

Errata

Alterations in the text have been underlined.

p.16,para.2,line 3 : ...Taylor[†] profile...

Add footnote:[†] This profile was first derived by Taylor,G.I.(Proc. Roy.Soc.A,84,1910) and is the steady-state solution of Burgers equation.

p.18,para.2,line 3 : ...procedure by choosing a new Reynolds number...

p.80,para.2,line 9 : ...radius of the tube (measured to the middle surface of the wall) and...

p.86,para.2,line 5 : ...non-linearity on the time scale of $\bar{\epsilon}'$ will...

p.88,para.1,line 11: ... $\bar{z} \rightarrow 0$, and by the method of characteristics
the solution is...

p.98,para.2,line 7,9,12 : To eliminate confusion with the longitudinal slope, $\tan \theta$, in equation (4.15), replace θ in T_θ , M_θ , $T_{z\theta}$ and $M_{z\theta}$ by γ (the angular co-ordinate for the tube). Note that the subscripts used here denote directions and not derivatives.

Non-linear waves in fluid-filled elastic tubes
and related problems

by

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Summary

The work described in this thesis is concerned with the propagation of waves in certain fluid systems. To present a theory for propagation on a fluid-filled elastic tube (as a model for the mammalian artery) it is first necessary and instructive to consider the classical shallow water wave problem. In Part I, this is discussed in some detail for a two dimensional viscous flow moving under gravity. The rôle of the various waves is examined, and compared with the well-known one dimensional theories. For small disturbances, a far-field asymptotic expansion is constructed which, in the first approximation, gives an equation incorporating non-linearity, damping and dispersion (the Korteweg-de Vries-Burgers equation). This same problem using a non-Newtonian stress law is discussed in Appendix I.

This new equation is examined in the steady-state, and for small damping a two-time asymptotic solution derived. The solution describes a surface profile which slowly changes from a solitary wave to cnoidal wave oscillations at a higher level. These waves decrease in amplitude and period so that eventually the surface level has changed, as is observed in the undular bore of some rivers. The asymptotic solution is compared with a numerical integration of the equation, the agreement being excellent as the damping coefficient approaches

zero.

Having reconsidered water wave theory, and incorporated two dimensional and viscous effects, the problem of wave propagation in an elastic tube filled with fluid is discussed in Part II. As a first simple theory, the inviscid problem with a Hooke's law describing the tube wall (which slowly tapers) is examined. The flow is assumed to have no shear across the radius, and the initial disturbance is small. However, eventually non-linearity becomes important and shocks may form. The results obtained agree with Whitham's theory for propagation in ray tubes of varying area.

The final chapter is concerned with a far more realistic model for the tube. The wall is described by Timoshenko's thin shell theory and the fluid is assumed viscous. By tethering the outside of the tube to restrict all longitudinal motion, and producing a flow by means of a body force, an examination along the lines of Part I is presented. There are many points of similarity, but a far-field theory produces a fourth order non-linear equation if the density of the wall material is large enough. This equation is discussed as in Part I and it is shown that an instability can occur.

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Notation

Only non-dimensional physical variables are listed (except where otherwise stated), and the appropriate non-dimensionalisation is given in brackets.

Part I

ρ - density of fluid

μ - viscosity of fluid

$\nu = \mu/\rho$ - kinematic viscosity

L - characteristic wave length

h_0 - undisturbed depth of channel

(g_1, g_2) - components of gravity

ε - amplitude parameter of motion

x - co-ordinate along channel bed (L)

y - co-ordinate upwards from bed (h_0)

t - time co-ordinate ($L/\sqrt{g_2 h_0}$)

$h(x, t)$ - disturbed surface profile (h_0)

$\bar{u}$ - x -component of velocity ($\sqrt{g_2 h_0}$)

$\bar{v}$ - y -component of velocity ($\frac{h_0}{L}\sqrt{g_2 h_0}$)

$\bar{p}$ - pressure relative to atmospheric ($\rho g_2 h_0$)

$U(y)$ - mean velocity profile ($\sqrt{g_2 h_0}$)

u - velocity perturbation, $\bar{u} = U(y) + \varepsilon u$

v - velocity perturbation, $\bar{v} = \varepsilon v$

p - pressure perturbation, $\bar{p} = 1 - y + \varepsilon p$

η - surface perturbation, $h = 1 + \epsilon \eta$

$\delta = h_0/L$, depth parameter

$\alpha = 3L/4h_0$, effect of slope

$Re = \frac{\sqrt{3}h_0 L}{\nu}$, flow Reynolds number

$F = \alpha \delta^2 Re$, Froude number

$c = c_1, c_2$, Burns propagation velocities

Δ - general small parameter

$\kappa = \Delta^2 \delta^2 / \epsilon$,

$R = Re/\Delta$

$$\left. \begin{aligned} X &= \Delta x \\ T &= \Delta t \\ V &= v/\Delta \end{aligned} \right\} \text{new scaled variables}$$

$\lambda = (\Delta Re \delta^2)^{-1}$ - parameter giving effect of viscous term

$\xi = X - cT, X - FT$, characteristic co-ordinates

$$\left. \begin{aligned} T &= \lambda^{1/2} T \\ Y &= y \lambda^{-1/2} \\ V &= V \lambda^{-1/2} \end{aligned} \right\} \text{boundary layer scaling}$$

$\zeta = -\frac{1}{c} \xi$

$H(\zeta, T) = -\frac{1}{c} \eta(\xi, T)$

$\tau = \epsilon T$

$W = U(y) - c$

$I_n(y) = \int_0^y W^{-n} dy, I_n = I_n(1)$

$\tau_1 = 2F\sqrt{2/\kappa} \tau$

$$\xi_1 = \sqrt{2/\kappa} \xi$$

$$S = (\xi_1 - \frac{1}{2} \eta_\infty \tau_1) / 2 \sqrt{2/\eta_\infty} - \text{steady-state variable}$$

$$R_1 = \frac{R}{3F} \sqrt{\kappa/2} (1 - \frac{2}{5} F^2)$$

$$v(s) = \eta_\infty^{-1} \eta(s)$$

$$\gamma = 2R_1 \sqrt{2/\eta_\infty} - \text{steady-state damping parameter}$$

$$\sigma = \gamma S$$

cn, sn, dn , Jacobian elliptic functions

$$\left. \begin{array}{l} a(\sigma) \\ b(\sigma) \\ \alpha(\sigma) \\ \nu(\sigma) \end{array} \right\} \text{ defined by } v(s) = a(\sigma) + b(\sigma) cn^2[\alpha(\sigma)S; \nu(\sigma)]$$

Part II

ρ - density of fluid

μ - viscosity of fluid

$\nu = \mu/\rho$ - kinematic viscosity

ω - characteristic frequency

R_H' - a mean radius

E - Young's modulus for the wall material

σ - Poisson's ratio

$$\tau = (1 - \sigma^2)^{-1}$$

$$B = E \tau$$

h - thickness of tube

ρ_0 - density of tube

$$\alpha = (E h / 2 \rho R_H')^{1/2} - \text{Moens propagation velocity}$$

ξ - amplitude parameter of motion

- z - co-ordinate along axis (a/ω)
 r - radial co-ordinate (R_n')
 t - time co-ordinate ($1/\omega$)
 R_0 - undisturbed radius of tube (R_n')
 $(= R_n'$ for no taper)
 R - disturbed radius of tube (R_n')
 A_0 - undisturbed area of tube ($R_n'^2$)
 A - disturbed area of tube ($R_n'^2$)
 $\bar{u}$ - z -component of velocity (a)
 $\bar{v}$ - r -component of velocity ($\omega R_n'$)
 $\bar{p}$ - pressure relative to undisturbed pressure (ϵa^2)
 $U(r)$ - mean velocity profile (a)
 u - velocity perturbation, $\bar{u} = U(r) + \epsilon u$
 v - velocity perturbation, $\bar{v} = \epsilon v$
 p - pressure perturbation, $\bar{p} = \epsilon p$
 η - radial tube perturbation ($\epsilon R_n'$)
 ζ' - longitudinal tube perturbation (dimensional)
 θ - local slope of the wall
 $Re = a^2/\nu\omega$ - flow Reynolds number
 $\delta = \omega R_n'/a$ - frequency parameter
 $\alpha = g/a\omega$ - body force parameter
 $F = \frac{1}{4}\alpha\delta^2 Re$ - Froude number
 $K = \delta^2/\epsilon$
 C - wave propagation velocity
 $\Delta = Re\tau/F$

$$\beta = \rho_0 h^2 / \rho R_n'$$

$$\gamma = \beta / 2\tau$$

$$s(z) = \varepsilon^{-1} R_0' \frac{dR_0}{dz}$$

$$s(z) = s(z/\varepsilon)$$

$$\xi = t - \gamma, \quad \gamma = F t, \quad \text{characteristic co-ordinates}$$

$$T = \varepsilon t \quad - \text{large time co-ordinate}$$

$$Z = \varepsilon \gamma \quad - \text{large distance co-ordinate}$$

$$X = \xi - \frac{kF}{\delta} T \quad - \text{steady-state co-ordinate}$$

$$h(x) = \frac{16}{k} \eta(\xi, T)$$

$$x = \Delta X$$

$$\left. \begin{array}{l} M_z, Q_z, T_z, T_\theta, \\ M_\theta, N_{z\theta}, T_{z\theta} \end{array} \right\} \text{moments and loads on element of tube (see } \\ \text{figure 13), dimensional}$$

Introduction

Fluid flow in elastic tubes has long been studied in an effort to gain a better understanding of the flow in the mammalian arterial system. As long ago as 1878, Moëns derived the simple formula for the speed of propagation, α , of pressure pulses along the tube ;

$$\alpha = \sqrt{\frac{Eh}{2\rho R_n'}} ,$$

where E , h , R_n' are respectively the Young's modulus, the thickness and the radius of the tube, and ρ the density of the fluid. This relation is valid for thin walled tubes ($h/R_n' \ll 1$). (Lambossy 1950,1951 gives an excellent review of the subject.) This particular propagation velocity is the dominant one for common materials. Two other important propagation modes, one in the fluid alone and one in the tube alone, are in general of much greater speed than α . Since only incompressible fluids will be considered, propagation in the fluid will be assumed to have an infinite velocity throughout this thesis.

More recently, numerous methods have been employed to examine the fluid-filled elastic tube. The bulk of these analyses are linearised theories, and no real attempt has been made to apply the relatively new advances in perturbation methods and non-linear wave theory. In particular, an excellent study of the linearised equations has been

made by Womersley (1957). This paper considered a simple thin shell theory to describe the wall and found suitable corrections to the wave velocity and flow rate due to viscosity, additional wall mass, boundary layer at the wall and non-linearity (amongst many other calculations). Three years earlier, Morgan and Kiely (1954) took a similar model and examined the effects of fluid viscosity on periodic waves. Both these papers considered only linearised small disturbance theories, and no analysis was given of any cumulative effects that non-linearity might be expected to produce.

In 1958, Lambert carried out a numerical integration of the full equations assuming a simple Hooke's law at the wall, the pressure change at the wall being made proportional to the change of radius. This assumption is also at the heart of thin shell theory, but in that theory the cross-coupling of radial and longitudinal strain are included by using a non-zero Poisson's ratio. Olsen and Shapiro (1967) have also integrated the full equations numerically, and used a complicated (non-linear) wall relation. They compared their results with an approximate non-linear theory requiring a judicious choice of the wall relation. Replacing the fluid viscosity by a wall friction, they obtained fair agreement between the numerical computations, their theory and some experimental results. Unfortunately, neither the experimental data nor the numerical integration went far enough to eluci-

date the possible formation of shock waves in the system. Finally, Seymour and Varley (1966) went one stage further and included the effects of viscoelasticity. They developed a non-linear wave theory, with a tube taper incorporated, and used an empirical integral relation defining the wall-pressure law. The final conclusions of the paper give some general comments on the peak pressure pulses obtainable in such fluid-tube systems.

The work described in this thesis uses standard perturbation methods to extend the knowledge that can be obtained about fluid flow in an elastic tube, from the theoretical standpoint. The general aim is to consider initial small disturbances at one end of the elastic tube (assumed semi-infinite), and to study the consequent wave motion along the tube. By small we mean to imply that initially the dominant equations of motion are linear, but any cumulative effect of non-linearity will be included causing the wave to eventually steepen and form a shock front. The steepening, however, can be limited by the presence of damping and/or dispersive effects in the propagating medium. Such effects are common features of non-linear wave motions and might be expected to be present in this particular physical situation.

Unfortunately, the complexity of the wall boundary conditions for the elastic tube makes the analysis rather com-

plicated and tends to obscure the basic features of the wave motion. Initially, attempts were made to simplify the problem by ad hoc or empirical approximations which, although leading to qualitatively correct descriptions of the flow, were regarded as unsatisfactory. It was realised, however, that a similar problem to that posed by the elastic tube was the motion of water waves on a two dimensional channel. The surface boundary conditions in this case are much simpler and the essential structure of the wave propagation easier to demonstrate. With this in mind, the water wave problem is re-examined in Part I of this thesis and some new results are derived. The same techniques are then applied to the elastic tube in Part II and many similarities will be noted.

The classical theories of water waves in channels (Lamb, 1958) lead to two distinctly different types of waves. One is the well-known discontinuity in surface level and velocity, often referred to as the 'hydraulic jump' or 'bore'. The other is a non-linear oscillatory motion described mathematically by one of the Jacobian elliptic functions and hence called the 'cnoidal wave'. This wave motion was first found by Korteweg and de Vries (1895) by making some special assumptions about the relative sizes of the depth of the channel, the wavelength of the motion and its amplitude.

A phenomenon often observed on rivers, especially during flood tides, is that of a bore followed by a train of waves - the so-called 'undular bore'. This particular motion has been studied by Benjamin and Lighthill (1954) who combined the two classical types of waves by suggesting that only some of the necessary energy need be dissipated at the front (bore). The rest of the energy is then transported away by a train of cnoidal waves.

To verify the nature of this undular bore, Chester (1966) considered the perturbation of a fully developed channel flow. He showed that a Taylor profile of the surface could give rise to oscillatory motion upstream, but a monotonic profile far downstream. This essentially describes the observed form of the undular bore (see Favre, 1935; Sturtevant, 1965; Sandover and Taylor, 1962). It is shown here that a single equation can be found which describes this phenomenon and a similar equation is derived for the wave propagation on the elastic tube.

The problem, in detail, is the study of small disturbances on a fully developed (Poiseuille) channel flow moving under gravity. A small disturbance will initially travel according to a linear wave equation. The cumulative effects of non-linearity will, however, so distort the wave profile that far away from the initial disturbance the linear theory is inadequate. It is shown that at first the wave moves, in

particular, according to the well-known linear theory of Burns (1953). In terms of the analysis presented by Lighthill and Whitham (1955), these waves are called 'dynamic' waves. Lighthill and Whitham show that for flood movement in long rivers their theory indicates that the dynamic waves decay leaving the main disturbance moving at the 'kinematic' wave velocity. This theory assumed a one dimensional flow together with a Chézy law for the friction on the channel bed.

The present work confirms that in suitably scaled large time and distance co-ordinates, the disturbance does indeed travel at the kinematic wave speed. To verify that the dynamic waves decay, the full equations are scaled to produce the appropriate terms which describe both the dynamic and kinematic waves. These equations can not be reduced to a single equation as was obtained by Lighthill and Whitham (1955). However, it is shown here that, by making use of simple boundary layer ideas, ~~that~~ a two dimensional model of the flow gives the result that the dynamic waves decay.

Having shown that the dynamic waves decay, attention may be focussed on the kinematic wave alone. By considering co-ordinates moving with this wave front, together with a large time co-ordinate, the effects of non-linearity, dispersion and damping on the wave profile can be described by an equation of the form

$$\eta_\tau + \eta \eta_x + \alpha \eta_{xxx} = \beta \eta_{xx}.$$

This is called the Korteweg-de Vries-Burgers (K-V-B) equation. In the limit $\alpha \rightarrow 0$ the equation reduces to Burgers equation (Burgers, 1948) which can be solved exactly by reducing it to the heat conduction equation (Hopf, 1950; Cole, 1951). The steady-state solution is the so-called Taylor shock profile. If the limit $\beta \rightarrow 0$ is taken then the equation becomes that due to Korteweg and de Vries (1895) to describe a special class of water waves. This equation is only just becoming amenable to analytic treatment (see Miura, 1970), but the steady-state solution gives the cnoidal waves mentioned previously. It is clear that the full K-V-B equation is probably the simplest that incorporates the effects of non-linearity ($\eta \eta_x$), damping ($\beta \eta_{xx}$) and dispersion ($\alpha \eta_{xxx}$), and as such is a very interesting equation to study in its own right.

The derivation of the equation follows directly by applying a formal expansion procedure to the full equations. The maximum generality is retained in the procedure by scaling the Reynolds number in such a way that the dominant damping terms remain throughout the analysis.

The K-V-B equation is studied in some detail and it is

shown that in the steady-state it exhibits all the characteristics of the undular bore. The steady-state form is firstly examined qualitatively in the phase-plane so that the general nature of the solution is obtained. It is then solved for small damping by making use of both averaging and perturbation methods (Johnson, 1970). The oscillatory solution obtained by the averaging procedure is successfully matched to a perturbation of the 'solitary wave'. This wave is well-known in water wave theory and is the non-oscillatory limit of the cnoidal wave. The final solution is then a wave of varying period passing through all the types of cnoidal wave, from the single solitary wave to the zero amplitude zero period cnoidal wave. The peak envelopes obtained from this theory are computed and compared with the results of numerical integrations of the full steady-state K-V-B equation. The agreement is such that the asymptotic theory for small damping (damping parameter $\rightarrow$ zero) is fully confirmed.

A closely related problem to those discussed in this thesis is the study of weak shock profiles in nearly collisionless plasmas. Such flows have much in common with classical water wave theory, and Grad and Hu (1967) have shown that the steady-state K-V-B equation can be used to describe the shock profile.

Since, in practice, many channel flows can not be fully described by the laminar Poiseuille velocity distribution

of the Navier-Stokes equations, a more general case is considered in Appendix I. With all the uncertainties associated with the turbulent motion usually encountered in such flows, it is difficult to postulate a satisfactory set of equations to describe the motion. The main problem is to construct a realistic mean flow. By introducing a generalisation of the stress law for the mean flow, it is shown that the techniques used in Part I can lead to essentially the same results.

Having examined and extended the work on two dimensional channel flows, the analysis of the elastic tube is undertaken in Part II. It is shown that many of the important features of the water wave problem also occur here.

As an extremely simple introduction to this type of flow, the problem of an inviscid fluid with a small uniform velocity profile across the tube is considered. The tube is permitted a small taper and the wall of the tube is described by the simplest form of Hooke's law (change in radius $\propto$ change in pressure). The results obtained are similar to those of Whitham (1956) for wave propagation in ray tubes.

The elastic tube is then studied in more detail in an attempt to construct a physically more realistic model. The main complication introduced in the channel flow was to incorporate the two dimensional nature of the flow. Even though

this was done, the analysis did not prove too difficult mainly because there was a steady mean flow. To study the flow through an elastic tube in a similar manner it is necessary to introduce a mean flow here, also. Now since the tube is elastic this can only be accomplished by making use of a body force aligned with the tube axis. Such a situation is rather infrequent for the human arterial system since pressure differences play an important rôle in this type of flow. The effect of a pressure gradient being the cause of the mean flow is clearly to produce a taper in the elastic tube. This in turn implies that a steady flow can not be produced. The situation of an unsteady mean flow is a little too difficult to handle at the present time. However, the advantages gained by including a steady average flow produced by a body force are thought to outweigh the obvious limitations of applying the theory directly to flows in mammalian arteries.

This work, then, considers the problem of a semi-infinite elastic tube of circular cross-section, no taper and with a steady flow of a viscous fluid moving through it under the action of a body force aligned with the axis of the tube. The wall of the tube can move both radially and longitudinally, but for simplicity it is assumed that all longitudinal motion is totally inhibited by some restraining mechanism outside the tube. In terms of an artery, this is the well-known phenomenon of 'tethering' (McDonald, 1960). This tethering

is assumed not to effect the radial motion at all. The tube itself is described by the thin shell theory of Timoshenko (1940) and the relevant results are derived here from first principles. The problem is then to examine the propagation of small disturbances to this fluid-tube system.

Analyses are presented for two extreme cases - one for infinite Reynolds number and the other for Reynolds number order unity. The small disturbance inviscid problem reduces to a set of equations analogous to those obtained by Burns (1953) for the water wave problem (see Part I). Consequently an expression is found for the propagation speed of the disturbance which incorporates the effects of a general shear flow in the tube and suitably alters the classical Moens value. The order unity Reynolds number case is reasonably close to typical values of the parameters encountered in arterial flow. For a particular choice of the order of magnitude of the frequency parameter (see Korteweg and de Vries, 1895) it is shown that the main disturbance travels only in one direction and can be described by a K-V-B type equation near the front. These results are exactly equivalent to the kinematic wave theory derived for the channel flow.

A slightly more general form of the K-V-B equation is obtained which includes a wall inertia parameter, β ,

$$\eta_{\tau} + \eta \eta_x + \eta_{xxx} = \Delta (\eta_{xx} + \beta \eta_{xxxx}).$$

$\beta=0$ corresponds to the K-V-B equation described earlier. Using similar techniques to those already implemented, an asymptotic solution is derived for $\Delta \ll 1$. It is found that an upper limit on β is necessary to ensure a stable profile. This conclusion is in full agreement with the work of Jacobs (1953) who also retained the wall inertia terms, but limited his analysis to the linearised problem.

Part I

Shallow water wave theory

and

the solution of the

Korteweg-de Vries-Burgers equation

1. Shallow water wave theory

This first chapter is concerned with the various types of small amplitude waves that can occur on the surface of shallow two dimensional channels. The possible importance of non-linear effects will not be ignored, and to this end formal scalings and asymptotic expansions will be used.

1.1 The equations and boundary conditions

The motion will be assumed to occur on a viscous incompressible fluid in two dimensions and the governing equations will be assumed to be the classical Navier-Stokes equations. The co-ordinates are given in figure 1. x' is measured along the bed of the channel; y' upwards from the bed; u' and v' are the x' - and y' -components of velocity, respectively. The undisturbed depth is h_0 , and disturbed $h'(x', t')$, where t' is a time co-ordinate. The mean velocity distribution is denoted by $U'(y')$, and the components of gravity are $(g_1, -g_2)$. The equations of motion are thus

$$u'_t + u'u'_{x'} + v'u'_{y'} + \frac{1}{\rho} p'_{x'} = g_1 + \nu(u'_{x'x'} + u'_{y'y'}) \quad (1.1)$$

$$v'_t + u'v'_{x'} + v'v'_{y'} + \frac{1}{\rho} p'_{y'} = -g_2 + \nu(v'_{x'x'} + v'_{y'y'}) \quad (1.2)$$

where p' is the pressure, ρ the density and ν the constant viscosity of the fluid. The equation of continuity is

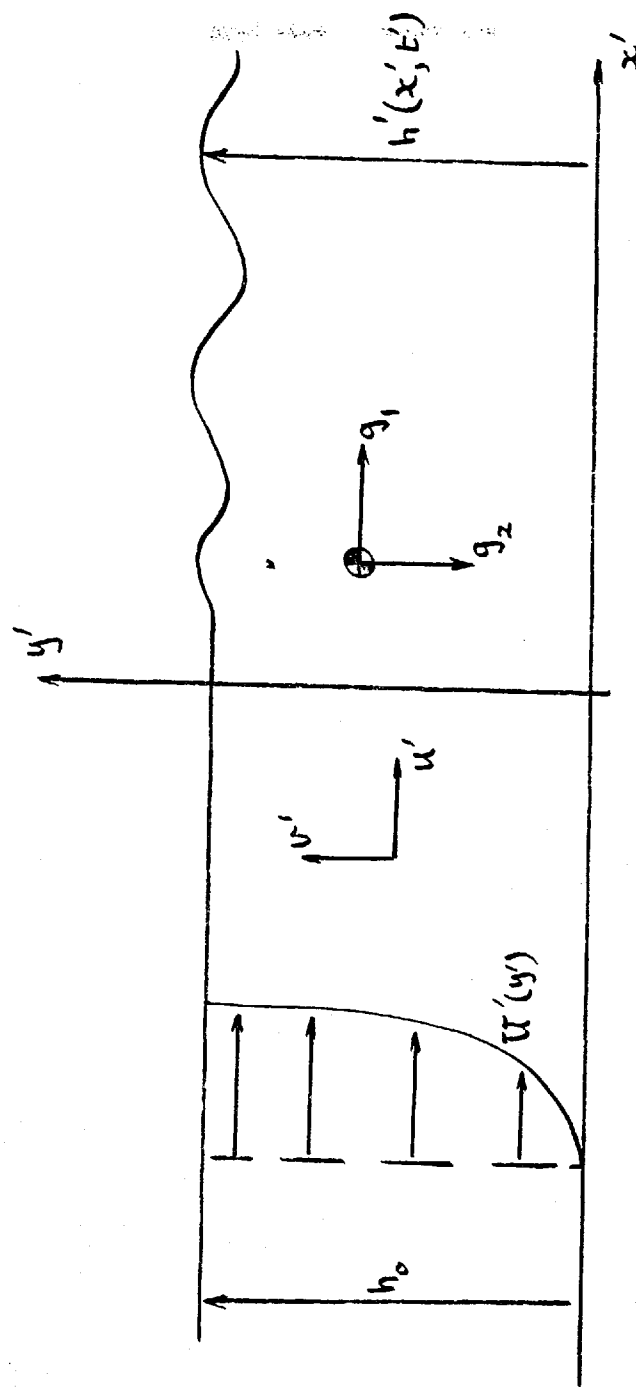


Figure 1 - Co-ordinates and variables for the channel flow

$$u'_{x'} + v'_{y'} = 0. \quad (1.3)$$

The boundary conditions on the bed of the flow are that no slip occurs there and that the bottom is a streamline,

$$u' = v' = 0 \quad \text{on} \quad y' = 0. \quad (1.4)$$

At the surface, the surface velocity and fluid velocity are continuous,

$$v' = h'_t + u'h'_{x'} \quad \text{on} \quad y' = h'(x', t'), \quad (1.5)$$

and the stress at the surface is continuous. This implies that the stress normal to the surface (τ_n) is constant (given by the atmospheric pressure) and that the stress tangential to the surface (τ_t) is zero. Thus we have

$$\left. \begin{array}{l} \tau_n = \text{constant} = -p'_a \\ \tau_t = 0 \end{array} \right\} \quad \text{on} \quad y' = h'(x', t') \quad (1.6)$$

where

$$\left. \begin{array}{l} \tau_n = \tau'_{yy'} \cos^2 \theta + \tau'_{xx'} \sin^2 \theta - \tau'_{xy'} \sin 2\theta \\ \tau_t = \tau'_{xy'} \cos 2\theta + \frac{1}{2} (\tau'_{yy'} - \tau'_{xx'}) \sin 2\theta \end{array} \right\} \quad (1.7)$$

and

$$\left. \begin{aligned} \tau_{x'x'} &= -p' + 2\mu u'_{x'}, \tau_{y'y'} = -p' + 2\mu v'_{y'}, \\ \tau_{x'y'} &= \mu(u'_{y'} + v'_{x'}) \text{ with } \tan \theta = h'_{x'}. \end{aligned} \right\} \quad (1.7)$$

μ is the coefficient of Newtonian viscosity and the surface tension is put equal to zero. The pressure p'_a is the pressure of the atmosphere above the fluid surface.

It is convenient at this stage to non-dimensionalise all the variables according to the scheme

$$\left. \begin{aligned} x &= \frac{x'}{L}, \quad y = \frac{y'}{h_0}, \quad t = \frac{\sqrt{g_2 h_0}}{L} t', \quad h = \frac{h'}{h_0}, \\ \bar{p} &= \frac{p' - p'_a}{\rho g_2 h_0}, \quad \bar{u} = \frac{u'}{\sqrt{g_2 h_0}}, \quad \bar{v} = \frac{L v'}{h_0 \sqrt{g_2 h_0}}, \quad \bar{u} = \frac{u'}{\sqrt{g_2 h_0}}, \end{aligned} \right\} \quad (1.8)$$

where a characteristic wavelength of the motion, L , and a characteristic velocity, $\sqrt{g_2 h_0}$, have been introduced. For a wave motion characterised by the parameter ϵ , we write

$$h = 1 + \epsilon \eta, \quad \bar{u} = \bar{U}(\eta) + \epsilon u, \quad \bar{v} = \epsilon v, \quad \bar{p} = 1 - \eta + \epsilon p, \quad (1.9)$$

where the pressure has been written as the perturbation from the hydrostatic pressure distribution and the velocity as the perturbation from the mean profile, $\bar{U}(\eta)$. Now writing

(1.8) and (1.9) into (1.1 - 1.7), and combining (1.6) and (1.7), gives the following non-dimensional scaled equations:

$$u_t + (\bar{u} + \varepsilon u)u_x + (\bar{u}' + \varepsilon u_y)v + p_x = \frac{\alpha}{\varepsilon} + \frac{1}{\varepsilon Re} \left[\varepsilon u_{xx} + \frac{\bar{u}'' + \varepsilon u_{yy}}{\delta^2} \right] \quad (1.10)$$

$$\delta^2 [v_t + (\bar{u} + \varepsilon u)v_x + \varepsilon v v_y] + p_y = \frac{\delta^2}{Re} \left[v_{xx} + \frac{1}{\delta^2} v_{yy} \right] \quad (1.11)$$

$$u_x + v_y = 0 \quad (1.12)$$

$$\bar{u} + \varepsilon u = v = 0 \quad \text{on} \quad y = 0 \quad (1.13)$$

$$v = \eta_t + (\bar{u} + \varepsilon u)\eta_x \quad (1.14)$$

$$1 - y + \varepsilon p = \frac{2\varepsilon [v_y + \varepsilon \delta^2 \eta_x^2 u_x - \eta_x (\bar{u}' + \varepsilon u_y + \varepsilon \delta^2 v_x)]}{Re (1 + \varepsilon^2 \delta^2 \eta_x^2)} \quad \left. \begin{array}{l} \text{on} \\ y = 1 + \varepsilon \eta \end{array} \right\} \quad (1.15)$$

$$(\bar{u}' + \varepsilon u_y + \varepsilon \delta^2 v_x)(1 - \varepsilon^2 \delta^2 \eta_x^2) - 2\varepsilon \delta^2 \eta_x (u_x - v_y) = 0 \quad (1.16)$$

In (1.10 - 1.16) we have introduced the parameters

$$\delta = \frac{h_0}{L}, \quad \alpha = \frac{g_1 L}{g_2 h_0}, \quad Re = \frac{\sqrt{g_2 h_0} L}{\nu}, \quad (1.17)$$

and the primes denote differentiation with respect to y .
The parameter δ indicates the 'shallowness' of the channel
and Re is the Reynolds number for the flow.

To obtain the fully developed flow in the limit $\varepsilon \rightarrow 0$, then we must have

$$\alpha + \frac{1}{\delta^2 Re} u'' = 0$$

which gives

$$u = F(y - \frac{1}{2} y^2). \quad (1.18)$$

This is the Poiseuille profile which satisfies the conditions of no shear at the surface ($y=1$) and no slip at the bottom ($y=0$). F is a Froude number defined by

$$F = \alpha \delta^2 Re = O(1) \quad (1.19)$$

which serves to define the size of α so that a fully developed flow is possible with F of order one. We note, for future reference, that this Froude number is exactly the one defined by Chester (1966),

$$F = \frac{g_1 L}{g_2 h_0} \cdot \frac{h_0^2}{L^2} \cdot \frac{\sqrt{g_2 h_0} L}{\nu} = \frac{U_0}{\sqrt{g_2 h_0}},$$

where $U_0 = \frac{g_1 h_0^2}{\nu}$ is the velocity used by Chester to give a stationary wave profile.

The velocity profile (1.18) is of the classical form when the fluid has been described by the Newtonian stress law. A more general way of describing the stress producing

the mean flow is given in Appendix I, which leads to differing forms for the velocity profile (~~e.g. the one-seventh power law~~).

1.2 The linearised problem

The equations (1.10 - 1.19) that have been developed in §1.1 are quite general. To proceed further it is necessary to say something about the relative sizes of the three parameters ε , δ and Re . The problem we are interested in here is such that

$$\varepsilon \ll 1, \delta \ll 1, Re \gg 1. \quad (1.20)$$

These statements, in turn, indicate that the initial disturbance to the flow is small, that the channel is shallow and finally that viscous effects do not dominate (for x and t order unity). We observe in (1.10 - 1.19) that the limiting process $\varepsilon \rightarrow 0$, δ and Re fixed, leads to the full linearised equations; and that $Re \rightarrow \infty$, ε and δ fixed, gives the inviscid problem. The situation in which $Re \rightarrow \infty$ and $\varepsilon = O(\delta^2)$ has been analysed in detail by Freeman and Johnson (1970) who derive the Korteweg-de Vries equation for the disturbance over a general shear flow. If δ and Re satisfy (1.20) and are such that

$$\delta^2 Re \gg 1, \quad (1.21)$$

then the limit $\varepsilon \rightarrow 0$, $\delta \rightarrow 0$ reduces the equations to those studied by Burns (1953) for the propagation of small disturbances on a general (inviscid) shear flow. The dominant equations are then

$$\left. \begin{aligned}
 u_t + U u_x + U' v + p_x &= 0 \\
 p_y &= 0 \\
 u_x + v_y &= 0 \\
 v = 0 \quad \text{on} \quad y = 0 \\
 v = \eta_t + U \eta_x \\
 p = \eta \\
 u_y + U'' \eta &= 0
 \end{aligned} \right\} \quad \text{on } y = 1 \quad (1.22)$$

Seeking a steady wave solution of (1.22) it is straightforward to show that

$$\left. \begin{aligned}
 \eta &= \eta(x - ct), \\
 u &= -\eta \frac{d}{dy} (w I_2); \quad I_n = \int_0^y \frac{dy}{w^n}; \quad w = U(y) - c,
 \end{aligned} \right\} \quad (1.23)$$

where C is the wave speed. C is given by the surface kinematic condition as the classical integral relation

$$\int_0^1 \frac{dy}{[U-c]^2} = 1. \quad (1.24)$$

The difficulties that arise when $U = c$ in the range of y have been considered by Velthuisen and Wijngaarden (1969). However, for $U(y)$ as the Poiseuille distribution there are no problems and (1.24) can be integrated to give

$$\frac{2}{a^2} \left[\frac{1}{a^2+1} + \frac{1}{a} \tan^{-1} \left(\frac{1}{a} \right) \right] = F^2 \quad \text{for } a > 0, \frac{C_1}{F} = \frac{a^2+1}{2} \quad (1.25)$$

and

$$\frac{1}{b^2} \left[\frac{1}{b} \ln \left| \frac{b+1}{b-1} \right| + \frac{2}{b^2-1} \right] = F^2 \quad \text{for } b > 1, \frac{C_2}{F} = \frac{1-b^2}{2}. \quad (1.26)$$

The two roots, C_1 and C_2 , of (1.24) thus satisfy the conditions given in Burns (1953), namely that

$$C_1 > U(1) = \frac{1}{2} F \quad \text{and} \quad C_2 < U(0) = 0.$$

We have thus seen that in the régime of the flow where x and t are of order unity (and with the parameters satisfying (1.20, 1.21)), the disturbance travels as two simple waves. These are in opposite directions and at the wave speeds given above, C_1 and C_2 .

If we now return to the full equations (1.10 - 1.19) and

rescale according to

$$\Delta x = X, \Delta t = T, \sigma = \Delta V, \quad (1.27)$$

so that $\Delta \rightarrow 0$ corresponds to large x and t , then for ε , δ and R_e fixed, the limit $\Delta \rightarrow 0$ gives

$$\left. \begin{aligned} u_{yy} &= 0 \\ u_x + v_y &= 0 \\ u = v &= 0 \quad \text{on } y = 0 \\ \left. \begin{aligned} v &= \eta_T + u \eta_x \\ u_y + u'' \eta &= 0. \end{aligned} \right\} & \text{on } y = 1 \end{aligned} \right\} \quad (1.28)$$

These relations can easily be solved to give

$$\left. \begin{aligned} u &= F \eta y, \quad v = -F \eta_x \cdot \frac{1}{2} y^2 \\ \eta &= \eta(x - F T). \end{aligned} \right\} \quad (1.29)$$

where

The wave described by (1.29) was named the 'kinematic' wave by Lighthill and Whitham (1955) in connection with a study on the flood movement in rivers. By assuming a one dimensional motion and including a friction term given by a Chézy law[†], Lighthill and Whitham construct an equation which includes all three propagation speeds (denoted here by c_1 , c_2 and F). This equation is of the form discussed by Whitham

[†] A comment on this analysis is given in § A.5, Appendix I.

(1959) and shows that for the channel flow problem the disturbances travelling at the dynamic wave speeds (C_1 and C_2) decay, leaving the main disturbance to travel at the kinematic wave speed (F). This conclusion appears to be borne out here where the limit $x, t \rightarrow \infty$ reduces the problem to the set of equations (1.28) with the kinematic wave solution (1.29).

To ensure that the main disturbance for the more complicated two dimensional problem does indeed travel at the kinematic wave speed, a closer examination of the equations must be made. Once this has been done, and the dynamic waves shown to decay, a detailed picture of the profile close to the kinematic wave front can be deduced.

1.3 The decay of the dynamic waves

Using the scaling (1.27) of § 1.2 and writing

$$\lambda = (\Delta R_e \delta^2)^{-1}$$

then the limit $\Delta \rightarrow 0$, $\varepsilon \rightarrow 0$ and retaining $O(1)$ and $O(\lambda)$ terms, gives the linearised set of equations which includes the appropriate terms for both the dynamic and kinematic waves. In other words we ensure that the dominant viscous effect is retained in (1.10 - 1.16) giving

$$\left. \begin{aligned}
 u_T + U u_x + U' v + p_x &= \lambda u_{yy} \\
 p_y &= 0 \\
 u_x + v_y &= 0 \\
 u = v = 0 &\quad \text{on } y = 0 \\
 \left. \begin{aligned}
 v &= \eta_T + U \eta_x \\
 p &= \eta \\
 u_y + U'' \eta &= 0.
 \end{aligned} \right\} &\quad \text{on } y = 1
 \end{aligned} \right\} \quad (1.30)$$

This set of equations corresponds to the single one dimensional equation deduced by Lighthill and Whitham (1955). In the present case, (1.30) can not readily be reduced to a single equation for $\eta(x, T)$. However, it is possible to obtain an asymptotic solution for $\lambda \ll 1$ (the scaling Δ is thus such that $\Delta Re \delta^2 \gg 1$). With this assumption it is now possible to examine the effect of the viscous term (λu_{yy}) on the dynamic waves. This is best done by looking in particular at the dynamic wave fronts. Thus introducing a characteristic co-ordinate measured from the dynamic wave front (ξ) and a new large time co-ordinate (T), where

$$\xi = x - cT, \quad \lambda^{1/2} T = T$$

and C is given by (1.25) or (1.26), then (1.30) become

$$\left. \begin{aligned} \lambda^{\frac{1}{2}} u_{\tau} + (u-c) u_{\xi} + u' v + \eta_{\xi} &= \lambda u_{\eta\eta} \\ u_{\xi} + v_{\eta} &= 0 \\ u = v = 0 &\quad \text{on } \eta = 0 \\ v = \lambda^{\frac{1}{2}} \eta_{\tau} + (u-c) \eta_{\xi} \\ u_{\eta} + u'' \eta &= 0. \end{aligned} \right\} \quad \text{on } \eta = 1 \quad (1.31)$$

We shall now show that the effect of retaining the (λ) term is to enable the dynamic waves to decay, leaving only the kinematic wave to carry the main disturbance. Eliminating

u_{ξ} and integrating across the depth of the channel, (1.31) is reduced to

$$\lambda^{\frac{1}{2}} \left[\int_0^1 \frac{u_{\tau}}{w^2} d\eta - \lambda^{\frac{1}{2}} \int_0^1 \frac{u_{\eta\eta}}{w^2} d\eta - \frac{\eta_{\tau}}{w_1} \right] = \eta_{\xi} (1 - I_{21}), \quad (1.32)$$

where we have used the notation of (1.23) and $w_1 = w(1)$, $I_{21} = I_2(1)$.

It is now necessary to obtain $u(\xi, \tau, \eta)$ valid across the depth of the channel. Clearly, for $\eta = O(1)$, then $\lambda \rightarrow 0$ indicates that u is given by the Burns theory (1.23) so that

$$u = -\eta [w I_2]' , \quad \eta = O(1). \quad (1.33)$$

Examination of (1.31) shows that u exhibits the expected boundary layer behaviour near $\eta = 0$. This is necessary to re-

adjust the inviscid Burns theory to include a no-slip condition on $y=0$. Now rescaling (1.31) for

$$y = \lambda^{1/2} Y, \quad v = \lambda^{1/2} V$$

and taking the limit $\lambda \rightarrow 0$ gives

$$-c u_{\xi} + \eta_{\xi} = u_{YY}$$

$$u = 0 \quad \text{on} \quad Y = 0$$

$$u \rightarrow \frac{\eta}{c} \quad \text{as} \quad Y \rightarrow +\infty.$$

This last condition ensures that u matches with the solution (1.33). Writing

$$u = \frac{\eta}{c} + w, \quad -\frac{\xi}{c} = \zeta$$

then the problem is transformed into

$$\left. \begin{aligned} w_{\zeta} &= w_{YY} \\ w &= -\frac{1}{c} \eta(\xi, \tau) = H(\zeta, \tau) \quad , \text{ say, on } Y=0. \end{aligned} \right\} \quad (1.34)$$

The solution to (1.34), using Duhamel's formulation, is

$$w = \frac{2}{\sqrt{\pi}} \int_0^{\infty} H\left(\zeta - \frac{Y^2}{4\sigma^2}, \tau\right) e^{-\sigma^2} d\sigma \quad (1.35)$$

which satisfies $w \rightarrow 0$ as $Y \rightarrow +\infty$ (exponentially). This result can be simplified when we note that ahead of the leading characteristics (for both $c > 0$ and $c < 0$) no disturbances occur. These regions are given by $\xi \geq 0$ ($c > 0$)

and $\xi < 0$ ($c < 0$) which can be combined as

$$H(\xi, \tau) \equiv 0, \quad \xi \leq 0 \quad (1.36)$$

so that

$$w = \frac{2}{\sqrt{\pi}} \int_{\frac{y}{2\xi^{1/2}}}^{\infty} H\left(\xi - \frac{y^2}{4\sigma^2}, \tau\right) e^{-\sigma^2} d\sigma.$$

Now using both (1.35) and (1.33) we can construct a composite (uniformly valid) solution for u ,

$$u = c H(w I_2)' + \frac{2}{\sqrt{\pi}} \int_0^{\infty} H\left(\xi - \frac{y^2}{4\lambda\sigma^2}, \tau\right) e^{-\sigma^2} d\sigma \quad (1.37)$$

for $0 \leq y \leq 1$, since the solution (1.35) becomes exponentially small as $y \rightarrow +\infty$. For convenience (1.37) is written in terms of $H(\xi, \tau)$ rather than $\eta(\xi, \tau)$.

The approximate equation ($\lambda \rightarrow 0$) for $H(\xi, \tau)$ may now be derived by using (1.37) in equation (1.32), ensuring that we retain only the dominant terms in the limit $\lambda \rightarrow 0$. It is straightforward to show that the contribution to the integral

$$\int_0^1 \frac{u_{\tau}}{w^2} dy$$

from (1.35) is $O(\lambda^{1/2})$. Integrating the contribution from (1.33) we obtain

$$\int_0^1 \frac{u_{\tau}}{w^2} dy = c H_{\tau} \left[2I_{31} - \frac{I_{21}}{w_1} \right] + O(\lambda^{1/2}). \quad (1.38)$$

The second integral is a little more complicated. Firstly we note that

$$\int_0^1 \frac{u_{yy}}{w^2} dy = \left[\frac{u_y}{w^2} \right]_0^1 - \left[\frac{u}{(w^2)'} \right]_0^1 + \int_0^1 \frac{u}{(w^2)''} dy$$

and as in (1.38) the final integral is comprised of terms of $O(1)$ and $O(\lambda^{1/2})$. Three of the other terms are $O(1)$ (or zero) when the evaluations on $y = 0, 1$ are performed. We are left with the one remaining term: $\left[-\frac{u_y}{w^2} \right]_{y=0}$. From (1.37)

we have that u_y can be rewritten such that

$$\begin{aligned} u_y &= c H(wI_2)'' + \frac{2}{\sqrt{\pi}} \int_0^\infty H_5\left(\beta - \frac{y^2}{4\lambda\sigma^2}, \tau\right) \left(-\frac{y}{2\lambda\sigma^2}\right) e^{-\sigma^2} d\sigma \\ &= c H(wI_2)'' - \frac{1}{\sqrt{\lambda\pi}} \int_0^\infty H_5(\beta - \tau, \tau) \cdot \frac{1}{\tau^{1/2}} e^{-\frac{y^2}{4\lambda\tau}} d\tau. \end{aligned}$$

Now evaluating on $y=0$, the dominant behaviour as $\lambda \rightarrow 0$ is

$$\int_0^1 \frac{u_{yy}}{w^2} dy \sim \frac{1}{c^2 \sqrt{\lambda\pi}} \int_0^\infty H_5(\beta - \tau, \tau) \cdot \frac{1}{\tau^{1/2}} d\tau. \quad (1.39)$$

The values of the integrals (1.38, 1.39) are substituted into the surface equation (1.32) and the fact that c satisfies the Burns condition $I_{2,1} = 1$, together with $\lambda \rightarrow 0$, leads to

$$H_{\tau} + \frac{\beta}{\sqrt{\pi}} \int_0^{\infty} H_{\zeta}(\zeta - \tau, \tau) \cdot \frac{1}{\tau^{1/2}} d\tau = 0. \quad (1.40)$$

The parameter $\beta = -(2c^3 I_3)^{-1}$ is always positive no matter which dynamic wave is being considered (e.g. $I_3 < 0$ for $c > 0$). Remembering that ahead of the leading characteristics ($\zeta - \tau \leq 0$) no disturbance to the surface occurs, then (1.40) can be written as

$$H_{\tau} + \frac{\beta}{\sqrt{\pi}} \int_0^{\zeta} H_{\zeta}(\zeta - \tau, \tau) \cdot \frac{1}{\tau^{1/2}} d\tau = 0. \quad (1.41)$$

Equation (1.41) can be solved by making use of the Laplace transform in τ . Writing the transform of $H(\zeta, \tau)$ as $\bar{H}(\rho, \tau)$ we obtain

$$\bar{H}_{\tau} + \frac{\beta}{\sqrt{\pi}} \rho \bar{H} \frac{\sqrt{\pi}}{\rho^{1/2}} = 0$$

so that

$$\bar{H} = \bar{f}(\rho) e^{-\beta \rho^{1/2} \tau}. \quad (1.42)$$

The arbitrary function $\bar{f}(\rho)$ is such that

$$\bar{H} \rightarrow \bar{f} \quad \text{as} \quad \tau \rightarrow 0,$$

or $H \rightarrow f(\zeta)$ as $\tau \rightarrow 0$. Again using the Faltung integral,

(1.42) can be written giving the final solution of equation (1.41) as

$$\left. \begin{aligned}
 H(\xi, T) &= \frac{\beta T}{2\sqrt{\pi}} \int_0^{\xi} f(\xi - \tau) e^{-\frac{\beta^2 T^2}{4\tau}} \tau^{-3/2} d\tau \\
 &= \frac{2}{\sqrt{\pi}} \int_{\frac{\beta T}{2\xi^{1/2}}}^{\infty} f\left(\xi - \frac{\beta^2 T^2}{4\sigma^2}\right) e^{-\sigma^2} d\sigma.
 \end{aligned} \right\} (1.43)$$

The solution given by (1.43) clearly decays as $T \rightarrow \infty$, and gives $H(\xi, 0) = f(\xi)$, which is some initial arbitrary disturbance. Thus we have satisfactorily shown that the two dimensional representation of the dynamic waves naturally leads to the conclusion that these waves decay.

1.4 The non-linear problem

Since the dynamic waves decay, the main disturbance must travel at the kinematic wave speed. We may thus limit our attention to the wave profile near to the kinematic wave front, and attempt to find a single equation to describe this profile in detail. In particular, it is hoped that such a model will be close to the observed nature of the undular bore (where the surface profile passes from one level to another with undulations at the upper level).

We shall develop an asymptotic solution based on the amplitude parameter, $\mathcal{E}$, and retain the dominant viscous terms by appropriately scaling the Reynolds number with respect to Δ .

As in § 1.2 we introduce

the scaled variables

$$\Delta x = X, \Delta t = T, \sigma = \Delta V$$

and we choose

$$\Delta^2 \delta^2 = \kappa \varepsilon ; \kappa = O(1) \quad (1.44)$$

which we may always do. To retain the viscous effects throughout the analysis, we write

$$\frac{R\varepsilon}{\Delta} = R \quad (1.45)$$

and treat R as independent of the limiting process. Taking the limit $\varepsilon \rightarrow 0$, R fixed, reproduces the kinematic wave theory given in equations (1.28). To enable the profile near to the kinematic wave front to be examined, we further introduce

$$\xi = x - FT, \quad \varepsilon T = \tau \quad (1.46)$$

where ξ is again a characteristic co-ordinate and τ is a large time co-ordinate. Substituting (1.44 - 1.46) into the basic equations (1.10 - 1.16) gives

$$(u - F)u_\xi + u'v + p_\xi = \frac{1}{R} \left(u_{\xi\xi} + \frac{1}{\kappa\varepsilon} u_{yy} \right) + O(\varepsilon) \quad (1.47)$$

$$p_y = \frac{1}{R} v_{yy} + O(\varepsilon) \quad (1.48)$$

$$u_\xi + v_y = 0 \quad (1.49)$$

$$u = v = 0 \quad \text{on} \quad y = 0 \quad (1.50)$$

$$v = \varepsilon \eta_\tau + (u - F) \eta_\xi + \varepsilon u \eta_\xi \quad (1.51)$$

$$1 - y + \varepsilon p = \frac{2\varepsilon}{R} [v_y - u' \eta_\xi] + O(\varepsilon^2) \quad (1.52)$$

$$u' + \varepsilon u_y + \kappa \varepsilon^2 v_\xi = O(\varepsilon^3). \quad (1.53)$$

A solution to the set (1.47 - 1.53) is sought as a formal expansion

$$Q(\xi, \tau, y; \varepsilon) = Q_0(\xi, \tau, y) + \varepsilon Q_1(\xi, \tau, y) + \dots$$

$$\eta(\xi, \tau; \varepsilon) = \eta_0(\xi, \tau) + \varepsilon \eta_1(\xi, \tau) + \dots$$

where $Q \equiv u, v, p$. The equations of the first approximation, as already mentioned, are the kinematic wave equations which result in

$$u_0 = F \eta_0 y, \quad v_0 = -F \eta_{0\xi} \cdot \frac{1}{2} y^2 \quad (1.54)$$

where $\eta_0(\xi, \tau)$ is undefined. This situation is common in non-linear wave problems. The difficulty is overcome by forming the next approximation in the course of which $\eta_1(\xi, \tau)$ will remain undefined, but an equation for $\eta_0(\xi, \tau)$ is found. From (1.47 - 1.53) we see that the equations for u_1, v_1 and η_1 are

$$(U-F)u_{0\zeta} + U'V_0 + P_{0\zeta} = \frac{1}{R}(u_{0\zeta\zeta} + \frac{1}{\kappa}u_{1\eta\eta}) \quad (1.55)$$

$$P_{0\eta} = \frac{1}{R}V_{0\eta\eta} \quad (1.56)$$

$$u_{1\zeta} + V_{1\eta} = 0 \quad (1.57)$$

together with the boundary conditions

$$u_1 = V_1 = 0 \quad \text{on} \quad \eta = 0 \quad (1.58)$$

$$P_0 - \tau_0 = \frac{2}{R}(V_{0\eta} - U'\tau_{0\zeta}) \quad (1.59)$$

$$\tau_1 U'' + \tau_0 u_{0\eta\eta} + u_{1\eta} + \kappa V_{0\zeta} = 0 \quad \left. \begin{array}{l} \text{on} \\ \eta = 1 \end{array} \right\} \quad (1.60)$$

$$V_1 + \tau_0 V_{0\eta} = \tau_{0\tau} + (U-F)\tau_{1\zeta} + U'\tau_0\tau_{0\zeta} + u_0\tau_{0\zeta\zeta} \quad (1.61)$$

Noting that $U(\eta)$ is the Poiseuille profile in this problem, then from (1.54, 1.56, 1.59) we have

$$P_0 = \tau_0 - \frac{F}{R}(1+\eta)\tau_{0\zeta}$$

From (1.55, 1.57, 1.58, 1.60) we obtain the solution for V_1 ,

$$\begin{aligned} \frac{1}{\kappa R}V_1 = & \tau_{0\zeta\zeta} \left[\frac{1}{2}\eta^2 - \frac{1}{6}\eta^3 - F^2 \left(\frac{1}{6}\eta^2 - \frac{1}{24}\eta^4 + \frac{1}{120}\eta^5 \right) \right] \\ & + \frac{F}{R}\tau_{0\zeta\zeta\zeta} \left[\frac{1}{12}\eta^4 + \frac{1}{6}\eta^3 - \frac{5}{4}\eta^2 \right] - \frac{F}{2\kappa R}\eta^2\tau_{1\zeta} \end{aligned}$$

Finally, using the surface kinematic condition (1.61), $\tau_1(\zeta, \tau)$ identically cancels leaving

$$\eta_{\tau} + 2F\eta_{\xi}\eta_{\xi\xi} + kF\eta_{\xi\xi\xi} = \frac{Rk}{3}\left(1 - \frac{2}{5}F^2\right)\eta_{\xi\xi\xi} \quad (1.62)$$

This is called the Korteweg-de Vries-Burgers (K-V-B) equation. We observe that for $R \rightarrow 0$, (1.62) becomes the Korteweg-de Vries equation (see Korteweg and de Vries, 1895), and for $k \rightarrow 0$ (Rk fixed) it reduces to the Burgers equation (see Burgers, 1948). In passing, we also note that there is a critical value of the Froude number at which the damping term (the second derivative in ξ) becomes zero. This occurs when

$$F = \sqrt{\frac{5}{2}}, \quad (1.63)$$

and is exactly the value obtained by Chester (1966) for the same problem (but by a different approach).

A detailed analysis of equation (1.62) will be given in chapter 2, which comprises the work described in Johnson (1970).

1.5 Conclusions

The analysis presented in this first chapter has attempted to re-examine the flow in a shallow two dimensional channel. In particular a Poiseuille velocity distribution for the fully developed flow has been considered and the formation of both dynamic and kinematic waves has been indi-

cated. Due to the presence of the no-slip condition on the bed of the channel, the dynamic (Burns) waves decay leaving the main disturbance travelling at the kinematic wave speed. Finally, the wave profile near to the kinematic wave front has been derived (with the dominant viscous effects retained) and shown to be described by the K-V-B equation (1.32). The various regions of the flow that we have examined are tabulated in Table I, and shown diagrammatically in figure 2.

Table I
(with $\delta \ll 1$, $\delta^2 R_e \gg 1$, $\epsilon \ll 1$)

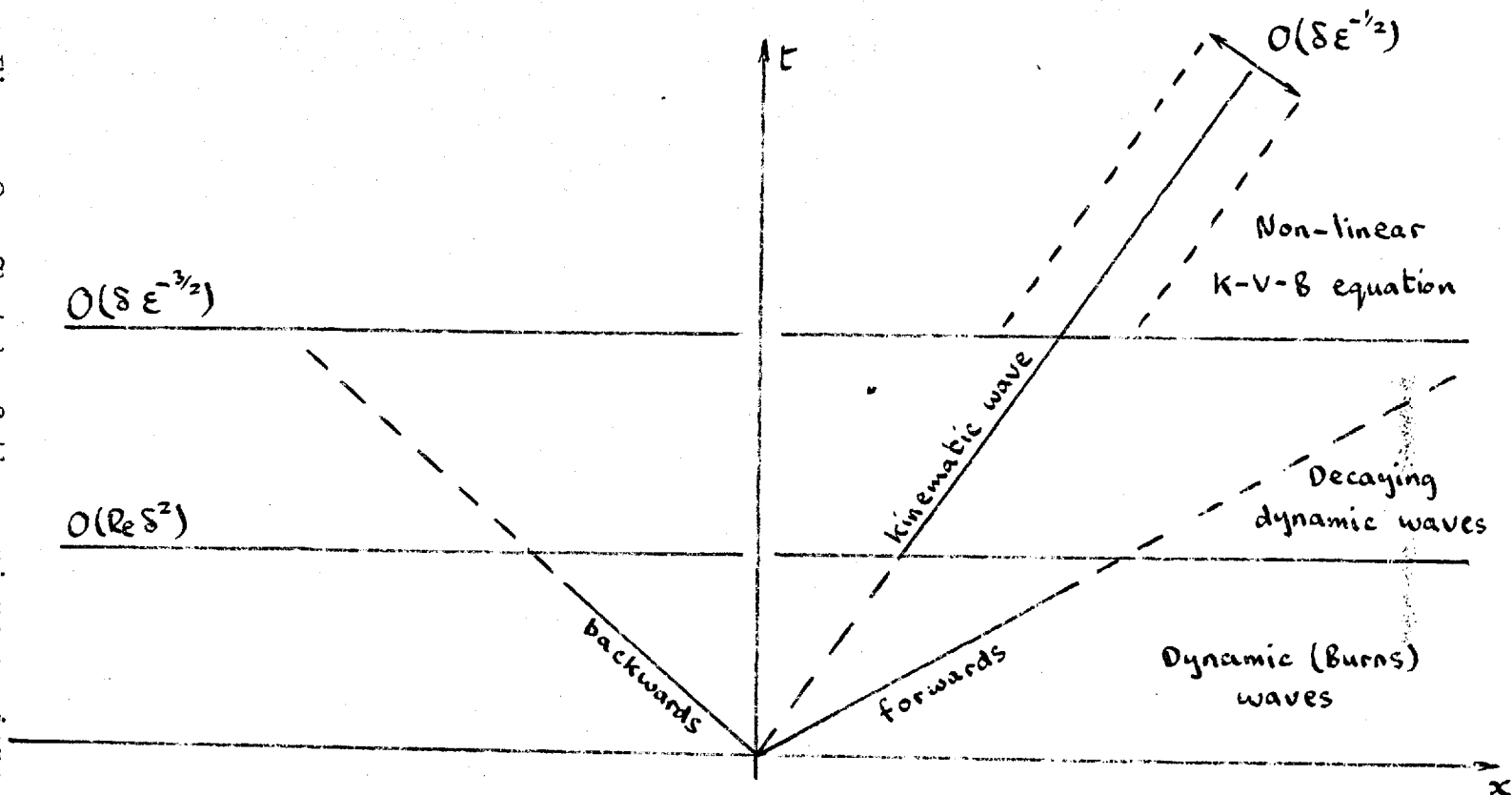
Region	Wave motion
$x, t = O(1)$	Dynamic (Burns) waves
$x, t \gg O(\delta^2 R_e)$	Decay of dynamic waves Dominance of kinematic wave
$x - Ft = O(\delta \epsilon^{-1/2})$ $t = O(\delta \epsilon^{-3/2})$	Near to kinematic wave front - non-linear K-V-B equation

For convenience, the regions in the figure have been separated by discrete lines. So that the non-linear K-V-B equation is valid in essentially a region of its own, we ask for simplicity that

$$O(\delta \epsilon^{-3/2}) > O(\delta^2 R_e).$$

Although such a restriction is not necessary, the form of the various waves is more clearly given in the figure (2). It is

Figure 2 - Sketch of the various regions of the flow



clear from the analysis that the dynamic waves will eventually decay (on a scale governed by Re), and that non-linearity will become important at the kinematic wave front (essentially on the scale of ϵ^{-1}).

The Korteweg-de Vries-Burgers equation will be studied shortly, but we have already noted that a critical value of Froude number ($\sqrt{\frac{5}{2}}$) exists. A critical value was also obtained by Lighthill and Whitham (1955) who studied this situation in some detail. They showed that this value was governed by the condition that the forward travelling dynamic wave was of the same speed as the kinematic wave. In such a situation their analysis became invalid. Although the present work does not directly give the same result (since the decay of the dynamic waves was not described by the type of equation discussed in Whitham (1959)), we see that the value at which $C_1 = F$ occurs at

$$F = 1.62 \quad (\text{from (1.25)}). \quad (1.64)$$

This is amazingly close to the critical value $F = \sqrt{\frac{5}{2}} = 1.58$ and occurs after it. Thus the analysis leading to the K-V-B equation would become invalid before $C_1 = F$. The interpretation of the condition $F^2 \gg \frac{5}{2}$ is given by Dressler (1949) as the instability condition heralding the formation of roll waves. This type of flow is not considered here.

Finally, we note that the damping factor, $Rk(1 - \frac{2}{5}F^2)$, can take reasonable numerical values even if R is large by considering values of F^2 close to $\frac{5}{2}$. It is also worth pointing out that the special case of $R = O(1)$ (e.g. $R_e = O(1)$, $\delta^2 = O(\epsilon)$), is essentially that discussed by Benjamin (1957). Benjamin included the effects of surface tension and sought a solution as a polynomial in y . Also for $R_e = O(1)$, Benney (1966) extended Benjamin's results obtaining an equation which can be reduced to the general form of the K-V-B equation far from the initial disturbance. Both these analyses were for thin films of fluid.

2. Solution of the K-V-B equation

The Korteweg-de Vries-Burgers equation as derived for the wave profile near to the kinematic wave front (1.62) is of the form

$$\eta_{\tau_1} + \eta_{\eta_1} + \eta_{\eta_1\eta_1} = R_1 \eta_{\eta_1\eta_1} \quad (2.1)$$

where

$$\tau = \frac{1}{2F} \sqrt{\frac{k}{2}} \tau_1, \quad \xi = \sqrt{\frac{k}{2}} \xi_1, \quad R_1 = \frac{R}{3F} \sqrt{\frac{k}{2}} \left(1 - \frac{2}{5} F^2\right).$$

Equation (2.1) would appear to be the simplest form of wave equation (uni-directional) in which non-linearity (η_{η_1}), dispersion ($\eta_{\eta_1\eta_1}$) and damping ($R_1 \eta_{\eta_1\eta_1}$) all occur. For the studies to be made of this interesting equation we shall assume throughout that $R_1 \geq 0$, so that the damping is always positive and $F^2 \leq \frac{5}{2}$. Only very recently (Miura, 1970) has any success been gained towards a general solution of the Korteweg-de Vries equation ($R_1 = 0$). Many aspects are still only partially understood, but for the type of problem we are considering (i.e. the undular bore) we expect that the steady-state solution will be satisfactory. Certainly most observed bores, once formed, do not change their form indicating that a steady solution has been reached. It is this phenomenon that we shall examine here.

For the steady-state equation we introduce the single independent variable .

$$S = \frac{\xi_1 - \frac{1}{2} \gamma_\infty \tau_1}{2 \sqrt{\frac{2}{\gamma_\infty}}}, \quad \gamma_0(-\infty) = \gamma_\infty, \quad (2.2)$$

and then write

$$v(s) = \frac{1}{\gamma_\infty} \gamma_0(s)$$

so that (2.1), upon integrating once, becomes

$$\frac{d^2 v}{ds^2} + 4v(v-1) = \gamma \frac{dv}{ds}, \quad \gamma = 2R_1 \sqrt{\frac{2}{\gamma_\infty}}. \quad (2.3)$$

γ is the new damping parameter, and the upstream and downstream conditions have been assumed to remain undisturbed as

$$v \rightarrow 0, \quad S \rightarrow \infty; \quad v \rightarrow 1, \quad S \rightarrow -\infty.$$

The undisturbed (zero) downstream conditions have been assumed since we are discussing the problem of disturbances moving into an undisturbed region. We shall now be studying the equation (2.3).

Techniques for the solution of the problem of a linear oscillator with a small non-linear term have been developed and are usually referred to as the Method of Averages (see Bogoliubov and Mitropolsky, 1961). Although these methods originate with Poincaré, the more modern approach as expounded by Bogoliubov and Mitropolsky is to form an asymp-

otic expansion in terms of the small parameter, ensuring that successive terms are periodic. These techniques have been further advanced by Kuzmak (1959) who shows that similar methods can be employed when the initial oscillation is itself non-linear.

For equation (2.3) we shall examine the form of the solution in the phase-plane and obtain similar results to Grad and Hu (1967). Grad and Hu suggested the steady-state equation to describe the weak shock profile in plasmas. In agreement with their work, we shall find that the form of the solution is exactly as observed for the undular bore (in a particular range of γ). This range extends from zero to some finite value, and throughout the range the type of solution is essentially unaltered. Since this is the case, and no solution is available for $\gamma = O(1)$, we develop an asymptotic solution for $\gamma \rightarrow 0$.

It is observed that for no damping ($\gamma = 0$), equation (2.3) is a non-linear equation with solutions which perform oscillations - the so-called 'cnoidal' waves of water wave theory. For small damping ($\gamma \ll 1$), similar application of averaging and periodicity conditions are possible as explained by Kuzmak (1959), but the analysis is somewhat more complicated (usually resulting in the integration of Jacobian elliptic functions). The solutions of the damped equation

extend over a large period range from the single solitary wave (infinite period), to the zero amplitude zero period cnoidal wave. Since averaging techniques rely on averaging over a large number of wavelengths such methods breakdown as the solitary wave is approached. It is then necessary to match this solution to a straightforward perturbation of the solitary wave.

2.1 The phase-plane analysis

Equation (2.3) can be studied in the phase-plane by introducing $w = \frac{dv}{ds}$. Since the equation does not contain S explicitly this reduces it to a simple first order equation for $w(v)$:

$$\frac{dw}{dv} = \frac{\gamma w - 4v(v-1)}{w}. \quad (2.4)$$

In the (w, v) plane (or phase-plane), there are two singular points

$w=0, v=0$; a saddle point

and $w=0, v=1$; $\begin{cases} \text{a stable node, } \gamma \geq 4 \\ \text{a stable spiral point, } \gamma < 4. \end{cases}$

A sketch of the integral curves for (2.4) with $4 > \gamma > 0$ is given in figure 3. The special case of $\gamma=0$ is the solitary wave, and clearly any other permitted value of γ causes the integral curves to spiral towards $v=1$. In the physical

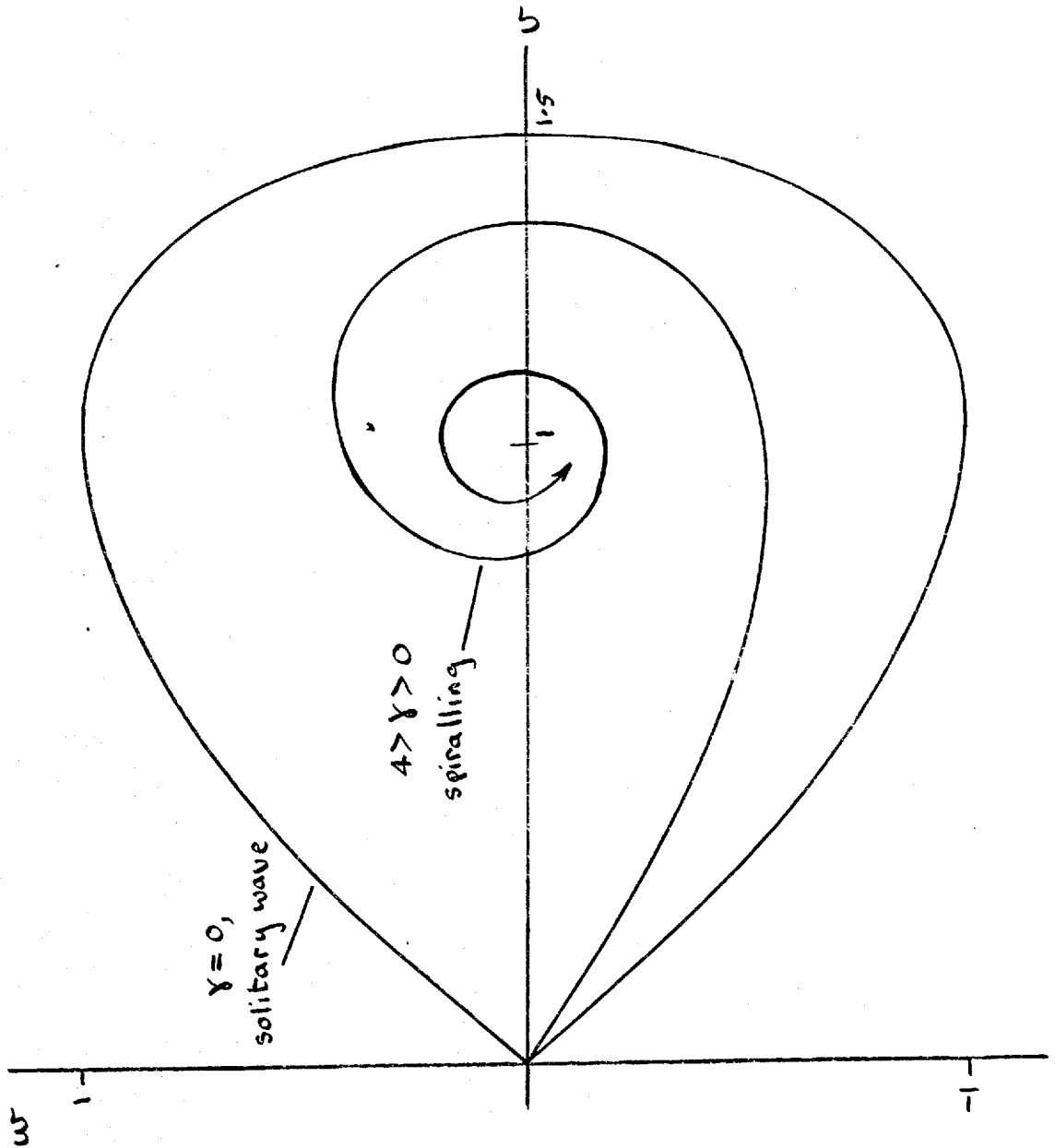


Figure 3 - Sketch of the phase-plane for $4 > \gamma \geq 0$

(σ, S) plane this corresponds to a profile which starts as a solitary wave and gradually deviates from it, eventually oscillating about $\sigma \approx 1$ (with ever decreasing period and amplitude).

To consider the case of large γ , we can write $\gamma w = w_1$, and then equation (2.4) becomes

$$\frac{1}{\gamma^2} \frac{dw_1}{d\sigma} = \frac{w_1 - 4\sigma(\sigma - 1)}{w_1}.$$

Consequently the integral curves now become as shown in figure 4. The special case of $\gamma \rightarrow \infty$ is the Taylor shock profile. As γ decreases from infinity, the integral curves deviate more and more from the Taylor profile while still remaining monotonic. When $\gamma < 4$ the curve spirals to $\sigma = 1$ as before. Physically, the solution for $\sigma(S)$ remains a monotonically increasing function from downstream to upstream conditions, until $\gamma < 4$ when the oscillatory motion about $\sigma \approx 1$ can occur.

Some typical results for the profile are shown in figure 5, where various numerical integrations of equation (2.3) confirm the phase-plane analysis. The figure shows on the same axes the two basically different types of solution already indicated. On the graph are a near-Taylor shock profile ($\gamma = 4.62$); a profile which is just oscillatory ($\gamma = 3.26$); a much larger amplitude oscillation ($\gamma = 0.1$); and finally for comparison the solitary wave ($\gamma = 0$). More details of

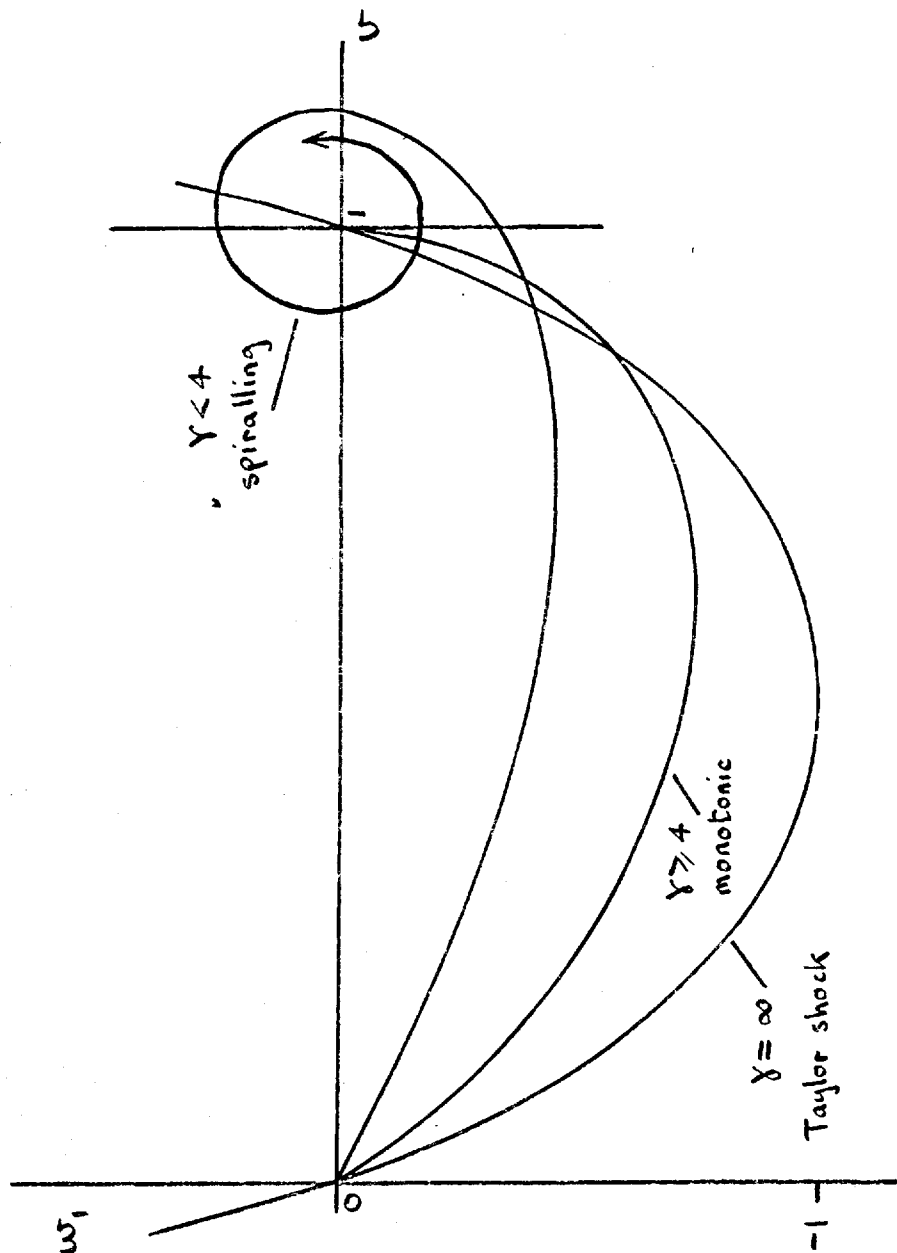


Figure 4 - Sketch of the phase-plane for $\gamma > 4$, $\gamma < 4$

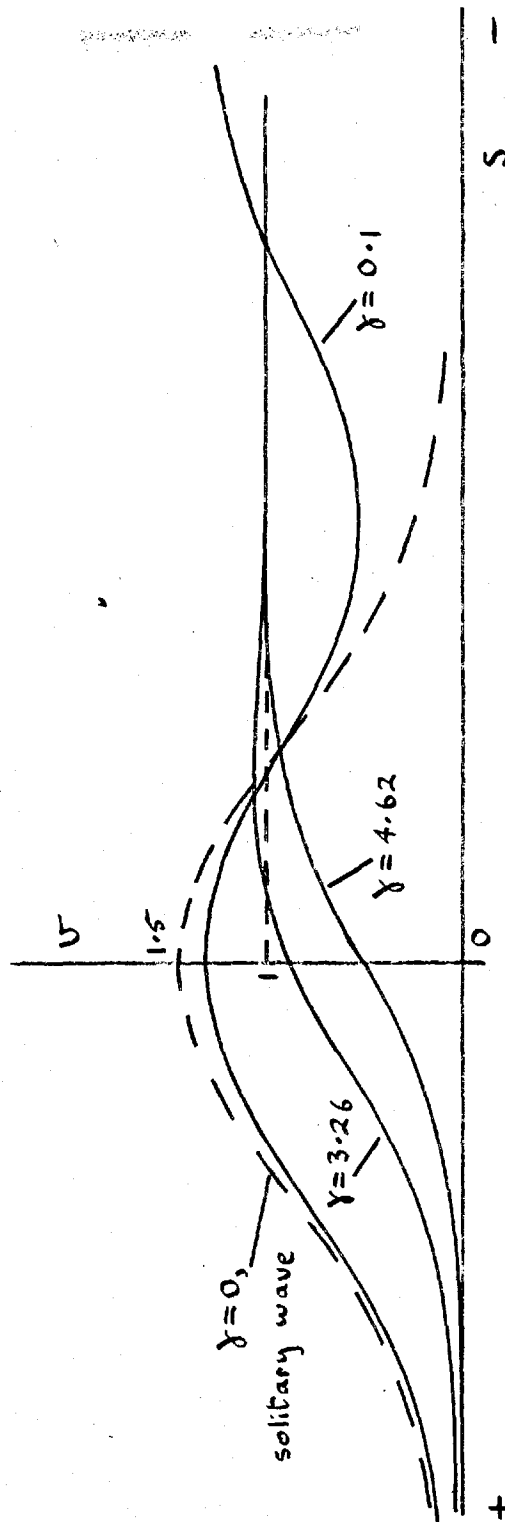


Figure 5 - Various integrations of the steady-state K-V-B equation

the numerical integration scheme employed will be given later.

2.2 Oscillatory asymptotic analysis for small damping

Figure 4 shows that for small damping ($\gamma \ll 1$) the solution starts very near to a solitary wave and then slowly departs from it, causing the profile to eventually oscillate about $u=1$. It is evident that the small damping causes both the period and amplitude to vary slowly - in fact the period must change from infinity to zero.

This suggests that a solution to equation (2.3), for small damping, can be found by allowing the amplitude and period of the undamped solution to slowly vary. This idea together with the insistence on periodicity is a particular case of a general technique developed by Kuzmak (1959), and a generalisation of earlier methods of averaging almost periodic solutions.

To utilise this method we must first find the undamped solution to the equation (see Lamb, 1956, Art. 253). Thus considering the general undamped equation,

$$\frac{d^2 u}{ds^2} + 4u(u-1) = A$$

where A is a constant, we integrate once

$$\begin{aligned} \frac{3}{8} \left(\frac{du}{ds} \right)^2 &= -u^3 + \frac{3}{2} u^2 + \frac{3}{4} A u + B \\ &= (u - u_1)(u - u_2)(u_3 - u), \end{aligned}$$

and write $\sigma_1 < \sigma_2 < \sigma_3$ as the roots of the cubic. The constants A and B can now be found by applying appropriate boundary conditions on the maximum and minimum heights of the motion. The general form of the cubic expression is sketched in figure 6 (curve A).

The only real solution of the equation occurs for $\left(\frac{d\sigma}{ds}\right)^2 \geq 0$, and thus the solution is either at $\sigma = \sigma_1$ or is a non-linear oscillation between σ_2 and σ_3 . Two special cases are worthy of note. One is $\sigma_2 \rightarrow \sigma_1$ (curve B) giving the solitary wave, and the other is $\sigma_2 \rightarrow \sigma_3$ (curve C) giving a discontinuous solution between σ_1 and σ_3 (the hydraulic jump).

The solution of the undamped equation can be written in terms of a Jacobian elliptic function $\text{cn}(u; \nu)$, by direct integration of the above expression, which gives

$$2\sqrt{\frac{2}{3}} S = \int_0^\sigma \frac{d\sigma}{\sqrt{(\sigma - \sigma_1)(\sigma - \sigma_2)(\sigma_3 - \sigma)}}.$$

Thus the solution for σ is

$$\sigma = \sigma_2 + (\sigma_3 - \sigma_2) \text{cn}^2 \left[S \sqrt{\frac{2(\sigma_3 - \sigma_1)}{3}} ; \nu \right] \quad (2.5)$$

where $\nu = \frac{\sigma_3 - \sigma_2}{\sigma_3 - \sigma_1}$ is the modulus and cn is defined by

$$u = \int_0^\phi \frac{d\theta}{(1 - \nu \sin^2 \theta)^{1/2}}, \quad \text{cn } u = \cos \phi.$$

See Abramowitz and Stegun (1965).

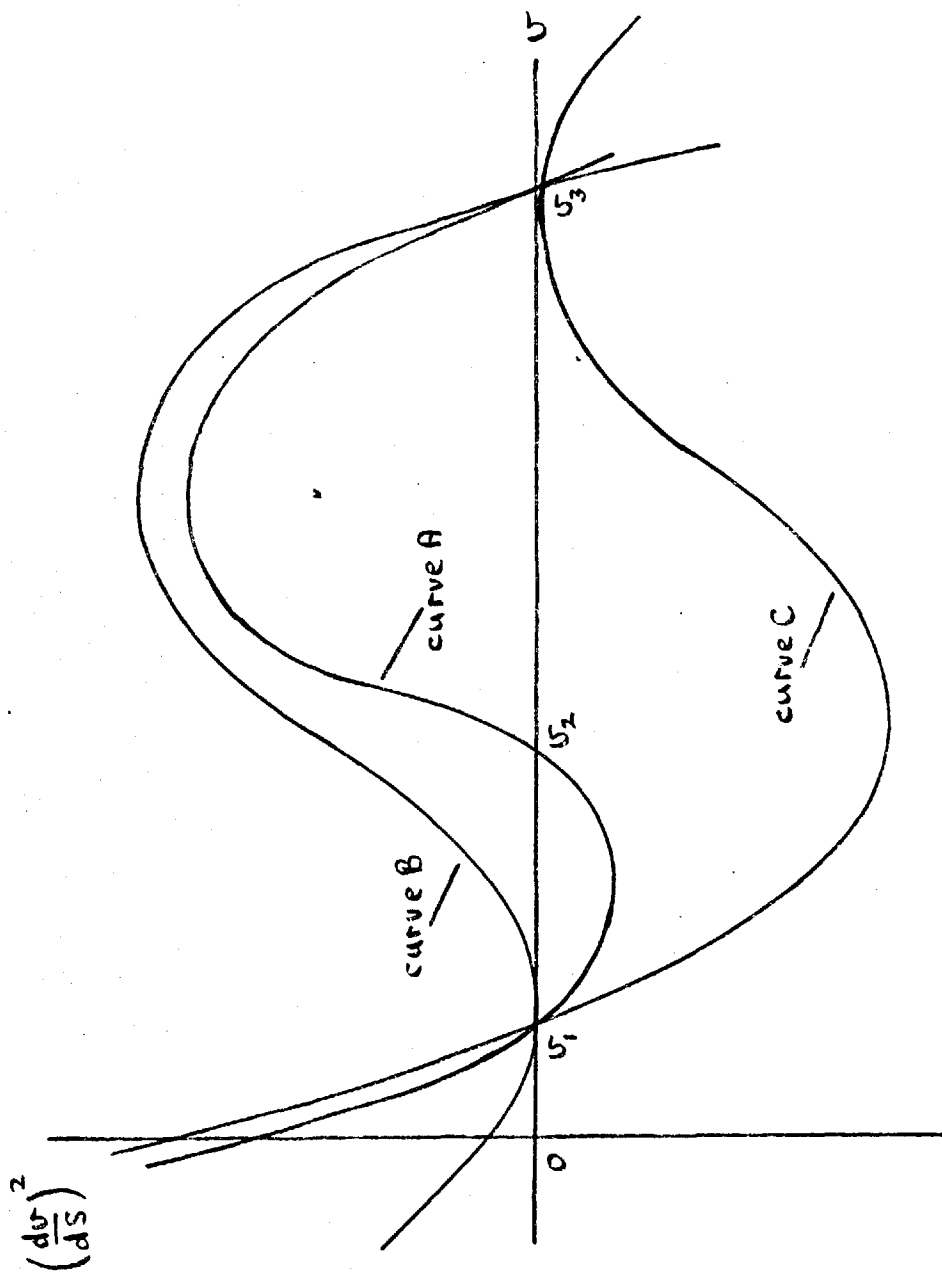


Figure 6 - Sketch of the cubic expression for the cnoidal wave

In the case of $A = B = 0$, then $\nu = 1$, and the solution becomes

$$v = \frac{3}{2} \operatorname{sech}^2 S, \quad (2.5a)$$

which is the solitary wave.

To consider the slow variation of amplitude and period, we introduce a slow time scale

$$\sigma = \gamma S, \quad \gamma \ll 1, \quad (2.6)$$

and a two parameter (S, σ) expansion is sought

$$v = v_0(S, \sigma) + \gamma v_1(S, \sigma) + \dots \quad (2.7)$$

where it will be assumed that each individual term is periodic in the 'fast' time scale, S .

The form of the undamped solution (2.5) suggests that

$$v_0 = a(\sigma) + b(\sigma) \operatorname{cn}^2 \left[S, \alpha(\sigma); \nu(\sigma) \right] \quad (2.8)$$

so that a , b , α and ν are allowed to vary on the 'slow' time scale, σ . We observe that $\alpha = \sqrt{\frac{2b}{3\nu}}$ to correspond exactly with expression (2.5). Now (2.8) describes the slowly varying nature of the solution, where the oscillation takes place between local values of a and $(a+b)$, at a local frequency determined by α .

Using relation (2.6) we have

$$\frac{d}{dS} \equiv \frac{\partial}{\partial S} + \gamma \frac{\partial}{\partial \sigma}, \quad \frac{d^2}{dS^2} \equiv \frac{\partial^2}{\partial S^2} + 2\gamma \frac{\partial^2}{\partial S \partial \sigma} + \gamma^2 \frac{\partial^2}{\partial \sigma^2},$$

where the operators $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial \sigma}$ are assumed to be order unity. With these operators, and the expansion (2.7), the equation (2.3) gives

$$O(1) : \sigma_{0ss} + 4\sigma_0(\sigma_0 - 1) = 0 \quad (2.9)$$

$$O(\gamma) : \sigma_{1ss} + 4\sigma_1(2\sigma_0 - 1) = \sigma_{0s} - 2\sigma_{0s\sigma} \quad (2.10)$$

Putting expression (2.8) into equation (2.9), and using the relations

$$-dn^2 + 1 - \nu = -\nu cn^2 = \nu(sn^2 - 1),$$

gives

$$\frac{4b^2}{3\nu} \left[1 - \nu + 2(2\nu - 1)cn^2 - 3\nu cn^4 \right] + 4(a + bcn^2)^2 - 4(a + bcn^2) = 0.$$

For this to be true for all values of s , the coefficients of cn^2 and cn^4 are equated to zero,

$$a = \frac{1}{2} - \frac{b}{3\nu}(2\nu - 1), \quad b = \frac{3\nu}{2}(\nu^2 - \nu + 1)^{-1/2} \quad (2.11)$$

A third equation is identically satisfied due to the choice of $\alpha(\sigma)$.

We now need one more relation to enable a , b and ν to

be determined. This is obtained from the condition that $v_1(s, \sigma)$ be periodic in S with the same period as v_0 . Firstly, we simplify equation (2.10) by writing

$$v_1 = v_{0s} \cdot f(s, \sigma), \quad (2.12)$$

which gives

$$f_s = \frac{1}{v_{0s}^2} \int_{b_0}^s (v_{0s}^2 - 2 v_{0s} v_{0s\sigma}) ds, \quad (2.13)$$

where b_0 is arbitrary. This can be written as

$$f_s = \frac{1}{v_{0s}^2} \left[\int_{b_0}^s v_{0s}^2 ds - \frac{\partial}{\partial \sigma} \int_{b_0}^s v_{0s}^2 ds \right].$$

Now v_1 is periodic if f_s is, so that evaluating across a period and integrating with respect to σ gives

$$c_1 e^{\sigma} = \int_{a_0 - \frac{1}{2} S_P}^{a_0 + \frac{1}{2} S_P} v_{0s}^2 ds, \quad (2.14)$$

where c_1 is a constant, S_P is the period of f and a_0 is any general point in the range of S . It should be noted that these expressions are to be interpreted with S and σ regarded as independent variables.

Expression (2.14) can now be rewritten by integrating over the quarter-period $\kappa(\nu)$ of v_{0s} ,

$$c_2 e^\sigma = b^2 \sqrt{\frac{2b}{3\nu}} \int_0^{k(\nu)} c n^2 y \cdot s n^2 y \cdot d n^2 y \cdot dy = b^2 \sqrt{\frac{2b}{3\nu}} L(\nu). \quad (2.15)$$

($K(\nu)$ is defined below.)

Using the definition of the Jacobian elliptic function, $L(\nu)$ becomes

$$L(\nu) = \int_0^{\pi/2} \cos^2 \phi \cdot \sin^2 \phi \cdot (1 - \nu \sin^2 \phi)^{1/2} \cdot d\phi. \quad (2.16)$$

With the well-known relations

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt, \\ (\Re c > \Re b > 0)$$

and

$$K(\nu) = \frac{1}{2} \pi F\left(\frac{1}{2}, \frac{1}{2}, 1; \nu\right) \\ E(\nu) = \frac{1}{2} \pi F\left(\frac{1}{2}, -\frac{1}{2}, 1; \nu\right),$$

where K and E are the complete elliptic integrals of the first and second kind, respectively, then (2.16) can be written as

$$\nu^2 L(\nu) \propto (\nu^2 - \nu + 1) E(\nu) - \frac{1}{2} (1-\nu)(2-\nu) K(\nu). \quad (2.17)$$

This expression, (2.17), together with

$$e^\sigma \propto \nu^2 (\nu^2 - \nu + 1)^{-5/4} L(\nu)$$

becomes the third condition which enables $a(\sigma)$, $b(\sigma)$ and $\nu(\sigma)$ to be determined. The constant of proportionality relating these expressions is found by requiring that

$$\nu = 1 \quad \text{at} \quad \sigma = \sigma_0 = \gamma S_0, \text{ say,}$$

so that

$$e^{\sigma - \sigma_0} = (\nu^2 - \nu + 1)^{-5/4} \left[(\nu^2 - \nu + 1) E - \frac{1}{2} (1 - \nu)(2 - \nu) K \right]. \quad (2.18)$$

So S_0 becomes the apparent position where the damping is "switched-on" for the oscillatory analysis. Hence S_0 is a constant of the asymptotic theory which must be fixed by matching to some initial solution. It should be noted that we have anticipated the undular bore type of solution by using the condition at $\nu = 1$ - the solitary wave limit.

The final solution given by (2.7, 2.12, 2.13) can be written as

$$\sigma = \sigma_0 + \gamma \sigma_s \int_{a_0}^S \frac{1}{\sigma_{s_2}^2} \left[\int_{b_0}^{S_2} \left(\sigma_{s_1}^2 - 2 \sigma_{s_1} \sigma_{s_1, \sigma} \right) dS_1 \right] dS_2 \dots \quad (2.19)$$

where a_0 is arbitrary. Kuzmak (1959, p. 737) points out that a solution like (2.19) is only valid provided that $S_p < \infty$ (see equation (2.14)) i.e. if σ_0 is truly oscillatory. Consequently (2.19) in some sense becomes invalid as $\nu \rightarrow 1$, since σ_0 then becomes the non-periodic solitary wave (see

expression (2.5a)). It should be noted that this type of breakdown of an asymptotic expansion is not that which is normally encountered in singular perturbation theory (see e.g. Van Dyke, 1964).

2.3 Perturbation of the solitary wave

Where the expansion (2.19) becomes invalid due to the non-oscillatory **behaviour** of ψ_0 , a non-oscillatory solution of equation (2.3) may be sought by performing a straightforward expansion in powers of γ , for fixed S . In fact we shall see that this amounts to a perturbation of the solitary wave.

It is more convenient to manipulate the integrated form of (2.3),

$$\frac{1}{2} \left(\frac{d\psi}{dS} \right)^2 = \gamma \int_{\infty}^S \left(\frac{d\psi}{dS_1} \right)^2 dS_1 - \frac{4}{3} \psi^3 + 2\psi^2, \quad (2.20)$$

where $\psi = \psi_S = 0$ at $S = +\infty$, so that the solution of (2.20) will be monotonic as $S \rightarrow +\infty$. Clearly expression (2.15), and hence the solution (2.19), is valid as $S \rightarrow -\infty$ describing the oscillatory nature of the solution. A straightforward expansion procedure is

$$\psi = V_0(S) + \gamma V_1(S) + \dots \quad (2.21)$$

which gives on substitution into equation (2.20),

$$O(1): \quad \frac{1}{2} V_0'^2 = 2V_0^2 - \frac{4}{3} V_0^3 \quad (2.22)$$

$$O(\gamma): V_0 V_1 = \int_{-\infty}^{\infty} V_0^2 dS_1 + 4V_0 V_1 (1 - V_0). \quad (2.23)$$

The solution to the $O(1)$ equation (2.22) is the solitary wave,

$$V_0 = \frac{3}{2} \operatorname{sech}^2 S \quad (2.24)$$

where the peak of the wave is fixed at $S=0$. Now substituting for V_0 in the $O(\gamma)$ equation, and carrying out a simple integration gives

$$\left. \begin{aligned} V_1 = \frac{2}{5} \operatorname{sech}^2 S (\tanh S - 1) + \frac{1}{20} \tanh S (7 \tanh S - 6) \\ - \frac{\tanh S}{10(1 + \tanh S)} + \frac{3}{4} S \tanh S \operatorname{sech}^2 S, \end{aligned} \right\} \quad (2.25)$$

where $V_1(0)=0$ ensures that the peak occurs at $S=0$ to $O(\gamma^2)$.

The asymptotic solution (2.24, 2.25) shows that this expansion procedure (2.21) is a perturbation about the solitary wave. Since relation (2.24) always stays bounded, this expansion certainly becomes invalid as $\tanh S \rightarrow -1$ ($S \rightarrow -\infty$), but we also note that it also becomes invalid as $S \rightarrow +\infty$. In both cases V_1/V_0 increases without limit.

2.4 Matching procedure

The breakdowns that occur in the asymptotic expansion which describes the perturbation of the solitary wave will

now be considered, and matching to other regions of the solution attempted. Firstly, the case for $S \rightarrow +\infty$ is a fairly elementary exercise. Since, as $S \rightarrow +\infty$, $U \rightarrow 0$, then we observe from equation (2.3) that

$$U \sim A e^{\beta S}, \quad \beta = \frac{1}{2} \left[\gamma - 4 \sqrt{1 + \gamma^2/16} \right], \quad S \rightarrow +\infty$$

$$\sim A e^{-2S} \left(1 + \frac{1}{2} \gamma S \dots \right)$$

where A is an unknown constant. This expression indicates the asymptotic exponential behaviour as $U \rightarrow 0$. Also for $S \rightarrow +\infty$, the solution given by (2.24, 2.25) can be expanded giving

$$U \sim \frac{3}{2} \cdot \frac{4}{(e^S + e^{-S})^2} \left[1 + \frac{1}{2} \gamma S \left(\frac{e^S - e^{-S}}{e^S + e^{-S}} \right) + \dots \right]$$

$$\sim 6 e^{-2S} \left(1 + \frac{1}{2} \gamma S \dots \right).$$

These two expansions match exactly if we choose $A = 6$. Thus we see that the solution to (2.3) continues to decay predominantly like a solitary wave even though (2.21) becomes invalid as $S \rightarrow +\infty$. The approach outlined here can be formalised by rescaling for $\gamma S = S_1$ and seeking a solution of the form $e^{-2S_1/\gamma} (f_0(S_1) + \dots)$. This expansion can then be matched (as above) to the perturbation of the solitary wave.

We may now turn our attention to the more complicated matching problem at the other end. It has already been men-

tioned that the averaged solution becomes invalid as $S \rightarrow +\infty$ since the period of the wave approaches infinity and averaging over a large number of waves becomes meaningless. However, it is possible to expand the oscillatory solution as $\nu \rightarrow 1$ - its non-oscillatory limit - so that by suitable choice of the undefined constants a_0 , b_0 and S_0 (equation (2.19)) it agrees term by term with the perturbation of the solitary wave given by equations (2.24, 2.25). This is therefore a matching procedure for these two very dissimilar expansions. The method outlined here does not follow the usual technique of matched asymptotic expansions (e.g. Van Dyke, 1964) where the breakdown of one expansion indicates a new scaling of the equations and consequently a new asymptotic expansion can be developed. Here there are, of course, obvious similarities between the two procedures but no justification can be given for the success of the technique. It can only be said that both expansions are presumably asymptotic expansions of the same complete solution of the equation (2.3).

We thus expand the oscillatory solution for $\nu \rightarrow 1$, $S \rightarrow S_0$, and in particular consider the region where

$$S - S_0 = O(1), \nu = 1 - \bar{\gamma}(\eta)\nu_1, \bar{\gamma}(\eta) \ll 1, \nu_1 > 0. \quad (2.26)$$

Equation (2.18) then gives

$$S \sim S_0 - \frac{33}{32} \nu_1^2 \quad (2.27)$$

where the only non-trivial value of $\bar{\gamma}$ has been used, namely

$$\bar{\gamma}^2 \ln \bar{\gamma} = -\gamma.$$

Thus we have $S < S_0$ for $\nu < 1$. Now from relation (2.8),

$$\begin{aligned} \sigma_{0s} &= -2\alpha b \operatorname{cn}(\alpha s) \operatorname{sn}(\alpha s) \operatorname{dn}(\alpha s) \\ &\sim -3 \operatorname{sech}^2 S \tanh S. \end{aligned} \quad (2.28)$$

Thus

$$\int_{b_0}^S (\sigma_{0s_1}^2 - 2\sigma_{0s_1} \sigma_{0s_1\sigma}) ds_1 \sim 9 \left[B + \frac{1}{3} \tanh^3 S - \frac{1}{5} \tanh^5 S \right],$$

since $\sigma_{s,\sigma} = 0(\bar{\gamma})$ and

$$B = \frac{1}{5} \tanh^5 b_0 - \frac{1}{3} \tanh^3 b_0. \quad (2.29)$$

Hence we have

$$\left. \begin{aligned} \int_{a_0}^S \frac{1}{\sigma_{0s_2}^2} \left[\int_{b_0}^{s_2} (\sigma_{0s_1}^2 - 2\sigma_{0s_1} \sigma_{0s_1\sigma}) ds_1 \right] ds_2 &\sim B \left[\frac{15}{8} S - \frac{1}{\tanh S} \right. \\ &+ \frac{\tanh S}{\operatorname{sech}^2 S} + \frac{1}{8} \frac{\tanh S}{\operatorname{sech}^4 S} + \frac{1}{8} \frac{\tanh^3 S}{\operatorname{sech}^4 S} \left. \right] + \frac{1}{12} \cosh^4 S - \frac{1}{20} \sinh^4 S + A \\ &= G(S) + A, \text{ say,} \end{aligned} \right\} \quad (2.30)$$

where

$$A = -G(a_0). \quad (2.31)$$

Finally

$$\sigma_0 \sim \frac{3}{2} \operatorname{sech}^2 S + O(\bar{\gamma}^2). \quad (2.32)$$

Thus expression (2.19), as $\nu \rightarrow 1$, can be written as

$$\left. \begin{aligned} v \sim & \frac{3}{2} \operatorname{sech}^2 S + \gamma \left[3B - \left(\frac{3}{20} + 3A \right) \tanh S \right. \\ & - \frac{45}{8} B \tanh^2 S + \left(3A - \frac{3}{20} \right) \tanh^3 S - \frac{1}{10} \frac{\tanh S}{\operatorname{sech}^2 S} \\ & \left. - \frac{3}{4} B \frac{\tanh^2 S}{\operatorname{sech}^2 S} - \frac{45}{8} B S \tanh S \operatorname{sech}^2 S \right]. \end{aligned} \right\} \quad (2.33)$$

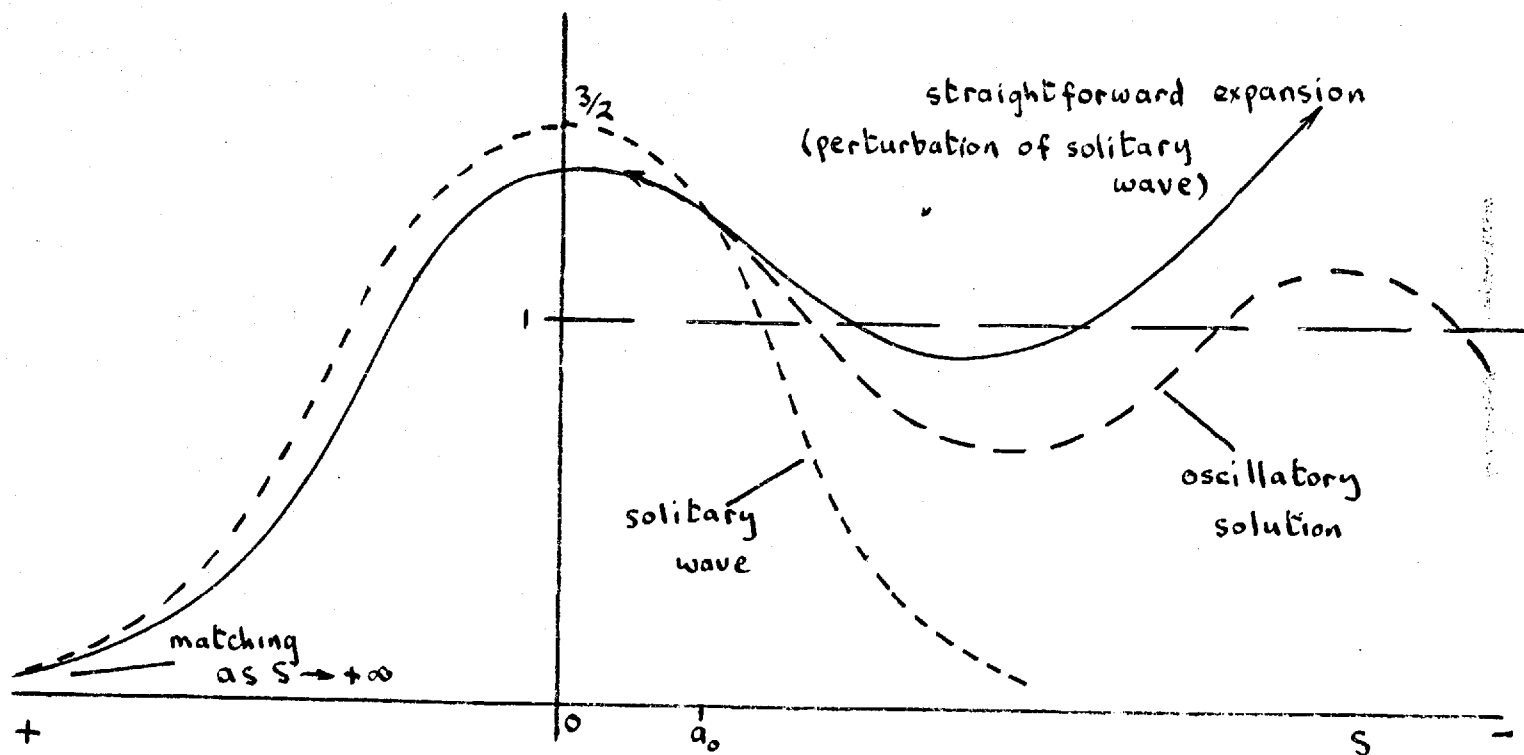
This matches exactly with the non-oscillatory solution (2.24, 2.25) if

$$A = -\frac{1}{12}, \quad B = -\frac{2}{15}. \quad (2.34)$$

Now from the definition of B , (2.29), and A , (2.30, 2.31), we have that $b_0 = +\infty$, and the one finite real solution for a_0 is $a_0 = -0.529$, obtained by a numerical computation. A sketch of the regions of validity for the asymptotic expansions, and their matching regions, are shown in figure 7. It is clear from the expansions themselves that, to $O(\gamma^2)$, two of them cross the solitary wave curve at $S = a_0$.

The matching of the oscillatory solution (2.33) with the perturbation of the solitary wave (2.24, 2.25) did not entail finding ν_1 . Thus since (2.32) is correct to $O(\bar{\gamma}^2)$, and ν_1 did not appear to this approximation, then it is impossible to define ν_1 , and hence S_0 . From the expressions (2.26) the limit of $\nu \rightarrow 1$ is taken such that $S - S_0 = O(1)$, and since

Figure 7 - Sketch of the asymptotic expansions



the straightforward expansion (2.24,2.25) is valid for $S=O(1)$, then we can say that $S_0 = O(1)$ and hence

$$e^{-\sigma} = e^{-\gamma S_0} = 1 + O(\gamma).$$

This result is not really unexpected. For the limit of small damping ($\gamma \ll 1$), the rate of dissipation of energy through the first wave (as we shall see shortly) is also small ($O(\gamma)$), and consequently the exact value of S_0 becomes irrelevant at the $O(1)$ level. It is now possible to compute $a(\sigma)$, $b(\sigma)$ and $\nu(\sigma)$ (and also $\alpha(\sigma)$ if required), and in figure 8 are given the computed curves of a , $a+b$ and ν as functions of σ . From the form of the oscillatory solution (2.6), it is seen that $(a+b)$ and a are the upper and lower envelopes respectively, and ν determines the period of the oscillation.

Also plotted on figure 8 are some peaks obtained by numerical integration of the original equation (2.3), using a predictor-corrector technique, on the Imperial College IBM 7094 computer. Details of this calculation are given in Appendix II. Integration for three values of γ ($= 0.1, 0.01, 0.001$) are shown. Agreement between the theory and computation is excellent for the two smaller values of γ , but not so good for $\gamma = 0.1$. The results, though, certainly confirm the validity of the asymptotic theory as $\gamma \rightarrow 0$.

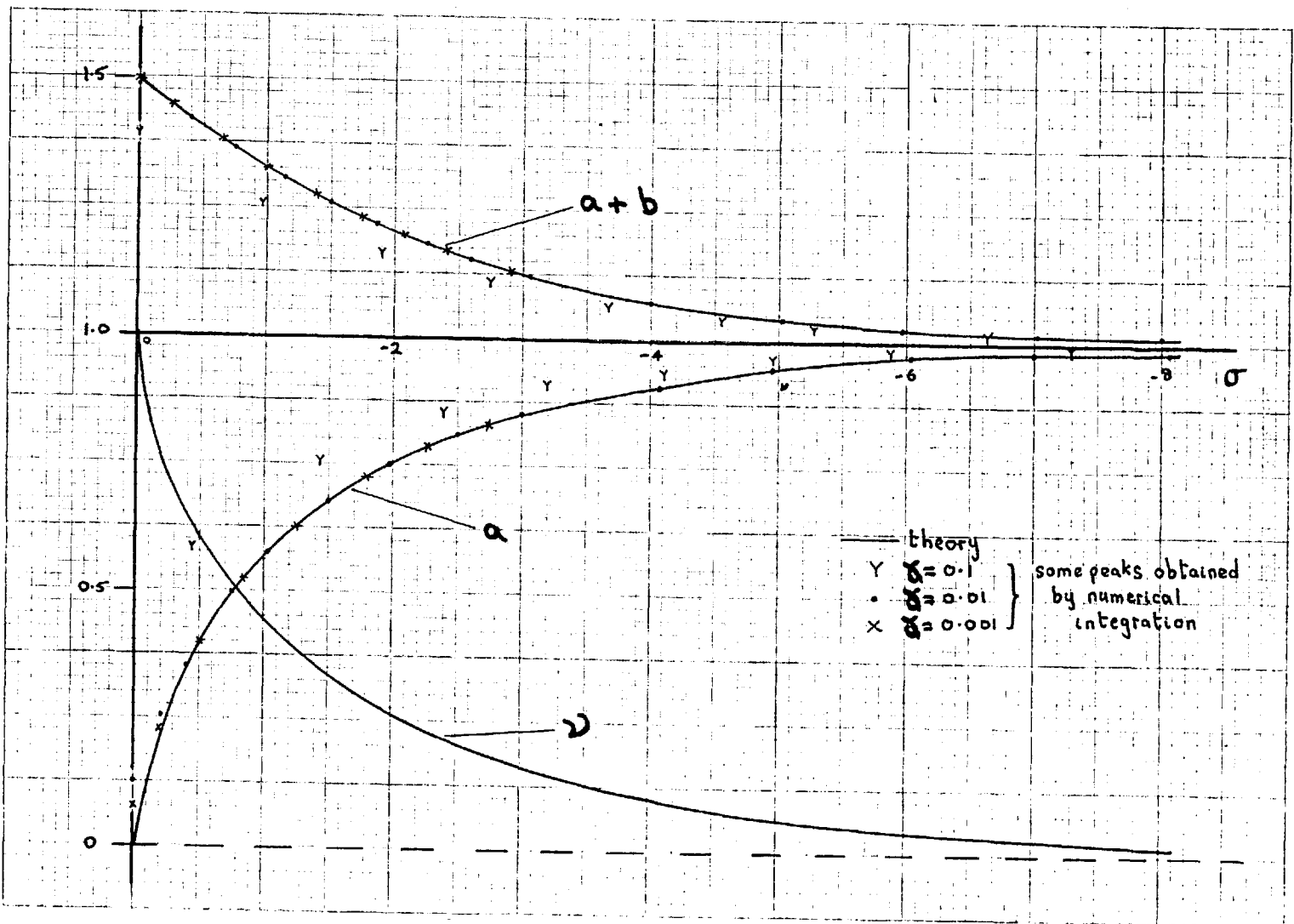


Figure 8 - Theoretical and numerical integration results

2.5 Discussion

The slowly varying (oscillatory) analysis has been successfully matched to a straightforward asymptotic expansion in γ , and this in turn has been matched to the asymptotic exponential behaviour of the solitary wave. The method leaves us with an arbitrariness in the oscillatory analysis - the point (S_0) where the damping effect appears to begin is not specified other than to $O(1)$. Including the next approximation (which would be exceedingly complicated and tedious) should enable this position to be defined. To obtain the envelope curves in the oscillatory region of the solution, $e^{-\sigma}$ has been approximated by unity^(its first approximation) and the results have been given in figure 8. As already noted, the numerical integration confirms the predicted envelope curves.

This analysis has shown that the steady-state solution, for small damping, to the K-V-B equation is a cnoidal wave with varying period passing through all the possible wave types, culminating in the solitary wave. Such a wave profile is observed in the undular bore. The measurements of Sandover and Taylor (1962) in fact show that the leading wave is very close indeed to the solitary wave.

In conclusion it is interesting to note the form of the total energy dissipated. Benjamin and Lighthill (1954) con-

sidered the general form of the undular bore, and suggested that the undular nature occurs when only some energy is dissipated at the front. The excess energy can then be carried away by a train of waves. The present model indicates just this phenomenon. The total energy dissipated is proportional to

$$\gamma \int_{-\infty}^S u_s^2 ds = E(S) \quad \text{from } +\infty \quad \text{upto } S.$$

Using equation (2.20) then

$$E(S) = \frac{1}{2} u_s^2 + \frac{4}{3} u^3 - 2 u^2.$$

Thus the energy dissipated upto a particular wave peak ($u_s = 0$) is

$$E_p(S_p) = \frac{4}{3} u_p^3 - 2 u_p^2 \quad (2.35)$$

where u_p is the peak at $S = S_p$. This function is plotted in figure 9. Denoting the total energy to be dissipated by E_T , the results are as expected. If the first peak is at $u_p = 1$ then all the energy loss has occurred and the peak is at $S = -\infty$ - a monotonic profile. If the peak is $u_p = \frac{3}{2}$ there is no energy loss at all and the profile is the solitary wave (which, of course, can exist on an inviscid flow).

The peak being $1 < u_p < \frac{3}{2}$ ($4 > \gamma > 0$), some energy is dissipated and following peaks move up the curve (figure 9) approaching $E_p/E_T = 1$. Thus the K-V-B equation certainly appears not only to describe the undular bore in detail but also to embody the ideas of Benjamin and Lighthill.

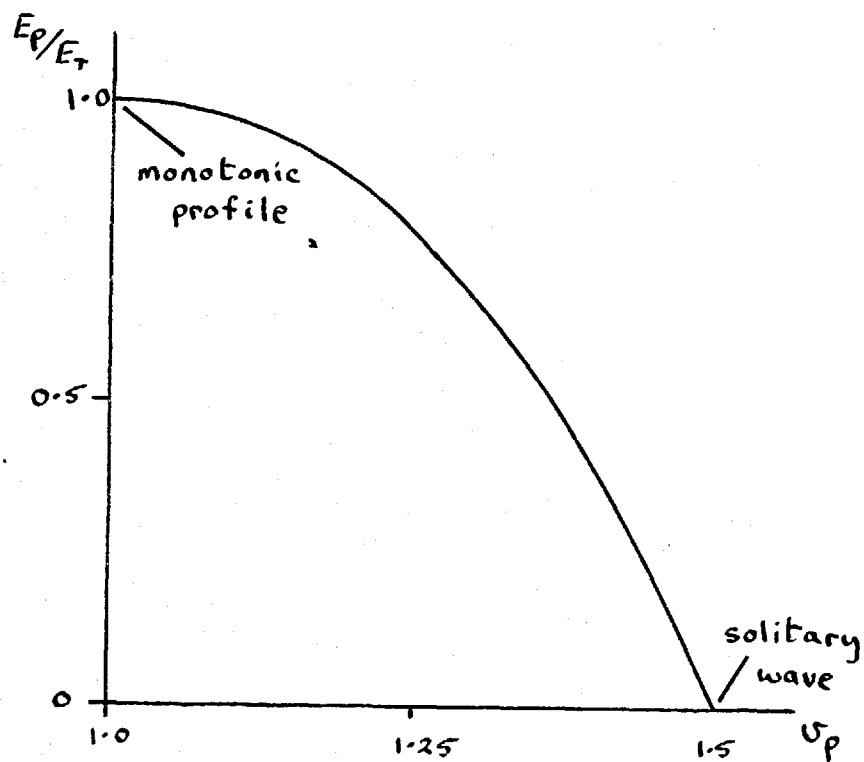


Figure 9 - Plot of the energy dissipated, against peak v_p

Part II

Fluid flow through an elastic tube

3. Inviscid small disturbance theory for the tube with taper

This first analysis, using the simplest form of Hooke's law at the wall, is intended as an introduction to the problem of flow through an elastic tube. Due to the simplicity of the analysis it is possible to permit the tube to have a slight taper in its unstressed state. In the more sophisticated work of chapter 4 the tube is assumed to have a constant (unstressed) radius.

The tube will be modelled in the most obvious way by considering it as an axi-symmetric, semi-infinite tube untethered and free in space. The disturbance to the tube wall will be caused by some form of piston at one end. The tube wall is thin (thickness/radius small) and the Hooke's law to describe its motion is written

$$p' - p'_0 = \frac{Eh}{R'_n} \left(\frac{R' - R'_0}{R'_0} \right) \quad (3.1)$$

where p' is the pressure in the fluid at the tube wall, R' is the radius of the tube and zero subscripts denote unstressed conditions. The Young's modulus for the wall material is E , the thickness is h and a mean radius is R'_n . The problem can now be written down in mathematical terms.

3.1 The equations and boundary conditions

The co-ordinate system and the variables are given in

figure 10. z' and r' are the axial and radial co-ordinates, respectively; u' and v' are the axial and radial components of velocity. The fluid is assumed incompressible and inviscid so that the Euler equations of motion for cylindrical axis-symmetric flow are

$$u'_t + u'u'_z + v'u'_r + \frac{1}{\rho} p'_z = 0 \quad (3.2)$$

$$v'_t + u'v'_z + v'v'_r + \frac{1}{\rho} p'_r = 0 \quad (3.3)$$

where t' is a time co-ordinate and ρ is the constant density of the fluid. The equation of continuity is

$$u'_z + \frac{1}{r'}(r'v')_r = 0. \quad (3.4)$$

Since the tube is allowed to taper, we also have that $R'_0 = R'_0(z')$ and the disturbed radius is given by $R'(z', t')$. The disturbance itself is produced by a piston at $z' = 0$, and is such that

$$u'(0, r', t') = \varepsilon f'(t'), \quad t' \geq 0, \quad \varepsilon \ll 1 \quad (3.5)$$

where $\left| \frac{f'(t')}{a} \right|_{\max} = 1$ and a is the characteristic velocity given by the Moëns value,

$$a = \sqrt{\frac{Eh}{2\rho R'_0}}. \quad (3.6)$$

The disturbance has been chosen to occur in velocity and we note that it is independent of r' . To the approxi-

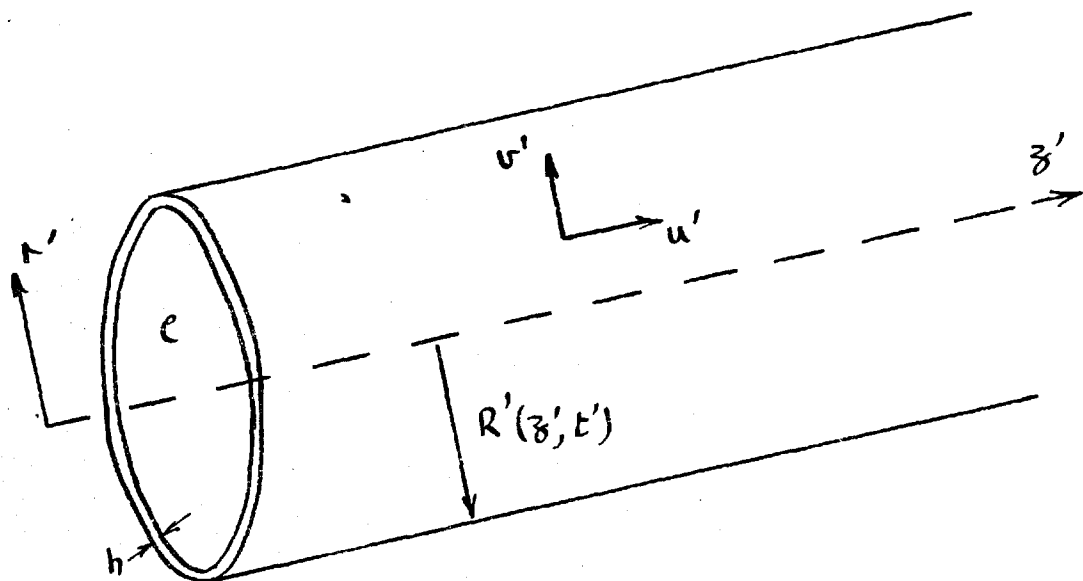


Figure 10 - Co-ordinates and variables for the tube with taper

mation that we shall be using here, the axial velocity will be assumed independent of the radial co-ordinate throughout. This results in a major simplification, but is not one of the restrictions imposed upon the analysis presented in chapter 4. With $\varepsilon \ll 1$, this work is thus a small disturbance theory.

The initial velocity function $f'(t')$ is assumed to be characterised by the frequency ω , and this introduction enables the variables to be non-dimensionalised,

$$\left. \begin{aligned} t &= \omega t', \quad z = \frac{\omega z'}{a}, \quad r = \frac{r'}{R_H'}, \quad u = \frac{u'}{\varepsilon a}, \quad v = \frac{v'}{\varepsilon \omega R_H'}, \\ p &= \frac{p' - p_0'}{\varepsilon \rho a^2}, \quad R = \frac{R'}{R_H'}, \quad R_0 = \frac{R_0'}{R_H'}, \quad f(t) = \frac{f'}{a}. \end{aligned} \right\} \quad (3.7)$$

The equations (3.2 - 3.4) now become

$$\left. \begin{aligned} u_t + \varepsilon u u_z + \varepsilon v u_r + p_z &= 0 \\ \delta^2 (v_t + \varepsilon u v_z + \varepsilon v v_r) + p_r &= 0 \\ u_z + \frac{1}{r} (r v)_r &= 0 \end{aligned} \right\} \quad (3.8)$$

where $\delta = \frac{\omega R_H'}{a}$ is a frequency parameter. To reduce the problem even further, the effects of wave dispersion relative to non-linearity can be neglected if we choose $\delta^2 < O(\varepsilon)$. This we do, and ask that u not be a function of the radial co-ordinate. The equations (3.8) thus simplify to

$$u_t + \varepsilon u u_z + p_z = 0 \quad (3.9)$$

$$p_r = O(\delta^2) = o(\epsilon) \quad (3.10)$$

$$\sigma = -\frac{1}{2} r u_z \quad (3.11)$$

where the velocity σ satisfies the symmetry condition that $\sigma = 0$ on $r = 0$. Similar assumptions to those used here were implemented by Seymour and Varley (1966) on the problem of fluid flow through a tube of visco-elastic material.

The fact that the fluid is inviscid, and $u_r = 0$, means that the two boundary conditions at the wall (on the axial and radial velocities) are replaced by the single condition that the wall is a stream surface. Thus

$$\sigma' \Big|_{\text{wall}} = R_t' + u' R_z' \quad (3.12)$$

in dimensional variables. But from (3.11) we see that

$$\sigma \Big|_{\text{wall}} = -\frac{1}{2} R u_z$$

so that non-dimensionalising (3.12) we obtain

$$R_t + \epsilon \left(u R_z + \frac{1}{2} R u_z \right) = 0, \quad (3.13)$$

or in terms of the non-dimensional area ($A = \pi R^2$),

$$A_t + \epsilon (u A)_z = 0. \quad (3.14)$$

Clearly, both (3.13) and (3.14) are alternative forms of the

continuity equation for the flow. In fact it is more convenient to work with the radius since from the formulation of Hooke's law, (3.1), we can introduce a non-dimensional radial perturbation, $\eta(z, t)$,

$$\frac{R' - R'_0}{R'_0} = \frac{R - R_0}{R_0} = \varepsilon \eta(z, t). \quad (3.15)$$

Eliminating R between (3.13) and (3.15) gives

$$\eta_t + \frac{1}{2} u_z + u \frac{\dot{R}_0}{R_0} + \varepsilon \left(u \eta_z + \frac{1}{2} \eta u_z + u \eta \frac{\dot{R}_0}{R_0} \right) = 0 \quad (3.16)$$

where $\dot{R}_0 \equiv \frac{dR_0}{dz}$. For the wall to have a small taper, we ask

in particular that it be defined by

$$\frac{\dot{R}_0}{R_0} = \varepsilon S(z), \quad S = O(1), \quad (3.17)$$

so that equation (3.16) becomes

$$\eta_t + \frac{1}{2} u_z + \varepsilon \left(u S + u \eta_z + \frac{1}{2} \eta u_z \right) = O(\varepsilon^2). \quad (3.18)$$

Finally, the non-dimensionalised equation of state for the wall is

$$p = 2\eta. \quad (3.19)$$

Now since $p_r =_0(\varepsilon)$, (3.9) can be rewritten as

$$u_t + \varepsilon u u_z + 2\eta_z = O(\varepsilon). \quad (3.20)$$

The problem has been reduced to that of solving the fairly simple pair of equations (3.18) and (3.20), for u and η , where ϵ is a small parameter denoting the scale of the non-linearity.

3.2 Solution

The pair of equations (3.18) and (3.20) are of a classical form often encountered in perturbation problems, and of course non-linear wave propagation problems. We expect that initially the disturbance will travel according to a linear wave equation. However, non-linearity on the scale of ϵ^{-1} will eventually cause the wave to distort to the extent of forming a discontinuous front. In this simple problem there is no dispersion or damping to alleviate the steepening.

The near-field (where z and t are of $O(1)$) can be analysed by seeking a solution of the form

$$\left. \begin{aligned} u(z, t; \epsilon) &= u_0(z, t) + \epsilon u_1(z, t) + o(\epsilon) \\ \eta(z, t; \epsilon) &= \eta_0(z, t) + \epsilon \eta_1(z, t) + o(\epsilon) \end{aligned} \right\} \quad (3.21)$$

Upon substituting into the equations we obtain

$$\eta_{0t} + \frac{1}{2} u_{0z} = 0 ; \quad u_{0t} + 2\eta_{0z} = 0,$$

and thus

$$u_{0tt} - u_{0zz} = 0. \quad (3.22)$$

Applying the initial condition at $z=0$ and noting that

no waves are coming in from infinity, we have the final solution

$$u_0 = f(t-z) ; \quad \eta_0 = \frac{1}{2} f(t-z). \quad (3.23)$$

The solution (3.23) indicates that the disturbance moves predominantly unchanged initially at a wave speed of unity in the positive z -direction. In dimensional variables this implies that the wave speed is the classical Moëns value.

The asymptotic expansions (3.21) are expected to become invalid when $z = O(\varepsilon^{-1})$ and $(t-z)$ is kept fixed. Thus we introduce the new large distance scale ($\bar{z}$) indicating the build-up of the non-linearity, and consider its effects near to the front of the wave by introducing a characteristic co-ordinate ($\bar{t}$),

$$\bar{z} = \varepsilon z, \quad \bar{t} = t - z. \quad (3.24)$$

Writing (3.24) into (3.18) and (3.20) gives

$$\left. \begin{aligned} \eta_{\bar{t}} - \frac{1}{2} u_{\bar{t}} + \varepsilon (u_{\bar{z}} - u \eta_{\bar{t}} - \frac{1}{2} \eta u_{\bar{t}} + \frac{1}{2} u_{\bar{z}}) &= O(\varepsilon^2) \\ u_{\bar{t}} - 2 \eta_{\bar{t}} + \varepsilon (2 \eta_{\bar{z}} - u u_{\bar{t}}) &= O(\varepsilon) \end{aligned} \right\} \quad (3.25)$$

where $S(\bar{z}) = S(\frac{\bar{z}}{\varepsilon})$. A solution is now sought to the pair (3.25) in the form of the expansions

$$\left. \begin{aligned} u(\xi, z; \epsilon) &= U_0(\xi, z) + \epsilon U_1(\xi, z) + o(\epsilon) \\ \eta(\xi, z; \epsilon) &= H_0(\xi, z) + \epsilon H_1(\xi, z) + o(\epsilon). \end{aligned} \right\} \quad (3.26)$$

The zeroth order does not define $U_0(\xi, z)$ and merely tells us that

$$U_0 = 2 H_0,$$

but forming the equations for U_1 and H_1 , and then eliminating both these gives a single equation for U_0 ,

$$U_{0z} - \frac{\xi}{4} U_0 U_{0\xi} + U_0 S = 0. \quad (3.27)$$

Equation (3.27) is non-linear and includes the effects of taper. In fact this equation incorporates the near-field solution as $z \rightarrow 0$, and thus it is straightforward to seek a solution of the form

$$U_0 = f(\phi) g(z) \quad (3.28)$$

where ϕ is a parameter identified with the characteristic surface defined by

$$\left(\frac{\partial \xi}{\partial z} \right)_\phi = -\frac{\xi}{4} U_0. \quad (3.29)$$

The function f is just that defining the piston motion, and to match with the initial conditions we must have $\phi = \xi$ on $z = 0$.

The taper term, $S(z)$, can be rewritten in terms of the

area change since we have

$$\varepsilon S(\xi) = \frac{1}{2A_0} \frac{dA_0}{d\xi}, \quad \pi R_0^2 = A_0,$$

and thus $S(\xi) = \frac{1}{2A_0} \frac{dA_0}{d\xi}$. Equation (3.27) now becomes

$$U_{0z} - \frac{\varepsilon}{4} U_0 U_{0\xi} + \frac{1}{2} \frac{\dot{A}_0}{A_0} U_0 = 0 \quad (3.30)$$

where $\dot{A}_0 \equiv \frac{dA_0}{d\xi}$.

Substituting (3.28) into (3.30) and using (3.29) gives

$$f'g\left(\phi - \frac{\varepsilon}{4} U_0 \phi_\xi\right) + f\left(g' + \frac{1}{2} \frac{\dot{A}_0}{A_0} g\right) = 0$$

and noting that $\left(\frac{\partial \xi}{\partial z}\right)_\phi = -\frac{\phi_z}{\phi_\xi}$, this immediately reduces to

$$g(z) = \sqrt{\frac{A_i}{A_0(z)}}, \quad (3.31)$$

where $A_i = A_0(0)$.

Now integrating equation (3.29), we obtain

$$\xi = \phi - \frac{\varepsilon}{4} f(\phi) \sqrt{A_i} \int_0^z \frac{dz}{\sqrt{A_0}}. \quad (3.32)$$

This expression is equivalent to that obtained by Whitham (1956) when examining the non-linear propagation of waves in ray tubes of varying area. As explained by Courant and Friedrichs (1967), the solution defined by (3.28) and (3.32) is not single valued in the (ξ, z) -plane where

$$\frac{\partial \xi}{\partial \phi} = 0,$$

that is, where

$$\frac{\xi}{4} f'(\phi) \sqrt{A_i} \int_0^z \frac{dz}{\sqrt{A_0}} = 1, \quad \xi = \phi - \frac{f}{f'}. \quad (3.33)$$

The path defined by expressions (3.33) is in general cusped and the solution is made single valued by the insertion of a Rankine-Hugoniot shock originating at the cusp (see figure 11). The cusp itself occurs where

$$\frac{\partial \xi}{\partial \phi} = \frac{\partial \xi}{\partial \phi} = 0, \text{ or } f''(\phi) = 0.$$

Inserting the Rankine-Hugoniot shock by equating the shock velocity to the average of the upstream and downstream propagation velocities (the weak shock wave result) gives

$$\left(\frac{\partial \xi}{\partial z} \right)_{\phi=\bar{\phi}} = -\frac{1}{2} \cdot \frac{\xi}{4} U_0.$$

This assumes that there is no disturbance upstream (and downstream the velocity is $\frac{\xi}{4} U_0$), and on the shock $\phi = \bar{\phi}(z)$. Differentiating (3.32) with respect to z and evaluating on the shock gives

$$\bar{\phi}'(z) = \frac{\xi}{4} f'(\bar{\phi}) \bar{\phi}'(z) \sqrt{A_i} \int_0^z \frac{dz}{\sqrt{A_0}} + \frac{\xi}{8} f(\bar{\phi}) \sqrt{\frac{A_i}{A_0}}.$$

After multiplying by $f(\bar{\phi})$, this equation can be integrated to give

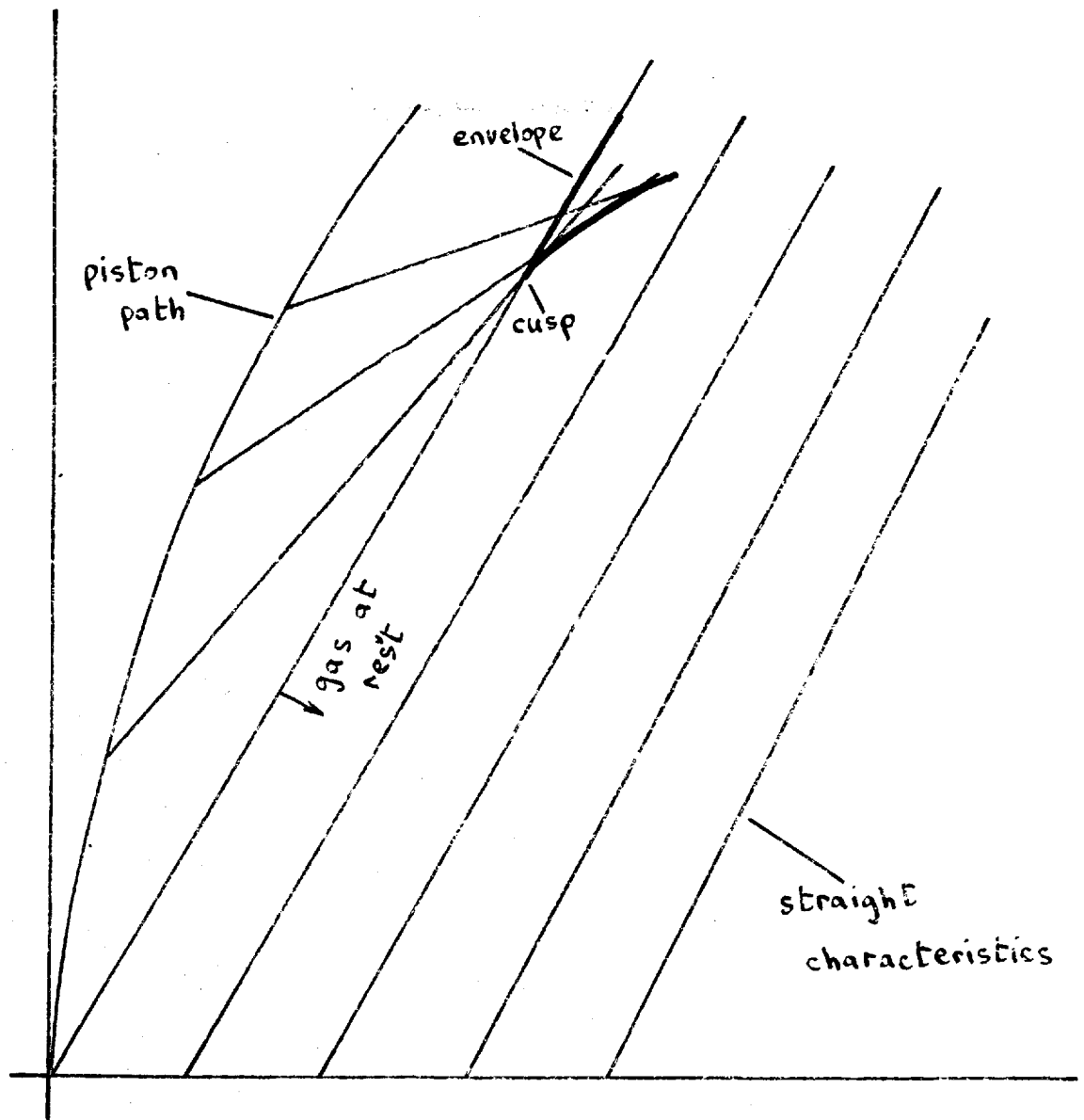


Figure 11 - The formation of a shock

$$\int_0^{\phi} f(\phi) d\phi = \frac{5}{8} f^2 \sqrt{A_i} \int_0^z \frac{dz}{\sqrt{A_0}} \quad (3.34)$$

which together with

$$\xi_w = \phi - 2 \frac{\int_0^{\phi} f(\phi) d\phi}{f(\phi)} \quad (3.35)$$

are parametric equations for the position of the shock at $\xi_w(z)$. Whitham (1956) obtained similar results when examining the particle velocity in a ray tube of area $A(z)$. He showed that an initial wave profile $F(t)$ gave a particle velocity of the form $F(\tau)/\sqrt{A}$, where $\tau(t, z)$ included the distortion of the wave profile. He also fitted a Rankine-Hugoniot shock into the flow and obtained a relation for its position defined by parametric equations like (3.34) and (3.35).

The elementary theory that has been presented here shows how the simplest possible model for the flow of a fluid through an elastic tube produces classical wave propagation results. The essential propagation velocity introduced by Moens is confirmed, and the ray tube work of Whitham (1956) reproduced for this particular problem. With only non-linearity included in the equations a shock wave can be formed, and its position found parametrically.

4. Wave propagation in a fluid-filled elastic tube with a mean flow produced by a body force

The simplified theory described in chapter 3 is now extended by the work presented in this chapter. The fluid will now be assumed viscous, although still incompressible, and a mean flow will be constructed in the tube. The easiest way to accomplish this, as for the channel flow, is to produce the steady mean flow by balancing the viscous stresses against a body force. This allows a fully developed Poiseuille pipe flow to be formed in the tube, without any effect on the motion of the tube wall.

The elastic tube is represented by a semi-infinite tube of circular cross-section without any taper. A body force is aligned with the axis of the tube and the external surface of the tube wall is restrained. This restraint, or tethering, is assumed to inhibit all longitudinal motion of the wall but leave the radial motion quite unaffected. As mentioned in the Introduction, this particular form of restraint is in fact an important phenomenon of arteries in situ (McDonald, 1960). To produce fairly realistic model for the tube wall, we shall make use of the thin shell theory of Timoshenko (1940), and full details will be given here.

4.1 The equations of motion for the fluid

The co-ordinates and variables for the flow are given in figure 12. z' is measured along the axis, and r' radially. u' and v' are the z' - and r' -components of velocity in the fluid, respectively. ξ' and η' are the longitudinal and radial components of the motion of the tube wall. ξ' will shortly be assumed zero in accordance with the tethering of the tube. R'_0 is the undisturbed radius of the tube under mean flow conditions, and h is the thickness of the wall. The constant density and kinematic viscosity of the fluid are ρ and ν respectively. The wall material has a density ρ_0 .

The axi-symmetric form of the Navier-Stokes equations is

$$u'_t + u'u'_z + v'u'_r = -\frac{1}{\rho}p'_z + g + \nu(u'_{zz} + \frac{1}{r'}u'_{rz} + u'_{rr}) \quad (4.1)$$

$$v'_t + u'v'_z + v'v'_r = -\frac{1}{\rho}p'_r + \nu(v'_{zz} + v'_{rr} + \frac{1}{r'}v'_r - \frac{1}{r'^2}v'), \quad (4.2)$$

where p' is the pressure in the fluid and g is the body force aligned with the tube axis. The equation of continuity is

$$u'_z + \frac{1}{r'}(r'v')_r = 0. \quad (4.3)$$

Introducing the Moëns velocity of propagation

$$a = \sqrt{\frac{Eh}{2\rho R'_0}} \quad (4.4)$$

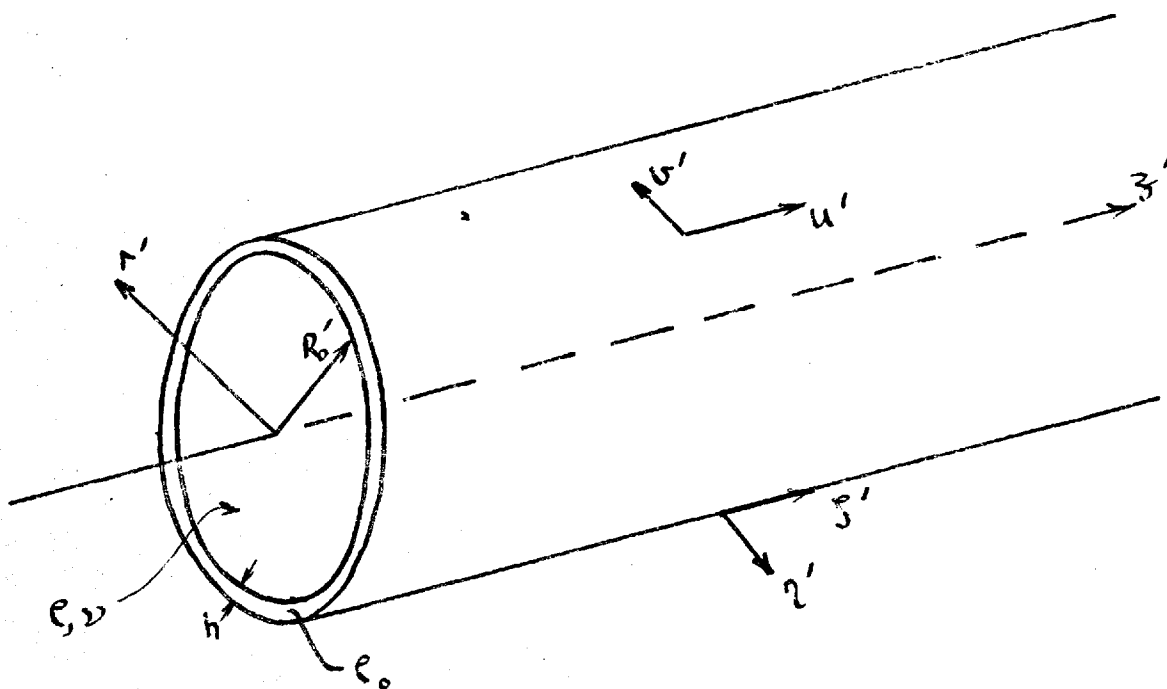


Figure 12 - Co-ordinates and variables for the elastic tube

where E is the Young's modulus for the wall material, and a characteristic frequency, ω , of the input disturbance at $z' = 0$, we non-dimensionalise the equations according to

$$t = \omega t', \quad z = \frac{\omega}{a} z', \quad r = \frac{r'}{R_0}, \quad \bar{u} = \frac{u'}{a}, \quad \bar{v} = \frac{v'}{\omega R_0}, \quad \bar{p} = \frac{p' - p_0'}{e a^2}. \quad (4.5)$$

p_0' is the constant pressure in the fluid when only the steady mean flow is occurring. The equations (4.1 - 4.3) thus become

$$\left. \begin{aligned} \bar{u}_t + \bar{u}\bar{u}_z + \bar{v}\bar{u}_r + \bar{p}_z &= \alpha + \frac{1}{Re} \left[\bar{u}_{zz} + \frac{1}{\delta^2} \left(\bar{u}_{rr} + \frac{1}{r} \bar{u}_r \right) \right] \\ \delta^2 [\bar{v}_t + \bar{u}\bar{v}_z + \bar{v}\bar{v}_r] + \bar{p}_r &= \frac{\delta^2}{Re} \left[\bar{v}_{zz} + \frac{1}{\delta^2} \left(\bar{v}_{rr} + \frac{1}{r} \bar{v}_r - \frac{1}{r^2} \bar{v} \right) \right] \\ \bar{u}_z + \frac{1}{r} (r\bar{v})_r &= 0. \end{aligned} \right\} \quad (4.6)$$

In (4.6) we have introduced the parameters

$$\alpha = \frac{g}{a\omega}, \quad \delta = \frac{\omega R_0'}{a}, \quad Re = \frac{a^2}{\nu\omega}.$$

α describes the size of the body force as compared to the characteristic acceleration $a\omega$; δ is the frequency parameter previously defined in chapter 3 and Re is a flow Reynolds number.

For a steady uniform flow through the tube, then we must have $\bar{u} = U(r)$ only, which gives

$$\alpha + \frac{1}{S^2 Re} (U'' + \frac{1}{r} U') = 0$$

so that

$$U(r) = F(1-r^2). \quad (4.7)$$

The primes denote differentiation, and we have defined $F = \frac{1}{4} \alpha S^2 Re = O(1)$ which prescribes the size of α so that a fully developed flow is possible with an order unity maximum velocity. The profile (4.7) satisfies the conditions of zero velocity at the wall ($r=1$) and zero shear at the centre line ($r=0$). We now consider a disturbance from these uniform conditions characterised by an amplitude parameter, ε , and write

$$\bar{u} = U(r) + \varepsilon u, \quad \bar{v} = \varepsilon v, \quad \bar{p} = \varepsilon p. \quad (4.8)$$

The equations of motion, (4.6), now become

$$u_t + (U + \varepsilon u) u_z + (U' + \varepsilon u_r) v + p_z = \frac{1}{Re} \left[u_{zz} + \frac{1}{S^2} (u_{rr} + \frac{1}{r} u_r) \right] \quad (4.9)$$

$$S^2 \left[v_t + (U + \varepsilon u) v_z + \varepsilon v v_r \right] + p_r = \frac{S^2}{Re} \left[v_{zz} + \frac{1}{S^2} (v_{rr} + \frac{1}{r} v_r - \frac{1}{r^2} v) \right] \quad (4.10)$$

$$u_z + \frac{1}{r} (rv)_r = 0. \quad (4.11)$$

For the flow to be axi-symmetric then the conditions at the centre line are

$$\left. \begin{array}{l} v = U' + \epsilon u_r = 0 \\ \text{or } v = u_r = 0 \end{array} \right\} \text{ on } r = 0. \quad (4.12)$$

4.2 The conditions at the tube wall

The detailed model of the tube wall is based on Timoshenko's thin shell theory (Timoshenko, 1940). Since small terms (which are normally neglected) may be of considerable importance here, the equations of motion for the tube wall are derived from first principles and all the relevant terms will be retained. Figure 13 shows an element of the tube.

M_z , Q_z and T_z are the bending moment, shear force and longitudinal force (per unit length), respectively, at any section of the tube. T_θ is the hoop force per unit length, and with the usual notation the assumption of axial symmetry gives

$$T_{z\theta} = T_{\theta z} = 0, \quad M_{z\theta} = M_{\theta z} = 0, \quad T_\theta|_{z',t'} = \text{const.}, \quad M_\theta|_{z',t'} = \text{const.}$$

All the forces on the element act either axially or radially.

$R'(z',t') = R_0 + \eta'$ is the radius of the tube and P_1 and P_2 are the loads exerted by the fluid on the inside face of the tube, per unit area. L is the load exerted by the restraining mechanism on the outer surface of the wall. P_1 is radial, P_2 axial and L is assumed to have an axial com-

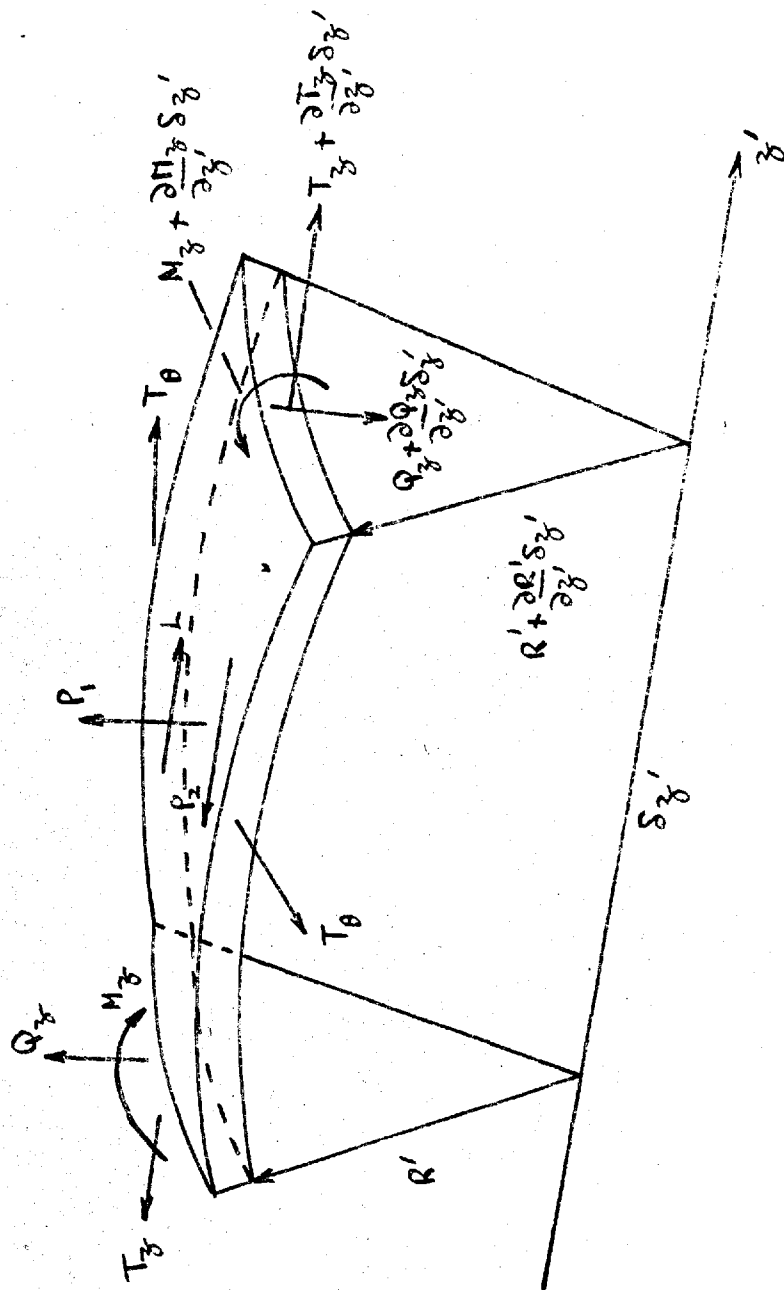


Figure 13 - Element of the tube wall

ponent only, so that the tethering does not effect the radial motion.

If η' and ξ' are the radial and longitudinal movements of the wall, then the general equations of motion for the wall become

$$L + \frac{\partial T_z}{\partial z'} - p_2 = e_0 h \frac{\partial^2 \xi'}{\partial t'^2} \quad (\text{long.}) \quad (4.13)$$

$$p_1 - \frac{T_\theta}{R'} - \frac{\partial Q_z}{\partial z'} = e_0 h \frac{\partial^2 \eta'}{\partial t'^2} \quad (\text{rad.}) \quad (4.14)$$

$$\frac{\partial M_z}{\partial z'} - Q_z - T_z \frac{\partial R'}{\partial z'} = \frac{e_0 h^3}{12} \frac{\partial^2 \theta}{\partial t'^2} \quad (\text{mom.}) \quad (4.15)$$

θ is such that $\tan \theta$ is the slope of the wall at any (z', t') . Now equation (4.13) serves to define $L(z', t')$ necessary to give no longitudinal motion i.e. $\xi' \equiv 0$. When this is the case, the slope of the wall is given by

$$\tan \theta = \eta'_{z'}. \quad (4.16)$$

The radial stress, p_1 , exerted by the fluid can be written as

$$\left. \begin{aligned} -p_1 &= (\tau_{rr'} + p'_0) \cos \theta - \tau_{r'z'} \sin \theta \\ \tau_{rr'} &= -p' + 2\mu \sigma'_{r'}, \quad \tau_{r'z'} = \mu(u'_{r'} + v'_{z'}) \end{aligned} \right\} \quad (4.17)$$

and μ is the viscosity of the fluid.

Since all the loads and extensions are defined either axially or radially, the application of Hooke's law gives

$$\left. \begin{aligned} T_{\theta} &= \frac{Eh}{1-\sigma^2} \left(\frac{\gamma'}{R_0'} + \sigma \frac{\partial S'}{\partial \gamma'} \right) = \frac{Eh}{1-\sigma^2} \cdot \frac{\gamma'}{R_0'} \\ T_{\gamma} &= \frac{Eh}{1-\sigma^2} \left(\frac{\partial S'}{\partial \gamma'} + \sigma \frac{\gamma'}{R_0'} \right) = \frac{Eh\sigma}{1-\sigma^2} \cdot \frac{\gamma'}{R_0'} \end{aligned} \right\} + O\left(\frac{Eh^3\gamma'}{R_0'^3}\right) \quad (4.18)$$

where $0 \leq \sigma \leq \frac{1}{2}$ is Poisson's ratio for the material of the wall. As derived in Timoshenko (1940), the two relations (4.18) are obtained by averaging the stresses across the (small) shell thickness. This is achieved by performing, say, the integration

$$T_{\gamma} = \int_{-h/2}^{h/2} \sigma_{\gamma} \left(1 - \frac{h}{R_0'}\right) dh$$

where σ_{γ} is the 'membrane' stress in the γ -direction. The error involved in this approximation is given by Timoshenko and quoted in (4.18).

The use of Hooke's law here is perhaps slightly restrictive especially since the type of tube we are envisaging is quite likely to exhibit non-Hookean properties. However, since we are considering small disturbances to the system we are asking only that the tube obeys Hooke's law locally, and this seems quite reasonable. The mammalian artery itself (McDonald, 1960) is known to agree fairly well with a linear law over most (but not all) its working range. The use of

a non-Hookean equation of state has been considered by Jacobs (1953), Seymour and Varley (1966) and Olsen and Shapiro (1967).

Similar to the results (4.18) we also have

$$M_z = -\frac{Bh^3}{12} \frac{\frac{\partial}{\partial z'} \left(\frac{\partial \eta'}{\partial z'} \right)}{\left[1 + \left(\frac{\partial \eta'}{\partial z'} \right)^2 \right]^{3/2}} \quad (4.19)$$

to the same approximation as in (4.18). Here, we have made use of the definition

$$B = E\tau = E(1-\sigma^2)^{-1}.$$

We can now eliminate Q_z between (4.14) and (4.15), and by using (4.16, 4.17) we obtain the single equation

$$\left. \begin{aligned} & -\frac{Bh^3}{12} \left[\frac{\eta'_{zz'}}{(1+\eta'^2_{z'})^{3/2}} \right]_{zz'} + \rho_0 h \eta'_{\theta\theta} + \frac{Bh\eta'}{R'_0(R'_0+\eta')} \\ & + (\rho'_0 - \rho' + 2\mu v'_{r'}) \cos \theta - \mu(u'_{r'} + v'_{z'}) \sin \theta \\ & - \frac{Bh\sigma}{R'_0} (\eta' \eta'_{z'})_{z'} = \frac{\rho_0 h^3}{12} \theta_{\theta\theta z'} \end{aligned} \right\} \quad (4.20)$$

which is to be evaluated at the wall, $r' = R'_0 + \eta'$. This equation can be non-dimensionalised by introducing the variables (4.5) together with

$$\eta = \frac{\eta'}{\epsilon R'_0}, \quad (4.21)$$

to give

$$\left. \begin{aligned} & -\frac{\tau}{6} \left(\frac{h}{R_0'}\right)^2 \delta^4 \left[\eta_{zz} (1 + \epsilon^2 \delta^2 \eta_z^2)^{-3/2} \right]_{zz} + \frac{\rho_0 h}{\epsilon R_0'} \delta^2 \eta_{tt} + 2\tau \eta (1 + \epsilon \eta)' \\ & + (-p + \frac{2}{R_e} v_r) \cos \theta - \frac{1}{\epsilon \delta R_e} (U' + \epsilon u_r + \epsilon \delta^2 v_z) \sin \theta \\ & - 2\epsilon \sigma \tau \delta^2 (\eta \eta_z)_z = \frac{1}{12} \frac{\rho_0}{\epsilon} \left(\frac{h}{R_0'}\right)^3 \frac{\delta^3}{\epsilon} \theta_{ttz} \text{ on } r = 1 + \epsilon \eta. \end{aligned} \right\} \quad (4.22)$$

The slope is

$$\tan \theta = \epsilon \delta \eta_z. \quad (4.23)$$

The final boundary condition states that the fluid velocity and wall velocity must be continuous at the wall. Since the wall is longitudinally restrained this becomes

$$\left. \begin{aligned} U + \epsilon u &= 0 \\ v &= \eta_t + (U + \epsilon u) \eta_z \end{aligned} \right\} \text{ on } r = 1 + \epsilon \eta. \quad (4.24)$$

We note that the first condition of (4.24) is only applicable to viscous fluids. If, at a later stage, we wish to examine the model of an inviscid fluid then this condition must be waived.

The problem that we are interested in is the small disturbance low frequency limit defined by

$$\epsilon \ll 1 \quad \text{and} \quad \delta \ll 1.$$

The equation describing the surface motion of the tube, with the fluid velocity defined, is given in its full generality by (4.22), within the framework of thin shell theory. Applying the limiting procedure to (4.22) and (4.23) we obtain

$$\theta = \varepsilon \delta \eta_z + O(\varepsilon^2 \delta^2) \quad (4.25)$$

and

$$\left. \begin{aligned} & -\frac{\tau}{6} \left(\frac{h}{R'_0}\right)^2 \delta^4 \eta_{zzzz} + \frac{e_0}{c} \frac{h}{R'_0} \delta^2 \eta_{tt} + 2\tau \eta (1 + \varepsilon \eta)' \\ & + \left(-\rho + \frac{2}{R_e} \sigma_r\right) \left(1 - \frac{\varepsilon^2 \delta^2}{2} \eta_z^2\right) - \frac{1}{R_e} (U' + \varepsilon U_r + \varepsilon \delta^2 \sigma_z) \left(\eta_z - \frac{\varepsilon^2 \delta^2}{6} \eta_z^3\right) \\ & - 2\sigma \tau \varepsilon \delta^2 (\eta \eta_z)_z = \frac{1}{12} \frac{e_0}{c} \left(\frac{h}{R'_0}\right)^3 \delta^4 \eta_{zztt} + O\left(\varepsilon \delta^4, \frac{e_0}{c} \left(\frac{h}{R'_0}\right)^3 \varepsilon \delta^5, \left(\frac{h}{R'_0}\right)^2 \varepsilon^2 \delta^6\right) \end{aligned} \right\} \quad (4.26)$$

on $r = 1 + \varepsilon \eta$,

where all the different types of error term have been written in the order of magnitude notation.

It is interesting, and serves as a check, to note that if the limit $\varepsilon \rightarrow 0$ is taken, and the remaining dominant terms in δ retained, then (4.26) reduces to

$$\left(\frac{\partial^2}{\partial t^2} + \frac{2\tau e R'_0}{e_0 h} \frac{\partial^2}{\partial z^2} \right) \eta_{zz} = 0. \quad (4.27)$$

This is an equation for $\eta(z, t)$ independent of the fluid equations. The equation is the high frequency infinitesimal disturbance limit of the problem, and two solutions are waves (travelling in opposite directions) at a dimensional speed

$$\sqrt{\frac{2e R'_0 \tau a^2}{e_0 h}} = \sqrt{\frac{\beta}{e_0}}.$$

The appearance of β here is due to longitudinal tethering with non-zero Poisson's ration(σ). If σ is zero, then the

wave speed becomes $\sqrt{\frac{E}{\rho_0}}$. This is the classical result for wave propagation in a bar of modulus E and density ρ_0 (see Coulson, 1965).

It is now clear that the problem as defined by the system of equations (4.9 - 4.12, 4.24, 4.26) bears many similarities to the two dimensional channel flow. The important difference, apart from the flow being axi-symmetric, is that the fluid surface condition is vastly more complicated. For the channel flow it was simply that the surface be a free surface.

When γ and τ are of order unity, and both ϵ and δ are small with Re large, a set of equations similar to those due to Burns (1953) for the channel flow are obtained. This indicates that 'dynamic' waves exist in this problem. Likewise, for large enough γ and τ , ~~two~~ of the viscous terms dominate and a 'kinematic' wave theory is derived, again similar to that for the channel flow. Finally, also in direct analogy to the water wave problem, it is possible to scale in such a way as to produce the terms which lead to both the dynamic and kinematic waves (see § 1.3). Since the damping is obviously positive it follows immediately that the dynamic waves can be shown to decay, leaving the main disturbance travelling at the kinematic wave speed. Due to its very close similarities to the work of Chapter 1, the analysis is not presented here but the individual forms

of the two types of waves will be examined.

The particular types of waves are most conveniently produced by considering the two separate problems given by

$$(i) \quad \varepsilon \ll 1, \quad \delta \ll 1, \quad R_e \rightarrow \infty; \quad z, t \text{ fixed},$$

$$(ii) \quad \varepsilon \ll 1, \quad \delta \ll 1, \quad R_e = O(1); \quad z, t \text{ fixed (for near-field)}.$$

These two limiting solutions will be given in § 4.3 and § 4.4.

4.3 Small amplitude, low frequency inviscid limit - the dynamic waves

By making the approximations indicated in (i) above, we are left with the following dominant equations

$$\left. \begin{aligned} u_t + U u_z + U' v + p_z &= 0 \\ p_r &= 0 \\ u_z + \frac{1}{r} (r v)_r &= 0, \end{aligned} \right\} \quad (4.28)$$

together with the boundary conditions

$$\left. \begin{aligned} p &= 2\tau\eta \\ v &= \eta_t + U \eta_z \end{aligned} \right\} \text{ on } r=1 \quad (4.29)$$

and

$$v = u_r = 0 \text{ on } r=0.$$

Where appropriate, the errors involved in the relations

(4.28) and (4.29) have been put equal to zero, so that we have performed the limiting process:

$$Re \rightarrow \infty; \quad \varepsilon, \delta \rightarrow 0.$$

The no-slip condition at the tube wall has been waived since the fluid has been assumed inviscid, and thus $U(r)$ can take any general form. As we see, the set (4.28,4.29) are exactly analogous to those obtained by Burns (1953), and thus we may seek a steady wave solution as a function of r and $\chi = z - ct$ only. χ is a characteristic co-ordinate and c is the (constant) wave speed which is to be found. Using these new co-ordinates, it is a simple matter to combine (4.28) and (4.29) to give

$$(U-c) \left[-\frac{1}{r} (rv)_r \right] + U'v + 2\tau\eta_\chi = 0$$

where all but the kinematic condition on $r=1$ have been applied. This relation can be integrated once

$$v = 2\tau\eta_\chi \frac{(U-c)}{r} \int_0^r \frac{r dr}{(U-c)^2}, \quad \eta = \eta(\chi),$$

where $v=0$ on $r=0$. Now applying the wall kinematic condition we obtain

$$\int_0^1 \frac{r dr}{(U-c)^2} = \frac{1}{2\tau}. \quad (4.30)$$

Expression (4.30) is the axi-symmetric form of the surface

condition given by Burns (1953) and serves to define the propagation speeds, C , in terms of the shear profile $U(r)$. If this profile is Poiseuille, then from (4.7) we have

$$\int_0^1 \frac{r dr}{[F(1-r^2) - c]^2} = \frac{1}{2\tau}$$

which can be directly integrated to give

$$c = \frac{1}{2} \left[F \pm \sqrt{F^2 + 4\tau} \right]. \quad (4.31)$$

This shows how the propagation speed of small disturbances depends upon the shear flow and the coupling between longitudinal and radial motion (when the longitudinal movement is totally restricted). If there was no coupling ($\sigma = 0$) and no mean flow in the tube ($F = 0$), then we obtain

$$c = \pm 1.$$

In dimensional form this is the classical Moens result for the propagation speeds of small disturbances. Thus even the unrealistic inviscid approximation leads to an interesting new result.

The wave speeds of the dynamic waves have been derived. From the two dimensional channel flow problem, we know that such waves eventually decay. To examine the dominant wave, we now consider the case in which the flow Reynolds number

is order unity. The same result can, in fact, be obtained by scaling for appropriately large z and t (see §1.2).

4.4 Small amplitude, low frequency viscous limit - the kinematic wave

Since we expect the main disturbance to be at the kinematic wave speed, we shall be particularly interested in the form of the wave profile at this front. For the water wave problem we derived the Korteweg-de Vries-Burgers equation and it seems reasonable that a similar result will be obtained for the elastic tube. To this end we shall retain the dominant terms in both ϵ and δ^2 , and in particular we define

$$\delta^2 = O(\epsilon) = k\epsilon ; \quad k = O(1). \quad (4.32)$$

This is the scaling that we have used previously and was first noted by Korteweg and de Vries (1895). Relation (4.32) ensures that non-linearity and dispersive effects are of the same order, and by virtue of assuming $Re = O(1)$ damping will also be important.

For convenience we will state all the governing equations, but retain only the terms necessary for our present purposes. Equations (4.9 - 4.11) become

$$u_t + Uu_z + U'v + p_z = \frac{1}{Re} \left[u_{zz} + \frac{1}{k\epsilon} (u_{rr} + \frac{1}{r} u_r) \right] + O(\epsilon) \quad (4.33)$$

$$p_r = \frac{1}{Re} \left(v_{rr} + \frac{1}{r} v_r - \frac{1}{r^2} v \right) + O(\epsilon) \quad (4.34)$$

$$u_z + \frac{1}{r} (rv)_r = 0. \quad (4.35)$$

The relevant form of the boundary conditions is

$$v = u_r = 0 \quad \text{on} \quad r = 0 \quad (4.36)$$

$$u + \epsilon u = 0 \quad (4.37)$$

$$v = \eta_t + (u + \epsilon u) \eta_z \quad (4.38)$$

$$\frac{\rho_0}{\epsilon} \frac{h}{\rho_0'} k \epsilon \eta_{tt} + 2\tau \eta - p + \frac{2}{Re} v_r - \frac{1}{Re} u' \eta_z = O(\epsilon). \quad (4.39)$$

The final condition, (4.39), incorporates the term $\frac{\rho_0}{\epsilon} \frac{h}{\rho_0'} k \epsilon \eta_{tt}$ so that even though $\frac{h}{\rho_0'} \ll 1$, we can have $\beta = \frac{\rho_0 h k \epsilon}{\epsilon \rho_0'} \leq O(1)$. This can occur if the density ratio ρ_0/ϵ is large enough.

In fact, for the case of arterial flow, we note that measurements give $\rho_0/\epsilon \approx 1$ (see McDonald, 1960), so that this term will be negligible. It is retained here to give more generality to the final result.

A near-field solution to the problem defined by (4.33 - 4.39) is sought as a formal expansion

$$\left. \begin{aligned} Q(z, t, r; \epsilon) &= Q_0(z, t, r) + \epsilon Q_1(z, t, r) + \dots \\ \eta(z, t; \epsilon) &= \eta_0(z, t) + \epsilon \eta_1(z, t) + \dots \end{aligned} \right\} \quad (4.40)$$

where $Q \equiv u, v, p$. From (4.33) we immediately obtain

$$u_{0rr} + \frac{1}{r} u_{0r} = 0$$

and the solution satisfying the boundary condition (4.36) is

$$u_0 = u_0(z, t).$$

Applying the condition at the wall, (4.37), then we have

$$U(\eta) + \varepsilon \eta_0 U'(\eta) + \varepsilon u_0 = 0$$

and since $U(r)$ is the Poiseuille profile, we obtain

$$u_0 = 2F\eta_0. \quad (4.41)$$

From the continuity equation we derive

$$v_0 = -rF\eta_{0z}$$

and ensuring that v_0 satisfies the kinematic wall condition gives

$$\eta_{0t} + F\eta_{0z} = 0 \quad ; \quad \eta_0 = \eta_0(z - Ft). \quad (4.42)$$

This result is exactly as obtained for the shallow water wave problem. As there, we know that the asymptotic expansions (4.40) will become invalid when $\varepsilon t = O(1)$ since this is the scale of the non-linearity. Mathematically, the breakdown is concerned with the occurrence of secular terms of the wave equation (4.42), as is well-known in this type of problem (see Cole, 1968).

The equations are rewritten in terms of a characteristic co-ordinate and a large time co-ordinate,

$$\xi = z - Ft, \quad \varepsilon t = T. \quad (4.43)$$

The equations and boundary conditions now become

$$(U-F)u_\xi + U'v + p_\xi = \frac{1}{Re} \left[u_{\xi\xi} + \frac{1}{\kappa\varepsilon} (u_{rr} + \frac{1}{r}u_r) \right] + O(\varepsilon) \quad (4.44)$$

$$p_r = \frac{1}{Re} (v_{rr} + \frac{1}{r}v_r - \frac{1}{r^2}v) + O(\varepsilon) \quad (4.45)$$

$$u_\xi + \frac{1}{r}(rv)_r = 0 \quad (4.46)$$

$$v = u_r = 0 \quad \text{on} \quad r=0 \quad (4.47)$$

$$U + \varepsilon u = 0 \quad (4.48)$$

$$v = \varepsilon \eta_T + (U-F)\eta_\xi + \varepsilon u \eta_\xi \quad \left. \vphantom{v = \varepsilon \eta_T + (U-F)\eta_\xi + \varepsilon u \eta_\xi} \right\} \quad \text{on} \quad (4.49)$$

$$p \eta_{\xi\xi} + 2\tau \eta - p + \frac{2}{Re} v_r - \frac{1}{Re} U' \eta_\xi = O(\varepsilon). \quad \left. \vphantom{p \eta_{\xi\xi} + 2\tau \eta - p + \frac{2}{Re} v_r - \frac{1}{Re} U' \eta_\xi = O(\varepsilon)} \right\} \quad r=1+\varepsilon \eta \quad (4.50)$$

As before, we seek a formal expansion solution

$$Q(\xi, T, r; \varepsilon) = Q_0(\xi, T, r) + \varepsilon Q_1(\xi, T, r) + \dots$$

$$\eta(\xi, T; \varepsilon) = H_0(\xi, T) + \varepsilon H_1(\xi, T) + \dots$$

where $Q \equiv u, v, p$. The zeroth order terms give

$$u_0 = 2FH_0, \quad v_0 = -rFH_{0\xi} \quad (4.51)$$

and $H_0(\xi, T)$ identically satisfies the kinematic condition so that H_0 remains undefined. To obtain an equation for H_0 we must examine the next approximation, in the course of which H_1 will not be defined. From (4.45, 4.50, 4.51) we have

$$\begin{aligned}
 p_o &= \beta H_o \xi \xi + 2\tau H_o - \frac{1}{Re} (-2F) H_o \xi + \frac{2}{Re} (-F) H_o \xi \\
 &= \beta H_o \xi \xi + 2\tau H_o
 \end{aligned} \tag{4.52}$$

so that if β is small the pressure at the wall is given by the Hooke's law result. The inclusion of the β term here indicates the effect of the wall inertia on the fluid pressure. Substituting into (4.44) we derive an equation for u_1 ,

$$\begin{aligned}
 [F(1-r^2) - F] 2FH_o \xi + (-2rF)(-rFH_o \xi) + \beta H_o \xi \xi \xi + 2\tau H_o \xi \\
 = \frac{1}{Re} 2F' H_o \xi \xi + \frac{1}{kRe} \frac{1}{r} (r u_{1,r})_r
 \end{aligned}$$

which simplifies to give

$$\frac{1}{kRe} u_{1,r} = \frac{r}{2} \left[\beta H_o \xi \xi \xi + 2\tau H_o \xi - \frac{1}{Re} 2FH_o \xi \xi \right].$$

Integrating once more and using the no-slip condition (4.48),

$$\frac{1}{kRe} u_1 = \left(\frac{r^2 - 1}{4} \right) \left[\beta H_o \xi \xi \xi + 2\tau H_o \xi - \frac{1}{Re} 2FH_o \xi \xi \right] + \frac{1}{kRe} [2FH_1 + FH_o^2].$$

Now using the continuity equation we can find u_1 and thus write it in the kinematic condition. u_1 identically cancels leaving

$$H_o \tau + 4FH_o H_o \xi + \frac{1}{8} kFH_o \xi \xi \xi = \frac{kRe}{16} \left[2\tau H_o \xi \xi + \beta H_o \xi \xi \xi \xi \right], \tag{4.53}$$

which is a single equation for $H_o(\xi, \tau)$. We note that if $\beta = 0$ in this approximation, then equation (4.53) becomes the K-V-B equation studied in chapter 2 (and Johnson, 1970).

If β is order unity, then we have a fourth order non-linear equation to solve and it is not immediately obvious in what way the β term will effect the solution. We shall study this equation in the next paragraph using the same technique as employed on the K-V-B equation.

4.5 Steady-state asymptotic solution of equation (4.53)

To reduce the equation to the steady-state form, we introduce

$$X = \xi - \frac{kF}{8} \tau, \quad H_0(\xi, \tau) = \frac{k}{16} h(x) \quad (4.54)$$

where X is a characteristic co-ordinate in which steady flow occurs. Substituting (4.54) into equation (4.53) and integrating once gives

$$h'' + h(h-1) = \Delta(h' + \gamma h''') \quad (4.55)$$

where $\Delta = \frac{Re\tau}{F}$ and $\gamma = \beta/2\tau$. The primes denote differentiation and the upstream and downstream conditions have been assumed to remain undisturbed as

$$\begin{aligned} h &\rightarrow 0 & \text{for } X &\rightarrow \infty \\ h &\rightarrow 1 & \text{for } X &\rightarrow -\infty \end{aligned} .$$

This latter condition can always be applied in general since any amplitude factor is automatically incorporated into the amplitude parameter, ξ .

Equation (4.55) can not be integrated further and consequently it is impossible to ascertain the form of the solution in the phase-plane. However, for $\Delta \ll 1$ we expect a solution similar to that obtained for $\gamma = 0$ in chapter 2. To this end we shall seek a solution as a slowly varying cnoidal wave and in particular note the effects of the wall inertia.

Before embarking upon the asymptotic analysis it is worthwhile examining the form of the solution as $x \rightarrow -\infty$ (where the profile is expected to be oscillatory). This is most easily accomplished by writing

$$h \sim 1 - A e^{\alpha x}, \quad x \rightarrow -\infty$$

where A and α are constants. Substituting into (4.55), and neglecting exponentially small terms, gives a cubic equation for α :

$$\alpha^2 + 1 = \Delta(\alpha + \gamma\alpha^3).$$

For $\Delta \ll 1$, the roots can be written as

$$\alpha \sim \frac{1}{\Delta\gamma}, \quad \pm i + \frac{\Delta(1-\gamma)}{2}$$

and we note that the oscillatory motion is given by two of these roots. If the oscillations are to be damped (stable) as $x \rightarrow -\infty$ then we must have

$$\gamma < 1.$$

If $\gamma = 1$, the oscillatory solution is neutrally stable.

The condition for stability can be expressed as

$$\frac{\beta}{2\tau} = \frac{\rho_0 \omega^2 R_0'^2}{E\tau} < 1$$

which states that the inertia force associated with the motion of the wall ($\rho_0 \omega^2 R_0'$) must be less than the elastic force due to the extension of the tube wall ($E\tau/R_0'$). This is exactly the restriction deduced by Jacobs (1953) from a linearised analysis of the problem with a non-zero mass of the wall.

Returning to the formal asymptotic solution, we introduce a 'slow' time scale, $x = \Delta X$ ($\Delta \ll 1$) and expand like

$$h(x) = h_0(x, x) + \Delta h_1(x, x) + \dots$$

where

$$h_0(x, x) = a(x) + b(x) c n^2 \left[\alpha(x) X; \nu(x) \right]. \quad (4.56)$$

See § 2.2. For a wholly periodic solution we ask that h_1 be periodic in X with the same period as h_0 , and as in chapter 2 we shall obtain a solution which alters from the solitary wave ($\nu = 1$) to the zero period zero amplitude cnoidal wave ($\nu = 0$).

Without going into the details, we observe that the zeroth order equation is

$$h_{0,xx} + h_0(h_0 - 1) = 0$$

and if (4.56) is to be a solution then

$$\left. \begin{aligned} \alpha &= \frac{1}{2} \left(\frac{2b}{3\nu} \right)^{1/2}, \quad a = \frac{1}{2} - \frac{1}{3} \left(\frac{b}{\nu} \right) (2\nu - 1), \\ b &= \frac{3}{2} \nu (\nu^2 - \nu + 1)^{-1/2} \end{aligned} \right\} \quad (4.57)$$

In figure 14 are given the curves of the upper ($a+b$) and lower (a) envelopes of the oscillation as functions of ν , obtained from (4.57). We now have three relations defining the quantities $\alpha(x)$, $a(x)$, $b(x)$, $\nu(x)$. The fourth is given by the periodicity of $h_1(x, x)$. The $O(\Delta)$ equation is

$$h_{1,xx} + 2h_{0,xx} + h_1(2h_0 - 1) = h_{0,x} + \gamma h_{0,xxx}$$

and for h_1 to be periodic in X then

$$\frac{\partial}{\partial x} \left[\int_0^{k(\nu)/\alpha} h_{0,x}^2 dx \right] = \int_0^{k(\nu)/\alpha} [(1 + \gamma - 2\gamma h_0) h_{0,x}^2] dx. \quad (4.58)$$

($k(\nu)$ is defined below.) If $\gamma = 0$, expression (4.58) can be integrated directly to give the result derived for the K-V-B equation. For $\gamma \neq 0$, then upon substitution for h_0 , the following complicated relation for $\frac{dx}{d\nu}$ is obtained:

$$\frac{dx}{d\nu} = \frac{-\frac{5}{4} S^4 (2\nu - 1) P + 2(2\nu - 1) E + 2S^4 \frac{dE}{d\nu} - (2\nu - 3) K - (1 - \nu)(2 - \nu) \frac{dK}{d\nu}}{2(1 + \gamma) P - 4\gamma (aP + \frac{3}{14} S^2 Q)} \quad (4.59)$$

where

$$S = (\nu^2 - \nu + 1)^{-1/4}, \quad P = 2(\nu^2 - \nu + 1) E - (1 - \nu)(2 - \nu) K$$

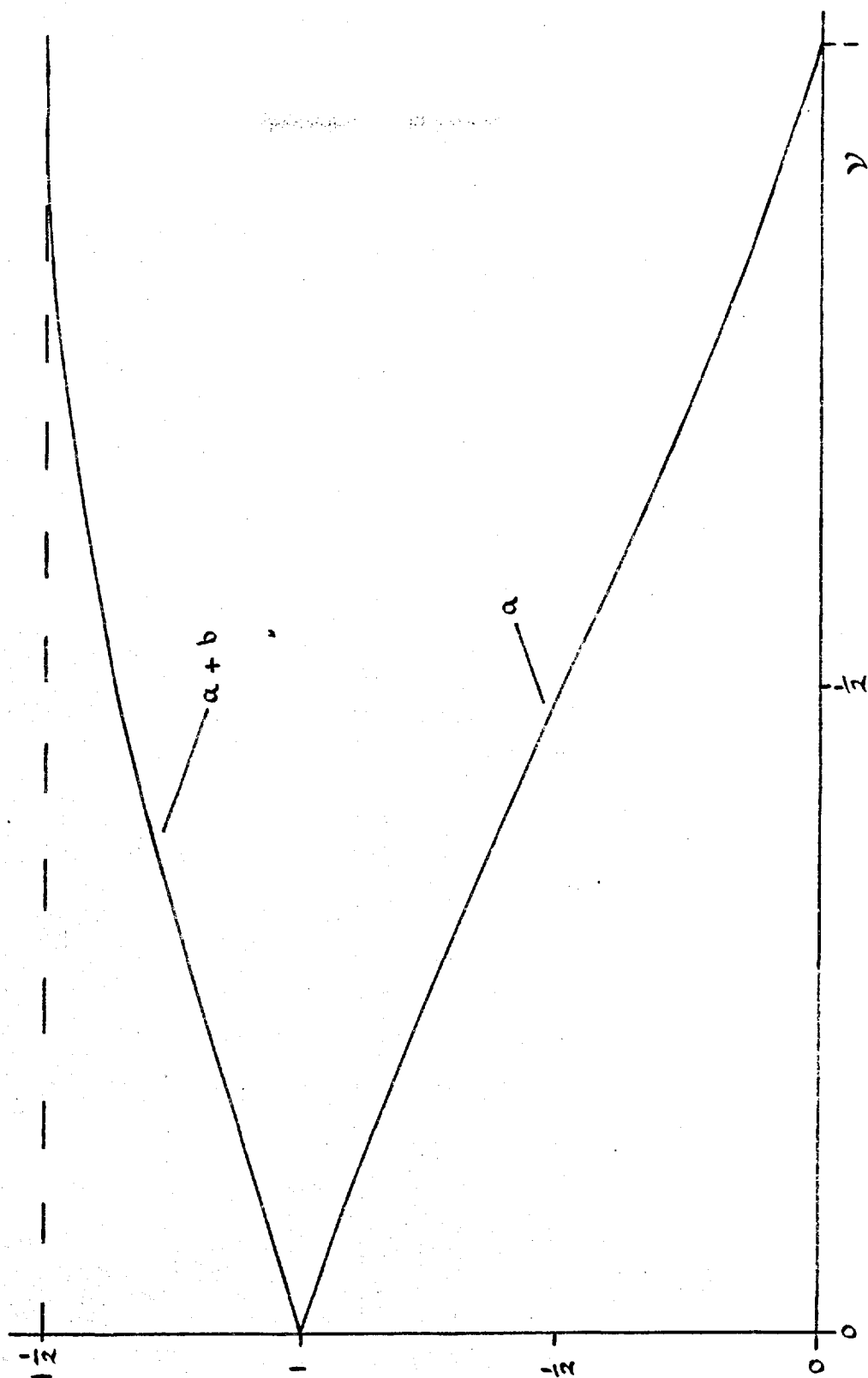


Figure 14 - Upper and lower envelope curves for the theoretical oscillations

$$Q = (2\nu - 1)(3\nu^2 - 3\nu + 8)E - (1 - \nu)(3\nu^2 - 15\nu + 8)K$$

and

$$\frac{dk}{d\nu} = \frac{E - (1 - \nu)K}{2\nu(1 - \nu)}, \quad \frac{dE}{d\nu} = \frac{E - K}{2\nu}.$$

Finally we have $k(\nu)$ and $E(\nu)$, the complete elliptic integrals of the first and second kind, respectively.

Relation (4.59) has been numerically integrated for $\gamma = 0, \frac{1}{2}, 1$ and the results are plotted in figure 15. The boundary conditions applied to ν is that $\nu = 1$ at $x = 0$. ν must then approach zero as $x \rightarrow -\infty$. The critical value of $\gamma = 1$ has been computed since it is possible to demonstrate that the profile still decays, although not exponentially. This is seen if we consider (4.59) as $\nu \rightarrow 0$,

$$\frac{dx}{d\nu} \sim \frac{1}{\nu(1 - \gamma) + B\nu^2},$$

where $B > 0$ is a complicated constant. Now if $0 \leq \gamma < 1$ then

$$\nu \sim \exp[(1 - \gamma)x], \quad x \rightarrow -\infty,$$

but if $\gamma = 1$ then

$$\nu \sim -\frac{1}{Bx}, \quad x \rightarrow -\infty.$$

For $\gamma > 1$ the theory predicts an exponential growth in ν as $x \rightarrow -\infty$ and corresponds to instability. This result is unacceptable. The computed results in figure 15 certainly show that even for $\gamma = 1$, ν approaches zero as $x \rightarrow -\infty$.

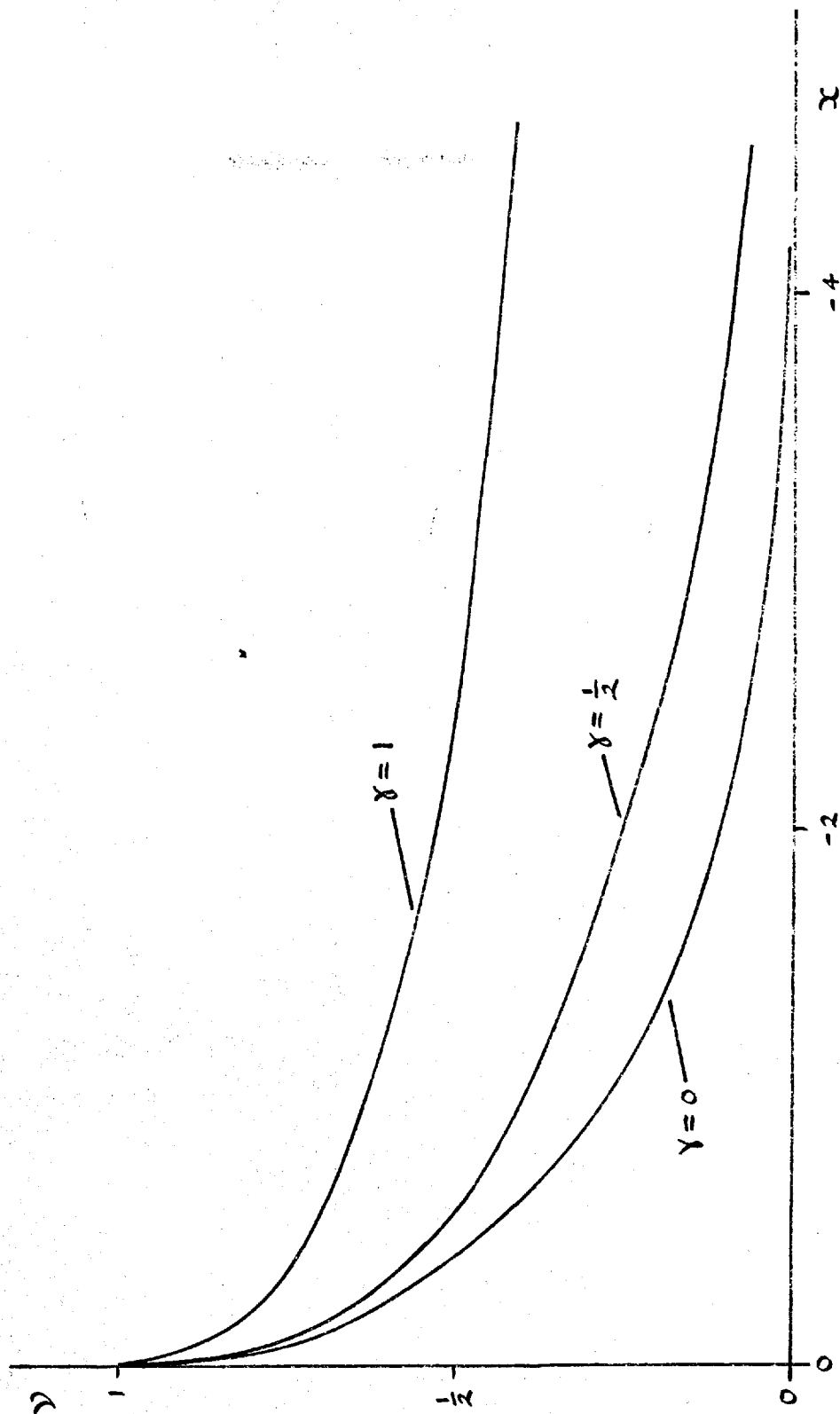


Figure 15 - Curves of ν against x for various γ ,
obtained by numerical integration

The case of $\gamma=0$ is exactly that given in more detail in chapter 2.

The matching of this asymptotic theory to the perturbation of the solitary wave, and this in turn matching with a simple exponential decay, is expected to follow directly. The details are not presented here but the results follow the lines of the analysis given in § 2.4.

4.6 Conclusion

A theory has been presented for the propagation of small disturbances on a fluid-filled elastic tube. The tube wall has been described by the thin shell theory of Timoshenko and a mean flow has been produced in the tube by introducing a body force. With the effects of shear in an inviscid flow, a comprehensive expression for the propagation speed of small disturbances has been derived. The relation is

$$\int_0^1 \frac{r dr}{[u(r)-c]^2} = \frac{1}{2\tau}$$

and is analogous to the water wave theory of Burns (1953). The appearance of τ ($= (1-\sigma^2)^{-1}$) is due to the fact that the tube wall was totally restrained longitudinally and the wall material was of non-zero Poisson's ratio (σ). For no mean flow and $\sigma=0$, the above integral relation for the wave speed, c , gives the classical Moens values (when

put into dimensional form).

For inviscid flow, the shear profile $U(r)$ can take any form. However, for viscous flow described by a Newtonian stress law, the mean profile must be that due to Poiseuille. If the flow Reynolds number is fairly large, the disturbance moves at the dynamic wave speed (C , above). These waves will decay as in the case of the two dimensional channel flow. The main disturbance is then left to travel at the kinematic wave speed. The form of this wave has been examined by considering the Reynolds number to be order unity.

In the 'far-field', non-linearity becomes important and it has been shown that a fourth order partial differential equation can describe the wave profile. Without the fourth derivative term (small wall inertia) the equation reduces to the K-V-B equation studied in chapter 2. Using the same form of averaging as employed there, an asymptotic solution is obtained which again is a slowly varying cnoidal wave (in the steady-state, for small damping). The effect of the wall inertia term is examined, and it is shown that a stable profile can occur only if

$$\rho_0 \omega^2 R_0'^2 / E_T \leq 1.$$

In the case of the 'less than' being applicable the stability is asymptotically exponential. However, for the equality, the stability is only algebraic ($\sim \frac{1}{x}$, $x \rightarrow \infty$). These results

confirm the simpler linearised analysis presented by Jacobs (1953), and also indicate that for arterial flow ($\frac{p_0}{p} \approx 1$) the profile is stable.

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Appendix I

The two dimensional channel flow analysis presented in chapter I was somewhat restrictive in that it assumed a Newtonian stress law. Such an assumption, as we have seen, immediately leads to the conclusion that the mean uniform flow must be a Poiseuille channel flow profile. It is known that many observed velocity profiles are considerably different in practice since the flow is turbulent. The work that has been presented in this thesis is, of course, for laminar flow. However, it seems reasonable to ask if the stress law describing the fluid can in any way be generalised so that other mean profiles are predicted, without necessitating analysing the full equations of turbulent motion.

If such a generalisation is possible, then the far-field region can be examined to see if the K-V-B equation (or one similar to it) can be derived. It will also be possible to average the equations by integrating over the depth of the channel and comparing the results with the work of Lighthill and Whitham (1955). We have mentioned that Lighthill and Whitham assumed that average equations described the flow, and used the Chézy law to give the friction on the channel bed. With the more general stress law, it may be possible to produce the same type of behaviour as predicted by the Chézy law.

A.1 A generalisation of the stress law

The equations of motion for an incompressible fluid can be written as

$$\left. \begin{aligned} \rho \frac{Du'_i}{Dt'} &= \rho F_i - \frac{\partial p'}{\partial x'_i} + \frac{\partial}{\partial x'_j} (d'_{ij}) \\ \frac{\partial u'_i}{\partial x'_i} &= 0 \end{aligned} \right\} \quad (\text{A.1})$$

where tensor (and summation) notation have been used, and the primes denote dimensional variables. The stress has been written as

$$\sigma'_{ij} = -p' \delta_{ij} + d'_{ij} \quad (\text{A.2})$$

where d'_{ij} is the deviatoric stress tensor. For symmetry and no motion we must have

$$d'_{ij} = d'_{ji} \quad , \quad d'_{ii} = 0. \quad (\text{A.3})$$

Now provided that (A.3) are satisfied, and that d'_{ij} also satisfy two axiomatic conditions, then d'_{ij} can take any form. The two axioms, often used in continuum mechanics, are that the equations of motion stay unchanged under rigid body translations and rotations. This implies that d'_{ij} can not be a function of u'_i or ω'_{ij} , where ω'_{ij} is the vorticity tensor. In particular, it may be a function of the rate of strain tensor

$$\frac{1}{2} \left(\frac{\partial u'_i}{\partial x'_j} + \frac{\partial u'_j}{\partial x'_i} \right) \quad (\text{A.4})$$

which is automatically symmetric. We note that the classical Newtonian theory is given by

$$d'_{ij} = \frac{1}{2} A_{ijkl} \left(\frac{\partial u'_k}{\partial x'_l} + \frac{\partial u'_l}{\partial x'_k} \right)$$

where A_{ijkl} is a constant rank four tensor. In this more general analysis, we shall assume that

$$d'_{ij} = f'_{ij} \left[\frac{1}{2} \left(\frac{\partial u'_i}{\partial x'_j} + \frac{\partial u'_j}{\partial x'_i} \right) \right], \quad (\text{A.5})$$

and thus exclude the possibility of d'_{ij} being dependent upon derivatives of the rate of strain tensor.

For simplicity, we shall restrict ourselves to the two dimensional case, $i, j = 1, 2$. If the body force (F_i) is due only to gravity, then we non-dimensionalise using (1.8), together with

$$d'_{12} = d'_{21} = \frac{\mu c}{h_0} f ; d'_{11} = -d'_{22} = \frac{\mu c}{L} g, \quad (\text{A.6})$$

where $g(\alpha)$ is an odd function. c is the characteristic velocity, $\sqrt{g_2 h_0}$, and μ is now a characteristic 'viscosity' enabling the non-dimensionalisation to be performed. The equations of motion thus become

$$\left. \begin{aligned} \frac{\partial \bar{u}}{\partial t} + \bar{p}_x &= \alpha + \frac{1}{Re} \left[\frac{\partial g}{\partial x} + \frac{1}{S^2} \frac{\partial f}{\partial y} \right] \\ S^2 \frac{\partial \bar{v}}{\partial t} + \bar{p}_y &= -1 + \frac{1}{Re} \left[-\frac{\partial g}{\partial y} + \frac{\partial f}{\partial x} \right] \\ \bar{u}_x + \bar{v}_y &= 0 \end{aligned} \right\} \quad (A.7)$$

where

$$g = g\left(\frac{\partial \bar{u}}{\partial x}\right) \quad \text{and} \quad f = f\left(\frac{1}{2} \frac{\partial \bar{u}}{\partial y} + \frac{S^2}{2} \frac{\partial \bar{v}}{\partial x}\right). \quad (A.8)$$

By virtue of the scaling (1.8) and (A.6), we assume that the parameter S only appears in (A.8) where explicitly indicated. Any constants of proportionality are to be order unity, independent of S . Since $g(\alpha)$ is an odd function the use of the continuity equation gives

$$g\left(\frac{\partial \bar{u}}{\partial x}\right) = g\left(-\frac{\partial \bar{v}}{\partial y}\right) = -g\left(\frac{\partial \bar{v}}{\partial y}\right)$$

so that $d'_{11} + d'_{22} = 0$. Hence with the restrictions outlined above, both f and g may be arbitrary functions. For the following analysis, both f and g will remain unspecified, but we will assume that Taylor expansions exist of these functions,

$$\left. \begin{aligned} f(\alpha + \beta) &= f(\alpha) + \beta f'(\alpha) + \dots \\ g(\alpha) &= g(0) + \alpha g'(0) + \dots \end{aligned} \right\} \quad (A.9)$$

For all non-dimensional quantities, primes will denote differentiation with respect to the appropriate arguments.

We expect that both f and g will satisfy the physical con-

dition that stress is a positive function of strain, so that in particular

$$f(0) = g(0) = 0. \quad (\text{A.10})$$

It is now possible to examine the solution to the problem of chapter 1, by seeking the kinematic wave solution and then the non-linear form in the far-field. The boundary conditions can be written as

$$\left. \begin{aligned} \bar{u} = \bar{v} = 0 \quad \text{on} \quad y = 0 \\ \bar{v} = h_t + \bar{u} h_x \\ \bar{p} = \frac{[(S^2 h_x^2 - 1)g - 2h_x f]}{Re(1 + S^2 h_x^2)} \\ (1 - S^2 h_x^2)f - S^2 h_x g = 0 \end{aligned} \right\} \quad \left. \begin{aligned} & \text{on} \\ & y = h \end{aligned} \right\} \quad (\text{A.11})$$

Without going into details, we shall firstly find the uniform flow solution, then the kinematic wave (near-field) and finally the far-field non-linear equation. The dynamic waves exist for this problem, being the solution for $Re \rightarrow \infty$, and they still decay for $Re > 0$ (positive damping).

A.2 The uniform flow

This is the solution for

$$\bar{u} = U(y), \quad \bar{v} = 0, \quad h = 1, \quad \bar{p} = 1 - y,$$

whence we obtain

$$\alpha + \frac{1}{Re \delta^2} \frac{d}{dy} f\left(\frac{1}{2} \frac{dU}{dy}\right) = 0 \quad (A.12)$$

with boundary conditions

$$f\left(\frac{1}{2} U'\right) = 0 \quad \text{on} \quad y = 1$$

$$U(y) = 0 \quad \text{on} \quad y = 0.$$

Defining, as before, the Froude number to be

$$F = \alpha Re \delta^2 = O(1),$$

then we have

$$f\left(\frac{1}{2} U'\right) = F(1-y). \quad (A.13)$$

This equation can be solved, in principle, to derive $U(y)$ for f a known function. For example, the classical Newtonian theory is

$$f\left(\frac{1}{2} U'\right) = 2 \cdot \frac{1}{2} U' \Rightarrow U(y) = F(y - \frac{1}{2} y^2).$$

We shall now perturb this uniform flow by introducing the amplitude parameter, ε .

A.3 Near-field solution - the kinematic wave

We seek a solution of the form

$$\bar{u} = U(y) + \varepsilon u, \quad \bar{v} = \varepsilon v, \quad h = 1 + \varepsilon \eta, \quad \bar{p} = 1 - y + \varepsilon p$$

where $\varepsilon \ll 1$, and also assume that $\delta^2 = k\varepsilon$. The variables

are expanded

$$Q(x, y, t; \varepsilon) = Q_0(x, y, t) + \varepsilon Q_1(x, y, t) + \dots$$

$$\eta(x, t; \varepsilon) = \eta_0(x, t) + \varepsilon \eta_1(x, t) + \dots$$

where $Q \equiv u, v, p$. Substituting the expansions into the full equations and boundary conditions, and expanding the two stress functions, we obtain the zeroth order:

$$\left. \begin{aligned} \frac{\partial}{\partial y} \left[u_{0y} f'(\tfrac{1}{2} U') \right] &= 0 \\ u_{0x} + v_{0y} &= 0 \\ u_0 = v_0 = 0 \text{ on } y=0 \\ v_0 &= \eta_{0t} + U \eta_{0x} \\ \eta_0 U'' f'(\tfrac{1}{2} U') + u_{0y} f'(\tfrac{1}{2} U') &= 0. \end{aligned} \right\} \begin{array}{l} \text{on} \\ y=1 \end{array} \quad (A.14)$$

It is straightforward to solve this set to give

$$\left. \begin{aligned} u_0 &= -\eta_0 [U'' f']_{\omega} \int_0^y \frac{dy}{f'(\tfrac{1}{2} U')} ; v_0 = \eta_{0x} [U'' f']_{\omega} \int_0^y \int_0^{y_1} \frac{dy_2}{f'(\tfrac{1}{2} U')} dy_1 \\ \eta_{0t} + \left\{ u_{0\omega} - [U'' f']_{\omega} \int_0^1 \int_0^{y_1} \frac{dy_2}{f'(\tfrac{1}{2} U'(y_2))} dy_1 \right\} \eta_{0x} &= 0 \end{aligned} \right\} \quad (A.15)$$

where

and $()_{\omega}$ indicates evaluation on $y=1$. Clearly, so that the integrals give us no trouble, then

$$f'(\tfrac{1}{2} U') \neq 0, \quad 0 \leq y \leq 1. \quad (A.16)$$

This we expect to be satisfied for a positive stress/strain

law. From the definition of $U(y)$, (A.13), we observe that

$$[U'' f']_{(0)} = -2F$$

so that the kinematic wave speed is

$$c_k = U(0) + 2F \int_0^1 \int_0^{y_1} \left[\frac{dy_2}{f'(\frac{1}{2}U')} \right] dy_1, \quad (A.17)$$

which is positive definite for $U(0) > 0$, $f' > 0$.

A.4 Far-field solution - the non-linear wave

To observe the effects of non-linearity and dispersion on the kinematic wave, we introduce characteristic and large time co-ordinates

$$\xi = x - c_k t, \quad \varepsilon t = T. \quad (A.18)$$

The Reynolds number, Re , will be kept fixed independent of the limiting process $\varepsilon \rightarrow 0$. Assuming the same expansions as in the near-field, but now as functions of (ξ, T, y) , we obtain the following problem for u_1, v_1, p_1 :

$$(U - c_k)u_{0\xi} + U'v_0 + p_{0\xi} = \frac{1}{Re} \left[\alpha u_{0\xi\xi} + \frac{1}{2k} \frac{\partial}{\partial y} \left\{ (u_{1y} + kv_{0\xi}) f' + \frac{u_{0y}^2}{4} f'' \right\} \right] \quad (A.19)$$

$$p_{0y} = \frac{1}{Re} \left[-\alpha u_{0\xi y} + \frac{1}{2} u_{0\xi y} f' \right] \quad (A.20)$$

$$u_{1\xi} + v_{1y} = 0 \quad (A.21)$$

$$u_1 = v_1 = 0 \quad \text{on} \quad y = 0 \quad (\text{A.22})$$

$$-\eta_0 + \rho_0 = \frac{1}{Re} \left[-\alpha_0 u_{0\tau} - 2f\eta_{0\tau} \right] \quad (\text{A.23})$$

$$\left. \begin{aligned} (\eta_1 u'' + \frac{\eta_0^2}{2} u''' + \eta_0 u_{0yy} + u_{1y} + k v_{0\tau}) f' + \frac{1}{4} (\eta_0 u'' + u_{0y})^2 f'' = 0 \\ \eta_0 v_{0y} + v_1 = \eta_{0\tau} + u_0 \eta_{0\tau} + (u - c_k) \eta_{1\tau} + \eta_0 \eta_{0\tau} u' \end{aligned} \right\} \text{on } y=1 \quad (\text{A.24})$$

$$(\text{A.25})$$

The argument of f , f' , f'' is $\frac{1}{2}U'$ throughout, and $g'(0) = \alpha$.

The zeroth order is given essentially by the near-field solution,

$$u_0 = 2F\eta_0 \int_0^y \frac{dy}{f'} ; \quad v_0 = -2F\eta_{0\tau} \int_0^y \int_0^{y_1} \frac{dy_2}{f'} dy_1, \quad (\text{A.26})$$

where $\eta_0(\tau, T)$ is undefined.

From (A.20) and (A.23) we obtain

$$\rho_0 = \eta_0 + \frac{1}{Re} \eta_{0\tau} \left[F(y-1) + 4\alpha F \int_0^y \frac{dy}{f'} - 2\alpha F \int_0^y \frac{dy}{f'} - 2f_{0\omega} \right]$$

where all subscripts $()_{\omega}$ indicate evaluation on $y=1$.

Substituting for ρ_0 into (A.19) and integrating twice with respect to y , together with the boundary conditions (A.22)

and (A.24), gives

$$\begin{aligned} \frac{1}{2Rek} u_1 = & \frac{F}{Re} \eta_{0\tau} \int_0^y \int_0^{y_1} \int_0^{y_2} \frac{(dy)^3}{f'} - \frac{F^2}{2Rek} \eta_0^2 \int_0^y \frac{f''}{f'^3} dy + 2F\eta_{0\tau} \int_0^y \frac{1}{f'} \int_0^y (u - c_k) \int_0^y \frac{(dy)^3}{f'} \\ & - 2F\eta_{0\tau} \int_0^y \frac{1}{f'} \int_0^y u \int_0^y \frac{(dy)^4}{f'} + \eta_{0\tau} \int_0^y \frac{y}{f'} dy + \frac{1}{Re} \eta_{0\tau} \int_0^y \frac{1}{f'} \int_0^y \left[F(y-1) + 4\alpha F \int_0^y \frac{dy}{f'} \right. \\ & \left. - 2\alpha F \int_0^y \frac{dy}{f'} - 2f_{0\omega} \right] (dy)^2 - \frac{2\alpha F}{Re} \eta_{0\tau} \int_0^y \frac{1}{f'} \int_0^y \int_0^y \frac{(dy)^3}{f'} + A(\tau, T) \int_0^y \frac{dy}{f'} \end{aligned}$$

where

$$\begin{aligned}
 A(\xi, T) = & -2F\eta_0 \int_0^1 (U - c_k) \int_0^y \frac{(dy)^2}{f'} + 2F\eta_0 \int_0^1 U' \int_0^y \int_0^y \frac{(dy)^3}{f'} - \eta_0 \xi \\
 & - \frac{1}{Re} \eta_0 \xi \int_0^1 [F(y-1) + 4\alpha F \int_0^1 \frac{dy}{f'} - 2\alpha F \int_0^y \frac{dy}{f'} - 2f_w] dy + \frac{2\alpha F}{Re} \eta_0 \xi \int_0^1 \int_0^y \frac{(dy)^2}{f'} \\
 & + \frac{1}{2Re\kappa} \left\{ -(\eta_0 U'' + \frac{1}{2}\eta_0^2 U''') f' - \frac{1}{4}(\eta_0 U'' + \frac{2F\eta_0}{f'} f'') f'' + F^2 \eta_0^2 \frac{f''}{f'^2} \right\} \Big|_0^1.
 \end{aligned}$$

It is now possible to integrate once more with respect to y and using the conditions (A.22, A.25) we derive a single equation for η_0 (η_1 identically cancels),

$$\eta_{0T} + 2F\rho\eta_0\eta_{0\xi} + F\kappa\eta_0\eta_{0\xi\xi} = Re\kappa\eta_0\eta_{0\xi\xi}$$

which is the K-V-B equation. The coefficients are given by

$$\rho \equiv \frac{1}{2F} U'_w + \left(\frac{c_k - U_w}{2F} \right) \left(F \frac{f''_w}{f'^2_w} - \frac{U''_w f'_w}{2F} \right) + 2J_1 - FI_1 \left[I \left(\frac{f''}{f'^3} \right) \right]$$

$$q \equiv \alpha \left(\frac{c_k - U_w}{F} \right)^2 + \left(\frac{c_k - U_w}{2F} \right) \left(1 + \frac{4}{F} f_w - 8\alpha J_1 \right)$$

$$+ I_1 \left\{ I \left[\frac{y}{f'} (y-2 - \frac{4f_w}{F}) + 2 \left(1 - \frac{4\alpha}{f'} \right) I(J) + 8\alpha I_1 \left(\frac{y}{f'} \right) \right] + \frac{2\alpha(c_k - U_w)}{F} J \right\}$$

$$\begin{aligned}
 r \equiv & \left(\frac{c_k - U_w}{F} \right) + I_1 \left\{ I \left[\frac{4F}{f'} I(U' I(J)) - \frac{4F}{f'} I(\overline{U - c_k} J) - \frac{2y}{f'} \right] \right. \\
 & \left. + \left(\frac{c_k - U_w}{2F} \right) [4F(U - c_k) J - 4F U' I(J)] \right\}
 \end{aligned}$$

where

$$J(y) \equiv \int_0^y \frac{dy}{f'}, \quad J(1) \equiv J_1; \quad I(\phi) \equiv \int_0^y \phi dy, \quad I_1(\phi) \equiv \int_0^1 \phi dy.$$

As a check, it is a fairly simple matter to take the classical Newtonian theory

$$f(\frac{1}{2}U') = U'; \quad U(y) = F(y - \frac{1}{2}y^2); \quad \alpha = 2$$

which gives

$$\rho = 1, \quad \eta = 1, \quad \tau = \frac{1}{3}(1 - \frac{2}{5}F^2).$$

(See equation (1.62).)

We have thus shown that the Korteweg-de Vries-Burgers equation is valid in the far-field even for a more general stress law. This is the case provided the deviatoric stress tensor is only a function of the rate of strain. To permit higher derivatives in these expressions will clearly lead to other than the K-V-B equation. It is now possible for a given mean velocity profile (that is, a given f) to derive the appropriate coefficients ρ , η , τ . In conclusion it is worth noting that any bound on the Froude number is now given by the general expression $\tau = 0$.

A.5 Average equations and a generalised stress law

In this final paragraph, we intend taking the equations of motion for the fluid, (A.7), and averaging them formally across the depth of the channel. The boundary conditions are as given in chapter 1. Clearly we hope to derive a solution to the problem (the surface disturbance) by only using averaging and boundary conditions. A knowledge of the

fine detail, it is hoped, will be unnecessary. Lighthill and Whitham (1955) assumed such average equations and managed to derive a single equation which described both the dynamic and kinematic waves.

The average equations will be found by integrating the local equations across the channel, and we will drop the bar notation of (A.7) for convenience. Hence from the continuity equation

$$\int_0^h u_x dy + [v]_0^h = 0$$

where $y = h(x, t)$ is the surface. But

$$v = h_t + u h_x \quad \text{on} \quad y = h$$

and $v = 0$ on $y = 0$, thus

$$\int_0^h u_x dy + h_t + u_h h_x = 0$$

where $u_h = u(x, h, t)$. Now transposing the integration and differentiation,

$$\frac{\partial}{\partial x} \int_0^h u dy - u_h h_x + h_t + u_h h_x = 0$$

and if we write

$$\int_0^h u dy = h \bar{u} \tag{A.27}$$

then

$$h_t + (h \bar{u})_x = 0 \tag{A.28}$$

and the bar notation now implies an average value across the depth.

Likewise averaging the x -momentum equation, and using (A.28), gives

$$\left. \begin{aligned} h \left[\bar{u}_t + (\bar{u}^2 - \frac{1}{2} \bar{u}^2)_x + \frac{1}{h} h_x (\bar{u}^2 - \bar{u}^2) \right] + \int_0^h p_x dy \\ = \alpha h + \frac{1}{Re} \int_0^h g_x dy + \frac{1}{Re \delta^2} [f]_0^h, \end{aligned} \right\} \quad (A.29)$$

and only in very trivial cases (e.g. $\bar{u} = \text{constant}$) can $\bar{u}^2 = \bar{u}^2$. We now seek an approximate solution to this problem as a perturbation from mean conditions as measured by the amplitude parameter ϵ ,

$$u = U(y) + \epsilon u_0 + \dots, \quad v = \epsilon v_0 + \dots$$

$$p = 1 - y + \epsilon p_0 + \dots, \quad h = 1 + \epsilon \eta_0 + \dots$$

From the y -momentum equation we immediately obtain

$$p_{0y} = 0$$

and since $p=0$ on $y=h$ we have $p_0 = \eta_0$. The average velocity is now given by

$$\begin{aligned} h \bar{u} &= \int_0^h (U + \epsilon u_0 + \dots) dy \\ &= \bar{U} + \epsilon \bar{u}_0 + O(\epsilon^2) \end{aligned}$$

and

$$\begin{aligned} h \bar{u}^2 &= \int_0^h [U + \epsilon u_0 + \dots]^2 dy = \int_0^h [U^2 + 2\epsilon U u_0 + \dots] dy \\ &= \bar{U}^2 + 2\epsilon \bar{U} \bar{u}_0 + O(\epsilon^2). \end{aligned}$$

Substituting in (A.29), and keeping only the dominant terms (including viscous) in the limit $Re \rightarrow \infty$, $\delta \rightarrow 0$, then

$$\varepsilon \bar{u}_0 + \varepsilon (2 \bar{u} \bar{u}_0 - \bar{u} \bar{u}_0)_x + \varepsilon \eta_0 (1 + \bar{u}^2 - \bar{u}^2) = \alpha (1 + \varepsilon \eta_0) + \frac{1}{Re \delta^2} [f]_0'.$$

It is now clear, without analysing further, that the average momentum equation involves not only knowing $\bar{u}(y)$ (which is assumed given) but also $u_0(x, y, t)$ so that the average $\bar{u}_0$ can be found. Such knowledge can only be obtained by a detailed analysis of the problem along the lines given in chapter 1. Hence we must conclude that the status of the equations used by Lighthill and Whitham (1955) can only be those of 'model equations'. However, it should be added that we could assume that $\bar{u}_0$ was of the form $\alpha \eta_0$, say, where α is an unknown constant. This enables an equation of the type found by Lighthill and Whitham to be derived.

If we accept the equations of motion given by Lighthill and Whitham, then we may Briefly examine how the Chézy law is constructed, and compare it with the Newtonian stress law (or the more general law presented here). Let us consider the mean Poiseuille channel flow as predicted by the Newtonian stress law. In dimensional variables this may be written as

$$u = \frac{1}{2\mu} \left| \frac{dp}{dx} \right| y(2h - y)$$

where the motion is produced by a constant pressure gradient.
Using the continuity equation then

$$v = -\frac{1}{2\mu} \left| \frac{dp}{dx} \right| h_x y^2$$

and with the surface condition

$$v(x, h, t) = h_t + u(x, h, t) h_x$$

we obtain

$$h_t + \frac{1}{\mu} \left| \frac{dp}{dx} \right| h^2 h_x = 0.$$

This equation immediately predicts the kinematic wave, and can be linearised if we consider small disturbances (η) from the mean height: $h = h_0 + \eta$,

$$\eta_t + \frac{1}{\mu} \left| \frac{dp}{dx} \right| h_0^2 \eta_x = 0.$$

However, we may also introduce the mean velocity

$$\bar{u} = \frac{1}{h} \int_0^h u dy = \frac{h^2}{3\mu} \left| \frac{dp}{dx} \right| \approx \frac{h_0^2}{3\mu} \left| \frac{dp}{dx} \right|$$

so that

$$h_t + 3\bar{u} h_x = 0.$$

These results may now be compared with the similar analysis for a Chézy friction law. If the wetted perimeter of the channel is L , and its cross-sectional area A , then we write

$$\begin{aligned}\text{force/unit length} &= \frac{L}{A} \left(\mu \frac{\partial u}{\partial y} \right) \Big|_{y=0} \\ &= \frac{L}{A} h \left| \frac{dP}{dx} \right|.\end{aligned}$$

But we have an average velocity $\bar{u}$,

$$\text{Force/unit length} = \frac{L}{A} h \cdot \frac{3\mu\bar{u}}{h^2}$$

and if we further note that

$$\frac{L}{A} \propto \frac{1}{h} \quad (\text{exactly true for a rectangular channel})$$

then

$$\text{force/unit length} \propto \frac{3\mu\bar{u}}{h^2} = \frac{3\mu}{\rho h \bar{u}} \cdot \frac{\bar{u}}{h} \cdot \rho \bar{u}.$$

For a Chézy law the Reynolds number defined as

$$Re = \frac{\rho h \bar{u}}{3\mu}$$

is assumed constant, and if the motion is produced by a constant pressure gradient (or gravity) then

$$\bar{u}^2 \propto h.$$

Using this relation in the exact average continuity equation (A.28), we obtain

$$h_t + \frac{3}{2} \bar{u} h_x = 0$$

which is the result used by Lighthill and Whitham (1955).

It is now evident that if a more general stress law is used (e.g. $f(\frac{1}{2}U')$) then the kinematic wave equation

$$h_t + k \bar{u} h_x = 0$$

can have any appropriate wave speed $k \bar{u}$. In particular we could choose the profile which would give the Chézy law results formally, without necessitating the use of average equations. In other words, we model the turbulent behaviour of the flow by using a non-Newtonian laminar theory. This approach then replaces the use of average 'model' equations.

Appendix II

In figures 16 and 17 is given the print out of the programme used for the integration of the steady-state K-V-B equation, on the Imperial College IBM 7094 computer. The technique used was a predictor-corrector (fifth order) scheme. Having read in the appropriate data for the run, the programme computes the first four values of y , $y'(x)$ and $y''(x)$ using the asymptotic expansion for the perturbation of the solitary wave (given in equations (2.24,2.25)). With these first four values, the next value is predicted and then corrected upto a specified accuracy. This is then repeated for as long as is required, always using the previous four values to compute the next.

The beginning of a typical print out of results is given in figure 18. The first line gives the relevant data for this particular run: H= step length, J= number of steps, X0= starting value of x , C= damping parameter, γ . The first four lines are the co-ordinates and the function values calculated from the asymptotic expansion. The rest of the results are obtained from the predictor-corrector routine, where the co-ordinate, function value and its first derivative are tabulated. The integer in the final column gives the number of iterative loops completed in the corrector sequence to reach the required accuracy.

C SOLUTION OF KORTEWEG-DE VRIES/BURGERS USING PREDICTOR-CORRECTOR

C /

DIMENSION X(5),Y(5),CY(5),C2Y(5)

READ(5,1) H,J,X0

1 FORMAT(F8.5,I5,F8.4)

READ(5,2) EPS

2 FORMAT(F8.5)

C

C WRITE OUT INPUT DATA

C

WRITE(6,3) F,J,X0,EPS

3 FORMAT(14H INPUT DATA IF, F8.5, 3H J=, I5, 4H X0=, F8.4, 3H C=, F8.5//)

C

C START CALCULATION WITH ASYMPTOTIC VALUE FOR LARGE NEGATIVE X

C

DO 201 I=1,4

X(I)=XC+FLCAT(I-1)*H

T=TANH(X(I))

Y(I)=1.5*(1.-T**2)+EPS*(0.4*(1.-T**2)*(T-1.)+0.05*T*(7.*T-6.)-

1 0.1*T/(1.+T)+0.75*X(I)*T*(1.-T**2))

DY(I)=2.0*T*(1.-T**2)+EPS*(0.4*(1.-T**2)*(1.+2.*T-3.*T**2)+

1 C.05*(1.-T**2)*(14.*T-6.)-0.1*(1.-T**2)/(1.+T)**2+

2 C.75*(1.-T**2)*T*X(I)*(1.-T**2)-2.*X(I)*T**2))

D2Y(I)=4.*Y(I)*(1.-Y(I))-EPS*CY(I)

WRITE(6,4) X(I),Y(I)

4 FORMAT(1H, F8.4, 5X, 1PE20.7//)

201 CONTINUE

C

C NEXT VALUE OF Y-PRIMED IS PREDICTED

C

K=0

K1=5

DO 202 M=1,100

DO 202 II=1,J

I=5

DY(I)=DY(I-4)+4./5.*H*(2.*C2Y(I-3)-D2Y(I-2)+2.*D2Y(I-1))

X(I)=XC+FLCAT(II+4+20000*(M-1))*H

CHECK=100000.

L=0

Figure 16 - Print out of the computer programme

C

C THIS VALUE IS NOW CORRECTED UPTO A SPECIFIED ACCURACY

C

105 $Y(I) = Y(I-2) + H/3 * (DY(I-2) + 4 * DY(I-1) + DY(I))$

IF (ABS(CHECK - Y(I)) / CHECK) .LT. .0000001) GO TO 106

103 $D2Y(I) = 4 * Y(I) * I1. - Y(I1)) - EPS + DY(I)$

$DY(I) = DY(I-2) + H/3 * (D2Y(I-2) + 4 * D2Y(I-1) + D2Y(I))$

CHECK = Y(I)

L = L + 1

GO TO 105

106 IF (I1.EQ.KT) GO TO 104

GO TO 107

C

C WRITE OUT THE SOLUTION

C

104 WRITE(6,7) Y(I), Y(I), DY(I), L

7 FORMAT(1P, F8.3, 5X, 1PE20.7, 5X, 1PE20.7, 5X, I3)

K = K + 1

KT = 5 + 5 * K

107 DO 202 I = 1, 4

$Y(I) = Y(I+1)$

$DY(I) = DY(I+1)$

$D2Y(I) = D2Y(I+1)$

$X(I) = X(I+1)$

202 CONTINUE

STOP

END

Figure 17 - Print out of the computer programme

0 EXECUTION
 INPUT DATA H= C.0100 J=2000 XC= -C.2000 C= 0.0100

-0.2000 1.4382205E-00

-0.1900 1.437194E-00

-0.1800 1.4409626E-00

-0.1700 1.4539461E-00

-0.110	1.4748285E-00	3.4869395E-01	1
-0.060	1.4887073E-00	2.0564811E-01	1
-0.010	1.4953240E-00	5.8418959E-02	1
0.040	1.4945344E-00	-8.9695777E-02	1
0.090	1.4863815E-00	-2.3615783E-01	1
0.140	1.4710005E-00	-3.7770972E-01	1
0.190	1.4427369E-00	-5.1207408E-01	1
0.240	1.4199539E-00	-6.3649991E-01	1
0.290	1.3852735E-00	-7.4337281E-01	1
0.340	1.3452457E-00	-8.4379266E-01	1
0.390	1.3006308E-00	-9.3403115E-01	1
0.440	1.2520950E-00	-1.0041365E-00	1
0.490	1.2004661E-00	-1.0593134E-00	1
0.540	1.1464139E-00	-1.0994876E-00	1
0.590	1.0907516E-00	-1.1256199E-00	1
0.640	1.0340762E-00	-1.1383008E-00	1
0.690	9.7712139E-01	-1.1389988E-00	1
0.740	9.2036078E-01	-1.1286656E-00	1
0.790	8.6441029E-01	-1.1090134E-00	1
0.840	8.0959641E-01	-1.0811406E-00	1
0.890	7.5640516E-01	-1.0467296E-00	1
0.940	7.0501505E-01	-1.0068187E-00	1
0.990	6.5578946E-01	-9.6293284E-01	1
1.040	6.0877380E-01	-9.1591934E-01	1
1.090	5.6422812E-01	-8.6706541E-01	1
1.140	5.2208779E-01	-8.1697382E-01	1
1.190	4.8253413E-01	-7.6667385E-01	1
1.240	4.4541750E-01	-7.1651950E-01	1
1.290	4.1086650E-01	-6.6730149E-01	1
1.340	3.7866991E-01	-6.1915051E-01	1
1.390	3.4852376E-01	-5.7265877E-01	1
1.440	3.2137723E-01	-5.2777234E-01	1
1.490	2.9611238E-01	-4.8493177E-01	1
1.540	2.7285322E-01	-4.4393915E-01	1

Figure 18 - Print out of the computer programme