Isotropic wave turbulence with simplified kernels: Existence, uniqueness, and mean-field limit for a class of instantaneous coagulation-fragmentation processes

Sara Merino-Aceituno

Citation: Journal of Mathematical Physics **57**, 121501 (2016); doi: 10.1063/1.4968814
View online: http://dx.doi.org/10.1063/1.4968814
View Table of Contents: http://aip.scitation.org/toc/jmp/57/12
Published by the American Institute of Physics

Articles you may be interested in

One-species Vlasov-Poisson-Landau system for soft potentials in #3
Journal of Mathematical Physics **57**, 121502121502 (2016); 10.1063/1.4971193

Non-Markovian quantum feedback networks I: Quantum transmission lines, lossless bounded real property and limit Markovian channels
Journal of Mathematical Physics **57**, 122101122101 (2016); 10.1063/1.4968817

Exactly solvable Hermite, Laguerre, and Jacobi type quantum parametric oscillators
Journal of Mathematical Physics **57**, 122107122107 (2016); 10.1063/1.4972293

Electromagnetism in terms of quantum measurements
Journal of Mathematical Physics **57**, 122104122104 (2016); 10.1063/1.4972287



CrossMark click for updates

JOURNAL OF MATHEMATICAL PHYSICS **57**, 121501 (2016)

Isotropic wave turbulence with simplified kernels: Existence, uniqueness, and mean-field limit for a class of instantaneous coagulation-fragmentation processes

Sara Merino-Aceituno[a)]
Imperial College London, London, United Kingdom

(Received 19 January 2016; accepted 12 November 2016; published online 5 December 2016)

The isotropic 4-wave kinetic equation is considered in its weak formulation using model (simplified) homogeneous kernels. Existence and uniqueness of solutions is proven in a particular setting where the kernels have a rate of growth at most linear. We also consider finite stochastic particle systems undergoing instantaneous coagulation-fragmentation phenomena and give conditions in which this system approximates the solution of the equation (mean-field limit). © *2016 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution 3.0 Unported License.* [http://dx.doi.org/10.1063/1.4968814]

NOMENCLATURE

$\mathbb{R}_+$ = $[0,\infty)$;
$\mathcal{B}$ = space of bounded measurable functions with bounded support;
D = $\{(\omega_1,\omega_2,\omega_3) \in \mathbb{R}_+^3 \,|\, \omega_1+\omega_2 \geq \omega_3\}$;
$\mathbf{k}$ = wavevector, it belongs to $\mathbb{R}^N$;
$\omega(\mathbf{k})$ = dispersion relation;
$\overline{T}$ = $\overline{T}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k})$ interaction coefficient;
$\mathcal{P}(\mathbb{R}_+)$ = space of probability measures in $\mathbb{R}_+$;
$\mathcal{M}(\mathbb{R}_+)$ = set of finite measures on $\mathbb{R}_+$.

I. INTRODUCTION

Wave turbulence (Refs. 17, 18, 10, and 14, [entry turbulence]) describes weakly non-linear systems of dispersive waves. The present work focuses in the case of 4 interacting waves.

We start with a brief presentation of the general 4-wave kinetic equation[15] and move quickly to consider the isotropic case with simplified kernels, which is the object of study of the present work, and present the main results.

A. The 4-wave kinetic equation

Using in shorthand $n_i = n(\mathbf{k}_i,t)$, $n_k = n(\mathbf{k},t)$, $\omega_i = \omega(\mathbf{k}_i)$, and $\omega = \omega(\mathbf{k})$, the 4-wave kinetic equation is given by

$$\frac{d}{dt}n(\mathbf{k},t) = 4\pi \int_{\mathbb{R}^{3N}} \overline{T}^2(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k})(n_1n_2n_3 + n_1n_2n_k - n_1n_3n_k - n_2n_3n_k) \times \delta(\omega_1+\omega_2-\omega_3-\omega)\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k})d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3, \tag{1}$$

where $\mathbf{k} \in \mathbb{R}^N$ is called wavevector; the function $n = n(\mathbf{k},t)$ can be interpreted as the spectral density (in $\mathbf{k}$-space) of a wave field and it is called energy spectrum; $\omega(\mathbf{k})$ is the dispersion relation;

a)s.merino-aceituno@imperial.ac.uk

 © 2016 Author(s)

and

$$\overline{T}_{123k} := \overline{T}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k})$$

is the interaction coefficient,

$$E = \int_{\mathbb{R}^N} \omega(\mathbf{k})\, n(\mathbf{k}) d\mathbf{k}, \quad W = \int_{\mathbb{R}^N} n(\mathbf{k}) d\mathbf{k}$$

correspond to the total energy and the waveaction (total number of waves), respectively. These two quantities are conserved formally.

Properties of the dispersion relation and the interaction coefficient. $\omega(\mathbf{k})$ and T_{123k} are homogeneous, i.e., for some $\alpha > 0$ and $\beta \in \mathbb{R}$

$$\omega(\xi\mathbf{k}) = \xi^\alpha \omega(\mathbf{k}), \qquad \overline{T}(\xi\mathbf{k}_1, \xi\mathbf{k}_2, \xi\mathbf{k}_3, \xi\mathbf{k}) = \xi^\beta \overline{T}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}) \qquad \xi > 0.$$

Moreover the interaction coefficient possesses the following symmetries:

$$\overline{T}_{123k} = \overline{T}_{213k} = \overline{T}_{12k3} = \overline{T}_{3k12}.$$

Example: shallow water. In the case of shallow water, we deal with weakly nonlinear waves on the surface of an ideal fluid in an infinite basin of constant depth h small. In this case (Ref. 16), we have that $\alpha = 1$, $\beta = 2$, dimension is 2, and

$$T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}) = -\frac{1}{16\pi^2 h} \frac{1}{(k_1 k_2 k_3 k)^{1/2}} \left[(\mathbf{k}_1 \cdot \mathbf{k}_2)(\mathbf{k}_3 \cdot \mathbf{k}) + (\mathbf{k}_1 \cdot \mathbf{k}_3)(\mathbf{k}_2 \cdot \mathbf{k}) + (\mathbf{k}_1 \cdot \mathbf{k})(\mathbf{k}_2 \cdot \mathbf{k}_3) \right]. \tag{2}$$

In general T will be given by very complex expressions, see, for example, Ref. 18.

Resonant conditions and the δ distributions. The delta distributions appearing in Equation (1) correspond to the so-called resonant conditions

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k},$$
$$\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) = \omega(\mathbf{k}_3) + \omega(\mathbf{k}).$$

This imposes the conservation of energy and momentum in the wave interactions.

B. The *simplified* weak isotropic 4-wave kinetic equation

We focus our study on the weak formulation of the isotropic 4-wave kinetic equation defined against functions in $\mathcal{B}(\mathbb{R}^N)$, the set of bounded measurable functions with bounded support in $\mathbb{R}^N$.

More specifically, we assume that $n(\mathbf{k}) = n(k)$ is a radial function (isotropic). Then, using the relation $\omega(\mathbf{k}) = k^\alpha$, we study the evolution of the angle-averaged frequency spectrum $\mu = \mu(d\omega)$ which corresponds to

$$\mu(d\omega) := \frac{|S^{N-1}|}{\alpha} \omega^{\frac{N-\alpha}{\alpha}} n(\omega^{1/\alpha}) d\omega,$$

where S^{N-1} is the N dimensional sphere. The total number of waves (waveaction) and the total energy are now expressed, respectively, as

$$W = \int_0^\infty \mu(d\omega), \tag{3}$$

$$E = \int_0^\infty \omega \mu(d\omega). \tag{4}$$

The weak form of the isotropic equation is given formally by

$$\mu_t = \mu_0 + \int_0^t Q(\mu_s, \mu_s, \mu_s)\, ds, \tag{5}$$

where Q is defined against functions $f \in \mathcal{B}(\mathbb{R}_+)$ as

$$\langle f, Q(\mu,\mu,\mu)\rangle = \frac{1}{2}\int_D \mu(d\omega_1)\mu(d\omega_2)\mu(d\omega_3)K(\omega_1,\omega_2,\omega_3) \times [f(\omega_1+\omega_2-\omega_3)+f(\omega_3)-f(\omega_2)-f(\omega_1)],$$

where $D := \{\mathbb{R}^3_+ \cap (\omega_1+\omega_2 \geq \omega_3)\}$. See Appendix B for the formal derivation of this equation.

Formally $K = K(\omega_1,\omega_2,\omega_3)$ is written as

$$K(\omega_1,\omega_2,\omega_3) = \frac{8\pi}{\alpha|S^{N-1}|^4}(\omega_1+\omega_2-\omega_3)^{\frac{N-\alpha}{\alpha}} \int_{(S^{N-1})^4} d\mathbf{s}_1 d\mathbf{s}_2 d\mathbf{s}_3 d\mathbf{s}\, \overline{T}^2(\omega_1^{1/\alpha}\mathbf{s}_1, \omega_2^{1/\alpha}\mathbf{s}_2, \omega_3^{1/\alpha}\mathbf{s}_3, (\omega_1+\omega_2-\omega_3)^{1/\alpha}\mathbf{s}) \times \delta(\omega_1^{1/\alpha}\mathbf{s}_1+\omega_2^{1/\alpha}\mathbf{s}_2-\omega_3^{1/\alpha}\mathbf{s}_3-(\omega_1+\omega_2-\omega_3)^{1/\alpha}\mathbf{s}). \tag{6}$$

Notice that formally K is homogeneous of degree

$$\lambda := \frac{2\beta-\alpha}{\alpha}. \tag{7}$$

Our starting point is Equation (5) considering *simplified kernels* K. In this work, we do not study the relation between the interaction coefficient $\overline{T}$ and K. Specifically, we will consider the following type of kernels.

Definition 1.1. We say that K is a model kernel if

- $K : \mathbb{R}^3_+ \to \mathbb{R}_+$;
- *K is continuous in* $\mathbb{R}^3_+ = [0,\infty)^3$;
- *K is homogeneous of degree* λ;
- $K(\omega_1,\omega_2,\omega_3) = K(\omega_2,\omega_1,\omega_3)$ *for all* $(\omega_1,\omega_2,\omega_3) \in \mathbb{R}^3_+$.

Some examples of model kernels are

$$\begin{aligned} K(\omega_1,\omega_2,\omega_3) &= \frac{1}{2}\left(\omega_1^p\omega_2^q\omega_3^r + \omega_1^q\omega_2^p\omega_3^r\right) \quad \text{with } p+q+r=\lambda, \\ K(\omega_1,\omega_2,\omega_3) &= (\omega_1\omega_2\omega_3)^{\lambda/3}, \\ K(\omega_1,\omega_2,\omega_3) &= \frac{1}{3}(\omega_1^\lambda+\omega_2^\lambda+\omega_3^\lambda). \end{aligned} \tag{8}$$

The main question we want to address is

For which types of kernels K there is existence and uniqueness of solutions for equation (5) and, moreover, can this solution(s) be taken as the mean-field limit of a specific stochastic particle system?

The present work gives a positive answer for a particular class of kernels as explained in Sec. I, but first, for the motivation of the problem, we need to answer the two following questions:

(a) Why is it relevant to study the weak isotropic 4-wave kinetic equation with simplified kernels?

The present work is inspired on the article[2] from the physics literature on wave turbulence. In Ref. 2 the author works with the 3-wave kinetic equation and considers its isotropic version also assuming simplified kernels. The idea is that the 3-wave kinetic equation can be interpreted as a process where particles coagulate and fragment. This interpretation allows to use numerical methods coming from the theory of coagulation-fragmentation processes, which can be applied to this type of simplified kernels.

As in Ref. 2, ignoring the specific shape of the interaction coefficient $\overline{T}$ is not uncommon in the wave turbulence literature; in general the shape of $\overline{T}$ is too complex, too messy to extract information. Moreover, the most important feature in wave turbulence, the steady states called KZ-spectrum, depends only on the parameters α, β, and N. That is why in the physics literature $\overline{T}$ plays a secondary role, sometimes no role at all.

It is believed that only the asymptotic scaling properties of the kernel will affect the asymptotic behaviour of the solution. This is similar to what happens in the case of the Smoluchowski's coagulation equation, where homogeneous kernels give rise to self-similar solutions (scaling solutions) in some cases. The hypothesis that solutions become self-similar in the long run under the presence of an homogeneous kernel is called dynamical scaling hypothesis, see Ref. 9 for more on this. It would be, therefore, interesting to prove the dynamical scaling hypothesis, under some conditions, for the coagulation-fragmentation process presented here and to link it to the existence of self-similar solutions for the simplified isotropic 4-wave kinetic equations.

(b) Why consider the isotropic case?

There are examples in the physics literature where the phenomena are considered to behave isotropically (like in Langmuir waves for isotropic plasmas and shallow water with flat bottom).

The main reason though to consider the isotropic case is that it makes easier to get a mean-field limit from discrete stochastic particle systems. Suppose that we want to find a discrete particle system that approximates the dynamics of (1). For given waves with wavenumbers $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$, we want to see if they interact. On one hand, due to the resonance conditions $\mathbf{k}$ defined as

$$\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3$$

is uniquely determined. On the other hand, on top we must add the constraint

$$\omega = \omega_1 + \omega_2 - \omega_3$$

and this in general will not be satisfied. Therefore, if we consider systems with a finite number of particles, in general, interactions will not occur and the dynamics will be constant.

We go around this problem by considering the isotropic case. By assuming that $n = n(k)$ is a rotationally invariant function, we add the degree of freedom that we need.

1. Summary of results and methodology

Next we summarise the main results in the present work. To prove them we use the techniques presented in Refs. 11 and 12 for the Smoluchowski equation (coagulation model). It is remarkable that this methodology can be applied to the study of equations from wave turbulence.

Remark 1.2 (Strategy). The main observation is that we can adapt the methods given in Refs. 11 and *12 for coagulation phenomena. In the original proof by Norris in Ref. 11, sublinear functions* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *are used, i.e.,*

$$\varphi(\lambda x) \le \lambda \varphi(x), \quad \lambda \ge 1,$$
$$\varphi(x + y) \le \varphi(x) + \varphi(y).$$

These functions are the key to get bounds because of the following property: let $(\mu_t^n)_{t\ge 0}$ *be a stochastic coagulation process with n particles, if initially*

$$\langle \varphi, \mu_0^n \rangle \le \Lambda$$

for some $\Lambda < \infty$*, for all* $n \in \mathbb{N}$*, then*

$$\langle \varphi, \mu_t^n \rangle \le \Lambda \text{ for all } n, t.$$

Actually, what we obtain is that

$$\langle \varphi, \mu_t^n \rangle \le \langle \varphi, \mu_0^n \rangle$$

thanks to the sublinearity of φ*; say that two particles of masses* $x, y \in \mathbb{R}_+$ *coagulate creating a particle of mass* $x + y$*, then*

$$\varphi(x + y) \le \varphi(x) + \varphi(y) \tag{9}$$

by sublinearity.

In general, this idea to get bounds cannot be applied to the type of stochastic particle processes that we are going to consider because they also include fragmentation phenomena; we will have

that in an interaction, two particles of masses $\omega_1, \omega_2 \in \mathbb{R}_+$ disappear and two particles of masses $\omega_1 + \omega_2 - \omega_3, \omega_3 \in \mathbb{R}_+$ are created.

To get bounds on this stochastic process using the method above, we need an expression analogous to (9), i.e.,

$$\varphi(\omega_1 + \omega_2 - \omega_3) + \varphi(\omega_3) \leq \varphi(\omega_1) + \varphi(\omega_2).$$

Therefore we can use the Norris method with the appropriate adaptations for the particular case where $\varphi(\omega) = \omega + c$ for a constant c, which we will take to be one.

Definition 1.3. Consider $\varphi(\omega) = \omega + 1$. We say that a kernel K is sub-multiplicative if

$$K(\omega_1, \omega_2, \omega_3) \leq \varphi(\omega_1)\varphi(\omega_2)\varphi(\omega_3). \tag{10}$$

a. Existence and uniqueness of solutions.

Definition 1.4 (Solution and types of solutions). We will say that $(\mu_t)_{t<T}$ is a local solution if it satisfies (5) for all bounded measurable functions f of bounded support and such that $\langle \omega, \mu_t \rangle \leq \langle \omega, \mu_0 \rangle$ for all $t < T$. If $T = +\infty$, then we have a solution. If moreover,

$$\int_0^\infty \omega \mu_t(d\omega)$$

is finite and constant, then we say that $(\mu_t)_{t<T}$ is conservative.

We call any local solution $(\mu_t)_{t<T}$ such that

$$\int_0^t \langle \varphi^2, \mu_s \rangle \, ds < \infty \quad \text{for all } t < T$$

a strong solution.

Theorem 1.5 (Existence and uniqueness of solutions). *Consider Equation (5) and a given μ_0 measure in $\mathbb{R}_+$. Define $\varphi(\omega) = \omega + 1$ and assume that K is submultiplicative model kernel. Assume further that $\langle \varphi, \mu_0 \rangle < \infty$ (i.e., initially the total number of waves (3) and the total energy (4) are finite). Then, if $(\mu_t)_{t<T}$ and $(\nu_t)_{t<T}$ are local solutions, starting from μ_0, and if $(\nu_t)_{t<T}$ is strong, then $\mu_t = \nu_t$ for all $t < T$. Moreover, any strong solution is conservative.*

Also, if $\langle \varphi^2, \mu_0 \rangle < \infty$, then there exists a unique maximal strong solution $(\mu_t)_{t<\zeta(\mu_0)}$ with $\zeta(\mu_0) = \langle \varphi^2, \mu_0 \rangle^{-1} \langle \varphi, \mu_0 \rangle^{-1}$.

The proof of this theorem will be an adaptation of Ref. 11 [Theorem 2.1].

Remark 1.6. In Ref. 2 the concept of finite capacity cascade in wave turbulence is shown to correspond to the appearance of gelation in coagulation-fragmentation phenomena. When gelation takes place, the total mass is not conserved. This corresponds to the formation of an "infinite" size cluster in finite time (see Ref. 11). In the theorem above, we prove that strong solutions are unique and we give conditions for its existence locally in time. When a strong solution exists, the total mass is conserved and gelation (finite capacity cascade) cannot take place. Therefore, gelation (finite capacity cascade) can only take place for weak solutions that are not strong. For this case, though, we cannot guarantee existence nor uniqueness of the weak solutions.

b. Mean-field limit (coagulation-fragmentation phenomena). We will consider a system of stochastic particles undergoing coagulation-fragmentation phenomena. The basic idea is that three particles $(\omega_1, \omega_2, \omega_3)$ with $\omega_1 + \omega_2 \geq \omega_3$ will interact at a given rate $K(\omega_1, \omega_2, \omega_3)$. In the interaction, first ω_1 and ω_2 coagulate to form $\omega_1 + \omega_2$ and then under the presence of ω_3 the coagulant splits into two other components which are $\omega_1 + \omega_2 - \omega_3$ and a new ω_3 (fragmentation). So interactions are

$$[\omega_1, \omega_2, \omega_3] \mapsto [\omega_1 + \omega_2 - \omega_3, \omega_3, \omega_3].$$

Note that we assume K is symmetric in the first two variables because in the interactions the role of ω_1 and ω_2 is symmetric.

We will define and build for each $n \geq 1$, $(X_t^n)_{t \geq 0}$ a instantaneous coagulation-fragmentation stochastic particle system of n particles (Section III A) following the previous ideas. The reason

for considering this particular stochastic particle system is because of its relation with the isotropic 4-wave kinetic equation. Particularly, we show that the dynamics of the particle system approximate the dynamics of the isotropic 4-wave kinetic equation (in a sense to be given below). This is of use, first, to have an interpretation of the dynamics of the isotropic 4-wave kinetic Equation (5); and second, these dynamics could be simulated numerically by using the particle system, similarly as done in Ref. 2.

We present here two mean-field limits each of them requiring a different set of assumptions.

Theorem 1.7 (First mean-field limit). *Assume that for* $\tilde{\varphi}(\omega) = \omega^{1-\gamma}$, $\gamma \in (0,1)$ *it holds that* K *is a model kernel with*

$$K(\omega_1,\omega_2,\omega_3) \le \tilde{\varphi}(\omega_1)\tilde{\varphi}(\omega_2)\tilde{\varphi}(\omega_3).$$

Assume also that $\langle \omega, X_0^n \rangle$ *is bounded uniformly in* n *by* $\langle \omega, \mu_0 \rangle < \infty$ *and*

$$X_0^n \to \mu_0 \quad \textit{weakly.}$$

Then the sequence of laws $(X_t^n)_{n\in\mathbb{N}}$ *is tight in the Skorokhod topology. Moreover, under any weak limit law,* $(\mu_t)_{t\ge 0}$ *is almost surely a solution of Equation (5). In particular, this equation has at least one solution.*

The proof of this theorem will be an adaptation of Ref. 11 [Theorem 4.1].

Denote by d some metric on $\mathcal{M}$, the set of finite measures on $\mathbb{R}_+$, which is compatible with the topology of weak convergence, i.e.,

$$d(\mu_n,\mu) \to 0 \quad \text{if and only if} \quad \langle f,\mu_n\rangle \to \langle f,\mu\rangle \tag{11}$$

for all bounded continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$. We choose d so that $d(\mu,\mu') \le \|\mu - \mu'\|$ for all $\mu,\mu' \in \mathcal{M}$.

Theorem 1.8 (Second mean-field limit). *Let* K *be a model kernel and let* μ_0 *be a measure on* $\mathbb{R}_+$. *Assume that for* $\varphi(\omega) = \omega + 1$ *it holds*

$$K(\omega_1,\omega_2,\omega_3) \le \varphi(\omega_1)\varphi(\omega_2)\varphi(\omega_3)$$

and that $\langle \varphi, \mu_0\rangle < \infty$ *and* $\langle \varphi^2, \mu_0\rangle < \infty$. *Denote by* $(\mu_t)_{t<T}$ *the maximal strong solution to (5) provided by Theorem 1.5. Let* $(X_t^n)_{n\in\mathbb{N}}$ *be a sequence of instantaneous coagulation-fragmentation particle system, with jump kernel* K. *Suppose that*

$$d(\varphi X_0^n, \varphi\mu_0) \to 0$$

as $n \to \infty$. *Then, for all* $t < T$,

$$\sup_{s\le t} d(\varphi X_s^n, \varphi\mu_s) \to 0$$

in probability, as $n \to \infty$.

The proof of this theorem will be an adaptation of Ref. 11 [Theorem 4.4].

Many mathematical works have been devoted to the study of the coagulation-fragmentation equation. We base our work on Refs. 11 and 12 but the reader is also referred to Refs. 4, 5, 8, and 15, as an example.

c. Applications. For the physical applications we consider K given by expression (8), i.e., $K(\omega_1,\omega_2,\omega_3) = (\omega_1\omega_2\omega_3)^{\lambda/3}$, which is submultiplicative (since $\omega^\lambda \le \omega + 1$, $\lambda \in [0,1]$).

If $\lambda \in [0,3)$, then we can apply all the previous theorems. For the case $\lambda = 3$, the theorems also apply with the exception of the first mean-field limit, Theorem 1.7.

Here are some examples:

- *Langmuir waves in isotropic plasmas and spin waves*: $\beta = 2$, $\alpha = 2$, so $\lambda = 1$ (the dimension is $N = 3$).
- *Shallow water* (isotropic in a flat bottom[16]): $\beta = 2$, $\alpha = 1$, so $\lambda = 3$ (dimension $N = 2$).
- *Waves on elastic plates*: $\beta = 3$, $\alpha = 2$, so $\lambda = 2$ (dimension $N = 2$).

However, these results cannot be applied to other systems like gravity waves on deep water, nonlinear optics, and Bose-Einstein condensates.

C. Relevance of these results

First, as mentioned before, we observe that the techniques in Refs. 11 and 12 for the coagulation equation can be applied to kinetic models in wave turbulence. Thanks to this, in this work we have dealt with the weak isotropic 4-wave kinetic equation with simplified kernels. When the kernels are at most linear, we have given conditions for the local existence and uniqueness of solutions. We have also derived the equation as a mean-field limit of interacting particle system given by a simultaneous coagulation-fragmentation: three particles interact with a coagulation-fragmentation phenomenon where one of the particles seems to act as a catalyst.

Second, using the interacting particle system, numerical methods could be devised to simulate the solution of the equation (as done by Ref. 2 for the 3-wave kinetic equation), by adapting the methods in Ref. 4.

Moreover, these numerical simulations would allow the study of self-similar solutions and steady state solutions. Particularly, it could be used to study some predictions made in wave turbulence, like the Kolmogorov-Zakharov spectra.

Finally, this theory can be applied to physical scenarios that include Langmuir waves, shallow water, and waves on elastic plates.

II. EXISTENCE OF SOLUTIONS FOR UNBOUNDED KERNEL

In this section we will follow the steps in Ref. 11 [Theorem 2.1] (see Remark 1.2).

A. Proof of theorem 1.5

The rest of this section will consist on the proof of this theorem, which we will split in different propositions. We will follow the idea and structure as in Ref. 11 [Theorem 2.1]. We want to apply a classical iterative scheme on Eq. (5) to prove the existence of solutions and for that we need estimates on Q of the type $\|Q(\mu)\| \le c\|\mu\|$ and $\|Q(\mu) - Q(\mu')\| \le c\|\mu - \mu'\|$. However, we do not have these estimates because the kernel is unbounded. To sort out this problem, we will consider a bounded domain $B \subset \mathbb{R}_+$ and an auxiliary process $(\mu_t^B, \lambda_t^B)_{t\ge 0}$, where μ_t^B gives a lower bound for μ_t, solution of Eq. (5), in the subset B and λ_t^B gives an upper estimate of the effect on μ_t^B on the particles outside B. For more details, see the particle system described in Sec. III C 1 which is built following this idea. The reason for using this auxiliary process is that the kernel restricted to a bounded set B becomes bounded, and so, we have the necessary estimates (Eqs. (15) and (16)) to perform an iterative scheme argument on the auxiliary process; this is the goal of Prop. 2.1 below. The rest of the proof of Th. 1.5 consists on making $B \uparrow \mathbb{R}_+$ and on comparing the auxiliary process with the process $(\mu_t)_{t\ge 0}$, the solution of Eq. (5).

Let $B \subset [0,\infty)$ be bounded. Denote by $\mathcal{M}_B$ the space of finite signed measures supported on B. We define $L^B : \mathcal{M}_B \times \mathbb{R} \to \mathcal{M}_B \times \mathbb{R}$ by the requirement

$$\begin{aligned}\langle (f,a), L^B(\mu,\lambda)\rangle = \frac{1}{2}\int_D (f(\omega_1+\omega_2-\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\in B} + a\varphi(\omega_1+\omega_2-\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\notin B}\\ + f(\omega_3) - f(\omega_1) - f(\omega_2))K(\omega_1,\omega_2,\omega_3)\mu(d\omega_1)\mu(d\omega_2)\mu(d\omega_3)\\ + (\lambda^2 + 2\lambda\langle\varphi,\mu\rangle)\int_0^\infty (a\varphi(\omega) - f(\omega))\,\varphi(\omega)\mu(d\omega)\end{aligned}$$

for all bounded measurable functions f on $(0,\infty)$ and all $a \in \mathbb{R}$ where $D = \{\mathbb{R}_+^3 \cap \omega_1 + \omega_2 - \omega_3 \ge 0\}$. We used the notation $\langle (f,a),(\mu,\lambda)\rangle = \langle f,\mu\rangle + a\lambda$.

Consider the equation

$$(\mu_t,\lambda_t) = (\mu_0,\lambda_0) + \int_0^t L^B(\mu_s,\lambda_s)\,ds. \tag{12}$$

We admit as a *local solution* any continuous map

$$t \mapsto (\mu_t, \lambda_t) : [0,T] \to \mathcal{M}_B \times \mathbb{R},$$

where $T \in (0,\infty)$, which satisfies Equation (12) for all $t \in [0,T]$.

Proposition 2.1 (Existence for the auxiliary process). Suppose $\mu_0 \in \mathcal{M}_B$ with $\mu_0 \geq 0$ and that $\lambda_0 \in [0,\infty)$. Equation (12) has a unique solution $(\mu_t, \lambda_t)_{t\geq 0}$ starting from (μ_0, λ_0). Moreover, $\mu_t \geq 0$ and $\lambda_t \geq 0$ for all t.

The proof is obtained by adapting the one in Ref. 11 [Proposition 2.2].

Proof. By assumption (10) it holds that for $\varphi(\omega) = \omega + 1$

$$K(\omega_1,\omega_2,\omega_3) \leq \varphi(\omega_1)\varphi(\omega_2)\varphi(\omega_3).$$

Observe that $\varphi \geq 1$. By a scaling argument we may assume, without loss, that

$$\langle \varphi, \mu_0 \rangle + \lambda_0 \leq 1,$$

which implies that

$$\|\mu_0\| + |\lambda_0| \leq 1.$$

We will show next by a standard iterative scheme that there is a constant $T > 0$ depending only on φ and B, and a unique local solution $(\mu_t, \lambda_t)_{t\leq T}$ starting from (μ_0, λ_0). Then we will see that $\mu_t \geq 0$ for all $t \in [0,T]$.

This will be enough to prove the proposition: if we put $f = 0$ and $a = 1$ in (12), we get

$$\begin{aligned}\frac{d}{dt}\lambda_t = \frac{1}{2}\int_D \varphi(\omega_1+\omega_2-\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\notin B}K(\omega_1,\omega_2,\omega_3)\mu(d\omega_1)\mu(d\omega_2)\mu(d\omega_3)\\ + (\lambda^2 + 2\lambda\langle\varphi,\mu\rangle)\int_0^\infty \varphi(\omega)^2\mu(d\omega). \qquad (13)\end{aligned}$$

So, since $\mu_t \geq 0$, we deduce that $\lambda_t \geq 0$ for all $t \in [0,T]$. Next, we put $f = \varphi$ and $a = 1$ to see that

$$\frac{d}{dt}\langle \varphi, \mu_t\rangle + \lambda_t = 0. \qquad (14)$$

Therefore,

$$\|\mu_T\| + |\lambda_T| \leq \langle \varphi, \mu_T\rangle + \lambda_T = \langle \varphi, \mu_0\rangle + \lambda_0 \leq 1.$$

We can now start again from (μ_T, λ_T) at time T to extend the solution to $[0,2T]$, and so on, to prove the proposition.

We use the following norm on $\mathcal{M}_B \times \mathbb{R}$:

$$\|(\mu,\lambda)\| = \|\mu\| + |\lambda|.$$

Note the following estimates: there is a constant $C = C(\varphi, B) < \infty$ such that for all $\mu, \mu' \in \mathcal{M}_B$ and all $\lambda, \lambda' \in \mathbb{R}$

$$\|L^B(\mu,\lambda)\| \leq C\|(\mu,\lambda)\|^3, \qquad (15)$$

$$\begin{aligned}\|L^B(\mu,\lambda) - L^B(\mu',\lambda')\| \leq C\Big(&\|\mu-\mu'\|\left(\|\mu\|^2 + \|\mu\|\|\mu'\| + \|\mu'\|^2\right) \qquad (16)\\ &+ (|\lambda| + |\lambda'|)|\lambda-\lambda'|\|\mu\| + |\lambda'|^2\|\mu-\mu'\|\\ &+ |\lambda-\lambda'|\|\mu\|^2 + |\lambda'|\left(\|\mu\|\|\mu-\mu'\| + \|\mu'\|\|\mu'-\mu\|\right)\Big).\end{aligned}$$

Observe that we get these estimates because we are working on a bounded set B.

We turn to the iterative scheme. Set $(\mu_t^0, \lambda_t^0) = (\mu_0, \lambda_0)$ for all t and define inductively a sequence of continuous maps

$$t \mapsto (\mu_t^n, \lambda_t^n) : [0,\infty) \to \mathcal{M}_B \times \mathbb{R}$$

by

$$(\mu_t^{n+1}, \lambda_t^{n+1}) = (\mu_0, \lambda_0) + \int_0^t L^B(\mu_s^n, \lambda_s^n)\, ds.$$

Set

$$f_n(t) = \|(\mu_t^n, \lambda_t^n)\|$$

then $f_0(t) = f_n(0) = \|(\mu_0, \lambda_0)\| \leq 1$ and by estimate (15) we have that

$$f_{n+1}(t) \leq 1 + C \int_0^t f_n(s)^3\, ds.$$

Hence

$$f_n(t) \leq (1 - 2Ct)^{-1/2} \quad \text{for } t < (2C)^{-1}.$$

Therefore, for all n setting $T = (4C)^{-1}$, we have

$$\|(\mu_t^n, \lambda_t^n)\| \leq \sqrt{2} \qquad t \leq T. \tag{17}$$

Next set $g_0(t) = f_0(t)$ and for $n \geq 1$

$$g_n(t) = \|(\mu_t^n, \lambda_t^n) - (\mu_t^{n-1}, \lambda_t^{n-1})\|.$$

By estimates (16) and (17), there is a constant $C = C(B, \varphi) < \infty$ such that

$$g_{n+1}(t) \leq C \int_0^t g_n(s)\, ds \quad t \leq T.$$

Hence by the usual arguments (Gronwall, Cauchy sequence), (μ_t^n, λ_t^n) converges in $\mathcal{M}_B \times \mathbb{R}$ uniformly in $t \leq T$, to the desired local solution, which is also unique. Moreover, for some constant $C < \infty$ depending only on φ and B, we have

$$\|(\mu_t, \lambda_t)\| \leq C \qquad t \leq T.$$

Finally, we are left to check that $\mu_t \geq 0$. For $t \leq T$, set

$$\theta_t(\omega_1) = \exp \int_0^t \left(\int_{\mathbb{R}_+^2 \cap (\omega_1 + \omega_2 \geq \omega_3)} K(\omega_1, \omega_2, \omega_3) \mu_s(d\omega_2) \mu_s(d\omega_3) + \left(\lambda_s^2 + 2\lambda_s \langle \varphi, \mu_s \rangle\right) \varphi(\omega_1) \right) ds$$

and define $G_t : \mathcal{M}_B \to \mathcal{M}_B$ by

$$\begin{aligned} \langle f, G_t(\mu) \rangle = \frac{1}{2} \int_D \big((f\theta_t)(\omega_1 + \omega_2 - \omega_3) \mathbb{1}_{\omega_1 + \omega_2 - \omega_3 \in B} + (f\theta_t)(\omega_3) \big) \\ \times K(\omega_1, \omega_2, \omega_3) \theta_t(\omega_1)^{-1} \theta_t(\omega_2)^{-1} \theta_t(\omega_3)^{-1} \\ \times \mu(d\omega_1) \mu(d\omega_2) \mu(d\omega_3). \end{aligned}$$

Note that $G_t(\mu) \geq 0$ whenever $\mu \geq 0$ and for some $C = C(\varphi, B) < \infty$ we have

$$\|G_t(\mu)\| \leq C \|\mu\|^3, \tag{18}$$

$$\|G_t(\mu) - G_t(\mu')\| \leq C \|\mu - \mu'\| \left(\|\mu\|^2 + \|\mu'\| \|\mu\| + \|\mu'\|^2 \right). \tag{19}$$

Now, concluding that $\mu_t \geq 0$ for all $t < T$ is a direct application of the end of the proof in Ref. 11 [Proposition 2.2], by proving first that $\tilde{\mu}_t = \theta_t \mu_t$ is positive. □

Proof of Theorem 1.5. We fix now $\mu_0 \in \mathcal{M}$ with $\mu_0 \geq 0$ and $\langle \varphi, \mu_0 \rangle < \infty$. For each bounded set $B \subset [0, \infty)$, let

$$\mu_0^B = \mathbb{1}_B \mu_0, \qquad \lambda_0^B = \int_{[0,\infty) \setminus B} \varphi(\omega) \mu_0(d\omega) \tag{20}$$

and denote by $(\mu_t^B, \lambda_t^B)_{t \geq 0}$ the unique solution to (12), starting from (μ_0^B, λ_0^B), provided by Proposition 2.1. We have that for $B \subset B'$,

$$\mu_t^B \leq \mu_t^{B'}, \qquad \langle \varphi, \mu_t^B \rangle + \lambda_t^B = \langle \varphi, \mu_t^{B'} \rangle + \lambda_t^{B'}.$$

The inequality will be proven in Proposition 2.2 and the equality is the consequence of expression (14) and the fact that

$$\langle \varphi, \mu_0^B \rangle + \lambda_0^B = \langle \varphi, \mu_0^{B'} \rangle + \lambda_0^{B'}$$

by expression (20).

Moreover, it holds that for any local solution $(\nu_t)_{t<T}$ of the 4-wave kinetic Equation (5), for all $t < T$,

$$\mu_t^B \leq \nu_t, \qquad \langle \varphi, \mu_t^B \rangle + \lambda_t^B \geq \langle \varphi, \nu_t \rangle. \tag{21}$$

We prove the first inequality in Proposition 2.3. The second inequality is the consequence of

$$\langle \varphi, \nu_t \rangle \leq \langle \varphi, \mu_0 \rangle \leq \langle \varphi, \mu_0 \rangle + \lambda_0^B = \langle \varphi, \mu_t^B \rangle + \lambda_t^B. \tag{22}$$

We now show how these facts lead to the proof of Theorem 1.5. Set $\mu_t = \lim_{B \uparrow [0,\infty)} \mu_t^B$ and $\lambda_t = \lim_{B \uparrow [0,\infty)} \lambda_t^B$. Note that

$$\langle \varphi, \mu_t \rangle = \lim_{B \uparrow [0,\infty)} \langle \varphi, \mu_t^B \rangle \leq \langle \varphi, \mu_0 \rangle < \infty.$$

So, by dominated convergence, using that K is submultiplicative, for all bounded measurable functions f,

$$\int_D f(\omega_1 + \omega_2 - \omega_3) \mathbb{1}_{\omega_1+\omega_2-\omega_3 \notin B} K(\omega_1, \omega_2, \omega_3) \mu_t^B(d\omega_1) \mu_t^B(d\omega_2) \mu_t^B(d\omega_3) \to 0,$$

and we can pass to the limit in (12) to obtain

$$\begin{aligned} \frac{d}{dt}\langle f, \mu_t \rangle = \frac{1}{2} \int_D & (f(\omega_1 + \omega_2 - \omega_3) + f(\omega_3) - f(\omega_1) - f(\omega_2)) \\ & \times K(\omega_1, \omega_2, \omega_3) \mu_t(d\omega_1) \mu_t(d\omega_2) \mu_t(d\omega_3) \\ & - (\lambda_t^2 + 2\lambda_t \langle \varphi, \mu_t \rangle) \langle f\varphi, \mu_t \rangle. \end{aligned}$$

For any local solution $(\nu_t)_{t<T}$ of the kinetic Equation (5) it holds that, for all $t < T$,

$$\mu_t \leq \nu_t, \qquad \langle \varphi, \mu_t \rangle + \lambda_t \geq \langle \varphi, \nu_t \rangle.$$

Hence, if $\lambda_t = 0$ for all $t < T$, then $(\mu_t)_{t<T}$ is also a local solution of Equation (5) and, moreover, is the only local solution on $[0,T)$, i.e., $(\mu_t)_{t<T} = (\nu_t)_{t<T}$. Using this, we prove next that if $(\nu_t)_{t<T}$ is a strong local solution of Equation (5), then it is unique. To do so, we just need to show that there exists a $T > 0$ such that $\lambda_t = 0$ for all $t < T$. If $(\nu_t)_{t<T}$ is a strong local solution, then

$$\int_0^t \langle \varphi^2, \mu_s \rangle \, ds \leq \int_0^t \langle \varphi^2, \nu_s \rangle \, ds < \infty$$

for all $t < T$; this allows us to pass to the limit in (13) to obtain

$$\frac{d}{dt}\lambda_t = (\lambda_t^2 + 2\lambda_t \langle \varphi, \mu_t \rangle) \langle \varphi^2, \mu_t \rangle \tag{23}$$

and to deduce from this equation that $\lambda_t = 0$ for all $t < T$. Therefore, we conclude that if a strong solution $(\nu_t)_{t<T}$ exists on $[0,T)$, then it is unique.

Next, we prove that if a strong solution $(\nu_t)_{t<T}$ exists, then it is conservative. We have that for any local solution $(\nu_t)_{t<T}$,

$$\begin{aligned} \int_0^\infty \omega \mathbb{1}_{\omega \leq n} \nu_t(d\omega) = & \int_0^\infty \omega \mathbb{1}_{\omega \leq n} \nu_0(d\omega) \\ & + \frac{1}{2} \int_0^t \int_D \{(\omega_1 + \omega_2 - \omega_3) \mathbb{1}_{\omega_1+\omega_2-\omega_3 \leq n} + \omega_3 \mathbb{1}_{\omega_3 \leq n} - \omega_1 \mathbb{1}_{\omega_1 \leq n} - \omega_2 \mathbb{1}_{\omega_2 \leq n}\} \\ & \times K(\omega_1, \omega_2, \omega_3) \nu_s(d\omega_1) \nu_s(d\omega_2) \nu_s(d\omega_3). \end{aligned} \tag{24}$$

Hence, if $(\nu_t)_{t<T}$ is strong, we have that

$$\int_0^t \langle \omega^2, \nu_s \rangle \, ds \leq \int_0^t \langle \varphi^2, \nu_s \rangle \, ds < \infty.$$

Then, by dominated convergence, the second term on the right tends to 0 as $n \to \infty$, showing that $(\nu_t)_{t<T}$ is conservative.

We conclude the proof of the theorem by showing that if $\langle \varphi^2, \mu_0 \rangle < \infty$, then a local strong solution exists. Suppose now that $\langle \varphi^2, \mu_0 \rangle < \infty$ and set $T = \langle \varphi^2, \mu_0 \rangle^{-1} \langle \varphi, \mu_0 \rangle^{-1}$. For any bounded set $B \subset [0, \infty)$, we have

$$\begin{aligned} \frac{d}{dt}\langle \varphi^2, \mu_t^B \rangle &\leq \frac{1}{2} \int_D \left\{ \varphi(\omega_1 + \omega_2 - \omega_3)^2 + \varphi(\omega_3)^2 - \varphi(\omega_1)^2 - \varphi(\omega_2)^2 \right\} \\ &\qquad \times K(\omega_1, \omega_2, \omega_3) \mu_t^B(d\omega_1) \mu_t^B(d\omega_2) \mu_t^B(d\omega_3) \\ &\leq \langle \varphi, \mu_t^B \rangle \langle \varphi^2, \mu_t^B \rangle^2 \leq \langle \varphi, \mu_0 \rangle \langle \varphi^2, \mu_t^B \rangle^2 \end{aligned} \tag{25}$$

so for $t < T$

$$\langle \varphi^2, \mu_t \rangle \leq (S - \langle \varphi, \mu_0 \rangle t)^{-1},$$

where $S = \langle \varphi^2, \mu_0 \rangle^{-1}$. Hence (23) holds and forces $\lambda_t = 0$ for $t < T$ as above, so $(\mu_t)_{t<T}$ is a strong local solution. □

Proposition 2.2. Suppose $B \subset B'$ and that $(\mu_t^B, \lambda_t^B)_{t \geq 0}$, $(\mu_t^{B'}, \lambda_t^{B'})_{t \geq 0}$ are the solutions of (12) for each one of these sets corresponding to the initial data given by (20). Then for all $t \geq 0$, $\mu_t^B \leq \mu_t^{B'}$.

The proof is obtained by adapting the one in Ref. 11 [Proposition 2.4].

Proof. Set

$$\theta_t(\omega_1) = \exp \int_0^t \left(\int_{\mathbb{R}_+^2 \cap (\omega_1 + \omega_2 \geq \omega_3)} K(\omega_1, \omega_2, \omega_3) \mu_s^B(d\omega_2) \mu_s^B(d\omega_3) + ((\lambda_s^B)^2 + 2\lambda_s^B \langle \varphi, \mu_s^B \rangle) \varphi(\omega_1) \right) ds.$$

Denote by $\pi_t = \theta_t(\mu_t^{B'} - \mu_t^B)$. Note that $\pi_0 \geq 0$. By Ref. 11 [Prop. 2.3], for any bounded measurable function f,

$$\begin{aligned} \frac{d}{dt}\langle f, \pi_t \rangle &= \langle f \frac{\partial \theta_t}{\partial t}, \mu_t^{B'} - \mu_t^B \rangle \\ &\quad + \langle (f\theta_t, 0), L^{B'}(\mu_t^{B'}, \lambda_t^{B'}) - L^B(\mu_t^B, \lambda_t^B) \rangle = I \\ &\quad + \int_D (f\theta_t)(\omega_1) K(\omega_1, \omega_2, \omega_3) \left(\mu_t^{B'}(d\omega_1) \mu_t^B(d\omega_2) \mu_t^B(d\omega_3) - \mu_t^{B'}(d\omega_1) \mu_t^{B'}(d\omega_2) \mu_t^{B'}(d\omega_3) \right) \\ &\quad + \left(((\lambda_t^B)^2 + 2\lambda_t^B \langle \varphi, \mu_t^B \rangle) - \left((\lambda_t^{B'})^2 + 2\lambda_t^{B'} \langle \varphi, \mu_t^{B'} \rangle \right) \right) \langle f\theta_t \varphi, \mu_t^{B'} \rangle, \end{aligned}$$

where

$$\begin{aligned} I &:= \frac{1}{2} \int_D (f\theta_t)(\omega_1 + \omega_2 - \omega_3) K(\omega_1, \omega_2, \omega_3) \\ &\qquad \times \left(\mathbb{1}_{\omega_1 + \omega_2 - \omega_3 \in B'} \mu_t^{B'}(d\omega_1) \mu_t^{B'}(d\omega_2) \mu_t^{B'}(d\omega_3) - \mathbb{1}_{\omega_1 + \omega_2 - \omega_3 \in B} \mu_t^B(d\omega_1) \mu_t^B(d\omega_2) \mu_t^B(d\omega_3) \right) \\ &\quad + \frac{1}{2} \int_D (f\theta_t)(\omega_3) K(\omega_1, \omega_2, \omega_3) \\ &\qquad \times \left(\mu_t^{B'}(d\omega_1) \mu_t^{B'}(d\omega_2) \mu_t^{B'}(d\omega_3) - \mu_t^B(d\omega_1) \mu_t^B(d\omega_2) \mu_t^B(d\omega_3) \right). \end{aligned}$$

Now, squaring the equality

$$\langle \varphi, \mu_t^B \rangle + \lambda_t^B = \langle \varphi, \mu_t^{B'} \rangle + \lambda_t^{B'}$$

we have that

$$\left((\lambda_t^B)^2 + 2\lambda_t^B \langle \varphi, \mu_t^B \rangle \right) - (\lambda_t^{B'})^2 - 2\lambda_t^{B'} \langle \varphi, \mu_t^{B'} \rangle = \langle \varphi, \mu_t^{B'} \rangle^2 - \langle \varphi, \mu_t^B \rangle^2$$

and therefore

$$\begin{aligned} \frac{d}{dt}\langle f, \pi_t \rangle = I + \int_{\mathbb{R}_+^3 \setminus D} &(f\theta_t)(\omega_1) \varphi(\omega_1) \varphi(\omega_2) \varphi(\omega_3) \\ &\left(\mu_t^{B'}(d\omega_1) \mu_t^{B'}(d\omega_2) \mu_t^{B'}(d\omega_3) - \mu_t^{B'}(d\omega_1) \mu_t^B(d\omega_2) \mu_t^B(d\omega_3) \right) \end{aligned}$$

$$+\int_D (f\theta_t)(\omega_1)(\varphi(\omega_1)\varphi(\omega_2)\varphi(\omega_3) - K(\omega_1,\omega_2,\omega_3)) \left(\mu_t^{B'}(d\omega_1)\mu_t^{B'}(d\omega_3)\mu_t^{B'}(d\omega_3) - \mu_t^{B'}(d\omega_1)\mu_t^{B}(d\omega_2)\mu_t^{B}(d\omega_3)\right).$$

Therefore, π_t satisfies an equation of the form

$$\frac{d}{dt}\pi_t = H_t(\pi_t),$$

where $H_t : \mathcal{M}_{B'} \to \mathcal{M}_{B'}$ and it holds $H_t(\pi) \geq 0$ whenever $\pi \geq 0$ and where we have estimates, for $t \leq 1$,

$$\|H_t(\pi)\| \leq C\|\pi\|$$

for some constant $C < \infty$ depending only on φ and B'. Therefore, we can apply the same sort of argument that we used for non-negativity to see that $\pi_t \geq 0$ for all $t \leq 1$ and then for all $t < \infty$. □

Proposition 2.3. Suppose that $(\nu_t)_{t<T}$ is a local solution of the 4-wave kinetic Equation (5), starting from μ_0. Then, for all bounded sets $B \subset [0,\infty)$ and all $t < T$, $\mu_t^B \leq \nu_t$.

Proof. Set θ_t as in proposition 2.2 and denote $\nu_t^B = \mathbb{1}_B\nu_t$ and $\pi_t = \theta_t(\nu_t^B - \mu_t^B)$. By applying Ref. 11 [Proposition 2.3] to π_t, we have that, for all bounded measurable functions f,

$$\frac{d}{dt}\langle f,\pi_t\rangle = \langle f\partial\theta/\partial t, \nu_t^B - \mu_t^B\rangle + \langle f\theta_t\mathbb{1}_B, Q(\nu_t)\rangle - \langle (f\theta_t,0), L^B(\mu_t^B,\lambda_t^B)\rangle.$$

The rest of the proof is obtained by adapting the one in Ref. 11 [Proposition 2.5]. Proceeding as in the proof of Proposition 2.2 we have that

$$\begin{aligned}
\frac{d}{dt}\langle f,\pi_t\rangle = {} & \chi_t \int_0^\infty (f\theta_t)(\omega_1)\varphi(\omega_1)\nu_t^B(d\omega_1) \\
& + \frac{1}{2}\int_D (f\theta_t)(\omega_1+\omega_2-\omega_3)K(\omega_1,\omega_2,\omega_3) \\
& \quad \times \mathbb{1}_{\omega_1+\omega_2-\omega_3\in B}\left(\nu_t(d\omega_1)\nu_t(d\omega_2)\nu_t(d\omega_3) - \mu_t^B(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3)\right) \\
& + \frac{1}{2}\int_D (f\theta_t)(\omega_3)K(\omega_1,\omega_2,\omega_3) \\
& \quad \times \left(\nu_t(d\omega_1)\nu_t(d\omega_2)\nu_t^B(d\omega_3) - \mu_t^B(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3)\right) \\
& + \int_{\mathbb{R}^3_+\setminus D} (f\theta_t)(\omega_1)\varphi(\omega_1)\varphi(\omega_2)\varphi(\omega_3)\left(\nu_t^B(d\omega_1)\nu_t(d\omega_2)\nu_t(d\omega_3) - \nu_t^B(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3)\right) \\
& + \int_D (f\theta_t)(\omega_1)(\varphi(\omega_1)\varphi(\omega_2)\varphi(\omega_3) - K(\omega_1,\omega_2,\omega_3)) \\
& \quad \times \left(\nu_t^B(d\omega_1)\nu_t(d\omega_2)\nu_t(d\omega_3) - \nu_t^B(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3)\right),
\end{aligned}$$

where $\chi_t = (\lambda_t^B)^2 + 2\lambda_t^B\langle\varphi,\mu_t^B\rangle + \langle\varphi,\mu_t^B\rangle^2 - \langle\varphi,\nu_t\rangle^2 \geq 0$.

Therefore, analogously as in the previous Proposition 2.2, we have that

$$\frac{d}{dt}\pi_t = \tilde{H}_t(\pi_t),$$

where $\tilde{H}_t : \mathcal{M}_B \to \mathcal{M}_B$ is linear and $\tilde{H}_t(\pi) \geq 0$ whenever $\pi \geq 0$. Moreover for $t \leq 1$

$$\|\tilde{H}_t(\pi)\| \leq C\|\pi\|$$

for some constant $C < \infty$ depending only on φ and B. □

III. MEAN-FIELD LIMIT

A. The instantaneous coagulation-fragmentation stochastic process

Define

$$D = \{(\omega_1,\omega_2,\omega_3) \in \mathbb{R}^3_+ \,|\, \omega_1+\omega_2 \geq \omega_3\}.$$

We consider X_0^n a probability measure on $\mathbb{R}_+$ written as a sum of unit masses

$$X_0^n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\omega_i}$$

for $\omega_1,\dots,\omega_n \in \mathbb{R}_+$. X_0^n represents a system of n waves labelled by their dispersion $\omega_1,\dots,\omega_n$.

We define a Markov process $(X_t^n)_{t\geq 0}$ of probability measures on $\mathbb{R}_+$. For each triple $(\omega_i,\omega_j,\omega_l) \in D$ of distinct particles, take an independent exponential random time T_{ijl}, $i<j$, with the parameter

$$\frac{1}{n^2}K(\omega_i,\omega_j,\omega_l). \tag{26}$$

Set $T_{ijk} = T_{jik}$ and set $T = \min_{ijl} T_{ijl}$. Then set

$$X_t^n = X_0^n \qquad \text{for } t < T$$

and

$$X_T^n = X_0^n + \frac{1}{n}(\delta_\omega + \delta_{\omega_l} - \delta_{\omega_i} - \delta_{\omega_j})$$

with $\omega = \omega_i + \omega_j - \omega_l$. Then begin the construction afresh from X_T^n.

We call the process $(X_t^n)_{t\geq 0}$ an instantaneous n-coagulation-fragmentation stochastic process.

Remark 3.1. Note that we should be careful not to pick the same particle twice as one particle cannot interact with itself. Suppose that $\omega_i = \omega_j = \omega_l$, then the Markov chain does not make a jump. The same happens with $\omega_i = \omega_l$ or $\omega_j = \omega_l$. Finally the case $\omega_i = \omega_j$ needs to be considered. For that we define

$$\mu^{(1)}(A\times B\times C) = \mu(A)\mu(B)\mu(C) - \mu(A\cap B)\mu(C)$$

as the counting measure of triples of particles with different particles in the first and second positions. Also, define

$$\mu^{(n)}(A\times B\times C) = \mu(A)\mu(B)\mu(C) - n^{-1}\mu(A\cap B)\mu(C). \tag{27}$$

Note that

$$n^3\mu^{(n)} = (n\mu)^{(1)}. \tag{28}$$

Generator of the Markov chain: For all $F \in C_b$,

$$GF(X) = \frac{n}{2}\int_D [F(X^{\omega_1,\omega_2,\omega_3}) - F(X)]\, K(\omega_1,\omega_2,\omega_3) X^{(n)}(d\omega_1, d\omega_2, d\omega_3),$$

where

$$X^{\omega_1,\omega_2,\omega_3} = X + \frac{1}{n}\left(\delta_{\omega_3} + \delta_{\omega_1+\omega_2-\omega_3} - \delta_{\omega_1} - \delta_{\omega_2}\right).$$

Interpretation of the stochastic process. Three different particles, say ω_1, ω_2, ω_3, interact at a random time given by rate (26).

The outcome of the interaction is that ω_1 and ω_2 merge and then, under the presence of ω_3, they split, creating a new particle ω_3 and another one with the rest $\omega = \omega_1 + \omega_2 - \omega_3$ (coagulation-fragmentation phenomena, which takes place instantaneously).

The martingale formulation. Now, for each function $f \in C_b(\mathbb{R}_+)$, the Markov chain can be expressed as

$$\langle f, X_t^n\rangle = \langle f, X_0^n\rangle + M_t^{n,f} + \int_0^t \langle f, Q^{(n)}(X_s^n)\rangle\, ds, \tag{29}$$

where $(M_t^{n,f})_{t\geq 0}$ is a martingale. Note that using (28) we have that

$$\begin{aligned}&\langle f, Q^{(n)}(\mu)\rangle\\ &= \frac{1}{2}\int_D (f(\omega_1+\omega_2-\omega_3) + f(\omega_3) - f(\omega_1) - f(\omega_2))K(\omega_1,\omega_2,\omega_3)\mu^{(n)}(d\omega_1,d\omega_2,d\omega_3).\end{aligned}$$

B. First result on mean-field limit

We will start working in the simpler case where K is bounded and see that the unbounded case will come as a "modification" of the bounded one.

1. Mean-field limit for bounded jump kernel

a. Uniqueness of solutions for bounded kernel.

Lemma 3.2. It holds that Q given in (5) is linear in each one of its terms and the following symmetry:

$$\langle f, Q(\mu,\nu,\tau)\rangle = \langle f, Q(\nu,\mu,\tau)\rangle.$$

Moreover,

$$Q(\mu,\mu,\mu) - Q(\nu,\nu,\nu) = Q(\mu+\nu,\mu-\nu,\mu) + Q(\mu+\nu,\nu,\mu-\nu) + Q(\mu,\nu,\nu-\mu). \tag{30}$$

Proof. The first part of the statement is immediate from the definition. The second part is proven using the symmetry property along with the linearity in each component. □

Proposition 3.3 (Uniqueness of solutions). Suppose that the jump kernel in (5) is bounded by Λ. Then for any given initial data, if there exists a solution for (5), then the solution is unique.

Proof. Suppose that we have $\mu_t, \nu_t \in \mathcal{P}(\mathbb{R}_+)$ solutions to (5) with the same initial data. We will compare these solutions in the total variation norm,

$$\|\mu_t - \nu_t\|_{TV} = \sup_{\|f\|_\infty = 1} \langle f, \mu_t - \nu_t\rangle = \sup_{\|f\|_\infty=1} \int_0^t \langle f, \dot\mu_t - \dot\nu_t\rangle.$$

Then by expression (30) we have that

$$\dot\mu_s - \dot\nu_s = Q(\mu_s+\nu_s,\mu_s-\nu_s,\mu_s) + Q(\mu_s+\nu_s,\nu_s,\mu_s-\nu_s) + Q(\mu_s,\nu_s,\nu_s-\mu_s).$$

Therefore, for any $f \in C_b(\mathbb{R}_+)$ such that $\|f\|_\infty = 1$, it holds

$$|\langle f, \dot\mu_s - \dot\nu_s\rangle| \le 24\Lambda \|\mu_s - \nu_s\|_{TV}.$$

Finally by Gronwall we conclude the result. □

Remark 3.4. Existence of solutions for this case can be proven directly using a classical argument of iterative scheme (as done previously for the unbounded case).

The following theorem is an adaptation of part of Ref. 11 [Theorem 4.1]. Much more detail is provided here than in the original reference. To give the details, the author was much guided by an unpublished report[1] that studied the homogeneous Boltzmann equation with bounded kernels.

Theorem 3.5 (Mean-field limit for bounded jump kernel). *Suppose that for a given measure μ_0, it holds that*

$$\langle \omega, X_0^n\rangle \le \langle \omega, \mu_0\rangle$$

and that as $n \to \infty$

$$X_0^n \to \mu_0 \qquad \text{weakly}.$$

Assume that the kernel is uniformly bounded

$$K \le \Lambda < \infty.$$

Then the sequence $(X^n)_{t\ge 0}$ converges as $n \to \infty$ in probability in $D([0,\infty)\times\mathcal{P}(\mathbb{R}_+))$. Its limit $(\mu_t)_{t\ge0}$ is continuous and it satisfies the isotropic 4-wave kinetic Equation (5). In particular, for all $f \in C_b(\mathbb{R}_+)$

$$\sup_{s\le t}\langle f, X_t^n\rangle \to \langle f, \mu_t\rangle,$$

$$\sup_{s\le t}|M_s^{f,n}|\to 0,$$
$$\sup_{s\le t}\int_0^t \langle f, Q^{(n)}(X_s^n)\rangle\, ds \to \int_0^t \langle f, Q(\mu_s)\rangle\, ds$$

all in probability. As a consequence, Equation (5) is obtained as the limit in probability of (29) as $n\to\infty$.

Corollary 3.6 (Existence of solutions for the weak wave kinetic equation). There exists a solution for (5) (expressed as the limit of the X_t^n*).*

Proof. We have that the limit $(\mu_t)_{t\ge 0}$ satisfies $\langle \omega, \mu_t\rangle \le \langle \omega, \mu_0\rangle$ by the following:

$$\langle \omega \mathbb{1}_{\omega\le k}, \mu\rangle = \lim_{n\to\infty}\langle \omega\mathbb{1}_{\omega\le k}, X_t^n\rangle$$

and we have that

$$\langle \omega\mathbb{1}_{\omega\le k}, X_t^n\rangle \le \langle \omega, X_t^n\rangle \le \langle \omega, \mu_0\rangle.$$

So by making $k\to\infty$ we get the bound. □

2. Proof of Theorem 3.5

We want to take the limit in the martingale formulation (29).

a. Step 1: Control on the martingale.

Proposition 3.7 (Martingale convergence). For any $f\in C_b(\mathbb{R}_+)$, $t\ge 0$

$$\sup_{0\le s\le t}|M_s^{n,f}|\to 0 \qquad \text{in } L^2(\mathbb{R})$$

in particular, it also converges in probability.

Proof of Proposition 3.7. We use Proposition 8.7 in Ref. 3 which ensures that

$$\mathbb{E}\left[\sup_{s\le T}|M_s^{n,f}|^2\right] \le 4\mathbb{E}\int_0^T \alpha^{n,f}(\mu_s)ds$$

as long as the right hand side is finite, where

$$\alpha^{n,f}(\mu_s) = \frac{1}{2}\int_D \left(\frac{1}{n}(f(\omega_1+\omega_2-\omega_3)+f(\omega_3)-f(\omega_1)-f(\omega_2))\right)^2 \times \frac{1}{n^2}K(\omega_1,\omega_2,\omega_3)(n\mu_s)^{(1)}(d\omega_1,d\omega_2,d\omega_3). \tag{31}$$

Therefore, using (28) we have that

$$\mathbb{E}\left[\sup_{s\le t}|M_s^{n,f}|^2\right] \le \frac{1}{n}32\|f\|_\infty^2\Lambda^2 t. \tag{32}$$

This implies the convergence of the supremum towards 0 in L^2 which implies also the convergence in probability. □

b. Step 2: Convergence for the measures.

Lemma 3.8. The sequence of laws of $(\langle f, X_t^n\rangle)_{n\in\mathbb{N}}$ *on* $D([0,\infty),\mathbb{R})$ *is tight.*

Lemma 3.9. The laws of the sequence $(X_t^n)_{n\in\mathbb{N}}$ *on* $D([0,\infty)\times\mathcal{P}(\mathbb{R}_+))$ *is tight.*

Proposition 3.10 (Weak convergence for the measures). There exists a weakly convergent subsequence $(X_t^{n_k})_{k\in\mathbb{N}}$ *in* $D([0,\infty)\times\mathcal{P}(\mathbb{R}_+))$ *as* $k\to\infty$.

Proof of Proposition 3.10. By Lemma 3.9 we know that the laws of the sequence $(X_t^n)_{n\in\mathbb{N}}$ are tight. This implies relative compactness for the sequence by Prokhorov's theorem. □

Proof of Lemma 3.8. We use Theorem A.7. To prove the first part (i) of the theorem we use that

$$|\langle f, X_t^n\rangle| = \left|\frac{1}{n}\sum_{i=1}^{n} f(\omega_t^{i,n})\right| \leq \frac{1}{n}\sum_{i=1}^{n} |f(\omega_t^{i,n})| \leq \|f\|_\infty$$

so for all $t \geq 0$, $\langle f, X_t^n\rangle \in [-\|f\|_\infty, \|f\|_\infty]$.

The second condition (ii) of the theorem will be consequence of the following inequalities:

$$\mathbb{E}\left[\sup_{r\in[s,t)} |M_r^{n,f} - M_s^{n,f}|^2\right] \leq \frac{1}{n} 32\|f\|_\infty^2 \Lambda^2 (t-s) \tag{33}$$

and

$$\mathbb{E}\left[\sup_{r\in[s,t)} \left(\int_s^r \langle f, Q^{(n)}(X_p^n)\rangle\, dp\right)^2\right] \leq 16\|f\|_\infty^2 \Lambda^2 (t-s)^2 \tag{34}$$

which imply that

$$\mathbb{E}\left[\sup_{r\in[s,t)} |\langle f, X_r^n - X_s^n\rangle|^2\right] \leq A\left((t-s)^2 + \frac{(t-s)}{n}\right) \tag{35}$$

for some $A > 0$ depending only on $\|f\|_\infty$ and Λ. □

Proof of Lemma 3.9. We will use Theorem A.6 to prove this. To check condition (i), we consider the compact set $W \in \mathcal{P}(\mathbb{R}_+)$ (compact with respect to the topology induced by the weak convergence of measures) defined as

$$W := \left\{\tau \in \mathcal{P}(\mathbb{R}_+) \,:\, \int_{\mathbb{R}_+} \omega\, \tau(d\omega) \leq C\right\}.$$

Consider $(\mathcal{L}^n)_{n\in\mathbb{N}}$ the family of probability measures in $\mathcal{P}(D([0,\infty); W))$ which are the laws of $(X^n)_{n\in\mathbb{N}}$. We have that

$$\mathcal{L}^n(D([0,\infty); W) = 1 \quad \text{for all } n \in \mathbb{N}$$

by the conservation of the total energy and its boundedness.

Now, condition (ii) in Theorem A.6 is fulfilled with the family of continuous functions in $\mathcal{P}(\mathbb{R}_+)$ defined as

$$\mathbb{F} = \{F \,:\, \mathcal{P}(\mathbb{R}_+) \to \mathbb{R} \,:\, F(\tau) = \langle f, \tau\rangle \text{ for some } f \in C_b(\mathbb{R}_+)\}.$$

So we are left with proving that for every $f \in C_b(\mathbb{R}_+)$ the sequence $\{\langle f, X^n\rangle\}_{n\in\mathbb{N}}$ is tight. This was proven in Lemma 3.8. □

c. Step 3: Convergence for the trilinear term.

Lemma 3.11 (Continuity of the limit). The weak limit of $(X_t^{n_k})_{t\geq 0}$ as $k \to \infty$ is continuous in time a.e.

Lemma 3.12 (Uniform convergence). For all $f \in C_b(\mathbb{R}_+)$, it holds

$$\sup_{s\leq t} |\langle f, X_s^{n_k} - \mu_s\rangle| \to 0 \quad \textit{weakly}$$

as $k \to \infty$.

Lemma 3.13. It holds that

$$\sup_{s\leq t} |\langle f, Q^{(n)}(X_s^{n_k}) - Q(\mu_s)\rangle| \to 0 \quad \textit{weakly}$$

as $k \to \infty$.

Proposition 3.14 (Convergence for the trilinear term). It holds that

$$\int_0^t \langle f, Q^{(n)}(X_s^{n_k})\rangle\, ds \to \int_0^t \langle f, Q(\mu_s,\mu_s,\mu_s)\rangle\, ds \quad \textit{weakly.}$$

Proof of Proposition 3.14. By Lemma 3.13 we can pass the limit inside the integral in time. □

Proof of Lemma 3.11. We have that for any $f \in C_b(\mathbb{R}_+)$

$$|\langle f, X_t^{n_k}\rangle - \langle f, X_{t^-}^{n_k}\rangle| \le \frac{4}{n_k}\|f\|_\infty$$

applying Theorem A.8 we get that $\langle f, \mu_t\rangle$ is continuous for any $f \in C_b(\mathbb{R}_+)$ and this implies the continuity of $(\mu_t)_{t\ge 0}$. □

Proof of Lemma 3.12. We know by Lemma 3.11 that the limit of $(X^{n_k})_{k\in\mathbb{N}}$ is continuous. The statement is the consequence of the continuity mapping theorem in the Skorokhod space (proven using the Skorokhod representation Theorem A.5) and the fact that $g(X)(t) = \sup_{s\le t}|X|$ is a continuous function in this space. □

Proof of Lemma 3.13. We abuse notation and denote by $(X_t^n)_{n\in\mathbb{N}}$ the convergent subsequence. We split the proof in two parts, we will prove for all $f \in C_b(\mathbb{R}_+)$,

(i) $\sup_{s\le t}|\langle f, (Q - Q^{(n)})(X_s^n)\rangle| \to 0$ as $n\to\infty$,
(ii) $\sup_{s\le t}|\langle f, Q(X_s^n) - Q(\mu_s)\rangle| \to 0$ as $n\to\infty$.

(i) is consequence of

$$\begin{aligned}|\langle f, \left(Q - Q^{(n)}\right)(X_s^n)\rangle| &= \frac{1}{2}\frac{1}{n}\int_{2\omega_2\ge\omega_3} (f(2\omega_2-\omega_3) + f(\omega_3) - 2f(\omega_2)) \\ &\qquad \times K(\omega_2,\omega_2,\omega_3)X_s^n(d\omega_2)X_s^n(d\omega_3) \\ &\le \frac{2}{n}\|f\|_\infty\Lambda. \end{aligned} \tag{36}$$

Now, for (ii) we compute that we have

$$\begin{aligned}\sup_{s\le t}|\langle f, Q(X_s^n) - Q(\mu_s)\rangle| \le \frac{1}{2}\int_D K(\omega_1,\omega_2,\omega_3)\,|f(\omega_1+\omega_2-\omega_3) + f(\omega_3) - f(\omega_1) - f(\omega_2)| \\ \times \sup_{s\le t}\left|X_s^n(d\omega_1)X_s^n(d\omega_2)X_s^n(d\omega_3) - \mu_s(d\omega_1)\mu_s(d\omega_2)\mu_s(d\omega_3)\right|. \end{aligned} \tag{37}$$

We conclude (ii) with an argument analogous to Lemma 3.12 and the fact that

$$X_t^n \otimes X_t^n \otimes X_t^n \to \mu_t \otimes \mu_t \otimes \mu_t$$

weakly (consequence of Lévy's continuity theorem). □

3. Proof of Theorem 1.7 (unbounded kernel)

Remark 3.15. The proof that we already wrote in the case of bounded kernels works here in most parts substituting Λ by M, where

$$\int_{\mathbb{R}_+} \omega X^n(d\omega) \le M = \langle \omega, \mu_0\rangle.$$

The only places where we need to be careful are Lemmas 3.12 and 3.13.

Lemma 3.16 (Convergence of a subsequence). There exists a subsequence $\left(X_t^{n_k}\right)_{k\in\mathbb{N}}$ that converges weakly in $D([0,\infty)\times\mathcal{P}(\mathbb{R}_+))$ as $k\to\infty$.

Proof. The proof is exactly the one as in Section III B 2 b and Proposition 3.10 using the bound on the jump kernel K, for example, in the proof of Lemma 3.8, in the bounds of expressions (33) and (34), the value of Λ will be substituted by M^3. □

Lemma 3.17. For any $f \in C_b(\mathbb{R}_+)$, $t \geq 0$, it holds that

$$\mathbb{E}\left[\sup_{s\leq t} |M_s^{n,f}|^2\right] \leq \frac{1}{n} 32 \|f\|_\infty^2 M^6 t.$$

Proof. The proof is the same one as in Proposition 3.7 using the bound on the jump kernel K. □

Lemma 3.18. It holds that for any $t \geq 0$

$$\sup_{s\leq t} |\langle f, Q^{(n)}(X_s^n) - Q(\mu_s)\rangle| \to 0 \quad \textit{weakly}$$

for f continuous and of compact support.

Proof. Here everything works as in Section III B 2 c, but we need to find bounds (36) and (37). This is an adaptation of the ideas in Ref. 11.

First, we will prove an analogous bound to (37).

Fix $\varepsilon > 0$ and define $p(\varepsilon) = \varepsilon^{-1/\gamma}$. Then for $\omega \geq p(\varepsilon)$ it holds

$$\frac{\tilde{\varphi}(\omega)}{\omega} \leq \varepsilon.$$

Now choose $\kappa \in (0, \gamma/[2(1-\gamma)])$. We split the domain into $F_1^p := \{(\omega_1,\omega_2,\omega_3) : \omega_1 \leq p^\kappa(\varepsilon), \omega_2 \leq p^\kappa(\varepsilon), \omega_3 \leq p^\kappa(\varepsilon)\}$ and F_2^p its complementary. In F_1^p the kernel is bounded and we have, with obvious notations,

$$\sup_{s\leq t} |\langle f, Q_1(X_s^n) - Q_1(\mu_s)\rangle| \to 0 \quad \text{weakly.}$$

On the other hand, in F_2^p, at least one of the components is greater than $p^\kappa(\varepsilon)$. Assume without loss of generality that $\omega_3 \geq p^\kappa(\varepsilon)$. Then

$$\begin{aligned}
|\langle f, Q_2(X_t^n)\rangle| &= \Big| \int_D \{f(\omega_1+\omega_2-\omega_3) + f(\omega_3) - f(\omega_1) - f(\omega_2)\} K(\omega_1,\omega_2,\omega_3) \\
&\qquad \times X_t^n(d\omega_1) X_t^n(d\omega_2) X_t^n(d\omega_3)\Big| \\
&\leq 4\|f\|_\infty \int_D \tilde{\varphi}(\omega_1)\tilde{\varphi}(\omega_2)\tilde{\varphi}(\omega_3) X_t^n(d\omega_1) X_t^n(d\omega_2) X_t^n(d\omega_3) \\
&\leq 4\|f\|_\infty \max\left\{(p^\kappa(\varepsilon))^{2(1-\gamma)}\varepsilon\langle\omega,\mu_0\rangle, (p^\kappa(\varepsilon))^{1-\gamma}\varepsilon^2\langle\omega,\mu_0\rangle^2, \varepsilon^3\langle\omega,\mu_0\rangle^3\right\} \\
&\leq c\varepsilon^\eta \qquad \text{for } \eta = 1 - 2\kappa(1-\gamma)/\gamma > 0
\end{aligned}$$

and analogously

$$|\langle f, Q_2(\mu_t)\rangle| \leq c\varepsilon^\eta.$$

This implies that

$$\limsup_{n\to\infty} \sup_{s\leq t} |\langle f, Q_2(X_s^n) - Q_2(\mu_s)\rangle| \leq 2c\varepsilon^\eta$$

but ε is arbitrary so the limit is proved.

We are left with proving an analogous estimate to (36), which is obtained straightforwardly since we restrict ourselves to continuous functions of compact support. □

Proof of Theorem 1.7. Thanks to the previous Lemmas we know that there exists convergent subsequence $X_t^{n_k} \to \mu_t$ weakly as $k \to \infty$ such that

$$\langle f, \mu_t\rangle = \langle f, \mu_0\rangle + \int_0^t \langle f, Q(\mu_s)\rangle ds$$

for any f is continuous of compact support. Now using the bounds on the jump kernel and that $\langle\omega,\mu_t\rangle \leq \langle\omega,\mu_0\rangle$ and a limit argument, we can extend this equation to all bounded measurable functions f. □

C. Second result on mean-field limit

1. A coupling auxiliary process

Write

$$X_0^n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\omega_i},$$

for $\omega_i \in \mathbb{R}_+$. Define for $B \subset \mathbb{R}_+$ bounded

$$X_0^{B,n} = \frac{1}{n}\sum_{i\,:\,\omega_i \in B}^{n} \delta_{\omega_i}.$$

Consider $\Lambda_0^{B,n}$ such that for each $B' \subset \mathbb{R}_+$ bounded such that $B \subset B'$ it holds

$$X_0^{B,n} \le X_0^{B',n}, \quad \langle \varphi, X_0^{B,n}\rangle + \Lambda_0^{B,n} = \langle \varphi, X_0^{B',n}\rangle + \Lambda_0^{B',n}. \tag{38}$$

Set

$$\nu^B = (\Lambda_0^{B,n})^2 + 2\Lambda_0^{B,n}\langle \varphi, X_0^{B,n}\rangle - \frac{1}{n^2}\sum_{k,j\,:\,\omega_j \notin B \text{ or } \omega_k \notin B} \varphi(\omega_j)\varphi(\omega_k).$$

Note that ν^B decreases as B increases and $\nu^{[0,\infty)} = (\Lambda_0^{B,n})^2 + 2\Lambda_0^{B,n}\langle \varphi, X_0^{B,n}\rangle \ge 0$.

For $i < j$ take independent exponential random variables T_{ijk} of parameter $K(\omega_i,\omega_j,\omega_k)/n^2$. Set $T_{ijk} = T_{jik}$. Also, for $i \ne j$, take independent exponential random variables S_{ijk} of parameter $\left(\varphi(\omega_i)\varphi(\omega_j)\varphi(\omega_k) - K(\omega_i,\omega_j,\omega_k)\right)/n^2$ (in all these cases we assume that $\omega_i + \omega_j \ge \omega_k$). We can construct, independently for each i, a family of independent exponential random variables S_i^B, increasing in B, with S_i^B having parameter $\varphi(\omega_i)\nu^B$.

Set

$$T_i^B = \min_{k,j\,:\,\omega_j \notin B \text{ or } \omega_k \notin B} \left(T_{ijk} \wedge S_{ijk}\right) \wedge S_i^B,$$

T_i^B is an exponential random variable of parameter

$$\frac{1}{n^2}\sum_{k,j\,:\,\omega_j \notin B \text{ or } \omega_k \notin B} \varphi(\omega_i)\varphi(\omega_j)\varphi(\omega_k) + \varphi(\omega_i)\nu^B = \varphi(\omega_i)\left((\Lambda_0^{B,n})^2 + 2\Lambda_0^{B,n}\langle \varphi, X_0^{B,n}\rangle\right).$$

For each B, the random variables

$$(T_{ijk}, T_i^B \,:\, i,j,k \text{ such that } \omega_i,\omega_j,\omega_k \in B,\ i < j)$$

form an independent family. Suppose that i is such that $\omega_i \in B$ and that j is such that $\omega_j \notin B$ or k is such that $\omega_k \notin B$, then we have

$$T_i^B \le T_{ijk}$$

and for $B \subset B'$ and all i, we have (as a consequence of (38))

$$T_i^B \le T_i^{B'}.$$

Now set

$$T = \left(\min_{i<j,k} T_{ijk}\right) \wedge \left(\min_i T_i^B\right).$$

We set $(X_t^{B,n}, \Lambda_t^{B,n}) = (X_0^{B,n}, \Lambda_0^{B,n})$ for $t < T$ and set

$$(X_T^{B,n}, \Lambda_T^{B,n}) = \begin{cases} (X_0^{B,n} - \frac{1}{n}\delta_{\omega_i} - \frac{1}{n}\delta_{\omega_j} + \frac{1}{n}\delta_{\omega_k} + \frac{1}{n}\delta_{\omega_i+\omega_j-\omega_k}, \Lambda_0^{B,n}) \\ \quad \text{if } T = T_{ijk},\ \omega_i, \omega_j, \omega_k, \omega_i + \omega_j - \omega_k \in B, \\ (X_0^{B,n} - \frac{1}{n}\delta_{\omega_i} - \frac{1}{n}\delta_{\omega_j} + \frac{1}{n}\delta_{\omega_k}, \Lambda_0^{B,n} + \frac{1}{n}\varphi(\omega_i + \omega_j - \omega_k)) \\ \quad \text{if } T = T_{ijk},\ \omega_i, \omega_j, \omega_k \in B,\ \omega_i + \omega_j - \omega_k \notin B, \\ (X_0^{B,n} - \frac{1}{n}\delta_{\omega_i}, \Lambda_0^{B,n} + \frac{1}{n}\varphi(\omega_i)), \quad \text{if } T = T_i^B,\ \omega_i \in B, \\ (X_0^{B,n}, \Lambda_0^{B,n}), \quad \text{otherwise.} \end{cases}$$

One can check that $X_T^{B,n}$ is supported on B and for $B \subset B'$

$$X_T^{B,n} \le X_T^{B',n}, \qquad \langle \varphi, X_T^{B,n} \rangle + \Lambda_T^B = \langle \varphi, X_T^{B',n} \rangle + \Lambda_T^{B'}. \tag{39}$$

We repeat the above construction independently from time T, again and again to obtain a family of Markov processes $(X_t^{B,n}, \Lambda_t^{B,n})_{t \ge 0}$ such that (39) holds for all time.

Remark 3.19. Notice that $\Lambda_0^{B,n}$ and $X_0^{B,n}$ in the definition of ν^B must be updated to $\Lambda_T^{B,n}$ and $X_T^{B,n}$ in the new step.

For a bounded set $B \subset [0, \infty)$, we will consider

$$X_0^{B,n} = \mathbb{1}_B X_0^n, \quad \Lambda_0^{B,n} = \langle \varphi \mathbb{1}_{B^c}, X_0^n \rangle.$$

a. Markov chain generator. For all $F \in C_b(\mathcal{M}^B)$, $\mu \in \mathcal{M}^B$ we have

$$\begin{aligned} \mathcal{G}F(\mu, \lambda) = {} & \frac{n}{2} \int_D \{F(\mu^{\omega_1,\omega_2,\omega_3}, \lambda) - F(\mu, \lambda)\} \mathbb{1}_{\omega_1+\omega_2-\omega_3 \in B} K(\omega_1, \omega_2, \omega_3) \mu^{(n)}(d\omega_1, d\omega_2, d\omega_3) \\ & + \frac{n}{2} \int_D \left\{F\left(\hat{\mu}^{\omega_1,\omega_2,\omega_3}, \lambda^{\omega_1+\omega_2-\omega_3}\right) - F(\mu, \lambda)\right\} \mathbb{1}_{\omega_1+\omega_2-\omega_3 \notin B} K(\omega_1, \omega_2, \omega_3) \mu^{(n)}(d\omega_1, d\omega_2, d\omega_3) \\ & + n \int_{\mathbb{R}_+} \{F(\mu^\omega, \lambda^\omega) - F(\mu, \lambda)\} \left(\lambda^2 + 2\lambda \langle \varphi, \mu \rangle\right) \varphi(\omega) \mu(d\omega), \end{aligned}$$

where

$$\begin{aligned} \mu^{\omega_1,\omega_2,\omega_3} &= \mu + \frac{1}{n}\left(\delta_{\omega_3} + \delta_{\omega_1+\omega_2-\omega_3} - \delta_{\omega_1} - \delta_{\omega_2}\right), \\ \hat{\mu}^{\omega_1,\omega_2,\omega_3} &= \mu + \frac{1}{n}\left(\delta_{\omega_3} - \delta_{\omega_1} - \delta_{\omega_2}\right), \\ \lambda^{\omega_1+\omega_2-\omega_3} &= \lambda + \frac{1}{n}\varphi(\omega_1 + \omega_2 - \omega_3), \\ \lambda^\omega &= \lambda + \frac{1}{n}\varphi(\omega), \\ \mu^\omega &= \mu - \frac{1}{n}\delta_\omega. \end{aligned}$$

b. Associated martingale. Remember the definition

$$\mu^{(n)}(A \times B \times C) = \mu(A)\mu(B)\mu(C) - n^{-1}\mu(A \cap B)\mu(C)$$

which has the property $n^3 \mu^{(n)} = (n\mu)^{(1)}$. Define for any bounded measurable function f on $\mathbb{R}_+$ and $a \in \mathbb{R}$,

$$\begin{aligned} L^{B,(n)}(\mu, \lambda)(f, a) &= \langle (f, a), L^{B,(n)}(\mu, \lambda) \rangle \\ &= \frac{1}{2} \int_{\mathbb{R}_+^3} \big(f(\omega_1 + \omega_2 - \omega_3) \mathbb{1}_{\omega_1+\omega_2-\omega_3 \in B} + a\varphi(\omega_1 + \omega_2 - \omega_3) \mathbb{1}_{\omega_1+\omega_2-\omega_3 \notin B} \\ &\qquad + f(\omega_3) - f(\omega_1) - f(\omega_2)\big) K(\omega_1, \omega_2, \omega_3) \mu^{(n)}(d\omega_1, d\omega_2, d\omega_3) \\ &\quad + \left(\lambda^2 + 2\lambda \langle \varphi, \mu \rangle\right) \int_{\mathbb{R}_+} (a\varphi(\omega) - f(\omega)) \varphi(\omega) \mu(d\omega) \end{aligned}$$

and

$$P^{B,(n)}(\mu,\lambda)(f,a) = \frac{1}{2n}\int_D \Big(f(\omega_1+\omega_2-\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\in B} + a\varphi(\omega_1+\omega_2-\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\notin B} + f(\omega_3) - f(\omega_1) - f(\omega_2)\Big)^2 K(\omega_1,\omega_2,\omega_3)\mu^{(n)}(d\omega_1,d\omega_2,d\omega_3) + \big(\lambda^2 + 2\lambda\langle\varphi,\mu\rangle\big)\int_{\mathbb{R}_+}(a\varphi(\omega)-f(\omega))^2\varphi(\omega)\mu(d\omega).$$

Then, for all f and a

$$M_t^n = \langle f, X_t^{B,n}\rangle + a\Lambda_t^{B,n} - \langle f, X_0^{B,n}\rangle - a\Lambda_0^{B,n} - \int_0^t L^{B,(n)}(X_s^{B,n},\Lambda_s^{B,n})(f,a)\,ds$$

is a martingale with previsible increasing process

$$\langle M\rangle_t = \int_0^t P^{B,(n)}(X_s^{B,n},\Lambda_s^{B,n})(f,a)\,ds.$$

2. Proof of Theorem 1.8

Remember the metric d in $\mathcal{M}^f$ defined around expression (11).

Proposition 3.20. Let $B\subset[0,\infty)$ be bounded and μ_0 be measure on $\mathbb{R}_+$ such that $\langle\varphi,\mu_0\rangle<\infty$ and that

$$\mu_0^{*n}(\partial B) = 0 \quad \text{for all } n\geq 1.$$

Assume that for $\varphi(\omega)=\omega+1$ it holds

$$K(\omega_1,\omega_2,\omega_3)\leq\varphi(\omega_1)\varphi(\omega_2)\varphi(\omega_3).$$

Consider $(\mu_t^B,\lambda_t^B)_{t\geq 0}$ the solution to (12) given by Proposition 2.1. Suppose that

$$d(X_0^{B,n},\mu_0^B)\to 0, \quad |\Lambda_0^{B,n}-\lambda_0^B|\to 0$$

as $n\to\infty$. Then for all $t\geq 0$,

$$\sup_{s\leq t} d(X_s^{B,n},\mu_s^B)\to 0, \quad \sup_{s\leq t}|\Lambda_s^{B,n}-\lambda_s^B|\to 0$$

in probability.

Proof of Proposition 3.20. Set $M=\sup_n\langle\varphi,X_0^{B,n}\rangle<\infty$. For all B and all continuous bounded functions f and all $a\in\mathbb{R}$

$$M_t^n = \langle f, X_t^{B,n}\rangle + a\Lambda_t^{B,n} - \langle f, X_0^{B,n}\rangle - a\Lambda_0^{B,n} - \int_0^t L^{B,(n)}(X_s^{B,n},\Lambda_s^{B,n})(f,a)\,ds \tag{40}$$

is a martingale with previsible increasing process

$$\langle M^n\rangle_t = \int_0^t P^{B,(n)}(X_s^{B,n},\Lambda_s^{B,n})(f,a)\,ds$$

(which is the analogous expression to (31)).

There is a constant $C<\infty$, depending only on B,Λ,φ, such that

$$|L^B(X_t^{B,n},\Lambda_t^{B,n})(f,a)|\leq C(\|f\|_\infty+|a|), \tag{41}$$

$$|(L^B-L^{B,(n)})(X_t^{B,n},\Lambda_t^{B,n})(f,a)|\leq Cn^{-1}(\|f\|_\infty+|a|), \tag{42}$$

$$|P^{B,(n)}(X_t^{B,n},\Lambda_t^{B,n})(f,a)|\leq Cn^{-1}(\|f\|_\infty+|a|)^2, \tag{43}$$

where L^B is defined in expression (12).

Hence by the same argument as in Theorem 3.5, the laws of the sequence $(X^{B,n},\Lambda^{B,n})$ are tight in $D([0,\infty),\mathcal{M}_B\times\mathbb{R})$ (inequality (43) is the analogous to (31); inequality (41) is analogous to (34)).

Similarly, the laws of the sequence $(X^{B,n}, \Lambda^{B,n}, I^n, J^n)$ are tight in $D([0,\infty), \mathcal{M}_B \times \mathbb{R} \times \mathcal{M}_{B\times B\times B} \times \mathcal{M}_{B\times B\times B})$, where

$$I_t^n(d\omega_1, d\omega_2, d\omega_3) = K(\omega_1,\omega_2,\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\in B} X_t^{B,n}(d\omega_1)X_t^{B,n}(d\omega_2)X_t^{B,n}(d\omega_3),$$
$$J_t^n(d\omega_1, d\omega_2, d\omega_3) = K(\omega_1,\omega_2,\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\notin B} X_t^{B,n}(d\omega_1)X_t^{B,n}(d\omega_2)X_t^{B,n}(d\omega_3).$$

Let (X, Λ, I, J) be some weak limit point of the sequence. Passing to a subsequence and using the Skorokhod representation theorem A.5, we can consider that the sequence converges almost surely, i.e., as a pointwise limit in $D([0,\infty), \mathcal{M}_B \times \mathbb{R} \times \mathcal{M}_{B\times B\times B} \times \mathcal{M}_{B\times B\times B})$. Therefore, there exist bounded measurable functions

$$I, J : [0,\infty) \times B \times B \times B \to [0,\infty)$$

symmetric in the first two components, such that

$$I_t(d\omega_1, d\omega_2, d\omega_3) = I(t,\omega_1,\omega_2,\omega_3)X_t(d\omega_1)X_t(d\omega_2)X_t(d\omega_3),$$
$$J_t(d\omega_1, d\omega_2, d\omega_3) = J(t,\omega_1,\omega_2,\omega_3)X_t(d\omega_1)X_t(d\omega_2)X_t(d\omega_3),$$

in $\mathcal{M}_{B\times B\times B}$ and such that

$$I(t,\omega_1,\omega_2,\omega_3) = K(\omega_1,\omega_2,\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\in B},$$
$$J(t,\omega_1,\omega_2,\omega_3) = K(\omega_1,\omega_2,\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\notin B},$$

whenever $\omega_1 + \omega_2 - \omega_3 \notin \partial B$ (notice that we assumed K to be continuous).

Now, passing to the limit in (40) we obtain, for all continuous functions f and all $a \in \mathbb{R}$, for all $t \geq 0$, almost surely

$$\begin{aligned}\langle (f,a),(X_t,\Lambda_t)\rangle = \langle (f,a),(X_0,\Lambda_0)\rangle &\\ + \frac{1}{2}\int_0^t \int_{\mathbb{R}_+^3} \big(f(\omega_1+\omega_2-\omega_3) + f(\omega_3) - f(\omega_1) - f(\omega_2)\big) &\\ \times I(s,\omega_1,\omega_2,\omega_3)X_s(d\omega_1)X_s(d\omega_2)X_s(d\omega_3)\,ds &\\ + \frac{1}{2}\int_0^t \int_{\mathbb{R}_+^3} (a\varphi(\omega_1+\omega_2-\omega_3) + f(\omega_3) - f(\omega_1) - f(\omega_2)) &\\ \times J(s,\omega_1,\omega_2,\omega_3)X_s(d\omega_1)X_s(d\omega_2)X_s(d\omega_3)\,ds &\\ + \int_0^t \big(\Lambda_s^2 + 2\Lambda_s\langle\varphi, X_s\rangle\big)\int_{\mathbb{R}_+} (a\varphi(\omega) - f(\omega))\,\varphi(\omega)X_s(d\omega)\,ds.&\end{aligned}$$

Consider now an analogous iterative scheme to the one done in Proposition 2.1 for this equation. Denote by $(\nu_t^n)_{n\in\mathbb{N}}$ the sequence approximating $(X_t)_{t\geq 0}$. We deduce that

$$\nu_t^0 = \mu_0, \quad \nu_t^{n+1} \ll \mu_0 + \int_0^t (\nu_s^n + \nu_s^n * \nu_s^n * \hat{\nu}_s^n)\,ds$$

for $\hat{\nu}(A) = \nu(-A)$ and for all $n \geq 0$ (notice that we have extended the measures in the previous expression to the whole $\mathbb{R}$ by taking value 0 in subsets of $(-\infty, 0)$).

By induction we have that

$$\nu_t^n \ll \gamma_0 = \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} \nu_0^{*k} * \hat{\nu}_0^{*l}.$$

This implies in our case (taking $n \to \infty$) that $X_t \otimes X_t \otimes X_t$ is absolutely continuous with respect to $\gamma_0^{\otimes 3}$ for all $t \geq 0$, almost surely. For $G = \{(\omega_1,\omega_2,\omega_3)\,|\,\omega_1+\omega_2-\omega_3 \in \partial B\}$, we have that $\gamma_0^{\otimes 3}(G) = 0$ because of the assumptions on μ_0 and that $\gamma_0^{\otimes 3}(G) = (\gamma_0 * \gamma_0 * \hat{\gamma}_0)(G)$.

Therefore we can replace $I(t,\omega_1,\omega_2,\omega_3)$ by $K(\omega_1,\omega_2,\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\in B}$ and $J(t,\omega_1,\omega_2,\omega_3)$ by $K(\omega_1,\omega_2,\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\notin B}$. Since the equation obtained after this substitution is the same as (12) and (μ_t^B, λ_t^B) is its unique solution, we conclude that the unique weak limit point of $(X^{B,n}, \Lambda^{B,n})$ in $D([0,\infty), \mathcal{M}_B \times \mathbb{R})$ is precisely $(\mu_t^B, \lambda_t^B)_{t\geq 0}$. □

The rest of the proof of Theorem 1.8 is exactly the same one as in Ref. 11 [Theorem 4.4] without modification.

ACKNOWLEDGMENTS

This work was carried during the author's Ph.D. at the CCA (Cambridge Centre for Analysis) Doctoral Training Centre. This work and the author are greatly indebted to James Norris for all his guidance, teaching, and corrections.

Thanks to Clément Mouhot for suggesting to work on the wave kinetic equation and useful discussions. Thanks to Martin Taylor for pointing at a useful report on "The Spatially Homogenous Boltzmann Equation and Particle Dynamics." Thanks to Amit Einav and Colm Connaughton for useful discussions.

During this project, the author was supported by the UK Engineering and Physical Sciences Research Council (EPSRC) Grant No. EP/H023348/1 for the University of Cambridge Centre for Doctoral Training, the Cambridge Centre for Analysis; and the ERC starting grant "Mathematical Aspects of Kinetic Theory (MATKIT)." She was also supported in part by EPSRC under Grant Reference No. EP/M006883/1 at Imperial College London.

Data statement: No new data were produced during this research.

APPENDIX A: SOME PROPERTIES OF THE SKOROKHOD SPACE

Theorem A.1 (Prohorov's theorem (Ref. 6, Chapter 3)). *Let (S,d) be complete and separable, and let $\mathcal{M} \in \mathcal{P}(S)$. Then the following are equivalent:*

1. *$\mathcal{M}$ is tight.*
2. *For each $\varepsilon > 0$, there exists a compact $K \in S$ such that*
$$\inf_{P\in\mathcal{M}} P(K^\varepsilon) \geq 1-\varepsilon,$$
where $K^\varepsilon := \{x \in S : \inf_{y\in K} d(x,y) < \varepsilon\}$.
3. *$\mathcal{M}$ is relatively compact.*

Let (E,r) be a metric space. The space $D([0,\infty);E)$ of càdlàg functions taking values in E is widely used in stochastic processes. In general we would like to study the convergence of measures on this space; however, most of the tools known for convergence of measures are for measures in $\mathcal{P}(S)$ for S a complete separable metric space. Therefore, it would be very useful to find a topology in $D([0,\infty)\times E)$ such that it is a complete and separable metric space. This can be done when E is also complete and separable; and the metric considered is the Skorokhod one. This is why in this case the space of càdlàg functions is called Skorohod space.

Some important properties of this space are the following.

Proposition A.2 (Ref. 6, Chapter 3). If $x \in D([0,\infty);E)$, then x has at most countably many points of discontinuity.

Theorem A.3 (Ref. 6, Chapter 3). *If E is separable, then $D([0,\infty);E)$ is separable. If (E,r) is complete, then $(D([0,\infty);E),d)$ is complete, where d is the Skorokhod metric.*

Theorem A.4. *The Skorokhod space is a complete separable metric space.*

Theorem A.5 (The almost sure Skorokhod representation theorem, Ref. 6, Theorem 1.8, Chapter 3). *Let (S,d) be a separable metric space. Suppose P_n, $n = 1,2,\ldots$ and P in $\mathcal{P}(S)$ satisfy $\lim_{n\to\infty}\rho(P_n,P) = 0$ where ρ is the metric in $\mathcal{P}(S)$. Then there exists a probability space $(\Omega,\mathcal{F},\nu)$ on which are defined S-valued random variable X_n, $n = 1,2,\ldots$ and X with distributions P_n, $n = 1,2,\ldots$ and P, respectively, such that $\lim_{n\to\infty} X_n = X$ almost surely.*

Theorem A.6 (Tightness criteria for measures on the Skorokhod space, Ref. 7 Theorem 3.1). *Let $(S,\mathcal{T})$ be a completely regular topological space with metrisable compact sets. Let $\mathbb{G}$ be a family of continuous functions on S. Suppose that $\mathbb{G}$ separates points in S and that it is closed under addition. Then a family $\{\mathcal{L}^n\}_{n\in\mathbb{N}}$ of probability measures in $\mathcal{P}(D([0,\infty);S)$ is tight if and only if the two following conditions hold:*

(i) *For each* $\varepsilon > 0$ *there is a compact set* $K_\varepsilon \subset S$ *such that*

$$\mathcal{L}^n(D([0,\infty); K_\varepsilon)) > 1 - \varepsilon, \quad n \in \mathbb{N}.$$

(ii) *The family* $\{\mathcal{L}^n\}_{n\in\mathbb{N}}$ *is* $\mathbb{G}$*-weakly tight.*

Theorem A.7 (Criteria for tightness in Skorokhod spaces (Ref. 6, Corollary 7.4, Chapter 3)). *Let* (E,r) *be a complete and separable metric space, and let* $\{X_n\}$ *be a family of processes with sample paths in* $D([0,\infty); E)$*. Then* $\{X_n\}$ *is relatively compact if and only if the two following conditions hold:*

(i) *For every* $\eta > 0$ *and rational* $t \geq 0$*, there exists a compact set* $\Lambda_{\eta,t} \subset E$ *such that*

$$\liminf_{n\to\infty} \mathbb{P}\{X_n(t) \in \Lambda^\eta_{\eta,t}\} \geq 1 - \eta.$$

(ii) *For every* $\eta > 0$ *and* $T > 0$*, there exists* $\delta > 0$ *such that*

$$\limsup_{n\to\infty} \mathbb{P}\{w'(X_n, \delta, T) \geq \eta\} \leq \eta,$$

where we have used the modulus of continuity w' *defined as follows for* $x \in D([0,\infty) \times E)$*,* $\delta > 0$*, and* $T > 0$*:*

$$w'(x,\delta,T) = \inf_{\{t_i\}} \max_i \sup_{s,t\in[t_{i-1},t_i)} r(x(s),x(t)),$$

where $\{t_i\}$ *ranges over all partitions of the form* $0 = t_0 < t_1 < \cdots < t_{n-1} < T \leq t_n$ *with* $\min_{1\leq i\leq n}$ $(t_i - t_{i-1}) > \delta$ *and* $n \geq 1$.

Theorem A.8 (Continuity criteria for the limit in Skorokhod spaces (Ref. 6, Theorem 10.2, Chapter 3)). *Let* (E,r) *be a metric space. Let* X_n*,* $n = 1,2,\ldots$*, and* X *be processes with sample paths in* $D([0,\infty); E)$ *and suppose that* X_n *converges in distribution to* X*. Then* X *is a.s. continuous if and only if* $J(X_n)$ *converges to zero in distribution, where*

$$J(x) = \int_0^\infty e^{-u}[J(x,u) \wedge 1]\, du$$

for

$$J(x,u) = \sup_{0\leq t\leq u} r(x(t), x(t-)).$$

APPENDIX B: FORMAL DERIVATION OF THE WEAK ISOTROPIC 4-WAVE KINETIC EQUATION

Suppose that $n(\mathbf{k}) = n(k)$ is a radial function (isotropic).

The waveaction in the isotropic case can be written as

$$W = \int_{\mathbb{R}^N} n(\mathbf{k}) d\mathbf{k} = \int_{\mathbb{R}_+\times S^{N-1}} n(k) k^{N-1} dk\, d\mathbf{s} = \frac{|S^{N-1}|}{\alpha} \int_0^\infty n(\omega)\omega^{\frac{N-\alpha}{\alpha}}\, d\omega,$$

where S^{N-1} is the $N-1$ dimensional sphere. From this expression, one can denote the angle-averaged frequency spectrum $\mu = \mu(d\omega)$ as

$$\mu(d\omega) := \frac{|S^{N-1}|}{\alpha} \omega^{\frac{N-\alpha}{\alpha}} n(\omega) d\omega.$$

The total number of waves (waveaction) and the total energy are, respectively,

$$W = \int_0^\infty \mu(d\omega),$$

$$E = \int_0^\infty \omega\mu(d\omega).$$

Next we explain the formal derivation of the weak isotropic 4-wave kinetic Equation (5). We have that

$$\begin{aligned}\int_{(0,\infty)} \partial_t \mu(\omega) d\omega &= \int_{\mathbb{R}^N} \partial_t n(\mathbf{k}) d\mathbf{k} \\ &= \int_{\mathbb{R}^4_+} d\omega_{0123} F(\omega_1,\omega_2,\omega_3,\omega)\delta(\omega_1+\omega_2-\omega_3-\omega) \\ &\qquad \times (\mu(\omega_1)\mu(\omega_2)\mu(\omega_3)\omega^{\frac{N-\alpha}{\alpha}} + \mu(\omega_1)\mu(\omega_2)\mu(\omega)\omega_3^{\frac{N-\alpha}{\alpha}} \\ &\qquad - \mu(\omega_1)\mu(\omega_3)\mu(\omega)\omega_2^{\frac{N-\alpha}{\alpha}} - \mu(\omega_2)\mu(\omega_3)\mu(\omega)\omega_1^{\frac{N-\alpha}{\alpha}})\end{aligned}$$

for $S^i = (S^{N-1})^i$, $d\omega_{0123} = d\omega d\omega_1 d\omega_2 d\omega_3$, $ds_{0123} = ds_1 ds_2 ds_3 ds$, and

$$\begin{aligned}F(\omega_1,\omega_2,\omega_3,\omega) = \frac{4\pi}{\alpha|S^{N-1}|^4}\int_{S^4} ds_{0123}\overline{T}^2(\omega_1^{1/\alpha}s_1,\omega_2^{1/\alpha}s_2,\omega_3^{1/\alpha}s_3,\omega^{1/\alpha}s) \\ \times \delta(\omega_1^{1/\alpha}s_1+\omega_2^{1/\alpha}s_2-\omega_3^{1/\alpha}s_3-\omega^{1/\alpha}s).\end{aligned}$$

Hence, μ_ω satisfies

$$\partial_t \mu(\omega) = \int_{\mathbb{R}^3_+} d\omega_{123} F(\omega_1,\omega_2,\omega_3,\omega)\delta(\omega_1+\omega_2-\omega_3-\omega) \tag{B1}$$

$$\begin{aligned}&\times (\mu(\omega_1)\mu(\omega_2)\mu(\omega_3)\omega^{\frac{N-\alpha}{\alpha}} + \mu(\omega_1)\mu(\omega_2)\mu(\omega)\omega_3^{\frac{N-\alpha}{\alpha}} \\ &- \mu(\omega_1)\mu(\omega_3)\mu(\omega)\omega_2^{\frac{N-\alpha}{\alpha}} - \mu(\omega_2)\mu(\omega_3)\mu(\omega)\omega_1^{\frac{N-\alpha}{\alpha}}).\end{aligned} \tag{B2}$$

Its weak formulation

$$\mu_t = \mu^{in} + \int_{\Omega^3} Q(\mu_s,\mu_s,\mu_s)\, ds$$

is defined against functions $g \in \mathcal{S}(\mathbb{R}_+)$ and using the symmetry of $\bar{T}$ we conclude that

$$\begin{aligned}\langle g, Q(\mu,\mu,\mu)\rangle = \frac{1}{2}\int_D d\omega_{123}\mu(\omega_1)\mu(\omega_2)\mu(\omega_3)K(\omega_1,\omega_2,\omega_3) \\ \times [g(\omega_1+\omega_2-\omega_3)+g(\omega_3)-g(\omega_2)-g(\omega_1)],\end{aligned} \tag{B3}$$

where

$$K(\omega_1,\omega_2,\omega_3) := 2(\omega_1+\omega_2-\omega_3)^{\frac{N-\alpha}{\alpha}} F(\omega_1,\omega_2,\omega_3,\omega_1+\omega_2-\omega_3).$$

Remark B.1. In Ref. 18 [Section 3.1.3], the authors state that even in isotropic medium, the interaction coefficient $\overline{T}$ in the 4-wave case cannot be considered to be isotropic too. We can rewrite

$$|\overline{T}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k})|^2 = \overline{T}_0^2 k^{2\beta} f_2\left(\frac{\mathbf{k}_1}{k},\frac{\mathbf{k}_2}{k},\frac{\mathbf{k}_3}{k}\right) \tag{B4}$$

for some dimensionless constant $\overline{T}_0$ and some dimensionless function f_2.

[1] Coca, A., Gataric, M., Maczynski, K., Taylor, M., and Wang, Z., "The spatially homogeneous boltzmann equation and particle dynamics," in *CCA Stochastic Project* (2012).
[2] Connaughton, C., "Numerical solutions of the isotropic 3-wave kinetic equation," Phys. D **238**(23), 2282–2297 (2009).
[3] Darling, R. W. R., Norris, J. R. *et al.*, "Differential equation approximations for markov chains," Probab. Surv. **5**, 37–79 (2008).
[4] Eibeck, A. and Wagner, W., "Approximative solution of the coagulation–fragmentation equation by stochastic particle systems," Stochastic Anal. Appl. **18**(6), 921–948 (2000).
[5] Escobedo, M., Mischler, S., and Rodriguez Ricard, M., "On self-similarity and stationary problem for fragmentation and coagulation models," Ann. Inst. Henri Poincare C Nonlinear Anal. **22**, 99–125 (2005).
[6] Ethier, S. N. and Kurtz, T. G., *Markov Processes: Characterization and Convergence* (John Wiley & Sons, 2009), Vol. 282.
[7] Jakubowski, A., "On the skorokhod topology," in *Annales de l'IHP Probabilités et Statistiques* (Elsevier, 1986), Vol. 22, pp. 263–285.
[8] Laurençot, P. and Mischler, S., "From the discrete to the continuous coagulation–fragmentation equations," Proc. R. Soc. Edinburgh: Sect. A Math. **132**(05), 1219–1248 (2002).

[9] Mischler, S. and Cañizo, J., "Uniqueness and regularity of scaling profiles for smoluchowski's coagulation equation," Rev. Mat. Iberoam. **27**(3), 803–839 (2011).
[10] Nazarenko, S., *Wave Turbulence* (Springer Science & Business Media, 2011), Vol. 825.
[11] Norris, J. R., "Smoluchowski's coagulation equation: Uniqueness, nonuniqueness and a hydrodynamic limit for the stochastic coalescent," Ann. Appl. Probab. **9**, 78–109 (1999).
[12] Norris, J. R., "Cluster coagulation," Commun. Math. Phys. **209**(2), 407–435 (2000).
[13] Peierls, R., "Zur kinetischen theorie der wärmeleitung in kristallen," Ann. Phys. **395**(8), 1055–1101 (1929).
[14] Scott, A. *et al.*, *Encyclopedia of Nonlinear Science* (Routledge, 2006).
[15] Wagner, W. *et al.*, "Explosion phenomena in stochastic coagulation–fragmentation models," Ann. Appl. Probab. **15**(3), 2081–2112 (2005).
[16] Zakharov, V., "Statistical theory of gravity and capillary waves on the surface of a finite-depth fluid," Eur. J. Mech.-B/Fluids **18**(3), 327–344 (1999).
[17] Zakharov, V., Dias, F., and Pushkarev, A., "One-dimensional wave turbulence," Phys. Rep. **398**(1), 1–65 (2004).
[18] Zakharov, V. E., L'Vov, V. S., and Falkovich, G., "Kolmogorov spectra of turbulence 1. Wave turbulence," in *Kolmogorov Spectra of Turbulence 1. Wave Turbulence*, edited by Zakharov, V. E., L'vov, V. S., and Falkovich, G. (Springer, Berlin, Germany, 1992), Vol. 1, p. 275, ISBN: 3-540-54533-6.