:

$$h(z) = \exp i z^A T_A$$

Moreover, the product of two elements of M gives a third element which also belongs to M , i.e.

$$h(z) \times h(z') = h(z'') \quad (2.1.4)$$

To prove this we have used (2.1.3), (1.6) and the following formula [3] .

$$e^A \cdot e^B = \exp \left[\sum_{n=1}^{\infty} \frac{1}{n!} C_n(A, B) \right] \quad (2.1.5)$$

We finally obtain:

$$Z'^A = (Z^A + Z'^A) + Z^B C_{BC}^A Z'^C \quad (2.1.6)$$

where

$$\begin{aligned} C_1 &= A + B \\ C_2 &= [A, B] \\ C_3 &= \frac{1}{2} [[A, B], B] + \frac{1}{2} [A, [A, B]] \end{aligned}$$

(due to the anticommuting nature of the θ 's, we will not need any higher coefficients in this expansion),
from which we can deduce the supertransformation (super diffeomorphism) on superspace:

$$\begin{aligned} \delta_Q \theta^a &= \epsilon^a \\ \delta_Q \bar{\theta}^i &= \bar{\epsilon}^i \\ \delta_Q x^a &= i \bar{\epsilon} \gamma^a \theta - i \bar{\theta} \gamma^a \epsilon \\ \delta_P \theta^a &= \delta_P \bar{\theta}^i = 0 \\ \delta_P x^a &= \xi^a \end{aligned} \quad (2.1.7)$$

Similarly, under the action of the Lorentz group the element $h(z)$ transforms as:

$$e^{i\lambda^{ab} M_{ab}} h(z) e^{-i\lambda^{ab} M_{ab}} = h(z') \quad (2.1.8)$$

Infinitesimally this will give:

$$\delta_M(\lambda)\theta^a = \frac{1}{2}\lambda^{ab}(\sigma_{ab}\theta)^a$$

$$\delta_M(\lambda)\bar{\theta}^{\dot{a}} = \frac{1}{2}\lambda^{ab}(\sigma_{ab}\bar{\theta})^{\dot{a}}$$

$$\delta_M(\lambda)x^a = \lambda^a_b x^b \quad (2.1.9)$$

(2.1.7) and (2.1.9) correspond to the induced representation of p , q and M on x^a , θ^a and $\bar{\theta}^{\dot{a}}$ where ξ is the parameter of the translations. In fact, equation (2.1.8) can be written infinitesimally as:

$$\delta(\Lambda)z^A = \Lambda z^A \quad (2.1.10)$$

which follows from the global left action of the supergroup on z^A , i.e. $g(\Lambda)z^A = z'^A := e^\Lambda z^A$ where Λ is defined as:

$$\Lambda = \Lambda^A \frac{\partial}{\partial z^A} = \Lambda^A \partial_A \quad (2.1.11)$$

Comparing (2.1.11) with (2.1.7) we find that

$$\Lambda = -i[R(p^a)\xi_a + \epsilon^{\dot{a}}R(Q_{\dot{a}}) + \bar{\epsilon}^{\dot{a}}R(\bar{Q}_{\dot{a}})]$$

where R is the differential operator representation of the superpoincaré algebra:

$$R(P_a) = i \partial_a$$

$$R(Q_\alpha) = i \frac{\partial}{\partial \theta_\alpha} - (\sigma^a \bar{\theta})_\alpha \partial_a$$

$$R(\bar{Q}_{\dot{\alpha}}) = -i \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + (\theta \sigma^a)_{\dot{\alpha}} \partial_a \quad (2.1.12)$$

Similarly, in the same way one finds:

$$R(M_{ab}) = i (x_a \partial_b - x_b \partial_a) + L_{ab}$$

$$+ \frac{1}{2} (\theta \sigma_{ab})^\alpha \frac{\partial}{\partial \theta^\alpha} - \frac{1}{2} (\bar{\theta} \sigma_{ab})^{\dot{\alpha}} \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} \quad (2.1.13)$$

where L_{ab} are a matrix representation of M_{ab} [9].

So far we have been discussing the left action of the supergroup on points z^A in flat superspace. One might also ask about the right action. The answer is simple since from group theory one discovers that the left and right action on z^A commute:

$$\text{i.e.} \quad (\Lambda z^A) \Sigma =: \Lambda(z^A \Sigma) \quad (2.1.14)$$

The solution of (2.1.14) for Σ is:

$$\Sigma = D^a \xi_a + \varepsilon^\alpha D_\alpha + \bar{\varepsilon}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}}$$

where

$$D_a = \partial_a \quad (a)$$

$$\bar{D}_{\dot{a}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{a}}} + i(\theta \sigma^a)_{\dot{a}} \partial_a \quad (b) \quad (2.1.15)$$

$$D_{\alpha} = \frac{\partial}{\partial \theta^{\alpha}} - i(\sigma^a \bar{\theta})_{\alpha} \partial_a \quad (c)$$

As one can see (2.1.15(a)) is the vector derivative while (2.1.15 (b) and (c)) are flat spinorial covariant derivatives. (2.1.15) can be summarized with the notation:

$$D_A = (D_a, D_{\alpha}, \bar{D}_{\dot{a}}) \quad (2.1.16)$$

Since these objects are defined on flat superspace they satisfy the following algebra:

$$[D_A, D_B] = T_{AB}^{\quad C} D_C \quad (2.1.17)$$

where $T_{AB}^{\quad C}$ is the torsion of flat superspace. Using (2.1.15) and (2.1.17) we deduce that in the absence of everything;

$$\begin{aligned} T_{ab}^{\quad c} &= T_{\alpha\beta}^{\quad \gamma} = 0 \\ T_{\alpha\beta}^{\quad \gamma} &= T_{\alpha\beta}^{\quad c} = 0 \\ T_{\alpha\beta}^{\quad c} &= 2i(\sigma^c)_{\alpha\beta} \end{aligned} \quad (2.1.18)$$

Now we discuss superfield theories in flat superspace. In

particular, we will study the constraints on S.Y.M. theories [9] and their solutions. We then show that the action density takes the form (1.28).

Because of the presence of the S.Y.M. field A_A all the covariant derivatives D_A are covariantized w.r.t. A_A . Thus we find

$$\nabla_A = D_A + A_A \quad (2.1.19)$$

Therefore the algebra (2.1.17) generalizes to:

$$[\nabla_A, \nabla_B] = T_{AB}^C \nabla_C + F_{AB} \quad (2.1.20)$$

where F_{AB} is the field strength of A_A .

If one imposes the constraint:

$$F_{\alpha\beta} = 0 \quad (2.1.21a)$$

then the connection $A_{\alpha\beta}$ can be expressed in terms of the spinor connections A_α and $A_{\dot{\alpha}}$. The constraint (2.1.21a) is equivalent to a redefinition of the connections A_A :

$$A_A \rightarrow A'_A = (A_\alpha, A_{\dot{\alpha}}, A_{\alpha\dot{\alpha}} - i F_{\alpha\dot{\alpha}}). \quad [7] \quad \text{For this reason}$$

(2.1.21a) is called a conventional constraint. Moreover, if we require that scalar multiplets described by chiral superfields be coupled to Super-Yang Mills theory, we must define covariantly chiral superfields ϕ i.e.

$$\bar{\nabla}_{\dot{\alpha}} \phi = 0$$

$$\{ \bar{\nabla}_{\dot{\alpha}}, \bar{\nabla}_{\dot{\beta}} \} \phi = 0$$

which implies

$$\bar{F}_{\dot{\alpha}\dot{\beta}} = 0 \quad (2.1.21b)$$

Similarly:

$$F_{\alpha\beta} = 0 \quad (2.1.21c)$$

The constraints (2.1.21b,c) are called representation-preserving constraints (for the chiral representation).

Equations (2.1.21 a,b,c) are called the matter constraints. Our objective now is to solve these constraints for A_a , $A_{\dot{\alpha}}$ and $\bar{A}_{\dot{\alpha}}$.

It is possible to guess these solutions. However, there exists a method which turns out to be very useful in solving the constraints (as we shall see in the third chapter). This method relies on finding "supergauge" transformations which preserve (2.1.21 a,b,c).

To find the form of such solutions we make use of the experience of invariance of four-dimensional Yang-Mills (Y.M.) theories. The field strength $F_{\mu\nu}$ is given by ($\mu = 1, 2, 3, 4$)

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + [A_{\mu}, A_{\nu}] \quad (2.1.22)$$

A_{μ} transforms as:

$$A'_{\mu} = \Omega A_{\mu} \Omega^{-1} + \Omega \partial_{\mu} \Omega^{-1} \quad (2.1.23)$$

under (2.1.23). The Y.M. action which is $\propto \text{Tr} F_{\mu\nu} F^{\mu\nu}$ is invariant while the field strength $F_{\mu\nu}$ transforms covariantly, i.e.:

$$F'_{\mu\nu} = \Omega F_{\mu\nu} \Omega^{-1} \quad (2.1.24)$$

Hence one immediately realizes that:

$$A_\mu = \Omega \partial_\mu \Omega^{-1} \quad (2.1.25)$$

is exactly the solution of the constraint:

$$F_{\mu\nu} = 0 \quad (2.1.26)$$

As we will see later on, in the case of S.Y.M. the solution of the constraint (2.1.21), in structure, would be identical to (2.1.25). However, in the case of S.Y.M. the situation is more subtle in the sense that the solution depends on the particular representation chosen of the gauge group. Effectively there are two important representations: the chiral and the vector representations. In the vector representation one insists on a Hermitian representation of the gauge group, i.e. let K be a Hermitian parameter

$$K(z) = K^\dagger T_i = K^\dagger(z) \quad (2.1.27)$$

where T_i are the generators of the gauge group.

Then a covariant superfield in the vector representation

transforms under a K transformation as:

$$\phi'_{(v)} = e^{i\Lambda} \phi_{(v)} \quad (2.1.28)$$

(The subscript (v) indicates that $\phi_{(v)}$ is the vector representation of the Y.M. group).

On the other hand, chiral superfields transform under local chiral gauge parameters defined with flat derivatives which are not Hermitian, i.e.:

$$\phi = e^{i\Lambda} \phi \quad (2.1.29)$$

However, we can introduce "convertors" e^u which convert Λ transformation to K transformation [7] .
i.e.

$$(e^u)' = e^{i\bar{\Lambda}} e^u e^{-i\kappa} \quad (a) \quad (2.1.30)$$

$$(e^{\bar{u}})' = e^{i\kappa} e^{\bar{u}} e^{-i\Lambda} \quad (b)$$

Using e^u and $e^{\bar{u}}$ we can construct a quantity which transforms independently of the K parameter, namely

$$e^v = e^u e^{\bar{u}} \quad (2.1.31)$$

which under (2.1.30) transforms as:

$$(e^v)' = e^{i\bar{\Lambda}} e^v e^{i\Lambda} \quad (2.1.32)$$

(2.1.32) corresponds to a different representation of the gauge group which is the chiral representation. Our objective is to find a U gauge transformation which leaves the constraint (2.1.21) invariant. Indeed, such a transformation is given by:

$$\delta A_\alpha = e^U D_\alpha (e^{-U}) \quad (2.1.33)$$

from (2.1.30) and the K invariance of (2.1.31). Therefore it is evident that we can choose a K-gauge transformation which can gauge away the imaginary part of u. In this gauge

$$e^U = e^{\bar{U}} = e^{\frac{1}{2}V} \quad (2.1.34)$$

where V is a real scalar superfield. Hence from (2.1.33) and (2.1.34) the solution of the constraint (2.1.21) can be pure gauge and given in terms of an unconstrained real scalar superfield V, i.e.

$$A_\alpha = e^{-V} D_\alpha e^V \quad (2.1.35)$$

In the gauge (2.1.34) and in the chiral representation $(\bar{D}_\alpha \Lambda = 0)$ we have:

$$A_\alpha = 0 \quad (2.1.36)$$

From (2.1.21a) we can deduce the solution for $A_{\alpha i}$ and hence we get

$$A_{\alpha i} = D_i A_\alpha \quad (2.1.37)$$

(Note that this is Hermitean only up to a Λ gauge transformation).

Therefore the solution of the covariant derivatives are given by:

$$\nabla_\alpha = e^{-V} D_\alpha e^V \quad (a)$$

$$\bar{\nabla}_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}} \quad (b)$$

$$\nabla_\alpha = -i\sigma_\alpha^{\dot{\alpha} i} \{ \nabla_{\dot{\alpha}}, \bar{\nabla}_i \} \quad (c) \quad (2.1.38)$$

Using (2.1.20) and (2.1.18) we deduce that:

$$\begin{aligned} F_{\alpha, \beta\dot{\beta}} &= i\epsilon_{\alpha\dot{\beta}} \bar{D}^2 (e^{-V} D_\beta e^V) \\ &:= i\epsilon_{\alpha\dot{\beta}} W_\beta \end{aligned} \quad (2.1.39)$$

such that:

$$W_\alpha = \bar{D}^2 (e^{-V} D_\alpha e^V) \quad (2.1.40)$$

where W_α is a chiral superfield, i.e. $\bar{D}^{\dot{\alpha}} W_\beta = 0$.

(2.1.32) is exactly the non-linear analogue of (1.27). The action (1.28) with the definition (2.1.32) constitutes the non-linear action of N=1 S.Y.M. theory.

2.2 Curved superspace

In this section we will briefly outline a formulation of N=1 S.G. theory in terms of N=1 superfields. The method that we will adopt is very similar to the formulation of N=1 S.Y.M. theory in the preceding section. It is our objective to construct covariant derivatives which are expressed in terms of the supervielbeins and connections of the curved superspaces [7].

Let us start by introducing the supervielbeins E_M^A , their inverses E_A^M and the Lorentz connections Ω_{MA}^B where $M, N \dots$ are world indices and $A, B \dots$ are flat Lorentz indices.

Under the action of the Lorentz group the vielbein E_M^A transforms as:

$$E_M^{A'} = L_B^{A'} E_M^B \quad (2.2.1)$$

where L_A^B is an element of the Lorentz group whose components are:

$$L_A^B = (L_{\alpha}^{\beta}, L_{\alpha}^{\dot{\beta}}, L_{\alpha\dot{\alpha}}^{\beta\dot{\beta}} = L_{\alpha}^{\beta} L_{\alpha\dot{\alpha}}^{\dot{\beta}}) \quad (2.2.2)$$

The components of the Lorentz connection Ω_{MA}^B are

$$\Omega_{MA}^B = (\Omega_{M\alpha}^{\beta}, \Omega_{M\alpha}^{\beta'}, \Omega_{M\alpha\alpha'}^{\beta\beta'})$$

where

$$\Omega_{M\alpha\alpha'}^{\beta\beta'} = (\Omega_{M\alpha}^{\beta} \delta_{\alpha'}^{\beta'} + \Omega_{M\alpha'}^{\beta'} \delta_{\alpha}^{\beta}) \quad (2.2.3)$$

Equations (2.2.2) and (2.2.3) imply that the Lorentz group acts reducibly on the tangent space of the curved superspace.

As in the case of S.Y.M. theory, we introduce an unconstrained real prepotential:

$$H = i H^M D_M \quad (2.2.4)$$

where $H^M = (H^m, H^\mu, \bar{H}^{\mu'})$ and D is the flat superspace covariant derivative.

Next we construct an element e^H which transforms under the action of the group as:

$$e^{H'} = e^{i\bar{\Lambda}} e^H e^{-i\Lambda} \quad * \quad (2.2.5)$$

where:

$$\Lambda = i \Lambda^M D_M$$

and Λ^μ is a chiral superfield. Moreover, being complex,

* The justification of using a real scalar superfield H goes along the same lines as in the case of S.Y.M.

it follows that $\Lambda^{\dot{\mu}} \neq \Lambda^{\mu\dot{\mu}}$. However, $(\Lambda^{\dot{\mu}})^{\dagger} = \bar{\Lambda}^{\dot{\mu}}$.
 Similarly, $(\Lambda^{\mu})^{\dagger} = \bar{\Lambda}^{\mu}$.

The infinitesimal form of (2.2.5) is:

$$\delta e^H = i \bar{\Lambda} e^H - i e^H \Lambda \quad (2.2.6)$$

To analyse the components of H we have to examine (2.2.6) carefully.

Consider the chiral superfield ϕ where:

$$\bar{D}_{\dot{\mu}} \phi = 0 \quad (2.2.7)$$

Since (2.2.7) should be preserved under general group transformations, then we should have:

$$\bar{D}_{\dot{\mu}} \delta \phi = \bar{D}_{\dot{\mu}} \Lambda \phi = 0 \quad (2.2.8)$$

which shows that Λ should be chiral.

(2.2.8) can be solved for Λ which gives

$$\Lambda^{\dot{\mu}} = \bar{D}^2 L^{\dot{\mu}}, \quad \Lambda^m = -i \bar{D}^{\dot{\mu}} L^{\dot{\mu}} \quad \text{and} \quad \Lambda^{\dot{\mu}} \text{ is arbitrary.}$$

Similarly

$$\bar{\Lambda}^{\dot{\mu}} = D^2 \bar{L}^{\dot{\mu}}, \quad \bar{\Lambda}^m = -i D^{\dot{\mu}} \bar{L}^{\dot{\mu}} \quad \text{and} \quad \bar{\Lambda}^{\dot{\mu}} \text{ is arbitrary.} \quad (2.2.9)$$

Expanding (2.2.6) in powers of H, one finds [7]:

$$\delta H = i(\bar{\Lambda} - \Lambda) + \frac{1}{2} [H, \Lambda + \bar{\Lambda}] + \mathcal{O}(H^2)$$

i.e.

$$\begin{aligned} \delta H^M = & i(\bar{\Lambda} - \Lambda)^M + \frac{1}{2} [H(\underline{\Lambda} + \bar{\Lambda})^M - (\underline{\Lambda} + \bar{\Lambda})H^M] \\ & + \mathcal{O}(H^2) \end{aligned} \quad (2.2.10)$$

From (2.2.9) we see that Λ^κ and $\bar{\Lambda}^\kappa$ are unconstrained and hence from (2.2.10) we can always gauge away H^κ and $\bar{H}^\kappa$. So we are only left with the vector component H^m .

To preserve the chiral representation, we go back to eqn. (2.2.6) and demand the cancellation of D_μ and $\bar{D}_\mu$ components. This can be achieved by choosing:

$$\bar{\Lambda}^\kappa = e^H \Lambda^\kappa e^{-H} \quad (a)$$

(2.2.11)

$$\Lambda^\kappa = e^H \bar{\Lambda}^\kappa e^{-H} \quad (b)$$

In this representation $\bar{\Lambda}^\kappa$ is the chiral representation Hermitean conjugate of Λ^κ and similarly for $\bar{\Lambda}^\kappa$.

We now construct the curved covariant derivatives. As we showed in the last section the covariant derivatives w.r.t. the Y.M. field transform as $\nabla'_A = e^{i\Lambda} \nabla_A e^{-i\Lambda}$ or infinitesimally $\delta \nabla_A = [i\Lambda, \nabla_A]$. In the presence of gravity, because of the local Lorentz transformations, such formulas have to be generalized. The new transformation reads:

$$\nabla'_A = L_A^B (e^{i\Lambda} \nabla_B e^{-i\Lambda}) \quad (a)$$

or

$$\delta \nabla_A = [i\lambda, \nabla_A] + \omega_A^B \nabla_B \quad (b) \quad (2.2.12)$$

where we have used the infinitesimal transformation:

$$L_A^B = \delta_A^B + \omega_A^B \quad (2.2.13)$$

with L_A^B as defined in (2.2.1) and (2.2.2).

The covariant derivative ∇_A takes the following form:

$$\nabla_A = E_A + \phi_{A\gamma}^{\beta} M_{\beta}^{\gamma} + \phi_{A\gamma}^{\beta'} \bar{M}_{\beta'}^{\gamma'} \quad (2.2.14)$$

Where $E_A = E_A^M D_M$, ϕ_A are Lorentz connections and M_{β}^{γ} are their associated Lorentz generators. Using (2.2.14), we can work out the torsion components T_{AB}^C , the Riemannian tensors $R_{AB\alpha}^{\beta}$ and $R_{AB\alpha'}^{\beta'}$ from:

$$[\nabla_A, \nabla_B] = T_{AB}^C \nabla_C + R_{AB}(M) + R_{AB}(\bar{M}) \quad (2.2.15)$$

The torsion can be expressed in terms of the Lorentz connections ϕ_A as:

$$T_{AB}^C = C_{AB}^C + \phi_{[AB]}^C + \bar{\phi}_{[AB]}^C \quad (2.2.16)$$

where C_{AB}^C is the anholonomy coefficient defined by:

$$[E_A, E_B] = C_{AB}^C E_C \quad (2.2.17)$$

Our objective is to determine E_A in such a way that it should transform covariantly under the action of the group defined by (2.2.12) and (2.2.2). Therefore we propose the following ansatz for E_M following the analogy of Y.M. (see eqn. (2.1.30) in the chiral representation .

$$E_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}} \quad (a)$$

$$E_{\alpha} = e^{-H} D_{\alpha} e^H \quad (b)$$

$$E_{\alpha} = -i \sigma_{\alpha}^{\dot{\alpha}} \{ E_{\alpha}, E_{\dot{\alpha}} \} \quad (c) \quad (2.2.18)$$

Although (2.1.30) transforms covariantly under the group action (2.1.34), unfortunately (2.2.18) does not transform covariantly under the group action (2.2.12) and (2.2.2). Thus, for example,

$$\delta E_{\dot{\mu}} = [i\Lambda, E_{\dot{\mu}}] + \omega_{\dot{\mu}}^{\nu} E_{\dot{\nu}} + \Sigma E_{\dot{\mu}} \quad (a)$$

where $\omega_{\dot{\mu}}^{\nu} = -\frac{1}{2} \bar{D}_{(\dot{\mu}} \Lambda^{\nu)}$ (b)

$$\Sigma = -\frac{1}{2} E_{\dot{\mu}} \Lambda^{\dot{\mu}} = -\frac{1}{2} \bar{D}_{\dot{\mu}} \bar{\Lambda}^{\dot{\mu}} \quad (c) \quad (2.2.19)$$

As it is obvious from (2.2.19 (a)), the last term is not an element of the gauge group, therefore it should be eliminated. This can be achieved by introducing a compensator ψ and its chiral representation hermitean conjugate $\tilde{\psi}$ following the same analogy of (2.2.11a):

$$\tilde{\psi} = e^{-H} \bar{\psi} e^H \quad (2.2.20)$$

By choosing ψ to transform as

$$\delta\psi = [i\lambda, \psi] - \Sigma\psi \quad (2.2.21)$$

(obviously ψ transforms non-covariantly as a density under the group action), the new vielbeins defined as:

$$\hat{E}_\mu = \psi E_\mu \quad (a)$$

$$\hat{E}_\mu = \tilde{\psi} E_\mu \quad (b) \quad (2.2.22)$$

do indeed transform covariantly. Hence we find

$$\delta\hat{E}_\mu = [i\lambda, \hat{E}_\mu] + \omega_\mu{}^\nu \hat{E}_\nu \quad (a)$$

$$\delta\hat{E}_\mu = [i\lambda, \hat{E}_\mu] + \tilde{\omega}_\mu{}^\nu \hat{E}_\nu \quad (b) \quad (2.2.23)$$

However, an immediate check reveals that $\hat{E}_m$ does not transform covariantly simply because of the existence of two terms of the form:

$$-i\omega_m{}^\nu \hat{E}_\nu - i\tilde{\omega}_m{}^\nu \hat{E}_\nu \quad (2.2.24)$$

which do not match the group structure (2.2.2). (For more details see [7]). However, it is found that the vector

vielbein

$$\begin{aligned} \check{E}_\alpha = \delta_\alpha^m \left[\hat{E}_m - \frac{i}{2} \hat{C}_{\mu\nu, \mu'}^{\nu\nu'} \hat{E}_\nu \right. \\ \left. - \frac{i}{2} \hat{C}_{\nu\mu', \mu}^{\nu\nu'} \hat{E}_{\nu'} \right] \end{aligned} \quad (2.2.25)$$

transforms covariantly according to:

$$\delta \check{E}_m = [i\lambda, \check{E}_m] + \check{\omega}_\mu{}^\nu \check{E}_{\nu\mu} + \omega_{\mu'}{}^{\nu'} \check{E}_{\mu'\nu'} \quad (2.2.26)$$

where $\hat{C}$ are anholonomy coefficients given by $[\hat{E}_A, \hat{E}_B] = \hat{C}_{AB}{}^C \hat{E}_C$.
Therefore equations (2.2.23) and (2.2.26) can be summarized by:

$$\delta \check{E}_M = [i\lambda, \check{E}_M] + \omega_M{}^N \check{E}_N \quad (2.2.27)$$

which is exactly the covariant transformation group law of (2.2.12) and (2.2.2).

By determining $\check{E}_A$ it is straightforward to determine the components of the anholonomy coefficient $C_{AB}{}^C$ using (2.2.17). However, one can see that specific linear combinations of these components transform exactly as Lorentz connections and hence they can be identified with the components of $\phi_{A\alpha}{}^\beta$ and $\phi_{A\alpha}{}^{\dot{\beta}}$ [7].

Now that we have determined the covariant derivatives ∇_A , we can determine from (2.2.15) all the torsion and Riemann components of N=1 curved superspace which are the ingredients

of the geometry of N=1 curved superspace.

As can be seen from (2.2.21), ψ transforms non-covariantly as a density. On the other hand, the object in S.G. which transforms as a density is the inverse superdeterminant of E_A^M , i.e.

$$\delta \hat{E}^{-1} = i \hat{E}^{-1} \overleftarrow{\lambda} \quad (2.2.28)$$

where:

$$\overleftarrow{\lambda} = i \lambda^M \overleftarrow{D}_M = i (-)^M (\overleftarrow{D}_M \lambda^M + D_M \lambda^M) \quad (a)$$

such that:

$$i \overleftarrow{\lambda} = i [\partial_m \lambda^m - D_\mu \lambda^\mu - \bar{D}_{\dot{\mu}} \bar{\lambda}^{\dot{\mu}}] \quad (b) \quad (2.2.29)$$

Therefore ψ should be related to E . The following ansatz has been proposed [7]:

$$\psi^{4n} = E^{n+1} \quad (2.2.30)$$

where n could take any real (or complex) value. In principle this equation can be solved for ψ . However, (2.2.30) restricts the choice of our gauge group. To see this we consider the variation of (2.2.30) using (2.2.28) and (2.2.21) to get the following restriction:

$$-4n \Sigma = -(n+1) (i \overleftarrow{\lambda}) \quad (a)$$

or

$$(3n+1)\bar{D}_{\dot{\mu}}\Lambda^{\dot{\mu}}=(n+1)(\partial_m\Lambda^m-D_{\dot{\mu}}\Lambda^{\dot{\mu}}) \quad (b) \quad (2.2.31)$$

This formula is interesting on its own, since for different values of n one can get different restrictions on the gauge group. For example, for $n = -\frac{1}{3}$ one obtains:

$$(\partial_m\Lambda^m-D_{\dot{\mu}}\Lambda^{\dot{\mu}})=0 \quad (2.2.32)$$

i.e. $\Lambda^{\dot{\mu}}$ is unrestricted and hence one can gauge away $H^{\dot{\mu}}$ and H^{μ} by complex conjugation (c.f. 2.2.9 - 2.2.11). In this Wess-Zumino gauge the field content of H^m is one graviton, one gravitino, an axial vector field and a complex scalar auxiliary field S given by:

$$S = \partial_m D^2 H^m \quad (2.2.33)$$

This super-multiplet $(e_m^a, \psi_{(\alpha\dot{\beta})\dot{\beta}}, A_{\alpha\dot{\alpha}}, S)$ is the off shell minimal N=1 S.G. multiplet. On the other hand, for $n \neq -\frac{1}{3}, 0$ one obtains the non-minimal off-shell N=1 S.G. model with the following content:

$(e_m^a, \psi_{(\alpha\dot{\beta})\dot{\beta}}, \text{real scalar } p, A_{\alpha\dot{\alpha}}; \text{divergence of an axial vector } p)$.

Finally, there is the $n=0$ model which gives an axial off-shell N=1 S.G. with content $(e_m^a, \psi_{(\alpha\dot{\beta})\dot{\beta}}, \chi_{\alpha}, V^a = \partial_b T^{[ab]})$.

2.3 Density action

In order to construct an action for off-shell S.G. theories one should be able to write down Lagrangians that transform as total derivatives;

$$\begin{aligned}
 \delta L &= (-)^M D_M (\Lambda^M L) \\
 &= (-)^M D_M \Lambda^M L + \Lambda^M L \overleftarrow{D}_M \\
 &= L (\overleftarrow{D}_M \Lambda^M + (-)^M D_M \Lambda^M) \\
 &= i L \overleftarrow{\Lambda}
 \end{aligned} \tag{2.3.1}$$

i.e. L transforms exactly as a density. Now we know that E^{-1} possesses exactly this property and since it also has the correct dimension it is then a very good candidate for a Lagrangian. Therefore the action for Poincare S.G. can be given up to a dimensional constant by:

$$I_{\text{S.G.}} = \frac{1}{\kappa^2} \int d^4 x d^4 \theta \hat{E}^{-1} \tag{2.3.2}$$

Using (2.2.22), one can calculate $\hat{E}$ in terms of E . The final answer for $n \neq 0$ is given by:

$$I_{\text{S.G.}}(n \neq 0) = \frac{1}{n \kappa^2} \int d^8 Z [E^n (1 \cdot e^{\overleftarrow{H}})^{\frac{n+1}{2}}] \tag{2.3.3}$$

The action for $n=0$ S.G. is more subtle and needs more explanation which lies outside the scope of this thesis[11].

2.4 Alternative Formulation of Supergravity, the notion of Complex Superspaces [10]

In the last three sections we have seen the approach of real, curved superspace to the formulation of off-shell $N=1$ S.G. The starting point of our construction were equations (2.2.12) and (2.2.2) which were supposed to be the gauge symmetry of S.G. theories in superspace. However, a close examination of such transformations ((2.2.2) and (2.2.12)) revealed the fact that they contained superfluous transformations* and hence the symmetry group (2.2.2) and (2.2.12) was larger than the required gauge group (Poincare or the conformal group) for supergravity theories. Therefore, we had to restrict the gauge symmetry group by getting rid of the superfluous transformations.

One way to accomplish this, was to demand the covariance of all the components of the vielbein E_A ; and the Lorentz connections Ω_A . The above mentioned restrictions were described by equation (2.2.31(b)). For the different values of n one obtained different types of restrictions on the full gauge group and hence different versions of off-shell $N=1$ S.G. In this section we will consider a type of superspace, the complex chiral superspace in which the gauge symmetry group is smaller than that of ordinary superspace. Consequently some of the constraints and restrictions imposed on the prepotentials in ordinary superspace naturally disappear in complex chiral superspace.

Complex chiral superspace is an $8+4$ dimensional superspace

* For example, from (2.2.19(a)) we can see that the transformation of the vielbein contains a Weyl scale transformation term ΣE_μ which is not in the Poincare gauge group, therefore it is a superfluous transformation.

$\{X_L^m, \theta_L^\mu, X_R^m, \bar{\theta}_R^{\dot{\mu}}\}$ (to be compared with the 4+4 dimensional real superspace $\{x^m, \theta^\mu, \bar{\theta}^{\dot{\mu}}\}$)

In this superspace:

$$X_R^m = (X_L^m)^\dagger \quad (a)$$

$$\bar{\theta}_R^{\dot{\mu}} = (\theta_L^\mu)^\dagger \quad (b) \quad (2.4.1)$$

where (2.4.1(a)) is a complex space-time 4-vector coordinate.

The gauge supergroup in this superspace is defined as the g.c.t. in the left (X_L^m, θ_L^μ) as well as in the right conjugate superspace $(X_R^m, \bar{\theta}_R^{\dot{\mu}})$.

Left superspace (l.s.s.)

Right superspace (R.s.s.)

$$X_L'^m = X_L^m + \lambda^m(X_L, \theta_L)$$

$$X_R'^m = X_R^m + \bar{\lambda}^m(X_R, \bar{\theta}_R)$$

(2.4.2)

$$\theta_L'^\mu = \theta_L^\mu + \lambda^\mu(X_L, \theta_L)$$

$$\bar{\theta}_R'^{\dot{\mu}} = \bar{\theta}_R^{\dot{\mu}} + \bar{\lambda}^{\dot{\mu}}(X_R, \bar{\theta}_R)$$

where

$$\bar{\lambda}^m(X_R, \bar{\theta}_R) = (\lambda^m(X_L, \theta_L))^\dagger, \quad \bar{\lambda}^{\dot{\mu}}(X_R, \bar{\theta}_R) = (\lambda^\mu(X_L, \theta_L))^\dagger$$

Since λ depends only on θ_L or $\bar{\theta}_R$ then the gauge group is already smaller than the g.c.t. group in the 4+4 dimensional real superspace.

The general group (2.4.2) corresponds to Weyl S.G. To obtain Einstein's S.G. we restrict (2.4.2) to a subgroup of it by requiring the supervolumes in the L.S.S. as well as in the R.S.S. to be preserved. Before preserving the supervolumes let us introduce the following notion. The chiral left-subspace is $\mathbb{C}^{4,2}$ and $\mathbb{C}^{*4,2}$ is its complex conjugate (the chiral right subspace), i.e.

$$\mathbb{C}^{4,2} = \{z_L\} = \{x_L^m, \theta_L^\mu\} \quad (a)$$

(2.4.3)

$$\mathbb{C}^{*4,2} = \{z_R\} = \{x_R^m, \bar{\theta}_R^\mu\} \quad (b)$$

Furthermore, consider the general complex superspace:

$$\mathbb{C}^{4,4} = \{z_L\} = \{x_L^m, \theta_L^\mu, \bar{\phi}_L^\mu\} \quad (a)$$

$$\mathbb{C}^{*4,4} = \{z_R\} = \{x_R^m, \bar{\theta}_R^\mu, \phi_R^\mu = (\bar{\phi}_L^\mu)^+\} \quad (b) \quad (2.4.4)$$

where $\bar{\phi}_L^\mu$ is a complex 4-spinor and $\mathbb{C}^{*4,4}$ is the complex conjugate of (2.4.4 (a)). The new complex 4-spinor $\bar{\phi}_L^\mu(\phi_R^\mu)$ transforms as:

$$\delta \bar{\phi}_L^\mu = \bar{f}^\mu(x_L, \theta_L, \bar{\phi}_L) \quad (a)$$

$$\delta \phi_R^\mu = f^\mu(x_R, \bar{\theta}_R, \phi_R) \quad (b) \quad (2.4.5)$$

(see 2.4.2).

(Note that in equation (2.4.2) λ^m and λ^μ ($\bar{\lambda}^m$ and $\bar{\lambda}^\mu$) are chiral (anti-chiral) parameters while $\bar{\mathcal{F}}$ is strictly arbitrary. The natural restriction which would reduce the transformations (2.4.2) and (2.4.4) to those which correspond to Poincaré or Einstein's S.G. is:

$$\left[\text{Ber} \left(\frac{\partial z'_L}{\partial z_L} \right) \right]^{3n+1} = \left[\text{Ber} \left(\frac{\partial \bar{z}'_L}{\partial \bar{z}_L} \right) \right]^{2n} \quad (2.4.6)$$

Infinitesimally this corresponds to:

$$(3n+1) \frac{\partial}{\partial \bar{\phi}_L^\mu} \bar{\mathcal{F}}^\mu = (n+1) \left(\frac{\partial \lambda^m}{\partial x_L^m} - \frac{\partial \lambda^\mu}{\partial \theta_L^\mu} \right) \quad (2.4.7)$$

This is identical to (2.2.31 (b)).

Evidently for different values of n one obtains different versions of off-shell $N=1$ S.G.

The interesting thing in this approach is that our earlier real superspace:

$$\mathbb{R}^{4,4} = \{x^m, \theta^\mu, \bar{\theta}^\mu\} \quad (2.4.8)$$

can be realized as a hypersurface in $\mathbb{C}^{4,4}$ and $\mathbb{C}^{*,4,4}$.

Mathematically this corresponds to writing

$$X^m = \text{Real } X_L^m = \frac{1}{2} (X_L^m + X_R^m) \quad (a)$$

$$H^m = \text{Im } X_L^m = \frac{1}{2i} (X_L^m - X_R^m) \quad (b)$$

$$H^\mu(x, \theta, \bar{\theta}) = \varphi_R^\mu - \theta_L^\mu \quad (c)$$

$$\bar{H}^{\dot{\mu}}(x, \theta, \bar{\theta}) = \bar{\varphi}_L^{\dot{\mu}} - \bar{\theta}_R^{\dot{\mu}} \quad (d) \quad (2.4.9)$$

$$\theta^\mu = \theta_L^\mu \quad (e)$$

$$\bar{\theta}^{\dot{\mu}} = \bar{\theta}_L^{\dot{\mu}} \quad (f)$$

Using the group transformations (2.4.2) and (2.4.5) we can determine the transformations of the prepotentials H^M i.e.

$$\delta H^m = H'^m(x', \theta', \bar{\theta}') - H^m(x, \theta, \bar{\theta}) = \frac{1}{2i} [\lambda^m(x_L, \theta_L) - \bar{\lambda}^m(x_R, \bar{\theta}_R)]$$

$$\delta H^\mu = H'^\mu(x', \theta', \bar{\theta}') - H^\mu(x, \theta, \bar{\theta}) = \varphi^\mu(x_R, \bar{\theta}_R, \varphi_R) - \lambda^\mu(x_L, \theta_L)$$

$$\delta H^{\dot{\mu}} = H'^{\dot{\mu}}(x', \theta', \bar{\theta}') - H^{\dot{\mu}}(x, \theta, \bar{\theta}) = \bar{\varphi}^{\dot{\mu}}(x_L, \theta_L, \bar{\varphi}_L) - \lambda^{\dot{\mu}}(x_R, \bar{\theta}_R) \quad (2.4.10)$$

Let us go back to (2.4.6) and consider it for different values of n . We obtain the following:

i) For $n = -\frac{1}{3}$, equation (2.4.6) becomes:

$$\text{Ber} \left(\frac{\partial \mathcal{Y}'_L}{\partial \mathcal{Y}_L} \right) = 1 \quad (2.4.11)$$

This means that the transformations (2.4.2) preserve the supervolume of $\mathbb{C}^{4,2}$. In this case the parameters

$\mathfrak{f}^\mu(\chi_R, \theta_R, \phi_R)$ and $\bar{\mathfrak{f}}^\mu(\chi_L, \theta_L, \bar{\phi}_L)$ are arbitrary, since, as can be seen from (2.4.7), they are unrestricted. Hence we can gauge away H^μ and $\bar{H}^\mu$.

Therefore, in this case, as expected, we get the minimal off-shell N=1 S.G. which has 12+12 fields.

ii) For $n=0$, equation (2.4.6) becomes:

$$\text{Ber} \left(\frac{\partial z'_L}{\partial z_L} \right) = 1 \quad (2.4.12)$$

which means that the supervolume of $\mathbb{C}^{4,4}$ is preserved. In this case the supervolume invariant element of $\mathbb{C}^{4,4}$ is:

$$d^8 z_L = d^4 \chi_L d^2 \theta_L d^2 \bar{\phi}_L \quad (a) \quad (2.4.13)$$

Similarly, the supervolume element for $\mathbb{C}^{*,4,4}$ is:

$$d^8 z_R = d^4 \chi_R d^2 \theta_R d^2 \phi_R \quad (b)$$

By making the change of variables, i.e.

$$Z_L \rightarrow Z \rightarrow Z_R$$

the supervolumes in

(2.4.13 a,b) are related by:

$$d^8 z_L = \text{Ber} \left(\frac{\partial z_L}{\partial z} \right) \text{Ber} \left(\frac{\partial z}{\partial z_R} \right) d^8 z_R \quad (2.4.14)$$

Therefore the quantity:

$$U(x, \theta, \bar{\theta}) = \text{Ber} \left(\frac{\partial z_L}{\partial z_R} \right) = \text{Ber} \left(\frac{\partial z_L}{\partial z} \right) \cdot \text{Ber}^{-1} \left(\frac{\partial z_R}{\partial z} \right) \quad (2.4.15)$$

is invariant under transformations (2.4.2) and (2.4.5).

From (2.4.15) it is evident that:

$$UU^\dagger = 1 \quad (2.4.16)$$

The $\hbar=0$ expansion of H^m can be easily obtained by using its transformation property under the chiral parameters λ^m and $\mathcal{J}$.

One finds the following:*

$$\begin{aligned} \frac{1}{\kappa} H^m(x, \theta, \bar{\theta}) &= \sigma^\mu \bar{\theta}^\mu e_{\mu\dot{\mu}}^m + \bar{\theta}^2 \theta^\mu \psi_{\mu}^m \quad (a) \\ &+ \theta^2 \bar{\theta}_{\dot{\mu}} \psi^{m\dot{\mu}} + \theta^2 \bar{\theta}^2 A^m. \end{aligned} \quad (2.4.17)$$

$$\begin{aligned} \frac{1}{\kappa} H^\mu(x, \theta, \bar{\theta}) &= i\theta^\mu A + \theta^\mu \bar{\theta}_{\dot{\mu}} \bar{\xi}^{\dot{\mu}} + \theta^2 \bar{\theta}_{\dot{\mu}} (V + iW)^{\mu\dot{\mu}} \\ &+ \bar{\theta}^2 \theta^\mu B + \theta^2 \bar{\theta}^2 \beta^\mu \end{aligned} \quad (b)$$

The remarkable feature of this model, as can be seen from (2.4.17 (a) and 2.4.17 (b)), is the existence of the fields A_m and $V_a = \frac{1}{2} (\sigma_a)_{\mu\mu'} V^{\mu\mu'}$ which transform as $U(1)$ gauge fields. However, they are not propagating fields (they are auxiliary), since their dimensions are cm^{-2} , which is not a physical dimension for $U(1)$ gauge fields.

* $[K] = \text{cm}^{+1}$, $[H^m] = \text{cm}^{+1}$, $[H^\mu] = [\theta^\mu] = \text{cm}^{1/2}$

Therefore they cannot have a normal kinetic term of the form $F_{mn} F^{mn}$. Hence the only way that one can make V_a and A_m couple to the pure S.G. model is through a coupling to a divergenceless (i.e. constrained) axial vector field. This corresponds to the constraint [10]

$$U(X, \theta, \bar{\theta}) = 1 \quad (2.4.18)$$

which gives a relation between A_m and V_a

$$A^m e_a^m V^a = \epsilon^{mnkl} \partial_n a_{kl} \quad ; a_{kl} = -a_{lk} \quad (2.4.19a)$$

($\delta a_{lk} = \partial_{[l} b_{k]}$), as well as:

$$A = \bar{\xi}^{\dot{\mu}} = B = 0 \quad , W_a = -\frac{1}{2} \partial_m e^m_a \quad , B_{\mu} = i \partial_m \psi^m_{\mu} \quad (2.4.19b)$$

Consider the following invariant integrals:

$$I_L = \int d^8 z_L \phi_L(z_L) = \int d^8 z \text{Ber}\left(\frac{\partial z_L}{\partial z}\right) \phi(z) \quad (a)$$

$$I_R = \int d^8 z_R \phi_R(z_R) = \int d^8 z \text{Ber}\left(\frac{\partial z_R}{\partial z}\right) \phi(z) \quad (b)$$

By using (2.4.18) we can see that: $I_R = I_L = I$, and hence we can identify

$$\text{Ber}\left(\frac{\partial z_L}{\partial z}\right) = E \quad (2.4.21)$$

where E is the Berzinian of the vielbeins for the curved superspace $\mathbb{R}^{4,4}$ with local $U(1)$ symmetry in the tangent space.

It is obvious, from the structure of the action (2.4.20), that $\phi(z)$ cannot be a constant since the integral would then vanish. In addition, all the real dimensionless scalar fields can either be constants or be proportional to u . But since u is constrained or equals 1 therefore ϕ cannot be a real scalar superfield. Therefore the action I_L is no more invariant. However, we can make I_L again invariant by demanding ϕ not to be a scalar, but transform as a scalar density, namely:

$$\delta\phi(x, \theta, \bar{\theta}) = L + R \quad (2.4.22)$$

where:

$$L = L(x_L, \theta_L)$$

$$R = R(x_R, \theta_R)$$

$$R = L^\dagger$$

are some left and right handed $\mathbb{C}^{4,2}$ (chiral) parameters. Then:

$$\begin{aligned} \delta \int d^8z E \cdot \phi &= \int d^8x_L d^2\theta_L d^2\bar{\theta}_L \cdot L(x_L, \theta_L) + h.c \\ &= 0 \end{aligned} \quad (2.4.23)$$

because $L(R)$ is independent of $\bar{\phi}_L(\phi_R)[10]$. Finally, the construction of $\phi(z)$ in terms of unconstrained pre-potentials H^m is not straightforward. However, the result is:

$$\phi = \ln \psi$$

$$\begin{aligned} \psi = & \det^{-\frac{1}{4}} \left(\frac{1}{4} [\Delta, \sigma_a \bar{\Delta}] H^m \right) \cdot \det^{-\frac{1}{8}} (\delta_n^m + \partial_n H^\kappa \partial_\kappa H^m) \\ & \times [\det (\delta_\nu^\mu + \Delta_\nu H^\mu) \cdot \det (\delta_\nu^\mu + \bar{\Delta}^\mu \bar{H}_\nu)]^{\frac{3}{8}} \end{aligned} \quad (2.4.24)$$

where ψ , as in the preceding section, is a density compensator, while Δ_ν is a covariant derivative:

$$\Delta_\mu = \partial_\mu + i \Delta_\mu H^m \partial_m \quad (a)$$

(2.4.25)

$$\bar{\Delta}_{\dot{\mu}} = -\bar{\partial}_{\dot{\mu}} - i \bar{\Delta}_{\dot{\mu}} H^m \partial_m \quad (b)$$

Therefore the final action of $n=0$ axial S.G. takes the following form in $\mathbb{R}^{4,4}$:

$$S = \frac{1}{\kappa^2} \int d^4 x d^2 \theta d^2 \bar{\theta} E \cdot \ln \psi \quad (2.4.26)$$

If one inserts (2.4.17), (2.4.19) and (2.4.24) into (2.4.26) one obtains in components the action discovered by [11].

THE STRUCTURE OF $N=2$ SUPERGRAVITY IN $N=1$ SUPREFIELDS

3.0 Introduction

The principal objective of this chapter is to complete the story of Chapter 2 by considering the next theory to $N=1$ S.G., i.e. $N=2$ S.G. As we know, $N=2$ S.G. is the only extended local supersymmetric theory whose off-shell superspace constraints are known. This is because for $N>3$, finite offshell representations of local supersymmetric theories cannot exist (because of the no-go theorems[24]), although recently it has been proved that such a barrier can be breached using an infinite set of auxiliary fields (which corresponds to an infinite off-shell representation in which the no-go theorem no longer applies). This is the mechanism of the so-called harmonic superspaces.

As we know, it seems to be very difficult to solve the off-shell $N=2$ S.G. constraints in a closed form in terms of unconstrained $N=2$ prepotentials.

One way of getting round this problem is to get rid of one of the symmetries and keep only the second supersymmetry. This is achieved by projecting on to $N=1$ superspace.

As will be described later, this is possible by considering the $\theta^{2\alpha}$ independent parts of the projected constraints $((\theta^{\alpha i} = \theta^{\alpha 1}, \theta^{\alpha 2}))$. In this way the $N=2$ S.G. constraints are entirely expressed in terms of $N=1$ superfields and the remaining $\theta^{\alpha 1}$ components of the constraints are nothing but those of pure $N=1$ S.G. coupled to $N=1$ matter fields.

Such a system is still invariant under the $N=2$ supersymmetry, however, not invariant under linearly realized $SU(2)$ symmetry.

In this chapter we will formulate the $N=2$ S.G. theory in terms of unconstrained $N=1$ superfields. We will indeed show that one can solve the projected $N=2$ constraints in terms of unconstrained $N=1$ superfields by fixing a particular gauge which is called the chiral gauge. In general it is not necessary to fix a gauge; however, it is very useful to look for gauges which would simplify the complexity of the analysis.

Our solutions are entirely expressed in terms of pure $N=1$ S.G. superfields, the matter prepotential $\mathcal{V}_\alpha$ and a spinor superfield λ [15], [16]. It is then straightforward to construct the action of the off-shell $N=2$ S.G. in terms of unconstrained $N=1$ superfields [29].

The plan of the rest of the chapter is as follows:

In section 3.1 we construct and solve the linearized $N=2$ S.G. constraints in chiral superspace. By comparing our solutions with those of [13] we discover that there exists a gauge which relates their solutions to ours at the linearized level.

In section 3.2 we describe the projection of $N=2$ S.G. constraints on to $N=1$ superspace, thus shedding light on the $N=1$ supersymmetry geometry of the $N=2$ theory.

In section 3.3 we prove that the gauge chosen in section 3.1 at the linear level can be consistently retained at the non-linear level. This will simplify a great deal the projected constraints. The $N=1$ structure of the projected constraints can be further exploited by redefining our $N=1$ projected constraints. Then one can easily extract the $N=1$ solutions of the constraints.

In section 3.4 we completely solve the constraints. In section 3.5 we construct the action and finally we draw our conclusions and speculations in section 3.6.

3.1 The Linear N=2 Theory

The action of N=2 supergravity was given in terms of N=2 superfields by Sokatchev, where the constraints in his formalism were partially solved. The remaining constraints are [13] :

$$T_{\alpha(i, \dot{\beta}j)}^c = 0 \quad (a)$$

$$\text{Im} (D_{\alpha i}^{\dot{\alpha}} T_{\alpha j} - T_{\alpha i}^{\dot{\alpha}} T_{\alpha j}) = 0 \quad (b) \quad (3.1.1)$$

such that

$$T_{\alpha i} \equiv T_{\alpha i, \dot{\beta}j}^{\dot{\beta}j} \quad (3.1.2)$$

in terms of prepotentials the constraint (3.1.1a) takes the following form:

$$F_{\alpha(i}^{-1} \bar{F}_{\dot{\beta}j)}^{\dot{\beta}} [\Delta_{\underline{r}}, \Delta_{\underline{r}}] H^m(\sigma_m)^{\gamma\dot{\gamma}} = 0 \quad (3.1.3)$$

where ($\underline{r} = (r, \kappa)$)

$$F_{\alpha}^{\dot{\alpha}} = S_{\alpha}^{\dot{\alpha}} + \Delta_{\alpha} H^{\dot{\alpha}}$$

$$\bar{\Delta}_{\underline{r}} = (\Delta_{\underline{r}})^{\dagger}$$

$$\Delta_{\underline{r}} = \partial_{\underline{r}} + i \Delta_{\underline{r}} H^m \partial_m \quad (3.1.4)$$

and D_α^i is the curved N=2 super covariant derivative.
 H^m , H^p and $H^{\dot{p}}$ are vector and spinor prepotentials
 respectively [13]. In fact, we will prove in Chapter 4 that
 the constraints (3.1.11) can be included in a single con-
 straint (3.1.5) if we introduce an extra coordinate in the
 N=2 superspace.

$$T_{\alpha(i} \hat{M}_{j)} = 0 \quad (3.1.5)$$

such that $\hat{M} = (M, 5)$

We expect the projected N=2 theory to describe an N=1
 supergravity theory coupled to (3/2,1) matter fields. To
 see this we use the field equations rather than the action
 (for simplicity). The N=2 superspace field equations are [18]

$$T_{\alpha i} = T_{\alpha j} = 0 \quad (3.1.6)$$

where $T_{\alpha i}$ is given by (3.1.2). Equation (3.1.6) implies
 that

$$T_\alpha = T_{\alpha 1} = 0 \quad (3.1.7)$$

$$\lambda_\alpha = T_{\alpha 2} = 0$$

We will prove that T_α is the equation of motion of N=1
 supergravity theory and λ_α contains the 0(3/2,1) multiplet.
 We start by linearizing the constraints (3.1.1a) and (3.1.1b),
 leaving the non-linear analysis to the next section.

Let

$$H^{\underline{t}} = \kappa h^{\underline{t}}$$

$$H^{\underline{t}'} = \kappa h^{\underline{t}'}$$

$$H^m = \sigma^{\alpha i} \sigma^m_{\alpha \dot{\beta}} \bar{\theta}^{\dot{\beta}'}_i + \kappa h^m \quad (3.1.8)$$

hence

$$\Delta_{\underline{\alpha}} = D_{\underline{\alpha}} + i\kappa D_{\underline{\alpha}} h^m \partial_m \quad (3.1.9)$$

where $D_{\underline{\alpha}} = \partial_{\underline{\alpha}} + i\sigma^m \bar{\theta} \partial_m$ is the flat super covariant derivative. By inserting (3.1.8) in (3.1.2) and keeping to first order in κ , we get (using the notation of [13])

$$T_{\underline{\alpha}} = \kappa [\bar{D}_{\underline{\alpha}} D_{\underline{\alpha}} h^{\underline{\alpha}} - \frac{1}{2} \epsilon_{ij} \bar{D}_{\underline{\alpha}} \kappa D_{\underline{\alpha}} h^{\underline{\alpha}j}] \quad (3.1.10)$$

(3.1.10) is invariant under the following linearized transformation in chiral superspace

$$\delta h^{\underline{\alpha}} = \lambda^{\underline{\alpha}} (\chi_+ iH, \theta)_+ \mathcal{F}^{\underline{\alpha}} (\chi_- iH, \theta_+ H, \bar{\theta}) \quad (3.1.11)$$

$\lambda^{\underline{\alpha}}$ is a chiral superfield and $\mathcal{F}^{\underline{\alpha}}$ satisfies [13] :

$$D_{\underline{\alpha}} (i \mathcal{F}^{\underline{\alpha}j}) = 0 \quad (3.1.12)$$

Next, linearizing (3.1.1a) and (3.1.1b), we obtain:

$$[D_{\alpha(i}, \bar{D}_{\dot{\beta}j)}] h^{\dot{\alpha}\dot{\beta}} = \bar{D}_{\dot{\beta}(i} h_{j)}^{\dot{\alpha}} - D_{\alpha(i} h_{j)}^{\dot{\beta}} \delta_{\dot{\beta}}^{\dot{\alpha}} \quad (a)$$

$$D_{\alpha(i} \bar{D}_{\dot{\beta}j)} h^{\alpha}_{\kappa)} = 0 \quad (b)$$

$$\bar{D}_{\dot{\alpha}(i} D_{\beta j)} H = \bar{D}_{\dot{\alpha}(i} h_{\beta j)} - D_{\beta(i} \bar{h}_{j), \dot{\alpha}} \quad (c) \quad (3.1.13)$$

It is now possible to split (3.1.10) into N=1 components by using (3.1.7). We project into N=1 superspace as follows [23] :

$$D_{\alpha i} = (D_{\alpha 1} = D_{\alpha}, \quad D_{\alpha 2} = d_{\alpha}) \quad (3.1.14)$$

$$\bar{D}_{\dot{\alpha} i} = (\bar{D}_{\dot{\alpha} 1} = -\bar{d}_{\dot{\alpha}}, \quad \bar{D}_{\dot{\alpha} 2} = \bar{D}_{\dot{\alpha}})$$

$$H = \kappa h, \quad h^{\alpha 1} = h^{\alpha}, \quad h^{\alpha 2} = \ell^{\alpha}, \quad \epsilon^{ij} = -\epsilon^{ji}$$

$$\{D_{\alpha}, \bar{d}_{\dot{\beta}}\} = 0$$

We can calculate $T_{\alpha 1}$ by using (3.1.14) and (3.1.13). The expression obtained for $T_{\alpha 1}$ will involve some unwanted terms (for example, terms linear and quadratic in d_{α} etc.). To get a pure expression involving only D_{α} (N=1 super covariant derivative) and h_{α} (the N=1 superfield) we eliminate the unwanted terms by using constraints (3.1.13) which lead to useful relations:

$$D_\alpha \bar{d}_{\dot{\beta}} \bar{h}^{\dot{\gamma}} = 0 \quad (a)$$

$$d_\alpha \bar{D}_{\dot{\beta}} \bar{l}^{\dot{\gamma}} = 0 \quad (b)$$

$$D_\alpha \bar{d}_{\dot{\beta}} \bar{l}^{\dot{\beta}} = D_\alpha \bar{D}_{\dot{\beta}} \bar{h}^{\dot{\beta}} - d_\alpha \bar{d}_{\dot{\beta}} \bar{h}^{\dot{\beta}} \quad (c)$$

$$d_\alpha \bar{d}_{\dot{\beta}} \bar{h}^{\dot{\beta}} + 2D_\alpha d_\beta l^\beta = -2D_\alpha d_\beta \bar{d}_{\dot{\beta}} h^{\beta\dot{\beta}} \quad (d)$$

$$\begin{aligned} [D_\beta \bar{D}_{\dot{\beta}} - d_\beta \bar{d}_{\dot{\beta}}] h^{\beta\dot{\beta}} &= (\bar{D}_{\dot{\beta}} \bar{h}^{\dot{\beta}} - \bar{d}_{\dot{\beta}} \bar{l}^{\dot{\beta}}) \\ &\quad + (d_\beta l^\beta - D_\beta h^\beta) \end{aligned} \quad (e) \quad (3.1.15)$$

From (d) and (e) we obtain another useful relation:

$$d_\alpha \bar{d}_{\dot{\beta}} \bar{h}^{\dot{\beta}} + 2D_\alpha \bar{d}_{\dot{\beta}} \bar{l}^{\dot{\beta}} = -2D_\alpha [D_\beta \bar{D}_{\dot{\beta}} h^{\beta\dot{\beta}} - \bar{D}_{\dot{\beta}} \bar{h}^{\dot{\beta}} + D_\beta h^\beta] \quad (3.1.16)$$

Using (3.1.16) and (3.1.15c) we get

$$D_\alpha \bar{d}_{\dot{\beta}} \bar{l}^{\dot{\beta}} = -2D_\alpha [D_\beta \bar{D}_{\dot{\beta}} h^{\beta\dot{\beta}} - \frac{1}{2} \bar{D}_{\dot{\beta}} \bar{h}^{\dot{\beta}} + D_\beta h^\beta] \quad (3.1.17)$$

Also

$$D_\alpha \bar{d}_{\dot{\beta}} \bar{l}^{\dot{\beta}} = -2D_\alpha [D_\beta \bar{D}_{\dot{\beta}} h^{\beta\dot{\beta}} - \frac{1}{2} \bar{D}_{\dot{\beta}} \bar{h}^{\dot{\beta}} + D_\beta h^\beta] \quad (3.1.18)$$

Going back to (3.1.10) we then have (putting $i=1$ and 2)

$$T_{\alpha 1} = T_{\alpha} = \kappa \left[-\frac{1}{2} D_{\alpha} \bar{D}_{\dot{\alpha}} \bar{h}^{\dot{\alpha}} - D_{\alpha} \bar{d}_{\dot{\beta}} \bar{l}^{\dot{\beta}} + \frac{1}{2} d_{\alpha} \bar{d}_{\dot{\beta}} \bar{h}^{\dot{\beta}} \right] \quad (3.1.19)$$

$$T_{\alpha 2} = \lambda_{\alpha} = \kappa \left[-\frac{1}{2} D_{\alpha} \bar{D}_{\dot{\alpha}} \bar{l}^{\dot{\alpha}} - d_{\alpha} \bar{D}_{\dot{\alpha}} \bar{h}^{\dot{\alpha}} - \frac{1}{2} d_{\alpha} \bar{d}_{\dot{\beta}} \bar{l}^{\dot{\beta}} \right] \quad (3.1.20)$$

Substituting (3.1.17) and (3.1.18) in (3.1.19) and using the redefined quantities:

$$\phi^{\alpha} = \frac{3}{2} \kappa h^{\alpha}$$

$$\mathcal{H}^{\alpha\dot{\alpha}} = \frac{3}{2} \kappa h^{\alpha\dot{\alpha}}$$

we finally have:

$$T_{\alpha} = D_{\alpha} \left[D_{\beta} \bar{D}_{\dot{\beta}} \mathcal{H}^{\beta\dot{\beta}} + 2 D_{\beta} \phi^{\beta} + \bar{D}^{\dot{\beta}} \bar{\phi}_{\dot{\beta}} \right] \quad (3.1.21)$$

As we have conjectured, (3.1.21) is the pure N=1 super-gravity field equation with $\mathcal{H}^{\alpha\dot{\alpha}}$, ϕ^{α} and $\phi^{\dot{\alpha}}$ being the pure N=1 vector and spinor prepotentials respectively.

To eliminate the d terms in (3.1.20) we need more identities which involve the scalar prepotential h. Putting i=1, j=2 in (3.1.13c) and then multiplying by $\bar{D}_{\dot{\alpha}}$ we get:

$$\begin{aligned} -d_{\beta} \bar{D}_{\dot{\alpha}} \bar{h}^{\dot{\alpha}} &= \bar{D}^{\dot{\alpha}} D_{\beta} \bar{l}_{\dot{\alpha}} - \bar{D}^{\dot{\alpha}} \bar{d}_{\dot{\alpha}} d_{\beta} H + \bar{D}^2 D_{\beta} H \\ &\quad + \bar{D}^2 l_{\beta} + \bar{D}^{\dot{\alpha}} \bar{d}_{\dot{\alpha}} h_{\beta} \end{aligned} \quad (3.1.22)$$

Putting $i=j=2$ in (3.1.13c) and then multiplying by $\bar{d}_\alpha$ we have:

$$\bar{D}_\alpha \bar{d}_\alpha h_\beta = \bar{D}^\alpha \bar{d}_\alpha d_\beta H - \bar{d}_\alpha d_\beta \bar{l}^\alpha \quad (3.1.23)$$

Inserting (3.1.23) in (3.1.22) we obtain:

$$-d_\beta \bar{D}_\alpha \bar{h}^\alpha = \bar{D}^2 D_\beta H - \bar{d}_\alpha d_\beta \bar{l}^\alpha + \bar{D}^2 l_\beta + \bar{D}^\alpha D_\beta \bar{l}_\alpha \quad (3.1.24)$$

Now putting $i=j=2$, $k=1$ in (3.1.13b) and contracting δ with β we obtain:

$$d_\alpha \bar{D}_\beta \bar{h}^\beta = \bar{D}_\beta D_\alpha \bar{l}^\beta - \bar{d}_\beta d_\alpha \bar{l}^\beta \quad (3.1.25)$$

where we have used

$$\bar{d}_\beta d_\alpha - \bar{D}_\beta D_\alpha = D_\alpha \bar{D}_\beta - d_\alpha \bar{d}_\beta$$

Adding (3.1.25) and (3.1.24) we get:

$$\bar{d}_\beta d_\alpha \bar{l}^\beta = \frac{1}{2} \bar{D}^2 D_\alpha H + \frac{1}{2} \bar{D}^2 l_\alpha \quad (3.1.26)$$

By inserting (3.1.24) and (3.1.26) in (3.1.20) and using the redefined quantities:

$$\psi_\alpha = -3\kappa l_\alpha$$

$$\bar{\psi}_{\dot{\alpha}} = -3\kappa \bar{l}_{\dot{\alpha}}$$

$$\mathcal{H} = -\frac{3}{2} i\kappa H \quad (3.1.27)$$

we finally obtain:

$$\begin{aligned} \tau_\alpha = \frac{1}{4} \lambda_\alpha = -\frac{1}{4} (\bar{D}^2 \psi_\alpha - 2 \bar{D}_{\dot{\beta}} D_\alpha \bar{\psi}^{\dot{\beta}} \\ - 2iW_\alpha - D_\alpha \phi) \end{aligned} \quad (3.1.28)$$

such that

$$W_\alpha = \bar{D}^2 D_\alpha \mathcal{H} \quad \text{satisfies} \quad D^\alpha W_\alpha = \bar{D}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}}$$

Here we have introduced a compensator ϕ for the system which under gauge transformations transforms as:

$$\delta \bar{\phi} = D^2 \Lambda \quad (3.1.29)$$

Similarly,

$$\delta \mathcal{H} = \frac{i}{2} (\bar{\Lambda} - \Lambda) \quad (a)$$

$$\delta \psi_\alpha = \Lambda_\alpha + D_\alpha \Lambda, \quad \bar{D}_{\dot{\alpha}} \Lambda = 0 \quad (b) \quad (3.1.30)$$

(3.1.28) contains the $0(3/2,1)$ matter multiplet, and agrees with the results of Siegel and Gates [20]. Therefore we have shown that the $N=2$ field equations, when projected into the $N=1$ superspace, do decompose into the correct field equations

of $0(1)$ supergravity and the $0(3/2,1)$ matter field couplings. We have checked the consistency of this result by repeating the above analysis to all orders in K .

3.2 The Non Linear N=2 Theory

The N=2 supergravity constraints in N=2 superspace were given in [21]. They are

$$\begin{aligned}
 \{D_{\underline{\alpha}}, D_{\underline{\beta}}\} &= C_{ij} C_{\alpha\beta} (Z + \bar{W}_{\dot{\gamma}}^{\dot{\delta}} \bar{M}_{\dot{\delta}}^{\dot{\gamma}}) \\
 \{D_{\underline{\alpha}}, \bar{D}_{\underline{\beta}}\} &= i\delta_i^j \hat{D}_{\alpha\dot{\beta}} - \frac{1}{4} (T_{\alpha(i} \bar{D}^{j)}_{\dot{\beta}} + \bar{T}^{(i}_{\dot{\beta}} D_{j\alpha}) \\
 &\quad - \frac{1}{2} (P_{(i}^{\quad j)}{}^{\dot{\gamma}}{}_{\dot{\beta}} M^{\gamma}_{\alpha} - P_{(i}^{\quad j)}{}_{\alpha\dot{\gamma}} \bar{M}^{\dot{\gamma}}_{\dot{\beta}}) \\
 [\hat{D}_{\alpha\dot{\alpha}}, D_{\underline{\beta}}] &= \frac{1}{4} (2T_{\underline{\beta}} \delta_{\alpha}^{\gamma} - T_{j\alpha} \delta_{\underline{\beta}}^j) \hat{D}_{\gamma\dot{\alpha}} \\
 &\quad + i[\frac{1}{4} \delta_{\alpha}^{\gamma} (3P_{\underline{\beta}}^{\quad \kappa}{}_{\dot{\beta}\dot{\alpha}} - 2\delta_{\underline{\beta}}^{\kappa} P_{i\dot{\beta}\dot{\alpha}}^i) + \delta_{\underline{\beta}}^{\gamma} (-P_{\underline{\beta}}^{\quad \kappa}{}_{\alpha\dot{\alpha}} \\
 &\quad + \frac{3}{4} \delta_{\underline{\beta}}^{\kappa} P_{i\alpha\dot{\alpha}}^i + \frac{1}{16} (T_{j\alpha} \bar{T}_{\alpha}^{\kappa} - 2\delta_{\underline{\beta}}^{\kappa} T_{i\alpha} \bar{T}_{\alpha}^i))] D_{\kappa\gamma} \\
 &\quad + i[C_{\alpha\beta} C_{ij} \bar{W}_{\alpha}^{\dot{\gamma}} + \frac{1}{8} \delta_{\alpha}^{\gamma} (D_{[\underline{\alpha}} T_{\underline{\beta}]} + 3D_{(\underline{\alpha}} T_{\underline{\beta}]} \\
 &\quad + \frac{1}{2} T_{j\alpha} T_{i\beta})] \bar{D}^i_{\dot{\gamma}} - \frac{3i}{4} C_{\alpha\beta} C_{ij} \bar{T}_{\alpha}^i Z + \hat{R}_{\alpha\dot{\alpha}\underline{\beta}}(M) \\
 \bar{W}_{\alpha\dot{\beta}} &= \bar{W}_{\dot{\beta}\alpha} \quad , \quad D_{\underline{\alpha}} \bar{W}_{\alpha\dot{\beta}} = 0 \\
 P_{\underline{\alpha}\underline{\beta}} &\equiv P_{i\alpha\dot{\beta}}^j \equiv D_{\underline{\alpha}} T_{\underline{\beta}} + \frac{1}{2} T_{\underline{\alpha}} \bar{T}_{\underline{\beta}} \\
 \bar{P}_{\underline{\alpha}\underline{\beta}} &\equiv P_{i\alpha\dot{\beta}}^j \equiv -(\bar{D}_{\underline{\beta}} T_{\underline{\alpha}} + \frac{1}{2} \bar{T}_{\underline{\beta}} T_{\underline{\alpha}}) \\
 P_{(i}^{\quad j)}{}^{\dot{\gamma}}{}_{\dot{\beta}} &= \bar{P}_{(i}^{\quad j)}{}_{\alpha\dot{\beta}} \\
 D_{\alpha(i} T_{j)\underline{\beta}} &= 0
 \end{aligned} \tag{3.2.1}$$

i is an $SU(2)$ index, α is a Lorentz spinor $SL(2, C)$ index, Z is a central charge and $M^{\alpha\beta}$ are $SL(2, C)$ generators. $R_{\alpha\dot{\alpha}\beta}$ of M are curvatures.

The main objective is to solve the constraints (3.2.1) or (3.1.1). A solution in terms of $N=2$ superfields (s.f.) is given in [13] using harmonic superspace. To solve them in terms of $N=1$ s.f. we have to project (3.1) into $N=1$ superspace [16]. Such projections have been described in detail in [16]. So we mention them briefly here. The reduction in principle is as described in the previous sections; however, the situation here is more difficult. The reduction is usually carried out using the Wess-Zumino gauge in which the $N=2$ gauge fields are expressed in terms of $N=1$ gauge fields, matter field strengths, torsion and curvatures.

To carry out the reduction of the constraints (3.2.1) we use a general method which can be applied to all $N \geq 2$ S.G., once the (on-shell or off-shell) extended superspace constraints are known. For convenience one starts by shifting the vector covariant derivative defined by $i D_{\alpha\dot{\alpha}} = \{D_{\alpha 2}, \bar{D}_{\dot{\alpha}}^2\}$ which breaks $SU(2)$ covariance and expresses (3.2.1) in terms of $D_{\alpha\dot{\alpha}}$.

In terms of this new derivative, then, (3.2.1) becomes

(3.2.2)

$$\{D_{\alpha}, D_{\beta}\} = C_{ij} C_{\alpha\beta} (Z + \bar{W}_{\gamma}^{\delta} \bar{M}_{\delta}^{\gamma'}) \quad (a)$$

$$\begin{aligned} \{D_{1\alpha}, D_{1\beta}\} = & i\nabla_{\alpha\beta} - \frac{1}{2} [T_{1\alpha} \bar{D}_{\dot{\beta}}^1 - T_{2\alpha} \bar{D}_{\dot{\beta}}^2 + \bar{T}_{\dot{\beta}}^1 D_{1\alpha} \\ & - \bar{T}_{\dot{\beta}}^2 D_{2\alpha}] - [(P_1^1{}_{\gamma\dot{\beta}} - P_2^2{}_{\gamma\dot{\beta}}) M^{\gamma}{}_{\alpha} \\ & - (P_1^1{}_{\alpha\dot{\gamma}} - P_2^2{}_{\alpha\dot{\gamma}}) \bar{M}^{\gamma'}{}_{\dot{\beta}}] \end{aligned} \quad (b)$$

$$\{D_{2\alpha}, \bar{D}^2_{\dot{\beta}}\} = iD_{\alpha\dot{\beta}} \quad (c)$$

$$\{D_{1\alpha}, \bar{D}^2_{\dot{\beta}}\} = -\frac{1}{2}(T_{1\alpha} \bar{D}^2_{\dot{\beta}} + \bar{T}^2_{\dot{\beta}} D_{1\alpha}) - P_1^2{}_{\gamma\dot{\beta}} M^\gamma_\alpha + P_1^2{}_{\alpha\dot{\gamma}} \bar{M}^{\gamma'}_{\dot{\beta}}$$
(d)

$$\{D_{2\alpha}, \bar{D}^1_{\dot{\beta}}\} = -\frac{1}{2}(T_{2\alpha} \bar{D}^1_{\dot{\beta}} + \bar{T}^1_{\dot{\beta}} D_{2\alpha}) - P_2^1{}_{\gamma\dot{\beta}} M^\gamma_\alpha + P_2^1{}_{\alpha\dot{\gamma}} \bar{M}^{\gamma'}_{\dot{\beta}}$$
(e)

$$\{D_{\alpha\dot{\alpha}}, D_{1\beta}\} = i \left[-\frac{1}{2} i T_{1\beta} D_{\alpha\dot{\alpha}} + \frac{1}{2} (\delta_{\dot{\alpha}}^{\gamma'} D_{2\alpha} T_{1\beta} - 2 C_{\alpha\beta} \bar{W}_{\dot{\alpha}}^{\gamma'}) \bar{D}^2_{\gamma'} \right.$$
(f)

$$+ \frac{1}{2} (P_1^2{}_{\beta\dot{\alpha}} \delta_{\alpha}^{\gamma'} - 2 P_1^2{}_{\alpha\dot{\alpha}} \delta_{\beta}^{\gamma'}) D_{2\gamma}$$

$$+ \frac{1}{2} (P_2^2{}_{\alpha\dot{\alpha}} - \frac{1}{2} T_{2\alpha} \bar{T}^2_{\dot{\alpha}}) D_{1\beta} + \frac{1}{2} \bar{T}^2_{\dot{\alpha}} C_{\alpha\beta} z]$$

$$+ R_{\alpha\dot{\alpha}1\beta}(M)$$

$$[D_{\alpha\dot{\alpha}}, D_{2\beta}] = i C_{\alpha\beta} \left[\frac{1}{2} i T_{2\gamma} D_{\alpha}^{\gamma'} + \frac{1}{4} (D_{2\gamma} T_2^{\gamma'} + \frac{1}{2} T_{2\gamma} T_2^{\gamma'}) \bar{D}^2_{\gamma'} \right.$$

$$+ \frac{1}{4} [D_{1\gamma} T_2^{\gamma'} + D_2^{\gamma'} T_{1\gamma}) \delta_{\dot{\alpha}}^{\gamma'} + 4 \bar{W}_{\dot{\alpha}}^{\gamma'}] \bar{D}^1_{\gamma'}$$

$$+ \frac{1}{2} (P_2^2{}_{\alpha}^{\gamma'} - 2 P_1^1{}_{\alpha}^{\gamma'}) D_{2\gamma}$$
(g)

$$+ P_2^1{}_{\alpha}^{\gamma'} D_{1\gamma} - \bar{T}^1_{\dot{\gamma}} z) + R_{\alpha\dot{\alpha}2\beta}(M)$$

The parameter K of a general $N=2$ gauge transformation is a real superfield (i.e. $K=K^+$):

$$K = K^A i D_A + i K \cdot M + i \bar{K} \cdot \bar{M} + K_Z \cdot i Z \quad (3.2.3)$$

A K gauge transformation of any covariant object Φ is (c.f. Chapter 2):

$$\delta \Phi = [iK, \Phi]$$

Consequently the covariant derivative D_A transforms as:

$$\delta D_A = [iK, D_A] \quad (3.2.4)$$

It is well known that the $N=0$ components of an $N=1$ superfield Φ can be obtained by projecting out the independent sector of various derivatives of the superfield. In our case we have a similar situation in which the $N=1$ s.f. components of an $N=2$ s.f. Φ can be obtained by projecting out the $\theta^{2\alpha}$ independent sector of various derivatives of the $N=2$ superfields.

Let us define:

$$\left. D_A \right|_{\theta^{2\alpha} = \bar{\theta}_2^{\dot{\alpha}} = 0} = \nabla_A + \psi_A^\mu \partial_{2\mu} + \psi_A^{\dot{\mu}} \bar{\partial}_{\dot{\mu}} \quad (a)$$

(3.2.5)

$$\nabla_A = E_A^M \partial_M + \varphi_A \cdot M + \bar{\varphi}_A \cdot \bar{M} + \varphi_A \cdot Z \quad (b)$$

where $\mathcal{D}_A = (\mathcal{D}_{\alpha\dot{\alpha}}, \mathcal{D}_{1\alpha}, \bar{\mathcal{D}}_{\dot{\alpha}})$ is the $\theta^{2\alpha}$ and $\bar{\theta}_2^{\dot{\alpha}}$ independent part of $\mathcal{D}_A$.

Also we define the components of the covariant derivatives $\mathcal{D}_A$ by $\mathcal{D}_A|$, $\mathcal{D}_{2\alpha}\mathcal{D}_A|$, $\mathcal{D}_{2\alpha}\mathcal{D}_{2\beta}\mathcal{D}_A|$ and so on. For example:

$$\begin{aligned} \mathcal{D}_{2\alpha}\mathcal{D}_A| &= (\mathcal{D}_{2\alpha} E_A^M)| \partial_M + (\mathcal{D}_{2\alpha} \varphi_{A\delta}^{\gamma})| M_{\gamma}^{\delta} \\ &\quad + (\mathcal{D}_{2\alpha} \bar{\varphi}_{A\delta}^{\gamma})| \bar{M}_{\gamma}^{\delta} (\mathcal{D}_{2\alpha} \bar{\varphi}_A)|. Z \end{aligned} \quad (3.2.6)$$

where we have used $\partial_M| = 0$,
 $\theta^{2\alpha} = \bar{\theta}_2^{\dot{\alpha}} = 0$

From (3.2.4) and by using (3.2.5a) and (3.2.5b) one can write

$$\begin{aligned} \delta \mathcal{D}_{2\alpha}| &= [i\kappa, \mathcal{D}_{2\alpha}]| = -i \partial_{2\alpha} \kappa| + \dots \\ &\equiv \kappa_{2\alpha}^{(1)} \end{aligned} \quad (3.2.7)$$

We can use $\kappa_{2\alpha}^{(1)}$ to go to a gauge where :

$$\mathcal{D}_{2\alpha}| \equiv \partial_{2\alpha} \quad (a)$$

$$\bar{\mathcal{D}}_{\dot{\alpha}}^2| \equiv \bar{\partial}_{\dot{\alpha}}^2 \quad (3.2.8)$$

(b)

This is the Wess-Zumino gauge for the spinor derivatives $\mathcal{D}_{2\alpha}$ and $\bar{\mathcal{D}}_{\dot{\alpha}}^2$.

Next consider $D_{2\alpha} D_{2\beta} |$. According to (3.2.4) one can write

$$\begin{aligned} \delta(D_{2\alpha} D_{2\beta}) | &= [iK, D_{2\alpha} D_{2\beta}] | \\ &= -i \partial_{2\alpha} \partial_{2\beta} K | + \dots \end{aligned} \quad (3.2.9)$$

Since $\{\partial_{2\alpha}, \partial_{2\beta}\} = 0$ we cannot gauge away $\{D_{2\alpha}, D_{2\beta}\} |$. However, we can choose a gauge

$$[D_{2\alpha}, D_{2\beta}] | = -C_{\alpha\beta} \partial_2{}^\gamma \partial_{2\gamma} | = 0 \quad (3.2.10)$$

where $\partial_2{}^\gamma \partial_{2\gamma} | = 0$ because of (3.2.6). The symmetric part of $D_{2\alpha} D_{2\beta} |$ is covariant. It can be expressed as:

$$\begin{aligned} D_{2\alpha} D_{2\beta} | &= \frac{1}{2} \{D_{2\alpha}, D_{2\beta}\} | \\ &= 0 \end{aligned} \quad (3.2.11)$$

This is zero because of (3.2.2a).

Similarly one finds

$$\begin{aligned} \bar{D}_{\dot{\alpha}}^2 D_{2\beta} | &= \frac{1}{2} \{\bar{D}_{\dot{\alpha}}^2, D_{2\beta}\} | \\ &= \frac{1}{2} i D_{\beta\dot{\alpha}} | \end{aligned} \quad (3.2.12)$$

Furthermore, for any s.f. Φ one can write

$$\begin{aligned}
\mathcal{D}_A \bar{\Phi}| &= (\mathcal{D}_A \bar{\Phi}) = \nabla_A \bar{\Phi}| + \psi_A^\mu \partial_{2\mu} \bar{\Phi}| + \psi_A^{\dot{\mu}} \bar{\partial}_{\dot{\mu}}^2 \bar{\Phi}| \\
&= \nabla_A \bar{\Phi}| + \psi_A^\mu (\mathcal{D}_{2\mu} \bar{\Phi})| + \psi_A^{\dot{\mu}} (\bar{\mathcal{D}}_{\dot{\mu}}^2 \bar{\Phi})|
\end{aligned}
\tag{3.2.13}$$

where we have used the definition (3.2.5a) and the gauge (3.2.8). On the other hand:

$$\mathcal{D}_{2\alpha} \mathcal{D}_B| = [\mathcal{D}_{2\alpha}, \mathcal{D}_B]| + (-)^b \mathcal{D}_B \mathcal{D}_{2\alpha}|$$

By substituting (3.2.5b), (3.2.11) and (3.2.12) and using (3.2.13) in the above expression, one obtains, upon using the gauge (3.2.8):

$$\begin{aligned}
\mathcal{D}_{2\alpha} \mathcal{D}_B| &= [\mathcal{D}_{2\alpha}, \mathcal{D}_B]| + (-)^b (C_{B\alpha}{}^\gamma \mathcal{D}_{2\gamma}| \\
&\quad + \frac{1}{2} i \psi_B^{\dot{\gamma}} \mathcal{D}_{\alpha\dot{\gamma}}|)
\end{aligned}
\tag{3.2.14}$$

As an application let B be 2β in (3.2.14); by using (3.2.2a) one finds:

$$\varphi_{2\beta}{}^\gamma = \psi_{2\beta}^{\dot{\gamma}} = 0$$

Similarly, by letting $B = 1\beta$ and using (3.2.2a) again we obtain:

$$\mathcal{D}_{2\alpha} \mathcal{D}_\beta| = -(C_{\alpha\beta} (Z + W.M) + \varphi_{\beta\alpha}{}^\gamma \mathcal{D}_{2\gamma}| + \frac{i}{2} \varphi_\beta{}^{\dot{\gamma}} \mathcal{D}_{\alpha\dot{\gamma}}|)
\tag{3.2.15}$$

Continuing this procedure we can find all the components of D_A using the definitions of N=1 superfields given in appendix A.

The N=1 superspace geometry is determined by constraints on N=1 field strengths, T_{AB}^C , $R_{AB}^{(M)}$, $R_{AB}^{(M)}$, F_{AB} and F_{AB} defined by:

$$\begin{aligned} [\nabla_A, \nabla_B] &= T_{AB}^C \nabla_C + R_{AB}^{\gamma} M_{\gamma}^{\gamma} \\ &+ R_{AB}^{\gamma} \bar{M}_{\gamma}^{\gamma} + F_{AB} Z \end{aligned} \quad (a) \quad (3.2.16)$$

$$F_{AB}^{\gamma} = \nabla_{[A} \psi_{B]}^{\gamma} - T_{AB}^D \psi_D^{\gamma} \quad (b)$$

We find the constraints by taking the $\Theta^{2\alpha}$ independent part of the N=2 constraint (3.2.2). In particular F_{AB} is evaluated as the coefficient of $D_{2\gamma} |$ and F_{AB}^{γ} follows as the coefficient of $\bar{D}_{\gamma}^2 |$ i.e.

$$[D_A, D_B] | = [\nabla_A, \nabla_B] + F_{AB}^{\gamma} D_{2\gamma} | + F_{AB}^{\gamma} \bar{D}_{\gamma}^2 |$$

To illustrate this procedure we consider (3.2.2a) from which we get an expression for $\{\nabla_{\alpha}, \nabla_{\beta}\}$, $F_{\alpha\beta}^{\gamma}$ and $F_{\alpha\beta}^{\gamma}$. Consider (3.2.2a) and let $A=1\alpha$, $B=1\beta$. This gives

$$\{D_{1\alpha}, D_{1\beta}\} = 0.$$

By substituting (3.2.5a) for $D_{1\alpha}$ and $D_{1\beta}$ in the above one obtains

$$\begin{aligned} & \nabla_{1(\alpha} \nabla_{\beta)} + [\nabla_{1(\alpha} (\psi_{\beta)}^{\delta} D_{2\delta} + \bar{\psi}_{\beta)}^{\delta} \bar{D}_{\delta}^2)] \\ & + \psi_{(\alpha}^{\gamma} D_{2\gamma} D_{1\beta)} + \psi_{(\alpha}^{\gamma} \bar{D}_{\gamma}^2 D_{1\beta)} \\ & = 0 \end{aligned} \quad (3.2.17)$$

We need to calculate $\bar{D}_{\delta}^2 D_{1\beta}$ which can be obtained in the same way as $D_{2\alpha} D_{\beta}$. Substituting (3.2.12), (3.2.11) and its conjugate in (3.2.17) gives:

$$\begin{aligned} \{ \nabla_{\alpha}, \nabla_{\beta} \} &= \psi_{(\alpha\beta)} (Z + \bar{W}_{\alpha}^{\dot{\beta}} \bar{M}_{\dot{\beta}}^{\alpha}) + \psi_{(\alpha}^{\gamma} \chi_{\gamma\dot{\gamma}} M_{\beta)}^{\dot{\gamma}} \\ &\quad - \psi_{(\alpha}^{\gamma} \chi_{\beta)\dot{\beta}} \bar{M}_{\dot{\gamma}}^{\dot{\beta}} + \frac{1}{2} \psi_{(\alpha}^{\gamma} \bar{\lambda}_{\dot{\gamma}} \nabla_{\beta)} \\ &\quad + i \psi_{(\alpha}^{\gamma} \psi_{\beta)}^{\gamma} \nabla_{\gamma\dot{\gamma}} \\ &\quad - [\nabla_{(\alpha} \psi_{\beta)}^{\gamma} - \frac{1}{2} \psi_{(\alpha}^{\gamma} \psi_{\beta)}^{\gamma} \bar{\lambda}_{\dot{\gamma}}] D_{2\gamma} \\ &\quad - [\nabla_{(\alpha} \psi_{\beta)}^{\gamma} - \frac{1}{2} \psi_{(\alpha}^{\gamma} T_{\beta)} \\ &\quad - \frac{1}{2} \psi_{(\alpha}^{\delta} \psi_{\beta)}^{\gamma} \bar{\lambda}_{\dot{\delta}}] \bar{D}_{\dot{\gamma}}^2 \end{aligned} \quad (3.2.18)$$

In order to complete the projection we must set the coefficients of $D_{2\gamma}|$ and $\bar{D}^2_{\dot{\gamma}}|$ to zero, i.e.

$$\nabla_{(\alpha} \psi_{\beta)}^{\gamma} - \frac{1}{2} \psi_{(\alpha}^{\dot{\gamma}} \psi_{\beta)}^{\gamma} \bar{\lambda}_{\dot{\gamma}} = 0 \quad (3.2.19)$$

Using (3.2.16b) we find that (3.2.19) is equivalent to:

$$F_{\alpha\beta}^{\gamma} = 0 \quad (3.2.20)$$

Similarly, by requiring the coefficient of $\bar{D}^2_{\dot{\gamma}}|$ to vanish in (3.2.18), one deduces that

$$F_{\alpha\beta}^{\dot{\gamma}} = \frac{1}{2} \psi_{(\alpha}^{\gamma} T_{\beta)}^{\dot{\gamma}} = 0. \quad (3.2.21)$$

By repeating this analysis for the remaining set of constraints (3.2.2) one obtains the rest of the projected S.G. and matter constraints. The complete list can be found in (3.3.1) and (A.1).

3.3 The chiral gauge and the non-linear N=2 constraints

At this stage* the reduced constraints look very messy, hence it is worthwhile looking for a special gauge which simplifies the structure of these constraints. Such a gauge has been found in [19] and it is called the chiral gauge (c.f. Chapter 2, page 33).

It is very simple to guess this gauge. Consider the

* See the Appendix for the reduced set of constraints (A.1)

linearized solution of the constraints (A.1). In particular consider the following [19] .

$$F_{\alpha\beta}^{\gamma} = F_{\alpha\beta}^{\dot{\gamma}} = F_{\alpha\dot{\beta}}^{\gamma} - \frac{1}{2} \bar{\lambda}_{\dot{\beta}} \delta_{\alpha}^{\gamma} = 0 \quad (a) \quad (3.3.1)$$

where

$$F_{AB}^{\gamma} = \nabla_{[A} \psi_{B]}^{\gamma} - T_{AB}^C \psi_C^{\gamma} \quad (b)$$

$$F_{\alpha\beta} = \psi_{(\alpha\beta)} \quad , \quad F_{\alpha\dot{\beta}} = \psi_{\alpha\dot{\beta}} + \psi_{\dot{\beta}\alpha} \quad (c)$$

$$F_{\alpha\dot{\alpha},\beta} = \psi_{\alpha\dot{\alpha},\beta} + \frac{i}{2} C_{\alpha\beta} \bar{\lambda}_{\dot{\alpha}}$$

with

$$F_{AB} = \nabla_{[A} \psi_{B]} - T_{AB}^C \psi_C \quad (d)$$

The constraints (3.3.1) are solved by (at linear level)

$$\psi_{\alpha\beta} = D_{\alpha} \psi_{\beta} + C_{\alpha\beta} \bar{\phi} \quad ; D_{\alpha} \bar{\phi} = 0 \quad (a) \quad (3.3.2)$$

$$\psi_{\alpha\dot{\beta}} = D_{\alpha} \phi_{\dot{\beta}} \quad (b)$$

$$\begin{aligned} \phi_{\gamma\dot{\gamma}} = & i [D_{\alpha} (\phi_{\dot{\beta}} - \psi_{\dot{\beta}}) + \bar{D}_{\dot{\beta}} (\phi_{\alpha} - \psi_{\alpha})] \\ & + \frac{1}{2} [\bar{D}_{\dot{\beta}}, D_{\alpha}] V \end{aligned} \quad (c)$$

$$\bar{\lambda}_{\dot{\alpha}} = D^{\alpha} \bar{D}_{\dot{\alpha}} (\psi_{\alpha} - \phi_{\alpha}) + D^2 (\psi_{\dot{\alpha}} - \phi_{\dot{\alpha}}) \quad (d)$$

$$+ \bar{W}_{\dot{\alpha}} + \bar{D}_{\dot{\alpha}} \phi$$

(3.1.28) represents another form of λ_{α} in the N=2 chiral superspace (used by Sokatchev [13]). By comparing this solution with (3.3.2d) one can deduce that solution (3.1.28) holds in a special gauge

$$\phi_{\alpha} = 0 \quad (3.3.3)$$

i.e. the chiral gauge. So without any loss of generality one can show (as we will see) that even at the non-linear level there always exists a gauge which takes one from the ordinary superspace to the chiral superspace. Equation (3.3.3) is such a gauge. To see the effect of this gauge on the supersymmetry parameter ε^{α} consider the variations of ϕ_{α} under supersymmetry (S.S.):

$$\delta \phi_{\alpha} = \varepsilon_{\alpha} - \Lambda_{\alpha} \quad ; \quad \bar{D}_{\dot{\alpha}} \Lambda_{\alpha} = 0 \quad (3.3.4)$$

hence by using (3.3.3) we deduce that $\varepsilon_{\alpha} = \Lambda_{\alpha}$. Therefore our supersymmetry representation is in the chiral representation. The chiral gauge (3.3.3) implies that its field strength is zero. As one can recognize we have:

$$\delta \psi_{\alpha}^{\beta} = D_{\alpha} \bar{\varepsilon}^{\beta} = D_{\alpha} \bar{\Lambda}^{\beta} = 0 \quad (3.3.5)$$

To see the non-linear analogue of (3.3.5) in the chiral gauge (3.3.3) consider the following constraint on ψ_{α}^{γ} :

$$F_{\alpha\beta}^{\dot{\gamma}} = \frac{1}{2} T_{(\alpha} \psi_{\beta)}^{\dot{\gamma}} \quad (3.3.6)$$

which in components of $F_{\alpha\beta}^{\dot{\gamma}}$ gives (using (3.3.1b) and (A.1))

$$\begin{aligned} \nabla_{(\alpha} \psi_{\beta)}^{\dot{\gamma}} - i \psi_{(\alpha}^{\dot{\delta}} \psi_{\beta)}^{\dot{\delta}} \psi_{\dot{\delta}\dot{\delta}}^{\dot{\gamma}} - \frac{1}{2} \bar{\lambda}_{\dot{\delta}} \psi_{(\alpha}^{\dot{\delta}} \psi_{\beta)}^{\dot{\gamma}} \\ = \frac{1}{2} T_{(\alpha} \psi_{\beta)}^{\dot{\gamma}} \end{aligned}$$

which is satisfied by setting $\psi_{\alpha}^{\dot{\gamma}} = 0$.

In the chiral gauge (3.3.3) the field strength transforms as [19]

$$\begin{aligned} \delta \psi_{\alpha}^{\dot{\beta}} &= (\nabla_{\alpha} - \frac{1}{2} T_{\alpha}) \bar{\epsilon}^{\dot{\beta}} - i \psi_{\alpha}^{\dot{\gamma}} \psi_{\dot{\gamma}\dot{\gamma}}^{\dot{\beta}} \bar{\epsilon}^{\dot{\gamma}} \\ &= 0 \end{aligned} \quad (3.3.7)$$

where we have used the supersymmetry variation of $\psi_{\alpha}^{\dot{\beta}}$ in (A.3). (3.3.7) must be an integrable equation.

From the structure of the constraints in (A.1), T_{α} comes into the combination:

$$\tilde{T}_{\alpha} = T_{\alpha} - \psi_{\alpha}^{\dot{\gamma}} \lambda_{\dot{\gamma}} \quad (3.3.8)$$

In terms of $\tilde{T}_{\alpha}$ (3.3.7) becomes

$$(\nabla_{\alpha} - \frac{1}{2} \tilde{T}_{\alpha}) \bar{\lambda}_{\dot{\beta}} + i \psi_{\alpha}^{\dot{\gamma}} W_{\dot{\gamma}\dot{\beta}}^{\dot{\gamma}} \bar{\lambda}_{\dot{\gamma}} = 0 \quad (3.3.9)$$

where

$$\psi_{\gamma}^{\dot{\gamma}}{}_{\dot{\beta}} = W_{\gamma}^{\dot{\gamma}}{}_{\dot{\beta}} - \frac{i}{2} \delta_{\dot{\beta}}^{\dot{\gamma}} \lambda_{\gamma} \quad (3.3.10)$$

Furthermore

$$\begin{aligned} W_{\gamma}^{\dot{\gamma}}{}_{\dot{\beta}} &= -W_{\gamma[\dot{\beta}}^{\dot{\gamma}]} + W_{\gamma\dot{\beta}}^{\dot{\gamma}} \\ &= -(-\delta_{\dot{\beta}}^{\dot{\gamma}} W_{\gamma}^{\dot{\delta}}{}_{\dot{\delta}}) + W_{\gamma\dot{\beta}}^{\dot{\gamma}} \\ \therefore W_{\gamma}^{\dot{\gamma}}{}_{\dot{\beta}} &= W_{\gamma\dot{\beta}}^{\dot{\gamma}} - \delta_{\dot{\beta}}^{\dot{\gamma}} W_{\gamma\dot{\delta}}^{\dot{\delta}} \end{aligned} \quad (3.3.11)$$

Also

$$\frac{1}{2} W_{\gamma(\dot{\beta}}^{\dot{\gamma})} \bar{\lambda}_{\dot{\gamma}} \equiv W_{\gamma\dot{\delta}}^{\dot{\gamma}} \bar{M}_{\gamma\dot{\delta}}^{\dot{\delta}} \bar{\lambda}_{\dot{\beta}} \quad (3.3.12)$$

where we have used the notations of [16] . Therefore (3.3.9) becomes:

$$(\Delta_{\alpha} - \frac{1}{2} \tilde{T}_{\alpha}) \bar{\lambda}_{\dot{\beta}} = 0 \quad (3.3.13)$$

where

$$\tilde{T}_{\alpha} = \tilde{T}_{\alpha} + i \psi_{\alpha}^{\gamma} W_{\gamma\dot{\delta}}^{\dot{\delta}} \quad (3.3.14)$$

$$\Delta_{\alpha} = \nabla_{\alpha} - \xi_{\alpha} Z + i \psi_{\alpha}^{\gamma} W_{\gamma\dot{\delta}}^{\dot{\gamma}} \bar{M}_{\gamma\dot{\delta}}^{\dot{\delta}} \quad (3.3.15)$$

Equations (3.3.14) and (3.3.15) are very important and they are the key ingredients for solving the N=2 supergravity constraints in terms of N = 1(S.F.). (3.3.14) and (3.3.15) are crucial in turning the reduced constraints (A.1) (in the chiral gauge) into the structure of N=1 constraints. As we will see later on, (3.3.13) is an integrable equation. In (3.3.15) Z represents the central charge and $\bar{\xi}_\alpha$ is a superfield to be determined. However, notice that this term in (3.3.15) will not affect (3.3.9) because

$$Z. \bar{\Lambda}_{\dot{\beta}} \equiv [Z, \bar{\Lambda}_{\dot{\beta}}] = 0 \quad (3.3.16)$$

(3.3.16) is true for any superfield because Z is abelian. Our previous discussions can be summarized in Table 1.

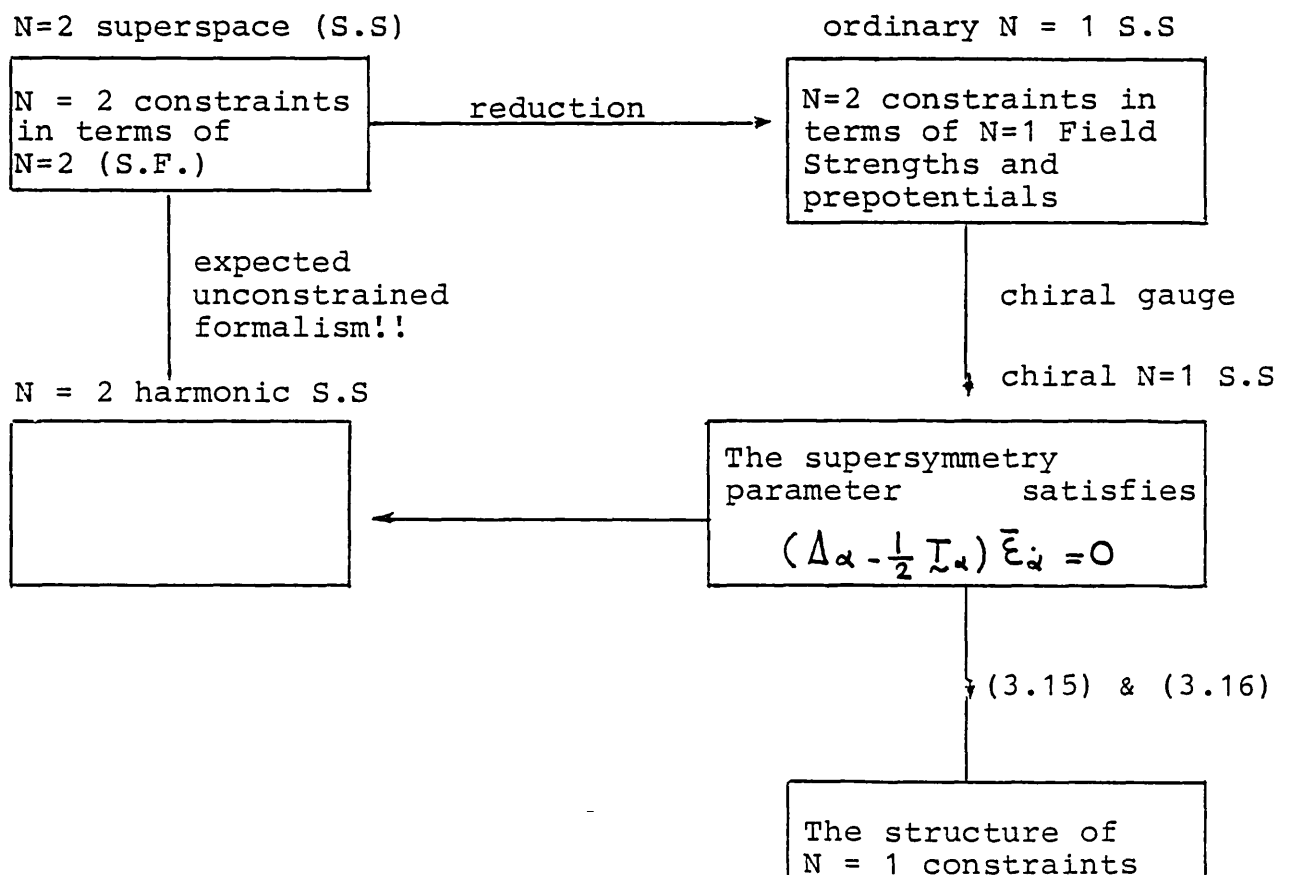


Table 1

After having described the strategy adopted in solving the constraints (which is summarized in table 1) we describe the mathematical details, which are divided into two parts. The first part is the explicit reduction into the structure of N=1 constraints, and the second part is to find the solution to the structure in terms of pure N = 1 S.Fs. This is the material of section 3.4. In the chiral gauge the constraints in (A.1) become

$$\{ \nabla_\alpha, \nabla_\beta \} = \psi_{(\alpha\beta)} (Z + \bar{W}_{\dot{\alpha}}^{\dot{\beta}} \bar{M}_{\dot{\beta}}^{\dot{\alpha}}) \quad (3.3.17)$$

$$\{ \nabla_\alpha, \bar{\nabla}_{\dot{\beta}} \} = i M_{\alpha\dot{\beta}}^{\dot{\gamma}\delta} \nabla_{\delta\dot{\gamma}} \quad (3.3.18)$$

$$\begin{aligned} & -\frac{1}{2} (\tilde{T}_\alpha \tilde{\nabla}_{\dot{\beta}} + \tilde{T}_{\dot{\beta}} \nabla_\alpha) - (\tilde{P}_{\beta\dot{\beta}} + \psi_\beta^{\gamma\delta} \bar{X}_{\gamma\dot{\beta}}) M_\alpha^\beta \\ & + \psi_\alpha^{\gamma\delta} \bar{X}_{\beta\dot{\beta}} M_\gamma^\beta + (\tilde{P}_{\alpha\dot{\alpha}} + \psi_\alpha^{\gamma\delta} X_{\gamma\dot{\delta}}) \bar{M}_{\dot{\beta}}^{\dot{\alpha}} - \psi_{\dot{\beta}}^{\gamma\delta} X_{\alpha\dot{\delta}} \bar{M}_{\dot{\gamma}}^{\dot{\alpha}} \\ [\nabla_{\alpha\dot{\alpha}}, \nabla_{\beta\dot{\beta}}] &= O_{\alpha\dot{\alpha}\beta\dot{\beta}}^{\gamma\delta} \nabla_{\gamma\dot{\gamma}} + i \psi_{\beta\alpha} L_{\dot{\alpha}}^{\dot{\gamma}} \bar{\nabla}_{\dot{\gamma}} \\ & + i (\psi_{\beta\alpha} \bar{X}_{\dot{\alpha}}^{\dot{\gamma}} + \frac{1}{2} \delta_\beta^{\gamma\delta} \tilde{H}_{\alpha\dot{\alpha}}) \nabla_{\gamma\dot{\gamma}} \\ & - i (M_{\beta\dot{\alpha}\alpha}^{\dot{\gamma}} \bar{\lambda}_{\dot{\gamma}} + i W_{\alpha\dot{\alpha}\beta} + \psi_{\beta\alpha} \tilde{T}_{\dot{\alpha}}) Z + R_{\alpha\dot{\alpha},\beta}(M) \end{aligned} \quad (3.3.19)$$

The matter constraints are

$$F_{\alpha\beta}^{\gamma} = 0, \quad F_{\alpha\beta}^{\dot{\gamma}} = 0 \quad (3.3.20 \quad a)$$

$$F_{\alpha\dot{\beta}}^{\gamma} = \frac{1}{2} (M_{\alpha\dot{\beta}}^{\gamma\dot{\delta}} \bar{\lambda}_{\dot{\delta}} + \psi_{\alpha}^{\gamma} \tilde{T}_{\dot{\beta}}) \quad (b)$$

$$F_{\alpha\dot{\alpha},\beta}^{\gamma} = -i \left\{ \psi_{\beta\alpha} (P_{\dot{\alpha}}^{\gamma} - \frac{1}{2} A_{\dot{\alpha}}^{\gamma}) + \right. \\ \left. (\delta_{\beta}^{\gamma} X_{\alpha\dot{\alpha}} - \frac{1}{2} \delta_{\alpha}^{\gamma} X_{\beta\dot{\alpha}}) \right\} \quad (c)$$

$$F_{\alpha\dot{\alpha},\beta}^{\dot{\gamma}} = -i C_{\alpha\beta} L_{\dot{\alpha}}^{\dot{\gamma}} + \frac{1}{2} \psi_{\alpha\dot{\alpha}}^{\dot{\gamma}} T_{\beta} \\ - \frac{i}{4} [2 \nabla_{\beta} \lambda_{\alpha} + \frac{1}{2} \psi_{\beta\alpha} \lambda^{\gamma} \lambda_{\gamma}] \delta_{\dot{\alpha}}^{\dot{\gamma}} \quad (d)$$

$$O_{\alpha\dot{\alpha}\beta}^{\delta\dot{\delta}} = i W_{\alpha\dot{\alpha}}^{\dot{\delta}} \psi_{\beta}^{\delta} + \frac{1}{2} \tilde{T}_{\beta} \delta_{\alpha}^{\delta} \delta_{\dot{\alpha}}^{\dot{\delta}} \quad (e)$$

$\tilde{T}_{\alpha}$ is as defined in (3.3.8), $W_{\alpha\dot{\alpha}}^{\dot{\beta}}$ and its conjugate $W_{\alpha\dot{\alpha}}^{\beta}$ as defined in (3.3.10); moreover [16]

$$\tilde{A}_{\alpha\dot{\alpha}} = A_{\alpha\dot{\alpha}} - i W_{\alpha\dot{\alpha}}^{\dot{\gamma}} \bar{\lambda}_{\dot{\gamma}} \quad (3.3.21 \quad a)$$

$$\tilde{P}_{\alpha\dot{\alpha}} = P_{\alpha\dot{\alpha}} - A_{\alpha\dot{\alpha}} - \psi_{\dot{\alpha}}^{\dot{\gamma}} X_{\alpha\dot{\gamma}} - \psi_{\alpha}^{\gamma} \bar{X}_{\gamma\dot{\alpha}} \quad (b)$$

$$L_{\dot{\alpha}}^{\dot{\beta}} = \bar{W}_{\dot{\alpha}}^{\dot{\beta}} + \frac{1}{4} \delta_{\dot{\alpha}}^{\dot{\beta}} [2S - \nabla^{\delta} \lambda_{\delta} + \psi^{\delta}_{\delta} R] \quad (c)$$

$$M_{\alpha\dot{\alpha}}^{\delta\dot{\delta}} = (\delta_{\alpha}^{\delta} \delta_{\dot{\alpha}}^{\dot{\delta}} + \psi_{\alpha}^{\delta} \psi_{\dot{\alpha}}^{\dot{\delta}}) \quad (d)$$

$$X_{\alpha\dot{\alpha}} = (\nabla_{\alpha} + \frac{1}{2} \tilde{T}_{\alpha}) \bar{\lambda}_{\dot{\alpha}} + \psi_{\alpha}^{\gamma} \tilde{A}_{\gamma\dot{\alpha}} + i \psi_{\alpha}^{\gamma} W_{\gamma\dot{\alpha}}^{\dot{\delta}} \bar{\lambda}_{\dot{\delta}} \quad (e)$$

$$P_{\alpha\dot{\alpha}} = (\nabla_{\alpha} + \frac{1}{2} \tilde{T}_{\alpha}) \bar{T}_{\dot{\alpha}} + \psi_{\alpha}^{\gamma} \bar{X}_{\gamma\dot{\alpha}} \quad (f)$$

From (3.3.21b, e and f) we find that

$$\tilde{\rho}_{\alpha\dot{\alpha}} = (\nabla_{\alpha} + \frac{1}{2}\tilde{T}_{\alpha})\tilde{T}_{\dot{\alpha}} - M_{\alpha\dot{\alpha}}{}^{\delta\dot{\delta}}\tilde{A}_{\delta\dot{\delta}} \quad (3.3.22)$$

Finally we have another constraint on $\tilde{T}_{\alpha}$ which is

$$\nabla_{(\alpha}\tilde{T}_{\beta)} = \psi_{(\alpha\beta)}L_{\dot{\alpha}}^{\dot{\alpha}} \quad (3.3.23)$$

Furthermore, we have the F_{AB} constraints which are the Z coefficients of (3.3.17 - 3.3.19). Before engaging in building the structure of $N=1$ constraints out of (3.3.17 - 3.3.19) we notice that some of them can be solved simply: from the $F_{\alpha\beta}$ constraint and (3.3.1d) we can solve for $\psi_{\alpha\beta}$ to obtain:

$$\psi_{\alpha\beta} = \nabla_{\alpha}\psi_{\beta} + C_{\alpha\beta}\bar{\phi} \quad ; \quad \nabla_{\alpha}\bar{\phi} = 0 \quad (3.3.24)$$

From (3.3.20b) and by using (3.3.1b) and the components of T_{AB}^C which can be read off from (3.3.18) we deduce that:

$$\psi_{\alpha\dot{\beta}}^{\gamma} = -i N_{\alpha\dot{\beta}}{}^{\delta\dot{\delta}} \nabla_{\dot{\delta}}\psi_{\delta}^{\gamma} + \frac{i}{2} \delta_{\alpha}^{\gamma} \bar{\lambda}_{\dot{\beta}} \quad (3.3.25)$$

where N is the inverse matrix of M in (3.3.21d). Hence

$$W_{\alpha\dot{\beta}}^{\gamma} = -i N_{\alpha\dot{\beta}}{}^{\delta\dot{\delta}} \nabla_{\dot{\delta}}\psi_{\delta}^{\gamma} \quad (3.3.26)$$

The solution of (3.3.20d) gives

$$L_{\dot{\alpha}}^{\dot{\beta}} = \frac{i}{2} N^{\alpha}_{\dot{\delta}\dot{\delta}}{}^{\dot{\beta}} [\nabla^{\dot{\delta}} W_{\alpha\dot{\alpha}}{}^{\dot{\delta}} + i W_{\alpha\dot{\alpha}}{}^{\dot{\gamma}} \psi^{\dot{\delta}\dot{\gamma}} W_{\dot{\gamma}\dot{\gamma}}{}^{\dot{\delta}}] \quad (3.3.27)$$

Therefore

$$W_{\dot{\alpha}\dot{\beta}} = L_{(\dot{\alpha}\dot{\beta})} \quad (3.3.28)$$

Also the $F_{\alpha\dot{\alpha},\beta}$ constraints yield

$$\begin{aligned} \bar{\lambda}_{\dot{\alpha}} = & -\frac{i}{2} N^{\alpha}_{\dot{\alpha}}{}^{\dot{\beta}\dot{\beta}} [(\nabla_{\dot{\alpha}\dot{\beta}} - \frac{i}{2} \tilde{A}_{\dot{\alpha}\dot{\beta}}) \psi_{\dot{\beta}} \\ & - \square_{\dot{\beta}} \phi_{\dot{\alpha}\dot{\beta}} - W_{\dot{\alpha}\dot{\beta}\dot{\beta}} - i \psi_{\dot{\beta}\dot{\alpha}} [(L_{\dot{\beta}}{}^{\dot{\gamma}} \bar{\psi}_{\dot{\gamma}} - \tilde{T}_{\dot{\beta}}) \\ & + \bar{X}{}^{\dot{\gamma}}_{\dot{\beta}} \psi_{\dot{\gamma}}]] . \end{aligned} \quad (3.3.29)$$

where

$$\square_{\dot{\beta}} \phi_{\dot{\alpha}\dot{\beta}} = \mathcal{D}_{\dot{\beta}} \phi_{\dot{\alpha}\dot{\beta}} + i W_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}} \psi_{\dot{\beta}}{}^{\dot{\gamma}} \phi_{\dot{\gamma}\dot{\gamma}} \quad (3.3.30)$$

with

$$\mathcal{D}_{\dot{\beta}} = (\nabla_{\dot{\beta}} + \frac{1}{2} \tilde{T}_{\dot{\beta}}) \quad (3.3.31)$$

Furthermore the solution of the $F_{\dot{\alpha}\dot{\beta}} = 0$ constraint and (3.3.20c)

give

$$\phi_{\alpha\dot{\alpha}} = -i N_{\alpha\dot{\alpha}}{}^{\gamma\delta} [\mathcal{D}_{\delta} \bar{\psi}_{\dot{\gamma}} + \bar{\mathcal{D}}_{\dot{\delta}} \psi_{\gamma}] + h_{\alpha\dot{\alpha}} \quad (3.3.32)$$

where

$$h_{\alpha\dot{\alpha}} = -\frac{1}{2} \{ [\nabla_{\alpha}, \nabla_{\dot{\alpha}}] + \frac{1}{2} (\tilde{T}_{\alpha} \bar{\nabla}_{\dot{\alpha}} - \tilde{T}_{\dot{\alpha}} \nabla_{\alpha}) \} V \quad (3.3.33)$$

where V is a compensator superfield for the matter system which can be gauged to zero. Moreover, (3.3.20c) gives:

$$\begin{aligned} \nabla_{\alpha\dot{\alpha}} \psi_{\beta}^{\gamma} &= \square_{\beta} W_{\alpha\dot{\alpha}}^{\gamma} + i C_{\alpha\beta} X^{\gamma}_{\dot{\alpha}} \\ &\quad - i \psi_{\beta\alpha} (\tilde{\rho}^{\gamma}_{\dot{\alpha}} + \psi^{(\gamma\delta)} \bar{X}_{\delta\dot{\alpha}} + \psi_{\dot{\alpha}}^{\gamma} X^{\gamma}_{\dot{\delta}}). \end{aligned} \quad (3.3.34)$$

Now we are in a position to exploit the $N = 1$ structure of the constraints in (3.3.17 - 3.3.19). As we have mentioned earlier, the key ingredients are equations (3.3.14) and (3.3.15). We start with (3.3.17). With the help of (3.3.28), (3.3.16) and formulas like (3.3.12) we find that

$$\{ \Delta_{\alpha}, \Delta_{\beta} \} = [\psi_{(\alpha\beta)} - \nabla_{(\alpha} \xi_{\beta)}] Z \quad (3.3.35)$$

It is interesting now to realize that if we choose

$$\psi_{(\alpha\beta)} = \nabla_{(\alpha} \xi_{\beta)} \quad (3.3.36)$$

we have the following remarkable result:

$$\{ \Delta_\alpha, \Delta_\beta \} = 0 \quad (3.3.37)$$

which has exactly the structure of the $N = 1$ constraints.

Using (3.3.24) and (3.3.15) again we have

$$\Delta_\alpha [\psi_\beta - \xi_\beta] = 0$$

which is immediately solved for ξ_β , we obtain

$$\xi_\beta = \psi_\beta + \frac{i}{2} \Delta_\beta V \quad (3.3.38)$$

We have chosen " $\frac{i}{2}$ " in front of $\Delta_\alpha V$ for convenience and V is as it was defined previously. Therefore eqn.(3.3.38) expresses the superfield ξ_β , which was first introduced in (3.16), in terms of the unconstrained prepotential ψ_β . (3.3.37) is very encouraging, so we proceed to express (3.3.17 - 3.3.19) in terms of $\Delta_\alpha, \bar{\Delta}_{\dot{\alpha}}$ and define

$$i \Delta_{\alpha\dot{\alpha}} = \{ \Delta_\alpha, \bar{\Delta}_{\dot{\alpha}} \} \quad (3.3.39)$$

(3.3.39) has been chosen such that together with (3.3.37) it has an $N = 1$ structure. Explicitly it has the following form:

$$i\Delta_{\alpha\dot{\alpha}} = iM_{\alpha\dot{\alpha}}{}^{\beta\dot{\beta}}\nabla_{\beta\dot{\beta}} - K_{\alpha\dot{\alpha}}{}^{\gamma}{}^{\dot{\gamma}}\bar{\Delta}_{\gamma}{}^{\dot{\gamma}} - K_{\alpha\dot{\alpha}}{}^{\gamma}\Delta_{\gamma} \quad (3.3.40)$$

$$-H_{\alpha\dot{\alpha}\beta}{}^{\gamma}M_{\gamma}{}^{\beta} + H_{\alpha\dot{\alpha}\beta}{}^{\gamma}{}^{\dot{\gamma}}\bar{M}_{\gamma}{}^{\dot{\gamma}}{}^{\beta}$$

where

$$K_{\alpha\dot{\alpha}}{}^{\gamma}{}^{\dot{\gamma}} = \frac{1}{2}\tilde{T}_{\alpha}{}^{\dot{\gamma}}\delta_{\dot{\alpha}}{}^{\gamma} - \frac{i}{2}\psi_{\alpha}{}^{\beta}W_{\beta(\dot{\alpha}}{}^{\dot{\gamma})} \quad (3.3.41 \text{ a})$$

$$K_{\alpha\dot{\alpha}}{}^{\gamma} = \frac{1}{2}\tilde{T}_{\dot{\alpha}}\delta_{\alpha}{}^{\gamma} + \frac{i}{2}\psi_{\dot{\alpha}}{}^{\dot{\beta}}W_{(\alpha\dot{\beta}}{}^{\gamma)} \quad (b)$$

$$H_{\alpha\dot{\alpha}\beta}{}^{\gamma} = [(\tilde{P}_{\beta\dot{\alpha}} + \psi_{\beta}{}^{\dot{\delta}}\bar{X}_{\dot{\alpha}\dot{\delta}})\delta_{\alpha}{}^{\gamma} - \psi_{\alpha}{}^{\dot{\delta}}\bar{X}_{\beta\dot{\delta}}] \\ + i(\nabla_{\alpha} + \frac{1}{2}\tilde{T}_{\alpha})\psi_{\alpha}{}^{\beta}W_{\beta\dot{\beta}}{}^{\gamma} \quad (c)$$

$$H_{\alpha\dot{\alpha}\beta}{}^{\dot{\gamma}} = [(\tilde{P}_{\alpha\dot{\beta}} + \psi_{\dot{\beta}}{}^{\dot{\delta}}\bar{X}_{\alpha\dot{\delta}})\delta_{\dot{\alpha}}{}^{\dot{\gamma}} - \psi_{\dot{\alpha}}{}^{\dot{\delta}}\bar{X}_{\alpha\dot{\delta}}] \\ + i(\nabla_{\dot{\alpha}} + \frac{1}{2}\tilde{T}_{\dot{\alpha}})\psi_{\alpha}{}^{\beta}W_{\beta\dot{\beta}}{}^{\dot{\gamma}} \quad (d)$$

Using (3.3.40), (3.3.19) and (3.3.15), we can work out

$[i\Delta_{\alpha\dot{\alpha}}, \Delta_{\beta}]$ hence we have:

$$[i\Delta_{\alpha\dot{\alpha}}, \Delta_{\beta}] = i\left\{-N_{\dot{\alpha}\dot{\beta}}{}^{\gamma\dot{\gamma}}\Delta_{\beta}M_{\alpha\dot{\alpha}}{}^{\dot{\beta}\dot{\gamma}} - K_{\alpha\dot{\alpha}}{}^{\gamma}{}^{\dot{\gamma}}\delta_{\beta}{}^{\gamma} \right. \\ \left. + N_{\gamma\dot{\gamma}}{}^{\delta\dot{\delta}}M_{\alpha\dot{\alpha}}{}^{\dot{\beta}\dot{\gamma}}G_{\dot{\beta}\dot{\delta}}{}^{\gamma\dot{\delta}}\right\}\Delta_{\gamma}{}^{\dot{\gamma}} \\ + \left\{\Delta_{\beta}K_{\alpha\dot{\alpha}}{}^{\gamma}{}^{\dot{\gamma}} - \frac{1}{2}H_{\alpha\dot{\alpha}(\beta}{}^{\gamma}{}^{\dot{\gamma}} - N_{\dot{\alpha}\dot{\beta}}{}^{\gamma\dot{\gamma}}\Delta_{\beta}M_{\alpha\dot{\alpha}}{}^{\dot{\beta}\dot{\gamma}}K_{\gamma\dot{\gamma}}{}^{\delta}{}^{\dot{\delta}}\right\}$$

$$\begin{aligned}
& + N_{\tau\dot{\tau}}^{\epsilon\dot{\epsilon}} M_{\alpha\dot{\alpha}}^{\delta\dot{\delta}} G_{\delta\dot{\delta}\beta}^{\tau\dot{\tau}} K_{\epsilon\dot{\epsilon}}^{\gamma} - M_{\alpha\dot{\alpha}}^{\delta\dot{\delta}} (\psi_{\beta\dot{\beta}} \bar{\chi}_{\dot{\delta}}^{\gamma} \\
& + \frac{1}{2} \delta_{\beta}^{\gamma} \tilde{A}_{\delta\dot{\delta}}) \} \Delta_{\gamma} + \{ \Delta_{\beta} K_{\alpha\dot{\alpha}}^{\gamma} - N_{\delta\dot{\delta}}^{\tau\dot{\tau}} \Delta_{\beta} M_{\alpha\dot{\alpha}}^{\delta\dot{\delta}} K_{\tau\dot{\tau}}^{\gamma} \\
& + N_{\tau\dot{\tau}}^{\epsilon\dot{\epsilon}} M_{\alpha\dot{\alpha}}^{\delta\dot{\delta}} G_{\delta\dot{\delta}\beta}^{\tau\dot{\tau}} K_{\epsilon\dot{\epsilon}}^{\gamma} \\
& - M_{\alpha\dot{\alpha}}^{\delta\dot{\delta}} \psi_{\beta\dot{\beta}} L_{\dot{\delta}}^{\gamma} \} \bar{\Delta}_{\dot{\gamma}} \\
& + M \text{ terms}
\end{aligned}$$

where

$$G_{\delta\dot{\delta}\beta}^{\gamma\dot{\gamma}} = O_{\delta\dot{\delta}\beta}^{\gamma\dot{\gamma}} - \frac{i}{2} \psi_{\beta}^{\epsilon} W_{\epsilon(\dot{\delta}}^{\dot{\gamma})} \delta_{\dot{\delta}}^{\gamma} \quad (3.3.43)$$

$O_{\delta\dot{\delta}\beta}^{\gamma\dot{\gamma}}$ is as defined in (3.3.20e). In fact, we can simplify the expressions of $G_{\delta\dot{\delta}\beta}^{\gamma\dot{\gamma}}$ and the K's in (3.3.41a,b) by using equation (3.3.14) and

$$\frac{1}{2} W_{\delta(\dot{\alpha}}^{\dot{\gamma})} = W_{\delta\dot{\alpha}}^{\dot{\gamma}} - \frac{1}{2} W_{\delta\dot{\beta}}^{\dot{\beta}} \delta_{\dot{\alpha}}^{\dot{\gamma}} \quad (3.3.44)$$

We get:

$$K_{\alpha\dot{\alpha}}^{\gamma\dot{\gamma}} = -i \psi_{\alpha}^{\delta} W_{\delta\dot{\alpha}}^{\dot{\gamma}} + \frac{1}{2} T_{\alpha}^{\gamma} \delta_{\dot{\alpha}}^{\dot{\gamma}} \quad (3.3.45a)$$

$$K_{\alpha\dot{\alpha}}^{\gamma} = i \psi_{\alpha}^{\delta} W_{\alpha\dot{\delta}}^{\gamma} + \frac{1}{2} T_{\alpha}^{\gamma} \delta_{\dot{\alpha}}^{\gamma} \quad (b)$$

$$G_{\delta\delta\beta}^{\gamma\dot{\gamma}} = iW_{\delta}^{\gamma\dot{\gamma}} \psi_{\beta\delta} + \frac{1}{2} T_{\beta}^{\gamma\dot{\gamma}} \delta_{\delta}^{\gamma\dot{\gamma}} \delta_{\delta}^{\gamma\dot{\gamma}} \quad (c)$$

The last constraint to convert is (3.3.23). By substituting (3.3.14), (3.3.15) and (3.3.27) into (3.3.23) we find the remarkable result:

$$\Delta_{(\alpha} T_{\beta)} = 0. \quad (3.3.46)$$

This is exactly the analogue of the pure $N = 1$ equation. Because of (3.3.37) and (3.3.46) one can prove that (3.3.13) is indeed integrable. Therefore we have proved that the $N = 2$ constraints expressed in terms of $N = 1$ superfield can be turned into the structure of pure $N = 1$ constraints by shifting the covariant derivatives ∇_{α} , $\nabla_{\dot{\alpha}}$ and the dimensional $\frac{1}{2}$ tensors $\tilde{T}_{\alpha}$ and $\tilde{T}_{\dot{\alpha}}$ according to equations (3.3.14) and (3.3.15). Armed with equations (3.3.37), (3.3.39), (3.3.40) and (3.3.43) we have solved a major block in the problem. Now to get the solution of (3.3.37 - 3.3.42) is relatively easy. In fact, since we know that (3.3.37 - 3.3.42) carries an $N = 1$ structure and also know how to solve the pure $N = 1$ supergravity constraints in terms of $N = 1$ superfields[14], then all we have to do is to relate the constraints (3.3.37 - 3.3.42) to those of pure $N = 1$ supergravity. By doing this we then have solved for quantities like Δ_{α} , T_{α} , ... etc. in terms of pure $N = 1$ supergravity superfields which are already unconstrained, since the $N = 1$ supergravity constraints are solved[14]. The obvious way to establish such a relation is through superfield gauge transformations (which are expressed in pure $N = 1$ superfields) and leave (3.3.37 - 3.3.42) invariant. Then such a gauge

transformation will be the solution of those constraints.

Let us illustrate with an example. Consider the case of $N = 1$ super Yang-Mills (as was discussed fully in Chapter 2).

The $F_{\alpha\beta}$ matter constraint is

$$F_{\alpha\beta} = \{D_\alpha, D_\beta\} = 0 \quad (3.3.47)$$

where $D_\alpha = d_\alpha + A_\alpha$ such that $\{d_\alpha, d_\alpha\} = 0$. (3.3.47) is the constraint on A_α which must be solved. We look for a gauge transformation which leaves (3.3.47) invariant. Hence we find that for

$$D_\alpha \rightarrow D_\alpha - (\bar{e}^V d_\alpha (e^V \cdot 1)) \quad (3.3.48)$$

(3.3.47) is invariant, where V is an unconstrained superfield and $d_\alpha (e^V \cdot 1) = d_\alpha e^V + e^V d_\alpha$. Therefore the solution for (3.3.47) is

$$D_\alpha = \bar{e}^V d_\alpha (e^V \cdot 1) \quad \text{i.e.} \\ A_\alpha = \bar{e}^V d_\alpha e^V \quad (3.3.49)$$

We now go back to our constraints (3.3.37) - (3.3.42). Consider (3.3.37). Then we let

$$\Delta_\alpha \rightarrow \Delta_\alpha - (D_\alpha + \Phi_\alpha \cdot \bar{M}) \quad (3.3.50)$$

such that D_α is the pure $N = 1$ supercovariant derivative of

$N = 1$ supergravity. Inserting (3.3.50) into (3.3.37) we find that the invariance is achieved if

$$\Delta_{(\alpha} \phi_{\beta)} \cdot \bar{M} = 0 \quad (3.3.51)$$

where $\phi_{\alpha} \cdot \bar{M} \equiv \phi_{\alpha\dot{\alpha}}^{\dot{\beta}} \bar{M}_{\dot{\beta}}^{\dot{\alpha}}$. In principle (3.3.51) can be solved in general as: (using 3.3.46)

$$\phi_{\beta\dot{\beta}}^{\dot{\gamma}} = (a R_{\beta} + b \tau_{\beta}) \delta_{\dot{\beta}}^{\dot{\gamma}} + \tau_{\dot{\beta}}^{-1\dot{\delta}} D_{\beta} \tau_{\dot{\delta}}^{\dot{\gamma}} \quad (3.3.52)$$

where R_{β} and $\tau_{\beta}^{\dot{\alpha}}$ are at this stage the unknown tensors which will be determined later on. Notice that in our case $\tau_{\dot{\alpha}}^{\dot{\beta}}$ stands for $e^{\dot{\nu}}$ in (3.3.49) and the first term in (3.3.52) has been introduced because it does not contribute when ϕ_{β} hits $\bar{M}$ since $\bar{M}_{\dot{\alpha}}^{\dot{\alpha}} = 0$ and also the fact that we are motivated by (3.3.46). Therefore in general a solution (3.3.52) exists for the constraint (3.3.51). Therefore we will take for granted that (3.3.50) is the gauge transformation which leaves (3.3.37) invariant and the solution of constraint (3.3.37) is given by:

$$\Delta_{\alpha} = D_{\alpha} + \phi_{\alpha} \cdot \bar{M} \quad (3.3.53)$$

The requirement that (3.3.53) leaves invariant the rest of the constraints (3.3.38 - 3.3.42) will entirely determine $\phi_{\alpha\dot{\alpha}}^{\dot{\beta}}$, τ_{α} , $\tilde{P}_{\alpha\dot{\alpha}}$, $\tilde{A}_{\alpha\dot{\alpha}}$, $\tilde{\lambda}_{\dot{\alpha}}$, $\phi_{\alpha\dot{\alpha}}$ etc. The process of gauge invariance is indeed equivalent to working out $[\Delta_{\alpha\dot{\alpha}}, \Delta_{\beta}]$ in terms of $\phi_{\alpha\dot{\alpha}}^{\dot{\beta}}$ (using equations (3.3.53) and (3.3.39) and equating

the corresponding coefficients of the vector derivatives the spinor derivative Δ_α and $\bar{\Delta}_{\dot{\alpha}}$ as will be shown later on. Now we will work out $[i\Delta_{\alpha\dot{\alpha}}, \Delta_\beta]$. Using the definition (3.3.39) and (3.3.53) we easily get:

$$i\Delta_{\alpha\dot{\alpha}} = iD_{\alpha\dot{\alpha}} + (\Delta_\alpha \bar{\Phi}_{\dot{\alpha}}) \cdot M \quad (3.3.54)$$

$$+ (\bar{\Delta}_{\dot{\alpha}} \phi_\alpha) \cdot M + \bar{\Phi}_{\dot{\alpha}\dot{\alpha}}{}^\gamma D_\gamma + \phi_{\alpha\dot{\alpha}}{}^\gamma \bar{D}_{\dot{\gamma}}$$

where

$$iD_{\alpha\dot{\alpha}} = \{D_\alpha, \bar{D}_{\dot{\alpha}}\} \quad (3.3.55)$$

From (3.3.54) and (3.3.53) we finally obtain:

$$\begin{aligned} [i\Delta_{\alpha\dot{\alpha}}, \Delta_\beta] &= [iD_{\alpha\dot{\alpha}}, D_\beta] + C_{\alpha\beta} [i\phi_{\gamma\dot{\alpha}}{}^{\dot{\beta}} \Delta_{\dot{\beta}}^\gamma \\ &\quad + D_\gamma \bar{\Phi}_{\dot{\alpha}}{}^\gamma \Delta_\gamma] - (\Delta_\beta \phi_{\alpha\dot{\alpha}}{}^\gamma \\ &\quad - \phi_{\beta\dot{\alpha}}{}^\gamma \phi_{\alpha\dot{\gamma}}) \bar{\Delta}_{\dot{\gamma}} + M \text{ terms.} \end{aligned} \quad (3.3.56)$$

The comutation relations of the 's are given in terms of (A.3 - A.8). From (A.5) and (3.55) we can express $[iD_{\alpha\dot{\alpha}}, D_\beta]$ in terms of $\Delta_{\gamma\dot{\gamma}}$, Δ_α and $\Delta_{\dot{\alpha}}$ (up to M terms), hence we have:

$$\begin{aligned}
[iD_{\alpha\dot{\alpha}}, D_{\beta}] = & \frac{1}{2} C_{\alpha\beta} [iS^{\gamma} \Delta_{\gamma\dot{\alpha}} - (\tilde{\rho}_{\dot{\alpha}}^{\gamma} - a_{\dot{\alpha}}^{\gamma}) \\
& + S^{\delta} \bar{\phi}_{\delta\dot{\alpha}}^{\gamma}] \Delta_{\gamma} + [-S^{\delta} \phi_{\alpha\delta}^{\gamma} + \frac{1}{2} (D^{\delta} + \frac{1}{2} S^{\delta}) \\
& S_{\delta} \delta_{\dot{\alpha}}^{\gamma}] \bar{\Delta}_{\gamma}]
\end{aligned} \tag{3.3.57}$$

$$+ r_{\alpha\dot{\alpha},\beta} (M) + M \text{ terms}$$

where S^{α} is the pure $N = 1$ supergravity tensor which satisfies (A.8). Therefore (3.3.56) becomes ($\tilde{D}^{\delta} = D^{\delta} + \frac{1}{2} S^{\delta}$)

$$\begin{aligned}
[i\Delta_{\alpha\dot{\alpha}}, \Delta_{\beta}] = & iC_{\alpha\beta} [-\phi_{\dot{\alpha}}^{\gamma} + \frac{1}{2} S^{\gamma} \delta_{\dot{\alpha}}^{\gamma}] \Delta_{\gamma\dot{\gamma}} \\
& - C_{\alpha\beta} [\tilde{D}^{\delta} \bar{\phi}_{\delta\dot{\alpha}}^{\gamma} + \frac{1}{2} (\tilde{\rho}_{\dot{\alpha}}^{\gamma} - a_{\dot{\alpha}}^{\gamma})] \Delta_{\gamma} \\
& - [(\Delta_{\beta} \phi_{\alpha\dot{\alpha}}^{\gamma} - \phi_{\beta\dot{\alpha}}^{\delta} \phi_{\alpha\dot{\delta}}^{\gamma}) \\
& + \frac{1}{2} C_{\alpha\beta} \{ S^{\delta} \phi_{\alpha\delta}^{\gamma} - \frac{1}{2} \tilde{D}^{\delta} S_{\delta} \delta_{\dot{\alpha}}^{\gamma} \}] \bar{\Delta}_{\gamma} \\
& + M \text{ terms} .
\end{aligned} \tag{3.3.58}$$

3.4 The Solution of the $N = 2$ Constraints in terms of $N = 1$ Superfields

Consider equations (3.3.42) and (3.3.58) which are the solutions of the $N = 2$ constraints in terms of $N = 1$ superfields. By equating the corresponding coefficients of $\Delta_{\gamma\dot{\gamma}}$, Δ_{γ} , $\bar{\Delta}_{\dot{\gamma}}$ we can determine $\phi_{\dot{\alpha}}^{\gamma}$, $\tilde{\rho}_{\dot{\alpha}}^{\gamma}$, $\tilde{A}_{\alpha\dot{\alpha}}$, $\bar{\chi}_{\alpha\dot{\alpha}}$, $\tilde{T}_{\alpha}$ and therefore Δ_{α} .

Hence (3.3.53) is indeed a solution for the constraints (3.3.30 - 3.3.42).

i) $\Delta_{\gamma\delta}$ Coefficients

$$C_{\alpha\beta} \left[-\phi_{\alpha}^{\gamma\delta} + \frac{1}{2} S^{\gamma} \delta_{\alpha}^{\delta} \right] = \quad (3.4.1)$$

$$- N_{\delta\delta}^{\gamma\delta} \Delta_{\beta} M_{\alpha\delta}^{\delta\delta} + N_{\tau\tau}^{\gamma\delta} M_{\alpha\delta}^{\delta\delta} G_{\delta\delta\beta}^{\tau\tau}$$

$$- K_{\alpha\delta}^{\gamma\delta} \delta_{\beta}^{\gamma}$$

This can be further simplified by using equations (3.3.45) leading to:

$$C_{\alpha\beta} \left[\phi_{\alpha}^{\gamma\delta} - \frac{1}{2} (S^{\gamma} - \tilde{T}^{\gamma}) \delta_{\alpha}^{\delta} \right] = \quad (3.4.2)$$

$$N_{\delta\delta}^{\gamma\delta} \Delta_{\beta} M_{\alpha\delta}^{\delta\delta} - i \psi_{\alpha}^{\delta} W_{\delta\delta}^{\gamma\delta} \delta_{\beta}^{\gamma}$$

$$- i N_{\tau\tau}^{\gamma\delta} M_{\alpha\delta}^{\delta\delta} W_{\delta}^{\tau\tau} \psi_{\beta\delta}$$

By inserting (3.3.15) in (3.3.26) and using the identity

$$M_{\alpha\beta}^{\beta\delta} W_{\beta\delta}^{\beta\delta} - M_{\alpha\delta}^{\delta\delta} W_{\delta\delta}^{\gamma\delta} = \psi_{\alpha}^{\delta} W_{\delta\beta}^{\delta\delta} \bar{M}_{\delta}^{\beta\delta} \psi_{\alpha}^{\gamma\delta} \quad (3.4.3)$$

we get

$$i M_{\alpha\beta}^{\beta\delta} W_{\beta\delta}^{\beta\delta} = \Delta_{\alpha} \psi_{\alpha}^{\gamma\delta} \quad (3.4.4)$$

which can be solved for W, i.e.

$$W_{\beta\delta}^{\beta\delta} = -i N_{\beta\delta}^{\alpha\beta} \Delta_{\alpha} \psi_{\alpha}^{\gamma\delta} \quad (3.4.5)$$

Inserting (3.4.5) in (3.4.2) and using the identities of Appendix B, (B.1 - B.4) we get

$$\begin{aligned} \Phi_{\dot{\alpha}}^{\gamma\dot{\gamma}} &= \frac{1}{2} (S^{\gamma} - \tilde{T}^{\gamma}) \delta_{\dot{\alpha}}^{\dot{\gamma}} \\ &\quad + \gamma_{\dot{\alpha}}^{\dot{\beta}}{}^{-1} D^{\gamma} \gamma_{\dot{\beta}}^{\dot{\gamma}} \end{aligned} \quad (3.4.6)$$

where

$$\gamma_{\dot{\alpha}}^{\dot{\beta}} = \frac{1}{2} M_{\alpha\dot{\alpha}} \delta^{\dot{\alpha}}_{\dot{\beta}} M^{\alpha}_{\dot{\beta}} \quad (3.4.7)$$

By comparing (3.4.6) with (3.3.52) we deduce that:

$$a = -b = \frac{1}{2} \quad , \quad R_{\beta} = S_{\beta} \quad (3.4.8)$$

(3.4.6) also implies that:

$$\tilde{T}_{\alpha} = S_{\alpha} + \gamma_{\dot{\alpha}}^{\dot{\beta}}{}^{-1} D_{\alpha} \gamma_{\dot{\beta}}^{\dot{\alpha}} \quad (3.4.9)$$

Inserting (3.4.6) into (3.3.53) we get:

$$\Delta_{\alpha} = D_{\alpha} + \gamma_{\dot{\alpha}}^{\dot{\beta}}{}^{-1} D_{\alpha} \gamma_{\dot{\beta}}^{\dot{\alpha}} \bar{M}_{\dot{\beta}}^{\dot{\alpha}} \quad (3.4.10)$$

(3.4.9) and (3.4.10) imply that we have completely solved

$$\tilde{T}_{\alpha} \quad , \quad \nabla_{\alpha} \quad , \quad \psi_{\alpha\dot{\beta}} \quad , \quad W_{\alpha\dot{\beta}}^{\dot{\gamma}} \quad , \quad L_{\dot{\alpha}}^{\dot{\beta}} \quad , \quad W_{\dot{\alpha}\dot{\beta}} \quad , \quad S \quad , \quad R \quad \text{and} \quad \Phi_{\alpha\dot{\alpha}}$$

in terms of pure $N = 1$ superfields. For example simplifies to:

$$\psi_{\alpha\beta} = D_\alpha \psi_\beta + C_{\alpha\beta} \bar{\phi} \quad ; D_\alpha \bar{\phi} = 0. \quad (3.4.11)$$

ii) Δ_γ Coefficients

$$\begin{aligned} -C_{\alpha\beta} \left\{ \tilde{D}^\delta \bar{\phi}_{\alpha\dot{\delta}} + \frac{1}{2} (\tilde{P}^\delta_{\dot{\alpha}} - a^\delta_{\dot{\alpha}}) \right\} = \\ \left\{ \Delta_\beta \delta_\alpha^\epsilon \delta_{\dot{\alpha}}^{\dot{\epsilon}} - N_{\alpha\dot{\alpha}}^{\epsilon\dot{\epsilon}} \Delta_\beta M_{\alpha\dot{\alpha}}^{\delta\dot{\delta}} \right. \\ \left. + N_{\tau\dot{\tau}}^{\epsilon\dot{\epsilon}} M_{\alpha\dot{\alpha}}^{\delta\dot{\delta}} G_{\delta\dot{\delta},\beta}^{\tau\dot{\tau}} \right\} K_{\epsilon\dot{\epsilon}}^\gamma \\ - M_{\alpha\dot{\alpha}}^{\delta\dot{\delta}} (\psi_{\beta\delta} \bar{\chi}_{\dot{\delta}}^\gamma + \frac{1}{2} \delta_\beta^\gamma \tilde{A}_{\delta\dot{\delta}}) \\ - \frac{1}{2} H_{\alpha\dot{\alpha}}^{\gamma\delta} \psi_\beta. \end{aligned} \quad (3.4.12)$$

Inserting (3.4.1) into (3.4.12) gives:

$$\begin{aligned} -C_{\alpha\beta} \left\{ \tilde{D}^\delta \bar{\phi}_{\alpha\dot{\delta}} + \frac{1}{2} (\tilde{P}^\delta_{\dot{\alpha}} - a^\delta_{\dot{\alpha}}) \right\} = \\ \left[\tilde{D}_\beta \delta_\alpha^\epsilon \delta_{\dot{\alpha}}^{\dot{\epsilon}} + \delta_\beta^\epsilon [\tau_{\alpha\dot{\alpha}}^{\dot{\epsilon}} - i\psi_\alpha^\delta W_{\delta\dot{\alpha}}^{\dot{\epsilon}}] \right] K_{\epsilon\dot{\epsilon}}^\gamma \\ - C_{\alpha\beta} \tilde{P}_{\dot{\alpha}}^\gamma - M_{\alpha\dot{\alpha}}^{\delta\dot{\delta}} \psi_{\beta\delta} \bar{\chi}_{\dot{\delta}}^\gamma + \psi_\alpha^\gamma \bar{\chi}_{\beta\dot{\alpha}} \\ - \psi_\beta^\delta \bar{\chi}_{\delta\dot{\alpha}} \delta_\alpha^\gamma - (\nabla_\alpha + \frac{1}{2} \tilde{T}_\alpha) K_{\alpha\beta}^\gamma. \end{aligned} \quad (3.4.13)$$

where we have used (3.3.22), (3.3.41c), (3.3.45b), (3.3.45a) and (3.4.6).

$$\tau_{\alpha\dot{\alpha}}^{\gamma\dot{\gamma}} = \gamma_{\dot{\alpha}}^{-1\dot{\beta}} \tilde{P}_{\dot{\beta}}^\gamma D_\alpha \tilde{P}_{\dot{\gamma}}^\gamma. \quad (3.4.14)$$

Now using (3.3.14), (3.3.15), (3.4.6) and (3.3.53), we can prove that:

$$\begin{aligned}
 (\nabla_\alpha + \frac{1}{2} \tilde{T}_\alpha) K_{\alpha\beta}{}^\gamma = \\
 [\tilde{D}_\alpha \delta_\beta{}^\epsilon \delta_{\dot{\alpha}}{}^{\dot{\epsilon}} + \\
 \delta_\beta{}^\epsilon [\tau_{\alpha\dot{\alpha}}{}^{\dot{\epsilon}} - i \psi_\alpha{}^\delta W_{\delta\dot{\alpha}}{}^{\dot{\epsilon}}]] K_{\epsilon\dot{\epsilon}}{}^\gamma
 \end{aligned}
 \tag{3.4.15}$$

By inserting (3.4.15) into (3.4.13) and using the formulas in Appendix A (eqn. A.7), (3.4.6), (3.3.45b) and (3.4.14) we can determine $\tilde{P}_{\alpha\dot{\alpha}}$ in terms of pure N = 1 superfields $P_{\alpha\dot{\alpha}}$, D^δ , S^δ etc. ... we obtain

$$\begin{aligned}
 \tilde{P}_{\alpha\dot{\alpha}} = \tilde{D}^\delta [\tau_{\delta\dot{\alpha}}{}^{\dot{\alpha}} + i \psi_\alpha{}^\delta W_{\delta\dot{\alpha}}{}^{\dot{\alpha}}] \\
 - M_{\alpha\dot{\alpha}}{}^{\delta\dot{\delta}} \psi_\delta{}^\gamma \bar{\chi}_{\gamma\dot{\delta}} + P_{\alpha\dot{\alpha}}.
 \end{aligned}
 \tag{3.4.16}$$

We can now determine $\tilde{A}_{\alpha\dot{\alpha}}$ from (3.4.16) by using (3.3.22). Finally we obtain $X_{\alpha\dot{\alpha}}$ by substituting (3.4.16) in (3.3.34) and setting $\beta = \gamma$.

i.e.

$$\begin{aligned}
 i M_{\alpha\dot{\beta}}{}^{\gamma\dot{\gamma}} M_{\beta\dot{\alpha}}{}^{\gamma\dot{\gamma}} \chi^{\beta\dot{\beta}} = \nabla_{\alpha\dot{\alpha}} \psi_\beta{}^\beta - \square_\beta W_{\alpha\dot{\alpha}}{}^\beta \\
 + i \psi_{\beta\dot{\alpha}} (P_{\alpha\dot{\alpha}}{}^\beta + \bar{Q}_{\alpha\dot{\alpha}}{}^\beta) + i \psi_{(\alpha\dot{\beta}} N_{\dot{\alpha}}{}^{\beta\dot{\delta}} (Q_{\delta\dot{\delta}} - \bar{Q}_{\delta\dot{\delta}})
 \end{aligned}
 \tag{3.4.17}$$

such that:

$$Q_{\alpha\dot{\alpha}} = \tilde{D}^{\dot{\delta}} [\tau_{\delta\alpha\dot{\alpha}} + i\psi_{\dot{\alpha}}^{\dot{\delta}} W_{\delta\alpha}] \quad (3.4.18)$$

To invert X in (3.4.17) consider the following term:

$$I_{\beta\dot{\beta}}^{\delta\dot{\delta}} = N^{\tau\dot{\tau}\dot{\alpha}\delta} N_{\tau\dot{\tau}}^{\alpha\dot{\delta}} M_{\alpha\dot{\beta}}^{\gamma\dot{\gamma}} M_{\beta\dot{\alpha}}^{\gamma\dot{\gamma}} \quad (3.4.19)$$

From the identity (B.1) we can show that

$$N_{\tau\dot{\tau}}^{\alpha\dot{\delta}} M_{\alpha\dot{\beta}}^{\gamma\dot{\gamma}} = N_{\alpha\dot{\tau}}^{\gamma\dot{\delta}} M_{\tau\dot{\beta}}^{\alpha\dot{\gamma}} \quad (3.4.20)$$

$$N^{\tau\dot{\tau}\dot{\alpha}\delta} M_{\beta\dot{\alpha}}^{\gamma\dot{\gamma}} = N^{\tau\dot{\tau}\dot{\alpha}\delta} M_{\beta\dot{\gamma}}^{\tau\dot{\alpha}} \quad (3.4.21)$$

Therefore:

$$\begin{aligned} I_{\beta\dot{\beta}}^{\delta\dot{\delta}} &= N_{\alpha\dot{\tau}}^{\gamma\dot{\delta}} M_{\beta\dot{\gamma}}^{\tau\dot{\alpha}} N^{\tau\dot{\tau}\dot{\alpha}\delta} M_{\tau\dot{\beta}}^{\alpha\dot{\gamma}} \\ &= N_{\alpha\dot{\tau}}^{\gamma\dot{\delta}} M_{\beta\dot{\gamma}}^{\tau\dot{\alpha}} (-C^{\delta\alpha} C_{\dot{\alpha}\dot{\beta}}) \\ &= - N_{\dot{\tau}}^{\delta\gamma\dot{\delta}} M_{\beta\dot{\gamma}}^{\tau\dot{\alpha}} \end{aligned} \quad (3.4.22)$$

$$\therefore I_{\beta\dot{\beta}}^{\delta\dot{\delta}} = \delta_{\beta}^{\delta} \delta_{\dot{\beta}}^{\dot{\delta}}$$

Hence:

$$X_{\alpha\dot{\alpha}} = N_{\alpha\dot{\beta}}^{\delta\dot{\delta}} N_{\beta\dot{\delta}}^{\gamma\dot{\gamma}} [\text{right hand side of (3.4.17)}]^{\beta\dot{\beta}} \quad (3.4.23)$$

So now we have determined $\tilde{P}_{\alpha\dot{\alpha}}$, $\tilde{A}_{\alpha\dot{\alpha}}$, $\bar{\lambda}_{\dot{\alpha}}$ and $X_{\alpha\dot{\alpha}}$ in

terms of completely pure $N = 1$ supergravity superfields.

Table 2 gives the full set of solutions of the constraints (3.3.17 - 3.3.23).

iii) $\bar{\Delta}$ Coefficients

$$\begin{aligned}
 & -\frac{1}{2} C_{\alpha\beta} [S^\delta \phi_{\alpha\delta}^{\gamma'} - \frac{1}{2} \tilde{O}^\delta S_\delta \delta_{\alpha}^{\gamma'}] \\
 & - (\Delta_\beta \phi_{\alpha\dot{\alpha}}^{\gamma'} - \phi_{\beta\dot{\alpha}}^{\delta'} \phi_{\alpha\dot{\delta}}^{\gamma'}) = \\
 & [\Delta_\beta S_\alpha^\epsilon \delta_{\alpha}^{\dot{\epsilon}} - N_{\delta\dot{\delta}}^{\epsilon\dot{\epsilon}} \Delta_\beta M_{\alpha\dot{\alpha}}^{\delta\dot{\delta}} \\
 & + N_{\tau\dot{\tau}}^{\epsilon\dot{\epsilon}} M_{\alpha\dot{\alpha}}^{\delta\dot{\delta}} G_{\delta\dot{\delta}\beta}^{\tau\dot{\tau}}] \kappa_{\epsilon\dot{\epsilon}}^{\gamma'} - M_{\alpha\dot{\alpha}}^{\delta\dot{\delta}} \gamma_{\beta\delta}^{\gamma'} L_{\dot{\delta}}^{\gamma'} \quad (3.4.24)
 \end{aligned}$$

Substituting (3.4.1) in (3.4.24) gives:

right hand side of (3.4.24) =

$$\begin{aligned}
 & [\tilde{O}_\beta S_\alpha^\epsilon \delta_{\alpha}^{\dot{\epsilon}} + S_\beta^\epsilon [\tau_{\alpha\dot{\alpha}}^{\dot{\epsilon}} - i \psi_\alpha^\delta W_{\delta\dot{\alpha}}^{\dot{\epsilon}}]] \kappa_{\epsilon\dot{\epsilon}}^{\gamma'} \\
 & - M_{\alpha\dot{\alpha}}^{\delta\dot{\delta}} \gamma_{\beta\delta}^{\gamma'} L_{\dot{\delta}}^{\gamma'} + \frac{1}{2} \phi_\beta^{(\delta\dot{\delta})} \kappa_{\alpha\dot{\alpha}\dot{\delta}}^{\gamma'}
 \end{aligned}$$

By using (3.3.14), (3.3.15), (3.4.9) and (3.4.10) we have:

$$\begin{aligned}
 & C_{\alpha\beta} [\tilde{O}^\delta \kappa_{\delta\dot{\alpha}}^{\gamma'} + \frac{1}{2} (S^\delta \phi_{\alpha\delta}^{\gamma'} - \frac{1}{2} \tilde{O}^\delta S_\delta \delta_{\alpha}^{\gamma'}) + \tau^{\delta(\alpha\dot{\delta})} \kappa_{\delta\dot{\alpha}\dot{\delta}}^{\gamma'}] \\
 & + \Delta_\beta \phi_{\alpha\dot{\alpha}}^{\gamma'} - \phi_{\beta\dot{\alpha}}^{\delta'} \phi_{\alpha\dot{\delta}}^{\gamma'} \\
 & + (\nabla_\alpha + \frac{1}{2} \tilde{T}_\alpha) \kappa_{\alpha\dot{\beta}}^{\gamma'} + \frac{i}{2} \psi_\alpha^\delta W_{\delta}^{(\delta\dot{\delta})} \kappa_{\beta\dot{\alpha}\dot{\delta}}^{\gamma'} \\
 & = M_{\alpha\dot{\alpha}}^{\delta\dot{\delta}} \gamma_{\beta\delta}^{\gamma'} L_{\dot{\delta}}^{\gamma'} .
 \end{aligned}$$

(3.4.25)

from (3.3.51)

$$\begin{aligned} \Delta_\beta \phi_{\alpha\dot{\alpha}}^{\gamma\dot{\gamma}} - \phi_{\beta\dot{\alpha}}^{\delta\dot{\delta}} \phi_{\alpha\dot{\delta}}^{\gamma\dot{\gamma}} = \\ -\frac{1}{2} C_{\alpha\beta} (\Delta_\delta \phi_{\dot{\alpha}}^{\delta\dot{\gamma}} - \phi_{\delta\dot{\alpha}}^{\delta\dot{\delta}} \phi_{\dot{\delta}}^{\delta\dot{\gamma}}) \end{aligned} \quad (3.4.26)$$

Finally we get:

$$\begin{aligned} \frac{1}{2} C_{\alpha\beta} \left[2\tilde{D}^\delta \kappa_{\delta\dot{\alpha}}^{\gamma\dot{\gamma}} + (S^\delta \phi_{\dot{\alpha}\delta}^{\gamma\dot{\gamma}} - \frac{1}{2} \tilde{D}^\delta S_\delta \delta_{\dot{\alpha}}^{\gamma\dot{\gamma}}) \right. \\ \left. - (\Delta_\delta \phi_{\dot{\alpha}}^{\delta\dot{\gamma}} - \phi_{\delta\dot{\alpha}}^{\delta\dot{\delta}} \phi_{\dot{\delta}}^{\delta\dot{\gamma}}) + 2\tau^{\delta(\dot{\delta}\dot{\gamma})} \kappa_{\delta\dot{\alpha}\dot{\delta}}^{\gamma\dot{\gamma}} \right] \\ + \frac{i}{2} \psi_\alpha^\delta W_\delta^{\delta(\dot{\delta}\dot{\gamma})} \kappa_{\beta\dot{\alpha}\dot{\delta}}^{\gamma\dot{\gamma}} + (\nabla_\alpha + \frac{1}{2} \tilde{T}_\alpha) \kappa_{\dot{\alpha}\beta}^{\gamma\dot{\gamma}} \\ = M_{\alpha\dot{\alpha}}^{\delta\dot{\delta}} \psi_{\beta\delta}^{\gamma\dot{\gamma}} L_{\dot{\delta}}^{\gamma\dot{\gamma}}. \end{aligned} \quad (3.4.27)$$

Since all the individual terms K , ϕ , L and Δ have already been solved completely in terms of pure $N = 1$ supergravity superfields, then eqn. (3.4.27) does not give us any new information. In fact, (3.4.27) is nothing more than an identity.

This is the end of our analysis of the solution of the $N = 2$ S.G. constraints in terms of $N = 1$ superfields, in particular the superfields $P_{\alpha\dot{\alpha}}$, S_α , D_α and in terms of ψ_α , V and Φ [15].

$$W_{\alpha\dot{\alpha}} = -i N_{\alpha\dot{\beta}} \Delta_{\dot{\beta}} \psi_{\alpha}^{\dot{\beta}}$$

$$\Delta_{\dot{\beta}} = D_{\dot{\beta}} + (\tau_{\alpha}^{-1}{}^{\dot{\beta}} D_{\dot{\beta}} \tau_{\dot{\beta}}^{\alpha}) \bar{M}_{\dot{\gamma}}^{\alpha}$$

$$\tau_{\alpha}^{\dot{\beta}} = \frac{1}{2} M_{\alpha\dot{\alpha}}^{\dot{\beta}\dot{\alpha}} M^{\alpha}{}_{\dot{\beta}\dot{\alpha}}$$

$$M_{\alpha\dot{\alpha}}^{\dot{\beta}\dot{\alpha}} = \delta_{\alpha}^{\dot{\beta}} \delta_{\dot{\alpha}}^{\dot{\alpha}} + \psi_{\alpha}^{\dot{\beta}} \psi_{\dot{\alpha}}^{\dot{\alpha}}$$

$$N_{\alpha\dot{\alpha}}^{\dot{\beta}\dot{\alpha}} = M^{-1}{}_{\dot{\beta}\dot{\alpha}}^{\alpha\dot{\alpha}}$$

$$\nabla_{\alpha} = \Delta_{\alpha} + (\psi_{\alpha} + \frac{i}{2} \Delta_{\alpha} V) Z - i \psi_{\alpha}^{\dot{\beta}} W_{\dot{\beta}\dot{\alpha}}^{\dot{\beta}} \bar{M}_{\dot{\beta}}^{\alpha}$$

$$\tilde{T}_{\alpha} = S_{\alpha} + \tau_{\alpha}^{-1}{}^{\dot{\beta}} D_{\alpha} \tau_{\dot{\beta}}^{\alpha} - i \psi_{\alpha}^{\dot{\beta}} W_{\dot{\beta}\dot{\alpha}}^{\dot{\beta}}$$

$$L_{\dot{\alpha}}^{\dot{\beta}} = \frac{i}{2} N_{\dot{\beta}\dot{\alpha}}^{\alpha\dot{\beta}} [\Delta_{\alpha} W^{\dot{\beta}}{}_{\dot{\alpha}}{}^{\dot{\gamma}} - i \psi_{\alpha}^{\dot{\gamma}} W_{\dot{\gamma}\dot{\alpha}}^{\dot{\gamma}} W^{\dot{\beta}}{}_{\dot{\gamma}}{}^{\dot{\delta}}]$$

$$W_{\dot{\alpha}\dot{\beta}} = \frac{1}{2} L_{(\dot{\alpha}\dot{\beta})}$$

$$\Phi_{\alpha\dot{\alpha}} = -i N_{\alpha\dot{\alpha}}^{\dot{\beta}\dot{\alpha}} [(\nabla_{\dot{\beta}} + \frac{1}{2} \tilde{T}_{\dot{\beta}}) \psi_{\dot{\beta}}^{\alpha} + (\bar{\nabla}_{\dot{\beta}} + \frac{1}{2} \tilde{\bar{T}}_{\dot{\beta}}) \bar{\psi}_{\dot{\beta}}^{\alpha}]$$

$$-\frac{1}{2} \{ [\nabla_{\alpha}, \bar{\nabla}_{\dot{\alpha}}] + \frac{1}{2} (\tilde{T}_{\alpha} \bar{\nabla}_{\dot{\alpha}} - \tilde{\bar{T}}_{\dot{\alpha}} \nabla_{\alpha}) \} V$$

$$X_{\alpha\dot{\alpha}} = N_{\alpha\dot{\beta}}^{\dot{\beta}\dot{\alpha}} N_{\dot{\beta}\dot{\alpha}}^{\alpha\dot{\beta}} \{ \nabla^{\dot{\beta}\dot{\beta}} \psi_{\dot{\beta}}^{\alpha} - \square_{\dot{\beta}} W^{\dot{\beta}\dot{\beta}}{}^{\dot{\gamma}} + i \psi_{\dot{\beta}}^{\dot{\beta}} (P^{\dot{\beta}\dot{\beta}} + \bar{Q}^{\dot{\beta}\dot{\beta}}) \\ + i \psi^{\dot{\beta}}{}_{\dot{\beta}} N^{\dot{\beta}\dot{\beta}}{}_{\dot{\gamma}\dot{\delta}} (Q^{\dot{\gamma}\dot{\delta}} - \bar{Q}^{\dot{\gamma}\dot{\delta}}) \}$$

$$Q_{\dot{\beta}\dot{\alpha}} = (D^{\alpha} + \frac{1}{2} S^{\alpha}) (\tau_{\alpha\dot{\beta}\dot{\alpha}} + i \psi_{\dot{\beta}}^{\alpha} W_{\alpha\dot{\alpha}})$$

$$\tau_{\alpha\dot{\beta}\dot{\alpha}} = \tau_{\alpha}^{-1}{}^{\dot{\gamma}} D_{\dot{\beta}} \tau_{\dot{\gamma}\dot{\alpha}}$$

$$\tilde{P}_{\alpha\dot{\alpha}} = \tilde{P}_{\alpha\dot{\alpha}} + Q_{\alpha\dot{\alpha}} - M_{\alpha\dot{\alpha}}^{\dot{\beta}\dot{\alpha}} \psi_{\dot{\beta}}^{\alpha} \bar{X}_{\dot{\gamma}\dot{\alpha}}$$

$$\tilde{A}_{\alpha\dot{\alpha}} = N_{\alpha\dot{\alpha}}^{\dot{\beta}\dot{\alpha}} [(\nabla_{\dot{\beta}} + \frac{1}{2} \tilde{T}_{\dot{\beta}}) \tilde{\bar{T}}_{\dot{\beta}} + N_{\dot{\beta}\dot{\alpha}}^{\dot{\gamma}\dot{\alpha}} \psi_{\dot{\gamma}}^{\dot{\beta}} \bar{X}_{\dot{\gamma}\dot{\alpha}} \\ + \tilde{P}_{\dot{\beta}\dot{\alpha}} + Q_{\dot{\beta}\dot{\alpha}}]$$

$$\begin{aligned}
\bar{\lambda}_\alpha &= -\frac{i}{2} N^\alpha_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} \left[(\nabla_{\alpha\dot{\beta}} - \frac{i}{2} \tilde{A}_{\alpha\dot{\beta}}) \bar{\psi}_\beta - i \psi_{\beta\alpha} \bar{\chi}^{\dot{\gamma}}_{\dot{\alpha}} \psi_\gamma \right. \\
&\quad \left. - W_{\alpha\dot{\beta}\beta} - \square_\beta \phi_{\alpha\dot{\beta}} - i \psi_{\alpha\beta} (L_{\dot{\beta}}^{\dot{\gamma}} \psi_\gamma - \tilde{T}_{\dot{\beta}}^{\dot{\gamma}}) \right] \\
\square_\beta \delta_\alpha^{\dot{\gamma}} \delta_{\dot{\alpha}}^{\dot{\gamma}} &= (\nabla_\beta + \frac{1}{2} \tilde{T}_\beta) \delta_\alpha^{\dot{\gamma}} \delta_{\dot{\alpha}}^{\dot{\gamma}} + i W_{\alpha\dot{\alpha}}^{\dot{\gamma}} \psi_\beta^{\dot{\gamma}} \\
\psi_{\alpha\beta} &= D_\alpha \psi_\beta + C_{\alpha\beta} \bar{\phi} \quad ; \quad D_\alpha \bar{\phi} = 0 .
\end{aligned}$$

Table -2-

The transformation rules for the prepotentials Φ and V are the same as in (3.1.29) and (3.1.30a), i.e.

$$\delta V = i (\Lambda - \bar{\Lambda}) \quad (3.4.28)$$

$$\delta \bar{\phi} = \nabla^2 \Lambda$$

$\delta \psi_\alpha$ can be derived using for $\delta \psi_M$, δE_α and $\delta E_{\alpha\dot{\alpha}}$ in (A.3a, c, d) and also noting that

$$\psi_M = E_M^\alpha \psi_\alpha + E_M^{\alpha\dot{\alpha}} \psi_{\dot{\alpha}} + E_M^{\alpha\dot{\alpha}} \phi_{\alpha\dot{\alpha}}.$$

We obtain

$$\delta \psi_\alpha = \Lambda_\alpha + \nabla_\alpha \Lambda + \bar{\Lambda}^{\dot{\alpha}} \left(\frac{1}{2} \bar{\lambda}_{\dot{\alpha}} \psi_\alpha + i \psi_\alpha^\beta \phi_{\beta\dot{\alpha}} \right) \quad (3.4.29)$$

For the sake of completeness we will briefly mention the content of the N=2 S.G. model which we have considered in this chapter.

It is well known that three different off-shell versions of N=2 S.G. exist [21] namely, the minimal Poincare S.G. types I, II and III. The type I version is constructed using the non-linear 8+8 multiplet, while the types II and III are constructed using the scalar and tensor multiplets respectively.

The model which we have considered in this thesis is the type I with the following 40+40 multiplet content.

$$(e_\mu^a, \psi_\mu^i, B_\mu, A_a, V_a^i, T_{ab}, S_{ij}, V_a, M, \chi^i, \lambda_i). \quad (3.4.30)$$

Where e_μ^a is the vielbein, ψ_μ^i is the gravitino, B_μ^{ij} is an abelian physical gauge vector field, A_a is an auxiliary field, V_a^i is an SU(2) auxiliary field, T_{ab} (anti-symmetric in its Lorentz indices) is an $so(2)$ anti-symmetric tensor field, S_{ij} are the triplet scalars, V_a is an auxiliary vector field, M_{ij} are complex scalars, χ^i and λ_i are auxiliary spinors.

The minimal off-shell N=1 S.G. content extracted from (3.4.30) (when written in terms of N=1 superfields), can be completely expressed in terms of the pure dimension 1/2 N=1 superfield S_α , the pure N=1 vector superfield $\mathcal{A}_\alpha$ and their derivatives (see A.6-A.8). This can be achieved by gauging away the scalar prepotential V, φ and setting the matter prepotential ψ_α (in table 2) to be zero. The remaining off-shell matter multiplets in (3.4.30) are expressed entirely in terms of ψ_α, V, φ and their derivatives (e.g. $\psi_{\alpha\beta}$). Further details can be found in [30].

3.5 The Action

It is not an easy task to find the action of N=2 S.G. in terms of N=1 superfields due to the complexity of the structure of the theory. However, there are essential ingredients which are sufficient to determine the action in terms of N=1 superfields. As one knows, it is not true that any coupling of pure N=1 S.G. to N=1 matter fields yields an off-shell N=2 S.G. theory. But the requirement of central charge and N=2 supersymmetry invariance can guarantee the equivalence of an N=2 theory with a particular coupling of N=1 S.G. to N=1 matter fields.

In superfield language this equivalence is more obvious because the N=2 off-shell S.G. constraints have been already solved in terms of unconstrained N=1 superfields. In terms of N=1 superfields the generic form of the N=2 action is:

$$I = \frac{1}{\kappa^2} \int d^8 z E^{-1} L. \quad (3.5.1)$$

where $L(\psi_\alpha, \psi_{\alpha\beta}, \psi_{\alpha\beta}^\gamma, h.c.)$ which corresponds to the N=1 matter part and κ is the gravitational coupling constant. E^{-1} is the pure N=1 super-vielbein which is already discussed in chapter 2 and reference[13] :

For the above action to correspond to the N=2 off-shell S.G., in addition to Lorentz and g.c.t. transformations, it must be invariant under U(1) and N=2 local supersymmetry, whose transformation rules are given in Appendix A.

Without any loss of generality the function L can be expanded as [29].

$$L = G(\psi_{\alpha\beta}, \psi\psi, \psi\bar{\psi}, h.c) \quad (3.5.2)$$

$$+ F(\psi_\alpha) \times H(\psi_{\alpha\beta}, \psi_{\alpha\beta}^\gamma, h.c) + h.c$$

In order for the action (3.5.1) to be U(1) invariant it is crucial that the L function given by (3.5.2) be invariant under U(1). However, the above decomposition of L will introduce further simplifications for the U(1) invariance of (3.5.2). The U(1) gauge transformation of ψ_α (which is the fundamental matter prepotential) is given by:

$$\delta\psi_\alpha = \nabla_\alpha \xi = D_\alpha \xi \quad (3.5.3)$$

where ξ is a real superfield and ∇_α is as defined in table 2. Furthermore, the scalar superfield $\bar{\phi}$ transforms as:

$$\delta\bar{\phi} = \nabla^2 \xi \quad (3.5.4)$$

Using (3.5.3) and (3.5.4), we find that:

$$\delta\psi_{\alpha\beta} = 0 \quad (a)$$

$$\delta\lambda_\alpha = 0 \quad (b)$$
(3.5.5)

where $\psi_{\alpha\beta}$ and λ_α as given in table 2. Consequently we find that H and G functions in (3.5.2) are invariant under U(1) gauge transformations

$$\delta H = \delta G = 0. \quad (3.5.6)$$

Finally, inserting (3.5.3) and (3.5.6) into (3.5.1), we find

that $U(1)$ invariance of the action I requires that:

$$(D_\alpha + S_\alpha)(F'H) + h.c. = 0 \quad (3.5.7a)$$

where

$$F' = \frac{\partial F}{\partial \psi_\alpha} \quad (3.5.7b)$$

In obtaining (3.5.7) we have used the following formula for integration by parts

$$\int d^8z E^{-1} X D_\alpha Y = \int d^8z E^{-1} Y (D_\alpha + S_\alpha) X \quad (3.5.8)$$

In general it is not straightforward to solve (3.5.6) (one equation with two unknowns F and H). However, since we already know that F is only a function of ψ_α and hence (in principle it has a polynomial expansion in terms of ψ_α) we consider the simplest choice:

$$F(\psi_\alpha) = \psi_\alpha \quad (3.5.9)$$

As a consequence of (3.5.9), equation (3.5.6) is a constraint on H which can be solved. Without any further detail the solution for H is given by [29] :

$$H = N_\alpha{}^\beta \lambda_\beta + \frac{1}{2}(\bar{X} + X\bar{B})\tilde{T}_\alpha - \frac{1}{2}(d_\alpha - \kappa_\alpha)(\bar{X} + X\bar{B}) \quad (3.5.10)$$

Furthermore, by requiring local $N=2$ supersymmetry invariance of (3.5.1), the G function can be entirely determined, as given by:

$$G = 2 \left[(1 - B\bar{B}) + \frac{1}{4} (\bar{\chi} + \chi B) \chi + \frac{1}{4} \bar{\chi} (\chi + \chi \bar{B}) \right]. \quad (3.5.11)$$

Hence, the complete N=2 S.G. action in terms of N=1 superfields is given as 29

$$S = -\frac{1}{2} \int d^8z e^{-1} \left[(1 - B\bar{B}) (2 + \psi^\alpha \lambda_\alpha + \bar{\psi}^{\dot{\alpha}}) + (\bar{\chi} + \chi \bar{B}) (\psi^\alpha T_\alpha + \bar{\Phi}) + (\chi + \bar{\chi} B) (\bar{\psi}^{\dot{\alpha}} \bar{T}_{\dot{\alpha}} + \Phi) \right]. \quad (3.5.12)$$

3.6 Conclusions

In this chapter we have analysed and studied, in great detail, the solution of N=2 S.G. constraints in terms of N=1 superfields. The procedure was to project the N=2 constraints into the ordinary N=1 superspace [16]. Then we noticed that this projection could go to the chiral N=1 superspace where the constraints took an easier structure.

We discovered that by shifting ∇_α and T_α according to (3.15) and (3.16) the projected constraints in the chiral N=1 superspace simplify and take the structure of the N=1 supergravity constraints (3.28-3.40-3.43). At this stage it was possible to find the solutions to these constraints. We looked for the gauge transformations which left (3.28-3.43) invariant. Then the solutions were exactly the gauge transformations. Such gauge transformations were found by shifting all objects in (3.28-3.40), in particular Δ_α , $\mathbb{T}_\alpha$, into the pure N=1 supergravity covariant derivatives D_α and the dimension half tensor S_α . Finally, we considered the structure of the action in terms of N=1 superfields.

One hopes to generalize the above results to higher N. This will be more complicated because there are no off-shell constraints for $N > 3$ supergravity theories (because of the

no go theorems [24]). A proper way of understanding the structure of the extended theories (in particular $N = 3$) starts by understanding the formulation of $N=2$ supergravity in terms of pure $N=1$ physical superfields. This means that all the $N=1$ auxiliary superfields should be eliminated via their equations of motion. As we know, the solution of the $N=2$ constraints in terms of $N=1$ superfields still contains auxiliary $N=1$ superfields. These superfields should be eliminated in order for $N=2$ supergravity to be written purely in terms of $N=1$ physical superfields. This is not an easy problem.

FORMULATION OF EXTENDED SUPERSYMMETRIC FIELD
THEORIES FROM HIGHER DIMENSIONAL POINT OF
VIEW

CHAPTER 4

4.0 Introduction

In the last chapter we dealt in great detail with the analysis of $N=2$ S.G. constraints in terms of $N=1$ superfields. We showed that, by projecting the $N=2$ S.G. constraints on to $N=1$ superspace and choosing a particular gauge (i.e. chiral gauge), the projected constraints took a particular form which was suitable for solving them in terms of unconstrained $N=1$ prepotentials.

On the other hand, Sokatchev [13] has achieved partial success in solving the $N=2$ constraints in terms of $N=2$ prepotentials by reducing the ordinary $N=2$ superspace to $N=2$ chiral subspaces. In addition, he was able to write down an action formula for off-shell $N=2$ S.G. However, it was only recently that a major breakthrough in solving the $N=2$ constraints in terms of unconstrained $N=2$ prepotentials was achieved. This was done using the powerful harmonic superspaces[17]. Unfortunately, it is yet unclear how to use harmonic superspaces as a tool to solve the $N=2$ constraints in higher dimensional spacetime. Therefore it is worthwhile to understand the $N=2$ S.G. constraints in higher dimensional space time before such an attempt.

It is the aim of this chapter to understand the four dimensional $N=2$ S.G. constraints from a higher dimensional point of view. We will show that the $N=2$ S.G. constraints in chiral superspace [13] are indeed equivalent to one $N=2$ S.G. constraint in five dimensions. We will work out in detail all the components of the torsion constraints using the Bianchi identities where the results are in agreement

with those of Sokatchev [25].

In section 4.2 we will show that in six dimensions one can solve the N=2 matter constraints as a power expansion in the N=2 coupling constant.

4.1 Realization of Sokatchev's N=2 S.G. constraints in chiral superspace from a single constraint in five dimensions

In this chapter we will use the two-component notation of where [13] :

$$\epsilon^{12} = -\epsilon_{12} = \epsilon^{i\bar{2}} = -\epsilon_{i\bar{2}} = \epsilon_{2\bar{1}} = \epsilon_{\bar{2}1} = i$$

$$\psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta$$

$$\psi_\alpha = \epsilon_{\alpha\beta} \psi^\beta \quad (4.1.1)$$

$$\epsilon^{\alpha\beta} \epsilon_{\alpha\beta} = -2 \quad , \quad \epsilon^{i\bar{j}} \epsilon_{i\bar{j}} = -2$$

$$(\epsilon^{\alpha\beta})^\dagger = \epsilon^{\alpha\beta} \quad , \quad (\epsilon^{i\bar{j}})^\dagger = -\epsilon_{i\bar{j}}$$

We also denote $\hat{A} = (5, A)$ as Lorentz flat index and $M = (5, M)$ as space world index, where 5 represents the extra fifth dimension corresponding to a central charge which will generate a U(1) vector field. We define, using the notation of ref. [13] :

$$\begin{aligned} \mathcal{D}_\alpha &= E_\alpha^{\hat{M}} \partial_{\hat{M}} + \Phi_\alpha \\ &= \nabla_\alpha + \Phi_\alpha + E_\alpha^5 \partial_5 \\ &= D_\alpha + E_\alpha^5 \partial_5 \end{aligned} \quad (4.1.2)$$

where

$$\phi_{\hat{A}} = \phi_{\hat{A}\beta}^{\delta} M_{\delta}^{\beta} + \bar{\phi}_{\hat{A}\beta}^{\delta} \bar{M}_{\delta}^{\beta} \quad (4.1.3)$$

is a Lorentz connection and D_{α} is the curved four dimensional super covariant derivative.

From the definition (4.1.2) we obtain:

$$\{\mathcal{D}_{\underline{\alpha}}, \mathcal{D}_{\underline{\beta}}\} = T_{\underline{\alpha}\underline{\beta}}^{\hat{A}} \mathcal{D}_{\hat{A}} + R_{\underline{\alpha}\underline{\beta}}(M, \bar{M}) \quad (4.1.4)$$

where the curvature terms can be shown to be equal to zero.

In four dimensions (4.1.4), in the presence of the central charge, is equivalent to:

$$\begin{aligned} \{\mathcal{D}_{\underline{\alpha}}, \mathcal{D}_{\underline{\beta}}\} &= T_{\underline{\alpha}\underline{\beta}}^A \mathcal{D}_A + F_{\underline{\alpha}\underline{\beta}} Z + R_{\underline{\alpha}\underline{\beta}}(M, \bar{M}) \\ \{D_{\underline{\alpha}}, D_{\underline{\beta}}\} &= T_{\underline{\alpha}\underline{\beta}}^A D_A + R_{\underline{\alpha}\underline{\beta}}(M, \bar{M}). \end{aligned} \quad (4.1.5)$$

Comparing (4.1.4) and (4.1.5), it is evident that:

$$\mathcal{D}_5 = Z \quad (a)$$

$$T_{\underline{\alpha}\underline{\beta}}^5 = F_{\underline{\alpha}\underline{\beta}} = \epsilon_{\alpha\beta} \epsilon_{ij} \quad (b) \quad (4.1.6)$$

Moreover, it is standard to set

$$\begin{aligned} T_{\underline{\alpha}\underline{\beta}}^A &= 0 \\ T_{\alpha i, \alpha j}^{\beta\dot{\beta}} &= -8i \epsilon_{ij} \delta_{\alpha}^{\beta} \delta_{\alpha}^{\dot{\beta}} \end{aligned} \quad (4.1.7)$$

Explicitly (4.1.4) corresponds to:

$$D_{(\alpha} E_{\beta)}^5 \partial_5 = \epsilon_{ij} \epsilon_{\alpha\beta} D_5 \quad (4.1.8)$$

where we have used (4.1.2) and (4.1.6b). The solution to (4.1.8) is given by:

$$E_{\underline{\alpha}}^5 = \frac{1}{2} (\theta_{\underline{\alpha}} + H_{\underline{\alpha}}) + \nabla_{\underline{\alpha}} H \quad (4.1.9)$$

The expression for $\nabla_{\underline{\alpha}}$ is given by :

$$\nabla_{\underline{\alpha}} = F_{\underline{\alpha}}^{-1}{}^{\mu} \Delta_{\mu}$$

where F is the inverse of the following matrix (as given in 2.5.25a):

$$F_{\mu i}{}^{vj} = \delta_{\mu}{}^v \delta_i{}^j + \Delta_{\mu i} H^{vj}$$

and

$$\Delta_{\mu i} = \partial_{\mu i} + i \Delta_{\mu i} H^m \partial_m$$

In (4.1.9) $\theta_{\underline{\alpha}}$ is the Grassmanian $N=2$ (superspace) coordinate, $H_{\underline{\alpha}}$ is the spinor prepotential [13] and H is a new scalar prepotential.

By substituting (4.1.9) in (4.1.8), we have:

$$\frac{1}{2} D_{(\alpha} [\theta_{\beta)} + H_{\beta)}] \partial_5 = \epsilon_{ij} \epsilon_{\alpha\beta} D_5 \quad (4.1.10)$$

where

$$D_{(\alpha} \nabla_{\beta)} H = \nabla_{(\alpha} \nabla_{\beta)} H = 0.$$

Therefore:

$$\begin{aligned} D_{\underline{\alpha}} (\theta_{\underline{\beta}} + H_{\underline{\beta}}) &= (1 + \Delta H)^{-1}_{\underline{\alpha}}{}^{\underline{\gamma}} \Delta_{\underline{\gamma}} (\theta_{\underline{\beta}} + H_{\underline{\beta}}) \\ &= (1 + \Delta H)^{-1}_{\underline{\alpha}}{}^{\underline{\gamma}} (C_{\underline{\gamma}\underline{\beta}} + \Delta_{\underline{\gamma}} H_{\underline{\beta}}) \\ &= C_{\underline{\alpha}\underline{\beta}} = C_{\underline{\beta}\underline{\alpha}} \\ &= \epsilon_{\underline{\alpha}\underline{\beta}} \epsilon_{ij} \end{aligned} \quad (4.1.11)$$

Hence:

$$\partial_5 = \mathcal{D}_5 \quad (4.1.12)$$

Therefore (4.1.9) is a solution of the constraint (4.1.4).

As we have mentioned previously, the N=2 S.G. constraints were partially solved by formulating the theory in chiral superspace. The remaining ones are:

$$T_{\alpha(i}{}_{\beta j)}{}^A = 0 \quad (a)$$

(b)

$$\text{Im} (D^{\alpha}_{(i} T_{\alpha j)} - T^{\alpha}_{(i} T_{\alpha j)}) = 0. \quad (4.1.13)$$

where $T_{\alpha j}$ is as defined in (3.1.2).

We will show that (4.1.13a,b) are equivalent to the single five dimensional N=2 S.G. constraint:

$$T_{\alpha(i}{}_{\beta j)}{}^{\hat{A}} = 0. \quad (4.1.14)$$

such that $\hat{A}=(5,A)$ [25] .

Obviously, (4.1.14) does indeed contain (4.1.13a) for $\hat{A} = A$. However, it still remains to show that it also includes (4.1.13b).

For convenience we will list some of the necessary torsion components needed in our calculations:

$$T_{\alpha i, \beta j}^5 = \epsilon_{\alpha\beta} \epsilon_{ij} \quad (4.1.15a)$$

$$T_{\alpha i, \dot{\alpha} j}^5 = 0 \quad (4.1.15b)$$

$$T_{\alpha i, \dot{\alpha} j}^{\beta\dot{\beta}} = 2i \epsilon_{ij} \delta_{\alpha}^{\beta} \delta_{\dot{\alpha}}^{\dot{\beta}} \quad (4.1.15c)$$

$$T_{\alpha i, \dot{\alpha} j}^{\beta\kappa} = \frac{1}{4} \delta_{\dot{\alpha}}^{\beta} (\delta_i^{\kappa} T_{j\alpha} + \delta_j^{\kappa} T_{i\alpha}) \quad (4.1.15d)$$

In order to establish the equivalence of $T_{\alpha i, j\dot{\beta}}^5 = 0$ to (4.1.13b) , it is necessary to find a relation between the torsion components in five dimensions and those in four dimensions. Using (4.1.15a,c) and one of the dimension zero Bianchi Identities (B.I), such a relation can be established

$$T_{\dot{\alpha}, j}^{\alpha} = \frac{3i}{4} \overline{T}_{\dot{\alpha} j} \quad (4.1.16)$$

The dimension 1/2 (B.Is) yield the following :

$$T_{\alpha i, \beta\dot{\beta}}^{\gamma\dot{\gamma}} = \frac{1}{4} \delta_{\dot{\beta}}^{\dot{\gamma}} (\epsilon_{\alpha\beta} T_i^{\gamma} + \delta_{\beta}^{\gamma} T_{i\alpha}) \quad (4.1.17a)$$

$$T_{\alpha\dot{\alpha}, i j}^{\beta\dot{\beta}} = \frac{i}{8} \delta_{\dot{\alpha}}^{\dot{\beta}} (\alpha_{i\dot{\alpha}}^{\beta} T_{j\alpha} - \frac{1}{4} T_{i\dot{\alpha}}^{\beta} T_{j\alpha}) \quad (4.1.17b)$$

At this stage it is important to carry out a consistency check on the above results by examining $[D_{\alpha i}, D_5]$ which must be zero. This follows as a consequence of the truncation of the theory to a true five dimensional theory as implied by the constraints, in which the five dimensional space time M^5 is given by;

$$M^5 = M^4 \times \mathbb{R}^1 \quad (4.1.18)$$

Using the definition of D_5 :

$$\{D_{\alpha i}, D_{\beta j}\} = \epsilon_{\alpha\beta} \epsilon_{ij} D_5 \quad (4.1.19)$$

and the Jacobi identity:

$$\begin{aligned} [D_{\alpha i}, \{\bar{D}_{\alpha}^{\dot{i}}, \bar{D}_{\dot{j}}^{\dot{i}}\}] + [\bar{D}_{\alpha}^{\dot{i}}, \{D_{\alpha i}, \bar{D}_{\dot{\alpha}}^{\dot{i}}\}] \\ + [\bar{D}_{\alpha}^{\dot{i}}, \{\bar{D}_{\dot{\alpha}}^{\dot{i}}, D_{\alpha i}\}] = 0 \end{aligned} \quad (4.1.20)$$

together with (4.1.15-4.1.17) it is straight forward to establish that:

$$[D_{\alpha i}, D_5] = 0 \quad (4.1.21)$$

$$[D_{\alpha\dot{\alpha}}, D_5] = 0$$

This implies that :

$$T_{\hat{A}, 5}^{\hat{B}} = 0 \quad (4.1.22)$$

and all the curvature terms with index 5 vanish.

Let us remark that in the in the Kaluza-Klein picture the E_5 component of the vielbein provides a source for the scalar fields. However, since we are considering trivial dimensional reduction in which we do not wish to have new additional physical degrees of freedom we must gauge this scalar to be a constant which we have chosen to be one.

Equipped with all the above results, we are ready to establish the equivalence of (4.1.14) and (4.1.13b). In fact it is enough to consider the particular dimension 1/2 Bianchi identity:

$$\begin{aligned} & \mathcal{D}_{\alpha i} \bar{T}_{\dot{\alpha} j, \beta \dot{\beta}}^5 + \bar{\mathcal{D}}_{\dot{\alpha} j} T_{\alpha i, \beta \dot{\beta}}^5 - \left[T_{\alpha i, \dot{\alpha} j}^{\hat{E}} T_{\hat{E}, \beta \dot{\beta}}^5 \right. \\ & \left. + T_{\beta \dot{\beta}, \alpha i}^{\hat{E}} T_{\hat{E}, \dot{\alpha} j}^5 + T_{\beta \dot{\beta}, \dot{\alpha} j}^{\hat{E}} T_{\hat{E}, \alpha i}^5 \right] \\ & = 0. \end{aligned} \quad (4.1.23)$$

Contracting with $\epsilon^{\beta \dot{\alpha}}$, $\epsilon^{\beta \alpha}$, symmetrising on i and j and using (4.1.15 - 4.1.17) we finally obtain:

$$\text{Im} [\mathcal{D}_{\alpha i} T_{j, \alpha} - T_{\alpha i} T_{j, \alpha}] = 0. \quad (4.1.24)$$

In the above analysis we have proved that the constant in front of the bilinear term in (4.1.13b) has been uniquely fixed by requiring the central charge invariance which is manifestly realized by the U(1) invariance of the B.I (4.1.16). This constant is :

$$a = -1 \quad (4.1.25)$$

which is in total agreement with ref[13] where the same result has been obtained by requiring that the imaginary part of the action to be zero.

However, the analysis of ref[13] was at the linearized level, while our analysis is at the full non-linear level[25].

There is a final remark about the transformations (3.1.30).

From ref. [13] we find that the vector prepotential H^M transforms as;

$$\delta H^{\hat{M}} = \frac{i}{2} (\bar{\lambda}^{\hat{M}} - \lambda^{\hat{M}}) (X + iH, \theta) - \left[\frac{1}{2} (\lambda^{\hat{N}} + \bar{\lambda}^{\hat{N}}) \partial_{\hat{N}} + \lambda^{\hat{\nu}} \partial_{\hat{\nu}} + \bar{\lambda}_{\hat{\nu}} \bar{\partial}^{\hat{\nu}} \right] H^{\hat{M}} \quad (4.1.32)$$

in our case ($M=5, m$). From (4.1.32) one can immediately deduce that at the linear level:

$$\delta H^5 = \delta \mathcal{H}^5 = \frac{i}{2} (\bar{\lambda}^5 - \lambda^5) \quad (4.1.33)$$

This is exactly (3.1.30a) with $\Lambda = \lambda^5$. On the other hand

$$\delta H^{\mu i} = -\lambda^{\mu i} (X + iH, \theta) + \mathcal{T}^{\mu i} - \left[\frac{1}{2} (\lambda^N + \bar{\lambda}^N) \partial_N + \lambda^{\hat{\nu}} \partial_{\hat{\nu}} + \bar{\lambda}_{\hat{\nu}} \bar{\partial}^{\hat{\nu}} \right]$$

at the linear level we have:

$$\delta H^{\alpha 2} = -\delta \psi^{\alpha} = -\lambda^{\alpha 2} + \mathcal{T}^{\alpha 2} \quad (4.1.34)$$

Using (3.1.12) we can always solve for $\mathcal{T}^{\alpha 2}$ such that

$$\mathcal{T}^{\alpha 2} = \nabla^{\alpha} \xi \quad (4.1.35)$$

However, by requiring $\delta \lambda_{\alpha} = 0$ (at the linear level) we can fix ξ to be $-\Lambda$ and redefine $\Lambda^{\alpha} = \lambda^{\alpha 2}$. We can now

get (3.1.30b).

4.2 The solution of N=2 S.Y.M. matter constraints in six dimensions

In this section we study the solution of the constraints of the vector multiplet of N=2 S.Y.M. theory in terms of unconstrained N=2 prepotentials in six dimensions. The techniques employed are very similar to those of N=2 S.Y.M. theory in four dimensions where it was shown by Howe, Stelle and Townsend that the non-linear solution of N=2 S.Y.M. constraints in four dimensions could be obtained as a power series in the S.Y.M. coupling constant[26]. In principle there are two ways of expressing the constraints in six dimensions. The first is the SU(2) covariant formalism in which the SU(2) symmetry is manifest. The second is where one breaks SU(2) covariance down to U(1); such a breaking allows the definition of the d=6 analogue of a chiral superfield ϕ which satisfies:

$$D_{\bar{\alpha}} \phi = 0 \Rightarrow D_{\alpha} \bar{\phi} = 0 \quad (4.2.1)$$

where $D_{\bar{\alpha}}$ and D_{α} are identifications of the first and second components of D_{α}^i ($i = 1, 2$ is an SU(2) index)** . Note that because of the SU(2) Majorana condition in d=6:

$$(\psi_{\alpha}^i)^* := \bar{\psi}_{\alpha i} = \epsilon_{ij} B_{\alpha}^{\beta} \psi_{\beta}^j \quad (4.2.2)$$

where the matrix B is unitary and satisfies $B^* B = -1$. The $D_{\bar{\alpha}}$ is related to the complex conjugate of D_{α}^i i.e.

** Unlike in the preceding section, we consider $D_{\underline{\alpha}}$ to be the flat covariant derivative.

$$(D_\alpha)^* := \bar{D}_\alpha \quad \text{by}$$

$$\bar{D}_\alpha := D_\alpha = -B_\alpha^*{}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}} \quad (4.2.3)$$

Although it is much simpler to use the U(1) notation, unfortunately it does not seem to be very useful in solving the Y.M. constraints beyond the linear level [28]. It is for this reason that we prefer to work in the full SU(2) covariant notation. The techniques of Stelle, Howe and Townsend are directly applicable. Our notation is exactly that of reference [27]. We introduce the notation:

$$D_\alpha{}^i D_\beta{}^j = \frac{i}{2} \epsilon^{ij} \partial_{\alpha\beta} + \epsilon^{ij} D_{\alpha\beta} + D_{\alpha\beta}{}^{ij} \quad (a)$$

$$D_{\alpha\beta} = \frac{1}{2} D_\alpha{}^i D_\beta{}^j = D_{\beta\alpha} \quad (b)$$

$$D_{\alpha\beta}{}^{ij} = D_{[\alpha}{}^i D_{\beta]}{}^j \quad (c) \quad (4.2.4)$$

for the product of two D's, where vector indices are represented by a pair of anti-symmetric spinor indices. We shall also define:

$$D^{ijkl} := D_\alpha{}^i D_\beta{}^j D_\gamma{}^k D_\delta{}^l \in \epsilon^{\alpha\beta\gamma\delta} \quad (4.2.5)$$

for the Lorentz scalar products of four D's. $\epsilon^{\alpha\beta\gamma\delta}$ is an invariant totally antisymmetric SU(4)* (Lorentz group in d=6) tensor. Using the properties of the D's, one can prove the following useful identity

$$D_{\alpha\beta} D^{\dot{\alpha}\dot{\beta}} = D^{\dot{\alpha}\dot{\beta}} D_{\alpha\beta} = 0 \quad (4.2.6)$$

Further identities can be found in Appendix C and ref.[27].

The d=6 Y.M. constraint in the SU(2) notation is very simple.

It reads:

$$F_{\alpha\beta}{}^{\dot{\alpha}\dot{\beta}} = 0 \quad (4.2.7)$$

In terms of a superspace gauge prepotential (4.2.7) can be rewritten as:

$$D_{\alpha}{}^i A_{\beta}{}^{\dot{i}} + D_{\beta}{}^{\dot{i}} A_{\alpha}{}^i + ig \{A_{\alpha}{}^i, A_{\beta}{}^{\dot{i}}\} - i\epsilon^{\dot{i}i} f_{\alpha\beta} = 0 \quad (4.2.8)$$

The antisymmetric part in ij is a conventional constraint determining $f_{\alpha\beta}$:

$$f_{\alpha\beta} = -i D_{[\alpha} A_{\beta]}{}^{\dot{i}} + \frac{1}{2} g \{A_{\alpha}{}^i, A_{\beta}{}^{\dot{i}}\} \quad (4.2.9)$$

However, the symmetric part in ij is a differential constraint on the prepotential $A_{\alpha}{}^i$:

$$D_{(\alpha} A_{\beta)}{}^{\dot{i}} + \frac{ig}{2} \{A_{(\alpha}{}^i, A_{\beta)}{}^{\dot{i}}\} = 0 \quad (4.2.10)$$

(To derive 4.2.8 one makes use of the (anti) commutator

$$\{ \mathcal{D}_A, \mathcal{D}_B \} = F_{AB}^z - \tau_{AB}^c \mathcal{D}_c \text{ where } \mathcal{D}_A = D_A + i[A_A,]$$

is S.Y.M. covariant derivative in the adjoint representation of some compact group G . Note that the only non-vanishing component of the (flat space torsion) is

$$\tau_{\alpha\beta}^{ij\gamma\delta} = -i\epsilon^{ij} \delta_{[\alpha}^{\gamma} \delta_{\beta]}^{\delta}$$

The linearized solution of constraint (4.2.10) is given by $(g=0)$ [27] :

$$A_{\alpha}^i = D_{\alpha}^i U + D_{\alpha}^m D_m^{ij\kappa} V_{i\kappa} \quad (4.2.11)$$

where U and $V_{i\kappa}$ are (a 1 and 3 of $SU(2)$) unconstrained $N=2$ superfields. To prove that (4.2.11) satisfies (4.2.10) at $g=0$ one makes use of (4.2.4) and (4.2.6). The above solution is very similar to that in $d=4$ dimensions [26], except for the fact that we do not have terms involving $\bar{D}$'s, since these are related to the D 's by the $SU(2)$ Majorana condition.

As we have mentioned previously, we can solve constraints (4.2.10) in power series of the S. Yang-Mills coupling constant g . However, an essential ingredient to obtain such a solution is to know beforehand the first order in g solution of the linearized solution. The power expansion technique can then be used by rewriting the non-linear constraint in a form identical to the linear constraint. This is normally achieved by differentiating the non-linear constraint with respect to coupling constant g . Hence, a solution can be obtained similarly to the linearized solution. Then the non-linear solution is obtained by integrating the former solution with respect to g .

Let $A_{\alpha}^i \rightarrow g A_{\alpha}^i$ then differentiating (4.2.10) with

respect to g gives:

$$\mathcal{D}_{(\alpha}{}^{ii} B_{\beta)}{}^j = 0 \quad (4.2.12)$$

where

$$B_{\alpha}{}^i = \frac{\partial}{\partial g} (g A_{\alpha}{}^i) \quad (4.2.13)$$

and

$$\mathcal{D}_{\alpha}{}^i = D_{\alpha}{}^i + ig \{ A_{\alpha}{}^i, \} \quad (4.2.14)$$

Equation (4.2.12) has indeed the same structure as the linearized constraint (4.2.10) with $g=0$, with the difference that flat derivatives have been replaced by covariant ones. Therefore, the solution of (4.2.12) is given by:

$$B_{\alpha}{}^i = \mathcal{D}_{\alpha}{}^i U + \mathcal{D}_{\alpha}{}^m \mathcal{D}_m{}^{ij} V_{ij} \quad (4.2.15)$$

The full non-linear solution for $A_{\alpha}{}^i$ may now be reconstructed as a power series in g from the integral equation:

$$A_{\alpha}{}^i = \frac{1}{g} \int_0^g d\kappa B_{\alpha}{}^i(\kappa) \quad (4.2.16)$$

The gauge transformations of the prepotentials V_{ij} and U can be obtained iteratively. For $A_{\alpha}{}^i$ the k gauge transformation

$$\delta A_{\alpha}{}^i = -\mathcal{D}_{\alpha}{}^i \kappa. \quad (4.2.17)$$

from which we deduce, to lowest order in g , that:

$$\delta U = -\kappa \quad (a)$$

$$\delta V_{ij} = 0 \quad (b) \quad (4.2.18)$$

(4.2.18a) shows that U is a pure gauge and can be set to zero by a proper choice of the parameter K . To determine such a choice we look for pregauge transformations which leave A invariant, i.e.:

$$\delta A_\alpha^i = 0 \quad (4.2.19)$$

We find to the lowest order that the pregauge transformations of U and V_{ij} are:

$$\delta V_{ij} = D_\alpha^\kappa \Lambda^\alpha_{ij\kappa}$$

$$\delta U = \frac{1}{3} D^{(r)ij\kappa} D_{\alpha\ell} \Lambda^\alpha_{ij\kappa}$$

$$\Lambda^\alpha_{ij\kappa} = \Lambda^\alpha_{(ij)\kappa} \quad (4.2.20)$$

Therefore, in order to stay in the gauge $U=0$, we must set:

$$K = \frac{1}{3} D^{(r)ij\kappa\ell} D_{\alpha\ell} \Lambda^\alpha_{ij\kappa} \quad (4.2.21)$$

It is interesting to write down the action of $N=2$ S.Y.M. in superspace and look for covariant quantization of the theory along the same lines of [26]. However, this lies outside the

scope of the thesis and will be reported elsewhere .

N=2 S.G. constraints projected onto N=1 superspace

Throughout our analysis in Chapter 3 we have used the notation of [16]. In this Appendix we list the projection of the N=2 constraints into N=1 superspace and the supersymmetry transformations of the prepotentials.[16]:

$$\{\nabla_\alpha, \nabla_\beta\} = i\psi_{(\alpha}^{\dot{\gamma}} \psi_{\beta)}^{\dot{\delta}} \nabla_{\dot{\gamma}\dot{\delta}} + \frac{1}{2} \bar{\lambda}_{\dot{\gamma}} \psi_{(\alpha}^{\dot{\gamma}} \nabla_{\beta)} \quad (\text{A.1})$$

$$+ \psi_{(\alpha\beta)} (Z + \bar{W}_{\dot{\gamma}}^{\dot{\delta}} \bar{M}_{\dot{\delta}}^{\dot{\gamma}}) - X_{\dot{\gamma}\dot{\delta}} \psi_{(\alpha}^{\dot{\gamma}} M_{\beta)}^{\dot{\delta}} + \psi_{(\alpha}^{\dot{\gamma}} X_{\beta\dot{\delta}} \bar{M}_{\dot{\gamma}}^{\dot{\delta}}.$$

$$\{\nabla_\alpha, \bar{\nabla}_{\dot{\beta}}\} = i\nabla_{\alpha\dot{\beta}} + i(\psi_\alpha^{\dot{\gamma}} \psi_{\dot{\beta}}^{\dot{\delta}} + \psi_\alpha^{\dot{\delta}} \psi_{\dot{\beta}}^{\dot{\gamma}}) \nabla_{\dot{\gamma}\dot{\delta}} - \frac{1}{2} (T_\alpha - \psi_\alpha^{\dot{\gamma}} \lambda_{\dot{\gamma}}) \bar{\nabla}_{\dot{\beta}}$$

$$- \frac{1}{2} (\bar{T}_{\dot{\beta}} - \psi_{\dot{\beta}}^{\dot{\gamma}} \bar{\lambda}_{\dot{\gamma}}) \nabla_\alpha + \psi_{\alpha\dot{\beta}} (Z + W_{\dot{\gamma}}^{\dot{\delta}} M_{\dot{\delta}}^{\dot{\gamma}})$$

$$+ \psi_{\dot{\beta}\alpha} (Z + \bar{W}_{\dot{\gamma}}^{\dot{\delta}} \bar{M}_{\dot{\delta}}^{\dot{\gamma}}) - (P_{\beta\dot{\beta}} - A_{\beta\dot{\beta}} - \psi_{\dot{\beta}}^{\dot{\gamma}} X_{\beta\dot{\gamma}}) M_\alpha^{\dot{\beta}}$$

$$+ \psi_\alpha^{\dot{\gamma}} \bar{X}_{\beta\dot{\beta}} M_\gamma^{\dot{\beta}} + (P_{\alpha\dot{\alpha}} - A_{\alpha\dot{\alpha}} - \psi_\alpha^{\dot{\gamma}} X_{\gamma\dot{\alpha}}) \bar{M}_{\dot{\beta}}^{\dot{\alpha}} - \psi_{\dot{\beta}}^{\dot{\gamma}} X_{\alpha\dot{\alpha}} \bar{M}_{\dot{\gamma}}^{\dot{\alpha}}.$$

$$[\nabla_{\alpha\dot{\alpha}}, \nabla_\beta] = \frac{1}{2} (T_\beta \nabla_{\alpha\dot{\alpha}} + \psi_{\beta\alpha} \lambda^{\dot{\gamma}} \nabla_{\dot{\gamma}\dot{\alpha}} + \psi_{\beta\dot{\alpha}} \bar{\lambda}^{\dot{\gamma}} \nabla_{\alpha\dot{\gamma}})$$

$$+ i(\psi_{\alpha\dot{\alpha}} \psi_\beta^{\dot{\gamma}} + \psi_{\alpha\dot{\alpha}}^{\dot{\gamma}} \psi_\beta^{\dot{\delta}}) \nabla_{\dot{\gamma}\dot{\delta}}$$

$$+ \frac{1}{2} (iA_{\alpha\dot{\alpha}} - \frac{i}{2} \lambda_\alpha \bar{\lambda}_{\dot{\alpha}} + \psi_{\alpha\dot{\alpha}}^{\dot{\gamma}} \bar{\lambda}_{\dot{\gamma}}) \nabla_\beta$$

$$+ i(\psi_{\beta\alpha} \bar{X}_{\dot{\alpha}}^{\dot{\gamma}} + \psi_{\beta\dot{\alpha}} W_\alpha^{\dot{\gamma}}) \nabla_{\dot{\gamma}}.$$

$$\begin{aligned}
& + \frac{i}{4} \psi_{\beta\dot{\alpha}} [2\bar{S} - \bar{\nabla}^{\dot{\gamma}} \bar{\lambda}_{\dot{\gamma}} - \psi^{\dot{\gamma}\dot{\gamma}} (A_{\dot{\gamma}\dot{\gamma}} - \frac{1}{2} \lambda_{\dot{\gamma}} \bar{\lambda}_{\dot{\gamma}}) \\
& + \psi^{\dot{\gamma}}_{\dot{\gamma}} \bar{R}] \bar{\nabla}_{\dot{\gamma}} - i (\psi_{\beta\dot{\alpha}} \chi_{\alpha}^{\dot{\gamma}} - \psi_{\beta\alpha} \bar{W}_{\dot{\alpha}}^{\dot{\gamma}}) \bar{\nabla}_{\dot{\gamma}} \\
& + \frac{i}{4} \psi_{\beta\alpha} [2S - \nabla^{\gamma} \lambda_{\gamma} + \psi^{\gamma\gamma} (\bar{A}_{\gamma\gamma} - \frac{1}{2} \lambda_{\gamma} \bar{\lambda}_{\gamma}) + \psi^{\gamma}_{\gamma} R] \bar{\nabla}_{\dot{\gamma}} \\
& + i [\frac{1}{2} C_{\alpha\beta} \bar{\lambda}_{\dot{\alpha}} - i \psi_{\alpha\dot{\alpha}\beta} - \psi_{\beta\alpha} \bar{T}_{\dot{\alpha}} - \psi_{\beta\dot{\alpha}} T_{\alpha}] Z \\
& + M \text{ terms.}
\end{aligned}$$

$$F_{\alpha\beta}^{\dot{\gamma}} = 0, \quad F_{\alpha\beta}^{\dot{\gamma}} = \frac{1}{2} T_{(\alpha} \psi_{\beta)}^{\dot{\gamma}} \quad (\text{A.2 a})$$

$$F_{\alpha\dot{\beta}}^{\dot{\gamma}} = \frac{1}{2} (\bar{\lambda}_{\dot{\beta}} S_{\alpha}^{\dot{\gamma}} - \psi_{\alpha}^{\dot{\gamma}} \bar{T}_{\dot{\beta}}) \quad (\text{b})$$

$$F_{\alpha\dot{\alpha},\beta}^{\dot{\gamma}} = -\frac{i}{2} S_{\alpha}^{\dot{\gamma}} \psi_{\beta\dot{\alpha}} (\bar{R} + \frac{1}{4} \bar{\lambda}^{\dot{\gamma}} \bar{\lambda}_{\dot{\gamma}}) \quad (\text{c})$$

$$-i (S_{\beta}^{\dot{\gamma}} \chi_{\alpha\dot{\alpha}} - \frac{1}{2} S_{\alpha}^{\dot{\gamma}} \chi_{\beta\dot{\alpha}}) - i \psi_{\beta\alpha} (P_{\dot{\alpha}}^{\dot{\gamma}} - \frac{1}{2} A_{\dot{\alpha}}^{\dot{\gamma}})$$

$$F_{\alpha\dot{\alpha},\beta}^{\dot{\gamma}} = -\frac{i}{2} C_{\alpha\beta} S_{\dot{\alpha}}^{\dot{\gamma}} (S + \frac{1}{2} \psi^{\dot{\gamma}}_{\dot{\gamma}} R) - i C_{\alpha\beta} \bar{W}_{\dot{\alpha}}^{\dot{\gamma}} \quad (\text{d})$$

$$+ \frac{1}{2} \psi_{\alpha\dot{\alpha}}^{\dot{\gamma}} T_{\beta} - \frac{i}{4} S_{\dot{\alpha}}^{\dot{\gamma}} [\nabla_{(\alpha} \lambda_{\beta)} + \frac{1}{2} \psi_{\beta\alpha}^{\dot{\gamma}} \lambda^{\dot{\gamma}} \lambda_{\dot{\gamma}}$$

$$- \psi_{(\alpha}^{\dot{\gamma}} (\bar{A}_{\beta)\dot{\gamma}} - \frac{1}{2} \lambda_{\beta} \bar{\lambda}_{\dot{\gamma}}] - \frac{i}{2} \psi_{\beta\dot{\alpha}} (\bar{A}_{\alpha}^{\dot{\gamma}} - 2 \bar{P}_{\alpha}^{\dot{\gamma}})$$

$$X_{\alpha\dot{\alpha}} = (\nabla_{\alpha} + \frac{1}{2} T_{\alpha}) \bar{T}_{\dot{\alpha}} - \psi_{\alpha\dot{\alpha}} \bar{R} + \psi_{\alpha}^{\dot{\gamma}} [A_{\dot{\gamma}\dot{\alpha}} - \frac{1}{2} \lambda_{\dot{\gamma}} \bar{\lambda}_{\dot{\alpha}}] \quad (\text{e})$$

$$\rho_{\alpha\dot{\alpha}} = (\nabla_{\alpha} + \frac{1}{2} T_{\alpha}) \bar{T}_{\dot{\alpha}} - \psi_{\alpha\dot{\alpha}} \bar{S} + \psi_{\alpha}^{\gamma} [\bar{\chi}_{\gamma\dot{\alpha}} - \frac{1}{2} \lambda_{\gamma} \bar{T}_{\dot{\alpha}}] \quad (f)$$

$$- \frac{1}{2} \psi_{\alpha}^{\gamma} [\nabla_{(\dot{\alpha}} \bar{\lambda}_{\gamma)} - \psi_{(\dot{\alpha}\gamma)} \bar{R} + \psi_{(\dot{\alpha}}^{\gamma} (A_{\gamma\gamma)} - \frac{1}{2} \lambda_{\gamma} \bar{\lambda}_{\gamma})].$$

$$\begin{aligned} \nabla_{(\alpha} T_{\beta)} &= \psi_{(\alpha\beta)} S + \psi_{(\alpha}^{\gamma} (\chi_{\beta)\gamma} - \frac{1}{2} T_{\beta}) \bar{\lambda}_{\gamma}) \\ &+ \frac{1}{2} \psi_{(\alpha}^{\gamma} \psi_{\beta)}^{\delta} [\nabla_{\gamma} \lambda_{\delta} - \psi_{\gamma\delta} R - \psi_{\gamma}^{\epsilon} (\bar{A}_{\delta\epsilon} - \frac{1}{2} \lambda_{\delta} \bar{\lambda}_{\epsilon})]. \end{aligned} \quad (g)$$

$$\rho_{\alpha\dot{\alpha}} - A_{\alpha\dot{\alpha}} = \bar{\rho}_{\alpha\dot{\alpha}} - \bar{A}_{\alpha\dot{\alpha}} \quad (h)$$

$$\delta\psi_M = E_M^{\alpha} \epsilon_{\alpha} + E_M^{\dot{\alpha}} \bar{\epsilon}_{\dot{\alpha}} + i E_M^{\alpha\dot{\alpha}} (\epsilon_{\alpha} \bar{T}_{\dot{\alpha}} + \bar{\epsilon}_{\dot{\alpha}} T_{\alpha}) \quad (A.3 \quad a)$$

$$\begin{aligned} \delta\psi_M^{\gamma} &= (\nabla_M - \frac{1}{2} E_M^{\dot{\alpha}} \bar{T}_{\dot{\alpha}}) \epsilon^{\gamma} + \\ &+ i E_M^{\alpha\dot{\alpha}} [(\epsilon_{\alpha} (\rho_{\dot{\alpha}}^{\gamma} - \frac{1}{2} A_{\dot{\alpha}}^{\gamma}) + \frac{1}{2} \bar{\epsilon}_{\dot{\alpha}} \delta_{\alpha}^{\gamma} (\bar{R} + \frac{1}{4} \bar{\lambda}^{\gamma} \bar{\lambda}_{\gamma})] \end{aligned} \quad (b)$$

$$\delta E_{\alpha} = \frac{1}{2} \bar{\epsilon}^{\dot{\alpha}} \bar{\lambda}_{\dot{\alpha}} E_{\alpha} + i (\epsilon^{\beta} \psi_{\alpha}^{\dot{\beta}} + \bar{\epsilon}^{\dot{\beta}} \psi_{\alpha}^{\beta}) E_{\beta\dot{\beta}} \quad (c)$$

$$\delta E_{\alpha\dot{\alpha}} = -i (\epsilon^{\beta} \psi_{\alpha\dot{\alpha}}^{\dot{\beta}} + \bar{\epsilon}^{\dot{\beta}} \psi_{\alpha\dot{\alpha}}^{\beta}) E_{\beta\dot{\beta}} \quad (d)$$

$$- \frac{1}{2} (\epsilon_{\alpha} \lambda^{\beta} E_{\beta\dot{\alpha}} + \bar{\epsilon}_{\dot{\alpha}} \bar{\lambda}^{\beta} E_{\alpha\dot{\beta}})$$

$$- i \epsilon_{\alpha} [\bar{\chi}_{\dot{\alpha}}^{\beta} E_{\beta} + \bar{W}_{\dot{\alpha}}^{\beta} \bar{E}_{\dot{\beta}} + \frac{1}{4} (2S + \psi^{\gamma} R - \nabla^{\gamma} \lambda_{\gamma} + \psi^{\gamma\delta} (\bar{A}_{\gamma\delta} - \frac{1}{2} \lambda_{\gamma} \bar{\lambda}_{\delta})) \bar{E}_{\dot{\alpha}}]$$

$$- i \bar{\epsilon}_{\dot{\alpha}} [\chi_{\alpha}^{\dot{\beta}} \bar{E}_{\dot{\beta}} + W_{\alpha}^{\dot{\beta}} E_{\beta} + \frac{1}{4} (2\bar{S} + \psi^{\gamma} \bar{R} - \bar{\nabla}^{\gamma} \bar{\lambda}_{\gamma} - \psi^{\gamma\delta} (A_{\gamma\delta} - \frac{1}{2} \lambda_{\gamma} \bar{\lambda}_{\delta})) \bar{E}_{\dot{\alpha}}]$$

Where:

$$\psi_M = E_M^A \psi_A, \quad \nabla_M = E_M^A \nabla_A, \quad \psi_M^{\gamma} = E_M^A \psi_A^{\gamma}$$

$$\delta\lambda_{\alpha} = \epsilon_{\alpha} R + \bar{\epsilon}^{\dot{\beta}} (\bar{A}_{\alpha\dot{\beta}} - \frac{1}{2} \lambda_{\alpha} \bar{\lambda}_{\dot{\beta}}) \quad (e)$$

$$\delta T_\alpha = \epsilon_\alpha S + \bar{\epsilon}^\beta (X_{\alpha\dot{\beta}} - \frac{1}{2} T_\alpha \bar{\lambda}_{\dot{\beta}}) \quad (f)$$

$$+ \frac{1}{2} \epsilon^\beta [\nabla_{(\alpha} \lambda_{\beta)} - \psi_{(\alpha\beta)} R - \psi_{(\alpha}{}^\gamma [\bar{A}_{\beta)\dot{\gamma}} - \frac{1}{2} \lambda_{\beta)} \bar{\lambda}_{\dot{\gamma}}]]$$

$$\delta E^{-1} = i E^{-1} (\epsilon^\alpha \psi_{\alpha\dot{\alpha}}{}^{\dot{\alpha}} + \bar{\epsilon}^{\dot{\alpha}} \psi_{\alpha\dot{\alpha}}{}^\alpha) \quad (g)$$

$$\{D_\alpha, D_\beta\} = 0 \quad (A.4)$$

$$\{D_\alpha, \bar{D}_{\dot{\beta}}\} = i D_{\alpha\dot{\beta}} \quad (A.5)$$

$$[D_{\alpha\dot{\alpha}}, D_\beta] = \frac{1}{2} C_{\alpha\beta} [S^\gamma D_{\gamma\dot{\alpha}} - \frac{i}{2} [(D^\gamma + \frac{1}{2} S^\gamma) S_\gamma] \bar{D}_{\dot{\alpha}}] \\ + i (\tilde{P}^\gamma_{\dot{\alpha}} - a^\gamma_{\dot{\alpha}}) D_\gamma + r_{\alpha\dot{\alpha},\beta}(M) \quad (A.6)$$

$$\tilde{P}_{\alpha\dot{\alpha}} = (D_\alpha + \frac{1}{2} S_\alpha) \bar{S}_{\dot{\alpha}} - a_{\alpha\dot{\alpha}} = \bar{\tilde{P}}_{\alpha\dot{\alpha}} \quad (A.7)$$

$$D_{(\alpha} S_{\beta)} = 0 \quad (A.3)$$

APPENDIX B

Some useful identities

Some useful identities have been used in obtaining the solutions in section 3.4 :

$$N^{\epsilon}_{\eta\eta} \beta^i M^{\alpha}_{\delta\delta} \eta^i = N^{\epsilon}_{\delta\eta} \eta^i M^{\alpha}_{\delta\eta} \beta^i \quad (\text{B.1})$$

$$N_{\beta\epsilon}^{\eta\delta} N^{\epsilon}_{\eta\eta} \beta^i \Delta^{\eta} = f^{-1}_{\eta}{}^{\delta} \Delta^{\eta} \quad (\text{B.2})$$

Where f^{-1} is the inverse of :

$$f_{\alpha}{}^{\beta} = \frac{1}{2} M_{\alpha\dot{\alpha}}^{\delta\delta} M^{\alpha}_{\delta\dot{\delta}} \beta^{\dot{\delta}} \quad (\text{B.3})$$

$$N_{\beta\epsilon}^{\eta\delta} N^{\epsilon}_{\delta\eta} \eta^i M^{\alpha}_{\delta\eta} \beta^i = 2 N_{\delta\dot{\delta}}^{\alpha\delta}$$

$$N_{\beta\epsilon}^{\eta\delta} N^{\epsilon}_{\eta\eta} \beta^i M_{\alpha\dot{\alpha}}^{\delta\dot{\delta}} \psi_{\delta}^{\eta} = 2 N_{\delta\dot{\alpha}}^{\alpha\delta} \psi_{\dot{\alpha}}^{\delta} \quad (\text{B.4})$$

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