

STATIC AND DYNAMIC ANALYSIS OF AXISYMMETRICAL
THIN-SHELL STRUCTURES BY A MIXED METHOD
OF FINITE ELEMENT FORMULATION

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SUMMARY

A mixed method isoparametric shell element using displacement and bending moment parameters at two nodal circles as independent variables is developed for the analysis of single and multiple-branched axisymmetrical thin shell structures under arbitrary static and dynamic loading, especially for shells with geometric discontinuity. The formulation ensures static and kinematic continuity at the point of discontinuity, thus resulting in an accurate stress distribution in that region which is of the great importance in the non-linear static analysis.

The small displacement theory is derived, first of all, and a special progressive elimination technique for the solution of branched shells discussed. Applications to various examples demonstrate excellent capability of this element for analysis of discontinuous shells.

An incremental linearised large displacement theory suitable for this formulation is then developed. It is shown to give a good estimate of the buckling loads for all types of axisymmetrical shells.

Finally, an application of this method to the linear dynamic analysis is presented. It is shown to be more economical than the existing displacement method elements with comparable accuracy.

Two computer programmes, one for the linear static and/or dynamic and the other for non-linear static analysis have been written, based on the theories presented. The flow charts of these are given, together with the more interesting computational procedures used.

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NOTATION

L	Meridional half length
S	True meridional distance
R	Radial coordinate
z	Axial coordinate
t	Time variable
$\bar{u}, \bar{v}, \bar{w}$	Displacements in the local coordinate system
u, v, w	Displacements in the global coordinate system
N_ϕ, N_θ, S	Membrane stresses
M_ϕ, M_θ, T	Bending stresses
Q_ϕ, Q_θ	Normal shear forces
η	Non-dimensional meridional distance
ϕ	Meridional angle
θ	Circumferential angle
ξ	Non-dimensional time variable
τ	Time interval
h	Row of Hermitian polynomials
r_I	Nodal geometry data in R - direction
z_I	Nodal geometry data in z - direction
κ	Material stiffness
ϕ	Material flexibility
$\hat{\delta}, \delta$	Vectors of displacements
ρ	Vector of nodal displacements
$\dot{\rho}$	Vector of nodal velocities
$\ddot{\rho}$	Vector of nodal accelerations
b	Vector of nodal bending moments
ϵ	Vector of strains
ω	Vector of surface rotations, strains and displacements
η	Vector of initial strains

α	Transformation matrix between strains and nodal displacements
D^t	Strain/displacement operator
S	Vector of stresses
β	Interpolation matrix
P	Kinematically equivalent element load vector
j	Vector of element initial loads
q	Vector of element initial curvatures
k	Element stiffness matrix
f	" flexibility "
t	" coupling "
m	" mass "
d	" damping "
a	Boolean transformation matrix
r	Structural displacement vector
$\dot{r}$	" velocity vector
$\ddot{r}$	" acceleration vector
B	" bending moment vector
R	" load vector
J	" initial load vector
Q	" initial curvature vector
K	" stiffness matrix
F	" flexibility matrix
T	" coupling "
M	" mass "
D	" damping "
Λ	Cartesian transformation matrix
L	Transformation matrix between ω and δ
C	Matrix of constraint equations

All other symbols are defined in the text.

INTRODUCTION

The development of the SABA element (Ref. 1) was motivated specifically for the small and large displacement analysis of the cooling towers. It has since been found to be ideally suitable for any continuous single-branched axi-symmetrical thin shells. However, there are many important engineering applications of axi-symmetrical thin shell structures where the SABA element need not be the best possible model. This applies generally to all structures where discontinuity exists in the geometry, such as a step in the thickness, or a change in the slope (e.g. a cylinder joining to a cone), or more drastically, a branched-shell where a generator suddenly becomes two at a point. An example of this last construction is the liquefied-natural-gas (LNG) container ships, where large spherical tanks are supported by cylindrical skirts. The reason for the short-coming is that, in the displacement method of formulation of the SABA element, the force resultants are obtained from the stress/strain relations in terms of the displacements, and equilibrium is satisfied only approximately in the Galerkin sense. These forces will be continuous across element boundaries when the structure is continuous, but will not be in equilibrium if there is any discontinuity in the geometry at a node. Consequently the result may not be sufficiently accurate, especially in a large displacement non-linear analysis. There is nothing unusual about this in the application of the finite element method; no element can claim to be suitable for all situations. This paper sets out to develop a different axi-symmetrical element specifically for the analysis of discontinuous thin-shell structures.

Each element is again generated from a segment of the meridian curve revolving around the central axis as the SABA element, and is continuous between the nodal circles at the ends. Discontinuity is allowed only at the nodes where two or more elements intersect. Variations of displacements and bending moments are separately described by a Fourier series in the circumferential direction, each harmonic associating with a polynomial of the meridional coordinate variable. This polynomial may be of any degree of complexity, but is most convenient and sufficiently accurate to be expressed as third order Hermitian functions in terms of the nodal values and the first derivative at the end nodes. The geometry of the meridian curve is similarly interpolated. The membrane strains are then derived from the displacements and the membrane stresses calculated from these strains. The bending strains on the other hand are given by the elastic stress/strain law directly from the assumed distribution of bending moments. Neither the equilibrium nor the compatibility conditions are satisfied exactly, but only approximately in the Galerkin sense by the application of both the Principles of virtual force and virtual displacements. Thus the formulation is based on a mixed method of approach, with the bending moments as well as the displacement prescribed by a cubic variation in the meridional direction, which is sufficient to ensure static and kinematic continuity, but is not overconformed. As the geometry is also similarly interpolated, the element is in fact isoparametric in more than one sense. The nodal parameters are the bending moments and the displacements, together with their first derivatives (equivalent to the normal shear forces, slopes and strains). These may be solved directly as unknowns, thereby enabling the continuity (static as well as kinematic) conditions to be directly imposed. There are therefore 12 unknowns at each node. To achieve a comparable moment

variation by the orthodox displacement method of approach, the SABA 7 element [1] would have to be used. The number of nodal unknowns are the same (displacements and their derivatives up to the third order), but equilibrium at the element boundaries is still not exactly satisfied which may make the results difficult to interpret for discontinuous structures such as branched shells.

The mixed method of finite element analysis, defined here as having the variations of stresses and displacements, and sometimes strains also, independently specified for each element, is of course no longer new. It has been receiving a lot of attention lately, if for no other reasons than the large variety of alternatives to the purely displacement or purely force idealisation it offers (See Refs. 2, 3, 4, 5). It also provides a fascinating field for the study of its convergence properties (See Refs. 6, 7). Wolf gave an excellent summary of the various possibilities in his thesis [8] and compared the results from some of these for a few simple plate bending problems, which seems to be the most popular subject for treatment by the mixed method. Prato derived a curved triangular element and used it for the linear and non-linear analysis of shells [9]. There is one application to the axial symmetrical shell small displacement problem by Gould and Sen [10], but they expressed the force parameters in terms of the nodal displacement parameters for each element, thereby giving up direct control over the inter-element equilibrium considerations. Some of the applications gave viable enough alternatives to the standard displacement method, but with no great sense of purpose.

In part I of this paper, the formulation of the small displacement theory is described and the method of solution discussed. Particular

attention is drawn to the case of the branched-shells, for which the banded-structure of the assembled matrix is destroyed. The solution procedure actually avoid this inconvenience by adopting a progressive elimination technique, which expresses the parameters at each node in terms of those at the next, starting from the unattached end of each branch towards the common point. The final solution at the branch junction and a backward substitution yields the complete set of nodal parameters. A few numerical examples are given to enable comparison with other methods of analysis, including the case of a spherical tank supported by a cylinder, typical of a LNG container. Part II formulates the so-called "geometric" stiffness matrix and performs the large displacement non-linear analysis by the incremental method. This method is used for the calculation of the buckling loads and examples are given for checking against known results. Amongst them is an example of a discontinuous shell for which no finite element has been obtained before. Finally, the mass matrix is developed and examples of small displacement dynamic analysis, including the case of a cooling tower subjecting to earthquake loading, are given in Part III.

PART 1 - Small Displacements

1.1 BASIC EQUATIONS IN THE THIN SHELL THEORY

Consider a frustum of an axi-symmetrical thin shell in the right hand cylindrical polar coordinate system (FIG. 1). Each point on the middle surface is defined by the radius R , the circumferential angle θ and axial variable z . The radius of the meridional curvature is denoted by R_ϕ and it forms angle ϕ with the z axis, measuring from the positive direction of z . The displacement of a point on the middle surface is uniquely determined, in a local system, by three components: $\bar{U}$ in the meridional direction s , $\bar{V}$ in the tangential direction t and $\bar{W}$ in the normal direction n , forming also a right hand system. The external surface loading on an infinitesimal shell element is conveniently represented by three components: P_ϕ in the meridional, P_θ in the tangential and P_n in the normal direction. This loading produces the state of stress which is completely described by six statically equivalent resultant forces and four bending moments, positive directions of which are shown in FIG. 1.

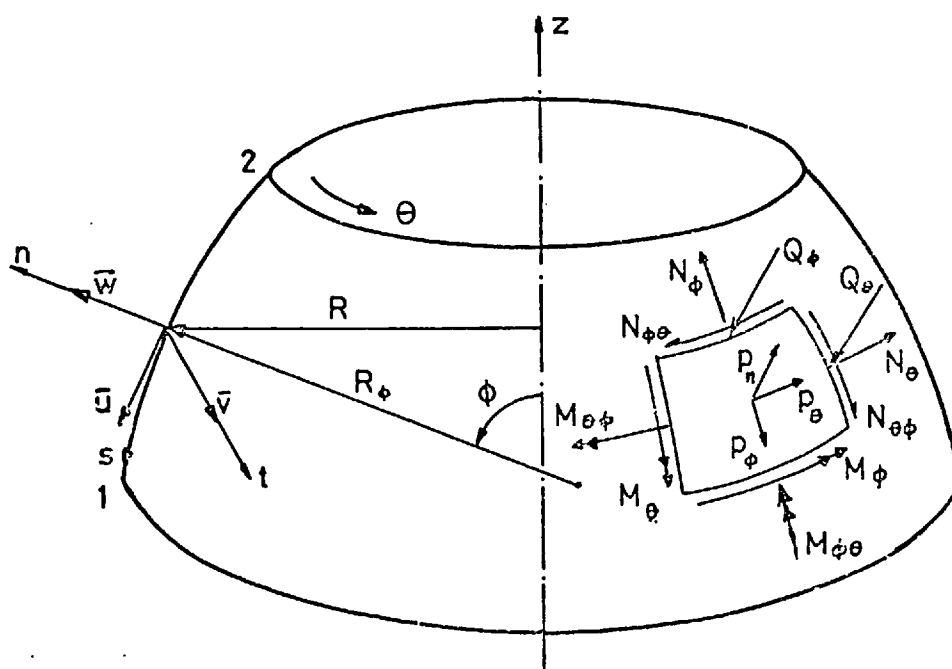


FIG. 1

A. Equilibrium

According to the thin shell theory derived by Novozhilov [11], the equilibrium equations in terms of stress resultants (membrane forces) and bending moments (FIG. 1) are as follows:

$$\begin{aligned}
 N'_{\phi} + \frac{\cos \phi}{R} N_{\phi} + \dot{N}_{\theta\phi} - \frac{\cos \phi}{R} N_{\theta} + \frac{1}{R_{\phi}} Q_{\phi} + P_{\phi} &= 0 \\
 N'_{\phi\theta} + \frac{\cos \phi}{R} N_{\phi\theta} + \dot{N}_{\theta} + \frac{\cos \phi}{R} N_{\theta\phi} + \frac{\sin \phi}{R} Q_{\theta} + P_{\theta} &= 0 \\
 Q'_{\phi} + \frac{\cos \phi}{R} Q_{\phi} + \dot{Q}_{\theta} - \frac{1}{R_{\phi}} N_{\phi} - \frac{\sin \phi}{R} N_{\theta} + P_n &= 0 \\
 M'_{\phi} + \frac{\cos \phi}{R} M_{\phi} - \frac{\cos \phi}{R} M_{\theta} + \dot{M}_{\theta\phi} - Q_{\phi} &= 0 \\
 M'_{\phi\theta} + \frac{\cos \phi}{R} M_{\phi\theta} + \frac{\cos \phi}{R} M_{\theta\phi} + \dot{M}_{\theta} - Q_{\theta} &= 0 \\
 N_{\phi\theta} - N_{\theta\phi} + \frac{1}{R_{\phi}} M_{\phi\theta} - \frac{\sin \phi}{R} M_{\theta\phi} &= 0 \\
 \text{where } ()' = \frac{\delta}{\delta s} (), \quad ()^{\cdot} = \frac{\delta}{R \delta \theta} () \quad \text{and } ds = R_{\phi} d\phi
 \end{aligned} \tag{1}$$

If shell is loaded at the edges, condition which is immediately realisable in the finite element idealisation where the applied loads are represented by equivalent forces acting at the edges of an element, then the following assumption can be made

$$N_{\phi\theta} - \frac{\sin \phi}{R} M_{\theta\phi} = N_{\theta\phi} - \frac{1}{R_{\phi}} M_{\phi\theta} = S \tag{2}$$

If a further assumption is introduced that the following moments are equal

$$M_{\phi\theta} = M_{\theta\phi} = T \tag{3}$$

then the set of equilibrium equations with ten unknowns can be greatly simplified. From the fourth and fifth equation of equilibrium it is possible to express normal shear forces Q_{ϕ} and Q_{θ} in terms of three moments M_{ϕ} , M_{θ} and T , so reducing the original set of unknowns to six

which are presented as a stress vector

$$\mathbf{S} = \{ N_\phi \ N_\theta \ S \ M_\phi \ T \ M_\theta \} \quad (4)$$

B. Compatibility

Corresponding to stress vector $\mathbf{S}$ there exist six strains

$$\mathbf{E} = \{ \epsilon_\phi \ \epsilon_\theta \ \epsilon_{\phi\theta} \ \kappa_\phi \ \kappa_{\phi\theta} \ \kappa_\theta \} \quad (5)$$

which, when expressed in terms of local displacements $\bar{U}$, $\bar{V}$ and $\bar{W}$, are

$$\begin{aligned} \epsilon_\phi &= \bar{U}' + \frac{1}{R_\phi} \bar{W} \\ \epsilon_\theta &= \frac{\cos \phi}{R} \bar{U} + \bar{V}' + \frac{\sin \phi}{R} \bar{W} \\ \epsilon_{\phi\theta} &= \bar{U}'' + \bar{V}' - \frac{\cos \phi}{R} \bar{V} \\ \kappa_\phi &= \frac{1}{R_\phi} \bar{U}' - \frac{R_\phi'}{R_\phi^2} \bar{U} - \bar{W}'' \\ \kappa_{\phi\theta} &= \frac{1}{R_\phi} \bar{U}'' + \frac{\sin \phi}{R} \bar{V}' - \frac{\sin \phi \cos \phi}{R^2} \bar{V} - \bar{W}''' + \frac{\cos \phi}{R} \bar{W}'' \\ \kappa_\theta &= \frac{\cos \phi}{RR_\phi} \bar{U} + \frac{\sin \phi}{R} \bar{V}'' - \bar{W}'' - \frac{\cos \phi}{R} \bar{W}' \end{aligned} \quad (6)$$

C. Stress/Strain Relationship

It is convenient at this stage to introduce notations for stresses and strains which are going to be used later on in the mixed method. The stresses $\mathbf{S}$ are partitioned into membrane forces $\mathbf{S}_M$, and bending moments $\mathbf{S}_B$

$$\mathbf{S} = \{ \mathbf{S}_M \ \mathbf{S}_B \} \quad (7)$$

where

$$\begin{aligned} \mathbf{S}_M &= \{ N_\phi \ N_\theta \ S \} \\ \mathbf{S}_B &= \{ M_\phi \ T \ M_\theta \} \end{aligned} \quad (8)$$

and the strains ϵ similarly partitioned into

$$\epsilon = \{ \epsilon_M \quad \epsilon_B \} \quad (9)$$

where

$$\begin{aligned} \epsilon_M &= \{ \epsilon_\phi \quad \epsilon_\theta \quad \epsilon_{\phi\theta} \} \\ \epsilon_B &= \{ \kappa_\phi \quad \kappa_{\phi\theta} \quad \kappa_\theta \} \end{aligned} \quad (10)$$

The elastic stress/strain relationship enables the membrane forces to be calculated in terms of the corresponding strains

$$S_M = \chi \epsilon_M \quad (11)$$

where

$$\chi = \frac{Et}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad (12)$$

and on the other hand, the bending strains can be determined from the corresponding bending moments

$$\epsilon_B = \phi S_B \quad (13)$$

where

$$\phi = \frac{12}{Et^3} \begin{bmatrix} 1 & 0 & -\nu \\ 0 & 1+\nu & 0 \\ -\nu & 0 & 1 \end{bmatrix} \quad (14)$$

1.2 GEOMETRY, MOMENT AND DISPLACEMENT INTERPOLATION

Let the shell be divided by nodal surfaces into a series of frusta with curved meridians, a typical element being shown in FIG. 2.

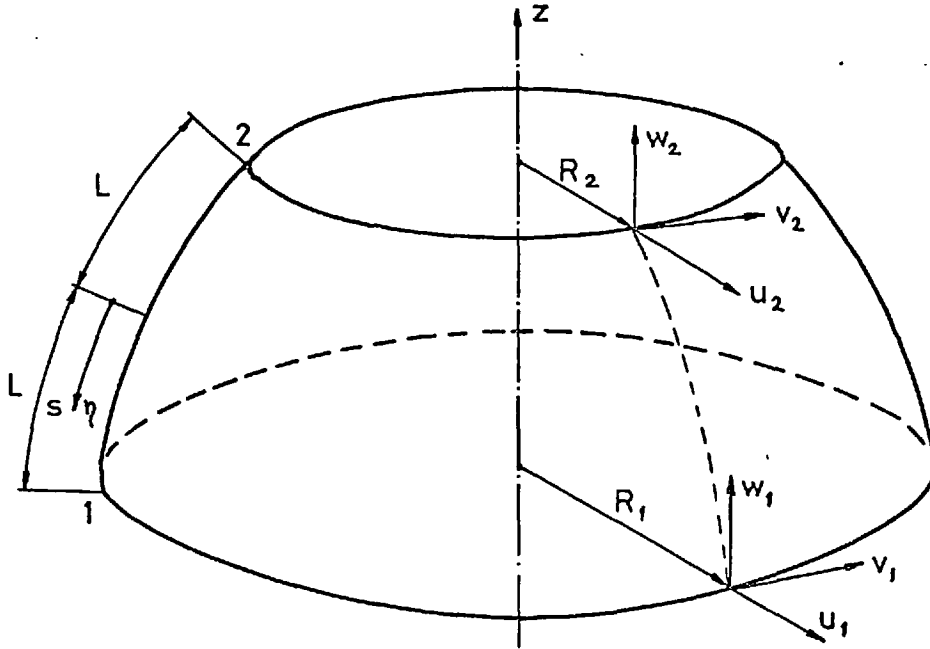


FIG. 2

The R and z coordinates on the element surface are defined by interpolation functions in terms of some known parameters at the two nodal circles. Since it is desired to preserve continuity of the meridian curve, the interpolation functions are chosen most conveniently to be the cubic Hermitian polynomials, thus the radius R is described as

$$R = \mathbf{h} \mathbf{r}_I \quad (15)$$

where

$$\mathbf{r}_I = \left\{ R_1 \left(\frac{dR}{ds} \right)_1, R_2 \left(\frac{dR}{ds} \right)_2 \right\} \quad (16)$$

$$\mathbf{h} = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 \end{bmatrix} \quad (17)$$

where

$$\begin{aligned}
 h_1 &= (2 + 3\eta - \eta^3) / 4 \\
 h_2 &= L(-1 - \eta + \eta^2 + \eta^3) / 4 \\
 h_3 &= (2 - 3\eta + \eta^3) / 4 \\
 h_4 &= L(1 - \eta - \eta^2 + \eta^3) / 4
 \end{aligned} \tag{18}$$

$\eta = S/L$ is the non-dimensional length parameter, measuring the distance S from the mid-point of the meridian curve of length $2L$, hence having values of $+1$ and -1 at the two end nodes 1 and 2 respectively. R_i and $(\frac{dR}{ds})_i$ are the values of R and its gradient along the meridian at node i . In exactly the same manner, the coordinate z is defined as

$$z = \mathbf{h} \mathbf{z}_I \tag{19}$$

where

$$\mathbf{z}_I = \left\{ z_1 \left(\frac{dz}{ds} \right)_1 \quad z_2 \left(\frac{dz}{ds} \right)_2 \right\} \tag{20}$$

This approximation is sufficient to describe a smooth shell. The importance of the geometry interpolation will be clear later on.

Supposing now that the moments M_ϕ , T and M_θ and their gradients are known at each node, then the moment variation along the meridian can be specified in terms of these end values and the same interpolation functions as for the geometry. For instance, the bending moment in the meridian direction can be described by

$$M_\phi = \mathbf{h} \mathbf{M}_\phi \tag{21}$$

where

$$\mathbf{M}_\phi = \left\{ M_{\phi 1} \left(\frac{\delta M_\phi}{\delta s} \right)_1 \quad M_{\phi 2} \left(\frac{\delta M_\phi}{\delta s} \right)_2 \right\} \tag{22}$$

The advantage of having moment derivatives at the nodes is not only because it is in line with the isoparametric approach, but because it can produce continuity of normal shear forces Q_ϕ and Q_θ . Generalising (21) for the complete set $\mathbf{S}_B$ one has

$$\mathbf{S}_B = \beta \mathbf{b} \quad (23)$$

where β is a diagonal matrix of Hermitian polynomials

$$\beta = [\mathbf{h} \ \mathbf{h} \ \mathbf{h}] \quad (24)$$

and vector $\mathbf{b}$ consists of the end values of the moments M_ϕ , T and M_θ and their first derivatives

$$\mathbf{b} = \{M_{\phi 1} \ M'_{\phi 1} \ M_{\phi 2} \ M'_{\phi 2} \ T_1 \ T'_1 \ T_2 \ T'_2 \ M_{\theta 1} \ M'_{\theta 1} \ M_{\theta 2} \ M'_{\theta 2}\} \quad (25)$$

where

$$(\)' = \frac{\delta}{\delta s} (\)$$

Having established continuous interpolation of geometry and moments, the same interpolation is now to be used for the displacements, thus giving a fully isoparametric element. Define (see FIG. 2)

$$\delta = \{u \ v \ w\} \quad (26)$$

as the displacement in the R , z and θ directions in the global coordinate system, and forming the element displacement vector

$$\rho = \{u_1 \ u'_1 \ u_2 \ u'_2 \ v_1 \ v'_1 \ v_2 \ v'_2 \ w_1 \ w'_1 \ w_2 \ w'_2\} \quad (27)$$

the displacements at any point on the element surface will be given as

$$\delta = \beta \rho \quad (28)$$

where β is defined by (24). Since the displacements and the geometry are interpolated by the same scheme, rigid-body movement can be truly

represented by appropriate displacements at the nodes, which will not produce any unwanted strains and stresses (see Ref. 1).

The strains are, however, expressed (e.g. 6) in terms of $\bar{u}$, $\bar{v}$ and $\bar{w}$, which may be collected into a local displacement vector

$$\hat{\delta} = \{\bar{u} \ \bar{v} \ \bar{w}\} \quad (29)$$

The transformation between these two sets of displacements is given by

$$\hat{\delta} = \Lambda \delta \quad (30)$$

where the transformation matrix Λ is given in the form

$$\Lambda = \begin{bmatrix} \cos \phi & 0 & -\sin \phi \\ 0 & 1 & 0 \\ \sin \phi & 0 & \cos \phi \end{bmatrix} \quad (31)$$

Once displacements δ are determined there will be no difficulties in calculating the strains. However, the essence of this mixed approach is that only the membrane strains ϵ_M will be calculated from δ , using the first three compatibility equations (6). The result of doing this is to produce a continuous membrane strain field which will then give, according to (11), continuous distribution of membrane forces S_M . The bending strains are to be determined from the bending moments S_B via the stress/strain relationship.

For an axi-symmetrical structure, the variation of forces and displacements in the circumferential direction can be decomposed into Fourier series. Separating the symmetrical and anti-symmetrical terms, the nodal parameters may be written as

$$\begin{aligned} \rho &= \rho_o + \sum_j \psi_j \rho_j + \sum_j \bar{\psi}_j \bar{\rho}_j \\ b &= b_o + \sum_j \psi_j b_j + \sum_j \bar{\psi}_j \bar{b}_j \end{aligned} \quad (32)$$

The matrices

$$\begin{aligned}\psi_j &= \begin{bmatrix} \cos j\theta & \mathbf{I}_4 & \sin j\theta & \mathbf{I}_4 & \cos j\theta & \mathbf{I}_4 \end{bmatrix} \\ \bar{\psi}_j &= \begin{bmatrix} \sin j\theta & \mathbf{I}_4 & \cos j\theta & \mathbf{I}_4 & \sin j\theta & \mathbf{I}_4 \end{bmatrix}\end{aligned}\quad (33)$$

distinguish between the symmetrical and anti-symmetrical parts, and $\mathbf{I}_4$ is a unit matrix of order 4×4 . For any point within an element, the displacements and moments for j^{th} harmonic can be written as

$$\delta_j = \beta \psi_j \rho_j \quad (34-a)$$

$$\mathbf{S}_{Bj} = \beta \psi_j \mathbf{b}_j \quad (34-b)$$

and

$$\delta = \sum_j \delta_j \quad (35-a)$$

$$\mathbf{S}_B = \sum_j \mathbf{S}_{Bj} \quad (35-b)$$

1.3 STRAIN-DISPLACEMENT RELATIONS

Having established the meridional and circumferential variations of displacements, it is straightforward to rewrite (6) in matrix form. For the j^{th} harmonic, the membrane strains will be given as

$$\epsilon_{Mj} = \mathbf{D}_M^t \hat{\delta}_j = \mathbf{D}_M^t \wedge \beta \psi_j \rho_j \quad (36)$$

and, similarly, the bending strains

$$\epsilon_{Bj} = \mathbf{D}_B^t \hat{\delta}_j = \mathbf{D}_B^t \wedge \beta \psi_j \rho_j \quad (37)$$

where the matrix operators D^t are defined by

$$D_M^t = \begin{bmatrix} \frac{\delta}{L\delta\eta} & 0 & \frac{1}{R_\phi} \\ \frac{\cos\phi}{R} & \frac{\delta}{R\delta\theta} & \frac{\sin\phi}{R} \\ \frac{\delta}{R\delta\theta} & \frac{\delta}{L\delta\eta} - \frac{\cos\phi}{R} & 0 \end{bmatrix} \quad (38)$$

and

$$D_B^t = \begin{bmatrix} \frac{1}{R_\phi L} \frac{\delta}{\delta\eta} - \frac{1}{R_\phi^2 L} \frac{\delta R_\phi}{\delta\eta} & 0 & -\frac{1}{L^2} \frac{\delta^2}{\delta\eta^2} \\ \frac{1}{R_\phi R} \frac{\delta}{\delta\theta} & \frac{\sin\phi}{RL} \frac{\delta}{\delta\eta} - \frac{\sin\phi\cos\phi}{R^2} & -\frac{1}{RL} \frac{\delta^2}{\delta\eta\delta\theta} + \frac{\cos\phi}{R^2} \frac{\delta}{\delta\theta} \\ \frac{\cos\phi}{R_\phi R} & \frac{\sin\phi}{R^2} \frac{\delta}{\delta\theta} & -\frac{1}{R^2} \frac{\delta^2}{\delta\theta^2} - \frac{\cos\phi}{RL} \frac{\delta}{\delta\eta} \end{bmatrix} \quad (39)$$

where the parameters L , R , R_ϕ , ϕ , θ and η are shown in FIGS. 1 and 2.

Since the matrices Λ , β and ψ_j are functions of ϕ , η and θ

respectively, the differentiation on the product $\Lambda \beta \psi_j$ can easily be performed and thus simplifying (36) and (37) to

$$\epsilon_{Mj} = \alpha_{Mj} \rho_j \quad (40)$$

$$\epsilon_{Bj} = \alpha_{Bj} \rho_j \quad (41)$$

The full expansion of matrices α is given as follows

$$\alpha_{Mj} = \begin{bmatrix} \frac{\cos\phi}{L} P_j^i h^i & 0 & -\frac{\sin\phi}{L} P_j^i h^i \\ \frac{1}{R} P_j^i h^i & \frac{1}{R} \bar{P}_j^i h^i & 0 \\ \frac{\cos\phi}{R} P_j^i h^i & (\frac{1}{L} h^i - \frac{\cos\phi}{R} h^i) \bar{P}_j^i & -\frac{\sin\phi}{R} P_j^i h^i \end{bmatrix} \quad (42)$$

and

$$\alpha_{Bj} = \begin{bmatrix} -\frac{\cos\phi_P}{R\phi L_j} h' - \frac{\sin\phi_P}{L^2} h'' & 0 & \frac{\sin\phi_P}{RL} h' - \frac{\cos\phi_P}{L^2} h'' \\ \frac{\sin\phi\cos\phi_P}{R^2} h' - \frac{\sin\phi_P}{RL} h' & -\frac{\sin\phi\cos\phi_P}{R^2} h + \frac{\sin\phi_P}{RL} h' & \frac{\cos^2\phi_P}{R^2} h' - \frac{\cos\phi_P}{RL} h' \\ -\frac{\sin\phi_P}{R^2} h'' - \frac{\sin\phi\cos\phi_P}{RL} h' & \frac{\sin\phi_P}{R^2} h' & -\frac{\cos\phi_P}{R^2} h'' - \frac{\cos^2\phi_P}{RL} h' \end{bmatrix} \quad (43)$$

where $P_j = \cos j\theta$ $\bar{P}_j = \sin j\theta$

or $P_j = \sin j\theta$ $\bar{P}_j = \cos j\theta$

$$P'_j = \frac{\delta P_j}{\delta\theta} \quad h' = \frac{\delta h}{\delta\eta} \quad \text{etc.}$$

1.4 FORMULATION OF THE MIXED METHOD OF APPROACH

In this mixed method of finite element idealisation, the bending moments $\mathbf{S}_B$ and the displacement $\hat{\delta}$ at any point are calculated by interpolation from the values of moments and displacements and their derivatives at the nodes (eqs. 23 and 28). The membrane strains

ϵ_M are derived from the displacements ρ (eq. 36). The bending strains ϵ_B and the membrane stress $\mathbf{S}_M$ are then calculated from the stress/strain relationship (eqs. 13 and 11). While the membrane strains ϵ_M are compatible with the displacements $\hat{\delta}$, the bending strains ϵ_B are not, and the compatibility equation

$$\epsilon_B - \mathbf{D}_B^t \hat{\delta} = 0 \quad (44)$$

can only be satisfied approximately by the application of the Principle of virtual force such that

$$\int \bar{\mathbf{S}}_B^t (\epsilon_B - \mathbf{D}_B^t \hat{\delta}) dA = 0 \quad (45)$$

The virtual forces $\bar{\mathbf{S}}_B$ may also be regarded as weighted integrals. On the other hand, the equilibrium conditions,

$$\mathbf{D}\sigma + \mathbf{w} = 0$$

and (46)

$$(\mathbf{D}n)\sigma = \varphi$$

(where $\mathbf{w}$ is the body force, φ the surface force, n the surface outward normal and $\mathbf{D}$ the linear differential operator), will not be satisfied exactly by the stresses σ from $\mathbf{S}_B$ and $\mathbf{S}_M$, but can be approximately enforced with a virtual displacement $\underline{u}$ serving as a weighting function, giving

$$\int \underline{u}^t (D\sigma + w) dv + \int \underline{u}^t [\varphi - (Dn)\sigma] dS = 0 \quad (47)$$

Application of the Gauss theorem (partial integration rule) changes this into the familiar form given by the Principle of virtual displacement

$$\int \underline{u}^t w dv + \int \underline{u}^t \varphi dS = \int (D^t u)^t \sigma dv = \int \underline{\epsilon}^t \sigma dv \quad (48)$$

which, in the context of this shell element, becomes

$$\underline{p}^t P = \int \underline{\epsilon}_M^t S_M dA + \int \underline{\epsilon}_B^t S_B dA \quad (49)$$

where P is the kinematically equivalent nodal force derived from the left hand side of eq. (48). As both the displacement field and the bending moment variation are chosen to be continuous across element boundaries, no "jump integral" need to be included in the approximate enforcement of compatibility and equilibrium in equations (45) and (49), and the integrations over the entire structure can be performed by summing over the individual elements. Since the terms of the Fourier series are orthogonal, integration around the circumference for different stress and strain harmonics will vanish, the harmonics are uncoupled and eqs. (45) and (49) may be applied to each harmonic separately.

For the j^{th} harmonic, the virtual system $\bar{S}_{Bj}$ can be expressed in terms of a virtual set of nodal moments $\bar{b}_j$ as in eq. (34-b)

$$\bar{S}_{Bj} = \beta \psi_j \bar{b}_j \quad (50)$$

hence equation (45) is satisfied if, for all harmonics

$$\int \psi_j^t \beta^t (\epsilon_{Bj} - D_B^t \hat{\delta}_j) dA = 0 \quad (45-a)$$

Since this mixed method of formulation treats the membrane and bending strains differently, any initial strain must also be separated into two parts and handled accordingly.

The total bending strain in the j^{th} harmonic is given by

$$\epsilon_{Bj} = \phi \sigma_{Bj} + \eta_{Bj} \quad (51)$$

where η_{Bj} is the initial bending strain.

Substitution of equations (34-a), (41), (50) and (51) into (45-a) gives

$$\int \psi_j^t \beta^t (\phi \beta \psi_j b_j + \eta_{Bj} - \alpha_{Bj} \rho_j) dA = 0 \quad (52)$$

or

$$f_j b_j + q_j - t_j \rho_j = 0 \quad (53)$$

where

$$f_j = \int \psi_j^t \beta^t \phi \beta \psi_j dA \quad (54)$$

is clearly a flexibility matrix derived from the interpolated bending moments,

$$t_j = \int \psi_j^t \beta^t \alpha_{Bj} dA \quad (55)$$

is a coupling matrix, and

$$q_j = \int \psi_j^t \beta^t \eta_{Bj} dA \quad (56)$$

is the vector of equivalent initial nodal curvatures.

Equation (53) is in fact a compatibility equation, i.e. it forces the bending strains to be approximately compatible with displacements. Similarly, virtual strains $\underline{\epsilon}_{Mj}$ and $\underline{\epsilon}_{Bj}$ can be expressed in terms of virtual displacements $\underline{\rho}$, as in (40) and (41). If the j^{th} harmonic of the total membrane strain ϵ_{Mj} includes some initial strains η_{Mj} , then the j^{th} harmonic of the membrane forces is given by

$$\mathbf{S}_{Mj} = \kappa (\boldsymbol{\varepsilon}_{Mj} - \boldsymbol{\eta}_{Mj}) \quad (57)$$

Substitution of this equation and eqs. (34b), (40) and (41) into eq. (49) gives

$$\mathbf{P}_j = \int \alpha_{Mj}^t \kappa \alpha_{Mj} dA \rho - \int \alpha_{Mj}^t \kappa \boldsymbol{\eta}_{Mj} dA + \int \alpha_{Bj}^t \beta \psi_j dA \mathbf{b}_j \quad (58)$$

or simply

$$\mathbf{P}_j = \mathbf{k}_j \rho_j + \mathbf{t}_j^t \mathbf{b}_j - \mathbf{j}_j \quad (59)$$

where

$$\mathbf{k}_j = \int \alpha_{Mj}^t \kappa \alpha_{Mj} dA \quad (60)$$

is the membrane stiffness matrix, as in the orthodox displacement method,

$$\mathbf{t}_j^t = \int \alpha_{Bj}^t \beta \psi_j dA \quad (61)$$

is the transpose of the coupling matrix (55), and

$$\mathbf{j}_j = \int \alpha_{Mj}^t \kappa \boldsymbol{\eta}_{Mj} dA \quad (62)$$

is the initial membrane load vector.

Eqs. (53) and (59) are the two equations from which the nodal parameters ρ_j and $\mathbf{b}_j$ can be found for a given system of applied forces and initial loads. There are two ways of solving these two equations, each with its advantages and disadvantages. Firstly, the element flexibility matrix $\mathbf{f}_j$ in (54) is clearly non-singular since the columns of β and all parameters in $\mathbf{b}_j$ are independent. It may be inverted to eliminate the moments $\mathbf{b}_j$ in terms of displacements ρ_j at the element level to obtain

$$\mathbf{b}_j = \mathbf{f}_j^{-1} (\mathbf{t}_j \rho_j - \mathbf{q}_j) \quad (63)$$

Substitution into (59) gives an equivalent element stiffness formulation

$$P_j = (k_j + t_j^t f_j^{-1} t_j) p_j - t_j^t f_j^{-1} q_j - j_j \quad (64)$$

which may be assembled and solved for the complete structure as in the displacement method. This is probably the most economical in computing time and space since the element flexibility matrix f_j is quite small. The number of unknown displacements in the final equation is confined to 6 at a node, and the stiffness matrix to be solved is banded. The big disadvantage is that the bending moments are no longer continuous across element boundaries after this treatment, as they are now calculated indirectly for each element from the nodal displacements. This disadvantage may be alleviated by the second treatment, which merges eqs. (53) and (59) into a single equation for each element and assembles them into the form

$$\begin{bmatrix} R_j + J_j \\ Q_j \end{bmatrix} = \begin{bmatrix} K_j & T_j^t \\ T_j & -F_j \end{bmatrix} \begin{bmatrix} r_j \\ B_j \end{bmatrix} \quad (65)$$

for the entire structure. This equation may then be solved as it stands simultaneously for the displacement and moments, or, the non-positive definite matrix may be condensed by eliminating the moments B_j in terms of the displacements r_j at the structural level. In spite of doubling the number of unknowns the former approach has been adopted here, because the equilibrium of moments between two or more adjacent elements can easily be introduced, as will be explained later. This is considered to be vital in the large displacement problem, where the solution is critically dependent on the stresses which should be as accurate as possible. The large displacement analysis will be given in Part II of this paper, and meanwhile, the details of the assembly of eq. (65) and the most convenient way of solution will be examined next.

1.5 ASSEMBLY OF ELEMENT PROPERTIES AND METHOD OF SOLUTION

In the small displacement theory there is no coupling between the harmonics in the square matrix in eq. (65). The response to each loading harmonic can therefore be solved separately. To do this, the equations (53) and (59) are written as one matrix equation

$$\begin{bmatrix} \mathbf{P}_j + \mathbf{j}_j \\ \mathbf{q}_j \end{bmatrix} = \begin{bmatrix} \mathbf{k}_j & \mathbf{t}_j^t \\ \mathbf{t}_j & -\mathbf{f}_j \end{bmatrix} \begin{bmatrix} \rho_j \\ \mathbf{b}_j \end{bmatrix} \quad (66)$$

or simply

$$\bar{\Gamma}_j = \bar{\mathbf{k}}_{Fj} \bar{\gamma}_j \quad (66-a)$$

The left hand side consists of nothing but the kinematically equivalent forces and initial loads, and may still be referred to as the force vector. The parameters on the right

$$\bar{\gamma}_j = \{ \rho_j \quad \mathbf{b}_j \} \quad (66-b)$$

contain displacement and moment parameters at the two nodes 1 and 2, as defined in eqs. (25) and (27). For convenience of assembly, all the parameters at each end are grouped together to form a new vector

$$\gamma_j = \{ \rho_1 \quad \mathbf{b}_1 \quad \rho_2 \quad \mathbf{b}_2 \}_j \quad (67)$$

where $\rho_i = \left\{ u \frac{\delta u}{\delta s} \quad v \frac{\delta v}{\delta s} \quad w \frac{\delta w}{\delta s} \right\}_i$

and $\mathbf{b}_i = \left\{ M\phi \frac{\delta M\phi}{\delta s} \quad T \frac{\delta T}{\delta s} \quad M\theta \frac{\delta M\theta}{\delta s} \right\}_i$

are the displacement and moment parameters at node i ($i=1,2$).

Clearly, the relationship between the old and new definitions is given by

$$\bar{\gamma}_j = \mathbf{C} \gamma_j \quad (68)$$

where the transformation (boolean) matrix $\mathbf{C}$ is defined as

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_1 \\ \mathbf{C}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2 \end{bmatrix} \quad (68)$$

in which

$$\mathbf{C}_1 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_2 & \mathbf{0} \end{bmatrix} \quad (69-a)$$

and

$$\mathbf{C}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{I}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_2 \end{bmatrix}$$

The forces corresponding to γ_j

$$\Gamma_j = \mathbf{C}^t \overline{\Gamma}_j = \{ \mathbf{P}_1 + \mathbf{j}_1 \quad \mathbf{q}_1 \quad \mathbf{P}_2 + \mathbf{j}_2 \quad \mathbf{q}_2 \}_j \quad (70)$$

will simply be a rearrangement of the kinematically equivalent nodal forces and the initial loads, and are related to γ_j by

$$\Gamma_j = \mathbf{C}^t \overline{\mathbf{k}}_{Fj} \overline{\gamma}_j = \mathbf{C}^t \overline{\mathbf{k}}_{Fj} \mathbf{C} \gamma_j = \mathbf{k}_{Fj} \gamma_j \quad (71)$$

For a single branched shell structure, the nodal parameters defined in eq. (67) can be identified from the entire set of nodal parameters

$$\mathbf{g}_j = \{ \gamma_1 \dots \gamma_k \dots \gamma_s \}_j \quad (72)$$

where γ_k is the parameters at the k^{th} node and s is the total number of nodes, by a Boolean matrix $\mathbf{a}_g$ for the g^{th} element

$$\gamma_{gj} = \mathbf{a}_g \mathbf{g}_j \quad (73)$$

Hence the complete set of applied forces on the structure, defined consistently with $\mathbf{g}_j$ as a super column of all nodal values

$$\mathbf{G}_j = \{ \mathbf{r}_1 \dots \mathbf{r}_k \dots \mathbf{r}_s \} \quad (74)$$

is related to $\mathbf{g}_j$ by

$$\mathbf{G}_j = \mathbf{K}_{Fj} \mathbf{g}_j \quad (75)$$

where the matrix $\mathbf{K}_{Fj}$ for the j^{th} harmonic is given as

$$\mathbf{K}_{Fj} = \sum_g \mathbf{a}_g^t \mathbf{k}_{Fgj} \mathbf{a}_g \quad (76)$$

This matrix can easily be inverted after suppression of rigid body freedoms and the solution obtained. A point to note is that the matrix $\mathbf{K}_{Fj}$, although well conditioned, is not a positive-definite matrix.

Alternatively, a positive-definite structural stiffness matrix may be obtained by the following procedure. Displacements and moments for a structure can be grouped separately as

$$\mathbf{r}_j = \{ \hat{\rho}_1 \dots \hat{\rho}_k \dots \hat{\rho}_s \}_j \quad (77)$$

$$\mathbf{B}_j = \{ \hat{\mathbf{b}}_1 \dots \hat{\mathbf{b}}_k \dots \hat{\mathbf{b}}_s \}_j$$

where $\hat{\rho}_k$ and $\hat{\mathbf{b}}_k$ are the displacement and moments at the k^{th} structural nodal point. Obviously, for a particular element

$$\begin{aligned} \rho_j &= \mathbf{a}_g \mathbf{r}_j \\ \mathbf{b}_j &= \mathbf{a}_g \mathbf{B}_j \end{aligned} \quad (78)$$

Substituting (78) into (53) and (59) give the following equations for the entire structure

$$\mathbf{F}_j \mathbf{B}_j + \mathbf{Q}_j - \mathbf{T}_j \mathbf{r}_j = \mathbf{O} \quad (79)$$

$$\mathbf{R}_j = \mathbf{K}_j \mathbf{r}_j + \mathbf{T}^t \mathbf{B}_j - \mathbf{J}_j \quad (80)$$

where

$$\begin{aligned}
 K_j &= \sum_g a_g^t k_{gj} a_g \\
 F_j &= \sum_g a_g^t f_{gj} a_g \\
 T_j &= \sum_g a_g^t t_{gj} a_g \\
 R_j &= \{ \hat{P}_1 \dots \hat{P}_k \dots \hat{P}_s \}_j \\
 J_j &= \{ j_1 \dots j_k \dots j_s \}_j \\
 Q_j &= \{ q_1 \dots q_k \dots q_s \}_j
 \end{aligned} \tag{81}$$

Now the nodal moments B_j can be calculated from

$$B_j = F_j^{-1} (T_j r_j - Q_j) \tag{82}$$

and eliminated from eq. (80) to yield

$$R_j = (K_j + T_j^t F_j^{-1} T_j) r_j - T_j^t F_j^{-1} Q_j - J_j \tag{83}$$

or

$$\hat{R}_j = \hat{K}_j r_j$$

Although the condensed matrix $\hat{K}_j$ is now positive-definite and the number of unknowns reduced by half, the solution by this approach is in fact less convenient since $\hat{K}$ is fully populated. For the large displacement analysis in particular, due to the coupling between harmonics introduced by geometrical non-linearities, the storage of all the large matrices involved would make the computation very inefficient to accomplish. The small displacement problem is therefore solved as it stands in eq. (75). It should be noticed that as the parameters at each node are grouped together the K_{Fj} matrix for every shell-branch is banded, having semi-band-width of 24 (each element spans 2 nodes with 12 parameters at each node). The solution for a single-shell structure is straight-forward; all the up-to-date developments and sophistications on solution of banded matrix equations may be involved.

The solution of branched shell structures is just as easy by expressing the parameters at one node in terms of those at the next node progressively, starting from the un-connected end of each of the branches towards the branched node. Finally the values of the parameters at the branch node are solved, having taken due consideration of the equilibrium requirements. The details of the treatment will be discussed later. The solution gives the complete set of nodal parameters, from which the displacements δ and bending moments $\mathbf{S}_B$ can be found by interpolation from eqs. (23) and (28). The membrane forces $\mathbf{S}_M$ are then calculated from eqs. (6) and (11), while the normal shears Q_ϕ and Q_θ from the equations of equilibrium (1).

1.6 KINEMATICALLY EQUIVALENT LOADS

From the formulation of this mixed method, eqs. (43) and (49), it is evident that the definition of the generalised forces $\mathbf{P}$ is exactly the same as in the displacement method. A distributed loading on the axisymmetric shell structure may be decomposed into Fourier series around the circumference and assumed to vary along the meridian in the same way as the displacements, moments and geometry. Then the equivalent nodal force for the j^{th} harmonic is given by

$$\mathbf{P}_j = \left(\int \psi_j^t \beta^t \beta \psi_j dA \right) \mathbf{p}_j \quad (84)$$

where the vector $\mathbf{p}_j$ is given by

$$\mathbf{p}_j = \{ p_{R1} \ p'_{R1} \ p_{R2} \ p'_{R2} \ p_{\theta 1} \ p'_{\theta 1} \ p_{\theta 2} \ p'_{\theta 2} \ p_{z1} \ p'_{z1} \ p_{z2} \ p'_{z2} \}_j \quad (85)$$

and represents the values of pressures at the nodal circles. The detail development has been given in Ref. [1] and will not be repeated here.

Two special cases are useful:

1. For a constant (along the meridian) normal pressure in the j^{th} harmonic

$$\mathbf{P}_j = p_j \int \psi_j^t \beta^t \{ \sin\phi \ 0 \ \cos\phi \} \cos j\theta dA \quad (86)$$

which after the integration round the circumference gives

$$\mathbf{P}_j = k\pi L p_j \int_{-1}^{+1} \{ \sin\phi \ h^t \ 0_4 \ \cos\phi \ h^t \} R d\eta \quad (87)$$

where $k = 2$ for $j = 0$ and $k = 1$ for all other values of j .

2. For a constant pressure in the z direction, e.g. the weight w

$$\mathbf{P} = 2\pi L W \int_{-1}^{+1} \{ 0_4 \ 0_4 \ h^t \} R d\eta \quad (88)$$

Both expressions can easily be evaluated once the shell geometry is known.

The forces given here correspond to the nodal displacements ρ_j defined in eq. (27). They must be transformed to Γ_j as defined in (70) and then assembled into

$$\mathbf{G}_j = \sum_g \mathbf{a}_g^t \Gamma_{gj} \quad (89)$$

to obtain the column of applied forces in the final equation (75).

1.7 TREATMENT OF BRANCHED SHELLS AND SHELLS WITH DISCONTINUITIES

The branched-shell structure considered here consists of simple shells having one common axis of symmetry (Fig. 3). The intersection of three or more shells is called a branch point, in fact a circle. When the shell contains more than one single branch, the simple banded structure of the K_{Fj} matrix in eq. (75) is unfortunately destroyed and it is no longer economical to solve this equation in the complete assembled form. Instead, the unknown parameters are partitioned into nodal groups, and, starting from the un-connected end of each branch, each group of nodal parameters is eliminated in terms of the parameters at the next node. This process is performed progressively towards the branch point, on whose parameters all other nodal parameters now depend. The final solution of the matrix equation at this point and a backward substitution then yields the complete set of wanted values. This method of solution is a special application of the so-called "frontal" technique. A schematic example is shown in Fig. 3-a, with arrows indicating the direction of the progressive elimination. Should the shell contain more than one branch point (Fig.3-b), the same method can still be applied. The solution of the

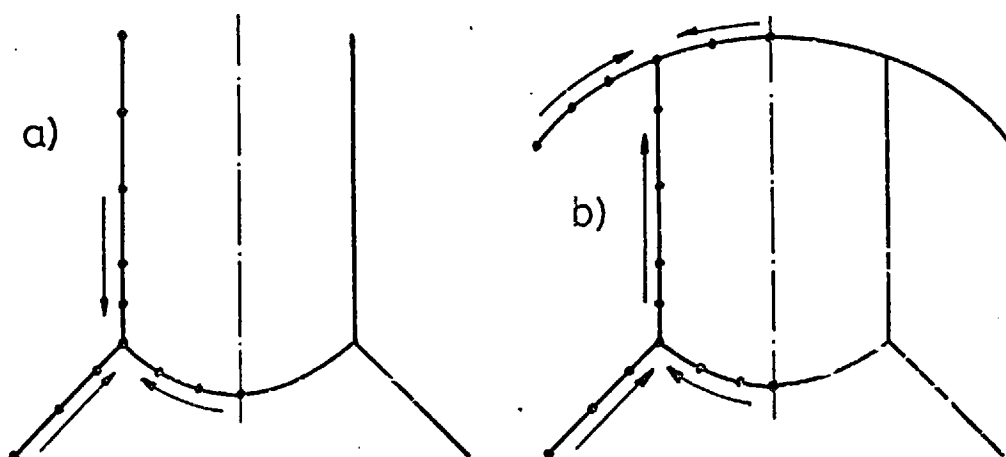


FIG. 3

parameters at branch points other than the last one will not be explicit and final, but will be in terms of the parameters at the next adjacent node.

Normally, at an ordinary nodal point of a continuous shell, continuity of displacements and moments between adjacent elements can be ensured simply by letting the two elements have the same set of nodal parameters at the common node, as has been done in section 1.5. This is no longer true at a point of discontinuity, especially at the branch point. In these circumstances, continuity (kinematic and static) requires that the displacement and the rotation (if the structure can be assumed to have rigid joints) at the joint are the same for all elements meeting there, and that the sums of all internal moments and resultant forces in two mutually perpendicular directions vanish. These requirements are implemented in the computing programme as constraints on the nominally independent parameters from the elements of different branches meeting at that point.

Firstly and most obviously, the displacements of all elements at the branch point are equal, which may be treated in the usual way as for the case of continuous shell. Next, the rotation is given by

$$\psi = \frac{\bar{u}}{R\phi} - \frac{\delta \bar{w}}{\delta s} = -\sin\phi \frac{\delta u}{\delta s} - \cos\phi \frac{\delta w}{\delta s} \quad (90)$$

after the substitution of eq. (30) and since $ds = R\phi \, d\phi$.

If ψ is introduced as an independent common parameter at the discontinuity, then (90) represents a constraint equation for the parameters $\frac{\delta u}{\delta s}$ and $\frac{\delta w}{\delta s}$ of each element meeting there. Note that this condition should be incorporated not only at branch points, but also at other points of discontinuity; for instance, a sudden change of

meridional direction, as when a cylinder is joined to a cone.

Consider now the moments and resultant forces at the point of discontinuity, and particularly at a branch point (Fig. 4).

Direct control can most easily

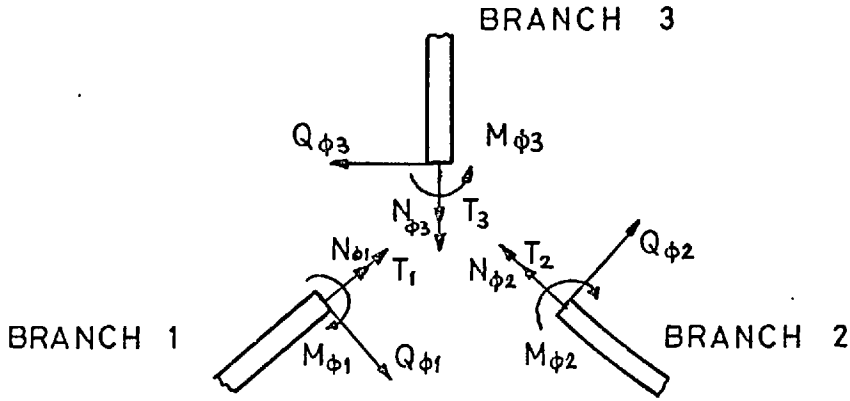


FIG. 4

be exerted over the equilibrium of the meridional moment M_ϕ with the constraint equation

$$\sum M_\phi = 0 \quad (91)$$

summing over all shell branches. Care must be taken on the sign of the moment M_ϕ . For instance, if M_ϕ is considered positive for edges of elements where $\eta = -1$ at the branch point (e.g. branches 1 and 2 in Fig.4), then it must be changed to have a negative sign for edge $\eta = +1$ (e.g. branch 3). The same remarks will apply to other forces referred to later on. It has been tacitly assumed that the actual applied loads are distributive in nature. In the unlikely event of a concentrated applied moment it should of course be incorporated in this equation.

Equilibria of the resultant membrane forces and shear forces are much more difficult, since the former are calculated from the displacements in the mixed method of approach. The constraint equations to be satisfied are, according to Novozhilov [11],

$$\begin{aligned}\Sigma X &= 0 \\ \Sigma Z &= 0 \\ \Sigma \left(S + 2 \frac{T}{R\theta} \right) &= 0\end{aligned}\tag{93}$$

where X and Z are the radial and axial force resultants

$$\begin{aligned}X &= N\phi \cos\phi + \left(Q\phi + \frac{\partial T}{R\partial\theta} \right) \sin\phi \\ Z &= N\phi \sin\phi - \left(Q\phi + \frac{\partial T}{R\partial\theta} \right) \cos\phi\end{aligned}\tag{94}$$

From the first and the third equation of (11)

$$N\phi = \frac{Et}{1-\nu^2} (\epsilon\phi + \nu \epsilon\theta)$$

and (95)

$$S = \frac{Et}{2(1+\nu)} \epsilon\phi\theta$$

where the strains $\epsilon\phi$, $\epsilon\theta$ and $\epsilon\phi\theta$ at the branch point can be expressed in terms of the displacement parameters as

$$\begin{aligned}\epsilon\phi &= \cos\phi u' - \sin\phi w' \\ \epsilon\theta &= \frac{1}{R} u + \frac{j}{R} v \\ \epsilon\phi\theta &= -j \frac{\cos\phi}{R} u - \frac{\cos\phi}{R} v + v' + j \frac{\sin\phi}{R} w\end{aligned}\tag{96}$$

The above expressions, where j is the j^{th} harmonic number, are derived from (40) and (42). Now the elasticity relations (95) can be written as

$$N_T^* = \frac{Et}{1-\nu^2} \left[\cos\phi u' - \sin\phi w' + \frac{\nu}{R} u + j \frac{\nu}{R} v \right] \quad (97)$$

$$S = \frac{Et}{2(1+\nu)} \left[-j \frac{\cos\phi}{R} u - \frac{\cos\phi}{R} v + v' + j \frac{\sin\phi}{R} w \right]$$

The normal shear force $Q\phi$ can now be expressed in terms of nodal force parameters, from the 4th equation of (1), as

$$Q\phi = M'\phi + \frac{\cos\phi}{R} M\phi - \frac{\cos\phi}{R} M\theta + \frac{j}{R} T \quad (98)$$

Equations (97) and (98) can now be substituted in the force equilibrium equation (93), which becomes a set of constraints relating a mixture of force and displacement parameters.

Thus, apart from the three displacements u , v and w which are common to all branches meeting at a point, the other nodal parameters from each branch may be regarded as nominally independent. With the introduction of a common rotation ψ , the parameters at a branch point consist of

$$\mathbf{r}_B^* = \{\psi \ u \ v \ w \ \mathbf{p}'_1 \ \mathbf{p}'_2 \ \mathbf{p}'_3 \ \dots \dots \} = \{\psi \ \mathbf{r}_B\} \quad (99)$$

where

$$\mathbf{p}'_k = \{ u' \ v' \ w' \ M\phi \ M'\phi \ T \ T' \ M\theta \ M'\theta \ }_k \quad (100)$$

are the parameters from the k^{th} branch. The progressive elimination so far yields a matrix equation

$$\mathbf{K}_B \ \mathbf{r}_B = \mathbf{R}_B \quad (101)$$

with the addition of ψ this is expanded to

$$\begin{bmatrix} 0 & 0 \\ 0 & \mathbf{K}_B \end{bmatrix} \begin{bmatrix} \psi \\ \mathbf{r}_B \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{R}_B \end{bmatrix} \quad (102)$$

or simply

$$\mathbf{K}_B^* \ \mathbf{r}_B^* = \mathbf{R}_B^*$$

Now $\mathbf{r}_B^*$ is constrained by n (number of branches) equations of the type in (90) and the six equations in (91), (92) and (93). Collectively they may be written as

$$\mathbf{C} \mathbf{r}_B^* = \mathbf{0} \quad (103)$$

The solution of (102) subject to constraints (103) may be found most easily from solving

$$\begin{bmatrix} \mathbf{K}_B^* & \mathbf{C}^t \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{r}_B^* \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_B^* \\ \mathbf{0} \end{bmatrix} \quad (104)$$

where $\boldsymbol{\lambda}$ consists of the generalised forces and displacements (as the case may be) necessary to enforce the constraints (or mathematically, the Lagrange multipliers).

In practice, it is found that the most important constraints are (90) and (91). The force equilibrium constraints in (92) and (93) are relatively un-important and may be ignored. Thus, for a branch point with 3 branches, the 31 parameters are constrained by only 4 equations, which affect only 10 of them

$(\psi, u'_1, u'_2, u'_3, w'_1, w'_2, w'_3, M\phi_1, M\phi_2, M\phi_3)$. The size of the matrix is enlarged only from 30×30 in (101) to 35×35 in (104). Since the constraints (90) and (91) are relatively simple, even this is unnecessary. The solution of (102) and (103) may be obtained in the original form by using (103) to reduce the number of independent variables (now $31 - 4 = 27$) in (102). Moreover, from the examples so far tried, the unaffected parameters $(v' M\phi' T T' M\theta M\theta')$ from the 3 branches may be constrained to be the same without introducing much error. If that is accepted, the number of independent parameters at the branch point (with 3 branches only) is only 15.

This may not be of a great deal of advantage in the small displacement analysis, but will make a very big difference in the large displacement problem where all harmonics are coupled.

1.8 APPLICATION OF THE MIXED METHOD OF APPROACH TO THE LINEAR ANALYSIS OF SHELLS

A computer programme based on the linear theory was tested first of all on two single-branch structures. One was a torus under constant internal pressure and the other a hyperbolic cooling tower under the wind loading. Having made sure that the results agree satisfactorily with other methods of solution, the programme was then used for the analysis of discontinuous and branched shells. The first example in this group is a cyclinder closed with a spherical segment and the second, a sphere supported by a cylindrical skirt. In all cases, the displacements, membrane stresses and particularly bending moments are obtained with an accuracy which clearly shows the advantage of this method for the analysis of the general axisymmetric shells.

a) Pressurised Torus

This problem has been chosen with the similar objectives as in Ref. [1], i.e. to prove the ability of the element to handle accurately rapid changes in bending moments. The torus is loaded by uniform external pressure of 1000 psi and its dimensions and the material properties are shown on Fig. 5a. The results are compared to those of Kalnins [12] who used a multisegment integration method to solve the equilibrium equations, and to the SABA 5 element of Ref. [1]. Ten elements are used over half of a structure, the division of which as well as the radial displacements are shown on Fig. 5b. The distributions of the membrane stress N_θ and bending moments M_ϕ and M_θ along the meridian line are shown on Figs. 6, 7 and 8, respectively. The disagreement in the bending moment distribution near the outer edge shown in Ref. [1] in comparison to Kalnins' results is not repeated here and the agreement with Ref. [12] is excellent. It must be emphasized however that only six elements were used in [1] in

comparison to ten elements by the mixed method. In order to investigate and compare the convergence of this method and the displacement method of [1], Fig. 9 is presented. It requires ten elements to converge to Kalnins' solution which is not as good as the six elements required by SABA 5, although the result seems to be nearer to those of Ref. [12]. The reason for the larger number of elements required by this mixed method probably lies in the inferior interpolation function used for the approximation of the geometry, namely the third order polynomial against the fifth order of SABA 5.

b) Hyperbolic Cooling Tower

The cooling tower shown in Fig. 10a has been used notably as an example of analysis by various methods, the finite difference method by Albasiny and Martin [13], the SABA 5 element of [1] and the SHEBA element of [14]. The tower is idealised in such a way that:

- i) the shell is of uniform thickness and of homogeneous, isotropic material
- ii) the base of the shell is fully clamped
- iii) the top edge of the shell is free
- iv) the shell is subjected to a pressure loading from the wind which is assumed to be constant along the meridian and circumferentially symmetrical, hence may be decomposed into

$$p(\theta) = \sum_0^n p_j \cos j\theta$$

In this analysis the same ten harmonics are used ($n=9$) as in [1] and [13], giving the pressure distribution shown on Fig.10b. The pressure coefficients are listed in Table 1.

j	P_j	j	P_j
0	.22892	5	-.12010
1	.27779	6	-.02678
2	.59821	7	.04443
3	.47010	8	.00180
4	.06269	9	-.01981

TABLE 1

This loading corresponds to a free stream dynamic head of 1 psf.

The distribution of radial displacement along the meridian is presented on Fig.11. Twelve elements are used in the analysis, the spacing of which is shown on the left hand side of Fig.11. The diagram of membrane forces N_ϕ and N_θ is shown on Fig.12. The agreement with aforementioned methods is good. The bending moment M_ϕ is given on Fig.13. Again, the agreement with [13] is excellent

and there are no "kinks" in the moment distribution mentioned in [1]. The reason for that is that the moment derivatives are continuous. The distribution of $M\phi$ round the circumference near the base (at $Z = -267.5\text{it}$) is shown on Fig. 14. Finally, the moment $M\theta$ is presented on Fig. 15. Although more elements have been used for this analysis in comparison with SABA 5 element of [1], this is compensated by the increased accuracy in the bending moment distribution obtained in the result.

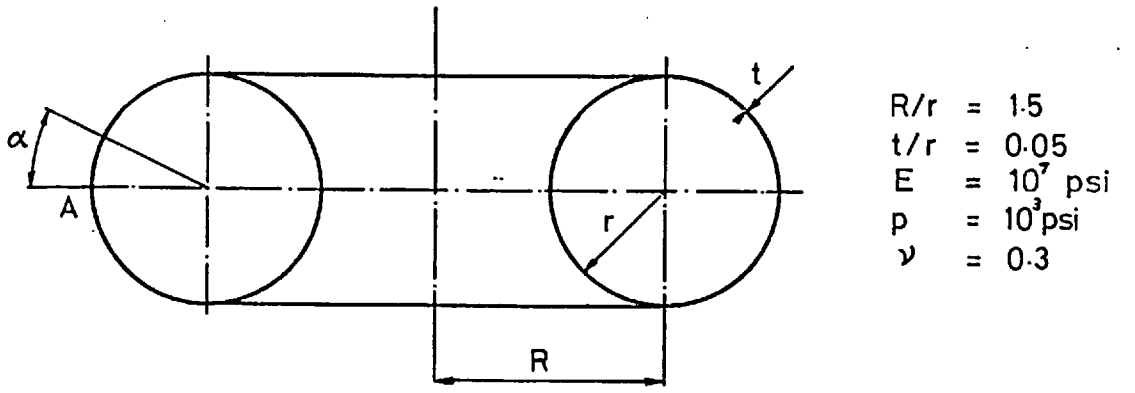
c) Cylinder with Spherical Cap

The objective of this analysis is to test the behaviour of the element at the point of discontinuity, and to compare it to the well known analytical solution. The structure is presented on Fig.16a and is loaded by an uniform internal pressure. It has the uneven element sub-division using 18 elements in all but concentrating near the junction point. At open end of the cylinder free expansion was permitted. The distribution of bending moment $M\phi$ is shown on Fig.16b, in comparison with the analytical solution by Flügge [15]. Overall agreement is excellent. The maximum value of moment $M\phi$ given by Flügge is $8.92 \times 10^{-3} \text{ pa}^2$ against $8.85 \times 10^{-3} \text{ pa}^2$ obtained by the mixed method, indicating an error of 0.77%.

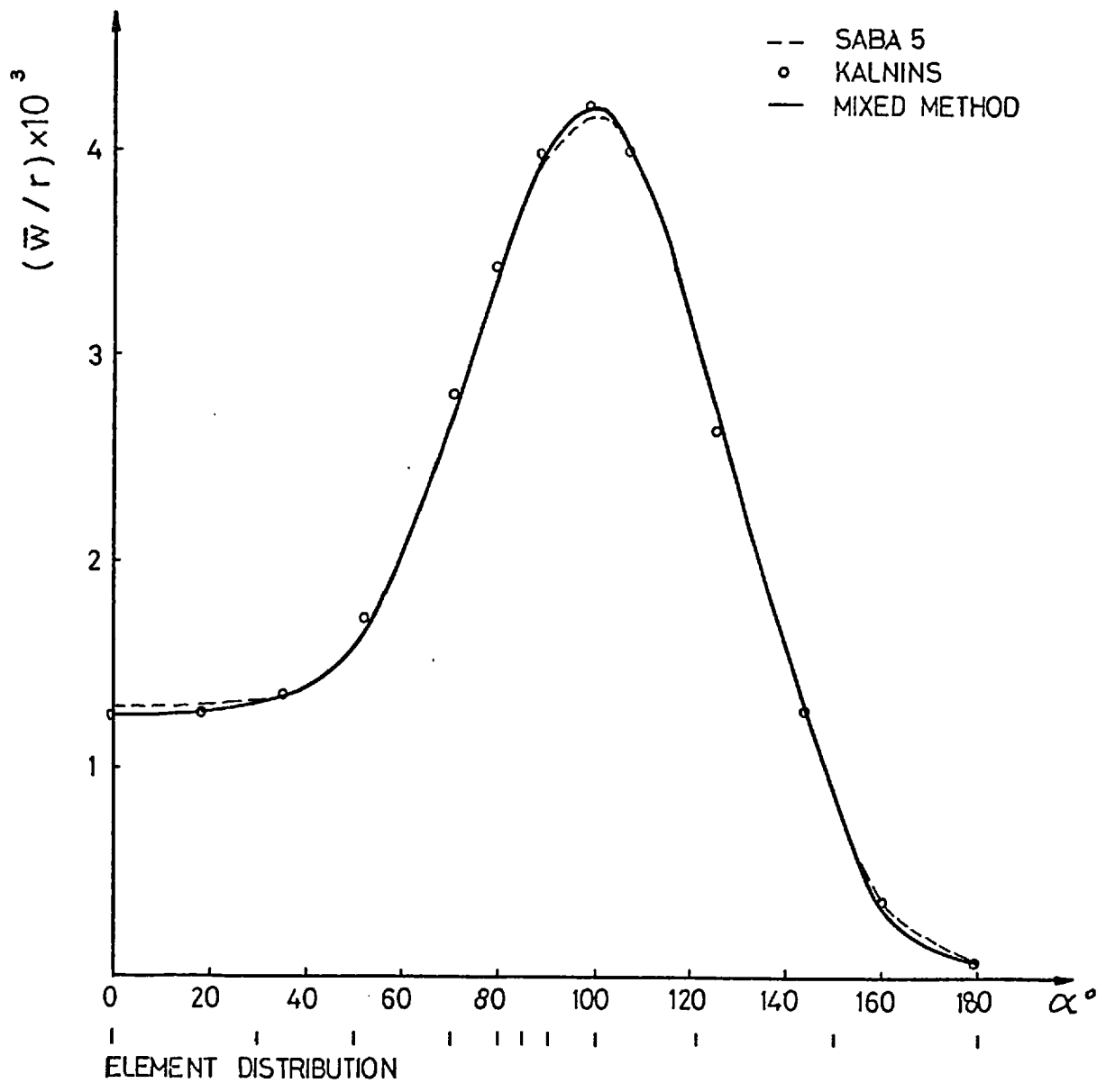
d) Spherical Tank Supported by a Cylindrical Skirt

This structure is a typical three-branch shell and is chosen due to its resemblance to LNG containers, Fig.17. The supporting cylindrical skirt was put at 45° from the bottom of the spherical tank, much lower than in conventional LNG container design in order to emphasize the discontinuities. The spherical tank was loaded by uniform internal pressure. The element sub-division in the analysis was uneven, concentrating more elements near the branch point. All constraints mentioned in section 7 were introduced at the branch point.

The results for moments $M\phi$, horizontal displacement and rotation at the branch point are compared with analytical solution in Table 2. The distribution of $M\phi$ for each branch is presented on Fig.18. The results achieved clearly show the ability of the element to handle this type of shell structures. The accuracy of the linear solution at the branch point is satisfactory which will be of a great importance in non-linear (buckling) analysis, as will be shown later.

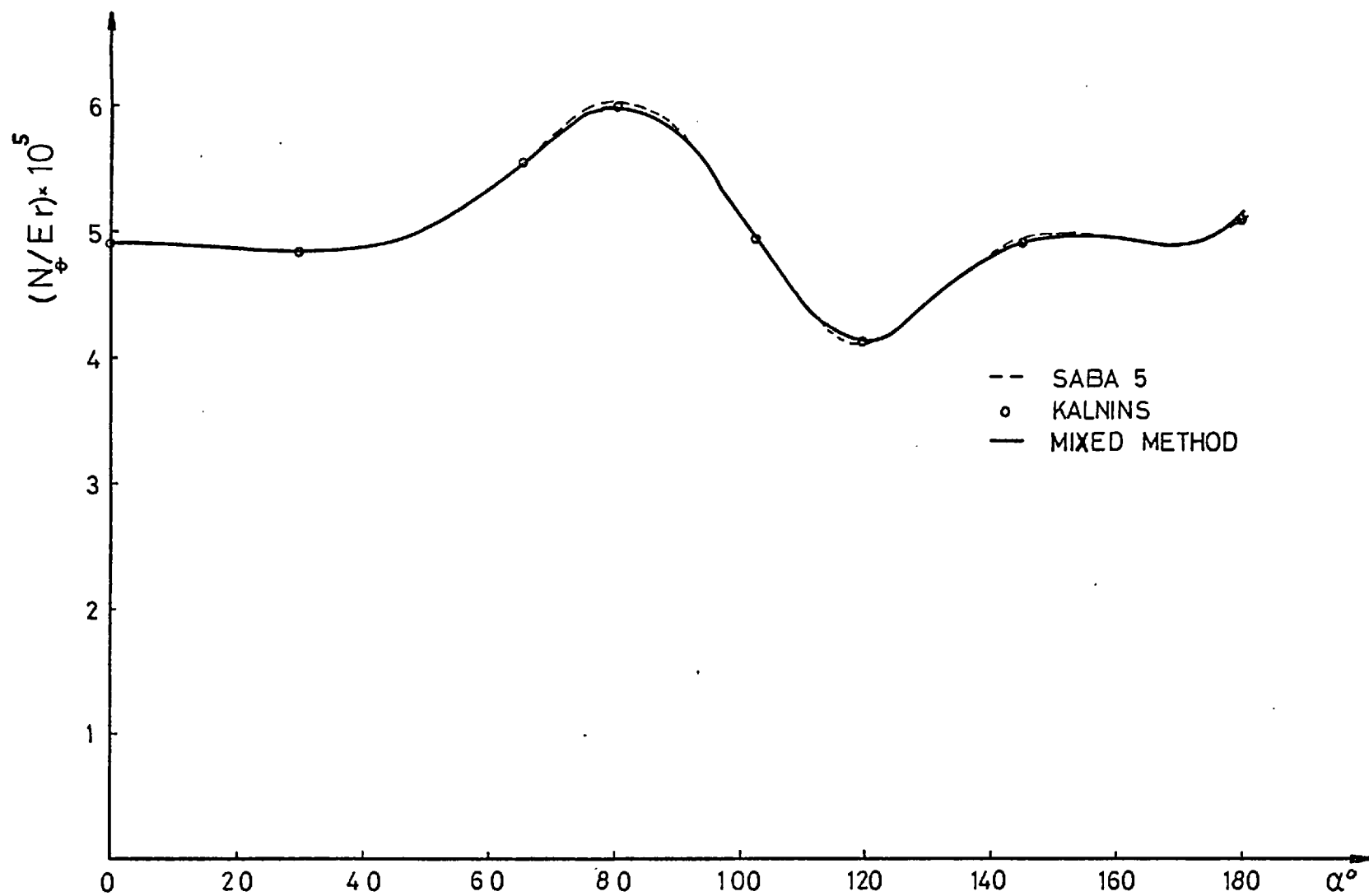


a) PRESSURISED TORUS



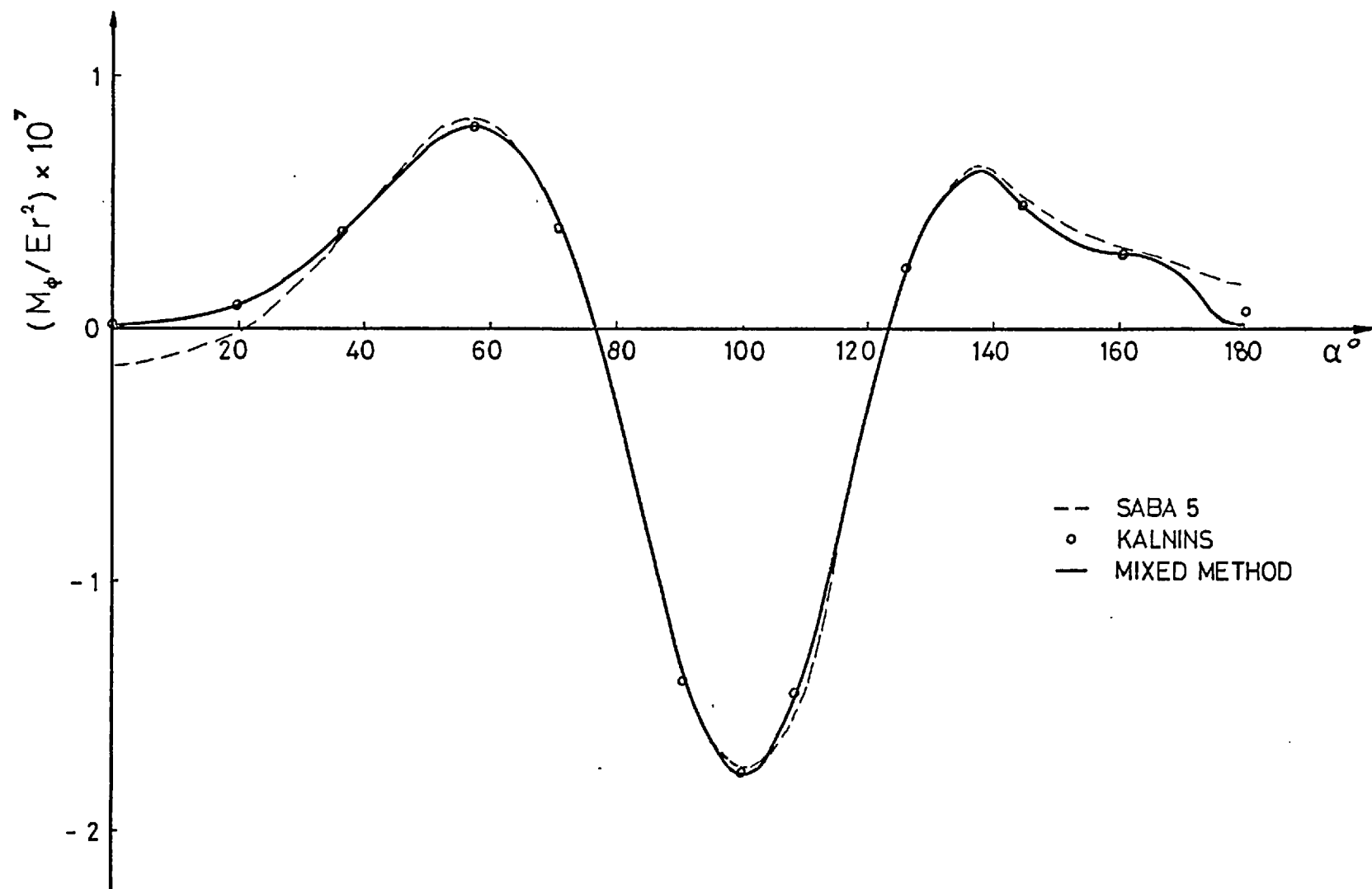
b) DISTRIBUTION OF RADIAL DISPLACEMENTS

FIG. 5

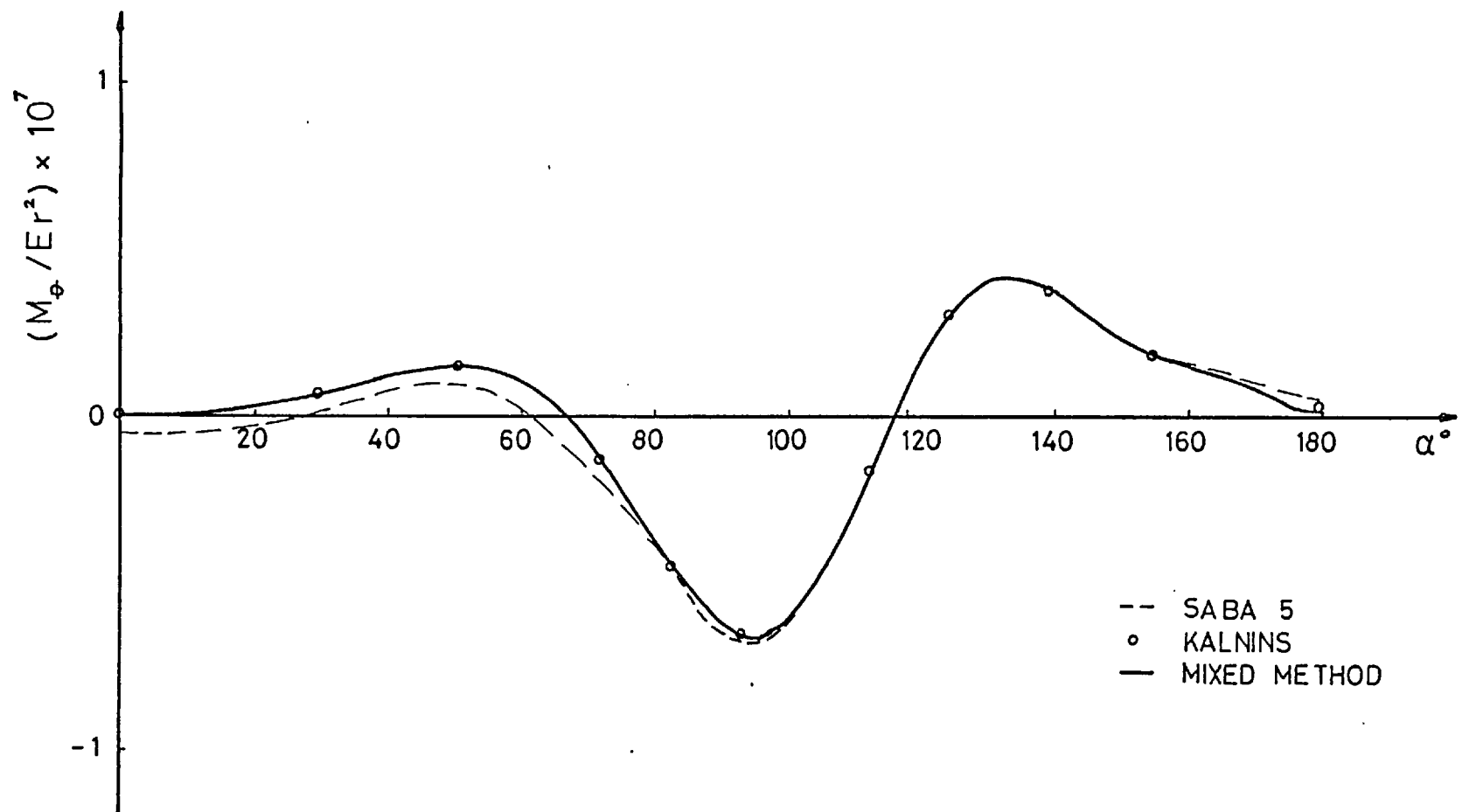


DISTRIBUTION OF MEMBRANE STRESSES N_ϕ

FIG. 6

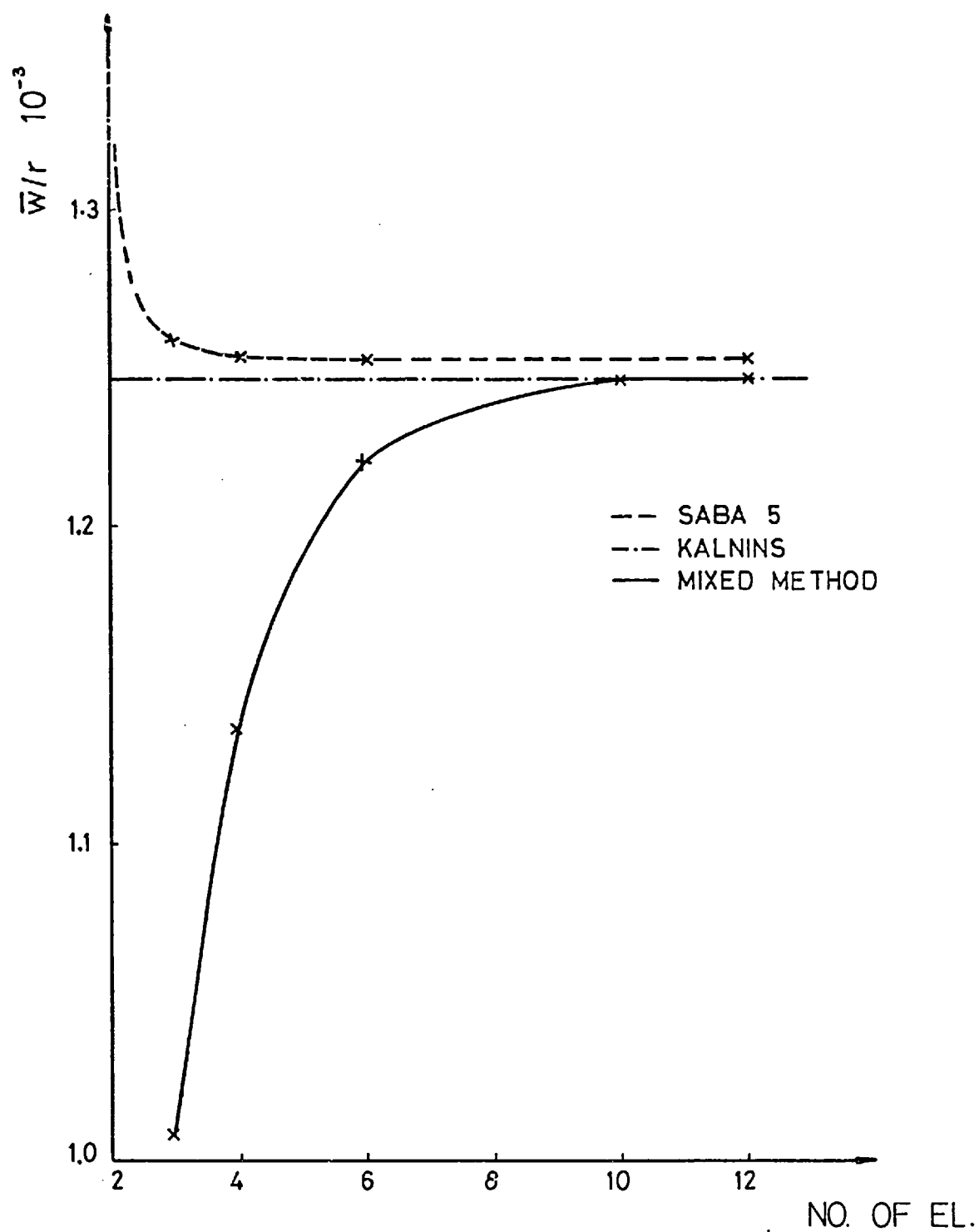


DISTRIBUTION OF BENDING MOMENT M_ϕ



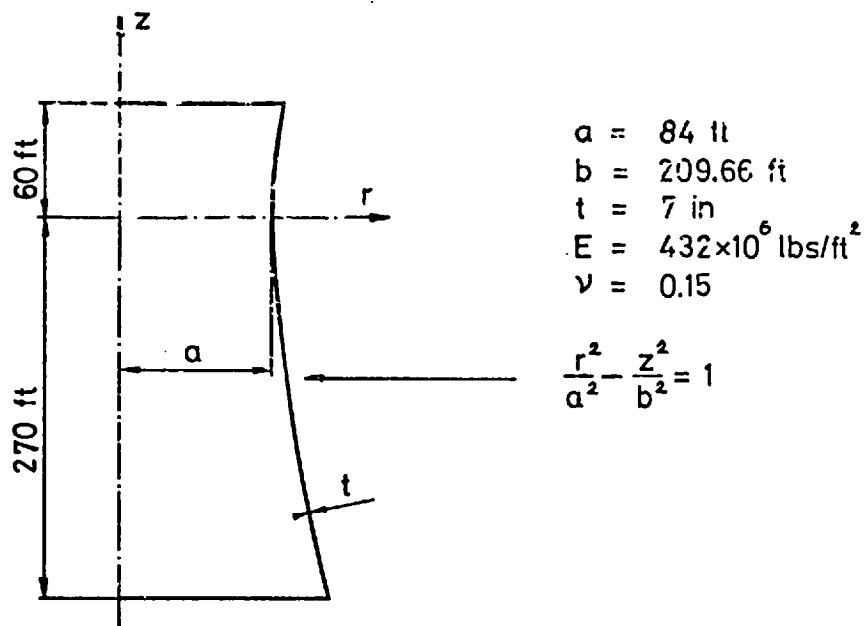
DISTRIBUTION OF BENDING MOMENT M_ϕ

FIG. 8

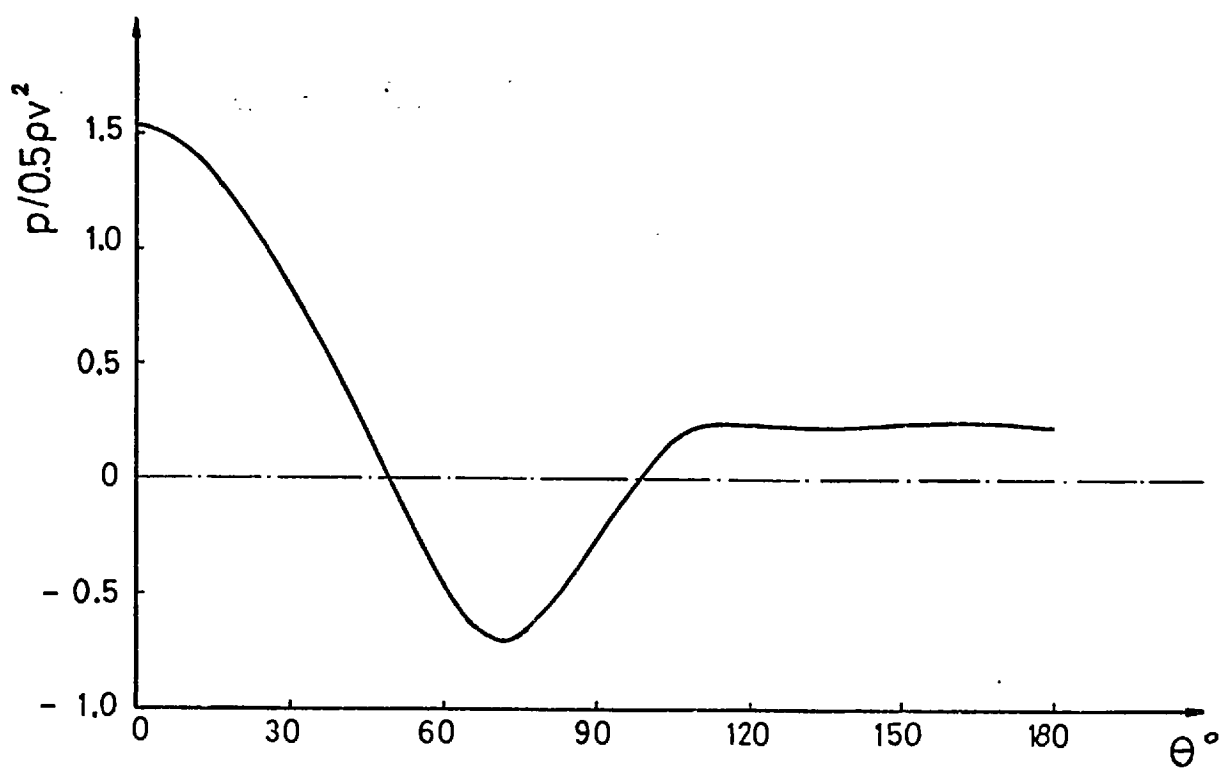


RADIAL DEFLECTION AT POINT A
VS NO. OF ELEMENTS

FIG. 9

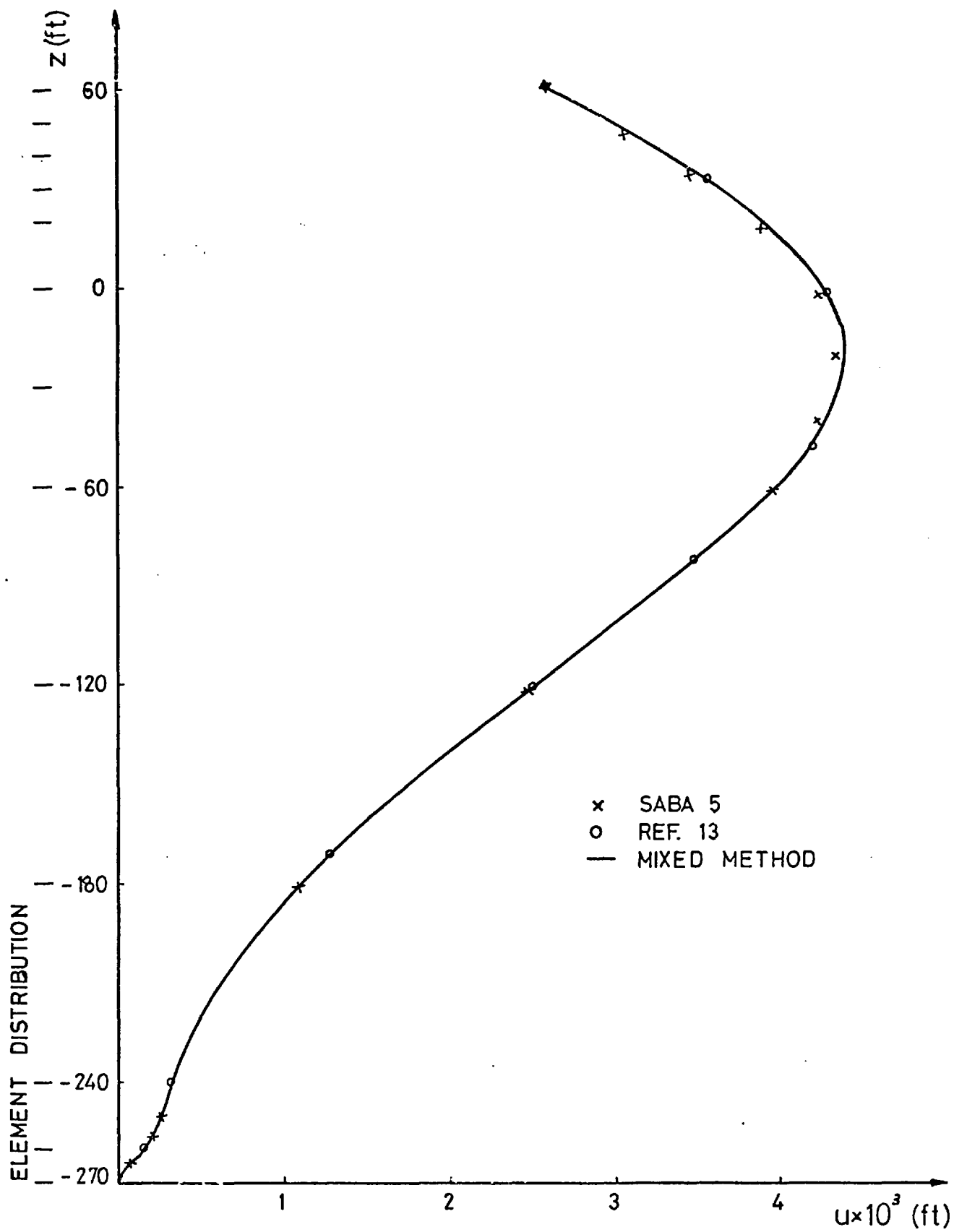


a) HYPERBOLIC COOLING TOWER



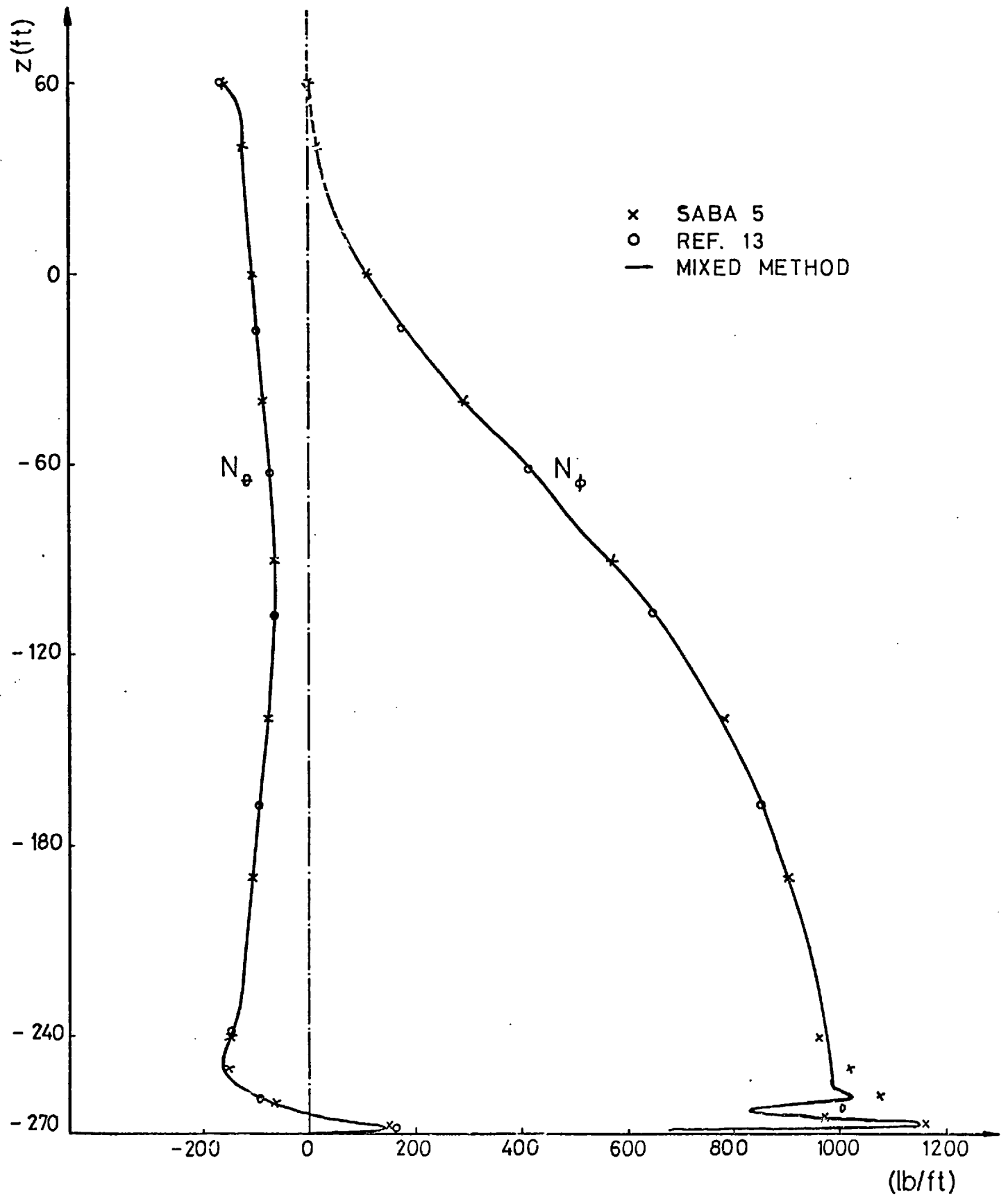
b) CIRCUMFERENTIAL DISTRIBUTION OF WIND PRESSURE

FIG. 10



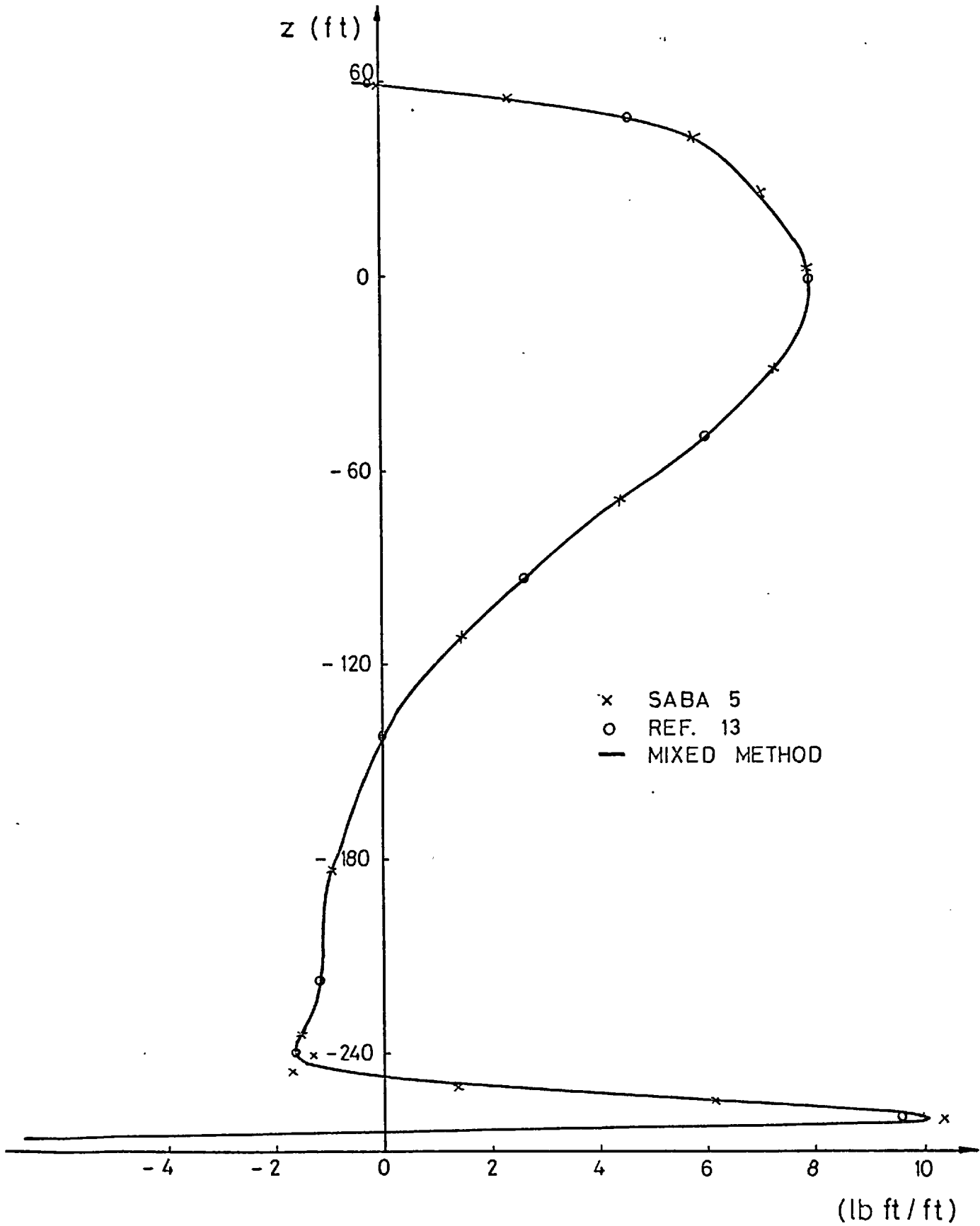
DISTRIBUTION WITH HEIGHT OF RADIAL
DEFLECTION AT $\theta = 0^\circ$

FIG. 11



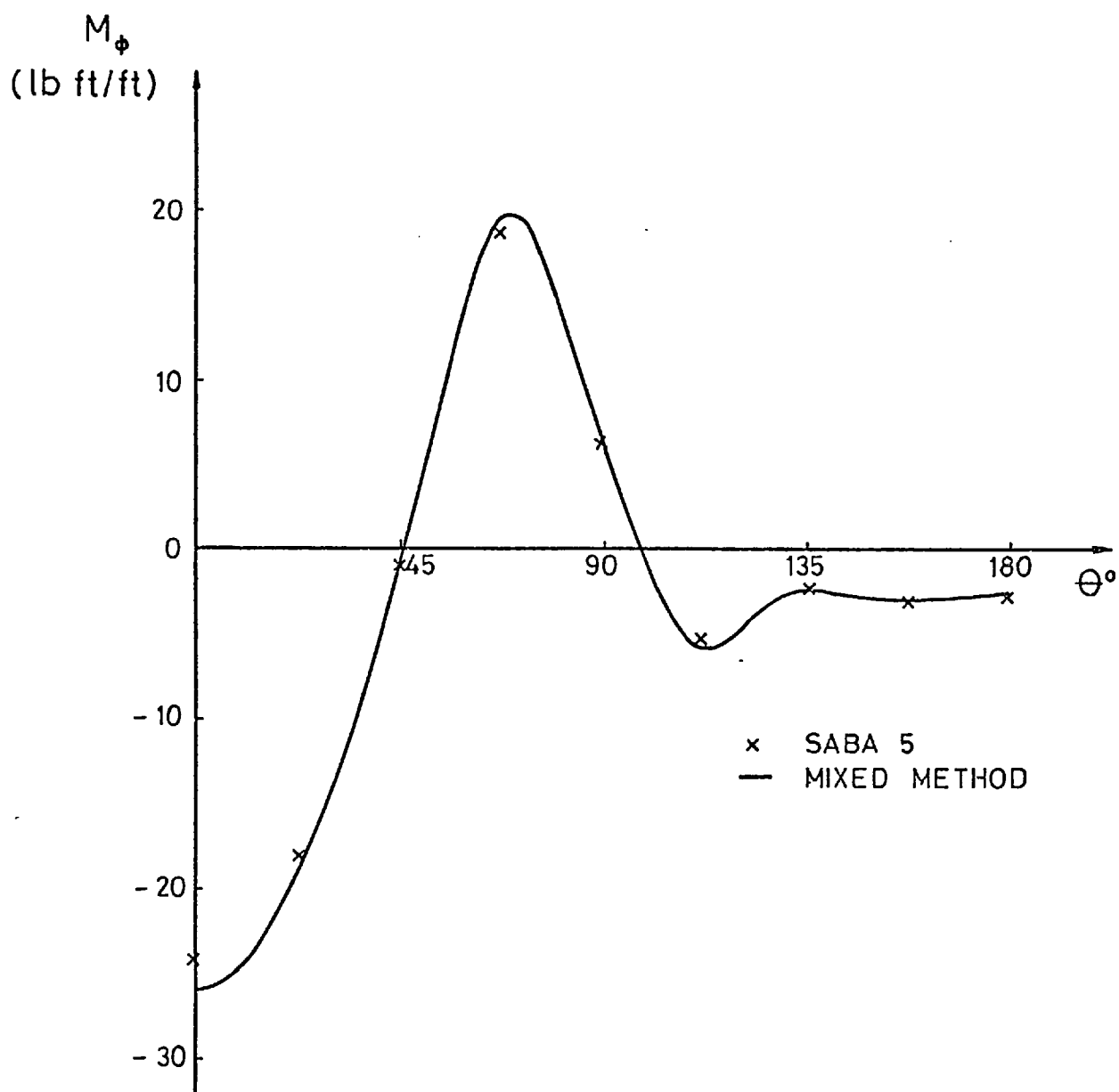
DISTRIBUTION WITH HEIGHT OF N_θ AND N_ϕ AT $\theta = 0^\circ$

FIG. 12



DISTRIBUTION WITH HEIGHT OF M_ϕ AT $\theta = 0^\circ$

FIG. 13



DISTRIBUTION WITH θ OF M_ϕ AT $Z=-267.5$ ft

FIG. 14

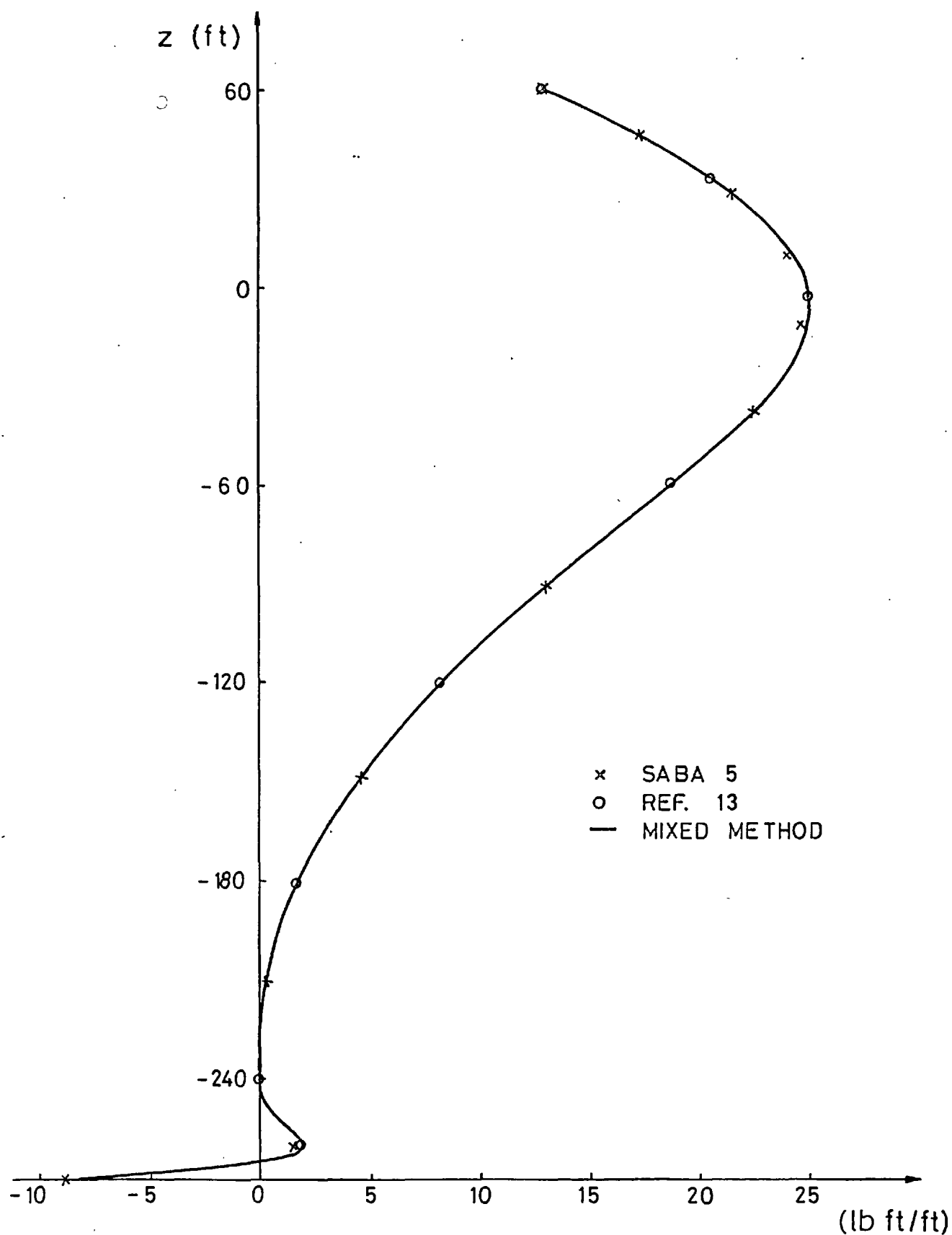
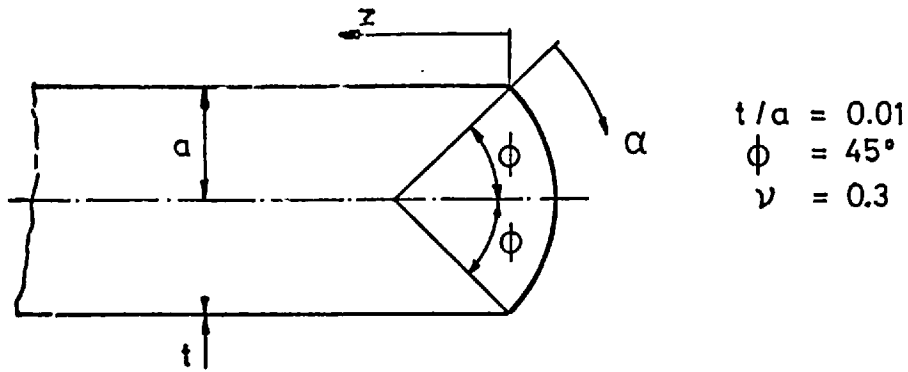
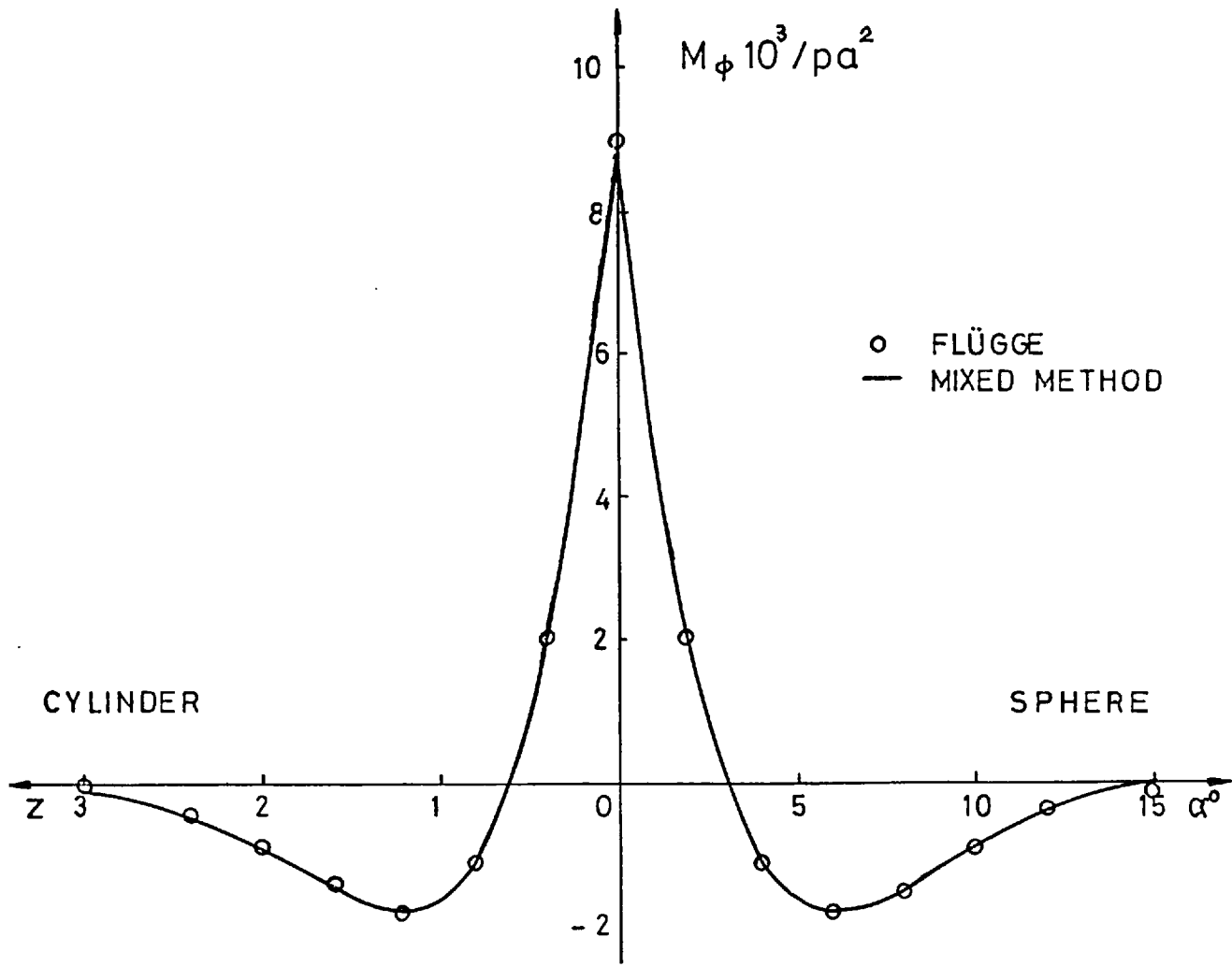
DISTRIBUTION WITH HEIGHT OF M_{θ} AT $\theta = 0^{\circ}$

FIG. 15



a) CYLINDER WITH SPHERICAL CAP



b) DISTRIBUTION OF M_ϕ

FIG. 16

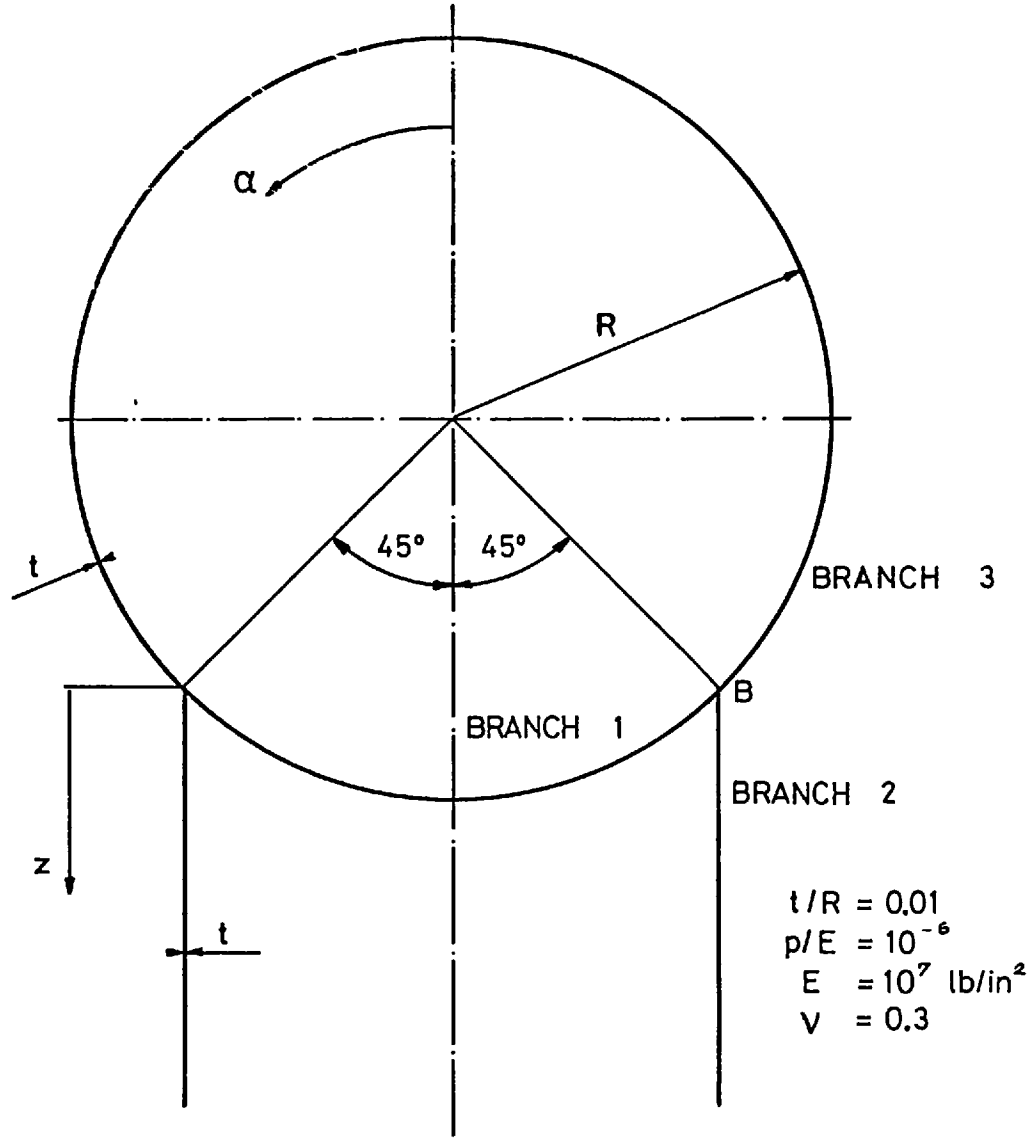
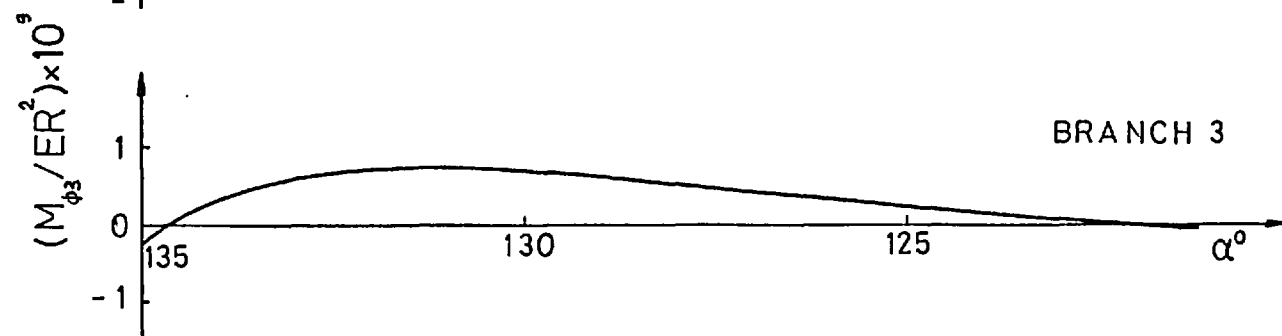
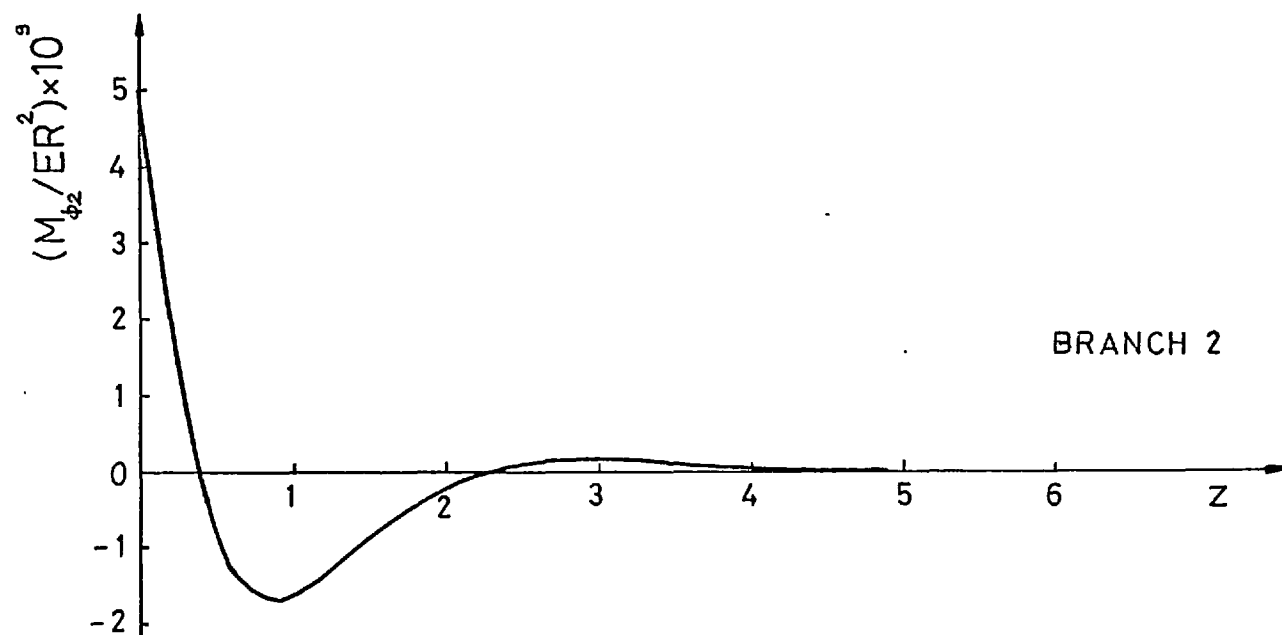
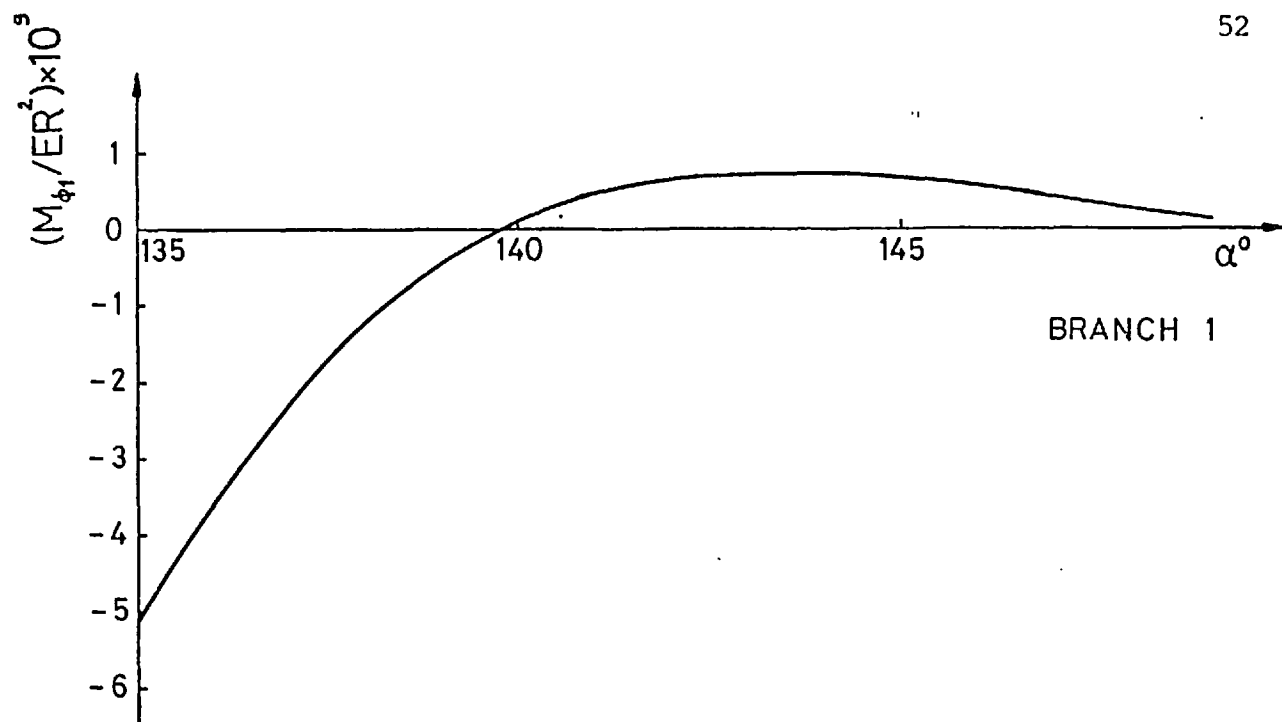


TABLE 2	RESULTS AT BRANCH POINT B	
	ANALYTICAL SOL	MIXED METHOD
$M_{\phi 1} / ER^2$	-5.1741×10^{-9}	-5.1264×10^{-9}
$M_{\phi 2} / ER^2$	4.9492×10^{-9}	4.9002×10^{-9}
$M_{\phi 3} / ER^2$	-0.2249×10^{-9}	$-0.226.1 \times 10^{-9}$
u_B / R	0.1844×10^{-3}	0.1856×10^{-3}
ψ	0.1051×10^{-2}	0.1011×10^{-2}

FIG. 17



DISTRIBUTION OF BENDING MOMENT M_ϕ

PART II - Large Displacements

2.1 INTRODUCTION

The large displacement non-linear analysis discussed in this part shares the same basic philosophy with that of the SABA element given in Part II of Ref. [1]. Much of the general observations contained in the introduction of that article is still relevant here. The non-linear behaviour arising from the effect of change of structural geometry is obtained in a sequence of small steps, each of which may either be linear or parabolic. The latter is achieved by an iterative scheme within each step. Material non-linearity has not been taken into account, but can easily be incorporated into the incremental procedure, as has been shown in Ref. [16]. The magnitude of the radial displacement is assumed to be of the same order as the shell thickness, large enough to invalidate the linear analysis and to demand a non-linear treatment, but not large enough (when compared with the radius) to alter the overall axi-symmetrical configuration of the structure. Hence the equations of equilibrium and compatibility for axi-symmetrical shells, given in Part I of this paper, are still applicable. The analysis is subjected therefore to the same limitation as the SABA element. It will not be able to give truly large deformations, say, in the post-buckling region, or near the point of total collapse. However, it will be perfectly adequate for giving a better understanding of the behaviour of shells with large radius to thickness ratio, for which the displacement can so easily be several times the thickness, and in particular, for predicting the initial buckling load.

The formal formulation of the large displacement theory by this mixed method of approach will be given in the next section. With the displacements and the bending moments interpolated independently as

in Part I, the relationship between the increments of the force and displacement parameters remain the same, to the first order of accuracy, as in the small displacement theory (eq.1-66), (equations will be numbered separately for each part of this paper, when reference is made to an equation of another part, the part number will precede the equation number), while the "stiffness" matrix, obtained in the application of the P.V.D. to enforce equilibrium between the force increments, is changed by the addition of a "geometric stiffness", arising from the effect of the existing forces and stresses at the beginning of the incremental step in a subsequent change of structural geometry. Thus the non-linear analysis turns out to require exactly the same additional matrix as in the displacement method, which may be obtained either by the inclusion of second order terms in the compatibility equations, or may be calculated by a direct application of the virtual work principle, examining the effects of each of the existing forces in turn. The latter method is adopted in Ref. [1] and can be utilised *in totum* here. It should be noticed that by using the present method of formulation of the geometric stiffness, no assumption on the magnitude of the rotations is necessary and no selection on the second order terms in the compatibility equations has to be made. The contribution to the geometric stiffness by all the existing forces, including the normal shear forces, can be made with the minimum of effort. Otherwise, since the normal shears do not appear in the stress matrix explicitly but are only given by the equations of equilibrium (I.1) from the bending moments and their gradients indirectly, their effects can so easily be missed out altogether if a strictly mathematical approach is taken (e.g. Ref. [17]).

The linearised small step theory bases the calculation on the properties at the beginning of the step, which makes the programme for

the non-linear analysis very simple, but the step size must necessarily be small. Better accuracy may be obtained by taking average of the stiffnesses at the beginning and at the end, using the linearised step as a first approximation and iterating until convergence is achieved. This is shown in Ref. [1] to be equivalent to the inclusion of second order terms in the incremental equation of equilibrium and compatibility, and the step size can be larger. However, the computing time required for convergence may be excessive if the step size is set too large, a happy medium can be struck by selecting a moderate step size and limiting the number of iterations to two. This compromise has been adopted in the computation of all the examples.

2.2. LINEARISED INCREMENTAL THEORY

To formulate the theory of the linearised incremental procedure for the large displacement non-linear analysis by the mixed method of approach, it is sufficient to consider the structure at the element level. For this purpose, the fundamental representation of the structural behaviour and the interpolation schemes are the same as before, but the analysis is divided into a series of small steps, for each of which the increments of the strains and the stresses are assumed to be linearly proportional to the increments of the displacements and force parameters respectively. Therefore, eqs. (I-40) and (I-23), which are true only for the small displacement analysis, become instead

$$d\epsilon_M = \alpha_M d\rho \quad (1)$$

and

$$dS_B = \beta db \quad (2)$$

where α_M and β are based on the current state of the structure and change with the loading and the geometry.

Thus, at the beginning of any incremental step, the applied force P induces displacement δ , strain ϵ and stress S , with force parameters b and displacement parameters ρ . As the result of a force increment dP , there are corresponding increments in the displacements $d\delta$, strain $d\epsilon$, stress dS and parameters $d\rho$ and db . Because of these changes, eqs. (1) and (2) also change to

$$d\epsilon_M = (\alpha_M + d\alpha_M) d\rho, \quad (3)$$

$$dS_B = (\beta + d\beta) db \quad (4)$$

at the end of the loading step.

Since the bending strains are calculated from the bending stresses, compatibility of the increment $d\epsilon_B$ with the displacement increment is not exact, but can be enforced approximately by the application of the P.V.F., giving

$$\int \beta^t [d\epsilon_B - D_B^t (d\hat{\delta})] dA = 0$$

from which

$$\int \beta^t (\phi \beta db + d\eta_B - \alpha_B d\rho) dA = 0$$

or

$$f db + dq - t d\rho = 0 \quad (5)$$

where dq is a column of initial curvature increments, if any, during the loading step. This equation relates the increments of the force and displacement parameters. The matrices involved are identical to those in the small displacement theory (eq. I-53) in form, but different in substance since β and α_B are based on the current geometry instead of the un-deformed geometry of the structure.

Equilibrium between the forces at any given stage is similarly satisfied approximately by repeated application of the P.V.D. At the beginning of the step

$$d\rho^t P = \int d\epsilon^t S dA \quad (6)$$

where $d\epsilon = D^t (d\hat{\delta}) = \alpha d\rho$ is a set of

virtual strains compatible with virtual displacement parameters $d\rho$, calculated according to the geometry at the beginning of the step.

This gives

$$P = \int \alpha^t S dA \quad (7)$$

At the end of the step, it becomes

$$d\rho^t (P + dP) = \int d\varepsilon^t (S + dS) dA \quad (8)$$

$$\text{where, } d\varepsilon = (\alpha + d\alpha) d\rho$$

which is now calculated according to the changed geometry, giving

$$P + dP = \int (\alpha^t + d\alpha^t) (S + dS) dA \quad (9)$$

Taking into account the equilibrium at the beginning expressed in eq.(7) and neglecting the second order terms, it yields the load increment

$$dP = \int \alpha^t dS dA + \int d\alpha^t S dA \quad (10)$$

Due to the fact that the bending and membrane stress increments are treated differently, the first term on the right hand side produces

$$\begin{aligned} \int \alpha^t dS dA &= \int (\alpha_M^t dS_M + \alpha_B^t dS_B) dA \\ &= \int [\alpha_M^t \kappa (\alpha_M d\rho - d\eta_M) + \alpha_B^t \beta db] dA \\ &= k_e d\rho + dj + t^t db \end{aligned} \quad (11)$$

where the k_e is the elastic stiffness matrix as in the small displacement theory, calculated from the membrane behaviour alone but according to the current state, dj is the increment of the initial membrane force. The second term on the right-hand side of eq. (10) arises from the effect of the existing stresses S at the beginning of the step on the change $d\alpha$ as a consequence of further deformation during the step. It may be rewritten as

$$\begin{aligned} \int d\alpha^t S dA &= \int \sum_i S_i d\alpha_i^t dA \\ &= \sum \int (S_i \frac{d\alpha_i^t}{d\rho} dA) d\rho = k_G d\rho \end{aligned} \quad (12)$$

where

$$\mathbf{k}_G = \sum \int \mathbf{S}_i \frac{d\alpha_i^t}{d\rho} dA \quad (12a)$$

gives the formal expression for the so-called geometric stiffness matrix. The summation is over all the components in the stress matrix $\mathbf{S}$, and $d\alpha_i^t$ is the i^{th} column of the matrix $d\alpha^t$ associated with the stress component S_i . Each derivative $d\alpha_i^t/d\rho$ is a $m \times m$ square matrix where m is the number of displacement parameters in ρ , six in the case of the present element.

Summing up, equation (10) becomes

$$d\mathbf{P} = (\mathbf{k}_e + \mathbf{k}_G) d\rho + \mathbf{t}^t d\mathbf{b} - d\mathbf{j} \quad (13)$$

combining this with eq. (5) in matrix form, the increments of the nodal parameters for each step of the non-linear analysis are given by

$$\begin{bmatrix} d\mathbf{P} + d\mathbf{j} \\ d\mathbf{q} \end{bmatrix} = \begin{bmatrix} \mathbf{k}_e + \mathbf{k}_G & \mathbf{t}^t \\ \mathbf{t} & -\mathbf{f} \end{bmatrix} \begin{bmatrix} d\rho \\ d\mathbf{b} \end{bmatrix} \quad (14)$$

which, apart from the addition of the geometric stiffness matrix $\mathbf{k}_G$ to the linear elastic stiffness $\mathbf{k}_e$, is governed by exactly the same matrices as in the small displacement theory. However, they are calculated according to the current state of the structural geometry and must be changed from step to step.

2.3 DETAIL DERIVATION OF THE GEOMETRIC STIFFNESS MATRIX

It transpires that the only additional matrix required for the incremental non-linear analysis over the small displacement case is the geometric stiffness matrix. In this respect, the mixed method of formulation is no different from the usual displacement method, and the method of detail calculations of this matrix employed by the SABA element in Ref. [1] can be applied equally well here.

The formal expression for the geometric stiffness matrix is given in eq. (12a). It accounts for the effect of the existing forces, represented by the 3 membrane forces and 3 bending moments in the matrix $\mathbf{S}$, on a further change of structural geometry. Equilibrium between these 6 stress components, and with other existing forces such as the normal shear forces and applied pressure, as demanded by eq. (I-1), is not exactly satisfied but approximately through the P.V.D. Thus, the geometric stiffness given by eq. (12a), although involving only 6 stress components would have the participation of all the existing forces by implication. However, the α matrix, as can be observed in eqs. (I-42) and (I-43), is a very complex function of the structural geometry. Its derivatives with respect to the nodal displacement parameters ρ , are by no means easy to obtain. Fortunately, instead of following straightly the formal mathematical derivation, it is possible to evaluate the geometric stiffness by a direct consideration of the effect of each component of the existing forces, examining physically how the original equilibrium state may be upset by a further deformation, and the additional forces required to restore equilibrium. The equivalent nodal forces are then expressed in terms of a stiffness (geometric) by the application of the Principle of virtual work. The details of this method of derivation have been given in

Part II of Ref. [1] and will not be repeated. Here, it is sufficient to equate the total virtual work of the additional nodal forces $d\mathbf{P}_G$ required to maintain equilibrium.

$$\begin{aligned} \delta \rho^t d\mathbf{P}_G = & \int \delta W = \int \left[\delta \omega_\phi N_\phi d\omega_\phi + \delta \omega_\theta N_\theta d\omega_\theta \right. \\ & + S (\delta \omega_\phi d\omega_\theta + \delta \omega_\theta d\omega_\phi) + Q_\phi (\delta \omega_\phi d\varepsilon_\phi + \delta \varepsilon_\phi d\omega_\phi) \\ & + Q_\theta (\delta \omega_\theta d\varepsilon_\theta + \delta \varepsilon_\theta d\omega_\theta) + \frac{1}{2} p (\delta \omega_\phi du + \delta u d\omega_\phi \\ & \left. + \delta \omega_\theta dv + \delta v d\omega_\theta) \right] dA \end{aligned} \quad (15)$$

The deformations involved in this equation may be related to the local displacement vector $\hat{\delta}$ by

$$\omega = \begin{bmatrix} \omega_\phi \\ \omega_\theta \\ \varepsilon_\phi \\ \varepsilon_\theta \\ u \\ v \end{bmatrix} = \begin{bmatrix} \frac{1}{R_\phi} & 0 & -\frac{\delta}{R_\phi \delta \phi} \\ 0 & \frac{\sin \phi}{R} & -\frac{\delta}{R \delta \theta} \\ \frac{\delta}{R_\phi \delta \phi} & 0 & \frac{1}{R_\phi} \\ \frac{\cos \phi}{R} & \frac{\delta}{R \delta \theta} & \frac{\sin \phi}{R} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \hat{\delta} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \\ L_6 \end{bmatrix} \hat{\delta} = \mathbf{L} \hat{\delta} \quad (16)$$

where the principal radii R_θ and R_ϕ are assumed to be unchanged by the deformations so that the structure remains practically a shell of revolution for the purpose of calculating the increments. Employing the definitions from eqs. (I-30) and (I-34a), the j^{th} harmonic of the increment of ω is given by

$$d\omega_j = \mathbf{L} \wedge \beta \psi_j d\rho_j \quad (17)$$

The j^{th} harmonic of the nodal force increment dP_{Gi} does virtual work on the i^{th} harmonic virtual displacement $\delta\rho_i$, which give rise to a compatible virtual deformation

$$\delta\omega_i = L \wedge \beta \psi_i \delta\rho_i \quad (18)$$

and from eq. (15), after cancelling the virtual quantity $\delta\rho_i$ from both sides,

$$dP_{Gi} = (k_G)_{ij} d\rho_j \quad (19)$$

where

$$\begin{aligned} (k_G)_{ij} = & \psi_i^t \beta^t \wedge^t [N_\phi L_1^t L_1 + N_\theta L_2^t L_2 \\ & + S (L_1^t L_2 + L_2^t L_1) + Q_\phi (L_1^t L_3 + L_3^t L_1) \\ & + Q_\theta (L_2^t L_4 + L_4^t L_2) + \frac{1}{2} P (L_1^t L_5 + L_5^t L_1 \\ & + L_2^t L_6 + L_6^t L_2)] \wedge \beta \psi_j \, dA \end{aligned} \quad (20)$$

This is the geometric stiffness matrix which couples the j^{th} harmonic of displacement increments to the i^{th} harmonic of the load increment. Notice that in this expression the normal shear forces and the applied pressure are directly involved, instead of the bending moments as given in eq. (12a). They are, in general, constructed from a series of harmonics, for example,

$$N_\phi = N_{\phi 0} + \sum_1^n N_{\phi k} \cos k\theta + \sum_1^n \bar{N}_{\phi k} \sin k\theta \quad (21)$$

so that for any particular pair of harmonics i and j , the coupling matrix $(k_G)_{ij}$ must be evaluated by summing through all the k stress harmonics. The integrands will always contain the multiplication of three harmonic functions, and are any one of the following types

- (a) $\sin \ell\theta \quad \sin m\theta \quad \sin n\theta$
- (b) $\sin \ell\theta \quad \sin m\theta \quad \cos n\theta$
- (c) $\sin \ell\theta \quad \cos m\theta \quad \cos n\theta$
- (d) $\cos \ell\theta \quad \cos m\theta \quad \cos n\theta$

where ℓ , m and n can be any one of i , j , or k . Therefore, the integration of eq. (20) around the circumference vanishes unless

$$\ell \pm m \pm n = 0 \quad (22)$$

and non-zero circumferential integral C arises only from the following combinations

- (1) All harmonics zero: $C = 2\pi$
- (2) One harmonic zero, the other two the same harmonic of sine or cosine: $C = \pi$
- (3) All harmonics non zero:

For type (b) : $C = \pi/2$ if $\ell - m = n$

or $C = -\pi/2$ if $\ell + m = n$

For type (d) : $C = \pi/2$ when (22) is satisfied.

A mistake was made in Ref. [1] for the value of the integral in combination (3) above. However, only the results with coupling between the non-zero harmonics were affected. The contribution from each term of eq. (20) to the geometric stiffness matrix and the table showing the couplings in the symmetrical case were correctly worked out in Ref. [1] and need not be repeated here.

Therefore, in the large displacement problem, there is, in general, coupling between the harmonics as shown. This means that each harmonic cannot be solved separately as in the small displacement solution. However, the progressive elimination technique developed for the analysis of branched-shells in Part I can be profitably

employed here for all cases, whether single or multiple branched. The displacement and moment parameters of all harmonics are grouped together at each nodal point and starting from an unconnected end, each group of nodal parameters is eliminated in terms of that at the next node. The coupling is thereby confined to two adjacent nodes. Only one element matrix is assembled and stored at a time. The complete structural matrix is never assembled in the full form. The solution at the last node, either at the other end of the shell or at a branched point, followed by a backward substitution yields the complete set of nodal parameters.

2.4 IMPROVEMENT OF THE LINEARISED INCREMENTAL THEORY

The results of the linearised incremental theory can be improved if an iteration technique is adopted. The linearised incremental theory becomes thus now non-linear.

Suppose that the total load at the end of the step $\mathbf{P}_0 + d\mathbf{P}$ is equilibrated by the stress $\mathbf{S}_0 + d\mathbf{S} + \frac{1}{2} d^2\mathbf{S}$. This assumption for stresses means that the second order term (in Taylor series expansion) is also taken into account, instead of just linear increment as in the linearised theory. Obviously, the strains at the end of the step will be more accurate if following relationship is used

$$d\mathbf{\epsilon} = (\alpha_0 + d\alpha + \frac{1}{2}d^2\alpha) d\mathbf{p} \quad (23)$$

The equilibrium between external and internal forces at the end of a step is now given by

$$\delta \underline{\mathbf{p}}^t (\mathbf{P}_0 + d\mathbf{P}) = \int \delta \underline{\mathbf{\epsilon}}^t (\mathbf{S}_0 + d\mathbf{S} + \frac{1}{2}d^2\mathbf{S}) dA \quad (24)$$

which after the substitution of eq. (23) and elimination of the virtual displacements $\delta \underline{\mathbf{p}}$ becomes

$$\mathbf{P}_0 + d\mathbf{P} = \int (\alpha_0^t + d\alpha^t + \frac{1}{2}d^2\alpha^t) (\mathbf{S}_0 + d\mathbf{S} + \frac{1}{2}d^2\mathbf{S}) dA \quad (25)$$

Taking into account the equilibrium at the beginning of the step (see eq.7), the above equation becomes

$$d\mathbf{P} = \int \left[\alpha_0^t d\mathbf{S} + d\alpha^t \mathbf{S}_0 + d\alpha^t d\mathbf{S} + \frac{1}{2}(d^2\alpha^t \mathbf{S}_0 + \alpha_0^t d^2\mathbf{S}) \right] dA \quad (26)$$

where only terms up to the second order are retained. The first two terms of eq.(26) give the elastic and the geometric stiffness, the coupling matrix based on conditions at the beginning of the step (see eq.13). If all values at the beginning of the step are denoted by index 0, it follows

$$\mathbf{k}_0 d\rho + \mathbf{t}_0^t db = \int (\alpha_0^t d\mathbf{S} + d\alpha^t \mathbf{S}_0) dA \quad (27)$$

where $\mathbf{k}_0$ is a total stiffness matrix. By taking the first order change of eq. (27), the following relationship is obtained

$$d\mathbf{k} d\rho + d\mathbf{t}^t db = \int (2d\alpha^t d\mathbf{S} + d^2\alpha^t \mathbf{S}_0 + \alpha_0^t d^2\mathbf{S}) dA \quad (28)$$

By comparison of the last three terms in eq. (26) with the right-hand side of eq. (28) it can easily be deduced that

$$d\mathbf{P} = (\mathbf{k}_0 + \frac{1}{2}d\mathbf{k}) d\rho + (\mathbf{t}_0^t + \frac{1}{2}d\mathbf{t}^t) db \quad (29)$$

The change of stiffness and coupling matrices within an incremental step is defined as

$$d\mathbf{k} = \mathbf{k}_1 - \mathbf{k}_0 \quad (30)$$

and

$$d\mathbf{t} = \mathbf{t}_1 - \mathbf{t}_0 \quad (30a)$$

where the index 1 corresponds to the values at the end of the step.

Substituting these two equations into eq. (29), yields

$$d\mathbf{P} = \frac{1}{2} (\mathbf{k}_0 + \mathbf{k}_1) d\rho + \frac{1}{2} (\mathbf{t}_0^t + \mathbf{t}_1^t) db \quad (31)$$

To prove the corresponding non-linear relationship for the compatibility equation (5), it is necessary to consider the relations at the beginning (see eq.2) and that at the end of the step (see eq.4) and to take the average geometry, thus it can be written

$$d\mathbf{S}_B = (\beta + \frac{1}{2}d\beta) db \quad (32)$$

The strains corresponding to these bending stresses will now be

$d\mathbf{E}_B + \frac{1}{2}d^2\mathbf{E}_B$ and should be compatible to the displacement increments $d\rho$

by enforcement of the following condition

$$d\epsilon_B + \frac{1}{2}d^2\epsilon - (\alpha_B + \frac{1}{2}d\alpha_B) d\rho = 0 \quad (33)$$

By the application of the Principle of virtual forces, it follows

$$\int (\beta^t + \frac{1}{2}d\beta^t) [(d\epsilon_B + \frac{1}{2}d^2\epsilon_B) - (\alpha_B + \frac{1}{2}d\alpha_B) d\rho] dA = 0 \quad (34)$$

Taking into account only terms up to the second order, the above equation yields

$$\begin{aligned} & \int [(\beta^t d\epsilon_B - \beta^t \alpha_B d\rho) + \frac{1}{2} (d\beta^t d\epsilon_B + \beta^t d^2\epsilon_B - \beta^t d\alpha_B d\rho - d\beta^t \alpha_B d\rho)] dA \\ & = 0 \end{aligned} \quad (35)$$

The first two terms of this equation give the well known compatibility relationship based on the geometry at the beginning of the step (thus index 0)

$$\mathbf{f}_0 d\mathbf{b} - \mathbf{t}_0^t d\rho = \int (\beta^t d\epsilon_B - \beta^t \alpha_B d\rho) dA \quad (36)$$

The first order change of eq. (36) gives

$$d\mathbf{f} d\mathbf{b} - d\mathbf{t}^t d\rho = \int (d\beta^t d\epsilon_B + \beta^t d^2\epsilon_B - \beta^t d\alpha_B d\rho - d\beta^t \alpha_B d\rho) dA \quad (37)$$

By the inspection of eqs. (35) and (37) it can be deduced that

$$(\mathbf{f}_0 + \frac{1}{2}d\mathbf{f}) d\mathbf{b} - (\mathbf{t}_0 + \frac{1}{2}d\mathbf{t}) d\rho = 0 \quad (38)$$

The change of the flexibility matrix within an incremental step can be defined as

$$d\mathbf{f} = \mathbf{f}_1 - \mathbf{f}_0 \quad (39)$$

substituting eqs. (30a) and (39) into (38) yields

$$\frac{1}{2} (\mathbf{f}_0 + \mathbf{f}_1) d\mathbf{b} - \frac{1}{2} (\mathbf{t}_0 + \mathbf{t}_1) d\rho = 0 \quad (40)$$

Combining eqs. (29) and (40) into one matrix equation gives

$$\begin{bmatrix} d\mathbf{P} + d\mathbf{j} \\ d\mathbf{q} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \mathbf{k}_0 + \mathbf{k}_1 & \mathbf{t}_0^t + \mathbf{t}_1^t \\ \mathbf{t}_0 + \mathbf{t}_1 & -(\mathbf{f}_0 + \mathbf{f}_1) \end{bmatrix} \begin{bmatrix} d\rho \\ d\mathbf{b} \end{bmatrix} \quad (41)$$

which is obviously non-linear relationship since all matrices with index 1 (at the end of the step) depend on $d\rho$. The solution can be obtained easily by applying the iteration technique. As the first approximation the linearised incremental theory is used (which is equivalent to stating that $\mathbf{k}_1 = \mathbf{k}_0$, $\mathbf{f}_1 = \mathbf{f}_0$ and $\mathbf{t}_1 = \mathbf{t}_0$) to obtain the displacement and moment increments at the end of the step. That solution is then used to calculate the values at the end of the step ($\mathbf{k}_1$, $\mathbf{f}_1$ and $\mathbf{t}_1$). The process is repeated until the results converge. In practice, however, it is found that, for problems of thin shells where the axial length of the structure is of the same order of magnitude as the typical radius, the deformations does not alter significantly the overall configuration and hence the changes in the elastic stiffness, the flexibility and the coupling matrices are all un-noticeably small throughout the step-by-step analysis and hence may be kept constant without affecting the ultimate result. The only matrix that needs to be updated is the geometric stiffness. This reduces the computing time required for the iterative step-by-step calculation. Furthermore, compromising on the loading step size and computing time, the number of iterations was limited to two, as in Ref. [1].

2.5 EXAMPLES

a) Torus Under External Pressure

The behaviour of the complete torus under uniform pressure is studied again to compare with the linear analysis in Part I, and with the results obtained by other methods of large displacement analysis. The torus has the same external dimensions and material properties as shown on Fig.5a, except the thickness to tube radius ratio, which is in this case $t/r = 0.01$. The shell is loaded by an external pressure $p = 100$ psi. The results of the linear and non-linear analysis are plotted on Figs. 19 to 21. Ten elements are used over a half of the structure and the total load applied in ten equal increments. The significant difference between stress resultants, moments and radial displacements in the linear and non-linear solution indicates the highly non-linear characteristics of the torus and demonstrates that a linear analysis would be quite inadequate in predicting these results. The results are in very good agreement with other methods of solution given in Refs. [1], [18] and [19], and provide a check for the computer programme.

b) Asymmetrically Loaded Spherical Cap

In the previous example, the loading and the deformations are both axi-symmetric although their relationship is non-linear. Hence, only the zero harmonic is involved in the analysis. In order to check the programme for a case of harmonic coupling, the shell (on Fig.22a) represents a clamped spherical cap, and the loading shown is a crude approximation of a uniform pressure over one half of its surface. Only three elements were used in the analysis as in Ref. [1].

The radial displacement as a function of the applied load is presented on Fig.22b. Point B corresponds to the position of the

maximum deflection. Harmonic displacements are given on Fig.23. Notice the significant displacement in the second harmonic due to coupling, although no corresponding load has been applied. The problem which was analysed first in [20] and [21] and then using the SABA element in Ref. [1] is again studied. A comparison of buckling loads obtained by various methods is given in Table 3.

TABLE 3

	P/Per
FAMILI, ARCHER	0.71
BALL	0.66
SABA 5	0.64
MIXED METHOD	0.705

The result predicted by the mixed method is considered as very good, being well between two finite difference solutions of Refs. [20] and [21]. It cannot be compared directly to the displacement method of Ref. [1], due to the slight mistake in the integration of the geometric stiffness matrix in [1], which tends to give lower values of buckling loads.

c) Buckling of Cylinders

This example was studied as a test case before proceeding to the buckling of discontinuous shells and hyperbolic cooling towers. All cylinders considered are built-in at one end and free at the other, as presented on Fig.24a. This case has been investigated by Wang and Billington [22], therefore a direct comparison can be made. Buckling under an uniform lateral pressure is studied first, which involves the deformation of the circumference of the cylinder into a symmetric harmonic waveform. In the computer programme, the actual geometrical imperfection which initiates the buckling of the structure was not implemented but simulated as an initial strain (or displacement) by a small perturbation load in the harmonic in which the buckling occurs.

In this case a perturbation load was of the order $10^{-6} \times$ uniform external pressure. Four cylinders of different lengths were analysed and the harmonic radial displacement presented on Fig.25. On the table on Fig.24b, these results are compared to those of [22]. Agreement is considered to be satisfactory.

The cylinder with $L/R = 4$ was chosen for a study of buckling under wind loading, because the cooling tower to be analysed later has roughly the same length to throat radius ratio. The wind loading was assumed constant along the meridian and expressed by a combination of eight harmonics round the circumference. Pressure coefficients for each harmonic are given in Table 5.

Harmonic	Pressure	Harmonic	Pressure
0	.11 13 90	4	.09 03 30
1	.26 97 45	5	-.09 74 18
2	.60 52 18	6	-.01 70 47
3	.49 00 69	7	.04 81 20

TABLE 5

Note that the constant pressure (zero harmonic) is obtained with a "vacuum factor" $c = 0.5$ of [22]. The results are given on Fig.26, where the harmonic radial displacement is plotted for the 3rd harmonic, which is the most influential and gives also largest displacements. The buckling pressure is about 21 psi, compared to 23.82 psi of Ref. [22]. This higher buckling load could probably be explained by the use of the membrane theory which could be inaccurate in cases where the harmonic number is small. It should be pointed out that this dynamic pressure corresponds to a wind speed of 1100 mph, which is a completely un-realistic value. Consequently the deformation of the shell is also un-realistic and should strictly be outside the realm of this type of analysis.

d) Buckling of a Cone Joining to a Cylinder

It has been shown in Part I that this method is most suitable for the analysis of discontinuous and/or branched shells, since equilibrium of forces at points of discontinuity can be explicitly imposed. This is an example chosen to study the buckling behaviour of such a shell, especially to investigate the effect of the implementation of the equilibrium conditions at the discontinuity on the buckling load.

For this purpose two types of shells are used in this example. The first is a combination of a cylinder and a cone, one half of the complete symmetrical structure is presented on Fig.27 and denoted as "shell A". The second, "shell B", is a modification of the first one from which the discontinuity is removed by the inserting of a segment of a torus between the cone and the cylinder. The smaller this segment is the closer the continuous shell resembles the discontinuous one, and the physical behaviour of the two should also become more and more similar. Thus shell B, with a small segment of torus, can be used as a "control specimen" for investigating the effect of various changes of the equilibrium conditions incorporated in the programme for the buckling analysis of shell A.

The buckling under an uniform pressure is then investigated. A perturbation load is applied to only one other harmonic, and the 4th harmonic is found to give the minimum buckling pressure. Harmonic displacements at the point of discontinuity of shell A and the geometrically corresponding point on the toroidal segment of shell B are presented on Fig.28. At first, all constraint equations (eqs. I-90, I-91, I-92 and I-93) are imposed at the point of discontinuity of shell A, and the buckling pressure of 1940 psi is obtained compared to 1908 psi of Ref. [23]. Shell B buckles at a pressure of 2040 psi (against 1921 psi from Ref. [23]).

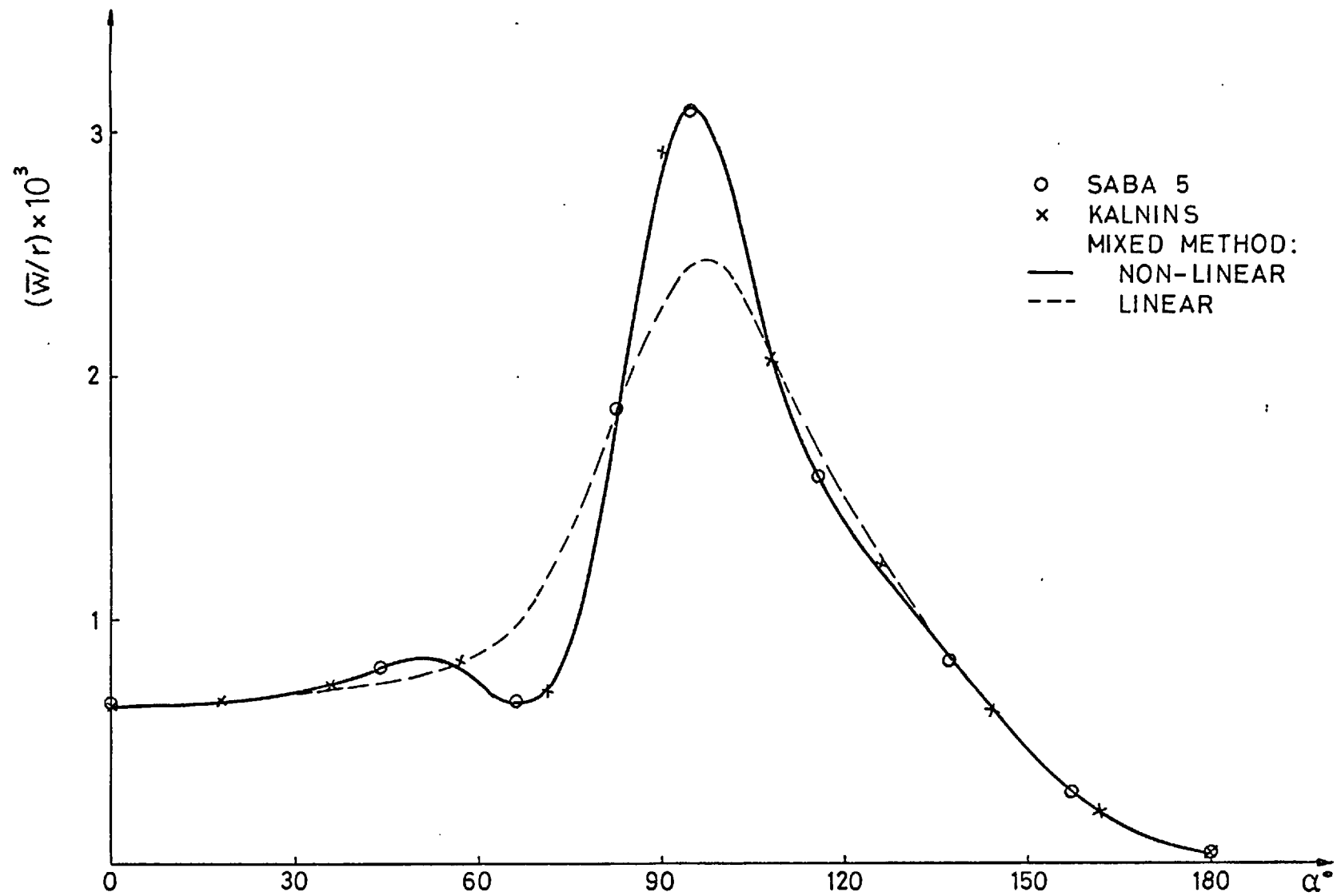
Next, only the constraint equations for rotations (displacement parameters, eq. I-90) and the equilibrium of meridional moments (eq. I-91) are imposed. The unaffected parameters from the two branches are treated in two ways: firstly, they are completely free to be different, and secondly, similar ones are assumed to be the same. The buckling load obtained is 1970 psi in both cases, which was expected, since the unaffected parameters have negligible influence in this type of buckling. It can be seen that the imposition of the equilibrium constraints to the membrane and normal shear forces has little influence, in particular, the enforcement of the equilibrium of twisting moments has practically no effect at all in this loading case. It might be worth mentioning en passant that the analysis of the same structure by the SABA element was not particularly satisfactory one to the presence of the discontinuity.

e) Buckling of Cooling Tower

The buckling analysis of hyperbolic cooling tower under uniform external pressure has been performed by many authors, most interesting being [24], [25], [26] and [1]. The last two references included buckling analysis due to wind load as well. Two towers here are studied (loaded by uniform lateral pressure), one a model and the other a full size tower. The properties of both are given on Fig. 29a. The magnitude of the perturbation load imposed to initiate buckling is 10^{-6} x external pressure. The boundary conditions of the base are considered as fully clamped (built-in). For the cooling tower model, agreement with the results of Ewing [26] is excellent. The buckling harmonic is the 5th one, its displacement given on Fig. 30. The same analysis was then performed for actual cooling tower, except that harmonics from $j = 2$ to $j = 9$ were tested one at a time, since the actual buckling load and waveform were unknown. The minimum value of

buckling pressure, $p = 260$ psf, was obtained with the waveform cos 7θ . Buckling in the adjacent waveforms require slightly higher pressures. For example, the 8th harmonic corresponds to a pressure $p = 270$ psf, and the 6th harmonic 275 psf. Harmonic radial displacements for the critical harmonic and two most influential are given on Fig.31.

For the buckling under steady wind loading, the pressure distribution similar to that given on Fig.10b was used, but the number of harmonics is limited to only first eight, instead of ten. The difference in the pressure distribution is negligible. The boundary condition at the base is taken as fully clamped, as in Ref. [1]. The results indicate the buckling pressure of dynamic head of about 430 psf, which corresponds to a wind speed of 410 m.p.h. Harmonic displacements for the 4th and the 5th harmonic are given on Fig.32, but the 6th and the 7th harmonics give comparable displacements as well. This indicates that the waveform is probably a combination of these harmonics. The radial displacement for $\theta = 0^\circ$ and two points at the tower are presented on Fig.33. Available results for comparison are those of [1] which gives the buckling pressure of 250 psf (314 m.p.h.) and of [26] with the pressure of 312 psf (350 m.p.h.). The discrepancy between the mixed method and [1] is great. But, as mentioned before, the calculation in the harmonic coupling of the geometric stiffness in [1] contains a mistake. As far as the results of Ref. [26] are concerned, the cooling tower there was simply supported at the base. If this is changed to the built-in conditions according to the estimate by Ewing, the buckling pressure would be about 402 psf (397 m.p.h.), which is close to the result obtained here.



DISTRIBUTION OF RADIAL DISPLACEMENT

FIG. 19

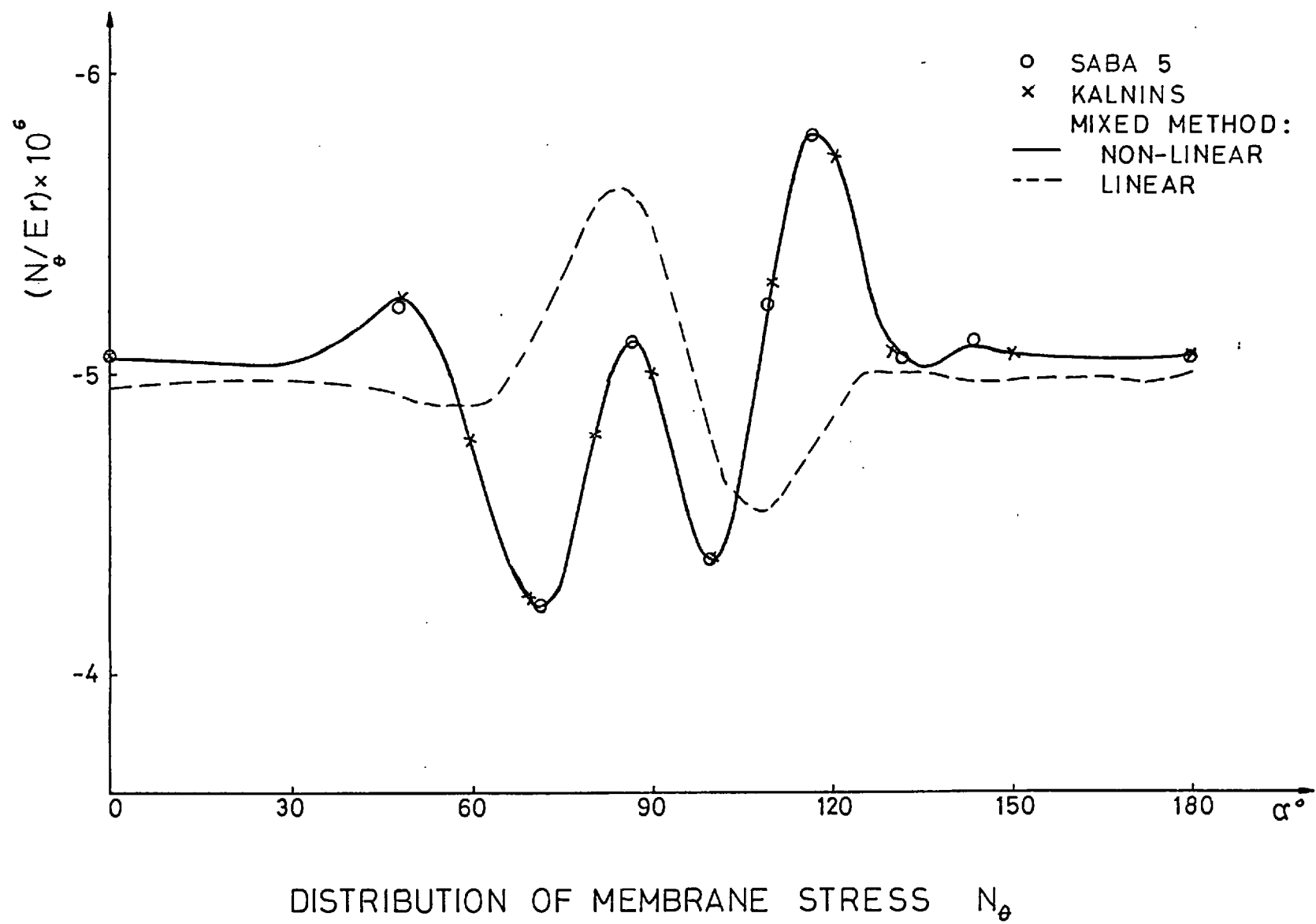
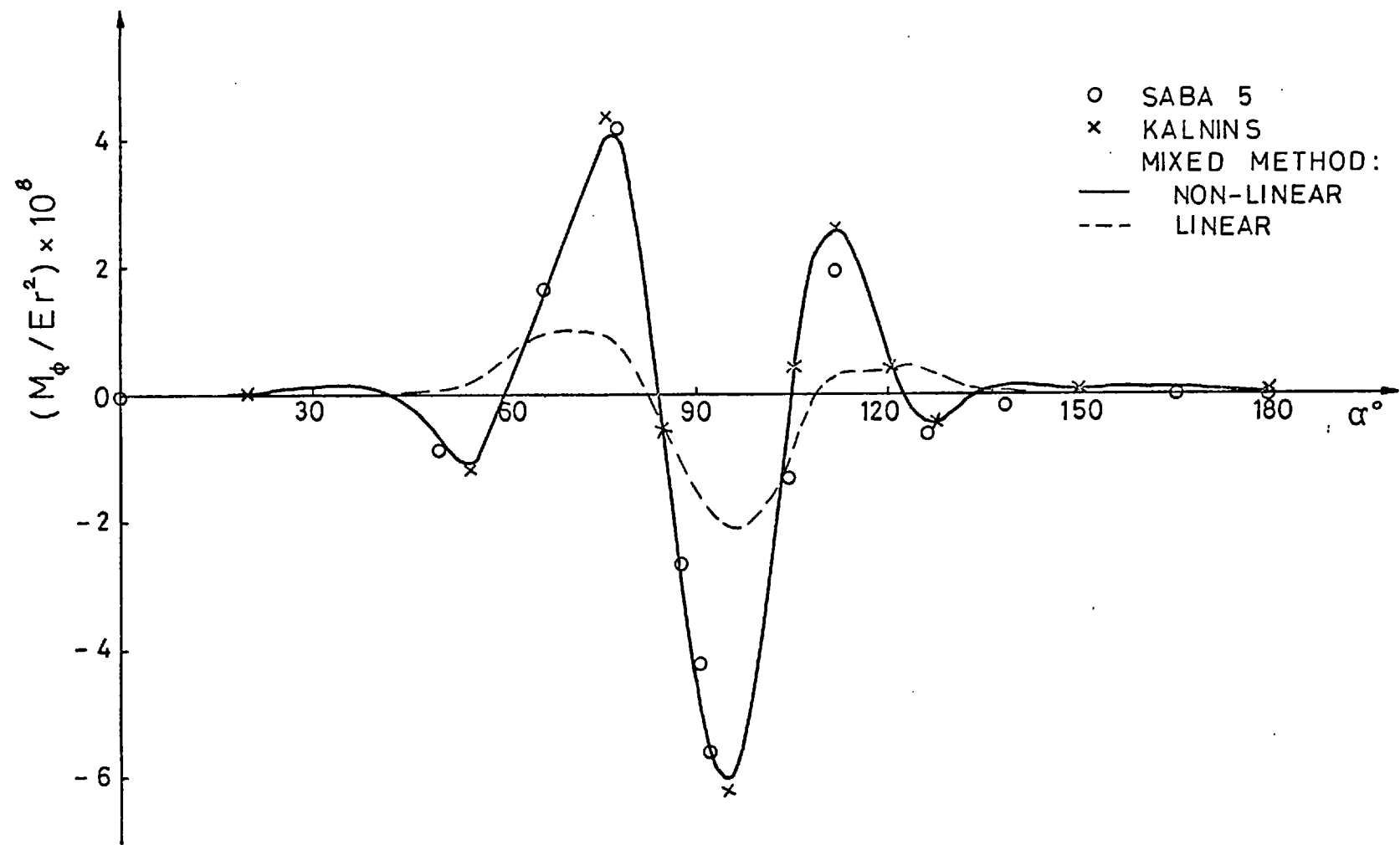
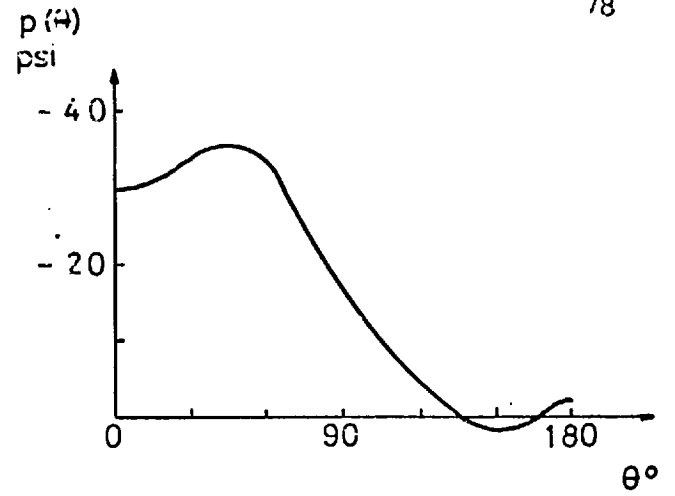
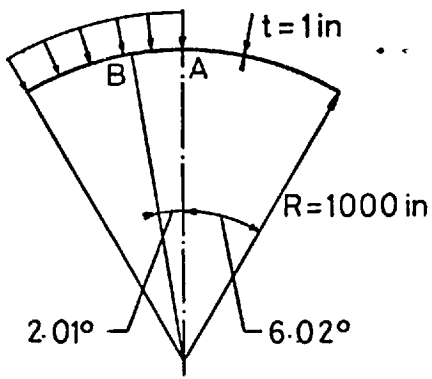


FIG. 20



DISTRIBUTION OF BENDING MOMENT M_ϕ

FIG. 21

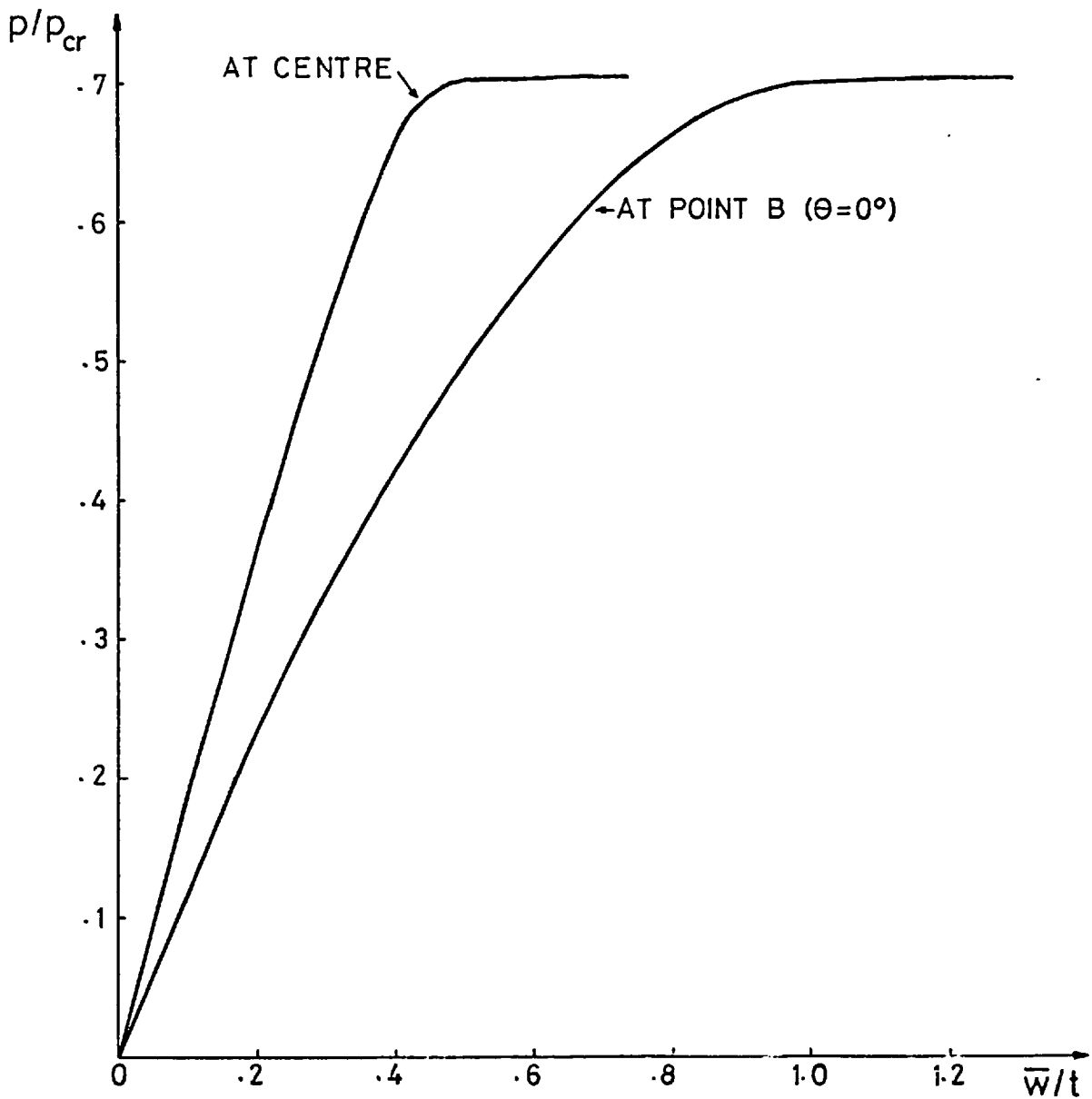


$$E = 27.3 \times 10^6 \text{ psi}$$

$$\nu = 0.3$$

$$p(\theta) = -16.55 - 21 \cos \theta + 7.04 \cos 3\theta$$

a) ASYMMETRICALLY LOADED CAP



b) LOAD-DEFLECTION CURVE

FIG. 22

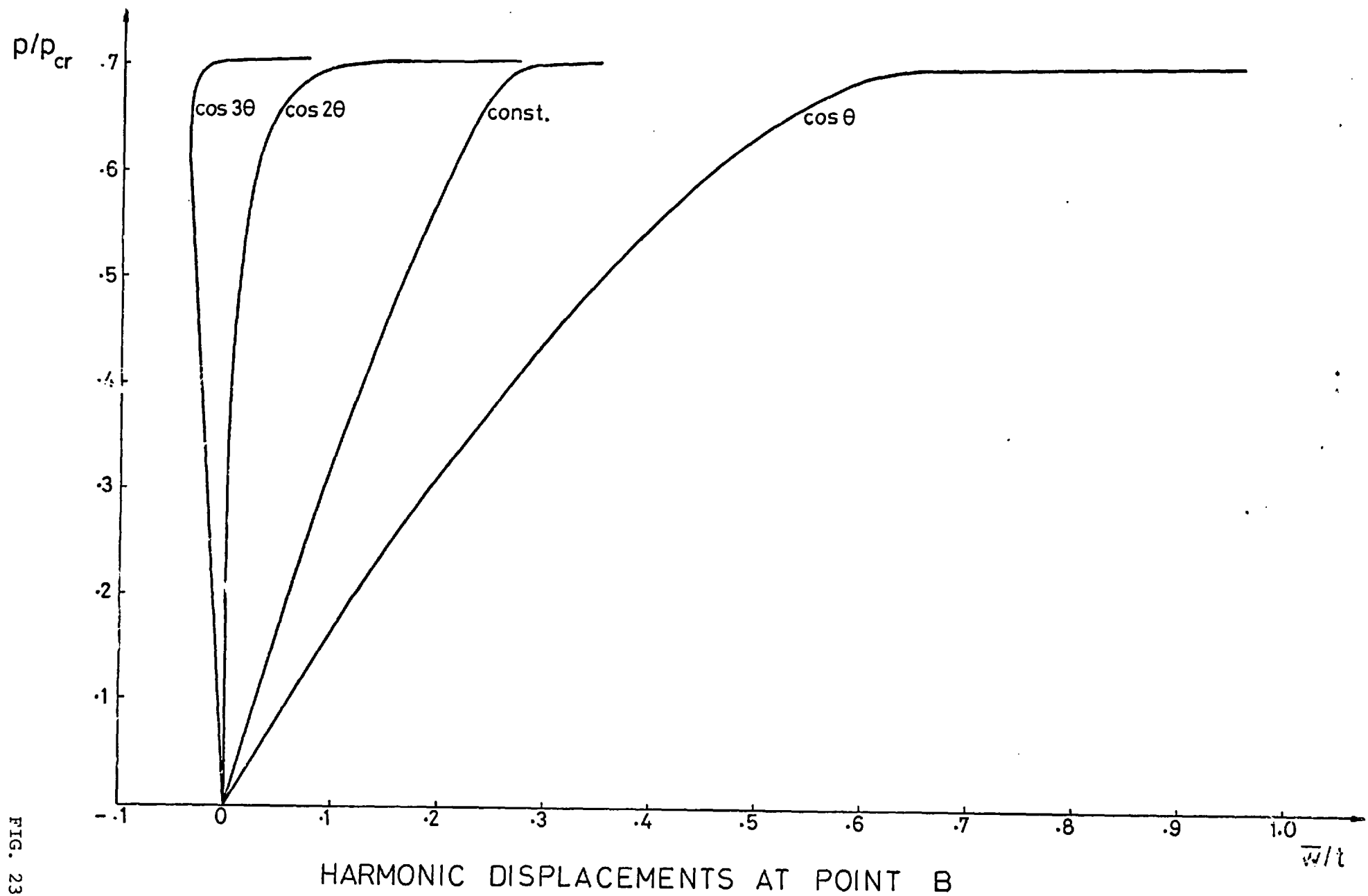
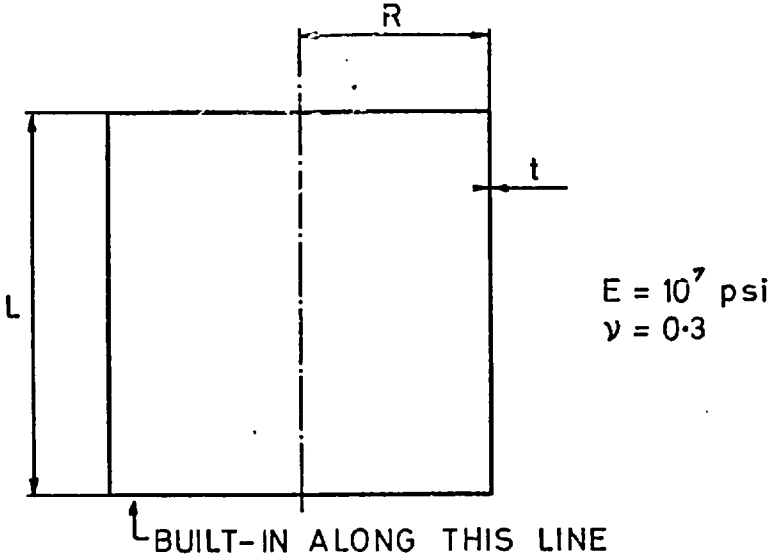


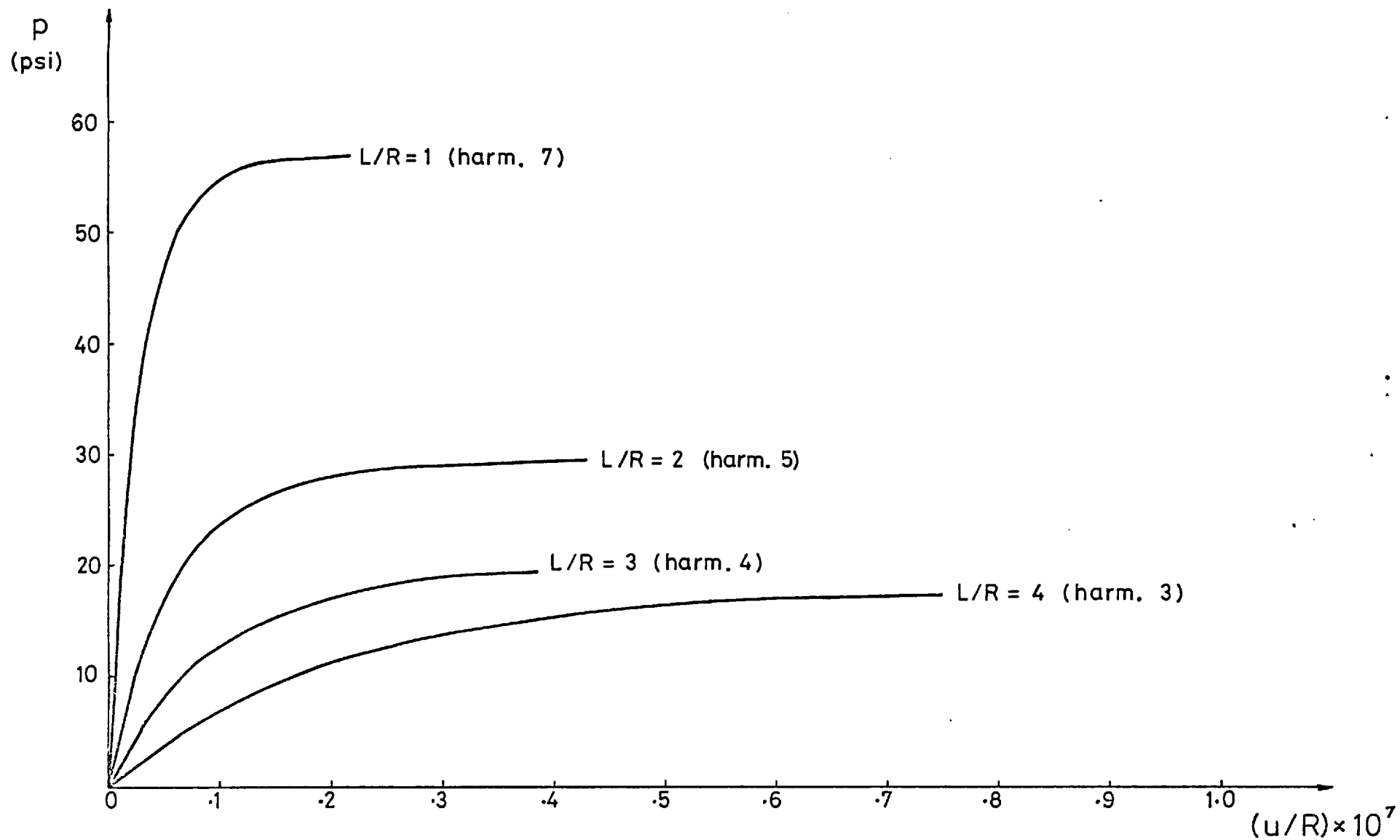
FIG. 23



a) CYLINDER UNDER EXTERNAL PRESSURE

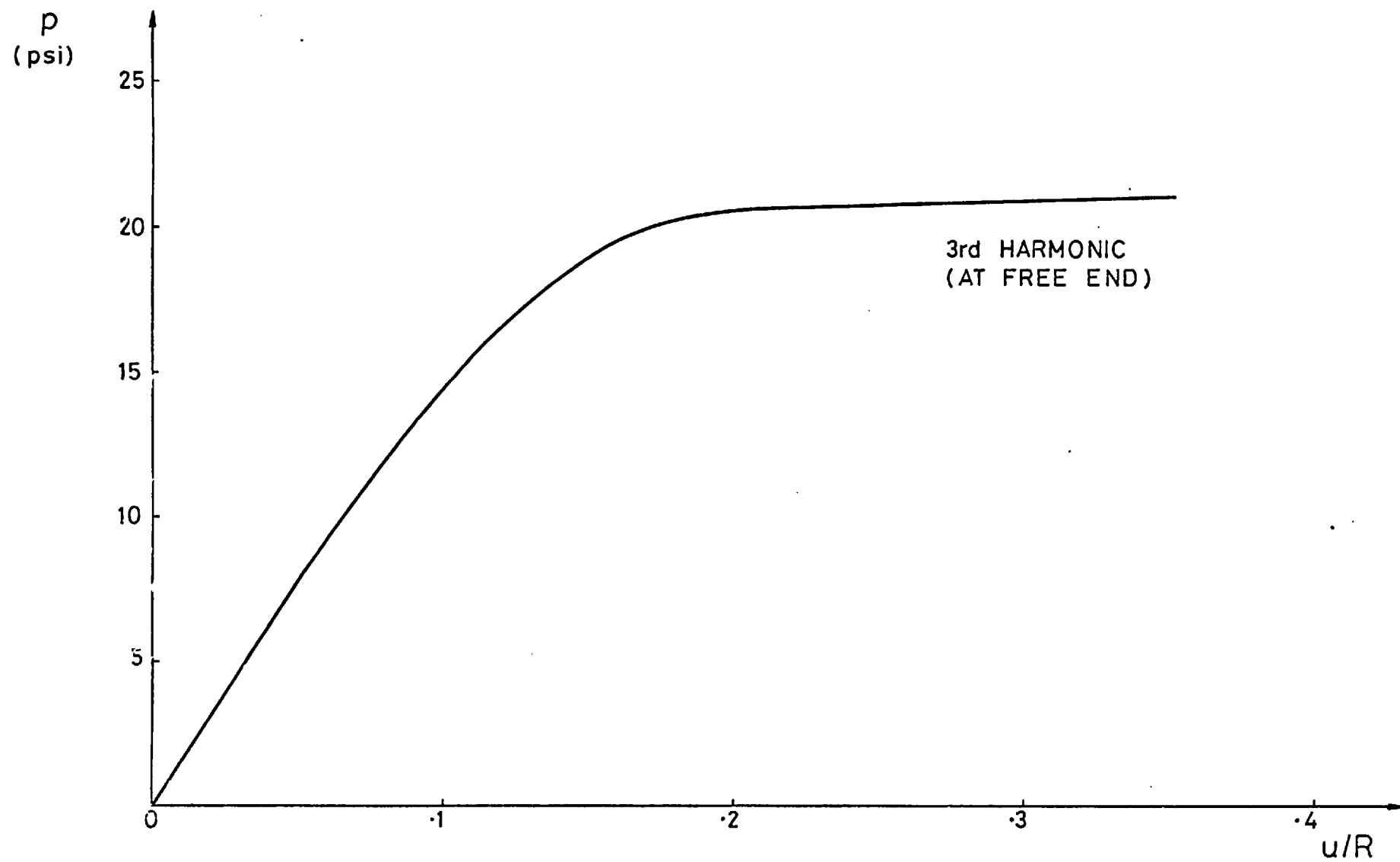
TABLE 4	R / t = 100			
	REF. [22]		MIXED METHOD	
L / R	HARM.	PRESS.	HARM.	PRESS.
1	7	63.27	7	57.
2	5	30.78	5	29.5
3	4	19.97	4	19.5
4	3	16.53	3	17.2

b) COMPARISON OF BUCKLING LOADS

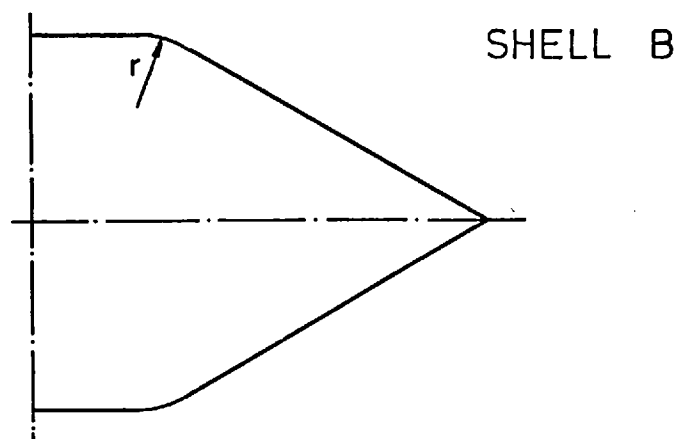
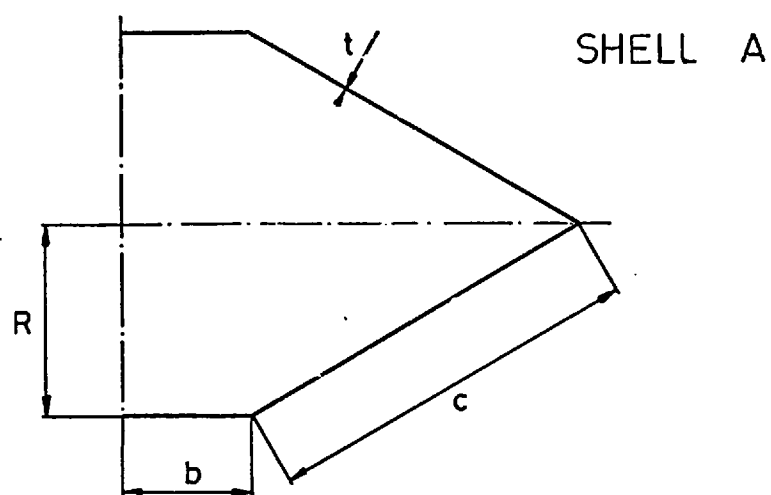


HARMONIC DISPLACEMENT AT FREE END OF CYLINDER

FIG. 25

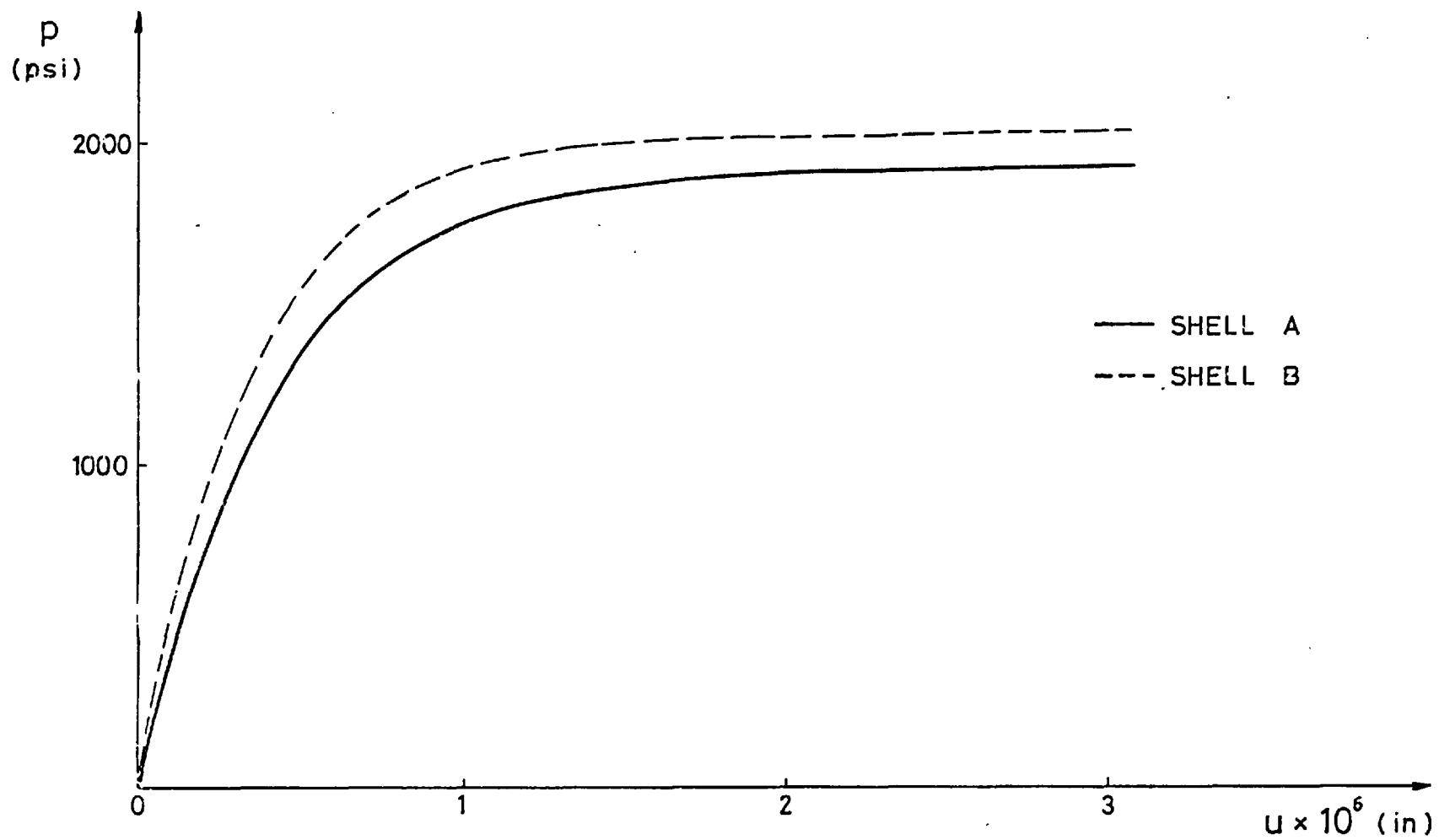


LOAD-DISPLACEMENT CURVE FOR CYLINDER ($L/R = 4$)



$$\begin{aligned}
 E &= 3 \times 10^7 \text{ psi} \\
 \nu &= 0.3 \\
 R &= 50 \text{ in} \\
 r &= 10 \text{ in} \\
 t &= 1 \text{ in} \\
 b &= 12.5 \text{ in} \\
 c &= 100 \text{ in}
 \end{aligned}$$

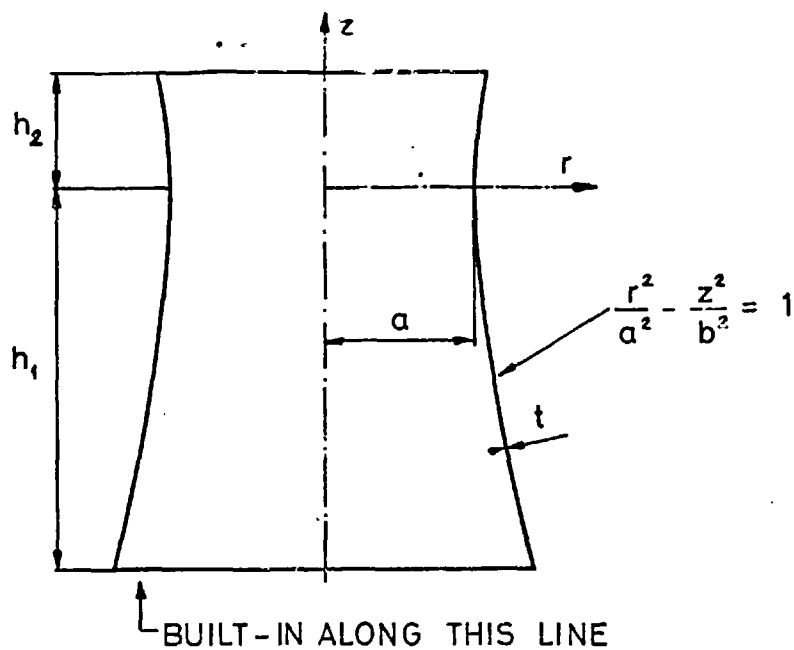
FIG. 27



HARMONIC DISPLACEMENT AT BRANCH POINT

FIG. 28

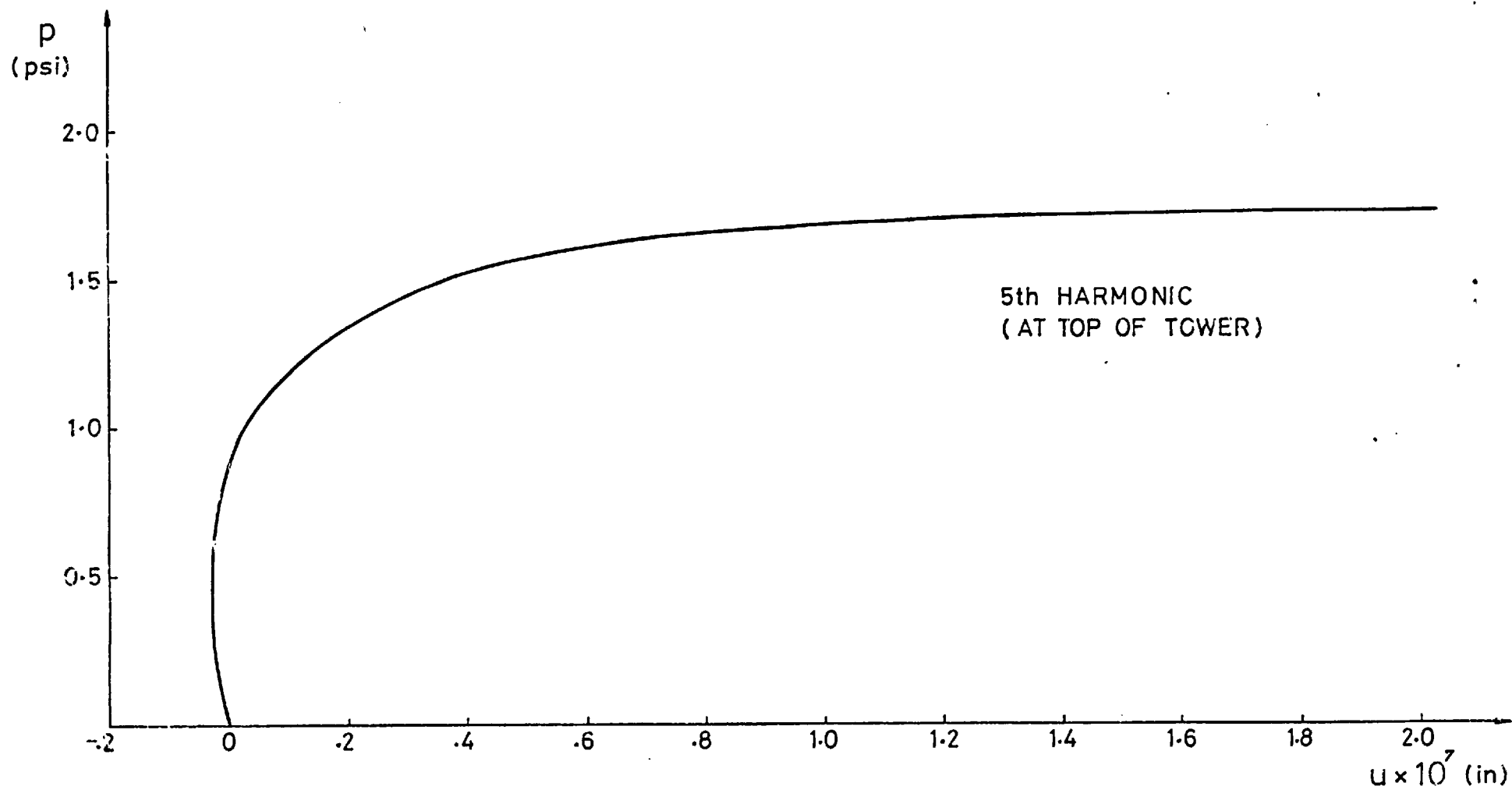
a) HYPERBOLIC COOLING TOWER



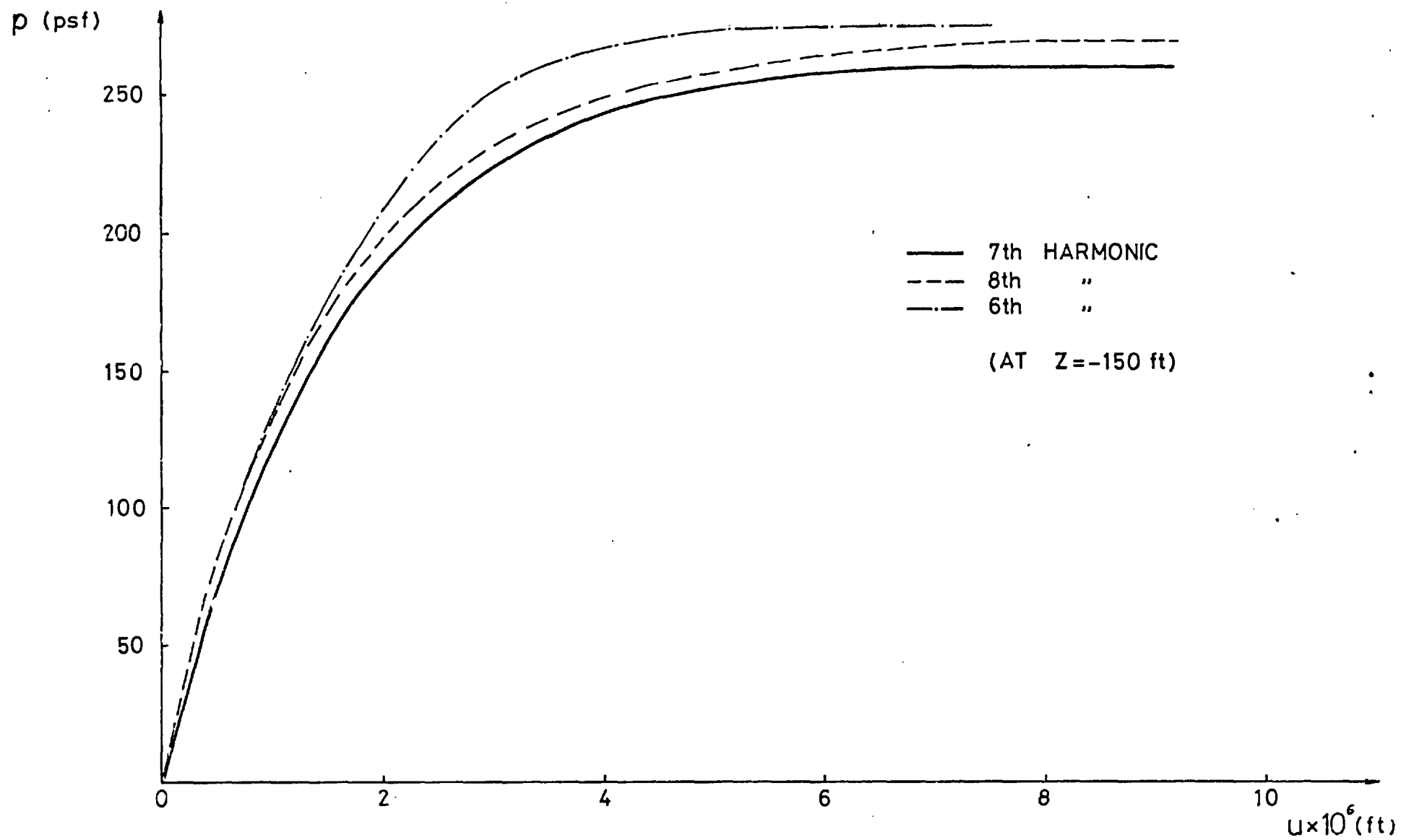
	a	t	h_1	h_2	a^2/b^2	ν	E
COOLING TOWER MODEL	4 in	0.38 in	11.92 in	3.67 in	0.16	0.4	5.5×10^5 psi
FULL-SIZE TOWER	84 ft	5 in	270 ft	60 ft	0.16	0.15	4.32×10^8 psf

b) COMPARISON OF BUCKLING LOADS FOR COOLING TOWER MODEL

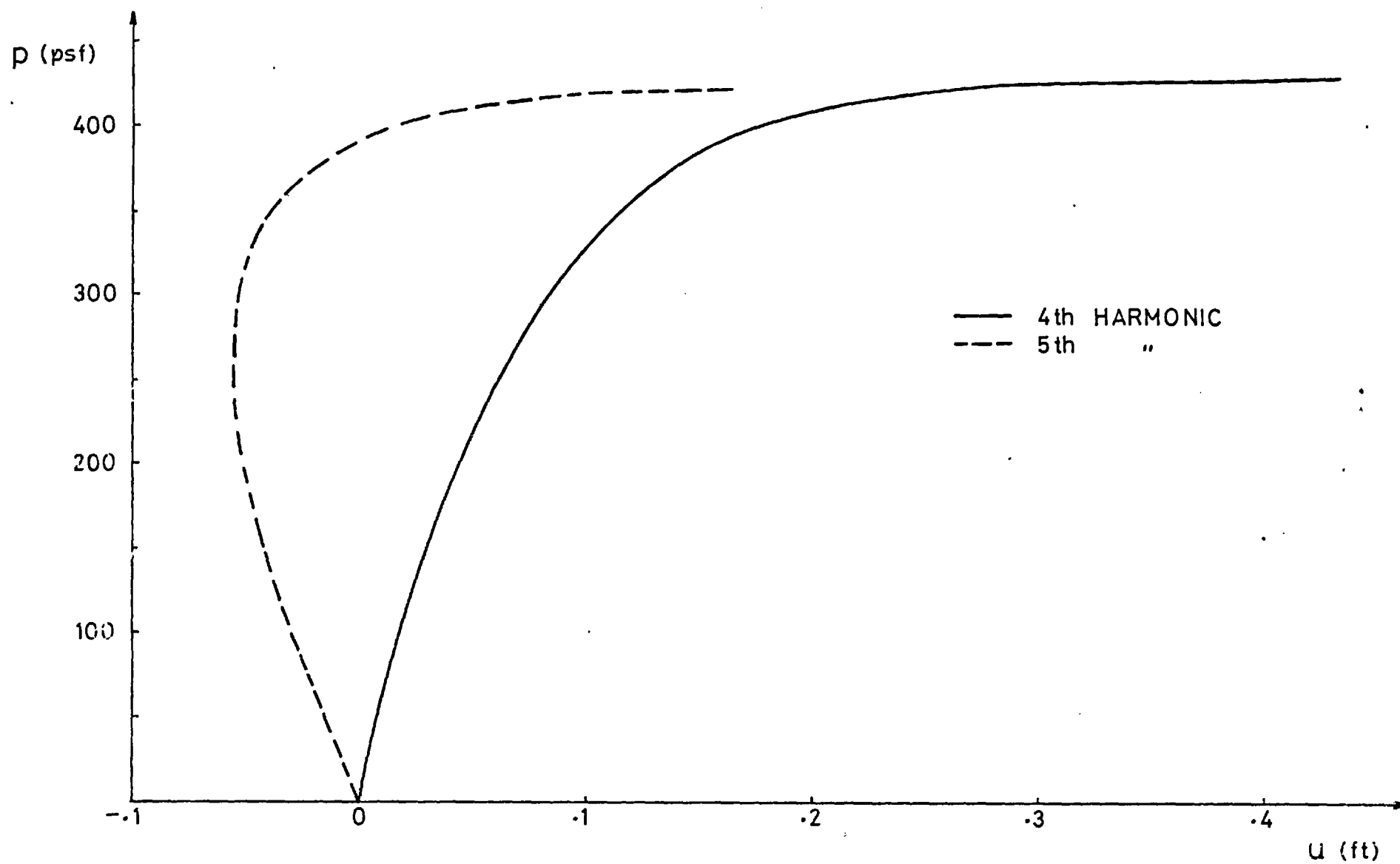
CONDITIONS AT THE BASE	REF [26]	SABA 5	MIXED METHOD
BUILT-IN	1.75 psi	1.71 psi	1.76 psi



LOAD-DISPLACEMENT CURVE FOR COOLING TOWER MODEL



LOAD-DISPLACEMENT CURVE FOR 5inch COOLING TOWER



HARMONIC DISPLACEMENT AT TOP OF TOWER

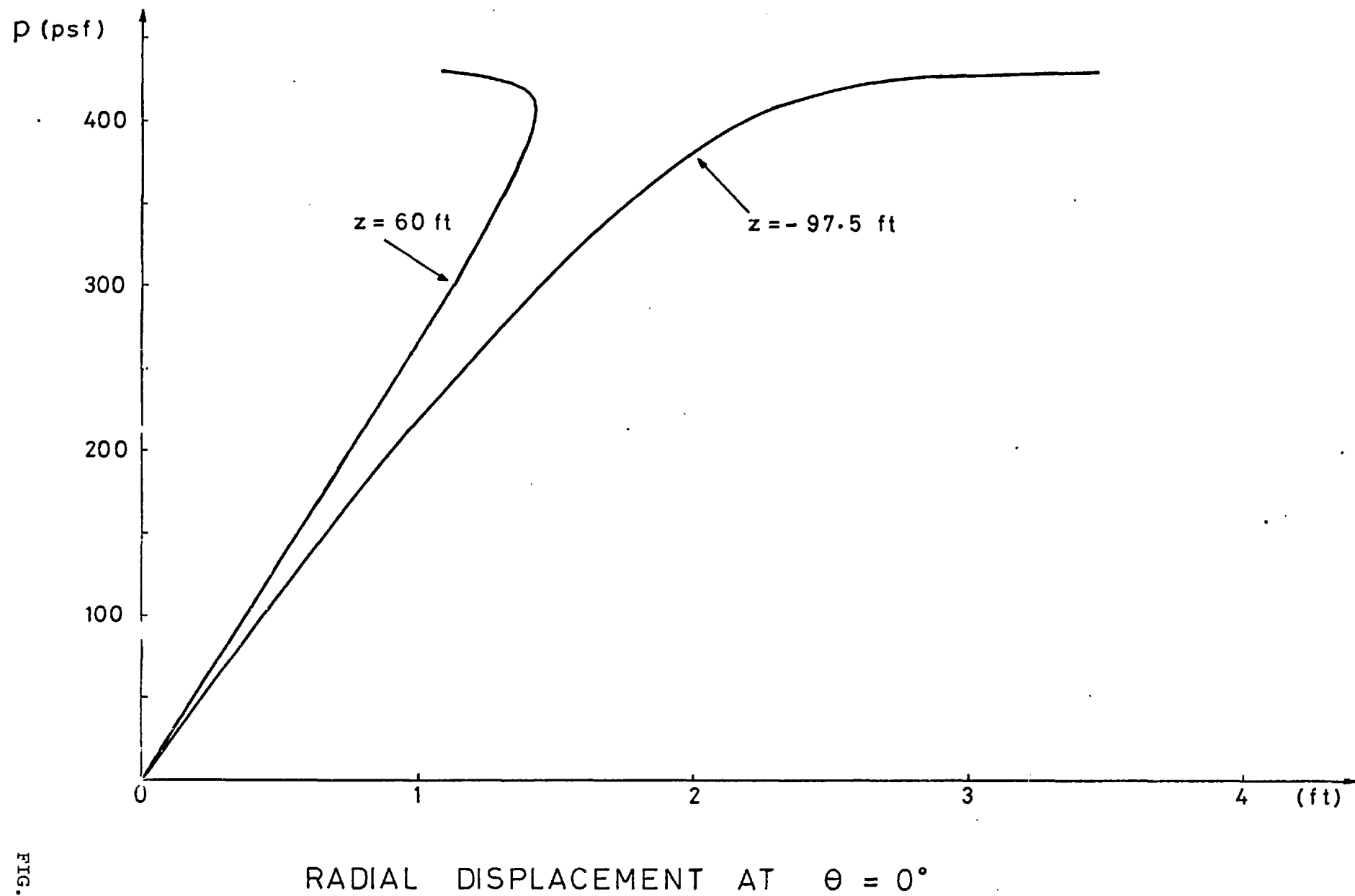


FIG. 33

PART III - Dynamic Analysis

3.1 INTRODUCTION

The determination of the transient dynamic response of shell structures to arbitrary loads has always been an important design problem. The equation of motion is a second order differential equation, for the solution of which various numerical procedures have been developed, usually referred to as the step-by-step integration methods. Two of the most frequently used procedures were reported by Newmark [27] and Argyris and Chan [16]. In the latter work, the inertia force was prescribed as a cubic function of time within a time step in terms of its values and the first derivatives at the beginning and at the end, giving in effect a fifth order displacement variation. Convergence within the time step is achieved using an iteration technique. The method was demonstrated on a complicated elasto-plastic dynamic impact problem in conjunction with a displacement method of finite element idealisation. Later it was utilised for an interesting problem of earthquake loading in [28]. A similar method, but with linear variation of acceleration was applied to linear and non-linear dynamic analysis of shells in Ref. [29].

In the formulation of the dynamic problem by the mixed method of approach, only the displacement parameters appear directly as function of the time variable in the form of velocities and accelerations in the equation of motion derived from the application of the P.V.D. for the approximate enforcement of equilibrium. The force parameters are only functions of time implicitly by their association with the displacement in the equation and in the application of the P.V.F. to enforce compatibility. It is therefore easiest to express the force parameters in terms of the displacement parameters. Then the equation

of motion may be integrated by the same method as used in Ref. [16] to give the displacements. This may appear to be contrary to the policy of Part I and II, where the displacement and force parameters are solved together. The reason there was to avoid having to solve a fully populated stiffness matrix. Here, in the procedure of the dynamic analysis, the stiffness matrix never required to be inverted. Indeed, the force parameters are always directly derivable at every stage of the calculation by a matrix multiplication and contribute only a column to the dynamic equation without changing the stiffness matrix as such. Hence the objection that was sustained in the static analysis never arises here. The step-by-step iterative scheme requires the inversions of the mass and the flexibility matrices at the very beginning of the calculation. Thereafter, only multiplications are necessary.

The advantage using this mixed method of approach in the dynamic analysis can now be appreciated. There are only six displacement parameters at each node, instead of twelve required by the displacement method (SABA 7) to give a comparable accuracy. The number of unknowns in the iterative procedure is thereby reduced by half, at the expense of only one additional matrix multiplication. For single-branched shell structures, the calculation follows almost exactly the same routine as for the displacement approach given in Ref. [16]. In the case of multi-branched shells, the constraints at the point of discontinuity necessitate a slightly different treatment. Since only the equality of rotation and the equilibrium of meridional bending moments are important, and they affect respectively only displacement and force parameters, they may be utilised to reduce the number of independent variables. In fact, the complete stiffness and damping matrices need never be assembled. Instead, the static and damping

forces may be calculated at the element level and "assembled for condensation. Even the progressive elimination technique may be applied if one so wishes. All these can be manipulated by ingenious programming, and the calculation can proceed just as easily as for the single-branched structures.

Although the method is quite capable of giving an accurate solution, it is very sensitive to the length of time step employed and in general requires a time step which is about a quarter of the shortest natural period of vibration of the equivalent mass system. If the time step is too large, the iterative calculation becomes divergent. The demand for a large amount of computer time due to the small step size imposes a limitation for really large problems. This is even more acute for non-linear problems, where the geometric stiffness has to be calculated for each iteration within a time step. Hence, it is thought prudent to exclude the non-linear dynamic analysis from this work, although the algorithm could be used directly for the purpose as soon as the subroutine for the calculation of geometric stiffness is included, and if the expense for the computation is no objection.

In the following chapters the matrix equations for dynamic analysis by the mixed method of finite element idealisation will be developed. The integration procedure will be briefly presented and the method for dealing with discontinuous structures discussed. The technique is then demonstrated on a number of sample problems.

3.2 FORMULATION OF THE MATRIX EQUATION FOR A DYNAMIC ANALYSIS

In the dynamic analysis the applied force $\mathbf{P}$ is equilibrated not only by the elastic force, but the damping and the inertia forces. If the damping force is assumed to be proportional to the velocity and the mass per unit volume is denoted by ρ , then the Principle of Virtual Work, which enforces equilibrium between the applied load and the internal forces is as follows .

$$\rho^t \mathbf{P} = \int \epsilon^t \mathbf{S} \, dA + \delta^t \int (c \dot{\delta} + \rho \ddot{\delta}) \, dV \quad (1)$$

Since the membrane and the bending stresses are obtained by different interpolation schemes in this mixed method in terms of the nodal displacement parameters ρ and force parameters $\mathbf{b}$ (eqs. I-23 and I-28), this equation becomes

$$\mathbf{P} = \mathbf{k} \rho + \mathbf{t}^t \mathbf{b} + \mathbf{d} \dot{\rho} + \mathbf{m} \ddot{\rho} \quad (2)$$

where the matrices $\mathbf{k}$ and $\mathbf{t}$ are the stiffness and coupling matrices as defined in Part I. (eqs. I-60 and I-55) and

$$\begin{aligned} \mathbf{d} &= k \pi L c \int \beta^t \beta \, R t d n \\ \mathbf{m} &= k \pi L \rho \int \beta^t \beta \, R t d n \end{aligned} \quad (3)$$

are the kinematically equivalent damping and the mass matrices respectively. Note, for the sake of simplicity, that the integration around the circumference is performed and contributes factor k , the value of which is 2 for zero harmonic and 1 for all other harmonics, and, therefore, harmonic number (j) is omitted. The damping matrix is proportional to the mass matrix in this formulation. A different assumption can make it proportional to the stiffness matrix instead (see Ref.16). More generally, it may be a linear combination of the two assumptions. In practice, a suitable lumped mass matrix and/or damping matrix may be used.

This is the equation of motion in the mixed method of formulation. It involves not only the displacement vector ρ and its time derivatives, but also the force vector $\mathbf{b}$. These two vectors are further related, however, by the compatibility equation (I-53).

$$\mathbf{f} \mathbf{b} - \mathbf{t} \dot{\rho} = \mathbf{0} \quad (4)$$

which applies to the dynamic as well as the static analysis.

Assembling eqs. (2) and (4) for all elements of the structure as in the static analysis gives

$$\mathbf{R}_a = \mathbf{K} \mathbf{r} + \mathbf{T}^t \mathbf{B} + \mathbf{D} \dot{\mathbf{r}} + \mathbf{M} \ddot{\mathbf{r}} \quad (5)$$

$$\mathbf{0} = \mathbf{F} \mathbf{B} - \mathbf{T} \mathbf{r} \quad (6)$$

where $\mathbf{R}_a$ is the applied force,

$$\mathbf{D} = \sum \mathbf{a}_g^t \mathbf{d}_g \mathbf{a}_g \quad (7)$$

$$\mathbf{M} = \sum \mathbf{a}_g^t \mathbf{m}_g \mathbf{a}_g \quad (8)$$

and the others are the same matrices defined in Part I. The sum of the first two terms on the right-hand side of eq. (5) is obviously static force and may be denoted by

$$\mathbf{R}_s = \mathbf{K} \mathbf{r} + \mathbf{T}^t \mathbf{B} \quad (9)$$

The problem now is to solve eqs. (5) and (6) simultaneous to find the nodal displacements $\mathbf{r}$ and nodal forces $\mathbf{B}$ as a function of time.

3.3 SOLUTION OF THE DYNAMIC PROBLEM

For this purpose the equation of motion can be rewritten as

$$M \ddot{\mathbf{r}} = \mathbf{R}_a - \mathbf{R}_s - \mathbf{D} \dot{\mathbf{r}} \quad (10)$$

In this form the equation is the same as in Ref. [16]. However, there is one essential difference: in the displacement approach of Ref. [16], the static force $\mathbf{R}_s$ is the elastic force arising from the displacement $\mathbf{r}$ alone. Here it is also dependent on the force parameters $\mathbf{B}$. It can be seen in eqs. (5) and (6) that $\mathbf{B}$ is a function of time only by its association with the displacement parameters $\mathbf{r}$. There is no time derivative of $\mathbf{B}$ in these equations. Providing those force parameters can be expressed in terms of the displacement parameters by solving eq. (6), the consequence of which will be discussed later on, eq. (10) can be solved by exactly the same method already described in Ref. [16]. Briefly, the inertia force

$$\mathbf{R} = M \ddot{\mathbf{r}} \quad (11)$$

is assumed to vary as a cubic function of time within a given time interval, defined in terms of the inertia force and its derivatives at the beginning and the end. If the time interval is denoted by τ , then the function can be expressed in terms of the non-dimensional time variable $\xi = t/\tau$, and the derivative of $\mathbf{R}$ with respect to ξ is

$$\mathbf{R}' = \frac{d\mathbf{R}}{d\xi} = \tau \frac{d\mathbf{R}}{dt} = \tau \dot{\mathbf{R}} \quad (12)$$

Thus

$$\mathbf{R} = \phi_1 \mathbf{R}_0 + \phi_2 \mathbf{R}'_0 + \phi_3 \mathbf{R}_1 + \phi_4 \mathbf{R}'_1 \quad (13)$$

The subscripts 0 and 1 denote the values at the beginning ($\xi = 0$) and the end ($\xi = 1$) of the time interval.

The functions ϕ_i are the Hermitian polynomials defined as

$$\begin{aligned}
 \phi_1 &= 1 - 3\xi^2 + 2\xi^3 \\
 \phi_2 &= \xi - 2\xi^2 + \xi^3 \\
 \phi_3 &= 2\xi^2 - 2\xi^3 \\
 \phi_4 &= -\xi^2 + \xi^3
 \end{aligned} \tag{14}$$

with their well known properties. This variation is found to be adequate in [16], later confirmed in [28], especially for the linear elastic case. The values of the displacements and velocities, $\mathbf{r}_1$ and $\dot{\mathbf{r}}_1$ at the end of the time step can be obtained by integrating eq. (11) twice with respect to ξ , and then putting $\xi = 1$, which results in

$$\dot{\mathbf{r}}_1 = \dot{\mathbf{r}}_0 + \frac{\tau}{12} \mathbf{M}^{-1} (6\mathbf{R}_0 + \tau\dot{\mathbf{R}}_0 + 6\mathbf{R}_1 - \tau\dot{\mathbf{R}}_1) = \dot{\mathbf{r}}_0 + d\dot{\mathbf{r}} \tag{15}$$

and

$$\mathbf{r}_1 = \mathbf{r}_0 + \tau\dot{\mathbf{r}}_0 + \frac{\tau^2}{60} \mathbf{M}^{-1} (21\mathbf{R}_0 + 3\tau\dot{\mathbf{R}}_0 + 9\mathbf{R}_1 - 2\tau\dot{\mathbf{R}}_1) = \mathbf{r}_0 + d\mathbf{r} \tag{16}$$

This is essentially a forward integration expressing the end results as their values at the beginning plus the increments during the time interval. The calculation involves the evaluation of the force vectors $\mathbf{R}$ and $\dot{\mathbf{R}}$ at the time "nodes" i from

$$\mathbf{R}_i = \mathbf{R}_{ai} - \mathbf{K} \mathbf{r}_i - \mathbf{T}^t \mathbf{B}_i - \mathbf{D} \dot{\mathbf{r}}_i \tag{17}$$

$$\dot{\mathbf{R}}_i = \dot{\mathbf{R}}_{ai} - \mathbf{K} \dot{\mathbf{r}}_i - \mathbf{T}^t \dot{\mathbf{B}}_i - \mathbf{D} \ddot{\mathbf{r}}_i \tag{17a}$$

where $\dot{\mathbf{R}}_a$ is the time rate of change of the applied force, and the force parameters $\mathbf{B}$ from the solution of eq. (6) given by

$$\mathbf{B} = \mathbf{F}^{-1} \mathbf{T}^t \mathbf{r} \tag{18}$$

$$\dot{\mathbf{B}} = \mathbf{F}^{-1} \mathbf{T}^t \dot{\mathbf{r}} \tag{18a}$$

Hence the expressions in eqs. (15) and (16) are not explicit because the values $\mathbf{R}_1$ and $\dot{\mathbf{R}}_1$ themselves depend on $\mathbf{r}_1$, $\dot{\mathbf{r}}_1$, $\ddot{\mathbf{r}}_1$. An iterative procedure is therefore necessary for every time step forward. Starting from a known state at the beginning of the step, the initial assumption that

$$\dot{\mathbf{R}}_1 = \dot{\mathbf{R}}_0 \quad (19)$$

and

$$\mathbf{R}_1 = \mathbf{R}_0 + \tau \dot{\mathbf{R}}_0 \quad (20)$$

enables the first approximation of $\dot{\mathbf{r}}_1$ and $\mathbf{r}_1$ to be calculated from eqs. (15) and (16). Then, with the help of eqs. (18) and (18a), the inertia force at the end of the step $\mathbf{R}_1$ may be re-calculated by eq. (17), giving an acceleration

$$\ddot{\mathbf{r}}_1 = \mathbf{M}^{-1} \mathbf{R}_1 \quad (21)$$

and then $\dot{\mathbf{R}}_1$ too may be re-calculated by eq. (17a). They are checked against the initial assumptions and the procedure reiterated until the results converge.

It may be seen that the calculation requires the inversion of the mass matrix $\mathbf{M}$ and the flexibility matrix $\mathbf{F}$ at the start of the iterative procedure. Thereafter, only matrix multiplications and additions are necessary. Each term on the right-hand side of eqs. (17) and (17a) contributes only one column to the inertia force. The execution can be extremely fast and economical, especially since the force parameters are expressed in terms of the displacement parameters (eq.18), the number of independent unknowns involved is effectively halved. It will be remembered that the static analysis (Part I) deliberately refrained from doing this, firstly, because the condensed stiffness matrix will be fully populated and cannot be efficiently solved, and secondly,

because equilibrium constraints at point of discontinuity cannot be easily introduced. The former objection never arises in the dynamic analysis since the stiffness matrix is not required to be inverted. The method for dealing with the latter will be discussed in the next section.

3.4 IMPOSITION OF CONSTRAINTS AT THE POINT OF DISCONTINUITY

These constraints arise from the requirement of static and kinematic continuity and are given in eqs. (I-90, 91, 92, 93). However, experience suggests that, apart from the obvious and vital but simple one of displacement equality, only the requirements of equal rotation and equilibrium of meridional bending moment are important. As each one affects one group of parameters separately, they may be conveniently interpreted as expressions of linear dependence and utilised to reduce the number of independent variables. To be more explicit, the constraints of eq. (I-90) and (I-91) may be written as

$$\mathbf{C} \mathbf{r} = \mathbf{0} \quad (22)$$

and

$$\mathbf{G} \mathbf{B} = \mathbf{0} \quad (23)$$

The displacements in $\mathbf{r}$ may be partitioned into independent ones $\mathbf{r}_L$ and dependent ones $\mathbf{r}_B$, and the constraint matrix $\mathbf{C}$ correspondingly, then eq. (22) becomes

$$\mathbf{r}_B = -\mathbf{C}_B^{-1} \mathbf{C}_L \mathbf{r}_L \quad (22a)$$

or

$$\mathbf{r} = \begin{bmatrix} \mathbf{r}_L \\ \mathbf{r}_B \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ -\mathbf{C}_B^{-1} \mathbf{C}_L \end{bmatrix} \mathbf{r}_L = \mathbf{p} \mathbf{r}_L \quad (24)$$

A similar treatment of eq. (23) expresses the force parameters $\mathbf{B}$ in terms of the independent ones $\mathbf{B}_L$ as

$$\mathbf{B} = \mathbf{q} \mathbf{B}_L \quad (25)$$

Substitution into the simultaneous equations (5) and (6) gives

$$\mathbf{R} = \mathbf{K} \mathbf{p} \mathbf{r}_L + \mathbf{T}^t \mathbf{q} \mathbf{B}_L + \mathbf{D} \mathbf{p} \dot{\mathbf{r}}_L + \mathbf{M} \mathbf{p} \ddot{\mathbf{r}}_L \quad (26)$$

$$\mathbf{T} \mathbf{p} \mathbf{r}_L - \mathbf{F} \mathbf{q} \mathbf{B}_L = \mathbf{O} \quad (27)$$

Application of the P.V.D. and P.V.F. respectively gives the generalised forces $\mathbf{R}_L$ corresponding to $\mathbf{r}_L$ and generalised displacement corresponding to $\mathbf{B}_L$, namely

$$\mathbf{R}_L = \mathbf{p}^t \mathbf{R} = (\mathbf{p}^t \mathbf{K} \mathbf{p}) \mathbf{r}_L + (\mathbf{p}^t \mathbf{T}^t \mathbf{q}) \mathbf{B}_L + (\mathbf{p}^t \mathbf{D} \mathbf{p}) \dot{\mathbf{r}}_L + (\mathbf{p}^t \mathbf{M} \mathbf{p}) \ddot{\mathbf{r}}_L \quad (28)$$

$$(\mathbf{q}^t \mathbf{T} \mathbf{p}) \mathbf{r}_L - (\mathbf{q}^t \mathbf{F} \mathbf{q}) \mathbf{B}_L = \mathbf{O} \quad (29)$$

which, of course, may be written as

$$\mathbf{R}_L = \mathbf{K}_L \mathbf{r}_L + \mathbf{T}_L^t \mathbf{B}_L + \mathbf{D}_L \dot{\mathbf{r}}_L + \mathbf{M}_L \ddot{\mathbf{r}}_L \quad (30)$$

and

$$\mathbf{T}_L \mathbf{r}_L - \mathbf{F}_L \mathbf{B}_L = \mathbf{O} \quad (31)$$

Each of the new matrix defined above is a condensed version of the corresponding antecedent. The same iterative routine may now be applied for the solution of this reduced problem, which is identical to the original one in form.

Computationally, only the mass and the flexibility matrices, which are to be inverted, have to be transformed formally. The stiffness, damping and the coupling matrices, which contribute only columns of forces in eq. (30) and displacement in eq. (31), need not be condensed at all. To illustrate better the actual calculation procedure, against the theoretical basis given above, the columns of forces and displacements in eqs. (5) and (6) may be denoted by:

$$\begin{aligned}
\mathbf{K} \mathbf{r} &= \mathbf{P}_M && \text{(membrane force)} \\
\mathbf{T}^t \mathbf{B} &= \mathbf{P}_B && \text{(bending force)} \\
\mathbf{D} \dot{\mathbf{r}} &= \mathbf{P}_D && \text{(damping force)} \\
\mathbf{T} \mathbf{r} &= \rho_M && \text{(membrane displacement)}
\end{aligned} \tag{32}$$

Then eqs. (28) and (29) also change to

$$\mathbf{R}_L = \mathbf{p}^t \mathbf{R} = \mathbf{p}^t \mathbf{P}_M + \mathbf{p}^t \mathbf{P}_B + \mathbf{p}^t \mathbf{P}_D + \mathbf{M}_L \ddot{\mathbf{r}}_L \tag{30a}$$

$$\mathbf{q}^t \rho_M - \mathbf{F}_L \mathbf{B}_L = \mathbf{0} \tag{31a}$$

Knowing the displacement, velocity and acceleration of the reduced vector $\mathbf{r}_L$ at any stage of the calculation, the full original set is also available (eq. 24). The last equation in (32) then gives ρ_M and eq. (31a) is solved for $\mathbf{B}_L$. Then the original set of force parameters $\mathbf{B}$ is given by eq. (25) and all columns of forces in (32) can now be calculated. (Incidentally they may be assembled from the element nodal forces if so wished. The full stiffness, damping and coupling matrices need not be assembled at all.) Eq. (30a) may now be used iteratively to determine the inertia force at the next stage. It can be seen that only the columns in (32) are condensed. The necessary operations for the condensation of these columns, as well as the mass and flexibility matrices, are in fact quite simple. First of all, it is seen that the constraints affect only the parameters at the branch point. Elsewhere, nothing is changed. Next, owing to the extremely simple nature of the equilibrium constraint on the meridional moments (eq. I-91), the condensation of $\mathbf{F}$ to $\mathbf{F}_L$ is affected by adding the row and column corresponding to the moment to be eliminated to those corresponding to the moments remaining. If the nodal points are numbered sensibly, as for example in Fig. 34.

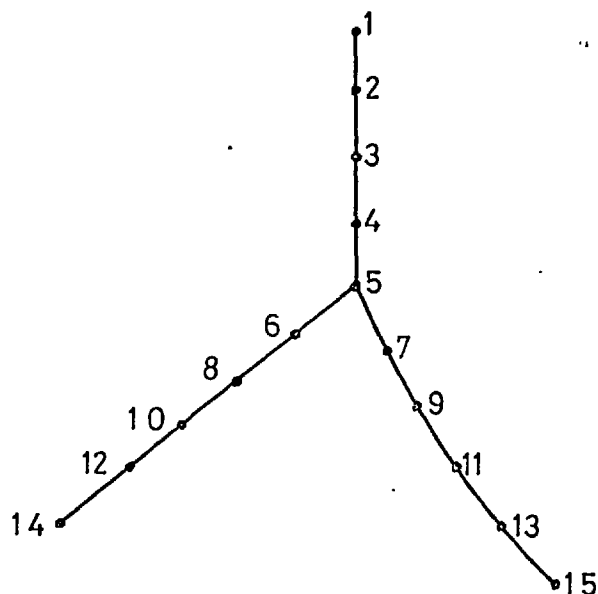


FIG. 34

the bandwidth of $\mathbf{F}$ and hence $\mathbf{F}_L$, remains manageably small. The mass matrix can be dealt with similarly, except that the row (and column) to be eliminated is multiplied by a factor before adding. All these can be achieved by simple programming devices instead of carrying out the formal matrix multiplications. The transformation matrices $\mathbf{p}$ and $\mathbf{q}$ do not have to be stored. With more sophistication, the progressive elimination technique employed in the static analysis of multi-branched shells can be adopted here to calculate the values of $d\mathbf{r}$ and $d\dot{\mathbf{r}}$ in equations (15) and (16), $\ddot{\mathbf{r}}_1$ of eq. (21) and $\mathbf{B}$ and $\dot{\mathbf{B}}$ of eqs. (18) and (18a). (This is applicable to both single and multi-branched structures). Then, the dynamic analysis can be performed just as conveniently and economically.

3.5 APPLICATIONS

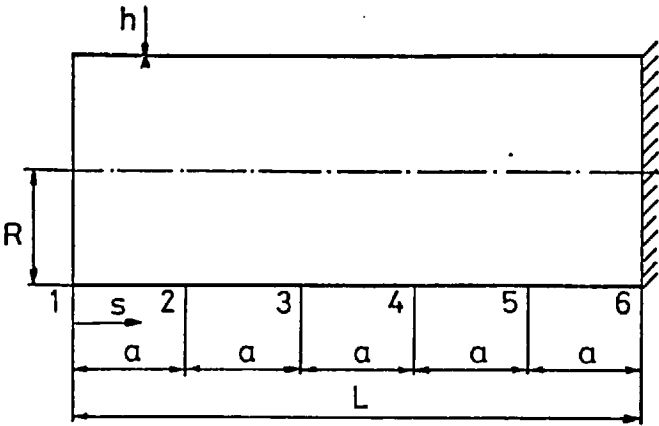
Two examples have been selected to illustrate the capabilities of this approach. The first example is the study of a blast loading on the cylindrical shell, the purpose of which is to present a comparison of current method with other methods of solution. The second example is the response of the hyperbolic cooling tower to an idealised earthquake loading. This problem shows not only the possibilities of this dynamic approach but also the advantage of the mixed method shell element. Unfortunately, no other result is available for comparison.

a) Blast Loading on Cylinder

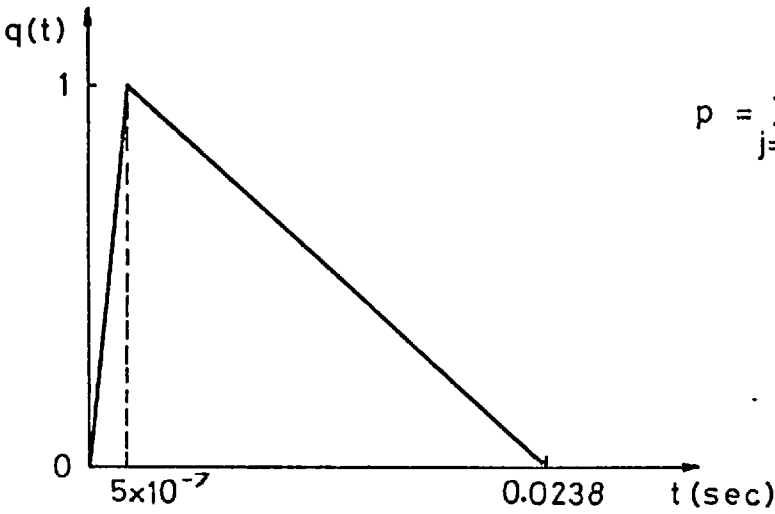
This problem was first reported in Ref. [30], by two numerical methods: an explicit and an implicit (Houbolt's) method. Stricklin [31] solved the same problem using Houbolt's method and a displacement method shell element. The cylindrical shell in this example (Fig.35) has geometric and material properties typical of those in missiles. The loading shown represents a blast loading and requires the use of the 0, 1 and 2 harmonics. The time history of the radial displacement harmonics at the free end is presented on Fig.36. The results obtained are identical to those of Refs. [30] and [31]. The time histories of the meridional moment M_ϕ at the built-in end of the cylinder are shown on Fig.37. It can be seen that the moment history is more irregular in shape than that of displacement. The high frequency wiggles in the moment curve are due to the finite element representation of the mass of the cylinder. Using the orthodox consistent mass matrix in the analysis, this phenomenon will always occur and the period of these wiggles will depend on the number of elements used in the analysis.

b) Earthquake Loading on the Cooling Tower

The cooling tower shown on Fig.29a, is used for this study. The earthquake loading is idealised as a sinusoidal radial acceleration in the first harmonic at the base of the tower. The amplitude of the acceleration is $g/2$ and the period is 0.1 sec. The time histories of the horizontal displacements are presented on Fig.38. As expected, the radial displacement at the base is in phase with the exciting acceleration, and is followed closely by the displacements at points B and C, which are near the base. The radial displacement at the throat (curve D) and at the top of the tower (curve E) are induced after approximately 0.04 sec which is about $2/5$ of the period of the exciting acceleration. These two curves (D and E) are not in phase with the base displacement (curve A) and their shapes are the result of wave action. The time histories of moment $M\phi$ are shown on Fig.39. Only the moment at the base is in phase with the acceleration. Curve B represents the moment at a point near the base (corresponds to the displacement curve B). The wiggles on both curves are caused by the finite mass and actually could be ignored, since only overall shape is meaningful.



$a = 4.5 \text{ in}$
 $L = 22.5 \text{ in}$
 $R = 9.2 \text{ in}$
 $h = 0.1 \text{ in}$
 $E = 10.5 \times 10^6 \text{ psi}$
 $\nu = 0.3$
 $\rho = 2.4 \times 10^{-4} \text{ lb-sec}^2/\text{in}^4$

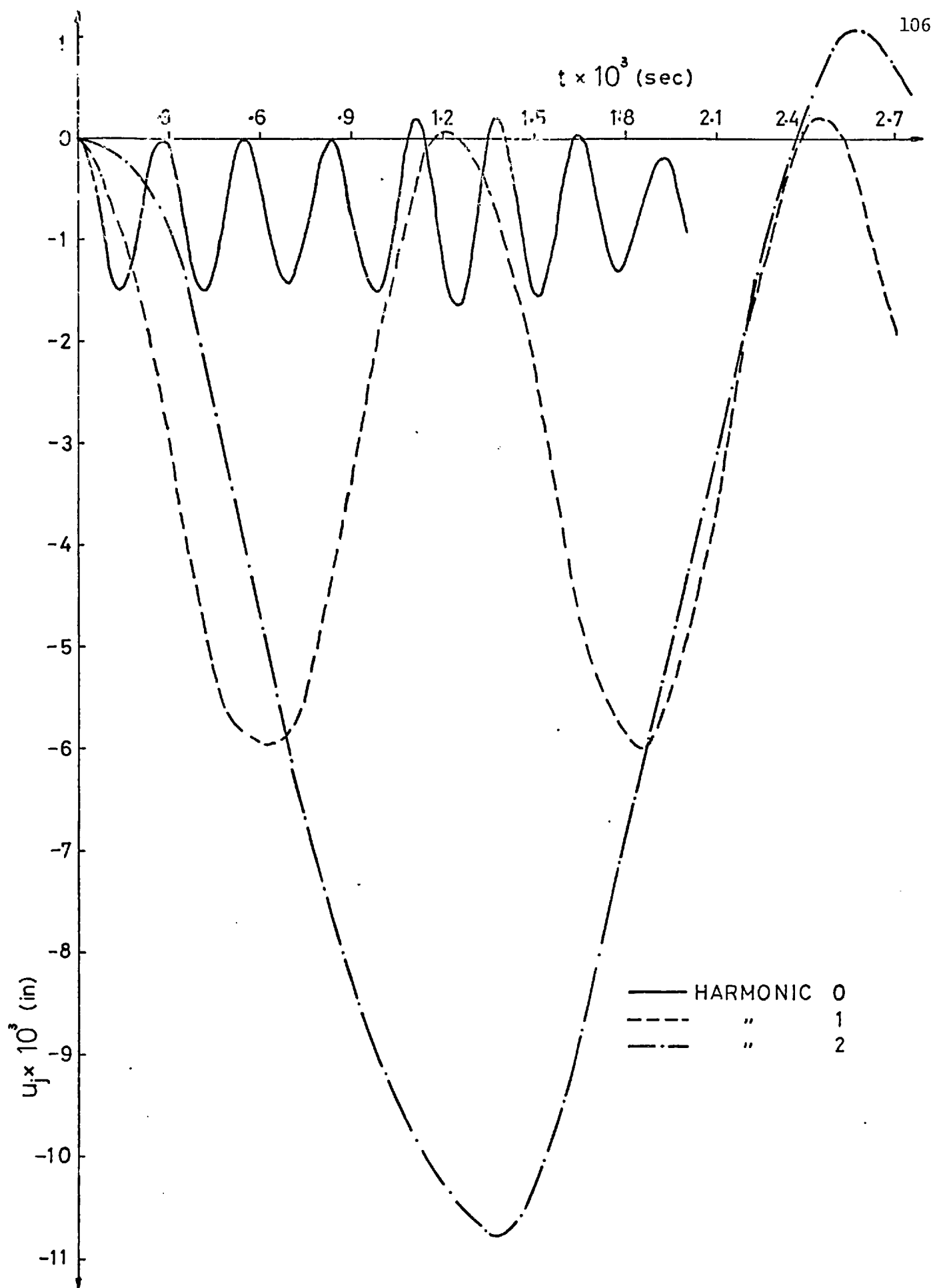


$$p = \sum_{j=0}^2 p_j(s) q(t) \cos j\theta$$

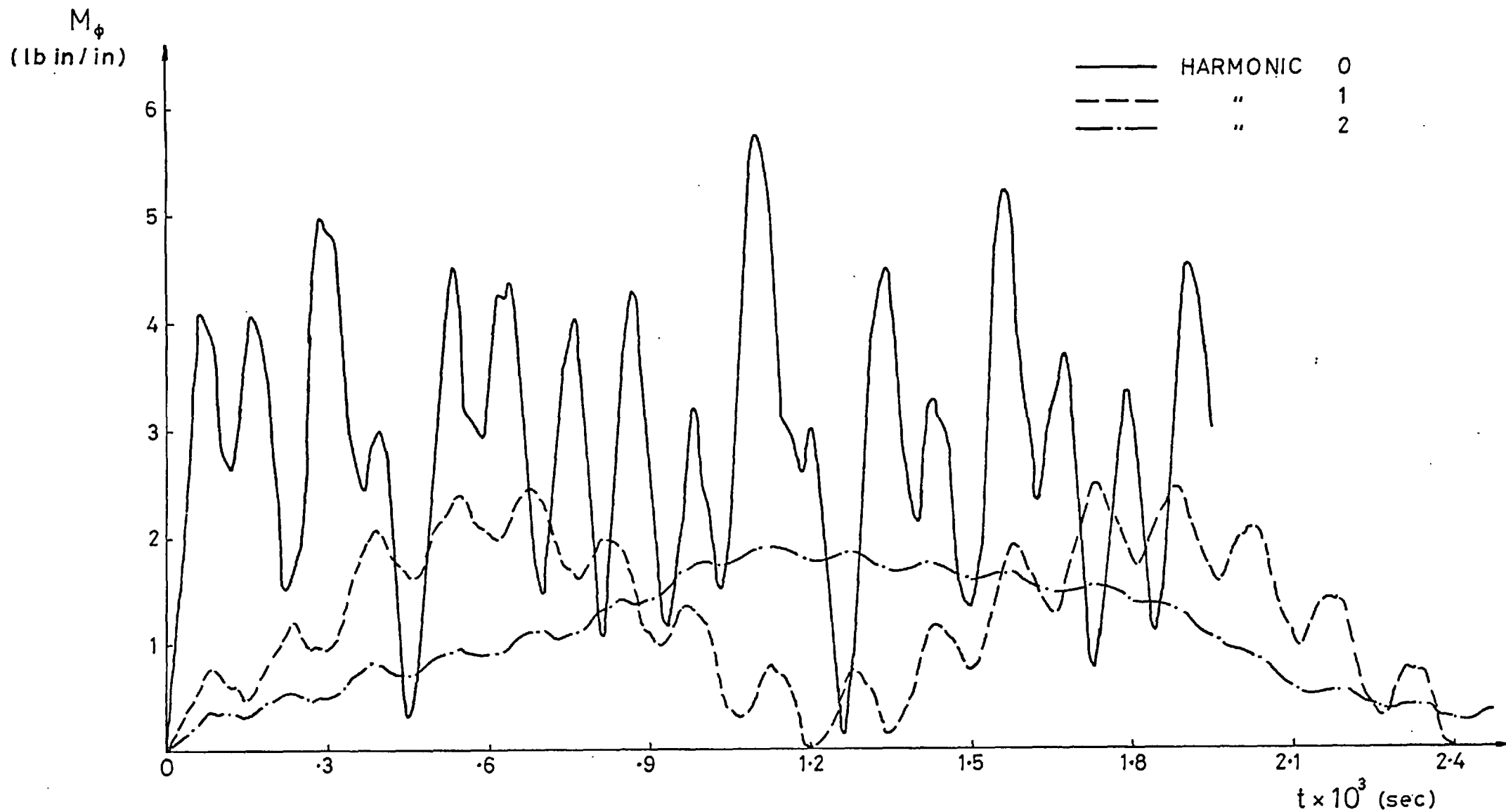
	$p_j(s) \quad [\text{psi}]$		
POINT	$j = 0$	$j = 1$	$j = 2$
1	9.378	4.529	0.528
2	8.769	3.149	0.594
3	8.160	1.770	0.660
4	7.890	1.440	0.780
5	7.767	1.320	0.777
6	8.448	1.947	0.798

BLAST LOADING ON A CLAMPED-FREE CYLINDER

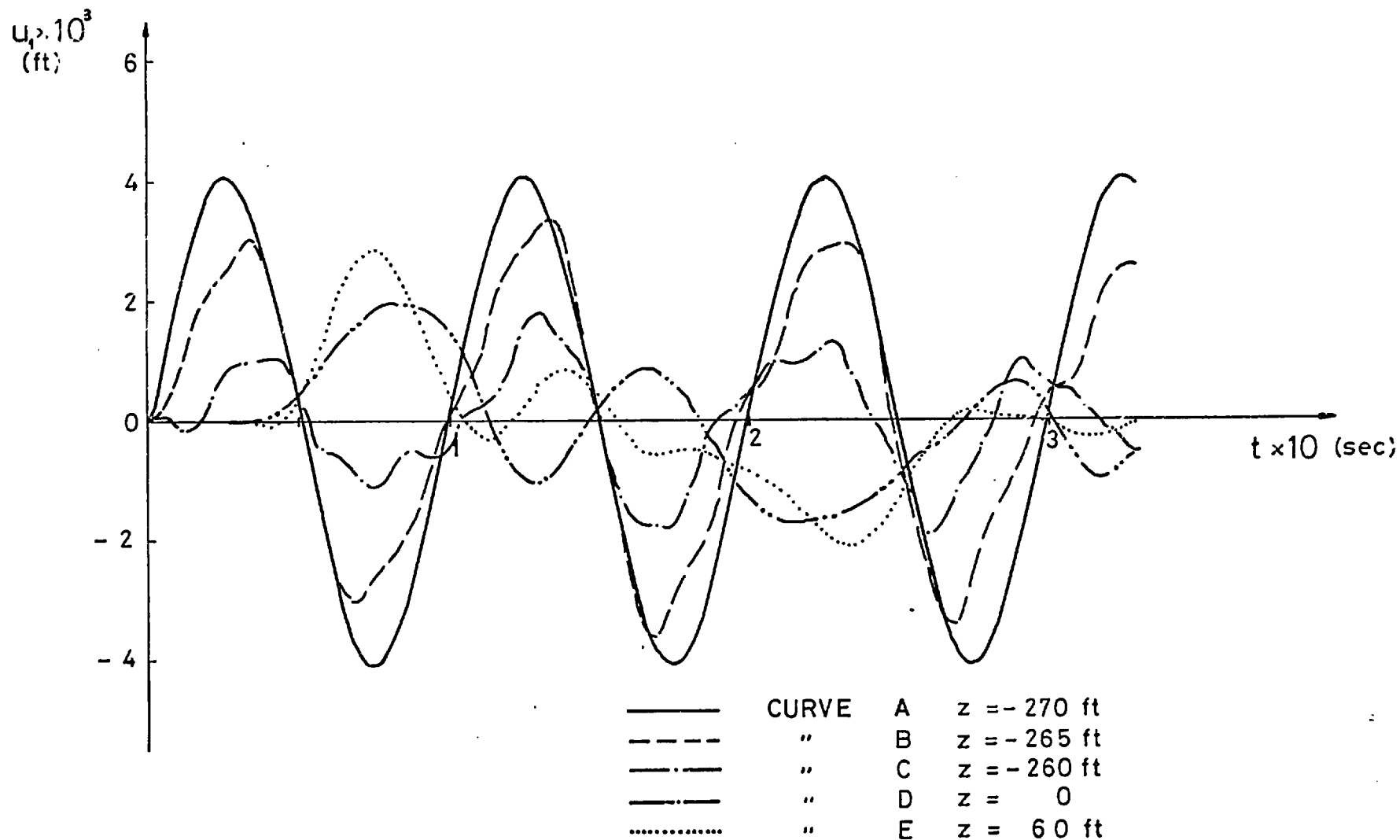
FIG. 35



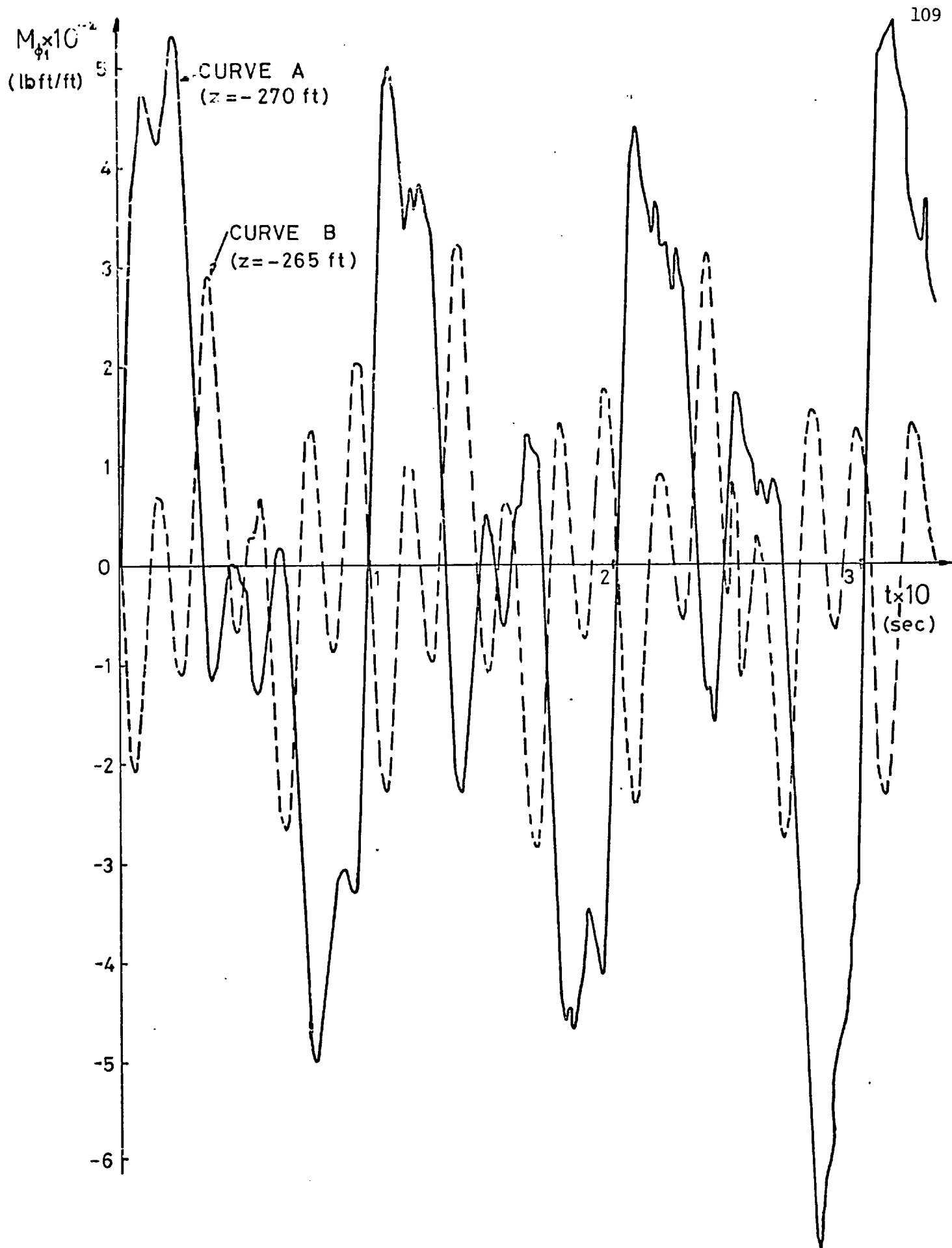
HARMONIC DISPLACEMENT AT FREE END



HARMONICS OF M_ϕ AT CLAMPED END



TOWER EXCITED BY EARTHQUAKE - DISPLACEMENTS



TOWER EXCITED BY EARTHQUAKE - MOMENTS

CONCLUSION

The derived mixed method shell element is ideally suitable for discontinuous and branched shell structures, but can also be applied with reliability to single-branched structures. The linear static analysis of single-branched shells by this method requires slightly more computing time than that by the SABA 5 element, since more elements are needed for convergence to an accurate solution, as shown in the example of the pressurised torus of Part I. The reason for this is possibly the inferior interpolation function used for the approximation of the shell geometry (i.e. the third order Hermitian polynomial against the fifth order of SABA 5). This disadvantage may be partly compensated by the greater accuracy of the results. The element is designed to give continuous bending moment which is confirmed in the results. In the case of branched shell structures, a comparison with the SABA element cannot be made, but the results are in close agreement with the analytical solution.

The results for the large displacement analysis of single-branched shell structures by this mixed method are comparable to those of Ref. [1] in all cases where the harmonic coupling is confined to two harmonics only (one of which is the zero harmonic). In the cases where there is coupling between various harmonics, the SABA element gave the lower values of the buckling load, as shown in the examples of Part II. This is, of course, due to the aforementioned oversight in the integration round the circumference of the geometric stiffness matrix in Ref. [1]. Otherwise, there is no reason to believe that the results would not be comparable. From the computing point of view, this formulation is faster than that of SABA 5, because it involves the numerical integration of smaller geometric stiffness matrix of order 12×12 against 18×18 of [1]. The saving in computing time is counterbalanced by the requirement for more elements.

The true advantage of this approach comes in the non-linear analysis of branched shells where SABA element does not give satisfactory results. The accurate stress distribution in the region of the point of discontinuity proved to be essential for the calculation of the correct geometric stiffness matrix in the incremental iterative analysis. From the evidence obtained in the examples, this mixed method of approach may confidently be applied to the static analysis of single or multi-branched thin axisymmetric shell structures. The number of harmonics used in the analysis should be kept to a reasonable number since the dimensions of the matrices involved increase as a multiple of the number of harmonics.

In the dynamic analysis by this mixed formulation, only displacement parameters are directly involved as functions of the time variable, resulting in only 6 parameters at each node against 12 by SABA 7 (which gives comparable accuracy) or 9 by SABA 5. This is achieved at the expense of one additional matrix multiplication (for the bending force) and one more initial matrix inversion (of the flexibility matrix). For large problems where computing time represents a limitation, this formulation may have considerable advantage.

This mixed method shell element has been shown to be a flexible and capable tool in the thin shell analysis, being proved to be more general than the existing elements, it may be a useful addition to the family of finite elements. The objectives in this work may be, therefore, considered to be fulfilled.

It may be interesting for future research to try to improve the convergence of this mixed method element. Most obviously the application of the higher order interpolation function for the representation of shell geometry might be worth investigating. It might be possible to achieve

the same convergence as that of the SABA 5 element by the use of the fifth order Hermitian polynomial for that purpose, in spite of the fact that, by doing so, the element might be over-conformed.

Another possibility for the general improvement of this element occurs in the examinations of the precise continuity requirement at the branch point. For most cases it is only necessary to satisfy the equality of rotations and equilibrium of meridional bending moments, in which case the present element is adequate. However, it may not be so for all cases. If complete static continuity is required, it might be more convenient to use a different set of nodal parameters. Suppose, not only bending moments are calculated by interpolating from the nodal values, but the membrane forces as well so

$$S = \beta b$$

then the compatibility equation must be extended to include all strains (i.e. to enforce the compatibility of membrane strains as well), but would still be of the form for the complete structure

$$F B - T r = 0$$

The P.V.D. would then give

$$R = T^t B$$

These two equations could be solved simultaneously in the form

$$\begin{bmatrix} R \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & T^t \\ T & -F \end{bmatrix} \begin{bmatrix} r \\ B \end{bmatrix}$$

It is now possible to enforce all equilibrium conditions at the branch point easily. This formulation, moreover, may be advantageous in cases where material non-linearity is required, but that is beyond the scope of this work.

APPENDIX

Two computer programmes finalise the theory presented in this work. The first is the programme for the large displacement incremental analysis of branched shell-structures. It can be used, of course, for the linear analysis as well, since the first loading step is linear. The second programme performs linear static and/or dynamic analysis of single shells. The flow charts of these programmes will be given and some interesting points discussed.

a) Large Displacement Analysis

The flow diagram for this programme is shown on Fig. A-1. The programme is organised in such a way that the incremental procedure can be restarted if so wished after one or more loading steps. The load increments can be set for each run after examining the previous results. In the case of unfortunate computer failure, the results from an analysis will not be completely lost. Before going into details of this programme some notations used should be explained.

NUMBR = Number of shell branches

NEL = Number of elements in a particular branch

NELMAX = Total number of elements

NUM 1 = Number of a loading step at which the incremental procedure starts

NUM 2 = Number of a loading step at which the incremental procedure ends for the current run.

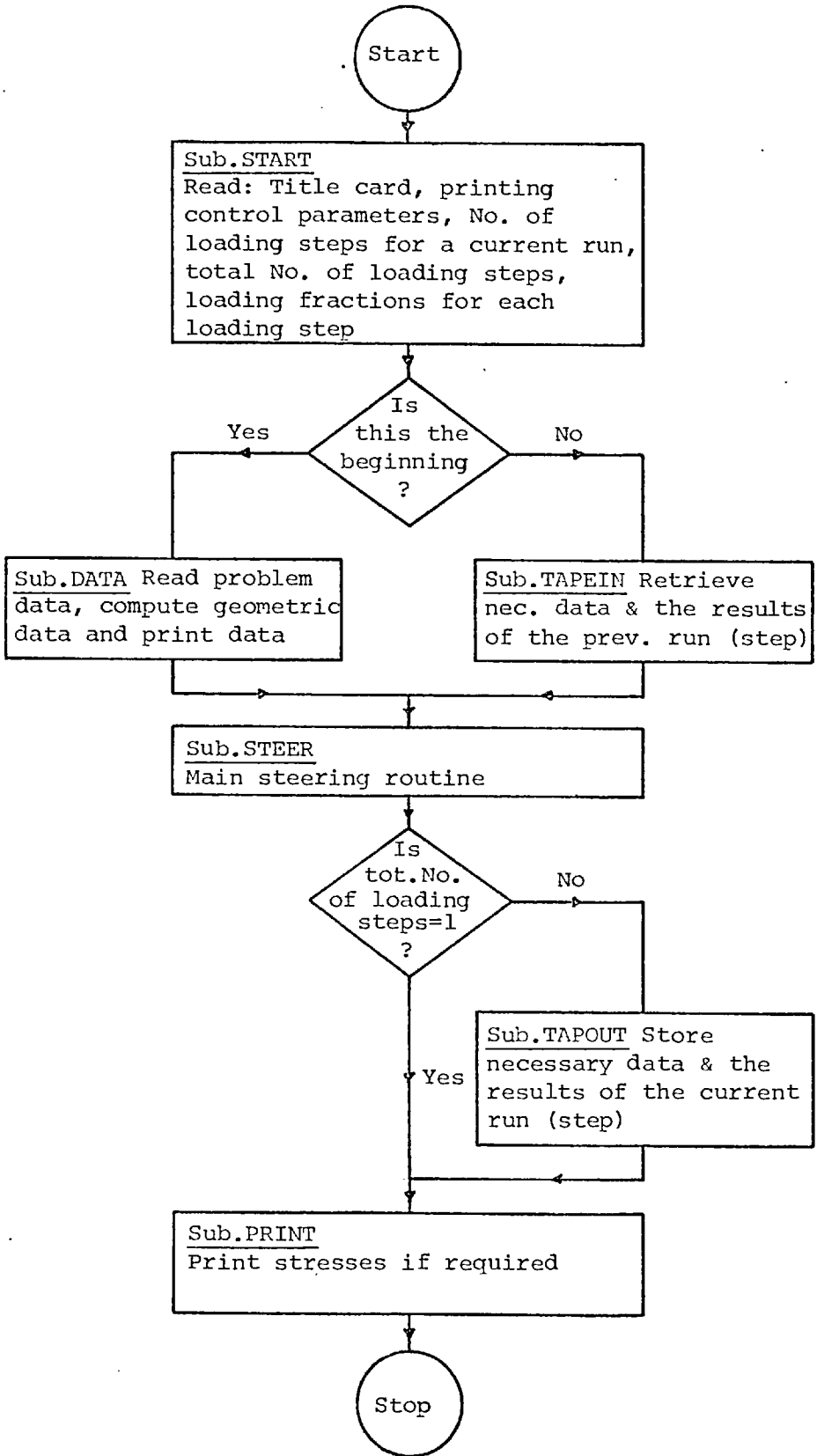
The main steering routine for the incremental analysis is presented on Fig. A-2. The routine is quite straightforward and follows closely the theory given in Part II. The "frontal" elimination routine is presented on Fig. A-3. Notice that the matrix k_F and vector Γ , for all harmonics, of an element are partitioned in the following way

$$k_F = \begin{bmatrix} k_A & k_{AB} \\ \text{Symm.} & k_B \end{bmatrix} \text{ and } F = \begin{bmatrix} P_A \\ P_B \end{bmatrix}.$$

where index A corresponds to node i , and B to node $i + 1$. Matrix k_S and vector P_S represent a working space. The backward substitution routine is shown on Fig. A-4. Notice that the matrix K_B^* and vector R_B^* for a branch point (see eq. I-102) are collected into k_A and P_A , respectively.

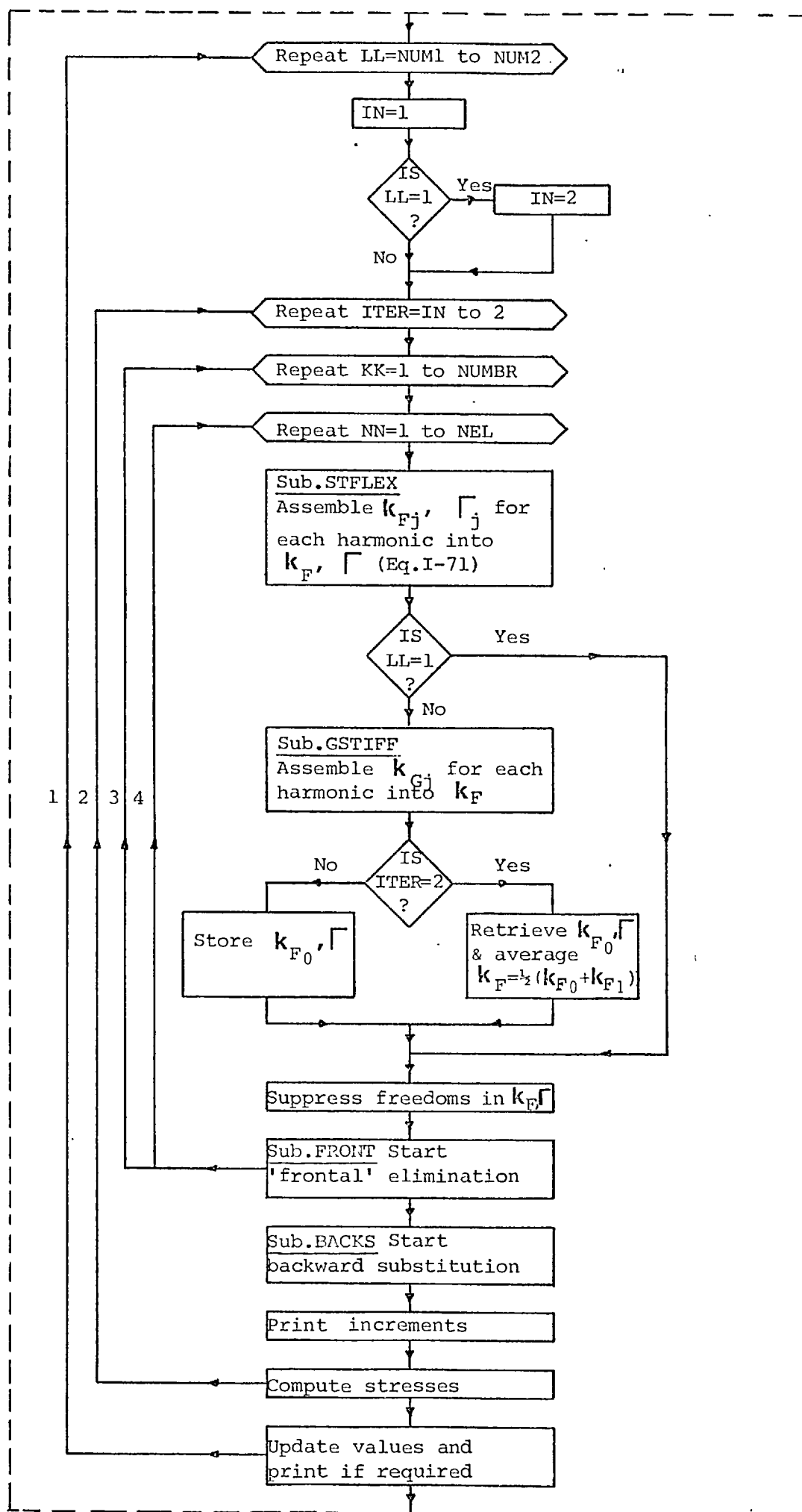
b) Linear Static and Dynamic Analysis

The flow diagram is presented on Fig. A-5. This programme in its present form can only be used for the analysis of single-branch shell structures. However, the theory presented in section 4. of Part III can easily be implemented. The programme can perform static, dynamic or static and dynamic analysis (NANAL = 1, 2 or 3, respectively). If the dynamic analysis is required it will assemble all necessary matrices (and invert M and F) and store all data onto a disc. The dynamic analysis starts then by reading in few parameters only: continuation parameter (NCONT = 1, otherwise at the very beginning NCONT = 0), parameters for a control of output, length of the time interval and time limit after which the programme stops automatically. The stresses can be printed after a chosen number of time steps, which is governed by NTEST (in that case equal 1, otherwise 0). The details of the routine for dynamic analysis are given on Fig. A-6. It starts by reading in the results of the previous analysis (and necessary data) and computes the dynamic response until a real computer time exceeds the specified time limit. All results are then stored for a next successive run.



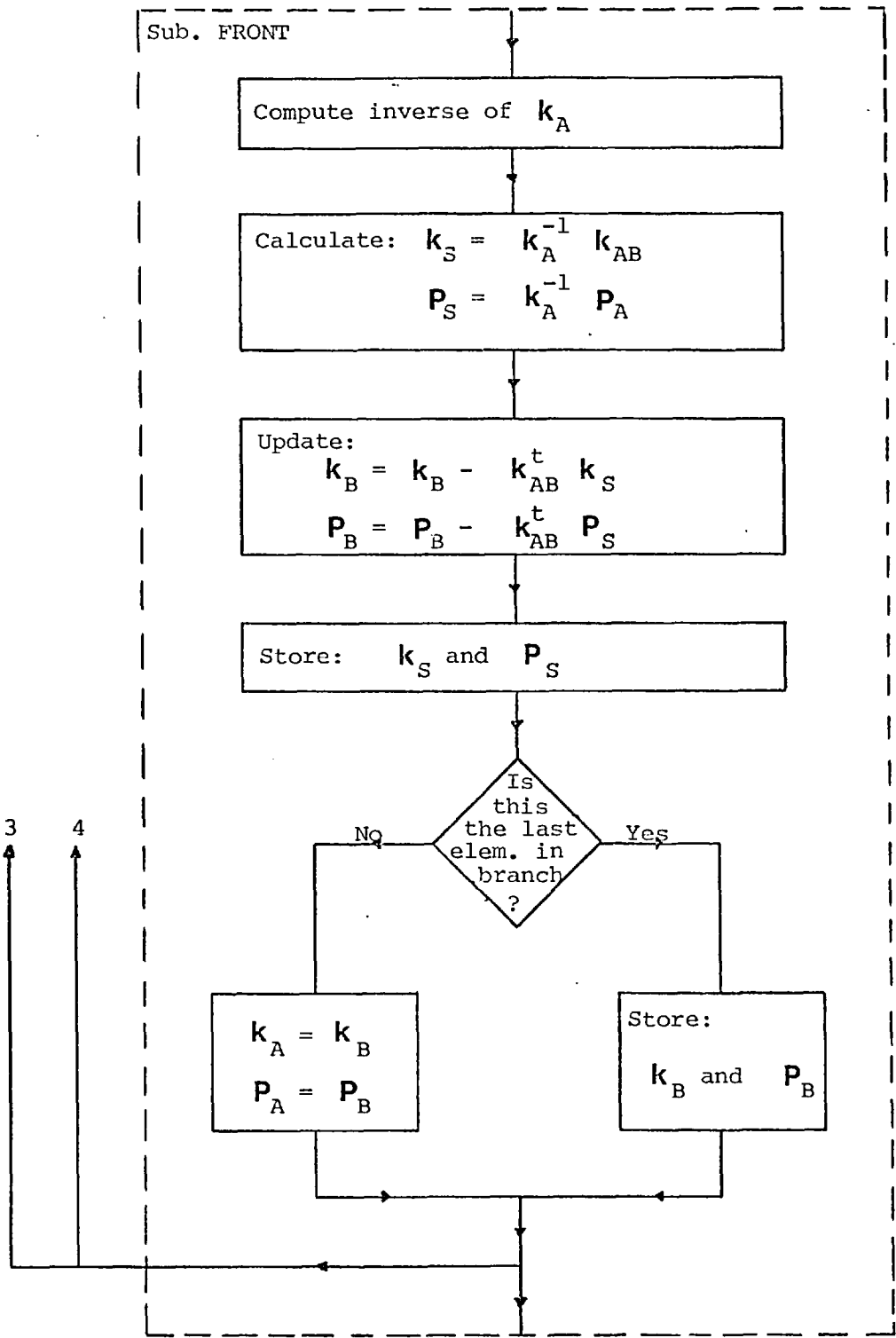
FLOW DIAGRAM FOR LARGE DISPLACEMENT ANALYSIS

FIG. A-1



MAIN STEERING ROUTINE

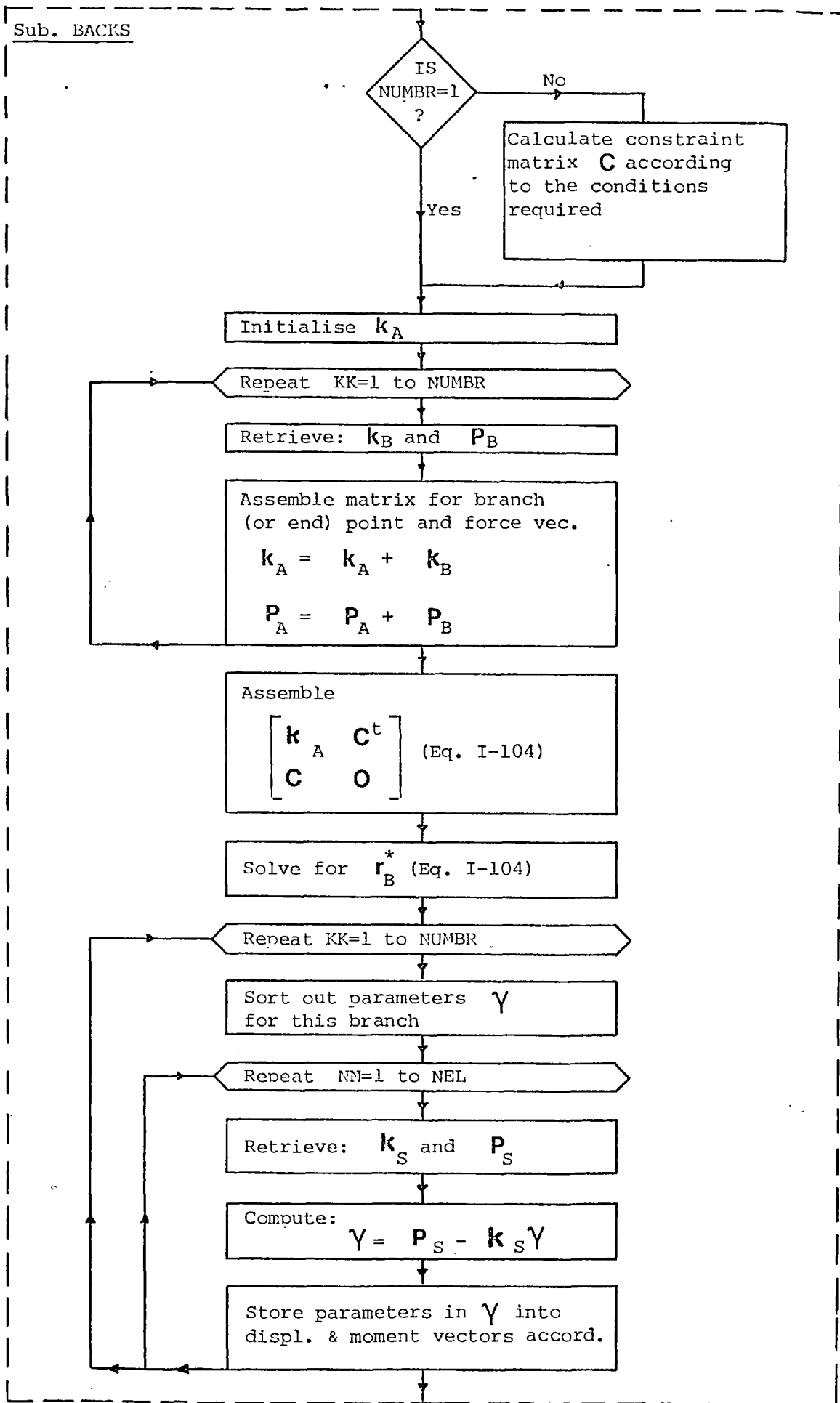
FIG. A-2



'FRONTAL' ELIMINATION ROUTINE

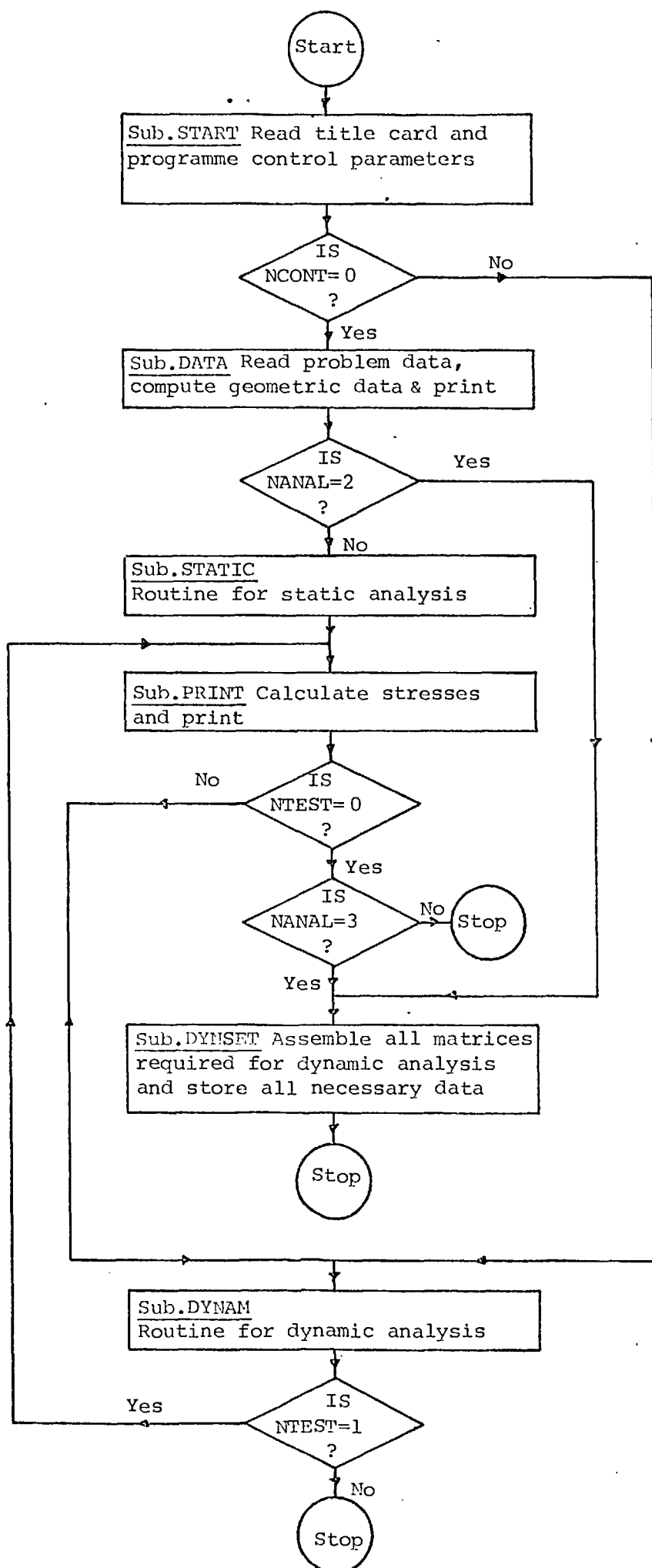
FIG. A-3

Sub. BACKS



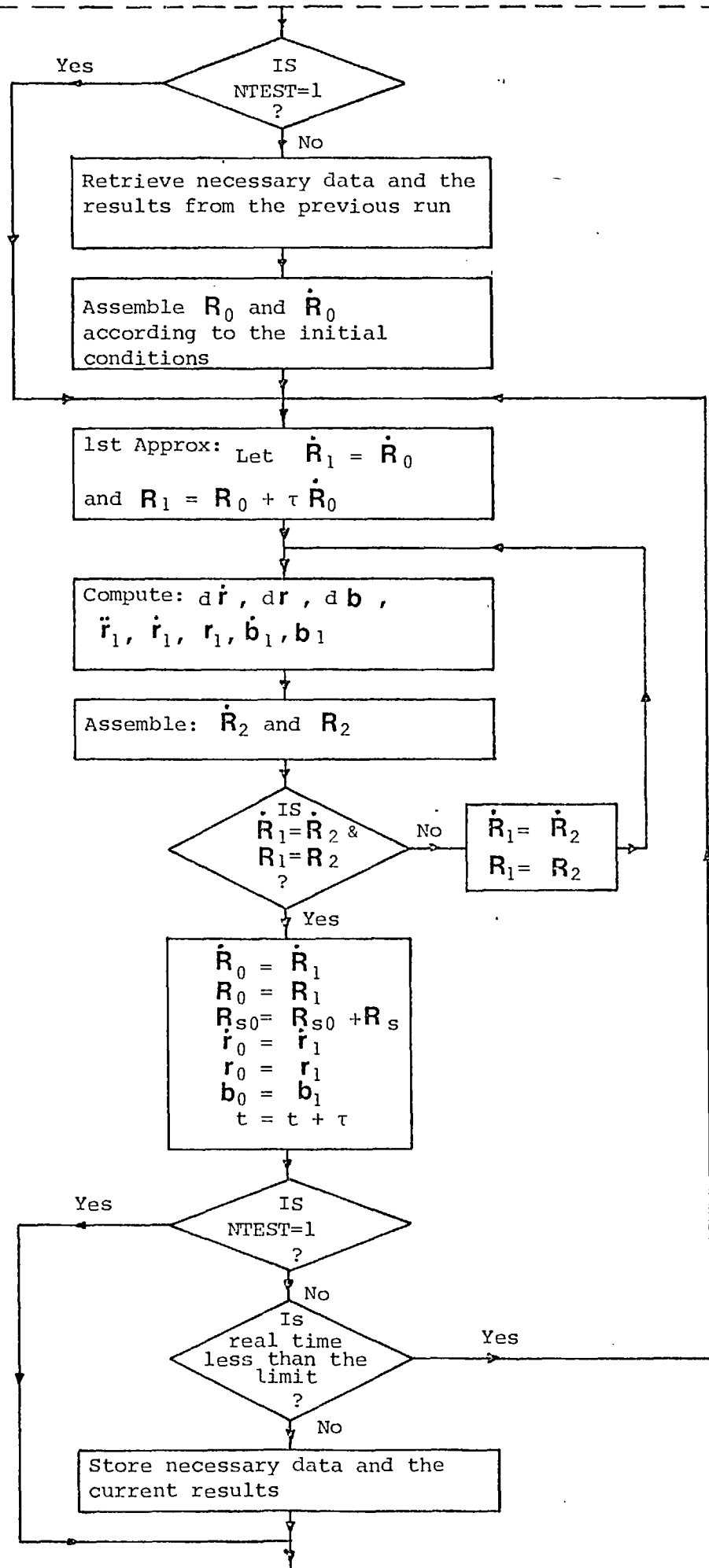
BACKWARD SUBSTITUTION ROUTINE

FIG. A-4



FLOW DIAGRAM FOR LINEAR STATIC AND DYNAMIC ANALYSIS

FIG.A-5



ROUTINE FOR DYNAMIC ANALYSIS

FIG. A-6

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